EXTREMAL ORIENTED GRAPHS AND ERDŐS-HAJNAL CONJECTURE

by

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Abstract

For a (finite or infinite) family $\mathbb{L}$ of oriented graphs, a new parameter called *the compressibility number* of $\mathbb{L}$ and denoted by $z(\mathbb{L})$ is defined. The main motivation is the application of this parameter in a special case of Turán-type extremal problems for digraphs, in which it plays the role of chromatic number in the classic extremal problems. We estimate this parameter for some special group of oriented graphs. Determining this parameter, in the most explicit possible form, for oriented graphs with bounded oriented coloring number (planar graph in particular) leads us to the famous Erdős-Hajnal conjecture.

Contents

Chapter 1

Introduction

1.1 Background and old results

By digraph we mean a directed graph which contains no loop and no multi-arcs: two or more arcs with the same direction between a pair of vertices. But it can have two opposite arcs between a pair of vertices, which are called symmetric arcs. For any digraph H we denote by $V(H)$ and $E(H)$, the set of vertices and arcs of H, respectively. The same definitions apply when replacing the notion digraph and arc with simple graph and edge, respectively. From now on by graph we mean simple graph. For $x, y \in V(H)$, we write $\overrightarrow{xy}$ to indicate that there is an arc in $E(H)$ oriented from x to y. The same notation also represents the arc itself. A digraph H is called a sub-digraph of another digraph H', and denoted by $H \subseteq H'$, if $V(H) \subseteq V(H')$ and $E(H) \subseteq E(H')$. For $S \subseteq V(H')$, we say the digraph H is called the induced sub-digraph of S in H', and denote by $H'|_S$, if $V(H) = S$ and $E(H)$ consists of all members of $E(H')$ with their endpoints in S.

A homomorphic mapping f from a digraph H to another digraph H' is a mapping $f : V(H) \to V(H')$ such that $\overrightarrow{f(x)f(y)}$ whenever $\overrightarrow{xy}$. In this case we write $H \to H'$ and say H is *homomorphic* to H'. Should the mapping f be a bijection with its inverse function being also a graph homomorphic mapping, then f is called a graph *isomorphic* mapping and we say H is *isomorphic* to H'. We say that H' contains H if H is isomorphic to a sub-digraph of H', otherwise we say H' avoids H or, equivalently, H' is H-free.

In their joint paper, [5], Brown and Harary initiated the study of extremal digraphs by considering the digraph analogues of the Paul Turán's classic extremal theorem [19, 20]: What is the maximum number of arcs in a digraph avoiding a certain forbidden digraph? And what are such extremal digraphs? Their study gives precise answers for the forbidden digraphs with four or less vertices.

Later on Brown, Erdős and Simonovits, in a series of papers [2, 3, 4], studied this problem and its extension to any family (finite or infinite) of forbidden digraphs. Given integer $n \geq 1$ and a square matrix A with zero or one along the main diagonal and zero or two anywhere else, in [2], they introduce a method for constructing a digraph $A(n)$ where $|V(A(n))| = n$. Then they show that for any family (finite or infinite) of forbidden digraphs, there is a matrix A such that for every $n \geq 1$, $A(n)$ contains non of the forbidden digraphs and $|E(A(n))|$ is very close to its optimal. Thus, this study ensures the existence of such a family of approximately extremal digraphs and gives some insight into its structure. But it does not answer the digraph version of the Paul Turán's extremal theorem, i.e., a clear estimation for the number of arcs of the extremal digraphs with n vertices. This is due to the unknown order of the matrix A which is used in the construction.

Attempts were made to find some estimations for this value. In [4] an algorithm is introduced to calculate it. In [16] some upper bounds were proved in terms of some parameters of the forbidden digraphs. These studies revealed the inherent complexity for finding any meaningful estimation of the number of extremal digraphs's arcs. This is in contrast to the undirected version in which a result of Erdős, Stone and Simonovits [9] gives us an exact asymptotic value for the number of edges of the extremal graphs in terms of the chromatic numbers of the forbidden graphs.

The situation becomes more tractable if we limit ourselves to the special case of *oriented graphs*: a family of digraphs which contain no pair of symmetric arcs. Actually this is suggested by Brown and Harary themselves in the last section of their paper [5] as a possible direction for future works. But this case, which we call the *extremal theory of oriented graphs*, was not considered in the later works of Brown et al. [2, 3, 4] and other subsequent works [16].

The work in the present thesis is along this unstudied direction. We say a oriented graph *H* is an *orientation* of a graph *G* if it is obtained from *G* by assigning an orientation to each edge of *G*. If *H* is acyclic (contains no directed cycle), then we say *H* is an *acyclic* orientation of *G*. An orientation of a complete graph is called tournament. An acyclic tournament is called transitive tournament. From extremal theory's point of view, the chromatic number of a graph *G* can be re-defined as the smallest number *k* such that *G* is homomorphic to the complete graph on *k* vertices. This definition can be naturally reformulated for oriented graphs by replacing the notion graph with the notion oriented graph, however the term 'complete graph on *k* vertices' cannot be simply replaced with 'complete oriented graph (=tournament) on *k* vertices' as we have many such tournaments. Thus the oriented analogues of this definition, after generalizing to a family of oriented graphs, will read:

Definition 1.1.1. *For any finite or infinite family $\mathbb{L}$ of oriented graphs, $z(\mathbb{L})$, the compressibility number of $\mathbb{L}$, is the smallest number k such that for any tournament on k vertices, at least one member of $\mathbb{L}$ is homomorphic to it. When $\mathbb{L}$ has only one member H, we may use the notations $z(\mathbb{L})$ and $z(H)$ interchangeably.*

Later in Theorem 2.1.1 we will see that $z(\mathbb{L})$ is well-defined, should $\mathbb{L}$ contain at least one acyclic oriented graph. We will show (Theorem 2.2.8) that this chromatic-type parameter replaces the role of chromatic number in the extremal oriented graph problem.

Definition 1.1.2. *For any oriented graph H, we define $p(H)$ to be the length (number of edges) of the longest directed path in H.*

One may find the definition of $z(H)$ similar to that of *one-color* oriented Ramsey number $r(H)$ (for example refer to [12]). $r(H)$ is defined to be the smallest integer *k* such that *H* is *isomorphic* to any tournament on *k* vertices. It must be noted that in calculating the Ramsey number $r(H)$, we look for an *isomorphic* copy of *H* in any arbitrary tournament while in the case of $z(H)$ we look for a *homomorphic* copy of *H*. This makes a huge difference as we will see. Thus, the known results for oriented Ramsey theory (see [12] for a list of references) cannot contribute significantly to the compressibility number as the concept of isomorphism is different from homomorphism except in some rare cases. One such case is when *H* is a transitive tournament. In this case a well known result of Erdős-Moser [8] (restated in Theorem 3.2.1) gives some estimation for the values of $z(H)$. Presently, this is the

only remarkable contribution of oriented Ramsey theory in determining a compressibility number.

Motivated by these relations we embark on determining $z(\mathbb{L})$ for any family of oriented graphs. Here we see that, in contrast to the oriented Ramsey problem which is hard even for orientations of paths and cycles (see [12]), compressibility number is more tractable. By employing various graph theoretic concepts (like chromatic number, planarity, etc) we well estimate $z(H)$ for some large and important groups of oriented graphs H, including orientations of trees (Theorem 3.1.1), cycles (Theorem 3.1.3), unions of internally disjoint paths with the same endpoints (Theorem 3.1.6). In these cases, the value of $z(H)$ is expressed in terms of $p(H)$. Additionally some estimations are given for acyclic orientations of complete graphs (Theorem 3.2.1) and complete bipartite graphs (Theorems 3.2.4,3.2.5).

The oriented coloring number of an arbitrary oriented graph H, denoted by $\chi_o(H)$, is the smallest number $k \geq 0$ such that H is homomorphic to an oriented graph on k vertices. Inspired by these results we target a more general family of oriented graphs which encompasses the previous ones: When H is an orientation of a planar graph. Based on the previous results we conjecture that $z(H)$ is upper bounded by $p(H)^d$ for some constant d (Conjecture 3.1.9). We show a direct connection (Theorem 4.0.17) between this conjecture and the so-called *oriented* Erdős-Hajnal conjecture (re-stated in Conjecture 4.0.10). Introduced in [1], this conjecture is about the existence of a transitive sub-tournament with polynomial order in any tournament which avoids a forbidden oriented graphs. We show (Corollary 4.0.18) that the oriented Erdős-Hajnal conjecture implies our planar graph conjecture and even a more general conjecture claiming the same upper bound when H has bounded oriented coloring number (of which planar graph orientations are a special case, see [14]).

This connection enables us to come up with good estimations of $z(H)$ in terms of $p(H)$, should we validate the oriented Erdős-Hajnal conjecture, at least for some certain forbidden oriented graphs (Theorem 5.0.22). This makes further motivation to study this conjecture . A result in [1] states that this conjecture is equivalent to the classic Erdős-Hajnal conjecture (re-stated in Conjecture 4.0.9) for graphs [7]. Moving along this direction we try to come up with the *oriented* analogous of some Erdős-Hajnal related problems stated in [7, 1] (Theorems 4.0.12,4.0.20,5.2.3). The result of this analysis is the validation of the oriented Erdős-Hajnal conjecture for some special classes of forbidden oriented graphs (Corollaries

5.1.5,5.1.6) along with a pseudo-polynomial upper bound in terms of $p(H)$ for $z(H)$, when H has bounded oriented number (Corollary 4.0.21).

