MODELING INVESTMENT RETURNS WITH A MULTIVARIATE ORNSTEIN-UHLENBECK PROCESS

by

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Abstract

A multivariate Ornstein-Uhlenbeck process is used to model the returns on different investment instruments. Model parameters are estimated under the principle of covariance equivalence. Fitted models can be used to price insurance products and analyze the risk associated with different asset allocation strategies. To illustrate the results obtained, an annuity is studied when assets are allocated between equity and two types of bonds. To show the importance of using a multivariate model, annuity prices are compared to those obtained from independent univariate processes.

Keywords: Annuity Pricing; Multivariate; Univariate; Ornstein-Uhlenbeck (OU) Process; AR(1) Process; Asset Allocation Strategy

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Chapter 1

Introduction

Modeling rates of return is a topic that has been studied for many years. Forecasting rates of return is important for insurance products, especially long term products, such as life insurance and annuities. Deterministic models are much easier to use, but stochastic models have become more and more popular. In Markowitz (1952), the rate of return on a security or a portfolio is considered as a random variable. In Boyle (1976), a White Noise is used to model the rate of return for each year. However, annual rates of return are independent from each other, which is not a very realistic assumption for most assets. In the stock market, one might consider the rates of return to be somewhat independent, but it is certainly not the case for the rates of return on assets such as long term bonds.

Panjer and Bellhouse (1980) and Bellhouse and Panjer (1981) talked about modeling interest rates and applications to life contingencies. The univariate stochastic models for interest rates they studied include White Noise, autoregressive process and OU process. They derived the moment generating functions and functions of mean, variance and covariance of the rate of return over time t in a general form for both continuous and discrete time frameworks. The stochastic models for interest rates were also applied in calculating net single premiums for a whole life insurance policy and a life annuity. They generalized the results found in Boyle (1976) by using models with dependent the interest rate fluctuations. Furthermore, in Bellhouse and Panjer (1981), they derived the results under a conditional autoregressive model for rate of return.

Unlike the previous authors, who modeled the rate of return, some people chose to model the rate of return accumulation function as a stochastic process. For example, Dhaene (1989) developed a method to calculate moments of rate of return and insurance functions when interest rates are assumed to follow an autoregressive integrated moving average process, ARIMA(p,d,q). In Parker(1994), different approaches are studied to model the rate of return and the rate of return accumulation function. Based on theoretical results and numerical values, modeling the rate of return directly seems like a more reasonable way than modeling the accumulation function.

In Parker (1995) a second order stochastic differential equation (SDE) is used to model the rates of return. The second order SDE is a continuous process whose discrete-time analogue is an ARMA(2,1) process. In his paper, Parker derived the expected values and autocovariance functions of the rates of return and of the rates of return accumulation function. Though this is still a univariate model, it is one step closer to a multivariate model, because in a bivariate vector AR(1) model, each variable has a univariate ARMA(2,1) model representation. An example of finding the univariate representation for a bivariate vector AR(1) model can be found in the book by Reinsel (1997, pp 30-34).

One common thing about those papers is that the rate of return is always modeled as a Gaussian process so the accumulation factor or the discounting factor follows a lognormal distribution. Also, all the models mentioned above are one dimensional. So, when we consider the rate of return of a portfolio made up of several assets, each asset is modeled separately and by doing so it is assumed that the assets are independent at all times. Such an assumption is not always realistic. Therefore, we would like to study a multivariate stochastic process to model rates of return for some assets together and compare the results with modeling each asset with a univariate model separately.

Let us consider two static asset allocation strategies. One is to determine an initial proportion to be invested in each asset and keeping the initial amount of money in that asset thereafter, which means that there is no rebalancing happening in the future. The other one is rebalancing the portfolio at a certain frequency according to a predetermined asset allocation. When the model for the rates of return is in a continuous time frame, we assume the investment is being rebalanced frequently. So a given percentage of the total asset invested in each asset is maintained over time. For example, consider n assets with a percentage w_i invested in asset i and a rate of return on asset i of δ_i. Therefore, for each dollar invested now, the accumulated value at time t is $\sum_{i=1}^{n} w_i e^{\int_0^t \delta_i dt}$ without rebalancing and $e^{\sum_{i=1}^{n} w_i \int_0^t \delta_i dt}$ if rebalancing frequently. In this project, we chose to rebalance the investment. Rebalancing can make sure that the investment stays well diversified overtime and therefore should be better for risk control.

Given the asset allocation strategy we chose, there are a few different approaches to model the rate of return of a portfolio with multivariate or univariate models. One way is to model the rate of return for each one of the assets by a univariate process, then assign a weight on each asset and calculate the total rate of return of the portfolio each year. This method is fast and easy. However, the correlation between the assets is ignored. Such correlation can make a significant impact on forecasting the rates of return. Another way is to calculate the total return of the portfolio each year with preassigned weight on each asset, then using a one dimensional stochastic process to model the portfolio's rate of return. This approach can somewhat take the correlation between the assets into consideration, but the characteristics of the portfolio are hard to capture fully with such a simple model. The third approach uses a multivariate stochastic process to model the rates of return for all the assets in a portfolio at once. Then we can consider not only the serial dependence for those assets, but also the correlations between assets. Introducing more parameters into the model allows more flexibility for the process as a better model of the portfolio's rates of return.

The Ornstein-Uhlenbeck (OU) process, also known as the Vasicek model, which is a mean reverting Gaussian process, has been used to model rates of return for many years. In this project, we use a multivariate Ornstein-Uhlenbeck process to model the rates of return for three assets. As a comparison, we model the rates of return of the three assets with three separate univariate Ornstein-Uhlenbeck processes. The rates of return are forecasted conditional on the starting value. Whole life annuities for males age 65 are priced using simulated rates of return with both univariate and multivariate models. Besides the models we considered for the rates of return, the asset allocation strategy is another important factor that affects the total rates of return of the portfolio. With different models for the rates of return, the asset allocation that results in the lowest annuity price is also different.

In Chapter 2, we present both the Ornstein-Uhlenbeck process and the AR(1) process. Also we derive formulas to convert an AR(1) model to its covariance equivalent OU process. Then, in Chapter 3 we estimate the parameters of the two models with the data we collected from three assets. Conditional on the starting value of the rates of return, we do some simulation and annuity pricing with the two different models and different asset allocation strategies in Chapter 4. At last, the conclusions are presented in Chapter 5.

Chapter 2

Models

A first order autoregressive model not only expresses a series' current value against its previous values, but it is also able to capture the series' characteristic of mean reversion, which is an important feature of interest rates. It is generally accepted that interest rates tend to move back towards a long term value, which could be the historical average or other reasonable values the user would like to choose. The Ornstein-Uhlenbeck process, also known as the Vasicek model, is the continuous-time analogue of the discrete-time AR(1) process. The parametric relations between these two models can be determined by the principle of covariance equivalence, which states that a discrete representation of a continuous system can be found by requiring that the covariance of the discrete model coincide with that of the continuous model at the sampling points. In other words, the two processes need to match their first two moments at all time. For a given OU process, a discrete representation by an AR(1) process can always be found, but the other way is not always possible. When modeling interest rates or rates of return, both processes, discrete or continuous, can be used. In this project, we consider interest rates and stock indices as continuous processes. However, such data can only be collected at a certain frequency in discrete-time frame. Therefore, when we estimate the parameters, we will use the collected data to fit an AR(1) model. Then based on our need, we can convert the AR(1) process to an equivalent continuous OU process, or just study the discrete process.

Using univariate AR(1) processes to model rates of return has been well studied. For example, we know the explicit expression for its covariance between the values at any two time points, t and s; we also know how to determine its covariance equivalent OU process and the criteria to make sure an AR(1) process has a covariance equivalent OU process and so on. However, there is not much done about modeling rates of return with multivariate AR(1) and OU process. In the next section, we will review univariate AR(1) and OU processes. Then we will extend some key results to multivariate AR(1) and OU processes, especially the covariance matrix over time and show how to apply the principle of covariance equivalence to convert a multivariate AR(1) process to an OU process.

2.1 Univariate Model

There are many textbooks and papers in the literature talking about univariate AR(1) and OU processes. The reference we used for this section is Pandit and Wu (1983). A brief review is given in this section, including the key properties of the processes and the parametric relations determined by the principle of covariance equivalence.

2.1.1 AR(1) process

Suppose that variable X_t is a time series with a mean of 0. If not, we would study the variable $X_t' = X_t - \mathrm{E}(X_t)$, which is centered around its mean.

If X_t is an AR(1) , then

$$X_t = \Phi X_{t-1} + a_t, \tag{2.1}$$

where a_t's are independent and identically distributed following the normal distribution with mean 0 and variance σ_a^2, i.e. N(0, σ_a^2).

If the starting value is given as X_0, then for $t = 1, 2, \ldots$

$$X_t = \Phi^t X_0 + \sum_{j=0}^{t-1} \Phi^j a_{(t-j)}. \tag{2.2}$$

Therefore,

$$\begin{aligned} \mathrm{E}(X_t|X_0) &= \Phi^t X_0, && (2.3) \\ \mathrm{Var}(X_t|X_0) &= \frac{1-\Phi^{2t}}{1-\Phi^2}\sigma_a^2 && (2.4) \end{aligned}$$

and

$$\mathrm{Cov}(X_t, X_{t-k}|X_0) = \Phi^k \frac{1-\Phi^{2(t-k)}}{1-\Phi^2}\sigma_a^2, \quad (2.5)$$

where $k \leq t$ and k is an integer. When $|\Phi| < 1$, the AR(1) process is stationary, which means its first and second moments exist as time t tends to infinity. And it is important to make sure the process we will study is stationary. Non stationary processes are not in the scope of the project.

2.1.2 Ornstein-Uhlenbeck Process

Velocity Process

Now let us take a look at the continuous Ornstein-Uhlenbeck process. The OU velocity process is also known as a Vasicek model. For X_t with mean 0, consider the following stochastic differential equation (SDE) with starting value X_0

$$dX_t = -\alpha X_t + \sigma dW_t, \quad (2.6)$$

where W_t is a standard Brownian Motion and α describes “the speed of the reversion”, i.e. how fast the process goes towards its long term mean from the given starting value. The larger the α is, the faster the reversion is. The parameter σ measures instant by instant the amplitude of randomness entering the system. The larger the value of σ, the higher the volatility of the system. The solution of this SDE is

$$X_t = e^{-\alpha t}X_0 + \sigma \int_0^t e^{-\alpha(t-s)}dW_s. \quad (2.7)$$

The first two moments are calculated as

$$\begin{aligned} \mathrm{E}(X_t) &= e^{-\alpha t}X_0, && (2.8) \\ \mathrm{Var}(X_t) &= e^{-2\alpha t}(\sigma^2 \frac{e^{2\alpha t}-1}{2\alpha}) && (2.9) \end{aligned}$$

and

$$\mathrm{Cov}(X_t, X_s) \quad = \quad e^{-\alpha(t+s)}(\sigma^2 \frac{e^{2\alpha \min(s,t)} - 1}{2\alpha}). \tag{2.10}$$

In the above expressions, we assume X_t's long term mean is 0. If not, suppose X_t's long term mean is C, then Equation (2.6) is changed to

$$dX_t = -\alpha(X_t - C) + \sigma dW_t. \tag{2.11}$$

And we should study the new variable $X'_t = X_t - C$ instead, whose long term mean is 0.

Position Process

The Ornstein-Uhlenbeck position process Y_t is obtained by integrating the velocity process X_t. So, we have

$$Y_t = Y_0 + \int_0^t X_s ds. \tag{2.12}$$

We calculate the first two moments of Y_t as

$$\begin{aligned} \mathrm{E}(Y_t) &= \mathrm{E}(Y_0 + \int_0^t X_s ds) \\ &= \mathrm{E}(Y_0) + \mathrm{E}(X_0)\frac{1 - e^{-\alpha t}}{\alpha} \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \mathrm{Cov}(Y_t, Y_s) &= \mathrm{Cov}(Y_0 + \int_0^t X_r dr, Y_0 + \int_0^s X_u du) \\ &= \mathrm{Var}(Y_0) + \mathrm{Var}(X_0)\frac{1 - e^{\alpha s} - e^{-\alpha t} + e^{-\alpha(t+s)}}{\alpha^2} + \frac{\sigma^2 min(s,t)}{\alpha^2} \\ &\quad + \frac{\sigma^2(-2 + 2e^{-\alpha t} + 2e^{-\alpha s} - e^{-\alpha|t-s|} - e^{-\alpha(t+s)})}{2\alpha^3}. \end{aligned} \tag{2.14}$$

To obtain the expression for Y_t, we consider the system of SDE

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} d\begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix}. \tag{2.15}$$

The solution is

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} e^{-\alpha t} & 0 \\ \frac{1-e^{-\alpha t}}{\alpha} & 0 \end{pmatrix}\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-\alpha(t-s)} & 0 \\ \frac{1-e^{-\alpha(t-s)}}{\alpha} & 0 \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} d\begin{pmatrix} W_{1,s} \\ W_{2,s} \end{pmatrix}. \tag{2.16}$$

Therefore, we have

$$Y_t = \frac{1 - e^{-\alpha t}}{\alpha} X_0 + \sigma \int_0^t \frac{1 - e^{-\alpha(t-s)}}{\alpha} dW_{1,s}. \tag{2.17}$$

2.1.3 Equivalent AR(1) and OU processes

Normally, interest rates and stock indices are observed at uniform sampling intervals. When the intervals are small enough, we can approximately consider them as continuous processes, which is what we did in this project. Therefore, the continuous process is obtained through its covariance equivalent discrete model. In this section, we will look at their parametric relations.

With explicit expressions for the covariance between the observations at any two time points for both AR(1) and OU processes, we can determine the parametric relations for those two processes by matching their covariance at all time. The covariance of the OU process is calculated as in Equation (2.10). Assume the system is sampled at a frequency of Δ. Then the covariance between the observations at time $t\Delta$ and time $t\Delta - k\Delta$ is

$$\mathrm{Cov}(X_{t\Delta}, X_{t\Delta - k\Delta}) = \frac{\sigma^2}{2\alpha} e^{-\alpha k \Delta}(1 - e^{-2\alpha(t\Delta - k\Delta)}). \tag{2.18}$$

For an AR(1) process, the covariance between observations t and $t - k$ is

$$\mathrm{Cov}(X_t, X_{t-k}) = \Phi^k \frac{\sigma_a^2}{1 - \Phi^2}(1 - \Phi^{2(t-k)}). \tag{2.19}$$

By matching the coefficients, we have

$$\Phi = e^{-\alpha\Delta} \tag{2.20}$$

and

$$\frac{\sigma_a^2}{1 - \Phi^2} = \frac{\sigma^2}{2\alpha}. \tag{2.21}$$

Alternatively, we can find the relation between Φ and α by matching the first moments of the two processes. As showed in Equations (2.3) and (2.8), we have $\Phi = e^{-\alpha\Delta}$. As we explained before, not every AR(1) process has a continuous representation. By looking at their parametric relations, we can see that to satisfy Equation (2.20), we must have $\Phi > 0$.

Since we limit our study to stationary processes, then we also need $\Phi < 1$ and $\alpha > 0$ to satisfy Equation (2.21).

The parametric relations and those conditions are easy to find for a univariate model. For a multivariate model, the principle and the general idea are the same, but the calculations are much more complicated.

2.2 Multivariate Model

When considering several series, instead of modeling each series with a univariate process, a multivariate model is used to model all series simultaneously. As an improvement of the combination of several univariate processes, this multivariate model can not only express the serial dependence, but also express the dependence among different series. In this section, both discrete and continuous multivariate models are studied, as well as their parametric relations to satisfy the principle of covariance equivalence.

2.2.1 Vector AR(1) Process

First, for the discrete model, consider the following vector AR(1) process

$$\underline{X}_t = \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ \vdots \\ X_{n,t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_{nn} \end{pmatrix} \begin{pmatrix} X_{1,(t-1)} \\ X_{2,(t-1)} \\ \vdots \\ X_{n,(t-1)} \end{pmatrix} + \begin{pmatrix} a_{1,t} \\ a_{2,t} \\ \vdots \\ a_{n,t} \end{pmatrix}, \tag{2.22}$$

where $\begin{pmatrix} a_{1,t} \\ a_{2,t} \\ \vdots \\ a_{n,t} \end{pmatrix}$ follows a multivariate normal distribution with mean $\underline{\mu} = \underline{0}$ and covariance matrix $\underline{\Sigma}_a$. If given a starting value of $\underline{X}_0$, the vector AR(1) process can be also written as

$$\underline{X}_t = \underline{\Phi}^t \underline{X}_0 + \sum_{j=0}^{t-1} \underline{\Phi}^j \underline{a}_{t-j}. \tag{2.23}$$

When conditioning on an initial value $\underline{X}_0$, the mean of $\underline{X}_t$ is

$$\mathrm{E}(\underline{X}_t|X_0) = \underline{\Phi}^t\underline{X}_0. \tag{2.24}$$

The covariance between $\underline{X}_t$ and $\underline{X}_{t-k}$, where k is a non negative integer, also conditional on $\underline{X}_0$, can be calculated as

$$\begin{aligned}
\mathrm{Cov}\left(\underline{X}_t, \underline{X}_{t-k}|\underline{X}_0\right) &= \mathrm{Cov}\left(\underline{\Phi}^t\underline{X}_0 + \sum_{j=0}^{t-1}\underline{\Phi}^j\underline{a}_{t-j}, \underline{\Phi}^{t-k}\underline{X}_0 + \sum_{i=0}^{t-k-1}\underline{\Phi}^i\underline{a}_{t-k-i}\middle| X_0\right)\\
&= \mathrm{Cov}\left(\sum_{j=0}^{t-1}\underline{\Phi}^j\underline{a}_{t-j}, \sum_{i=0}^{t-k-1}\underline{\Phi}^i\underline{a}_{t-k-i}\right)\\
&= \sum_{j=0}^{t-1}\sum_{i=0}^{t-k-1}\mathrm{Cov}\left(\underline{\Phi}^j\underline{a}_{t-j}, \underline{\Phi}^i\underline{a}_{t-k-i}\right)\\
&= \sum_{j=0}^{t-1}\sum_{i=0}^{t-k-1}E\left(\left(\underline{\Phi}^j\underline{a}_{t-j}\right)\left(\underline{\Phi}^i\underline{a}_{t-k-i}\right)^T\right)\\
&= \sum_{i=0}^{t-k-1}\underline{\Phi}^{k+i}\underline{\Sigma}_a\left(\underline{\Phi}^i\right)^T.
\end{aligned}$$

So we have

$$\mathrm{Cov}\left(\underline{X}_t, \underline{X}_{t-k}|\underline{X}_0\right) = \sum_{i=0}^{t-k-1}\underline{\Phi}^{k+i}\underline{\Sigma}_a\left(\underline{\Phi}^i\right)^T. \tag{2.25}$$

2.2.2 Ornstein-Uhlenbeck Process

In this section, a multivariate Ornstein-Uhlenbeck process is studied in continuous time.

Velocity Process

Consider the following general multivariate OU process,

$$d\begin{pmatrix}X_{1,t}\\X_{2,t}\\\vdots\\X_{n,t}\end{pmatrix} = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \dots & \alpha_{1n}\\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n}\\ \vdots & \vdots & \dots & \vdots\\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn}\end{pmatrix}\begin{pmatrix}X_{1,t}\\X_{2,t}\\\vdots\\X_{n,t}\end{pmatrix}dt + \begin{pmatrix}\sigma_{11} & \sigma_{12} & \dots & \sigma_{1n}\\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n}\\ \vdots & \vdots & \dots & \vdots\\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn}\end{pmatrix}d\begin{pmatrix}W_{1,t}\\W_{2,t}\\\vdots\\W_{n,t}\end{pmatrix}, \tag{2.26}$$

where $\underline{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix}$ is the diffusion matrix, and $\begin{pmatrix} W_{1,t} \\ W_{2,t} \\ \vdots \\ W_{n,t} \end{pmatrix}$ is a vector of n independent standard Brownian Motions. For each standard Brownian Motion, $W_{i,t}$, in a small time step dt, $dW_{i,t}$ follows a normal distribution with mean 0 and variance dt. With this result, it is possible to use a lower triangular matrix $\underline{\sigma}'$, instead of the $\underline{\sigma}$ in Equation (2.26), so that $\underline{\sigma}' \cdot d\underline{W}_t$ has the same distribution as $\underline{\sigma} \cdot d\underline{W}_t$.

We have

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix} \begin{pmatrix} dW_{1,t} \\ dW_{2,t} \\ \vdots \\ dW_{n,t} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} \sigma_{1k} \cdot dW_{k,t} \\ \sum_{k=1}^{n} \sigma_{2k} \cdot dW_{k,t} \\ \vdots \\ \sum_{k=1}^{n} \sigma_{nk} \cdot dW_{k,t} \end{pmatrix}, \tag{2.27}$$

where $dW_{k,t}$, for all integers k ($k = 1, 2, \dots, n$), are i.i.d. normal distributions with mean 0 and variance dt. Each element in $\underline{\sigma} \cdot d\underline{W}_t$ is a linear combination of normal distributions $dW_{k,t}$ that still follows a normal distribution. For the ith element in $\underline{\sigma} \cdot d\underline{W}_t$, it follows a normal distribution with mean 0 and variance $\sum_{k=1}^{n} \sigma_{ik}^2 \cdot dt$. The result for $\underline{\sigma}' \cdot d\underline{W}_t$ with the lower triangular matrix $\underline{\sigma}'$ is almost the same. Each element in $\underline{\sigma}' \cdot d\underline{W}_t$ follows a normal distribution with mean 0 and variance $\sum_{k=1}^{n} {\sigma'_{ik}}^2 \cdot dt$, except $\sigma'_{ik} = 0$ when $i < k$. Let $\sum_{k=1}^{n} {\sigma'_{ik}}^2 = \sum_{k=1}^{n} {\sigma_{ik}}^2$, then $\underline{\sigma}' \cdot d\underline{W}_t$ has the same distribution as $\underline{\sigma} \cdot d\underline{W}_t$. The purpose of doing this is not to do the conversion between $\underline{\sigma}'$ and $\underline{\sigma}$, but to state the fact that we can use a lower triangular matrix with fewer parameters to obtain the same distribution. Using a lower triangular matrix is more convenient when we determine the parametric relations between $\underline{\Phi}$, $\underline{\Sigma}_a$ in the AR(1) process and $\underline{A}$, $\underline{\sigma}$.

Let

$$\underline{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix}.$$

The solution of the system of SDE in Equation (2.26) is

$$\underline{X}_t = e^{\underline{A}t}\underline{X}_0 + \int_0^t e^{\underline{A}(t-s)}\underline{\sigma} d\underline{W}_s. \tag{2.28}$$

$\underline{A}$ is related to the time that the process will take to go back towards its long term mean from the given starting value, except that in the multivariate model, the speed is a combined effect of all series included in the system. And $\underline{\sigma}$ measures the instant randomness from all series.

The mean of $\underline{X}_t$, conditional on the initial value $\underline{X}_0$, is

$$\mathrm{E}(\underline{X}_t|X_0) = e^{\underline{A}t}\underline{X}_0. \tag{2.29}$$

The covariance of $\underline{X}_s$ and $\underline{X}_t$ can be calculated as

$$\begin{aligned}\mathrm{Cov}\left(\underline{X}_s, \underline{X}_t\right) = e^{\underline{A}s} \Big\{ & \mathrm{E}\left[\left(\left(\underline{X}_0 - \mathrm{E}\left(\underline{X}_0\right)\right)\left(\underline{X}_0 - \mathrm{E}\left(\underline{X}_0\right)\right)^T\right)\right] \\ & + \int_0^{\min(s,t)} \left(e^{\underline{A}u}\right)^{-1} \underline{\sigma} \cdot \underline{\sigma}^T \left(\left(e^{\underline{A}u}\right)^{-1}\right)^T du \Big\} (e^{\underline{A}t})^T.\end{aligned} \tag{2.30}$$

When $\underline{X}_0$ is constant, then $\mathrm{E}\left[\left(\left(\underline{X}_0 - \mathrm{E}\left(\underline{X}_0\right)\right)\left(\underline{X}_0 - \mathrm{E}\left(\underline{X}_0\right)\right)^T\right)\right] = 0$. Then define $\underline{\Sigma}_{OU}$ by $\underline{\Sigma}_{OU} = \underline{\sigma} \cdot \underline{\sigma}^T$. So the covariance matrix simplifies to

$$\mathrm{Cov}\left(\underline{X}_s, \underline{X}_t\right) = e^{\underline{A}s} \left\{ \int_0^{min(s,t)} \left(e^{\underline{A}u}\right)^{-1} \underline{\Sigma}_{OU} \left(\left(e^{\underline{A}u}\right)^{-1}\right)^T du \right\} (e^{\underline{A}t})^T. \tag{2.31}$$

2.2.3 Position Process

The multivariate Ornstein-Uhlenbeck position process is the integral of the velocity process $\underline{X}_t$, so

$$\underline{Y}_t = \underline{Y}_0 + \int_0^t \underline{X}_s ds. \tag{2.32}$$

Consider the system of stochastic differential equation

$$d\begin{pmatrix}\underline{X}_t \\ \underline{Y}_t\end{pmatrix} = \begin{pmatrix}\underline{A} & \underline{0} \\ \underline{E} & \underline{0}\end{pmatrix}\begin{pmatrix}\underline{X}_t \\ \underline{Y}_t\end{pmatrix} dt + \begin{pmatrix}\underline{\sigma} & \underline{0} \\ \underline{0} & \underline{0}\end{pmatrix} d\underline{W}_t, \tag{2.33}$$

where $\underline{E}$ is the n dimensional identity matrix. Let

$$\underline{B} = \begin{pmatrix}\underline{A} & \underline{0} \\ \underline{E} & \underline{0}\end{pmatrix};$$

then the solution of the above system of stochastic differential equation is

$$\begin{pmatrix} \underline{X}_t \\ \underline{Y}_t \end{pmatrix} = e^{\underline{B}t} \begin{pmatrix} \underline{X}_0 \\ \underline{Y}_0 \end{pmatrix} + \int_0^t e^{\underline{B}(t-s)} \begin{pmatrix} \underline{\sigma} & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix} d\underline{W}_s. \tag{2.34}$$

2.2.4 Equivalent Vector AR(1) and Multivariate OU Processes

One may notice that the multivariate model has mathematical expressions similar to those for the univariate model, except that the expressions are expanded from one dimensional to n dimensional. However, determining the parametric relations between vector AR(1) and multivariate OU processes is a challenge. With univariate model, the solution is pretty intuitive, but with multivariate model, we have encountered significant computational problem when increasing the number of series in vector $\underline{X}_t$.

