ACTUARIAL APPLICATIONS OF
THE LINEAR HAZARD TRANSFORM

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Lingzhi Jiang

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APPROVAL

Name: Lingzhi Jiang

Degree: Master of Science

Title of Project: Actuarial Applications of the Linear Hazard Transform

Examining Committee: Dr. Derek Bingham, Simon Fraser University
Chair

Dr. Cary Chi-Liang Tsai, Simon Fraser University,
Senior Supervisor

Dr. Gary Parker, Simon Fraser University,
Supervisor

Dr. Joan Hu, Simon Fraser University,
External Examiner

Date Approved: JAN 1 5 2010

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Abstract

In this thesis, we study the linear hazard (LH) transform and its applications in actuarial science. Under the LH transform, the survival function of a risk is distorted, which provides a safety margin for pricing insurance products. Combining the assumption of α-approximation, the net single premium of a continuous insurance policy can be approximated in terms of the net single premiums of discrete insurance ones. We also find that the LH transform is good at fitting by regression between two mortality curves. With the method of mortality fitting, the mortalities for the future years can be predicted as well. Finally, the applications of the LH transform for an insurance company's asset managements, such as mortality swap, risk ordering and optimal reinsurance, are explored.

Keywords: Linear Hazard Transform, Proportional Hazard Transform, Mortality Fitting, Mortality Prediction, Mortality Swap, Risk Ordering, Optimal Reinsurance

Dedication

To my parents.

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Chapter 1

Introduction

The proportional hazard (PH) transform has been proposed by Wang (1995) to calculate the risk adjusted premium. It is a remarkable milestone since it possesses many desirable properties. It can be applied to areas such as ambiguous risks, excess-of-loss coverages, risk portfolios and increased limits. Let random variable X represent a continuous and non-negative risk, and $\mu_X(t)$ be the associated hazard rate. Under the PH transform, the hazard rate becomes

$$\mu_{X_\bullet}(t) = \alpha \mu_X(t), \ \alpha > 0. \tag{1.1}$$

The proportional hazard transform takes a proportional rate of the underlying force of mortality to provide a safety margin, and can be used for the purpose of pricing, estimation, prediction, etc. The proportional hazard transform can be extended by adding a constant term to form the linear hazard transform. This project focuses on the linear hazard transform and its applications.

Because only discrete survival probabilities are available in practice, pricing continuous insurance products is difficult. To solve this difficulty, fractional age assumptions have been made during the past for the purpose of approximating survival probabilities at fractional ages, among which are the linear, exponential and harmonic approximations. With the help of these techniques, the underlying survival functions can be approximated and therefore be used to price insurance products.

Also, these three assumptions are generalized and studied in the framework of α-approximation. When the linear hazard (LH) transform is applied to life insurance pricing, α-approximation is a powerful tool to evaluate the continuous insurance and annuity products.

This paper is organized as follows: Chapter 2 is a literature review. It is an overview of the past research conducted on the PH transform, α-approximation, mortality study, asset management of insurance companies, risk ordering and optimal insurance. In Chapter 3, basic actuarial concepts and formulas are reviewed. With the help of α-approximation, formulas for pricing continuous insurance products are proposed under the PH and LH transforms. The chapters that follow explore the applications of the LH transform. Chapter 4 applies the LH transform to mortality regression. Some practical techniques are suggested and examples are illustrated. In Chapter 5, the LH transform is applied to predict future mortality based on the historical data. Different prediction methodologies are discussed and applied to actual mortality. In Chapter 6, we study risk ordering and optimal reinsurance under the LH transform. The relationships among the LH order and other risk orders are explored. Explicit formulas are proposed to solve an optimal reinsurance problem under the LH transform. Chapter 7 studies the asset management of insurance companies. Mortality swap is a possible approach and its pricing method under the LH transform is suggested. Finally, we summarize the findings in Chapter 8.

Chapter 2

Literature Review

Applications of the proportional hazard (PH) transform in insurance were proposed by Wang (1995). For a continuous and non-negative random variable X, its survival function and hazard rate are denoted by $S_X(t)$ and $\mu_X(t)$, respectively; the hazard rate after the PH transform, denoted by $\mu_{X_\bullet}(t)$, satisfies

$$\mu_{X_\bullet}(t) = \alpha\mu_X(t),$$

where $\alpha > 0$ and $X_\bullet$ is the corresponding random variable, which implies

$$S_{X_\bullet}(t) = [S_X(t)]^\alpha.$$

Wang (1996) showed that the PH transform resembles the risk-neutral valuation in financial economics. When $\alpha < 1$, $E[X_\bullet]$ is called the risk adjusted premium because it involves a safety margin

$$E[X_\bullet] - E[X] = \int_0^\infty [S_X(t)]^\alpha dt - \int_0^\infty S_X(t)dt > 0$$

for the pure premium $E[X]$. The PH transform also preserves the stop loss order of risks with increasing concave utility functions. He applied the PH transform to risk ordering and introduced the PH transform order. Moreover, he explored the relationship between the dangerous order and PH transform order, and further connected the PH order, the dangerous order with the stochastic dominance order and the stop

loss order. The relationships among these orders were also discussed. Later, Wang (1998) applied this method to insurance rate making. Examples were illustrated with respect to the excess-of-loss coverages, increased limits, risk portfolios, etc. This project extends the PH transform to the linear hazard (LH) transform, that is,

$$\mu_{X_*}(t) = \alpha\mu_X(t) + \beta,$$

where X_* is the corresponding random variable. An LH order will be introduced. The connection of the LH order with other risk orders will also be established. Moreover, the LH transform can be applied to mortality fitting and mortality prediction based on existing mortality rates.

As mentioned in Chapter 1, the linear, exponential and harmonic approximations are three common assumptions for fractional age mortality. Frostig (2002) conducted a comparison study of the three assumptions above with unknown survival functions. Jones and Mereu (2000) introduced a broader concept of fractional age assumption called α-approximation. It is a unified approach that incorporates and generalizes all the three approximations. The α-approximation assumes that the α-power of the survival function at a fractional age is the linear interpolation of the α-power of the survival functions at two adjacent integer values. They also studied the smoothing of force of mortality under the fractional age assumptions, and did an application using actual mortality data. Later, Frostig (2003) studied different approximations with respect to the stochastic ordering. She also derived properties of the fractional age assumptions. Yi and Weng (2006) combined α-approximation and copula, and applied them to multiple life insurance; two kinds of approximation approaches were constructed, and results were derived for risk ordering in the context of multiple life insurance. This paper considers applications of the α-approximation in the pricing of insurance products under the LH transform, with comparisons to the PH transform. Explicit approximation formulas are given and the relationship between the pricing of discrete and continuous risks is studied.

Asset and liability management, which helps match liabilities with assets in order to stabilize cash flows in the future, is an interesting research area for insurance companies. When the actual mortality differs from the expected one, on one hand,

the values of life insurance assets and liabilities change, and may cause losses for life insurance issuers. It is the same case for life annuity issuers. This is called mortality risk. On the other hand, the values of liabilities of life insurance and annuities liabilities move in opposite directions, which provides a possible approach to hedging against mortality risks. Therefore, when insurance companies write life insurance products, it is a common practice for them to sell annuities at the same time to hedge against future potential losses due to mortality risks, and vice versa. Cox and Lin (2004) studied this natural hedging strategy. Based on some empirical evidence, they showed that adopting a natural hedging strategy leads to lower premium charges. The idea of survivor bonds, raised and discussed in several papers, is that the government could issue a new bond, namely survivor bond, to help annuity issuers hedge against mortality risks. The coupon of this bond is contingent on the percentage of retirees who are still alive at a certain age. For example, if mortality improves, i.e., more people survive than the expected, annuity companies need to pay more benefits. However, they receive more coupons from the survivor bonds to offset the impact of mortality improvement. As a result, the company's cash flow is stabilized. Blake, Cairns and Dowd (2006) discussed how companies can hedge against mortality risks by mortality-linked securities, including survivor bonds, swaps, futures and options. Cox and Lin (2005) studied the pricing of such securities and the pricing of mortality risk bonds under the PH transform in particular. This paper will discuss the pricing of a mortality swap under the LH transform.

Optimal insurance and reinsurance is another important issue for the insurance companies. Many researches have been conducted on this topic in the past. Young (1999) studied the optimal insurance assuming that the price is given by Wang's premium principle. In that paper, a mixed random variable model was assumed for analysis, and its distribution function is given by

$$F_X(x) = (1 - q) + q \int_0^x f(t)dt$$

for $x \geq 0$, where $q \in (0, 1]$ is the probability that X is positive and f is the probability density function of $X|X > 0$. Based on this model, Young (1999) determined the optimal contact for a risk-averse company that wishes to optimize its expected

utility function. Later, Promislow and Young (2005) considered the optimal insurance for a general risk X and a general set of premium principles. On the part of reinsurance, Kaluszka (2001) investigated an optimal reinsurance problem under the mean-variance premium principle. Both global and local reinsurance were studied. Later, Kaluszka (2005) also proposed a general approach to solving optimal reinsurance problems. He assumed that the reinsurer's premium is fixed. Reinsurance companies decide to optimize different indexes based on their needs. Examples of the exponential, p-mean value, semi-deviation, semi-variance, Dutch and Wang's premium principles were given. This paper will focus on the optimal reinsurance under the LH transform. Explicit formulas are proposed.

Chapter 3

Linear Hazard Transform

3.1 Preliminaries

3.1.1 Actuarial Mathematics Concepts

Before studying the linear hazard transform, some definitions and symbols regarding actuarial mathematics are introduced in the following. Let $T(x)$ be the future lifetime of an individual aged x,

$$S_{T(x)}(t) = Pr\{T(x) > t\} \triangleq {}_tp_x$$

be the survival function of $T(x)$,

$$F_{T(x)}(t) = Pr\{T(x) \leq t\} \triangleq {}_tq_x = 1 - {}_tp_x$$

be the distribution function of $T(x)$, and

$$\mu_x(t) = \frac{f_{T(x)}(t)}{S_{T(x)}(t)} = -\frac{d}{dt}lnS_{T(x)}(t)$$

be the force of mortality, where $f_{T(x)}(t) = {}_tp_x\mu_x(t)$ is the probability density function of $T(x)$. Moreover, let X be the time of death for an individual aged x, and F be the associated distribution function. Then we have $X = x + T(x)$. As a result,

$$_tp_x = Pr\{T(x) > t\} = \frac{S(x+t)}{S(x)}$$

and

$$_tq_x = Pr\{T(x) \le t\} = 1 - \ _tp_x = \frac{S(x) - S(x+t)}{S(x)}.$$

In insurance products pricing, the net single premium (NSP) is an important concept. Now we give the definitions for some life insurance policies and their net single premiums.

Definition 1. *Term life insurance is a life insurance that provides a fixed payment of death for a specified time period. The net single premium of an n-year discrete term life insurance that pays a benefit of 1 at the end of the year of death of the insured within n years is denoted by $A^1_{x:\overline{n}|}$. On the other hand, the net single premium of an n-year continuous term life insurance that pays a benefit of 1 at the time of death of the insured within n years is denoted by $\overline{A}^1_{x:\overline{n}|}$.*

Definition 2. *Annuity is a stream of payments made continuously or at equal intervals for a specified time period or a life time while a given life survives. Annuity due is made at the beginning of each year. Annuity immediate is made at the end of each year. The net single premiums of n-year discrete annuities due and immediate are denoted by $\ddot{a}_{x:\overline{n}|}$ and $a_{x:\overline{n}|}$, respectively. The net single premium of an n-year continuous annuity is denoted by $\overline{a}_{x:\overline{n}|}$.*

Definition 3. *Endowment is an instrument that provides a fixed payment for death for a specified time period and a benefit for survival beyond the specified time period. The net single premium of an n-year discrete endowment insurance that pays a benefit of 1 at the end of the year of death of the insured within n years and 1 for survival beyond n years is denoted by $A_{x:\overline{n}|}$. The net single premium of an n-year continuous endowment insurance that pays a benefit of 1 at the time of death of the insured within n years and 1 for survival beyond n years is denoted by $\overline{A}_{x:\overline{n}|}$.*

Definition 4. *Curtate future lifetime of a person aged x is the number of future years completed by the time of death, and is denoted by $K(x)$.*

Definition 5. *Temporary expected lifetime is the life expectancy of a person aged x over an n-year time period, and is denoted by $\overset{\circ}{e}_{x:\overline{n}|}$. By letting n go to infinity, we*

have the complete expectation of life of an individual aged x, $E[T(x)]$, denoted by $\overset{\circ}{e}_x$. Similarly, the expected curtate life time of a person aged x over an n-year time period is denoted by $e_{x:\overline{n}|}$ while the curtate expectation of life of a person aged x, $E[K(x)]$, is denoted by e_x.

Definition 6. *The actuarial present value of an n-year term annuity of 1 per year, payable in installments of $\frac{1}{m}$ at the beginning of each m-th of the year while a person of age x is still alive, is denoted by $\ddot{a}^{(m)}_{x:\overline{n}|}$.*

3.1.2   Basic Formulas

The NSP actuarially discounts all future cash flows and adds them up to get a lump sum payment that is paid by the policyholder at the start of a policy. We can see that the net single premium is a big payment needed to be made at the beginning. It may not be realistic for policyholders to do so due to budget constraints. An alternate approach is the net level premium (NLP) which annuitizes the NSP over a specific time period by dividing the NSP of the policy by the NSP of an annuity. For example, the NLP of an n-year discrete term life insurance, denoted by $P^1_{x:\overline{n}|}$, with each of n payments made at the beginning at the year whenever the insured is alive is $A^1_{x:\overline{n}|} / \ddot{a}_{x:\overline{n}|}$. The NSP's for the standard insurance products discussed in the previous subsection are given below.

- The NSP of a discrete n-year term life insurance is

$$A^1_{x:\overline{n}|} = \sum_{k=1}^{n} {}_{k-1|}q_x v^k = \sum_{k=1}^{n} ({}_{k-1}p_x - {}_kp_x)v^k$$

  where ${}_{k-1|}q_x = {}_{k-1}p_x - {}_kp_x$ is the probability that an individual of age x dies between times $(k-1)$ and k.

- The NSP of a continuous n-year term life insurance is

$$\overline{A}^1_{x:\overline{n}|} = \int_0^n {}_tp_x\mu_x(t)v^t dt.$$

- The NSP of a discrete n-year annuity due is

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} {}_kp_x v^k.$$

- The NSP of a discrete n-year annuity immediate is

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x v^k.$$

- The NSP of a continuous n-year annuity is

$$\overline{a}_{x:\overline{n}|} = \int_0^n {}_tp_x v^t dt.$$

- The NSP of a discrete n-year endowment is

$$A_{x:\overline{n}|} = \sum_{k=1}^{n} {}_{k-1|}q_x v^k +{}_n E_x.$$

where ${}_nE_x \stackrel{\triangle}{=} {}_np_x v^n$ is the NSP for the survival benefit payable when the insured survives to the end of n years.

- The NSP of a continuous n-year endowment is

$$\overline{A}_{x:\overline{n}|} = \int_0^n {}_tp_x \mu_x(t) v^t dt +{}_n E_x.$$

Let the random variable

$$T^*(x) = \begin{cases} T(x), & 0 < T(x) \leq n, \\ n, & n < T(x), \end{cases}$$

and denote $E[T^*(x)]$ by $\overset{\circ}{\mathbf{e}}_{x:\overline{n}|}$. This expectation is the expected lifetime of an individual aged x over the next n years. Let the random variable

$$K^*(x) = \begin{cases} K(x), & K(x) = 0, 1, 2, ..., n-1, \\ n, & K(x) = n, n+1, ..., \end{cases}$$

and denote $E[K^*(x)]$ by $\mathbf{e}_{x:\overline{n}|}$. This expectation is the expected curtate lifetime of an individual aged x over the next n years. These two expectations are calculated by

$$\overset{\circ}{\mathbf{e}}_{x:\overline{n}|} = \int_0^n {}_t p_x dt$$

and

$$\mathbf{e}_{x:\overline{n}|} = \sum_{k=1}^n {}_k p_x,$$

respectively.

3.1.3 Approximation of Survival Probabilities at Fractional Points

In actuarial mathematics, the number of people who survive at the end of each year (integer value) and the number of deaths during that year can be expected based on the mortality table. Assuming that p_x or q_x is given for all x's, actuaries can calculate the quantities such as survival probabilities, probability of people dying in a given period, etc., at the integer time points. However, the exact survival probabilities at the fractional ages are not available.

To solve this problem, actuaries make use of survival probabilities at the integer values and make appropriate assumptions on survival functions. The following three assumptions are common approaches in actuarial practice.

Definition 7. *Linear approximation* (*or UDD assumption*) : ${}_s p_x$ *is said to be linearly approximated if*

$${}_s p_x = (1 - s) \, {}_0 p_x + s \, p_x = (1 - s) + s \, p_x$$

for $0 \leq s < 1$ and $x = 0, 1, 2....$

Remark 1.

Linear approximation implies that

$$1 - {}_s q_x = (1 - s) + s(1 - q_x)$$

which can be rearranged as follows:

$$_sq_x = s \, q_x.$$

In this case, the force of mortality is

$$\mu_x(s) = \frac{d(_sq_x)/ds}{_sp_x} = \frac{d(s \, q_x)/ds}{_sp_x} = \frac{q_x}{1 - s \, q_x}.$$

Definition 8. *Exponential approximation (or constant force of mortality assumption) :* $_sp_x$ *is said to be exponentially approximated if*

$$ln \, _sp_x = (1 - s) \, ln \, _0p_x + s \, lnp_x,$$

or equivalently,

$$_sp_x = \,_0p_x^{1-s} \cdot p_x^s = p_x^s$$

for $0 \le s < 1$ *and* $x = 0, 1, 2....$

Remark 2.

Exponential approximation implies that the force of mortality is

$$\mu_x(s) = \frac{d(_sq_x)/ds}{_sp_x} = \frac{d(1 - p_x^s)/ds}{p_x^s} = \frac{(-logp_x) \, p_x^s}{p_x^s} = -logp_x,$$

a constant force.

Definition 9. *Harmonic approximation (or Balducci assumption) :* $_sp_x$ *is said to be harmonically approximated if*

$$\frac{1}{_sp_x} = (1 - s)\frac{1}{_0p_x} + s\frac{1}{p_x},$$

or equivalently,

$$_sp_x = \frac{1}{(1 - s) + \frac{s}{p_x}} = \frac{p_x}{(1 - s)p_x + s}$$

for $0 \le s < 1$ *and* $x = 0, 1, 2....$

Remark 3. *Harmonic approximation implies that*

$$_sq_x = 1 - \ _sp_x = 1 - \frac{p_x}{(1-s)p_x + s} = \frac{s\,q_x}{(1-s)p_x + s}.$$

In this case, the force of mortality is

$$\mu_x(s) = \frac{d(\frac{s\,q_x}{(1-s)p_x + s})/ds}{_sp_x} = \frac{\frac{q_x((1-s)p_x+s)-s\,q_x^2}{((1-s)p_x+s)^2}}{\frac{p_x}{(1-s)p_x+s}} = \frac{q_x}{(1-s)p_x + s}.$$

The three assumptions above can be generalized and summarized in the framework of α-approximation.

