POLYADIC MODAL LOGICS WITH APPLICATIONS IN NORMATIVE REASONING

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Abstract

The study of modal logic often starts with that of unary operators applied to sentences, denoting some notions of necessity or possibility. However, we adopt a more general approach in this dissertation. We begin with object languages that possess multi-ary modal operators, and interpret them in relational semantics, neighbourhood semantics and algebraic semantics. Some topics on this subject have been investigated by logicians for some time, and we present a survey of their results. But there remain areas to be explored, and we examine them in order to gain more knowledge of our territory. More specifically, we propose polyadic modal axioms that correspond to seriality, reflexivity, symmetry, transitivity and euclideanness of multi-ary relations, and prove soundness and completeness of normal systems based on these axioms. We also put forward polyadic classical systems determined by classes of neighbourhood frames of finite types such as superset-closed frames, quasi-filtroids and filtroids. Equivalences between categories of modal algebras and categories of relational frames and neighbourhood frames are demonstrated. Furthermore some of the systems studied in this dissertation are shown to be translationally equivalent.

While the first part of our study is purely formal, we take a different route in the second part. The multi-ary modal operators, previously interpreted in classes of mathematical structures, are given meanings in ordinary discourse. We read them as modalities in normative thinking, for instance, as the "ought" when we say "you ought to visit your parents, or at least call them if you cannot visit them". A series of polyadic modal logics, called systems of deontic residuation, are proposed. They represent real-life situations involving, for example, normative conflicts and contrary-to-duty obligations better than traditional deontic logics based on unary modal operators do.

Keywords: polyadic modal logic; relational semantics; neighbourhood semantics; translational equivalence; deontic logic; deontic residuation

In memory of my brother

Kam Yuen

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Chapter 0

Introduction

Modern modal logic since C.I. Lewis's *A Survey of Symbolic Logic* (1918) has been predominantly monadic in character: it posits unary modal operators, usually labelled by their intended meanings such as necessity, possibility, impossibility and contingency. The advent of relational semantics in the 1950's has not changed the dominance of the monadic language in the study of modal logic. In spite of some exceptions, most contemporary systems of modal logic (for example, the weakest normal modal system K, and its extensions such as KT and KD) deal with a single unary operator, but there are no reasons, other than simplicity, to insist on this self-imposed constraint on the arity of modal operators. It may be argued, as we do here, that a more general approach is preferred, both from a formal and from an interpretational point of view.

Mathematicians have long been working on functions and relations of finite ranks (i.e. functions taking n arguments and relations consisting of n-tuples, for some finite number n). Logic, as the formal study of reasoning, has had a close relationship with mathematics. (We note here that Boole, in the 19th-century, conceived propositional logic as an algebra of propositions.) Thus viewed, there are good reasons to develop a general theory of modal logic, in which unary operators are merely special cases of multi-ary operators that can take finitely many sentences as their operands. In fact such an approach has already been suggested by Jónsson and Tarski's paper "Boolean Algebras with Operators. Part I" (1951), which shows that every Boolean algebra supplemented with finitary functions satisfying the conditions of normality and additivity can be represented as a subalgebra of the complex algebra of a relational structure. Effectively their work provides a multi-ary relational semantics for the polyadic modal language although its implications for modal logic had not been recognized for some time after the publication of their paper.

But our interest in modal logic is not purely formal. We also want to investigate how modal vocabulary is deployed in ordinary discourse. Modalities used in natural languages are quite often polyadic in character. For example, "X until Y" in discourse about time, and "Obligatorily, if P then Q" when we deliberate on obligation. And there is no reason to limit modal expressions to dyadic ones. Accepted norms of colloquy dictate that we should refrain from being long-winded, but we can easily imagine that an artificial agent (or robot) is perfectly capable of handling iterated constructions such as "Obligatorily, if P then if Q then if ... then R." Indeed, the application of modal logic in formalizing notions of time, obligation, knowledge, etc. (in fields such as computer science, linguistics, economics) often requires a language that is polyadic modal. In other words, there is a need for polyadic modal logic from an interpretational point of view.