To find the covariance equivalent OU process of a vector AR(1) process is to determine the relationship that must exist between the matrices $\underline{A}$, $\underline{\sigma}$ and $\underline{\Phi}$, $\underline{\Sigma}_a$ to satisfy the principle of covariance equivalence. Explicit parametric relations between a vector AR(1) process and multivariate OU process haven't been given before. This section mainly shows how the parameter relations are determined in the multivariate case. Assuming a vector AR(1) process is given, there are two matrices in the corresponding OU process that need to be determined, $\underline{A}$ and $\underline{\sigma}$.

Determine $\underline{A}$

We start with $e^{\underline{A}t}$, since it is very straight forward to determine. Then we can solve for $\underline{A}$ by using the eigenvalues and eigenvectors of $e^{\underline{A}t}$. To satisfy the principle of covariance equivalence, the OU process and the AR(1) process need to match their first two moments at all time. To match their first moments, Equation (2.29) should equal to Equation (2.24). Therefore, for all t, we have

$$\mathrm{E}(X_t|X_0) \quad = \quad e^{\underline{A}t}\underline{X}_0 \quad = \quad \underline{\Phi}^t\underline{X}_0$$

which implies that

$$e^{\underline{A}t} \quad = \quad \underline{\Phi}^t.$$

So the solution is

$$e^{\underline{A}} = \underline{\Phi}. \tag{2.35}$$

If one is only interested in the velocity process then having $e^{\underline{A}}$ is sufficient. $\underline{A}$ is required when the position process needs to be studied. The explicit expression for $e^{\underline{A}t}$ can be obtained by using the eigenvalues and eigenvectors of $\underline{A}$. Assume the eigenvalues of $\underline{A}$ are $\mu_1, \mu_2, \dots, \mu_n$ with corresponding eigenvectors $\begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} v_{12} \\ v_{22} \\ \vdots \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ 1 \end{pmatrix}$. Then,

$$\mathrm{E}(\underline{X}_t | X_0) = e^{\underline{A}t}\underline{X}_0 = \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \cdot e^{\mu_1 t} \\ c_2 \cdot e^{\mu_2 t} \\ \vdots \\ c_n \cdot e^{\mu_n t} \end{pmatrix}, \tag{2.36}$$

where $c_1, c_2, \dots, c_n$ are constants. Since $\underline{X}_t$ is a continuous process, when $t = 0$, we must have $\mathrm{E}(\underline{X}_t|_{t=0}) = \underline{X}_0$, where $\underline{X}_0$ is the initial value that is already given. From Equation (2.36) with $t = 0$, we have

$$\underline{X}_0 = \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}. \tag{2.37}$$

The constants, c_1, c_2, …, c_n, are obtained by solving the system of linear equations in (2.37). So we have $\underline{c} = \underline{V}^{-1}\underline{X}_0$. Then plug the solutions in Equation (2.36) to calculate the expressions for $\mathrm{E}(\underline{X}_t)$. After that, we can determine $e^{\underline{A}t}$ by matching the coefficients in $\mathrm{E}(\underline{X}_t)$. Let $\underline{\Xi} = e^{\underline{A}t}$ and the element in row i and column j in the matrix $e^{\underline{A}t}$ be denoted by $\Xi[i, j]$. The element in row i of vector $\mathrm{E}(\underline{X}_t)$ is $\sum_{s=1}^{n} \Xi[i, s] X_{s,0}$. On the right hand side of Equation (2.36), the eigenvalues and eigenvectors are known and c_1, c_2, …, c_n are also expressed with $X_0[s, 1]$ and some constants. Therefore, the elements in $e^{\underline{A}t}$ can be determined by matching the coefficients of $X_{s,0}$.

To further determine $\underline{A}$, we need the eigenvalues and eigenvectors of the matrix $\underline{\Phi}$. Assume the $n \times n$ symmetric matrix $\underline{\Phi}$ has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since we have $e^{\underline{A}} = \underline{\Phi}$, the eigenvalues of $\underline{A}$ are $\mu_1 = \log(\lambda_1), \mu_2 = \log(\lambda_2), \ldots, \mu_n = \log(\lambda_n)$ and the corresponding eigenvectors are the same as those of $\underline{A}$. For example, if $\underline{V}_1$ is the eigenvector corresponding to the eigenvalue λ_1 of matrix $\underline{\Phi}$, then $\underline{V}_1$ is the eigenvector corresponding to the eigenvalue $\mu_1 = \log(\lambda_1)$ of matrix $\underline{A}$. By the definition of eigenvalue and eigenvector, assuming λ is one eigenvalue and its corresponding eigenvector, $\underline{V}$, they should satisfy the system of linear equations

$$(\lambda E - \underline{\Phi})\, \underline{V} = 0. \tag{2.38}$$

We decompose Φ by using its eigenvalues and eigenvectors, $\underline{\Phi} = \underline{V \Lambda V^{-1}}$, where $\underline{\Lambda}$ is a diagonal matrix made up with eigenvalues and each column in $\underline{V}$ is corresponding eigenvector. To determine $\underline{A}$, the eigenvalues of $\underline{\Phi}$ on the diagonal of matrix $\underline{\Lambda}$, $\underline{\lambda}$, need to be replaced with the eigenvalues of $\underline{A}$, $\underline{\mu} = \log(\underline{\lambda})$, which are described before. The columns of $\underline{V}$ are still the eigenvectors of $\underline{\text{A}}$, which are the same as the ones of $\underline{\Phi}$.

Determine matrix $\underline{\sigma}$

Assume the vector AR(1) process is already fitted, so we can first determine $e^{\underline{A}}$. Then we can solve for the matrix $\underline{\sigma}$ using the principle of covariance equivalence, mathematically, by matching $\text{Cov}(\underline{X}_t, \underline{X}_s)$ for the OU and AR(1) processes at any given time points, t and s. Instead of solving $\underline{\sigma}$ directly, we solve for $\underline{\Sigma}_{OU}$ first, which is $\underline{\sigma} \cdot \underline{\sigma}^T$. Then we can find $\underline{\sigma}$ by using the Cholesky decomposition. The goal is to find all the elements in the $n \times n$ matrix, $\underline{\Sigma}_{OU}$, which means that we need as many as $\frac{n(n+1)}{2}$ equations to solve the $\frac{n(n+1)}{2}$ unknowns. For simplicity, instead of using $\text{Cov}(\underline{X}_t, \underline{X}_s)$, we first focus on $\text{Var}(\underline{X}_t)$. By setting the variances of $\underline{X}_t$ equal at time t for the OU and AR(1) processes, we got ourselves a system of linear equations, which has $\frac{n(n+1)}{2}$ equations and $\frac{n(n+1)}{2}$ unknown variables in $\underline{\Sigma}_{OU}$ matrix. Since we solve the system of equations that match the variances of $\underline{X}_t$ for both the OU process and the AR(1) process at time t, we need to verify that the $\underline{\sigma}$ we just found is the right solution. One way is to arbitrarily randomly choose some integers as t and s and test if the covariances between $\underline{X}_t$ and $\underline{X}_s$ of the OU process match the one of the AR(1) processes.

Finding explicit solutions, even with a symbolic software, can be really complicated in this case. When we work with numerical values, the question becomes to use this system of linear equations to solve for each element in matrix $\underline{\Sigma}_{OU}$. To make the system of linear equations simple, we started with $t = s = 1$, which means $k = 0$. According to Equation (2.25), when $t = 1$ and $k = 0$, the variance of $\underline{X}_1$ for vector AR(1) process is $\underline{\Sigma}_a$. And let the element in row k column l in matrix $\underline{\Sigma}_a$ be $\Sigma_{a_{kl}}$.

Then let us look at the OU process. According to Equation (2.31), when $t = s = 1$,

$$\mathrm{Var}\left(\underline{X}_1\right) = e^{\underline{A}} \left\{ \int_0^1 \left(e^{\underline{A}u}\right)^{-1} \underline{\Sigma}_{OU} \left(\left(e^{\underline{A}u}\right)^{-1}\right)^T du \right\} (e^{\underline{A}})^T. \tag{2.39}$$

Assume the element in row i column j of $\underline{\Sigma}_{OU}$ is $\Sigma_{OU_{ij}}$ and plug in the numerical value of $e^{\underline{A}}$ that we already solved, then each element in the covariance matrix of $\mathrm{Var}\left(\underline{X}_1\right)$ is a linear combination of $\Sigma_{OU_{ij}}$. For the OU process, assume the element in row k column l of $\mathrm{Var}\left(\underline{X}_1\right)$ is Var_{kl}, which can be calculated as:

$$\mathrm{Var}_{kl} = \sum_{i=1}^{n} \sum_{j=1}^{n} (c^{kl})_{ij} \Sigma_{OU_{ij}}, \tag{2.40}$$

where all $(c^{kl})_{ij}$ are coefficients determined from Equation (2.39). The upper script kl in $(c^{kl})_{ij}$ indicates that different elements have different constants. To satisfy the principle of covariance equivalence $\Sigma_{a_{kl}}$ must be equal to $\sum_{i=1}^{n} \sum_{j=1}^{n} (c^{kl})_{ij} \Sigma_{OU_{ij}}$. There are $n \times n$ elements in the matrix of $\mathrm{Var}(X_1)$ for both OU process and vector AR(1) process. Since $\underline{\Sigma}_{OU} = \underline{\sigma} \cdot \underline{\sigma}^T$, by doing a Cholesky decomposition, we can determine the $\underline{\sigma}$ matrix.

If the system has a unique solution, we can be sure that the vector AR(1) process has a continuous representation. Since we need to calculate the eigenvectors of $\underline{A}$ as $\log(\underline{\lambda})$, all eigenvalues of $\underline{\Phi}$ need to be greater than 0. Further, if we want to limit our study to stationary processes, we need to make sure all the eigenvalues of $\underline{\Phi}$ are less than 1 in absolute value. Therefore, all eigenvalues of $\underline{\Phi}$, $\underline{\lambda}$, should be greater than 0 and less than 1.

The method described above is not the only way to solve for $\underline{\Sigma}_{OU}$. An alternative approach is provided in Appendix B.

2.3 Example

The following is an example of how to convert a vector AR(1) process into an equivalent OU process. Given the following vector AR(1) process of two variables with starting value $\underline{X}_0$ like in Equation (2.23)

$$\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}^t \begin{pmatrix} X_{1,0} \\ X_{2,0} \end{pmatrix} + \sum_{j=0}^{t-1} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}^j \begin{pmatrix} a_{1(t-j)} \\ a_{2(t-j)} \end{pmatrix},$$

where $\begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$ follows a multivariate normal distribution with mean $\mu = \underline{0}$ and covariance matrix $\underline{\Sigma}_a$. From Equation (2.35), we have $e^{\underline{A}} = \underline{\Phi}$. Assume the eigenvalues of $e^{\underline{A}}$ are λ_1 and λ_2 and their corresponding eigenvectors are $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} b \\ 1 \end{pmatrix}$. As we mentioned in Equation (2.29), the mean of an OU process is

$$\mathrm{E}(\underline{X}_t) = e^{\underline{A}t}\mathrm{E}(\underline{X}_0) = e^{\underline{A}t}\underline{X}_0 = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}, \tag{2.41}$$

and when $t = 0$, c_1 and c_2 can be solved in terms of $\underline{X}_0 = \begin{pmatrix} X_{1,0} \\ X_{2,0} \end{pmatrix}$. That is

$$c_1 = \frac{X_{1,0} - aX_{2,0}}{b-a} \tag{2.42}$$

and

$$c_2 = \frac{-X_{1,0} + bX_{2,0}}{b-a}. \tag{2.43}$$

By matching the coefficients of $X_{1,0}$ and $X_{2,0}$, we can have the following explicit expression for e^{At}

$$e^{At} = \begin{pmatrix} \frac{b\lambda_2^t - a\lambda_1^t}{b-a} & \frac{ab(\lambda_1^t - \lambda_2^t)}{b-a} \\ \frac{\lambda_2^t - \lambda_1^t}{b-a} & \frac{b\lambda_1^t - a\lambda_2^t}{b-a} \end{pmatrix}. \tag{2.44}$$

Then, using Equation (2.31) and symbolic computation, we can calculate $OU_{cov} = \mathrm{Cov}\left(\underline{X}_s, \underline{X}_t\right)$.

Let $\underline{\Sigma}'_{OU} = \begin{pmatrix} \Sigma_{OU_{11}} \\ \Sigma_{OU_{12}} \\ \Sigma_{OU_{21}} \\ \Sigma_{OU_{22}} \end{pmatrix}$. After simplifications, the resulting elements of the covariance matrix are:

$$
\begin{aligned}
OU_{cov}[1,1] &= \frac{\underline{\tau} \cdot \underline{\pi}_{11} \cdot \underline{\Sigma}'_{OU}}{2\log(\lambda_1)\log(\lambda_2)(\log(\lambda_1)+\log(\lambda_2))(a-b)^2}, \\
OU_{cov}[1,2] &= \frac{\underline{\tau} \cdot \underline{\pi}_{12} \cdot \underline{\Sigma}'_{OU}}{2\log(\lambda_1)\log(\lambda_2)(\log(\lambda_1)+\log(\lambda_2))(a-b)^2}, \\
OU_{cov}[2,1] &= \frac{\underline{\tau} \cdot \underline{\pi}_{21} \cdot \underline{\Sigma}'_{OU}}{2\log(\lambda_1)\log(\lambda_2)(\log(\lambda_1)+\log(\lambda_2))(a-b)^2}, \\
OU_{cov}[2,2] &= \frac{\underline{\tau} \cdot \underline{\pi}_{22} \cdot \underline{\Sigma}'_{OU}}{2\log(\lambda_1)\log(\lambda_2)(\log(\lambda_1)+\log(\lambda_2))(a-b)^2},
\end{aligned}
$$

where

$$
\underline{\tau}^T = \begin{pmatrix}
\lambda_1^s \lambda_2^t \log(\lambda_1)\log(\lambda_2) \\
\lambda_1^t \lambda_2^s \log(\lambda_1)\log(\lambda_2) \\
\lambda_1^{t-s} \log(\lambda_1)\log(\lambda_2) \\
\lambda_2^{t-s} \log(\lambda_1)\log(\lambda_2) \\
\lambda_1^{t+s} \log(\lambda_1)\log(\lambda_2) \\
\lambda_2^{t+s} \log(\lambda_1)\log(\lambda_2) \\
\lambda_1^{t-s} (\log(\lambda_2))^2 \\
\lambda_2^{t-s} (\log(\lambda_1))^2 \\
\lambda_1^{t+s} (\log(\lambda_2))^2 \\
\lambda_2^{t+s} (\log(\lambda_1))^2
\end{pmatrix},
$$

$$\underline{\pi}_{11} = \begin{pmatrix} -2ab & 2a^2b & 2ab^2 & -2a^2b^2 \\ -2ab & 2ab^2 & 2a^2b & -2a^2b^2 \\ a(2b-a) & ab(a-2b) & -a^2b & a^2b^2 \\ b(2a-b) & ab(b-2a) & -ab^2 & a^2b^2 \\ a^2 & -a^2b & -a^2b & a^2b^2 \\ b^2 & -ab^2 & -ab^2 & a^2b^2 \\ -a^2 & a^2b & a^2b & -a^2b^2 \\ -b^2 & ab^2 & ab^2 & -a^2b^2 \\ a^2 & -a^2b & -a^2b & a^2b^2 \\ b^2 & -ab^2 & -ab^2 & a^2b^2 \end{pmatrix},$$

$$\underline{\pi}_{12} = \begin{pmatrix} -2a & 2a^2 & 2ab & -2a^2b \\ -2b & 2b^2 & 2ab & -2ab^2 \\ (2b-a) & -b(2b-a) & -ab & ab^2 \\ (2a-b) & -a(2a-b) & -ab & a^2b \\ a & -ab & -ab & ab^2 \\ b & -ab & -ab & a^2b \\ -a & ab & ab & -ab^2 \\ -b & ab & ab & -a^2b \\ a & -ab & -ab & ab^2 \\ b & -ab & -ab & a^2b \end{pmatrix},$$

$$\underline{\pi}_{21} = \begin{pmatrix} -2b & 2ab & 2b^2 & -2ab^2 \\ -2a & 2ab & 2a^2 & -2a^2b \\ a & -ab & a(b-2a) & ab(2a-b) \\ b & -ab & b(a-2b) & ab(2b-a) \\ a & -ab & -ab & ab^2 \\ b & -ab & -ab & a^2b \\ -a & ab & ab & -ab^2 \\ -b & ab & ab & -a^2b \\ a & -ab & -ab & ab^2 \\ b & -ab & -ab & a^2b \end{pmatrix},$$

and

$$\underline{\pi}_{22} = \begin{pmatrix} -2 & 2a & 2b & -2ab \\ -2 & 2b & 2a & -2ab \\ 1 & -b & (b-2a) & -b(b-2a) \\ 1 & -a & (a-2b) & -a(a-2b) \\ 1 & -b & -b & b^2 \\ 1 & -a & -a & a^2 \\ -1 & b & b & -b^2 \\ -1 & a & a & -a^2 \\ 1 & -b & -b & b^2 \\ 1 & -a & -a & a^2 \end{pmatrix}.$$

We decided to solve for $\underline{\Sigma}_{OU}$ numerically. Letting $s = t = 1$, we have matrix $\mathrm{Var}(X_1)$ of the OU process, which is set to be equal to matrix $\underline{\Sigma}_a$ of the vector AR(1) process element by element. And we can get a system of four linear equations to find the four elements in $\underline{\Sigma}_{OU}$. As long as this system of linear equations has a unique solution, $\underline{\Sigma}_{OU}$ can be determined and $\underline{\sigma}$ of OU process can be obtained by a Cholesky decomposition $\underline{\Sigma}_{OU}$. An explicit expression for $\underline{\Sigma}_{OU}$ can be obtained from symbolic calculations, but the expressions for the solution are really long and complicated. For the 2×2 matrix in this example, the explicit solutions might still be available, but when the dimensions of the matrices are increased, the complication of the solutions grows exponentially. Also for the purpose of this project, having the numerical solution is sufficient.

Another thing we would like to mention is the calculation problem that we have encountered. The symbolic calculation for the theoretical covariance matrix of the multivariate OU process gets extremely complicated as the dimension of the matrix is increased. Actually, for six variables, the expressions of the covariance matrix given by Equation (2.31) cannot be calculated by R. One problem we had is that we must do symbolic calculation and simplification for Equation (2.31) before doing numerical calculations. For example, we tried calculating the integral in Equation (2.31) first with symbols, then plugging in numerical values to get the final numerical result. This approach doesn't work. When numerical calculations are done manually, some terms can be canceled out before further calculation. However, that is not the case with computers. Unless doing all the simplification before hand, computers will do the calculation step by step, regardless of the fact that some terms can be canceled or simplified. For example, consider the expression $\log(s) \cdot \frac{1}{\log(s)}$, which is, of course, equal to 1. For s very small, say 0.1^{500}, a computer would first calculate 0.1^{500}, then take its logarithm, which results in negative infinity in R, and the expression cannot be evaluated. Given the nature of this project, it is very much possible we eventually encounter such situations, especially when the time unit is one day and we are looking at a time period of 50 to 100 years. In our case, the integral could result in some infinitely small values in the denominator, which can be canceled out when multiplied by the two matrices outside the integral, but because the simplification is not done before hand, the calculation ends up giving unreasonable results. Therefore, symbolic calculation and simplification would have to be done until reaching a final result for $\text{Cov}(X_s, X_t)$. Plugging in numerical values too early could result in unreasonable results or no result at all.

Chapter 3

Investment Models

In this chapter, we consider first order univariate and multivariate models for the rates of return of three different assets. Using real data, models for the rates of return of the three different assets are estimated. The processes studied in Chapter 2 are used to model $\underline{X}_t$, the rate of return at time t, and $\underline{Y}_t$, the cumulated rate of return until time t. These three assets will constitute the universe of financial instruments. In the next chapter, we analyze investment strategies that consist of investing different amounts in these assets.

3.1 Data Collection

In order to estimate the parameters of our model, we collected daily data for the past 35 years of the US market. We made an assumption that there are only three assets available for investment. The three assets we chose are 10-year long term bond, 3-month treasury bill and S&P 500 Index. For both univariate and multivariate models we estimated the AR(1) model from the discrete data, then converted it to its covariance equivalent continuous process. The continuous process is used to model future rates of return. The S&P 500 Index is used to calculate the return from equity, which represents the high volatility asset. The 10-year long term bond represents the low volatility long term asset. The 3-month treasury bill represents an asset with moderate volatility.

The data for long term bond and short term bill is collected from the released statistics of the Board of Governors of the Federal Reserve System of the United States. The data is collected at a daily frequency from early 1974 to June 2009. The rate of return on equity is calculated from the S&P 500 index. For example, if S&P 500 closed at I_t on day t and I_{t-1} on the previous day, then the rate of return for equity on day t is $\log(I_t) - \log(I_{t-1})$. There is something we need to point out. One would expect the average annual equity return calculated from S&P 500 to be higher than the average annual return for a 10-year long term bond. However, for this particular set of data, we found that the average return from equity is slightly lower than long term bond. We later find that this specific result affects the optimal asset allocation strategy in our simulated results.

The purpose of this project is to study one multivariate OU process and compare it with a combination of several univariate OU processes. As a result, we chose its discrete-time analogue AR(1) process to model the daily data. To determine whether an AR(1) model is a good choice or not, we take a look at the plots of the autocorrelation function (ACF) and partial autocorrelation function (PACF) for the rates of return of the three assets in Figure 3.1. The autocorrelation function is defined as $\rho = \frac{\mathrm{Cov}(X_t, X_s)}{\sqrt{\mathrm{Var}(X_t) Var(X_s)}}$. The partial autocorrelation at lag k may be regarded as the correlation between X_1 and X_{k+1}, adjusted for the intervening observations $X_2, \ldots, X_k$. A more detailed definition of the partial autocorrelation function is given in Brockwell and Davis (1991, page 98). From Figure 3.1 we can see that, the autocorrelation is quite strong for long term bond and short term bill. And even after a 40-day lag, the autocorrelation is still significant. In their PACF plots, we see that the first partial autocorrelation coefficient, which equals to Φ in an AR(1) model, is close to 1 and others are relatively small. Therefore, a first order autoregressive process looks like a reasonable model for the rates of return for the long term bond and the short term bill. Based on the partial autocorrelation coefficients we saw, we expect the process to be close to a random walk. For the equity, other than the autocorrelation coefficient at lag 0, which is always 1, we see very weak autocorrelations between the observations at two different time points. This actually indicates that the process modeling equity's rates of return is close to White Noise. Although, goodness-of-fit was not checked in this project, we assumed both univariate and multivariate AR(1) processes are reasonable.

3.2 Estimation

In this section, we describe how the parameters of the univariate and multivariate models are estimated. The estimated models are shown in this section as well.

3.2.1 Estimation Method

For the AR(1) process, there are two parameters that need to be determined, $\underline{\Phi}$ and $\underline{\sigma_a}$. With the ordinary least square method in R, the estimation of the two parameters can be done after subtracting the mean of the collected historical data. As the initial value $\underline{X}_0$, we use the last observation of $\underline{X}_t$. To study the characteristics of the model, we also considered other initial values. Also, the mean of the historical data is used as the long term mean for the OU process. Each asset's historical mean is used in both univariate and multivariate models, so that, for the same asset, the two models should result in two processes reverting to the same mean level.