Definition 10. *Let F be the distribution function of the time of death X for an individual aged x and α be a real number. Then the survival function of X, $S = 1-F$, is said to be α-approximated if S satisfies*

$$S(x+s)^\alpha = (1-s)S(x)^\alpha + s\,S(x+1)^\alpha \tag{3.1}$$

for $0 \le s < 1$, $x = 0, 1, 2, \ldots$, and $\alpha \ne 0$.

To obtain an expression for $\alpha = 0$, we rewrite (3.1) as

$$\begin{aligned} S(x+s) &= [(1-s)S(x)^\alpha + s\,S(x+1)^\alpha]^{\frac{1}{\alpha}} \\ &= e^{\frac{1}{\alpha}ln[(1-s)S(x)^\alpha + s\,S(x+1)^\alpha]}. \end{aligned}$$

With the help of L'Hopital's rule, we have

$$\begin{aligned} \lim_{\alpha \to 0} S(x+s) &= \lim_{\alpha \to 0} e^{\frac{(1-s)S(x)^\alpha lnS(x)+s\,S(x+1)^\alpha lnS(x+1)}{(1-s)S(x)^\alpha+s\,S(x+1)^\alpha}} \\ &= e^{(1-s)lnS(x)+slnS(x+1)}. \end{aligned}$$

That is, for $\alpha = 0$, F satisfies

$$lnS(x+s) = (1-s)lnS(x) + slnS(x+1),$$

or equivalently,

$$S(x+s) = S(x)^{1-s} \cdot S(x+1)^s. \tag{3.2}$$

Since $_sp_x = \frac{S(x+s)}{S(x)}$, from (3.1) and (3.2) we get

$$
_sp_x = \begin{cases} \frac{[(1-s)S(x)^\alpha + s\,S(x+1)^\alpha]^{\frac{1}{\alpha}}}{[S(x)^\alpha]^{\frac{1}{\alpha}}} = [(1-s) + sp_x^\alpha]^{\frac{1}{\alpha}}, & \alpha \neq 0, \\ S(x)^{-s} \cdot S(x+1)^s = p_x^s, & \alpha = 0. \end{cases}
$$

With α equaling -1, 0 and 1, the α-approximation reduces to the special cases of the harmonic, exponential and linear approximations, respectively.

3.2 Linear Hazard Transform

Wang (1995) introduced the proportional hazard transform. Under the PH transform, the force of mortality, known as a hazard rate, is multiplied by a constant.

Definition 11. *Given a force of mortality $\mu_x(t)$, the proportional hazard transform of $\mu_x(t)$ is defined as $\mu_{x_\bullet}(t) = \alpha_x \mu_x(t)$ for some $\alpha_x > 0$ where the subscript $\bullet$ of x denotes the proportional hazard transform.*

As a result, the transformed survival probability can be expressed as

$$
_tp_{x_\bullet} = Pr\{T(x_\bullet) > t\} = e^{-\int_0^t \mu_{x_\bullet}(s)ds} = e^{-\alpha_x \int_0^t \mu_x(s)ds} = (e^{-\int_0^t \mu_x(s)ds})^{\alpha_x} = (_tp_x)^{\alpha_x}.
$$

The idea of the PH transform is to calculate risk-adjusted premium by changing the weight of right tail. In the case where rare events take place and cause large losses, the PH transform will charge higher premium portion for large tail loss.

Definition 12. *Given a force of mortality $\mu_x(t)$, the linear hazard transform of $\mu_x(t)$ is defined by*

$$
\mu_{x_*}(t) = \alpha_x \mu_x(t) + \beta_x \tag{3.3}
$$

for some $\alpha_x > 0$ where the subscript $*$ of x denotes the linear hazard transform.

Similarly, the LH transformed survival function can be expressed as

$$
_tp_{x_*} = e^{-\int_0^t \mu_{x_*}(s)ds} = e^{-\int_0^t [\alpha_x \mu_x(s) + \beta_x]ds} = \left[e^{-\int_0^t \mu_x(s)ds}\right]^{\alpha_x} e^{-\beta_x t} = [_tp_x]^{\alpha_x} e^{-\beta_x t}. \tag{3.4}
$$

Generally, we need $\alpha_x > 0$, and β_x could be negative. To ensure that $\mu_{x_*}(t) > 0$ for all $t \geq 0$, we require $\beta_x > -\alpha_x \inf\{\mu_x(t) : t \geq 0\}$. Since the force of mortality is a hazard rate, $\mu_{x_*}(t)$ is a linear hazard transform of $\mu_x(t)$. The LH transform (3.4), like the PH transform, is the adjusted force of mortality creating a safety margin as well for pricing life insurance ($\alpha_x > 1$) or life annuity ($\alpha_x < 1$). Comparing it with the PH transform, we can see that the difference is that an extra constant term is added to the transformed force of mortality (hazard rate). When $\beta_x = 0$, $\mu_{x_*}(t)$ is the proportional hazard transform of $\mu_x(t)$. In this case, the transformed hazard rate is denoted by $\mu_{x_\bullet}(t) = \mu_{x_*}(t)|_{\beta_x=0}$. When $\alpha_x = 0$, we have $\mu_{x_*}(t) = \beta_x$, a constant force of mortality. To simplify these symbols, we use α and β for α_x and β_x, respectively, and we will use them throughout the project.

First, rewrite $t = k + s$ where k is an integer and $s \in [0, 1)$. Then from (3.4), we get $_{k+s}p_{x_*} = [_{k+s}p_x]^\alpha e^{-\beta(k+s)} = [_kp_x \, _sp_{x+k}]^\alpha e^{-\beta(k+s)}$. Applying α-approximation in (3.2) to $[_{k+s}p_x]^\alpha$ for $\alpha \neq 0$ yields

$$
\begin{aligned}
_{k+s}p_{x_*} &= [(1-s) + s(p_{x+k})^\alpha](_kp_x)^\alpha e^{-\beta(k+s)} \\
&= \{(1-s)[_kp_x]^\alpha + s[_{k+1}p_x]^\alpha\}e^{-\beta(k+s)} \\
&= (1-s)[_kp_{x_*}]e^{-\beta s} + s[_{k+1}p_{x_*}]e^{\beta(1-s)}.
\end{aligned}
\tag{3.5}
$$

Taking natural logarithm and differentiating with respect to s leads to

$$
\begin{aligned}
-\mu_{x_*}(k+s) &= \frac{d\ln[_{k+s}p_{x_*}]}{ds} = \frac{d_{k+s}p_{x_*}}{ds}\frac{1}{_{k+s}p_{x_*}} \\
&= \frac{\{[_{k+1}p_x]^\alpha - [_kp_x]^\alpha\}e^{-\beta(k+s)} - \beta \cdot_{k+s}p_{x_*}}{_{k+s}p_{x_*}}
\end{aligned}
$$

or

$$
\begin{aligned}
_{k+s}p_{x_*}\,\mu_{x_*}(k+s) &= \{[_kp_x]^\alpha - [_{k+1}p_x]^\alpha\}e^{-\beta(k+s)} + \beta \cdot_{k+s}p_{x_*} \\
&= [_kp_{x_*}]e^{-\beta s} - [_{k+1}p_{x_*}]e^{\beta(1-s)} + \beta \cdot_{k+s}p_{x_*}.
\end{aligned}
\tag{3.6}
$$

Let $\bar{A}^1_{x_*:\overline{n}|i}$ and $\bar{a}_{x_*:\overline{n}|i}$ be the net single premiums of the continuous n-year term life and n-year temporary life annuity, respectively, based on the adjusted force of mortality $\mu_{x_*}(t)$. Also, let $\delta_\beta = \delta + \beta$, where δ satisfies $e^{-\delta} = v = (1+i)^{-1}$. Then the

corresponding discount factor v_β and interest rate i_β which satisfy $(1 + i_\beta)^{-1} = v_\beta = e^{-\delta_\beta}$ can be solved as $v_\beta = e^{-(\delta+\beta)} = ve^{-\beta}$ and $i_\beta = (1 + i)e^\beta - 1$, respectively. Also, we define $d_\beta = i_\beta \cdot v_\beta = 1 - v_\beta$,

$$X_0 = \int_0^1 v_\beta^s ds = \frac{1 - v_\beta}{\delta_\beta}, \tag{3.7}$$

and

$$X_1 = \int_0^1 s v_\beta^s ds = \frac{1 - v_\beta}{\delta_\beta^2} - \frac{v_\beta}{\delta_\beta}. \tag{3.8}$$

The following proposition gives an expression for $\bar{a}_{x_*:\overline{n}|\,i}$ in terms of $\ddot{a}_{x_*:\overline{n}|\,i}$ and $a_{x_*:\overline{n}|\,i}$.

Proposition 1. *Under the α-approximation assumption,*

$$\bar{a}_{x_*:\overline{n}|\,i} = (X_0 - X_1)\,\ddot{a}_{x_*:\overline{n}|\,i} + \frac{X_1}{v_\beta}\,a_{x_*:\overline{n}|\,i} = \frac{\delta_\beta - d_\beta}{\delta_\beta^2}\,\ddot{a}_{x_*:\overline{n}|\,i} + \frac{i_\beta - \delta_\beta}{\delta_\beta^2}\,a_{x_*:\overline{n}|\,i}.$$

Proof:

$$
\begin{aligned}
\bar{a}_{x_*:\overline{n}|\,i} &= \int_0^n {}_tp_{x_*}v^t dt = \int_0^n [{}_tp_x]^\alpha e^{-\beta t} v^t dt \\
&= \sum_{k=0}^{n-1} \int_0^1 [{}_{k+s}p_x]^\alpha e^{-\beta(k+s)} v^{k+s} ds \\
&= \sum_{k=0}^{n-1} v_\beta^k \int_0^1 \{(1-s)[{}_kp_x]^\alpha + s[{}_{k+1}p_x]^\alpha\} v_\beta^s ds \\
&= \sum_{k=0}^{n-1} [{}_kp_x]^\alpha e^{-\beta k} v^k \int_0^1 (1-s)v_\beta^s ds + \frac{1}{e^{-\beta}v} \sum_{k=0}^{n-1} [{}_{k+1}p_x]^\alpha e^{-\beta(k+1)} v^{k+1} \int_0^1 s v_\beta^s ds \\
&= (X_0 - X_1) \sum_{k=0}^{n-1} {}_kp_{x_*} v^k + \frac{X_1}{v_\beta} \sum_{k=0}^{n-1} {}_{k+1}p_{x_*} v^{k+1} \\
&= \frac{\delta_\beta - d_\beta}{\delta_\beta^2}\,\ddot{a}_{x_*:\overline{n}|\,i} + \frac{i_\beta - \delta_\beta}{\delta_\beta^2}\,a_{x_*:\overline{n}|\,i}.
\end{aligned}
$$

Note that since $A^1_{x:\overline{n}|i} = v\,\ddot{a}_{x:\overline{n}|i} - a_{x:\overline{n}|i}$, or $\ddot{a}_{x:\overline{n}|} = (1+i)[A^1_{x:\overline{n}|i} + a_{x:\overline{n}|i}]$, $\bar{a}_{x_*:\overline{n}|\,i}$ can also be expressed in terms of $A^1_{x_*:\overline{n}|i}$ and $a_{x_*:\overline{n}|\,i}$. That is,

$$\bar{a}_{x_*:\overline{n}|\,i} = (X_0 - X_1)\,(1+i)[A^1_{x_*:\overline{n}|\,i} + a_{x_*:\overline{n}|\,i}] + \frac{X_1}{v_\beta}\,a_{x_*:\overline{n}|\,i}. \tag{3.9}$$

Corollary 1. *Under the α-approximation assumption,*

$$\bar{a}_{x_*} = (X_0 - X_1)\,\ddot{a}_{x_*} + \frac{X_1}{v_\beta}\,a_{x_*} = (X_0 - X_1)\,(1+i)[A_{x_*} + a_{x_*}] + \frac{X_1}{v_\beta}\,a_{x_*}.$$

Proof: Letting n go to infinity in Proposition 1 and (3.9) yields the result.

From (3.6), a relationship between $\bar{A}^1_{x_*:\overline{n}|i}$ and $\bar{a}_{x_*:\overline{n}|i}$ can be derived as well.

Proposition 2. *Under the α-approximation assumption,*

$$\bar{A}^1_{x_*:\overline{n}|\,i} = (X_0 + \beta X_0 - \beta X_1)\,\ddot{a}_{x_*:\overline{n}|\,i} + \frac{\beta X_1 - X_0}{v_\beta}\,a_{x_*:\overline{n}|\,i}.$$

Proof: From (3.6), we have

$$
\begin{aligned}
\bar{A}^1_{x_*:\overline{n}|\,i} &= \int_0^n {}_t p_{x_*}\mu_{x_*}(t)v^t dt \\
&= \sum_{k=0}^{n-1}\int_0^1 {}_{k+s}p_{x_*}\mu_{x_*}(k+s)v^{k+s}ds \\
&= \sum_{k=0}^{n-1} v^k \int_0^1 v^s\{[{}_k p_{x_*}]e^{-\beta s} - [{}_{k+1}p_{x_*}]e^{\beta(1-s)} + \beta\cdot{}_{k+s}\,p_{x_*}\}ds \\
&= \sum_{k=0}^{n-1} v^k \int_0^1 \{{}_k p_{x_*}v_\beta^s - {}_{k+1}\,p_{x_*}v_\beta^s e^\beta + \beta v^s\cdot{}_{k+s}\,p_{x_*}\}ds. \quad (3.10)
\end{aligned}
$$

Then with (3.7) and (3.8), equation (3.10) can be written as

$$
\begin{aligned}
\bar{A}^1_{x_*:\overline{n}|\,i} &= \sum_{k=0}^{n-1} v^k\{[{}_k p_{x_*} - {}_{k+1}\,p_{x_*}e^\beta]X_0\} + \beta\sum_{k=0}^{n-1}v^k\int_0^1 v^s\cdot{}_{k+s}\,p_{x_*}ds \\
&= X_0\sum_{k=0}^{n-1}({}_k p_{x_*})v^k - \frac{X_0}{v_\beta}\sum_{k=0}^{n-1}({}_{k+1}p_{x_*})v^{k+1} + \beta\,\bar{a}_{x_*:\overline{n}|\,i} \\
&= \beta\,\bar{a}_{x_*:\overline{n}|\,i} + X_0\,\ddot{a}_{x_*:\overline{n}|\,i} - \frac{X_0}{v_\beta}\,a_{x_*:\overline{n}|\,i}, \quad (3.11)
\end{aligned}
$$

which is a relationship between $\bar{A}^1_{x_*:\overline{n}|i}$ and $\bar{a}_{x_*:\overline{n}|\,i}$. Then by Proposition 1,

$$
\begin{aligned}
\bar{A}^1_{x_*:\overline{n}|\,i} &= \beta[(X_0 - X_1)\,\ddot{a}_{x_*:\overline{n}|\,i} + \frac{X_1}{v_\beta}\,a_{x_*:\overline{n}|\,i}] + X_0\,\ddot{a}_{x_*:\overline{n}|\,i} - \frac{X_0}{v_\beta}\,a_{x_*:\overline{n}|\,i} \\
&= (X_0 + \beta X_0 - \beta X_1)\,\ddot{a}_{x_*:\overline{n}|\,i} + \frac{\beta X_1 - X_0}{v_\beta}\,a_{x_*:\overline{n}|\,i}.
\end{aligned}
$$

Corollary 2. *Under the α-approximation assumption,*

$$\bar{A}_{x_*} = (X_0 + \beta X_0 - \beta X_1)\,\ddot{a}_{x_*} + \frac{\beta X_1 - X_0}{v_\beta}\,a_{x_*}.$$

This corollary follows from Proposition 2 by letting n go to infinity.

Note that $\ddot{a}_{x_*:\overline{n}|\,i}$ in Propositions 1 and 2 can be written as

$$\ddot{a}_{x_*:\overline{n}|\,i} = \sum_{k=0}^{n-1}(_k p_x)^\alpha e^{-k\beta}v^k = \sum_{k=0}^{n-1}(_k p_x)^\alpha v_\beta^k = \sum_{k=0}^{n-1}(_k p_{x_\bullet})v_\beta^k = \ddot{a}_{x_\bullet:\overline{n}|\,i_\beta}.$$

Similarly, we have

$$a_{x_*:\overline{n}|\,i} \;=\; a_{x_\bullet:\overline{n}|\,i_\beta}.$$

Therefore, $\bar{a}_{x_*:\overline{n}|\,i}$ in Proposition 1 can be rewritten as

$$\bar{a}_{x_*:\overline{n}|\,i} = (X_0 - X_1)\,\ddot{a}_{x_\bullet:\overline{n}|\,i_\beta} + \frac{X_1}{v_\beta}\,a_{x_\bullet:\overline{n}|\,i_\beta},$$

and $\bar{A}^1_{x_*:\overline{n}|\,i}$ in Proposition 2 can also be rewritten as

$$\bar{A}^1_{x_*:\overline{n}|\,i} = (X_0 + \beta X_0 - \beta X_1)\,\ddot{a}_{x_\bullet:\overline{n}|\,i_\beta} + \frac{\beta X_1 - X_0}{v_\beta}\,a_{x_\bullet:\overline{n}|\,i_\beta}.$$

Next, we apply the α-approximation to $\overset{\circ}{e}_{x_*:\overline{n}|}$, the expected lifetime of a person aged x over an n-year time period under the linear hazard transform. Before we give the proposition, we introduce the following notations for the purpose of expression:

$$Y_0 \;=\; \int_0^1 e^{-\beta s}ds = \frac{1 - e^{-\beta}}{\beta}, \tag{3.12}$$

and

$$Y_1 \;=\; \int_0^1 se^{-\beta s}ds = \frac{1 - e^{-\beta}(1+\beta)}{\beta^2}. \tag{3.13}$$

Proposition 3. *Under the linear hazard transform and α-approximation assumption, the expected lifetime of a person aged x over an n-year time period can be approximated as*

$$\overset{\circ}{e}_{x_*:\overline{n}|} = (Y_0-Y_1)(1+e_{x_*:\overline{n-1}|})+Y_1e^{\beta}e_{x_*:\overline{n}|} = \frac{\beta-1+e^{-\beta}}{\beta^2}(1+e_{x_*:\overline{n-1}|})+\frac{e^{\beta}-1-\beta}{\beta^2}e_{x_*:\overline{n}|}$$

where $e_{x_*:\overline{n}|}$ is the expected curtate lifetime of a person aged x over an n-year time period under the linear hazard transform.

