GLIMPSES OF INFINITY
INTUITIONS, PARADOXES, AND COGNITIVE LEAPS

by

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ABSTRACT

This dissertation examines undergraduate and graduate university students' emergent conceptions of mathematical infinity. In particular, my research focuses on identifying the cognitive leaps required to overcome epistemological obstacles related to the idea of actual infinity. Extending on prior research regarding intuitive approaches to set comparison tasks, my research offers a refined analysis of the tacit conceptions and philosophies which influence learners' emergent understanding of mathematical infinity, as manifested through their engagement with geometric tasks and two well known paradoxes – Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum. In addition, my research sheds new light on specific features involved in accommodating the idea of actual infinity.

The results of my research indicate that accommodating the idea of actual infinity requires a leap of imagination away from 'realistic' considerations and philosophical beliefs towards the 'realm of mathematics'. The abilities to clarify a separation between an intuitive and a formal understanding of infinity, and to conceive of 'infinite' as an answer to the question 'how many?' are also recognised as fundamental features in developing a normative understanding of actual infinity. Further, in order to accommodate the idea of actual infinity it is necessary to understand specific properties of transfinite arithmetic, in particular the indeterminacy of transfinite subtraction.

One afternoon I sat relaxing on the grass, avoiding my responsibilities. Nearby, a group of children played under the stern eye of a chaperone. As the children ran closer and closer to where I sat, seeking ways to avoid their chaperone, I became curious and asked one child:

"If you could do anything you wanted today,
what would it be?"

The child replied with a question of his own:

"You mean beyond what I could actually achieve?"

***

This dissertation is dedicated to the people in my life who helped me achieve beyond what I ever imagined, and whom I love dearly:

To my family

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CHAPTER 1:

INTRODUCTION

Commemorating Georg Cantor's contributions to the theory of mathematical infinity is a plaque that reads: "The essence of mathematics lies in its freedom" (Aczel, 2000, p.228).

1.1 History, Rationale, and Research Questions

Infinity has played a central role in the historical development of mathematics and mathematical thought. From as early as 450 BC, mathematicians and philosophers have been intrigued by the ethereal dance of infinity. Over the centuries, as an understanding of infinity developed and changed, mathematics too evolved, reflecting the community's emerging understanding of a concept so heavily shrouded in mystery. With new discoveries and inventions it eventually became clear that not one, but many, concepts of infinity had a place in mathematics.

One of the early contributions to the idea of infinity was made by Aristotle when he distinguished between two types of infinity: *potential* infinity and *actual* infinity. Aristotle defined potential infinity as "that whose infinitude is spread over time" (Moore, 1995, p.114), that which is inexhaustible. One may think of potential infinity as a process, which at every instant in time is finite, but which goes on forever. In contrast, actual infinity describes a completed entity that encompasses what was potential. It is "the infinite present at a moment in time" (Dubinsky, Weller, McDonald, & Brown, 2005a, p.341).

Aristotle considered potential infinity a "fundamental feature of reality" (Moore, 1995, p.114), however, he refuted the existence of actual infinity. He believed that since an infinite totality, such as that of the natural numbers, could never be enumerated, infinite quantities could not exist. To Aristotle, the idea of a 'completed' infinite entity – the actually infinite – was "incomprehensible, because the underlying process of such an actuality would require the whole of time" (Dubinsky et al., 2005a, p.341). The idea of actual infinity and the interplay between it and potential infinity inspired some of the earliest and most resilient controversy in mathematics – much of which stemmed from mystifying paradoxes created by Zeno of Elea in 450 BC. Aristotle's rejection of the possibility of the actually infinite had a profound and lasting influence in mathematics: for two thousand years mathematical infinity was conceived of mainly as potential.

Of course, a concept such as infinity cannot help but attract some renegade thinkers. Archimedes, for instance, demonstrated through one-to-one correspondence that two infinite sets could be "equal in multitude" (Netz & Noel, 2007, p.201). The *Archimedes Palimpsest* presents an argument which corresponds an infinite set of triangles, which make up a prism, with an infinite set of lines, which make up a rectangle, and as such is the earliest known mathematical document to consider actual infinity. Another advocate of actual infinity was Galileo Galilei. In his 1638 manuscript *Discorsi e Dimonstrazioni Matematiche, intorno a due nuove scienze* a dialogue between the characters Simplicio and Salviati illustrated some of the anomalies of actual infinity, although it fell short of resolving them. Some years later, Bernard Bolzano made a compelling case for the actually infinite by critiquing definitions of infinity that he deemed too narrow, such as Spinoza's understanding that "those things alone are infinite which are incapable of further increase" (Bolzano, 1950, p.82). Bolzano, unlike Aristotle, could conceive of an infinite collection as a totality without having to imagine each element individually. He conceived of infinite sets by describing their elements, and as such could compare their cardinalities. Although Bolzano's work was influential in advancing the notion of actual infinity, it lacked the consistency and rigour of Georg Cantor's theory of transfinite numbers. Cantor's work with transfinite cardinals and ordinals was revolutionary, and contributed significantly to the foundations of set theory.

Whereas Bolzano and Cantor grappled with the infinitely large, Gottfired Wilhelm Leibniz had his mind on the infinitesimally small. Leibniz's infinitesimals played a fundamental role in the discovery and development of calculus, and are central to mathematical analysis, the "symphony of the infinite" (Hilbert, 1925, p.138). Despite the important ideas linked to infinitesimals, the infinitesimally small went without rigorous definition until the 20th century, with the introduction of nonstandard analysis by Abraham Robinson.

This historical outline hints at the central role infinity has, and continues, to play in the pursuit of knowledge and mathematical understanding. However, infinity also has a role in the pursuit of knowledge *of* mathematical understanding.

As a new researcher in mathematics education, I was drawn to the concept of infinity for several reasons. I was, and still am, intrigued by the infinite – by its properties, its mystique, its freedom from the constraints of 'reality'. It was exactly that 'freedom' which captured me: that distinctive quality which rouses the imagination, provoking controversy, and challenging some of the fundamental ideas intuited as truth. Further, in meeting these challenges and controversy an individual is invited to think in often new and complex ways – to engage in 'advanced mathematical thinking'.

The term 'advanced mathematical thinking' carries with it many descriptions. Although there is no agreement on the definition, many of the characteristics describing advanced mathematical thinking are exemplified in the concept of infinity.

One working description suggests advanced mathematical thinking (AMT) involves abstract, deductive thought (Robert & Schwarzenberger, 1991; Tall 1991, 1992), and includes "proving in a logical manner based on definitions" (Tall, 1991, p.20). Alternatively, ideas that exercise advanced mathematical thinking may be considered as ones that are not "entirely accessible to the five senses" (Edwards, Dubinsky, & McDonald, 2005, p.18), and lack "an intuitive bases founded on experience" (Tall, 1992, p.495). The abstract and intangible nature of actual infinity epitomises both of these descriptions. The ideas discussed in this research dissertation illustrate just a few of the infinity-related problems that rely on the abstract, formal definitions of concepts for which intuition and the senses have no foundation. Further, the ability to engage in AMT about infinity depends on a related complex mental process: that of abstraction. As an illustration, consider the definition for equivalence of infinite sets in Cantorian set theory: it depends on the abstraction of particular attributes of the set elements. As will become clear in the following chapters, in order to think meaningfully about actual infinity, an individual must be able to extract and isolate relevant properties and relations. Abstraction requires a shift in attention from the objects of thought to their structure or relationships (Harel & Tall, 1991). As is illustrated in this research dissertation, investigating the equivalence between two sets involves abstracting from the particular numbers of each set (the objects), to consider, for instance, whether the sets are countable (the structure) or whether a correspondence exists between the sets (the relationship).

Another alternative definition of AMT is Harel and Sowder's (2005) relativistic term 'advanced mathematical-thinking' (AM-T), which describes the cognitive processes at work when overcoming an epistemological obstacle. Harel and Sowder reference Brousseau (1997) in his description of Duroux's idea of an epistemological obstacle. In this sense, an epistemological obstacle must satisfy three conditions: (1) it has occurred as a cognitive obstacle in the history of mathematics, (2) it is knowledge or conception that produces inconsistencies in different contexts, and (3) it may withstand "both occasional contradictions and the establishment of a better piece of knowledge" (Brousseau, 1997, p.99). The epistemological obstacles related to infinity and learners' attempts to cope with these obstacles and the corresponding abstractions required for their conception, are of interest in my research. These obstacles, and the leaps involved in overcoming them, motivated the overarching questions addressed in this dissertation.

Specifically, I attend to the following interrelated questions: (1) What can be learned about university students' emergent conceptions of infinity through their engagement with geometric tasks and paradoxes? (2) What are the specific features of accommodating actual infinity? (3) What are the cognitive leaps connected to the idea of mathematical infinity?

1.2 Thesis Organisation

The journey to answer these questions that is presented in this dissertation mirrors my journey as a researcher. It begins with a consideration of the theoretical background of infinity within mathematics and mathematics education. Chapter 2 examines various mathematical disciplines in which infinity plays an important role. A historical account of Cantor's theory of transfinite numbers, Robinson's nonstandard analysis, and topics in calculus are among the issues touched upon. Chapter 3 extends on these issues by exploring and resolving a selection of well-known paradoxes that deal with the counter-intuitive nature of actual infinity. In Chapter 4, the role of infinity within mathematics education is addressed through an exposition of prior research concerning learners' understanding of infinite sets and infinite cardinals. Chapter 5 offers an account of the theoretical perspectives that guided my research.

In Chapters 6 through 8 I present three separate studies, which identify and discuss different issues pertaining to the research questions expressed above. Within each of the studies, I present the particular research methodologies that I followed, and introduce the groups of participants. A guiding theme in these studies was to explore not only what an individual *knows* about infinity, but also what an individual *is willing to* learn about infinity. The first study, *Intuitions of 'Infinite Numbers': Infinite Magnitude* *vs. Infinite Representation*, explores the naïve and emergent conceptions of infinity of undergraduate university students, as manifested in their engagement with a series of geometric tasks. Chapter 7, *Paradoxes as a Window to Infinity*, examines approaches to infinity of two groups of university students with different mathematical background: undergraduate students in liberal arts programs and graduate students in a mathematics education master's program. Data for this study was drawn from participants' engagement with two of the well-known paradoxes explored in Chapter 3: Hilbert's Grand Hotel paradox, and the Ping-Pong Ball Conundrum. As a follow up, Chapter 8 presents the study *Accommodating the Idea of Actual Infinity*, which seeks to provide a refined account of both naïve and sophisticated conceptions of infinity. This study considers the conceptions of mathematics majors, graduates, and doctoral candidates, as they engaged with the Ping-Pong Ball Conundrum and one of its variations.

Chapter 9 is devoted to a discussion and analysis of the main themes and challenges which emerged throughout my research, and which transcend the individual studies. It identifies the epistemological obstacles participants faced, and frames them within the context of the *cognitive leaps* necessary for developing an understanding of mathematical infinity. The dissertation closes with a summary of the outcomes and main contributions of my research, which are offered in Chapter 10. Overall, my research offers a refined understanding of individuals' struggles with infinity, the obstacles that are faced, and ways in which those obstacles can be overcome.

CHAPTER 2:

INFINITY IN MATHEMATICS

This chapter attempts to illustrate the relevance and importance of infinity to mathematics by exploring some of the major disciplines to which infinity contributes: calculus, analysis, and set theory. Although the chapter does not exhaust the many and varied contributions and conceptions of infinity within mathematics, it paints a picture of the diverse understanding and applicability of infinity by focusing on the areas in which the infinite is most prominent.

In the two sections that follow, some of the fundamental properties and theory related to infinity are discussed and framed within an historical context. The first section of this chapter explores various aspects of Cantorian set theory. Properties of cardinal and ordinal infinity are developed to give an idea of the inherent anomalies of actual infinity, as well as to provide a foundation for forthcoming chapters in which normative and intuitive understandings of actual infinity are discussed and contrasted. The second section briefly delves into the 'symphony of infinities' of calculus, analysis, and nonstandard analysis. It offers a broad perspective of 'non-Cantorian' infinity, and speculates on how an understanding of different infinities can contribute to an overall understanding of calculus or analysis. The mathematics developed in this chapter sets the stage for the paradoxes regarding infinity that are explored in Chapter 3.

2.1 Cantorian Set Theory

Cantorian set theory is a rich area of investigation whose complexity is matched by its elegance, intrigue, and beauty. Cantor's characterization of actual infinity and his theory of transfinite numbers have had a significant impact on mathematics over the past hundred years. The profoundness of his ideas inspired David Hilbert to praise them as providing mathematics "with the deepest insight into the nature of the infinite" (Hilbert, 1925, p.138-9) procured by "a discipline which comes closer to a general philosophical way of thinking" (ibid). Cantor's theory, though controversial at the time, added depth and rigour to emerging conceptions of infinity, which included Bolzano's progressive views that the infinite is more than "that *which has no end*" (Bolzano, 1950, p.82).

Cantor's theory of transfinite numbers established two types of transfinite, or 'infinite', number: cardinals, numbers which quantify the sizes of sets, and ordinals, generalized natural numbers used in indexing. Both transfinite cardinals and ordinals developed from Cantor's consideration of the quantity of elements in sets; they extrapolate from everyday uses of number in counting and ordering, respectively. The notation adopted in this chapter differs from Cantor's notation, but is used for the sake of clarity and easy reading. For a set M, denote the cardinal number of M by the symbol $|M|$, and the ordinal number, or ordinal type, of M by the symbol $\overline{M}$. Properties of transfinite cardinals are developed at considerable length in the following sections so as to establish a shared understanding of concepts relevant to the normative resolutions of the paradoxes explored in Chapter 3, as well as to the empirical studies presented in Chapters 6 – 8.

2.1.1 Transfinite Cardinals

In order to investigate the cardinality of a set M, Cantor considered each of the elements m in M. However, rather than being interested in which elements were in M, or what their particular magnitudes may be, Cantor abstracted from these elements to identify each with a 'unit'. He defined the cardinality of M, $|M|$, as "a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M" (Cantor, republished in 1955, p.86). In other words, by projecting each element $m \in M$ to an abstract unit, Cantor could focus on the magnitude of sets without the distraction of the particular elements in the set – a distraction which mislead many of his predecessors. This set of units was then quantified to describe the 'size' of the set M, that is, to define its cardinality.

Cantor also defined an equivalence relation on two sets M and N, $M \sim N$, as a correspondence between each element of one with exactly one element of the other. By definition, the equivalence is: reflexive, $M \sim M$; symmetric, $M \sim N$ implies $N \sim M$; and transitive, if M, N, and P are sets such that $M \sim N$ and $N \sim P$, then $M \sim P$. It is through this equivalence relation that Cantor established how two sets could have the same cardinality. He wrote, "Of fundamental importance is the theorem that two aggregates M and N have the same cardinal number if, and only if, they are equivalent" (Cantor, 1955, p.87). Thus, M and N have the same cardinality (or are 'equinumerous') if, and only if, there exists a one-to-one correspondence, denoted $\varphi : M \rightarrow N$.

To appreciate the significance of this classification of cardinality, the following three typical examples of transfinite cardinality are considered.

*Example 1: Cardinalities of **N** and **E***

Let **N** denote the set of all natural numbers {1, 2, 3, ...}, and **E** the set of all even numbers {2, 4, 6, ...}. To show |**N**| = |**E**|, it is sufficient to establish an equivalence between the two sets, **N** ~ **E**. The bijective map $\varphi : \mathbf{N} \rightarrow \mathbf{E}$, for which $\varphi(n) = 2n$, for every $n \in \mathbf{N}$, determines the required equivalence. Clearly, φ is one-to-one, since every natural number has a unique double. Also, φ is onto since every even number is a product of 2 and a specific natural number. The following Figure 2.1 illustrates this:

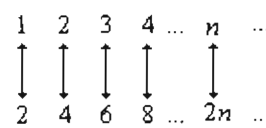


Figure 2.1: Corresponding **N** and **E**

Thus, **N** ~ **E**, and |**N**| = |**E**|.

*Example 2: Cardinalities of **N** and **Q***

As before, let **N** denote the set of natural numbers {1, 2, 3, ...}. Also, let **Q** denote the set of rational numbers, with $\mathbf{Q} = \{\frac{a}{b} : a, b \in \mathbf{Z}, b \neq 0\}$, where **Z** denotes the set of integers.

To determine whether or not these two sets are equinumerous, it is helpful to make use of a particular property of **N**, that is, its countability. The natural numbers are an example of a countably infinite set, or as Cantor referred to them, a 'denumerable' set. Their cardinality is denoted by the symbol $\aleph_0$. An arbitrary set is considered countable if, and only if, it can be put in a one-to-one correspondence with a subset of **N**; all finite sets are countable, as are some infinite sets. To establish that an infinite set has cardinality equivalent to $|\mathbf{N}| = \aleph_0$, it is enough to show that its elements can be systematically listed in such a way that all of the elements are guaranteed to be included in the list without any repetition. In the case of the rational numbers, this is best illustrated with the following diagram:

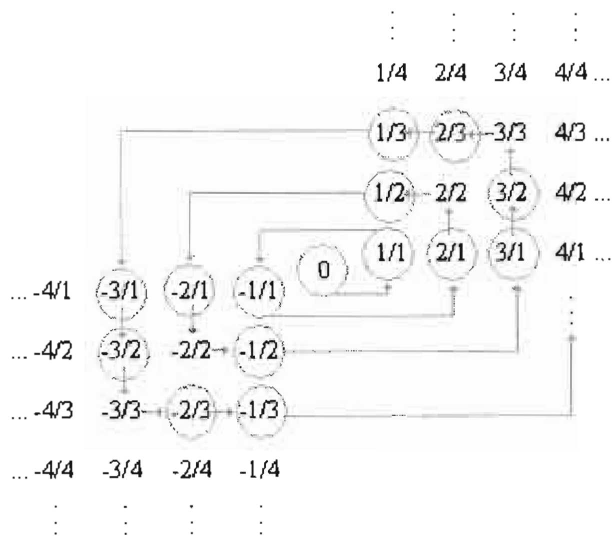


Figure 2.2: Corresponding **N** and **Q**

Unwinding the spiral yields a countable list in which each rational number can be paired with exactly one natural number: 0 in **Q** is paired with $1 \in \mathbf{N}$, 1/1 is paired with 2, -1/1 is paired with 3, 2/1 is paired with 4, 1/2 is paired with 5, and so on.

*Example 3: Cardinalities of **N** and **R***

The real controversy over Cantor's theory of transfinite numbers stemmed from his assertion that there were infinities of different magnitude. This was very difficult for his contemporaries to digest, and indeed it was something that even Cantor had difficulty believing (Cavaillès, 1962). The prestigious mathematician Leopold Kronecker, believed mathematics should be based only on natural numbers (Lavine, 1994), and apparently went to some lengths to try to suppress Cantor's work (Aczel, 2000; Rucker, 1982).

This example shows that the cardinality of **R**, where **R** denotes set of real numbers, is larger than $\aleph_0$. Suppose to the contrary that $|\mathbf{R}| = \aleph_0$. For this to be true, it should be possible to express the set of real numbers as a countable list (as in example 2 with **Q**). In Cantor's words, it should be possible to "bring the totality [of **R**] into the form of a simply infinite series $\gamma_1, \gamma_2, \ldots, \gamma_\upsilon, \ldots$ such that $\{\gamma_\upsilon\}$ would represent the totality of [**R**]" (Cantor, 1955, p.171). To use Cantor's notation, consider $\gamma_\upsilon \in \mathbf{R}$ for every $\upsilon \in \mathbf{N}$. As a real number, γ_υ has a (non-unique) decimal representation. If, for example, γ_υ is less than 1, then its representation can be expressed as $\gamma_\upsilon = 0.\gamma_{\upsilon1}\gamma_{\upsilon2}\gamma_{\upsilon3}\ldots$ The indexing of the decimal digits serves two purposes: the first index number identifies to which real number the decimal digit corresponds, and the second digit identifies the position in the decimal expansion that the digit occupies. For example, γ_{29} is the ninth digit in the decimal expansion of γ_2 (for whatever number γ_2 might be). The argument proceeds as follows. Assume each $\gamma_\upsilon \in (0, 1)$[1]. Then, the infinite sequence $\gamma_1, \gamma_2, \ldots, \gamma_\upsilon, \ldots$ can be written as:

$$
\begin{array}{l}
0.\gamma_{11}\gamma_{12}\gamma_{13}\ldots \\
0.\gamma_{21}\gamma_{22}\gamma_{23}\ldots \\
0.\gamma_{31}\gamma_{32}\gamma_{33}\ldots \\
\quad\ldots \\
0.\gamma_{\upsilon1}\gamma_{\upsilon2}\gamma_{\upsilon3}\ldots \\
\quad\ldots
\end{array}
$$

This list, however, is not exhaustive. It is possible to construct a number in (0, 1) that is distinct from each of the numbers in the list. This is done by Cantor's diagonal argument.

[1] It is then possible to determine $|\mathbf{R}|$ by either generalizing the same argument, or establishing a one-to-one correspondence between (0, 1) and **R**. See Burger & Starbird (2000), or Goldberg (1965) for this proof.

Imagine a number $0.\overline{\gamma}_{11}\overline{\gamma}_{22}\overline{\gamma}_{33}\ldots\overline{\gamma}_{\upsilon\upsilon}\ldots$, where $\overline{\gamma}_{11}$ is different from γ_{11}, and $\overline{\gamma}_{22}$ is different from γ_{22}, and similarly, each of the $\overline{\gamma}_{\upsilon\upsilon}$ is different from each of the $\gamma_{\upsilon\upsilon}$, and where the $\overline{\gamma}_{\upsilon\upsilon}$ are not 0's or 9's. Then, $0.\overline{\gamma}_{11}\overline{\gamma}_{22}\overline{\gamma}_{33}\ldots\overline{\gamma}_{\upsilon\upsilon}\ldots$ is a real number in the interval (0, 1) which is not on the list since it is different by at least one digit from each of the γ_υ above – it differs from γ_1 by the first decimal digit, γ_2 by the second digit, γ_υ by the υ-th digit, and so on. Further, restricting $\overline{\gamma}_{\upsilon\upsilon}$ from being 0 or 9 speaks to the non-uniqueness of decimal representations of real numbers – it prevents a situation where, say, 0.100… is on the list, and 0.099… is the number with each $\overline{\gamma}_{\upsilon\upsilon}$ different from $\gamma_{\upsilon\upsilon}$. This construction contradicts the assumption $|\mathbf{R}| = \aleph_0$, and establishes the real numbers as an *uncountably* infinite set, a set whose cardinality is larger than $\aleph_0$.

Properties: Transfinite Arithmetic

As can be deduced from the above examples, transfinite cardinals have different properties from finite ones. Here I illustrate some aspects of transfinite cardinal arithmetic by considering the smallest transfinite cardinal, $\aleph_0$, and the set **N**. Imagine adding to **N** a new element, say β. Then the union (**N**, β) is equivalent to **N** by a correspondence that matches $1 \in \mathbf{N}$ with β, $2 \in \mathbf{N}$ with the element 1 in (**N**, β), and every $\upsilon \in \mathbf{N}$ greater than 1 with $\upsilon - 1 \in (\mathbf{N}, \beta)$. The following diagram illustrates the pairing:

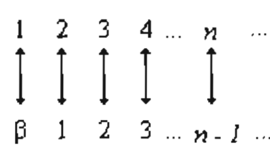


Figure 2.3: Corresponding **N** and (**N**, β)

This establishes the property that $\aleph_0 = \aleph_0 + 1$. Order is not an issue with cardinalities, so $\aleph_0 + 1 = \aleph_0$. Adding one to both sides of the equation gives, $\aleph_0 + 2 = \aleph_0 + 1 = \aleph_0$. By repeating this one finds: $\aleph_0 = \aleph_0 + \upsilon$, for any $\upsilon \in \mathbf{N}$, and further $\aleph_0 = \aleph_0 + \aleph_0$. This equality can be rewritten as $\aleph_0 \cdot 2 = \aleph_0$. Repeatedly adding $\aleph_0$ to itself yields $\aleph_0 \cdot \upsilon = \aleph_0$, for any $\upsilon \in \mathbf{N}$.

It is also true that $\aleph_0 \cdot \aleph_0 = \aleph_0$. To verify this, consider $\Omega = \{(a, b) : a, b \in \mathbf{N}\}$. The sets $\{(a, 0) : a \in \mathbf{N}\}$ and $\{(0, b) : b \in \mathbf{N}\}$ correspond with $\mathbf{N}$ in a natural way. Thus, Ω can be thought to have cardinality $\aleph_0 \cdot \aleph_0$. The goal now is to show Ω is countable. This is accomplished by showing that all the elements in Ω can be systematically listed. One way to do so is in the following list, which proceeds from top to bottom, left to right:

(1, 1)
(1, 2), (2, 2), (2, 1)
(1, 3), (2, 3), (3, 3), (3, 2), (3, 1)
...
$(1, \upsilon), (2, \upsilon), \ldots, (\upsilon, \upsilon), (\upsilon, \upsilon - 1), \ldots, (\upsilon, 1)$
...

This listing establishes Ω is countable, so it is equivalent to $\mathbf{N}$, and $\aleph_0 \cdot \aleph_0 = \aleph_0$.

From this equality, $\aleph_0^2 = \aleph_0$. Multiplying both sides of the equation by $\aleph_0$ yields $\aleph_0^3 = \aleph_0^2 = \aleph_0$. Repeatedly multiplying by $\aleph_0$ gives the equality $\aleph_0^\upsilon = \aleph_0$, valid for every $\upsilon \in \mathbf{N}$. These properties extend to other transfinite cardinals such as $\aleph_1$ – the cardinality of the smallest uncountable infinite set. Cantor established that for every cardinal number, there is a next-larger cardinal, $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\upsilon, \ldots$. Further, he showed that no largest cardinal exists. This claim is justified following the next section, which turns to the other of Cantor's transfinite numbers: ordinals.

2.1.2 Transfinite Ordinals

An ordinal number is defined in an analogous way to cardinal number, only with added criteria. Consider a set A to be *ordered* if there is a relation $<$, such that:

(a) For any $a, b \in A$, then $a < b$ or $b < a$ or $a = b$;

(b) For any $a, b, c \in A$, if $a < b$ and $b < c$, then $a < c$.

The set **N** is ordered, as is the set of integers, **Z**. An ordered set A is called *well-ordered* if every non-empty subset S of A has a least element, that is, there exists an element $l \in S$ such that $l \leq s$, for all $s \in S$ (Ciesielski, 1997). The natural numbers therefore are also well-ordered, whereas the integers are not since there is no least integer. As before, Cantor abstracted from the elements of a well-ordered set, though this time retaining the order among the elements: the abstracted elements are units with the same order as their corresponding elements in A (Cantor, 1955, p.112).

As in the cardinal case, Cantor established an equivalence relation between well-ordered sets to determine if their ordinal numbers were equal. He defined two well-ordered sets as equivalent, denoted by $A \approx B$, if there exists an *order-preserving* bijection, $\psi : A \rightarrow B$, for which $\psi(a) < \psi(b)$, if $a < b$. It follows that two well-ordered sets have the same ordinal, $\overline{A} = \overline{B}$, if and only if, they are equivalent, $A \approx B$. Recalling example 1 from section 2.1.1, the bijection $\varphi(n) = 2n$, for all $n \in \mathbf{N}$, established an equivalence between **N** and **E**. This bijection is also order-preserving, so $\overline{\mathbf{N}} = \overline{\mathbf{E}}$.

Cantor's theory of transfinite numbers included a description of addition and multiplication for ordinals, not all of which is within the scope of this dissertation. The impact of order on this branch of transfinite arithmetic can be illustrated by comparing the ordinals of the sets **N**, (ρ, **N**), and (**N**, ρ), for $\rho \notin \mathbf{N}$. By definition, the set (ρ, **N**) has least element ρ, followed by the regular ordering of **N**. Conversely, (**N**, ρ) has an ordering which places ρ as its largest element. Thus, if $\omega = \overline{\mathbf{N}}$, then $1 + \omega = \overline{(\rho, \mathbf{N})}$, and $\omega + 1 = \overline{(\mathbf{N}, \rho)}$. There exists an order-preserving bijection $\psi : \mathbf{N} \to (\rho, \mathbf{N})$, such that $\psi(1) = \rho$, and $\psi(n) = n - 1$, for every $n > 1$. Consequently, $\mathbf{N} \approx (\rho, \mathbf{N})$ and $\omega = 1 + \omega$. On the contrary, the sets **N** and (**N**, ρ) are not equivalent: **N** has no greatest element but (**N**, ρ) does, namely ρ. Thus, $\omega + 1$ is different from $\omega = 1 + \omega$.

The last bit of Cantor's work examined in this chapter addresses the possibility of an infinite quantity of infinite quantities.

2.1.3 Cantor's Theorem and the Continuum Hypothesis

An elegant result of Cantor's work with transfinite numbers is the discovery that there is an unending and strictly increasing sequence of cardinal numbers. This result is a consequence of the following theorem:

Theorem: There is no greatest cardinal number.