3.2.2 Estimation Results

First, we look at the historical data we collected from 1974 to 2009 for the three assets we are analyzing. As we can see from Figure 3.2, the rate of return of the long term bond looks positively correlated with the rate of return of the short term bill. However, from the graph it is hard to see whether the rate of return of equity is correlated or not with either the long term bond or the short term bill. Such correlations among different assets cannot be captured when modeling each asset by a univariate model. That is the main reason to study the multivariate model as an improvement of the univariate model. The graphs also indicate that the rate of return on equity has the highest volatility, followed by the short term bill, and the rate of return of long term bond has the lowest volatility. We first estimated the parameters for both of the univariate and multivariate AR(1) models at a daily frequency

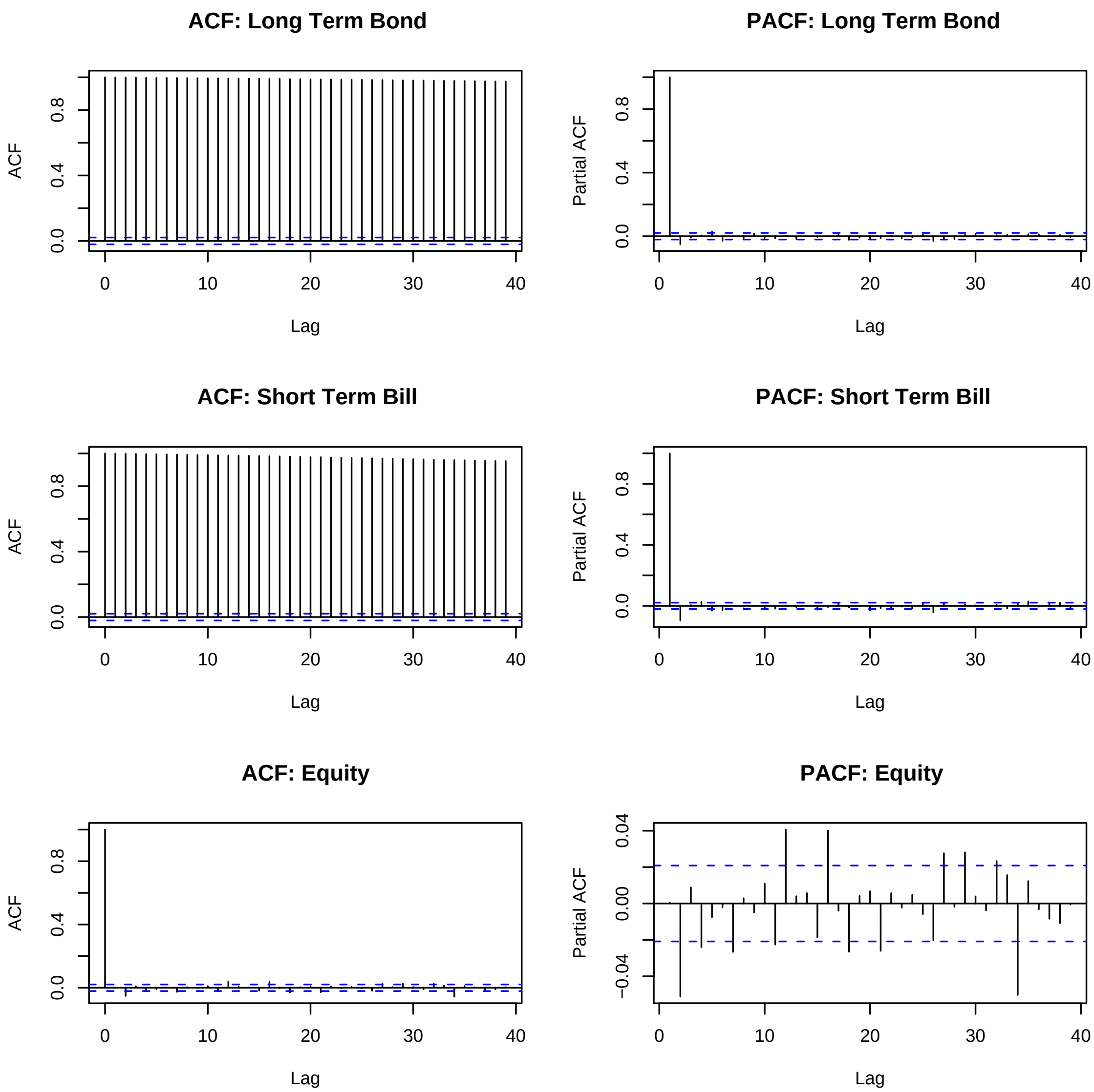


Figure 3.1: Plots of autocorrelation function and partial autocorrelation function for rates of return of three assets at daily frequency

as that is the frequency of our data. Then we converted the discrete time AR(1) process to its continuous time analogue, the Ornstein-Uhlenbeck process. And since our application is mainly to price life annuities in this project, we will focus on the annual rates of return of each asset.

Now let us look at the estimates we got from R for the three assets using both univariate and multivariate models.

Estimated Parameters of Univariate Model

- **Long Term Bond**

$$X_{Lt} - 0.0002843438 = 0.9997242(X_{L(t-1)} - 0.0002843438) + a_{Lt}, \tag{3.1}$$

  where 0.0002843438 is the long term mean of the daily rates of return for long term bonds, and the standard deviation of a_{Lt} is 2.806297e-06.

- **Short Term Bill**

$$X_{St} - 0.0002212772 = 0.999575(X_{S(t-1)} - 0.0002212772) + a_{St}, \tag{3.2}$$

  where 0.0002212772 is the long term mean of the daily rates of return for short term bill, and the standard deviation of a_{St} is 3.989231e-06.

- **Equity**

$$X_{Et} - 0.0002776229 = 0.0004515107(X_{E(t-1)} - 0.0002776229) + a_{Et}, \tag{3.3}$$

  where 0.0002776229 is the long term mean of the daily rates of return for equity, and the standard deviation of a_{Et} is 0.01110829.

From the estimated results, we can see that all of the three AR(1) processes are stationary. For both long term bond and short term bill, their Φ's are quite close to 1, which means that the rates of return of those two assets for one day are very much correlated with the rates of return for the previous day. However, this is not the case for the equity's

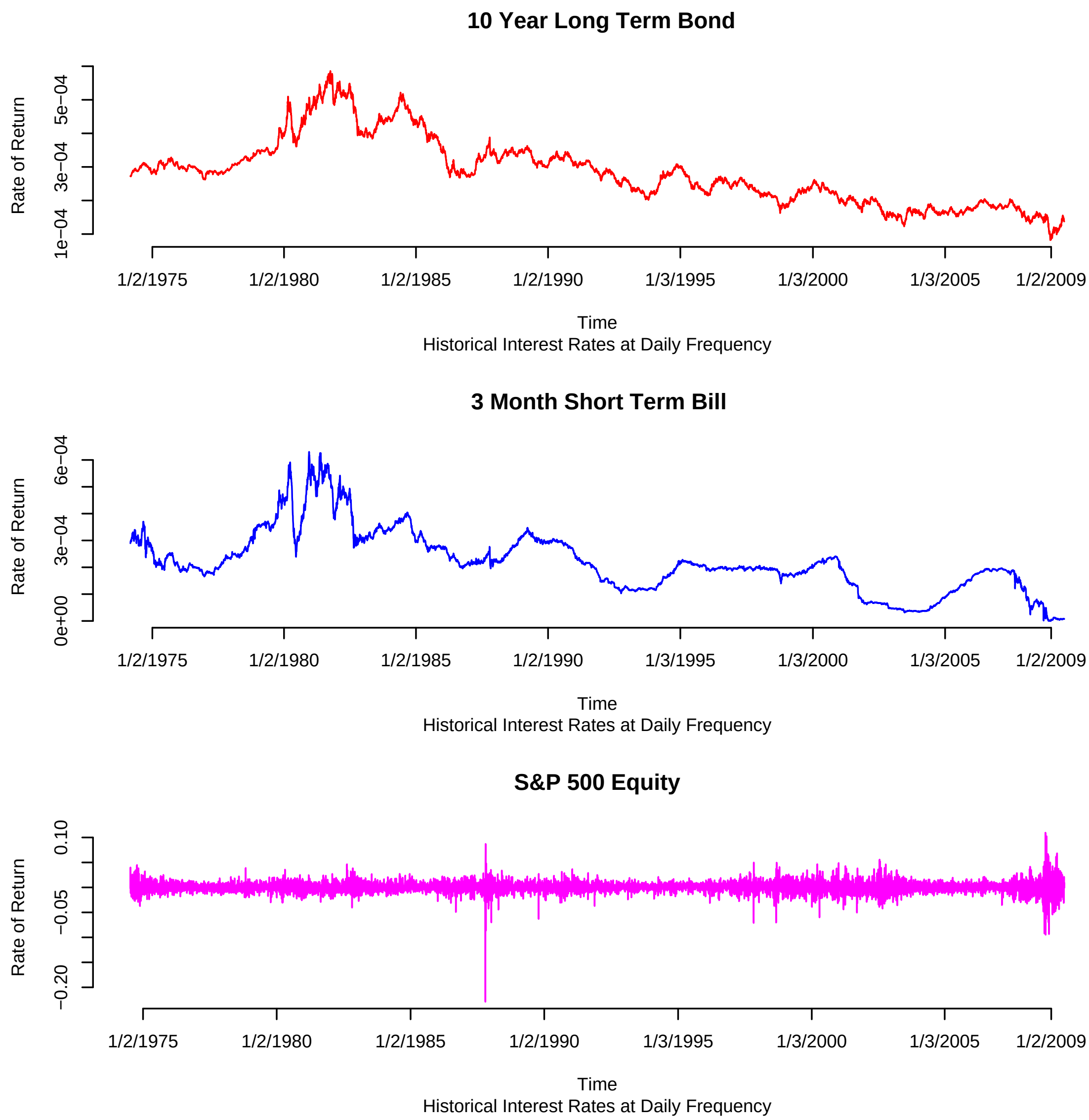


Figure 3.2: Historical rate of return for three assets at daily frequency

rates of return. For the volatility, as shown in Figure 3.2, the long term bond has the lowest volatility, followed by the short term bill. The equity's volatility is much higher than the other two assets.

Estimated Parameters of Multivariate Model

To be consistent with the expressions used in the univariate model, the subscrip "L" represents the long term bond and "S" for the short term bill then "E" for equity. So now let us look at the estimated parameters for the vector AR(1) model when we combine the three assets together.

$$\begin{pmatrix} X_{Lt} - 0.0002843438 \\ X_{St} - 0.0002212772 \\ X_{Et} - 0.0002776229 \end{pmatrix}$$
$$= \begin{pmatrix} 0.998519 & 0.001143 & 6.640e-06 \\ 0.001513 & 0.998382 & 7.454e-06 \\ 1.103105 & 0.285538 & 2.880e-04 \end{pmatrix} \begin{pmatrix} X_{L(t-1)} - 0.0002843438 \\ X_{S(t-1)} - 0.0002212772 \\ X_{E(t-1)} - 0.0002776229 \end{pmatrix} + \begin{pmatrix} a_{Lt} \\ a_{St} \\ a_{Et} \end{pmatrix}. \quad (3.4)$$

The covariance matrix for $\begin{pmatrix} a_{Lt} \\ a_{St} \\ a_{Et} \end{pmatrix}$ is $\Sigma_a = \begin{pmatrix} 7.866e-12 & 5.411e-12 & 1.941e-10 \\ 5.411e-12 & 1.590e-11 & 1.788e-10 \\ 1.941e-10 & 1.788e-10 & 1.234e-04 \end{pmatrix}$.

From the result, we can see that the interest rates for long term bond and short term bill will affect the equity's rate of return. However, the other way isn't true. Take the long term bond for example, the centered interest rate of one day $(X_{L(t)} - 0.0002843438)$ is 0.998519 of the interest rate of previous day's long term bond $(X_{L(t-1)} - 0.0002843438)$ plus 0.001143 of the previous day's short term bill's interest rate $(X_{S(t-1)} - 0.0002212772)$ plus 6.640e-06 of the previous day's equity rate of return $(X_{E(t-1)} - 0.0002776229)$ plus a random term a_{Lt}. We can see for long term bond, from the coefficients, that its interest rates are greatly dependent on the previous day's interest rate and a small portion comes from the previous day's short term bill's interest rate. Compared to the impact of the other two assets, the impact of the equity's return is almost negligible, given the coefficient is 6.640e-06. The explanation of the expressions for short term bill is similar to the one for long term bond. It is greatly affected by the previous day's return of itself and a little affected by long term bond, with almost no impact from equity. However, the case is slightly different for equity. The rate of return for equity depends on both long term bond and short term bill's rates for the last time period; more on long term bonds (coefficient is 1.103105) than short term bill (coefficient is 0.285538). However, the coefficient for equity itself is still very small compared to the other two.

3.3 Equivalent AR(1) and OU processes

After we have our estimated AR(1) model for both univariate and multivariate processes, we can convert the AR(1) model to its covariance equivalent OU process as we discussed in Chapter 2. To determine the OU process, we need to find $e^{\underline{A}}$ and $\underline{\sigma}$. As we showed in Section 2.3, $e^{\underline{A}}$ is easier to determine and $e^{\underline{A}} = \underline{\Phi}$. So in our multivariate case,

$$e^{\underline{A}} = \begin{pmatrix} 0.998519 & 0.001143 & 6.640e-06 \\ 0.001513 & 0.998382 & 7.454e-06 \\ 1.103105 & 0.285538 & 2.880e-04 \end{pmatrix}. \tag{3.5}$$

Further, we can solve the matrix $\underline{A}$ by using the eigenvalues and eigenvectors of $\underline{\Phi}$, which are given below.

- **Eigenvalues:**

$$0.9997768991 \quad 0.9971336583 \quad 0.0002785119$$

- **Eigenvectors:**

$$\begin{pmatrix} -0.4881086 \\ -0.5331138 \\ -0.6910425 \end{pmatrix}, \begin{pmatrix} 0.5720872 \\ -0.6961795 \\ 0.4336478 \end{pmatrix}, \begin{pmatrix} 6.640515e-06 \\ 7.461558e-06 \\ -1.000000e+00 \end{pmatrix}$$

Here we have

$$\underline{A} = \begin{pmatrix} -0.001535351 & 0.001130987 & 5.435783e-05 \\ 0.001456321 & -0.001635677 & 6.107815e-05 \\ 9.041391375 & 2.332559643 & -8.185972 \end{pmatrix}. \tag{3.6}$$

Using the method we described in Chapter 2, we can solve for the $\underline{\Sigma}_{OU}$ matrix and using the Choleski decomposition we find $\underline{\sigma}$ for the OU process. In our case,

$$\underline{\sigma} = \begin{pmatrix} 2.803571e-06 & 0.000000 & 0.00000000 \\ 1.923492e-06 & 3.494203e-06 & 0.00000000 \\ -1.851517e-03 & -7.404694e-04 & 0.04489877 \end{pmatrix}. \tag{3.7}$$

We now have the covariance equivalent OU process for the vector AR(1) process.

The univariate case is much simpler than the multivariate case. The general expression is

$$X_t - X_{mean} = e^{-\alpha t}(X_0 - X_{mean}) + \int_0^t e^{-\alpha(t-s)}\sigma dW_s. \tag{3.8}$$

Again we have $e^{-\alpha} = \Phi$, which means $\alpha = -\log(\Phi)$. Also σ is much easier to determine,

$$\sigma = \sqrt{\frac{-2\alpha\sigma_a^2}{(1-\Phi^2)}}, \tag{3.9}$$

where σ_a and Φ come from the AR(1) process. Therefore, we have the following table of parameters for the univariate OU processes.

Asset	α	σ
Long Term Bond	0.0002758579	2.806684e-06
Short Term Bill	0.0004251711	3.990080e-06
Equity	7.702911	0.04360034

Table 3.1: Univariate OU process: α and σ_X for three assets

In the OU process, we know that α is the parameter that determines how fast the process will return to the long term mean. The larger the absolute value of α is, the faster the process will go towards the long term mean. From this table, we see that the rates of return for equity go towards the long term mean very fast, but long term bond and short term bill will not. It is much easier to explain such characteristics for a univariate OU process than for a multivariate process. Those characteristics for the three assets will be shown in both multivariate and univariate models in the simulations.

3.4 The Ornstein-Uhlenbeck Position Process

The AR(1) process can be obtained directly from the data collected. Then the discrete time AR(1) process can be converted to its covariance equivalent continuous time OU process. Those processes can be used to calculate the daily rate of return for the three assets. However, since our goal is to calculate annuity prices, the annual rate of return is more useful and efficient. Therefore, we also want to look at the OU position process $\underline{Y}_t$, which is the integration of the velocity process $\underline{X}_t$, so as showed in Chapter 2,

$$\underline{Y}_t = \underline{Y}_0 + \int_0^t \underline{X}_s ds, \tag{3.10}$$

where Y_0 is $\underline{0}$. Let the $\underline{\sigma}$ matrix for the OU velocity process be $\underline{\sigma}_X$; then for the corresponding OU position process, the $\underline{\sigma}$ matrix is $\underline{\sigma}_Y = \begin{pmatrix} \underline{\sigma}_X & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix}$. So the SDE is

$$\begin{pmatrix} \underline{X}_t - \underline{X}_{mean} \\ \underline{Y}_t - \underline{Y}_{mean} \end{pmatrix} = e^{\underline{B}t} \begin{pmatrix} \underline{X}_0 - \underline{X}_{mean} \\ \underline{Y}_0 - \underline{Y}_{mean} \end{pmatrix} + \int_0^t e^{\underline{B}(t-s)} \underline{\sigma}_Y d\underline{W}_s. \tag{3.11}$$

For the multivariate model, we have the following $\underline{B}$ and $\underline{\sigma}_Y$

$$\underline{B} = \begin{pmatrix} \underline{A} & \underline{0} \\ \underline{E} & \underline{0} \end{pmatrix} = \begin{pmatrix} -0.001535351 & 0.001130987 & 5.435783e-05 & 0 & 0 & 0 \\ 0.001456321 & -0.001635677 & 6.107815e-05 & 0 & 0 & 0 \\ 9.041391375 & 2.332559643 & -8.185972 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\underline{\sigma}_Y = \begin{pmatrix} \underline{\sigma}_X & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix} = \begin{pmatrix} 2.803571e-06 & 0 & 0 & 0 & 0 & 0 \\ 1.923492e-06 & 3.494203e-06 & 0 & 0 & 0 & 0 \\ -1.851517e-03 & -7.404694e-04 & 0.04489877 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the univariate model, the idea is the same. The matrix $\underline{B}$ is $\begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix}$ and the matrix $\underline{\sigma}_Y$ is $\begin{pmatrix} \sigma_X & 0 \\ 0 & 0 \end{pmatrix}$. Table 3.2 shows the estimated $\underline{B}$ and $\underline{\sigma}_Y$ matrices for the three univariate OU processes.

Asset	$\underline{B}$	$\underline{\sigma}_Y$
Long Term Bond	$\begin{pmatrix} -0.0002758579 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2.806684e-06 & 0 \\ 0 & 0 \end{pmatrix}$
Short Term Bill	$\begin{pmatrix} -0.0004251711 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 3.990080e-06 & 0 \\ 0 & 0 \end{pmatrix}$
Equity	$\begin{pmatrix} -7.702911 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.04360034 & 0 \\ 0 & 0 \end{pmatrix}$

Table 3.2: Univariate model: matrix $\underline{B}$ and $\underline{\sigma}_Y$ for three assets

Chapter 4

Applications

4.1 Integrated Wiener Process

To simulate a continuous time OU process, such as the ones described in Equations (3.8) and (3.11), we need to generate values from an integrated Wiener process of the type: $\int_0^t f(s)dW_s$, where $f(s)$ is a function of time s. We arbitrarily choose to approximate such stochastic integral by the following sum

$$\sum_{i=0}^{t/dt-1} dt f\left(\frac{i+(i+1)}{2}dt\right)\sqrt{dt}\eta_i, \tag{4.1}$$

where i is an integer, dt is a very small time step, $\frac{t}{dt}$ is an integer and η_i $(i = 0, 1, \ldots, \frac{t}{dt}-1)$ are i.i.d. random variables normally distributed with mean 0 and variance 1. To simulate observations at any time t for an OU process, we need to take the exponential of the matrices in the expressions for the OU processes.

For the multivariate case, let us look at the following SDE defined in Equation (3.11),

$$\begin{pmatrix} \underline{X}_t - \underline{X}_{mean} \\ \underline{Y}_t - \underline{Y}_{mean} \end{pmatrix} = e^{\underline{B}t} \begin{pmatrix} \underline{X}_0 - \underline{X}_{mean} \\ \underline{Y}_0 - \underline{Y}_{mean} \end{pmatrix} + \int_0^t e^{\underline{B}(t-s)} \underline{\sigma}_Y d\underline{W}_s,$$

where $\underline{B} = \begin{pmatrix} \underline{A} & \underline{0} \\ \underline{E} & \underline{0} \end{pmatrix}$ and $\underline{\sigma}_Y = \begin{pmatrix} \underline{\sigma}_X & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix}$. We can have the explicit expression for $e^{\underline{B}t}$ in terms of the eigenvalues and eigenvectors of $\underline{B}$. Let the eigenvalues of $\underline{B}$ be $\underline{\lambda} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix}$ and the corresponding eigenvectors be

$$\begin{pmatrix} 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & d & e & f \\ 0 & 0 & 0 & g & h & i \\ 1 & 0 & 0 & j & k & l \\ 0 & 1 & 0 & m & n & o \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then with the method we described in Chapter 2, using Equations (2.36) and (2.37) and matching the coefficients of elements in $\underline{X}_0$ we have the following explicit expression for $e^{\underline{B}t}$. Define the following determinants:

$$|P_1| = \begin{vmatrix} i & h \\ f & e \end{vmatrix}, \quad |P_2| = \begin{vmatrix} d & f \\ g & i \end{vmatrix}, \quad |P_3| = \begin{vmatrix} g & h \\ d & e \end{vmatrix},$$

$$|P_4| = \begin{vmatrix} i & h \\ c & b \end{vmatrix}, \quad |P_5| = \begin{vmatrix} a & c \\ g & i \end{vmatrix}, \quad |P_6| = \begin{vmatrix} h & g \\ b & a \end{vmatrix},$$

and

$$|P_7| = \begin{vmatrix} c & b \\ f & e \end{vmatrix}, \quad |P_8| = \begin{vmatrix} a & c \\ d & f \end{vmatrix}, \quad |P_9| = \begin{vmatrix} e & d \\ b & a \end{vmatrix}.$$

We solve $e^{\underline{B}t}$ in terms of $|P_1|$, $|P_2|$, $|P_3|$, $|P_4|$, $|P_5|$, $|P_6|$, $|P_7|$, $|P_8|$, $|P_9|$ as

$$e^{\underline{B}t}[,1] = \frac{1}{\Gamma}\begin{pmatrix} a|P_1|e^{\lambda_4 t} - b|P_2|e^{\lambda_5 t} - c|P_3|e^{\lambda_6 t} \\ d|P_1|e^{\lambda_4 t} - e|P_2|e^{\lambda_5 t} - f|P_3|e^{\lambda_6 t} \\ g|P_1|e^{\lambda_4 t} - h|P_2|e^{\lambda_5 t} - i|P_3|e^{\lambda_6 t} \\ j|P_1|(e^{\lambda_4 t}-1) - k|P_2|(e^{\lambda_5 t}-1) - l|P_3|(e^{\lambda_6 t}-1) \\ m|P_1|(e^{\lambda_4 t}-1) - n|P_2|(e^{\lambda_5 t}-1) - o|P_3|(e^{\lambda_6 t}-1) \\ |P_1|(e^{\lambda_4 t}-1) - |P_2|(e^{\lambda_5 t}-1) - |P_3|(e^{\lambda_6 t}-1) \end{pmatrix},$$

$$e^{\underline{B}t}[,2] = \frac{1}{\Gamma}\begin{pmatrix} -a|P_4|e^{\lambda_4 t} + b|P_5|e^{\lambda_5 t} - c|P_6|e^{\lambda_6 t} \\ -d|P_4|e^{\lambda_4 t} + e|P_5|e^{\lambda_5 t} - f|P_6|e^{\lambda_6 t} \\ -g|P_4|e^{\lambda_4 t} + h|P_5|e^{\lambda_5 t} - i|P_6|e^{\lambda_6 t} \\ -j|P_4|(e^{\lambda_4 t}-1) + k|P_5|(e^{\lambda_5 t}-1) - l|P_6|(e^{\lambda_6 t}-1) \\ -m|P_4|(e^{\lambda_4 t}-1) + n|P_5|(e^{\lambda_5 t}-1) - o|P_6|(e^{\lambda_6 t}-1) \\ -|P_4|(e^{\lambda_4 t}-1) + |P_5|(e^{\lambda_5 t}-1) - |P_6|(e^{\lambda_6 t}-1) \end{pmatrix},$$

$$e^{\underline{B}t}[,(3:6)] = \frac{1}{\Gamma}\begin{pmatrix} -a|P_7|e^{\lambda_4 t} - b|P_8|e^{\lambda_5 t} + c|P_9|e^{\lambda_6 t} & 0 & 0 & 0 \\ -d|P_7|e^{\lambda_4 t} - e|P_8|e^{\lambda_5 t} + f|P_9|e^{\lambda_6 t} & 0 & 0 & 0 \\ -g|P_7|e^{\lambda_4 t} - h|P_8|e^{\lambda_5 t} + i|P_9|e^{\lambda_6 t} & 0 & 0 & 0 \\ -j|P_7|(e^{\lambda_4 t}-1) - k|P_8|(e^{\lambda_5 t}-1) + l|P_9|(e^{\lambda_6 t}-1) & 1 & 0 & 0 \\ -m|P_7|(e^{\lambda_4 t}-1) - n|P_8|(e^{\lambda_5 t}-1) + o|P_9|(e^{\lambda_6 t}-1) & 0 & 1 & 0 \\ -|P_7|(e^{\lambda_4 t}-1) - |P_8|(e^{\lambda_5 t}-1) + |P_9|(e^{\lambda_6 t}-1) & 0 & 0 & 1 \end{pmatrix},$$

where $\Gamma = iae - idb - gce - haf + hdc + gbf$ and $e^{\underline{B}t}[,col]$ denotes column col of the matrix $e^{\underline{B}t}$.

