

**CROSSINGS AND NESTINGS IN FOUR  
COMBINATORIAL FAMILIES**

by

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B.Sc.H., Acadia University, 2007

THESIS SUBMITTED IN PARTIAL FULFILLMENT  
OF THE REQUIREMENTS FOR THE DEGREE OF  
MASTER OF SCIENCE  
IN THE DEPARTMENT  
OF  
MATHEMATICS

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SIMON FRASER UNIVERSITY  
Summer 2009

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# Abstract

The combinatorial families of matchings, set partitions, permutations and graphs can each be represented by a series of vertices along a horizontal line with arcs connecting them. Such a representation is referred to as an arc annotated sequence. A natural crossing and nesting structure arises in each of these representations, and remarkably enough, equidistribution between these two statistics has been shown for both matchings and partitions. To show this, tools such as RSK and several bijections are required. Furthermore, other useful bijections to lattice paths, and Ferrers diagrams give additional information, and aid the enumeration for each of the four families according to these two statistics.

*I would like to dedicate this thesis to my three brothers: Nigel,  
Adrian and Seth.*

# Acknowledgments

This work was funded in part by NSERC PGS-M. I would like to thank to Dr. Cedric Chauve, Dr. Eric Fusy, Jacob Post, Mariolys Rivas and Dr. Lily Yen for their many discussions on this subject, and give a particularly large thank you to my supervisor, Dr. Marni Mishna, who not only helped me at every turn, but provided invaluable support, guidance, and encouragement throughout the completion of this thesis. And of course, thank you Mom and Dad.

# Contents

<b>Approval</b>	<b>ii</b>
<b>Abstract</b>	<b>iii</b>
<b>Dedication</b>	<b>iv</b>
<b>Acknowledgments</b>	<b>v</b>
<b>Contents</b>	<b>vi</b>
<b>List of Tables</b>	<b>ix</b>
<b>List of Figures</b>	<b>x</b>
<b>I Background</b>	<b>1</b>
<b>1 Introduction</b>	<b>2</b>
1.1 Four combinatorial families . . . . .	3
1.2 Statistics in arc annotated sequences . . . . .	4
<b>2 Combinatorial structures</b>	<b>8</b>
2.1 Ferrers diagrams and tableaux . . . . .	8
2.2 Walks in the lattice . . . . .	13
2.3 Chord diagrams . . . . .	15

<b>3</b>	<b>Enumerative techniques</b>	<b>17</b>
3.1	Generating functions . . . . .	17
3.2	Continued fractions . . . . .	18
<b>II</b>	<b>Families of arc annotated sequences</b>	<b>22</b>
<b>4</b>	<b>Matchings</b>	<b>23</b>
4.1	Representation as an arc annotated sequence . . . . .	23
4.2	Distribution of nestings and crossings . . . . .	24
4.3	Further bijective results . . . . .	29
4.4	Enumerative results . . . . .	41
<b>5</b>	<b>Set partitions</b>	<b>47</b>
5.1	Representation as arc annotated structure . . . . .	47
5.2	Distribution of nesting and crossing statistics . . . . .	50
5.3	Further bijective results . . . . .	58
5.4	Enumerative results . . . . .	64
<b>6</b>	<b>Labeled Graphs</b>	<b>71</b>
6.1	Representation as an arc annotated structure . . . . .	71
6.2	Distribution of nesting and crossing statistics . . . . .	72
<b>7</b>	<b>Permutations</b>	<b>76</b>
7.1	Representation as an arc annotated sequence . . . . .	76
7.2	Distribution of crossings and nestings . . . . .	77
7.3	Bijections to bicolored weighted Motzkin paths . . . . .	84
7.4	Distribution of other permutations statistics . . . . .	87
7.5	Enumerative results . . . . .	93
<b>III</b>	<b>Conclusion</b>	<b>97</b>
7.6	Big Picture . . . . .	98
7.7	Map . . . . .	100



<i>CONTENTS</i>	viii
7.8 Open questions . . . . .	102
<b>Bibliography</b>	<b>104</b>
<b>Appendix</b>	<b>107</b>

# List of Tables

3.1	All bijections . . . . .	20
3.2	A description of all objects. . . . .	21
4.1	Total number of nestings and crossings for matchings of size $n$ . . . . .	24
4.2	$\Phi_6 : \mathcal{W}_6^{(1,5)} \rightarrow \mathcal{D}_3^{1,5}$ . . . . .	40
4.3	$\Phi_7 : \mathcal{B}_6 \rightarrow \mathcal{C}_6$ . . . . .	42
4.4	The triangular table of $[y^k x^{2n}]$ for matchings on $\{1, 2, \dots, 2n\}$ with $k$ nestings. . . . .	44
7.1	$C_+(i), C_-(i), N_+(i), N_-(i)$ . . . . .	77
7.2	Nesting and Crossing values for all permutation in $S_4$ . . . . .	79
7.3	$C_+[n], C_-[n], N_+[n], N_-[n]$ . . . . .	80
7.4	$NE(n, k)$ (and also $CR(n, k)$ ) for small $n$ and $k$ . . . . .	83
7.5	Triangular sequence of $[y^k x^n]$ for permutations on $\{1, 2, \dots, n\}$ with $k$ nestings . . . . .	95
7.6	Summary table. . . . .	99
7.7	Objects represented as growth diagrams, and interpretations of chains. . . . .	100

# List of Figures

1.1	A crossing . . . . .	5
1.2	A nesting. . . . .	5
1.3	A 3-crossing . . . . .	6
1.4	A 3-nesting . . . . .	6
2.1	Two arrangements of 3 chords on a circle. . . . .	16
4.1	The arc annotated sequence for $A$ . . . . .	32
4.2	The arc annotated sequence for $B$ . . . . .	32
4.3	The Dyck path corresponding to matchings $A$ and $B$ . . . . .	32
4.4	The weighted Dyck path corresponding to matching $A$ . . . . .	32
4.5	The weighted Dyck path corresponding to matching $B$ . . . . .	33
4.6	All matchings on $\{1, 2, 3, 4, 5, 6\}$ . . . . .	45
5.1	Average number of nestings according to crossings. . . . .	66
5.2	As the maximum height of the corresponding Motzkin path increases from 1 to 10, so does the average number of enhanced nestings in the set partition. . . . .	68
7.1	The arc annotated sequences of $\sigma_1$ and $\sigma_2$ with the $3\ 1-2$ patterns illustrated by the dotted lines. . . . .	88
7.2	Each curve is a different maximum height of the Motzkin path corresponding to the permutation under $\Phi_{16a}$ . . . . .	96

**Part I**

**Background**

# Chapter 1

## Introduction

In 2005, Chen, Deng, Du, Stanley and Yan examined matchings and set partitions in what they referred to as [3] ‘standard representation.’ For both matchings and partitions in this representation, they showed the remarkable result of equidistribution between two naturally arising statistics, **nestings** and **crossings**. Before this, results were limited to matchings, with enumeration understood according to only crossings for many years, and equidistribution was only shown in the smallest cases for matchings. Little was understood of partitions. Thus with their paper, ‘Crossings and nestings of matchings and partitions,’ [3], Chen et al. opened the door for further questions about not only the distribution and enumeration of matchings and partitions, but for other combinatorial objects that might be placed in this standard representation as well. The ‘standard representation’ that Chen et al. referred to is what we call an **arc annotated sequence**.

**Definition** An *arc annotated sequence* is a series of ordered points on a horizontal line that are connected with a set of arcs located either above the line, or both above and below.

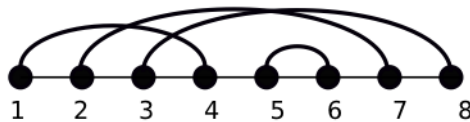
Each of the four combinatorial families that we examine: matchings, partitions, graphs and permutations, can be represented with arc annotated sequences. It is well known that matchings, set partitions, permutations, and to a certain extent graphs, can be easily counted according to size. Our goal is to understand the distribution of these objects according to other statistics that naturally arise when these families are placed in arc annotated sequences. These statistics that arise are called nestings and crossing. To accomplish our goal we employ:

1. bijections with other combinatorial objects to understand the distribution,
2. generating functions for enumeration, and
3. continued fractions for computational purposes.

## 1.1 Four combinatorial families

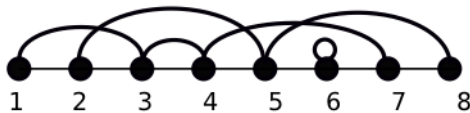
Matchings, set partitions, graphs and permutations are each a class of combinatorial objects that can be represented with these arc annotated sequences. A few examples are given to get a taste for how they are used, and formal definitions will follow.

**Example 1.1.1.** *The matching  $\{\{1,4\},\{2,7\},\{5,6\},\{3,8\}\}$  can be represented in an arc annotated sequence as follows:*



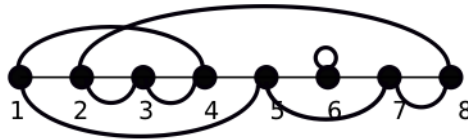
A matching on a set is a set partition in which all blocks are required to be of size 2. With this in mind, the more generalized set partition has an obvious natural shape:

**Example 1.1.2.** *The set partition  $\{\{1347\},\{258\},\{6\}\} = 1347|258|6$  is represented in the following arc annotated sequence:*



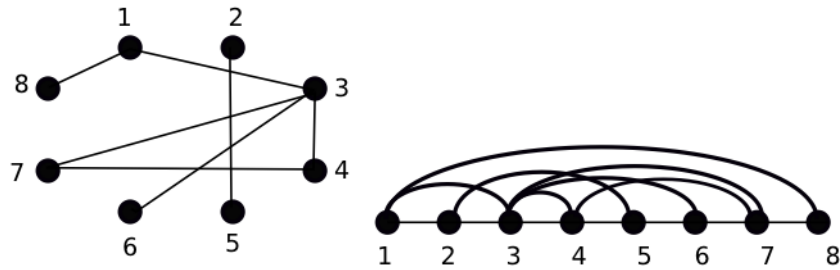
Matchings are the more restricted version of these arc annotated sequences, with each vertex only being connected to exactly one arc, while in partitions a vertex may be connected to two. However, in each of these cases all of the arcs must remain above the horizontal line. When a permutation is represented with an arc annotated sequence, although each vertex must be connected to exactly two arcs, the restriction of staying above the horizontal line is broken.

**Example 1.1.3.** Let  $A = \begin{bmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{bmatrix}$  be a representation of a permutation of  $\{1, 2, \dots, n\}$  in this notation. This means that  $\sigma(1) = a_1$ ,  $\sigma(2) = a_2$ , and in general  $\sigma(i) = a_i$ . With this in mind, the permutation  $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 8 & 2 & 3 & 1 & 6 & 5 & 7 \end{bmatrix}$  is represented with an arc annotated sequence as follows:



Finally, a graph  $G$  is the least restrictive with regards to its arc annotated sequence. Although the convention of staying above the horizontal line is still followed, because of the nature of a graph, there is not necessarily an obvious limit on the number of arcs that will be connected to a vertex.

**Example 1.1.4.** Consider the following labelled graph on 8 vertices and its corresponding arc annotated sequence:



## 1.2 Statistics in arc annotated sequences

There are many statistics that can be observed in arc annotated sequences. The main two examples that we study here are nestings and crossings.

**Definition** A *crossing* in an arc annotated sequence occurs when two arcs cross in the diagram. For example, the following is a crossing:

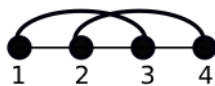


Figure 1.1: A crossing

Formally, a crossing between arcs  $(i_1, j_1)$  and  $(i_2, j_2)$  occurs when  $i_1 < i_2 < j_1 < j_2$ . In Figure 1.1, there is a crossing between  $(1, 3)$  and  $(2, 4)$ .

One or more crossings may occur in each of the combinatorial structures above and our goal will be to understand their distribution. The other core statistic in arc annotated sequences is the *nesting*.

**Definition** A *nesting* in an arc annotated sequence occurs when two arcs are concentric, forming a ‘rainbow-like’ substructure as follows:



Figure 1.2: A nesting.

Formally, a nesting between arcs  $(i_1, j_1)$  and  $(i_2, j_2)$  occurs when  $i_1 < j_1 < j_2 < i_2$ . In Figure 1.2, there is a nesting between  $(1, 4)$ , and  $(2, 3)$ .

Crossing and nesting structures arise in all of the arc annotated sequence representations of matchings, partitions, permutations and graphs, so naturally we wonder about their distributions, and if it is possible to enumerate based on them. Also, if we consider examples 1.1.1, 1.1.2, 1.1.3 and 1.1.4 we realize that often there are multiple arcs crossing and nesting, and it is with this in mind that the following definition is required:

**Definition** A *k-crossing* in an arc annotated sequence is a set of  $k$  arcs that are pairwise crossing. For example, a 3-crossing in the matching  $\{(1, 4), \{2, 5), \{3, 6)\}$  is depicted in Figure 1.3.

Formally, a *k-crossing* is a set of  $k$  arcs  $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$  such that  $i_1 < i_2 < \dots < i_k < j_1 < j_2 < \dots < j_k$ .



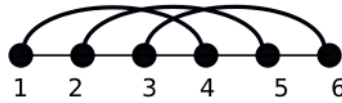


Figure 1.3: A 3-crossing

We should note that a 2-crossing is by definition, equivalent to a crossing. A  $k$ -crossing implies that a sequence has at least  $k$  arcs involved, and because of this we expect  $k$ -crossings with  $k > 2$  will not be as numerous as 2-crossings. We can also point out that a 1-crossing is simply one (mutually crossing) arc, and so any non-empty arc annotated sequence would have a 1-crossing. Any sequence with the empty set of arcs is an example of a 0-crossing.

There is a corresponding nesting statistic called  $k$ -nestings.

**Definition** A  $k$ -nesting in an arc annotated sequence is a set of  $k$  arcs that are pairwise concentric, forming a ‘rainbow-like’ picture. An example of a 3-nesting in the matching  $\{\{1,6\}, \{2,5\}, \{3,4\}\}$  has the following arc annotated sequence:

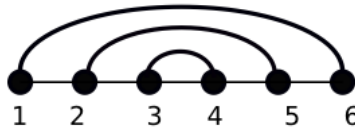


Figure 1.4: A 3-nesting

If we consider Figure 1.4 for a moment, we realize that although it contains a 3-nesting, it also technically contains a 2-nesting between arcs  $\{3,4\}$  and  $\{2,5\}$ . Furthermore, there are actually *three* sets of 2-nestings, and technically 3 sets of 1-nestings as well. Since a maximal condition is not specified in a  $k$ -nesting, it can be useful to consider a sequence with regards to what it does *not* contain.

**Definition** An arc annotated sequence is  $k$ -noncrossing ( $k$ -nonnesting) if none of the arcs form a  $k$ -crossing ( $k$ -nesting).

Although these crossings and nestings are very easy to see in an arc annotated sequence, counting with regards to them or even understanding their distribution proves to be quite

a bit more difficult. Defining a maximality condition for crossings and nestings in the sequence will help. Thus, for matchings  $M \in \mathcal{B}$ , we define  $cr(M)$  to be the maximal  $i$  such that  $M$  has an  $i$ -crossing and  $ne(M)$  to be the maximal  $j$  such that  $M$  has a  $j$ -nesting.

**Theorem 1.2.1.** (Chen et al.) [3] For all  $n, i$  and  $j$ , let  $f_n(i, j)$  be the number of matchings  $M$  on  $\{1, 2, \dots, 2n\}$  with  $cr(M) = i$  and  $ne(M) = j$ .

$$f_n(i, j) = f_n(j, i)$$

This tells us that in matchings,  $k$ -crossings and  $k$ -nestings are *equidistributed*  $\forall k$ .

There is a more general result for set partitions that is proved in [3] as well. One might be tempted to think that such a result is quite natural and perhaps a simple bijection is all that is needed to show this theorem. In reality, it is quite difficult and requires the use of tableaux and the Robinson-Schensted-Knuth algorithm from algebraic combinatorics. Results for the other combinatorial objects require bijections with walks, paths in the lattice, and fillings of Ferrers diagrams which are summarized in Table 3.1. Because of this, in the next section there are many definitions and examples of many combinatorial structures, which are very valuable in Chapters 2 and 3. With all of these combinatorial structures at our disposal, we outline a proof of Theorem 1.2.1 in Chapter 2.1.

# Chapter 2

## Combinatorial structures

Many of the other combinatorial structures we will be dealing with can be grouped into their own families. It is our goal to show that in most cases, each of our four main combinatorial objects can be mapped to one or more members of the same family. We describe the members of these other combinatorial family structures here.

### 2.1 Ferrers diagrams and tableaux

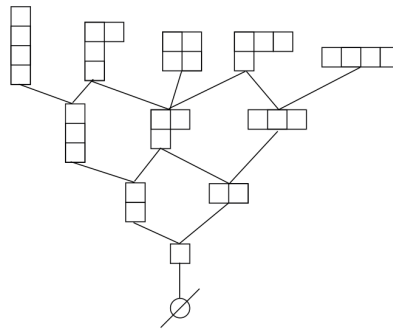
Central to understanding the nesting and crossing distributions in our combinatorial diagrams are *Ferrers diagrams*. They are critical in proving Theorem 1.2.1, as they will be used as elements in tableaux, and in Section 6.2 they will be used to prove equidistributive results independently.

**Definition** An *Ferrers diagram of shape*  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  (also called a Young diagram) is the arrangement of squares from left to right and top to bottom having  $\lambda_i$  cells in row  $i$ . Furthermore, a *filling* of a Ferrers diagram is Ferrers diagram in which each cell contains a nonnegative integer. When 0 is assigned, by convention the cell is left empty.

**Example 2.1.1.** Here is a filling of a Ferrers diagram of shape  $(3, 2)$ : 

There is an ordering that arises regarding the size of Ferrers diagrams, with a Ferrers diagram  $F_1$  that fits inside another Ferrers diagram  $F_2$  having the relation  $F_1 \geq F_2$ , with

$F_1 = F_2$  if they are the same diagram. A general partial ordering on a set can be represented with a *Hasse diagram* where if  $x < y$ , then  $x$  is shown on a lower level than  $y$  and there is path of line segments between them. When Ferrers diagrams are the partially ordered set being represented with a Hasse diagram it is called *Young's lattice*. We show the first five levels of Young's lattice below:



A tableaux is a walk on Young's lattice, with each step taken between comparable diagrams. Because of this, movement within a tableaux is first restricted by moving on the lines between the Ferrers diagrams, and then by further conditions depending on the type of tableaux being defined. We define first vacillating tableaux:

**Definition** A *Vacillating tableau* is a sequence  $\lambda^0, \lambda^1, \dots, \lambda^{2n}$  of Ferrers diagrams. A tableau  $V$  is denoted  $V_\lambda^{2n}$  if it has shape  $\lambda$  and length  $2n$ . It is also required that  $\lambda^0 = \emptyset$ ,  $\lambda^{2n} = \lambda$ ;  $\lambda^{2i+1}$  comes from  $\lambda^{2i}$  by doing nothing or deleting square; and  $\lambda^{2i}$  comes from  $\lambda^{2i-1}$  by doing nothing or adding a square.

**Example 2.1.2.** What follows is an example of a vacillating tableau of length 9 and shape  $\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$ :



By convention, this tableau in shorter notation is called a tableau of length 9 and shape 21 and is represented as follows:

$$\emptyset, \emptyset, 1, \emptyset, 1, 1, 2, 2, 21$$

A list of integers  $a_1, a_2, \dots, a_n$  represents a tableau with  $a_i$  squares in the  $i^{\text{th}}$  row.

Similar to the vacillating tableaux are the hesitating tableaux and oscillating tableaux.

**Definition** A *hesitating tableau* of shape  $\emptyset$  and length  $2n$  is a path on Young’s lattice from  $\emptyset$  to  $\emptyset$  where each step consists of a pair of moves:

1. do nothing and then add a square;
2. remove a square and then do nothing; or
3. add a square and then remove a square.

It should be noted that Example 2.1.2 is **not** a hesitating tableau because although the first step of doing nothing and then adding a square is legal, the next step in which a square is removed and then added is not.

**Example 2.1.3.** *There are 5 hesitating tableaux of shape  $\emptyset$  and length 6. They are:*

$\emptyset, 1, \emptyset, 1, \emptyset, 1, \emptyset$	$\emptyset, 1, \emptyset, \emptyset, 1, \emptyset, \emptyset$	$\emptyset, \emptyset, 1, 11, 1, \emptyset, \emptyset$	$\emptyset, \emptyset, 1, \emptyset, \emptyset, 1, \emptyset$	$\emptyset, \emptyset, 1, 2, 1, \emptyset, \emptyset$
---	---	--	---	---

**Definition** A *oscillating tableau* of shape  $\emptyset$  and length  $2n$  is a sequence  $\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset$  where  $\lambda^i$  is obtained from  $\lambda^{i-1}$  by either adding one square or removing one.

Only the first hesitating tableau in Example 2.1.3 is also considered to be a oscillating tableau.

Recall that each of these tableaux is actually a sequence of Ferrers diagrams, and recall that a filling of a Ferrers diagram is completed by placing integers in each square of the diagram. Considering individual Ferrers diagrams becomes very useful when we consider the ways in which they are filled. These fillings of Ferrers diagrams will indeed be critical in understanding the crossings and nestings in arc annotated sequences.

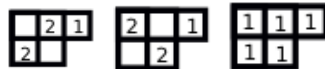
Firstly, note that each filling of a Ferrers diagram can be viewed as numbers in an arrangement of squares, reminiscent of a matrix. From this, the notion of a matrix being contained in a Ferrers diagram follows.

**Definition** A filling of a Ferrers diagram  $T$  contains a  $s \times t$  0–1 matrix  $M$  if there is some selection  $\{r_1, r_2, \dots, r_s\}$  of increasing rows and  $\{c_1, c_2, \dots, c_t\}$  increasing columns such that if  $M_{i,j} = 1$ , then the cell  $(r_i, c_j)$  is nonempty. Otherwise, the filling *avoids*  $M$ .

**Example 2.1.4.** *The  $2 \times 2$  identity matrix,  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  is contained in the Example 2.1.1.*

A further restriction that can be placed on Ferrers diagrams is to stipulate the row and column sums. This is referred to as a Ferrers diagram with *prescribed row and column sums* and is described by two sequences of nonnegative integers  $\rho_i$  and  $\gamma_j$ . Such a diagram is only fillable when the row and column sums are given by  $\rho_i$  and  $\gamma_j$ .

**Example 2.1.5.** *F, the Ferrers diagrams of shape  $\lambda = (3, 2)$  with the prescribed row and column sums  $\rho = (3, 2)$  and  $\gamma = (2, 2, 1)$  has three distinct fillings:*



When prescribed row and column sums are applied to Ferrers diagrams of shape  $\lambda$ , there may be more than one filling possible, as demonstrated in Example 2.1.5. With this in mind, the question can be asked about which matrices are contained and avoided in a Ferrers diagram with a prescribed row and column sum.

**Definition** Two matrices are *equirestrictive* if for all Ferrers diagrams  $T$  with prescribed row and column sums, the number of fillings of  $T$  that avoid one is the same as the number of fillings of  $T$  that avoid the other.

**Example (continued)** *For the example above, consider the  $2 \times 2$  identity and antiidentity matrices:  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . With the prescribed row and column fillings as above, they are both avoided once, and so it follows that they are equirestrictive for all Ferrers diagram of shape  $\lambda = (3, 2)$  with perscribed row and column sums  $\rho = (3, 2)$  and  $\gamma = (2, 2, 1)$ . This notion of being equirestrictive will be of utmost importance in tying together the equidistribution of three of our four families.*

Finally, we define here Young tableaux (*not* Young’s diagram) because they are necessary for performing the Robinson-Schensted-Knuth algorithm, which is critical to the understanding of the major result of Theorem 1.2.1 by Chen et at. [3] regarding the equidistribution of nestings and crossings in matchings and partitions.

*Young tableaux* can be thought of as a subset of fillings of Ferrers diagrams, requiring that the Ferrers diagram be filled with elements that increase weakly along rows and strictly down columns.

**Example 2.1.6.** *What follows is a Young tableau of shape  $\lambda = (3,2)$ . Although the boxes have been drawn in here, for simplicity in the future we will just illustrate the array of numbers.*

1	2	2
2	3	

Finally, one very important tool in the proof by Chen et al. in [3] of Theorem 1.2.1 is the *Robinson-Schensted-Knuth algorithm*, also known as RSK. Although the details of how RSK works are beyond the scope of this project, we are interested in the fact that it establishes the correspondence between permutations and pairs of semi-standard Young tableaux of the same shape, we give a brief description of the algorithm and one example.