1.2   New results and the thesis structure

A simple argument (Theorem 2.1.1) in the first part of Chapter 2 ensures the existence of $z(\mathbb{L})$ with the condition that $\mathbb{L}$ contains at least one acyclic oriented graph. The famous Erdős-Stone-Simonovits theorem [9] (re-stated in Theorem 2.2.2) states that the maximum number of edges of a graph on n vertices avoiding a family of forbidden graphs is asymptotically $\frac{r-2}{r-1}\binom{n}{2}$ with r being the minimum chromatic number among the forbidden graphs. In the second part of Chapter 2 we show the oriented version of this theorem (Theorem 2.2.8) with the chromatic number r substituted with the compressibility number parameter: the maximum number of arcs of any oriented graph on n vertices which avoids all member of $\mathbb{L}$ is asymptotically (Definition 2.2.3): $\left(\frac{z(\mathbb{L})-2}{z(\mathbb{L})-1}\right)\binom{n}{2}$. Furthermore the optimal matrix for $\mathbb{L}$ (Definition 2.2.4) has order $z(\mathbb{L})-1$ and the oriented graphs generated by this matrix are certain orientations of the Turán graphs with $z(\mathbb{L})-1$ parts (Theorem 2.2.8).

In Chapter 3 we well-estimate the value of $z(H)$, in terms of $p(H)$ in most cases, when H is any orientation of the following classes of graphs: trees (Theorem 3.1.1), cycles (Theorem 3.1.3), unions of internally disjoint paths with shared endpoints (Theorem 3.1.6), complete (Theorem 3.2.1) and complete bipartite graphs (Theorem 3.2.4). Based on these results we conjecture that in general $z(H)$ is upper bounded by $p(H)^d$ for some constant d when H is any acyclic orientation of a planar graph (Conjecture 3.1.9).

Chapter 4 is dedicated to the latter conjecture. We show a close connection between the compressibility number and the order of the biggest transitive sub-tournament in a tournament avoiding some forbidden tournament (Theorem 4.0.17). This way we link (Corollary 4.0.18) our conjecture for the planar graph orientations and a more general class of graphs (i.e. the class of oriented graphs with bounded oriented coloring numbers) with the oriented version of Erdős-Hajnal conjecture formulated in [1] (re-stated in Conjecture 4.0.10).

In Chapter 5 we study some Erdős-Hajnal related problems motivated by the results of the previous chapter. With a very straightforward argument we convert some results from [7] (re-stated in Theorem 4.0.19) and [1] (re-stated in Theorem 5.2.2) to their oriented versions

(Theorems 4.0.20,5.2.3). The conversion of the first theorem (Theorem 4.0.20) ensures a transitive sub-tournament with pseudo-polynomial order for restricted tournaments. This leads to the result that $z(H)$ is upper bounded by $p(H)^{d\log p(H)}$ (for some constant d) for any oriented graph H with bounded oriented coloring number (Corollary 4.0.21). The conversion of the second theorem (Theorem 5.2.3) gives us a very powerful blow-up construction method to validate the oriented Erdős-Hajnal conjecture for more oriented graphs composed of the known ones. We prove this conjecture for any oriented graph of order at most four as well as any directed cycle as the forbidden oriented graph. Armed with these results we can validate the conjecture for even more oriented graphs, i.e., all tournaments with the length of its biggest directed cycle at most four (Corollary 5.2.4).

1.3   Notations

For an oriented graph H, we denote by $|V(H)|$ the order of H. Let $v \in V(H)$, we denote by $N^+(v)$ and $N^-(v)$ the set $\{u \in V(H)|\overrightarrow{vu}\}$ and $\{u \in V(H)|\overrightarrow{uv}\}$, respectively. $|N^+(v)|$ and $|N^-(v)|$ are called out-degree and in-degree of v in H, respectively. The sum of in-degree and out-degree of v is called the degree of v. The single arc oriented graph is an oriented graph with two vertices and an arc between them. A directed walk $W = (v_0, v_1, \cdots, v_l)$ of order l in H is defined as a sequence of vertices of H (not necessarily distinct) such that for every $0 \le i < l$, $\overrightarrow{v_iv_{i+1}}$. We call v_0 and v_l the starting and ending vertex of W, respectively. At some occasions during this work, we may treat W as a set of its vertices (for instance when we define a mapping on W or talk about its membership), in this occasions consider it as the set $\{v_0, v_1, \cdots, v_l\}$. We say W is a directed path if all of its vertices are distinct. A directed path with an additional arc going from its ending vertex to its starting vertex is called a directed cycle. Any orientation of a path is called oriented path.

Chapter 2

The nature of compressibility number

2.1 The existence

In this section we show that when $z(\mathbb{L})$ is well-defined:

Theorem 2.1.1. *a) For any finite or infinite family $\mathbb{L}$ of oriented graphs, $z(\mathbb{L})$ has a finite value if and only if $\mathbb{L}$ has at least one acyclic oriented graph. b) Given an acyclic oriented graph H, $z(H) \geq p(H) + 1$*

Proof. a) If $\mathbb{L}$ contains no acyclic oriented graph, then every transitive tournament avoids all its members, so $z(\mathbb{L})$ if not defined. Now suppose $H \in \mathbb{L}$ is an acyclic oriented graph. Then according to Erdős-Moser theorem [8], H is homomorphic to any tournament of order $2^{|V(H)|-1}$. This means $z(\mathbb{L})$ is upper bounded by $2^{|V(H)|-1}$. b) Obviously H cannot be homomorphic to the transitive tournament of order $p(H)$, as the latter has no oriented walk of order $p(H) + 1$, so the inequality follows. $\qquad \square$

2.2 Oriented Erdős-Stone-Simonovits

In this section we show the role of compressibility number in extremal oriented graph theory.

Definition 2.2.1. *For any family $\mathbb{L}$ of graphs and $n \geq 1$, we define $ex(n, \mathbb{L})$ to be the biggest integer k such that there is a graph G with n vertices and k edges which has no member of*

$\mathbb{L}$ *as sub-graph. All such graphs G which yield this maximum are called extremal graphs. We define* $\overline{ex}(n, \mathbb{L})$ *similarly with the notions graph and edge replaced with digraph and arc, respectively. We define* $\overline{exo}(n, \mathbb{L})$ *similar to* $\overline{ex}(n, \mathbb{L})$ *with the extra condition of digraph G being an oriented graph.*

Erdős-Stone-Simonovits theorem ([9]) gives the following asymptotic estimation for $ex(n, \mathbb{L})$:

Theorem 2.2.2. *For any family* $\mathbb{L}$ *of graphs, let* $r = \min_{F \in \mathbb{L}} \{\chi(F)\}$. *Suppose* $r \geq 2$. *Then*

$$ex(n, \mathbb{L}) = \left(\frac{r-2}{r-1} + o(1) \right) \binom{n}{2} \tag{2.1}$$

where $o(1) \to 0$ *as* $n \to \infty$

Finding a similar estimation for $\overline{ex}(n, \mathbb{L})$ is more challenging (at least up until now no good estimation is found):

Definition 2.2.3. *([2]) The sequence* $S = (H_1, H_2, \cdots)$ *of digraphs with* $|V(H_n)| = n$ *for every* $n \geq 1$, *is called a sequence of asymptotic extremal digraphs for* $\mathbb{L}$, *if no member of S contains any member of* $\mathbb{L}$ *and:*

$$\lim_{n \to \infty} \frac{|E(H_n)|}{\overline{ex}(n, \mathbb{L})} = 1.$$

Definition 2.2.4. *([2]) Given* $n \geq 1$ *and a matrix* $A = (a_{i,j})_{i,j \leq r}$ *with* 0 *or* 1 *along the main diagonal and* 0 *or* 2 *outside of the diagonal. The* optimal matrix graph $A(n)$ *is a directed graph on n vertices with maximum number of arcs such that* $V(A(n))$ *can be partitioned into r classes* $C_1, C_2, \cdots, C_r$ *satisfying two conditions: First, for all* $1 \leq i \neq j \leq r$: $\overrightarrow{xy}$ *for every* $x \in C_i$ *and* $y \in C_j$ *if and only if* $a_{i,j} = 2$, *second, for all* $1 \leq i \leq r$: *The induced sub-digraph of* C_i *is a transitive tournament if* $a_{i,i} = 1$ *and an empty graph (no arcs) otherwise. If there are several choices for* $A(n)$, *pick one of them arbitrary.*

Now the main result of [2] concerning the value $\overline{ex}(n, \mathbb{L})$ is as follows:

Theorem 2.2.5. *([2]) For any finite or infinite family* $\mathbb{L}$ *of digraphs there exists a matrix* $A = (a_{i,j})_{i,j \leq r}$ *such that* $\{A(n)\}_{n \geq 1}$ *is a sequence of asymptotic extremal digraphs for* $\mathbb{L}$.

This result and the techniques used in its proof are not used anywhere in this thesis. So I do not include its long full proof here and instead, I give a general outline of the proof. Please refer to the original paper [2] for the complete proof.