The first part of this dissertation (Chapters 1 through 8) is concerned with the formal study of modal logic with polyadic modal languages as our object languages. We study normal systems in the context of relational semantics (Chapters 2, 3 and 4), and classical systems in the context of neighbourhood semantics (Chapter 5). Equivalences between descriptive relational frames and normal modal algebras, and between descriptive neighbourhood frames and modal algebras are presented in Chapters 6 and 7. Moreover some of the polyadic modal systems are shown to be translationally equivalent in Chapter 8. The scope of these chapters is broad. But they far from exhaust the subject-matter of polyadic modal logic. We limit our attention to the n-adic generalizations of monadic formulas that are well known not just for historical reasons but also for the mathematical reason that they correspond to basic relational properties such as reflexivity, symmetry and transitivity. Soundness and completeness of selected polyadic systems are demonstrated in these chapters. However, we have to forgo many other modal formulas such as the Geach formula, and neglect other topics such as decidability and complexity. A thorough investigation of polyadic modal logic requires many more chapters than we could afford in a single dissertation. Our aim here is to make the area accessible to philosophers who are interested in n-ary necessity.

As noted earlier, studying modalities of ordinary discourse is part of our motivation for investigating polyadic modal logic. Accordingly we apply some of the results of the first part to normative reasoning in the second part of this dissertation. More specifically, we provide a survey of modern deontic logic in Chapter 9, and then put forward in Chapter 10 the

n-adic system D_n as a deontic logic, which is extended by principles of deontic residuation. The systems of deontic residuation we propose provide a better formalization of the notion of contrary-to-duty imperatives than the traditional Standard Deontic Logic does. However we do not claim that our systems solve every problem in deontic logic: for example, the occurrence of normative conflicts is allowed in our systems but only if the conflicting obligations are not unshirkable. Nonetheless, the final chapter of this dissertation exemplifies the resources that n-ary necessity can offer to philosophical logicians and philosophers who are willing to make the effort.

Chapter 1

Modal Languages and Set-Theoretic Semantics

We begin our dissertation by introducing the object languages we are going to study. They extend the languages of propositional logic with multiple modal operators, each of which takes finitely many formulas as its arguments. Various semantic idioms for our polyadic multi-modal languages are examined: the relational semantics, the neighbourhood semantics and hybrids of them. We call these semantic idioms "set-theoretic" since evaluations of the truth of formulas according to them are essentially set-theoretic operations.

1.1 Object languages

1.1.1 The languages of propositional logic

To specify a formal language, we need first a set of symbols (called its alphabet), then a set of rules (called its syntax) for concatenating symbols into formulas. The modal languages we are going to study in this dissertation are extensions of the language of propositional logic, the alphabet of which consists of atoms p_n (where n is a non-negative integer), truth-functional connectives $\neg$, $\vee$ and $\bot$, and punctuation marks (and). While most of the time we work with the set of atoms mentioned above, we shall on occasion deal with other sets of atoms (either finite or denumerable). So the notion of the language of propositional logic is generalized to that of a language of propositional logic over a countable set P of atoms.

Definition 1.1.1 (Propositional languages)**.** A *propositional language* over a countable set P of atoms, denoted by $\mathcal{L}(P)$, has the following primitive symbols:

- atoms p, all of which are members of P;
- connectives $\bot$ (falsity), $\neg$ (negation), and $\vee$ (disjunction);
- punctuation marks (and).

Formulas of $\mathcal{L}(P)$ are defined inductively as follows:

- every atom p is a formula;
- $\bot$ is a formula;
- if α is a formula, then so is $\neg\alpha$;
- if α and β are formulas, then so is $(\alpha \vee \beta)$;
- if α is a formula, then it is so in virtue of the above clauses. $\dashv$

The above inductive definition of formulas is often given in a more concise form called the Backus-Naur Form (BNF):

$$\alpha ::= p|\bot|\neg\alpha|(\alpha \vee \alpha),$$

where p ranges over the elements of P. (Note that each occurrence of α to the right of ::= stands for any already constructed formula. So the two occurrences of α in $(\alpha \vee \alpha)$ may be replaced by different formulas. Some authors emphasize this by using expressions such as $(\alpha \vee \beta)$ although this is not strictly required.)