The Robinson-Schensted-Knuth algorithm works by the row insertion of a positive integer  $k$  into a standard Young tableau, which recall is an array of numbers weakly increasing along rows and strictly increasing down columns. To do this, if  $k$  is larger than the largest element of the first row,  $k$  is added to the end of the first row. If not, then  $k$  is placed in the first row such that it is still a weakly increasing row. However,  $k$  then ‘bumps out’ the next largest element in that row, say  $k'$ , which is then inserted in the second row. The same process is repeated in the second row with  $k'$  now.

**Example 2.1.7.** *Consider the standard Young tableau  $T =$*

1	2	2	3
2	2	3	3
3	4		
5			

*and insert  $k = 1$ .*

*Then 1 replaces the first 2 in the first row. Next 2 replaces the first 3 in the second row. Then 3 bumps the 4 in the third row, and 4 bumps 5 down to create a 5<sup>th</sup> row, which gives*

1	1	2	3
2	2	2	3
3	3		
4			
5			

*the result:*

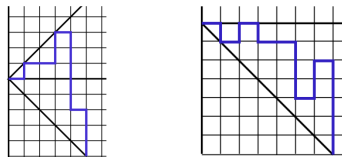
Although the combinatorial structures defined in this section are of great importance in establishing equidistribution for the statistics of crossings and nestings in matchings, partitions, and graphs, they are of limited use for the task of enumerating. It is with this in mind that we turn to the next combinatorial family that lends itself quite nicely to both bijections and enumeration: lattice paths.

## 2.2 Walks in the lattice

A *lattice* is an arrangement of points regularly spaced in the plane, and a walk in the lattice is a sequence of line segments connecting regularly spaced points in an array. The area and the type of step that is allowed may be restricted to give different combinatorial results. We first consider *partially directed walks* in wedges, because they are in bijection with matchings and permutations.

**Definition** A *partially directed self avoiding walk* in the *symmetric wedge* is a walk in the plane that starts at the origin and can take steps either north, south, or east, but a south step may not immediately precede or follow a north step and is confined to the lattice between  $y = \pm x$ . A partially directed walk in the *asymmetric wedge* is confined to the lattice between  $y = 0$  and  $y = -x$ .

**Example 2.2.1.** *The following are partially directed self avoiding walks in the symmetric wedge, and asymmetric wedge respectively.*



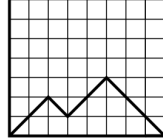
Here, the statistics that are of greatest concern in partially directed self avoiding walks are the number of north steps  $N$ , the number of east steps  $n$ , and the length of the last descent,  $M$ . In Example 2.2.1 with the symmetric wedge,  $n = 5$ ,  $N = 3$  and  $M = 3$ . For the asymmetric wedge,  $n = 7$ ,  $N = 3$  and  $M = 5$ .

The other two main types of lattice walks we use are *Dyck paths* and *Motzkin paths*. Let  $\mathcal{D}_{2n}$  be the set of Dyck paths of length  $2n$  and  $\mathcal{M}_n$  be the set of Motzkin paths of length  $n$ .



**Definition** A *Dyck path* is a lattice path of length  $2m$  in the plane from the origin  $(0,0)$  to  $(2m,0)$  with steps taken from  $\{(1,1), (1,-1)\}$  that stays strictly above but may touch the  $x$ -axis.

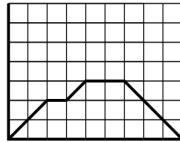
**Example 2.2.2.** The following figure is an example of a Dyck path, from  $\mathcal{D}_8$ ,



The set of Dyck paths, is in fact a subset of Motzkin paths, which are more general paths in the lattice. Motzkin paths also allow a horizontal step east. Dyck paths are simply Motzkin paths in which horizontal steps do not exist. We take advantage of this later for enumerative purposes. Formally:

**Definition** A *Motzkin path* is a lattice path of length  $m$  in the plane from the origin  $(0,0)$  to  $(m,0)$  with steps taken from  $\{(1,1), (1,0), (1,-1)\}$  that stays strictly above but may touch the  $x$ -axis.

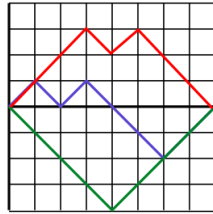
**Example 2.2.3.** The following is an example of a Motzkin path of length  $M_9 \in \mathcal{M}_n$ :



Both Dyck paths and Motzkin paths have been well studied and their enumeration is well known. In fact, Motzkin paths are counted by ‘Motzkin numbers’ and Dyck paths are just one example of the many objects that are counted by Catalan numbers which are defined recursively as  $C_{n+1} = \sum_{i=1}^n C_i C_{n-i}$ , or with the binomial coefficient  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . With this in mind, in order for Dyck and Motzkin paths to be of more use to us we can slightly alter some of their requirements.

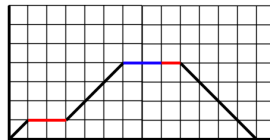
For example, *free* Dyck paths are Dyck paths which relax the restriction of not crossing the  $x$ -axis. Also, if  $A$  and  $B$  are two Dyck paths, they are called *noncrossing* if  $A$  never goes below  $B$ . These two concepts can be combined to form  $k$ -tuples  $(P_1, P_2, \dots, P_k)$  of *noncrossing free* Dyck paths of length  $2n$ , also known as a ‘fan’ of Dyck paths.

**Example 2.2.4.** *The following is an example of a 3-tuple of noncrossing free Dyck paths where  $P_1$  is red,  $P_2$  is blue and  $P_3$  is green.*



Here though, we require more than these simple lattice paths in order to perform some of our bijections. To achieve this, certain steps may be assigned two different colors, making the paths bicolored. Also, we can add a weight vector  $w = (w_1, w_2, \dots, w_{2n})$  to a Motzkin path such that a weight on step  $i$  for an east or south east step may have weight  $w_i$  from 0 to the maximum height at that step,  $h_i$ , minus one. By convention, a step that has no weight assigned to it has a weight of  $w_i = 0$  implied.

**Example 2.2.5.** *A bicoloured weighted Motzkin paths may have two different colored east steps and a weight vector  $w = (0, 0, 0, 0, 0, 0, 0, 1, 2, 0, 3, 2, 1, 0)$*



Lattice paths and tableaux are our primary families, but other structures are useful as well. We describe them here.

### 2.3 Chord diagrams

And finally, we consider circles with  $n$  chords arranged on them. We are particularly interested in this combinatorial structure because a natural crossing structure also occurs.

**Example 2.3.1.** *Consider two arrangements of 3 chords on a circle as seen in Figure 2.1. In each of these examples a natural crossing structure occurs that will be preserved in when it is represented as an arc annotated sequence.*

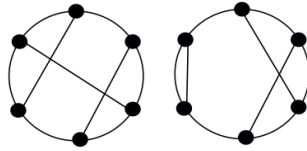


Figure 2.1: Two arrangements of 3 chords on a circle.

Chord diagrams are in bijection with matchings as seen in Chapter 4.

Now with an understanding of these combinatorial objects we may move forward and consider enumerative techniques.

# Chapter 3

## Enumerative techniques

### 3.1 Generating functions

To address the counting problems, we use *generating functions* whose coefficients are the number of objects of a certain size in a class. We define  $\mathcal{A}$  to be a class of objects where  $A_n$  is the number of objects in  $\mathcal{A}$  that are of size  $n$ . Then  $A_n$  is a sequence and can be represented by a formal power series, which is called its *generating function*,  $A(z)$ . Formally, the *ordinary* generating function of a sequence is the formal power series:

$$A(z) = \sum_{n \geq 0} A_n z^n,$$

while the *exponential* generating function is the formal power series:

$$A(z) = \sum_{n \geq 0} \frac{A_n z^n}{n!}.$$

Exponential generating functions are most often used in the labeled universe.

What follows is two example of combinatorial objects that are both very important in this thesis, one being counted by an ordinary generating functions and the other by an exponential generating function.

**Example 3.1.1.** *Consider the set of Dyck paths. As already stated, it is well known that Dyck paths are counted using Catalan numbers. We let  $\mathcal{D}$  be the class of Dyck paths, and*

$D_n$  be the number of Dyck paths of length  $2n$ . Then, the coefficient of  $z^n$  in  $D(z)$ ,  $[z^{2n}]D(z)$  is the number Dyck paths of length  $2n$ . The ordinary generating function for Dyck paths is:

$$D(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z} = 1 + z^2 + 2z^4 + 5z^6 + 14z^8 + 42z^{10} + 132z^{12} \dots$$

and so,

$$D_n = \frac{1}{2n+1} \binom{2n}{n}.$$

(A000108 [23])

Indeed, we will be counting according to both the length of an arc annotated sequence, and according to the nesting and crossing statistics. In order to do this, we will require the use of *multivariate* generating functions that do in fact keep track of more than one parameter. If only two parameters are being used, such a generating function is called *bivariate*.

Let  $a_{n,k}$  be the number of objects in some combinatorial class  $\mathcal{A}$  such that one parameter is equal to  $n$  (usually the size) and another is equal to  $k$ . Then the ordinary bivariate generating function of  $A(z, u)$  is:

$$A(z, u) = \sum_{n,k} f_{n,k} z^n u^k,$$

### 3.2 Continued fractions

Many of our generating functions can be expressed as continued fractions. Flajolet [12] illustrated a simple way to express Motzkin paths with continued fractions, and this turns out to be extremely useful for computing the initial terms of generating series for several of our classes.

$$\frac{a_0}{b_0 - \frac{a_1}{b_1 - \frac{a_2}{b_2 - \frac{a_3}{\ddots}}}}$$

Here is his construction for the class of Motzkin paths,  $\mathcal{M}$ .

**Example 3.2.1.** Let  $\mathcal{M}_n$  be the class of Motzkin paths of length  $n$ . The continued fraction

representation of  $M$  will be built by assigning the variable  $a_i$  to all north east steps of starting height  $i$ ,  $b_i$  to all south east steps of starting height  $i$  and  $c_i$  to all east steps of height  $i$ .

If we restrict the maximum height to be 0, we call this class  $\mathcal{M}_n^0$ . Such a path is simply a set of east steps at height 0, therefore:

$$M_n^0 = \frac{1}{1 - c_0} = 1 + c_0 + c_0^2 + c_0^3 + \dots$$

Let  $\mathcal{M}_n^1$  be the set of Motzkin paths of maximum height 1. These are made up with only east steps at height 0,  $c_0$ , or there can be a north step followed by some east steps and a down step.

$$c_0 \rightarrow c_0 + \frac{a_0 b_1}{1 - c_1}$$

therefore:

$$M_n^1 = \frac{1}{1 - c_0 - \frac{a_0 b_1}{1 - c_1}}.$$

Using this method, Motzkin paths of maximum height  $h$  are counted by:

$$M_n^h = \frac{1}{1 - c_0 - \frac{a_0 b_1}{1 - c_1 - \frac{a_1 b_2}{1 - c_{h-1} - \frac{\ddots}{1 - c_h}}}}$$

$$M(a, b, c; z) = 1 + cz + (c^2 + ab)z + \dots$$

From this, the  $\mathcal{M}_n$  can be counted using an *infinite* continued fraction. Each step adds to the length which is counted by  $z$  and so each  $a_i, b_i, c_i \rightarrow z$  to get the continued fraction generating function. Although computers can not handle these infinite continued fractions, for practical enumeration computation purposes, a high maximum height is sufficient, and in fact we use these continued fraction expansions to give a single multivariate generating function for matchings, partitions and permutations according to both length and number of nestings.

Table 3.1 gives a summary of all the bijections in this thesis and where they are located. On the following page in Table 3.2 each combinatorial object is explained.

Bijection name	Object 1	Object 2	Page number
$\Phi_1$	$\mathcal{B}_{2n}^{<nc>}$	$\mathcal{D}_{2n}$	24
$\Phi_2$	$\mathcal{B}_{2n}^{<nm>}$	$\mathcal{D}_{2n}$	25
$\Phi_3$	$\mathcal{B}_{2n}^{<nm>}$	$\mathcal{B}_{2n}^{<nc>}$	25
$\Phi_4$	$\mathcal{B}_{2n}$	$\mathcal{O}S_0$	27
$\Phi_5$	$\mathcal{B}_{2n}$	$\mathcal{D}_{2n}^{<w1>}$	29
$\Phi_6$	$\mathcal{D}_{2n}^{N,M}$	$\mathcal{W}_n^{(N,M)}$	33
$\Phi_7$	$\mathcal{B}_{2n}$	$\mathcal{C}_n$	39
$\Phi_8$	$\mathcal{P}_n^{<nm>}$	$\mathcal{B}_{2n}^{<nm>}$	50
$\Phi_9$	$\mathcal{P}_n$	$\mathcal{V}_\lambda^{2n}$	52
$\Phi_{10}$	$\mathcal{P}_n^{en}$	$\mathcal{H}$	58
$\Phi_{11}$	$\mathcal{P}_n^{<nc>}$	$\mathcal{D}_{2n}^{**}$	58
$\Phi_{12}$	$\mathcal{P}_n$	$\mathcal{M}_n^{<w1>}$	61
$\Phi_{13}$	$\mathcal{G}_n$	$\mathcal{F}_\lambda$	72
$\Phi_{14a}$	$\mathcal{S}_n$	$\mathcal{M}_n^{<w3>}$	84
$\Phi_{14b}$	$\mathcal{S}_n$	$\mathcal{M}_n^{<w4>}$	85
$\Phi_{15}$	$\mathcal{B}_{n,k,l,m}$	$\mathcal{D}_{n,k,l,m}$	88
$\Phi_{16}$	$\mathcal{S}_n^{N,M}$	$\mathcal{W}_n^{<N,M>}$	91

Table 3.1: All bijections

Object	Description
Main objects:	(with variations)
$\mathcal{B}_{2n}$ $\mathcal{B}_{2n}^{<nm>}$ $\mathcal{B}_{2n}^{<nc>}$ $\mathcal{P}_n$ $\mathcal{P}_n^{en}$ $\mathcal{P}_n^{<nm>}$ $\mathcal{P}_n^{<nc>}$ $\mathcal{G}_n$ $\mathcal{E}_n$ $B(n, k, l, m)$ $D(n, k, l, m)$	matchings of $\{1, 2, \dots, 2n\}$ nonnesting matchings of $\{1, 2, \dots, 2n\}$ noncrossing matchings of $\{1, 2, \dots, 2n\}$ set partitions of $\{1, 2, \dots, n\}$ set partitions of $\{1, 2, \dots, n\}$ with enhanced crossings and nestings nonnesting set partitions of $\{1, 2, \dots, n\}$ noncrossing set partitions of $\{1, 2, \dots, n\}$ graphs on $n$ vertices with no singletons, multiple edges allowed. permutations of $\{1, 2, \dots, n\}$ permutations of $\{1, 2, \dots, n\}$ with $k$ weak exceedances, $l$ crossings, and $m$ nestings. permutations of $\{1, 2, \dots, n\}$ with $n - k$ descents, $l$ occurrences of the pattern $2 - 3 1$ and $m$ of the pattern $3 1 - 2$ .
Tableaux:	
$\mathcal{V}_\lambda^{2n}$ $\mathcal{H}$ $\mathcal{O}_{s_\lambda}$	vacillating tableaux of length $2n$ with shape $\lambda$ hesitating tableaux oscillating tableaux
Walks in the lattice:	
$\mathcal{D}_{2n}$ $\mathcal{D}_{2n}^{<w1>}$ $\mathcal{D}_{2n}^{**}$ $\mathcal{D}_{2n}^{N, M}$ $\mathcal{M}_{2n}^{<wi>}$ $\mathcal{W}_n^{(N, M)}$ $\mathcal{W}_n^{<N, M>}$	Dyck paths of length $2n$ . weighted Dyck paths of length $2n$ with weight vector $w1$ . pairs of non crossing Dyck paths of length $2n$ . Dyck paths of length $2n$ with total weight $N$ and $M$ the position of the first south step with weight 0. bicolored weighted Motzkin paths of length $n$ and weight vector $wi$ . partially directed walks in the symmetric wedge with $N$ north steps, $n$ east steps and length of the last descent equal to $M - 1$ . partially directed walks in the asymmetric wedge with $N$ north steps, $n$ east steps and length of the last descent equal to $M$ .
$\mathcal{F}_\lambda$	fillings of Ferrers diagrams of shape $\lambda$ .
$\mathcal{C}_n$	circles with $n$ chords placed on them.

Table 3.2: A description of all objects.



## **Part II**

# **Families of arc annotated sequences**

# Chapter 4

## Matchings

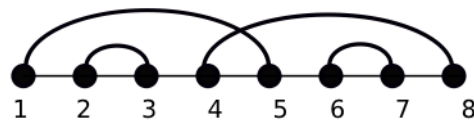
### 4.1 Representation as an arc annotated sequence

A *matching* on the set  $\{1, 2, \dots, 2n\}$  is the partition of the set into blocks of size two. Let  $\mathcal{B}_{2n}$  be the set of matchings on  $\{1, 2, \dots, 2n\}$ . We give the notation  $\{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_n, b_n\}\}$  for a matching where each  $a_i, b_i \in \{1, 2, \dots, 2n\}$ .

A matching is represented as an arc annotated sequence of length  $2n$  where each vertex has exactly one arc extending from it. An arc connects the two elements in a common block, with the smaller vertex  $a_i$  being a left hand point, also called an ‘opener’, and the larger vertex in the block  $b_i$  being a right hand end point, also called a ‘closer.’

Crossing and nesting structures follow the definitions given in Section 1.2.

**Example 4.1.1.** Let  $M = \{\{1, 5\}, \{2, 3\}, \{4, 8\}, \{6, 7\}\}$  be a matching.  $M$  is depicted with an arc annotated sequence as follows:



This example includes a crossing between arcs  $\{1, 5\}$  and  $\{4, 8\}$ , and two nestings:  $\{1, 5\}$  with  $\{2, 3\}$  and  $\{4, 8\}$  with  $\{6, 7\}$ . This matching  $M$  is a 3-noncrossing and a 3-nonnesting, and by extension for  $k \geq 3$  it is a  $k$ -noncrossing and a  $k$ -nonnesting. It follows that  $cr(M) = 2$  and  $ne(M) = 2$ .

The number of matchings on the set  $\{1, 2, \dots, 2n\}$ ,  $B_{2n}$ , is well known to be  $B_{2n} = (2n - 1)!! = (2n - 1)(2n - 3) \dots (5)(3)(1)$ . This is Sequence A001147 in the On-line Encyclopedia of Integer Sequences [23]. We would like to enumerate based on the number of crossings and nestings. Without going into detail, we give a table for the number of nestings and crossings of matchings of length 4, 6, 8 and 10.

$n$	Number of crossings	Number of nestings
2	0	0
4	1	1
8	15	15
10	214	214

Table 4.1: Total number of nestings and crossings for matchings of size  $n$ .

Based on this table, we have support for equidistribution of crossings and nestings over matchings on  $\{1, 2, \dots, n\}$ . We now turn to bijections that prove this, first for noncrossing and nonnesting matchings, and then for matchings in general.

## 4.2 Distribution of nestings and crossings

We first examine a well known bijection  $\Phi_1$  between  $\mathcal{B}_{2n}^{<nc>}$ , noncrossing matchings on  $\{1, 2, \dots, 2n\}$  and  $\mathcal{D}_{2n}$ , Dyck paths of length  $2n$ . There exists a very similar bijection  $\Phi_2$  between nonnesting matchings on  $\{1, 2, \dots, 2n\}$ , so we will then be able to give a bijection  $\Phi_3$  between noncrossing and nonnesting matchings on  $\{1, 2, \dots, 2n\}$ .

**Bijection  $\Phi_1$**  (Well known result)





- $\mathcal{B}_{2n}^{<nc>}$ : noncrossing matchings on  $\{1, 2, \dots, 2n\}$
- $\mathcal{D}_{2n}$ : Dyck paths of length  $2n$ .

**Example 4.2.1.** The noncrossing matching  $B = \{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$  represented as an arc annotated sequence and image under  $\Phi_1$  follow:



**Steps**

1. Traverse the arc annotated path from left to right
2. Generate the Dyck path with the following dictionary:

**Theorem 4.2.1.** *The map  $\Phi_1$  is a bijection from noncrossing matchings on  $\{1, 2, \dots, 2n\}$  to Dyck paths of length  $2n$ .*

*Proof.* The proof is an immediate consequence of the construction. For each opener, there is a north east step in the Dyck path, for each closer there is a south east step. This is a one to one and onto mapping, and there is exactly one way in which openers and closers are connected such that no crossings occur. Also, we can note that both noncrossing matchings and Dyck paths are counted by Catalan numbers,  $C_n = \frac{1}{n+1} \binom{2n}{n}$ .

□

**Remark:** Always close the most recently opened arc.

**Bijection  $\Phi_2$  (Folklore)**

- $\mathcal{B}_{2n}^{<nn>}$ : nonnesting matchings on  $\{1, 2, \dots, 2n\}$
- $\mathcal{D}_{2n}$ : Dyck paths of length  $2n$ .

**Steps**

1. Proceed as in  $\Phi_1$ .

**Theorem 4.2.2.** *The mapping  $\Phi_2$  is a bijection between nonnesting matchings on  $\{1, 2, \dots, 2n\}$  and Dyck paths of length  $2n$ .*

*Proof.* There is exactly one way in which openers and closer can be connected such that the matchings is nonnesting. Thus, this is a one to one and onto correspondence.  $\square$

**Remark** Always close least recently opened arc.

**Bijection  $\Phi_3$**  (Folklore)

- $\mathcal{B}_{2n}^{<nm>}$ : nonnesting matchings on  $\{1, 2, \dots, 2n\}$
- $\mathcal{B}_{2n}^{<nc>}$ : noncrossing matchings on  $\{1, 2, \dots, 2n\}$

**Example 4.2.2.** *The matching  $\{\{1, 3\}, \{2, 5\}, \{4, 6\}\}$  is in bijection with  $\{\{1, 6\}, \{2, 3\}, \{4, 5\}\}$  under  $\Phi_3$ :*



**Steps**

1. Disconnect all arcs of the nonnesting (noncrossing) matching.
2. Reconnect all arcs in the only possible way such that the new matching is noncrossing (nonnesting).

**Theorem 4.2.3.** *Based on bijection  $\Phi_3$ , the number of noncrossing matchings and nonnesting matchings on  $\{1, 2, \dots, 2n\}$  are equal for all  $n$ .*

*Proof.* This is a direct consequence of bijection  $\Phi_3$ , and also  $\Phi_1$  and  $\Phi_2$  which gave the counting sequence for noncrossing and nonnesting matchings to be the  $n^{th}$  Catalan number.  $\square$

In her Ph.D. thesis, M. de Sainte-Catherine [8] showed that the distribution of 2-crossings and 2-nestings in matchings was the same. A generalized result, the main result from Chen et al.’s ‘Crossings and nestings of matchings and partitions’ [3], shows that crossing and nesting statistics are equidistributed in matchings.

**Theorem 4.2.4.** *(Chen et. al. [3]) The number of matchings  $M$  on  $\{1, 2, \dots, 2n\}$  where  $cr(M) = i$  and  $ne(M) = j$  equals the number of matchings  $M$  on  $\{1, 2, \dots, 2n\}$  where  $cr(M) = j$  and  $ne(M) = i$ .*

As a result of Theorem 4.2.4, we get:

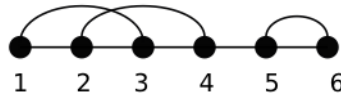
**Theorem 4.2.5.** (Chen et al. [3]) *The number of matchings  $M$  on  $\{1, 2, \dots, 2n\}$  with  $cr(M) = k$  is equal to the number of matchings  $M$  on  $\{1, 2, \dots, 2n\}$  with  $ne(M) = k$ .*

In order to prove this, a bijection  $\Phi_4$  between matchings on  $\{1, 2, \dots, 2n\}$  and oscillating tableaux,  $\mathcal{O}_s$  must first be described. Here we only show  $\Phi_4 : \mathcal{B}_{2n} \rightarrow \mathcal{O}_s$ , and leave the more complicated  $\Phi_4 : \mathcal{O}_s \rightarrow \mathcal{B}_{2n}$  because it is a simplified version of a similar bijection  $\Phi_{12}$  done between set partitions and vacillating tableaux.