Definition 2.2.6. *For an $n \times n$ matrix A we define $g(A)$, the density of the matrix A, as* $\max\{uAu^{-1}\}$ *where the maximum is taken over all vectors* $u = (u_1, u_2, \cdots, u_n)$ *with* $u_i \geq 0$ *and* $\sum_{i=1}^{n} u_i = 1$.

Proof. (Theorem 2.2.5, A short summary from [2]) Given a matrix A and a family of digraphs $\{G^n\}$, we shall say that A is weakly contained by the family $\{G^n\}$ if the maximum $m = m_n$ for which $A(m) \subseteq G^n$ is unbounded as $n \to \infty$. Suppose $\mathbb{L} = \{L_1, L_2, \cdots\}$. First, a sequence $\{Z^n\}_{n=1,2,\cdots}$ of digraphs with following extremal property is defined: for each n, Z^n avoids the members of $\mathbb{L}$ and each of its vertices has degree $\geq d$ where d is the maximum integer for which such a sequence of digraphs exists. The aim is to construct a matrix B which is weakly contained in $\{Z^n\}_{n=1,2,\cdots}$ and has maximum density. They show that this condition guarantees $\{B(n)\}_{n=1,2,\cdots}$ to be a sequence of asymptotic extremal digraphs. To find such a matrix B, they use the following recursive construction: Suppose the matrix A is weakly contained by the family $\{Z^n\}$ and $g(A) < \limsup_{n \to \infty} d_n/n$ (d_n is the minimum degree in Z^n). Then they construct (in a lemma) a matrix A' which is weakly contained in $\{Z^n\}_{n=1,2,\cdots}$ and $g(A') > g(A)$. Now they start from the matrix $D_r = (2 - \delta_{ij})_{i,j \leq r}$ where δ_{ij} is the Kronecker symbol: 1 if $i = j$ and 0 otherwise and r is the largest integer such that D_r is weakly contained in $\{Z^n\}_{n=1,2,\cdots}$. They show that such an r exists. At each step this construction is applied to the matrix obtained in the last step, until we arrive at a matrix B where $g(B) = \limsup_{n \to \infty} d_n/n$. They show that this matrix is obtainable in finite steps. Note that $B(n)$ avoids all the members of $\mathbb{L}$ since B is weakly contained in $\{Z^n\}_{n=1,2,\cdots}$ and Z^n avoids all the members of $\mathbb{L}$. Next they show that the condition $g(B) = \limsup_{n \to \infty} d_n/n$ indeed grantees its being a sequence of asymptotic extremal digraphs. □

In our attempt to estimate the value $\overline{\mathrm{ex}}(n, \mathbb{L})$, we need the oriented graph version of Definition 2.2.3:

Definition 2.2.7. *The sequence $S = (H_1, H_2, \cdots)$ of oriented graphs with $|V(H_n)| = n$ for every $n \geq 1$, is called a sequence of asymptotic extremal oriented graphs for $\mathbb{L}$ if no member of S contains any member of $\mathbb{L}$ and:*

$$\lim_{n \to \infty} \frac{|E(H_n)|}{\overline{exo}(n, \mathbb{L})} = 1.$$

We formulate the oriented graph version of Theorem 2.2.5 which contains more information about the structure of matrix A owing to the simpler structure of oriented graphs

compared to digraphs. This extra clarification helps us to come up with a clear asymptotic estimation for $\overline{\mathrm{exo}}(n, \mathbb{L})$ which is similar to that of Theorem 2.2.2:

Theorem 2.2.8. *a) For any finite or infinite family $\mathbb{L}$ of oriented graphs, with at least one acyclic member with more than one arc and $z(\mathbb{L}) \geq 2$, there exists a matrix $A = (a_{i,j})_{i,j \leq z(\mathbb{L})-1}$ such that $\{A(n)\}_{n \geq 1}$ is a sequence of asymptotic extremal oriented graphs for $\mathbb{L}$.*

b) Furthermore the matrix A has two properties: first, all the entries along the main diagonal are zero and second, $a_{i,j} + a_{j,i} = 2$ for every $1 \leq i \neq j \leq z(\mathbb{L}) - 1$. In other words, $A(n)$ ($n \geq 1$) is an orientation of the Turán graph $T_{n,z(\mathbb{L})-1}$. Thus:

$$\overline{exo}(n, \mathbb{L}) = \left(\frac{z(\mathbb{L}) - 2}{z(\mathbb{L}) - 1} + o(1) \right) \binom{n}{2} \tag{2.2}$$

where $o(1) \to 0$ as $n \to \infty$

The comparison of (2.2) in Theorem 2.2.8 to (2.1) in Theorem 2.2.2 implies that compressibility number is the exact substitute for chromatic number when it comes to extremal problems for graphs and oriented graphs, respectively. The part (a) of Theorem 2.2.8 is obtained by applying Theorem 2.2.5 to the family $\mathbb{L} \cup \{A\}$ of digraphs where A is the digraph consisting of two vertices and a pair of symmetric arcs between them. So our proof mainly targets part (b). We need a Ramsey-type argument before proving this theorem:

Lemma 2.2.9. *There exists an unbounded non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ such that, for every $n \geq 1$ and an arbitrary orientation H of a complete bipartite graph, with parts A and B having n vertices each, there exist subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $|A_1| = |B_1| = f(n)$ and all the arcs between A_1 and B_1 have the same orientation.*

Proof. Suppose n is large enough such that $m = \lfloor \log n - \log \log n \rfloor < n$ (define $f(n) = 1$ for small ns). Choose m arbitrary vertices $v_1, v_2, \cdots, v_m$ in part A. Make the sets $C_m \subseteq C_{m-1} \subseteq \cdots \subseteq C_1 \subseteq C_0$ recursively as follows: Set $C_0 = B$, for any $m \geq i \geq 1$, C_i is either $N^+(v_i) \cap C_{i-1}$ or $N^-(v_i) \cap C_{i-1}$ depending on which one has bigger size. Then obviously $|C_m| \geq |B|/2^m \geq \log n$. For each v_i, all the arcs between this vertex and C_m have the same orientation. Then according to the Pigeon hole principle we can find a subset $D \subseteq \{v_1, v_2, \cdots, v_m\}$ with $|D| \geq m/2$ such that, all the arcs between D and C_m have the same orientation. Since these sets both have the size larger than $m/2$ we can safely set $f(n) = \lfloor m/2 \rfloor$. $\qquad \square$

Proof. (Theorem 2.2.8) Let $T_{z(\mathbb{L})-1}$ be a tournament on $z(\mathbb{L}) - 1$ vertices avoiding all the members of $\mathbb{L}$. Let $V(T_{z(\mathbb{L})-1}) = \{v_1, v_2, \cdots, v_{z(\mathbb{L})-1}\}$. Let $A = (a_{i,j})_{i,j \leq z(\mathbb{L})-1}$ be a matrix with zero along the main diagonal and $a_{i,j} = 2$ if $\overrightarrow{v_i v_j}$ and zero otherwise. We claim that this is the desirable matrix. Given an arbitrary $n \geq 1$. Let $C_1, C_2, \cdots, C_{z(\mathbb{L})-1}$ be the vertex classes of $V(A(n))$ as defined in Definition 2.2.4. Suppose for some $H \in \mathbb{L}$, $H \to A(n)$. Let $f : V(H) \to V(A(n))$ be the corresponding homomorphic mapping. Define the homomorphic mapping g from H to $T_{z(\mathbb{L})-1}$ as follows: for every $h \in V(H)$ let i be the index such that, $f(h) \in C_i$, then define $g(h) = v_i$. This mapping contradicts the definition of $T_{z(\mathbb{L})-1}$. Thus $A(n)$ avoids all members of $\mathbb{L}$. Let B be a $(z(\mathbb{L}) - 1) \times (z(\mathbb{L}) - 1)$ matrix such that $\{B(n)\}_{n \geq 1}$ is a sequence of asymptotic extremal oriented graphs for $\mathbb{L}$. Suppose $\{B(n)\}_{n \geq 1}$ is not a sequence of asymptotic extremal oriented graphs for $\mathbb{L}$. Then we will have:

$$\limsup_{n \to \infty} \frac{|E(B(n))|}{|E(A(n))|} = b > 1 \tag{2.3}$$

This implies that there is an infinite set $N_1 \subseteq \mathbb{N}$ such that:

$$\lim_{n \in N_1 \to \infty} \frac{|E(B(n))|}{|E(A(n))|} = b > 1 \tag{2.4}$$

Note that according to the Turán's theorem [19, 20] $A(n)$, in order to have the maximum number of arcs, should be an orientation of the Turán graph $T_{n, z(\mathbb{L})-1}$. Pick a large enough $n \in N_1$ (later in the proof we will see how large). According to Erdős-Stone theorem ([10]), (2.4) implies that there exists an unbounded monotone function $t : \mathbb{N} \to \mathbb{N}$ such that, $B(n)$ contains a certain orientation of the complete $z(\mathbb{L})$-partite graph with $t(n)$ vertices in each part. Call this oriented graph W and its parts $W_1, W_2, \cdots, W_{z(\mathbb{L})}$. Run the following algorithm: Consider an arbitrary order for all the pairs (i, j) with $1 \leq i < j \leq z(\mathbb{L})$. Then according to this order for each pair (i, j), in its turn, replace the sets W_i and W_j with the sets A_1 and B_1 respectively where the last two sets are obtained by applying Lemma 2.2.9 to $A = W_i$ and $B = W_j$. Let f be the function defined in this lemma. After this process, the original oriented graph W will have changed to a smaller oriented graph with all its arcs between every two parts W_i and W_j having the same orientation. Let $T_{z(\mathbb{L})}$ be a tournament on $z(\mathbb{L})$ vertices $w_1, w_2, \cdots, w_{z(\mathbb{L})}$ such that $\overrightarrow{w_i w_j}$ $(i \neq j)$ iff the common orientation of the arcs between W_i and W_j in W, is from the latter to the former set. Then there exists $H \in \mathbb{L}$ which possesses a homomorphic mapping $f : V(H) \to V(T_{z(\mathbb{L})})$. We can easily define a one-to-one homomorphic mapping $g : V(H) \to V(W)$ as follows:

for any $h \in V(H)$ suppose $f(h) = w_i$. Then assign an arbitrary vertex in W_i, which has not been assigned yet, to $g(h)$. Such assignment is possible if n is large enough such that $|W_i| = f^{z(\mathbb{L})-1}(t(n)) \geq |V(H)|$ (recall f is the function defined in Lemma 2.2.9). Since both f and t are unbounded monotone functions, we can pick up a large enough n to fulfill the latter inequality. The consequence is, the mapping g introduces a sub-oriented graph of W isomorphic to H, in other words $B(n)$ will contain H, contradiction. Thus the assumption stated in (2.3) is wrong and thus we conclude that $\{A(n)\}_{n\geq 1}$ is a sequence of asymptotic extremal oriented graphs.   □

Open Question 2.2.10. *Adopting the notations defined in this section, in the Erdős-Simonovits theorem [9], the notion $z(\mathbb{L})$ is replaced with $\min_{G\in\mathbb{L}}\{\chi(G)\}$. We conjecture that it is not true in the oriented version, i.e. $z(\mathbb{L}) \neq \min_{H\in\mathbb{L}}\{z(H)\}$ for some family $\mathbb{L}$ of acyclic oriented graphs.*

Chapter 3

Compressibility number for some oriented graphs

3.1 Sparse graphs

The second part of theorem 2.1.1 gives $p(H) + 1$ as a lower bound for $z(H)$. The results below show that for some sparse graphs, $z(H)$ is very close, and sometimes equal to this minimum:

Theorem 3.1.1. *Suppose H is an orientation of a tree, then $z(H) = p(H) + 1$.*

To prove this theorem and some others in this thesis, we need the following useful lemma:

Lemma 3.1.2. *Assume H is an oriented acyclic graph, then it is homomorphic to transitive tournament with order $p(H) + 1$.*

Proof. For every $v \in V(H)$ define $l(v)$ to be the length of the longest directed path ending at v. Obviously $0 \leq l(v) \leq p(H)$. Additionally if $\overrightarrow{uv}$ then $l(v) > l(u)$, since due to H being acyclic, the longest directed path P ending at u of length $l(u)$ can be further extended to a directed path of length $l(u) + 1$ by adding the arc $\overrightarrow{uv}$ to P. Now given the transitive tournament T of order $p(H) + 1$ with the vertex sequence $< t_0, t_1, \cdots, t_{p(H)} >$ where $\overrightarrow{t_i t_j}$ when $i < j$, define the following homomorphic mapping f from H to T: For all $v \in H$, $f(v) = t_{l(v)}$. $\qquad \square$

Proof. (Theorem 3.1.1) Let T be a tournament of order $p(H) + 1$. If T is transitive, then $H \to T$ according to Lemma 3.1.2. Otherwise T contains a directed cycle C of order 3.

One can easily see $H \to C$ (assign one of the labels $0, 1, 2$ recursively to the vertices of H as follows: assign 0 to the root. For a vertex v if its parent w is assigned the label i, then assign $i + 1 \pmod 3$ to v if $\overrightarrow{wv}$ and $i - 1 \pmod 3$ otherwise. Let c_0, c_1, c_2 be vertices of C in clockwise order $(\overrightarrow{c_i c_{i+1}}, i = 0, 1, 2$, index calculations are mod 3). Then map all the vertices of H with label i [$i = 0, 1, 2$] to c_i). $\qquad \square$

Theorem 3.1.3. *Suppose H is an acyclic orientation of a cycle, then $z(H) \leq \max\{p(H) + 1, 6\}$.*

Definition 3.1.4. *Let $(s, t) \in V(T) \times V(T)$ be a pair of vertices in a tournament T. For $l \geq 0$ we say (s, t) accepts a directed walk of order l if and only if there exists an order l directed walk W in T with s and t as its starting and ending vertex, respectively. According to this, define the accepting set of pair (s, t) as $\{l | (s, t) \text{ accepts a directed walk of order } l\}$ and denote it by $D_T(s, t)$ (note that if $s = t$ then $0 \in D_T(s, t)$)*

Proof. (Theorem 3.1.3) $V(H)$ can be arranged in a sequence $A = (h_0, h_1, \cdots, h_{n-1})$ such that, for $0 \leq i < n$, either $\overrightarrow{h_i h_{i+1}}$ or $\overrightarrow{h_{i+1} h_i}$ (index calculations are mod n). Since H itself is acyclic, w.l.o.g. we can assume $\overrightarrow{h_0 h_1}$ and $\overrightarrow{h_0 h_{n-1}}$. To each vertex h_i ($0 \leq i < n$), assign the letter 'f' if $\overrightarrow{h_i h_{i+1}}$ and 'b' otherwise. In this case h_0 and h_{n-1} are assigned 'f' and 'b' respectively. Considering this fact, it's easy to see that there exists a number $t > 0$ and $2t$ non-empty sequences $F_1, B_1, F_2, B_2, \cdots, F_t, B_t$ of vertices of H such that $A = F_1 \oplus B_1 \oplus \cdots F_t \oplus B_t$ (operation $\oplus$ stands for sequence concatenation) and all vertices in F_i and B_i ($1 \leq i \leq t$) are assigned 'f' and 'b', respectively. Let T_m be a tournament with $m = \max\{p(H) + 1, 6\}$ vertices. Suppose there exist $(s, t) \in V(T_m) \times V(T_m)$ such that $\{1, 2, \cdots, m - 1\} \subseteq D_{T_m}(s, t)$, then we $H \to T_m$, since we can define a homomorphic mapping $f : V(H) \to V(T_m)$ as follows: for every $v \in V(H)$ consider two cases:

1. $v \in F_i$ for some $1 \leq i \leq t$

   Suppose $F_i = (u_1, \cdots, u_{m_i})$ and $v = u_j$. The definition of $p(H)$ yields $m_i \leq m - 1$, thus $m_i \in D_{T_m}(s, t)$, i.e. there is an order m_i directed walk from s to t in T_m. Let $W = (w_1, w_2, \cdots, w_{m_i+1})$ be such a walk. Define $f(v) = w_j$.

2. $v \in B_i$ for some $1 \leq i \leq t$

   Suppose $B_i = (u_1, \cdots, u_{m_i})$ and $v = u_j$. With the same argument as the previous case we conclude that there is a walk $W = (w_1, w_2, \cdots, w_{m_i+1})$ from s to t. This time define $f(v) = w_{m_i+2-j}$.

One can easily check that f is indeed a homomorphic mapping. This argument brings down our task to proving that there is indeed a pair $(s,t) \in V(T_m) \times V(T_m)$ such that $\{1, 2, \cdots, m-1\} \subseteq D_{T_m}(s,t)$. It is easy to see the following additive property: For $(s,t), (t,u) \in V(T_m) \times V(T_m)$, $D_{T_m}(s,t) + D_{T_m}(t,u) \subseteq D_{T_m}(s,u)$ (for two sets X and Y define $X + Y = \{x + y | x \in X, y \in Y\}$). Now we take the last step. Suppose T_m contains a directed cycle of length 3 and let v be one of its vertices. Then clearly $\{3k | k \geq 0\} \subseteq D_{T_m}(v,v)$. Suppose $|N^+_{T_m}(v)| \geq |N^-_{T_m}(v)|$ (the opposite case can be handled similarly). Let F be the oriented graph induced by $N^+_{T_m}(v)$ ($N^-_{T_m}(v)$), then $|V(F)| \geq 3$ since $m \geq 6$. Thus F contains three distinct vertices x, y, z such that $\overrightarrow{xy}$ and $\overrightarrow{yz}$. Then clearly $D_{T_m}(v,z)$ ($D_{T_m}(x,v)$) contains $\{1, 2, 3\}$. By applying the additive property, we can deduce that this set contains $\{1, 2, 3\} + D_{T_m}(v,v) \supset \{1, 2, 3\} + \{3k | k \geq 0\} \supset \{1, 2, \cdots, m-1\}$. So we may assume that T_m is free from directed 3-cycles, and thus a transitive tournament. Let $(v_0, v_1, \cdots, v_{m-1})$ be an ordering of T_m such that $\overrightarrow{v_i v_j}$ if and only if $i < j$. In this case it is quit trivial that (v_0, v_{m-1}) is the desired pair. $\qquad\square$

Corollary 3.1.5. *For any H being an acyclic orientation of a cycle with $p(H) \geq 5$, we have $z(H) = p(H) + 1$*

Note that the case $p(H) < 5$ in the above corollary may result in $z(H)$ bigger than $p(H) + 1$. For example when H is a transitive tournament of order 3 ($p(H) = 2, z(H) = 4$).