To evaluate the stochastic integral in Equation (3.11) more efficiently, we will pre-calculate $\Omega = e^{\underline{B}(t-s)}\underline{\sigma}_Y$. Letting σ_{Xij} be element (i,j) of matrix σ_X, we find that the first three columns of Ω are:

$$
\begin{aligned}
\Omega[1,1] &= \frac{1}{\Gamma}\sigma_{X11}\left(a|P_1|e^{\lambda_4(t-s)} - b|P_2|e^{\lambda_5(t-s)} - c|P_3|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X21}\left(-a|P_4|e^{\lambda_4(t-s)} + b|P_5|e^{\lambda_5(t-s)} - c|P_6|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X31}\left(-a|P_7|e^{\lambda_4(t-s)} - b|P_8|e^{\lambda_5(t-s)} + c|P_9|e^{\lambda_6(t-s)}\right),\\
\Omega[1,2] &= \frac{1}{\Gamma}\sigma_{X22}\left(-a|P_4|e^{\lambda_4(t-s)} + b|P_5|e^{\lambda_5(t-s)} - c|P_6|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X32}\left(-a|P_7|e^{\lambda_4(t-s)} - b|P_8|e^{\lambda_5(t-s)} + c|P_9|e^{\lambda_6(t-s)}\right),\\
\Omega[1,3] &= \frac{1}{\Gamma}\sigma_{X33}\left(-a|P_7|e^{\lambda_4(t-s)} - b|P_8|e^{\lambda_5(t-s)} + c|P_9|e^{\lambda_6(t-s)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Omega[2,1] &= \frac{1}{\Gamma}\sigma_{X11}\left(d|P_1|e^{\lambda_4(t-s)} - e|P_2|e^{\lambda_5(t-s)} - f|P_3|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X21}\left(-d|P_4|e^{\lambda_4(t-s)} + e|P_5|e^{\lambda_5(t-s)} - f|P_6|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X31}\left(-d|P_7|e^{\lambda_4(t-s)} - e|P_8|e^{\lambda_5(t-s)} + f|P_9|e^{\lambda_6(t-s)}\right),\\
\Omega[2,2] &= \frac{1}{\Gamma}\sigma_{X22}\left(-d|P_4|e^{\lambda_4(t-s)} + e|P_5|e^{\lambda_5(t-s)} - f|P_6|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X32}\left(-d|P_7|e^{\lambda_4(t-s)} - e|P_8|e^{\lambda_5(t-s)} + f|P_9|e^{\lambda_6(t-s)}\right),\\
\Omega[2,3] &= \frac{1}{\Gamma}\sigma_{X33}\left(-d|P_7|e^{\lambda_4(t-s)} - e|P_8|e^{\lambda_5(t-s)} + f|P_9|e^{\lambda_6(t-s)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Omega[3,1] &= \frac{1}{\Gamma}\sigma_{X11}\left(g|P_1|e^{\lambda_4(t-s)} - h|P_2|e^{\lambda_5(t-s)} - i|P_3|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X21}\left(-g|P_4|e^{\lambda_4(t-s)} + h|P_5|e^{\lambda_5(t-s)} - i|P_6|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X31}\left(-g|P_7|e^{\lambda_4(t-s)} - h|P_8|e^{\lambda_5(t-s)} + i|P_9|e^{\lambda_6(t-s)}\right),\\
\Omega[3,2] &= \frac{1}{\Gamma}\sigma_{X22}\left(-g|P_4|e^{\lambda_4(t-s)} + h|P_5|e^{\lambda_5(t-s)} - i|P_6|e^{\lambda_6(t-s)}\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X32}\left(-g|P_7|e^{\lambda_4(t-s)} - h|P_8|e^{\lambda_5(t-s)} + i|P_9|e^{\lambda_6(t-s)}\right),\\
\Omega[3,3] &= \frac{1}{\Gamma}\sigma_{X33}\left(-g|P_7|e^{\lambda_4(t-s)} - h|P_8|e^{\lambda_5(t-s)} + i|P_9|e^{\lambda_6(t-s)}\right),
\end{aligned}
$$

$$\begin{aligned}
\Omega[4,1] &= \frac{1}{\Gamma}\sigma_{X11}\left(j|P_1|(e^{\lambda_4(t-s)}-1)-k|P_2|(e^{\lambda_5(t-s)}-1)-l|P_3|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X21}\left(-j|P_4|(e^{\lambda_4(t-s)}-1)+k|P_5|(e^{\lambda_5(t-s)}-1)-l|P_6|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X31}\left(-j|P_7|(e^{\lambda_4(t-s)}-1)-k|P_8|(e^{\lambda_5(t-s)}-1)+l|P_9|(e^{\lambda_6(t-s)}-1)\right),\\
\Omega[4,2] &= \frac{1}{\Gamma}\sigma_{X22}\left(-j|P_4|(e^{\lambda_4(t-s)}-1)+k|P_5|(e^{\lambda_5(t-s)}-1)-l|P_6|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X32}\left(-j|P_7|(e^{\lambda_4(t-s)}-1)-k|P_8|(e^{\lambda_5(t-s)}-1)+l|P_9|(e^{\lambda_6(t-s)}-1)\right),\\
\Omega[4,3] &= \frac{1}{\Gamma}\sigma_{X32}\left(-j|P_7|(e^{\lambda_4(t-s)}-1)-k|P_8|(e^{\lambda_5(t-s)}-1)+l|P_9|(e^{\lambda_6(t-s)}-1)\right),
\end{aligned}$$

$$\begin{aligned}
\Omega[5,1] &= \frac{1}{\Gamma}\sigma_{X11}\left(m|P_1|(e^{\lambda_4(t-s)}-1)-n|P_2|(e^{\lambda_5(t-s)}-1)-o|P_3|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X21}\left(-m|P_4|(e^{\lambda_4(t-s)}-1)+n|P_5|(e^{\lambda_5(t-s)}-1)-o|P_6|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X31}\left(-m|P_7|(e^{\lambda_4(t-s)}-1)-n|P_8|(e^{\lambda_5(t-s)}-1)+o|P_9|(e^{\lambda_6(t-s)}-1)\right),\\
\Omega[5,2] &= +\frac{1}{\Gamma}\sigma_{X22}\left(-m|P_4|(e^{\lambda_4(t-s)}-1)+n|P_5|(e^{\lambda_5(t-s)}-1)-o|P_6|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X32}\left(-m|P_7|(e^{\lambda_4(t-s)}-1)-n|P_8|(e^{\lambda_5(t-s)}-1)+o|P_9|(e^{\lambda_6(t-s)}-1)\right),\\
\Omega[5,3] &= \frac{1}{\Gamma}\sigma_{X33}\left(-m|P_7|(e^{\lambda_4(t-s)}-1)-n|P_8|(e^{\lambda_5(t-s)}-1)+o|P_9|(e^{\lambda_6(t-s)}-1)\right),
\end{aligned}$$

$$\begin{aligned}
\Omega[6,1] &= \frac{1}{\Gamma}\sigma_{X11}\left(|P_1|(e^{\lambda_4(t-s)}-1)-|P_2|(e^{\lambda_5(t-s)}-1)-|P_3|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X21}\left(-|P_4|(e^{\lambda_4(t-s)}-1)+|P_5|(e^{\lambda_5(t-s)}-1)-|P_6|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X31}\left(-|P_7|(e^{\lambda_4(t-s)}-1)-|P_8|(e^{\lambda_5(t-s)}-1)+|P_9|(e^{\lambda_6(t-s)}-1)\right),\\
\Omega[6,2] &= +\frac{1}{\Gamma}\sigma_{X22}\left(-|P_4|(e^{\lambda_4(t-s)}-1)+|P_5|(e^{\lambda_5(t-s)}-1)-|P_6|(e^{\lambda_6(t-s)}-1)\right)\\
&\quad+\frac{1}{\Gamma}\sigma_{X32}\left(-|P_7|(e^{\lambda_4(t-s)}-1)-|P_8|(e^{\lambda_5(t-s)}-1)+|P_9|(e^{\lambda_6(t-s)}-1)\right),\\
\Omega[6,3] &= \frac{1}{\Gamma}\sigma_{X33}\left(-|P_7|(e^{\lambda_4(t-s)}-1)-|P_8|(e^{\lambda_5(t-s)}-1)+|P_9|(e^{\lambda_6(t-s)}-1)\right),
\end{aligned}$$

The elements in the last three columns of Ω are all 0.

Finally, we have $\int_0^t e^{\underline{B}(t-s)}\underline{\sigma}_{xy}d\underline{W}_s = \int_0^t \Omega d\underline{W}_s$, which can be written as

$$\int_0^t \Omega d\underline{W_s} = \begin{pmatrix} \int_0^t \Omega[1,1]dW_{1,s} + \int_0^t \Omega[1,2]dW_{2,s} + \int_0^t \Omega[1,3]dW_{3,s} \\ \int_0^t \Omega[2,1]dW_{1,s} + \int_0^t \Omega[2,2]dW_{2,s} + \int_0^t \Omega[2,3]dW_{3,s} \\ \int_0^t \Omega[3,1]dW_{1,s} + \int_0^t \Omega[3,2]dW_{2,s} + \int_0^t \Omega[3,3]dW_{3,s} \\ \int_0^t \Omega[4,1]dW_{1,s} + \int_0^t \Omega[4,2]dW_{2,s} + \int_0^t \Omega[4,3]dW_{3,s} \\ \int_0^t \Omega[5,1]dW_{1,s} + \int_0^t \Omega[5,2]dW_{2,s} + \int_0^t \Omega[5,3]dW_{3,s} \\ \int_0^t \Omega[6,1]dW_{1,s} + \int_0^t \Omega[6,2]dW_{2,s} + \int_0^t \Omega[6,3]dW_{3,s} \end{pmatrix}. \tag{4.2}$$

For the univariate case, the general expression for Y_t is

$$Y_t = \frac{1-e^{-\alpha}}{\alpha}(X_0 - X_{mean}) + \sigma \int_0^t \frac{1-e^{-\alpha(t-s)}}{\alpha} dW_s. \tag{4.3}$$

So with the Equations (3.11), (4.1), (4.2) and (4.3), we can do the simulation for both univariate and multivariate OU processes. We used the same method, as described in Equation (4.1), to evaluate the stochastic integrals in both the multivariate model and the univariate model. However, for the univariate model, there is a much faster and more accurate method to evaluate the stochastic integrals. By calculating the covariance between Y_t and Y_{t+k}, we can obtain explicit results for the stochastic integral in Equation (4.3). For the multivariate model, we encountered numerical problems when working with the matrices of dimension 6.

4.2 Simulated Results

We simulated 5000 sets of realizations independently for our study. For each realization, we first simulate the annual rates of return from now to 100 years later with the latest observations from the historical data as the starting values for the three assets with both univariate model and multivariate model. Then we change the starting value and analyze the differences between the simulated results. With the simulated rates of return, we can price annuities under different asset allocation strategies. One goal is to study annuity prices and optimal asset allocation strategies for different models and different starting values.

4.2.1 Simulated Rates of Return

First let us look at the simulated rates of return for the three assets considered with their last values of the historical data we collected as the starting values. In this case, the starting values are lower than their long term means.

The mean of the simulated results are shown in Figure 4.1. For each graph, the black line is the result produced by the multivariate model and the colored line is the result produced by the univariate model. The graphs show a typical mean reverting process. For each asset, both univariate model and multivariate model show that the processes return to the same long term mean, which we choose to be the mean of each asset's historical data we collected. Long term bond's annualized historical mean is about 7.14%, short term bill's annualized historical mean is about 5.55% and equity's annualized historical mean is about 6.97%. Those numerical values are the ones we applied in the model as the three assets long term means for this project. As the graphs show, our simulated results are consistent with the models we set. Also the three assets show one common characteristic that it takes more time for the multivariate model to go back towards the long term mean. However, in the "short term", the processes produced by different models have different trajectories to approach the long term mean. The "short term" is actually not that short. For example, in the multivariate model, it takes the equity rates of return about 50 years to go back to the long term mean, while in the univariate model, it takes less than a day to return to the long term mean and stay at that level. Another thing one may notice is that for long term bond and short term bill, both the univariate model and the multivariate model start from the same value and end up with the same long term mean for each asset. The trajectory of these two assets using a multivariate model is quite similar with the one using a univariate model. This is consistent with the estimated drift term of the two models. In the multivariate case, for example, the rate of return of long term bond at time t mainly depends on its rate of return at time $t-1$, but is not affected by short term bill very much and even less influenced by the equity's rate of return at time $t-1$.

Also from the graph, it looks like equity shows a very different pattern. This can be

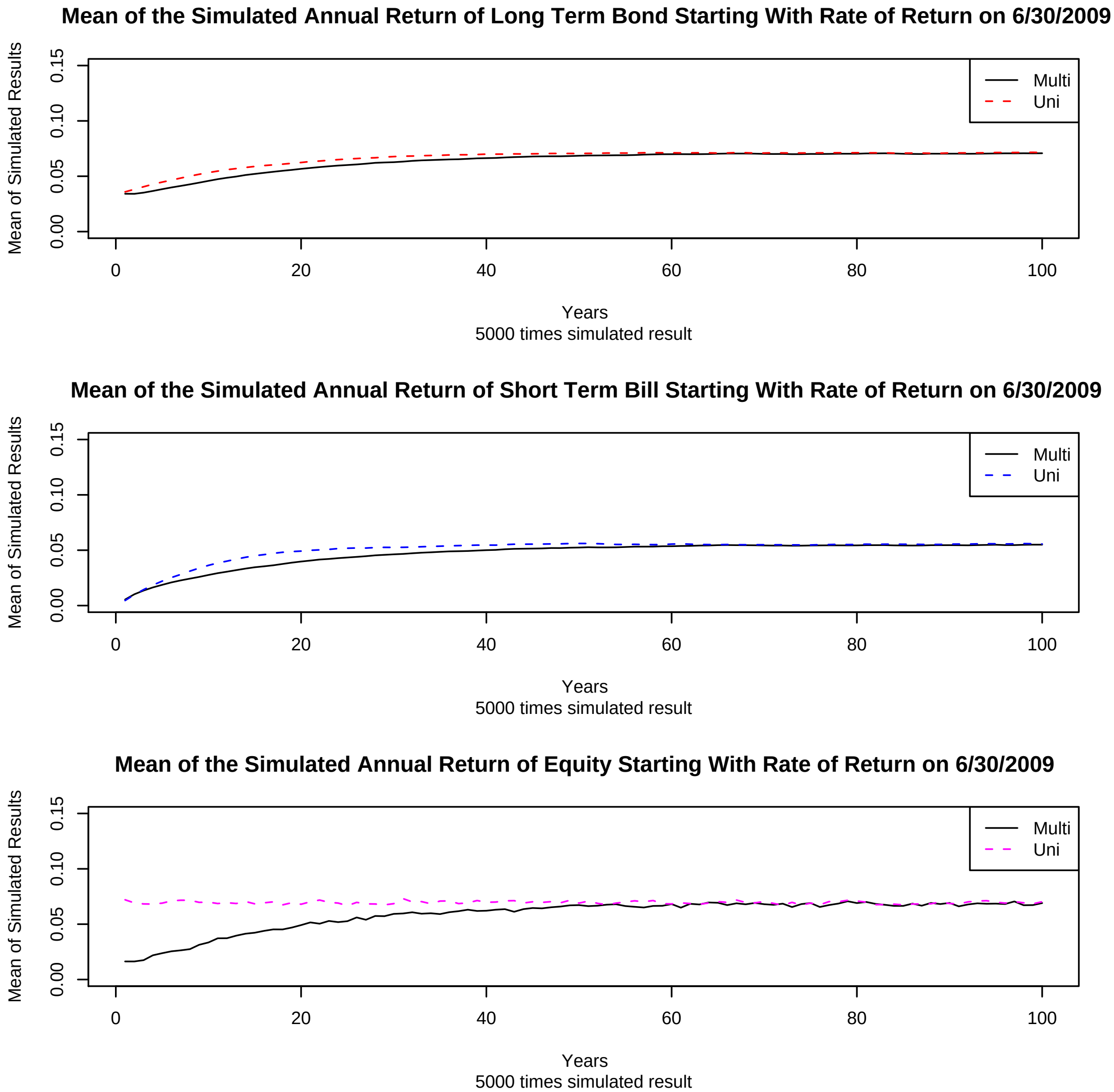


Figure 4.1: Mean of the simulated annual rate of return with starting value lower than long term mean

explained by the parameters we estimated for the model. As discussed in Chapter 3, α determines how fast the process is expected to go back towards the long term mean. When time t goes to infinity, the process should approach its long term mean. To get a general idea of how long it will take a process to go back towards its long term mean, we use $\frac{1}{\alpha}$. After $\frac{1}{\alpha}$ units of time, we can expect the process to be about 63%, which is $1 - e^{-1}$, closer to its long term mean. For both the long term bond and the short term bill, their α is at the level of 10^{-4} or $\frac{1}{\alpha}$ is at the level of 10^4 days. This means it will take decades before the process is expected to be even 63% closer to the long term mean. So at the end of the first year, the rates of return simulated by the two models for both long term bond and short term bill are not significantly different. However, when we look at the equity, for the first 40 to 50 years, the simulated rate of return is very different for the two models. In the univariate model, the equity α is about 7.7, which means it will take less than one day for its rate of return to go back to the long term mean. However, if we look at the parameters for the vector AR(1) process, the equity rate of return is greatly affected by rates of return of the long term bond and the short term bill. The centered rate of return of equity at time t is the sum of 1.103105 times of previous day's long term bond return and 0.285538 times previous day's short term bill return and 2.88×10^{-4} times of the previous day's equity return and a random term. So we can expect that in the multivariate model, when the long term bond and short term bill's rates of return are below the long term mean, so will the equity. So basically, if the rates of return of long term bond and short term bill are low, one can expect the same pattern for equity.

We will now look at the variance of the simulated results. Figure 4.2 shows that both the multivariate model and the univariate model are stationary processes with finite variance. We show $\mathrm{Var}(\int_t^{t+1} \underline{X}_t dt)$, but one could also look at $\mathrm{Var}(\int_0^t \underline{X}_t dt)$. As we can see, 100 years is long enough for the process variance to become stable. Unfortunately, we were not able to obtain the theoretical variance of $\underline{Y}_t$ for a multivariate OU process in a form that can be useful in numerical calculation. We can look at the variance of X_t for both multivariate and univariate processes, these graphs are shown in Appendix A. The following table shows

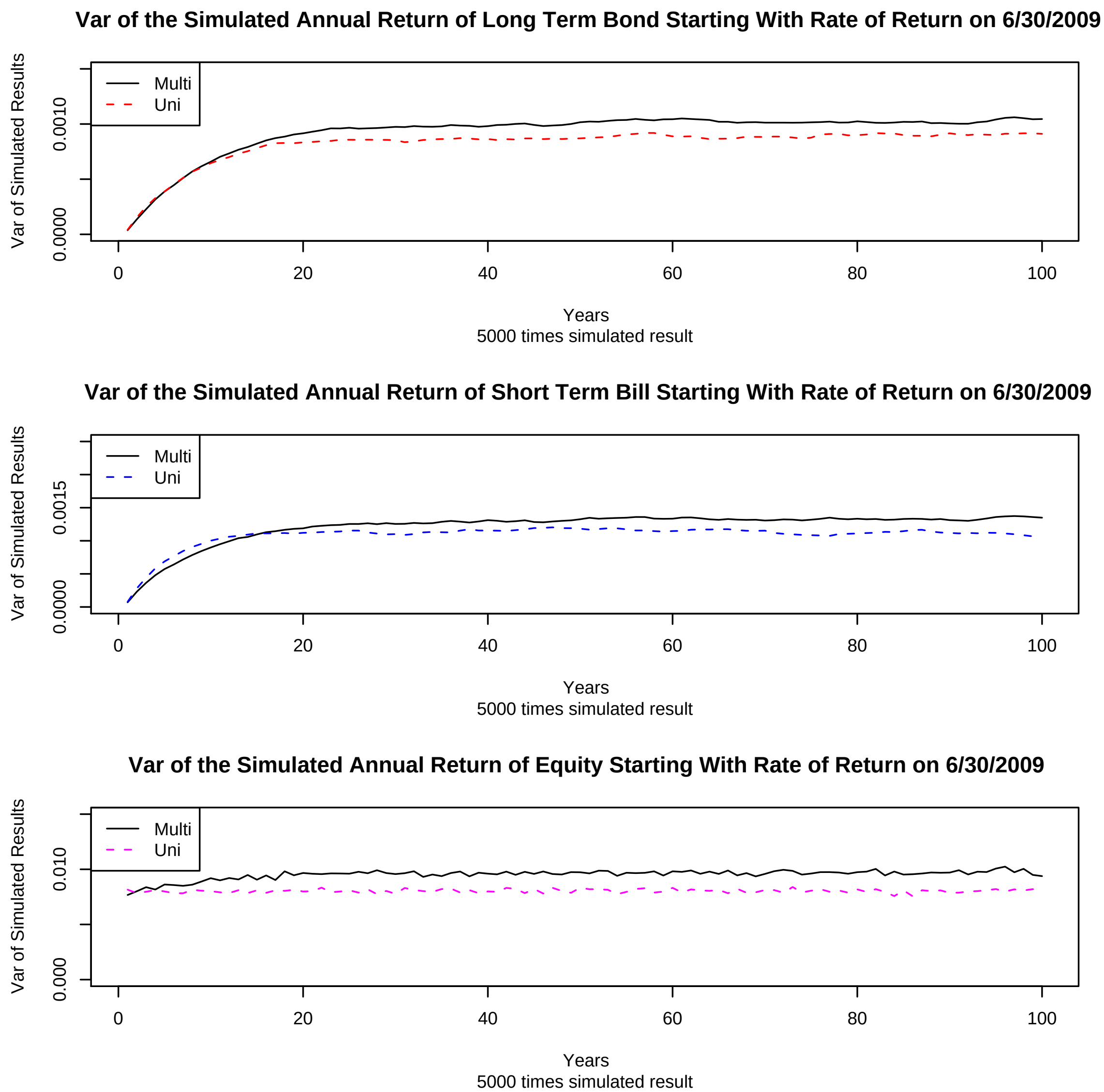


Figure 4.2: Variance of the simulated annual rate of return with starting value lower than long term mean

$\text{Var}(X_t)$ after 100 years for the three assets with different models and the theoretical values are consistent with our simulated values.

	$\text{Var}(X_t)$	
Asset	Multivariate Model	Univariate Model
Long Term Bond	1.683343e-08	1.427813e-08
Short Term Bill	2.208272e-08	1.872274e-08
Equity	0.0001234083	0.0001233942

Table 4.1: $\text{Var}(X_t)$ with multivariate and univariate models for three assets

The trajectories are quite similar for the two models. However, there are two differences one may notice. First, in the "short term", the univariate model has a higher variance than the multivariate model. This actually applies to all three assets, but here the graph for short term bill is the most obvious one. If we zoom in on only the first 10 years, which is shown in Appendix A, we can see the same pattern more clearly for the other two assets. The other difference is in the long term, the multivariate model always has a higher variance than the univariate model for all three assets. This can be explained by the characteristics of the models themselves. The univariate model will consider each asset independently, while the multivariate model will take into account the covariance between assets. This eventually makes the variance of the multivariate model higher than the univariate model. Assuming a series is truly independent from other series, then fitting in a univariate model is sufficient. But if the series is really dependent on some other series, the existence of the correlations between different assets cannot be fully captured by a univariate model. This might result in a lower long term variance in the univariate case compared to the multivariate one, since it is missing some covariances with other assets. Also when the series is dependent of other series, its sample data has more volatilities, which has to be picked up by the model somehow. So as a compensation, the univariate model has a higher variance for the short term. As we can see from the previous chapters, the multivariate model introduced more parameters than the univariate model, which allows more flexibility when fitting the model.