**Bijection  $\Phi_4$**  (Chen et al. [3], Sundaram [25])

- $\mathcal{B}_{2n}$ : matchings on  $\{1, 2, \dots, 2n\}$ ,
- $\mathcal{O}_s$ : oscillating tableaux.

**Example** Consider the matching  $B = \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$  and its arc annotated sequence representation:



Under  $\Phi_4$ , the matching  $B$  becomes:  $\emptyset, 1, 11, 1, \emptyset, 1, \emptyset$ .

**Steps** For the direction  $\Phi_4 : \mathcal{B}_{2n} \rightarrow \mathcal{O}_s$ :

1. First we make a series of Young tableaux,  $T_i$  from the arc annotated sequence.
2. Let  $T_{2n} = \emptyset$ ,
3. Construct  $T_j$  from  $T_{j+1}$ , from right to left.
4. If  $j$  is a closer of an arc  $(i, j)$ , insert  $i$  by the RSK algorithm into the tableau.
5. If  $j$  is an opener or an arc  $(j, k)$ , remove  $j$  by the RSK algorithm from the tableau.
6. Repeat until  $T_j$  is defined for all  $j$

7. Take the sequence of shapes of the Young tableaux to be the oscillating tableaux.

**Theorem 4.2.6.** *The map  $\Phi_4$  is a bijection from matchings on  $\{1, 2, \dots, 2n\}$  to oscillating tableaux.*

*Proof.* Here we do not give the other direction because this is a simplification of a more general bijection,  $\Phi_{12}$  that is proven in Section 5.2. □

**Example (continued)** *We flesh out the details of Example 4.2 done above:*

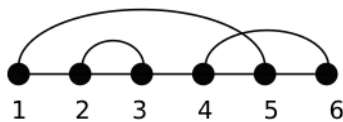
*We construct the standard Young tableau. First we start with  $T_{2n} = \emptyset$ . Then we consider vertex 6 and see that it is a closer to arc  $\{5, 6\}$  and so insert 5 into the tableau. Next, we see that 5 the corresponding closer to arc  $\{5, 6\}$  and so 5 is deleted. At this stage we have  $T_4, T_5, T_6 = \emptyset, 5, \emptyset$ . Then we examine vertex 4 which is a closer to arc  $\{2, 4\}$ , so 2 is inserted. Similarly, 1 is inserted, then 2 is deleted, and finally 1 is deleted. Our standard Young tableau is:*

$$\emptyset, 1, \frac{1}{2}, 2, \emptyset, 5, \emptyset.$$

*From this we get the corresponding oscillating tableaux is:*

$$\emptyset, 1, 11, 1, \emptyset, 1, \emptyset$$

**Example 4.2.3.** *Consider the matching  $\{\{1, 5\}, \{2, 3\}, \{4, 6\}\}$  and its arc annotated sequence:*



*This is represented by first the standard Young tableau and then the corresponding oscillating tableau:*

$$\emptyset, 1, 12, 1, \frac{1}{4}, 4, \emptyset$$

$$\emptyset, 1, 2, 1, 11, 1, \emptyset$$

Through these examples we gain the intuition that a nesting is indicated by a row of length greater than 1 and a crossing is indicated by column of height greater than 1. A more generalized version of this bijection,  $\Phi_{12}$  will be outlined in Section 5.2 which will lead to the proof of the equidistribution between crossing and nesting statistics in matchings.

### 4.3 Further bijective results

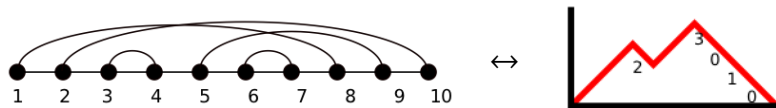
Because of bijections  $\Phi_1, \Phi_2$  between noncrossing (nonnesting) matchings, the next question that arises becomes: *Is there a bijection between  $\mathcal{B}_{2n}$ , matchings on  $\{1, 2, \dots, 2n\}$  and Dyck paths of length  $2n$ ,  $\mathcal{D}_{2n}$ ?* Clearly the rules would have to change.

Indeed, in [21], Rubey gives consequences of a bijection done by Kasraoui and Zeng [16] which is a variation on bijections done by Flajolet [12] and Viennot [27] between  $\mathcal{B}_{2n}$  and *weighted* Dyck paths. The bijection  $\Phi_5$ , described below, gives a nice correspondence with nestings and crossings in the original arc annotated diagram.

**Bijection  $\Phi_5$**  ( Kasraoui-Zeng[16], Flajolet [12], Rubey [21], Viennot [27])

- $\mathcal{B}_{2n}$ : matchings on  $\{1, 2, \dots, 2n\}$
- $\mathcal{D}_{2n}^{<w_1>}$ : Dyck paths of length  $n$  with weight vector  $w_1 = (w_1, w_2, \dots, w_{2n})$  where each  $w_i < h_i$ , the height at step  $i$ .

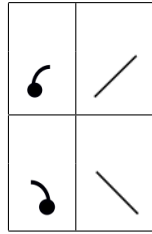
**Example 4.3.1.** *The matching  $B = \{\{1, 8\}, \{2, 10\}, \{3, 4\}, \{5, 9\}, \{6, 7\}\}$ , its arc annotated sequence and its image under  $\Phi_5$  follows:*



#### Steps

1. Traverse the arc annotated path from left to right
2. Generate the Dyck path with the following dictionary:





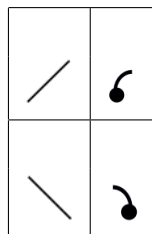
3. Generate the weight vector  $w = (w_1, w_2, \dots, w_{2n})$  as follows:
  - (a) Assign weight  $w_i = 0$  to all  $\diagup$  steps.
  - (b) Let  $j$  be left most available closer in the arc annotated sequence.
  - (c) Label all available openers to its left with labels  $\{0, 1, \dots, j-1\}$
  - (d) If  $k$  is the vertex that  $j$  is connected to on the left and  $k$  has label  $a_k$ , assign  $j$  weight  $w_j = a_k$ .
  - (e) Now  $k$  and  $j$  are considered unavailable.
  - (f) Repeat for the next available left most closer vertex until all steps are weighted.

**Theorem 4.3.1.** *The map  $\Phi_5$  is a bijection from the arc annotated sequence of matchings on  $\{1, 2, \dots, 2n\}$  to weighted Dyck paths of length  $2n$ .*

*Proof.* We show the inverse direction,  $\Phi_5 : \mathcal{D}_{2n}^{<w1>} \rightarrow \mathcal{B}_{2n}$ .

**Steps**

1. Traverse the Dyck path  $D \in \mathcal{D}_{2n}^{<w1>}$  left to right
2. Generate the arc annotated sequence with the following dictionary, leaving the arcs unconnected:

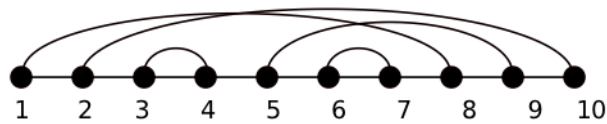


3. Take the left most closer  $j$ ; consider step  $j$  in  $D_{2n}$  with weight  $w_j$ .

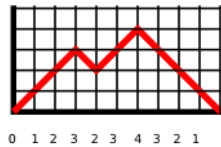
4. Label every available opener from left to right with labels  $0, 1, \dots, w_j, \dots, h_j - 1$  where  $h_j$  is the height of the corresponding step in the Dyck path.
5. Connect  $j$  to the opener with label  $w_j$ .
6. Now connected vertices are considered unavailable. Repeat.

□

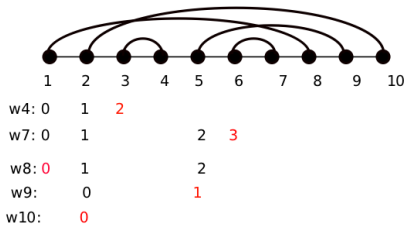
**Example 4.3.2.** Consider that matching  $\{\{1, 8\}, \{2, 10\}, \{3, 4\}, \{5, 9\}, \{6, 7\}\}$  which is represented with an arc annotated sequence as follows:



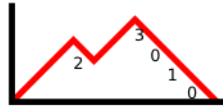
The shape of part of the Dyck path with the possible heights listed along the bottom gives:



Based on these heights, the possible values of weights can be assigned. For example,  $w_4$  could be 0, 1 or 2 because its height at step 4 is 3. What follows is the arc annotated sequence of the matching with the corresponding weight possibilities underneath for the right hand end points. The corresponding left hand end point gives its weight which is shown in red.



Finally, the selected weights are assigned to the corresponding south steps in the Dyck path to get the final weighted Dyck path for the matching.



To illustrate how important the weighting is to a successful bijection, consider the following two matchings, that both have the same Dyck shape, but different weightings in order to understand what a crossing and a nesting looks like in its Dyck path incarnation.

**Example 4.3.3.** Let  $A = \{\{1, 4\}, \{2, 5\}, \{3, 6\}, \{7, 9\}, \{8, 10\}, \{11, 13\}, \{12, 15\}, \{14, 16\}\}$  and  $B = \{\{1, 6\}, \{2, 5\}, \{3, 4\}, \{7, 10\}, \{8, 9\}, \{11, 16\}, \{12, 13\}, \{14, 15\}\}$  be two matchings on  $\{1, 2, \dots, 16\}$ .



Figure 4.1: The arc annotated sequence for  $A$ .



Figure 4.2: The arc annotated sequence for  $B$

Although their arc annotated sequences are different, as can be seen below, the shape of their Dyck paths is the same.

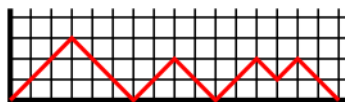


Figure 4.3: The Dyck path corresponding to matchings  $A$  and  $B$ .

It is only with adding the weights that a difference is found.

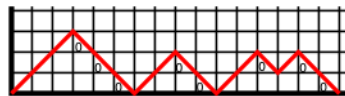


Figure 4.4: The weighted Dyck path corresponding to matching  $A$ .

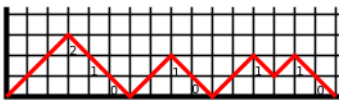


Figure 4.5: The weighted Dyck path corresponding to matching B.

From this we get the intuition that a nesting occurs whenever a weighting  $w_i$  is assigned greater than 0 and a crossing occurs whenever a weighting  $w_i$  is assigned that is less than  $h_i - 1$ , the starting height of that particular step. Clearly the easier statistic to spot under  $\Phi_5$  is a nesting, which indeed is found wherever a weighting  $w_i > 0$  is assigned. Fortunately, due to the work by Chen et al. in [3], if we can enumerate matchings according to both length and the number of nestings, the result will hold for crossings.

Before moving on to other bijective results, we summarize 4 of the bijections by showing each of the 15 matchings on  $\{1, 2, 3, 4, 5, 6\}$  under  $\Phi_1, \Phi_2, \Phi_4, \Phi_5$  where applicable. See table.

Recall that a partially directed self avoiding walk is a walk in the lattice in which north, east and south steps are allowed, with the condition that north and south steps may not be consecutive. Further recall that the symmetric wedge is defined between the lines  $y = \pm x$ . In his paper, ‘Nestings of matchings and permutations and north steps in PDSAWs’, [21] Rubey gives a bijection between matchings and partially directed self avoiding walks that end with an east step in the symmetric wedge according to three parameters.

Let  $n$  be the number of east steps,  $N$  be the number of north steps, and  $M$  be the altitude of the last east step. Once we realize that a walk is entirely determined by the heights of the east steps, it is clear that the number of such walks with  $n$  east steps is  $a_n = (2n - 1)!!$  which is also the number of matchings on  $\{1, 2, \dots, 2n\}$ . From this, we look for a bijection between partially directed self avoiding walks in the symmetric plane with  $n$  east steps and matchings of  $\{1, 2, \dots, 2n\}$ .

#### **Bijection $\Phi_6$ (Rubey [21])**

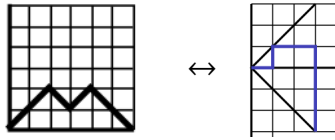
- $\mathcal{W}_n^{(NM)}$ : Partially directed walks in the symmetric wedge with  $N$  north steps,  $n$  east steps and length of the last descent equal to  $M - 1$ .
- $\mathcal{D}_{2n}^{N,M}$ : Dyck paths of length  $2n$  with total weight  $N$  and  $M$  the positions of the first down step with weight 0.

Type	Arc annotated sequence	$\Phi_1$	$\Phi_2$	$\Phi_4$	$\Phi_5$
nonnesting	<p>1 2 3 4 5 6</p>			$\emptyset, 1, \emptyset, 1, \emptyset, 1, \emptyset$	<p><math>w = (0, 0, 0, 0, 0, 0)</math></p>
	<p>1 2 3 4 5 6</p>			$\emptyset, 1, \emptyset, 1, 11, 1, \emptyset$	<p><math>w = (0, 0, 0, 0, 0, 0)</math></p>
	<p>1 2 3 4 5 6</p>			$\emptyset, 1, 11, 1, \emptyset, 1, \emptyset$	<p><math>w = (0, 0, 0, 0, 0, 0)</math></p>
	<p>1 2 3 4 5 6</p>			$\emptyset, 1, 11, 1111, 11, 1, \emptyset$	<p><math>w = (0, 0, 0, 0, 0, 0)</math></p>
	<p>1 2 3 4 5 6</p>			$\emptyset, 1, 11, 1, 11, 1, \emptyset$	<p><math>w = (0, 0, 0, 0, 0, 0)</math></p>

Type	Arc annotated sequence	$\Phi_1$	$\Phi_2$	$\Phi_4$	$\Phi_5$
1 nesting				$\emptyset, 1, 2, 1, \emptyset, 1, \emptyset$	
				$\emptyset, 1, 2, 1, 11, 1, \emptyset$	
				$\emptyset, 1, 11, 1, 2, 1, \emptyset$	
				$\emptyset, 1, \emptyset, 1, 2, 1, \emptyset$	
				$\emptyset, 1, 11, 21, 2, 1, \emptyset$	
				$\emptyset, 1, 2, 21, 11, 1, \emptyset$	

Type	Arc annotated sequence	$\Phi_1$	$\Phi_2$	$\Phi_4$	$\Phi_5$
2 nestings				$\emptyset, 1, 11, 21, 11, 1, \emptyset$	 $w = (0, 0, 0, 2, 0, 0)$
				$\emptyset, 1, 2, 1, 2, 1, \emptyset$	 $w = (0, 0, 1, 1, 0, 0)$
				$\emptyset, 1, 2, 21, 2, 1, \emptyset$	 $w = (0, 0, 1, 1, 0, 0)$
3 nestings				$\emptyset, 1, 2, 3, 2, 1, \emptyset$	 $w = (0, 0, 0, 2, 1, 0)$

**Example 4.3.4.** Consider the following partially directed self avoiding walk  $W$  and its Dyck path under  $\Phi_6$ :














The weight vector for the Dyck path is  $(0, 0, 1, 0, 0, 0)$ .

**Steps**

1. The Dyck path is built recursively following a series of steps.
2. Define  $D^i$  to be the Dyck path at iteration  $i$ .
3. Start with the partially directed self avoiding walk (PDSAW) with one east step. This corresponds to the first iteration of the Dyck path,  $D^1$ , the only Dyck path of length 2 ( $\nearrow \searrow$ ).
4. Consider the next east step in the PDSAW. Prepend the step  $\nearrow \searrow$  to  $D^1$ .
5. If the second east step is at altitude  $h_2$ , perform the following set of steps  $h_2 - 1$  times.



	$D^i$	$D^{i+1}$
I	 (0,0,0)	 (0,0,0)
II	 (0,0,k)	 (k,0,0)
III	 (0,0,0)	 (0,1,0)
IV	 (l,0,k)	 (l,k+1,0)
V	 (k,0,0)	 (0,k+1,0)

6. Prepend  to the Dyck path of length 4 that resulted from the above table.
7. If the third east step is at altitude  $h_3$ , perform the steps above  $h_3$  times.
8. Repeat for all east steps.

**Theorem 4.3.2.** *The mapping  $\Phi_6$  is a bijection between partially directed walks in the symmetric wedge with  $N$  north steps,  $n$  east steps and length of the last descent equal to  $M - 1$  and Dyck paths of length  $2n$  with total weight  $N$  and  $M$  the position of the first down step of with weight 0.*

*Proof.* We direct the reader to [21]. □

**Example 4.3.4 (continued)** *We start with the PDSAW with 1 east step,  $W^1$  and  $D^1$  is the only Dyck path of length 2. Next we consider the PDSAW with 2 east steps,  $P^2$ , the second*

being at altitude 1.  $D^1$  gets  $\diagup \diagdown$  prepended to become  $D^2$ . The second east step in the original  $W$  has altitude 3, thus we perform two iterations of steps from the table.  $D^2$  has form I, so we perform I to get  $D^3$ .  $D^3$  is of the form III so we perform III to get  $D^4$ .

Then we have performed 2 iterations, so we prepend  $\diagup \diagdown$  to  $D^4$  to get  $D^5$ . Because the third east step in the original  $W$  has altitude 4, we perform 3 iterations of the table. First we perform I to get  $D^6$ , and the I again to get  $D^7$ , and finally our last iteration uses II and gives  $D^8$  which is the weighted Dyck path corresponding to the original PDSAW  $W$ . We summarize this in Table 4.2:

For the final bijection with matchings we examine entry A067311 in the On-line Encyclopedia of Integer Sequences [23] which counts the number of  $n$  chord arrangements on a circle according to simple crossings. This sequence is

$$1, 1, 1, 2, 1, 5, 6, 3, 1, 14, 28, 28, 20, 4, 1 \dots$$

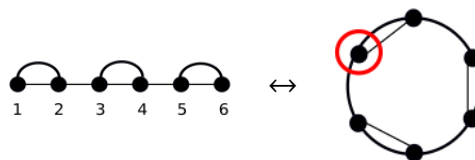
which is later shown to be the same as our sequence for the number of matchings on  $\{1, 2, \dots, 2n\}$  with  $k$  2-nestings (it is a triangular sequence), see Table 4.4.

We immediately suspect that there is a bijection  $\Phi_{10}$  from these circular chord arrangements and our matchings with  $k$  nestings. Due to Chen et al. we know that crossings and nestings are equidistributed and therefore we may consider crossings directly here.

**Bijection  $\Phi_7$**

- $\mathcal{B}_{2n}$ : matchings on  $\{1, 2, \dots, 2n\}$ .
- $\mathcal{C}_n$ : arrangements of  $n$  chords on a circle.

**Example 4.3.5.** Let  $M =$  be a matching on  $\{1, 2, 3, 4, 5, 6\}$ , represented first as an arc annotated sequence and then as 3 chords placed in a circle.




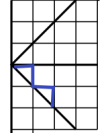
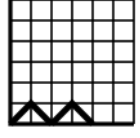
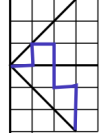
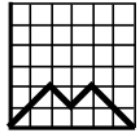
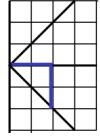
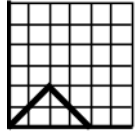
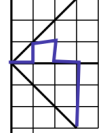
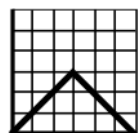
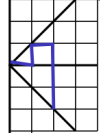
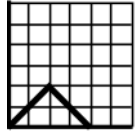
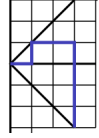
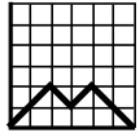
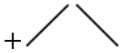
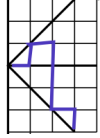
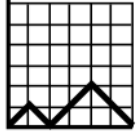
Step	$i$	$W^i$	$D^i$	Step	$i$	$W^i$	$D^i$
	2		 (0,0,0,0)	I	6		 (0,0,0,0,1,0)
I	3		 (0,0,0,0)	I	7		 (0,0,0,0,1,0)
III	4		 (0,0,1,0)	II	8		 (0,0,1,0,0,0)
	5		 (0,0,0,0,1,0)				

Table 4.2:  $\Phi_6 : \mathcal{W}_6^{(1,5)} \rightarrow \mathcal{D}_3^{1,5}$

**Steps**

1. For  $\mathcal{B}_{2n} \rightarrow \mathcal{C}_n$  simply extend the horizontal line from 1 around to the largest vertex and rearrange so a circle is formed.
2. For  $\mathcal{C}_n \rightarrow \mathcal{B}_{2n}$ , choose a particular endpoint of a chord, for example the top left most vertex, give it a label 1, and assign the labels  $\{2, 3, \dots, 2n\}$  in a clockwise direction to the rest of the chord endpoints.

3. Split the circle between endpoint 1 and endpoint  $2n$  and rearrange so that a horizontal line with arcs above the line is formed.

Note that this labeling must occur on the circles otherwise isomorphisms occur.

**Theorem 4.3.3.** *The mapping  $\Phi_7$  is a bijection between matchings on  $\{1, 2, \dots, 2n\}$  and the placement of  $n$  chords on a circle.*

*Proof.* Both combinatorial objects have the same counting sequence A067311. Both directions of the bijection are already given in the Steps.  $\square$

We illustrate the correspondence between matchings on  $\{1, 2, \dots, 2n\}$  and  $n$  chords on a circle under the bijection  $\Phi_7$ . As indicated in the first case by the red circle, the top left most vertex is given the first label of 1, see Table 4.3.

We notice upon inspection that a crossing in a circle corresponds to a crossing in the arc annotated sequence. Because of this, and the equidistribution, we understand why the number of matchings according to nestings and these circles with  $n$  chords have the same integer sequence. A nesting in these chord placements is more of a challenge to spot, and requires picking out chords with labeled endpoints more extremal (larger for one end, smaller for the other) than another chord.

## 4.4 Enumerative results

Matchings with 2-crossings have been studied in the past by both Touchard [26] and Riordan [20]. They gave the generating function for the number of matchings on  $\{1, 2, \dots, 2n\}$  according to the number of 2-crossings:

**Theorem 4.4.1.** *(Toucard-Riordan) The number of matchings with  $k$  2-crossings is counted by the coefficient of  $q^k$  in:*

$$M_{2n} = \frac{1}{(1-q)^n} \sum_{i \geq 0} (-1)^i \left( \binom{2n}{n-1} - \binom{2n}{n-i-1} \right) q^{\binom{i+1}{2}}.$$

Arc annotated sequence	Circle	Arc annotated sequence	Circle

Table 4.3:  $\Phi_7 : \mathcal{B}_6 \rightarrow \mathcal{C}_6$

When  $n = 4$ , the Touchard-Riordan formula gives:

$$M_{2n} = 14 + 28q + 28q^2 + 20q^3 + 10q^4 + 4q^5 + y^6,$$

indicating 14 matchings on  $\{1, 2, \dots, 8\}$  that had no 2-crossing, 28 that had one 2-crossing, 28 that had two 2-crossings, 20 with three 2-crossings, 10 with four 2-crossings, 4 with five, and one with six 2-crossings. We recover these numbers via a continued fraction computation, a method that will be very useful towards counting partitions and permutations according to the number of 2-crossings.

In the introduction we found **continued fraction** representation for Motzkin paths of length  $n$  to be:

$$M = \frac{1}{1 - c_0 - \frac{a_0 b_1}{1 - c_1 - \frac{a_1 b_2}{1 - c_2 - \frac{a_2 b_3}{\ddots}}}}$$

If  $\mathcal{D}_{2n}$  is the set of all Dyck paths of length  $2n$ , then by mapping  $c_i \rightarrow 0$  we get the continued fraction representation for regular Dyck paths. However, the bijection  $\Phi_1$  required a weighting on the Dyck paths. We will let  $x$  track the length of the Dyck path and  $y$  track the height at each step. By this method, a north step,  $a_i$  will only add to length, so  $a_i \rightarrow x$ , but a south step may have a weighting up to  $i - 1$  where  $i$  is the height at that step, thus  $b_i \rightarrow x(1 + y + y^2 + \dots + y^{i-1})$ .

From this we are able to use a computer to find the number of matchings according to the number of nestings, since a nesting corresponds to a weighting greater than 0. Because of the work done by [3] that proved equidistribution, we only need to work with nestings, and the results will hold for crossings. Please see the Appendix for the Maple code that was used.