Proof: From (3.5) and the definition of expected curtate lifetime, we have

$$
\begin{aligned}
\overset{\circ}{e}_{x_*:\overline{n}|} &= \int_0^n {}_tp_{x_*}dt \\
&= \sum_{k=0}^{n-1}\int_0^1 {}_{k+s}p_{x_*}ds \\
&= \sum_{k=0}^{n-1}\int_0^1 [(1-s){}_kp_{x_*}e^{-\beta s} + s\,{}_{k+1}p_{x_*}e^{\beta(1-s)}]ds \\
&= \sum_{k=0}^{n-1}\int_0^1 [se^{-\beta s}({}_{k+1}p_{x_*}e^{\beta} - {}_kp_{x_*}) + {}_kp_{x_*}e^{-\beta s}]ds.
\end{aligned}
$$

With the notations introduced in (3.12) and (3.13), we get

$$
\begin{aligned}
\overset{\circ}{e}_{x_*:\overline{n}|} &= \sum_{k=0}^{n-1}[Y_1({}_{k+1}p_{x_*}e^{\beta} - {}_kp_{x_*}) + Y_0\,{}_kp_{x_*}] \\
&= (Y_0 - Y_1)\sum_{k=0}^{n-1}{}_kp_{x_*} + Y_1e^{\beta}\sum_{k=0}^{n-1}{}_{k+1}p_{x_*} \\
&= (Y_0 - Y_1)(1+\sum_{k=1}^{n-1}{}_kp_{x_*}) + Y_1e^{\beta}\sum_{k=1}^{n}{}_kp_{x_*} \\
&= \frac{\beta-1+e^{-\beta}}{\beta^2}(1+e_{x_*:\overline{n-1}|}) + \frac{e^{\beta}-1-\beta}{\beta^2}e_{x_*:\overline{n}|}.
\end{aligned}
$$

Corollary 3. *Under the α-approximation assumption,*

$$\overset{\circ}{e}_{x_*} = \frac{\beta-1+e^{-\beta}}{\beta^2} + \frac{e^{\beta}+e^{-\beta}-2}{\beta^2}e_{x_*}.$$

Proof: This corollary follows directly from Proposition 3 by letting n go to infinity.

A special case: when $\beta = 0$, Y_0 and Y_1 in (3.12) and (3.13) become $Y_0 = \int_0^1 ds = 1$ and $Y_1 = \int_0^1 s\,ds = \frac{1}{2}$, and Proposition 3 reduces to

$$\overset{\circ}{\mathbf{e}}_{x_\bullet:\overline{n}|} = \frac{1 + \mathbf{e}_{x_\bullet:\overline{n-1}|} + \mathbf{e}_{x_\bullet:\overline{n}|}}{2}.$$

Moreover, under the linear hazard transform and α-approximation assumption, it is easy to obtain formulas for deferred m-year continuous n-year life annuities and life insurance from Propositions 1 and 2 because ${}_{m|n}\bar{a}_{x_*}$ and ${}_{m|n}\bar{A}_{x_*}$ can be written as

$$_{m|n}\bar{a}_{x_*} = {}_{m}p_{x_*} v^m \bar{a}_{x_*+m:\overline{n}|\,i},$$

and

$$_{m|n}\bar{A}_{x_*} = {}_{m}p_{x_*} v^m \bar{A}^1_{x_*+m:\overline{n}|\,i},$$

respectively. By letting n go to infinity, we can also get formulas for deferred m-year whole life insurance and annuity.

Next, we apply α-approximation to $\ddot{a}^{(m)}_{x_*:\overline{n}|}$ and explore its relationship with the continuous annuity.

Proposition 4. *Under the α-approximation assumption,*

$$\ddot{a}^{(m)}_{x_*:\overline{n}|i} = \ddot{a}_{x_*:\overline{n}|i}\left[\ddot{a}^{(m)}_{\overline{1}|\,i_\beta} - \frac{1}{m}(I\ddot{a})^{(m)}_{\overline{1}|\,i_\beta}\right] + \frac{1}{mv_\beta}(I\ddot{a})^{(m)}_{\overline{1}|\,i_\beta} a_{x_*:\overline{n}|i}. \tag{3.14}$$

Proof:

$$
\begin{aligned}
\ddot{a}^{(m)}_{x_*:\overline{n}|i} &= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \frac{1}{m} v^{k+\frac{j}{m}} {}_{k+\frac{j}{m}}p_{x_*} \\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \frac{1}{m} v_\beta^{k+\frac{j}{m}} \left({}_{k+\frac{j}{m}}p_x\right)^\alpha \\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \frac{1}{m} v_\beta^{k+\frac{j}{m}} \left[(1-\frac{j}{m})\,{}_kp_x^\alpha + \frac{j}{m}\,{}_{k+1}p_x^\alpha \right] \\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \frac{1}{m} v_\beta^{k+\frac{j}{m}}\,{}_kp_x^\alpha + \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \frac{1}{m} v_\beta^{k+\frac{j}{m}}\frac{j}{m}\left({}_{k+1}p_x^\alpha - {}_kp_x^\alpha\right) \\
&= \sum_{k=0}^{n-1} {}_kp_x^\alpha v_\beta^k \frac{1}{m}\sum_{j=0}^{m-1} v_\beta^{\frac{j}{m}} + \sum_{k=0}^{n-1}\left({}_{k+1}p_x^\alpha - {}_kp_x^\alpha\right)v_\beta^k \frac{1}{m}\sum_{j=0}^{m-1}\frac{j}{m}v_\beta^{\frac{j}{m}} \\
&= \ddot{a}_{x_*:\overline{n}|i}\ddot{a}^{(m)}_{\overline{1}|\,i_\beta} + \left[\frac{1}{v_\beta}a_{x_*:\overline{n}|i} - \ddot{a}_{x_*:\overline{n}|i}\right]\frac{1}{m}(I\ddot{a})^{(m)}_{\overline{1}|\,i_\beta} \qquad\qquad (3.15) \\
&= \ddot{a}_{x_*:\overline{n}|i}\left[\ddot{a}^{(m)}_{\overline{1}|\,i_\beta} - \frac{1}{m}(I\ddot{a})^{(m)}_{\overline{1}|\,i_\beta}\right] + \frac{1}{mv_\beta}(I\ddot{a})^{(m)}_{\overline{1}|\,i_\beta}a_{x_*:\overline{n}|i}.
\end{aligned}
$$

Proposition 5. *Under the α-approximation assumption, if we let m go to infinity, (3.14) becomes*

$$
\bar{a}_{x_*:\overline{n}|\,i} = (X_0 - X_1)\,\ddot{a}_{x_*:\overline{n}|\,i} + \frac{X_1}{v_\beta}\,a_{x_*:\overline{n}|\,i}
$$

which is the same as the one in Proposition 1.

Proof: First, $\lim_{m\to\infty}\ddot{a}^{(m)}_{x_*:\overline{n}|i} = \bar{a}_{x_*:\overline{n}|\,i}$. Next, by (3.15) and l'Hopital's rule,

$$
\begin{aligned}
\lim_{m\to\infty}\ddot{a}^{(m)}_{\overline{1}|\,i_\beta} &= \lim_{m\to\infty} \frac{\sum_{j=0}^{m-1} v_\beta^{\frac{j}{m}}}{m} \\
&= (1-v_\beta)\lim_{m\to\infty} \frac{\frac{1}{m}}{1 - v_\beta^{\frac{1}{m}}} \\
&= (1-v_\beta)\lim_{m\to\infty} \frac{-\frac{1}{m^2}}{-(\ln v_\beta)\,v_\beta^{\frac{1}{m}}\left(-\frac{1}{m^2}\right)} \\
&= -(1-v_\beta)\lim_{m\to\infty} \frac{1}{(\ln v_\beta)\,v_\beta^{\frac{1}{m}}} \\
&= \frac{1-v_\beta}{\beta+\delta}.
\end{aligned}
$$

Moreover, from (3.15),

$$
\begin{aligned}
\lim_{m \to \infty} \frac{1}{m} (I\ddot{a})_{\overline{1}|\, i_\beta}^{(m)} &= \lim_{m \to \infty} \sum_{j=0}^{m-1} \frac{j v_\beta^{\frac{j}{m}}}{m^2} \\
&= \lim_{m \to \infty} \frac{-m v_\beta (1 - v_\beta^{\frac{1}{m}}) + v_\beta^{\frac{1}{m}} (1 - v_\beta)}{m^2 (1 - v_\beta^{\frac{1}{m}})^2} \\
&= \frac{-v_\beta (\beta + \delta) + (1 - v_\beta)}{(\beta + \delta)^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\bar{a}_{x_*:\overline{n}|\, i} &= \lim_{m \to \infty} \ddot{a}_{x_*:\overline{n}|}^{(m)} \\
&= \frac{1 - v_\beta}{\beta + \delta} \ddot{a}_{x_*:\overline{n}|\, i} + \frac{-v_\beta (\beta + \delta) + (1 - v_\beta)}{(\beta + \delta)^2} \left[\frac{1}{v_\beta} a_{x_*:\overline{n}|\, i} - \ddot{a}_{x_*:\overline{n}|\, i} \right] \\
&= \left[\frac{1 - v_\beta}{\beta + \delta} - \frac{-v_\beta (\beta + \delta) + (1 - v_\beta)}{(\beta + \delta)^2} \right] \ddot{a}_{x_*:\overline{n}|\, i} \\
&\quad + \frac{-v_\beta (\beta + \delta) + (1 - v_\beta)}{v_\beta (\beta + \delta)^2} a_{x_*:\overline{n}|\, i}.
\end{aligned}
$$

It is easy to check that

$$
\frac{1 - v_\beta}{\beta + \delta} - \frac{-v_\beta (\beta + \delta) + (1 - v_\beta)}{(\beta + \delta)^2} = \frac{\delta_\beta - d_\beta}{\delta_\beta^2} = X_0 - X_1,
$$

and

$$
\frac{-v_\beta (\beta + \delta) + (1 - v_\beta)}{v_\beta (\beta + \delta)^2} = \frac{i_\beta - \delta_\beta}{\delta_\beta^2} = \frac{X_1}{v_\beta},
$$

which completes the proof.

Chapter 4

Mortality Fitting under the LH Transform

4.1 Mortality Improvement Fitting

Intuitively, one application of the linear hazard transform is the fitting of different sets of mortalities for pricing insurance and annuities. Due to the improvement of medical conditions, living environment and health care system, people tend to live a longer life. Table 4.1 illustrates a portion of male mortalities from 1980 CSO and 2001 CSO.

As Table 4.1 and Figure 4.1 demonstrate, we see that over decades, the probability of dying at a given age is gradually decreasing (the probability of survival is gradually increasing), indicating the extended longevity of people on average.

In order to capture the improvement of mortality, the linear hazard transform can be used to model the improvement of force of mortality over decades and predict the trend of future mortality.

Throughout this study, mortality tables from 1980 CSO and 2001 CSO are used unless indicated otherwise. We want to model the mortality improvement from 1980 CSO to 2001 CSO. Suppose we are selling insurance and annuity products with a term of n years. We have the options of fitting $_k p_x$ or p_{x+k}, $k = 1, 2, ..., n$, between

Table 4.1: Male mortalities from 1980 CSO and 2001 CSO

Age x	q_x: 1980 Male	q_x: 2001 Male	Improvement of mortality [1]
30	0.00173	0.00114	0.00059
31	0.00178	0.00113	0.00065
32	0.00183	0.00113	0.00070
33	0.00191	0.00115	0.00076
34	0.00200	0.00118	0.00082
35	0.00211	0.00121	0.00090
36	0.00224	0.00128	0.00096
37	0.0024	0.00134	0.00106
38	0.00258	0.00144	0.00114
39	0.00279	0.00154	0.00125
40	0.00302	0.00165	0.00137
41	0.00329	0.00179	0.00150
42	0.00356	0.00196	0.00160
43	0.00387	0.00215	0.00172
44	0.00419	0.00239	0.00180
45	0.00455	0.00265	0.00190
46	0.00492	0.00290	0.00202
47	0.00532	0.00317	0.00215
48	0.00574	0.00333	0.00241
49	0.00621	0.00352	0.00269

[1]Improvement of mortality is the difference between q_x's from
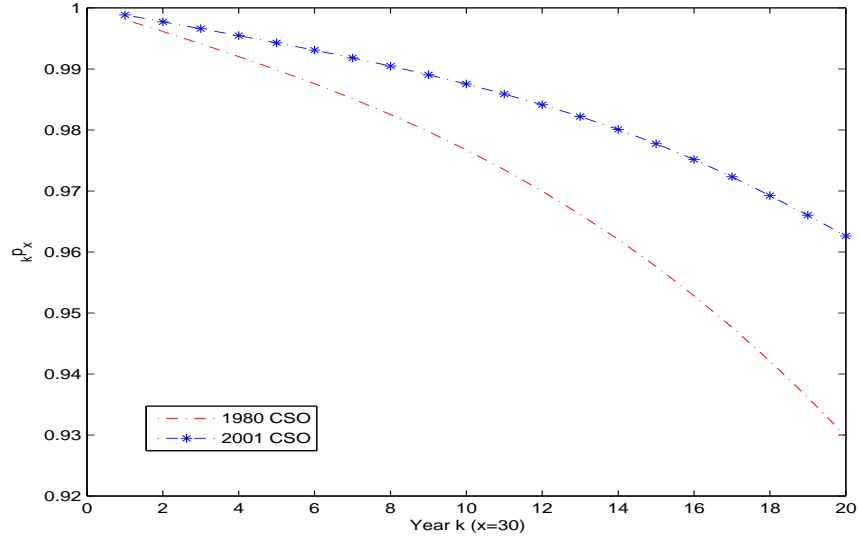1980 CSO male and 2001 CSO male

Figure 4.1: $_kp_x$: 1980 CSO and 2001 CSO male mortalities, $x = 30$, $n = 20$

1980 CSO and 2001 CSO. In this paper, fitting $_kp_x$ rather than p_{x+k} is studied. The reason will be discussed later. Let $_kp_{x,A}$ be the source survival probability (1980 CSO in this context), $_kp_{x,B}$ be the target survival probability we want to fit (2001 CSO), and $_kp_{x_*,A}$ be the fitted survival probability by fitting $_kp_{x,A}$ to $_kp_{x,B}$ under the linear hazard transform (3.4). What we need to do is to obtain the values of α and β such that fitted values $_kp_{x_*,A}$ are as close to $_kp_{x,B}$ as possible.

Let's have a look at the plot of $_kp_x$, which is shown in Figure 4.1. The plot shows that the curve of $_kp_x$ looks like an exponential function. In fact,

$$_kp_x = e^{-\int_0^k \mu_x(s)ds}.$$

This observation prompts us to consider the model

$$\mu_{x,B}(t) = \alpha\mu_{x,A}(t) + \beta + \epsilon(t), \tag{4.1}$$

where $\epsilon(t)$ is a white noise at t, which implies

$$_tp_{x,B} = {_tp_{x,A}^{\alpha}} e^{-\beta t} e^{-\int_0^t \epsilon(s)ds}.$$

In order to obtain the estimated values of α and β in the regression, we take the natural logarithm on both sides to produce

$$\ln(_tp_{x,B}) = \alpha \ln(_tp_{x,A}) - \beta t - \int_0^t \epsilon(s)ds.$$

We need to minimize the sum of square errors

$$\sum_{k=1}^n [\int_0^k \epsilon(s)ds]^2 = \sum_{k=1}^n \{\ln(_kp_{x,B}) - [\alpha \ln(_kp_{x,A}) - \beta k]\}^2. \tag{4.2}$$

Based on the reasoning above, the following method is proposed.

- Take the natural logarithm on $_kp_{x,A}$, and $_kp_{x,B}$, $k = 1, 2, ..., n$, respectively;

- Do regression based on $\ln {}_kp_x$. Correspondingly, the sum of square errors is

$$S_{LH} \triangleq \sum_{k=1}^n [\ln(_kp_{x,B}) - \ln(_kp_{x*,A})]^2 = \sum_{k=1}^n [\ln(_kp_{x,B}) - \alpha \ln(_kp_{x,A}) + \beta k]^2.$$

- Obtain values of α and β such that S_{LH} is minimized.

With the help of this method, explicit formulas can be obtained for α and β. To minimize S_{LH}, take the derivatives with respect to α and β, respectively, let the resulting expressions equal 0, and solve them for α and β. That is,

$$\frac{\partial S_{LH}}{\partial \alpha} = -2 \sum_{k=1}^n \ln(_kp_{x,A}) [\ln(_kp_{x,B}) - \alpha \ln(_kp_{x,A}) + \beta k] = 0, \tag{4.3}$$

and

$$\frac{\partial S_{LH}}{\partial \beta} = 2 \sum_{k=1}^n k [\ln(_kp_{x,B}) - \alpha \ln(_kp_{x,A}) + \beta k] = 0. \tag{4.4}$$

Solving equations (4.3) and (4.4) for α and β gives

$$\alpha = \frac{cd - be}{ad - b^2} \tag{4.5}$$

and

$$\beta = \frac{bc - ae}{ad - b^2} \tag{4.6}$$

where

$$a = \sum_{k=1}^{n} (\ln {}_kp_{x,A})^2, \tag{4.7}$$

$$b = \sum_{k=1}^{n} k \, (\ln {}_kp_{x,A}),$$

$$c = \sum_{k=1}^{n} (\ln {}_kp_{x,A})(\ln {}_kp_{x,B}), \tag{4.8}$$

$$d = \sum_{k=1}^{n} k^2,$$

and

$$e = \sum_{k=1}^{n} k \, (\ln {}_kp_{x,B}).$$

We are also going to fit ${}_kp_{x,B}$ on ${}_kp_{x,A}$ under the proportional hazard transform, and compare the performance of the LH and PH transforms. Adopting the same methodology as mentioned above, similar formula can be obtained for the value of α under the PH transform by minimizing the following sum of square errors

$$S_{PH} \triangleq \sum_{k=1}^{n} [\ln ({}_kp_{x,B}) - \alpha \, \ln ({}_kp_{x,A})]^2.$$

Here α is the only variable to be determined. Taking the derivative with respect to α and then setting to 0 yields

$$\frac{\partial S_{PH}}{\partial \alpha} = -2 \sum_{k=1}^{n} \ln ({}_kp_{x,A})[\ln ({}_kp_{x,B}) - \alpha \, \ln ({}_kp_{x,A})] = 0. \tag{4.9}$$

Solving (4.9) for α gives

$$\alpha = \frac{c}{a} \tag{4.10}$$

where a and c are defined by (4.7) and (4.8).

We apply the methodologies above to 1980 CSO and 2001 CSO mortalities for an individual aged x with a term of $n = 20$ years. Comparisons are made among (α, β) for the LH transform, α for the PH transform and the corresponding values achieved by running software R package. Standard error (S.E.) is calculated by $S.E. = \sqrt{\frac{SSE}{n}}$, where SSE is defined by $\sum_{k=1}^{n}[\int_0^k \epsilon(s)ds]^2$. Tables 4.2 and 4.3 summarize these regression results.