Theorem 3.1.6. *Suppose H is an acyclic orientation of a union of vertex disjoint paths with the same endpoints. Then $z(H) = O(p(H)^d)$ for some constant d (independent of H).*

The main tool to prove this theorem is the following lemma:

Lemma 3.1.7. *Let L be the tournament consisting of two order 3 directed cycles b_0, b_1, b_2 and b'_0, b'_1, b'_2 with all arcs oriented from b_i to b'_j for $0 \leq i, j \leq 2$. Let H be an acyclic orientation of a union of vertex disjoint paths with the same endpoints, then $H \rightarrow L$.*

Proof. Let u and v be the shared endpoints of the oriented paths in H. w.l.o.g., by choosing the right labels for the endpoints, we can assume that there is no directed path from v to u in H. We show that for any oriented path P of order $k \geq 1$ with endpoints u and v (except any directed path from v to u), there is homomorphic mapping f from P to L such that $f(u) = b_0$ and $f(v) = b'_0$. Obviously by putting all such mappings together for all oriented paths P in H we get $H \rightarrow L$. Let $< u = w_0, w_1, \cdots, w_{k-1}, w_k = v >$ be an arrangement of

$V(P)$ where for each $1 \leq i \leq k$ either $\overrightarrow{w_{i-1}w_i}$ or $\overrightarrow{w_i w_{i-1}}$. There should be an index r such that $\overrightarrow{w_{r-1}w_r}$, otherwise P will be a directed path from v to u. Define the oriented paths P_1, P_2 to be the induced sub-oriented graphs of the subsets $\{w_0, \cdots, w_{r-1}\}$ and $\{w_r, \cdots, w_k\}$ in P, respectively. Clearly P_1 is homomorphic to the order 3 directed cycle b_0, b_1, b_2 in L. Let f_1 be homomorphic mapping corresponding to this homomorphism. We can assume $f_1(w_0) = b_0$. Similarly let f_2 be the homomorphic mapping from P_2 to the order 3 directed cycle b'_0, b'_1, b'_2 such that $f_2(w_k) = b'_0$. Then by putting f_1 and f_2 together we obtain the desired homomorphic mapping f from P to L. $\qquad\square$

Proof. (Theorem 3.1.6) To complete the proof we need to use some concepts and theorems expounded in Sections 4 and 5. So we postpone the proof to the end of Section 5. $\qquad\square$

Open Question 3.1.8. *For which oriented graphs H, $z(H) = p(H) + 1$? or $z(H) = O(p(H) + 1)$?*

The above oriented graphs are all special cases of planar graphs:

Conjecture 3.1.9. *There is a constant d such that for every acyclic orientation H of a planar graph, we have: $z(H) = O(p(H)^d)$.*

This conjecture will be discussed in the next chapter.

3.2 Dense graphs

A well known result of Erdős-Moser [8] can be translated as follows:

Theorem 3.2.1. *([8]) Suppose H is a transitive tournament with n vertices, then $2^{\frac{n-1}{2}} \leq z(H) \leq 2^{n-1}$.*

Proof. ([8]) By induction on n we show that every tournament of order 2^{n-1} contains a transitive sub-tournament of order 2^{n-1}. For $n = 1$ it is trivial. Suppose the claim holds for all tournaments with order 2^{k-1} with k less than $n \geq 2$. Let T be an arbitrary tournament on n vertices. Let $v \in V(T)$ such that $|N^+(v)| \geq |N^-(v)|$. Such a vertex exists because $\sum_{v \in V(T)} |N^+(v)| = \sum_{v \in V(T)} |N^-(v)|$. Then the set $N^+(v)$ has a at least 2^{n-2} vertices. According to the induction hypothesis it should contain a transitive sub-tournament of order $n - 1$. Adding the vertex v to this tournament, we arrive at a transitive sub-tournament of order n.

Now suppose that every tournament of order m contains a transitive sub-tournament of order n. Using a probabilistic argument we will find a lower bound on m. Consider m distinct vertices. Every tournament on these vertices contains a transitive sub-tournament of order n which must have one of the $\binom{m}{n}$ subsets as its vertices. Any one of these subsets in order to be transitive, can be ordered in $n!$ ways. Having fixed the transitive subset (including its order) we observe that such a transitive subset can appear in exactly $2^{\binom{m}{2}-\binom{n}{2}}$ tournaments, since such a tournament is determined by the orientations on its $\binom{m}{2}$ edges with $\binom{n}{2}$ of which have already been fixed. Finally, since each of $2^{\binom{m}{2}}$ oriented graphs has a transitive sub-tournament of n vertices we have:

$$\binom{m}{n}n!2^{\binom{m}{2}-\binom{n}{2}} \geq 2^{\binom{m}{2}}$$

By using $\binom{m}{n} \leq m^n/n!$ we get $m \geq 2^{\frac{n-1}{2}}$. □

For $a, b \geq 1$ the (a, b)-graph is the complete bipartite (simple) graph with a vertices on one part and b vertices on the other part. We consider acyclic orientations of the (a, b)-graph. In this case the compressibility number varies widely depending on the orientation. We attempt to find the worst case:

Definition 3.2.2. *For integers a, b, define $z(a, b) = \max\{z(H)\}$ where the maximum is taken over all acyclic orientations H of the (a, b)-graph.*

Definition 3.2.3. *For every integer a, $T(a, a + 1)$ is an orientation of the $(a, a + 1)$-graph with two parts A and B such that the elements of A and B can be arranged in a sequence $(x_1, x_2, \cdots, x_a)$ and $(y_1, y_2, \cdots, y_{a+1})$ respectively with these properties: for every $1 \leq i \leq a + 1$ and $i \leq j \leq a$, $\overrightarrow{y_i x_j}$ and for every $1 \leq i \leq a$ and $i < j \leq a + 1$, $\overrightarrow{x_i y_j}$. The oriented graph $T(a, a)$ is obtained from $T(a, a + 1)$ by eliminating the vertex y_{a+1}.*

Theorem 3.2.4. *Suppose $b \geq a \geq 1$, then $z(a, b) = z(T(a, a + \epsilon))$ where $\epsilon = 1$ if $b > a$ and zero otherwise.*

Proof. Let G be the (a, b)-graph with parts A ($|A| = a$) and B. Obviously G can be oriented in such a way that it contains $T(a, a + \epsilon)$. So $z(a, b) \geq z(T(a, a + \epsilon))$. Now we show $H \to T(a, a + \epsilon)$, for any acyclic orientation H of the (a, b)-graph. H being acyclic, one can arrange its vertices in a sequence $A = < v_1, v_2, \cdots, v_{a+b} >$ such that there are no indexes $i < j$ such that $\overrightarrow{v_j v_i}$. In a similar way define the arrangement $B = < u_1, u_2, \cdots, u_{2a+\epsilon} >$ for

the oriented acyclic graph $T(a, a + \epsilon)$. Now define the mapping $f : V(H) \to V(T(a, a + \epsilon))$ recursively as follows: $f(v_1) = u_1$ and for $i = 2, \cdots, a + b$, suppose $f(v_{i-1}) = u_j$, then define $f(v_i)$ to be u_j if v_i and v_{i-1} both belong to the same part of H (i.e. either A or B) and u_{j+1} otherwise. It is easy to see that this recursion works its way to the end without encountering any problem and the resulted mapping f is indeed a homomorphic mapping from H to $T(a, a + \epsilon)$. This implies that $z(H) \leq T(a, a + \epsilon)$ for every orientation H of the (a, b)-graph. Thus $z(a, b) \leq z(T(a, a + \epsilon))$ and by combining with the first inequality we get $z(a, b) = z(T(a, a + \epsilon))$. $\square$

It is interesting to note that $z(T(a, a + \epsilon))$ equals to $r(T(a, a + \epsilon)))$ for $\epsilon \in \{0, 1\}$ (one color Ramsey number of $T(a, a+\epsilon)$, see [12]). A similar probabilistic argument to [8] yields:

Theorem 3.2.5. $z(T(a, a + \epsilon)) \geq 2^{a/2}$.