Other familiar truth-functional connectives—$\top$ (truth), $\wedge$ (conjunction), $\rightarrow$ (conditionality), and $\leftrightarrow$ (biconditionality)—are introduced by the following identities.

$$\begin{aligned} \top &= \neg\bot \\ (\alpha \wedge \beta) &= \neg(\neg\alpha \vee \neg\beta) \\ (\alpha \rightarrow \beta) &= (\neg\alpha \vee \beta) \\ (\alpha \leftrightarrow \beta) &= ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)) \end{aligned}$$

In writing formulas of $\mathcal{L}(P)$, we usually omit the outermost parentheses. More parentheses can be dropped without ambiguity by adopting the following rule: among the binary connectives, $\vee$ and $\wedge$ bind more strongly than the others.

We mention here some of the metalinguistic conventions used in this dissertation. English letters p, q, r, ... (with or without subscripts) stand for atoms. Lower case Greek letters α, β, γ, ... (with or without subscripts) denote formulas of the object languages, whereas upper case Greek letters Γ, Δ, Σ, ... denote sets of formulas.

1.1.2 Polyadic modal languages

A propositional modal language (or simply a modal language) extends a propositional language with operators that are characterized as "modal" (so called because they tell us something about the mode in which their operands are true). In this dissertation, we consider not just unary operators, i.e. those that are applied to one formula. Operators that take finite numbers of arguments are also studied. As a matter of convention, specific symbols are used to denote modal operators in some applications of modal logic. For example, in temporal logic the future tense and the past tense modalities are often written as G and H (for "it is always going to be the case" and "it has always been the case", respectively). However when we are studying the general theory of modal languages and logics, more generic symbols are desirable. For that purpose, we use the symbols $\Box_0$, $\Box_1$, ..., $\Box_\xi$, ..., where ξ is an ordinal, for our primitive modal operators. We dub them "squares" or "boxes".

The above preliminary remarks make it clear that in defining a modal language, we need to specify, in addition to the base language, the ordinal which contains the smaller ordinals used to index the modal operators and the number of arguments each operator accepts. This leads us to the notion of a modal type.

Definition 1.1.2 (Modal types)**.** A *modal type* is a pair $\tau = \langle \zeta, \rho \rangle$ where ζ is an ordinal such that $1 \leq \zeta \leq \omega$, and ρ is a function assigning each $\xi < \zeta$ a natural number $\rho(\xi)$. $\dashv$

Definition 1.1.3 (Modal languages)**.** Let $\tau = \langle \zeta, \rho \rangle$ be a modal type and P a set of atoms. The *modal language* of type τ over P, denoted by $\mathcal{L}_\tau(P)$, is the extension of the language $\mathcal{L}(P)$ with modal operators $\Box_\xi$'s where $\xi < \zeta$. Formulas of the language is specified by the following rule:

$$\alpha ::= p|\bot|\neg\alpha|(\alpha \vee \alpha)|\Box_\xi(\underbrace{\alpha, \ldots, \alpha}_{\rho(\xi)\text{ times}}),$$

where p ranges over the elements of P. ⊣

As in the case of truth-functional connectives, we introduce an often-used abbreviation of modality called the duals of "squares" and dubbed "diamonds" as follows.

$$\Diamond_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)}) = \neg\Box_\xi(\neg\alpha_1, \ldots, \neg\alpha_{\rho(\xi)})$$

For a language of modal type $\tau = \langle\zeta, \rho\rangle$, ordinals smaller than ζ are used to index modal operators, and, for each $\xi < \zeta$, the number of arguments $\Box_\xi$ takes is the finite number $\rho(\xi)$. Note that ζ is required to be greater than zero to ensure that there is at least one operator (viz. $\Box_0$) present in every modal language we have occasions to study in this dissertation. Moreover, since $\zeta \leq \omega$, our modal languages contain countably many modal operators and so are themselves countable languages (given that the base language is also countable).

The number of arguments $\Box_\xi$ takes, viz. $\rho(\xi)$, is called the *rank* or *arity* of the modal operator. Operators with arities one, two, three, ..., are often described as unary, binary, ternary, etc. We call modal languages or types with unary operators only "monadic", those with binary operators only "dyadic", those with ternary operators only "triadic", and so on. Observe that we adopt Latin-based prefixes for operators (and functions, relations, etc.) and Greek-based prefixes for languages (and types, logics, etc.).