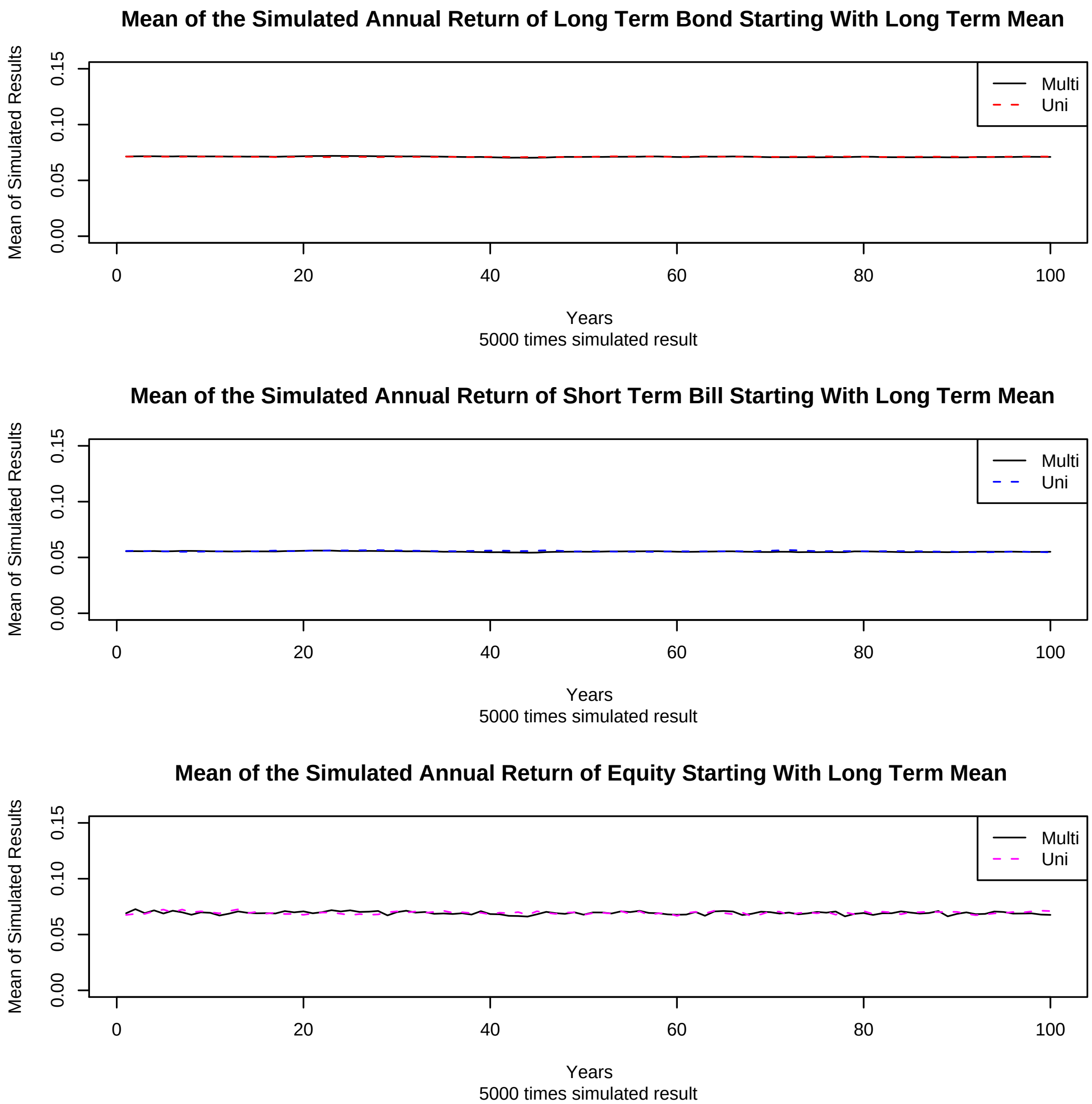


Figure 4.3: Mean of the simulated annual rate of return starting with the long term mean

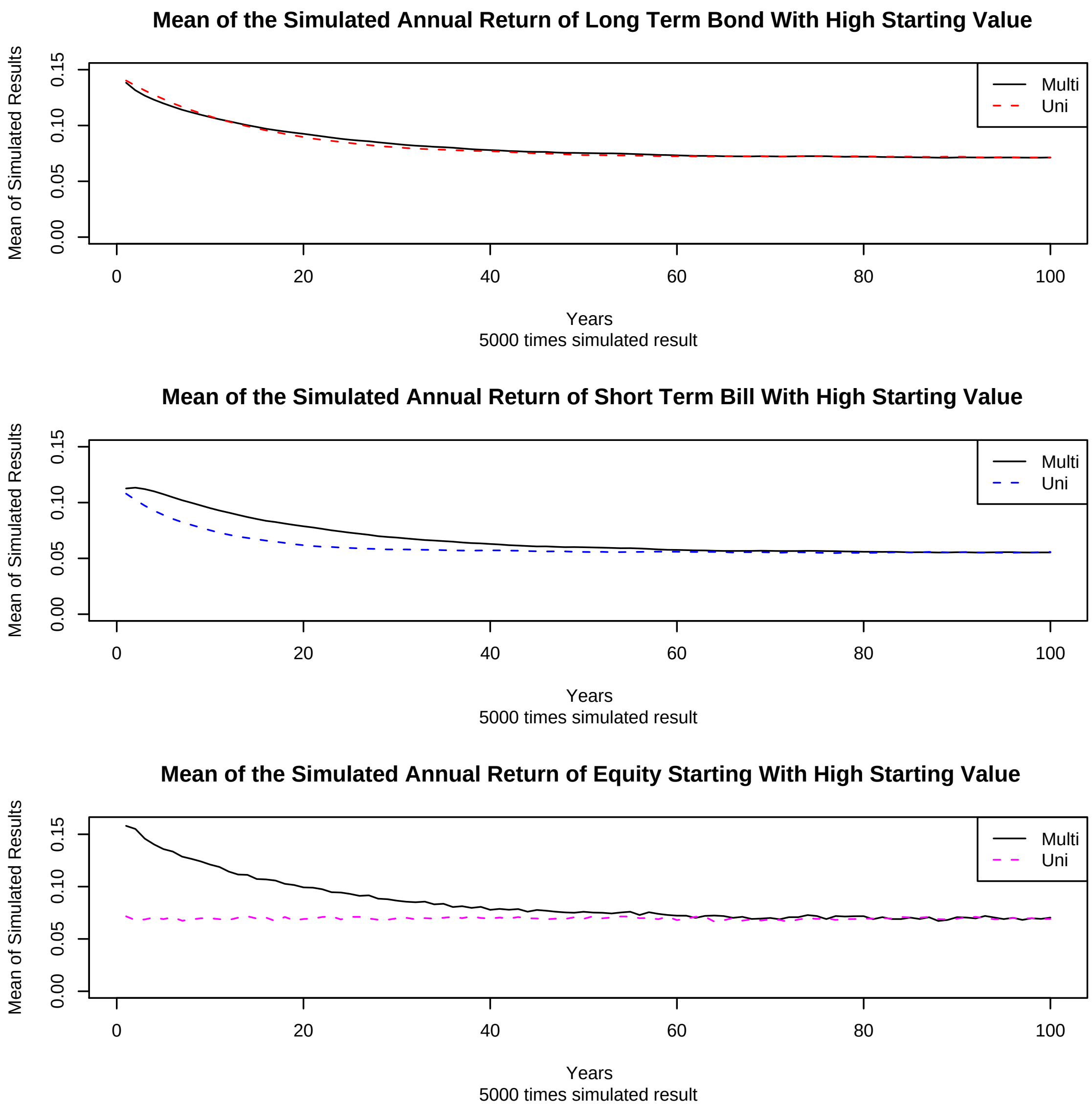


Figure 4.4: Mean of the simulated annual rate of return with starting value higher than long term mean

Figure 4.3 and 4.4 show the mean of simulated annual rates of return with different starting values. One set of the results is using the long term mean of each asset as the starting value. The other set is using a starting value that is higher than the long term mean. We arbitrarily chose twice the long term mean as the starting value for each asset. As we can see, when starting with the long term mean, the annual rate of return tends to stay there. In our case, both univariate model and multivariate model result in a flat line at the long term mean for each asset. When the starting value is higher than the long term mean, we notice that, similar to the case when the starting value is lower than the long term mean, it takes more time for the multivariate model to go back to the long term mean. The graphs for the variances of the simulated results are in the Appendix A, mainly because the pattern of the variance is almost the same regardless of the starting value.

When using the simulated annual rates of return to price an annuity, we will see that the differences created by the nature of two models can result in a different optimal asset allocation strategy. Even when the starting value is the long term mean and the mean of the simulated annual rates of return for both models are overlapping, the two models can still produce different results for the annuity prices.

4.3 Annuity Pricing

We now look at annuity prices produced by our simulated rates of return. The annuity is priced for a 65-year old male with the 1994 Uninsured Pensioner Mortality Table (UP94 Table). First, we review some functions useful for annuity pricing and then we look at some asset allocation strategies. At last, we will show some results with the simulated rates of return.

4.3.1 Discount Factor and Actuarial Present Value

Define the rate of return accumulation function as $Y(t) = \int_0^t \delta_s ds$. So the discount factor is $e^{-Y(t)}$. Since $Y(t)$ is a Gaussian process, the discount factor, $e^{-Y(t)}$, follows a lognormal distribution. Therefore, we have

$$\mathrm{E}(e^{-Y(t)}) = e^{-\mathrm{E}(Y(t))+0.5\mathrm{Var}(Y(t))}, \tag{4.4}$$

$$\mathrm{E}(e^{-(Y(t)+Y(s))}) = e^{-(\mathrm{E}(Y(t))+\mathrm{E}(Y(s)))+0.5(\mathrm{Var}(Y(t))+\mathrm{Var}(Y(s))+2\mathrm{Cov}(Y(t),Y(s)))}, \tag{4.5}$$

and

$$\begin{aligned}\mathrm{Var}(e^{-Y(t)}) &= \mathrm{E}(e^{-2Y(t)}) - \mathrm{E}^2(e^{-Y(t)}) \\ &= e^{-2\mathrm{E}(Y(t))+2\mathrm{Var}(Y(t))} - e^{-2\mathrm{E}(Y(t))+\mathrm{Var}(Y(t))} \\ &= e^{-2\mathrm{E}(Y(t))+\mathrm{Var}(Y(t))}(e^{\mathrm{Var}(Y(t))} - 1).\end{aligned} \tag{4.6}$$

The actuarial present value of future payments is the function we will use to price an annuity. Assume a \$1 payment is made at the beginning of year t, $t = 1, 2, \ldots$, if the annuitant is alive. The probability that the annuitant now age x will be alive at the beginning of year t is ${}_{t-1}p_x$. So the actuarial present value of this contingent \$1 payment at time $t-1$ is $1 \cdot_{t-1} p_x \cdot e^{-\int_0^{t-1}\delta_s ds}$. Let ω be the maximum age of the mortality table. Then the present value of a whole life annuity sold to a person age x can be calculated as

$$\ddot{a}_x = \sum_{t=0}^{\omega-x-1} {}_tp_x e^{-\int_0^t \delta_s ds} = \sum_{t=0}^{\omega-x-1} {}_tp_x e^{-Y(t)}. \tag{4.7}$$

The ${}_tp_x$ is calculated from the UP94 mortality table and the discount factor $e^{-Y(t)}$ is simulated by the univariate and multivariate OU processes.

4.3.2 Asset Allocation Strategy

As mentioned earlier, for simplicity, we are assuming that only three kinds of assets are available in the market, namely, 10-year long term bond, 3-month short term bill and equity. We are using a continuous process to model the interest rate and we also assume the total asset is rebalanced frequently. The investment strategy consists of allocating the total asset between bond, bill and equity. An example could be 60% invested into long term bond, 10% of the total asset into short term bill and 30% into equity. Since we are rebalancing frequently, the portion that is allocated to each asset will remain the same all the time.

We know that different investment strategies will result in different prices and risks of annuities. So we will study different asset allocation strategies with different proportion invested in the three assets. The higher the rate of return is, the lower the annuity price is. We are trying to find the asset allocation strategy that yields the lowest annuity price, which in this project we consider as the optimal asset allocation strategy. Of course, one can also consider other criteria, such as value at risk, to find their own optimal asset allocation strategy, but those are not in the scope of this project. We will see from our results that multivariate and univariate models will provide different results. Also different starting values of the rate of return can make a difference on the optimal asset allocation as well.

The annuity prices and the variances for different asset allocation strategies are shown in the following 3-D graphs. The graphs only show the percentage invested in long term bond and short term bill since the rest is invested in equity. The total proportion invested in the three assets is 100%. The blank part of the surfaces in those 3-D graphs are the ones where the proportion invested in long term bond and short term bill together would be greater than 100%. To make sure of a clear view of the plots, not all the graphs are viewed from the same angle.

First, let us look at the scenario where the starting value of the rate of return is below its long term mean. The multivariate model shows that the lowest annuity price is when 100% of the total asset is put in long term bond. However, the univariate models shows the best way is to put 100% in equity. And this can be explained by the behavior of the two models. From figure 4.1, we see that it takes a much longer time for equity to go back to its long term mean with a multivariate model than a univariate model. And mainly due to this characteristic, the two models lead to different optimal asset allocation strategies. With the univariate model, the annual rates of return for equity are approximately at the level of its long term mean, which is higher than the annual interest rates for long term bond during the first 40 years. This characteristic makes it reasonable to invest 100% in equity to reach the lowest annuity price. Also the sample variances from the simulated results are quite different for the two models. When the multivariate model starts with a relatively low rate of return for equity, it will stay there for a long time. As a result, the rate of return can be even negative. However, the univariate model for the equity's rate of return can revert back to its long term mean within a day. And that is why the simulated annuity values generated by the multivariate model have variances that are much higher than the ones generated by the univariate model.

From figure 4.8, we see that the annuity value has a higher variance when assets are invested 100% in equity than when 100% are invested in short term bill, although short term bill is a less risky asset than equity. One reason is that the mean of Y_t for the short term bill is much smaller than the equity mean of Y_t. Therefore, $e^{-\mathrm{E}(Y_t)}$ is larger when $\mathrm{E}(Y_t)$ is smaller. Also, in the univariate model, the equity rate of return is almost a White Noise, so daily rates of return are independent from each other. However, there is great serial dependence between short term bill and long term bond. After a certain number of years, both short term bill and long term bond can have a larger variance for Y_t than equity. From Equation (4.6), which shows the variance of each payment, we can tell that the variance of the annuity value depends on the mean, variance and auto-covariance of Y_t. So these are the reasons that, although the short term bill is considered almost a risk free asset, investing 100% of the assets in short term bill can be a riskier asset allocation strategy than 100% assets invested in equity. However, such distortion can be avoided by using a multivariate model which takes the correlation among the three assets into consideration.

Then we changed the starting value for the multivariate and univariate model. The mean of the rates of return generated by the two models are almost the same, as shown in Figure 4.3. The only thing that is different for the two models are the variances. Still, for our asset allocation strategies, we get totally different results from the two models. For the multivariate model, the optimal strategy is still to invest 100% in long term bond, which makes sense because long term bond has the highest long term mean among the three assets. For the univariate model, the optimal strategy is to invest 80% in long term bond and 20% in equity. This result is not quite consistent with that of Blake et al (2001). One of their conclusion is "conservative bond-based asset-allocation strategies require substantially

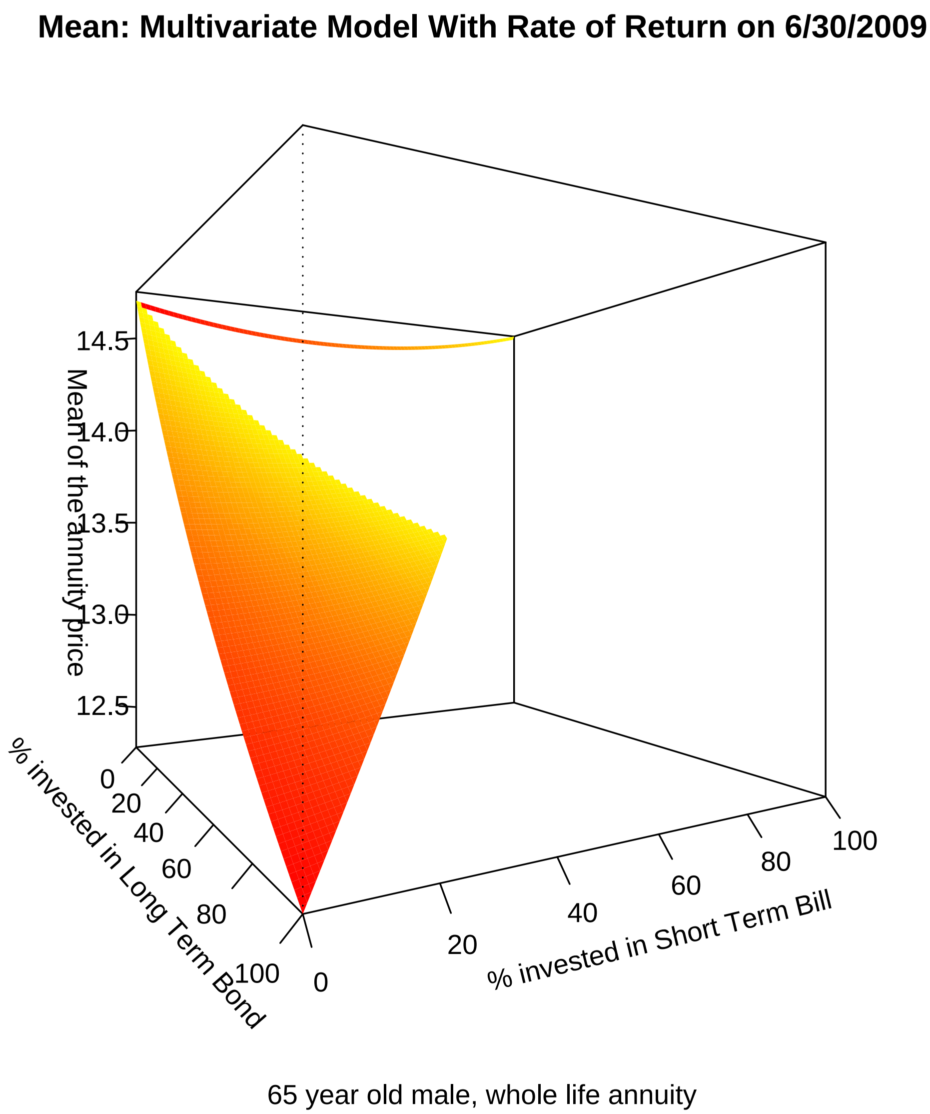


Figure 4.5: Mean of the annuity prices with the simulated rate of return from multivariate model starting with rate of return on 6/30/2009

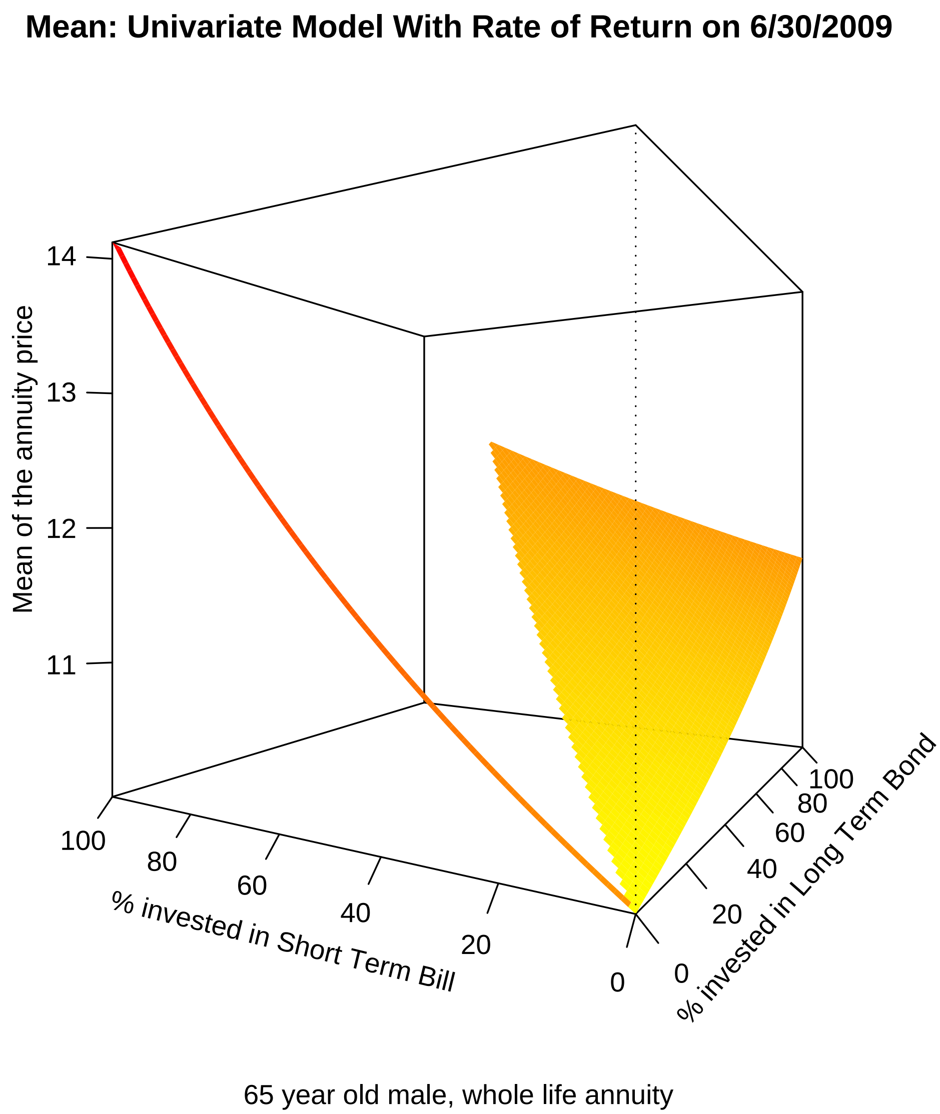


Figure 4.6: Mean of the annuity prices with the simulated rate of return from univariate model starting with rate of return on 6/30/2009

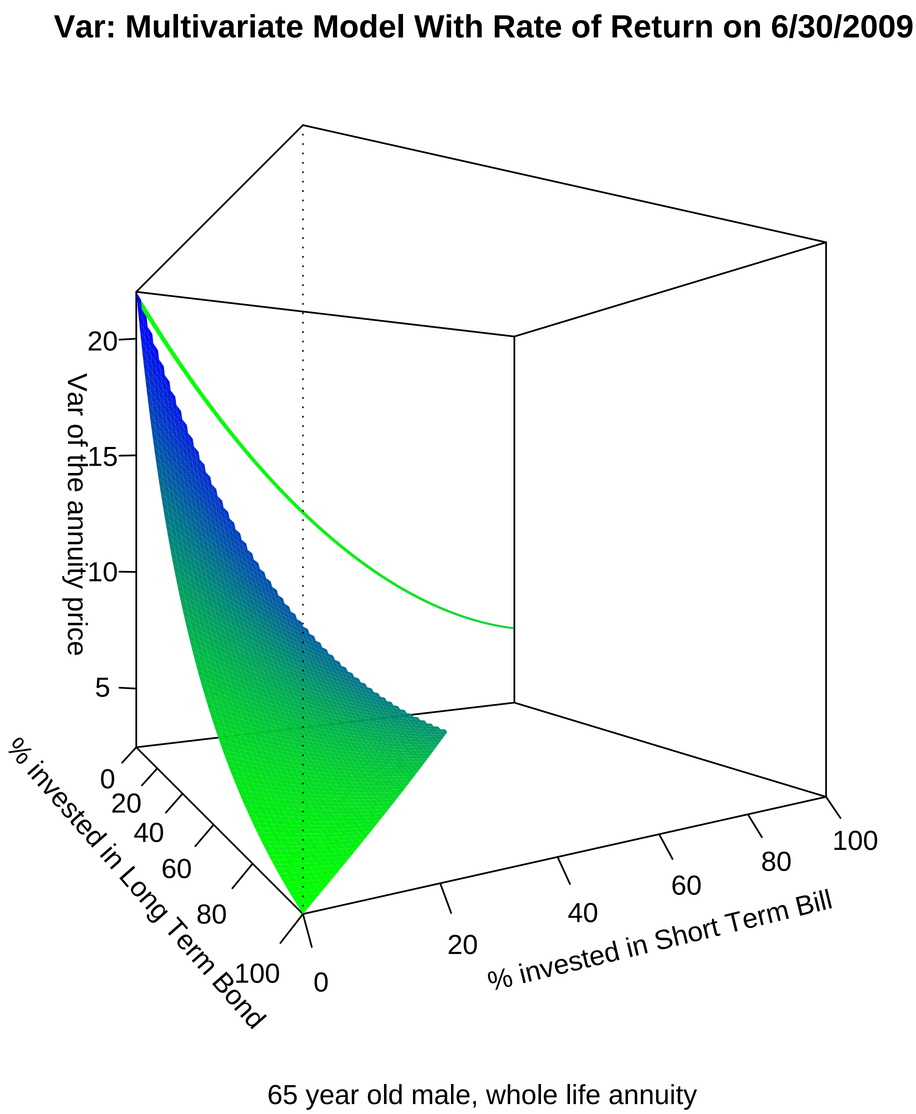


Figure 4.7: Variance of the annuity prices with the simulated rate of return from multivariate model starting with rate of return on 6/30/2009

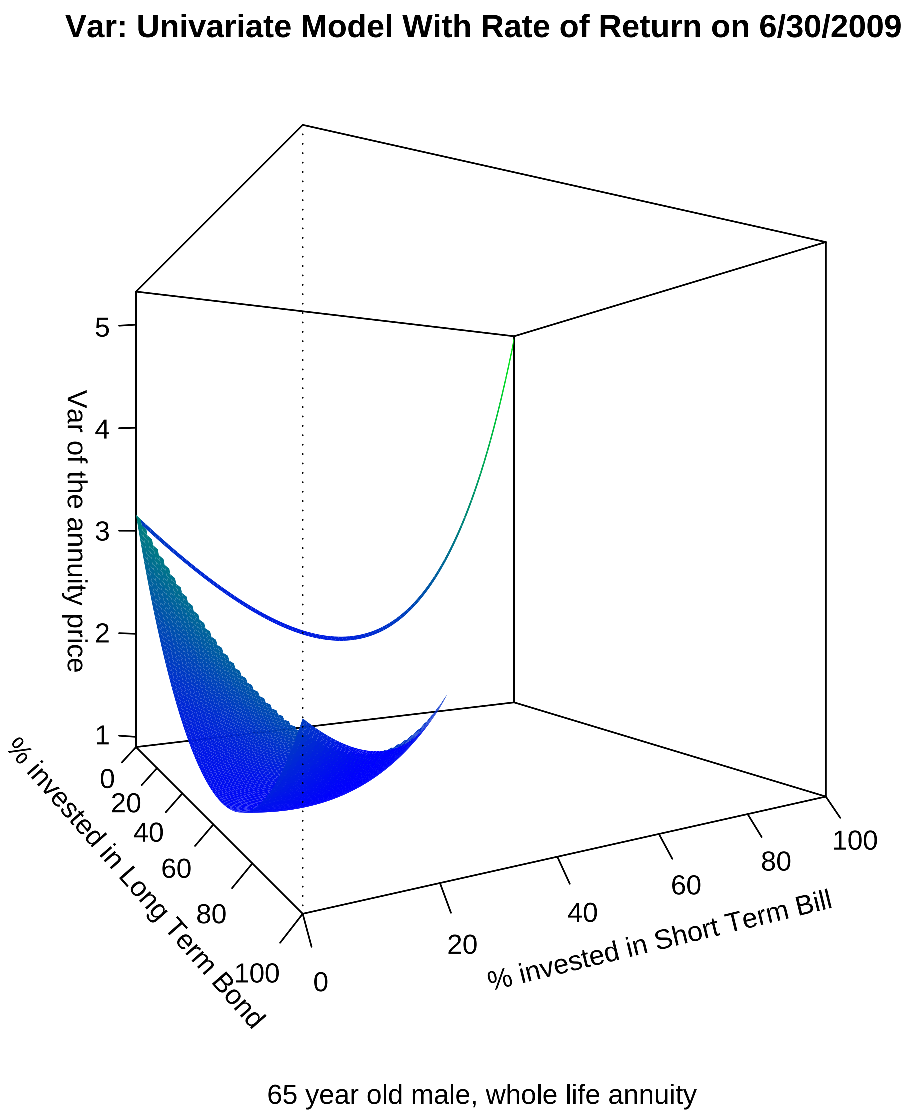


Figure 4.8: Variance of the annuity prices with the simulated rate of return from univariate model starting with rate of return on 6/30/2009

higher contribution rates than riskier equity-based strategies if the same retirement pension is to be achieved.” Translating this into our case, it means that conservative bond-based asset-allocation will result in a higher annuity price than equity-based strategies. However, this can be explained by the nature of the data we collected. As we pointed out in Chapter 3, the average rate of return of long term bond is higher than equity, but in Blake et al (2001) the rate of return of bond is much lower than the rate of return of equity. Blake et al (2001) also mentioned that a static asset-allocation strategy with a high equity weighting delivers better result. Because of the high rate of return of the long term bond in this project, the conclusion that an asset allocation strategy with a high weight in higher return asset is the best for a long term investment still stands, except in our case the highest return asset seems to be the long term bond.