Proof. (a slight modification of the theorem 3.2.1's proof given in [8]) Suppose every tournament of order m contains a $T(a, a+\epsilon)$. We want to find a lower bound on m. Consider m distinct vertices. Every tournament on these vertices contains a $T(a, a+\epsilon)$ which must have one of the $\binom{m}{2a+\epsilon}$ subsets as its vertices. Any one of these subsets in order to be a $T(a, a+\epsilon)$, first must be divided into two parts A and B, having a and b vertices respectively. This can be done in $\binom{2a+\epsilon}{a}$ ways. Then each part must be ordered, this gives $a!(a + \epsilon)!$ alternatives. Having fixed the $T(a, a + \epsilon)$ subset (including its order) we observe that such a subset can appear in exactly $2^{\binom{m}{2}-a(a+\epsilon)}$ tournaments, since such a tournament is determined by the orientations on its $\binom{m}{2}$ edges with $a(a + \epsilon)$ of which have already been fixed. Finally, since each of $2^{\binom{m}{2}}$ oriented graphs contains a $T(a, a + \epsilon)$ we have:

$$\binom{m}{2a + \epsilon}\binom{2a + \epsilon}{a}a!(a + \epsilon)!2^{\binom{m}{2}-a(a+\epsilon)} \geq 2^{\binom{m}{2}}$$

By using $\binom{m}{2a+\epsilon} \leq m^{2a+\epsilon}/(2a + \epsilon)!$ we get $m \geq 2^{a/2}$. $\square$

Conjecture 3.2.6. $z(T(a, a + \epsilon)) \leq 2^{a+O(1)}$ *for* $\epsilon \in \{0, 1\}$

Chapter 4

Relationship with Erdős-Hajnal conjecture

For a graph G we say the subset $S \subseteq V(G)$ is a homogeneous set if its induced subgraph is either a clique or a graph with no edges. We denote by $\hom(G)$ the size of the largest homogeneous set in G. Similarly we define $\hom(T)$ for a tournament T as the size of its biggest transitive sub-tournament. The classic Erdős-Hajnal [7] conjecture deals with the value of $\hom(G)$ when G avoids a certain forbidden graph. We say the graph G avoids another graph F, or is F-free, if it has no *induced* subgraph isomorphic to F (see [7]).

Definition 4.0.7. *The family of monotone functions $\{g_f : \mathbb{N} \to \mathbb{N}\}_{f=1,2,\cdots}$ is called an Erdős-Hajnal function family if for every $f \geq 1$, $n \geq 1$ and every graph F with size at most f the following condition holds: For every size n F-free graph G we have $\hom(G) \geq g_f(n)$. The same definition applies to* oriented *Erdős-Hajnal function family, with F being an arbitrary oriented graph (with size at most f) and the notions F-free graph G and $\hom(G)$ replaced with F-free tournament T and $\hom(T)$, respectively.*

Definition 4.0.8. *We say the oriented graph F is an oriented* Erdős-Hajnal Restriction *(EHR), if there exists a constant $c(F)$ such that for every F-free tournament T on n vertices $\hom(T) \geq n^{c(F)}$. The same definition can be applied to the case when F is a graph. In this case we say F is an EHR, if there exists a constant $c(F)$ such that for every F-free graph G on n vertices $\hom(G) \geq n^{c(F)}$.*

Based on the above definitions, the Erdős-Hajnal conjecture can be formulated as follows:

Conjecture 4.0.9. *([7]) Every graph is an EHR, or equivalently, for every integer f, there exists a positive integer $c(f)$ such that $\{n^{c(f)}\}_{f=1,2,\ldots}$ is an Erdős-Hajnal function family.*

Alon et al., in [1], introduce the oriented version of the above conjecture:

Conjecture 4.0.10. *([1]) Every oriented graph is an oriented EHR, or equivalently, for every integer f, there exists a positive integer $c(f)$ such that $\{n^{c(f)}\}_{f=1,2,\ldots}$ is an oriented Erdős-Hajnal function family.*

They proved that these two conjectures are equivalent.

Theorem 4.0.11. *([1]) Conjectures 4.0.9 and 4.0.10 are equivalent*

Based on their techniques, one can come up with a slightly more generalized statement:

Theorem 4.0.12. *Let $\{g_f : \mathbb{N} \to \mathbb{N}\}_{f=1,2,\ldots}$ be a family of monotone functions. The function family $\{g'_f : \mathbb{N} \to \mathbb{N}\}_{f=1,2,\ldots}$ is defined as $g'_f = g_{\lceil cf^3 \log^2 f \rceil}$ for every $f \geq 1$ where c is a big enough constant (during the proof we will see how big it should be). Then $\{g'_f\}$ is an oriented Erdős-Hajnal function family if $\{g_f\}$ is an Erdős-Hajnal function family. Similarly, $\{g'_f\}$ is an Erdős-Hajnal function family if $\{g_f\}$ is an oriented Erdős-Hajnal function family.*

The proof of the above theorem is based on two results from [1] and [15]:

Definition 4.0.13. *A graph G with a linear order $<$ on its vertices is called an ordered graph and denoted by $(G, <)$. An ordered graph $(G, <)$ is a subgraph of another ordered graph $(G', <')$, if there is a mapping $f : V(G) \to V(G')$ with these two conditions: 1. $f(u) <' f(v)$ if and only if $u < v$, 2. $(f(u), f(v)) \in E(G')$ if and only if $(u, v) \in E(G)$.*

Definition 4.0.14. *A tournament T with a linear order $<$ on its vertices is called an ordered tournament and denoted by $(T, <)$. An ordered tournament $(T, <)$ is a sub-tournament of another ordered tournament $(T', <')$, if there is a mapping $f : V(T) \to V(T')$ with these two conditions: 1. $f(u) <' f(v)$ if and only if $u < v$, 2. $\overrightarrow{f(u)f(v)}$ if and only if $\overrightarrow{uv}$.*

Theorem 4.0.15. *([15]) For any ordered graph $(G, <)$ on n vertices, there exists a graph G' on $O(n^3 \log^2 n)$ vertices such that, for every ordering $<'$ of G', $(G, <)$ is a subgraph of $(G', <')$.*

Theorem 4.0.16. *(The oriented version of 4.0.15 [1]) For any ordered tournament $(T, <)$ on n vertices, there exists a tournament T' on $O(n^3 \log^2 n)$ such that, for every ordering $<'$ of T', $(T, <)$ is a sub-tournament of $(T', <')$.*

The proofs of these two theorems are based on probabilistic method. I do not include these proofs as they are very long and the techniques are not related to this thesis. The reader may refer to the original references for the complete proofs.

Proof. (theorem 4.0.12) Suppose $\{g_f\}$ is an Erdős-Hajnal function family. Let F be an arbitrary oriented graph on f vertices. w.l.o.g. we can assume that F is a tournament. Define an arbitrary ordering $<_F$ on the vertices of F. Let $< v_1, v_2, \cdots, v_f >$ be the linear arrangement of $V(F)$ where $v_i <_F v_j$ with $i < j$. Let $G(F, <_F)$ be the ordered graph obtained from $(F, <_F)$ by replacing all arcs $\overrightarrow{v_i v_j}$ for $i < j$, with an edge and eliminating the rest of arcs. Let G' be the graph obtained by applying Theorem 4.0.15 to the ordered graph $G(F, <_F)$. Now suppose the order n tournament T avoids F. Let $<_T$ be an arbitrary ordering on the vertices of T and define the ordered graph $G(T, <_T)$ in a similar way to $G(F, <_F)$. Then the graph $G(T, <_T)$, without considering its vertex ordering, should avoid G', otherwise $G(T, <_T)$ will contain the ordered sub-graph $(G', <_{G'})$ for some ordering $<_{G'}$ on the vertices of G'. The definition of G' implies $G(F, <_F)$ is an ordered sub-graph of $G(T, <_T)$ which, in turn, means T contains F, contradiction. So $G(T, <_T)$ contains a homogeneous set of order $g_k(n)$, where k is any upper bound on the number of vertices of G'. According to Theorem 4.0.15 there is a constant d such that $k \leq \lceil df^3 \log^2 f \rceil$. By choosing the constant c (in the theorem's statement) bigger than d, we conclude that the order of this homogenoues set is at least $g'_f(n)$ which induces a transitive tournament of the same order in T. Thus $\{g'_f\}$ is an oriented Erdős-Hajnal function family.

The proof of the second part is very similar. Suppose $\{g_f\}$ is an oriented Erdős-Hajnal function family and F be an arbitrary simple graph on f vertices. For an arbitrary ordering $<_F$ on the vertices of F. Let $T(F, <_F)$ be the ordered tournament obtained from the ordered graph $(F, <_F) = < v_1, v_2, \cdots, v_f >$ as follows: for every $1 \leq i < j \leq f$ place the arc $\overrightarrow{v_i v_j}$ if there is an edge between v_i and v_j (and remove this edge too), otherwise place the arc $\overrightarrow{v_j v_i}$. Let T' be obtained by applying Theorem 4.0.16 to the ordered tournament $T(F, <_F)$. Now suppose the order n graph G avoids F. A similar argument to the above show that $T(G, <_G)$ (defined similar to $T(F, <_F)$) should avoid T', and thus it contains a transitive sub-tournament of order $g_k(n)$ with k being any upper bound on the number of vertices of T'. By choosing the constant c big enough (in relation to Theorem 4.0.16) one concludes that G contains a homogeneous set of order $g'_f(n)$. $\qquad\square$

The following very general theorem shows a direct link between the 'Erdős-Hajnal-type'

problems and the compressibility number:

Theorem 4.0.17. *Let $\{g_f\}_{f=1,2,\ldots}$ be an oriented Erdős-Hajnal function family and H be an acyclic oriented graph whose oriented coloring number is at most r. Define k as the smallest integer such that $g_r(k) \geq p(H) + 1$ Then we have $z(H) \leq k$.*

Proof. Let T be an arbitrary tournament of size k. Since $\chi_o(H) \leq r$, there should be an oriented graph H' on at most r vertices such that $H \to H'$. If T contains H' then it means $H \to T$ and we are done. If T avoids H', then it contains a transitive sub-tournament T' with $g_r(k) \geq p(H) + 1$ vertices. Then according to Lemma 3.1.2 $H \to T'$ which again means $H \to T$. □

Corollary 4.0.18. *Assuming Conjecture 4.0.9 (or equivalently 4.0.10), for every class of oriented graphs H with oriented coloring number not exceeding a constant b, there is a constant $d(b)$ such that $z(H) = O(p(H)^{d(b)})$. In particular the conjecture 3.1.9 follows as planar graph orientations have bounded oriented coloring number (see [14]).*

Proof. Let $\{n^{c(f)}\}_{f=1,2,\ldots}$ be an oriented Erdős-Hajnal function family as described in Conjecture 4.0.10. Applying Theorem 4.0.17 for this function family, we get $k = \lceil (p(H) + 1)^{1/c(b)} \rceil$ and the corollary follows by placing $d(b) = 1/c(b)$. □

Additionally the above corollary holds for other families of oriented graphs with bounded oriented coloring number such as oriented graphs with bounded degree, or tree width, or genus (see [11, 13, 17, 18]).