The proof develops in two parts. The first part establishes an increasing sequence of infinite cardinals, and the second part addresses the possibility of a greatest cardinal. Consider a set S and its power set $P(S)$, which consists of all the subsets of S. The cardinality $|S|$ is less than or equal to the cardinality $|P(S)|$, since the map from S to $P(S)$ that takes every $x \in S$ to the subset $\{x\} \in P(S)$ is a bijection onto a subset of $P(S)$. To show the cardinality $|S|$ is strictly less than the cardinality $|P(S)|$, that is $|S| < |P(S)|$, consider any map $\tau : S \to P(S)$. By considering a specific subset of $P(S)$ it is possible to show that no map τ will map onto the entire set $P(S)$. Let K be the subset of $P(S)$ that contains only the elements which do not correspond to an element in the domain of τ; formally, let $K = \{x \in S : x \notin \tau(x)\}$, where $\tau(x) \subset P(S)$. There are two possibilities for K: either K is in the image of τ, or else it is not. If K is not in the image of τ, then τ cannot be bijective since $P(S)$ would contain an element for which no corresponding element in S exists. Conversely, if K is in the image of τ, then there must exist some element $y \in S$, for which $K = \tau(y)$. In this case, either $y \in \tau(y)$ or $y \notin \tau(y)$ must be true. The former case establishes y in the image of τ and implies $y \notin K$, contradicting the assumption $K = \tau(y)$. On the other hand, if y is not in the image of τ – i.e. $y \notin \tau(y)$ – then $y \in K$, which again contradicts the assumption that $K = \tau(y)$. Thus, no map from S to its power set $P(S)$ can be bijective, and so, the cardinality of $P(S)$ must be strictly greater than the cardinality of S, $|P(S)| > |S|$. This result is known as Cantor's Power Set Theorem. It gave rise to Russell's paradox regarding the cardinality of the power set of the set of all sets, which, unfortunately, is outside the scope of this chapter.

The remainder of the proof that no greatest cardinal exists follows from Cantor's Power Set Theorem. Suppose, in order to derive a contradiction, there exists a largest cardinal number $|C|$. There must also exist a corresponding set C for which $|C|$ is the cardinality. Furthermore, C must have a power set, $P(C)$. By Cantor's Power Set Theorem, $|P(C)|$ is strictly greater than $|C|$. However, this contradicts the assumption that $|C|$ is the largest cardinal number. QED.

A consequence of these theorems is the endless sequence of transfinite cardinals, $\aleph_0$, $\aleph_1$, $\aleph_2$, ..., $\aleph_\upsilon$, ... However, it is an open question whether this list includes transfinite cardinalities of all sizes. The Continuum Hypothesis proposes there is no cardinality between the cardinality of the natural numbers, $|\mathbf{N}| = \aleph_0$, and the cardinality of the continuum, $|P(\mathbf{N})| = \aleph_1$. As it stands, the Continuum Hypothesis can neither be proved nor disproved: in 1940 Kurt Gödel proved that it is impossible to *disprove* the Continuum Hypothesis, and conversely, in 1963, Paul Cohen proved that it is impossible to *prove* it (Burger & Starbird, 2000). The uncertainty in the Continuum Hypothesis has led some mathematicians to question whether there might be alternative means to deal with infinity. Such a theory developed from the work of Abraham Robinson in the 1960s. His treatment of infinitesimals and infinitely large numbers is a focus of the next part of this chapter.

2.2 Nonstandard Analysis, Infinitesimals, and Calculus

Infinitesimals have played an important role in the founding and teaching of calculus. In the 1600s Gottfried Leibniz and Sir Isaac Newton independently developed a calculus that used infinitesimal numbers to intuitively describe derivatives, integrals, and rates of change (Loeb, 2000; Keisler, 2000). At the time, mathematicians were more concerned with expanding and developing results than rigorous proof (Kleiner, 2001). However, as the trend shifted from intuitive geometric reasoning to more analytic reasoning, an emphasis on rigour began to develop. Leibniz's and Newton's intuitive uses of the infinitesimally small were harshly criticized. George Berkeley, for instance, described infinitesimals sardonically as "the ghosts of departed quantities" (as quoted by Lavine, 1994, p.24), and criticized the general lack of consistency within calculus and analysis.

Nearly two hundred years later, in the 1820s, Augustin Cauchy introduced rigour into analysis, and did so through the concept of limits (Lavine, 1994). Yet, Cauchy's definition of limit was still murky and made use of infinitesimals in an intuitive way, despite their ambiguity. As the focus of analysis shifted more towards limits, the need for complete rigour and consistency became more prominent. Finally, in 1870, Karl Weierstrass formulated a completely rigorous treatment of calculus by introducing the ε-δ definition of limit, and consequently, eliminating infinitesimals from the foundations of analysis (Kleiner, 2001; Lavine, 1994). The insight gained from considering limits in terms of infinitesimals, however, was too powerful to ignore completely. Even today, mathematicians appeal to infinitesimals on an intuitive level to help their students develop an understanding of the ε-δ limit.

In 1960, Abraham Robinson gave a rigorous mathematical foundation for the use of infinitesimals in calculus (Loeb, 2000). His work in nonstandard analysis has catalyzed research in areas beyond calculus as well, including probability, mathematical physics, and finance. Nonstandard analysis established a rigorous calculus that builds naturally on the intuitiveness of infinitesimals. This section examines some of the properties of infinite numbers and infinitesimals in the set of nonstandard numbers, the hyperreals. It also explores how a 'non-standard' approach to infinity may contribute to students' understanding of calculus, and in particular, limits. The chapter then closes with a look at properties of infinite series and sequences, and demonstrates how different conceptions of infinity can play a part in students' understanding of calculus.

2.2.1 Nonstandard Numbers

The set of hyperreal numbers, denoted **R***, is an extension of the set of real numbers that includes both 'standard' real numbers and 'nonstandard' numbers – numbers that are infinitely big or infinitesimally small. The extension is praised for its natural generalization of properties of real numbers to hyperreal numbers (Tall, 2001). All of the standard rules of arithmetic, ordering, and applying functions in **R** are also true in **R***, they are merely extended to apply to all of the elements in **R***. The field of hyperreal numbers consist of three types of nonstandard number: infinitesimals, positive infinite numbers, and negative infinite numbers. By definition, a positive infinite number is one that is *greater* than every real number, and likewise, a negative infinite number is one that is *less* than every real number. An element $\varepsilon \in \mathbf{R}^*$ is called a positive infinitesimal if $0 < \varepsilon < \alpha$, for every positive real number α. Similarly, ε is called a negative infinitesimal if $\alpha < \varepsilon < 0$, for every real $\alpha < 0$. In general, an infinitesimal is a number ε that is either positive or negative infinitesimal or zero. Keisler (2000) uses the metaphor of the real line to help establish an intuition of infinitesimals. He suggests thinking about the real line zoomed in at zero with something like a super microscope; infinitesimals are the very small, negligible numbers around and including zero.

Properties of infinite numbers and infinitesimals are more or less what might be expected from extending properties of real numbers. For instance, with real numbers, the reciprocals of very small numbers are themselves quite large. Similarly, if ε is positive infinitesimal, then $1/\varepsilon$ is a positive infinite number. Conversely, the reciprocal of an infinite number is infinitesimal. In order to gain a flavour for the intuitive treatment of infinity within nonstandard analysis, some of the properties of infinitesimals and infinite numbers are illustrated in the following examples. A more complete introduction to hyperreals can be found in Keisler (2000).

For each of the following examples consider ε, δ as two infinitesimals, a, b as two real numbers, and H, K as two infinite numbers.

Example 1. Products of Hyperreals

(i) $a \cdot \varepsilon$ is infinitesimal. Intuitively, picture an infinitely thin rectangle of length a, and infinitesimal area. If ε is a positive infinitesimal, then $6 \cdot \varepsilon$ is infinitesimal and greater than ε. Likewise, $28 \cdot \varepsilon$ is infinitesimal and greater than $6 \cdot \varepsilon$.

(ii) $\varepsilon \cdot \delta$ is infinitesimal. For example, if $0 < \varepsilon < \alpha$, for every $\alpha > 0$, then the product $\varepsilon \cdot \delta$ satisfies $0 < \varepsilon \cdot \delta < \alpha \cdot \delta < \alpha$ for every $\alpha > 0$, and is thus infinitesimal by definition.

(iii) H·K is infinite. H and K can be thought of as the reciprocals of some infinitesimal numbers, so, $H \cdot K = 1/\varepsilon \cdot 1/\delta = (\varepsilon \cdot \delta)^{-1}$. Since $\varepsilon \cdot \delta$ is infinitesimal by (i), its reciprocal must be infinite. A similar argument shows $a \cdot H$ is infinite.

(iv) $H \cdot \varepsilon$ is indeterminate. For example, if $\varepsilon = 1/H^2$, then $H \cdot \varepsilon$ will be infinitesimal; if $\varepsilon = 1/H$, then the product $H \cdot \varepsilon$ is equal to 1; and if $\varepsilon = \frac{1}{\sqrt{H}}$ then the product $H \cdot \varepsilon$ will be an infinite number.

Example 2. Quotients of Hyperreals

(i) The quotient of any infinite or real number by a non-zero infinitesimal is an infinite number: H/ε and a/ε are infinite.

(ii) The quotient of any infinitesimal or real number by an infinite number is infinitesimal: ε/H and a/H are infinitesimal.

(iii) b/ε and H/b are infinite numbers, provided ε and b are non-zero. Similarly, ε/b and b/H are infinitesimal.

(iv) The quotient of two infinitesimals, ε/δ, or of two infinite numbers, H/K, is indeterminate. As in the case of products, the relative size of infinitesimals and infinite numbers determines the value of the quotient.

Example 3. Calculations

(i) The quotient $\frac{5\varepsilon^4 - 8\varepsilon^3 + \varepsilon^2}{3\varepsilon}$ is infinitesimal provided ε is non-zero. Notice that $\frac{5\varepsilon^4 - 8\varepsilon^3 + \varepsilon^2}{3\varepsilon}$ simplifies as $\frac{5}{3}\varepsilon^3 - \frac{8}{3}\varepsilon^2 + \frac{\varepsilon}{3}$. Since each of the terms is infinitesimal, the sum is also infinitesimal.

(ii) The quotient $\frac{2H^2 + H}{H^2 - H + 2}$ is finite, but not infinitesimal. To see this, rewrite the quotient as: $\frac{2 + 1/H}{1 - 1/H + 2/H^2}$. As 1/H and $2/H^2$ are infinitesimal, the numerator and denominator are both finite (non-infinitesimal) numbers, and so is the quotient.

(iii) If H is positive infinite then $\sqrt{H + 1} - \sqrt{H - 1}$ is infinitesimal. Although the root of a positive infinite number is positive infinite (Keisler, 2000), this difference can be simplified to show that it is infinitesimal. Multiplying $\sqrt{H + 1} - \sqrt{H - 1}$ by $\frac{\sqrt{H + 1} + \sqrt{H - 1}}{\sqrt{H + 1} + \sqrt{H - 1}}$ yields the product $\frac{2}{\sqrt{H + 1} + \sqrt{H - 1}}$. The denominator is a sum of two positive infinite numbers, and hence is itself positive infinite. Thus the quotient, and its equivalent expression $\sqrt{H + 1} - \sqrt{H - 1}$, is infinitesimal.

The goal of this section was to demonstrate some of the properties attributed to nonstandard number systems. These properties, as well as others, contributed to a rigorous and consistent foundation for a use of infinity that is different from Cantor's cardinal and ordinal infinity. Although it is not yet as widely studied as Cantorian set theory, researchers in nonstandard analysis promote its applicability to all branches of mathematics and claim it can do much to enrich and direct new investigations in mathematics (Loeb, 2000). This use of infinity and infinitesimals also has consequences for mathematics educators, particularly with respect to understanding limits (Tall, 1981, 1992, 2001), as limits and nonstandard numbers are calculated in analogous ways.

2.2.2 Infinitesimals and Limits

The ε-δ definition of a limit, formulated by Weierstrass, contributed to the first completely rigorous treatment of calculus. With the introduction of the ε-δ definition, infinitesimals gradually faded from formal use (until recently with Robinson's work in the 1960s), though it is still customary to argue informally in terms of them. The informal appeal to infinitesimals that appears in limiting language such as 'approaches' or 'gets as close as you'd like' has been linked to students' inappropriate mental model of a limit as unreachable (Tall, 1980; Tall & Vinner, 1981). Williams (1991) observed a persistent notion of limits as unreachable in college students who were familiar with the formal definition. Intuitions of limits as dynamic, potentially infinite processes, along with the abstractness of the ε-δ definition have contributed to the extensive difficulties students encounter in calculus (Cottrill, et al., 1996; Davis & Vinner, 1986; Sierpinska, 1987; Tall, 1981, 1992; Tall & Vinner, 1981; Williams, 1991).

Robinson's nonstandard analysis and its corresponding calculus make use of the intuition of infinitesimals to establish a notion of limit that is in the spirit of the founding fathers of calculus, and that may help students overcome the challenges of ε-δ analysis (Tall, 2001). The intuitive ease with which a nonstandard limit is defined, and the explicit attention to the role of infinitesimals contribute to a formulation of the limit concept that is at once similar and distinct from the ε-δ one. For instance, standard calculus texts, such as Stewart (1999), define the limit of a function as:

Let f be a function defined on some open interval (a, b), and let $x \in (a, b)$. The limit of $f(x)$ as x approaches a is equal to the value L, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.

Conversely, a nonstandard definition reads:

$f(x) - L$ is infinitesimal whenever $x - a$ is infinitesimal.

The latter definition is less abstract, and naturally extends students' prior experiences with limiting computations. In nonstandard calculus, limits of functions are typically computed by determining the 'standard' part of the function. This is exemplified in the following example.

Example 1. Nonstandard Limits

Find the limit as $x \to \infty$ of $f(x) = \frac{2x^2 + x}{x^2 - x + 2}$. Let H be a positive infinite number. Then the limit of $f(x)$ is the standard part of $f(\mathrm{H})$. That is: $\mathrm{st}(\frac{2\mathrm{H}^2 + \mathrm{H}}{\mathrm{H}^2 - \mathrm{H} + 2})$. Recalling example 3(ii) from the previous section, $\frac{2\mathrm{H}^2 + \mathrm{H}}{\mathrm{H}^2 - \mathrm{H} + 2}$ simplifies to $\frac{2 + 1/\mathrm{H}}{1 - 1/\mathrm{H} + 2/\mathrm{H}^2}$, and taking the standard part of $f(\mathrm{H})$ essentially amounts to neglecting the infinitesimal values of $1/\mathrm{H}$ and $2/\mathrm{H}^2$. Thus the limit is $\mathrm{st}(f(\mathrm{H})) = 2$.

Nonstandard and standard approaches to limit computations are carried out in very similar ways. The standard approach determines the limit as x approaches infinity by examining each of the terms 2, 1, $1/x$, $-1/x$, and $2/x^2$ individually. The last four of these terms 'tend' to zero, and the limit is determined to be 2. The major differences in these approaches lie in how the limits are conceived. Defining the limit as the standard part of a hyperreal number maintains the dynamic nature attributed to limits without obscuring the property of attainability. The concept of limit is fundamental to calculus and analysis. It is closely linked to important ideas in both fields, including infinite series and sequences.

2.2.3 Series and Sequences

The importance of series and sequences in and beyond calculus is irrefutable, and stems from Newton's early description of functions as sums of infinite series (Lavine, 1994). In modern use, it is important to distinguish between the properties of convergent and divergent series and sequences. Infinite sums such as $1 + x + x^2 + x^3 \cdots$ were problematic for Newton since for some values of x the series converged, but for others, the series diverged. Newton generally dismissed divergent series as useless, and was more interested in developing concepts around series that converged (Lavine, 1994). Leonhard Euler, on the contrary, rejected the idea that attention should be restricted to convergent series, and developed several techniques for computing infinite sums of both convergent and divergent series, many of which are still widely used.

Series and the limits of their corresponding sequences are fundamentally interconnected: limits are used in order to determine convergence, and convergence can be used in order to determine limits. A series $\sum_{n=0}^{\infty} a_n$ is defined as convergent if the sequence of its partial sums $\{s_n\}$, where $s_n = a_0 + a_1 + \ldots + a_n$, is convergent and the limit as n tends to infinity of $\{s_n\}$ exists as a real number. Otherwise, the series diverges. The sum of a convergent series is equal to the limit value of $\{s_n\}$. Additionally, if the series is convergent, then the limit as n approaches infinity of the sequence $\{a_n\}$ is zero. To show this last implication, one need only express the term a_n as the difference of the two partial sums $s_n - s_{n-1}$, and then take their limits. Since $\{s_n\}$ converges, the limits of s_n

and s_{n-1} are equal, and thus their difference is zero. These properties of series illustrate an instance where understanding infinity strictly as potential – an inexhaustible process – can lead to inaccurate conceptions, such as that of limit as unreachable and sum as unattainable. As a result of conceiving of a potentially infinite process of summing, infinite series may be thought to either all 'sum to infinity,' or to sum very close, but not equal, to the limit of the partial sums.

Infinite series have also played an important role in the history and use of integration. Newton developed a method of integrating functions that involved integrating each term of its infinite series representation. This method of integration is still used today for such functions as e^{-x^2}, for which no other means is appropriate. Functions, such as $f(x) = e^{-x^2}$, whose antiderivative is not an elementary function (i.e. one that is built from basic operations such as addition, exponentials, or logarithms), are expressed as polynomials with infinitely many terms. Integrating these functions then involves integrating the series representation term by term (up to a point, to establish the pattern) and then determining the new sum. In addition to integration, expressing a function as a sum of infinitely many terms is also useful for solving differential equations, and approximating functions by polynomials. Approximating by polynomials is used in computer science, for example, to represent functions on calculators or computers. Replacing functions with infinite series is also common practice in mathematical physics and chemistry, where phenomena are analyzed based on the behaviour of the series that represent them. An understanding of infinity is not only central for an understanding of mathematics, but also of the mathematical foundation of many scientific disciplines.

CHAPTER 3:

PARADOXES OF THE INFINITE

> "More than once in history the discovery of paradox has been the occasion for major reconstruction at the foundations of thought." (Quine, 1966, p.3)

Concepts of infinity are at the centre of many mathematical paradoxes. As the renowned mathematician and philosopher Bernard Bolzano observed:

> "Certainly most of the paradoxical statements encountered in the mathematical domain ... are propositions which either immediately contain the idea of the infinite, or at least in some way or other depend upon that idea for their attempted proof" (Bolzano, 1950, p.75).

Paradoxes involving the infinite are unlike many philosophical paradoxes that are comprised of self-contradictions or absurd assumptions, such as the barber who shaves all and only the village men who do not shave themselves (an informal variation of Russell's paradox), or Epimenides the Cretan, who said that all Cretans were liars (Quine, 1966).

Instead, paradoxical statements regarding the infinite stem from the seemingly impossible attributes of mathematical infinity, and tend to expose preconceptions that were once believed to be fundamental. Quine classified such a paradox as falsidical – one that "not only seems at first absurd but also is false, there being a fallacy in the purported proof" (1966, p.5). These fallacies might arise from erroneously extending familiar properties of finite concepts to the infinite case, or from the belief that infinity is synonymous with eternity.

Properties of infinity have puzzled and intrigued minds for centuries, with the earliest conundrums dating back to Zeno of Elea circa 450 BC. Zeno's paradoxes highlighted the inherent anomalies of the infinite, and had such a profound impact on mathematics and mathematical thought that Bertrand Russell attributed to them "the foundation of a mathematical renaissance" (1903, p.347). Today, there are several paradoxes concerning the infinite, though most stir up the same tensions first noted by Zeno – namely the conflict between intuition and formal mathematics, and the interplay between potential and actual infinity.

The paradoxes explored in this chapter build on the mathematical understanding of infinity that was developed in Chapter 2. They are arranged into three sections, each of which begins with the presentation and normative resolutions of related paradoxes. The common themes in their resolutions are then summarised. Section 3.1 examines two famous paradoxes attributed to Zeno, including the famous incident which pit the athlete Achilles against a tortoise in an impossible race. Section 3.2 includes Hilbert's Grand Hotel, a building that can continually accommodate new guests despite having no vacancy. Finally, section 3.3 is dedicated to the 'super-task' involved in the Ross-

Littlewood paradox. Alternatively known as the Ping-Pong Ball Conundrum, this paradox extends the issues raised by Zeno by requiring the coordination of three infinite entities: two sets of ping-pong balls, and a collection of intervals of time.

3.1 Infinite Series, Finite Sums

One of the seemingly paradoxical properties of infinity encountered by students is during some of their first experiences with calculus: that an infinite series may sum to a finite quantity. The counter-intuitive aspect of an endless calculation summing to a finite number is the basis for some of the earliest recorded paradoxes. Zeno of Elea famously toiled with this mysterious property of what are now known to be convergent series. He devised the two following paradoxes, which seem to defy physics and experience, and which went without rigorous resolution until the advent of modern calculus.

3.1.1 The Dichotomy Paradox

> *A man wishes to walk the length of a room. Before he can travel the entire distance, he must first walk half the distance. After that, he must walk half the remaining distance, and then again half the remaining distance. Continuing in this way, can the man walk the entire length of the room?*

Everyday experience seems to resolve this question quickly enough: surely it is possible to travel the length of a room. The Dichotomy Paradox, however, lies in the way the distance between the two opposing walls of the room is subdivided. The sequence of the successive distances can be represented as $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$; since each term is a power of $\frac{1}{2}$ and no power of $\frac{1}{2}$ is equal to zero, the end of the room seems to be unreachable. In modern mathematical language, this paradox can be resolved by considering the total distance as the sum of the half distances, that is, the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ This series is convergent, and so the limit of its partial sums – $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{1}{4}$, $s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, and so on – will be equivalent to its sum. Alternatively, a formula devised by Euler for verifying infinite sums shows that summing the total number of half distances results in the entire length of the room, that is, $\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{½}{1 - ½} = 1$.

3.1.2 Achilles and the Tortoise

> *Achilles and a tortoise agree to race. Since Achilles is the faster runner, the tortoise is given a head start. Can Achilles overtake the tortoise to win the race?*

Here again everyday experience suggests an obvious resolution to this problem: clearly the faster runner will gain a lead and win the race. As in The Dichotomy Paradox, however, this "catch 22" stems from the subdivision of space. Since the tortoise starts ahead of Achilles, by the time Achilles has made up the tortoise's head start, the tortoise will have travelled a further distance. Achilles must therefore make up for this new distance. Again, however, by the time he has, the tortoise has moved still further. Although this new distance is shorter, Achilles must again make up for it. This continual progression of making up a seemingly endless sequence of distances suggests Achilles can never pass the tortoise because he cannot catch up infinitely many times.

Consider a representation of this paradox in terms of sequences, series, and convergence. Unlike Zeno's dichotomy above, Achilles is not travelling towards a static object (the wall). His attempt to catch the moving tortoise necessitates an examination of two series: the distances Achilles travels, $\{a_n\}$, and the distances the tortoise travels, $\{t_n\}$. Here a_1 represents Achilles' position at time 1 (the start of the race), a_2 his position at time 2, and a_n his position at time n, and similarly for the tortoise and t_n. This is summarized in the following diagram:

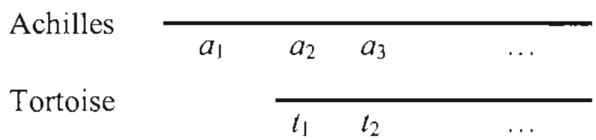


Figure 3.1: Sequences $\{a_n\}$ and $\{t_n\}$

As the race progresses and the finish line approaches, the positions a_n and t_n become closer and closer. Mathematically, both sequences are said to be monotonic and bounded. It is well known (e.g. Stewart, 1999) that every bounded, monotonic sequence of real numbers is convergent. Now, consider the relationship between $\{a_n\}$ and $\{t_n\}$. Evidently, $a_n < t_n$ for all $n \in \mathbf{N}$ – hence Achilles' perpetual need to catch up – however, notice that for every n, $t_n = a_{n+1}$. The sequences are the same aside from one term, and the added initial term in $\{a_n\}$ does not affect the limit to which the sequence converges. Therefore, the limits of the two sequences must be same:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1} = P, \text{ for some } P \in \mathbf{N}.$$

This limit P is precisely where Achilles will catch up to the tortoise.

3.1.3 Common Themes

The understanding of infinity required for both of the above resolutions hinges on the distinction between potential and actual infinity. Recall, potential infinity may be thought of as inexhaustible – a process, which at every instant in time is finite but which continues forever. Whereas, actual infinity is thought of as a complete and existing entity, one that encompasses what was potential. With respect to the process of crossing a room or catching up to a tortoise, the traveller is faced with a *potentially* infinite number of subregions – there is always more distance to travel, an end point never seems to be reached. However, the set of successive halves {½, ¼, ⅛, …}, for example, is an *actually* infinite set – it is a complete entity that contains an infinite number of elements. Consider again the limit of partial sums discussed in The Dichotomy Paradox resolution. The dynamic nature of a limit relates to potential infinity – the partial sums *approach* 1. However, the actually infinite set {½, ¼, ⅛, …}, as well as the set $\{a_n\}$ from Achilles and the Tortoise paradox, represent bounded sequences. The set of distances to cross the room or catch up to the tortoise are completed infinite sets whose elements may be summed to attain a definite result. The infinite number of intervals that seemed to extend endlessly while in potential is actually encompassed within a finite distance. It is toward the actually infinite that we now turn.

3.2 Cardinality and Infinite Sets

Cantor's (1915) *Contributions to the Founding of the Theory of Transfinite Numbers* revolutionized infinity's role within mathematics. The existence of actual infinity, and the relationship an infinite set has with a proper subset of itself, were concepts that were grappled with and poorly understood for many years prior to, and following, Cantor's publication. Bolzano, for instance, recognized the actual infinity of natural numbers, although his ideas conflicted with what is now accepted as the formal mathematical definition of infinite sets. Bolzano's inconsistent characterization of infinite sets illustrates the challenges in the paradoxes of this subsection.

Bolzano reasoned that two sets, such as the sets of rational numbers in $A = [0, 5]$ and $B = [0, 12]$, could be coupled in a one-to-one correspondence. However, he argued that at the same time the numbers in [0, 12] were obviously more numerous than the ones in the set [0,5]. He wrote:

"Although every quantity in *A* [the set of numbers from 0 to 5] or *B* [the set from 0 to 12] allows of coupling with one and only one in *B* or *A*, yet the set of quantities in *B* is other and greater than in *A*, since the *distance* between the two quantities in *B* is other and greater than the *distance* between the corresponding quantities in *A*" (Bolzano, reprinted 1950, p.100).

Bolzano constructed the map $5y = 12x$, for $x \in [0, 5]$ and $y \in [0, 12]$, and argued that although it coupled each element in [0, 5] with exactly one element in [0, 12] and vice versa, the two set were not equinumerous. He reasoned that since the interval [0, 12] is longer than [0, 5], the 'distance' between values for y must be larger than the 'distance' between values for x. Thus, he claimed the set [0, 12] was more numerous than [0, 5]. Contrary to what is accepted today, Bolzano warned that when addressing infinite sets, one-to-one correspondence

"never justifies us, we now see, in inferring the *equality of the two sets, in the event of their being infinite*, with respect to the multiplicity of their members – that is, when we abstract from all individual differences" (1950, p.98).

He reasoned instead that "two sets can still stand in a relation of inequality, in the sense that the one is found to be a whole and the other a part of that whole" (ibid).

3.2.1 Galileo's Paradox

There are as many perfect squares as there are natural numbers.

Galileo's paradox first appeared in his 1638 manuscript *Discorsi e Dimonstrazioni Matematiche, intorno a due nuove scienze* as a topic of conversation between the characters Simplicio and Salviati. Salviati suggested the sets were equinumerous since:

> "there are as many [squares] as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square" (Galilei, reprinted 1914, p.32).

This argument effectively establishes a one-to-one correspondence between the two sets, which by Cantor's definition, guarantees that the cardinalities of both sets are the same. However, this seems to conflict with practical experience: how can a set that is properly contained in another set be equal to it in size? Extending Bolzano's line of reasoning, one would have to conclude the set of natural numbers {1, 2, 3, ...} was greater (more numerous) than the set of squares {1, 4, 9, ...} because "one is found to be a whole and the other a part of that whole" (1950, p.98). Galileo, however, reasoned the sets must be equal in size because in his understanding of infinity, "the attributes 'equal,' 'greater,' and 'less,' are not applicable to infinite, but only to finite, quantities" (1914, p.32-3). In fact, neither argument satisfactorily resolves Galileo's paradox.

It was Cantor's method of abstracting from the elements of a set and identifying them each with a 'unit' that enabled him to define the cardinal number of a set as "a definite aggregate of units ... [which exists] as an intellectual image or projection of the given aggregate" (1915, p.86). In other words, Cantor countered Bolzano's 'distance' argument by ignoring the particular magnitude of the numbers and identifying them each with an abstract 'unit'. With this abstraction, Cantor established that two sets are equivalent if, and only if, their cardinalities are equal – that is, if, and only if, there is a one-to-one correspondence between the 'units' in one set and the 'units' in the other. In this paradox, it is the correspondence that Galileo's Salviati describes that establishes equivalence. However, Galileo's argument that infinite sets are incomparable is inconsistent with today's convention. Recall that Cantor's diagonal argument, discussed in Chapter 2, showed that the real numbers, for instance, are more numerous than the natural numbers, dispelling Galileo's claim that comparisons such as 'greater' or 'less' are not applicable to infinite quantities.