The definition of modal languages above (Definition 1.1.3) is completely general—it defines a polyadic multimodal language, i.e. a language possibly with multiple modal operators, which may have different arities. However the study of modal logics is greatly simplified by limiting our attention to those based on unimodal languages, those with a single modal operator (which we shall denote by $\Box$ for simplicity). The reason is that results in a unimodal setting can straightforwardly be applied to a multimodal language. In the following we describe, as examples, unimodal languages which we will regularly meet in this dissertation. Note that the set of atoms is suppressed, as will often be the case if it is clear what set of atoms we are working with.

Example 1.1.4 (Polyadic unimodal languages)**.** The simplest unimodal language is the one that has a single unary operator $\Box$. We call it the basic modal language or simply $\mathcal{L}_1$. In general, a modal language with a single n-ary operator $\Box$ is called $\mathfrak{L}_n$. Thus we have the following sequence of modal languages: $\mathfrak{L}_1$, $\mathfrak{L}_2$, $\mathfrak{L}_3$, ..., or in English: monadic, dyadic, triadic, ... (uni)modal languages. $\dashv$

1.2 Semantics for propositional languages

The simplest model for a propositional language is an assignment of truth-value (either T or F) to each atom of the language. However to prepare ourselves for the models of modal propositional languages, we adopt a slightly more complicated approach — we take a propositional model to be a collection of points, at which atoms are either true or false. These points can be taken as possible states of some system we are describing. To specify the points at which an atom is true, we use a function called a valuation. A propositional model is thus a set of points together with a valuation.

Definition 1.2.1 (Propositional models)**.** A *model* for a propositional language $\mathfrak{L}(P)$ is a pair $\langle U, V\rangle$, where

- U, the universe of the model, is a non-empty set of points;
- V, the valuation function of the model, assigns to each atom $p \in P$ a set of points in U. $\dashv$

Definition 1.2.2 (Truth in propositional models)**.** Let $\mathfrak{M} = \langle U, V\rangle$ be a model of a propositional language $\mathfrak{L}(P)$. An $\mathfrak{L}(P)$-formula α is said to be *true* at a point x in $\mathfrak{M}$ (notation: $\mathfrak{M}, x \models \alpha$) according to the following inductive definition, where $\mathfrak{M}, x \not\models \alpha$ means that α is false at x in $\mathfrak{M}$:

- $\mathfrak{M}, x \models p_i$ if $x \in V(p_i)$; otherwise $\mathfrak{M}, x \not\models p_i$.
- $\mathfrak{M}, x \not\models \bot$.
- $\mathfrak{M}, x \models \neg\alpha$ if $\mathfrak{M}, x \not\models \alpha$; otherwise $\mathfrak{M}, x \not\models \neg\alpha$.
- $\mathfrak{M}, x \models (\alpha \vee \beta)$ if either $\mathfrak{M}, x \models \alpha$ or $\mathfrak{M}, x \models \beta$; otherwise $\mathfrak{M}, x \not\models (\alpha \vee \beta)$.

If $\mathfrak{M}, x \models \alpha$ for all $x \in U$, α is said to *hold* in $\mathfrak{M}$ (notation: $\mathfrak{M} \models \alpha$). $\dashv$

Truth conditions for the defined truth-functional connectives can easily be derived:

- $\mathfrak{M}, x \models \top$.
- $\mathfrak{M}, x \models (\alpha \wedge \beta)$ if both $\mathfrak{M}, x \models \alpha$ and $\mathfrak{M}, x \models \beta$; otherwise $\mathfrak{M}, x \not\models (\alpha \wedge \beta)$.
- $\mathfrak{M}, x \models (\alpha \rightarrow \beta)$ if either $\mathfrak{M}, x \not\models \alpha$ or $\mathfrak{M}, x \models \beta$; otherwise $\mathfrak{M}, x \not\models (\alpha \rightarrow \beta)$.
- $\mathfrak{M}, x \models (\alpha \leftrightarrow \beta)$ if *either* both $\mathfrak{M}, x \models \alpha$ and $\mathfrak{M}, x \models \beta$ *or* both $\mathfrak{M}, x \not\models \alpha$ and $\mathfrak{M}, x \not\models \beta$; otherwise $\mathfrak{M}, x \not\models (\alpha \leftrightarrow \beta)$.

The above truth conditions can be recast in set-theoretic language. First we define the notion of the truth-set of a formula in a model.