Erdős and Hajnal, in [7], proved a weaker version of conjecture 4.0.9:

Theorem 4.0.19. *([7]) For every graph F, there exists a constant $c(F)$ such that for every F-free graph G on n vertices: $hom(G) \geq 2^{c(F)\sqrt{\log n}}$.*

The techniques used in the proof of this theorem are not used anywhere in this thesis. So I do not include its long full proof here and instead, I give a general outline of the proof. Please refer to the original paper [7] for the complete proof.

Proof. (Theorem 4.0.19, A short summary from [7]) They define by induction a family Γ of graphs, which is called co-graphs, as follows: Initially Γ contains the graph with one vertex. Assume $G_1, G_2 \in \Gamma$. Then $G \in \Gamma$ where $V(G) = V(G_1) \cup V(G_2)$ and $E(G)$ is either $E(G_1) \cup E(G_2)$ or $E(G_1) \cup E(G_2) \cup E^*$, in which $E^* = \{(u_1, u_2) : u_1 \in V(G_1), u_2 \in$

$V(G_2)\}$. Let $f = |V(F)|$. A simple observation is for $H \in \Gamma$, $\hom(H) \geq |V(H)|^{1/2}$. Thus it is sufficient to show that every F-free graph G on n vertices contains a members of Γ with order at least $g(n) = e^{c\sqrt{\log n}-K}$ for some constant c (depending on F) and $K = (f \log 2 + 2 \log f)/f$. This claim is proofed by induction on n. The key lemma used for this induction states that if an order n graph G avoids F, then it contains two large (in terms of n, f and a free variable used for optimization) disjoint vertex subsets A and B such that either each vertex of A is connected to many vertices of B (i.e. almost complete bipartite graph) or each vertex of A is connected to a few vertices of B (i.e. almost an empty bipartite graph). Then applying the induction hypothesis to A, we get that A contains an order $g(|A|)$ subset A_1 with its induced subgraph being in Γ. Then, based on the property of A, B, they find a large subset B_1 of B which either each vertex of A_1 is connected to each vertex of B_1 or no vertex of A_1 is connected to any vertex of B_1. Now applying the induction hypothesis to B_1, we get an order $g(|B_1|)$ subset B_2 of B_1 with its induced subgraph being in Γ. Putting A_1 and B_2 together gives us a member of Γ again (see the definition of Γ) of order $|A_1| + |B_2| = g(|A|) + g(|B_1|)$. Then the values of $|A|$ and $|B_1|$ being large enough, they prove that $g(|A|) + g(|B_1|)$ is indeed $\geq g(n)$ and we are done.   □

One can easily convert it to its oriented analogous:

Theorem 4.0.20. *For every oriented graph F, there exists a constant $c'(F)$ such that every F-free tournament T with n vertices contains a transitive sub-tournament whose size is at least $2^{c'(F)\sqrt{\log n}}$*

Proof. Theorem 4.0.19 states that there is a function $d : \mathbb{N} \to \mathbb{R}$ such that, $\{g_f(n) = \lceil 2^{d(f)\sqrt{\log n}} \rceil\}_{f=1,2,\cdots}$ is an Erdős-Hajnal function family. Applying Theorem 4.0.12, we are led to the oriented Erdős-Hajnal function family $\{g'_f(n)\}$ defined as $g'_f(n) = g_{\lceil cf^3 \log^2 f \rceil}(n) = \lceil 2^{d(\lceil cf^3 \log^2 f \rceil)\sqrt{\log n}} \rceil$. Thus we just need to define $d'(f) = d(\lceil cf^3 \log^2 f \rceil)$ and $c'(F) = d'(|V(F)|)$.   □

This provides us with the oriented Erdős-Hajnal function family $\{g_f(n) = 2^{c'(f)\sqrt{\log n}}\}_{f=1,2,\cdots}$ where $c'(f) = \min\{c'(F)\}$ with minimum taken over all oriented graphs F with at most f vertices. By plugging this function family into theorem 4.0.17 we get:

Corollary 4.0.21. *For every oriented graph H with oriented coloring number bounded by a constant b (in particular planar graphs' orientations), there is a constant $d(b)$ such that $z(H) = O(p(H)^{d(b) \log p(H)})$.*

Proof.

$$g_b(k) = 2^{c'(b)\sqrt{\log k}} \geq p(H) + 1 \Rightarrow k = \lceil (p(H) + 1)^{\frac{\log(p(H)+1)}{c'(b)^2}} \rceil \Rightarrow d(b) = \frac{\log 3}{\log 2 \cdot c'(b)^2}$$

□

Chapter 5

Some oriented Erdős-Hajnal problems

Theorem 4.0.17 motivates us to find as many oriented EHRs as possible. The following theorem gives a more specific relation between conjecture 3.1.9 and oriented EHRs:

Theorem 5.0.22. *For every family $\mathbb{L}$ of oriented graphs, if there exists a finite set $\mathbb{X}$ of oriented EHRs such that, for every $H \in \mathbb{L}$: $H \to X$ for some $X \in \mathbb{X}$, then there exists a constant d such that $z(H) = O(p(H)^d)$ for every $H \in \mathbb{L}$.*

Proof. For every $X \in \mathbb{X}$ let $c(X)$ be the constant that is defined in Definition 4.0.8. Let $c = \min_{X \in \mathbb{X}}\{c(X)\}$ and define $d = \lceil 1/c \rceil$. For an arbitrary $H \in \mathbb{L}$, let $X \in \mathbb{X}$ be the oriented graph such that $H \to X$. Let T be any tournament of order $(p(H) + 1)^d$. If T contains X, then $H \to T$. Otherwise, X being an oriented EHR, T contains a transitive sub-tournament T' of order $|V(T)|^{c(X)} \geq p(H) + 1$. Then according to Lemma 3.1.2, H will be homeomorph to T' and thus $H \to T$ again. $\qquad\square$

Open Question 5.0.23. *Is there any finite set $\mathbb{X}$ of oriented EHRs such that, for every acyclic orientation H of a planar graph, $H \to X$ with $X \in \mathbb{X}$(The affirmative answer to this question immediately implies Conjecture 3.1.9).*

5.1 Partitioning restrictions

Definition 5.1.1. *The vertex cut (A, B) of the oriented graph H, with none of A and B being empty, is called an* oriented cut*, if all the arcs of the cut, if there is any, are oriented*

from A to B.

Definition 5.1.2. *The oriented graph F of order $f \geq 3$ is called a* partitioning restriction *if every F-free tournament of order at least f has an oriented cut.*

For example every directed cycle is a partitioning restriction (see theorem 5.1.4). The implication is:

Theorem 5.1.3. *Every partitioning restriction is an oriented EHR.*

Proof. By induction on n we show that every tournament T with $n \geq 2$ vertices avoiding a partitioning restriction F of order $f \geq 2$, contains a transitive sub-tournament of order $\lfloor \frac{n}{f} \rfloor + 2$ and thus F should be an oriented EHR. It's easy to check it when $n < f$, since every such tournament has a 2-transitive sub-tournament. Now suppose T is of order at least f and the claim holds for all smaller tournaments. Since T avoids F, according to the definition of F, T has an oriented cut (A, B). Let T_1 and T_2 be the largest transitive sub-tournament of the sub-tournaments induced by A and B respectively. Let T^* be the induced sub-tournament of $V(T_1) \cup V(T_2)$. Trivially T^* is transitive. w.l.o.g. assume $V(T_1) \geq V(T_2) \geq 1$. If $|V(T_2)| = 1$ then we must have $|B| = 1$. This means $|A| = n - 1 \geq 2$. Thus by applying the induction hypothesis to the induced graph of A we get: $|V(T^*)| = |V(T_1)| + |V(T_2)| \geq \lfloor \frac{n-1}{f} \rfloor + 2 + 1 \geq \lfloor \frac{n}{f} \rfloor + 2$. Now suppose $|V(T_2)| \geq 2$. The apply the induction hypothesis to both induced graphs of A and B, we'll have: $|V(T^*)| = |V(T_1)| + |V(T_2)| \geq \lfloor \frac{|A|}{f} \rfloor + 2 + \lfloor \frac{|B|}{f} \rfloor + 2 \geq \lfloor \frac{n}{f} \rfloor + 2$. So the inductive argument follows. □