3.2.2 Hilbert's Paradox: The Grand Hotel

> *The Grand Hotel has infinitely many rooms and no vacancy. If only one person is allowed per room, how can the hotel accommodate a new guest?*

Unlike in a hotel with finitely many rooms, in the Grand Hotel, 'no vacancy' does not prohibit a new guest from being accommodated. The idea is simply to free up an already occupied room by rearranging the accommodations. This can be done in different ways, one possibility is to have the guest in room one move to room two and displace the person there. This guest moves from room two to room three. The guest in room three moves to room four, and so on. Since there are infinitely many rooms, each guest can displace his neighbour, and leave the first room vacant for the new arrival.

The resolution of this paradox relies on the one-to-one correspondence between the sets $\mathbf{N} = \{1, 2, 3, \dots\}$ and $\tilde{\mathbf{N}} = \{2, 3, 4, \dots\}$. The map sending $x \in \mathbf{N}$ to $x + 1 \in \tilde{\mathbf{N}}$ is bijective, and thus the cardinalities of the two sets are the same. Identifying the set of guests with $\mathbf{N}$, and the set of occupied rooms after the shift with $\tilde{\mathbf{N}}$, it is clear that even when each guest has moved to his neighbour's room, there are still enough rooms for all. Analogous arguments extend to variations of Hilbert's Grand Hotel that attempt to accommodate arbitrary, or even countably infinite, amounts of new guests.

3.2.3 Common Themes

Both Galileo's paradox and Hilbert's Grand Hotel paradox are resolved through Cantor's theory of transfinite numbers. In order to resolve these paradoxes it is necessary to appreciate that infinite quantities have distinct properties from finite ones. Galileo's claim that 'greater,' 'less,' and 'equal' are not applicable to infinite quantities did have some truth to it – 'greater,' 'less,' and 'equal' as they apply to *finite* quantities are inappropriate for comparing *infinite* ones. Adding one more element to a finite set will change its cardinality, but the same is not true when dealing with infinite sets. As discussed in Chapter 2, it is possible even to double or triple, for example, the quantity of elements in an infinite set without altering its cardinality. All of the comparisons between infinite sets and cardinalities hinge on an ability to correspond each element of one set with exactly one element of another. If a one-to-one correspondence exists between two infinite sets, they are, by definition, equinumerous. It is this elegance, along with the profound insight of Cantor's theory of transfinite numbers, which prompted Hilbert to praise Cantor's work as "the finest product of mathematical genius and of the supreme achievements of purely intellectual human activity" (1925, p.138).

3.3 Transfinite Subtraction

In this section the issues raised by the great minds of Zeno, Galileo, and others, are extended even further. The paradoxes explored in this section are deviations of the Ross-Littlewood paradox, and centre around a 'super-task' – a task which occurs within a finite interval of time, yet which involves infinitely many steps (Thompson, 1954). The following 'super-task' contributes to the Ping-Pong Ball Conundrum and its variations, which illustrate aspects of transfinite arithmetic that are quite different from the properties of finite arithmetic.

3.3.1 The Ping-Pong Ball Conundrum

> *An infinite set of numbered ping-pong balls and a very large barrel are instruments in the following experiment, which lasts 60 seconds. In 30 seconds, the task is to place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time (and working more and more quickly), balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. After the experiment is over, at the end of the 60 seconds, how many ping-pong balls remain in the barrel?*

In this thought experiment there are three infinite sets to consider: the in-going ping-pong balls, the out-going ping-pong balls, and the intervals of time. The necessity to coordinate three infinite sets, along with the counterintuitive (and unavoidable) boundedness of one of them, creates a level of complexity in this paradox that is absent in Hilbert's Grand Hotel, for example. The infinite sequence of time intervals {½, ¼, ⅛, …} is bound between 0 seconds and 1 minute, and the sum of the corresponding series is 1 (½ + ¼ + ⅛ + … = 1). The conflict between an 'unlimited' number of time intervals and a 'limited' time of 1 minute (or 60 seconds) underscores the interplay between potential and actual infinity. In order to make sense of the normative resolution to this paradox, an understanding of actual infinity is necessary. Despite the fact that at every time interval there are more in-going than out-going balls, at the end of the experiment the barrel will be empty. An important aspect in the resolution of this paradox is the one-to-one correspondence between each of the infinite sets and the set of natural numbers.

The sets of in-going and out-going balls, being numbered as they are, both correspond to the set of natural numbers. This correspondence ensures that at the end of the experiment, as many balls were removed from the barrel as went in. The set of out-going balls and the set of time intervals, which can be represented as B ={1, 2, 3, … }, and T = {½, ¼, ⅛, …}, respectively, can also be put into a one-to-one correspondence by pairing any $x \in B$ with $(½)^x \in T$. This correspondence assures that when the 60 seconds runs out, so do the balls. These facts are necessary but not sufficient to guarantee an empty barrel.

An essential feature of this thought experiment is the ordering of the out-going balls. It is not enough that the amount of out-going balls corresponds to the amount of time intervals. In order for the barrel to be empty at the end of the experiment the ping-pong balls must be removed consecutively, beginning from ball #1. Consequently, there will be a specific time for which each of the in-going balls is removed. The issue of order and its effect on the paradox resolution is addressed in the following paradox. The Ping-Pong Ball Conundrum and its variation constantly engage the minds of mathematicians and philosophers, attempting both to provoke controversy (Van Bendegem, 1994) and to lay controversy to rest (Allis & Koetsier, 1995).

3.3.2 The Ping-Pong Ball Variation

> *An infinite set of numbered ping-pong balls and a very large barrel are instruments in the following experiment, which lasts 60 seconds. In 30 seconds, the task is to place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, the next 10 balls are placed in the barrel and ball number 11 is removed. Again, in half the remaining time, balls numbered 21 to 30 are placed in the barrel, and ball number 21 is removed, and so on. At the end of the 60 seconds, how many ping-pong balls remain in the barrel?*

This thought experiment is very similar to the Ping-Pong Ball Conundrum previously discussed. The two paradoxes serve as an illustration of the anomalous properties of transfinite subtraction – while transfinite addition is a well-defined extension of addition of natural numbers, transfinite subtraction is indeterminate. In the Ping-Pong Ball

Variation, the ping-pong balls are not removed in consecutive order. Instead, the experiment calls for the removal of balls numbered 1 at time one, ball number 11 at time two, ball number 21 at time three, and so on. Thus, despite the bijection between the natural numbers, the powers of ½, and the set {1, 11, 21, ...}, infinitely many balls remain in the barrel at the end of the 60 seconds. In this experiment there will never be a time interval wherein balls 2 to 10, 12 to 20, 22 to 30, and so on, are removed. The seemingly minor distinction between removing balls consecutively versus removing them in a different ordering has a profound impact on the resolution of the paradoxes: while in one instance subtracting $\aleph_0$ from itself yielded zero, in the other it yielded $\aleph_0$.

Another way to vary this ping-pong dilemma is to introduce a bit of bedlam: rather than removing the balls in a specific order, consider the consequences of removing the balls randomly. If balls were removed randomly, it would be impossible to determine precisely how many balls were left in the barrel, or which ones for that matter. Perhaps all of the balls were removed, perhaps one or two balls were 'skipped' and left behind, or, perhaps infinitely many balls remain.

3.3.3 Common Themes

The Ping-Pong Ball Conundrum highlights more of the subtleties of Cantor's theory of transfinite numbers – namely, the ambiguity with subtracting infinite quantities. Although Cantor established transfinite addition as a well-defined operation, where $\aleph_0 + n = \aleph_0$ for any $n \in \mathbf{R}$, subtraction cannot be uniquely defined, since $\aleph_0 - \aleph_0$ could be any real number (or indeed $\aleph_0$ itself). Due, in part, to the ambiguity of transfinite subtraction, in these ping-pong experiments, it was not the number of balls that were removed that is the only important feature – it is also *which* balls and *how*.

The importance of order in the ping-pong conundrum is unavoidable, although it is something that is easily overlooked in a learner's first encounter with these paradoxes. Common experience with subtraction, and the corresponding intuitions, may leave a learner unprepared to address the indeterminacy of transfinite subtraction – that is, unprepared to recognise that subtracting a cardinality from itself might not yield zero. A detailed look at the specific challenges associated with transfinite arithmetic, and in particular the challenges connected to the expression '∞ - ∞', occurs in Chapter 9: *Cognitive Leaps toward Understanding Infinity*.

The guiding intuitions and conceptions that contributed to the historical controversy around the Ping-Pong Ball Conundrum, its variations, and also around Hilbert's Grand Hotel paradox, persist in learners' naïve approaches to these paradoxes. As discussed in the empirical studies in Chapters 7 and 8, many of these conceptions emerged as coercive influences in participants' resolutions, and either neglected or were at odds with the accepted mathematical properties of actual infinity. Whether it was the indeterminacy of subtracting infinitely many ping-pong balls from infinitely many ping-pong balls, or the confusion around a 'completely full' infinite hotel, the resistance toward accepting properties of actual infinity illuminated participants' conceptions regarding the nature of mathematics.

CHAPTER 4:

INFINITY IN MATHEMATICS EDUCATION RESEARCH

Students' reasoning concerning the counterintuitive nature of cardinal infinity has been a popular focus of current research (see among others: Dreyfus & Tsamir 2004; Fischbein, Tirosh, & Hess, 1979; Tall 2001; Tsamir, 1999, 2001; Tsamir & Dreyfus, 2002; Tsamir & Tirosh, 1999; Weller, Brown, Dubinsky, McDonald, & Stenger, 2004). The body of literature ranges from explorations of learners' intuitive understanding of infinity to developing pedagogical tasks that will encourage a deliberate use of formal definitions. Learners' notions of infinity have also contributed to the development of several epistemological frameworks (e.g. Brown, McDonald, & Weller, in press; Dreyfus & Tsamir, 2004; Lakoff & Nunez, 2000; Tall, 1980).

This exposition of the mathematics education literature regarding learners' understanding of infinity begins with a review of the research which addresses the intuition of infinity as an 'endless' entity. Fischbein, Tirosh, and Melamed (1981) established that problems relating to actual infinity appear as contradictory to the basic intuition that infinity means inexhaustible. Accordingly, section 4.1 surveys some of the tasks regarding actual infinity that elicited intuitions of a single, 'endless' infinite. Extending on early work by Fischbein and his collaborators, a current trend in research has been to examine learners' conceptions when comparing cardinalities of different infinite sets. Their conceptions have been analysed based on the techniques or principles they apply to the task. As such, the focus of section 4.2 takes account of learners' naïve resolutions to set comparison tasks. In contrast, section 4.3 surveys the informed conceptions of students with varying levels of mathematical sophistication, as well as those with instruction on Cantorian set theory. The approaches and conceptions of informed participants, along with the persistent intuitions of naïve participants, have prompted several pedagogical considerations and strategies, which are addressed in the final section of this chapter, section 4.4.

4.1 Intuitions of a Single, 'Endless' Infinite

The concept of infinity carries with it a "surprisingly rich intuitive base that many students seem naturally to be endowed with" (Mamona-Downs, 2002, p.49). Research into the nature of learners' intuitions of infinity has shown "that infinity appears intuitively as being equivalent with inexhaustible" (Fischbein, 2001, p.324). Specifically, learners are naturally inclined to conceive of a potential or 'dynamic' infinity – a process for which every step is finite, but which continues endlessly. Intuitions of an 'endless infinite' have been observed in students of all levels, from middle school to university (e.g. Tirosh, 1991). In resonance with the general characteristics of intuitions, the idea of an endless infinite tends to be resilient: it is seen as self-evident, intrinsically certain, coercive, and resolute (Fischbein, 1987).

An interesting illustration of learners' intuition of an endless infinity appeared in Fischbein et al.'s 1981 research with middle school students (grades 8 and 9). One of the research tasks involved determining the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, which appeared geometrically as a series of line segments of decreasing length. The majority of participants confidently concluded that the magnitude of the resulting sum would be infinite, while only a small minority of participants realised the sum of the series was equal to 2. Typical justifications for the infinite result included:

> "A line segment can be extended endlessly;"
> "We shall always be able to add another part;" and
> "The process can be continued endlessly" (Fischbein et al., 1981, p.505).

Similarly, in Tirosh's 1991 study, participants of varying ages and math background who reasoned that the set of even numbers was infinite did so with justifications such as:

> "If you add 2 to an even number, you get another even number, and you can always add an even number" (p.343).

The association of infinity with inexhaustibility was suggested by Fischbein to be "the essential reason for which, intuitively, there is only one kind, one level of infinity. An infinity which is equivalent with inexhaustible cannot be surpassed by a richer infinity" (2001, p.324).

The belief that there is only a single, endless, infinite has also surfaced as students considered and compared the quantity of natural numbers with the quantity of real numbers (Fischbein et al., 1981). In a study that analysed the conceptions, levels of confidence, and degrees of obviousness associated with middle school students' responses to set comparison tasks, Fischbein et al. (1981) asked participants to compare the number of elements in the set {1, 2, 3, …} with the number of points on a line. The goal of the study was to quantify the level of confidence and degree of obviousness a student expressed in his or her solution in order to estimate the intuitive acceptance of that response. A high level of confidence and a high degree of obviousness corresponded to a high degree of intuitiveness. In response to the comparison, a small minority of participants reasoned that it was not possible to match every point with a different natural number. However, these students demonstrated little confidence in their responses and tended to base conclusions on interpretations that are deemed incorrect by mathematical convention, such as "The line has no beginning but the numbers start with 1" (Fischbein et al., 1981, p.507). In contrast, Fischbein et al. noted that the large majority of students who answered incorrectly – that the two sets were equinumerous – accepted that solution as "highly evident and reliable" (ibid). The typical response by these students was "there is an infinity of points on the line, and there is an infinity of natural numbers" (Fischbein et al., 1981, p.506).

4.2 Naïve Resolutions to Set Comparison Tasks

Current research suggests students' approaches to tasks regarding infinity tend to develop by reflecting on knowledge of related finite concepts and extending these familiar properties to the infinite case (Dreyfus & Tsamir, 2004; Fischbein, 2001; Fischbein et al., 1979; Tall, 2001). As Fischbein (2001) observed, when learners attempt to establish an understanding of abstract concepts, their tacit mental representations in the reasoning process replace the abstract concepts by more accessible and familiar ones. In particular, when analysing infinite sets, students may apply familiar methods for comparing sets that are acceptable in the case of finite sets, such as the inclusion (or part-whole) method, but which result in contradictions in the infinite case (Dreyfus & Tsamir, 2004; Fischbein et al., 1979; Tall, 2001).

Consider, for example, the two finite sets A = {1, 2, 3, 4} and B = {2, 4}. The inclusion method of comparison demonstrates that set B is a subset of A, and not equal to A, and thus has fewer elements. Attempting to correspond each element in A with a unique element in B also shows that the two sets contain a different number of elements. Conversely, take the sets **N** = {1, 2, 3, ...} and **E** = {2, 4, 6, ...}. In this case, the inclusion method and the correspondence method give way to contradictory results. Recall from Chapter 2 that by definition, an infinite set can be put into a one-to-one correspondence with one of its proper (infinite) subsets, such as with the sets **N** and **E**.

Tirosh and Tsamir (1996), who investigated high school students' conceptions of infinity, observed that the majority of participants reasoned that **N** had greater cardinality than **E**. The main argument offered by students drew on the inclusion method of comparison: since **E** was contained in **N**, **N** was clearly more numerous. Similar results were obtained by Fischbein et al. (1979) when they surveyed middle school students. Again, the majority of participants reasoned that **N** was the larger set since **E** was included in **N**. Fischbein et al. (1979) also observed that the inclusion method was the preferred method for comparing the set of points on a line segment and the set of points on a square, as many participants argued that the segment was a part of the square and thus the two could not have the same quantity of points. Other students objected to the possibility that sets of different dimension could have the same cardinality. Analogous responses were also observed for the comparison of points on a square with points on a cube. The argument that objects of different dimension must have different cardinality can also be viewed as an implicit inclusion argument. Properties of three-dimensional objects discussed in grade school geometry tend to be described in terms of the two-dimensional shapes that form them: a square forms a face of a cube, and as such might be considered as 'included' in the cube.

The inconsistencies observed in students' responses to these questions versus their comparison of the natural numbers with points on a line (discussed in section 4.1) is proposed by Fischbein et al. to stem from the "highly labile" nature of the intuition of infinity, one that depends both on "conjectural and contextual influence" (1979, p.32). Tsamir noted additional contradictions in students "declaring that infinite sets are incomparable and then proceeding to compare them," (1999, p.228), or "stating that all infinite sets are equal (have the same number of elements) and then proceeding to provide 'unequal' as a solution" (ibid). Dreyfus and Tsamir observed that their "students frequently reached contradictory conclusions when comparing the same pair of sets given in different representations. Unfortunately, [the students] usually remained unaware of these contradictions" (2002, p.4).

Inconsistencies in students' responses have also been linked to irrelevant visual aspects. Tirosh and Tsamir (1996, as well as Dreyfus & Tsamir, 2004; Stavy & Tirosh, 2000; Tsamir, 2001; Tsamir & Dreyfus, 2002; Tsamir & Tirosh, 1999) observed that the presentation of infinite sets had an impact on high school students' choices of comparison methods. If two sets were presented side-by-side, students were more likely to conclude the sets were of different magnitude than if the same sets were presented one above the other. These studies made use of four different visual representations of sets and observed which were more likely to elicit a controlled use of the one-to-one method of set comparison. The different representations presented to students are exemplified in Figure 4.1 (below); they include the numerical-vertical, numerical-explicit, geometric, and numerical-horizontal representations.

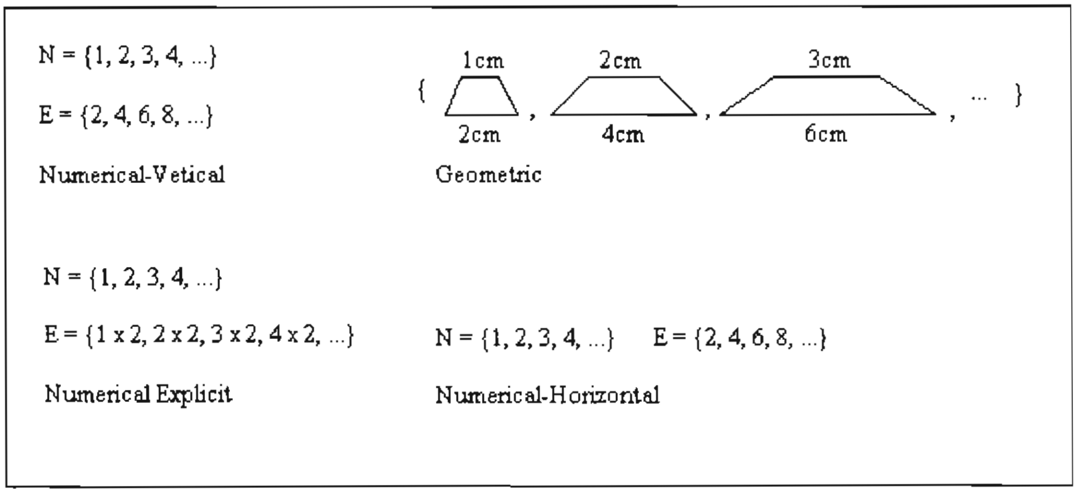


Figure 4.1: Set Representations

Results of these studies consistently found that both the numerical-explicit and geometric representations elicited a more instinctive use of one-to-one correspondences when comparing sets than the other two representations. In contrast, the numerical-horizontal representation gave an impression of nested sets, and was found to encourage an erroneous use of the inclusion method. As one student reflected, the geometric representation provided "a graphic, a visual image of the possible way to pair the numbers in the two infinite sets" (Dreyfus & Tsamir, 2004, p.284). This student also felt that the numerical-explicit representation put "extra emphasis on the way we can pair matching elements. It shows beyond any doubt that there is an equivalence correspondence. It also emphasizes the way that this correspondence determines the equality of the sets' size" (Dreyfus & Tsamir, 2004, p.285).

4.3 Instruction and Mathematical Sophistication

A series of studies has been conducted on the conceptions of students of varying levels of mathematical sophistication as they attended to the cardinalities of finite and infinite sets (e.g. Tirosh, 1991; Stavy & Tirosh, 2000). Tirosh (1991) surveyed students from elementary school to university to investigate the intuitive methods used to determine whether a given set was infinite or finite. Surprisingly, Tirosh observed that of the various groups of students, university students most frequently gave incorrect responses when shown practical examples of finite sets, such as the number of drops of water in a cup or in the Pacific Ocean. Participants described the "infinite divisibility of drops of water" and "the unlimited space of the Pacific Ocean" (Tirosh, 1991, p.346) to justify their infinite responses, with the former argument being by far the most common justification for infiniteness among all groups of students.

In a similar study, Stavy and Tirosh (2000) investigated the relationship between students' mental models regarding the particulate nature of matter and the infinite nature of geometrical figures. Students in grades 7, 8, 10, and 12 were asked about the divisibility of copper wire and line segments. In resonance with Tirosh's (1991) observations, the frequency of incorrect infinite responses – that a copper wire may be divided 'endlessly' – increased with age. Tirosh (1999) summarized: more mathematical background did not correspond to more correct categorizations of infinite sets. Nevertheless, Tirosh did note a correlation between students' responses and their mathematics experiences: students with a more sophisticated mathematical background demonstrated a more systematic use of logical schemes – which Fischbein et al. (1979) observed are naturally adapted to finite objects. Tirosh also noted that students with more mathematical experience made more frequent use of 'intuitive rules' – which are taken as self-evident and viewed as true without justification (Stavy & Tirosh, 2000). These observations hint at the possibility that understanding properties of infinite sets requires a step away from the intuitive and a realisation that prior experience with finite entities and schemes may not be generalizable to the infinite case.

The relationship between students' conceptions of infinity and their mathematical background, particularly with respect to their knowledge of Cantorian set theory, continues to be of interest. Research has shown that knowledge of Cantorian set theory and the preferred method of set comparison did not prevent secondary and even college students from oscillating between one-to-one correspondence and inclusion methods. In fact, these students even seemed unaware of the necessity of avoiding incompatible methods of comparison (Borasi, 1985; Duval, 1983; Fischbein et al., 1979; Fischbein et al., 1981; Tirosh, 1991; Tirosh & Tsamir, 1996; Tsamir, 1999, 2003; Tsamir & Tirosh, 1999). One such instance occurred in Tsamir (2003), where prospective secondary school teachers were asked to assess the applicability of different methods of set comparison when dealing with infinite sets. They were presented with three methods of comparison: one-to-one correspondence, inclusion, and the 'single infinity' argument. The majority of participants had previously taken a class in Cantorian set theory, and despite formal knowledge and their tendency to accept the one-to-one correspondence criterion as an appropriate method of comparison, a substantial number of these participants also found the other criteria acceptable. Tsamir noted, "Even after studying set theory, participants still failed to grasp one of its key aspects, that is, that the use of more than one … criteria for comparing infinite sets will eventually lead to contradiction" (2003, p.90).

In a recent study, Brown, McDonald, and Weller (in press) also observed that formal instruction had little impact on the approaches of university students as they addressed a question of cardinality comparison. These students had recently completed an upper year mathematics course which addressed Cantorian set theory, and they were asked to prove or disprove the equality $\bigcup_{k=1}^{\infty} P(\{1, 2, ..., k\}) = P(N)$, where P represents the power set operator. With a similar argument to Cantor's diagonal proof (Chapter 2), it is possible to show that $P(N)$, the set of all subsets of N, is uncountable; whereas, the infinite union on the left of the equal sign is a countable union of finite sets. Of the thirteen students interviewed by Brown et al., only one was able to correctly solve the task, and then only with considerable prompting from the interviewer. Interestingly, although the students "demonstrated knowledge of the definitions of the objects involved, all of the students tried to make sense of the infinite union by constructing one or more infinite processes" (McDonald & Brown, 2008, p.61) – that is, the students relied on intuitive, rather than formal, approaches. Furthermore, each of the students attempted to resolve this problem by constructing an infinite iterative process, even though the problem itself was stated in terms of static objects, not processes (Weller et al., 2004).

4.4 Pedagogical Strategies

Students' well-documented struggle to understand and appreciate aspects of cardinal infinity has motivated efforts to improve and refine pedagogical strategies. The geometric representation pictured in Figure 4.1 (section 4.2) offers one example of how researchers and instructors can make use of the tangible nature of a visual image. In the set activities administered by Dreyfus and Tsamir (2004, also Tsamir & Dreyfus, 2002; Tsmair &

Tirosh, 1999), geometric figures were used to emphasize a correspondence between numerical sets, as well as to draw students' attention to the inconsistencies of comparing infinite sets with different methods. For instance, an activity designed by Tsamir and Tirosh (1999) had students examine the set of natural numbers, which were enumerated on a card up to the number 21. The task began by circling all of the multiples of four that appeared in the set and then creating a new set with these circled numbers. Students were later asked to compare the cardinalities of {1, 2, 3, ...} and {4, 8, 12,...}. Many students relied on the inclusion method for comparison and concluded that the set of natural numbers was greater than the set of multiples of four. One of the goals of this study was to elicit cognitive conflict, thus this task was followed by an analysis of the geometrical representation of the corresponding sets, which was intended to emphasise the one-to-one correspondence. Students were asked to consider a set of line segments with increasing lengths (Figure 4.2a), and then to imagine constructing squares in such a way that the segments were of the same lengths as the sides of the squares (Figure 4.2b).

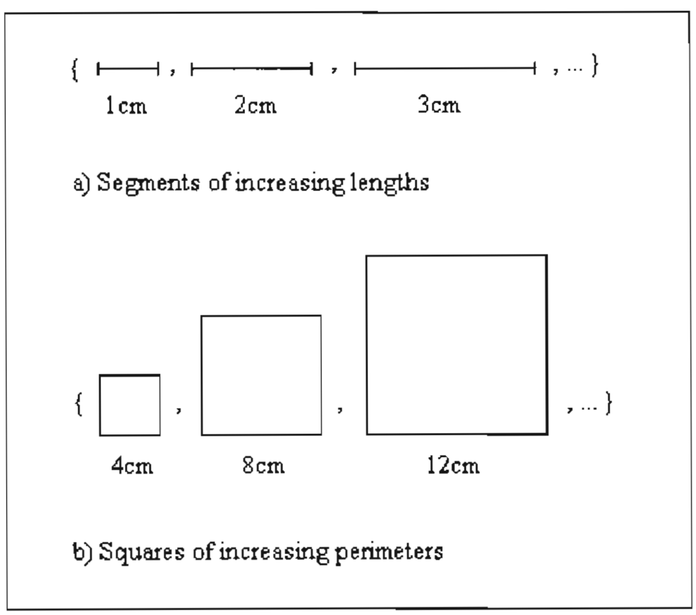


Figure 4.2: Geometry as Analogy

As in the diagram, the lengths and perimeters were written below each segment and square, respectively. Students were asked whether more than one square could be constructed for each segment. They were also prompted to speculate on the relationship between the length of a side and the length of the perimeter of any square. Tsamir and Tirosh concluded that this series of activities "has the potential for raising students' awareness of incompatibilities in their own solutions to the same mathematical problem" (1999, p.216). Tsamir (2003) emphasised the importance of drawing to students' awareness the inconsistencies resulting from different methods of comparison. McDonald and Brown (2008) concurred with Tsamir's suggestion that eliciting cognitive conflict in learners offers a means to help "students understand what to do and why, and that using more than one criteria to compare infinite sets leads to contradictions" (McDonald & Brown, 2008, p.59). Similarly, Sierpinska (1987) suggested eliciting cognitive conflict in learners might be a starting point to overcoming epistemological obstacles related to limits and infinity, such as the problematic view that mathematics should avoid dealing with infinity and instead restrict its attention to finite numbers.

The geometrical presentation of Figure 4.2 above, offered Tsamir and Tirosh's participants a one-to-one pairing that was much more obvious to see than in numeric presentations. The sets {1, 2, 3, ...} and {4, 8, 12, ...} were thought of as the set of lengths of the segments and the set of perimeters of the corresponding squares, respectively. This use of a geometric image provided an alternative way for students to consider the numbers and sets they were comparing. Attributing numbers to lengths and perimeters may be a way to detach the numbers from their magnitudes and create an analogy with which to consider the two sets in question. The meaning of the numbers in this context would then change from their meaning in the abstract set notation, and students could attend, not to the specific numbers themselves, but to the natural correspondence between a side and a perimeter of a square. In addition, through this analogy, students could extrapolate their experiences with, and ideas about, finite sets of squares in order to imagine an infinite set that is consistent with their prior knowledge. Tsamir and Tirosh recognised the use of analogy with a familiar experience as an effective instructional tool for triggering "the spontaneous use of one-to-one correspondence" (1999, p.216). However, as noted by Tsamir (2003), and as mentioned in section 4.3, participants must come to appreciate one-to-one correspondence as the only appropriate method of infinite cardinality comparison, and it is unclear whether reasoning by analogy will contribute to such an understanding.