When a maximum height of  $h = 10$  was used for the corresponding Dyck path, we got the following results for  $B_{n,k}(x, y)$  where the coefficient of  $x^n y^k$  represented the number of matchings on the set  $\{1, 2, \dots, 2n\}$  with  $k$  2-crossings.

$$B(x, y) = 1 + (1)x^2 + (2 + y)x^4 + (5 + 6y + 3y^2 + y^3)x^6 + (14 + 28y + 28y^2 + 20y^3)$$

$$\begin{aligned}
 &+10y^4 + 4y^5 + y^6)x^8 + (42 + 120y + 180y^2 + 195y^3 + 165y^4 \\
 &\quad +117y^5 + 70y^6 + 35y^7 + 15y^8 + 5y^9 + y^{10})x^{10} \\
 &+(132 + 495y + 990y^2 + 1430y^3 + 1650y^4 + 1617y^5 + 1386y^6 + 1056y^7 + 726y^8 + 451y^9 \\
 &\quad +252y^{10} + 126y^{11} + 56y^{12} + 21y^{13} + 6y^{14} + y^{15})x^{12} + \dots
 \end{aligned}$$

We can arrange these results into a triangular table to show the sequence of coefficients in Table 4.4.

	1	y	y <sup>2</sup>	y <sup>3</sup>	y <sup>4</sup>	y <sup>5</sup>	y <sup>6</sup>	y <sup>7</sup>	y <sup>8</sup>	y <sup>9</sup>	y <sup>10</sup>	...
1	1											
x <sup>2</sup>	1											
x <sup>4</sup>	2	1										
x <sup>6</sup>	5	6	3	1								
x <sup>8</sup>	14	28	28	20	10	4	1					
x <sup>10</sup>	42	120	180	195	165	117	70	35	15	5	1	
x <sup>12</sup>	132	495	990	1430	1650	1627	1386	1056	726	451	252	...
⋮						...						

Table 4.4: The triangular table of  $[y^k x^{2n}]$  for matchings on  $\{1, 2, \dots, 2n\}$  with  $k$  nestings.

As expected due to the Touchard-Riordan formula, this triangular series is already found in the On-line Encyclopedia of Integers [23] as Sequence A067311: “ Triangle read by rows:  $T(n, k)$  gives number of ways of arranging  $n$  chords on a circle with  $k$  simple intersections (i.e. no intersections with 3 or more chords) - positive values only. ” Because of this, we examined the bijection between  $n$  chords on a circle with  $k$  intersections and matchings in the next section.

When we draw out all 15 matchings on  $\{1, 2, \dots, 6\}$ , we see as expected that there are five matchings with no nestings, six with one nesting, three with 2 nestings and one with 3 nestings. This is summarized in Figure 4.6.

Once we have found this bivariate continued fraction, we can begin to ask questions about the average number of 2-crossings and 2-nestings in a matching on  $\{1, 2, \dots, 2n\}$ . The solution is:

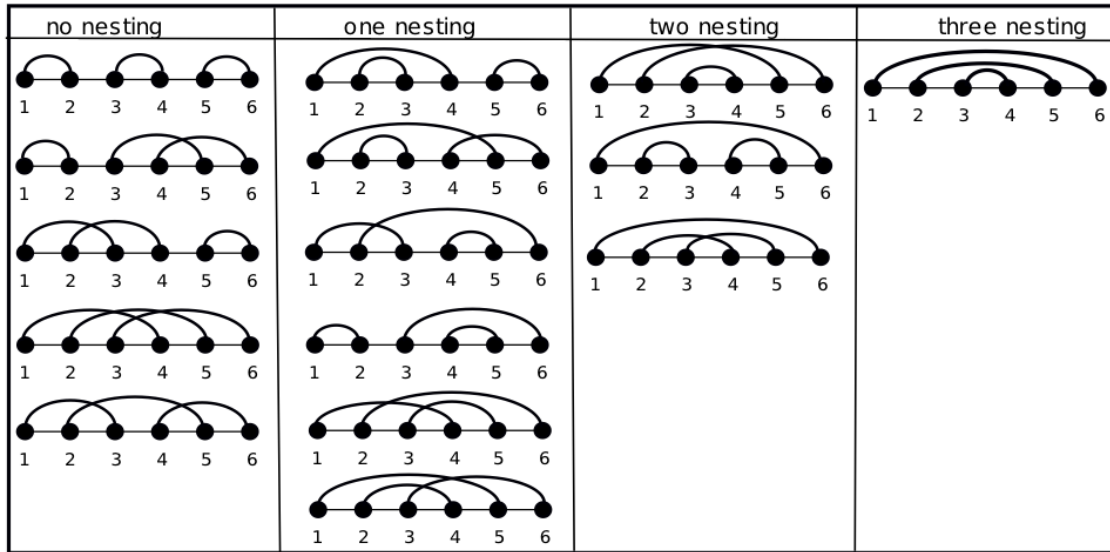


Figure 4.6: All matchings on  $\{1, 2, 3, 4, 5, 6\}$ .

$$\mathbb{E}_{\mathcal{A}_n} = \frac{[x^n] \partial_y A(x, y)|_{y=1}}{[x^n] A(x, 1)}$$

Again, we employ Maple for this calculation. Please see the appendix for the code that gives the following sequence of the average number of 2-nestings in matchings of length  $[0, 2, 4, 6, \dots]$  whose corresponding Motzkin path has a maximum height of  $h = 10$ :

$$\left[0, \frac{1}{3}, 1, 2, \frac{10}{3}, 5, \frac{405}{59}, \frac{6940}{789}, \frac{128661561}{11983066}, \frac{2117687955}{167123684}, \frac{68262246956}{4680410057}, \frac{179028481288}{10894070099}, \frac{4093917286979}{225600144748}, \frac{119740386127205}{6091261880829}, \dots\right]$$

Chen et al. [3] give the following further enumerative results regarding  $k$ -noncrossings matchings of  $\{1, 2, \dots, 2n\}$ :

**Theorem 4.4.2.** *The number of  $k$ -noncrossing matchings of  $\{1, 2, \dots, 2n\}$  is equal to the number of closed lattice walks of length  $2n$  in the set:*

$$V_k = \{(a_1, a_2, \dots, a_{k-1}) : a_1 \geq a_2 \geq \dots \geq a_{k-1} \geq 0, a_i \in \mathbb{Z}\}$$

*from the origin to itself with unit steps in any coordinate direction or its negative.*



For  $k = 2$  this is simply  $\Phi_2$ , and for the case of  $k = 3$  they show:

**Theorem 4.4.3.** *The set of 3-noncrossing matchings is in one-to-one correspondence with the set of pairs of noncrossing Dyck paths.*

Define  $f_k(m)$  to be the number of  $k$ -noncrossing matchings of  $\{1, 2, \dots, 2n\}$ , so the generating function for  $k$ -noncrossing matchings is:

$$F_k(x) = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}$$

Then, Chen et al. [3] note that this equation has already been determined to be:

$$F_k(x) = \det[I_{i-j}(2x) - I_{i+j}(2x)]_{i,j=1}^{k-1},$$

where  $I_n(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$ . From this, substituting in  $k = 2$  we return to the case of  $C_n$  (Catalan numbers) which count the number of 2-noncrossing matchings. When  $k = 3$ ,

$$f_3(n) = C_n C_{n+2} - C_{n+1}^2.$$

# Chapter 5

## Set partitions

### 5.1 Representation as arc annotated structure

Set partitions of  $\{1, 2, \dots, n\}$ ,  $\mathcal{P}_n$ , are a generalization of matching in which the blocks no longer are restricted to being of size 2.

**Definition** Consider a set  $\mathcal{A}$ . A set partition,  $P$  is the division of this set into blocks that must cover the entire set and do not contain repetition. We give the notation

$$a_1 a_2 \dots, a_i | b_1 b_2 \dots b_j | \dots | k_1 k_2 \dots, k_m$$

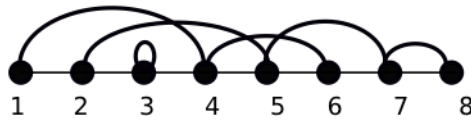
to represent the set partition

$$\{\{a_1, a_2, \dots, a_i\}, \{b_1, b_2, \dots, b_j\}, \dots, \{k_1, k_2, \dots, k_m\}\}.$$

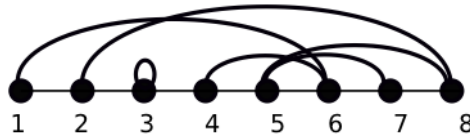
**Example 5.1.1.** Let the set  $A = \{3, 4, 6, 7, 8, 9\}$ . Then  $P = \{\{3, 6\}, \{4, 7, 9\}, \{8\}\} = 36|479$  is a valid set partition of  $A$ .  $Q = \{3, 4\}, \{6, 7, 9\} = 34|679$  is not, because it does not contain 8.

Set partitions are represented with arc annotated sequences by drawing arcs connecting elements in a block in increasing order.

**Example 5.1.2.** The set partition  $P = 146|2578|3$  is represented with the following arc annotated sequence:



In the arc annotated representation of set partitions, vertices that are both left hand endpoints and right hand endpoints exist. These types of vertices are referred to as *transitory*. Because of these transitory vertices, we need to specify that arcs are drawn between closest elements in a block, that is no element is skipped and then visited. For example, Example 5.1.2 would always be represented as Example 5.1.2 and never as the following diagram:



This is an incorrect representation of  $P$  because the 1 should be connected to the 4, not the 6. The convention of having the arcs connected in increasing order is broken twice here.

The set of set partitions of length  $n$ ,  $\mathcal{P}_n$  has been well studied. Indeed, it is well known that the number of set partitions is counted by the Bell numbers  $B_n$  (Sloane [23]) (A000110) which satisfy the following recursion:

$$P_{n+1} = B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_k = 1, 1, 2, 5, 15, 52, 203, 877, \dots$$

It is also known that their exponential generating function is:

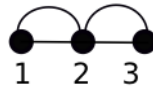
$$P(x) = B(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x} - 1.$$

We will go beyond this understood enumeration and count according to both the size of the set partition and other statistics, such as number of crossings and nestings. We give a table now that counts the total number of crossings and nestings across all partitions of length  $n$ .

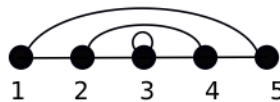
$n$	Number of crossings	Number of nestings
2	0	0
3	0	0
4	1	1
5	11	11
6	89	89

It should be noted that unless stated otherwise, singletons and transitory vertices do not contribute to either a crossing or a nesting. However, in some instances, specifically when it comes to enumeration, we sometimes would like to include a singleton below an arc as a nesting, or a transitory vertex as a crossing. With this in mind we define:

**Definition** An *enhanced  $k$ -crossing* of a set partitions  $P$  is a set of  $k$  edges,  $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$  such that  $i_1 < i_2 < \dots < i_k \leq j_1 < j_2 < \dots < j_k$ . Pictorially, the following is an enhanced 2-crossing:



**Definition** An *enhanced  $k$ -nesting* of a set partition  $P$  is a set of  $k$  edges,  $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$  such that  $i_1 < i_2 < \dots < i_k \leq j_k < \dots < j_2 < j_1$ . Pictorially, the following diagram is an enhanced 3-nesting:



Beyond these enhanced statistics, we will find that in order to fully understand some of the results regarding set partitions there are a few special statistics that need to be defined:

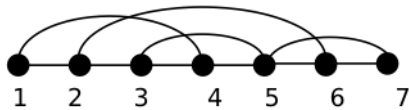
**Definition**  $\min(P) = \{\text{minimal block element of } P\}$  and  $\max(P) = \{\text{maximal block element of } P\}$ .

**Example 5.1.3.** Let  $P = 138|245|67$ . Then  $\min(P) = \{1, 2, 6\}$  and  $\max(P) = \{5, 7, 8\}$ .

**Definition** Let  $P_n(S, T) = \{P \text{ is a partition: } \min(P) = S, \max(P) = T\}$ , then  $f_{n,S,T} = |\{P \in P_n(S, T) : cr(P) = i, ne(P) = j\}|$ .

These maximal and minimal statistics are crucial in defining the set partitions version of Theorem 1.2.1.

We note that there are different definitions of  $k$ -crossings and  $k$ -nonnestings for partitions in the literature. In particular, Klazar [17] requires that there be 3 mutually crossing *blocks* in a 3-crossing, while in our definition we need 3 mutually crossing *arcs*. Similarly, a  $k$ -noncrossing partition according to Klazar is a partition which does not have  $k$  mutually crossing blocks, while we define a  $k$ -noncrossing partition to be a partition without  $k$  mutually crossing arcs. To make this clear, consider the following arc annotated sequence representation of  $14|26|357$ :



This partition according to our definition does not have a 3-crossing because no three arcs are mutually crossing. However, according to Klazar, the block  $\{3,5,7\}$  crosses both arcs  $\{1,4\}$  and  $\{1,6\}$  and according to both definition  $\{1,4\}$  and  $\{2,6\}$  cross, so it is a 3-crossing by this alternative definition.

## 5.2 Distribution of nesting and crossing statistics

As was the case for matchings, the number of 2-noncrossings partitions of  $\{1,2,\dots,n\}$  is known to be the  $n$ th Catalan number,  $C_n = \frac{1}{2n+1} \binom{2n}{n}$ . Because of this, we can describe a bijection  $\Phi_8 : \mathcal{B}_{2n}^{<nc>} \rightarrow \mathcal{P}_{2n}^{<nc>}$ .

**Bijection  $\Phi_8$  (Folklore)**

- $\mathcal{B}_{2n}^{<nc>}$ : noncrossing matchings on  $\{1,2,\dots,2n\}$
- $\mathcal{P}_n^{<nc>}$ : noncrossing partitions on  $\{1,2,\dots,n\}$

**Example** *The noncrossing matching  $\{\{1,4\},\{2,3\},\{5,6\}\}$  is in bijection with the set partition  $12|3$ :*



**Steps**

1. Traverse the partition (matching) from left to right.
2. Generate the arcs for the matching (partition) based on the following dictionary:

Partition	Matching

3. Connect the arcs in the only way possible such that a noncrossing matching (partition) occurs.

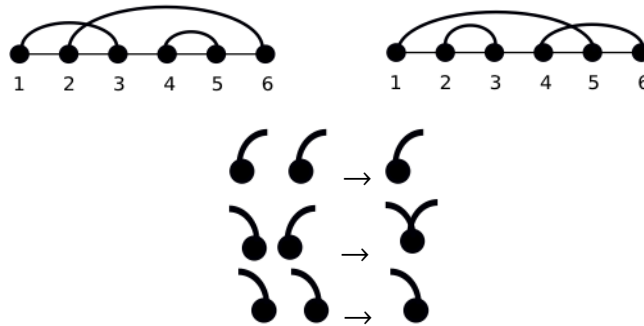
**Theorem 5.2.1.** *The map  $\Phi_8$  is a bijection from noncrossing matchings on  $\{1, 2, \dots, 2n\}$  to noncrossing partitions on  $\{1, 2, \dots, n\}$ .*

*Proof.* Read the dictionary in the reverse direction. The structure of a noncrossing matching allows two consecutive vertices to only be aligned in these four possible ways.  $\square$

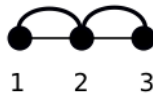
After seeing this bijection, one might be tempted to try this bijection  $\Phi_8$  on matchings and partitions, rather than on noncrossing matchings and noncrossing partitions. To illustrate why this is not that simple we use the following example.

**Example 5.2.1.** *Let  $B_1 = \{\{1, 3\}, \{2, 6\}, \{4, 5\}\}$  and  $B_2 = \{\{1, 5\}, \{2, 3\}, \{4, 6\}\}$  be matchings, represented with arc annotated sequences below:*

*In each case, traversing the sequence left to right we get:*



So that the resulting partitions under  $\Phi_8$  are  $P_1 = P_2 =$



Because a transitory vertex does not contribute to crossings, and a singleton does not contribute to a nesting, we know through  $\Phi_8$  that noncrossing and nonnestings are equidistributed among partitions of the set  $\{1, 2, \dots, n\}$ .

In order to prove the more general result of equidistribution by Chen et al. [3] stated in Theorem 1.2.1, we first need a bijection  $\Phi_9$  between partitions and vacillating tableaux.

Recall that a vacillating tableau is a sequence of arrays or squares in Ferrers diagrams, but we can represent these arrays with integers  $a_i a_{i-1} \dots a_2 a_1$  where  $a_j$  indicates  $a_j$  squares in the  $j^{th}$  row.

**Bijection  $\Phi_9$**  (Chen et al., [3])

- $\mathcal{P}_n$ : set partitions on  $\{1, 2, \dots, n\}$
- $\mathcal{V}_\lambda^{2n}$ : vacillating tableaux of shape  $\lambda$  and length  $2n$ .

**Example 5.2.2.** Let  $V = \emptyset, \emptyset, 1, 1, 2, 2, 21, 11, 11, 1, 1, \emptyset, \emptyset$  be a vacillating tableau of shape  $\emptyset$  and length 12. Under  $\Phi_9$ ,  $V$  becomes the partition  $16|235|4$ .

**Steps**

1.  $V_\lambda = \{\emptyset = \lambda^0, \lambda^1, \lambda^2, \dots, \lambda^{2n} = \lambda\}$

2. Consider the vacillating tableaux from left to right.
3. Notes:
  - (a) We construct a partition  $P$  and a corresponding standard Young tableau  $T$ .
  - (b) The pairs  $(P_0, T_0), (P_1, T_1), \dots, (P_n, T_n) = (P, T)$  are constructed recursively.
  - (c)  $P_0$  is the identity partition:  $1|2|3|\dots|n-1|n$ .
  - (d)  $T_0$  is the empty standard Young tableau:  $\emptyset$ .
  - (e) At most one arc is added at each  $P_i$ . The  $T_i$  stipulate which arc to add.
4. Consider  $\lambda^i$  and compare it to  $\lambda^{i-1}$ :
  - (a) If  $\lambda^i = \lambda^{i-1}$ :
    - i.  $P_i = P_{i-1}$  and,
    - ii.  $T_i = T_{i-1}$ .
  - (b) If  $\lambda^i \supset \lambda^{i-1}$ :
    - i.  $i$  must be even, so  $i = 2k$  for some  $k$ .
    - ii. Insert  $k$  into  $T_i$  in the square that must have been added to  $\lambda^{i-1}$  to get  $\lambda_i$
    - iii.  $P_i = P_{i-1}$
  - (c) If  $\lambda^i \subset \lambda^{i-1}$ :
    - i.  $i$  must be odd. Let  $i = 2k - 1$  for some  $k$ .
    - ii. Consider  $T_{i-1}$  of shape  $\lambda^{i-1}$  which must be larger than the shape  $\lambda^i$  of  $T_i$ .
    - iii. There must be some  $j$  that when ‘deleted’ from  $T_{i-1}$  gives  $T_i$  of shape  $\lambda$ .
    - iv. RSK is not considered in reverse, so determine which  $j$  needs to be inserted in  $T_i$  to give  $T_{i-1}$  which is already determined.
    - v.  $j$  must be less than  $k$ .
    - vi.  $P_i$  is  $P_{i-1}$  plus the pair (arc)  $(j, k)$ .
5.  $(P_n, T_n)$  is the set partition and standard Young tableaux associated with the vacillating tableau  $V$ .



**Theorem 5.2.2.** *The map  $\Phi_9$  is a bijection from vacillating tableau of shape  $\lambda$  and length  $2n$  to set partitions on  $\{1, 2, \dots, n\}$ .*

*Proof.* Here we only show the inverse direction for vacillating tableaux of shape  $\lambda = \emptyset$ . We direct the reader to Chen et al.’s [3] for the proof with tableaux with arbitrary shape.

**Steps**

1. Consider the arc annotated sequence of the set partition from *right to left*.
2. We build the series of standard Young tableaux starting with  $T_{2n} = \emptyset$ .
3. The shapes of this series of standard Young tableaux will be the vacillating tableau.
4. For every vertex  $j$  in the arc annotated sequence,  $T_{2j-1}$  and  $T_{2j-2}$  are defined as follows:

$j$	arcs	$T_{2j-1}$	$T_{2j-2}$
opener	$(j, k)$	$-j$	$-$
closer	$(i, j)$	$-$	$+j$
singleton	$(j, j)$	$-$	$-$
transitory	$(i, j), (j, k)$	$-j$	$+i$

5. **Note:** ‘ $+j$ ’ indicates that  $j$  has been inserted by the RSK algorithm into the standard Young tableau, ‘ $-j$ ’ indicates that  $j$  has been removed from the tableau, and ‘ $-$ ’ means there was no change to the tableau at that step.

□

We illustrate each direction with a detailed example. First, reconsider Example 5.2.2:

**Example (continued)** We start with

$$V = (\emptyset = \lambda^0, \lambda^1, \lambda^2, \dots, \lambda^{12} = \lambda) = (\emptyset, \emptyset, 1, 1, 2, 2, 21, 11, 11, 1, 1, \emptyset, \emptyset)$$

To be clear, this is just notation for the shapes of the tableaux that actually appear as:

$$V = \emptyset, \emptyset, \square, \square, \square\square, \square\square, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \square, \square, \square, \square, \emptyset, \emptyset$$

Our initial partition  $P_0$  is the identity, and initial standard Young tableau is the empty set,  $T_0 = \emptyset$ . We increment along the vacillating tableau, starting with  $\lambda^1$ , comparing it with  $\lambda^0$ . Clearly both are equal to the empty set, so according to step 4(a),  $P_1$  remains the identity and  $T_1 = 0$ .

Next we compare  $\lambda^2$  with  $\lambda^1$ . Notice that  $\lambda^2$  is a square while  $\lambda^1$  is the empty set so we are at step 4(b),  $\lambda^2 \supset \lambda^1$ . Then  $2 = 2k$  implies  $k = 1$  and  $P_2$  remains the identity and  $T_2$  has 1 inserted, thus,  $T_2 = 1$ . We summarize the of this example in a table:

$i$	$\lambda^i$	$\lambda^{i-1}$	Relation	Step	$P_i$	$T_i$
1	$\emptyset$	$\emptyset$	=	4(a)	-	-
2	$\square$	$\emptyset$	$\supset$	4(b)	-	+1
3	$\square$	$\square$	=	4(a)	-	-
4	$\square \square$	$\square$	$\supset$	4(b)	-	+2
5	$\square \square$	$\square \square$	=	4(a)	-	-
6	$\begin{array}{ c } \hline \square \\ \hline \square \end{array}$	$\square \square$	$\supset$	4(b)	-	+3
7	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \end{array}$	$\subset$	4(c)	+(2,3)	-2
8	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \end{array}$	=	4(a)	-	-
9	$\square$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \end{array}$	$\subset$	4(c)	+(3,5)	-3
10	$\square$	$\square$	=	4(a)	-	-
11	$\emptyset$	$\square$	$\subset$	4(c)	+(1,6)	-1
12	$\emptyset$	$\emptyset$	=	4(a)	-	-

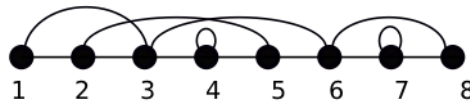
From this table it is easy to see that  $P = 16|235|4$  and the standard Young tableaux are:

$$T := \emptyset, 1, 1, 12, 12, \begin{array}{|c|} \hline 1 & 2 \\ \hline 3 & \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 3 \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 3 \end{array}, 1, \emptyset, \emptyset$$

Taking the shapes of  $T$  we see that we still have the vacillating tableau:

$$V = (\emptyset, \emptyset, 1, 1, 2, 2, 21, 11, 11, 1, \emptyset, \emptyset)$$

**Example 5.2.3.** Consider the partition  $P = 1368|25|4|7$  of  $\{1, 2, \dots, 8\}$  represented as follows:



Then, working from right to left, the 8 steps are:

1. Do nothing then insert 6.
2. Do nothing twice.
3. Delete 6 then insert 3.
4. Do nothing then insert 2.
5. Do nothing twice.
6. Delete 3 then insert 1.
7. Delete 2 then do nothing.
8. Delete 1 then do nothing.

This gives the following corresponding SYTs:

$$\emptyset, \emptyset, 1, 1, \begin{matrix} 1 \\ 2 \end{matrix}, 2, \begin{matrix} 2 \\ 3 \end{matrix}, \begin{matrix} 2 \\ 3 \end{matrix}, \begin{matrix} 2 \\ 3 \end{matrix}, 3, 3, \emptyset, 6, 6, 6, \emptyset, \emptyset$$

So the vacillating tableau is:  $\emptyset, \emptyset, 1, 1, 11, 1, 11, 11, 11, 1, 1, \emptyset, 1, 1, 1, \emptyset, \emptyset$ .