At last, we can take a look at the case where starting values are higher than the long term means. The multivariate model shows that the optimal asset allocation strategy is to invest 100% in equity, while the univariate model shows that investing 100% in long term bond is the optimal strategy. Again, this is related to the fact that the equity rates of return from the multivariate model takes more time to return to its long term mean. As Figure 4.4 shows, even if long term bond has a slightly higher long term mean than equity, in the first few years equity still delivers higher rate of return than long term bond for the multivariate model. That is why investing 100% in equity is the optimal strategy for the multivariate model.

Another thing that the three different starting value scenarios have in common is their variance graphs. The univariate model shows that a diversified portfolio can reduce the variance, which means the risk. However, in the multivariate model, the lowest variance is always reached when 100% asset is invested in long term bond, the least risky asset.

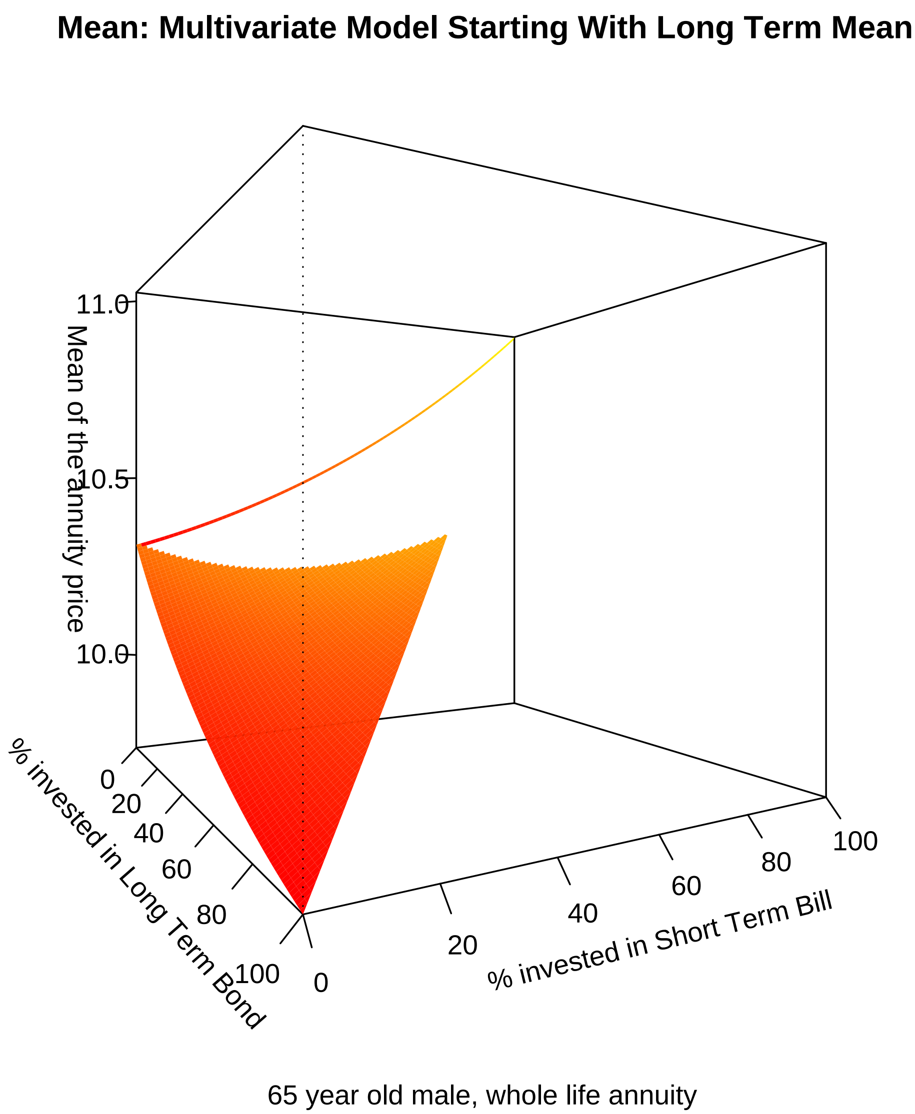


Figure 4.9: Mean of the annuity prices with the simulated rate of return from multivariate model starting with the long term mean

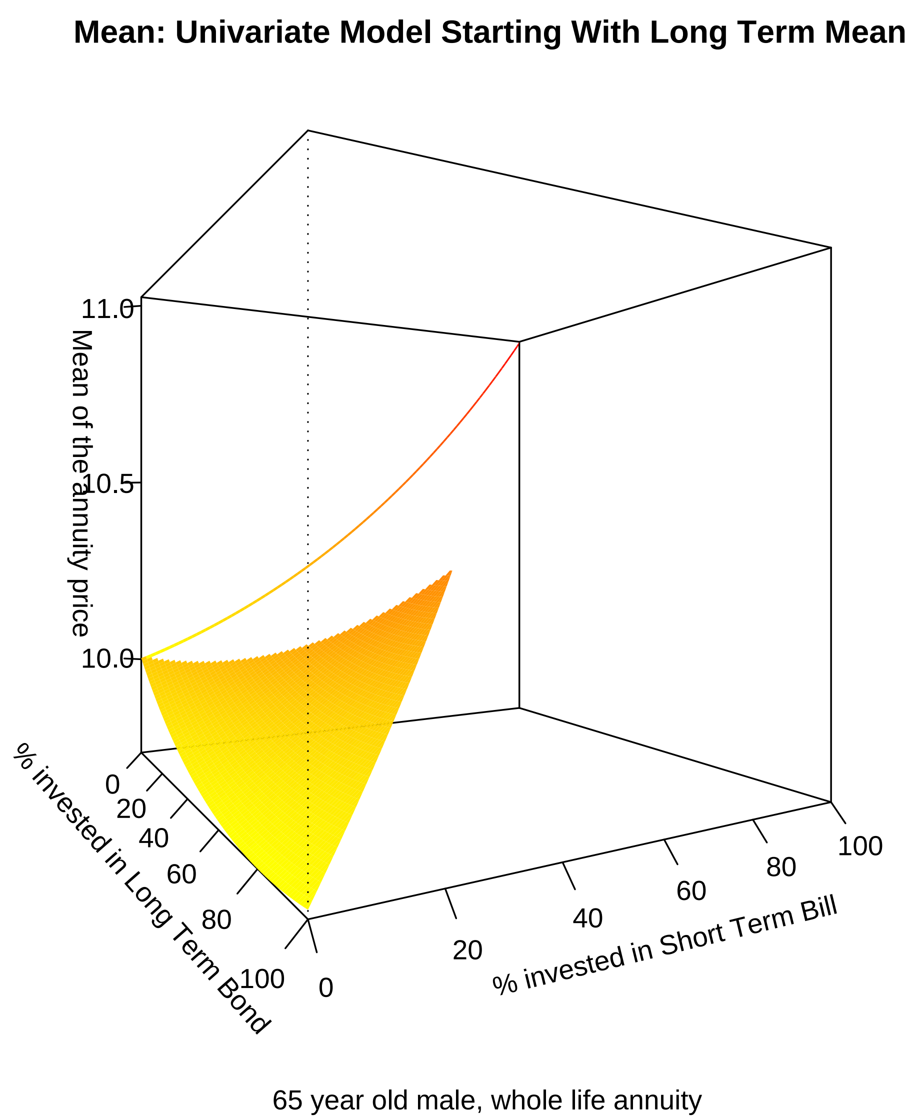


Figure 4.10: Mean of the annuity prices with the simulated rate of return from univariate model starting with the long term mean

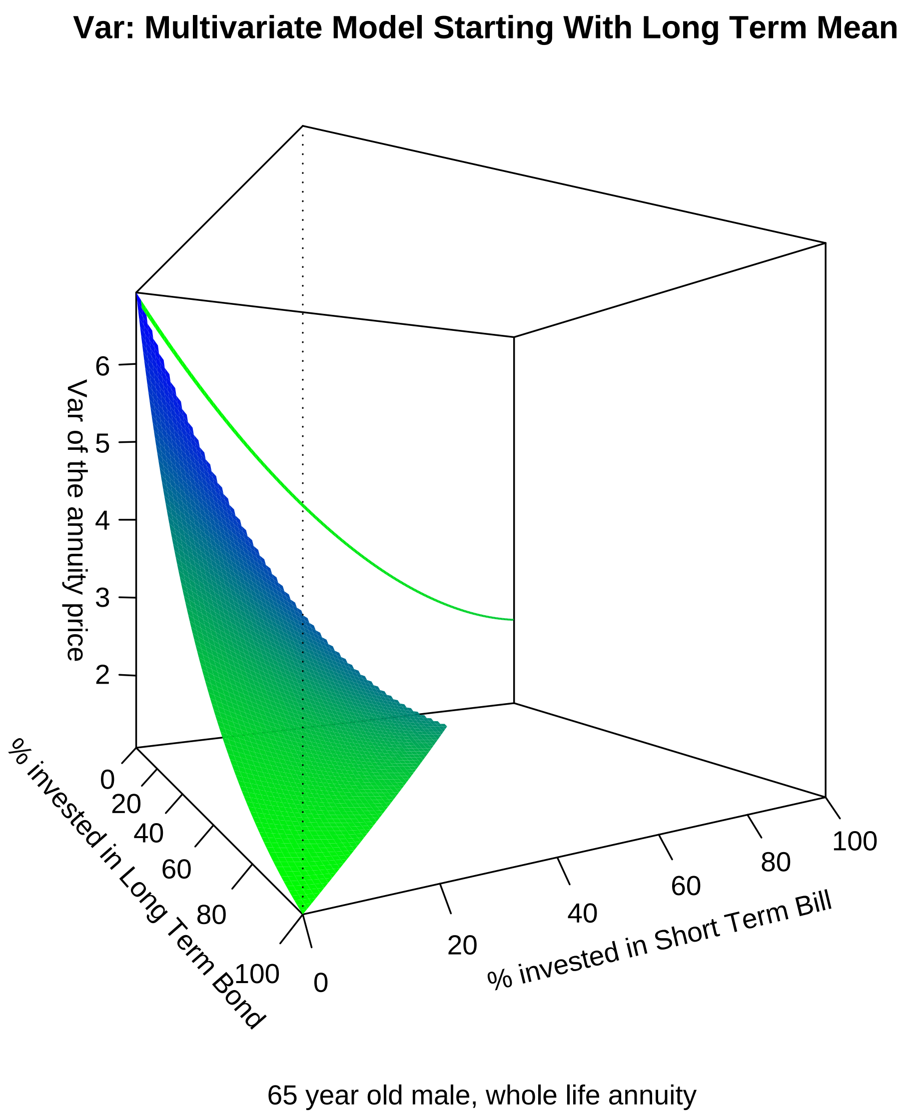


Figure 4.11: Variance of the annuity prices with the simulated rate of return from multivariate model starting with the long term mean

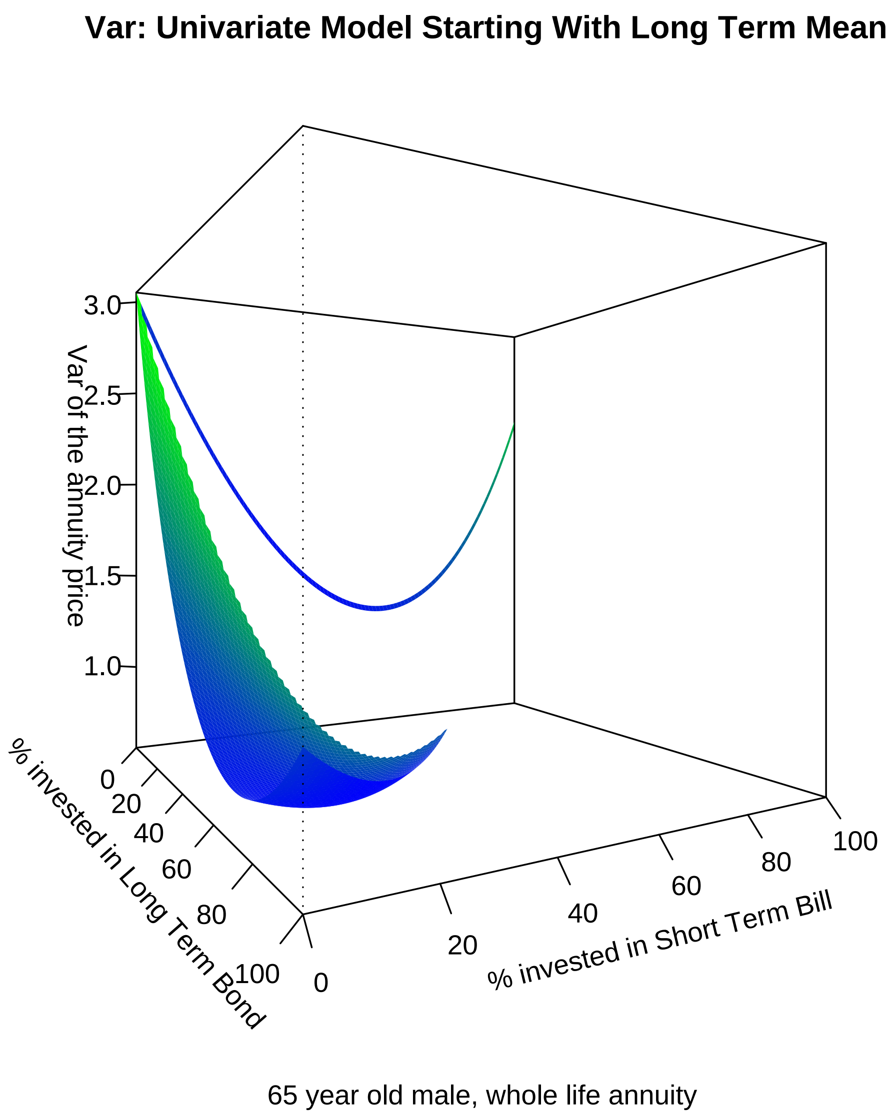


Figure 4.12: Variance of the annuity prices with the simulated rate of return from univariate model starting with the long term mean

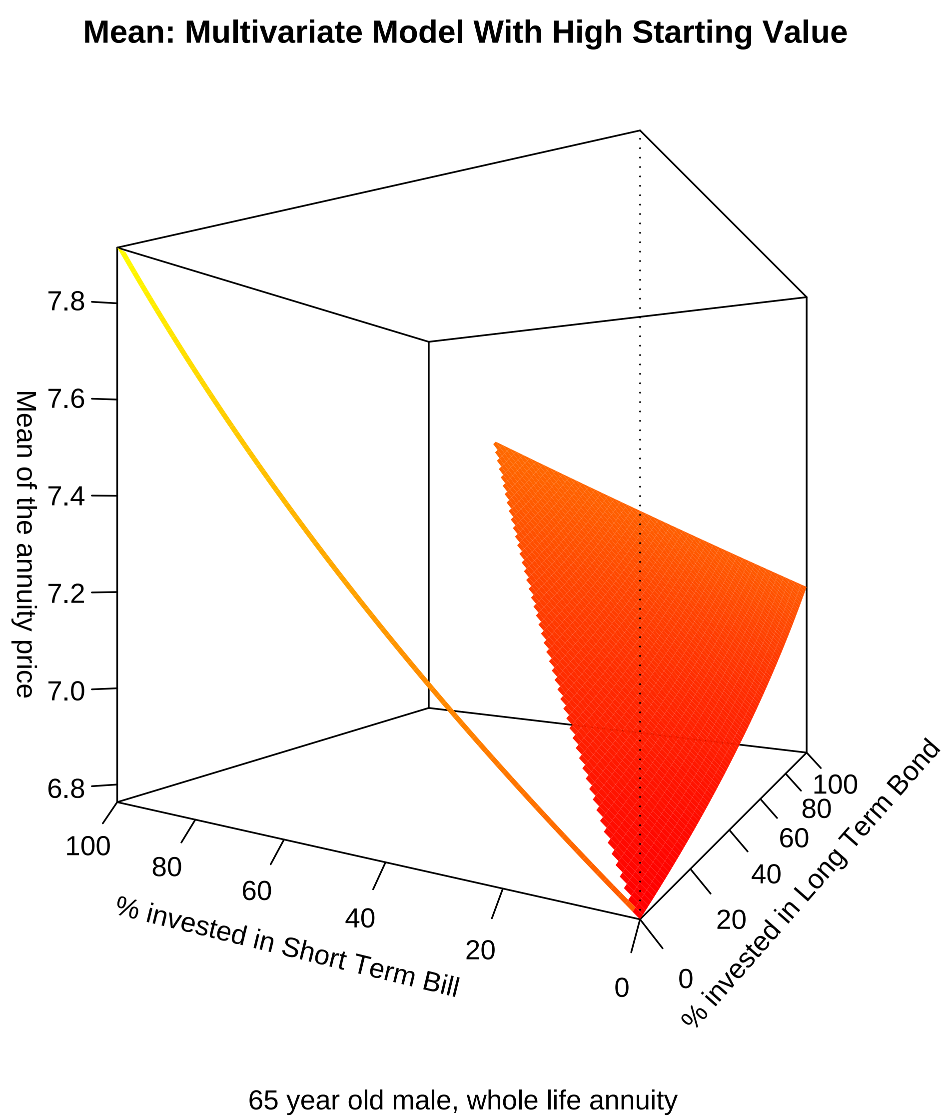


Figure 4.13: Mean of the annuity prices with the simulated rate of return from multivariate model with high starting value

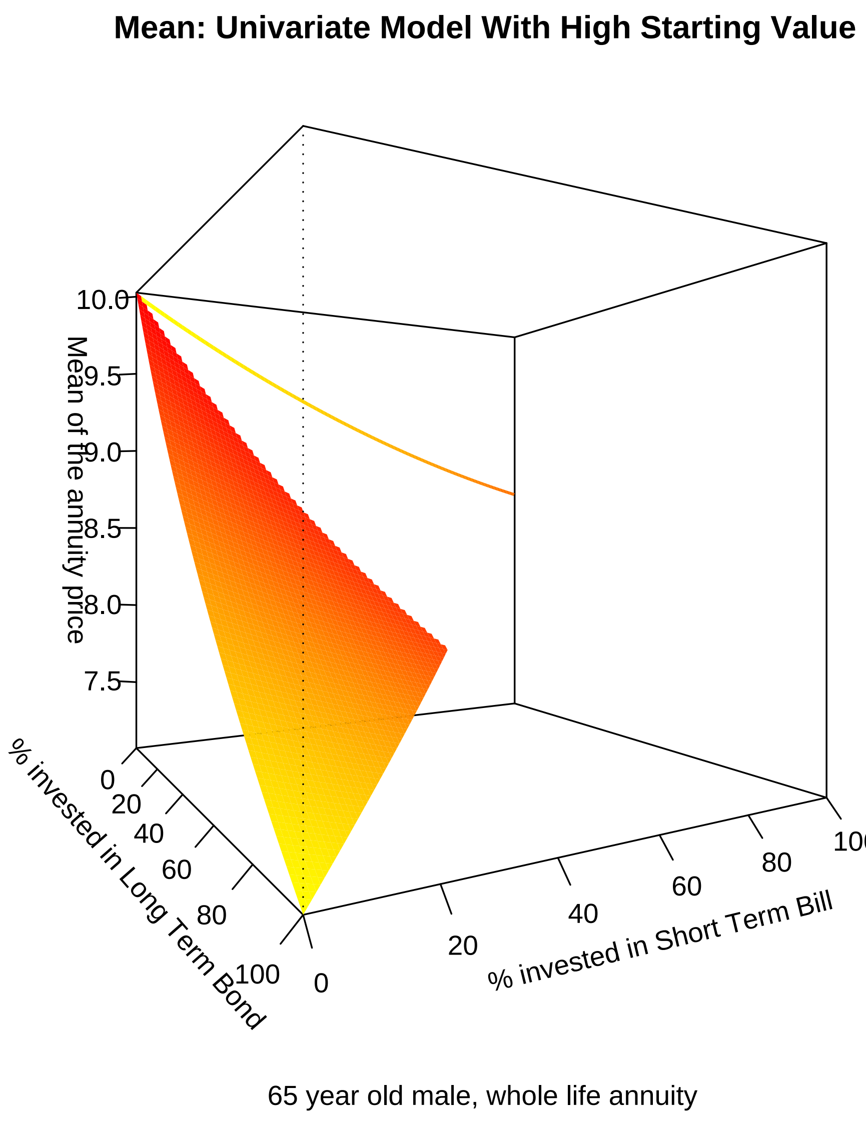


Figure 4.14: Mean of the annuity prices with the simulated rate of return from univariate model with high starting value