Having now established some of the key results relating to infinity in mathematics and mathematics education, the next chapter discusses the theoretical perspectives that guided my research. The studies that are presented in Chapters 6, 7, and 8, as well as the integrated analysis and discussion that follows in Chapter 9, stemmed from my consideration of the interrelated theoretical underpinnings of 'reducing levels of abstraction' (Hazzan, 1999), APOS Theory (Dubinsky & MacDonald, 2001), and 'measuring infinity' (Tall, 1980).

CHAPTER 5:

THEORETICAL PERSPECTIVES

The theoretical perspectives discussed in this chapter contributed to the overarching framework which guided the empirical investigations that are presented in Chapters 6, 7, and 8. Three interrelated frameworks are discussed and connected in this chapter, and the means with which they informed the analysis of participants' responses to tasks and paradox resolutions are illustrated. Hazzan's (1999) perspective of reducing the level of abstraction in order to establish meaning about a concept is connected to Tall's framework of 'measuring infinity', and then related to Dubinsky and McDonald's (2001) APOS – Action, Process, Object, Schema – Theory. This inter-connected theoretical foundation is used to interpret participants' approaches to resolving tasks regarding actual infinity, as well as to analyse the conceptions that were elicited by these tasks.

The concept of actual infinity can present considerable challenges to learners, and as exemplified in Chapter 4, a more sophisticated mathematical background does not necessarily correspond to a normative approach to problems and paradoxes of actual infinity. The anomalies of actual infinity – some of which were illustrated in Chapter 2 – are difficult to grasp for various reasons, including their inaccessibility, abstraction, and formal structure. Compounding these difficulties are the inconsistencies learners may face when trying to reconcile properties of actual infinity with their prior mathematical knowledge, experiences, and intuitions. As such, it is not uncommon for learners to experience a state of *cognitive conflict* as they grapple with an entity that must be treated as a completed totality, but which appears to our intuitions as endless. Cognitive conflict is regarded as a state in which learners become aware of inconsistent or competing ideas. Piaget (1985) described an analogous state of 'disequilibration' as an essential aspect of cognitive growth. In Piaget's perspective, cognitive structures are developed as learners integrate information into their existing structures – that is, new knowledge is thought to develop in conjunction with prior understanding, building on, extending, or revamping existing knowledge. During the process of cognitive growth, Piaget hypothesised that learners experience temporary stages in a cycle of equilibration, disequilibration, and reequilibration (Kamii, 1986). In other words, an individual is constantly seeking to maintain a state of cognitive coherence (equilibration), however this state of equilibration may be unhinged by the recognition of discrepancies between the individual's existing cognitive structure and external information. This 'unhinging' relates to Piaget's stage of disequilibration; it instantiates cognitive conflict. The recognition of discrepancies is necessary to trigger a stage of cognitive conflict, which can in turn motivate the individual to reconcile the conflict by seeking new information or by attempting to restructure existing information. Reconciling the cognitive conflict, or 'disequilibration', corresponds to Piaget's 'reequilibration', the final stage in his cycle of cognitive growth.

In certain cases, inconsistencies in an individual's cognitive structure may go unnoticed. In such a case, if a learner's incompatible and inconsistent ideas are recognized by the instructor, but not yet recognised by the individual, the individual is said to face a *potential* cognitive conflict (Zazkis & Chernoff, 2008). Zazkis and Chernoff argue that a potential cognitive conflict can develop into a cognitive conflict in an instructional situation. For instance, problems and paradoxes regarding infinity present learners with a potential cognitive conflict – a discrepancy exists between properties of actual infinity and a learner's intuition and prior (finite) experiences. In the studies described in Chapters 6, 7, and 8, activities were designed and employed as means to draw to the attention of participants the inconsistencies in their ideas. The intent of the activities was to invoke a cognitive conflict from the potential conflict that was presented by properties of infinity as they appeared in paradoxes or geometric tasks.

Participants' reactions to the potential cognitive conflict offered by properties of actual infinity, as well as their naïve and emergent conceptions of infinity, were of interest in the forthcoming studies. The following sections explore the interrelated theoretical frameworks that were used to interpret participants' conflict resolution as well as their emergent conceptions of infinity. In the first section, the perspective of reducing the level of abstraction (Hazzan, 1999) is discussed in connection with Tall's (1980) framework of 'measuring infinity'. Following that, the second section considers the APOS (Action, Process, Object, Schema) Theory (Dubinsky & McDonald, 2001).

5.1 Reducing Abstraction and Measuring Infinity

As learners engage in novel problem solving situations, their attempts to make sense of unfamiliar and abstract concepts can be described through the framework of reducing levels of abstraction (Hazzan 1999). In Hazzan's (1999) perspective, learners will attempt to cope with novel concepts through different means of reducing abstraction. For instance, Hazzan described "students' tendency to work with canonical procedures in problem solving situations" (1999, p.80) as a means of reducing abstraction. She observed that working with familiar entities was a common strategy for learners who were faced with problems for which an understanding of the mathematical entities involved were not yet constructed. As an example, Hazzan noted that when learning abstract algebra, students would "often treat groups as if they were made only of numbers and of operations defined on numbers" (1999, p.77). By basing arguments on familiar mathematical entities, such as numbers, in order to cope with unfamiliar concepts, such as groups, students lower the level of abstraction of those concepts. In the context of infinity, one such example is students' use of familiar (finite) measuring properties to interpret infinite quantities of measurable entities, such as the quantity of points on a line segment. This example of reducing the level of abstraction of infinitely many points on a line segment relates to Tall's (1980) notion of 'measuring infinity.'

5.1.1 Measuring Infinity

Tall (1980) suggested intuitions of infinity can develop by extrapolating measuring, rather than cardinal, properties of numbers. Many of our everyday experiences with measurement and comparison associate 'longer' with 'more.' For example, a longer inseam on a pair of pants corresponds to more material. Likewise, a longer distance to travel corresponds to more steps one must walk. Tall (1980) proposed extrapolating this notion can lead to an intuition of infinities of 'different sizes.' A measuring intuition of infinity coincides with the notion that although any line segment has infinitely many points, the longer of two line segments will have a 'larger' infinite number of points. Tall (1980) called this notion 'measuring infinity' and suggested it is a reasonable and natural interpretation of infinite quantities, especially when dealing with measurable entities such as line segments. Although Tall's perspective of 'measuring infinity' focuses on geometric entities such as line segments, I would like to suggest that the notion of 'measuring infinity' extends further. Relating Tall's perspective to the Ping-Pong Ball Conundrum, an intuition of 'measuring infinity' can be seen in arguments that connect to the different rates of ping-pong balls that are placed into and removed from the barrel. In this context, the rate of in-going ping-pong balls is measured relative to the rate of out-going balls, with the faster rate accumulating a 'larger' infinite number of balls. An intuition of 'measuring infinity', with respect to line segments, as Tall suggests, and also regarding rates, can be interpreted as an attempt to familiarize the unfamiliar by basing arguments on known relationships, and might develop as a consequence of learners' attempts to lower the level of abstraction of comparing infinite cardinalities.

5.1.2 Coping with the Unfamiliar

In addition to relying on familiar entities to reduce the level of abstraction of novel ones, Hazzan (1999) observed that learners' use of personal language, as well as the complexity of the entities with which they choose to deal, are also indicative of attempts to reduce levels of abstraction. With respect to personal language, Hazzan interprets "students' personalization of formal expressions and logical arguments by using first-person language" as an attempt to reduce the level of abstraction of that expression. For instance, language such as "I can find" or "I want to find" (Hazzan, 1999, p.80), indicate, in Hazzan's perspective, ways that a student may cope with unfamiliar terminology and concepts. In the context of abstract algebra and relating to the complexity of mathematical entities, Hazzan builds on the assumption that "the more compound an entity is, the more abstract it is" (1999, p.82). For example, she describes a set of groups as being more compound, and hence more complex and abstract, an entity than a single group. As such, a student may attempt to reduce the level of abstraction of that compound entity, say a set of groups, by examining only one element, one group, in that set. An analogous attempt to reduce the level of abstraction might occur with respect to infinity through the generalization of properties of a finite cardinality to draw conclusions about an infinite cardinality.

Hazzan (1999) relates her framework of reducing levels of abstraction to the APOS (Action, Process, Object, Schema) Theory of Dubinsky and McDonald (2001) through the observation that process conceptions of a mathematical entity may be considered on a lower level of abstraction than their corresponding conceptions as objects. She also argues that a learner's attempt to reduce the level of abstraction of a mathematical entity through, for instance, the use of first-person language, or by working with canonical procedures when problem solving, indicate that the learner holds a process (rather than object) conception of that entity. Process and object conceptions are in the centre of the third framework considered in this study, that of the APOS (Action, Process, Object, Schema) Theory (Dubinsky & McDonald, 2001).

5.2 APOS Theory

The APOS Theory postulates a framework for interpreting learners' understanding of tertiary mathematics. Through the mechanisms of *internalisation* and *encapsulation* the learner is said to construct meaning for mathematical entities that are conceptualised with the 'structures' of the APOS Theory: *actions, processes, objects,* and *schemas* (Dubinsky & McDonald, 2001). In the terminology of the APOS Theory, an understanding of a mathematical entity begins with an *action* conception of that entity. Action conceptions are recognised by an individual's need for an explicit expression to manipulate or evaluate. Eventually, an action may be *interiorised* as a mental *process*. That is, once an action has been interiorised, the individual can imagine performing an action without having to directly execute each and every step. A process conception is recognised by qualitative descriptions which may describe actions though not execute them. If the individual realises the process as a completed totality, then *encapsulation* of that process to an *object* is said to have occurred. Encapsulation of a process is a sophisticated step in an individual's conceptualisation. It requires appreciating the mathematical entity as a completed object that can be acted upon. In other words, the entity is conceived of as an object upon which transformations or operations may be applied. These three structures of the APOS Theory – the action, process, and object – describe how the idea of a single mathematical entity may develop. However, it is possible that a mathematical concept may be composed of more than one entity, involving several actions, processes, and objects that must be coordinated into a mental *schema*. It is then this schema that "provides an individual with a way of deciding which mental structures to use in dealing with a mathematical problem situation" (Dubinsky, Weller, McDonald, & Brown, 2005a, p.339).

5.2.1 Infinity as Process and Object

Dubinsky, Weller, McDonald, and Brown (2005a) proposed an APOS analysis of two conceptions of infinity: actual and potential. They suggested that interiorising infinity to a process corresponds to an understanding of potential infinity, that is, infinity is imagined as performing an endless action, though without imagining the implementation of each step. Encapsulating this endless process to a completed object, in turn, corresponds to a conception of actual infinity. Relating this distinction to the appearances of infinity explored in the up-coming studies, one may consider geometric representations of infinity as either processes or objects. For instance, a conception of potential infinity, with respect to points on a line segment, might correspond to a process of marking or 'creating' points on a line segment that is imagined to continue indefinitely. While actual infinity might be illustrated by the idea that the infinite number of points on a line segment exists as a completed entity, without needing to be marked. Further, the process-object duality of infinity may also be instantiated in certain paradoxes of the infinite. In the case of the Ping-Pong Ball Conundrum, the action of cutting the remaining time in half can be imagined to continue indefinitely, and would thus describe potential infinity. Whereas actual infinity would entail the completed infinite process of halving time intervals, and would describe the set of time intervals as a completed entity, where each interval exists within the 60 seconds. Similarly, in Hilbert's Grand Hotel paradox, the hotel itself corresponds to actual infinity – it is a completed, infinite entity. On the contrary, a potentially infinite hotel might be one that can continually create new rooms in order to accommodate new guests.

As in the more general case, encapsulation of infinity is considered to have occurred once the learner is able to think of infinite quantities "as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied" (Dubinsky et al., 2005a, p.346). Dubinsky et al. also observed that "in the case of an infinite process, the object that results from encapsulation transcends the process, in the sense that it is not associated with nor is it produced by any step of the process" (2005a, p.354). Brown, McDonald, and Weller (in press) introduced this possibility, and termed the encapsulated object of infinity a *transcendent object*. An object which transcends any individual step of its corresponding process may require, in Dubinsky et al.'s perspective, "a radical shift in the nature of one's conceptualisation" (2005a, p.347).

Dubinsky et al. (2005a) suggested further that the conceptions of infinity as a process or an object and the relationship between them contributes to the individual's infinity schema. In the context of the previous examples, an infinity schema that coordinates object and process conceptions of infinity would help the individual identify that the paradoxes and infinite cardinalities are normatively accepted as instances of actual infinity, rather than potential infinity. Dubinsky, Weller, McDonald, and Brown recommend that pedagogical strategies "should focus on helping students to interiorize actions repeated without end, to reflect on seeing an infinite process as a completed totality, and to encapsulate the process to construct the state at infinity, with an understanding that the resulting object transcends the process" (2005b, p.264).

As previously mentioned, the frameworks that guided this dissertation – Hazzan's theory of reducing levels of abstraction, Tall's 'measuring infinity', and the APOS Theory – are interrelated. Hazzan's theory aims to identify the techniques with which learners attempt to make sense of novel and abstract mathematical ideas – e.g. through the use of familiar concepts or personal language. Tall's framework is interpreted as a special case of reducing the level of abstraction of those mathematical ideas by applying familiar measuring techniques to cope with properties of infinite quantities. The APOS Theory presents a hierarchical framework that decomposes how those mathematical ideas are understood – e.g. as process, or objects – and is connected to the aforementioned perspectives by Hazzan's observation that a "process conception of a mathematical concept can be interpreted as on a lower level of abstraction than its conception as an object" (1999, p.79).

Extending these ideas, the studies presented in the following chapters interpret university students' naïve and informed ideas, as well as their attempts to reduce the level of abstraction of infinity. Chapter 6 provides an account of participants' emergent conceptions as they engaged in a series of geometric tasks designed to elicit personal reflection and provoke cognitive conflict. Chapters 7 and 8 use the lens of paradoxes to interpret participants' conflict resolution. In Chapter 7, participants' engagement with Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum sparked cognitive dissonance, while in Chapter 8 the Ping-Pong Ball Conundrum and a variation of it, confronted participants with different conceptual challenges. In particular, Chapter 8 takes a closer look at the relationship between 'encapsulation' of infinity, as described by the APOS Theory, and the conceptual accommodation of actual infinity.

CHAPTER 6:

INTUITIONS OF 'INFINITE NUMBERS': INFINITE MAGNITUDE VS. INFINITE REPRESENTATION

This study explores the naïve and emerging conceptions of university students as they address properties of cardinal infinity and transfinite arithmetic, and as they attempt to coordinate intuition and reflection with formal instruction. In what follows, participants' engagement with geometric representations of infinity are described and used as a lens to their understanding of infinity and 'infinite numbers'. In particular, participants' conceptions as they attended to the number of points 'missing' from the shorter of two line segments are of interest. In addition, this study explores what sort of connection, if any, participants made between a geometric representation of infinity and a numeric one. The following specific research questions are addressed: 1) What connections do participants make between geometric and numeric representations of infinity, i.e. between points on a line and real numbers? 2) What can be learned about participants' conceptions as they confront a bound infinity? 3) What can be learned about participants' conceptions of infinity as they address properties of transfinite subtraction?

6.1 Setting and Methodology

The participants in this study were 24 undergraduate university students in an applied science program. They were enrolled in "Foundations of Analytic and Quantitative Reasoning", a course which I taught and which was designed to develop students' mathematical thinking through inquiry and problem solving. The class met twice a week for two-hour seminars in which fundamental topics and concepts of mathematics were reviewed. Topics were explored through problem solving and small group activities, and included ideas related to patterns and numbers, properties of fractions and decimals, and properties of lines. One of the objectives of the course was to provide an opportunity for students to engage in critical analysis and personal reflection regarding some of the fundamental ideas in mathematics that were novel to them. The topic of infinity was included as one of these fundamental ideas.

Data collection relied on three main sources: (i) individual written responses to 'reflection activities', (ii) arguments presented during class discussions, about which field notes were taken that were summarized immediately after class observations, and (iii) follow up interviews with two of the participants. The 'reflection activities' consisted of a series of written questionnaires administered over the span of the course that were designed to elicit participants' naïve conceptions and then to encourage them to reconsider, develop, and critique the underlying ideas through further individual questioning. This form of questioning echoed the general atmosphere of the class: participants were regularly challenged to critique, explain, and justify their ideas. The reflection activities were used on several occasions throughout the term on a variety of topics, not all of which are within the scope of this study. Tasks were formulated based on participants' previous responses, as well as the common themes which emerged from the class. Participants' familiarity with this method of questioning contributed to the reliability of their responses.

It was important, both for research and instructional purposes, that participants' responses were not affected by seemingly correct solutions or the desire to appease their instructor. The reflections were not judged for grades, but rather, were used to develop discussion in subsequent classes. In order to avoid swaying participants' responses, very little instruction was provided initially, and it was made clear that there was no one 'right' answer being sought. The activities were designed to complement this approach and took different forms, such as recalling participants' previous responses and presenting them with a slight twist so as to encourage participants to challenge the issues they had unearthed. Other questions presented participants with a dubious argument that claimed to be from one of their peers in order to provoke a critique of the ideas involved. The basis for both styles of question was to avoid presenting an authoritative position. Participants addressed each issue based on its appeal to their own emerging ideas.

As mentioned, in addition to participants' written responses, data was collected from class discussions, including an instructional discussion on cardinality and infinite sets, which occurred at the end of the course. The discussion included comparing cardinalities of countable and uncountable infinite sets through one-to-one correspondences, or the idea of 'coupling'. Some of the specific conceptions that arose in participants' reflections were also addressed. Details concerning the scope of instruction are provided following an exploration of participants' responses to the reflection activities. Data was also collected from follow up interviews which were conducted with two participants, Lily and Jack. The interviews further explored their emerging and lasting conceptions of infinity, and also probed the inconsistencies that surfaced in their responses to discussion topics and previous questionnaires.

The interactive design of this study does not follow the typical format of research reports. As such, the majority of the specific tasks addressed by participants are presented in the body of the Results and Analysis section. The study began with a two-part question, Q_0 (below). This task set the stage for exploring participants' connection between numeric and geometric representations of infinity.

Q_0. (a) How many fractions can you find between the numbers $\frac{1}{19}$ and $\frac{1}{17}$? How do you know? (b) How many points are there on a line segment? How do you know?

Later questionnaires developed in response to participants' reactions to Q_0 and its follow-ups, and focused on the sets of points on line segments of varying lengths. They were intended to investigate ideas regarding 'infinite numbers' as well as 'infinite number' properties.

6.2 Results and Analysis

In the spirit of capturing emerging conceptions of infinity, the data presented in this section follows an atypical format. The story of participants' engagement in this study is unfolded much as it occurred: as a journey of developing understanding. This section is organised into four parts beginning with participants' initial reactions, and ending with their informed reflections after months of engagement. Between these two ends lies an analysis of the themes which surfaced in participants' emergent conceptions of infinity.

In particular, three main threads weave through the following sections. The first thread speaks to participants' reluctance to engage with the idea of infinity – realistic experience and practical needs were persuasive factors in some participants' reasoning. A second thread is recognised in participants' inconsistent intuitions and approaches toward infinity – one such example that surfaced involves participants' conflicting notions of potential and 'measuring' infinity. A third thread relates to an observed disconnect in participants' conceptions of real numbers and their geometric representation as points on a line.

6.2.1 Bound Infinity: Numbers and Points

From the early stages of the study, a clear disconnect in participants' conceptions of numbers and points on a real number line was observed. Typical arguments to item $Q_0(a)$, which concerned the number of fractions between $\frac{1}{19}$ and $\frac{1}{17}$, are exemplified by the following two responses:

> "Infinite. Because there are endless numbers that can be put into the numerator or the denominator and still making sure the fraction is larger than $\frac{1}{19}$ and smaller than $\frac{1}{17}$"; and "You can find an infinite amount of fractions in between $\frac{1}{17}$ and $\frac{1}{19}$ because you can continue to add digits after the decimal point forever (e.g. $\frac{1}{18}$, $\frac{1.3}{18}$, $\frac{1.3625}{18}$, etc.) making the fractions a little bigger or smaller."

A common idea in these responses is that of potential infinity. The notions of "endless numbers" or adding "digits after the decimal point forever" imply infinity is conceived of as a process. The idea of changing the numerator or denominator corresponds to an action that is imagined to continue "forever", and is consistent with Fischbein et al.'s (1981) suggestion that infinity is intuitively thought of as inexhaustible.

Of the responses to item $Q_0(a)$, only two of the 24 described an inconsistency with an 'endless' infinite existing within a bound. Robbie described a conflict between an 'endless' infinite contained in a 'finite' interval, and refused the possibility of infinitely many numbers between $\frac{1}{19}$ and $\frac{1}{17}$ "because otherwise [$\frac{1}{19}$] will be the only number in the universe." Similarly, Neil also objected to a bound infinite: he expressed difficulty with the idea of "getting to" the end of the interval since the process of adding decimal digits has "no stopping point". Instead, he suggested the quantity of numbers depended on the "real life situation". For instance, Neil remarked, "If you're counting like, atoms or something, then… you're allowed to go really small in the decimal place. But, let's say you're counting computers, you're not going to have 0.1 or 0.2 computers." He explained further that it was possible to attain the end of the interval "when you have restrictions, such as you're only allowed to go to a certain number of decimal places." Through his idea of restrictions, Neil distinguished between "real life" possibilities and mathematical or "theoretical" ones. He suggested that "theoretically" there could be infinitely many numbers but "in terms of life there isn't, because life has restrictions on it," and "in real life situations, I don't really see it being necessary. Like, even if it was there, it's not necessary because it just confuses things." Neil seemed to resist engaging with infinity in the "theoretical" realm of mathematics because of its impracticality, suggesting he prefers "just kind of ignoring it [infinity], kind of, to make the situation easier."

In response to item $Q_0(b)$, regarding the number of points on a line segment, the majority of participants (17 out of 24) indicated that points were either the places that a line segment begins and ends, or else they were markers that partition a line segment into equal units. These responses were surprising in light of participants' responses to item

Q_0(a), and their ideas regarding the infinite quantity of numerical 'values' on any line segment. Participants' arguments supporting an infinite number of 'values' on a line segment were similar in nature to their arguments regarding item Q_0(a) above. They described processes of finding "as many values as we want", however they distinguished between the finite number of points that 'existed' on a line segment and the infinite number of points that could be "given a value" or "labelled". Participants suggested that until points were "labelled" on a line segment they were not there. As before a process conception of infinity is recognised in participants' responses. Further, the idea of 'finding values', or 'creating points' by assigning them numerical values, may be interpreted as an attempt to reduce the level of abstraction of an infinite yet bounded quantity to a finite number of 'marked' points.

Participants' distinction between 'point' and 'value' prompted a class discussion regarding the geometry of points and lines in order to establish a shared understanding (to use the term loosely) of the infinite magnitude of points (rather than 'values') on a line segment. As participants attempted to accommodate this possibility some initial resistance was observed. Some participants suggested that the word 'infinity' in this context was used as a way to identify the unknown, claiming "it is not humanly possible to figure out the number of points… so it is said to be infinite." Similarly, Jim proposed the possibility that 'infinity' was a label attributed to unknown quantities for which there does not exist a means of measurement. He related the conversation to a radio discussion he had heard:

> "They were talking about miracles on the radio today, and how people put the label 'miracle' on something that's wonderful but that they don't understand necessarily, and it helps us to kind of put it into a category that our brains can then figure out.

> And that might be something the same with the word 'infinity', where something that keeps going past any way that we can measure, we put this word onto it because then it's settled and we can push it out of the way and move on with our lives."

Jim's suggestion that using 'infinity' as a label for immeasurable quantities as a way of settling controversy and to "push it out of the way and move on with our lives" resonates with Neil's suggestion of "ignoring" actual infinity "to make things easier." In both cases, participants attempted to cope with the cognitive conflict elicited by counterintuitive properties of infinity by avoiding engaging with the "theoretical" realm of mathematics and restricting their attention to the "realistic" domain of "measurable" entities.

Other participants attempted to accommodate the idea of an infinite number of points on a line segment by shifting the process of an infinite extension (e.g. 'adding' or 'creating' points) to a process of infinite zoom by introducing the notion of point size. For instance, Lily remarked, "In a line, there can be many points present because the size of the points have no limit. It could be an extremely big point or a microscopic size point." Similarly, Hank reasoned that "there should be able to be an infinite number of points on a line segment as you hypothetically zoom in infinitely", suggesting that "there is no end to how far you can zoom in at a microscopic, atomic, or subatomic level and beyond". Hank was reluctant to commit to the idea that infinitely many points existed on a line segment "because of the lack of research [he had] done in this area." His reluctance is further illustrated by the comment that "there should be able to be an infinite number of points", rather than 'there are an infinite number'.

The idea of 'microscopic' or 'subatomic' sized points might have developed as an extrapolation of scientific knowledge pertaining to the composition of physical entities, such as atoms from subatomic particles. Tall (1980) also suggested the notion of point size might develop from physical knowledge, noting that "[p]hysical points have size when they are marked with the stroke of a pen" (p.272). Alternatively, some participants' ideas of point size stemmed from an association that participants were making between point and number. This perspective was exemplified in Dylan's statement:

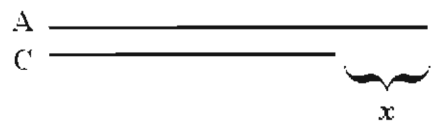
> "0, 1, 2 those would be big points, or you could have 0, 0.5, 1, 1.5, then those would be smaller points. And you could go smaller or bigger depending on what you want to do."

Thus, a microscopic point might be associated with the number 0.00…001, whereas "big points" were associated with whole numbers; much as the gradients on a ruler distinguish between whole measures and fractional measures with marks of different sizes.

The association between point size and numeric value, although different from the conventional one, was nevertheless an early connection between real numbers and their representation on the number line. Further, it seemed to indicate a change in participants' conceptions as they had begun to connect geometric and numeric representations of infinity. However, subsequent questionnaires revealed that participants' point-number correspondence was flawed and inconsistent, if it was made at all.

The questionnaire following this discussion related to the number of points on line segments of different lengths, and prompted participants to reflect on the number of points 'missing' from the shorter of the two segments. The following specific question was posed:

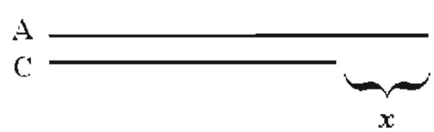
Q_1. Consider line segments A and C again. Suppose that the length of A is equal to the length of C + x, where x is some number greater than zero, as depicted below. What can you say about the number of points on the portion of A whose length is x?



As in previous questionnaires, participants offered process-oriented responses to Q_1, such as Levon's argument that the segment with length x "has an infinite amount of points… because you can put as many points as you wish on that tiny portion." Participants reasoned that even a very small line segment was composed of an infinite number of points: "Although the portion of x is small, it still contains an infinite number of points within it", and "there are an infinite number of points between the start and end point… Regardless of length". In order to investigate both participants' rationale when comparing the number of points on line segments of different lengths, and also participants' intuitions regarding subtracting infinite quantities, Q_2 (below) presented their conclusions with a slight twist.

6.2.2 Subtracting Infinity: Intuition of 'Measuring Infinity'

Q_2. On a previous question, you reasoned that two line segments A and C both have infinitely many points.



Suppose that the length of A is equal to the length of C + x, where x is some number greater than zero. You also previously suggested that the segment with length x has infinitely many points. That is, the ∞ points on A minus the ∞ points on C leaves an ∞ number of points on the segment with length x. Put another way, $\infty - \infty = \infty$. Do you agree with this statement? Please explain.

Participants' responses to Q_2 revealed inconsistencies in their conceptions, as well as a strong intuitive resistance to the idea of subtracting infinite quantities. Jack, for example, experienced a conflict as a conception of infinity emerged that contrasted his intuition. Previously, Jack had described infinity as a "hypothetical number" that is "the biggest number you can get", and for which "you'd have to count your whole life and you still would never get there." Intuitively, Jack seemed to conceive of infinity as an unattainable extension of 'very big'. His comment that counting your whole life "still would never get [you] there" typifies a process conception of infinity. However, this fundamental notion of infinity was challenged by the visual representation of the two line segments. In response to Q_2 Jack explicitly acknowledged his confusion, and wrote:

> "What I'm thinking is that if you got infinite points on A and if you got infinite on C, well, you're seeing that they're not equal. So how can you say that infinite points are equal? Like, visually, you're seeing that A is bigger, so therefore the infinite number has to be bigger on A than the infinite number on C. But then again, infinite is the largest you can get, so that's kind of confusing."

Jack observed that the two line segments are not equal in length, and thus concluded that the two could not have an equal amount of infinite points despite his insistence that infinity is "the largest you can get." The conflict in Jack's conceptions might be attributed to an attempt to extrapolate everyday experiences with finite measurements, where length and quantity are often directly proportional. Using familiar experiences to make sense of novel situations is considered by Hazzan (1999) as an attempt to reduce the level of abstraction of the new concept. In the case of infinity, extrapolating experiences with measurement can be deemed as a conception of 'measuring infinity' (Tall, 1980). Jack's conception of 'measuring infinity' was inconsistent with his intuition of a single, never-ending infinity, and his recognition of this created a cognitive conflict that he was unable to resolve.