In the examples we saw in Section 4.2 with matchings and oscillating tableaux, we gained the intuition that a tableau with more than one row indicates crossings and with more than one column indicates a nesting. Our suspicions are confirmed with the following results by Chen et al. in [3].

In fact, the maximal statistics of  $cr(P)$  and  $ne(P)$  that we defined in the introduction and saw similar versions of in Section 4.1 can be found under this bijection  $\Phi_9$ :

**Theorem 5.2.3.** (Chen et al. [3]) *Let  $P$  be a partition and denote  $\Phi_9(P)$  to be the vacillating tableau that results through the bijection above. Then  $cr(P)$  is the most number of rows of any  $\lambda^i$  and  $ne(P)$  is the most number of columns of any  $\lambda^i$ .*

**Example 5.2.4.** (Continued) In Example 5.2.3 seen above, pictorially it can be seen that  $cr(P) = 2$ , which is the maximum number of rows, and  $ne(P) = 1$  which is the maximum number of columns.

So although we have support for Theorem 5.2.3, in order to prove this, an external theorem will be required.

**Theorem 5.2.4.** (Schensted [22])  $\sigma_i$  has a decreasing subsequence of length,  $k$  if and only if the partition  $\lambda^i$  in  $\phi(P)$  has at least  $k$  rows.

*Proof.* (Sketch of Theorem 5.2.3) First, interpret a  $k$ -crossing of  $P$  in terms of the entries of SYTs in  $\Phi_9(P)$  and prove that the arcs form a  $k$ -crossing if and only if  $1, \dots, n$  leave in increasing order. Then, define a permutation of these tableaux by adding and deleting entries to create  $\sigma_n = w_1, w_2, \dots, w_r$  which gives that the content( $T_i$ ) leaves in the order  $w_r, \dots, w_1$ .

Then apply RSK  $\sigma_i$  and get  $(T_i, B_i)$ . Finally use Schensted's theorem to connect RSK with increasing and decreasing subsequences and the result follows. □

As a corollary to this main theorem, Chen et al. showed results for  $k$ -nonesting and  $k$ -noncrossing partitions:

**Theorem 5.2.5.** [3] Let  $NCN_{k,l}(n)$  be the number of partitions of  $\{1, 2, \dots, n\}$  that are  $k$ -noncrossings and  $l$ -nonnestings; then

$$NCN_{k,l}(n) = NCN_{l,k}(n).$$

**Theorem 5.2.6.** [3] Let  $NC_k(n)$  be the number of  $k$ -noncrossing partitions of  $\{1, 2, \dots, n\}$  and  $NN_k(n)$  be the number of  $k$ -nonnesting partitions of  $\{1, 2, \dots, n\}$ . Then,

$$NC_k(n) = NN_k(n).$$

It should also be noted that there is a bijection between set partitions, in which enhanced crossings and nestings are used, and a different type of tableaux, called the *hesitating tableaux*,  $\mathcal{H}$ , defined earlier. This is of interest because this gives that equidistribution holds between the statistics of enhanced crossings and nestings as well.

The bijection  $\Phi_{10} : \mathcal{P}_n^{en} \rightarrow \mathcal{H}$  is very similar to the bijection  $\Phi_9$ , with only changes required when  $j$  is a singleton or a transitory vertex. Indeed, if  $j$  is a singleton, insert and then delete  $j$ . In the other case, when  $j$  is both a closer of  $(i, j)$  and an opener of  $(j, k)$ , first insert  $i$  and then delete  $j$ . We discuss enhanced crossings and nestings further when we consider enumerating our set partitions.

Although we have been able to prove equidistribution using these tableaux, and although we are indeed able to recognize nesting and crossing structures, the very visual and natural translation to the crossing and nesting that we understood so clearly to begin with has become buried in the tableaux and RSK. Furthermore, enumerating these tableaux based on finding the maximum number of rows and columns in any one of the  $T_i$  is by no means a natural nor obvious task. It is with this in mind that we turn to other bijections.

### 5.3 Further bijective results

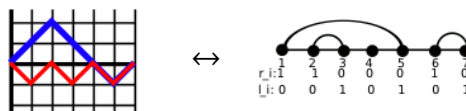
We again consider the noncrossing and nonnesting cases. In [4] a direct bijections between pairs of noncrossing free Dyck paths  $\mathcal{D}_{2n}^{**}$  and noncrossing partitions,  $\mathcal{P}_n^{<nc>}$  is shown.

Before we describe the bijection, recall that the right degree sequence  $(r_1, r_2, \dots, r_n)$  of the vertices  $\{1, 2, \dots, n\}$  in a arc annotated sequence is the number of arcs extending to the right of a vertex  $i$  and the *left degree sequence*  $(l_1, l_2, \dots, l_i)$  of the vertices  $\{1, 2, \dots, n\}$  represents the number of arcs extending to the left of a vertex  $i$ . Also, the pairs  $(P, Q) = (p_1, q_1), (p_2, q_2), \dots, (p_{2n}, q_{2n})$ , can be represented with a sequence of pairs of  $U$ s and  $D$ s to indicate north east and south east steps respectively.

**Bijection  $\Phi_{11}$**  (Chen et al.) [4]





- $\mathcal{P}_n^{<nc>}$ : noncrossing partitions of the set  $\{1, 2, \dots, 2n + 1\}$  into  $n + 1$  blocks.
- $\mathcal{D}_{2n}^{**}$ : pairs of noncrossing free Dyck paths of length  $2n$ .

**Example 5.3.1.** Consider the following pair of noncrossing free Dyck paths  $(P, Q)$  and the noncrossing partition under  $\Phi_{11}$  represented as an arc annotated sequence:



**Steps**

1. Traverse the pairs of free Dyck paths left to right
2. Generate the left and right degree sequences for the partition according to the following dictionary:





$(p_i, q_i)$	Picture	$r_i$	$l_{i+1}$
(U,U)		1	1
(U,D)		1	0
(D,U)		0	1
(D,D)		0	0

3. Draw an opener at vertex  $i$  if  $r_i = 1$  and a closer if  $l_i = 1$ , a transitory if both are 1 and a singleton if both are 0.
4. Connect the openers and closers in the only way such that a noncrossing arc annotated sequence representation of a partition occurs.

**Theorem 5.3.1.** *The map  $\Phi_{11}$  is a bijection from pairs of noncrossing free Dyck paths to noncrossing partitions.*

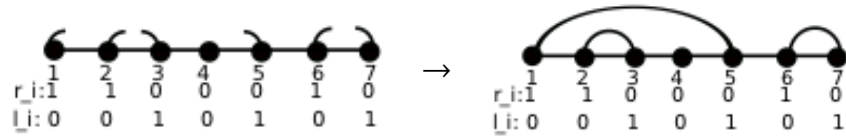
*Proof.* This construction can be reversed,  $\Phi_{11} : \mathcal{P} \rightarrow \mathcal{D}^{**}$ :

1. Traverse the arc annotated sequence from left to right.
2. Label each vertex with its left and right degree sequences.
3. The right degree of vertex  $i$  and the left degree of vertex  $i + 1$  gives the corresponding steps at  $(p_i, q_i)$ :

$r_i$	$l_{i+1}$	Steps
1	1	
1	0	
0	1	
0	0	

□

**Example 5.3.1 (continued).** From the pairs of Dyck paths we get the right degree sequence  $(1, 1, 0, 0, 0, 0, 1, 0)$  and the left degree sequence  $(0, 0, 1, 0, 1, 0, 1)$ . We construct the openers, closers, singletons and transitory vertices of the arc annotated sequence based on this, and then connect such that no crossing occurs:



Clearly this bijection  $\Phi_{11}$  could be rewritten for nonnesting partitions of the set  $\{1, 2, \dots, 2n + 1\}$  into  $n + 1$  blocks by altering step (4). This gives rise to questions regarding the number and size of blocks in a partition:

1. Are crossings and nestings equidistributed among partitions of  $\{1, 2, \dots, kn\}$  with blocks of size  $k$ ?
2. Are crossings and nestings equidistributed among partitions of  $\{1, 2, \dots, n\}$  with  $k$  blocks?

The answer to question (1) is **no**. Upon examining all 10 partitions of  $\{1, 2, 3, 4, 5, 6\}$  with blocks of size 3 we see that there are 10 2-nestings (N) and 11 2-crossings (C):

Partition	Notes	Partition	Notes	Partition	Notes
			+C		+C,+N
	+2N		+C,+N		+3C
	+2C,+N		+C,+2N		+2C,+N
	+2N				

However, when we also examine all remaining partitions of  $\{1, 2, 3, 4, 5, 6\}$  with 2 blocks, we see that there are an equal number (23) of 2-crossings and 2-nestings. We believe that the answer to question (2) is yes.

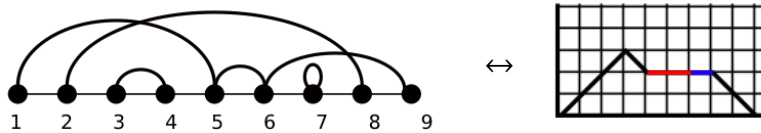
**Conjecture 1.** *2-crossings and 2-nestings are equidistributed among set partitions of  $\{1, 2, \dots, n\}$  with  $k$  blocks.*

In Section 4.3 which considered lattice paths in bijection with matchings, we were only concerned with weighted Dyck paths. In order to deal with partitions, we use with a generalized Dyck path, the Motzkin path, in particular, bicolored weighted Motzkin paths. Motzkin paths have a very nice continued fraction representation, and through this bijection we get computational results.

**Bijection  $\Phi_{12}$**  (Kasraoui-Zeng [16])

- $\mathcal{P}_n$ : Partitions on the set  $\{1, 2, \dots, n\}$ .
- $\mathcal{M}_n^{<w_2>}$ : Bicolored weighted Motzkin paths of length  $n$  with weight vector  $w_2 = (w_1, w_2, \dots, w_n)$  where  $w_i < h_i$ , the height at step  $i$ .

**Example 5.3.2.** *Let  $P = 1\ 5\ 6\ 9\ |2\ 8\ |3\ 4\ |5$  be a partition of  $\{1, 2, \dots, 9\}$ . We show  $P$  as an arc annotated sequence and as a bicolored weighted Motzkin path under  $\Phi_{12}$ :*













*The weight vector for the bicolored Motzkin path is  $w_2 = (0, 0, 0, 2, 0, 1, 0, 0, 0)$*

**Steps**

1. Traverse the arc annotated sequence from left to right.
2. Generate the bicolored Motzkin paths according the following dictionary:



Vertex type	Step
	
	
	
	

3. Generate the weight vector  $w_1 = (w_1, w_2, \dots, w_n)$  as follows:
  - (a) Assign weight  $w_i = 0$  to all  and  steps.
  - (b) Let  $j$  be the left most available closer or transitory in the arc annotated sequence.
  - (c) Label all available openers and transitories to the left of  $j$  with the labels  $\{0, 1, \dots, j-1\}$
  - (d) If  $k$  is the vertex that  $j$  is connected to on the left and  $k$  has label  $a_k$ , assign  $j$  weight  $w_j = a_k$ .
  - (e) Now  $k$  and  $j$  are considered unavailable.
  - (f) Repeat for the next available left most closer or transitory until all steps are weighted.

**Theorem 5.3.2.** *The map  $\Phi_{12}$  is a bijection from the arc annotated sequence of partitions on  $\{1, 2, \dots, n\}$  and bicolored weighted Motzkin paths of length  $n$ .*

*Proof.* We show the inverse direction  $\Phi_{12} : \mathcal{M}_{2n}^{<w_1>} \rightarrow \mathcal{P}_n$ .

**Steps**

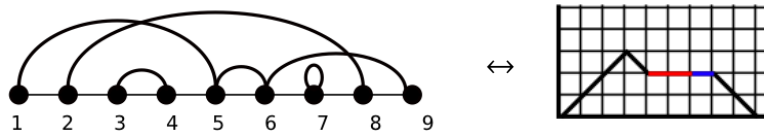
1. Traverse the Motzkin path  $M \in \mathcal{M}_n^{<w_1>}$  from left to right
2. Generate the arc annotated sequence with the following dictionary, leaving the (non-singleton) arcs unconnected:


3. Take the left most closer or transitory  $j$ ; consider step  $j$  in  $M_n$  with weight  $w_j$ .
4. Label every available opener or transitory from left to right with labels  $a1 = (0, 1, \dots, w_j, \dots, h_j - 1)$  if  $j$  is a closer or  $a2 = (0, 1, \dots, w_j, \dots, h_j)$  if  $j$  is transitory where  $h_j$  is the height of the corresponding step in the Motzkin path.
5. Connect  $j$  to the opener or transitory  $k$  that has with label  $w_j$ .
6. Now connected vertices  $k, j$  are considered unavailable. Repeat.

□

Notice that this weight vector  $w1$  was used in bijection  $\Phi_5$  between matchings and Dyck paths, and to this end  $\Phi_{12}$  is a generalization of  $\Phi_5$ , just as set partitions are a generalization of matchings.

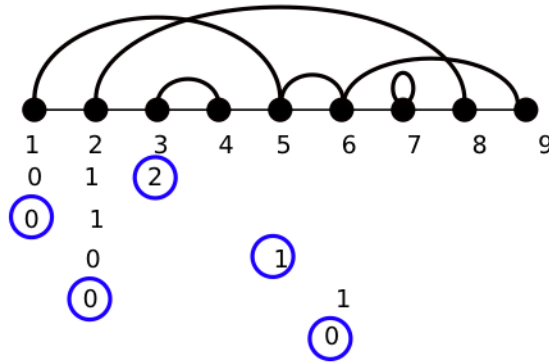
**Example (continued)** We illustrate how the weight vector was assigned in Example 5.3.2. The partition  $P = 1569|28|34|5$  is first represented by its arc annotated sequence and then by the shape of the bicolored Motzkin path:



Then we examine the arc annotated sequence and find that vertex 4 is the first closer. Then we label the openers (and transitories) to the left of 4 with the  $(0, 1, 2)$ . Vertex 4 is

connected to vertex 3 which has a label of 3 assigned to it, so step 4 in the corresponding bicolored Motzkin path has weight  $w_4 = 2$ .

The next available closer or transitory is vertex 5. All available openers and transitories to the left of 5 are labeled  $(0, 1)$ , and we see that 5 is connected to vertex 1 which has label 0. Thus, step 5 in the corresponding Motzkin path under  $\Phi_{12}$  has weight 0. We summarize how the remaining steps are weighted with the following figure:



Therefore the corresponding weight vector for our bicolored Motzkin path is  $w_2 = (0, 0, 0, 2, 0, 1, 0, 0, 0)$ .

### 5.4 Enumerative results

We saw in Section 5.3 that through the bijection  $\Phi_{12}$ , partitions on  $\{1, 2, \dots, n\}$  are in bijection with bicolored weighted Motzkin paths. As was shown in Section 3.2, the continued fraction for Motzkin paths is:

$$Mn^{[k]} = \frac{1}{1 - c_0 - \frac{a_0 b_1}{1 - c_1 - \frac{a_1 b_2}{\ddots \frac{1}{1 - c_k}}}}$$

Where  $a_i$  is a north east step with starting height  $i$ ,  $b_i$  is a south east step of starting height  $i$ , and  $c_i$  is an east step at height  $i$ . We use this bijection and continued fraction to compute the number set partitions according to both length,  $x$ , and number of nestings,  $y$ .

To do this,  $a_i$  contributes only to the length since under  $\Phi_{12}$  a north step has weight 0. Thus,  $a_i \rightarrow x$ . A step  $b_i$  may have a weight of up to  $i - 1$  and contributes to the length, so,  $b_i \rightarrow (1 + y + y^2 + \dots, y^{i-1})x$ . And finally, under  $\Phi_{12}$  an east step  $c_i$  either has no weight and contributes only to length, or has weight up to  $i - 1$ . Therefore,  $c_i \rightarrow x + (1 + y + y^2 + \dots, y^{i-1})x$ .

Using this, we can use a Maple procedure to compute  $P(x, y)$  the number of set partitions according to length and number of nestings. Let  $[x^n y^k]P(x, y)$  be the number of set partitions of length  $n$  with  $k$  nestings. Please see Appendix for the code that was used to get the following results. When dealing with a continued fraction that is not in closed form such as this, we must specify a maximum height  $h$  that the Motzkin path can reach. For  $h = 10$  we get the following result:

$$\begin{aligned}
 P(x, y) = & 1 + x + 2x^2 + 5x^3 + (y + 14)x^4 + (y^2 + 9y + 42)x^5 + (2y^3 + 14y^2 + 55y + 132)x^6 \\
 & + (y^5 + 6y^4 + 35y^3 + 120y^2 + 286y + 429)x^7 + (y^7 + 7y^6 + 35y^5 + 119y^4 + 364y^3 \\
 & \quad + 819y^2 + 1365y + 1430)x^8 + (2y^9 + 13y^8 + 59y^7 + 203y^6 \\
 & \quad + 586y^5 + 1394y^4 + 2940y^3 + 4900y^2 + 6188y + 4862)x^9 + \dots
 \end{aligned}$$

We can find the average number of partitions on  $\{1, 2, \dots, n\}$  according to the number of nestings by using:

$$\mathbb{E}_{\mathcal{A}_n} = \frac{[x^n] \partial_y A(x, y)|_{y=1}}{[x^n] A(x, 1)}$$

Through Maple, we get the sequence:

$$0, 0, 0, \frac{1}{15}, \frac{11}{52}, \frac{89}{203}, \frac{660}{877}, \frac{959}{828}, \frac{11689}{7049}, \frac{52301}{23195}, \frac{2001593}{678569}, \frac{7885513}{2106589}, \frac{127819330}{27615419}, \dots$$

which corresponds the coefficients of  $x^n$  where  $n$  starts at 0. These results are as expected because they agree with the arc annotated sequence representations for partitions according to both length and nestings when they are drawn out for up to  $n = 4$ . We plot the average number of nestings vs. length of the arc annotated sequence, and vary the maximum height of the corresponding Motzkin path:

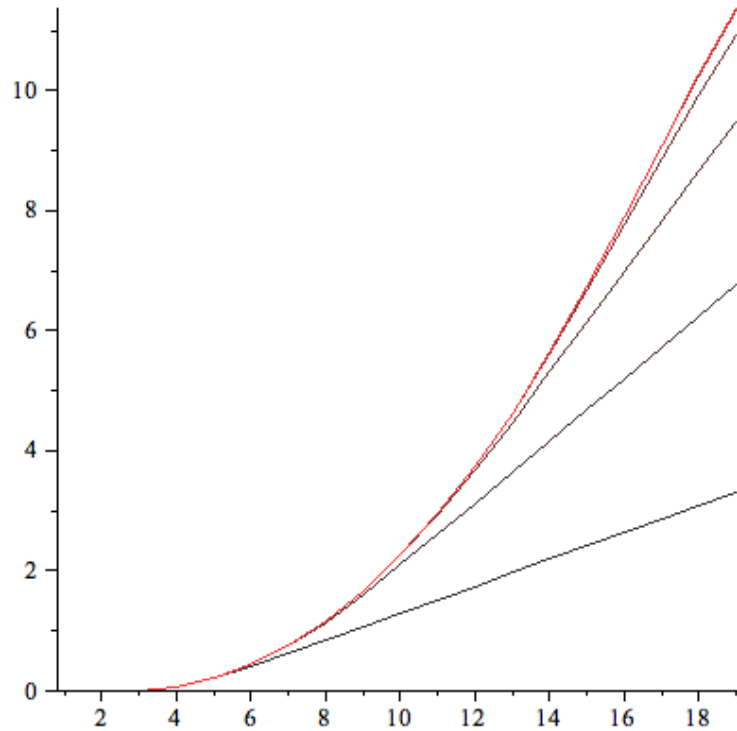


Figure 5.1: Average number of nestings according to crossings.

In Figure 5.1 we see the average number of nesting according to crossings for various corresponding Motzkin paths of different maximum heights for up to  $t = 20$  terms. As the corresponding Motzkin path maximum height increases from 1 to 10, so does the average number of nestings in the arc annotated sequence. Based on the shapes of those with higher maximum height, the average number of nestings appears to have an exponential shape.

We can also find the variance for the number of 2-nestings on partitions through computer calculations:

$$0, 0, 0, \frac{14}{225}, \frac{555}{2704}, \frac{18266}{41209}, \frac{618554}{769129}, \frac{4507111}{3427920}, \frac{99262064}{49688401} \dots$$

Recall that an *enhanced* nesting also considers a singleton beneath an arc to be a nesting. We can use the continued fraction representation to count the number of partitions according to enhanced nestings by changing how the weights are assigned to  $a_i, b_i$  and  $c_i$ .

As before,  $a_i \rightarrow x$ , and  $b_i \rightarrow (1 + y + y^2 + \dots, y^{i-1})x$ , but now a singleton under and

arc counts as a nesting so clearly a singleton at height  $i$  contributes  $i$  nestings to the arc annotated sequence of the partition while the transitory steps are as before. Thus,  $c_i \rightarrow y^i x + (1 + y + y^2 + \dots, y^{i-1})x$ .

Let  $P^{en}(x, y)$  count the number of partitions of the set  $\{1, 2, \dots, n\}$  according to the number of crossings. See the Appendix for the code used to find the following result:

$$\begin{aligned}
 P^{en}(x, y) = & 1 + ((1))x + ((2))x^2 + ((4 + y))x^3 + ((9 + 5y + y^2))x^4 + (21 + 20y + 9y^2 \\
 & + 2y^3)x^5 + (51 + 71y + 51y^2 + 23y^3 + 6y^4 + y^5)x^6 \\
 & + (127 + 238y + 238y^2 + 161y^3 + 77y^4 + 28y^5 + 7y^6 + y^7)x^7 \\
 & + ((323 + 770y + 994y^2 + 896y^3 + 610y^4 + 334y^5 + 147y^6 + 51y^7 + 13y^8 + 2y^9))x^8 \\
 & + (835 + 2436y + 3870y^2 + 4356y^3 + 3810y^4 + 2748y^5 + 1671y^6 + 863y^7 + \dots
 \end{aligned}$$

We can also find and plot the average number of enhanced nestings in set partitions:

$$0, 0, \frac{1}{5}, \frac{7}{15}, \frac{11}{13}, \frac{271}{203}, \frac{1694}{877}, \frac{1213}{460}, \frac{72994}{21147}, \frac{253760}{57987}, \frac{1832653}{339081}, \frac{27246158}{4189541}, \frac{102309959}{13468383}, \dots$$

We note that computational results in both the regular nesting and enhanced nesting cases are as expected. Because the coefficient of  $x^4$  of  $P(x, y)$  is  $(y + 14)$ , we expect that there will be 14 nonnesting partitions on  $\{1, 2, 3, 4\}$ , and 1 partition with a single nesting on  $\{1, 2, 3, 4\}$ . The coefficient of  $x^4$  in  $P^{en}(x, y)$  is  $(9 + 5y + y^2)$  indicating that there are 9 partitions with no enhanced nestings, 5 partitions with one enhanced nesting and one with two enhanced nestings. We draw out the 15 partitions to verify that this is true:

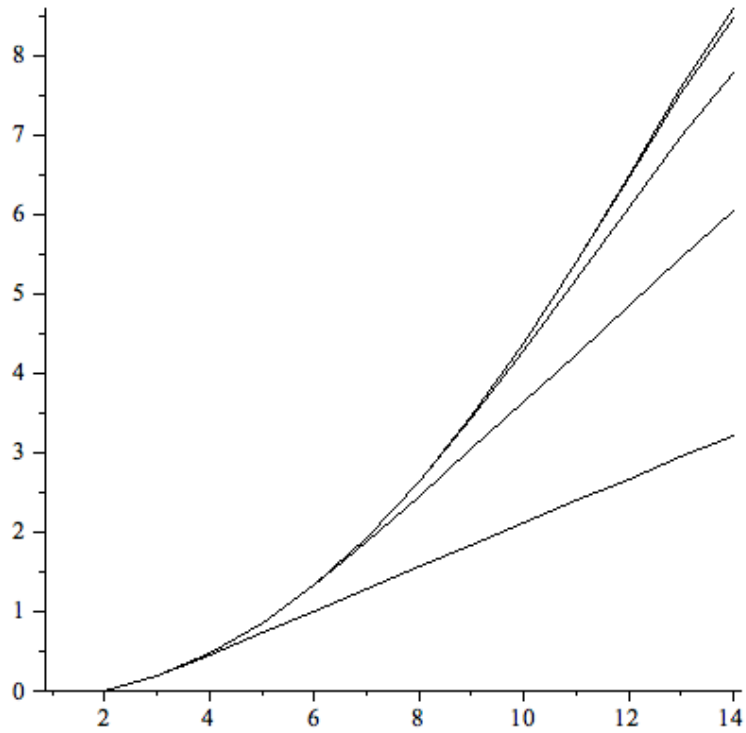
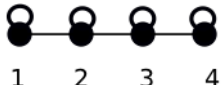
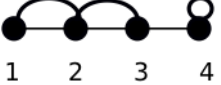
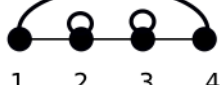

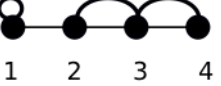


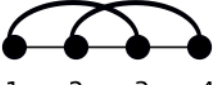



Figure 5.2: As the maximum height of the corresponding Motzkin path increases from 1 to 10, so does the average number of enhanced nestings in the set partition.