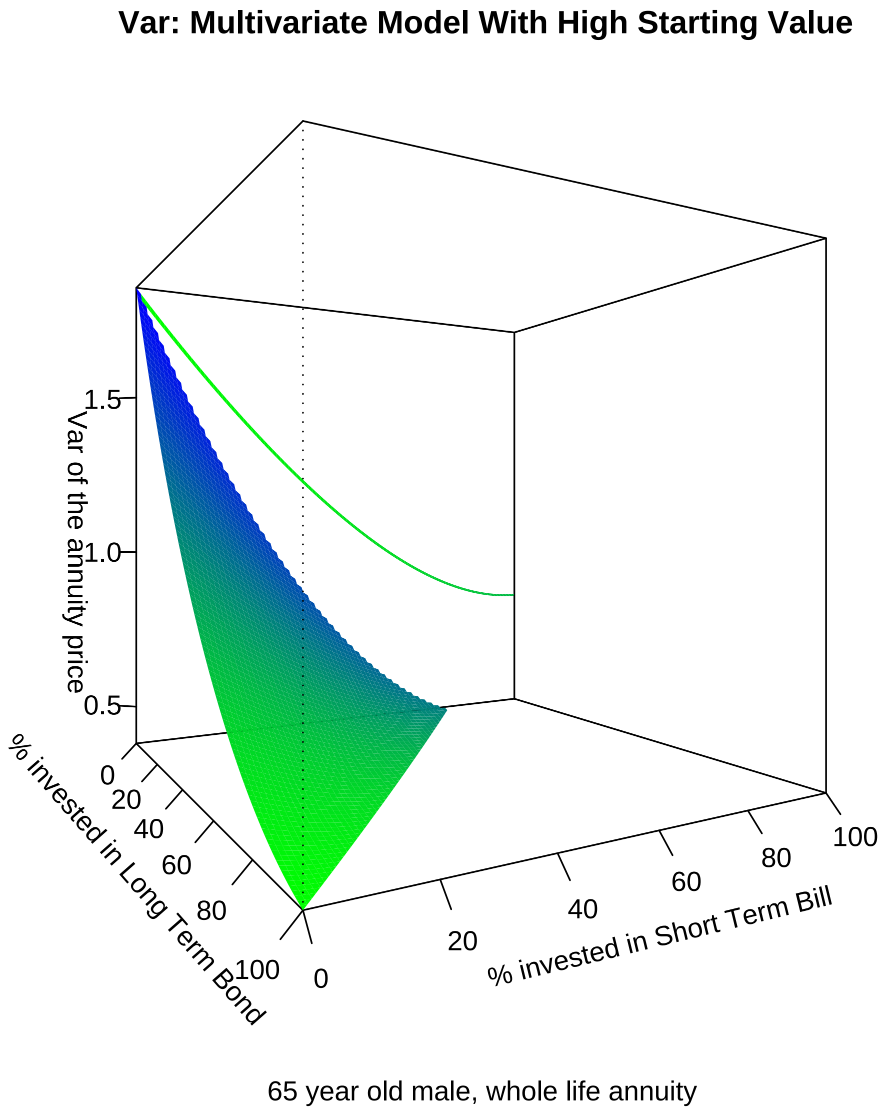


Figure 4.15: Variance of the annuity prices with the simulated rate of return from multivariate model with high starting value

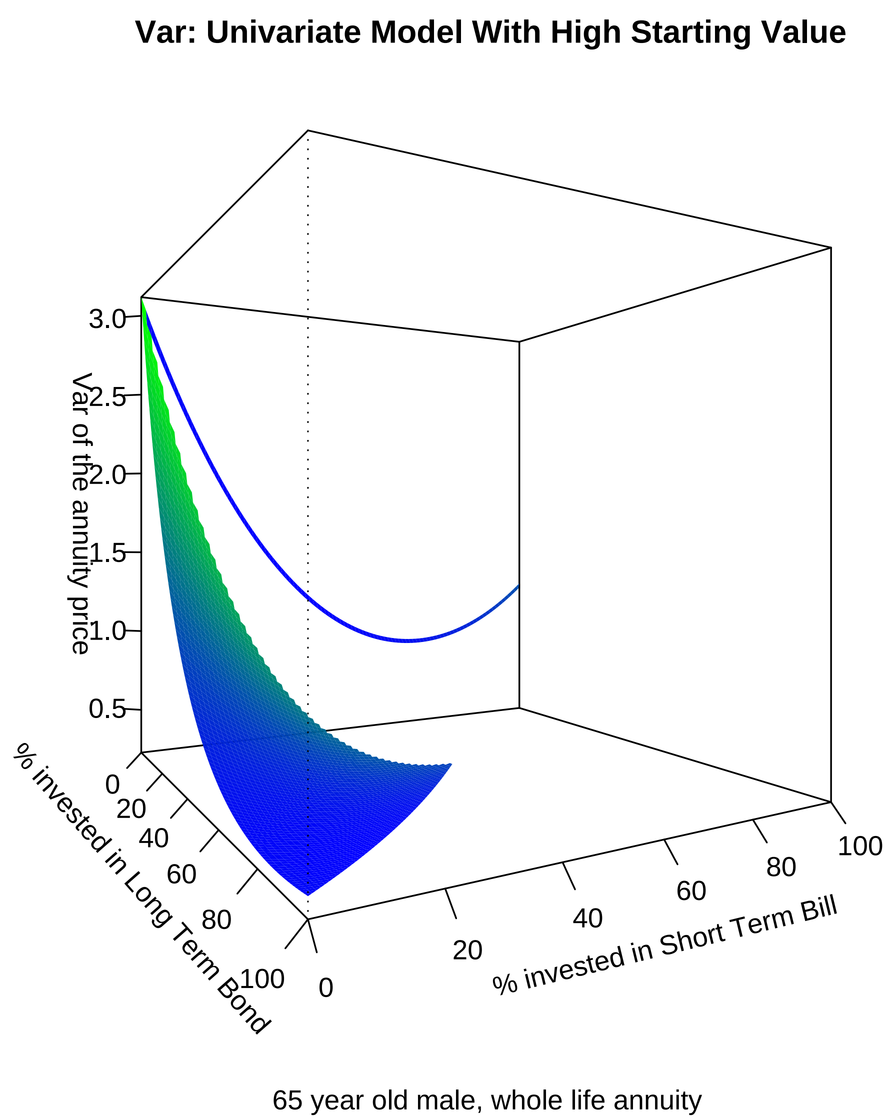


Figure 4.16: Variance of the annuity price with the simulated rate of return from univariate model with high starting value

Chapter 5

Conclusions

This project focused on the multivariate and univariate Ornstein-Uhlenbeck process to model the rates of return of three assets. The advantage of the multivariate model is that by introducing more parameters to the model, more characteristics of the rates of return can be captured and expressed than can be with the univariate model. Most importantly the correlation between different assets is taken into consideration in the multivariate model. In the multivariate model, although the rates of return of long term bond and short term bill don't depend on the rates of return of equity, equity's rates of return are greatly affected by long term bond and short term bill. This is also showed in the graphs in Chapter 4. When the long term bond and short term bill's rates of return are above (below) the long term mean of theirs, the equity's return is also above (below) its long term mean. However, the univariate model shows the mean of equity's annual rates of return will almost stay at the same level. Since the multivariate model also captures the covariance between one asset and another, in the long term, the variance of the annual rates of return is slightly higher than what the univariate model shows. However, in the short term, the univariate model shows a higher variance than the multivariate model. As for the asset allocation strategy to achieve the lowest annuity price for a 65 year old male, the two models show different results even with the same starting value and same long term mean.

Of course, using a multivariate OU process to model the rates of return is much more complicated and time consuming than using a univariate model. We can calculate the theoretical covariance matrix for a three dimensional process. However, when we combine the velocity and position process together as one system of stochastic differential equations, the matrix is raised to six dimensions, in which case we encountered significant calculation problem and we couldn't calculate the theoretical covariance matrix anymore. So we used sample variance and covariance of our simulated result to do the studies.

Also, there could be some improvements for annuity pricing. In this project, we only considered stochastic interest rates and took mortality rate as determined, but mortality rate could also change over time. So we can also consider using stochastic mortality rates. Besides, our calculation for annuity price is for pure premium, not including expenses or taxes, etc. As for the multivariate OU process, the most important future work is solving the computational problems, with which one could calculate the theoretical covariance matrix and also make the simulation much faster. At last, there is still a lot to learn about the properties of the multivariate OU process modeling the rates of return.

Appendix A

Other Simulated Results

In Appendix A, we show more graphs of simulated results related to Chapter 4. Figure A.1 zooms in on the first 10-year period of Figure 4.2 to show that the simulated results from an univariate model have higher variances than the ones from the multivariate model in the short term. Figure A.2 shows the variance of the simulated annual rate of return starting with the long term mean. The mean of these simulated results was shown in Figure 4.3. Figure A.3 shows the variance of the simulated annual rate of return with high starting value. See Figure 4.4 for the mean of these simulated results.

In addition to simulating the annual rate of return, which is $\int_i^{i+1} X_s ds$, we also simulated the rates of return on the last day of each year. Figures A.4 to A.9 show the means and variances of those simulated rates of return on the last day of each year for 3 different scenarios of starting values. Figures A.4 and A.5 show the case where the simulation is using the rate of return on June 30th, 2009 as the starting value. Figures A.6 and A.7 are for the case where the starting value is the same as the long term mean. Figures A.8 and A.9 are for the case where the starting value is higher than the long term mean.

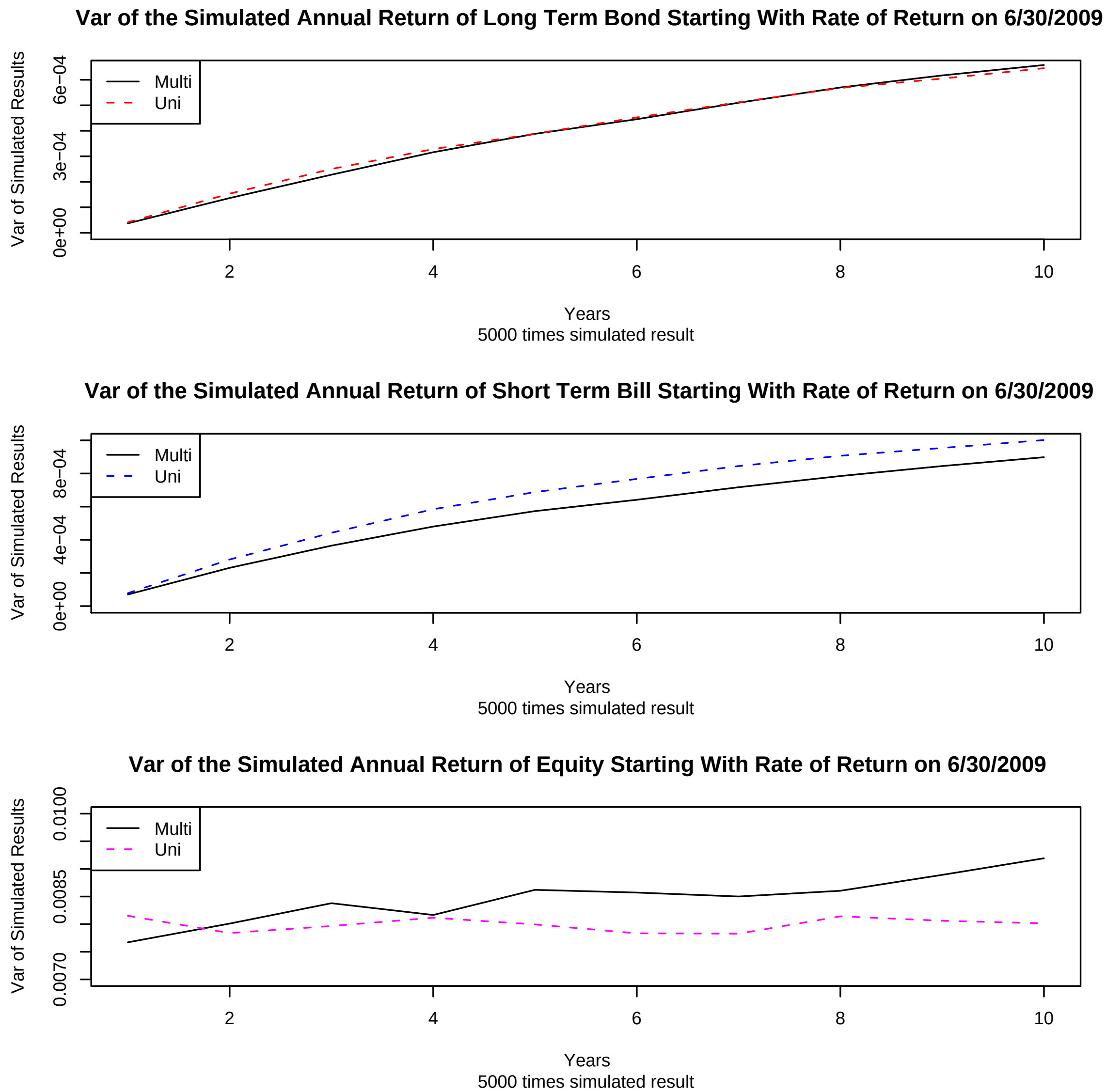


Figure A.1: Variance of the first 10 years simulated annual rate of return with starting value lower than long term mean

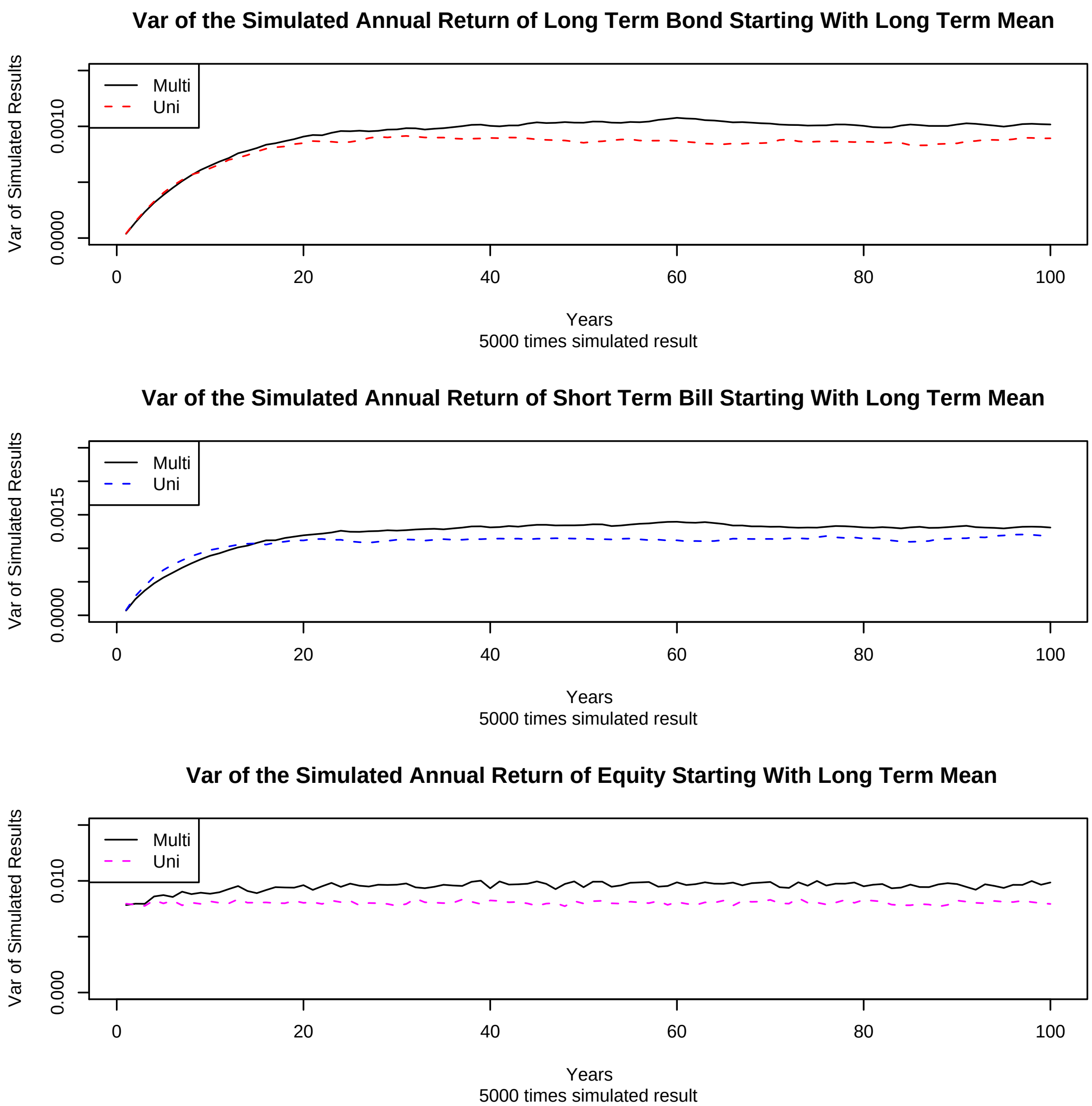


Figure A.2: Variance of the simulated annual rate of return starting with the long term mean

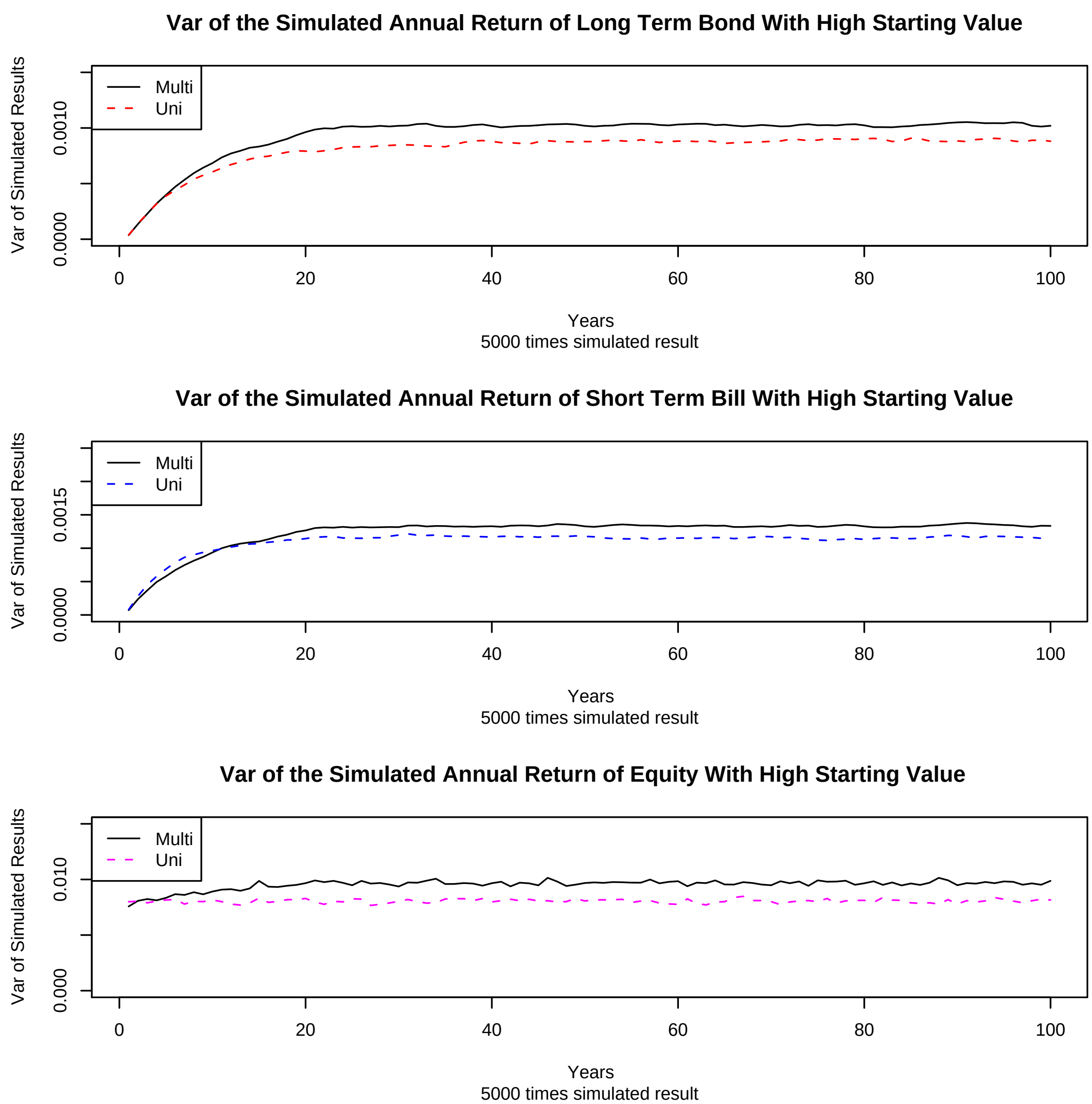


Figure A.3: Variance of the simulated annual rate of return with starting value higher than long term mean

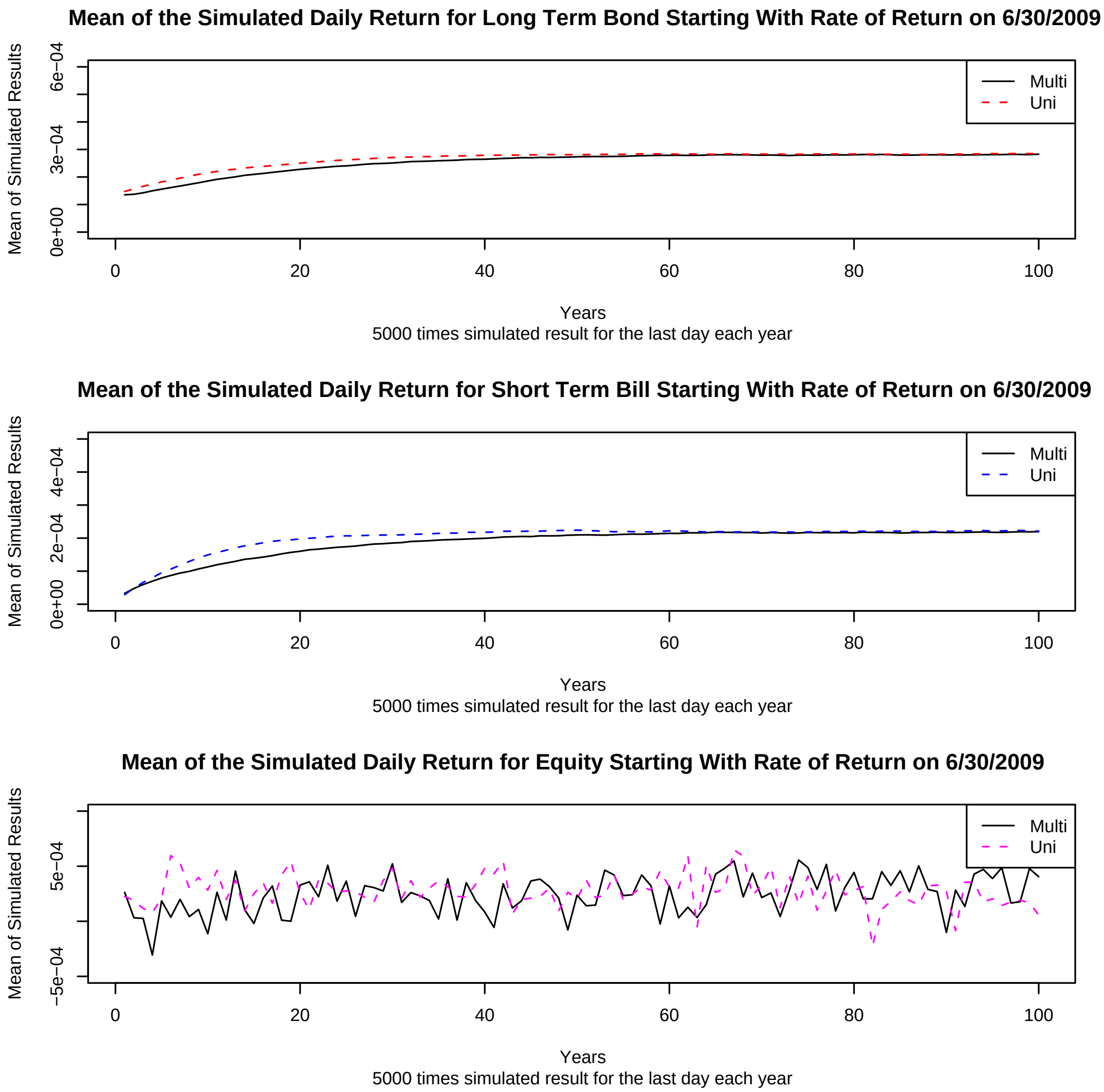


Figure A.4: Mean of the simulated daily rate of return at the end of each year with starting value lower than long term mean