The notion of 'measuring infinity' surfaced in several participants' responses to Q_2, however most participants neglected the inconsistency between it and their intuition of potential infinity. This is exemplified in Rosemary's response. Rosemary rationalized the expression "$\infty - \infty = \infty$" by arguing that while any line segment will have infinitely many points, a longer segment would have a larger infinite number of points. She also claimed that subtracting an infinite quantity from another (albeit "larger") infinite quantity would leave "a lot of points… extending into infinity" and "it will take forever" to count them. The discrepancy between her process conception of infinity, as exhibited by Rosemary's description of "extending into infinity" and taking "forever", and her measuring conception of a "larger" infinity went unnoticed.

In addition to the contextual influence of considering 'measurable' entities such as line segments, part of the intuitive appeal of 'measuring infinity' may be attributed to participants' understanding of subtraction. An intuition of 'measuring infinity' is not only consistent with the extrapolation of finite measurement to conclude longer segments must have more points, but also it is consistent with participants' experiences subtracting nonnegative numbers. As Nina noted, "an infinite number subtracted by itself will equal 0 because anything subtracted by itself will be zero." The possibility that subtracting infinite quantities is different from subtracting finite ones only occurred to one participant, Levon, who suggested that "although mathematically it should equal zero, points on a line do not follow math reasoning." Levon reasoned that since "every line has infinite points… when you subtract infinite from infinite you will still get infinite."

Although Levon seemed to consider "math reasoning" as restricted to finite entities, his statement suggested a dawning awareness of a distinction between arithmetic properties of numbers and transfinite numbers. Conversely, Nina and the majority of participants did not consider the possibility that an arithmetic operation might have different properties when applied to infinity. Rather, Nina's intuition that "anything subtracted by itself will be zero" was coercive, and may have encouraged expanding her notion of infinity to include the idea of 'measuring infinity'.

Of the various responses to Q_2, Lily's was unique. In her response, she disagreed with the possibility that $\infty - \infty = \infty$. She wrote:

> "I disagree with this statement. For example, π is an infinite (on going) number. If we subtract $\pi - \pi$ the answer is 0, NOT ∞. But, if there is a restriction that says we can't subtract by the same number it could still be an infinite number, but just a smaller value. For example, $\pi - 2\pi = -\pi$, is still an infinite number, only negative."

Lily appeared to conceive of infinity as potential – her use of the qualifier "on going" to describe her notion of an "infinite number" corresponds to a process conception of infinity. However, the on-going process in Lily's conception is applied, not to the magnitude of her "infinite number", but to its infinite decimal representation. Lily's objection to Q_2 seems to stem from confusion between an infinite magnitude, such as the number of points on a line segment, and the infinite number of digits in the decimal representation of π. Her use of π to justify claims about infinite magnitudes is indication of a disconnect between points on a line and real numbers – Lily seemed not to associate π with an individual point on the number line.

Another interesting aspect of Lily's response was her use of "restrictions." She proposed that the difference between two 'infinite numbers' might be another 'infinite number' if there are appropriate restrictions placed on the quantities. By restricting the 'values of infinity' she reasoned that it is possible to attain "an infinite number, it [will] just be a smaller value." Appending "restrictions" allowed Lily to conceive of 'infinite numbers' with different sizes, despite the conflict with her description of infinity as "on going". The notion of infinities with 'different values' is consistent with an intuition of measuring infinity (Tall, 1980), and serves as an example of reducing the level of abstraction. According to Hazzan, this can be seen as another case of using familiar procedures to cope with novel and abstract concepts: Lily applies the familiar procedure of subtracting real numbers to cope with the concept of subtracting transfinite ones.

6.2.3 'Infinite Numbers': On-going Decimals

Lily's confusion between an infinite number of elements and an infinite number of digits in one particular element emphasised the disconnect observed in the early stages of the study between numeric and geometric representations of infinity. Lily's attempt to formulate an argument that was consistent with her experiences and intuitions about number and magnitude prompted a follow up to Q_2. This follow up questionnaire (Q_3) recalled Q_2, presented Lily's argument verbatim, as well as a similar one, and asked participants to elaborate on whether or not they agreed with the arguments.

> **Q_3.** Recall [Q_2 as quoted above].
>
> Student X: [Lily's response as quoted above]
>
> Student Y: I disagree with this statement. You can subtract two infinite numbers and NOT end up with ∞. For example, $\frac{1}{3}$ is an infinite number, but $\frac{1}{3} - \frac{1}{3} = 0$, NOT ∞. Also, $\frac{4}{6}$ and $\frac{1}{6}$ are both infinite (on going) numbers, but if we subtract $\frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2} =$ 0.5, which is not an infinite number. But sometimes it's possible to subtract two

infinite numbers and get an infinite number. For example, $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$, which is infinite and smaller than $\frac{1}{3}$. So, sometimes ∞ - ∞ = ∞, but usually not.

What may appear as a rather provocative line of questioning was a part of the class milieu where many ideas were challenged by offering controversial examples for participants' critique. The rationale for this question was to identify whether Lily's ideas associating magnitude and decimal representation were recognised by participants as inaccurate, or whether they were readily taken up. The intent was also to investigate if participants were connecting real numbers with their representations as points on a line, and to distinguish between potential confusion with the magnitude of irrational numbers and the magnitude rational numbers.

Most participants (22 out of 24) agreed with at least one of the arguments in Q_2, which came as a surprise in light of the common description of infinity as the "largest you can get". Confusion between infinite magnitude and infinite decimal representation revealed two distinct interpretations of 'infinite numbers', and also confirmed a disconnect in participants' conceptions of geometric and numeric representations of infinity. For the participants who agreed with both arguments, confusion between magnitude and representation was broad: they ignored the finite magnitude of both rational and irrational numbers. For instance, Janis wrote:

> $\frac{4}{6}$ and $\frac{1}{6}$ are both infinite (on going) numbers but when subtracting them your result is $\frac{1}{2}$ which is not infinite. This proves that an infinite number subtracting by another infinite number is not always another infinite number. As a result the statement ∞ - ∞ = ∞ is not true because sometimes the result is infinite but a different value and other times the result is not infinite.

In her response, Janis readily accepted the arguments of students X and Y, neglecting the differences between a particular (finite) value and an infinite quantity. Janis used the infinity symbol to represent numbers of different magnitudes, and as such, exemplified participants' notions that infinity has no 'specific value'. The dynamic nature of this conception can be interpreted as an attempt to reduce the level of abstraction of an entity that is beyond the realm of her imagination. Janis's attempt to extrapolate her experiences with finite quantities, and also to use them explicitly (though perhaps unknowingly) to justify her notions of infinity, is further indication of an attempt to reduce the level of abstraction of the expression '$\infty - \infty$'.

Other participants held a slightly different conception of 'infinite number' – they recognised rational numbers as finite quantities and associated them with points on a number line, but did not make the same association with irrational numbers, considering them infinite quantities. This interpretation was exemplified in Rosemary's response to Q_3. When addressing student X, Rosemary remarked:

> $\pi - \pi = 0$ that is correct because one is taking away the same amount of points from what they initially began with will give 0, but in the line segment question, the amount of points in x (which is ∞ amount) is much less than the amount of points in A and C. Which because of this, I agree with Student X's second statement of how there should be restrictions. In this case, points in x are less than points in A or C.

As in Q_2, Rosemary's response is consistent with the idea of 'measuring infinity', using Lily's notion of 'restrictions' to accommodate the possibility that a longer segment will have a greater number of points. Further, Rosemary identified with Lily's argument regarding $\pi - \pi$, alluding to the possibility of a line segment having π-many points. Her remark that $\pi - \pi = 0$ is correct because "one is taking away the same amount of points from what the initially began with" suggests she imagined the magnitude of π as analogous to the quantity of points on the segment and exemplifies participants' general confusion regarding the magnitude of irrational numbers and their geometric representation as points on a number line.

Additional evidence of Rosemary's attempts to reduce the level of abstraction of subtracting transfinite numbers is seen in her response to student Y:

> Student Y states: $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$ (which is an ∞ number) but $\frac{4}{6} - \frac{1}{6} = \frac{3}{6}$ (which is only 0.5 and not an ∞ number). Well, when we represent these numbers on a number line *[drew two line segments, one from 0 to $\frac{1}{6}$ and one from 0 to $\frac{1}{2}$, and labelled the segments A and B, respectively]* then won't both line segments have ∞ points? (But of course segment B will have more than segment A)

Once again, Rosemary appealed to her intuition of 'measuring infinity' as she related student Y's numeric example to its geometric representation. In contrast to her use of π, Rosemary distinguished rational numbers from infinite quantities. Although she stated that $\frac{1}{6}$ was an "infinite number," she observed its specific value on the number line. Similarly, she remarked that though $\frac{1}{2}$ was not infinite itself (it "is only 0.5"), when represented on a number line she acknowledged there were still infinitely many points between 0 and $\frac{1}{2}$. This distinct handling of rational and irrational numbers suggests a misconception about real numbers: whereas rational numbers were associated with points, irrational numbers were not. Worthy of note is Rosemary's use of the words "infinite number": both to represent a number with infinitely many (nonzero) digits in a decimal representation, as well as to represent the infinite quantity of points on a line segment. It would be interesting to see if Rosemary's measuring conception would be so persuasive had she not applied the same terminology to two different notions.

6.2.4 After Instruction: Robbie and Grace

At the end of the course, a class discussion was held on equivalences of infinite sets, as well as on the distinction between an infinite decimal expansion and an infinite quantity. The instructional discussion regarding correspondences included the following well-known geometric construction of a bijection between two line segments AB and CD. The construction begins by connecting the endpoints of AB and CD with line segments that extended past the endpoints of CD to meet at a point labelled p, as depicted in Figure 6.1.

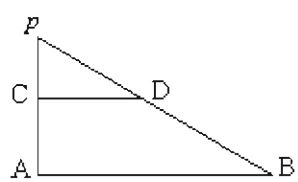


Figure 6.1: Triangle Construction

An arbitrary point, w, can be labelled on AB and connected to the point p by a line segment. The connecting segment will intersect CD at a point r, as depicted in Figure 6.2.

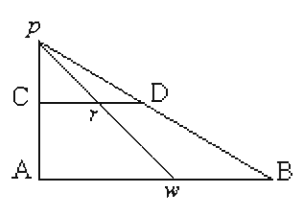


Figure 6.2: Coupling Points

With this construction, it is possible to pair up each point on AB with exactly one point on CD. Conversely, a ray from p to any point on CD can be extended to meet a point on AB in a unique way. In this manner, every point on CD is paired with exactly one point on AB. Thus a one-to-one correspondence is constructed between the set of points on AB and the set of points on CD. Most participants easily followed the construction, though there was significant intuitive resistance to the idea that the longer line segment would not have more points. Lily, for example, suggested that "maybe it would look like it [the ray from *p* to AB] touched every point in that line, but if we zoom in, maybe there would be one point that it didn't touch" since the segment "could have holes" or there "might be some kind of minor error" in the correspondence construction.

Participants who accepted the argument did so by focusing on the process of constructing the correspondence. Robbie, who had consistently described infinity as 'endless', reasoned:

> "We can draw as many lines as we want from *p* to B [AB], and each time each line will only intersect with one point on lines [CD] and [AB]. When that happens, they paired up to make a set coordinate."

Similarly, Grace remarked:

> "There is a one-to-one ratio of points on each of the line segments and since each line segment has an infinite number of points, the one-to-one ratio will stay constant forever."

Both Robbie and Grace were comfortable with the correspondence argument and accepted it as the means for comparing these sets of points. Establishing one-to-one correspondences between infinite sets is seen by Dubinsky et al. (2005a) as 'acting' on those sets and, as such, indicates an object conception of infinity. While Robbie and Grace were able to 'act' on the two sets, they nevertheless referred to ideas related to potential infinity, such as the process of "drawing as many lines as we want", and the suggestion that the "ratio will stay constant forever". Rather than treating the sets as completed objects, Robbie and Grace attributed the infinite process in their conceptions to the bijection – shifting the process from one aspect of the sets to another.

6.2.5 After Instruction: Lily and Jack

After the end of the course, follow up interviews were conducted with two participants: Lily and Jack. Lily and Jack were chosen for interviews because of their enthusiasm toward engaging with, and critiquing, notions of infinity. Both participants were vocal during discussions, and were comfortable sharing their intuitions and explaining their reasoning in the written portion of the study. In addition, Lily and Jack exhibited a desire to clarify their thinking, and were eager to continue exploring properties of infinity through further questioning and conversation.

The interview with Lily readdressed her conception of π as an 'infinite number' after she had been instructed on the distinction between infinite magnitude and infinite representation, as well as on the finite value of π. Since it was the number of decimal digits that gave π its infinite quality, Lily was asked to speculate on the number of decimal digits of a rational scalar of π. She reasoned, "if we times it [π] by 3 it'll just be a bigger number, with more digits." As with the line segments, Lily expressed ideas consistent with 'measuring infinity': she associated "bigger" with "more," believing that 3π would be infinite but a "bigger infinite" with "more digits" than π.

Lily's perception of the "infinite size" of π persisted despite instruction and also in conflict with her ideas regarding 3.14 as an approximation of π. She claimed that 3π was "3 times a number that's really big." To determine the magnitude of 3π, Lily used the familiar number 3.14, yet she was surprised to calculate that triple this number was only about 9: "let's say π is 3.14, then times 3 is going to be big. Well, not big, but (pause) well, kind of triple?" Notwithstanding Lily's attempts to reduce the level of abstraction of π by working with 3.14, it seemed difficult for her to accept π as a small number. When asked about the possibility of measuring a length of π cm, she claimed that one would need "a really big ruler" with huge spaces between each whole number calibration to accommodate all of π's decimal digits. She argued that since π's expansion was infinite and never-ending, then any segment of length π would have to be "really long, until, if possible, there's an end to it." Lily seemed to ignore the actual magnitude of each of π's decimal digits, which, together with her process conception of a never-ending infinite, might have contributed to her notion of π as very large, despite the relatively small magnitude of 3.14.

The struggle to accommodate conflicting ideas, such as Lily faced with her conceptions of π, also surfaced in the interview with Jack. In his written responses, Jack had toiled with his competing conceptions of potential and measuring infinity. Following instruction, Jack continued to express inconsistent notions of infinity as he attempted to reconcile his naïve understanding with a normative one. The interview with Jack began by recalling class instruction on the correspondence between points on line segments of different lengths.

Jack was easily able to recreate the bijective argument presented above. However, he insisted, "that A [AB] is bigger, so therefore the infinite number has to be bigger on A [AB] than the infinite number on C [CD]". Jack's conception of measuring infinity was compelling, and he continued to struggle with the conflict between it and his intuition that infinity "is the largest you can get" and is "never-ending". In an attempt to challenge his intuition of a 'larger' infinite number of points on segment CD, Jack was asked to consider the number of points on two circles of different circumference. He claimed there were an infinite number of points because "drawing a line from the centre to the side [*drew the radius of the circle*], you can draw infinite of them." He noted that the circles would have the same number of points because "you're not caring about the length of the radius, which makes your circle bigger or smaller. You're caring about the 360 degrees," that is, the number of radii, which is the same in both circles. As Jack attended to the number of points on a circle, a strong analogy emerged between his approach in this context and his approach with line segments. In both cases, Jack extrapolated his experience with measurement. Consequently, Jack's 'measuring' conception of infinity seemed to be influenced by the measurable entity itself – line segment or circle – and the manner in which that entity is typically measured. Whereas lengths of segments are determined by measuring from start to end, circumferences of circles are typically calculated in terms of radii. Interestingly, while the approach was the same in both contexts, Jack's conclusions were not: he concluded that the line segments had 'different' infinite quantities of points, while the circles did not. Further, when applied to the number of points on circles of different radii, Jack's conception of 'measuring infinity' did not conflict with his conception of infinity as "the largest you can get", which seemed to reinforce Jack's certainty of an 'endless', unsurpassable infinite.

In an attempt to draw to Jack's attention the inconsistencies stemming from his intuition of 'measuring infinity', we then proceeded to 'cut open' and 'flatten' each circle, as in Figure 6.3 below.

Figure 6.3: Flattening the Circle

Jack judged that even though the shape of the circles was now different, the number of points had not changed[1]. Jack reasoned that the two 'flattened circles' would still have an equinumerous set of points because "you still have that imaginary [centre] point, and all the [radii] connecting to it." This construction is essentially the same as the triangle argument above: the point p corresponds to the 'imaginary centre point', and the rays extending from p, which correspond to the radii, intersect with the longer and shorter line segments an equal amount of times. The visual representation of this construction had a significant effect on Jack's perceptions. Comparing and equating the number of radii of two circles was canonical, even when they were flattened. However, Jack noted "if you go back to this [lines AB and CD], still, if you look at it this way it still doesn't make sense. The circle way kind of does. Well, not kind of, it actually does." Surprisingly, despite the attempt to bring to the forefront the analogy between Jack's 'measuring' approaches in both cases, he did not recognise it. Although Jack was troubled by the discrepancies he observed in his reasoning, he did not find problematic the intuition of 'measuring infinity', which yielded the inconsistencies. Rather, when his 'measuring' intuition was in agreement with his intuition of potential infinity Jack was able to overlook visual cues that otherwise prompted conflict, as in the case with 'flattened'

[1] Topologically, the line segment and circle do differ: an open line segment is isomorphic to $S^1 \setminus \{N\}$, for some point N. However, since the goal was to compare two circles in their 'new form' and not to compare the line segment with the circle, this fact was not addressed at that moment in the conversation.

circles. Eventually, Jack accepted that line segments of different lengths may have the same quantity of points, stating it was "hard to believe, but it makes sense."

6.3 Discussion

Understanding the connection between different representations of a mathematical entity can be problematic for learners. Peled and Hershkovitz (1999), for instance, noted difficulties in learners' appreciation of different representations of irrational numbers. Similarly, Sirotic and Zazkis observed that the "geometric representation of irrational numbers was strangely absent" (2007, p.479) from the conceptualisations of many pre-service teachers who participated in their research. In resonance with these findings, the undergraduate students who participated in this study had difficulty identifying specific numbers, both rational and irrational, as points on a number line. Further, confusion between the infinite magnitude of points on a line segment and the infinite decimal representation of numbers was identified as an obstacle to a conventional understanding of mathematical infinity. Participants' use of finite quantities to explain phenomena of transfinite ones misguided their emerging conceptions of infinity, and illustrated a disconnect in their conceptions of numeric and geometric representations of infinity.

As participants addressed first numeric and then geometric presentations of infinity, a difference in their approaches was observed. Whereas participants drew mainly on an intuition of potential infinity to describe the process of constructing infinitely many rational numbers, they were more likely to conjure an image of measuring infinity when addressing the number of points on a line segment. As participants grappled with different properties and presentations of infinity, a struggle emerged between competing and inconsistent notions of endlessness and a large, unknown number. Participants' responses support the argument that infinity is conceived of intuitively as an inexhaustible process, an endless potential. However, the conception of measuring infinity, which also emerged spontaneously during participants' comparison of infinite sets of points, was a persuasive factor in their reasoning, and at times overshadowed the association of infinity with endlessness. The conflict between coexisting intuitions of potential and measuring infinities, along with the seeming incompatibility of intuitive and normative approaches to infinity emerged as a common theme during participants' engagement with geometric tasks, as well as with paradoxes (Chapters 7 and 8), and will be addressed in detail in Chapter 9: *Cognitive Leaps Toward Understanding Infinity*.

This study shed new light on participants' emerging conceptions of infinity as manifested in their engagement with geometric tasks. In particular, it contributed new insight on participants' conceptions of infinity by explicitly addressing the questions of participants' strategies when confronted with a bound infinite set and with properties of transfinite subtraction – two important questions which invited continued investigations in Chapters 7 and 8. As participants were challenged to accommodate the possibility that an infinite, 'endless' quantity of points could be bound within a finite length – or that an infinite number of decimal digits could be bound within a finite number – two approaches were observed: either participants denied the possibility of a bound infinite, or else they avoided dealing with the bound altogether. Participants who avoided the bound demonstrated a resistance toward the idea of the uniform and 'infinitely small' size of points. Instead, they introduced the idea of an 'infinite zoom', and thereby were able to focus on an infinite process without explicitly acknowledging the bound. Attributing different sizes to points enabled participants to imagine an infinite which continued indefinitely, though not as an extension of the idea of 'very big', but as an extension of 'very small'. Interestingly, although some participants attributed magnitudes of increasingly smaller size to points with the idea of 'infinite zoom', they did not make the same association for the place value of decimal digits of irrational numbers.

An important contribution of this study relates to uncovering participants' naïve and emergent conceptions of transfinite subtraction. The indeterminacy of transfinite subtraction was problematic for most participants. In particular, resistance toward the possibility that '$\infty - \infty = \infty$' surfaced despite participants' previous assertions that the number of points on a segment was infinite "regardless of length". As a means to cope with the indeterminacy of transfinite subtraction, the intuition of 'measuring infinity' emerged. Participants were more readily able to introduce the idea of infinities of 'different sizes' than they were to conceive of an arithmetic operation whose properties conflicted with their prior understanding of subtraction. This study offers a first glimpse at learners' attempts to make sense of arithmetic properties of 'infinite numbers'. It also opens the door for further investigation regarding the specific conceptual challenges of transfinite subtraction, which are considered in Chapter 9: *Cognitive Leaps toward Understanding Infinity*.

CHAPTER 7:

PARADOXES AS A WINDOW TO INFINITY

This study examines approaches to infinity of two groups of university students with different mathematical background: undergraduate students in liberal arts programs and graduate students in a mathematics education master's program. Data are drawn from participants' engagement with two paradoxes – Hilbert's Grand Hotel and the Ping-Pong Ball Conundrum – the results of which have been presented in part in Mamolo and Zazkis (2008).

Paradoxes of infinity have provoked controversy and discussion since Zeno of Elea began speculating on the possibility of infinite subdivisions of space. The discussion and debate provoked by mathematical paradoxes, such as those of Zeno, can be used by educators as an important instructional tool to help bridge the gap between mathematics and education, and to offer an opportunity for participants to develop their mathematical thinking (Movshovitz-Hadar & Hadass, 1990). A helpful instructional tool can be also used as a research tool (Zazkis & Leikin, 1987). As such, this study uses paradoxes as a research tool to investigate participants' intuitive and emerging understanding of infinity. Participants' responses to paradoxes regarding infinity reveal potential cognitive conflicts – situations in which inconsistencies in reasoning are noticed by the instructor but not yet by the individual (Zazkis & Chernoff, 2008).

This study attends to learners' conceptions before and after instruction, as well as their methods for addressing the (potential) cognitive conflict invoked by Hilbert's Grand Hotel and the Ping-Pong Ball Conundrum. The following three questions are addressed: (1) What can be learned about participants' conceptions of infinity by considering their responses to the paradoxes? (2) In what ways do responses differ with mathematical background? (3) What specific features of the paradoxes are challenging for participants?

7.1 Setting and Methodology

7.1.1 Paradoxes

The two paradoxes considered in this study are presented below. They were selected because of their varying level of difficulty, and for the different qualities of infinity involved in each: while Hilbert's Grand Hotel engages participants in conceiving of the infinitely large, the Ping-Pong Ball Conundrum requires coordinating three infinite sets, one of which encompasses the infinitesimally small. The normative resolutions to both Hilbert's Grand Hotel and the Ping-Pong Ball Conundrum are found in Chapter 3.

Hilbert's Paradox: The Grand Hotel

> *The Grand Hotel has infinitely many rooms and no vacancy. If only one person is allowed per room, how can the hotel accommodate a new guest?*

The Ping-Pong Ball Conundrum

> *An infinite set of numbered ping-pong balls and a very large barrel are instruments in the following experiment, which lasts 60 seconds. In 30 seconds, the task is to place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time (and working more and more quickly), balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. After the experiment is over, at the end of the 60 seconds, how many ping-pong balls remain in the barrel?*

7.1.2 Participants

Thirty-six university students participated in this study. Group 1 (G1) consisted of 16 practicing high school mathematics teachers enrolled in a master's program in mathematics education. These participants held Bachelor's degrees in mathematics or science, but had no prior formal exposure to Cantorian set theory. The graduate students were enrolled in 'Foundations of Mathematics', a course for practicing teachers that was taught by an instructor in the Faculty of Education. The course explored some of the foundations of mathematics and mathematical thought, and focused on 'big ideas' and 'great theorems'. Cantor's theory of transfinite numbers was one of the 'big ideas' presented in the course, following the 'great theorem' establishing that the rational numbers have the same cardinality as the natural numbers.

Group 2 (G2) consisted of 20 undergraduate students in liberal arts and social sciences who had no mathematical background beyond high school. The undergraduate students were enrolled in a course that I taught called 'Foundations of Academic Numeracy', which was designed to develop quantitative reasoning and critical analysis. The topic of infinity was included in order to introduce participants to some of the fundamental ideas in mathematics. In both groups, the paradoxes were used to elicit participants' ideas and to provoke discussion about some of the surprising qualities of mathematical infinity. Though, understandably, the subsequent mathematical discussion and the level of formalism in the presented material varied significantly, a similar approach of engaging participants with the paradoxes was used in both groups.

7.1.3 Data Collection

Data are drawn from two main sources: 1) individual written responses before and after instruction, and 2) arguments presented during class discussions, about which field notes were taken that were summarized immediately after class observations. Regular meetings with the instructor of the graduate course were held in order to develop tasks, and for debriefing purposes after those tasks were carried out. In order to access the data, the study began by presenting participants with Hilbert's Grand Hotel and asking them to record their ideas individually. Group and class discussions followed, during which participants' resolutions, and the normative mathematical solution, were presented. Following the instructional discussion, participants readdressed the paradox and were asked to explain why they agreed or disagreed with the normative resolution.

A similar method of data collection was used when participants were presented the Ping-Pong Ball Conundrum. After participants recorded their initial responses, group and class discussion included formal instruction on cardinality and infinite sets. The instructional tasks included comparing infinite countable sets using one-to-one correspondence, or 'coupling', and the conventional mathematical resolution presented in Chapter 3 was explained. Participants then were asked to readdress in writing the original question – At the end of the experiment, how many ping-pong balls are left in the barrel? The analysis focused on identifying common threads in participants' individual written responses as well as their arguments presented during the discussion. In what follows the themes that emerged are presented and exemplified with excerpts from participants' work. The excerpts chosen were the most illustrative ones that exemplified the common themes emergent in both groups; they came mostly from the liberal arts students who were a very expressive group.

7.2 Results and Analysis

7.2.1 Hilbert's Grand Hotel

Despite the varied levels of mathematical background and skill amongst the participants, initial reactions to Hilbert's Grand Hotel paradox were fairly consistent throughout. Both groups of participants provided naïve responses that were strongly influenced by practical experiences. An underlying theme involved the conceptual difficulties associated with the hotel's lack of vacancy.

Participants' responses before instruction: holding on to reality

The leap of imagination necessary for conceiving of an infinite hotel and for resolving the paradox was difficult for a significant number of participants to make. Nearly half of the participants in both G1 and G2 initially provided responses that reflected practical experience, but which avoided resolving the mathematics. Such responses included recommending the new guest sleep in the lobby, having the manager vacate his own room, or putting 2 or more guests in the same room, despite the fact that this contradicted the 'givens' of the problem, that is, accommodating the new guest in a personal room and allowing only one occupant per room.

Participants' realistic experiences also influenced their thinking as they considered the premise of the paradox, objecting, for example, to the feasibility of infinitely many guests since "the world only has a couple billion people *[sic]*." One G2 student reasoned, "In order for every room to be full there would have to be infinite guests, which is impossible."

Other participants looked for loopholes, such as the possibility that the rooms are occupied, but not by guests. For instance, Jimmy (G2) argued:

> "I don't understand how infinitely many rooms could be full. The manager says they are full, but full of what? Maybe they are filled with boxes or furniture and the manager could clear one of them out for [the guest]... It just doesn't make sense that if there are infinitely many rooms that they all could be full. It defies logic!"

Interestingly, although participants criticized the idea of infinitely many guests, they did not object to an infinite number of rooms in the hotel. This may be attributed in part to the possibility that participants could conceptualise a hotel that extends into space without attending to the actual amount of space such a hotel would encompass. However, the idea of a *full hotel* was problematic for many participants due to the conceptual challenges associated with *filling the hotel.*

Participants' responses before instruction: filling the hotel

A common difficulty that arose for both groups of participants – those with a formal mathematical background and those without – was the idea of completely filling an infinite hotel. Several participants accused the manager of false advertising (some even threatened to sue). They insisted if "there are infinitely many rooms, you can never really be completely full, $\infty + 1$ is still ∞" or "if all the rooms are full there's a set number of rooms." Typical responses from both groups also included remarks such as, "Infinity is an always increasing number, so there should be a room available." These remarks suggest infinity is conceived of as a dynamic entity, an "always increasing number" that is never attained, supporting the similar proposition of Fischbein (2001). Furthermore, participants attributed the quality of completion to a finite entity or "set number", and some could not address the question of accommodating a new guest, as they were unable to overcome the perceived impossibility of filling the hotel. Resistance to a completed infinite entity highlights participants' difficulty in accepting the idea of actual infinity embraced in a 'full' or 'completely filled' hotel. Further, the attention to filling the hotel demonstrates participants' process conceptions corresponding to potential infinity.