Definition 1.2.3 (Truth-sets)**.** Let $\mathfrak{M} = \langle U, V \rangle$ be a propositional model for a propositional language $\mathcal{L}(P)$ and α a formula of $\mathcal{L}(P)$. The *truth-set* of α in $\mathfrak{M}$, denoted by $\|\alpha\|^{\mathfrak{M}}$, is the set of points of U at which α is true in $\mathfrak{M}$. ⊣

For a propositional model $\mathfrak{M} = \langle U, V \rangle$, the truth-sets of formulas are as below.

$$\begin{aligned}
\|p_i\|^{\mathfrak{M}} &= V(p_i) \\
\|\bot\|^{\mathfrak{M}} &= \emptyset \\
\|\top\|^{\mathfrak{M}} &= U \\
\|\neg\alpha\|^{\mathfrak{M}} &= U - \|\alpha\|^{\mathfrak{M}} \\
\|\alpha \vee \beta\|^{\mathfrak{M}} &= \|\alpha\|^{\mathfrak{M}} \cup \|\beta\|^{\mathfrak{M}} \\
\|\alpha \wedge \beta\|^{\mathfrak{M}} &= \|\alpha\|^{\mathfrak{M}} \cap \|\beta\|^{\mathfrak{M}} \\
\|\alpha \rightarrow \beta\|^{\mathfrak{M}} &= (U - \|\alpha\|^{\mathfrak{M}}) \cup \|\beta\|^{\mathfrak{M}} \\
\|\alpha \leftrightarrow \beta\|^{\mathfrak{M}} &= \|\alpha \rightarrow \beta\|^{\mathfrak{M}} \cap \|\beta \rightarrow \alpha\|^{\mathfrak{M}}
\end{aligned}$$

Evidently, if α holds in $\mathfrak{M}$ then $\|\alpha\|^{\mathfrak{M}}$ is simply U.

In propositional logic we are not so much interested in truth in a model as truth in *every* model. Put it another way, our interest lies in formulas that are true independently of whether their atoms are true or false rather than in formulas that happen to be true on some assignments of truth values but false on other assignments. So we generalize the notion of truth in a model to the notion of validity in a class of models. Note that the concept of validity and other related ones we are going to define are applicable not just to propositional languages but also to their extensions such as modal languages (and indeed many other formal languages). We indicate this generality by not mentioning any particular language in our definitions.

Definition 1.2.4 (Validity in classes of models)**.** Let $\mathbb{C}$ be a class of models. A formula α is said to be *valid* in $\mathbb{C}$ if it holds in every model in $\mathbb{C}$ (notation: $\models_{\mathbb{C}} \alpha$). If it is valid in the class of all models, we simply say it is valid and write $\models \alpha$.

The class of valid propositional formulas is exactly the class of tautologies, which are formulas true on every assignment of truth values to atoms. In the following we define the important notion of semantic entailment. Intuitively it is the idea that truth of a set of hypotheses guarantees truth of its conclusion.

Definition 1.2.5 (Semantic entailment)**.** A set Σ of formulas is said to *semantically entail* a formula α in a class $\mathbb{C}$ of models (notation: $\Sigma \models_{\mathbb{C}} \alpha$) if for every model $\mathfrak{M}$ in $\mathbb{C}$ and every point x in $\mathfrak{M}$, we have $\mathfrak{M}, x \models \alpha$ whenever $\mathfrak{M}, x \models \sigma$ for every $\sigma \in \Sigma$. If $\mathbb{C}$ is the class of all models, we simply say Σ semantically entails α (notation: $\Sigma \models \alpha$). $\dashv$

Note that α is entailed by the empty set of formulas if and only if it is a valid formula. Thus we write $\models \alpha$ for $\emptyset \models \alpha$.

Although we adopt a particular interpretation of the truth-functional connectives of propositional languages in Definition 1.2.2, it is by no means the only interpretation of them. In general a language may be interpreted in different types of models, and, given the same type of models, it may be interpreted according to different sets of truth conditions. With this in mind, we define the notion of an idiom which allows us to talk about different interpretations of a language.