No enhanced nestings	1 enhanced nesting	2 enhanced nestings
		
		
		
		

Only the last partition in the second column has a nesting by our usual definition.

Another major result about set partitions exploits their connection with closed lattice walks in order to enumerate noncrossing partitions.

**Theorem 5.4.1.** (Chen et al. [3]) Denote the  $i$ -th unit coordinate vector in  $\mathbb{R}^{k-1}$  with  $\epsilon_i$ . Then the number of  $k$ -noncrossing partitions of  $[n]$  is equal to the number of closed lattice walks in the region

$$V_k = \{(a_1, a_2, \dots, a_{k-1}) : a_1 \geq a_2 \geq \dots \geq a_{k-1} \geq 0, a_i \in \mathbb{Z}\}$$

from the origin back to itself of length  $2n$  with steps  $\pm\epsilon_1$  or the zero vector, where the walk goes backwards or stands still after an even number of steps and goes forward and stands still after an odd number of steps.

This result allowed Bousquet-Melou and Xin in [1] to enumerate 3-noncrossing and 3-nonnesting partitions of  $\{1, 2, \dots, n\}$ , first finding the asymptotics and then enumerating explicitly.

**Theorem 5.4.2.** (Bousquet Mélou, Xin, [1]) Asymptotically, as  $n$  tends to infinity,

$$C_3(n) \sim \frac{3^9 5 \sqrt{3} * 9^n}{2^5 \pi n^7}$$

**Theorem 5.4.3.** (Bousquet Mélou, Xin, [1]) For  $n \geq 1$ , the number of 3-noncrossings partitions of  $[n]$  is

$$C_3(n) = \sum_{j=1}^n \frac{4(n-1)!n!(2j)!}{(j-1)!j!(j+1)!(j+4)!(n-j)!(n-j+2)!} P(j, n)$$

where

$$P(j, n) = 24 + 18n + (5 - 13n)j + (11n + 20)j^2 + (10n - 2)j^3 + (4n - 11)j^4 - 6j^5.$$

Upon evaluation, we find:

$$C_3(n) = 1, 2, 5, 15, 52, 202, 859, 3930, 19095, 97566, 520257, 2877834, 16434105, \dots$$



Which is entry A108304 in the Online Encyclopedia of Integer Sequences [23]. Recall that a 3-noncrossing may either be a 2-crossing or a noncrossing sequence. Through the continued fraction method we counted according to the number of 2-crossings, this sequence counts the number of partitions with at most 2-crossings.

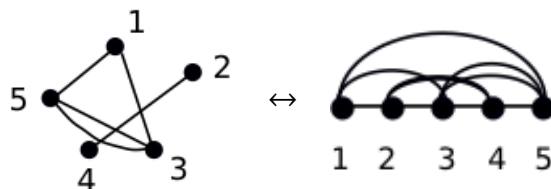
# Chapter 6

## Labeled Graphs

### 6.1 Representation as an arc annotated structure

A labeled graph on  $n$  vertices,  $\mathcal{G}_n$  can be represented by an even more generalized arc annotated sequence. Set partitions and matchings are each simplified cases of labeled graphs on  $n$  vertices,  $\mathcal{G}_n$ . Such a graph  $G_n \in \mathcal{G}_n$  can also form an arc annotated sequence. Our class of graphs  $G$  on  $\{1, 2, \dots, n\}$  allows use of multiple edges and singletons (isolated vertices), but not loops. This differs from previous nestings and crossings diagrams because singletons have been represented with a small loop.

**Example 6.1.1.** Consider the graph  $G$  on 5 vertices and its representation as an arc annotated sequence:



Crossings and nestings arise when vertices are placed on a horizontal line in increasing order. Crossings, nestings,  $k$ -crossings,  $k$ -nestings,  $k$ -noncrossings and  $k$ -nonnestings are defined as before.

*Simple* graphs  $\mathcal{G}_n^*$  on  $n$  vertices have the additional restriction that there are no multiple edges allowed. In such a case we can enumerate  $G_n^*$  because between every two vertices an

edge exists, or does not. Therefore,

$$G_n^* = 2^{\binom{n}{2}}.$$

However, by allowing multiple edges there is no upper bound on the number of possible graphs in our class  $G_n$ , so we consider instead the distribution of nestings and crossings. Because of this lack of limit on the edges adjacent to a vertex  $i$ , lattice path and tableaux bijections are not used. Instead, we turn to fillings of Ferrers diagrams.

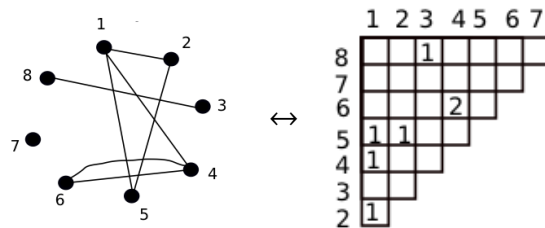
## 6.2 Distribution of nesting and crossing statistics

Recall that a filling of a Ferrers diagram consists of assigning a nonnegative entry to each of the cells in the diagram; by convention a cell filled with 0 is left empty. Let  $\mathcal{F}_\lambda$  be the set of fillings of Ferrers diagrams of shape  $\lambda = (n-1, n-2, \dots, 1)$  and  $\mathcal{G}_n$  be the set of multigraphs with no loops on  $n$  vertices. We will now describe the bijection:

**Bijection  $\Phi_{13}$**  (de Mier [6])

- $\mathcal{G}_n$ : the set of graphs on  $n$  vertices with multiple edges allowed but no loops.
- $\mathcal{F}_\lambda$ : the set of Ferrers diagrams of shape  $\lambda = (n-1, n-2, \dots, 1)$ .

**Example 6.2.1.** Consider the following graph  $G$  on 8 vertices and its corresponding Ferrers filling:



### Steps

1. For a graph  $G$  on  $n$  vertices, assign a Ferrers diagram of shape  $\lambda = (n-1, n-2, \dots, 2, 1)$
2. Consider two vertices  $i$  and  $j$ ,  $i < j$  connected by  $d \geq 0$  edges.

3. Fill the square of column  $i$  and row  $n - j + 1$  (the  $j^{\text{th}}$  row labeled from the bottom) with  $d$ .
4. Repeat until all pairs of vertices are considered.

**Theorem 6.2.1.** *The mapping  $\Phi_{13}$  from graphs on  $n$  vertices with multiple edges to Ferrers diagrams of shape  $\lambda = (n - 1, n - 2, \dots, 2, 1)$  is a bijection.*

*Proof.* We show the inverse direction,  $\Phi_{13} : \mathcal{F}_\lambda \rightarrow \mathcal{G}_n$ .

### Steps

1. Consider a Ferrers diagram of shape  $\lambda = (n - 1, n - 2, \dots, 2, 1)$ .
2. Given a graph  $G$  with  $n$  vertices, start with no edges.
3. Connect vertices  $i$  and  $j$ ,  $i < j$  with  $d$  edges if  $d$  is in the square in column  $i$  and the  $j^{\text{th}}$  row from the bottom.

□

From this bijection a great deal of information can be gained due to the filling of the corresponding Ferrers diagram regarding the crossings and nestings of its arc annotated sequence. First, however, recall that a Ferrers diagram may *contain* an  $s \times t$  0-1 matrix  $A$  if there is a selection of  $s$  rows and  $t$  columns in the Ferrers diagram such that if  $A_{i,j} = 1$ , then the square  $(r_i, c_j)$  in the Ferrers diagram is also nonempty. In figure 6.2.1, the squares in columns 1 and 2 and rows 4 and 5 show that both the  $2 \times 2$  identity and anti-identity matrices are contained.

Of greatest interest are Ferrers diagrams that do not contain the identity matrix,  $I_k$  and the anti-identity matrix,  $J_k$ . This is because, if the edges  $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$  are a  $k$ -nesting of the arc annotated sequence of  $G$ , then this filling of the Ferrers diagram of  $G$  contains the  $k \times k$  identity matrix in columns  $i_1, i_2, \dots, i_k$  and rows  $n - j_1 + 1, n - j_2 + 1, \dots, n - j_k + 1$ , and if  $G$  has a  $k$ -crossing then its Ferrers diagram contains the anti-identity matrix  $J_k$ .

Although results about the fillings of Ferrers diagrams were previously known by Krattenthaler [18], it was Anna de Mier [6] who noticed the connections between fillings of

Ferrers diagrams and the arc annotated sequences of labelled graphs. Krattenthaler had been studying growth diagrams, de Mier was able to take his results and extend them to give very nice results about the arc annotated sequences of graphs.

**Theorem 6.2.2.** *(De Mier [6]) For any Ferrers diagram and any  $n \in \mathbb{Z}$ , consider fillings of that diagram that add to  $n$ . Then for every  $k > 1$ , the number of such fillings that do not contain the identity matrix  $I_k$  equals the number of fillings that do not contain the anti-identity matrix  $J_k$ .*

From this the following results are immediate:

**Corollary 6.2.3.** *(De Mier [6]) The number of  $k$ -noncrossing graphs with  $n$  vertices and  $m$  edges is equal to the number of  $k$ -nonnesting such graphs.*

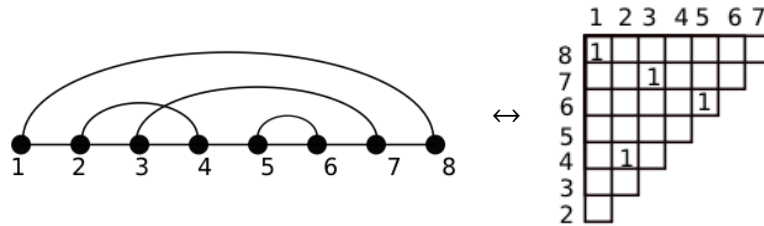
**Corollary 6.2.4.** *(De Mier [6]) The number of  $k$ -noncrossing simple graphs on  $n$  vertices with  $m$  edges equals the number of such  $k$ -nonnesting simple graphs.*

This result shows us that indeed, the remarkable property of equidistribution for crossings and nestings also holds in graphs.

Even when just considering the family of simple graphs, because each vertex may be adjacent to  $n - 1$  vertices, in the corresponding arc annotated sequence there are many different ‘types’ of vertices. For example, if we just consider all graphs on 4 vertices and only examine vertex 1, it could either have 0 arcs connected to it, be an opener as before, or could be an opener of ‘degree 2’ or of ‘degree 3’ if it were connected to two other, or all three of the other vertices. Thus even simple graphs will be a challenge to break down into a few cases to be bijected with paths in the lattice or tableaux. For this reason, these Ferrers diagrams are very important in proving equidistribution of nestings and crossings in  $\mathcal{G}_n$ .

This bijection  $\Phi_{11}$  between  $\mathcal{G}_n$  and  $\mathcal{F}_\lambda$  helps to illustrate the commonalities between graphs, partitions, and matchings. In particular, the concept of ‘matchings’ draws heavily from graph theory. With this in mind, a matching of  $\{1, 2, \dots, 2n\}$  can be viewed as a graph on  $2n$  vertices where each vertex is adjacent to exactly (at most for imperfect matchings) one edge. Then under  $\Phi_{11}$  the filling of the Ferrers diagram contains  $n$  1s and a no row or column may contain more than one filled square.

**Example 6.2.2.** *Consider the matching  $M = \{1, 8\}, \{2, 4\}, \{3, 7\}, \{5, 6\}$  represented as an arc annotated sequence, as a graph with 8 vertices, and as a Ferrers diagram of shape  $\lambda = (8, 7, \dots, 2, 1)$ :*



Partitions of  $\{1, 2, \dots, n\}$  can also be redrawn as graphs where edges connect consecutive elements in the same block. In this case, the corresponding graph must have no cycles. Under  $\Phi_{11}$  the filling of the Ferrers diagram has the condition that there is at most 1 square filled in each row and column. As few as 0 squares may be filled (the case  $P = 1|2|3|\dots|n-1|n$ ), and as many as  $n$  (when the partition is  $P = 123 \dots n-1n$ ).

**Example 6.2.3.** Consider the partition  $P = 1378|246|5$  and its arc annotated sequence. The arc annotated sequence has 5 nonsingleton arcs, and so the corresponding Ferrers diagram has five blocks filled with 1s.



# Chapter 7

## Permutations

### 7.1 Representation as an arc annotated sequence

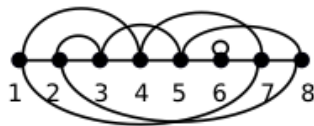
The final of our four families is the set of permutation of  $\{1, 2, \dots, n\}$ ,  $\mathfrak{S}_n$ . In this instance the convention of keeping the arcs above the horizontal line is abandoned. For this reason, we to define special additional statistics to study the arc annotated sequences of permutations.

Let  $\mathfrak{S}_n$  be the set of permutations on  $\{1, 2, \dots, n\}$ . Let  $\sigma \in \mathfrak{S}_n$  be denoted:

$$\sigma = \begin{bmatrix} & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{bmatrix}.$$

If  $i \leq \sigma(i)$  then an arc is drawn as before, above the horizontal line connecting  $i$  and  $\sigma(i)$ . This means that the element  $i$  is mapped to an element of  $\{1, 2, \dots, n\}$  of equal or greater size. Alternatively, if  $\sigma(i) < i$ , the arc connecting  $\sigma(i)$  and  $i$  is drawn *below* the horizontal line.

**Example 7.1.1.** Consider the permutation  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 5 & 7 & 8 & 6 & 1 & 2 \end{bmatrix}$  represented with an arc annotated sequence as follows:



## 7.2 Distribution of crossings and nestings

### Nesting and crossing statistics

We can count the number of nestings and crossings for each vertex:

$C_+(i)$ :	the number of crossing above the axis with their left end point $< i$ and their right end point $\geq i$ .
$C_-(i)$ :	the number of crossings below the axis with their left end point $< i$ and their right end point $> i$ .
$N_+(i)$ :	the number of nestings above the axis with their left end point $< i$ and their right end point $> i$ ( $i$ may be a singleton).
$N_-(i)$ :	the number of nestings below the axis with their left end point $< i$ and their right end point $> i$ .

These statistics are illustrated visually in Table 7.1:

$C_+(i)$	$C_-(i)$	$N_+(i)$	$N_-(i)$

Table 7.1:  $C_+(i), C_-(i), N_+(i), N_-(i)$

Note that nesting statistics are closely related to another statistic in permutations called *alignments* that were defined in [28] and used in [5] where they are referred to as  $A_+(i)$  and  $A_-(i)$ . Next we count the total number of upper and lower crossings and nestings in a permutation  $\sigma \in \mathfrak{S}_n$ :

$C_+(\sigma) = \sum_{i=0}^n  C_+(i) $	$C_-(\sigma) = \sum_{i=0}^n  C_-(i) $
$N_+(\sigma) = \sum_{i=0}^n  N_+(i) $	$N_-(\sigma) = \sum_{i=0}^n  N_-(i) $

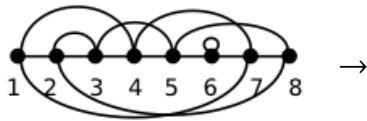


The total number of nestings and crossings in one permutation  $\sigma$ :

$$N(\sigma) = N_+(\sigma) + N_-(\sigma)$$

$$C(\sigma) = C_+(\sigma) + C_-(\sigma).$$

**Example 7.1.1 (continued)** We show the arc annotated sequence and previously defined nesting and crossing statistics for the permutation  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 5 & 7 & 8 & 6 & 1 & 2 \end{bmatrix}$ :



$i$	$C_+(i)$	$C_-(i)$	$N_+(i)$	$N_-(i)$
1	0	0	0	0
2	0	0	1	0
3	2	0	0	0
4	2	0	0	0
5	2	0	0	0
6	0	0	1	0
7	0	1	0	0
8	0	0	0	0
$\sigma$	6	1	2	0

We can count all of the crossings and nestings over all permutations of  $\{1, 2, \dots, n\}$ :

$C_+[n]$ : the total number of above crossings for all permutation on  $\{1, 2, \dots, n\}$ .

$$C_+[n] = \sum_{\sigma \in \mathfrak{S}_n} C_+(\sigma)$$

$C_-[n]$ : the total number of below crossings for all permutation on  $\{1, 2, \dots, n\}$ .

$$C_-[n] = \sum_{\sigma \in \mathfrak{S}_n} C_-(\sigma)$$

$N_+[n]$ : the total number of above nestings for all permutation on  $\{1, 2, \dots, n\}$ .

$$N_+[n] = \sum_{\sigma \in \mathfrak{S}_n} N_+(\sigma)$$

$N_-[n]$ : the total number of below nestings for all permutation on  $\{1, 2, \dots, n\}$ .

$$N_-[n] = \sum_{\sigma \in \mathfrak{S}_n} N_-(\sigma)$$

To see this, we first illustrate all the permutations of the set  $\{1,2,3,4\}$ , noting their statistics in a table. We use the vector notation  $[C_+(\sigma), C_-(\sigma); N_+(\sigma), N_-(\sigma)]$ .

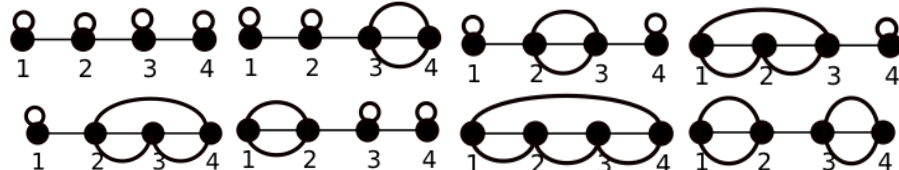
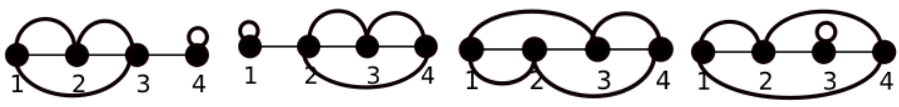
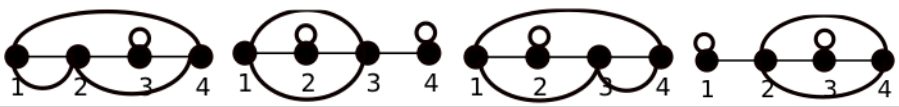
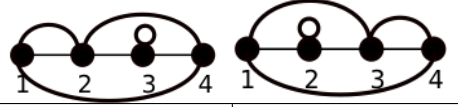
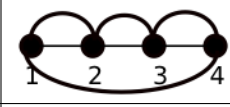
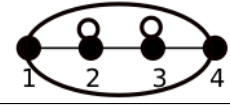
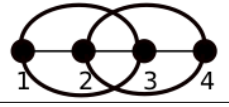
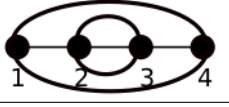
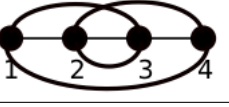
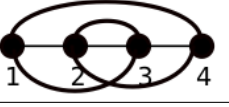
$[0,0;0,0]$							
							
$[1,0;0,0]$							
							
$[0,0;1,0]$							
							
$[1,0;1,0]$		$[2,0;0,0]$		$[0,0;2,0]$			
							
$[1,1;0,0]$		$[0,0;1,1]$		$[1,0;0,1]$		$[0,1;1,0]$	
							

Table 7.2: Nesting and Crossing values for all permutation in  $S_4$

From this, we start a table of  $C_+[n], C_-[n], N_+[n], N_-[n]$  as seen in Table 7.3.

In her paper, ‘Crossings and alignments of permutations,’ [5] Corteel proves equidistribution between crossing and nesting statistics, with the condition that the number of arcs above the horizontal axis are the same. Define the number of *weak exceedances* of a permutation  $\sigma$  to be the number of arcs above the horizontal line.

$n$	$C_+[n]$	$C_-[n]$	$N_+[n]$	$N_-[n]$
1	0	0	0	0
2	0	0	0	0
3	1	0	1	0
4	10	2	10	2
5	104	16	104	16

Table 7.3:  $C_+[n], C_-[n], N_+[n], N_-[n]$

**Theorem 7.2.1.** [5] *The number of permutations with  $k$  weak exceedances and  $l$  crossings and  $m$  nestings is equal to the number of permutations with  $k$  weak exceedances and  $l$  nestings and  $m$  crossings.*

This is done by noting the relationship between  $C_+(i)$  and  $N_+(i)$ , and  $C_-(i)$  and  $N_-(i)$  respectively. Then, through infinite continued fractions, the generating function is found. This is further examined in Section 7.3.

One major difference between permutations and the other three combinatorial families that we have examined is that extending the crossing and nesting statistics to become  $k$ -crossings or  $k$ -nonnestings causes some complications due to the set of arcs beneath the axis. Similar to the regular crossing and regular nesting case, however, we can naturally define:

$k_+$ -crossing:	a set of $k$ arcs above the axis that are mutually crossing. (a transitory vertex above the axis is considered a crossing.)
$k_-$ -crossing:	a set of $k$ arcs below the axis that are mutually crossing.
$k_+$ -nesting:	a set of $k$ arcs above the axis that are mutually nesting. (a singleton below an arc is considered a nesting.)
$k_-$ -nesting:	a set of $k$ arcs below the axis that are mutually crossing

We say a permutation  $\sigma$  has a  $k$ -crossing ( $k$ -nesting) if it has either a  $k_+$ -crossing or a  $k_-$ -crossing ( $k_+$ -nesting or  $k_-$ -nesting).

For example, the largest  $k$ -crossing that can occur on a permutation of length 4 or less is still a crossing as defined before, a 2-crossing. The first 3-crossing that occurs in a permutation of length 5 is a  $3_+$ -crossing and must have the form  $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & * & * \end{bmatrix}$ . Then  $\sigma(4)$  and  $\sigma(5)$  can either be 1 or 2 in one of two orders. Similarly for the first 3-nesting, it

occurs at on a permutation of length 5 as  $\left[ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & * & * \end{array} \right]$ . Again, the arcs on the bottom of the arc annotated sequence can be filled in in two difference ways. Thus, we know that the number of 3-crossings for permutations of length 5 equals the number of 3-nestings in permutations of length 5.

We generalize this.

**Theorem 7.2.2.** (Burrill and Mishna) Define:

$NE(n,k)$  : the number of permutations of  $\{1,2,\dots,n\}$  with largest nesting a  $k$ -nesting.

$CR(n,k)$ : the number of permutations of  $\{1,2,\dots,n\}$  with largest crossing a  $k$ -crossing.


Then,


$$NE\left(n, \left\lceil \frac{n}{2} \right\rceil\right) = \begin{cases} m! & \text{if } n = 2m + 1; \\ (m - 1)!(2m^2 - 1) + 2(m!) - 1 & \text{if } n = 2m. \end{cases}$$

*Proof.* Consider the first case when  $n = 2m + 1$ . We are looking for the number of permutations with largest nesting  $\left\lceil \frac{2m+1}{2} \right\rceil = m + 1$ . In order to have a such a permutation, we must use a singleton nested so the  $m + 1$  crossing occurs above the horizontal line. Therefore, the array will have the form:

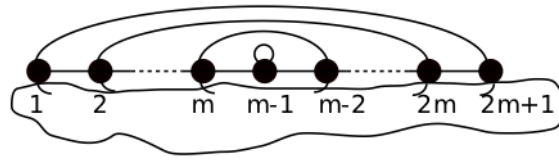
$$\sigma = \left[ \begin{array}{cccccccccc} 1 & 2 & \dots & m-1 & m & m+1 & \dots & 2m & 2m+1 \\ 2m+1 & 2m & \dots & m+1 & m & * & \dots & * & * \end{array} \right]$$

In the corresponding arc annotated sequence of a permutation, if the top of the a vertex

is an opener, then the vertex must have the shape: . The first  $m$  vertices look like this.