An argument given by the liberal arts students relating to the 'completed', full hotel suggested they had difficulty separating philosophical beliefs with mathematics. Some participants reasoned that if a hotel could have infinitely many rooms, then the new guest would already have a room because they "must be part of the infinite." These participants seemed to associate infinity with an all-encompassing entity – a conception that has yet to be expressed explicitly in participants' reasoning regarding the infinite of numbers or points, yet which may have a tacit influence on their resistance toward encapsulating actual infinity as an object. The all-encompassing infinite was a persuasive line of reasoning, and was, for some, the preferred argument even after instruction.

Participants' responses after instruction: an endless shift

Overall, the mathematics education students readily accepted the normative resolution, which involves shifting the set of guests by one room to free up a space. They were also able to extend the argument to variations of the paradox that included accommodating arbitrary finite numbers of guests into the hotel, such as accommodating The Beatles or the Vancouver Canucks. Further, the graduate students could extend their reasoning to the 'Infinite Towers' variation, which involves accommodating a countably infinite number of guests after a fire causes an evacuation of one of the two towers. In contrast, the liberal arts students, even those who accepted the normative resolution, were more resistant to it. Their responses after instruction tended to be based on their struggles to make sense of the mathematics, as well as on the 'real life' practicality of the situation.

A process conception of infinity is recognized in typical responses of both G1 and G2 participants that accepted the normative solution in their references to "shifting over" rooms, which would "never stop since there's an infinite amount of people." These participants described an endless "chain reaction" that was set in motion when the guest in the first room displaced his neighbour – a notion that is further consistent with a process conception of infinity. Such responses can also be interpreted as attempts to reduce the level of abstraction of operating on infinitely many objects. Rather than applying the transformation of shifting rooms to the entire set of guests, participants applied the transformation guest by guest. This "chain reaction" coincides with Hazzan's (1999) observation that students will attempt to reduce the level of abstraction of set transformations by operating on a single element in a set rather than the entire set.

After participants were exposed to the normative solution, a struggle emerged between conflicting notions of infinity as inexhaustible and also as a large, unknown, number. Some G2 participants questioned what would happen to the 'last' guest, while at the same time acknowledging there could not be a 'last' guest since infinity was never ending. Eric, a liberal arts student, expressed difficulty with the paradox, and initially reasoned that the "rooms would go on forever" and "you could keep on adding people forever to fill them." After reflecting on the normative resolution, Eric remarked:

> "This works because although the infinite rooms are infinitely full, it makes space for you by making one of those rooms free. I was first troubled by the idea of one

'last' person not having a room, but then I realized that the last person would ask me to shift rooms, and so on, so there would be a constant rotation."

Eric's descriptions of "rooms that go on forever," "adding people forever to fill them," and the "constant rotation" of switching rooms correspond to a process conception of infinity. At first, he imagined a hotel that extends indefinitely and to which new guests can always be added to the next empty room in sequence. Analogous to conceiving of the natural numbers as infinite because it is always possible to add one more to the last number, a fundamental aspect of Eric's initial image of an infinite hotel seems to be linked to the possibility (and process) of always adding one more guest. As Eric tried to incorporate the normative solution, a cognitive conflict emerged between the idea of a completed "infinitely full" hotel and the process-conception of "adding people forever." In an attempt to resolve the conflict, Eric introduced the idea of an infinite "rotation" – the infinite process in Eric's conception shifted from the process of adding guests to the process of moving them. Attributing a cyclical structure to the hotel may be Eric's attempt to reduce the level of abstraction of a completed, yet endless, entity.

While the majority of both groups of participants acknowledged the normative solution on some level, G2 students continued to question and criticize the impracticality and inconvenience of moving so many people. For instance, Stan (G2) wrote:

"Although I understand and agree to an extent the idea of switching rooms to make room #1 available, I don't think it is logical because I know that I wouldn't want to move rooms (call me "high maintenance")... In all reality, I would just like to move on to another hotel, where I can settle in for my length of stay and not be bothered by moving at any given point in time."

Similarly, Clyde (G2) explained:

"Well, mathematically, that answer works but realistically, the suggestion is unfeasible. However, I guess this isn't a realistic scenario anyway so that answer does satisfy the question. I just get a funny mental image of the guest getting sound sleep while everyone else has to continue to shift rooms infinitely."

The reluctance to let go of 'realistic' responses illustrates the resilience and coerciveness of intuitions described by Fischbein et al. (1979). Clyde's "funny mental image" of a continuous room shift is similar to Eric's "constant rotation" in that the infinite process is attributed to the transformation of moving guests, despite the fact that each guest's transformation is actually finite – each guest moves only once, but there is an infinite amount of guests who move. Clyde's recognition that the paradox "isn't a realistic scenario" and that the "answer works" suggest he has realized a gap between his intuitive understanding of infinity and a formal one, and he is able to clarify this distinction.

Following the relative ease with which GI participants accepted the normative resolution of Hilbert's Grand Hotel paradox, a more challenging task was sought. The Ping-Pong Ball Conundrum provided such an engagement and was presented to both groups in the fashion described above.

7.2.2 Ping-Pong Ball Conundrum

Striking similarities were found in the responses of both groups of participants regarding the Ping-Pong Ball Conundrum that persisted throughout their engagement. Participants' initial solutions to the possible number of balls remaining in the barrel at the end of the 60 seconds can be clustered around two main themes, focusing on the rates of change and the possibility of ending the experiment, respectively:

"There are infinitely many balls left in the barrel;" and

"The process is impossible since the time interval is halved infinitely many times, so the 60 seconds never ends."

Participants' responses before instruction: rates of infinity

The argument that infinitely many balls remain in the barrel was most frequently justified by appealing to the different rates of in-going and out-going balls: at each time interval 10 balls go into the barrel, but only one is removed. Nine out of 20 liberal arts and 13 out of 16 mathematics education students reasoned that the number of balls remaining in the barrel must be a multiple of nine or "9∞." The typical response being:

"There is 9× more balls in the barrel than out of the barrel at all times. At the end of the 60 seconds there are 9∞ balls in and ∞ balls out."

Sheila (G1) elaborated on the effect of rates:

"Every time the remaining time is halved, the equivalent change (+10 – 1) = 9 balls are added. So there will be an infinite of balls in the basket. Some may say that an infinite amount of balls have been taken out of the basket, which is true, but it is not an equivalent infinity to what is put in... There will be 9 times as many in the basket as you took out."

The notion of different rates of infinities seems to extrapolate common (finite) experiences with rates of change. As many participants correctly observed, at every n-th time interval, $9n$ balls remain in the barrel. This is consistent with the observation that participants' conceptions of infinity tend to arise by reflecting on their knowledge of finite concepts and extending these familiar properties to the infinite case (Dubinsky et al., 2005; Dreyfus & Tsamir, 2004; Fischbein, 2001). Experience with finite rates of change and realistic possibilities also surfaced in comments such as Jimmy's (G2), who observed that the experiment was "definitely outside the realm of possibility!" Similarly, Timmy (G1) recognised an inconsistency with his finite experiences and his conclusion regarding the remaining number of ping-pong balls. He reasoned:

> "There are infinitely many time periods, therefore infinitely many times during which 10 balls are put in and one thrown out. So – there are an infinite number of balls in the basket as well as an infinite number thrown out. It doesn't make sense to have this result, but there it is."

Attending to the different rates of in-going and out-going ping-pong balls is an approach which instantiates the use of familiar procedures to cope with novel and abstract concepts. According to Hazzan (1999), such an approach occurs as a method of reducing the level of abstraction. The rate argument might also be a consequence of a process-oriented approach to resolving the Ping-Pong Ball Conundrum, as the argument focuses on iterating individual steps indefinitely. In fact, as mentioned, the argument that the total number of in-going balls is nine times larger than the number of out-going balls holds at every point in time; it fails only at the completion of the process at infinity – a concept that was, in itself, problematic for participants.

Participants' responses before instruction: an endless 60 seconds

Another conception of infinity surfaced as participants addressed the possibility of a 'completed 60 seconds,' arguing that the experiment could never end. As Quine (1966) noted, during a person's attempts to resolve certain paradoxes regarding infinity, a "fallacy emerges [which is] the mistaken notion that an infinite succession of intervals of time has to add up to all eternity" (1966, p.5). This 'fallacy' highlights the distinction between potential and actual infinity. In terms of the ping-pong balls, conceiving of an inexhaustible experiment corresponds to potential infinity – a process, which at every instant is finite but which goes on forever. Whereas, actual infinity describes a complete and existing entity of time intervals within 60 seconds, and which encompasses what was potential. The 'fallacy,' to use Quine's term, lies not in the conception of an endless infinite, but rather in conceiving of potential infinity when the entity is actually infinite.

The process conception of infinity expressed by the idea of an inexhaustible 60 seconds surfaced in the initial responses of three out of 16 graduate students and 15 out of 20 undergraduate students. Participants reasoned that since the intervals of time could be continually divided to smaller and smaller amounts without reaching zero, the experiment would never end. This argument is exemplified in Kenny's (G2) statement:

> "Even with 1 second left we can still divide this amount of time into infinitely small amounts of time (if physics does not apply). Therefore, the experiment will continue into eternity and the number of [ping-pong] balls will be infinite in the barrel."

There are at least two points of interest in Kenny's remark. The first is related to limits and series. Series and the limits of their corresponding sequences are fundamentally interconnected: limits are used in order to determine convergence, and convergence can be used in order to determine limits. A series $a_0 + a_1 + \cdots + a_n + \cdots$ is defined as convergent if the sequence of its partial sums $\{s_n\}$, where $s_n = a_0 + a_1 + \cdots + a_n$, is convergent and the limit as n tends to infinity of $\{s_n\}$ exists as a real number. Otherwise, the series diverges. In Kenny's argument a confusion is identified between the convergent series of "infinitely small amounts of time" that sum to 60 seconds and a divergent series that "will continue into eternity." This confusion might stem from an informal understanding of limits as unreachable – a common conception of college students (Williams, 1991), and one that is linked to a process conception of infinity (Cottrill et al., 1996).

The second interesting aspect of Kenny's argument lies in his conclusion that the barrel should be infinitely full. If the experiment were to go on endlessly, then at no moment will the barrel contain infinitely many balls; instead it will always (endlessly) contain a finite quantity of balls – $9n$ balls. Kenny seems to hold an inconsistent conception of infinity: on one hand, infinity is viewed as endless, yet on the other hand, it is used to describe a large unknown quantity. These competing notions of infinity, that surfaced also in G1 and G2 participants' responses to Hilbert's Grand Hotel, present a potential cognitive conflict, and support the suggestion that an understanding of infinity depends both on "conjectural and contextual influence" (Fischbein et al., 1979, p.32).

Participants' responses after instruction: rates of infinity

As mentioned, the instruction included the idea of set comparison via one-to-one correspondence. Also, the normative resolution to the Ping-Pong Ball Conundrum was presented as an alternative for consideration. Recalling the resolution presented in Chapter 3, correspondences exist between the sets of in-going balls, out-going balls, time intervals, and the natural numbers. As such, and as a consequence of the order in which the out-going balls were removed, the barrel ends up empty at the completion of the experiment. This resolution was problematic for the majority of participants. Indeed, the number of G2 participants who appealed to the rate argument in their responses increased by four participants after discussion. G1 students also found the argument for different rates coercive. Roughly two thirds of them maintained this conception despite instruction.

Resistance toward the normative resolution was, for some participants, quite strong. The result of an empty barrel conflicted with participants' experiences as well as their expectations, and as a consequence, there were participants in both G1 and G2 who refused to accept the resolution as a logically consistent possibility. For instance, Randy (G2) persisted with the rate argument, stating:

> "I can't agree with 0 balls remaining. You put in more number of balls than you take out. I still think my original answer is correct!"

Similarly, Sheila (G1) insisted:

> "I will not accept a logical argument that the basket is empty. Such an argument would be flawed."

Other participants took a more moderate approach to the normative resolution. Shelly (G2), for example, also continued to argue in favour of the rate argument, though she reflected on the normative resolution: "I'm sure it makes sense if you're comfortable with the concept of infinity."

As part of the instructional conversation, participants were challenged to name a ball remaining in the barrel if indeed the barrel was not empty. This challenge was given in order to shift the focus away from the process of inserting and removing balls, and toward a final result. However, both groups of participants continued to demonstrate an overwhelming intuitive resistance to the possibility of an empty barrel. As Kyle (G2) explained:

> "There is an infinite number of balls in the barrel, however it is impossible to name a specific ball. As soon as a number is chosen, it is possible to determine the exact time… that ball was removed… I can't name a numbered ball that remains but then I also couldn't tell you how many balls we began with because there were infinity. Since you are always adding more than you are taking out, you can move at lightning speed, and you have infinity time intervals, I believe the task never ends."

With regard to the quantity of ping-pong balls, Kyle exemplified the typical conceptions that emerged in participants regardless of their mathematical sophistication. Kyle seemed to treat infinity as a large unknown number that could be scaled, but that would always remain large and unknown, and hence "infinite." Kyle also concluded that experiment "never ends," that is, by imagining the experiment being carried out, "infinite" is perceived as synonymous with "never ending."

Following instruction on cardinality equivalences, a quarter of the liberal arts participants and the majority of the mathematics education students were able to explicitly construct a one-to-one correspondence between in-going and out-going balls. Yet, none of the G2 participants understood the correspondence to mean the barrel would be empty – instead ideas of an infinitely full barrel persisted. For instance, Wendy wrote:

> "There are still infinitely many balls left in the barrel, because even though there is a one to one correspondence between the sets {1, 2, 3, 4, ...}, {9, 18, 27, 36, ...}, the rate at which you are putting in is more than you are taking out. So even if there are just as many numbers in each set, they will never even out, because the process continues infinitely and you continue to put more in than you take out."

The inherent contradiction in Wendy's and similar responses went unnoticed.

Only 4 participants in G1 (out of 36 participants) suggested that the number of balls in the barrel was zero after instruction, but added a comment that pointed to the distinction between what they "learned" and what they "believed". Timmy (G1), for example, conceded:

> "I can now entertain the idea that there are no balls in the basket, but I don't like it."

Likewise, Leopold (G1) commented,

"There are conflicting views and now I am not sure whether there is none or infinite balls in the basket. My gut feeling seems to want to say that there are an infinite number but there seems to be none as well."

7.3 Discussion

Paradoxes have played an important role in the history of mathematics and mathematical thought. The cognitive conflict elicited by a paradox can be difficult for a learner to resolve, particularly, as observed, when the resolution depends on notions that defy intuition, experience, and reality. Nevertheless, the impulse to resolve a paradox can be powerful motivation for a learner to refine his or her understanding of the concepts involved (Movshovitz-Hadar & Hadass, 1990).

As participants responded to Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum, cognitive conflicts emerged between competing naïve conceptions of infinity as endless or as a large number, and also between intuitions and normative solutions. Interestingly, while participants could conceive of infinitely many ping-pong balls within a barrel, they expressed difficulty with the idea of filling the hotel with infinitely many guests. This observation illustrates how the novel lens of paradoxes can help identify specific difficulties inherent in conceiving of actual infinity. This study invites a more refined account of participants accommodating the idea of actual infinity, which is the focus of the follow up study presented in Chapter 8: *Accommodating the Idea of Actual Infinity*.

This study offers new insight on the question of how learners' responses to Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum differ with mathematical background. Data revealed that despite different levels of mathematical sophistication, both groups of participants attended to, and were challenged by, similar features of the paradoxes. Responses of participants in both groups to the Ping-Pong Ball conundrum were surprisingly similar before, during, and after instruction, while the mathematics education (G1) students, unlike the liberal arts (G2) students, found the resolution of Hilbert's Grand Hotel paradox unproblematic. Further, both groups of participants expressed notions corresponding to a process conception of infinity; however G2 participants were more likely to find problematic the idea of a bounded infinite set, such as infinitely many time intervals within a minute. This difficulty exemplifies the resistance towards the idea of actual infinity and may also be attributed to specific conceptual challenges regarding the 'infinitely small', in comparison to infinity as an extension of the idea of 'very big'.

Three distinct trends were observed in the data:

(i) Participants dismissed the normative solution and found refuge in non-mathematical considerations. Attending to the practical impossibility of the presented problems served as a cognitive conflict resolution, or, more likely, cognitive conflict avoidance.

(ii) Participants attempted to reconcile the normative solution with their naïve conceptions. For these participants the cognitive conflict was apparent and presented a considerable frustration.

(iii) Participants distinguished between their intuitive tendency and formal knowledge. The cognitive conflict resolution for these participants consisted of separation rather than reconciliation.

Fischbein et al. (1981) suggested that the intuition of infinity might be so deeply rooted that it may be extremely difficult to produce a lasting effect on it through instruction. However, through participants' engagement with paradoxes, some changes in their intuitive approach to resolving infinity-related problems were observed, and their challenges have been articulated. Furthermore, participants who acknowledged the gap between their intuitive and formal understandings of infinity may have taken an important first step toward encapsulating infinity as an object. The ability to separate intuitive from formal understanding will be discussed in further detail in Chapter 9: *Cognitive Leaps Toward Understanding Infinity*.

Several researchers have asserted that paradoxes offer a fruitful lens for investigating conceptions of infinity (e.g. Dubinsky et al., 2005a); however reported research using paradoxes is limited (e.g. Mamolo & Zazkis, 2008). By considering the question of participants' responses to paradoxes, this study has confirmed the findings of prior research related to process conceptions of infinity and inconsistency in participants' reasoning. In addition, this study has shed new light on conceptions that might influence an understanding of actual infinity in broader contexts as well. For instance, the philosophical belief connecting infinity with an all-encompassing entity, and the challenges connected to the idea of a bounded infinite, are conceptions which have not been exposed by more conventional set comparison tasks, but which may nevertheless influence learners' responses in these contexts. Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum served as beneficial research tools for eliciting participants' ideas, provoking cognitive conflict, and identifying perceptions and intuitions that might present obstacles in adopting a 'conventional' understanding of actual infinity.

CHAPTER 8:

ACCOMMODATING THE IDEA OF ACTUAL INFINITY

This third and final study developed from the premise that an understanding of actual infinity goes beyond the basic requirements of encapsulation. In Dubinsky et al.'s (2005a) perspective, encapsulation is recognised by a learner's treatment of infinite sets as completed entities upon which bijections can be applied to determine cardinalities. Without doubt, these aspects are necessary, however they alone do not appear to be sufficient in accommodating the idea of actual infinity – a concept for which our intuitions and experience offer no consistent guidance. As illustrated in Chapters 6 and 7, participants were able to act on infinite sets and establish correspondences while still describing notions that were consistent with process conceptions of infinity. For instance, Robbie (Chapter 6) corresponded the sets of points on two line segments by expressing a way to 'couple' points and imagining the coupling process to continue indefinitely. Similarly, Clyde (Chapter 7) understood and accepted the normative resolution to

Hilbert's Grand Hotel paradox, yet described an 'endless chain reaction' to explain the transformation of shifting guests, which was in fact a finite transformation. These two cases offered an early indication that the manner in which participants operated on infinite sets – *how* they acted on the objects – was significant.

Designed as a follow up to *Paradoxes as a Window to Infinity* (Chapter 7), this study delves into the conceptions of mathematics majors, graduates, and doctoral candidates with the intent to shed light on the mental constructs and cognitive leaps that are necessary and sufficient to establish meaning about actual infinity. The Ping-Pong Ball Conundrum and its variation, recalled from Chapter 3, are used as a lens to identify specific features involved in accommodating the idea of actual infinity. Specifically, this study identifies three main themes relating to cognitive leaps, cardinality, and *how* infinite cardinals are dealt with, and addresses their roles in participants' conceptions of infinity. In particular, this study explores the question: What are the necessary and sufficient features of accommodating actual infinity?

8.1 Setting and Methodology

8.1.1 Participants

Data for this study were collected from eight participants with advanced mathematical backgrounds. Each of the participants had prior experience with Cantor's theory of transfinite numbers through formal instruction during upper level undergraduate mathematics courses. In particular, they were familiar with comparing infinite sets via one-to-one correspondence, such as corresponding the sets of natural numbers and rational numbers, and also with Cantor's diagonal argument establishing the set of real numbers as having larger cardinality than the set of natural numbers. These ideas were outlined previously in Chapter 2. Participants also had substantial experience with infinity in calculus. The participants that were questioned in this study included students enrolled in undergraduate degrees in mathematics, as well as participants who had completed at least a master's degree in mathematics or mathematics education.

- Jan was a mathematics major in a south eastern state university in the USA. She was in her final year of the program and was very interested in the concept of infinity both from a mathematical and philosophical point of view. In addition to her background with cardinal infinity, she had informally explored aspects of Robinson's 'nonstandard infinity' (Chapter 2). Jan anticipated pursuing a graduate degree in mathematics.
- Maria was a classmate of Jan's in the mathematics program. Her familiarity with Cantor's theory included an awareness of the Continuum Hypothesis (Chapter 2), as well as some properties of transfinite ordinal numbers.
- Joey was in his fourth year of an undergraduate degree in mathematics and physics at a university in eastern Canada. Joey had taken upper year courses in set theory and analysis, both of which touched on Cantor's theory. Since his participation in this study, Joey had completed his degree and was searching for employment in industry.
- Marc was a doctoral student in mathematics at large south eastern university in the USA. His research interests included aspects of algebraic topology. He was also interested in mathematical logic, and was familiar with the idea of a 'super-task' (Chapter 3).

- Vince was a doctoral candidate in mathematics at a university in eastern Canada. Since his participation in this study, he had completed his degree, and began a professional research career in cryptography.
- Jenny was a doctoral candidate in mathematics education at a university in eastern Canada. Her area of research was in didactiques des mathematics, and her scholarly background included an undergraduate degree in mathematics and physics.
- Dion was an instructor at a university in eastern Canada. He held a master's degree in mathematics education and a bachelor's degree in mathematics. Dion taught prospective secondary school teachers in mathematics and didactiques, the curriculum for which included aspects of Cantor's theory, such as establishing a bijection between the sets of natural and even numbers.
- Veronica was a mathematics instructor at a private secondary school in eastern Canada. She had recently completed a master's degree in applied mathematics.

8.1.2 Data Collection

For the data collection, interviews were conducted with three of the participants, while data from the other five participants were collected through email correspondence. The two methods of data collection provided different information regarding participants' conceptions. Whereas the interviews offered responses that were more spontaneous, email correspondence offered participants the opportunity to put their thoughts in writing, which contributed to more precise and balanced responses. Emails were exchanged with the two doctoral candidates in mathematics – Marc and Vince – and also with the three undergraduate mathematics students – Jan, Maria, and Joey. Interviews were conducted with Dion and Veronica – the two instructors – as well as with Jenny, the doctoral candidate in mathematics education.

Both methods of data collection began by presenting participants with the Ping-Pong Ball Conundrum, which is recalled below. As in the previous study (Chapter 7), participants were asked to determine how many ping-pong balls remained in the barrel at the end of the 60-second experiment, and to explain their reasoning. The Ping-Pong Ball Conundrum was chosen because of its level of complexity, its amenability to variations, and also because of the necessity to address a bound infinite set in the paradox resolution. In order to probe what aspects beyond set comparison might influence an understanding of actual infinity, the Ping-Pong Ball Variation (also recalled below) was presented to three of the participants. Jan, Dion, and Veronica were eager to engage in further problems regarding infinity, and as such were presented with the Ping-Pong Ball Variation after discussing the normative resolution to the Ping-Pong Ball Conundrum.

The Ping-Pong Ball Conundrum

> *An infinite set of numbered ping-pong balls and a very large barrel are instruments in the following experiment, which lasts 60 seconds. In 30 seconds, the task is to place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time (and working more and more quickly), balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. After the experiment is over, at the end of the 60 seconds, how many ping-pong balls remain in the barrel?*

The Ping-Pong Ball Variation

> *An infinite set of numbered ping-pong balls and a very large barrel are instruments in the following experiment, which lasts 60 seconds. In 30 seconds, the task is to place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, the next 10 balls are placed*

in the barrel and ball number 11 is removed. Again, in half the remaining time, balls numbered 21 to 30 are placed in the barrel, and ball number 21 is removed, and so on. At the end of the 60 seconds, how many ping-pong balls remain in the barrel?

Both the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation invite a thought experiment in which infinitely many balls are place into, and removed from, a barrel. The difference between the two thought experiments is a subtle matter of which balls get removed – balls numbered 1, 2, 3, ... in the first experiment, and balls numbered 1, 11, 21, ... in the second experiment. The consequence of this distinction is that although both experiments essentially involve the same task – inserting and removing infinitely many ping-pong balls – the results are quite different. Whereas the Ping-Pong Ball Conundrum ends with an empty barrel, the Ping-Pong Ball Variation ends with infinitely many balls in the barrel. As mentioned in Chapter 3 (where a more detailed account of the paradoxes' normative resolutions can be found), the two thought experiments illustrate the indeterminacy of transfinite subtraction. It will be argued that awareness of this property is one of the important elements to accommodating the idea of actual infinity.

8.2 Results and Analysis

Surprisingly, despite the sophisticated mathematical knowledge of participants, only three – Jan, Marc, and Dion – provided a resolution to the Ping-Pong Ball Conundrum that was consistent with the normative one. Indeed, as participants attempted to reconcile properties of actual infinity with the notions that were elicited by the Ping-Pong Ball Conundrum, and also by the Ping-Pong Ball Variation, several common themes arose. These themes can be gathered around three main characteristics, which relate to (1) the leap of imagination required to conceive of actual infinity, (2) the identification of infinite cardinality with 'how many', and (3) the manner in which transfinite cardinals are acted upon. Each of the following subsections identifies and examines one of these integral steps to accommodating the idea of actual infinity.

8.2.1 Leaps of Imagination

Conceiving of actual infinity exemplifies mathematical thinking that "extrapolates beyond the practical experience of the individual" (Tall, 1980, p.1). As such, problems addressing infinity require a leap of imagination away from practical experience. In resonance with observations made in Chapter 7, participants resisted extrapolating beyond their practical or realistic experiences when addressing the ping-pong experiments. Letting go of realistic considerations was problematic for participants despite their considerable experience with advanced and abstract mathematics – mathematics that is inaccessible to the five senses (Edwards et al., 2005) and that lacks an intuitive basis (Tall, 1992). The inability to take the cognitive leap into the realm of mathematical infinity manifested in participants' responses in striking similarity to the reactions of participants in Chapter 7, despite significant differences in mathematical sophistication. There were those who were unable to 'leap', others who recognised the need to 'leap' but resisted, and some who could 'leap' to work within the realm of mathematics and clarify a separation between 'real' possibilities and mathematical truth.

The inability to leap toward the imaginative surfaced in participants' reluctance to distance themselves from practical or realistic concerns such as physical possibilities and constraints. For instance, Vince, a doctoral student in mathematics, objected to the feasibility of the experiment, and refuted the possibility of completing the experiment. He remarked that the "first thing that comes to mind is that the problem is not really that well-defined as the time left, 1/2n, never reaches zero". Vince went on to consider the processes of inserting and removing balls, and concluded that:

> "You'll have lots of balls in the barrel when you reach 0 [end of the experiment], which you won't. And this is clearly a ridiculous answer if you consider the whole thing to be something that could take place. So my final answer is 136."

Vince's desire to consider the experiment as "something that could take place" suggests a reluctance to engage with the thought experiment in the Ping-Pong Ball Conundrum, which is by no means an experiment that could actually take place. Vince's resistance to let go of practical experience is also recognised in his comment that "lots of balls in the barrel" is "clearly a ridiculous answer", which suggest he was unable or unwilling to conceive of a barrel that could contain infinitely many balls. Also, Vince's notion of an endless experiment corresponds to a conception of potential infinity and is suggestive of a process conception, in terms of the APOS Theory. A conception of potential infinity, together with resilient practical concerns, seemed to prevent Vince from considering the infinite sets of balls or time intervals as completed objects.

Imagining the experiment being carried out also influenced Joey, an undergraduate student in mathematics. Joey's response began by describing the physical items in the paradox – such as the barrel and the balls – and speculating on the outcome if he were to actually perform the experiment. Joey wrote:

> "Well, at first I'm thinking about a massive collection of white ping-pong balls. And an actual wooden barrel. Clearly thinking about actually performing the experiment and then realising there is no way I can actually move that fast in real life so I realise the final ping-pong ball count would be finite. However thought experiment... so.. mathematically..."

Joey's instinct was to consider the experiment in a 'realistic' way, and like Vince, he initially approached the paradox as though it were an experiment that he could perform. Once Joey realised that the physical constraints of reality restricted his solution to a finite "count", he distinguished between what was possible practically versus mathematically. This distinction suggests Joey was, to a degree, aware of a conflict between practical experiences and the realm of infinity. Nevertheless, Joey's realistic approach of "thinking about actually performing the experiment" seemed to influence his deliberations even as he addressed the thought experiment "mathematically". For instance, he continued to describe the experiment as though it were being carried out:

> "Since I keep halving the time I add and remove ping-pong balls, I will never reach 60 seconds. So the experiment should never end, really. Meaning I have an infinite number of ping-pong balls, and yet there are more in the barrel. Since infinity is not an actual number, you can't say I have infinity here, but 9 times infinity there."