Table 4.2: Summary of 1980 CSO Male fitting 2001 CSO Male, $n = 20$

Age x	Method	LH			PH	
		α	β	S.E.	α	S.E.
30	Formula	0.5496741	0.0000757	0.0001894	0.5767702	0.0002358
	R	0.5491000	0.0000774	0.0001893	0.5768290	0.0002358
40	Formula	0.6036607	-0.0001786	0.0002310	0.5745926	0.0004301
	R	0.6030000	-0.0001750	0.0002309	0.5743650	0.0004298
50	Formula	0.6916951	-0.0010723	0.0002425	0.6185154	0.0022611
	R	0.6915000	-0.0010690	0.0002425	0.6170160	0.0022504

Table 4.3: Summary of 1980 CSO Female fitting 2001 CSO Female, $n = 20$

Age x	Method	LH			PH	
		α	β	S.E.	α	S.E.
30	Formula	0.5372490	0.0000159	0.0001358	0.5445070	0.0001389
	R	0.5371000	0.0000163	0.0001358	0.5445140	0.0001389
40	Formula	0.8836010	-0.0011710	0.0004622	0.6210066	0.0020379
	R	0.8815000	-0.0011620	0.0004620	0.6198000	0.0020370
50	Formula	0.9086175	-0.0011806	0.0010160	0.7768785	0.0023616
	R	0.9140314	-0.0012276	0.0010127	0.7753600	0.0023573

As illustrated by Tables 4.2 and 4.3, we can see that estimates of α and β under the LH transform in (4.5), (4.6), and the estimate of α under the PH transform in (4.10), yield good approximation compared with exact values obtained by R software. The standard errors are very close to the exact values obtained by R as well, which
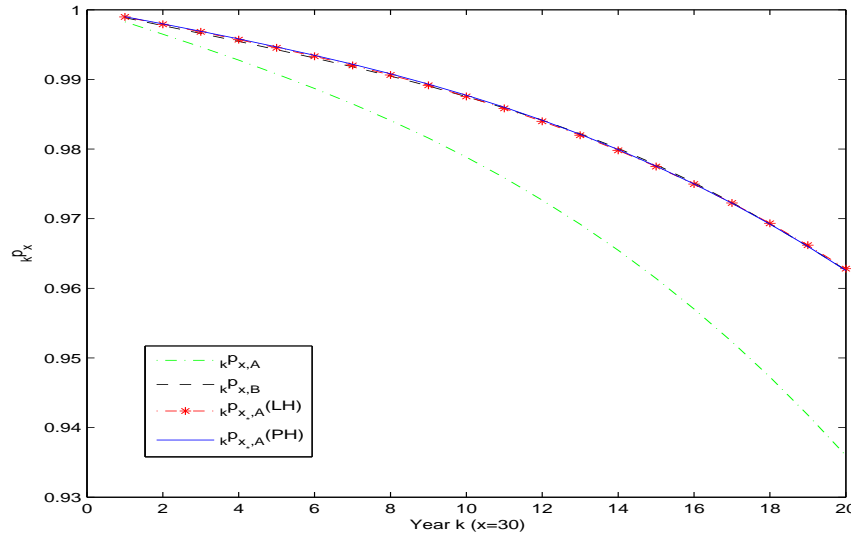
Figure 4.2: $_kp_x$: A (1980 CSO male) fits B (2001 CSO male), $x = 30$, $n = 20$

justifies us to use those formulas to estimate values of α and β that minimize the sum of square errors.

Figures 4.2-4.5 are regression plots of 1980 CSO male mortality fitting 2001 CSO male mortality for ages 30 and 40 over a time period of 20 years. q_{x+k}'s in Figures 4.3 and 4.5 are obtained by $q_{x+k} = 1 - \frac{k+1p_x}{kp_x}, k = 0, 1, \ldots, n - 1$, where $_kp_x$'s are from Figures 4.2 and 4.4, respectively.

Observing Figures 4.2-4.5 and Tables 4.2 and 4.3, we can tell that the LH transform produces smaller standard errors than the PH transform. Adding one more parameter β to the proportional hazard transform does yield more accurate regression results. By fitting $_kp_{x,A}$ to $_kp_{x,B}$ under the LH transform, we can minimize the error.

From the results as shown above, we see that as long as two mortalities from either two different years or different genders are available, we can use this methodology to regress one on the other to get the values of α and β. The relationship between two sets of mortalities can be determined by these two parameters α and β, which serve as the foundation of fitting of mortalities, pricing of life insurance and annuity product,
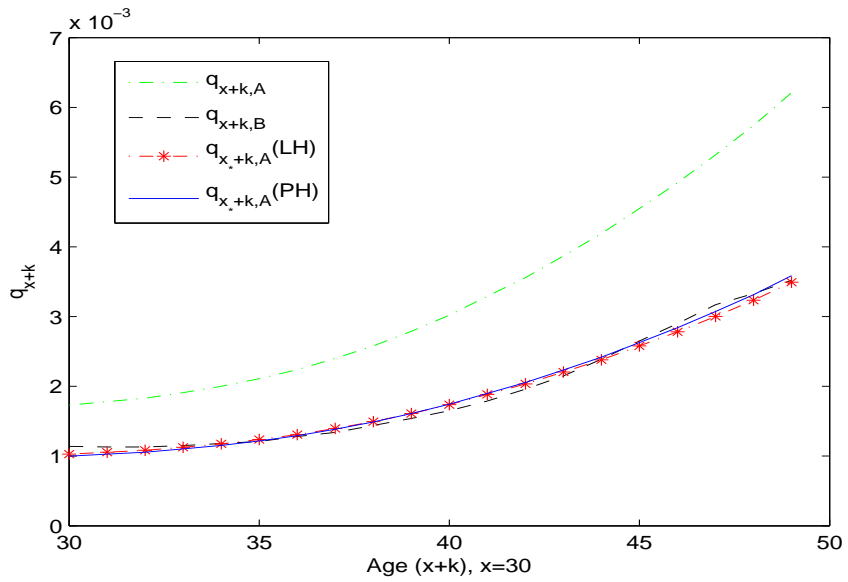
Figure 4.3: q_{x+k}: A (1980 CSO male) fits B (2001 CSO male), $x = 30$, $n = 20$
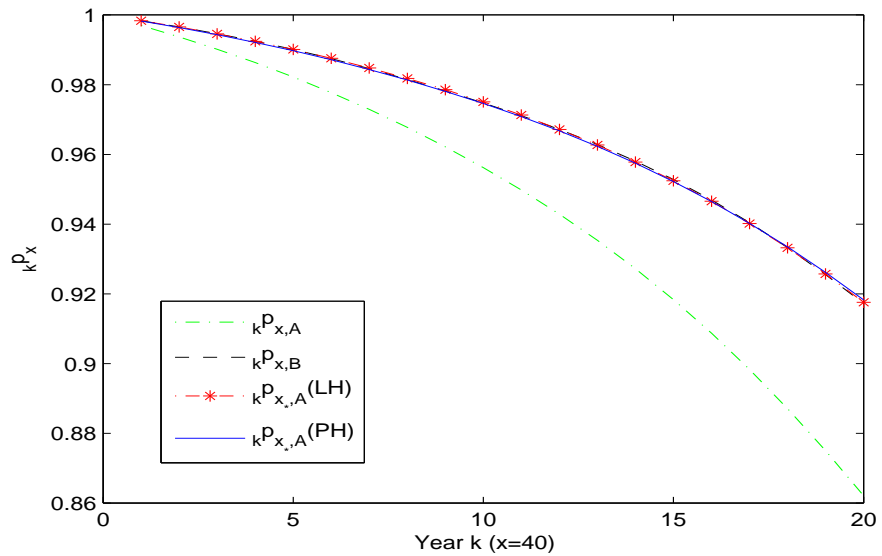

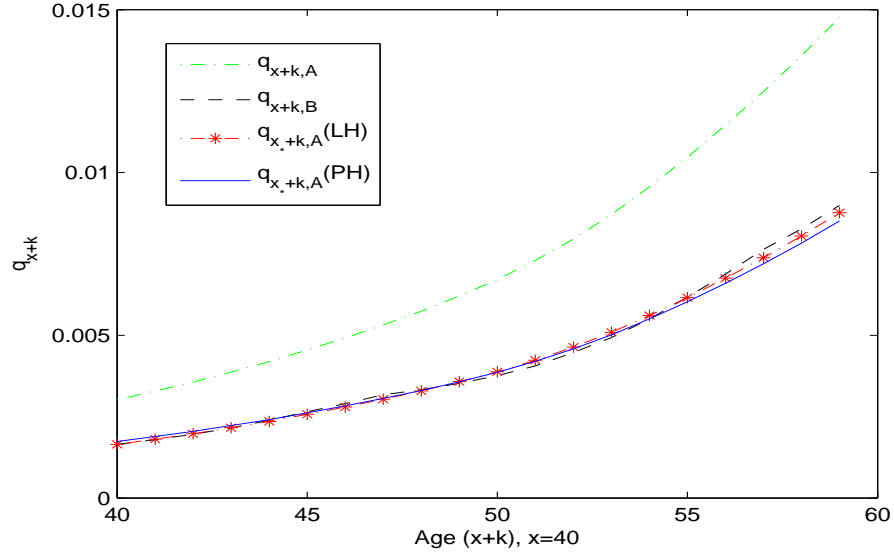Figure 4.4: $_k p_x$: A (1980 CSO male) fits B (2001 CSO male), $x = 40$, $n = 20$

Figure 4.5: q_{x+k}: A (1980 CSO male) fits B (2001 CSO male), $x = 40$, $n = 20$

and prediction of mortality improvement.

4.2 Fitting $_kp_x$ Versus Fitting p_{x+k}

As mentioned in the previous section, fitting p_{x+k} under the linear hazard transform is one alternative approach to conduct regression. That is, finding α and β such that the sum of square errors

$$\sum_{k=1}^{n}[\int_{k-1}^{k}\epsilon(s)ds]^2 = \sum_{k=0}^{n-1}[\ln(p_{x+k,B}) - \alpha\ln(p_{x+k,A}) + \beta]^2 \tag{4.11}$$

is minimized. Fitting $_kp_x$ will minimize the sum of square errors in (4.2) calculated based on $_kp_x$ while fitting p_{x+k} will minimized the sum of square errors in (4.11) based on p_{x+k}; each method has its own advantages. In this project, fitting $_kp_x$ is thoroughly studied for the following reasons:

- The net single premium of an n-year life annuity policy is evaluated by the

formula

$$\ddot{a}_{x_*:\overline{n}|\,i} = \sum_{k=0}^{n-1} {}_kp_{x_*} v^k, \tag{4.12}$$

or

$$a_{x_*:\overline{n}|\,i} = \sum_{k=1}^{n} {}_kp_{x_*} v^k, \tag{4.13}$$

both of which are expressed in terms of ${}_kp_{x_*}$. Once we know the interest i and obtain the fitted ${}_kp_{x_*}$ values, this annuity can be evaluated accordingly.

- The net single premium of an n-year term life insurance policy can be expressed in terms of $\ddot{a}_{x_*:\overline{n}|}$ and $a_{x_*:\overline{n}|}$ as follows:

$$A^1_{x_*:\overline{n}|} = v\,\ddot{a}_{x_*:\overline{n}|} - a_{x_*:\overline{n}|}.$$

Therefore, fitting ${}_kp_{x_*}$ is sufficient to calculate the net single premium of this insurance product.

- When we compute the deferred annuity and deferred insurance, the discounting factor ${}_mp_{x_*} v^m$ is needed, where m is the term of deferral. The discounting factor is a function of ${}_kp_{x_*}$ as well.

- When calculating the net level premium of a life insurance or annuity policy, we just take the ratio of one net single premium to the other. These two net single premiums are all related to ${}_kp_{x_*}$ rather than p_{x_*+k}.

- Although the error terms in (4.2) are not independent of each other, there is a benefit of doing so. When companies price life insurance and annuity products, the accuracy of ${}_kp_x$'s for the first few years are very important. Fitting ${}_kp_x$ stresses more on the accuracy of estimates in the near future, i.e., ${}_kp_x$'s for small k. The errors in ${}_kp_x$ for large k can be largely reduced by the discount factor v^k as in (4.12) and (4.13). Table 4.4 compares the estimates of ${}_kp_x$'s under the ${}_kp_x$ fitting and p_{x+k} fitting, where ${}_kp_x$ fitting error is the difference between ${}_kp_{x_*,A}$ (${}_kp_x$ fitting) and ${}_kp_{x,B}$ divided by ${}_kp_{x,B}$. As we can see, ${}_kp_x$ fitting produces more accurate estimates of ${}_kp_x$'s for small k.

Table 4.4: $_kp_x$: A (2001 CSO male) fits B (2001 CSO female), $x = 30$, $n = 20$, based on $_kp_x$ fitting and p_{x+k} fitting

k	$_kp_{x,B}$	$_kp_{x*,A}$ ($_kp_x$ fitting)	$_kp_{x*,A}$ (p_{x+k} fitting)	$_kp_x$ fitting error	p_{x+k} fitting error
1	0.998860	0.998973	0.998991	0.011313%	0.013101%
2	0.997731	0.997920	0.997954	0.018872%	0.022358%
3	0.996604	0.996840	0.996891	0.023677%	0.028773%
4	0.995458	0.995717	0.995783	0.026079%	0.032641%
5	0.994283	0.994547	0.994625	0.026527%	0.034394%
6	0.993080	0.993318	0.993407	0.023919%	0.032893%
7	0.991809	0.992019	0.992116	0.021158%	0.031007%
8	0.990480	0.990634	0.990738	0.015590%	0.026026%
9	0.989054	0.989154	0.989259	0.010117%	0.020819%

For the reasons above, it justifies us to fit $_kp_x$. Tables 4.5 and 4.6 are illustrations of fitting $_kp_x$ versus fitting p_{x+k} for various premiums compared with true values. In these tables, 2001 CSO male mortality is used to fit 2001 CSO female mortality. Fitted values are used to price life insurance and annuity products that were sold in 2001, and compared with true values (based on 2001 CSO female mortality). Interest rate is assumed to be 5%.

Table 4.5: Comparison of fitting 2001 CSO female mortality by male mortality between $_kp_x$ fitting and p_{x+k} fitting, $x = 30$, $n = 20$

	Female	$_kp_x$ fitting	change (%)	p_{x+k} fitting	change (%)
$A^1_{x:\overline{n}|}$	0.0155563	0.0156102	**0.3465%**	0.0156318	0.4857%
$A_{x:\frac{1}{n}|}$	0.3663534	0.3663579	0.0012%	0.3663534	**0.0000%**
$A_{x:\overline{n}|}$	0.3819096	0.3819681	**0.0153%**	0.3819852	0.0198%
$\ddot{a}_{x:\overline{n}|}$	12.9798979	12.9786700	**-0.0095%**	12.9783113	-0.0122%
$P^1_{x:\overline{n}|}$	0.0011985	0.0012028	**0.3560%**	0.0012045	0.4980%
$P_{x:\frac{1}{n}|}$	0.0282247	0.0282277	**0.0107%**	0.0282281	0.0122%
$P_{x:\overline{n}|}$	0.0294232	0.0294304	**0.0248%**	0.0294326	0.0320%
$\mathbf{e}_{x:\overline{n}|}$	19.7706840	19.7694184	**-0.0064%**	19.7688000	-0.0095%

Table 4.6: Comparison of fitting 2001 CSO female mortality by male mortality between $_kp_x$ fitting and p_{x+k} fitting, $x = 40$, $n = 20$

	Female	$_kp_x$ fitting	change (%)	p_{x+k} fitting	change (%)
$A^1_{x:\overline{n}|}$	0.0360356	0.0360953	0.1658%	0.0360737	**0.1058%**
$A_{x:\frac{1}{n}|}$	0.3516501	0.3515586	-0.0260%	0.3516501	**0.0000%**
$A_{x:\overline{n}|}$	0.3876857	0.3876539	**-0.0082%**	0.3877238	0.0098%
$\ddot{a}_{x:\overline{n}|}$	12.8586006	12.8592680	**0.0052%**	12.8577999	-0.0062%
$P^1_{x:\overline{n}|}$	0.0028024	0.0028069	0.1606%	0.0028056	**0.1120%**
$P_{x:\frac{1}{n}|}$	0.0273475	0.0273389	-0.0312%	0.0273492	**0.0062%**
$P_{x:\overline{n}|}$	0.0301499	0.0301459	**-0.0134%**	0.0301548	0.0161%
$\mathbf{e}_{x:\overline{n}|}$	19.4929997	19.4935685	**0.0029%**	19.4916554	-0.0069%

As Tables 4.5 and 4.6 show, fitting $_kp_x$ is better than fitting p_{x+k} when we price life insurance or annuity products, or calculate the expected life time (generally has a smaller error margin). Although $_kp_x$ fitting is not as good at pricing some insurance products for some age, the overall accuracy confirms that fitting $_kp_x$ is a good approach.

4.3 Fitting Mortality on Separate Intervals

Previous subsections deal with mortality fitting for people aged 30 and 40. A reason for this is that most of life insurance products are sold to adults. Except some special circumstances such as global epidemic which causes great changes in mortality structure, mortality rates for adult groups are usually increasing with age. Fitting mortality for adult groups between two sets of mortalities usually gives very good estimation.

For infant or teenage groups, however, the mortality rates are a little bit more complicated. Figures 4.6 and 4.7 give fitted curves for 1980 CSO (male) fitting 2001 CSO (male) for $x = 5$ and $n = 20$. Both q_{x+k} curves have slightly similar but different
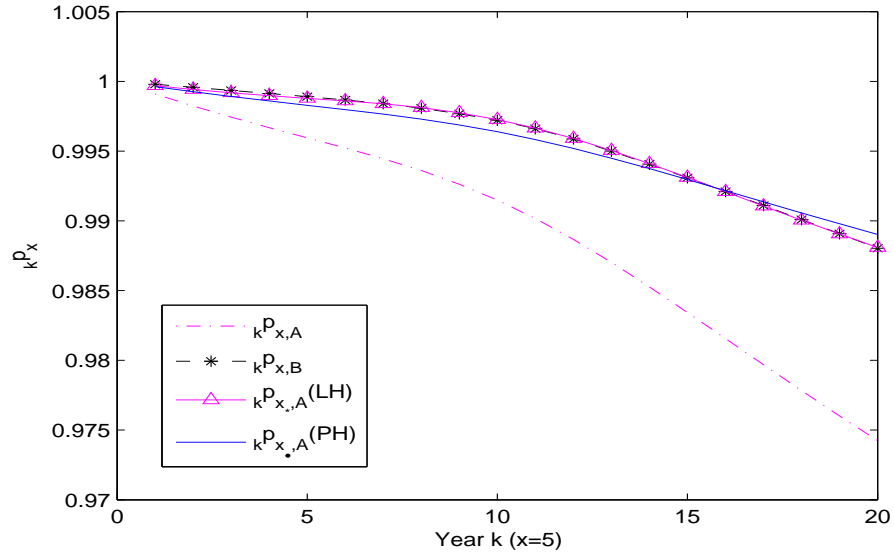
Figure 4.6: $_kp_x$: A (1980 CSO male) fits B (2001 CSO male) for $x = 5$ and $n = 20$
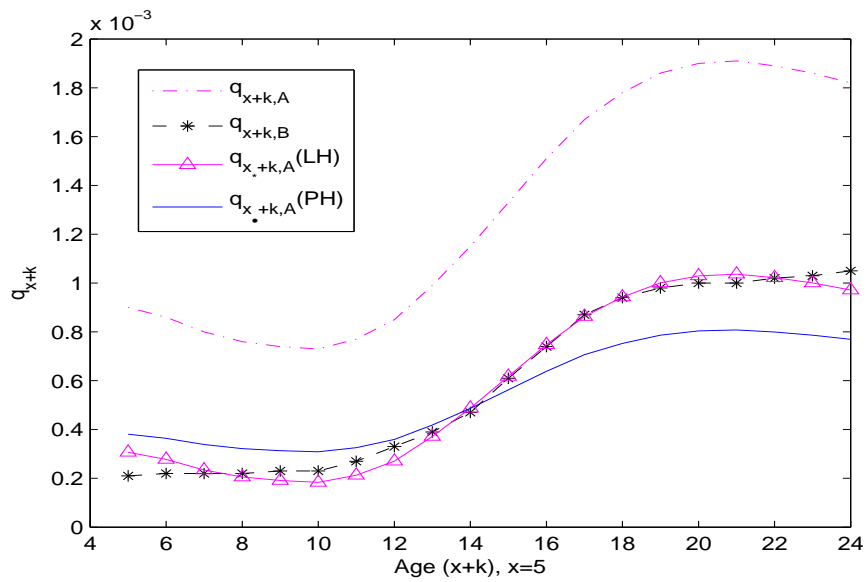

Figure 4.7: q_{x+k}: A (1980 CSO male) fits B (2001 CSO male) for $x = 5$ and $n = 20$

shapes. The curve for 1980 CSO is not monotone. It goes down first and then starts going up at around age 10, and steadily increases until the age reaches around 20 where the curve starts going down again. On the other hand, 2001 CSO curve is increasing; it increases slowly over intervals [5,10] and [19, 24], but increases faster over the interval [10,19].