Here we introduce one family of partitioning restrictions:

Theorem 5.1.4. *Every directed cycle C of order $c \geq 3$ is a partitioning restriction.*

Proof. Let T be a tournament of order $n \geq 2$ avoiding C. If $c = 3$ then C is transitive and trivially we are done. So we may assume that $c \geq 4$ and T has a cycle of length 3. Let k be the largest number $\leq c$, such that T contains a cycle K of order k. Obviously $k < c$. Let $v_0, v_1, \cdots, v_{k-1}$ be the vertices of K in the clockwise order. For each $v \in V(T) - V(K)$, either $N^+(v) \cap V(K) = \emptyset$ or $N^-(v) \cap V(K) = \emptyset$. For otherwise, there are $x, y \in V(K)$ such that $\overrightarrow{xv}$ and $\overrightarrow{vy}$. K being a cycle, we can easily deduce that there is an index $0 \leq i < k$ such that $\overrightarrow{v_i v}$ and $\overrightarrow{v v_{i+1}}$ (index calculations are all mod k). Then we have the directed cycle $v, v_{i+1}, v_{i+2}, \cdots v_i$ of order $k+1$. Contradicting the definition of k. Let A^+ and A^- be the set of vertices v for which $N^-(v) \cap V(K) = \emptyset$ and $N^+(v) \cap V(K) = \emptyset$, respectively. The previous

argument states $A^+ \cup A^- = V(T) - V(K)$. We claim that all the arcs between A^+ and A^- are oriented from the former to the latter. Otherwise there are $u \in A^+$ and $w \in A^-$ such that $\overrightarrow{wu}$. Then the directed cycle $u, v_{k-1}, v_{k-2}, \cdots, v_1, w$ has length $k+1$. Again the definition of k is violated. Either A^+ or A^- is nonempty, since $|A^+ \cup A^-| = |V(T) - V(K)| = n - k \geq 1$. Call the nonempty one A. Then it's easy to see that either $(A, V(T) - A)$ or $(V(T) - A, A)$ is an oriented cut. □

Corollary 5.1.5. *Every directed cycle is an oriented EHR.*

Corollary 5.1.6. *Every oriented graph F with at most four vertices is an oriented EHR.*

Proof. W.L.G we can assume F is a tournament. It is trivial when $|V(F)| \leq 2$. When $|V(F)| = 3$, F is either a transitive tournament (an obvious oriented EHR) or a directed cycle. In the latter case $c(F) = 1$. Now suppose $|V(F)| = 4$. Suppose F contains a vertex v with either in-degree or out-degree zero. Let F' be obtained from F by deleting v. Let A be the single arc oriented graph. Then F is isomorphic to $A(v, F')$ (refer to 5.2.1 for the definition). Knowing that both A and F' are oriented EHRs, by applying Theorem 5.2.3 (in the next subsection) we conclude that so is F. Let T be an arbitrary tournament on n vertices. If it avoids the order 4 directed cycle, then according to Corollary 5.1.5 it contains a transitive sub-tournament of order n^c for some constant c. Now suppose T contains a directed cycle with vertex set $X = \{x_0, x_1, x_2, x_3\}$ where $\overrightarrow{x_i x_{i+1}}$ for all $i = 0, \cdots, 3$ (index calculations are mod 4). Then if F does not contain such a vertex v, a simple argument shows that $F \to T|_X$. Thus we can set $c(F) = c$. □

5.2 A constructive method

A powerful constructive method in [1] enables us to make new EHRs, out of the known EHRs.

Definition 5.2.1. *For any graph G with $V(G) = \{v_1, v_2, \cdots, v_k\}$ and k other graphs $F_1, F_2, \cdots, F_k$ we define $G(F_1, F_2, \cdots, F_k)$ as the graph obtained from G by replacing each v_i with a copy of F_i, and joining any vertex of the copy of F_i to any vertex of a copy of F_j, $j \neq i$, if and only if $v_i v_j \in E(H)$. Similarly we define $H(F_1, F_2, \cdots, F_k)$, for oriented graphs $H, F_1, F_2, \cdots, F_k$ with $V(H) = \{v_1, v_2, \cdots, v_k\}$, by replacing each v_i with a copy of F_i and joining each vertex in F_i to each vertex F_j, $i \neq j$, with an arc oriented from the former set to the latter set if and only if $\overrightarrow{v_i v_j}$.*

Theorem 5.2.2. *([1]) If $G, F_1, \cdots, F_k$ are simple EHRs, then so is $G(F_1, F_2, \cdots, F_k)$.*

Similar to theorem 4.0.20, one can come up with its oriented version:

Theorem 5.2.3. *If $H, F_1, \cdots, F_k$ are oriented EHRs, then so is $H(F_1, F_2, \cdots, F_k)$.*

The proof of the above theorem is easily obtained from the proof of Theorem 5.2.2 by replacing the terminologies for graphs with that of oriented graphs. More precisely, for a vertex v in a graph, the notions of set $\{u : u \text{ and } v \text{ are adjacent}\}$ and set $\{u : u \text{ and } v \text{ are not adjacent}\}$ must be replaced with the notions $N^+(v)$ and $N^-(v)$, respectively in the oriented version of the proof (in which v will be a vertex in an oriented graph).

Proof. (Theorem 5.2.3. Oriented version of the theorem 5.2.2's proof given in [1]) Let $V(H) = \{v_1, v_2, \cdots, v_k\}$. Obviously, it is sufficient to show that $H(F_1, v_2, \cdots, v_k)$ is an oriented EHR. Let H_0 denote the oriented graph obtained from H by the deletion of v_1. For simplicity, write $H(F_1)$ for $H(F_1, v_2, \cdots, v_k)$. Let T be an $H(F_1)$-free tournament with n vertices, and assume that $\hom(T) < n^{c(H)\delta}$ ($c(H)$ is the constant defined in Definition 4.0.8). We would like to get a contradiction, provided that $\delta > 0$ is sufficiently small. Let $m = \lceil n^\delta \rceil > k$. By the definition of $c(H)$, any m-element subset of $U \subseteq V(T)$ must contain H. Otherwise, we would find a transitive sub-tournament of order $> m^{c(H)}$ which is impossible. Therefore, T has at least $\binom{n}{m}/\binom{n-k}{m-k}$ induced sub-oriented graphs isomorphic to H. For each of these sub-graphs, fix an isomorphic mapping from H into T. Since the number of mappings from H_0 to T is smaller than $n(n-1)\cdots(n-k+2)$, there exists a mapping, which can be extended to an isomorphic mapping of H to T in at least

$$M = \frac{\binom{n}{m}}{\binom{n-k}{m-k}n(n-1)\cdots(n-k+2)} \tag{5.1}$$

different ways. In other words, there are $k-1$ vertices $v_2', \cdots, v_k' \in V(T)$, and there exists an at least M-element subset $W \subseteq V(T)$ such that, for every $w \in W$:

$$f(v_1) = w, f(v_i) = v_i'(i = 2, \cdots, k)$$

is an isomorphic mapping from H to T. Consider now the sub-tournament $T|_W$ of T induced by W. This tournament must be F_1-free, otherwise T would not be $H(F_1)$-free. Since F_1 an oriented EHR, we know that the order of

$$\hom(T|_W) \geq |W|^{c(F_1)} \geq M^{c(F_1)}$$

On the other hand:

$$n^{c(H)\delta} > \hom(T) \geq \hom(T|_W)$$

Comparing the last two inequalities and plugging in the value (5.1) for M, we obtain that:

$$n^{\delta c(H)/c(F)} > \frac{\binom{n}{m}}{\binom{n-k}{m-k}n(n-1)\cdots(n-k+2)} = \frac{n-k+1}{m(m-1)\cdots(m-k+1)} > n^{1-k\delta}$$

which gives the desired contradiction, provided that:

$$\delta < \frac{c(F)}{c(H)+k\cdot c(F)}$$

$\square$

Corollary 5.2.4. *Every tournament F, with the order of its largest directed cycle at most four, is an oriented EHR.*

Proof. Induction on the order of F. For $|V(F)| \leq 4$ the claim holds according to Corollary 5.1.6. Assume $|V(F)| \geq 5$. Since F avoids the length 5 directed cycle, then according to Theorem 5.1.4 F contains an oriented cut (B,C). B and C are both tournaments with order smaller than the order of F and contain no directed cycle larger than 4. So according to the induction hypothesis both are oriented $EHRs$. Let A be the single arc oriented graph. Then applying Theorem 5.2.3 we conclude that $F = A(B,C)$ (Definition 5.2.1) is also an oriented EHR. $\square$

Conjecture 5.2.5. *Every oriented graph F, with the order of its largest directed cycle less than four, is an oriented EHR.*

Now we can give the proof of Theorem 3.1.6:

Proof. (Theorem 3.1.6) Let A and B be the single arc oriented graph and the order 3 directed cycle, respectively. Then we observe that the tournament L defined in Lemma 3.1.7 is exactly $A(B,B)$ (Definition 5.2.1). Note that both A and B are oriented EHRs, so according to Theorem 5.2.3, $L = A(B,B)$ is an oriented EHR too. Now applying Theorem 5.0.22, for the family of oriented graphs H and $\mathbb{X} = \{L\}$, guarantees the existence of such constant d. $\square$

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