Joey maintained a personal connection to the experiment, and described his own involvement in terms of the actions he would take and the outcomes he would face. The use of personal language to cope with an abstract concept is, in Hazzan's (1999) perspective, an attempt to reduce the level of abstraction of that concept, and as such is indication of a process conception of infinity. Further, Joey's description of infinity as "more like a destination, an indication of an unlimited amount", and his remark that the "experiment should never end", are consistent with a process conception of infinity.

A use of personal language was also identified in Jenny's response, as she too resisted letting go of 'practical' concerns. Jenny, a doctoral candidate in mathematics education, also commented on physical limitations regarding the experiment. She found the issues of speed and time problematic, suggesting, "there is not enough time to work so fast". She also noted that "the fastest speed is light speed" and that if she could work at the speed of light then time would slow down and there would be "infinity time, so there will never be a last ball". She described herself as being stuck in an endless experiment, left to insert and remove ping-pong balls for eternity, lamenting "but I don't want to do that with my life".

The cognitive leap from the 'realistic' to the realm of mathematics was a source of difficulty for Vince, Joey, and Jenny. Their resistance to engage with the realm of imagination and mathematics noticeably impacted each of their resolutions, and in the case of Vince prevented him from resolving the paradox beyond giving an arbitrary number as his solution. Other students attempted to bridge their realistic concerns with the surreal thought experiment by introducing assumptions. These students reconciled reality and infinity by assuming the impossible was possible. For instance, Maria, an undergraduate student in mathematics, reasoned there would be an infinite number of ping-pong balls remaining in the barrel at the end of the experiment "assuming the barrel has an infinite volume and can house an infinite number of ping-pong balls". Maria imagined an infinite iterative process, noting that "per iteration, the barrel gains an additional 9 ping pong balls than it had previously… so to determine the number of ping pong balls in the barrel at the end of the experiment, we can simply determine the number of iterations and then multiply this by 9." Maria seemed to treat infinity as a very large number, and as such, needed to assume that the physical constructs could accommodate this 'infinite size'. Interestingly, Maria did not assume the existence of infinitely many ping-pong balls, only that they could be housed in the barrel. Recalling the normative solution, Maria's assumption is superfluous as the barrel at no moment contains infinitely many ping-pong balls.

Marc, a doctoral candidate in mathematics, was also compelled to introduce assumptions before addressing the paradox. As will be discussed in the following section, Marc reasoned through two different approaches to the paradox before settling on the normative resolution, which he observed was the only way to produce a consistent result. Marc preceded his responses by "assuming that it's [the experiment is] physically possible, i.e. assuming that time is continuous and that you can work fast enough". Despite Marc's eventual discovery of the normative solution to the paradox, his initial approach was process-oriented as he described "the overall rate growth of the partial sum" of ping-pong balls, as well as his own actions of inserting and removing balls.

In accordance with Chapter 7, participants who resisted distancing themselves from reality by clinging to practical concerns or by introducing capricious assumptions tended to approach the paradox in intuitive, process-oriented ways, such as focusing on infinite iterations. As Fischbein (1987) observed, properties of actual infinity contradict the finiteness of mental schemas and intuitions. In contrast, the ability to take a leap of imagination away from the realistic or the intuitive corresponds to an ability to engage effectively in advanced mathematical thinking – thinking which lacks "intuitive bases founded on experience" (Tall, 1992, p.495). Clarifying the limitations that finite experience has on an understanding of infinity seems to be a fundamental aspect of accommodating properties of actual infinity – properties which by all means lack intuitive bases founded on practical experience.

Distinguishing between an intuitive and a formal understanding of infinity was suggested previously to be an important first step to accommodating the idea of actual infinity. Such a distinction was an essential aspect in Jan's reasoning as she addressed both the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation, the latter of which required an understanding that went beyond recognising infinite sets as completed entities, and will be discussed in subsection 8.2.3. Jan, an undergraduate mathematics student was one of the three participants who came up with the normative resolution to the paradox, and she was the only participant to clarify a separation between her realistic intuitions and her mathematical thinking. Jan's solution is discussed in detail in the following section, however it is worth noting now that her awareness of the limitations of intuition seemed to contribute significantly to her understanding of actual infinity. After discussing her solution, Jan reflected, "Intuitively, it seems that the number of balls SHOULD blow up to infinity (though intuition frequently fails us when it comes to the infinite)". Her willingness to distinguish between intuitive and formal understandings can be linked to her ability to take the leap of imagination necessary for accommodating actual infinity. Indeed, holding on to realistic, finite experiences and intuitions seemed to hinder the encapsulation of a process for which there is no final step, but for which a completed totality does exist. A discussion of further aspects involved in accommodating the idea of actual infinity – which involves encapsulating infinity to a completed object – continues in the following two sections. The next subsection focuses on the importance of distinguishing between formal understandings of infinity and their applicability to different areas of mathematics, such as potential infinity in calculus, non-standard infinity in nonstandard analysis, and cardinal infinity in set theory.

8.2.2 Cardinal Infinity as 'How Much'

Fischbein (2001) suggested, "intuitively, there is only one kind, one level of infinity" (p.324), that of potential infinity – the inexhaustible. While it may be that our intuition knows only one infinite, within mathematics there are several. A glimpse of different infinities was offered in Chapter 2 through an exploration of properties of cardinal infinity in set theory, non-standard infinity in nonstandard analysis, and potential infinity in calculus. Identifying the infinity with which the Ping-Pong Ball Conundrum deals is an important step in accommodating its normative resolution, and one that only three participants were able to make.

Marc was one of the participants who distinguished between different views of infinity, and who was able supply the normative resolution to the paradox. He identified two ways to think about the Ping-Pong Ball Conundrum:

> "So there are two ways to think about it... The different ways of dealing with it, involving different concepts of infinity, that have different properties, and so give different answers."

In his response, Marc not only recognised different concepts of infinity, but he also distinguished between their respective properties, and acknowledged how the distinct interpretations affected the paradox resolution. Marc's first approach was to consider the paradox from the perspective of limits and series:

> "One way is to think of the balls as actual numbers, having magnitudes, and when you throw in a bunch of balls into the barrel, you are forming a partial sum in a series. And when you take away a ball, you are subtracting from that partial sum. So if you check the overall rate growth of the partial sum, it is always increasing, and so the series diverges and you get an infinite number of balls in the barrel."

Marc described an action of throwing in and removing balls that he then imagined as a process whose "overall rate growth... is always increasing". These notions are consistent with potential infinity, and though potential infinity has a distinguished place in mathematics, it does not help resolve this paradox. The normative resolution, which was presented in Chapter 3, relies on an understanding of cardinal infinity and set comparison. Recalling the solution, the Ping-Pong Ball Conundrum ends with an empty barrel, in part because there exists a one-to-one correspondence between each of the infinite sets involved and the set of natural numbers. Marc's approach using series and partial sums lead to inconsistencies, as he observed:

> "But something feels wrong about this, for every number, shouldn't you have a ball outside the barrel with that number on it, and so shouldn't every ball be outside the barrel? This makes me feel that this is the wrong way to think about it."

Marc identified a problem with his approach using series and partial sums and, after reflecting on it, considered a "second way of thinking about it, which is in terms of set theory." In Marc's set theoretic approach to the paradox resolution, he reasoned that "from this point of view, there should be no balls in the barrel. Because if you look at the set of balls that you have to remove, it is the set of all balls."

With Marc's second resolution came a second way of describing infinity: in terms of completed entities. Marc considered the "set of balls that you have to remove" as a totality when he compared it with the "set of all balls" and concluded that they were the same. By identifying the two sets as equivalent, Marc recognised the existence of a correspondence between the set of out-going balls and the entire set of balls. Further, his means of treating each set as a totality is consistent, in terms of the APOS Theory, with an object conception of infinity (Dubinsky et al., 2005a). Accommodating the idea of actual infinity involves, in Dubinsky et al.'s point of view, encapsulating infinity to an object, which is said to occur once the individual is able to think of infinite quantities as completed entities, or "as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied" (2005a, p.346). Marc's set theoretic resolution to the Ping-Pong Ball Conundrum indicates he has taken this important step towards encapsulation.

As will be discussed in the following subsection, treating an infinite set as a completed object upon which one may act is a necessary, but not sufficient, aspect of accommodating actual infinity. The aspects of accommodating actual infinity that go beyond Dubinsky et al.'s (2005a) description of encapsulation are illustrated in Dion's response to the Ping-Pong Ball Variation, as he grappled with *how* to act on the object of an infinite set. Dion, a university instructor in mathematic education, had concisely and easily resolved the Ping-Pong Ball Conundrum by establishing the appropriate one-to-one correspondences between sets of balls and time intervals (see Chapter 3 for the normative resolution). Dion recognised immediately that the paradox dealt with infinite sets and appealed to his knowledge of Cantor's theory of transfinite numbers to identify equivalent cardinalities.

Similar features to Marc's and Dion's replies were observed in Jan's response, as she too was able to identify that the paradox dealt with cardinal infinity. She remarked:

> "The question was 'how many', which is a question of cardinality, and equal cardinality of two sets is entirely determined by the existence or non-existence of a bijection between the two sets in question."

Jan drew on her formal understanding of cardinal infinity in her response to the paradox, stating the conditions that must be met to determine 'how many' and then addressing those conditions by explicitly constructing a bijection between the sets of balls:

> "So, first note that every ball that is put into the barrel is removed. For example, if a ball is placed in the barrel in the nth step, then it is removed in one of the steps 10n – 9, 10n – 8, …, 10n – 1, 10n. So if a ball is placed in the barrel during the minute, it will be taken out. Conversely, if a ball was taken out of the barrel, it must have been put in at some point during the minute… This establishes a bijection between the balls put in the barrel and those taken out. More concretely, we can assume that some ball does remain after the minute is up, and without loss of generality, let's say it's the nth ball. But we know that this ball is taken out during one of the steps 10n – 9, 10n – 8, …, 10n – 1, 10n, and all of these steps occur within one minute due to the fact that the series: (Sum from k = 1 to k = infinity of $(½)^k$) converges to 1. But then the aforementioned ball is NOT in the barrel at the end of the minute, which contradicts our original assumption that it was. Therefore there are no balls left in the barrel at the end of the minute."

Jan's response began by considering the sets of in-going and out-going balls. Once a one-to-one correspondence between the sets was established, she addressed the set of time intervals, describing a correspondence between the set of out-going balls and the set of time intervals by noting that "all of these steps occur within one minute" due to the convergence of the corresponding series. By constructing bijections and describing the behaviour of all the balls by the behaviour of the *n*th ball, Jan acted upon totalities – she treated infinite sets "statically" (Dubinsky et al., 2005b, p.257) as cognitive objects. The ability to conceive of an infinite process "as a totality, a whole capable of being acted upon" (ibid) is indicative, in terms of the APOS Theory, of an object conception of infinity. Jan's response offered further indication of an object conception of infinity when she observed that the resulting set after the completion of the experiment was not simply a generalization of an individual step:

> "We seem to have obtained a strictly increasing function (namely, the number of balls as a function of the number of time steps) that is bounded below by zero, but that is 'discontinuous at infinity', and somehow equals zero 'at infinity'."

As mentioned previously, Dubinsky et al. suggest encapsulation of infinity requires a realisation "that the state at infinity is not directly produced by any step of the process. Instead… the state at infinity reflects the totality of the process rather than any of its individual aspects" (2005b, p.260). In their opinion, an object conception of infinity "stands apart from or transcends the process" (ibid). Jan's acknowledgement of the "discontinuity at infinity" speaks to this transcendence.

8.2.3 Acting on Actual Infinity

Dubinsky et al.'s proposition that encapsulation has occurred once "the notion of potentiality is transformed into an instance of actual infinity, a mathematical entity to which actions can be applied" (2005a, p.346), takes for granted *how* actions might be applied to infinite sets. In this section, it will be argued that an understanding of the particular properties of transfinite actions, in particular transfinite subtraction, and the ways in which those properties differ compared to their corresponding finite actions is a necessary aspect of accommodating the idea of actual infinity. This section considers three participants' engagement with the Ping-Pong Ball Variation and interprets their reactions to properties of transfinite subtraction.

As mentioned, three participants – Dion, Veronica, and Jan – engaged with the Ping-Pong Ball Variation after responding to the Ping-Pong Ball Conundrum and discussing its normative resolution. Recall that Dion and Jan provided resolutions that were consistent with the normative solution, while Veronica did not. During their engagement with the Ping-Pong Variation, Dion and Veronica, the two instructors, as well as Jan addressed the issue of remaining ping-pong balls when the experiment called for the removal of balls numbered 1, 11, 21, 31, … Dion, who was familiar with Cantor's theory, having taught aspects of it to prospective teachers in the past, easily recognised the similarities between the Ping-Pong Ball Conundrum and the variation, and he commented on the relevance of Cantor's theory to his solution. When addressing the Ping-Pong Variation, Dion reasoned that, as before, there existed bijections between the sets of ping-pong balls and the set of time intervals. He concluded that the variant and the "ordered case" should yield the same result: an empty barrel.

Dion's observations regarding the one-to-one correspondences between sets of balls and time intervals were correct: the infinite set of ping-pong balls removed from the barrel can be put in a one-to-one correspondence with the natural numbers, as can the set of in-going balls, and the set of time intervals. However, in this thought experiment only the balls numbered 1, 11, 21, 31, … were removed. As a result, there is no time interval for which the balls numbered 2 to 10, 12 to 20, 22 to 30, and so on, are removed – thus leaving a barrel containing infinitely many balls. Dion, however, argued that the barrel would be empty because "after you go [remove] 1, 11, 21, 31, …, 91, etc, you go back to 2". He described a "strong leaning to Cantor's theorem" (i.e. Cantor's theory of transfinite numbers), and although he insisted "at some point we'll get back to 2", he could not justify the claim. During the interview, Dion grappled with the possibility of a nonempty barrel, stating "if ball number 2 is there, so is 2 to 10, etc… so, infinite balls there? I have trouble with that". Dion went on to observe that while "on one hand infinite minus infinite equals 0, on the other it's infinite" – a property of transfinite arithmetic that was absent in his prior knowledge. Engaging with the two paradoxes contributed to Dion's discovery of the indeterminacy of transfinite subtraction. Eventually, Dion conceded he was "convinced" of the normative solution to the Ping-Pong Ball Variation.

Dion's "strong leaning to Cantor's theorem" and his ready acknowledgement of bijections between relevant sets, suggests he had developed secondary intuitions (in the sense of Fischbein, 1987) with respect to correspondences and infinite cardinality problems. However when faced with a problem that required going beyond simply constructing correspondences, Dion was unable to apply his understanding of "Cantor's theorem" in an appropriate way. In order to make sense of the problem, Dion seemed to fall back on his intuition, not of infinity, but of subtraction. His insistence that "at some point we'll get back to 2" may be attributed to his experiences with arithmetic: in the case of finite quantities, subtracting a quantity from itself will yield zero – something which is not so when dealing with infinite quantities. Extrapolating properties of finite subtraction to properties of transfinite subtraction serves as an example of reducing abstraction – Dion applied a familiar concept to cope cognitively with a novel situation. Dion's revelation that "on one hand infinite minus infinite equals 0, on the other it's infinite" suggests that accommodating actual infinity goes beyond the ability to act on an object, and includes an understanding of *how* to act on that object.

In contrast to Dion's response was Veronica's approach to the Ping-Pong Ball Conundrum and the Ping-Pong Variation. In both cases, Veronica appealed to an intuitive understanding of infinity as potential. When addressing the Ping-Pong Ball Conundrum, Veronica reasoned that since the rate of in-going balls was greater than the rate of out-going balls, the barrel must contain infinitely many balls at the end of the experiment. Veronica connected the concept of infinity with "eternity", and had difficulty accepting the argument that the one-to-one correspondences between sets of in-going balls, out-going balls, and time intervals, guaranteed the barrel would end up empty. After some discussion, Veronica reflected that "if you don't think about one-to-one correspondences, the instinct is there are 9 left every time you take one out, so it's 9 infinity". The normative resolution to the Ping-Pong Ball Variation came much more easily to Veronica, who readily acknowledged there would be balls left in the barrel – although her instinct was that there would be a "bigger" infinity of balls remaining in, than removed from, the barrel. Veronica's resolutions are indicative of a process conception of infinity.

Veronica's intuitive approach to the two paradoxes contributed both to her resistance and acceptance of the normative resolutions of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation, respectively. Similarly (and conversely!), Dion's learned approach contributed to his acceptance and resistance to the normative resolutions of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation, respectively. These two cases illustrate the importance of separating intuitive reasoning from formal reasoning when dealing with actual infinity. Indeed, as mentioned earlier, the participant who seemed to have the most success accommodating the idea of actual infinity was able to distinguish between intuitive and formal approaches both when considering correspondences and when considering properties of transfinite subtraction. Jan realised that intuitions she developed with respect to subtracting finite quantities did not extend to infinite ones. When addressing the Ping-Pong Ball Variation, she remarked:

> "So at first, one might guess that 'the infinity of balls put in is somehow greater than the infinity of balls removed'. However! Here we get into the indeterminacy of the 'quantity' infinity minus infinity. That is, transfinite cardinal arithmetic doesn't work exactly like finite cardinal arithmetic."

Jan was further able to connect her understanding of correspondences between infinite sets to explain the indeterminacy of transfinite subtraction:

> "Even though there is a bijection between the set of balls put into the barrel and the set of balls removed, there are still an infinite number of balls left in the barrel after the minute is up! This is because N [the set of balls] is... equinumerous with a proper subset of itself. To show that there are infinitely many balls left after the minute is up, we can easily create an infinite sequence of balls that are not removed, namely: 2, 12, 22, 32, ... = $\{10n + 2 \mid n=0,1,2,\ldots\}$. This set is clearly infinite, and represents a subset of the balls left after the minute. Since the set of all balls left after the minute contains an infinite subset, it too must be infinite."

Jan's awareness that what "one might guess" was not sufficient to address problems regarding infinity, and her ability to deduce the consequences stemming from a set being equinumerous with one of its proper subsets contributed to her understanding of the indeterminacy of transfinite subtraction. Jan went on to reflect again on the relationship between intuition and actual infinity, as well as on her experiences with the paradoxes:

> "So, if we think about both the original question [the Ping-Pong Ball Conundrum] and its variation, we seem to have done the exact same thing (physically) in both cases, but due to some arbitrary numbering system that we have imposed upon the set of balls removed, we have changed the remaining number from zero to infinity! But why should numbering matter? We seem to have done the same thing in both cases. This is another case where the intuition we've learned from the physical world fails us when it comes to the infinite."

Jan also remarked, "it is nearly impossible to talk about it [actual infinity] informally for too long without running into entirely too much weirdness". Her ability to distinguish between intuitive and formal responses, in addition to her extrapolation of the formal structure of infinite cardinalities to make sense of properties of transfinite arithmetic contributed to a profound understanding of actual infinity.

8.3 Discussion

This study extended prior research regarding the use of paradoxes as a lens to learners' conceptions of infinity. In Chapter 7, the Ping-Pong Ball Conundrum and Hilbert's Grand Hotel paradox were found to be effective tools in eliciting cognitive conflict and encouraging a refinement of participants' understanding of infinity. In this follow up study, the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation were used as a lens to ascertain a more refined account of the necessary and sufficient aspects of accommodating the idea of actual infinity. In particular, participants' engagement with both paradoxes revealed the important factors beyond conceiving of an infinite set as a completed entity, which contributed to an understanding of actual infinity. Acknowledging the distinction between *how* actions, such as arithmetic operations, behave differently when applied to transfinite entities versus finite entities was an integral part of Jan's accommodation of actual infinity. Other key factors in accommodating actual infinity included the ability to distinguish between intuitive and formal knowledge, to understand infinite cardinality as 'how many', and to deduce the indeterminacy of transfinite subtraction from the formal definition of an infinite set. Identifying these key factors can be seen as the main contributions of this study. A detailed discussion of these factors and their corresponding challenges is reserved for Chapter 9: *Cognitive Leaps*.

CHAPTER 9:

COGNITIVE LEAPS TOWARD UNDERSTANDING INFINITY

This chapter explores some of the common themes which emerged in participants' responses, transcended the individual studies, and created obstacles to the understanding of actual infinity. Overcoming an epistemological obstacle "means that the student will have to rise above his convictions, to analyse from outside the means he had used to solve problems in order to formulate the hypotheses he had admitted tacitly so far, and become aware of the possible rival hypotheses" (Sierpinska, 1987, p.374). In some instances, the only way to overcome an obstacle – to rise, as Sierpinska wrote, above convictions, prior experience, and intuition – is through a *cognitive leap*. The themes and obstacles explored in this chapter are framed in terms of the cognitive leaps required to overcome them. The cognitive leaps facing an individual as he or she attempts to develop an understanding of actual infinity include a leap from the philosophical to the mathematical (section 9.1), a leap from the intuitive to the formal (section 9.2), and a leap toward accommodating transfinite arithmetic (section 9.3). The term 'philosophical' is used in this thesis in a broad sense to include emotions, beliefs, or personal worldview.

9.1 The Mathematical Entity of Infinity

An early cognitive leap that is required for developing an understanding of actual infinity involves accepting infinity as a *mathematical entity*. Throughout history, the concept of infinity has been tied to the philosophical and the theological. Aczel (2000) described the connection between God and infinity that the Kabbalah, a mystic sect of Judaism, established centuries ago. Aczel summarized the Kabbalists' view of the infinite light of "the great entity that is God" (2000, p.34). He wrote:

> "That entity [of God] is so large, so supreme, so far beyond description, that it is given the only name the Kabbalists could possibly use to describe it: *Ein Sof*. The two words mean *Infinity*. God is infinite" (Aczel, 2000, p.34, emphasis in original).

Throughout my research, participants expressed ideas in line with the philosophical and theological perspectives exemplified in Aczel's interpretation of *Ein Sof*. A more precise literal translation of *ein sof* is "endless" (Zazkis, personal communication). However, when considering the tasks of this research, the perspectives that may relate infinity to God and endlessness, diverge from the mathematical conception of actual infinity. As such, in order to develop an understanding of actual infinity, a leap away from the philosophical is necessary. Once distinction between the philosophical and mathematical is made, and infinity is conceived of as an entity with mathematical properties, another leap toward recognising that there are several infinities within mathematics, which also differ from just that which is "far beyond description" (Aczel, 2000, p.34), is important.

This section considers the philosophical notions of infinity that emerged across my three studies. These notions emerged in conflict with the normative mathematical properties of actual infinity, and created obstacles in participants' understanding. Section 9.1.1 identifies the philosophical perspective that infinity is an impossibility that should not be of concern to mathematics. Section 9.1.2 discusses participants' confusion between the infinite and the unknown, while section 9.1.3 explores the beliefs that infinity is all encompassing and eternal.

9.1.1 The Impossibility of Infinity

Across the three studies, resistance toward imagining an entity that was infinite manifested in participants' resilient realistic considerations. The reluctance to extrapolate beyond the physically or practically possible surfaced in the responses of both naïve and mathematically sophisticated participants. For some participants, disbelief in the existence of an infinite entity surfaced as a response to the premise of the task and prevented participants' acceptance or acknowledgment of normative facts.

In resonance with Sierpinska's (1987) identification of an epistemological obstacle relating to learners' belief that mathematics should concern itself only with the finite, many participants who addressed Hilbert's Grand Hotel paradox in Chapter 7, restricted their attention to realistic 'resolutions'. Both the liberal arts undergraduates and the mathematics education graduate students avoided tackling the idea of infinitely many full rooms. Instead, they preferred to offer the new guest accommodation in the lobby or a shared room. Some of the liberal arts students clarified that their recommendation stemmed from an inability to imagine a hotel with infinitely many guests. Jimmy, for example, expressed his confusion that "it just doesn't make sense that if there are infinitely many rooms that they all could be full. It defies logic!"

Further evidence of participants' resistance toward conceiving of an infinite entity surfaced in Chapter 6 as they addressed the number of fractions in a closed interval. Neil, for instance, reasoned that while "theoretically" there could be infinitely many fractions within the bound, "in terms of life there isn't". Neil also believed that such a "theoretical" possibility was "not necessary because it just confuses things" and that it was better "just kind of ignoring it [infinity]".

9.1.2 Infinite is Unknown

In a similar vein to the belief that infinity is impossible is the view that infinity is an entity that is *impossible to know* – or in Aczel's words "far beyond description" (2000, p.34). For many participants, an infinite quantity corresponded to an unknown quantity, and one which was beyond human means to determine. For instance, Kyle, a liberal arts student whose response to the Ping-Pong Ball Conundrum was discussed in Chapter 7, reasoned that he "couldn't tell you how many balls we began with because there were infinity". Similarly, Joey, who in Chapter 8 tried to clarify his ideas regarding the quantity of ping-pong balls, described infinity as a "count" that is so large it is "impossible to write down". He opined that infinity might be "A number so large it could not be verified. Thus escaping our perception. Perhaps infinity is a veil draped over that which we do not know. Something that we cannot perceive."

Joey's philosophical speculation is echoed in Jim's proposition from Chapter 6 that the term 'infinity' is simply a label attributed to unknown quantities for which there does not exist a means of measurement. In Jim's consideration of the quantity of points on a line segment, he suggested that labelling something as 'infinite' was a way "to kind of put it into a category that our brains can then figure out." Akin to Kyle and Joey, Jim imagined infinity as "something that keeps going past any way that we can measure, [so] we put this word onto it because then it's settled and we can push it out of the way and move on with our lives."

For those participants who believed that if "it is not humanly possible to figure out the number… it is said to be infinite", the distinction between large finite quantities that are 'too large to perceive' and infinite quantities went unnoticed. Further, these participants displayed a tendency to describe infinite quantities as entities that 'could go on forever' or 'could end' but it was unknown. Although it is impossible to enumerate an infinite quantity of elements, that quantity is still 'known' to be infinite. Appreciating this distinction is an important leap toward understanding actual infinity.

9.1.3 An Eternal or All-encompassing Entity

Participants' engagement with Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum brought to light some philosophical conceptions that, though they are not made explicit, may influence learners' consideration of infinity in other contexts as well. The idea that infinity is an all-encompassing entity surfaced in the responses of liberal arts students to Hilbert's Grand Hotel paradox. Participants argued against the need to accommodate a 'new' guest since every current and future guest would already have a room because they "must be part of the infinite". These participants believed that as soon as a new guest asked for a room his pre-assigned room would be instantly available. The belief in an all-encompassing infinite was coercive and resilient. Not only did it persist despite instruction, but it also swayed other participants' ideas.

An analogous philosophical perspective that emerged in the data links the idea of infinity with eternity. This connection emerged as participants attended to the 60-second experiment in the Ping-Pong Ball Conundrum. It also influenced how they addressed the infinite quantity of ping-pong balls. For example, Veronica, one of the participants in Chapter 8, described a connection between her ideas of infinity and 'eternity'. This association hindered her understanding of the one-to-one correspondence method of comparison when it yielded the counter-intuitive result of an empty barrel. Further, it suggests that Veronica's conception of infinity is related to endless time. A conception of infinity that is restricted by a temporal component limits an individual to a process-oriented conception of infinity, and obstructs understanding 'completed' infinite sets.

9.2 Separating the Intuitive from the Formal

Fischbein et al. (1981) suggested that intuitive interpretations are active during individuals' attempts to solve, understand, or create in mathematics. However primary intuitions, which are generally rooted in everyday life and previous practical experience, often hinder students' functioning in a new mathematical field (Tsamir, 1999). Tsamir suggested "instructors should be attentive to the relations among formal and intuitive knowledge and to the conflicts which may arise in the mismatching applications of these different types of knowledge" (1999, p.231-2). Dubinsky and Yiparaki (2000) suggested that using 'real life' intuitive context to teach evaluation of mathematical statements can be more harmful than helpful. They observed that "the conventional wisdom to teach by making analogies to the real world can fail dramatically", and advised, "to remain in the mathematical realm" (2000, p.283). Relating this advice to infinity – a concept for which no 'real world' analogy can do justice – my research suggests that the ability to clarify a separation between intuitive and formal knowledge is an important leap toward accommodating the idea of actual infinity. Fischbein (1987) observed that intuitive knowledge is coercive, intrinsically certain, and resolute. As such, emotional conflicts, in addition to cognitive ones, were observed in participants' reactions to actual infinity. However, emotions connected to the concept of infinity are out of the scope of my dissertation as I am interested primarily in the cognitive aspect.

This section begins with a review of participants' intuitions of infinity and the representations that elicited them (9.2.1). It also examines some of the struggles participants faced in trying to bridge their intuitions with formal properties (9.2.2). The section concludes by presenting an argument in support of encouraging learners to take the leap that separates their intuitive from their formal knowledge when addressing properties of actual infinity (9.2.3).

9.2.1 Intuitions of Infinity

The naïve ideas and intuitive strategies that emerged during participants' engagement in my research are consistent with those observed in prior research (e.g. Fischbein et al., 1981; Tall, 1980). Participants related the idea of infinity to endlessness, relied on prior experience with number and measurement, and remained in most cases unaware of the inconsistencies between competing intuitions and also between naïve and formal notions.