One major problem with fitting one curve to the other under the hazard transforms is that the fitted curve usually inherits the shape of the original curve. The fitted curve minimizes the sum of square errors without changing its original shape. Apparently, the problem here is that the q_{x+k} curves for 1980 CSO and 2001 CSO have different shapes. Therefore, there will be significant differences between the fitted curve and the curve for 2001 CSO as illustrated in Figure 4.7. Although the linear hazard transform gives a better fit than the proportional hazard transform, there is still room for improvement. Note that although q_{x+k} plot shows the drawback, $_kp_x$ plot in Figure 4.6 indicates that the fitted curve under the LH transform looks good. The reason is that the error of q_{x+k} at each step offsets each other's impact since $_kp_x$ is the product of $(1 - q_{x+i})$, $i = 0, 1, ..., k - 1$. Therefore, the deviation of the fitted $_kp_x$ curve is less significant than that of the fitted q_{x+k} curve.

In order to solve this problem, we consider partitioning the age interval [5,25] into two or more intervals, over each of which both q_{x+k} curves for 1980 CSO and 2001 CSO look more similar to each other. First of all, we have two options to split the interval, either non-overlapping subintervals [5,10], [11, 20], [21, 24], or overlapping subintervals [5,10], [10, 20], [20, 24]. The idea behind this is that we make both curves monotone and look as similar to each other as possible over each subinterval, and make the number of partitioned subintervals as few as possible to save time and effort. Regressions are done under the PH and LH transforms. Both $_kp_x$ and p_{x+k} fitting are applied for the purpose of comparison. The sum of square errors based on the deviation of $_kp_x$ is calculated. The results are summarized in Table 4.7. From Table 4.7, it can be concluded that the non-overlapping subintervals gives a better fit overall. Figures 4.8-4.11 are regression on the non-overlapping subintervals. Table 4.8 is a summary of fitting on the entire interval $[5, 24]$ and the non-overlapping subintervals $[5, 10]$, $[11, 20]$ and $[21, 24]$. Table 4.8 tells us that when we do regression

Table 4.7: Comparison of different methods of fitting over partitioned subintervals

S.S.E.	Transform	Non-overlapping	Overlapping
$_kp_x$ fitting	LH	4.34E-09	4.41E-09
	PH	3.11E-07	2.66E-07
p_{x+k} fitting	LH	4.30E-09	4.63E-09
	PH	1.20E-06	1.26E-06

Table 4.8: Standard error comparison between the entire interval and non-overlapping subintervals based on $_kp_x$ fitting

Regression on	LH method	PH method
Entire interval	7.979E-05	59.9E-05
Non-overlapping subintervals	4.337E-09	310.7E-09

over the non-overlapping subintervals, the standard error will be greatly reduced for both the PH and LH transforms. Figures 4.8 and 4.10 are the improved fitting plots for $_kp_x$. As mentioned earlier, due to the cumulative offset impact of errors, the differences are not easily noticed. When we look at Figures 4.9 and 4.11, however, we observe some interesting results. One thing worth noting is that the curve under the LH transform well fits the curve for 2001 CSO while the curve under the PH transform does not fit as well, and has obvious jumps at the points of partition, 10 and 20. The explanation for this is that the PH transform does not have a constant term, and therefore does not have as much flexibility as the LH transform. Regression over non-overlapping subintervals under the LH transform makes these two curves have similar properties and, as a result, fitting error can be greatly reduced. However, regression over non-overlapping subintervals under the PH transform may cause a non-smooth curve at the partitioning points between subintervals, and the sum of square errors is still far larger than that under the LH transform, which is caused by its inflexibility in the transform.

Another interesting observation is that the increasing monotonicity of q_{x+k} may change after applying both the LH and PH transforms over some intervals. For

example, in Figure 4.9, the q_{x+k}'s are increasing on the interval $[11, 20]$ before and after the LH transform. However, the decreasing q_{x+k}'s on the intervals $[5, 10]$ and $[21, 24]$ become increasing after the LH transform because the corresponding α's are negative (we allow negative α for the purpose of mortality fitting) for these intervals. Consider the sequence $\{q_{x+k} : k = n_1, n_1 + 1, \ldots, n_2\}$ where $0 \le n_1 \le n_2 \le n$. Then

$$
\begin{aligned}
q_{x+k} - q_{x+k-1} &= 1 - p_{x+k} - (1 - p_{x+k-1}) \\
&= e^{-\int_{k-1}^{k} \mu_x(s)ds} - e^{-\int_{k}^{k+1} \mu_x(s)ds}.
\end{aligned}
$$

Therefore,

$$
q_{x+k} - q_{x+k-1} \ge 0 \quad \Leftrightarrow \quad \int_{k-1}^{k} \mu_x(s)ds \le \int_{k}^{k+1} \mu_x(s)ds.
$$

For the fitted sequence $\{q_{x_*+k} : k = n_1, n_1 + 1, \ldots, n_2\}$,

$$
\begin{aligned}
q_{x_*+k} - q_{x_*+k-1} &= e^{-\int_{k-1}^{k} \mu_{x_*}(s)ds} - e^{-\int_{k}^{k+1} \mu_{x_*}(s)ds} \\
&= e^{-\beta}\left(e^{-\alpha \int_{k-1}^{k} \mu_x(s)ds} - e^{-\alpha \int_{k}^{k+1} \mu_x(s)ds}\right).
\end{aligned}
$$

Therefore,

$$
q_{x_*+k} - q_{x_*+k-1} \ge 0 \quad \Leftrightarrow \quad
\begin{cases}
\int_{k-1}^{k} \mu_x(s)ds \le \int_{k}^{k+1} \mu_x(s)ds, & if \ \alpha \ge 0; \\
\int_{k-1}^{k} \mu_x(s)ds \ge \int_{k}^{k+1} \mu_x(s)ds, & if \ \alpha \le 0.
\end{cases}
$$

For the reasons above, we have

$$
q_{x+k} \le q_{x+k-1} \Leftrightarrow
\begin{cases}
q_{x_*+k} \le q_{x_*+k-1} \ and \ \alpha \ge 0; \\
q_{x_*+k} \ge q_{x_*+k-1} \ and \ \alpha \le 0.
\end{cases}
$$

Therefore, the increasing (decreasing) monotonicity of q_{x+k} is preserved under both the LH and PH transforms if α is non-negative (non-positive).
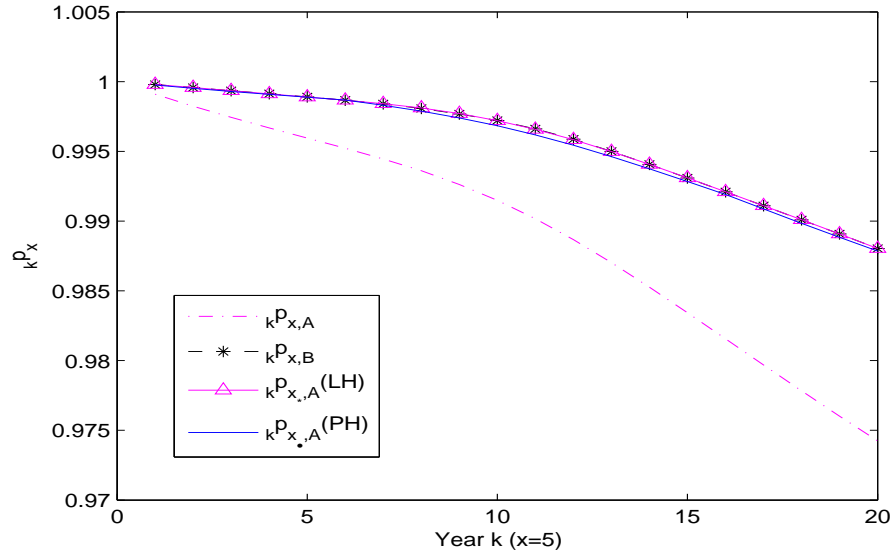
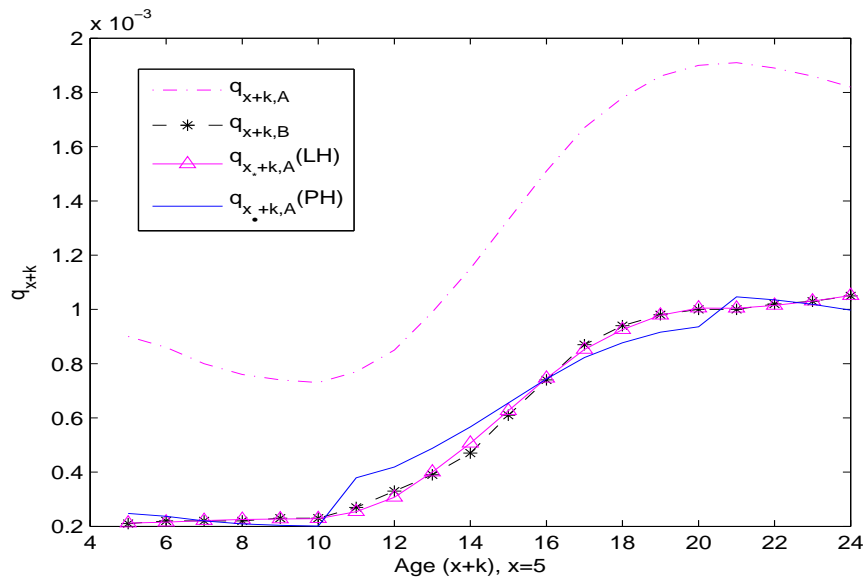Figure 4.8: $_k p_x$: A (1980 CSO male) fits B (2001 CSO male) for $x = 5$ and $n = 20$ based on p_{x+k} fitting

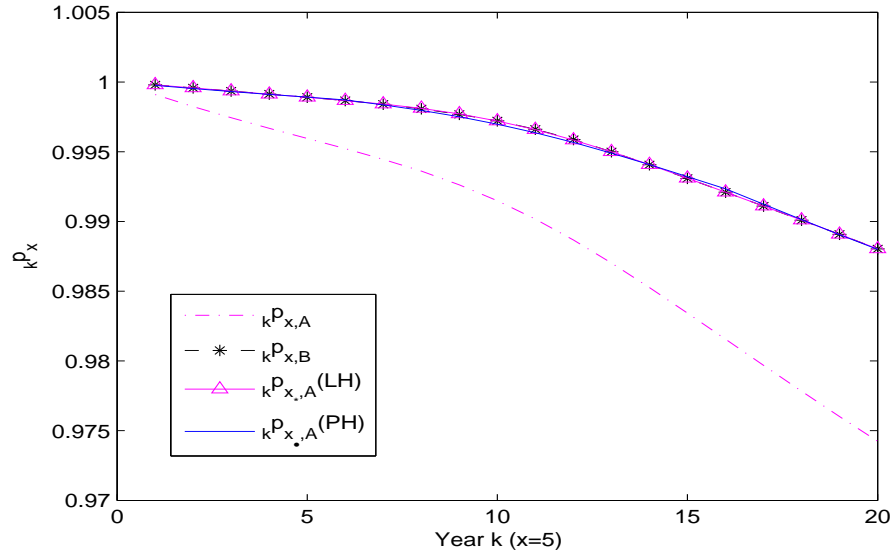Figure 4.9: q_{x+k}: A (1980 CSO male) fits B (2001 CSO male) for $x = 5$ and $n = 20$ based on p_{x+k} fitting

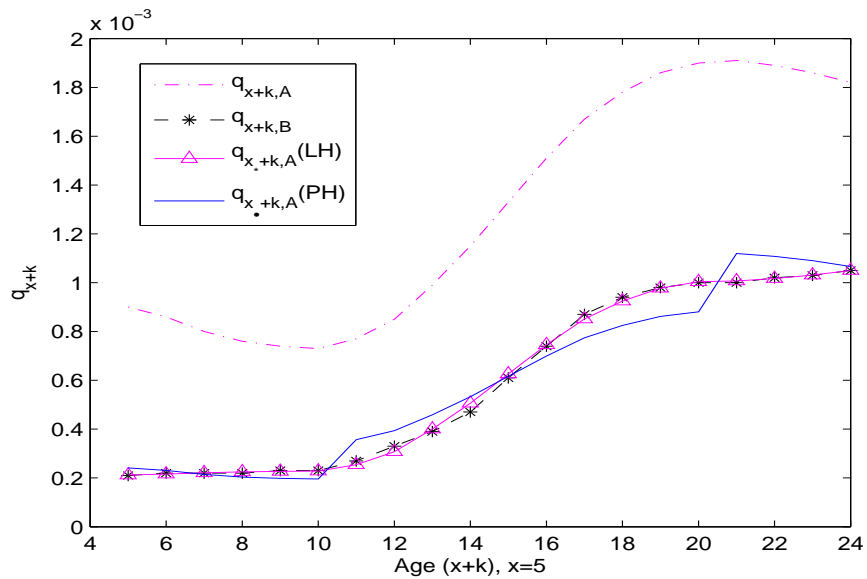Figure 4.10: $_kp_x$: A (1980 CSO male) fits B (2001 CSO male) for $x = 5$ and $n = 20$ based on $_kp_x$ fitting

Figure 4.11: q_{x+k}: A (1980 CSO male) fits B (2001 CSO male) for $x = 5$ and $n = 20$ based on $_kp_x$ fitting

Chapter 5

Mortality Prediction

5.1 Linear Interpolation of Parameters α and β of the LH Transform

We have showed that fitting $_kp_x$ under the LH transform is better since it has a smaller sum of square errors than the PH transform. The fitting of historical data works well. Intuitively, we can also predict future mortality given current estimates of the parameters of the LH transform. Suppose two mortality tables for years $(y - m)$ and y are available. By fitting year $(y - m)$'s mortality to year y's, estimates of α and β can be obtained. We assume that mortality improvement from year y to $(y + m)$ follows the same trend, i.e., $\mu_x^{(y+m)}(t) = \alpha\mu_x^{(y)}(t) + \beta$. We need to find approximate values of α and β for year $(y + k)$, $k = 0, 1, ...,$ i.e., α_{y+k} and β_{y+k} such that $\mu_x^{(y+k)}(t) = \alpha_{y+k}\mu_x^{(y)}(t) + \beta_{y+k}$ with $\alpha_y = 1$, $\alpha_{y+m} = \alpha$, $\beta_y = 0$ and $\beta_{y+m} = \beta$. In this project, linear interpolation is proposed to calculate α_{y+k} and β_{y+k}, that is,

$$\alpha_{y+k} = (1 - \frac{k}{m}) \times 1 + \frac{k}{m} \times \alpha, \tag{5.1}$$

$$= 1 + \frac{k}{m}(\alpha - 1) \tag{5.2}$$

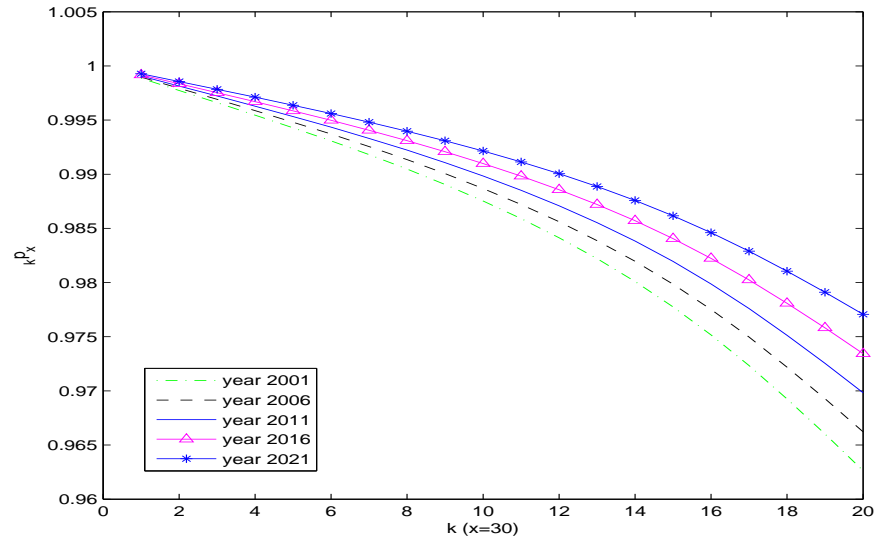and

$$\beta_{y+k} = \frac{k}{m} \times \beta, \tag{5.3}$$

Figure 5.1: Predicted $_kp_x$: use 2001 CSO male to predict future mortality, $x = 30$, $n = 20$

$k = 0, 1, 2, \ldots$

Formulas (5.1) and (5.3) use the linear interpolation to estimate values of α_{y+k} and β_{y+k}. It is reasonable to do so because of the observation that mortality improves steadily over time. Given an individual aged x, a time period m and a particular future year $y + k$, we can estimate the values of α_{y+k} and β_{y+k} with the help of formulas (5.1) and (5.3). Then we can project the mortality curve for the individual and the particular year. Note that from (5.2), if $\alpha < 1$, predicted α_{y+k}'s will be less than 1; if $\alpha > 1$, predicted α_{y+k}'s will be greater than 1.

For the purpose of illustration, several scenarios are discussed and plots are drawn to show the effect of the linear interpolation of α and β. 1980 CSO and 2001 CSO mortality tables are available. Regression of these two sets of mortalities are illustrated previously in Tables 4.2 and 4.3. We are going to estimate mortality for year 2001 and the following years. Therefore, m equals 21 and y equals 2001 in this case.