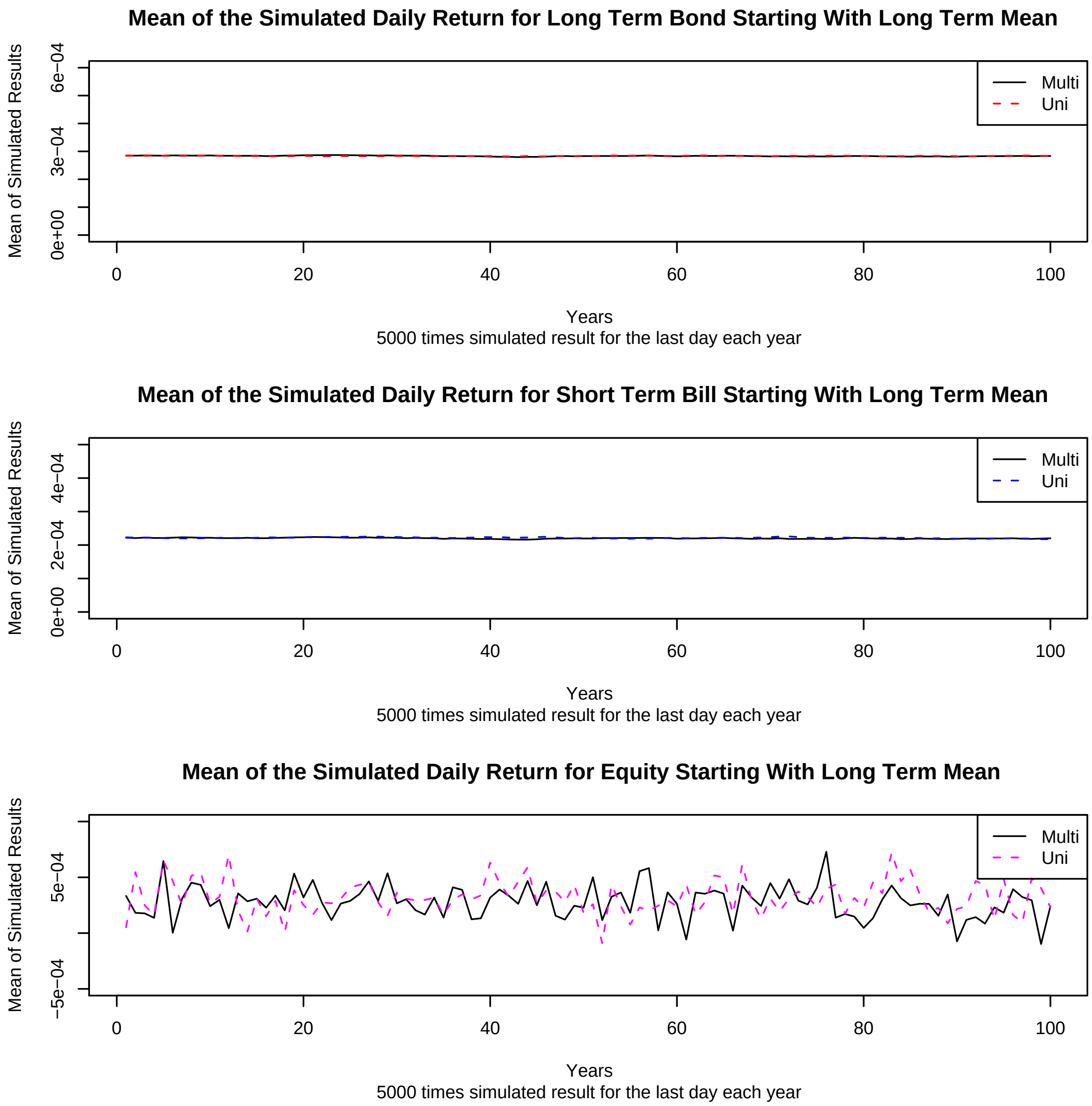


Figure A.5: Mean of the simulated daily rate of return at the end of each year starting with long term mean

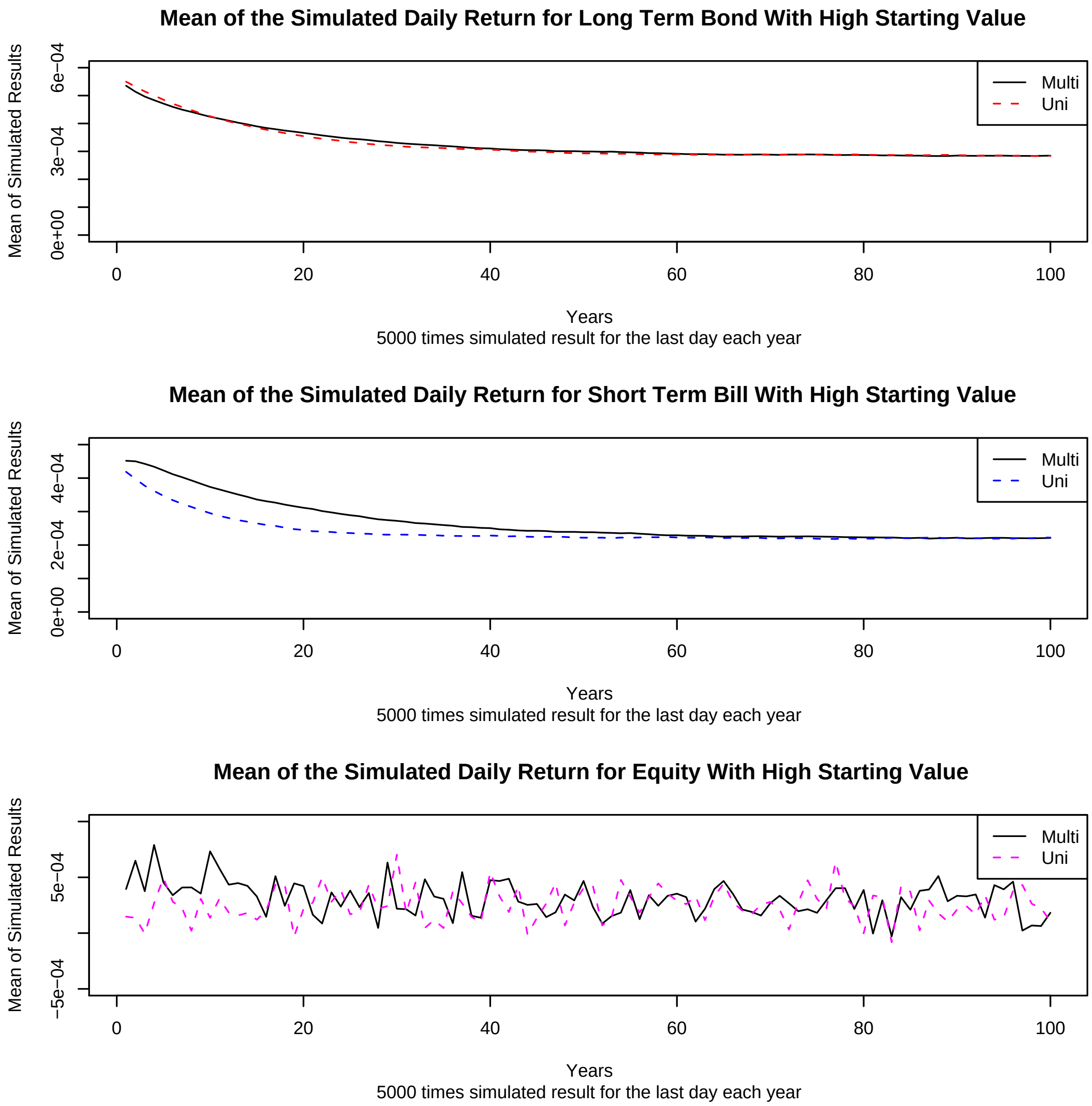
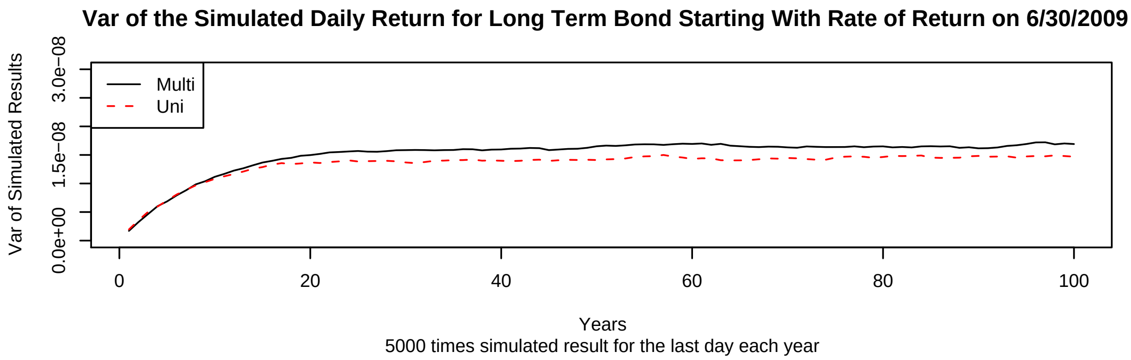
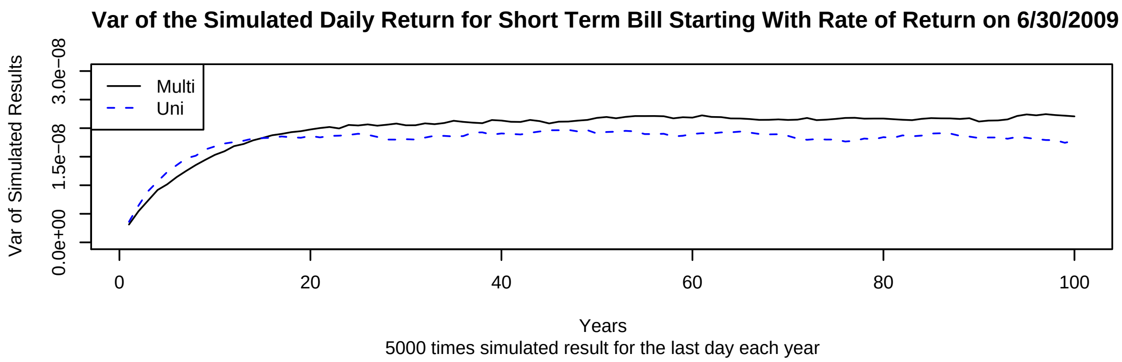


Figure A.6: Mean of the simulated daily rate of return at the end of each year with starting value higher than long term mean





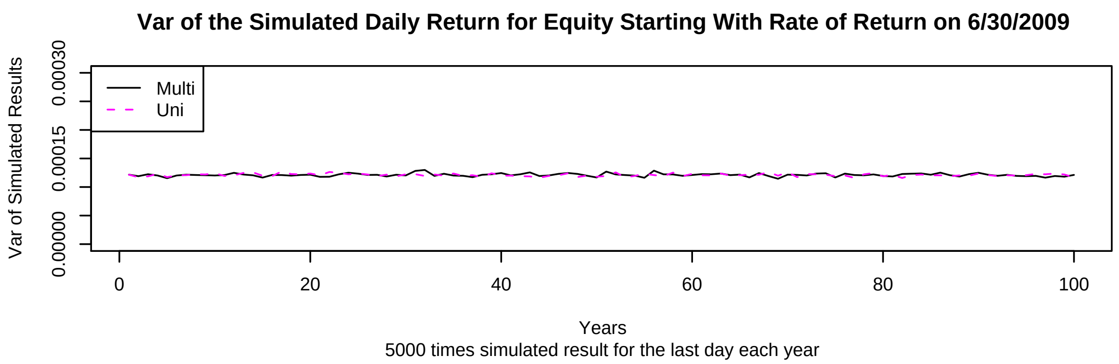
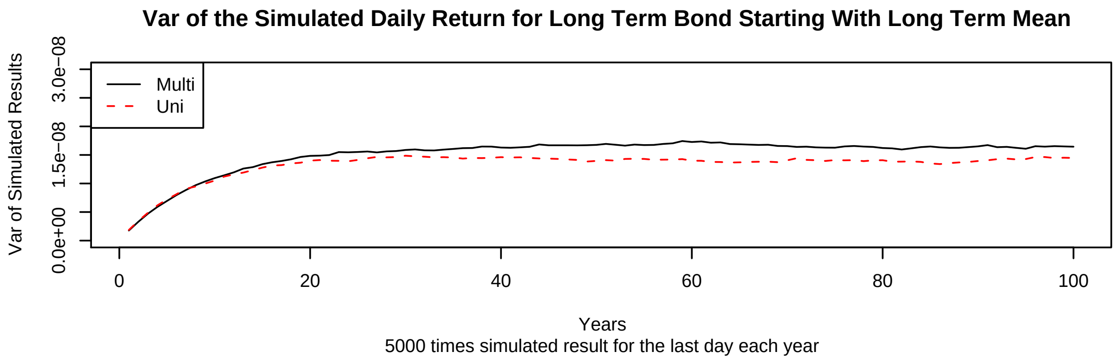
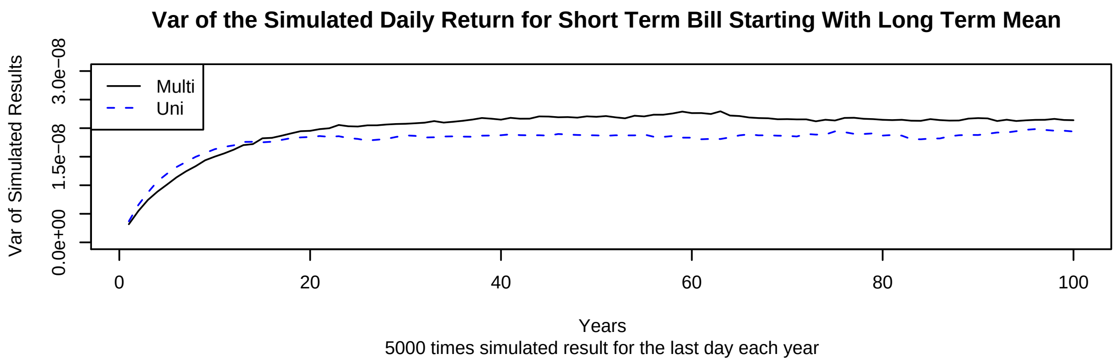


Figure A.7: Variance of the simulated daily rate of return at the end of each year with starting value lower than long term mean





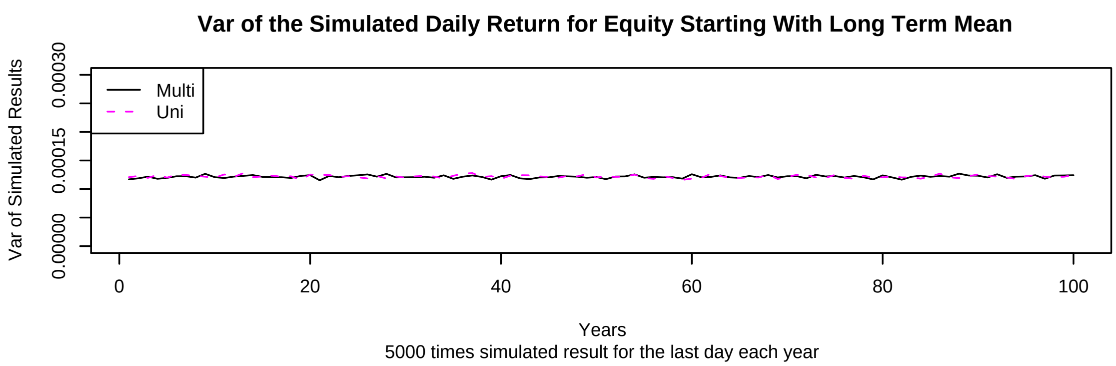


Figure A.8: Variance of the simulated daily rate of return at the end of each year starting with long term mean

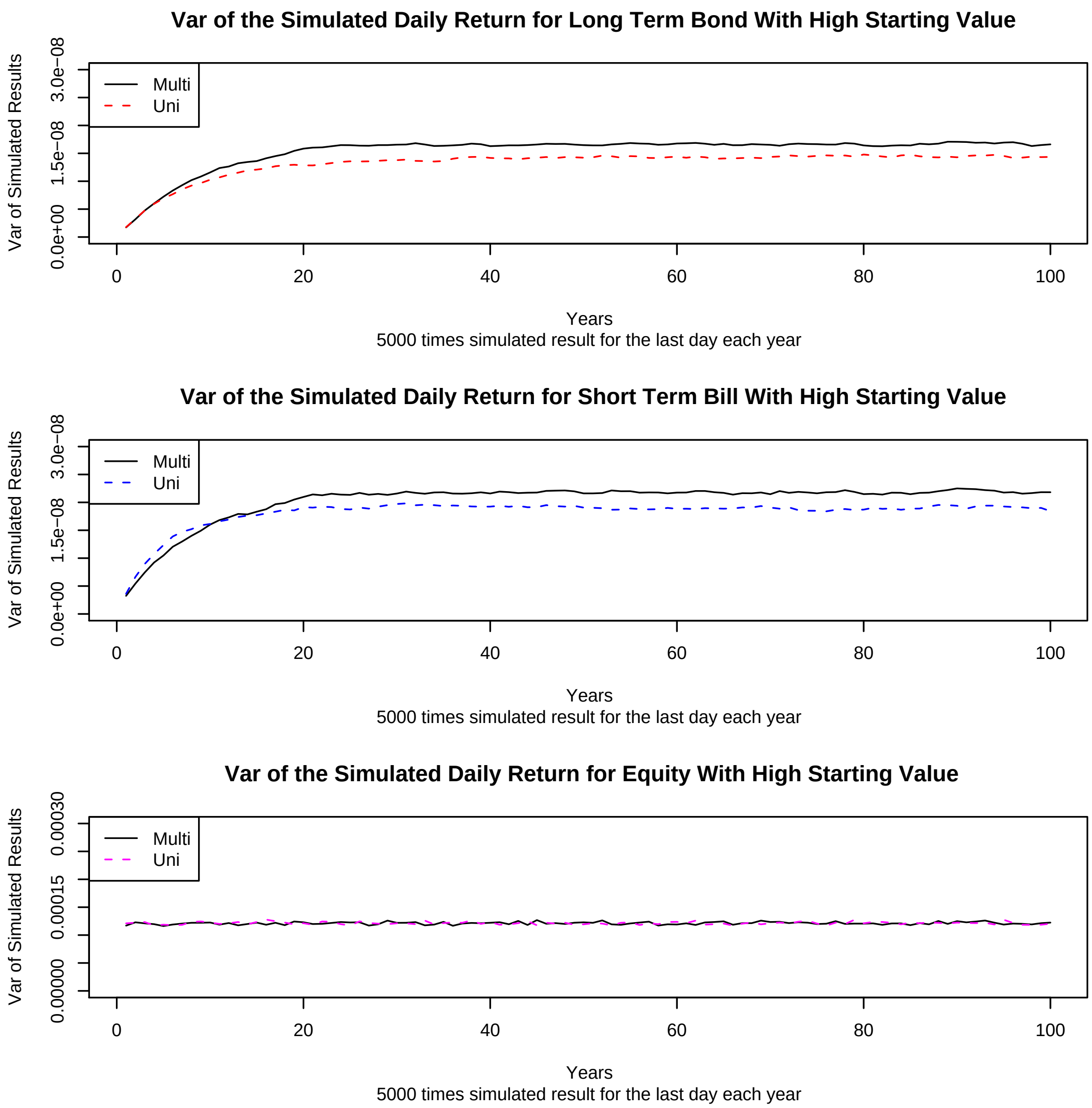


Figure A.9: Variance of the simulated daily rate of return at the end of each year with starting value higher than long term mean

Appendix B

Equivalent Processes: Alternative Approach

In section 2.2.4 we found $\underline{\Sigma}_{OU}$ for the covariance equivalent multivariate OU process by matching the variances of the AR(1) and OU processes at time 1. In this appendix, we show an alternative approach to solve for $\underline{\Sigma}_{OU}$ by matching the variances of AR(1) and OU processes when time t goes to infinity.

Assume that $\mathrm{Var}(\underline{X}_t) = \mathrm{Cov}(\underline{X}_t, \underline{X}_t)$ has a limit M as $t \to \infty$, which would be true for a stationary process. For the vector AR(1) process, when $t \to \infty$, according to Equation (2.25), we have

$$\lim_{t\to\infty} \mathrm{Var}(X_t) = \sum_{i=0}^{\infty} \underline{\Phi}^i \underline{\Sigma}_a \left(\underline{\Phi}^i\right)^T. \tag{B.1}$$

Assume $\underline{\Phi} = e^{\underline{A}}$ has eigenvalue decomposition $\underline{\Phi} = \underline{V \Lambda V^{-1}}$, where $\underline{\Lambda}$ is a diagonal matrix $\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$. Let $\underline{\mu} = \log(\underline{\Lambda}) = \begin{pmatrix} \log(\lambda_1) & 0 & \dots & 0 \\ 0 & \log(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \log(\lambda_n) \end{pmatrix}$. We have

$$\underline{A} = \underline{V \mu V^{-1}}, \tag{B.2}$$

$$\underline{\Phi}^i = \underline{V \Lambda^i} \underline{V}^{-1} \tag{B.3}$$

and

$$\left(\underline{\Phi}^i\right)^T = \left(\underline{V}^{-1}\right)^T \underline{\Lambda}^i \left(\underline{V}\right)^T. \tag{B.4}$$

Therefore, Equation (B.1) can be written as

$$\lim_{t\to\infty} \mathrm{Var}(X_t) = \underline{V}\left(\sum_{i=0}^{\infty} \underline{\Lambda}^i \underline{V}^{-1} \underline{\Sigma}_a \left(\underline{V}^{-1}\right)^T \underline{\Lambda}^i\right) \underline{V}^T. \tag{B.5}$$

Define $\underline{F} = \underline{V}^{-1}\underline{\Sigma}_a\left(\underline{V}^{-1}\right)^T$ and let the element in row k column l of $\underline{F}$ be F_{kl}. Then the element in row k column l of $\underline{\Lambda}^i\underline{F}\underline{\Lambda}^i$ is $\lambda_k^i F_{kl}\lambda_l^i$. Let $\underline{R} = \sum_{i=0}^{\infty} \underline{\Lambda}^i \underline{V}^{-1}\underline{\Sigma}_a \left(\underline{V}^{-1}\right)^T \underline{\Lambda}^i = \sum_{i=0}^{\infty} \underline{\Lambda}^i \underline{F}\underline{\Lambda}^i$ and the element in row k column l of $\underline{R}$ be $R_{kl} = \sum_{i=0}^{\infty} \lambda_k^i F_{kl}\lambda_l^i$. Since, for a stationary process, we have $|\lambda_k| < 1$ for all k, we can calculate R_{kl} as

$$R_{kl} = \frac{F_{kl}}{1-\lambda_k\lambda_l}. \tag{B.6}$$

Therefore, we have

$$\lim_{t\to\infty} \mathrm{Var}(X_t) = \underline{V}\underline{R}\underline{V}^T. \tag{B.7}$$

Now for the OU process, with Equation (2.31), we can calculate $\lim_{t\to\infty} \mathrm{Var}(X_t)$. Since $\underline{A} = \underline{V\mu V^{-1}}$, we have $e^{\underline{A}t} = \underline{V}e^{\underline{\mu}t}\underline{V}^{-1}$. So $\lim_{t\to\infty}\mathrm{Var}(X_t)$ can be calculated as

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{Var}(X_t) &= \lim_{t\to\infty} e^{\underline{A}t}\int_0^t \left(e^{\underline{A}n}\right)^{-1} \underline{\Sigma}_{OU} \left(\left(e^{\underline{A}n}\right)^{-1}\right)^T dn \left(e^{\underline{A}t}\right)^T \\
&= \lim_{t\to\infty} \underline{V}e^{\underline{\mu}t}\underline{V}^{-1}\int_0^t \left(\underline{V}e^{\underline{\mu}n}\underline{V}^{-1}\right)^{-1} \underline{\Sigma}_{OU} \left(\underline{V}e^{-\underline{\mu}n}\underline{V}^{-1}\right)^T dn \left(\underline{V}^{-1}\right)^T e^{\underline{\mu}t}\underline{V}^T \\
&= \lim_{t\to\infty} \underline{V}\int_0^t e^{(t-n)\underline{\mu}}\underline{V}^{-1}\underline{\Sigma}_{OU}\left(\underline{V}^{-1}\right)^T e^{(t-n)\underline{\mu}} dn \underline{V}^T.
\end{aligned}$$

Let $t-n=y$, then

$$\lim_{t\to\infty} \mathrm{Var}(X_t) = \lim_{t\to\infty} \underline{V}\int_0^t e^{y\underline{\mu}}\left(\underline{V}^{-1}\underline{\Sigma}_{OU}\left(\underline{V}^{-1}\right)^T\right) e^{y\underline{\mu}} dy \underline{V}^T.$$

Define $\underline{F}_{OU} = \underline{V}^{-1}\underline{\Sigma}_{OU}\left(\underline{V}^{-1}\right)^T$, we have

$$\lim_{t\to\infty} \mathrm{Var}(X_t) = \lim_{t\to\infty} \underline{V}\int_0^t e^{y\underline{\mu}}\underline{F}_{OU}e^{y\underline{\mu}} dy \underline{V}^T. \tag{B.8}$$

From Equation (B.8) for the OU process and Equation (B.7) for the AR(1) process, for them to be covariance equivalent we need that

$$\underline{R} = \int_0^t e^{y\underline{\mu}}\underline{F}_{OU}e^{y\underline{\mu}} dy. \tag{B.9}$$

If the element in row k column l of $\underline{F}_{OU}$ is $(F_{OU})_{kl}$, then the element in row k column l of $e^{y\underline{\mu}}\underline{F}_{OU}e^{y\underline{\mu}}$ is $e^{y\mu_k}(F_{OU})_{kl}e^{y\mu_l}$ and

$$\int_0^\infty e^{y\mu_k}(F_{OU})_{kl}e^{y\mu_l}dy = -\frac{(F_{OU})_{kl}}{\mu_k+\mu_l}. \tag{B.10}$$

For a stationary process, $|\lambda_k| < 1$ and $\mu_k = \log(\lambda_k) < 0$ for all k. Therefore, from Equations (B.9) and (B.10), we want

$$\begin{aligned} R_{kl} &= -\frac{(F_{OU})_{kl}}{\mu_k+\mu_l} \\ &= \frac{F_{kl}}{1-\lambda_k\lambda_l}. \end{aligned}$$

So, we get

$$(F_{OU})_{kl} = -\frac{\log(\lambda_k)+\log(\lambda_l)}{1-\lambda_k\lambda_l}F_{kl}. \tag{B.11}$$

Since $\underline{F}_{OU} = \underline{V}^{-1}\underline{\Sigma}_{OU}\left(\underline{V}^{-1}\right)^T$, we obtain $\underline{\Sigma}_{OU}$ as

$$\underline{\Sigma}_{OU} = \underline{V}\,\underline{F}_{OU}\underline{V}^T. \tag{B.12}$$

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