Connecting the idea of infinity to an 'endless', 'on-going' entity was prevalent in participants' responses to each of the paradoxes and geometrical tasks. The intuition of 'endlessness' corresponds to the idea of *potential infinity* – a process which is finite at every instant, but which goes on indefinitely. Describing infinite entities in terms of the process required to establish those entities surfaced in the responses of participants regardless of their level of mathematical sophistication. For instance, in Chapter 6, the undergraduate students who addressed infinite subsets of the set of real numbers described processes when those subsets were presented both numerically and geometrically. As participants attended to the quantity of fractions in the interval $[\frac{1}{19}, \frac{1}{17}]$, they imagined a process in which "endless numbers can be put into the numerator or the denominator". Similarly, when attending to a geometric presentation – the number of points on a line segment – participants imagined 'finding' or 'creating' an unlimited quantity of points in order to account for their infinite number.

In Chapter 7, intuitions of potential infinity emerged, for example, in participants' descriptions of a Grand Hotel with "an always increasing number" of rooms, and in their resistance to the idea of a 'completely filled' hotel. Responses to the Ping-Pong Ball Conundrum, in Chapters 7 and 8, also focused on endless processes, such as halving the time intervals. Both liberal arts undergraduate students and doctoral candidates in mathematics objected to the time limit of 60 seconds because they imagined that "since the time interval is halved infinitely many times… the 60 seconds never ends".

As participants attended to the comparison of two or more infinite sets, an intuition of infinity that extrapolated measuring properties of numbers emerged. Tall (1980) introduced the idea of 'measuring infinity' as a metaphor to describe learners' intuition that a longer line segment would have more (infinitely many) points than a shorter segment. Tall speculated that when presented with measurable entities, such as geometric objects, learners would intuitively appeal to the idea of 'measuring infinity', extrapolating ideas such as 'longer means more'. In Chapter 6, participants confronted properties of infinity through the geometric context of comparing line segments of different lengths. As participants reasoned about the number of points on line segments of different lengths and the number of points 'missing' from the shorter segment, many appealed to an intuition of 'measuring infinity'. As Rosemary summarised: "The amount of points in A is greater than C, even though each line has infinite amount of points."

Intuitions of 'measuring infinity' also appeared in participants' responses to the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation, in Chapters 7 and 8, as they attended to the different rates of in-going and out-going ping-pong balls. A common resolution to the Ping-Pong Ball Conundrum suggested that "the process of putting balls in at a higher rate than taking balls out" would result in a barrel that contained infinitely many balls and from which a 'smaller' infinite number of balls was removed. Attending to the different rates of in-going and out-going balls evoked arguments of a "bigger infinity" since there are "9× more balls in the barrel than out of the barrel at all times. At the end of the 60 seconds there are 9∞ balls in and ∞ balls out." Attending to the measurable entity of a rate of change and deducing from it ideas of "larger" and "smaller" infinities serves as an example of an intuition of 'measuring infinity'.

The intuitions of potential infinity and of measuring infinity surfaced as competing ideas in participants' responses to both geometric tasks and paradoxes, despite the inconsistency of a 'never-ending' entity that may be 'smaller' or 'larger' than another 'never-ending' entity. A common trend in participants' conceptions is illustrated in Rosemary's comparison of the number of points on line segments of different lengths; task Q_2 from Chapter 6. As she addressed this task, Rosemary reached the contradictory conclusion that although infinity "keeps going and going", the longer of the two line segments would have a "larger" infinite number of points.

As mentioned, the conflict between potential and measuring infinities usually was not recognised by participants. For instance, Joey, one of the mathematics majors who addressed the Ping-Pong Ball Conundrum from Chapter 8, reasoned inconsistently that an infinite number of time intervals is endless, but an infinite number of ping-pong balls could be exceeded by a larger infinite amount. He wrote:

> "I will never reach 60 seconds. So the experiment should never end, really. Meaning I have an infinite number of ping-pong balls, and yet there are more in the barrel."

Similarly, Kenny, a liberal arts student from Chapter 7, argued that the ping-pong ball experiment "will continue into eternity and the number of [ping-pong] balls will be infinite in the barrel". With respect to time, Kenny imagined an endless, potential infinite, however, with respect to measuring the amount of balls, Kenny imagined a large, unknown number. The flexible use of these incompatible notions, which were elicited by different presentations of equinumerous infinite sets, illustrates a hazard of relying on an intuitive understanding of a counterintuitive concept, and motivates the significance of a leap away from the intuitive.

9.2.2 Attempts to Coordinate Intuitive and Formal Knowledge

As participants' naïve conceptions were challenged by directing their attention to some of the formal properties of actual infinity, the conflict between an intuitive understanding of potential or measuring infinity, and the normative properties of actual infinity was realised by some participants. A trend that emerged as a consequence of this realisation involved participants' attempts to appreciate formal properties on an intuitive level, and thus 'bridge the gap'. In resonance with observations by Fischbein (1987), participants who attempted to reconcile intuitive and formal understandings tended to adapt the formal notions to establish consistency with their intuitions.

The common strategy in participants' attempts toward reconciliation involved what I refer to as 'shifting the process'. Shifting the process occurs when there is a change in aspect, or quality, of infinity to which participants attribute a process. This is recognised, for example, in Eric's consideration of Hilbert's Grand Hotel paradox. Eric initially reasoned that "you could keep on adding people forever to fill" Hilbert's Grand Hotel because the rooms in the hotel "would go on forever". When this conception was challenged by the normative resolution to the paradox, Eric refined his idea of the hotel. In his attempt to reconcile the intuition of an endless process with the idea of a completely full hotel, Eric explained:

> "Although the infinite rooms are infinitely full, it makes space for you by making one of those rooms free. I was first troubled by the idea of one 'last' person not having a room, but then I realized that the last person would ask me to shift rooms, and so on, so there would be a constant rotation."

The conflict in a hotel that should 'go on forever' but that is 'full' was resolved for Eric through his introduction of the idea of an infinite "rotation" of guests. The infinite process in Eric's conception shifted from adding guests to the process of moving them. Similarly, Clyde's approach to Hilbert's Grand Hotel demonstrated a shift in processes, when he reasoned that the new guest would get "sound sleep while everyone else has to continue to shift rooms infinitely." As Clyde addressed the normative resolution to the paradox, the infinite process in his conception was shifted to the transformation of moving guests, despite the fact that each guest moves only once. Clyde's response is also interesting because in addition to attempting to bridge intuition with formal properties, he discerned that the problem was 'unrealistic' and that the solution was reasonable despite his intuitions. Such discernment is important; it will be explored further in Section 9.2.3.

Other instances which revealed a shift in the process of participants' conceptions emerged in Chapter 6 as participants addressed the geometric construction corresponding the sets of points on line segments of different lengths. As participants attempted to make sense of the one-to-one argument, they attended to the process of constructing the correspondence. For instance, participants reasoned that between the segments it was possible to "draw as many lines as we want", and that the "one-to-one ratio [of points] will stay constant forever." Participants' description of the process of establishing a one-to-one correspondence is seen as a shift from prior notions that drew on the possibility of "creating as many [points] as you want" to account for the equipotence of two sets.

Attempts to reconcile an intuition of measuring infinity with formal properties was also recognised in participants' responses to correspondence arguments. This was particularly apparent in the conversation with Jack, in Chapter 6, which took place after instruction. Jack understood the idea of corresponding points through 'coupling', and was easily able to recreate the geometric construction that established the bijection, however he struggled with the discrepancy between it and his intuition. He stated that "visually, you're seeing that A [line segment AB in the construction] is bigger, so therefore the infinite number has to be bigger on A [AB] than the infinite number on C [line segment CD]." In contrast, when Jack was presented with an analogous construction using circles, his intuition of measuring infinity yielded a result that was consistent with the normative one. Until Jack could make a connection between the normative result and his measuring intuition, he could not accept that the two sets of points were equinumerous.

Using the same measuring approach yielded inconsistent solutions when applied by Jack to different entities. When the inference from measuring infinity was consistent with the normative result, Jack was at ease. However, when it was inconsistent, Jack experienced considerable frustration. Despite the inconsistent implications of Jack's approach, he continued to rely on his intuition of measuring infinity, which would inevitably lead to further conflict. Jack's eventual recognition that the normative resolution was "hard to believe, but it makes sense" suggests a dawning realisation that intuition is unreliable with respect to infinity.

9.2.3 Separating the Intuitive from the Formal

Aczel (2000) shared the story of Rabbi Ben Zoma, a rabbi who strove through meditation to witness the robed figure of God as He had appeared to Moses. As Rabbi Ben Zoma achieved his goal, his experience was so intense that he allegedly "glanced at the infinite light of God's robe and lost his mind, for he could not reconcile ordinary life with his vision" (2000, p.27). Although none of the participants of my studies lost their minds (to the best of my knowledge), they did face considerable frustration trying to reconcile the intuitions stemming from 'ordinary life' with an understanding of infinity, and in the end were unable to do so in a way that would yield consistent conclusions.

The concept of actual infinity is so far removed from our experiences that relying on intuition can be treacherous, even when those intuitions develop from experience with Cantor's theory; that is, even when the reliance is on what Fischbein (1987) termed 'secondary intuitions'. A case in point is Dion's attempt to resolve the Ping-Pong Ball Variation, in Chapter 8. Dion had demonstrated a solid understanding of corresponding infinite sets, however when he instinctively applied this knowledge to the Ping-Pong Ball

Variation it resulted in a conflict. As discussed in section 9.3, part of Dion's conflict emerged because of intuitions regarding subtraction and their inapplicability to transfinite numbers. Dion's experience with these paradoxes exemplifies a situation where an appropriate formal understanding might still be led astray by the influence of intuition.

The one participant who consistently resisted being led astray by intuition was Jan - an undergraduate student in mathematics from Chapter 8. Jan demonstrated a very profound understanding of actual infinity, and an important idea that she often returned to regarded the separation of intuition and formal knowledge. When addressing the Ping-Pong Ball Conundrum, Jan realised that through the experiment

> "we seem to have obtained a strictly increasing function (namely, number of balls as a function of number of time steps) that is bounded below by zero, but that is 'discontinuous at infinity', and somehow equals zero 'at infinity'."

She observed that "intuitively, it seems that the number of balls SHOULD blow up to infinity (though intuition frequently fails us when it comes to the infinite)".

Similarly, when considering the Ping-Pong Ball Variation, Jan clarified a distinction between her intuition that "the infinity of balls put in is somehow greater than the infinity of the balls removed" and the normative property of "the indeterminacy of the 'quantity' infinity minus infinity". She went on to reflect that the indeterminacy of transfinite arithmetic "is another case where the intuition we've learned from the physical world fails us when it comes to the infinite". Jan's sophisticated understanding of actual infinity seemed to hinge on her awareness that "it is nearly impossible to talk about it [infinity] informally for too long without running into entirely too much weirdness... The subject literally seems to force itself into an Axiom System."

9.3 Transfinite Arithmetic

This section addresses conceptions of transfinite arithmetic, in particular addition and subtraction, which were of interest in Chapter 6 as participants attended to the number of points 'missing' from the shorter of two line segments, and in Chapters 7 and 8 as participants engaged with the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation. From both a mathematical and an educational perspective, this section contemplates the cognitive leaps required for an understanding of transfinite arithmetic.

Recalling part of the discussion from Chapter 2, arithmetic on the class of transfinite numbers emerges in part as an extension of arithmetic on the set of natural numbers. Natural numbers may be identified with cardinalities of finite sets, and as such one definition of addition over the natural numbers involves determining the cardinality of the union of two disjoint sets. More formally, if A and B are two disjoint sets with cardinalities a, b in **N**, then the sum of the two natural numbers $a + b$ is equal to the cardinality of the union set of A and B – the set (A$\cup$B). An analogous definition extends to transfinite numbers, with the difference that at least one of the cardinalities of A, B is a transfinite number. There is, however, an important difference between finite and transfinite cardinal arithmetic. As explained in Chapter 2, there are non-unique sums when adding transfinite numbers, and this has significant consequences for transfinite subtraction. Recall, for instance Figure 2.3, which demonstrated the correspondence between the two sets **N** and (**N**, β), and which illustrated the property $\aleph_0 + 1 = \aleph_0$.

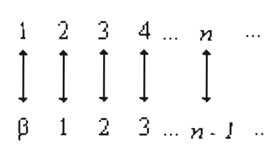


Figure 2.3: Corresponding **N** and (**N**, β)

By extending this construction, it is possible to illustrate the property that $\aleph_0 + 2 = \aleph_0$, and similarly $\aleph_0 + 3 = \aleph_0$, and $\aleph_0 + 4 = \aleph_0$, and so on. Consequently, $\aleph_0 - \aleph_0$ is indeterminate, since it could be equal to 1, or 2, or 3, or 4, ... The indeterminacy of transfinite subtraction emerges in sharp contrast to subtraction on the set of natural numbers, which is well defined. Subtraction on the set of natural numbers can be considered simply as the opposite of addition, however transfinite subtraction does not have an analogous and well-defined structure, precisely because of the non-unique sums.

Despite the many parallels in the definitions of natural number and transfinite arithmetic, the two are distinct. A source of confusion is that both addition of natural numbers and of transfinite numbers use the same symbol: "+"; similarly for subtraction. In order to distinguish between natural number and transfinite arithmetic, it is useful to introduce some notation: the symbols $+_N$ and $-_N$ will be used to represent addition and subtraction, respectively, over the set of natural numbers, and the symbols $+_\infty$ and $-_\infty$ will be used to represent addition and subtraction, respectively, over the class of transfinite (cardinal) numbers. The following two subsections examine some of the conceptual difficulties of transfinite arithmetic: section 9.3.1 takes a look at participants' conceptions regarding domain dependence of arithmetic, and in particular the expression "∞ - ∞", while section 9.3.2 considers the expression '∞ + 1'.

9.3.1 Domain Dependence

The operations of addition and subtraction can be considered as multivariable functions, and as such their definitions depend on the domain to which they apply. As participants addressed the expression '∞ - ∞ = ∞' in the context points missing from the shorter of two line segments (Chapter 6), the majority did not consider the possibility that an arithmetic operation might have different properties when applied to different domains; that is, when applied to infinite quantities rather than finite ones. This came as a surprise in light of participants' typical assertions that 'infinity is not really a number'. Further, participants' resistance toward the possibility that '$\infty - \infty = \infty$' surfaced despite the geometric presentation of the line segments and participants' previous assertions that the number of points on a line segment was infinite "regardless of length". When faced with the expression '$\infty - \infty = \infty$', participants seemed to fall back on their intuitions of $+_N$ and $-_N$, reasoning as Nina did, that "an infinite number subtracted by itself will equal 0 because anything subtracted by itself will be zero." Interestingly, an intuition of $-_N$ seemed to be more coercive than the basic intuition of potential infinity, as participants were more likely to accommodate the idea of 'infinities with different sizes' than to conceive of a domain-specific arithmetic operation, such as $-_\infty$. Indeed, it was in response to the expression '$\infty - \infty = \infty$' that participants formulated the notion of 'measuring infinity'.

The tacit influence of the intuition of $-_N$ also emerged during participants' attempts to resolve the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation in Chapter 8. For instance, as mentioned in the previous section, Dion, who had a sophisticated background in mathematics, experienced a conflict as he address the Ping-Pong Ball Variation. Dion recognised that the quantity of balls removed from the barrel was equal to the quantity of balls placed into the barrel, and resisted the idea that there could be an equally infinite quantity of balls remaining in the barrel. He argued that "after you go [remove] 1, 11, 21, 31, ..., 91, etc, you go back to 2", insisting without justification that "at some point we'll get back to 2". Dion, like Nina, seemed to think that 'anything' subtracted by itself should be zero, and had self-described "trouble" with the idea that infinitely many balls would remain in the barrel. With much resistance, Dion eventually accepted that "on one hand infinite minus infinite equals 0, on the other it's infinite". In spite of Dion's prior understanding of transfinite cardinals and the theory behind infinite set comparison, when he was faced with a non-routine problem that addressed transfinite subtraction, Dion resorted to an intuition of $-_N$, and struggled with the indeterminacy of $-_\infty$.

The conceptual challenges associated with properties of transfinite arithmetic may be attributed to several factors. One factor relates to participants' inconsistent and informal notions of infinity. It may be expected that learners who conceive of infinity as a 'big number', would use $+_N$ and $-_N$ as the default operation for 'infinite numbers'. However, reluctance to conceive of a domain specific operation that would have different properties than $+_N$ and $-_N$ was wide spread. Participants who distinguished properties of infinity from properties of natural numbers were nonetheless disinclined to imagine that transfinite arithmetic would be distinct as well. This observation suggests that another contributing factor to participants' difficulties with the normative properties of transfinite arithmetic may be credited to the fact that prior experiences with different domains did not introduce inconsistencies. For example, $3 +_N 2$ is equal to $3.0 +_Q 2.0$, where $+_Q$ represents addition over the set of rational numbers. Although participants might realise that the rational number 3.0 has distinct properties from the natural number 3, and that arithmetic algorithms are different for rational numbers and natural numbers, addition and subtraction over the two domains nevertheless yield consistent sums and differences, respectively. Further, it is also possible that confusion regarding transfinite arithmetic stems from the inherent ambiguity of expressions such as '$\infty + 1$', or '$\aleph_0 + 1$'.

9.3.2 The Creature '$\infty + 1$'

The conceptual challenges with the issue of arithmetic being domain dependent are further confounded when considering the creature '$\infty + 1$', or '$\aleph_0 + 1$'. From a naïve perspective, it would seem that the expression '$\infty + 1$' involves summands which belong to two distinct domains – '∞' belongs to the domain of 'infinite numbers' and 1 to the domain of natural numbers. With this naïve perspective, the question of how to evaluate an expression where the two summands are conceived of as belonging to different domains would rise as a difficult conceptual challenge to overcome, and one for which an individual may lack prior experience.

Mathematically, the entity '$\infty + 1$', or more formally '$\aleph_0 + 1$' (for example), is seen as a sum of two *cardinal numbers*, numbers which quantify the sizes of sets. Recalling ideas discussed in Chapter 2, the cardinal number $\aleph_0$ is considered the cardinality of a set with $\aleph_0$ elements, and the cardinal number 1 is considered the cardinality of a set with one element. As such, the two summands are not elements of different domains, but are both considered cardinalities of sets. Thus, the sum '$\aleph_0 + 1$' is also considered the cardinality of a set: it is the cardinality of the disjoint union of a set with $\aleph_0$ elements and a set with 1 element.

Conceptually, the entity '$\infty + 1$' may be quite different from the mathematical description. The intuitions connected to the number 1, to addition, to infinity, and the imagery elicited by that symbolic representation might contribute to an understanding far removed and incompatible with the normative one. The relevance of understanding transfinite arithmetic to accommodating the idea of actual infinity was motivated in Chapter 8, however as of yet, what learners' conceptions are of the creature '$\infty + 1$' remains an open question.

The cognitive leaps that were outlined in this chapter were presented as ways to overcome epistemological obstacles related to philosophical beliefs, intuitions, and arithmetic properties of actual infinity. These obstacles emerged during participants' engagement with the tasks and paradoxes presented in Chapters 6, 7, and 8. They transcended the different appearances of infinity, and stalled participants with both naïve and sophisticated mathematical backgrounds. The considerable frustration faced by participants who were not able to overcome these obstacles by reconciling their understanding of infinity with a normative one, motivates the need for a cognitive leap to imagine beyond conviction, intuition, and prior experience, and toward an accommodation of actual infinity.

CHAPTER 10:

CONCLUSION

The renowned mathematician David Hilbert wrote:

> "From time immemorial, the infinite has stirred men's *emotions* more than any other question. Hardly any other idea has stimulated the mind so fruitfully" (1925, p.136, emphasis in the original).

In resonance with Hilbert's observation, the idea of infinity stirred the emotions and stimulated the imaginations of participants with naïve or sophisticated understanding of mathematics. As participants endeavoured to make sense of the counterintuitive and abstract nature of actual (cardinal) infinity, the persuasive philosophical beliefs and prior mathematical knowledge that influenced their conceptions emerged in contrast to the normative properties. Many participants experienced considerable cognitive conflict as they attempted to reduce the level of abstraction of actual infinity to make accessible the inaccessible. Participants' resolute attempts to reconcile finite reality with the infinite emphasised their desire to appreciate on an intuitive level an entity which is totally beyond the reach of our finite intuitions. In contrast, participants who were able to clarify a separation between their intuitive and formal knowledge achieved what many minds throughout history have strived for: a glimpse of the deep and mysterious nature of infinity.

10.1 Summary of Results and Contributions

The journey of my research began with an investigation of learners' emergent conceptions of infinity. Whereas much research has focused on stable or persistent conceptions of infinity, my research attended to the developing ideas of university students. A guiding theme in my investigations extended beyond the focus of what an individual *knows* about infinity, to what an individual *can* or *is willing to* learn about infinity. As such, one of the overarching research questions that motivated each of the individual studies presented in Chapters 6, 7, and 8, related to identifying the emergent conceptions of participants as they engaged in situations or activities through which their ideas and intuitions about infinity could be challenged and developed.

In Chapter 6, undergraduate applied science students' naïve and emergent conceptions of infinity were explored via their engagement with a series of geometrical tasks. The tasks were designed in response to participants' developing understanding, and delved into their conceptions of infinity, transfinite arithmetic, and real numbers. The interactive design of the questionnaires offered a fresh approach to data collection and serves as one of the methodological contributions of my research. Throughout participants' engagement with the tasks, surfaced a disconnect between numeric and geometric representations of infinity, as well as confusion between the infinite magnitude of entities and the infinite representation of non-terminating decimals. The conceptual challenges that participants faced regarding comparisons of infinite sets, bound infinite sets, and transfinite subtraction were exemplified by Lily's struggle to reconcile her understanding of the real number π with observed properties of infinite quantities and with her understanding of the decimal 3.14. One of the specific contributions of this study relates to Lily's confusion regarding magnitude and representation, and her resistance toward the idea that an infinite number of decimal digits could be 'bound within a finite number'. Clarifying the associations, analogies, and confusions upon which participants relied and struggled is an important contribution that will inform instructional choices in the future. This study also offered a first glimpse at learners' attempts to make sense of arithmetic properties of 'infinite numbers', which set the stage for the subsequent investigation presented in Chapter 8.

Chapters 7 and 8 introduced the use of paradoxes as a research tool for investigating conceptions of infinity. In Chapter 7, undergraduate liberal arts students and graduate students in a mathematics education master's program addressed Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum. During their engagement, participants were confronted with the indeterminacy of transfinite subtraction and with the idea of a bound and completed infinite entity – such as the completely filled hotel, or the set of infinitely many time intervals bound within 60 seconds. Surprisingly, despite the different levels of mathematical sophistication, both the liberal arts students and the mathematics education students attended to, and were challenged by, similar features of the paradoxes. One of the contributions of this study was in identifying paradoxes as beneficial research tools for eliciting participants' ideas, provoking cognitive conflict, and clarifying perceptions and intuitions that might present obstacles in adopting a 'conventional' understanding of actual infinity. Through the use of paradoxes, a refined understanding of learners' intuitions was achieved, extending prior knowledge regarding the tacit influences that contribute to learners' conceptions of infinity.

In addition to identifying emergent conceptions of infinity, my research also sought to clarify the specific features involved in accommodating the idea of actual infinity. Along this line, the studies in Chapters 6 – 8 were intended to delve progressively deeper into the intricacies of understanding actual infinity. In particular, the investigation presented in Chapter 8, which explored the conceptions of mathematics majors, graduates, and doctoral candidates as they attended to the Ping-Pong Ball Conundrum and one of its variations, offered a refined account of the necessary and sufficient aspects of accommodating the idea of actual infinity. One of the main contributions of this study is in the identification of features which go beyond the APOS description of encapsulation. One important feature involves a leap of imagination away from the 'realistic' and the intuitive. The need for such a leap surfaced in the previous studies as well, and prompted the realisation that accommodating the idea of actual infinity seems to rely on the ability to separate the intuitive from the formal. In addition, understanding actual infinity as a cardinal includes conceiving of a completed object that describes 'how many', and which may be acted upon in the sense of the APOS Theory. A subtlety related to acting on infinity was brought to light by Dion's and Jan's responses to the Ping-Pong Ball Conundrum and its variation. Although Dion had 'leapt' to the realm of mathematics and could conceive of infinity as 'how many', his understanding of infinity nevertheless lacked one of the fundamental features that contributed to Jan's profound understanding: the knowledge of *how* infinite cardinals are dealt with. An important contribution of this study identifies the necessity of understanding properties of transfinite arithmetic in order to accommodate the idea of actual infinity. Furthermore, my research lays the foundation for an extension to the theoretical framework of the APOS Theory. The APOS Theory connects a learner's ability to apply actions to a mathematical entity to his or her encapsulation of that entity as an object. However, this framework overlooks the different ways in which actions may be applied. It also neglects to consider what, if anything, can be inferred about an individual's conceptualisation based on *how* that individual applies actions and *which* actions are applied. My research suggests a refinement of the APOS Theory which includes a consideration of *how* actions are applied, and it opens the door to future investigations regarding the extent to which this refinement may be appropriate.

The third question that guided my research considered the cognitive leaps connected to the idea of mathematical infinity. As discussed in Chapter 9, an important leap toward understanding actual infinity involves a willingness to consider infinity as a mathematical entity. A leap away from the philosophical toward the mathematical was recognised as a way to overcome the epistemological obstacles corresponding to the ideas that (i) infinity is impossible and therefore should not be of concern to mathematics, (ii) infinity is 'impossible to know' and the term 'infinite' serves a label to describe the unknown, and (iii) infinity is eternal or all-encompassing. Once infinity is accepted as a mathematical entity, a distinction between an intuitive understanding of infinity and a formal one is needed. A leap away from the intuitive toward the formal is suggested as a way to rise above inconsistencies between competing intuitions of potential and 'measuring' infinities, and also between naïve and formal notions. Unsuccessful attempts to coordinate intuitive and formal knowledge, and the frustration encountered by participants who tried, speak to the significance that a leap toward the formal has on the realisation of a normative understanding of infinity. Further to this end, my research presents a first look at the conceptual challenges regarding transfinite arithmetic, and the cognitive leaps that may be required to establish a normative understanding of adding and subtracting infinite cardinals. One important leap is connected to the understanding of addition and subtraction as a multivariable functions whose properties depend on the domain to which they apply. The intuitive resistance regarding the idea that properties of arithmetic are domain-dependent created a serious obstacle toward appreciating, and even acknowledging, the indeterminacy of transfinite subtraction.

10.2 Limitations

I briefly note limitations of this research, relating to methodology. Foremost, a limitation of my research stems from my sample of participants. The samples were convenience samples, particularly with respect to the undergraduate participants of the studies presented in Chapters 6 and 7. The applied science and liberal arts students were chosen because of availability, rather than for a purpose directly connected to the parameters of the studies. In addition, a limitation regarding the choice of research tools is noteworthy. The Ping-Pong Ball Conundrum offered a paradox that required conceptualising infinity as a completed object, yet which presented the experiment as a process. The language of the paradox invited participants to imagine themselves engaged in the infinite process of inserting and removing balls from a barrel. Thus on some level it is not surprising that the paradox elicited process oriented responses and conceptions. However, based on the ubiquity of process oriented responses, particularly with respect to presentations of infinity that did not suggest any process, it seems unlikely that a phrasing which described a completed experiment would have a significant impact on participants' conceptions.

10.3 Future Directions

My research invites future investigation in several directions. Relating to methodology, the use of paradoxes as effective instructional tools in mathematics has been well documented (e.g. Movshovitz-Hadar & Hadass, 1990), however the usc of paradoxes as research tools is limited. The design of methodology for which mathematical paradoxes are integral data collection instruments may offer researchers new insight on learners' conceptions of various mathematical concepts relevant to school and university curricula.

Pertaining to content, my research suggests a need for further investigation into learners' conceptions of both transfinite arithmetic, and the domain-dependence of arithmetic operations. A possible extension of my research includes investigating the specific conceptual challenges associated with the indeterminacy of transfinite subtraction. This study compared the conceptions of participants with different levels of mathematical sophistication and found surprisingly similar approaches to the tasks. I would be interested to see if the same is also true when participants address issues related to transfinite subtraction.

Another extension of my research that is of interest involves learners' conceptions of the expression '$\infty + 1$', as well as the epistemological obstacles associated with that entity. As discussed in Chapter 9, it is possible that learners might see '$\infty + 1$' as the addition of two incompatible summands. If so, an area of investigation relates to how learners cope with such a situation, and if they alter their conceptions of infinity or addition in order to make sense of that sum.

A more whimsical avenue for future research might be an investigation of the coerciveness of certain intuitions. In Chapter 6, for example, the intuition of 'measuring infinity' at times overshadowed the basic intuition of infinity as endless, and it would be interesting to explore in the direction of what contributes to the coerciveness of one intuition over another.

With respect to the domain dependence of arithmetic, this study revealed learners' reluctance to imagine subtraction as dependent on the set of entities upon which it operates. Whether learners' reluctance to consider arithmetic as domain-dependent occurs in other instances as well, such as with arithmetic over the rational numbers or over a polynomial ring, remains an open question that invites future investigation.

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