If the vertex is a closer above the axis, then it must have shape . The last  $m$  vertices look like this. The final vertex, the  $m + 1^{th}$  vertex is a singleton, and has no lower half. The figure below illustrates what we understand of the arc annotated sequence, with the cloud underneath the axis representing the unknown.

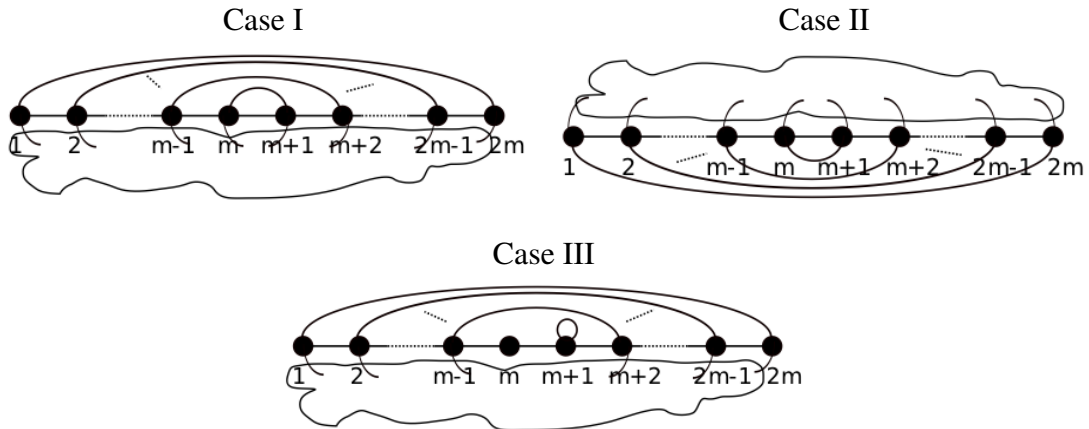
In order to enumerate the number of permutations with  $m + 1$  blocks, we simply count the number of ways of joining the arcs below the axis. Each of first  $m$  openers must join with one of the last  $m$  closers. Each time a connection is made, there is one less possibility.



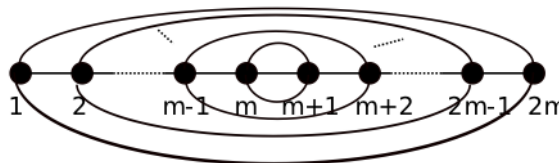
Thus:

$$NE(2m + 1, m + 1) = m!$$

Let  $n = 2m$ . We find the number of permutations of  $\{1, 2, \dots, 2m\}$  with a  $m$ -nesting. There are three cases:



In Case I there are  $m$  openers and  $m$  closers on the bottom that may be joined in  $m!$  ways. This is also true for Case II, with the arcs being joined on top. Note that we have overcounted the following permutation:



Case III still has an  $m$ -nesting using  $2m - 1$  arcs on the top, leaving a bonus vertex, depicted as being vertex  $m$  in the diagram above. This bonus vertex can occur before any of the other nesting, after all of the other nestings, or in any of the  $2m - 2$  positions, so there are  $2m$  possibilities for the bonus vertex to be in.

Without the bonus vertex, there are  $m - 1$  openers and  $m - 1$  closers that can be connected in  $(m - 1)!$  ways. However, for every possible set of arcs, the bonus vertex can also

be included in that block, or it can be in a block by itself as a singleton, so  $m$  possibilities. Then we have overcounted the case where there are two singletons in positions  $m$  and  $m - 1$ , where the  $m - 1$  openers and  $m - 1$  closers can be connected in  $(m - 1)!$  ways. Putting this together we have the result:

$$\begin{aligned} NE(2m, m) &= (m - 1)!(2m)(m) - (m - 1)! + m! + m! - 1 \\ &= (2m^2 - 1)(m - 1)! + 2(m!) - 1. \end{aligned}$$

□

Because this result can also be completed for permutations of size  $\{1, 2, \dots, n\}$  with  $\lceil \frac{n}{2} \rceil$  crossings by using transitory vertices instead of singletons for the final crossing, we get the following:

**Theorem 7.2.3.** For all  $n > 2$ ,  $NE\left(n, \lceil \frac{n}{2} \rceil\right) = CR\left(n, \lceil \frac{n}{2} \rceil\right)$ .

By running a computer procedure that counts the  $NE(n, k)$  and  $CR(n, k)$  for  $n$  up to 9 we have evidence to conjecture that:

**Conjecture** For all  $n > 2, k > 0, NE(n, k) = CR(n, k)$

Table 7.4 gives the results of the procedure:

$n \backslash k$	1	2	3	4	5	6
4	14	10				
5	42	76	2			
6	132	543	45			
7	429	3904	701	6		
8	1430	29034	9623	233		
9	4862	225753	126327	5914	24	

Table 7.4:  $NE(n, k)$  (and also  $CR(n, k)$ ) for small  $n$  and  $k$

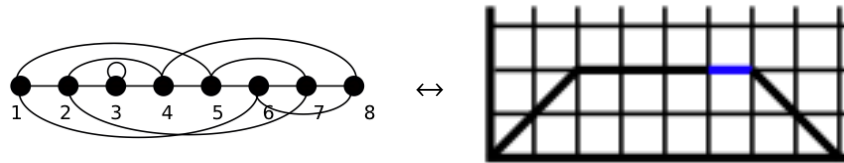
### 7.3 Bijections to bicolored weighted Motzkin paths

Next we will describe a bijection  $\Phi_{14a}$  between permutations of  $\{1, 2, \dots, n\}$  and bicolored weighted Motzkin paths of length  $n$ . This bijection will provide good computational results through continued fractions, and will aid in proving a later bijection  $\Phi_{16}$  between partially directed self avoiding walks and permutations.

**Bijection  $\Phi_{14a}$**  [21, 13]

- $\mathfrak{S}_n$ : permutations of  $\{1, 2, \dots, n\}$ .
- $\mathcal{M}_n^{<w_3>}$ : bicolored Motzkin paths of length  $n$  with weight vector  $w_3 = (w_1, w_2, \dots, w_n)$ . Each  $w_i$  may be up to the height of step  $i$ .

**Example 7.3.1.** Let  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 8 & 7 & 1 & 2 & 6 \end{bmatrix}$ . Then  $\sigma$  is represented with the following arc annotated sequence and bicolored weighted Motzkin path under  $\Phi_{14a}$ .



Above is  $\Phi_{14a}(\sigma)$  with weight vector  $w_3 = (0, 0, 2, 1, 0, 0, [1, 0], [0, 0])$ .

**Steps**

1. Traverse the arc annotated sequence of the permutation from left to right.
2. Generate the shape of the bicolored Motzkin path according to the following dictionary:

Vertex	Step	Vertex	Step

3. Assign weights to each step of the Motzkin path as follows:
  - (a) Assign weight 0 to all north steps/ transitory vertices.
  - (b) Consider the left most closer or transitory vertex  $i$ .
  - (c) If it is a closer, give labels  $0, 1, \dots, a_i$  to all available openers and transitories above the horizontal line to its left, and labels  $0, 1, \dots, b_i$  to all openers and transitories below the horizontal line to its left.
  - (d) If  $j$  is the vertex  $i$  is connected to on top with label  $a_j$  and  $k$  is the vertex connected to  $i$  on the bottom with label  $b_k$ , give step  $i$  the weight  $[a_j, b_k]$ .
  - (e)  $j, i, k$  are all considered unavailable. Repeat.
  - (f) If  $i$  is a transitory vertex (top or bottom), label all available openers and other transitory vertices (only top if  $i$  is top, only bottom if  $i$  is bottom) to the left and up to  $i$  with labels  $0, 1, \dots, a_i$ .
  - (g) If  $j$  is the vertex connected to  $i$  with label  $a_j$ , assign step  $i$  weight  $a_j$  in the Motzkin path.
  - (h)  $i, j$  are considered unavailable. Repeat until all available east and south east steps have been assigned weights.

**Theorem 7.3.1.** *The mapping  $\Phi_{14a}$  from permutations to bicolored weighted Motzkin paths is a bijection.*

*Proof.* The reverse mapping is similar to those done for  $\Phi_5$  and  $\Phi_{12}$  between Dyck paths and matchings, Motzkin paths and partitions. For further details, we direct the reader to [21]. □

In [5], Corteel assigns a different weight vector  $w_4$  to the the bijection  $\Phi_{14a}$  that keep track of both nestings and crossings, and also weak exceedances We call this bijection  $\Phi_{14b}$  because only the weight vector changes, and it is essentially the same bijection. Although more information is explicit with this new weight vector, we will can easily use  $\Phi_{14a}$  for computational purposes.

**Bijection  $\Phi_{14b}$**  [21, 13, 5]



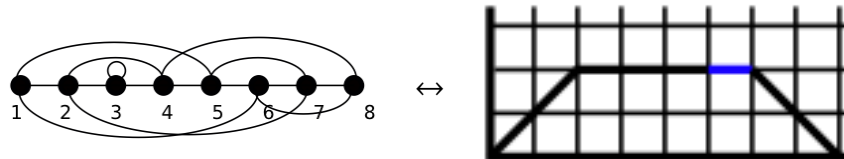
- $\mathfrak{S}_n$ : permutations of  $\{1, 2, \dots, n\}$ .
- $\mathcal{M}_n^{<w_4>}$ : bicolored Motzkin paths of length  $n$  with weight vector  $w_4 = (w_1, w_2, \dots, w_n)$  assigned based on the number of crossings and nestings at each vertex  $i$ .

If a step  $i$  in the Motzkin path under  $\Phi_{14a}$  is a north step or the colored east step corresponding to a transitory vertex above the axis in the Matching, then  $w_i = yp^{|A_+(i)|}q^{|C_+(i)|}$ . The  $y$  counts the weak exceedances,  $p$  keeps track of nestings above the axis and  $q$  keeps track of nestings below the axis. If  $i$  is a south step or an east step corresponding to a transitory vertex below the axis, then the weight assigned is  $w_i = p^{|A_-(i)|}q^{|C_-(i)|}$ . We illustrate the different weight vectors:

**Example 7.3.2.** Consider the permutation see in Example 7.3.1,

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 8 & 7 & 1 & 2 & 6 \end{bmatrix}.$$

Under  $\Phi_{14a}$  the weight vector  $w_3 = (0, 0, 2, 1, 0, 0, [1, 0], [0, 0])$  is given. The shape of the Dyck path is as before:



To get the weighting assigned by Corteel, consider each vertex of the arc annotated sequence from left to right. The arc  $\{1, 5\}$  is not crossed or nested by any arc that starts before it. Thus  $w_1 = y$ . Vertex 2 corresponds to a north step so we examine its arc above the axis,  $\{2, 4\}$  which is nested by  $\{1, 5\}$ . Thus  $w_2 = yp$ . We summarize the rest of the weights in a table:

Vertex	Type of vertex	Above or below axis	Arc	Nested with	Crossed with	$w_i$
1	opener	above	{1,5}	–	–	$y$
2	opener	above	{2,4}	{1,5}	–	$yp$
3	transitory	above	{3,3}	{1,5}, {2,4}	–	$yp^2$
4	transitory	above	{4,8}	–	{2,4}, {1,5}	$yp^2$
5	transitory	above	{5,7}	{4,8}	{1,5}	$ypq$
6	transitory	below	{1,6}	–	{2,7}	$q$
7	closer	below	{2,7}	–	{6,8}	$q$
8	closer	below	{6,8}	–	–	1

Thus,  $w_4 = (y, yp, yp^2, yp^2, yq^2, ypq, q, q, 1)$  is the vector associated to the Motzkin path of the permutation  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 8 & 7 & 1 & 2 & 6 \end{bmatrix}$  under  $\Phi_{14b}$ .

### 7.4 Distribution of other permutations statistics

Patterns also arise in permutations. These patterns are important in results about distribution of crossings and nestings. Specifically, in Theorem 7.4.1, the permutations with patterns  $2-31$  and  $31-2$  are shown to be in bijection with permutations with crossings and nestings. The pattern  $31-2$  is in a permutation if there is some  $i < j$  such that  $\sigma(i) > \sigma(j) > \sigma(i+1)$ . This means that there are two consecutive numbers in the permutation  $\sigma$  such that  $\sigma(i)$  is larger than  $\sigma(i+1)$ , and they are followed by the mapping of some  $i$  to  $j$  larger than  $i$  to between them.

**Example 7.4.1.** Let  $\sigma_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{bmatrix}$  and  $\sigma_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{bmatrix}$  and. For  $\sigma_1$ ,  $51-4$  and  $51-2$  are both occurrences of the  $31-2$  pattern. In  $\sigma_2$ ,  $53-4$  and  $41-2$  are both examples of the  $31-2$  pattern.

The pattern  $2-31$  occurs in a permutation if there is an  $i < j$  such that  $\sigma(j+1) < \sigma(i) < \sigma(j)$ . As was seen in the arc annotated sequences of permutations with  $31-2$  patterns, finding these patterns in arc annotated sequences is not immediate, so spotting them in the usual array representation of a permutations is the method of choice.

Finally, we define *descent* and *ascent* in permutations.

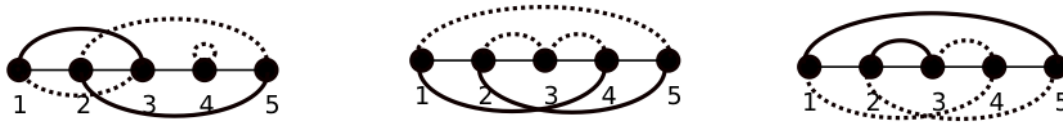


Figure 7.1: The arc annotated sequences of  $\sigma_1$  and  $\sigma_2$  with the  $31-2$  patterns illustrated by the dotted lines.

descent: an index  $i$  such that  $\sigma(i) > \sigma(i+1)$   
 ascent: an index  $i$  such that  $\sigma(i) < \sigma(i+1)$

**Example 7.4.2.** Consider  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{bmatrix}$ . There is a descent in positions 3 because  $\sigma(3) = 6 > 4 = \sigma(4)$ . Other descents occur at positions 4, 5 and 6. There is an ascent in position 1 because  $\sigma(1) = 3 < 5 = \sigma(2)$ . There is another ascents in position 2.

The distribution of these patterns in permutations are used to further understand the distribution of nestings and crossings in permutations.

**Theorem 7.4.1.** [5] The number  $B(n,k,l,m)$  of permutations  $\sigma$  of  $\{1,2,\dots,n\}$  with  $k$  weak exceedances,  $l$  crossings and  $m$  nestings is equal to the number  $D(n,k,l,m)$  of permutations of  $\{1,2,\dots,n\}$  with  $n-k$  descents,  $l$  occurrences of the pattern  $2-31$  and  $m$  occurrences of the pattern  $31-2$ .

Now we illustrate the bijection  $\Phi_{15}$  between these two types of permutations.

**Bijection  $\Phi_{15}$  (Corteel [5])**

- $B(n,k,l,m)$ : permutations of  $\{1,2,\dots,n\}$  with  $k$  weak exceedances,  $l$  crossings and  $m$  nestings.
- $D(n,k,l,m)$ : permutations of  $\{1,2,\dots,n\}$  with  $n-k$  descents,  $l$  occurrences of the pattern  $2-31$  and  $m$  occurrences of the pattern  $31-2$ .

**Example 7.4.3.** Under  $\Phi_{15}$ , the permutation  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 1 & 4 \end{bmatrix}$  is in bijection with

$$\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 4 & 2 & 3 \end{bmatrix}.$$

**Steps**

1. The permutation  $\tau$  is built from two tableaux,  $\tau_-$  which controls underneath the axis, and  $\tau_+$  which controls above the line, each with 2 rows.
2. Build  $\tau_-$  first:
  - (a) The first row contains all the entries of  $\sigma$  that are the beginning of a descent.
  - (b) Order the first row in increasing order.
  - (c) Give the second row all entries of  $\sigma$  that are the end of a descent, unordered.
  - (d) Find the  $(31-2)(i)$  sequence for  $\sigma$ .
  - (e) For each increasing entry  $i$  in the first row, place below it the  $((31-2)(i)+1)^{th}$  smallest entry of the second row not yet chosen.
3. Fill in those values of  $\tau_-$  into a partially completed  $\tau$ .
4. Partially build the arc annotated sequence.
5. Then build  $\tau_+$ :
  - (a) The first row contains all entries of  $\sigma$  that are the beginning of an ascent (we let  $\sigma(0) = 0$  and  $\sigma(n+1) = n+1$  so the last entry is considered the beginning of an ascent).
  - (b) Order the first row in increasing order.
  - (c) Give the second row all entries of  $\sigma$  that are not the end of a descent, unordered.
  - (d) Find the  $(2-31)(i)$  sequence of  $\sigma$ .
  - (e) For an entry  $i$  in the first row, if  $(2-31)(i) = k$ , that is the number of crossings that occur above the axis with vertex  $i$ ,  $C_+(i)$ .
6. Fill in the remaining entries of  $\tau$  from  $\tau_+$  such that the  $(2-31)(i) = C_+(i)$  sequence is satisfied. This is a unique.

**Theorem 7.4.2.** *The mapping  $\Phi_{15}$  is a bijection between permutations on  $\{1, 2, \dots, n\}$  with  $k$  descents,  $l$  occurrences of the  $31-2$  pattern,  $m$  occurrences of the  $2-31$  pattern and permutations of  $\{1, 2, \dots, n\}$  with  $k$  weak exceedances,  $m$  crossings and  $l$  nestings.*

*Proof.* The mapping  $\Phi_{15}$  can be reversed by changing the sequence of patterns to sequences of  $C(i)$  and  $N(i)$ , crossings and nestings.  $\square$

**Example 7.4.3 (continued)** We start with the permutation  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 1 & 4 \end{bmatrix}$ . It has  $k = 3$  descents,  $l = 2$  occurrences of the  $31 - 2$  pattern and  $m = 2$  occurrences of the  $2 - 31$  pattern. These are spotted from the array and noted explicitly below:

Descents	$31 - 2$	$2 - 31$
53	53 - 4	5 - 62
62	62 - 4	3 - 62
21		

From this table we can construct the  $(31 - 2)(i)$  and  $(2 - 31)(i)$  sequences. Note that we define  $\sigma(0) = 0$  and  $\sigma(n + 1) = n + 1$  so that the patterns can be understood for all elements of the permutation:

$$((31 - 2)(1), (31 - 2)(2), \dots, (31 - 2)(6)) = (0, 0, 0, 2, 0, 0)$$

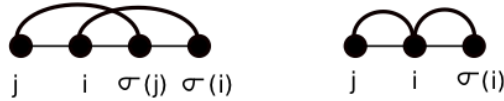
$$((2 - 31)(1), (2 - 31)(2), \dots, (2 - 31)(6)) = (0, 0, 1, 0, 1, 0)$$

If an integer in the permutation is the ‘2’ in  $d \geq 0$  appearances of the pattern  $31 - 2$  then the  $i^{\text{th}}$  entry in the  $(31 - 2)(i)$  sequence is  $d$ . Similarly for the pattern  $2 - 31$ .

Now we construct our tableaux,  $\tau_-$ , calling it  $\tau_-^*$  until it is completed. From our table, we know that 5, 6 and 2 are each the beginning of a descent:  $\tau_-^* = \begin{bmatrix} 5 & 6 & 2 \\ * & * & * \end{bmatrix}$ . We order the first row and place in the unordered bottom row the ends of the descents, 3, 2 and 1:  $\tau_-^* = \begin{bmatrix} 2 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}$ . Next, since  $(31 - 2)(2) = 0$ , 5 is given the smallest available entry. Also  $(31 - 2)(5) = 0$  and  $(31 - 2)(6) = 0$  so  $\tau_-$  has the second row simply in increasing order:  $\tau_- = \begin{bmatrix} 2 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ .

We start to construct  $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ * & 1 & * & * & 2 & 3 \end{bmatrix}$ . Now  $\tau_+^*$  is has its first row made up of all ascents in the permutation  $\sigma$  in increasing order. Thus:  $\tau_+^* = \begin{bmatrix} 1 & 3 & 4 \\ * & * & * \end{bmatrix}$ .

Because of  $(2 - 31)(1) = 0$ ,  $(2 - 31)(3) = 1$  and  $(2 - 31)(4) = 0$  we know that only  $i = 3$  will have a crossing that looks as follows:



We construct the partial arc annotated sequence and realize that indeed there is only one way for three arcs to connect on the top such that the arc extending from 3 crosses with an arc that started before it. Thus:



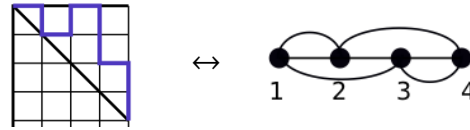
Therefore, our final permutation  $\tau$  as an array is  $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 4 & 2 & 3 \end{bmatrix}$ . Inspecting the arc annotated sequence we see that there are indeed  $n - k = 6 - 3 = 3$  weak exceedances,  $m = 2$  crossings and  $l = 2$  nestings as expected.

There exists a bijection like  $\Phi_6$  between permutations and partially directed walks in the symmetric wedge, however it does not give a meaningful relationship of the crossings in the arc annotated to a statistic in the walk. However, in his paper, [21], Rubey gives a direct bijection by Phillippe Nadeau between walks in the asymmetric wedge with permutations according to patterns :

**Bijection  $\Phi_{16}$**  (Nadeau [21])

- $\mathcal{W}_n^{N,M}$ : Partially directed walks in the asymmetric wedge with  $N$  north steps,  $n$  east steps and length of the last descent equal to  $M$ .
- $\mathfrak{S}_n^{N,M}$ : Permutations of  $\{1, 2, \dots, n\}$  with  $N$  occurrences of the patter  $31 - 2$  and 1 mapped to  $M$ .

**Example 7.4.4.** *The following partially directed self avoiding walk is in bijection with the permutation  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix}$ :*



**Steps**

1. Let  $W$  be a partially directed self avoiding walk with  $n$  east steps.
2. Let  $R = \{1, 2, \dots, n\}$ .
3. Let  $k_i$  be the  $y$ -coordinate of the  $(n - i + 1)^{st}$  east step of  $W$
4. For  $i \in R$  let  $h_i = 1 - k_i$
5.  $\sigma(i)$  is the  $h$  **largest** element of  $R$ .
6. Delete  $\sigma(i)$  from  $R$ .

**Theorem 7.4.3.** *The mapping  $\Phi_{16}$  is a bijection between  $\mathfrak{S}_n^{N,M}$  and  $\mathcal{W}_n^{N,M}$*

*Proof.* We direct the reader to [21] □

**Example 7.4.4 (continued)** In our partially directed walk we have  $n = 4$  east steps. Start with  $R = \{1, 2, 3, 4\}$ . The  $(4 - 1 + 1)^{st} = 4^{th}$  east step is at  $y$ -coordinate  $k_1 = -2$ . Then,  $h_1 = 1 - (-2) = 3$ , so the 3rd largest element of  $R$  is 2, thus  $\sigma(1) = 2$ . We summarize the rest of the steps with a table:

$i$	$y$	$(n - i + 1)$	$k_i$	$R$	$h_i$	$\sigma(i)$
1	0	4	-2	{1, 2, 3, 4}	$1 - (-2) = 3$	2
2	-1	3	0	{1, 3, 4}	$1 - 0 = 1$	4
3	0	2	-1	{1, 3}	$1 - (-1) = 2$	1
4	-2	1	0	{3}	$1 - 0 = 1$	3

So indeed the walk  $W$  under  $\Phi_{16}$  gives the permutation  $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix}$  as already stated.

## 7.5 Enumerative results

The only closed form for a counting of permutations according to nestings and crossings that we located in the literature actually made no mention of crossings, and instead counted according to alignments and weak exceedances:

**Theorem 7.5.1.** [28] *The number of permutations of  $\{1, 2, \dots, n\}$  with  $k$  weak exceedances and  $l$  alignments is the coefficient of  $q^{(k-1)(n-k)-l}$  in*

$$E_{n,k}(q) = q^{k-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n q^{k(i-1)} \left( \binom{n}{i} q^{k-i} + \binom{n}{i-1} \right).$$

In Section 3.2 we described the continued fraction representation of Motzkin path according to maximum height. We can use this representation to count permutations of  $\{1, 2, \dots, n\}$  according to number of nestings through bijection  $\Phi_{14a}$ . Recall that when  $i$  indicates the starting height of a step:

1.  $a_i$  corresponds to north east steps,
2.  $b_i$  to south east steps,
3. and  $c_i$  to east steps.