Future projections are illustrated by Figures 5.1-5.4 for people aged 30 and 40 over a time period of 20 years. Actual mortality (2001 CSO) is also included in the figures for the purpose of comparison. For each age x, the first plot predicts
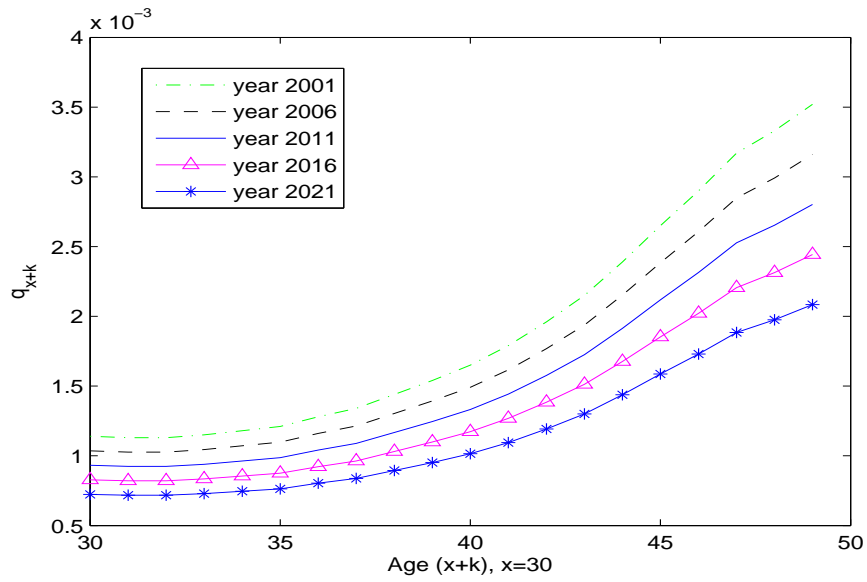
Figure 5.2: Implied q_{x+k}: use 2001 CSO male to predict future mortality, $x = 30$, $n = 20$
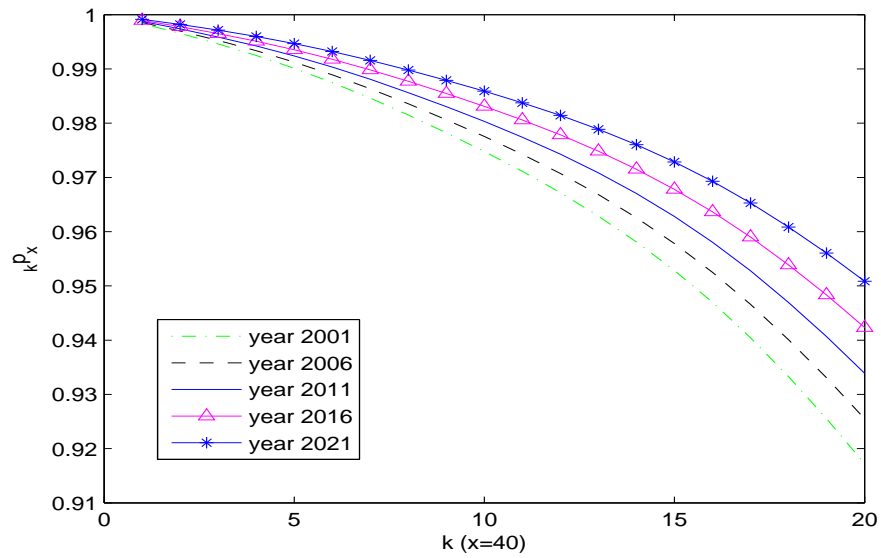

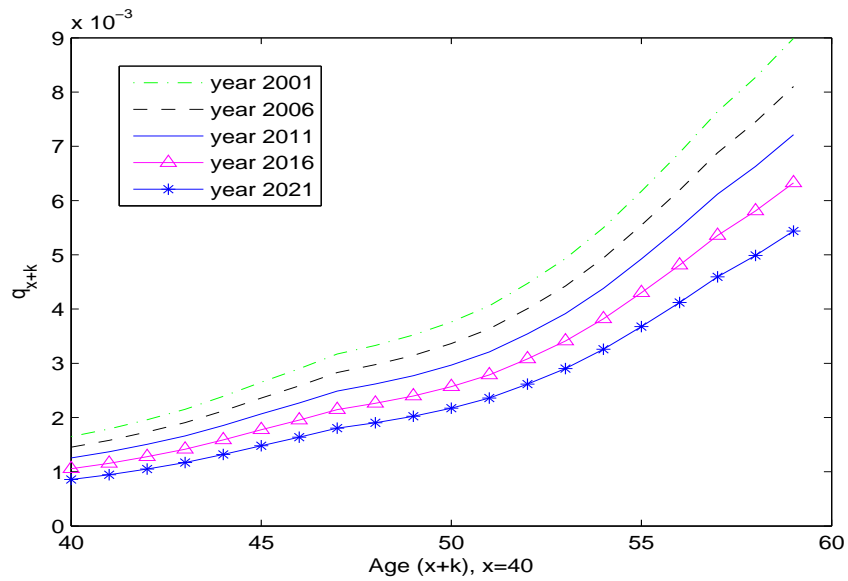Figure 5.3: Predicted $_k p_x$: use 2001 CSO male to predict future mortality, $x = 40$, $n = 20$

Figure 5.4: Implied q_{x+k}: use 2001 CSO male to predict future mortality, $x = 40$, $n = 20$

survival probability $_{k}p_{x}$ for future years accompanied by another plot of q_{x+k} implied by $q_{x+k} = 1 - \frac{_{k+1}p_x}{_{k}p_x}$.

These figures show that the linear interpolation adjustment of α and β are reflected in the trend of mortality improvement for the future. Actually, this method can also be applied to obtain the mortalities for the years in the past where the mortality tables are not available. As a matter of fact, making a mortality table is a time-consuming process because lots of work involving data collection and processing needs to be done. So when we try to obtain a mortality table for a year in the past, it is not practical to do all the necessary work to get one. Instead, we can rely on two most recent mortality tables available (1980 CSO and 2001 CSO) and the associated values of α and β. Then the methodology above is applied to predict mortalities for given x, n and a certain year between 1980 and 2001. This could save a lot of time and efforts. The linear hazard transform and the linear interpolation of parameters do provide us a tool to forecast future and past mortalities.

5.2 Future Diagonal q_x

The last section discusses the method of the linear interpolation of α and β to predict mortality curves for the years in the past or future. But when we price insurance products, this method does not take the mortality risk into account with respect to time. That is to say, each mortality curve for a year is estimated based on a time origin for that year. When a person ages, his or her mortality should change accordingly, i.e., the mortality improves to a new one based on that particular year. We call this the diagonal mortality method. Taking diagonal q_x captures the improvement of mortality each year and takes into account the effect of time. It is a dynamic mortality scheme instead of a static one where all mortality rates are based on only one specific year's table. Table 5.1 is an illustration of how this method works.

Table 5.1: Method of Taking Diagonal Mortality

Year	Mortality Table						
y	$\mathbf{q_x^{(y)}}$	$q_{x+1}^{(y)}$	$q_{x+2}^{(y)}$	$q_{x+3}^{(y)}$	$q_{x+4}^{(y)}$	$q_{x+5}^{(y)}$	$\cdots$
y+1	$q_x^{(y+1)}$	$\mathbf{q_{x+1}^{(y+1)}}$	$q_{x+2}^{(y+1)}$	$q_{x+3}^{(y+1)}$	$q_{x+4}^{(y+1)}$	$q_{x+5}^{(y+1)}$	$\cdots$
y+2	$q_x^{(y+2)}$	$q_{x+1}^{(y+2)}$	$\mathbf{q_{x+2}^{(y+2)}}$	$q_{x+3}^{(y+2)}$	$q_{x+4}^{(y+2)}$	$q_{x+5}^{(y+2)}$	$\cdots$
y+3	$q_x^{(y+3)}$	$q_{x+1}^{(y+3)}$	$q_{x+2}^{(y+3)}$	$\mathbf{q_{x+3}^{(y+3)}}$	$q_{x+4}^{(y+3)}$	$q_{x+5}^{(y+3)}$	$\cdots$
y+4	$q_x^{(y+4)}$	$q_{x+1}^{(y+4)}$	$q_{x+2}^{(y+4)}$	$q_{x+3}^{(y+4)}$	$\mathbf{q_{x+4}^{(y+4)}}$	$q_{x+5}^{(y+4)}$	$\cdots$
y+5	$q_x^{(y+5)}$	$q_{x+1}^{(y+5)}$	$q_{x+2}^{(y+5)}$	$q_{x+3}^{(y+5)}$	$q_{x+4}^{(y+5)}$	$\mathbf{q_{x+5}^{(y+5)}}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

As we see in Table 5.1, for a person aged x over a time period of n, we project the future mortalities for the next n years. Then the mortality $q_{x+k}^{(y+k)}$, $k = 0, 1, \ldots, n-1$, on the diagonal line are selected. This makes sense because as a person ages, his or her mortality might change due to mortality improvement. This progressive approach
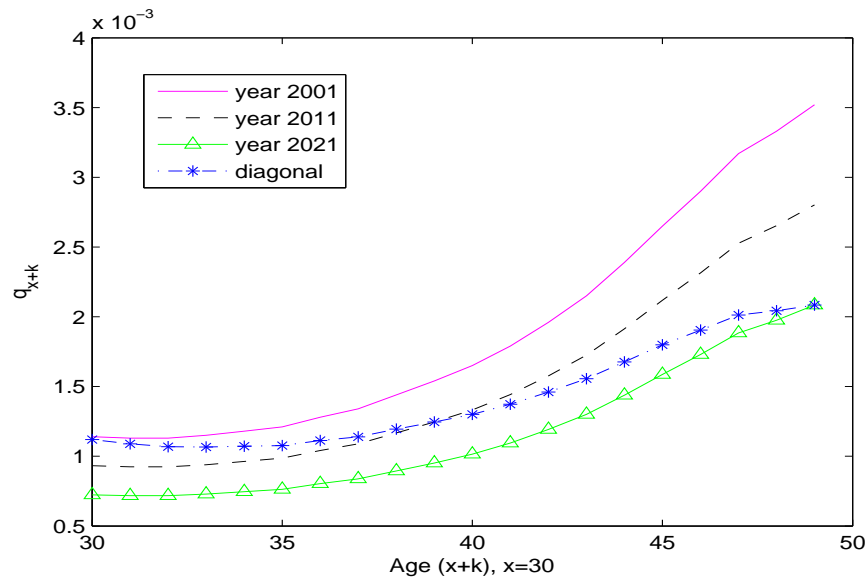
Figure 5.5: q_{x+k} with the diagonal method, male, $x = 30$, $n = 20$

help us better understand the dynamic of future mortalities.

As an application, the diagonal method is applied to the 2001 CSO male mortality, aged 30 and 40 over a time period of 20 years. Based on the mortality for year 2001, we are going to predict the mortalities for years 2011 and 2021. The plots include the mortality curves obtained in section 5.1 as well as diagonal mortality curve. The following plots are diagonal projections of future mortalities based on the 2001 CSO mortality. As observed in Figures 5.5 and 5.7, the diagonal mortality curve gradually moves from year 2001 curve towards year 2021 curve, colliding with year 2021 curve at the end. This demonstrates that diagonal mortality curve represents the mortality improvement over time. To approximate this diagonal curve, we can also apply the LH transform to year 2001 mortality to obtain the best estimate. The adopted methodology is the same as the one in Chapter 4. Figures 5.9-5.12 are regression plots. As seen from Figures 5.9-5.12, the LH transform gives a good fit of the future diagonal mortality curve based on the 2001 CSO.

Tables 5.2 and 5.3 are premiums for insurance products based on 2001 CSO, the diagonal mortality and the mortality fitted by the LH transform. As we can see from
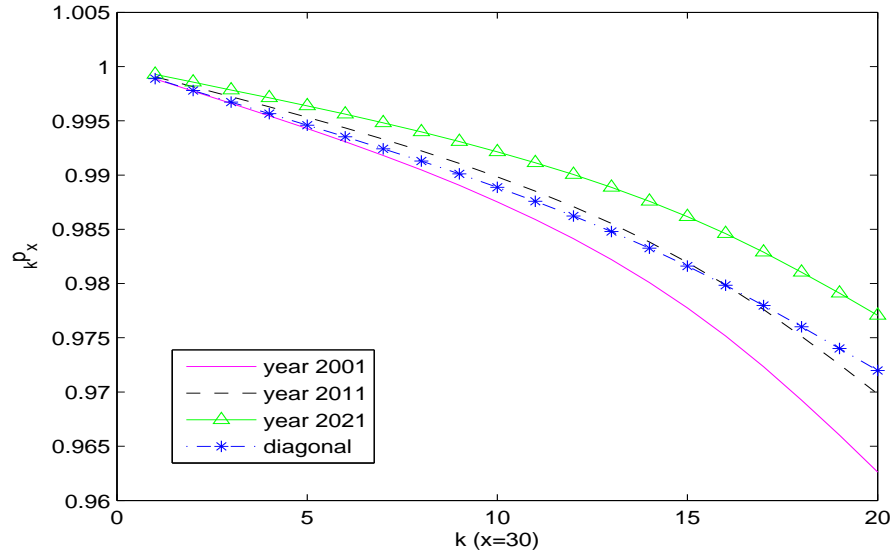
Figure 5.6: $_kp_x$ with the diagonal method, male, $x = 30$, $n = 20$
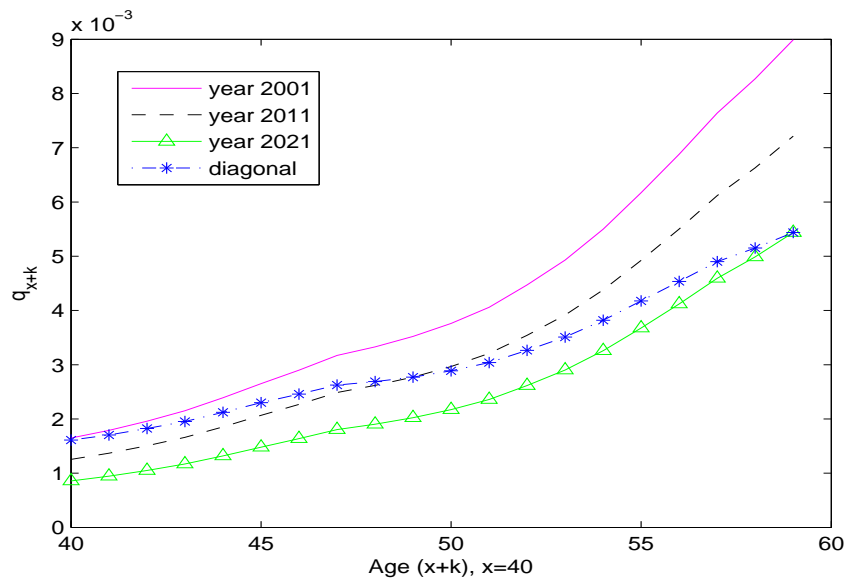
Figure 5.7: q_{x+k} with the diagonal method, male, $x = 40$, $n = 20$
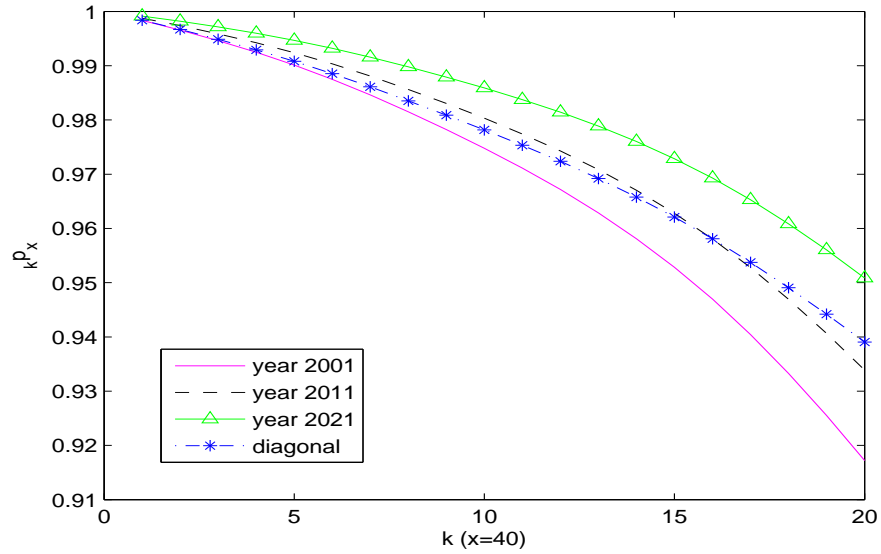
Figure 5.8: $_k p_x$ with the diagonal method, male, $x = 40$, $n = 20$
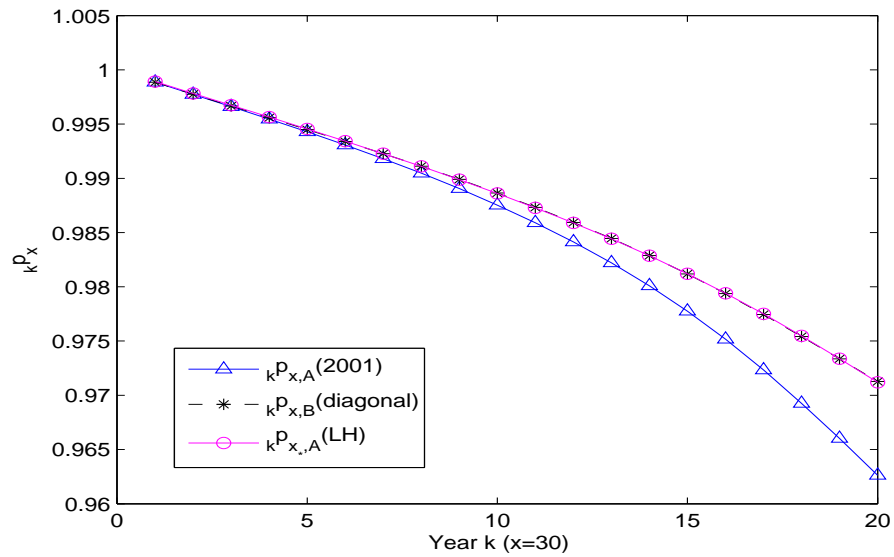
Figure 5.9: $_k p_x$: fitting 2001 CSO male to the diagonal projection mortality, $x = 30$, $n = 20$
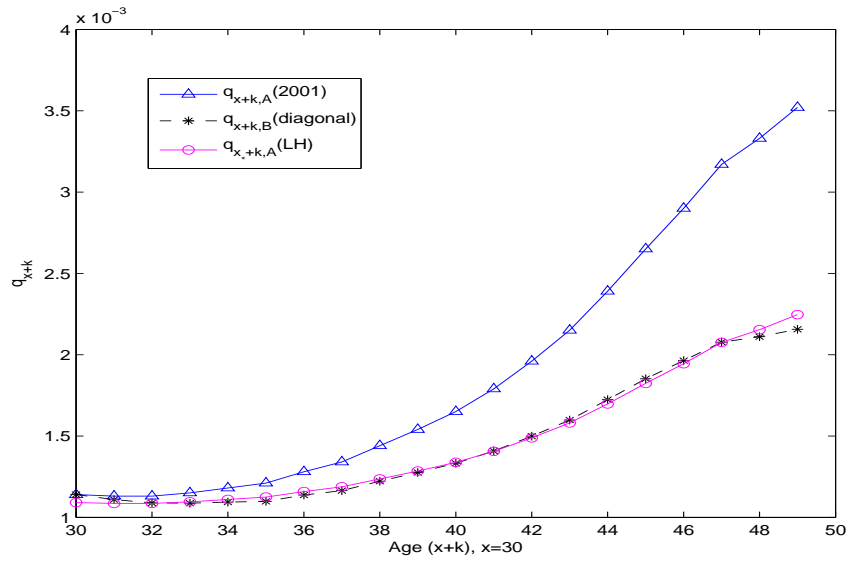
Figure 5.10: q_{x+k}: fitting 2001 CSO male to the diagonal projection mortality, $x = 30$, $n = 20$
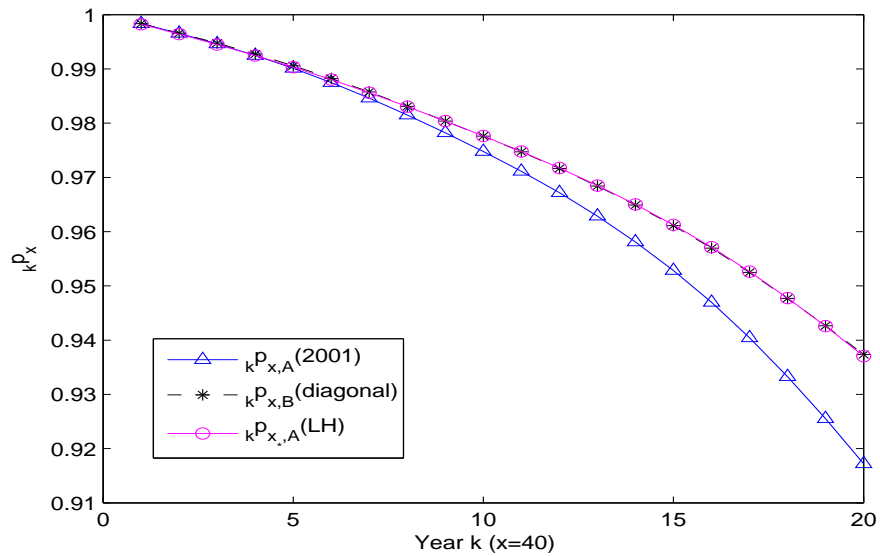
Figure 5.11: $_kp_x$: fitting 2001 CSO male to the diagonal projection mortality, $x = 40$, $n = 20$

Figure 5.12: q_{x+k}: fitting 2001 CSO male to the diagonal projection mortality, $x = 40$, $n = 20$

these tables, the LH fitting gives very good approximations to these premiums based on the diagonal mortalities.

Table 5.2: Comparison of premiums based on 2001 CSO, diagonal mortality and LH fitted mortality, $x = 30$, $n = 20$

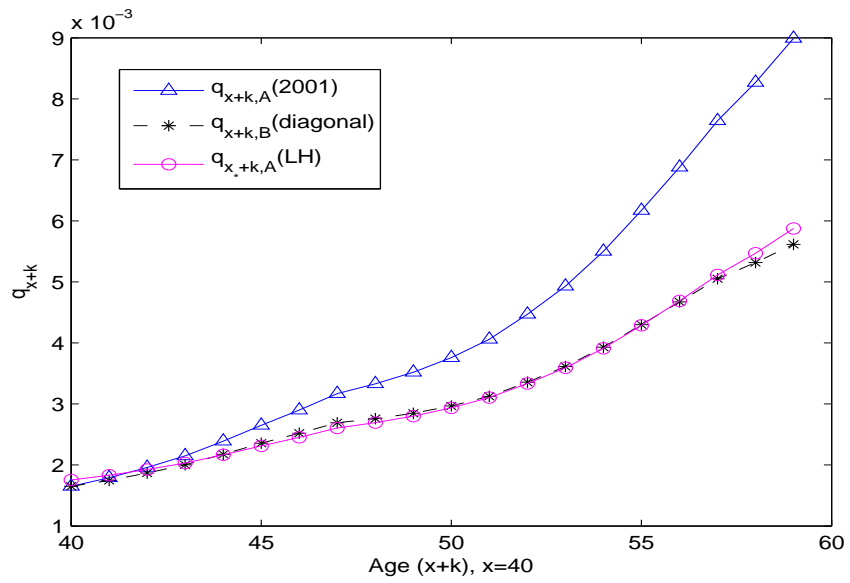|	2001 CSO	Diagonal	LH fitting	Error (%)
$A^1_{x:\overline{n}|}$	0.0209139	0.0167868	0.0168035	0.0992%
$A_{x:\overline{n}|}^{1}$	0.3628028	0.3660589	0.3660311	-0.0076%
$A_{x:\overline{n}|}$	0.3837167	0.3828457	0.3828345	-0.0029%
$\ddot{a}_{x:\overline{n}|}$	12.9419495	12.9602411	12.9604748	0.0018%
$P^1_{x:\overline{n}|}$	0.0016160	0.0012953	0.0012965	0.0974%
$P_{x:\overline{n}|}^{1}$	0.0280331	0.0282448	0.0282421	-0.0094%
$P_{x:\overline{n}|}$	0.0296491	0.0295400	0.0295386	-0.0047%
$\mathbf{e}_{x:\overline{n}|}$	19.6903784	19.7373572	19.7375670	0.0011%

Table 5.3: Comparison of premiums based on 2001 CSO, diagonal mortality and LH fitted mortality, $x = 40$, $n = 20$

	2001 CSO	Diagonal	LH fitting	Error (%)
$A^1_{x:\overline{n}|}$	0.0450985	0.0354123	0.0355665	0.4354%
$A_{x:\overline{n}|}^{1}$	0.3456883	0.3532805	0.3531635	-0.0331%
$A_{x:\overline{n}|}$	0.3907868	0.3886928	0.3887300	0.0096%
$\ddot{a}_{x:\overline{n}|}$	12.7934773	12.8374510	12.8366693	-0.0061%
$P^1_{x:\overline{n}|}$	0.0035251	0.0027585	0.0027707	0.4416%
$P_{x:\overline{n}|}^{1}$	0.0270207	0.0275195	0.0275121	-0.0270%
$P_{x:\overline{n}|}$	0.0305458	0.0302780	0.0302828	0.0157%
$\mathbf{e}_{x:\overline{n}|}$	19.3541276	19.4660930	19.4651640	-0.0048%

Chapter 6

Risk Ordering and Optimal Reinsurance

6.1 Risk Ordering

In insurance industry, it is sometimes difficult or not necessary to quantify the exact amount of a risk. Instead, actuaries are more interested in comparing two risks based on some criteria. This is called risk ordering. As long as they have the ordering of the underlying risks, actuaries can further compare the ordering of other quantities of interest, such as ruin probabilities, prices of insurance products, etc.

For a risk Z, let $S_Z(t)$ be its survival function, $t \geq 0$. The PH transform of random variable Z is denoted by $Z_\alpha^\bullet$ and the LH transform of Z is denoted by $Z_{\alpha,\beta}^*$. It is straightforward to see that the survival function of $Z_\alpha^\bullet$ is $S_Z(t)^\alpha$ while the survival function of $Z_{\alpha,\beta}^*$ is $S_Z(t)^\alpha e^{-\beta t}$. Define $\pi_\alpha(Z) = \int_0^\infty S_Z(t)^\alpha dt = E(Z_\alpha^\bullet)$ and $\pi_{\alpha,\beta}(Z) = \int_0^\infty S_Z(t)^\alpha e^{-\beta t} dt = E(Z_{\alpha,\beta}^*)$ for $\alpha > 0$ and $\beta \geq 0$. We give a variety of definitions of ordering in the following.

Definition 13. *A risk Y stochastically dominates a risk X (written as $X \leq_{st} Y$) if and only if $S_X(t) \leq S_Y(t)$ for $t \geq 0$.*

Definition 14. *A risk X is smaller than a risk Y in the hazard rate order (written as $X \leq_{hr} Y$) if and only if $r_X(t) \geq r_Y(t)$ for $t \geq 0$, where $r_Z(t) = \frac{f_Z(t)}{S_Z(t)}$, $Z = X, Y$.*

Since $S_Z(t) = e^{-\int_0^t \mu_Z(s)ds}$, $Z = X, Y$, it is easy to see that $X \leq_{hr} Y$ implies $X \leq_{st} Y$.

Definition 15. *A risk X is smaller than a risk Y in the PH-transform (proportional hazard transform) order (written as $X \leq_{ph} Y$) if and only if $\pi_\alpha(X) \leq \pi_\alpha(Y)$ for $\alpha > 0$.*

Definition 16. *A risk X is smaller than a risk Y in the LH-transform (linear hazard transform) order (written as $X \leq_{lh} Y$) if and only if $\pi_{\alpha,\beta}(X) \leq \pi_{\alpha,\beta}(Y)$ for $\alpha > 0$ and $\beta \geq 0$.*

It is trivial that $X \leq_{lh} Y$ implies $X \leq_{ph} Y$ by letting $\beta = 0$.

Definition 17. *A risk X is smaller than a risk Y in the Laplace transform order (written as $X \leq_{lt} Y$) if and only if $E(e^{-sX}) \geq E(e^{-sY})$ for $s \geq 0$.*

Since $E(e^{-sZ}) = \int_0^\infty e^{-st} f_Z(t)dt = -\int_0^\infty e^{-st} dS_Z(t) = 1 - s \int_0^\infty e^{-st} S_Z(t)dt$ for $s \geq 0$ and $Z = X, Y$, we have $X \leq_{lt} Y \Leftrightarrow \int_0^\infty e^{-st} S_X(t)dt \leq \int_0^\infty e^{-st} S_Y(t)dt$ for $s \geq 0$.

Definition 18. *A risk X is less dangerous than a risk Y (written as $X \leq_D Y$) if and only if (1) $E(X) \leq E(Y)$ and (2) there exists $c \geq 0$ such that*

$$S_X(t) \begin{cases} \geq S_Y(t), & 0 \leq t < c, \\ \leq S_Y(t), & c \leq t. \end{cases}$$

Proposition 6. *For risks X and Y, $X \leq_{st} Y$ if and only if $X^*_{\alpha,\beta} \leq_{st} Y^*_{\alpha,\beta}$, which implies $X \leq_{lh} Y$, or equivalently, $X^\bullet_\alpha \leq_{lt} Y^\bullet_\alpha$.*

Proof: For all $t \geq 0$, $\alpha > 0$ and $\beta \geq 0$,

$$
\begin{aligned}
X \leq_{st} Y \quad &\Leftrightarrow \quad 0 \leq S_X(t) \leq S_Y(t) \\
&\Leftrightarrow \quad 0 \leq S_X(t)^\alpha e^{-\beta t} \leq S_Y(t)^\alpha e^{-\beta t} \\
&\Leftrightarrow \quad S_{X^*_{\alpha,\beta}}(t) \leq S_{Y^*_{\alpha,\beta}}(t) \\
&\Leftrightarrow \quad X^*_{\alpha,\beta} \leq_{st} Y^*_{\alpha,\beta},
\end{aligned}
$$

which implies

$$
\int_0^\infty S_X(t)^\alpha e^{-\beta t} dt \leq \int_0^\infty S_Y(t)^\alpha e^{-\beta t} dt \tag{6.1}
$$

or $X \leq_{lh} Y$. From (6.1), we also have that $X_\alpha^\bullet \leq_{lt} Y_\alpha^\bullet$, completing the proof.

Note that letting $\alpha = 1$ and $\beta \geq 0$ in (6.1) gives that $X \leq_{lh} Y$ implies $X \leq_{lt} Y$.

The following proposition explores the relationship among the dangerous order, Laplace transform order and LH transform order.

Proposition 7. $X \leq_D Y$ and $X \leq_{lt} Y \Rightarrow X \leq_{lh} Y$ and $X^*_{\alpha,\beta} \leq_D Y^*_{\alpha,\beta}$.

Proof: Let $g(x) = x^\alpha$, $\alpha \in (0,1]$; it is easy to see that $g(x)$ is concave and non-decreasing. Therefore, for any point $(S_Y(t), g(S_Y(t)))$, there exists a tangent line $L(u)$ touching the curve $g(x)$ at $(S_Y(t), g(S_Y(t)))$ such that

$$
\begin{cases}
L(u) = g'(S_Y(t))(u - S_Y(t)) + g(S_Y(t)), \\
L(u) \geq g(u),
\end{cases} \quad \forall\, u \in [0,1].
$$

Therefore, we know that $g(u) - g(S_Y(t)) \leq g'(S_Y(t))(u - S_Y(t))$. Now let $u = S_X(t)$; we have

$$
g(S_X(t)) - g(S_Y(t)) \leq g'(S_Y(t))[S_X(t) - S_Y(t)] \tag{6.2}
$$

which means

$$
g(S_Y(t)) - g(S_X(t)) \geq g'(S_Y(t))[S_Y(t) - S_X(t)]. \tag{6.3}
$$

Since $g(x)$ is concave and non-decreasing, we have that $g'(x)$ is non-increasing and non-negative. Therefore, for $t > c$,

$$
S_Y(t) < S_Y(c) \Rightarrow g'(S_Y(t)) \geq g'(S_Y(c)). \tag{6.4}
$$

By (6.3) and (6.4),

$$g(S_Y(t)) - g(S_X(t)) \geq g'(S_Y(c))[S_Y(t) - S_X(t)]$$

$$\Rightarrow \quad [g(S_Y(t)) - g(S_X(t))]e^{-\beta t} \geq g'(S_Y(c))[S_Y(t) - S_X(t)]e^{-\beta t}$$

$$\Rightarrow \quad \int_c^\infty [g(S_Y(t)) - g(S_X(t))]e^{-\beta t}dt$$

$$\geq \int_c^\infty g'(S_Y(c))[S_Y(t) - S_X(t)]e^{-\beta t}dt. \tag{6.5}$$

On the other hand, for $t \leq c$,

$$S_Y(t) > S_Y(c) \Rightarrow g'(S_Y(t)) \leq g'(S_Y(c)). \tag{6.6}$$

From (6.2) and (6.6), we get

$$g(S_X(t)) - g(S_Y(t)) \leq g'(S_Y(c))[S_X(t) - S_Y(t)]$$

$$\Rightarrow \quad [g(S_X(t)) - g(S_Y(t))]e^{-\beta t} \leq g'(S_Y(c))[S_X(t) - S_Y(t)]e^{-\beta t}$$

$$\Rightarrow \quad \int_0^c [g(S_X(t)) - g(S_Y(t))]e^{-\beta t}dt$$

$$\leq \int_0^c g'(S_Y(c))[S_X(t) - S_Y(t)]e^{-\beta t}dt. \tag{6.7}$$

Combining (6.5) and (6.7) leads to

$$\int_0^\infty g(S_Y(t))e^{-\beta t}dt - \int_0^\infty g(S_X(t))e^{-\beta t}dt$$

$$= \int_c^\infty [g(S_Y(t)) - g(S_X(t))]e^{-\beta t}dt - \int_0^c [g(S_X(t)) - g(S_Y(t))]e^{-\beta t}dt$$

$$\geq \int_c^\infty g'(S_Y(c))[S_Y(t) - S_X(t)]e^{-\beta t}dt - \int_0^c g'(S_Y(c))[S_X(t) - S_Y(t)]e^{-\beta t}dt$$

$$= g'(S_Y(c))[\int_0^\infty S_Y(t)e^{-\beta t}dt - \int_0^\infty S_X(t)e^{-\beta t}dt]. \tag{6.8}$$

As a result, (6.8) can be written as

$$\pi_{\alpha,\beta}(Y) - \pi_{\alpha,\beta}(X) \geq g'(S_Y(c)) \int_0^\infty [S_Y(t) - S_X(t)]e^{-\beta t}dt. \tag{6.9}$$

Since $X \leq_{lt} Y$, we have $\int_0^\infty S_X(t)e^{-\beta t}dt \leq \int_0^\infty S_Y(t)e^{-\beta t}dt$ for $\beta \geq 0$. Moreover, $g'(x)$ is non-negative. Therefore, we conclude from (6.9) that

$$\pi_{\alpha,\beta}(Y) - \pi_{\alpha,\beta}(X) \geq 0$$

for $\alpha > 0$ and $\beta \geq 0$, which means $X \leq_{lh} Y$.

Next, the definition of the dangerous ordering for $X \leq_D Y$ implies

$$(S_X(t))^\alpha e^{-\beta t} \begin{cases} \geq (S_Y(t))^\alpha e^{-\beta t}, & 0 \leq t < c, \\ \leq (S_Y(t))^\alpha e^{-\beta t}, & c \leq t. \end{cases}$$

Together with

$$E(X_{\alpha,\beta}^*) = \int_0^\infty (S_X(t))^\alpha e^{-\beta t} dt = \pi_{\alpha,\beta}(X) \leq \pi_{\alpha,\beta}(Y) = \int_0^\infty (S_Y(t))^\alpha e^{-\beta t} dt = E(Y_{\alpha,\beta}^*),$$

we reach that $X_{\alpha,\beta}^* \leq_D Y_{\alpha,\beta}^*$ for $\alpha > 0$ and $\beta \geq 0$.

All the ordering relationships above are summarized in Diagram 1.

Diagram 1:

$$\boxed{\begin{array}{ccccc} X \leq_{hr} Y \implies & X \leq_{st} Y(\equiv X_{\alpha,\beta}^* \leq_{st} Y_{\alpha,\beta}^*) \implies & X \leq_{lt} Y \\ & \Downarrow & \Uparrow \Downarrow + X \leq_D Y \\ X \leq_{ph} Y & \Longleftarrow X \leq_{lh} Y(\equiv X_\alpha^\bullet \leq_{lt} Y_\alpha^\bullet) \end{array}}$$

6.2 Optimal Reinsurance

Let X be a non-negative random variable representing the amount of claims in a certain time period. An insurance company, or known as the cedent, is faced with the risk X and decides to purchase a reinsurance contract from a reinsurance company. The reinsurance contract is written on X; the reinsurance company collects a risk adjusted premium P and promises to pay $R(X)$, $0 \leq R(X) \leq X$. Therefore, the remaining amount $X - R(X)$ is paid by the ceding company. $R(X)$ is also known as a compensation function. Under the LH transform, the risk adjusted premium P satisfies

$$P = E[R(X)^*] = \int_0^\infty S_{R(X)^*}(t) dt = \int_0^\infty S_{R(X)}(t)^\alpha e^{-\beta t} dt, \tag{6.10}$$

where $S_{R(X)}(t)$ is the survival function of $R(X)$, and $R(X)^*$ is the corresponding random variable of $R(X)$ under the LH transform.

This study extends the results of Kaluszka (2005). We tackle an optimization problem of reinsurance based on the utility function by maximizing the expected utility function $E\,u(R(X) - X)$. Since it is assumed that the cedent is risk averse, we know that u is an increasing and concave function. If we substitute $w(x) = -u(-x)$, the problem becomes to minimize $E\,w(X - R(X))$ where w is an increasing and convex function. First, we define $\mathcal{R}_0 = \{0 \le R(X) \le X; R(\cdot) \text{ is nondecreasing and}$ left continuous$\}$. We will find an optimal reinsurance contract from the set $\mathcal{R}_0$.

Let Π be a mapping from non-negative random variables to real numbers. Π is called a convex functional if for non-negative random variables X and Y,

$$\Pi(Y) \ge \Pi(X) + E(\Pi'(X)(Y - X)),$$

where $\Pi'(X)$ is a derivative of Π at X.

At a reinsurance premium P, the cedent needs to minimize a convex functional $\Pi(X - R(X))$. So we consider the following reinsurance problem

$$\min \Pi(X - R(X)) \ \ s.t. \ f(P) = H(R(X)), \ \ 0 \le R(X) \le X, \tag{6.11}$$

where P is a real value given by some premium principle, H is a convex function and f is an increasing function. Theorem 1 provides a solution to problem (6.11).

Theorem 1. *Assume there exist a compensation function $\hat{R}(X)$, derivatives $H' = H'(\hat{R}(X))$ and $\Pi' = \Pi'(X - \hat{R}(X))$, and a real $c > 0$ such that*
(1) $(\Pi' - c \inf H')_+ = c\,(H' - \inf H')$,
(2) if $H'(w) = \inf H'$ then $\hat{R}(X)(w) = 0$,
(3) $f(P) = H(\hat{R}(X))$, and
(4) $E[X]$, $E[|\Pi'X|]$, and $E[|H'X|]$ are finite.
Then $\hat{R}(X)$ is a solution of (6.11).

Proof: Refer to Kaluszka (2005).