Let  $P^{[h]}$  be the set of Motzkin paths of height at most  $h$ , where  $h \geq 0$ .

$$P^{[h]} = \frac{1}{1 - c_0 - \frac{a_0 b_1}{1 - c_1 - \frac{a_1 b_2}{\ddots \frac{1}{1 - c_h}}}}$$

If we are simply looking for the number of Motzkin paths based on maximum height and length, then at each  $P^{[i]}$  the  $a_i, b_i$  and  $c_i \rightarrow x$ . For example, the coefficient of  $x^n$  in

$$P^{[2]}(x) = \frac{1}{1 - x - \frac{x^2}{1 - x - \frac{x^2}{1 - x}}}$$

gives the number of Motzkin paths of length  $n$  with maximum height 2.



But we count the permutations according to both length and number of nestings. We assign the coefficient of  $x^n y^i$  to be the number of permutations of length  $n$  with  $i$  nestings. In order to understand this each type of step will receive a different mapping.

Let  $h_i$  be the height at each step  $i$  in the path and  $w_i$  be its weight. The weight of every north east step will be  $w_i = 0$  because an opener will not create a nesting above it. An east step could have a weighting of 0 up to  $h_i$ . This is because there can be up to  $h_i$  nesting occur. Finally, a south east step is assigned two weights, each between  $[0, h_i - 1]$ . This weighting is summarized as follows:

$$\begin{aligned} a_i &\rightarrow x \\ b_i &\rightarrow (1 + y + y^2 + \dots + y^{i-1})^2 x \\ c_i &\rightarrow (1 + y + y^2 + \dots + y^i) x \end{aligned}$$

We direct the reader to the appendix for the Maple code that was used to get the continued fraction in this section, and the results in the next section.

The number of permutation on  $\{1, 2, \dots, n\}$  with  $k$  nestings is counted by the coefficient of  $[x^n y^k] S(x, y)$  in the following series, whose result comes from the corresponding Motzkin path of maximum height  $h = 10$ :

$$\begin{aligned} S(x, y) = & 1 + (1)x + (2)x^2 + (5 + y)x^3 + (14 + 8y + 2y^2)x^4 + (y^4 + 7y^3 + 25y^2 + 45y + 42)x^5 + \\ & (132 + 220y + 198y^2 + 112y^3 + 44y^4 + 12y^5 + 2y^6)x^6 + \\ & (y^9 + 9y^8 + 42y^7 + 140y^6 + 352y^5 + 700y^4 + 1092y^3 + 1274y^2 + 1001y + 429)x^7 \\ & + (1430 + 4368y + 7280y^2 + 8400y^3 + 7460y^4 + 5392y^5 + 3262y^6 + 1664y^7 + 716y^8 + \dots) \end{aligned}$$

When the arc annotated sequences are drawn for the first few  $n$ , there is agreement with our continued fraction. All 24 permutations of  $\{1, 2, 3, 4\}$  were seen in Table 7.2. All permutations in the sections 1), 2) 5) and 10) had no nestings. There are 14 such permutations. In sections 3), 4), 7) and 8) each permutation had 1 nesting. There are 8 permutations in these categories. In sections 6) and 9) the permutations had 2 nestings, for a total of two such permutations. Because of this, we agree that the coefficient of  $x^4$  should

be  $(14 + y^8 + 2y^2)$  as was found computationally.

Our computation gave us  $(2y^2 + 8y + 14)$  and indeed, there are 14 permutations with no nestings, 8 with one nestings and 2 with 3 nestings as expected.

Ideally we would be able to have our continued fraction extend of indefinitely, so that our maximum height condition was not required. As this is not possible computationally, we should continue to search for a closed form for the multivariate generating function of permutations with nestings as a constraint.

Nevertheless we can now create the table to show the triangular sequence for the coefficients of  $y^k x^n$ :

	1	$y$	$y^2$	$y^3$	$y^4$	$y^5$	$y^6$	$y^7$	$y^8$	$y^9$
1	1									
$x$	1									
$x^2$	2									
$x^3$	5	1								
$x^4$	14	8	2							
$x^5$	42	45	25	7	1					
$x^6$	132	220	198	112	44	12	2			
$x^7$	429	1001	1274	1092	700	352	140	42	9	1

Table 7.5: Triangular sequence of  $[y^k x^n]$  for permutations on  $\{1, 2, \dots, n\}$  with  $k$  nestings

As was the case with matchings and set partitions that we also represented with weighted paths in the lattice in bijections  $\Phi_5$  and  $\Phi_{12}$ , we can also find the average number of permutations according to both length and the number of nestings using

$$\mathbb{E}_{\mathcal{A}_n} = \frac{[z^n] \partial_y A(x, y)|_{y=1}}{[z^n] A(x, 1)}.$$

Then using a Maple procedure, see Appendix, we get the following sequence of averages for permutations of length  $1, 2, 3, 4, \dots, 9$  whose corresponding Motzkin path has maximum height  $h = 10$ :

$$0, 0, \frac{1}{6}, \frac{1}{2}, 1, \frac{5}{3}, \frac{5}{2}, \frac{7}{2}, \frac{14}{3}, \frac{10886380}{1814399}, \frac{149615414}{19954839}, \frac{545006175}{59689787}, \frac{21958832860}{2034464481}, \dots$$

As the maximum height of the corresponding Motzkin path under  $\Phi_{14a}$  increases, so do

the average number of nestings. We plot the average number of nestings verses length of the permutation for corresponding Motzkin paths of height 1 through 10:

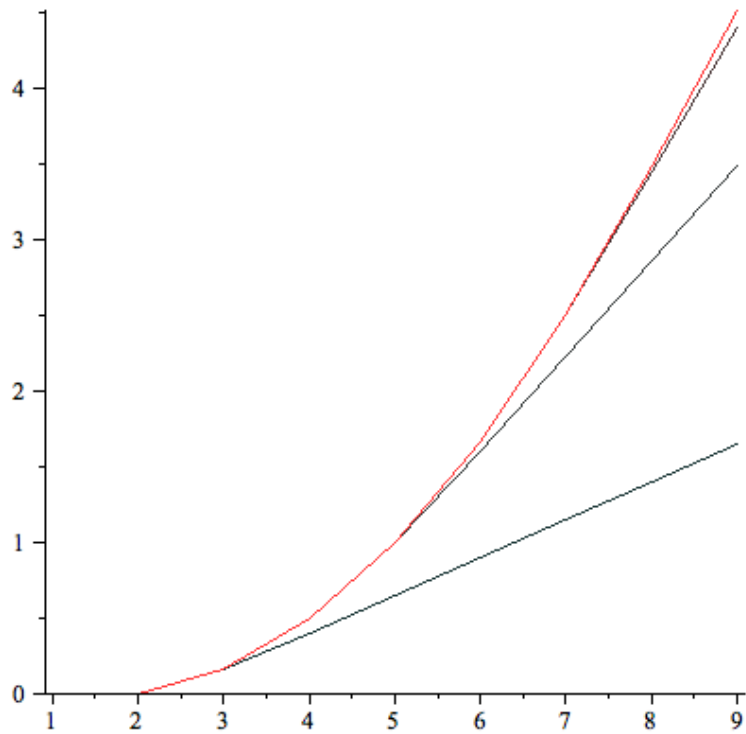


Figure 7.2: Each curve is a different maximum height of the Motzkin path corresponding to the permutation under  $\Phi_{16a}$ .

## **Part III**

# **Conclusion**

## 7.6 Big Picture

Each of matchings, partitions and permutations can be represented with a continued fraction expansion of Motzkin paths of maximum height. Using this, we give a **single** multivariate generating function by way of the following substitutions:

$$\begin{aligned} a_i &\rightarrow x \\ b_i &\rightarrow \frac{1}{2}q(q-1)(1-y^i)\frac{x}{(1-y)} - q(q-2)(1-y^i)\frac{x}{(1-y)} + \frac{1}{2}(q-1)(q-2)(1-y^i)^2\frac{x}{(1-y)^2} \\ c_i &\rightarrow -q(q-2)\left(x + \frac{x(1-y^i)}{(1-y)}\right) + \frac{1}{2}(q-1)(q-2)\left((1-y^{(i+1)})\frac{x}{(1-y)} + \frac{(1-y^i)x}{(1-y)}\right) \end{aligned}$$

Using these substitutions into the continued continued fraction, see Appendix, we get the following series expansions by varying  $q$  which agree with our previously seen computations:

Matchings  $q = 2$ :

$$B(x, y) = 1 + 1x^2 + (2 + y)x^4 + (5 + 6y + 3y^2 + y^3)x^6 + \dots$$

Partitions  $q = 1$ :

$$P(x, y) = 1 + x + 2x^2 + 5x^3 + (y + 14)x^4 + (y^2 + 9y + 42)x^5 + \dots$$

Permutations  $q = 0$ :

$$S(x, y) = 1 + 1x + 2x^2 + (5 + y)x^3 + (14 + 8y + 2y^2)x^4 + \dots$$

This single generating function highlights the common lattice path bijection and illustrates how useful this continued fraction machinery is. By changing the variable substitutions based on varying definitions of nestings and crossings, we are able to enumerate matchings, partitions, and permutations accordingly.

Furthermore, we can summarize the main results regarding equidistribution into a table using generic theorems as follows:

**Theorem A:** The number of objects of size  $n$  with no 2-crossing is in bijection with the number of objects of size  $n$  with no 2-nesting.

**Theorem B:** The number of objects of size  $n$  with  $k$  2-crossings is in bijection with the number of objects of size  $n$  with  $k$  2-nesting.

**Theorem C:** The number of objects of size  $n$  with  $i$  crossings and  $j$  nestings is equal to the number of objects of size  $n$  with  $j$  nestings and  $i$  crossings.

**Theorem D:** Let  $N(n, i, j)$  be the number of objects of size  $n$  with maximum nesting of size  $i$  and maximum crossing of size  $j$ . Then  $N(n, i, j) = N(n, j, i)$ .

Object	Theorem A	Theorem B	Theorem C	Theorem D
Matchings	Folklore page 26	de Sainte-Catherine [8] page 26	Chen et al. [3] page 26	Chen et al. [3] 26
Partitions	Folklore page 62	Foata & Zeilberger (variation) [13] page 51	Chen et al. [3] page 7	Chen et al. [3] page 7
Graphs			de Mier [6] page 74	
Permutations *WEX		Foata & Zeilberger [13] page 85	Corteel [5] page 79	Conjecture page 83

Table 7.6: Summary table.

\*Note that WEX indicates that weak exceedances must also be equivalent in order for results of equidistribution.

These collections of objects beg for a more general theory. One such first attempt is the study of chains in growth diagrams of Krattenthaler [18]. To enter into details is beyond the scope of this project, however we can state his main theorem and try to put it in context:

**Theorem 7.6.1.** *Let  $N(F; n; NE = s, SE = t)$  be the number of  $0 - 1$  fillings of the Ferrers shape  $F$  with exactly  $n$  1's such that there is at most one 1 in each column and in each row and such that the longest NE-chain has length  $s$  and the longest SE chain, the smallest*

rectangle containing the chain being contained in  $F$ , has length  $t$ . Then, for any Ferrers shape  $F$  and positive integers  $s$  and  $t$  we have:

$$N(F; n; NE = s, SE = t) = N(F; n; NE = t; SE = s).$$

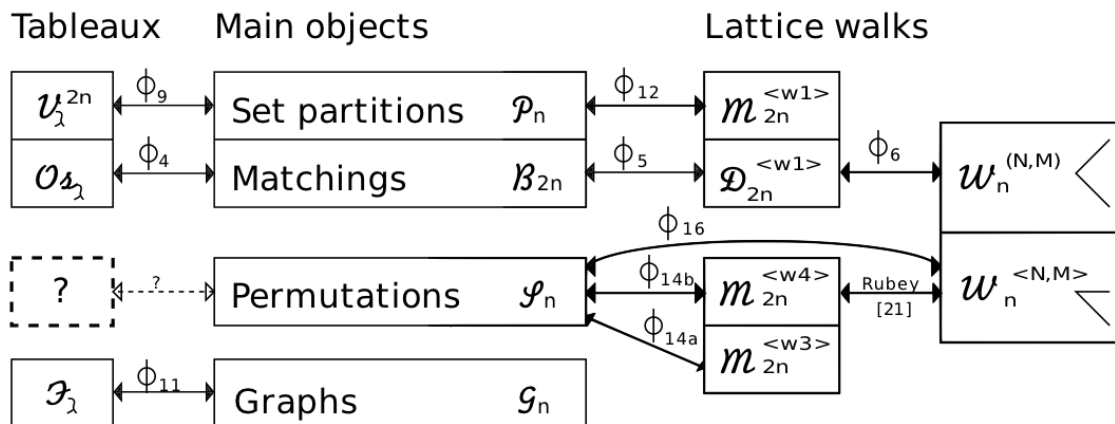
Again, we will not give precise details of NE chains, but rather list what they represent in several key cases. Let

Object	Diagram shape	NE chain	SE chain	Restrictions
Matchings	triangular	$k$ -nesting	$k$ -crossing	one $1 \in (\text{row } i \cup \text{column } i)$
Set partitions	triangular	$k$ -nesting	$k$ -crossing	one $1 \in \text{row } i$ ; one $1 \in \text{column } i$
Graphs	triangular	$k$ -nesting	$k$ -crossing	no restriction
Permutations	square	longest increasing subsequence	longest decreasing subsequence	

Table 7.7: Objects represented as growth diagrams, and interpretations of chains.

## 7.7 Map

We give a map that encompasses many of the bijections seen in Part II, followed by a table that gives the description of the combinatorial objects involved:



Main objects:	
$\mathcal{B}_{2n}$	matchings of $\{1, 2, \dots, 2n\}$
$\mathcal{B}_{2n}^{<nm>}$	nonnesting matchings of $\{1, 2, \dots, 2n\}$
$\mathcal{B}_{2n}^{<nc>}$	noncrossing matchings of $\{1, 2, \dots, 2n\}$
$\mathcal{P}_n$	set partitions of $\{1, 2, \dots, n\}$
$\mathcal{P}_n^{en}$	set partitions of $\{1, 2, \dots, n\}$ with enhanced crossings and nestings
$\mathcal{P}_n^{<nm>}$	nonnesting set partitions of $\{1, 2, \dots, n\}$
$\mathcal{P}_n^{<nc>}$	noncrossing set partitions of $\{1, 2, \dots, n\}$
$\mathcal{G}_n$	graphs on $n$ vertices with no singletons, multiple edges allowed.
$\mathcal{S}_n$	permutations of $\{1, 2, \dots, n\}$
Tableaux:	
$\mathcal{V}_\lambda^{2n}$	vacillating tableaux of length $2n$ with shape $\lambda$
$\mathcal{H}_\lambda$	hesitating tableaux
$\mathcal{O}_{s_\emptyset}$	oscillating tableaux



Walks in the lattice:	
$\mathcal{D}_{2n}$	Dyck paths of length $2n$ .
$\mathcal{D}_{2n}^{\langle w1 \rangle}$	weighted Dyck paths of length $2n$ with weight vector $w1$ .
$\mathcal{D}_{2n}^{**}$	pairs of non crossing Dyck paths of length $2n$ .
$\mathcal{D}_{2n}^{N,M}$	Dyck paths of length $2n$ with total weight $N$ and $M$ the position of the first south step with weight 0.
$\mathcal{M}_{2n}^{\langle wi \rangle}$	bicolored weighted Motzkin paths of length $n$ and weight vector $wi$ .
$\mathcal{W}_n^{(N,M)}$	partially directed walks in the symmetric wedge with $N$ north steps, $n$ east steps and length of the last descent equal to $M - 1$ .
$\mathcal{W}_n^{\langle N,M \rangle}$	partially directed walks in the asymmetric wedge with $N$ north steps, $n$ east steps and length of the last descent equal to $M$ .
$\mathcal{F}_\lambda$	fillings of Ferrers diagrams of shape $\lambda$ .

## 7.8 Open questions

There are still many questions that have arisen that we would like to answer. While  $k$ -noncrossing matchings were in bijection with fans of Dyck paths for enumerative results, is there a set of objects that  $k$ -noncrossing (nonnesting) set partitions are in bijection with that may lead to enumerative results? A closed form for the number of partitions according to nestings (crossings) still remains open. Likewise for permutations according to nestings. Finding a method to extend the weighted Dyck/Motzkin path bijections and continued fraction computational results to count according to 3-nestings ( $k$ -nestings?) in matchings, set partitions, and permutations would be very beneficial. For graphs, enumerative results are still elusive, so perhaps if we restricted to the simple case we could enumerate according to both the number of vertices and nestings. We have two conjectures:

**Conjecture 2.** When  $NE(n,k)$  counts the number of permutations on  $\{1,2,\dots,n\}$  with a  $k$  nestings, and  $CR(n,k)$  counts the number of permutations on  $\{1,2,\dots,n\}$  with a  $k$ -crossing,

$$NE(n,k) = CR(n,k).$$

**Conjecture 3.** *2-crossings and 2-nestings are equidistributed among set partitions of  $\{1, 2, \dots, n\}$  with  $k$  blocks.*

Finally, are there other structures that can be represented with arc annotated sequence that might take advantage of the results of Krattenthaler to show equidistribution?

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# Appendix

## Matchings

The following code was used to find the average number of nestings in matchings of  $\{1, 2, \dots, 2n\}$ :

```
with(plots);
t := 20;
M1 := proc (h)
local current, i, M5, M, L, L1, A, P, B, B1, A1, Q, Dn, R;
option remember;
current := 0;
for i from h by -1 to 1 do
current := a[i-1]*b[i]/(1-c[i]-current)
end do;
M5 := 1/(1-c[0]-current);
for i from 0 to h do
M := subs(a[i] = x, M5);
M5 := M;
M := subs(b[i] = (1-y^i)*x/(1-y), M5);
M5 := M;
M := subs(c[i] = 0, M5);
M5 := M
end do;
L := map(series, series(M5, x, t), y, t);
```

```

L1 := convert(L, polynom);
A := convert(L1, polynom);
P := seq(coeff(A, x, i), i = 1 .. t-1);
B := diff(A, y);
B1 := subs(y = 1, B);
B := B1;
A1 := subs(y = 1, A);
A := A1;
Q := seq(coeff(B, x, i), i = 1 .. t-1);
P := seq(coeff(A, x, i), i = 1 .. t-1);
Dn := seq(Q[2*i]/P[2*i], i = 1 .. (1/2)*t-1)
end proc

```

Next we plot this using the following procedure:

```

for h to 10 do
R[h] := listplot([seq([2*i, M1(h)[i]], i = 1 .. (t-1)*1/2)],
    color = COLOR(RGB, (1/11)*h, .2000, .2000), legend
    = cat("max height=", h))
end do;
display(seq(R[k], k = 1 .. 10))

```

Using  $t=26$  and this code:

```

for h to 15 do
R[h] := listplot([seq([2*i, M1(h)[i]], i = 1 .. (t-1)*1/2)],
    color = COLOR(RGB, (1/11)*h, .2000, .2000))
end do;
display(seq(R[k], k = 1 .. 15))

```

we get the following plot for the average number of nestings according to the length of the matching.

## Partitions

The following code was used to find the average number of nestings in partitions of  $\{1, 2, \dots, n\}$ :

```

with(plots);
t := 20;
M1 := proc (h)
local current, i, M5, M, L, L1, A, P, B, B1, A1, Q, Dn, R;
option remember;
current := 0;
for i from h by -1 to 1 do
current := a[i-1]*b[i]/(1-c[i]-current)
end do;
M5 := 1/(1-c[0]-current);
for i from 0 to h do
M := subs(a[i] = x, M5);
M5 := M;
M := subs(b[i] = (1-y^i)*x/(1-y), M5);
M5 := M;
M := subs(c[i] = x+(1-y^i)*x/(1-y), M5);
M5 := M
end do;
L := map(series, series(M5, x, t), y, t);
L1 := convert(L, polynom);
A := convert(L1, polynom);
P := seq(coeff(A, x, i), i = 1 .. t-1);
B := diff(A, y); B1 := subs(y = 1, B);
B := B1;
A1 := subs(y = 1, A);
A := A1;
Q := seq(coeff(B, x, i), i = 1 .. t-1);
P := seq(coeff(A, x, i), i = 1 .. t-1);
Dn := seq(Q[i]/P[i], i = 1 .. t-1)
end proc

```

Then using the following code we get a plot for the average number of nestings in partitions of  $\{1, 2, \dots, n\}$ .



```

for h to 10 do
R[h] := listplot([seq([2*i, M1(h)[i]], i = 1 .. (t-1)*1/2)],
  color = COLOR(RGB, (1/11)*h, .2000, .2000), legend
  = cat("max height=", h))
end do;
display(seq(R[k], k = 1 .. 10))

```

## Permutations

```

with(plots);
t := 10;
M1 := proc (h)
local current, i, M5, M, L, L1, A, P, B, B1, A1, Q, Dn, R;
option remember;
current := 0;
for i from h by -1 to 1 do
current := a[i-1]*b[i]/(1-c[i]-current)
end do;
M5 := 1/(1-c[0]-current);
for i from 0 to h do
M := subs(a[i] = x, M5);
M5 := M;
M := subs(b[i] = (1-y^i)^2*x/(1-y)^2, M5);
M5 := M;
M := subs(c[i] = (1-y^(i+1))*x/(1-y)+(1-y^i)*x/(1-y), M5);
M5 := M end do;
L := map(series, series(M5, x, t), y, t);
L1 := convert(L, polynom);
A := convert(L1, polynom);
P := seq(coeff(A, x, i), i = 1 .. t-1);
B := diff(A, y); B1 := subs(y = 1, B);
B := B1;

```

```

A1 := subs(y = 1, A);
A := A1;
Q := seq(coeff(B, x, i), i = 1 .. t-1);
P := seq(coeff(A, x, i), i = 1 .. t-1);
Dn := seq(Q[i]/P[i], i = 1 .. t-1)
end proc

```

And then the plot of averages again for permutations:

```

for h to 15 do
R[h] := listplot([seq([i, M1(h)[i]], i = 1 .. t-1)],
    color = COLOR(RGB, (1/11)*h, .2000, .2000))
end do;
display(seq(R[k], k = 1 .. 15))

```

## Matchings, partitions and permutations

We use a  $q$  polynomial to show the connection between matchings, partitions, and permutations. The code used to find both the number of matchings ( $q=2$ )/partitions ( $q=1$ )/permutations ( $q=0$ ) according to the number of nestings and the average number of nestings (Maple):

```

with(plots); t := 20;
M1 := proc (h, q) l
local current, i, M5, M, L, L1, A, P, B, B1, A1, Q, Dn, R;
option remember;
current := 0; for i from h by -1 to 1 do
current := a[i-1]*b[i]/(1-c[i]-current)
end do;
M5 := 1/(1-c[0]-current);
for i from 0 to h do
M := subs(a[i] = x, M5);
M5 := M;

```

```

M := subs(b[i] = (1/2)*q*(q-1)*(1-y^i)*x/(1-y)-q*(q-2)*(1-y^i)*x/(1-y)+
(1/2)*(q-1)*(q-2)*(1-y^i)^2*x/(1-y)^2, M5);
M5 := M;
M := subs(c[i] = -q*(q-2)*(x+(1-y^i)*x/(1-y))+
(1/2)*(q-1)*(q-2)*((1-y^(i+1))*x/(1-y)+(1-y^i)*x/(1-y)), M5);
M5 := M
end do;
L := map(series, series(M5, x, t), y, t);
L1 := convert(L, polynom);
A := convert(L1, polynom);
P := seq(coeff(A, x, i), i = 1 .. t-1);
print('total*nestings', P);
B := diff(A, y);
B1 := subs(y = 1, B);
B := B1;
A1 := subs(y = 1, A);
A := A1;
Q := seq(coeff(B, x, i), i = 1 .. t-1);
P := seq(coeff(A, x, i), i = 1 .. t-1);
print('deriv', Q);
print('total', P);
if q = 2 then
Dn := seq(Q[2*i]/P[2*i], i = 1 .. (1/2)*t-1)
else
Dn := seq(Q[i]/P[i], i = 1 .. t-1)
end if;
print('Average', Dn)
end proc

```