In order to solve this optimal reinsurance under the LH transform, we consider the following problem

$$\min E[w(X - R(X))] \ \ s.t. \ f(P) = E[h(R(X))Z], \ \ 0 \le R(X) \le X, \tag{6.12}$$

where Z is a fixed positive random variable, h is a strictly convex, increasing and differentiable function, P is a real number and f is an increasing function. Put

$$v_c(t) = t + h'^{-1}[(\frac{1}{c\mu_Z}w'(t) - h'(0))_+ + h'(0)] \tag{6.13}$$

where $\mu_Z = E[Z] > 0$. Let $v_c^{-1}(t)$ be the inverse function of $v_c(t)$, $t \geq 0$. Theorem 2 provides a solution to problem (6.12).

Theorem 2. *Assume that $E[X]$, $E[Xh'(X)Z]$ and $E[Xw'(X)]$ are finite. Since $w'(0) \geq 0$, $h'(0) \geq 0$ and $\mu_Z > 0$, there exists the smallest non-negative real number, say c_0, such that $w'(0) \leq c_0\mu_Z h'(0)$. Suppose also that $\mu_Z h(0) < f(P) < E[h(L)Z]$, where L stands for the right limit of $X - v_c^{-1}(X)$ at c_0. Then a solution of problem (6.12) is given by $\hat{R}(X) = X - v_c^{-1}(X)$ with $c > c_0$ being such that $E[h(\hat{R}(X))Z] = f(P)$.*

Proof: We use Theorem 1 with $\Pi(X) = E[w(X)]$ and $H(X) = E[h(X)Z]$. First, condition (1) of Theorem 1 is equivalent to the following

$$[w'(x - R(x)) - c\mu_Z h'(0)]_+ = c\mu_Z h'(R(x)) - c\mu_Z h'(0), \ x \geq 0, \tag{6.14}$$

where $\inf H' = \inf h'(x)E[Z] = h'(0)\mu_Z > -\infty$. Arguing as in Theorem 2 in Kaluszka (2005), we can conclude that there exists a non-decreasing function R, being the solution of problem (6.12), if $w'(0) \leq c\mu_Z h'(0)$. Moreover, $x \mapsto x - R(x)$ is non-decreasing. Since both $R(x)$ and $x - R(x)$ are non-decreasing, we have $0 = R(0) < R(x)$ and $0 = 0 - R(0) \leq x - R(x)$. Therefore, $0 \leq R(x) \leq x$ for $x \geq 0$. It is easy to see that (6.14) can be expressed as $v_c(x - R(x)) = x$, where $v_c(x)$ is given by (6.13). So $R(x) = x - v_c^{-1}(x)$.

Second, since $\inf H' = h'(0)\mu_Z$ and h' is increasing, it is obvious that condition (2) of Theorem 1 holds.

Third, put $\phi(c) = f(P) - E[h(X - v_c^{-1}(X))Z]$, $c > c_0$. Because

$$E[h(X)Z] = E[Z \int_0^X h'(t)dt] + \mu_Z h(0) \leq E[Zh'(X)X] + \mu_Z h(0) < \infty$$

according to the dominated convergence theorem, $\phi(c)$ is continuous at c. Moreover, $\phi(\infty) = f(P) - \mu_Z h(0) > 0$ and $\phi(c_0+) = f(P) - E[h(L)Z] < 0$. Therefore, there is a real c such that $E[h(R(X))Z] = f(P)$. Condition (3) of Theorem 1 holds.

Last, condition (4) of Theorem 1 obviously holds. Proof is completed.

Theorem 3. *The optimal reinsurance problem* (6.12) *under the LH transform for* $\beta < 0$ *is given by* $\hat{R}(X) = X - v_c^{-1}(X)$, *where*

$$v_c(t) = t - \frac{1}{\beta} \ln[(\frac{1}{c\mu_Z}w'(t) - 1)_+ + 1],$$

$\mu_Z = E[g'(1 - F(X))]$, *F is the distribution function of X,* $g(x) \triangleq x^\alpha$, $v_c^{-1}(t)$ *is the inverse of* $v_c(t)$, *and* $c > c_0$ *is such that* $E[h(\hat{R}(X))g'(1 - F(X))] = P$ *with* $h(x) \triangleq \frac{1-e^{-\beta x}}{\beta}$.

Proof: From (6.10),

$$\begin{aligned}
P &= \int_0^\infty g(Pr(R(X) > t))e^{-\beta t}dt \\
&= \int_0^\infty \int_0^1 1_{[0, Pr(R(X)>t)]}(x)e^{-\beta t}dg(x)dt \\
&= \int_0^1 \int_0^\infty 1_{[x, \infty]}(Pr(R(X) > t))e^{-\beta t}dtdg(x) \quad (6.15)
\end{aligned}$$

where

$$1_{[x, \infty]}(Pr(R(X) > t)) = \begin{cases} 1, & if \ Pr(R(X) > t) > x, \\ 0, & if \ Pr(R(X) > t) \le x. \end{cases}$$

And since $F_{R(X)}^{-1}(t) = R(F^{-1}(t))$ for every $R(X) \in \mathcal{R}_0$ (see Rolski et al., 1999, p.97),

(6.15) can be written as

$$\int_0^1 \int_0^{F_{R(X)}^{-1}(1-x)} e^{-\beta t} \, dt \, dg(x)$$

$$= \int_0^1 \frac{1 - e^{-\beta F_{R(X)}^{-1}(1-x)}}{\beta} \, dg(x)$$

$$= \int_0^1 \frac{1 - e^{-\beta R(F^{-1}(1-x))}}{\beta} \, dg(x)$$

$$= \int_0^\infty \frac{1 - e^{-\beta R(x)}}{\beta} \, d[1 - g(1 - F(x))]$$

$$\triangleq E[h(R(X))g'(1 - F(X))]$$

where $h(x) \triangleq \frac{1 - e^{-\beta x}}{\beta}$. Note that $h(x)$ is strictly convex, increasing and differentiable since $h'(x) = e^{-\beta x} > 0$ and $h''(x) = -\beta e^{-\beta x} > 0$. Let $Z = g'(1 - F(X))$, $f(P) = P$ and apply Theorem 2; then $\hat{R}(X) = X - v_c^{-1}(X)$ is the solution to (6.12) under the LH transform where $v_c^{-1}(t)$ is the inverse of $v_c(t)$ given by (6.13) with $h'^{-1}(t) = -(1/\beta) \ln t$ and $h'(0) = 1$. In this case,

$$v_c(t) = t - \frac{1}{\beta} \ln[(\frac{1}{c\mu_Z} w'(t) - 1)_+ + 1],$$

and $c > c_0$ is the one such that $E[h(\hat{R}(X))g'(1 - F(X))] = P$.

Chapter 7

Mortality Swap

7.1　Background

When insurers sell insurance or annuity products, they face mortality risks - actual versus expected mortality. The pricing of insurance or annuity products is based on a predetermined mortality table, interest rate and loading. If the actual mortality experience differs from the expected, the company that sells the product may suffer losses or gain profits from the unexpected mortality change. For example, when the mortality improves, i.e., more people survive than the originally expected, we have a situation where annuity companies suffer losses because they have to pay more survival benefits while insurance companies gain since they pay fewer death benefits during the term of the policy. On the other hand, when the mortality worsens, i.e., more people die than the expected, it will result in losses for life insurance company because more death benefits need to be paid while creating gains for annuity companies since not as many survival benefits as expected need to be paid out.

In order to stabilize the cash flows for companies, the idea of mortality swap is proposed. Within the framework of mortality swaps, the annuity insurer is willing to pay floating payments on the counter-party's life insurance while receiving floating payments on its own annuity policies.

Upon entering the mortality swap, suppose no payment is made at the beginning. These two parties agree that the value for each of the parties must be based on the

equal present values of their future cash flows. Each year, the annuity insurer pays a floating amount to the life insurer based on the actual number of deaths; the face value is F per death. On the other hand, the life insurer pays a floating amount to the annuity insurer based on the actual number of survivals; the payout is b per year per annuitant.

7.2 Main Results of Mortality Swap

Lemma 1. *Suppose the underlying force of mortality is $\mu_{x_*}(t) = \alpha\mu_x(t) + \beta$, and there is a random mortality shock ϵ on the parameter α, i.e., $\mu_{x_*}^{(\epsilon,0)}(t) = (\alpha + \epsilon)\mu_x(t) + \beta$, where the random shock ϵ is assumed to follow a normal distribution with mean c_1 and variance σ_1^2. To factor the random shock into pricing, the annuity can be priced as*

$$\ddot{a}_{x_*:\,n|}^{(\epsilon,0)} = E_\epsilon\left[\ddot{a}_{(K(x_*)+1)\wedge n|}^{(\epsilon,0)}\right] = \sum_{k=0}^{n-1} v^k\,_kp_{x_*}e^{-c_1\,\Lambda(k)+\frac{\sigma_1^2}{2}\,\Lambda^2(k)}$$

where

$$\Lambda(k) = \int_0^k \mu_x(t)dt = -\ln(_kp_x).$$

Proof: In the case of the mortality shock ϵ on parameter α, assume that $h(w)$ is the

probability density function of ϵ; then we have

$$
\begin{aligned}
E_\epsilon\left[\ddot{a}^{(\epsilon,0)}_{(K(x_*)+1)\wedge n|}\right] &= E_\epsilon\left[\sum_{k=0}^{n-1} v^k \; {}_k p^{(\epsilon,0)}_{x_*}\right] \\
&= \int_{-\infty}^{\infty} \left(\sum_{k=0}^{n-1} v^k \; ({}_k p_x)^{\alpha+w} e^{-\beta k}\right) h(w)dw \\
&= \sum_{k=0}^{n-1} v^k \; ({}_k p_x)^\alpha e^{-\beta k} \int_{-\infty}^{\infty} ({}_k p_x)^w \; h(w)dw \\
&= \sum_{k=0}^{n-1} v^k \; ({}_k p_x)^\alpha e^{-\beta k} \int_{-\infty}^{\infty} e^{-w\,\Lambda(k)} \; h(w)dw \\
&= \sum_{k=0}^{n-1} v^k \; ({}_k p_x)^\alpha e^{-\beta k} \; M_\epsilon[-\Lambda(k)] \\
&= \sum_{k=0}^{n-1} v^k \; {}_k p_{x_*} \; e^{-c_1\,\Lambda(k)+\frac{\sigma_1^2}{2}\Lambda^2(k)}.
\end{aligned}
$$

Lemma 2. *Suppose the underlying force of mortality is $\mu_{x_*}(t) = \alpha\mu_x(t)+\beta$, and there is a random mortality shock ξ on the parameter β, i.e., $\mu^{(0,\xi)}_{x_*}(t) = \alpha\,\mu_x(t) + (\beta+\xi)$, where the random shock ξ is assumed to follow a normal distribution with mean c_2 and variance σ_2^2. To factor the random shock into pricing, the annuity can be priced as*

$$
E_\xi\left[\ddot{a}^{(0,\xi)}_{(K(x_*)+1)\wedge n|}\right] = \sum_{k=0}^{n-1} v^k \; {}_k p_{x_*} e^{-c_2\,k+\frac{\sigma_2^2}{2}\,k^2}.
$$

Proof: In the case of the mortality shock ξ on parameter β, assume that $g(u)$ is the

probability density function of ξ; then we have

$$
\begin{aligned}
E_\xi\left[\ddot{a}^{(0,\xi)}_{(K(x_*)+1)\wedge n|}\right] &= E_\xi\left[\sum_{k=0}^{n-1} v^k \, {}_kp^{(0,\xi)}_{x_*}\right] \\
&= \int_{-\infty}^{\infty}\left(\sum_{k=0}^{n-1} v^k \, ({}_kp_x)^\alpha e^{-(\beta+u)\,k}\right) g(u)du \\
&= \sum_{k=0}^{n-1} v^k \, ({}_kp_x)^\alpha e^{-\beta k}\int_{-\infty}^{\infty} e^{-u\,k}\, g(u)du \\
&= \sum_{k=0}^{n-1} v^k \, {}_kp_{x_*}\, M_\xi(-k) \\
&= \sum_{k=0}^{n-1} v^k \, {}_kp_{x_*}\, e^{-c_2\,k+\frac{\sigma_2^2}{2}\,k^2}.
\end{aligned}
$$

Theorem 4. *Suppose the underlying force of mortality is $\mu_{x_*}(t) = \alpha\mu_x(t) + \beta$, and there are random mortality shocks ϵ and ξ on the parameters α and β, respectively, i.e., $\mu^{(\epsilon,\xi)}_{x_*}(t) = (\alpha + \epsilon)\mu_x(t) + (\beta + \xi)$, where the random shocks $\epsilon \sim N(c_1,\sigma_1^2)$ and $\xi \sim N(c_2,\sigma_2^2)$. To factor the random shocks into pricing, the annuity can be priced as*

$$
E\left[\ddot{a}^{(\epsilon,\xi)}_{((K(x_*)+1)\wedge n)|}\right] = \sum_{k=0}^{n-1} v^k \, {}_kp_{x_*}\, e^{-c_1\,\Lambda(k)+\frac{\sigma_1^2\,\Lambda^2(k)}{2}}\, e^{-c_2\,k+\frac{\sigma_2^2}{2}k^2}.
$$

Proof: According to the iterated expectation for continuous random variables, we have

$$
E\left[\ddot{a}^{(\epsilon,\xi)}_{((K(x_*)+1)\wedge n)|}\right] = E_\epsilon\left[E_\xi\left(\ddot{a}^{(\epsilon,\xi)}_{((K(x)+1)\wedge n)|}\,\Big|\,\epsilon\right)\right]. \tag{7.1}
$$

With the help of Lemmas 1 and 2, it is easy to obtain the result.

Based on the theorems and statements above, it is straightforward for us to obtain the following proposition.

Proposition 8. *Suppose that an insurance-written company and an annuity-written company swap mortality for the purpose of hedging potential mortality shocks. The life issuer and annuity issuer enter a mortality swap, where the annuity issuer pays a floating benefit to the life issuer based on the actual number of deaths for the face value*

of F per death at the end of the year, and gets a floating benefit from the life issuer based on the actual number of survivors for the amount of b per year per annuitant at the end of the year. The life issuer swaps N_1 insureds with the annuity issuer for N_2 annuitants. For the annuity issuer, the underlying force of mortality is $\mu_x(t)$ while the underlying force of mortality for the life issuer is assumed a linear hazard transform of $\mu_x(t)$, that is, $\mu_{x_*}(t) = \alpha \mu_x(t) + \beta$. The corresponding forces of mortality for the annuity and life issuers with random mortality shocks are

$$\mu_x^{(\epsilon_1, \xi_1)}(t) = (1 + \epsilon_1)\,\mu_x(t) + \xi_1$$

and

$$\mu_{x_*}^{(\epsilon_2, \xi_2)}(t) = (\alpha + \epsilon_2)\,\mu_x(t) + (\beta + \xi_2),$$

respectively, where the mortality shocks follow the normal distributions given by

$$\epsilon_1 \sim N(d_1, \sigma_{1a}^2),$$
$$\xi_1 \sim N(d_2, \sigma_{2a}^2),$$
$$\epsilon_2 \sim N(c_1, \sigma_{1l}^2),$$

and

$$\xi_2 \sim N(c_2, \sigma_{2l}^2).$$

Then the value of the swap can be calculated by

$$F \times N_1 \times \left(\sum_{k=0}^{n-1} v^{k+1}\,{}_k p_{x_*}\, q_{x_*+k}\, C_{1,k}\, C_{2,k} \right) = b \times N_2 \times \left(\sum_{k=1}^{n} v^k\,{}_k p_x\, D_{1,k}\, D_{2,k} \right)$$

where

$$C_{1,k} = e^{-c_1 \Lambda_l(k) + \frac{\sigma_{1l}^2 \Lambda_l^2(k)}{2}},$$
$$C_{2,k} = e^{-c_2 k + \frac{\sigma_{2l}^2 k^2}{2}},$$
$$D_{1,k} = e^{-d_1 \Lambda_a(k) + \frac{\sigma_{1a}^2 \Lambda_a^2(k)}{2}},$$
$$D_{2,k} = e^{-d_2 k + \frac{\sigma_{2a}^2 k^2}{2}},$$
$$\Lambda_a(k) = \int_0^k \mu_x(t)dt = -\ln({}_k p_x),$$

and

$$\Lambda_l(k) \;=\; \int_0^k \mu_{x_*}(t)dt = \alpha\Lambda_a(k) + \beta k = -\alpha\,\ln(_kp_x) + \beta k.$$

In the notations above, the subscripts a and l stand for companies that provide annuities and life insurance, respectively.

How mortality swap works is of interest to many. If mortality improves and fewer people than expected die, the life issuer pays less benefits than expected and makes money out of its insurance business line. But upon entering this swap, it is obliged to pay the annuity benefits sold by the annuity issuer. Since the annuity policies are issued to similar population, similar shocks strike that population and therefore more people survive. So the life issuer needs to pay survival benefits. The risk is transferred from the annuity issuer to the life issuer. One the other hand, when mortality worsens, the annuity issuer makes money out of its annuity business since fewer survival benefits are paid out. It is obliged, however, to pay the death benefits from the insurance policies issued by the life insurance company. In this case, the risk is transferred from the life issuer to the annuity issuers. To sum up the discussion above, the mortality risk is shared between the life and the annuity issuers. As a result, negative impacts can be offset, and their future annual cash flows are more stable.

Chapter 8

Conclusion

The PH transform and its properties were proposed and investigated by Wang (1995). This marked a milestone in actuarial research. The PH transform distorts the survival function and adjusts weights in the tail of the survival function. Its applications to a variety of areas, such as insurance premium calculation, rate making, and risk measure, have become widely known in actuarial practice.

In this project, we study the linear hazard transform. As a complimentary tool, we adopt the assumption of α-approximation. With the help of α-approximation assumption, we obtain explicit formulas for pricing continuous insurance products under the LH transform. Such formulas are expressed in terms of the net single premiums of discrete insurance products.

The LH and PH transforms are also useful for fitting two mortality curves by regression. Comparison is made between the LH transform fitting and the PH transform fitting. It is concluded that the LH transform fitting gives better results with a smaller sum of square errors than the PH transform fitting does because the LH transform fitting has two parameters which provide more flexibility and accuracy for regression. Due to the good fit under the LH transform, the relationship between two sets of mortalities can be determined by these two parameters of the LH transform.

Mortality prediction is explored under the LH transform. The diagonal mortality method is proposed and applied for mortality prediction. Plots of different prediction methods are drawn for the purpose of comparison. It is found that the diagonal

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method well represents the mortality improvement and is useful for mortality prediction.

Finally, the LH transform is applied to asset management, such as mortality swap, risk ordering and optimal reinsurance. When insurance or annuity companies sell products, they bear the risk of unfavorable mortality change. In order to stabilize the cash flows, mortality swap between the insurance issuer and the annuity issuer is considered. The value of a mortality swap incorporates the LH transform as well. Under the LH transform, an optimal reinsurance strategy is studied and an explicit formula is suggested. The LH transform order is also introduced and its connections with other risk orders are explored. The future work includes the potential extension of the LH transform $\mu_{x_*}(t) = \alpha\mu_x(t) + \beta$ to $\mu_{x_*}(t) = \alpha\mu_x(t) + \beta(t)$, where $\beta(t)$ is a function of t.

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