Definition 1.2.6 (Semantic idioms)**.** A *semantic idiom* $\mathfrak{I}$ for a language is a class $\mathbb{C}$ of models together with a set of truth conditions which collectively defines truth of every formula in a model belonging to $\mathbb{C}$. $\dashv$

Validity and entailment in an idiom are just the same as validity and entailment in a class of models. Thus we write $\models_{\mathfrak{I}} \alpha$ if α is valid in $\mathfrak{I}$, and $\Sigma \models_{\mathfrak{I}} \alpha$ if Σ entails α in $\mathfrak{I}$. As is often the case, the truth theory for a class $\mathbb{C}$ of models is clear in the context, and we revert to the earlier notation, viz. $\models_{\mathbb{C}} \alpha$ and $\Sigma \models_{\mathbb{C}} \alpha$.

1.3 Relational semantics for modal languages

Binary relational semantics is often attributed to Kripke, who published several influential papers in the late 1950's and early 1960's (Kripke (1959, 1963, 1965)). However, as many writers on the history of modern modal logic point out, the idea of using a binary relation to study monadic modal languages had already been nurtured among logicians before the 1960's, for example, Carnap, Meredith, Prior, Smiley, Kanger and Hintikka. (Copeland (2002) provides a survey on the development of possible worlds semantics up to the mid 1960's.) Generalizing binary relational semantics to multi-ary relational semantics is well-known in the literature. See, for instance, Gabbay (1976), Johnston (1976) and Blackburn et al. (2001). The idea of using multi-ary relational structures to analyze polyadic modal languages was already hinted at in Jónsson and Tarski's paper "Boolean Algebras with Operators. Part I" (Jónsson and Tarski (1951)). However, relevance of the paper to modal logic had not been recognized for some time after its publication.

Definition 1.3.1 (Relational models)**.** Let $\tau = \langle \zeta, \rho \rangle$ be a modal type and P a set of atoms. A *relational model* for the language $\mathcal{L}_\tau(P)$ is a triple $\langle U, \mathcal{R}, V \rangle$ where

- U, the universe of $\mathfrak{M}$, is a non-empty set of points;
- $\mathcal{R}$ is a set of relations R_ξ's such that $\xi < \zeta$ and R_ξ is an $(\rho(\xi)+1)$-ary relation on U;
- V is a valuation assigning to each atom p a set $V(p)$ of points. ⊣

Definition 1.3.2 (Truth in relational models)**.** Let $\mathfrak{M} = \langle U, \mathcal{R}, V \rangle$ be a relational model for a modal language $\mathcal{L}_\tau(P)$ where $\tau = \langle \zeta, \rho \rangle$ is a modal type and P a set of atoms. Truth conditions for $\mathcal{L}_\tau(P)$-formulas are those of Definition 1.2.2 plus the following one for modal formulas:

- $\mathfrak{M}, x \models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$ if $\forall y_1, \ldots, y_{\rho(\xi)},\ Rxy_1 \cdots y_{\rho(\xi)} \implies \exists i : \mathfrak{M}, y_i \models \alpha_i$; otherwise $\mathfrak{M}, x \not\models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$. ⊣

Truth condition for $\Diamond_\xi$, the dual of $\Box_\xi$, is thus:

- $\mathfrak{M}, x \models \Diamond_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$ if $\exists y_1, \ldots, y_{\rho(\xi)} : Rxy_1 \cdots y_{\rho(\xi)}$ & $\forall i,\ \mathfrak{M}, y_i \models \alpha_i$; otherwise, $\mathfrak{M}, x \not\models \Diamond_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$.

Recall that in Section 1.1.2 we announce that the most common modal languages we deal with in this dissertation are the unimodal ones: $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, ..., each with a modal operator of rank one, two, three, ..., respectively. We describe below, as an example, their models and the truth conditions for $\Box$ and its dual $\Diamond$.

Example 1.3.3 (Relational models for unimodal languages)**.** Let $\mathcal{L}_n$ be a modal language with a single modal operator $\Box$ of rank n. A relational model for $\mathcal{L}_n$ is a triple $\langle U, R, V\rangle$ where U is a non-empty set of points, R an $(n+1)$-ary relation on U, and V a valuation. Truth conditions for $\Box$ and $\Diamond$ are as follows:

- $\mathfrak{M}, x \models \Box(\alpha_1, \ldots, \alpha_n)$ if $\forall y_1, \ldots, y_n,\ Rxy_1 \cdots y_n \implies \exists i : \mathfrak{M}, y_i \models \alpha_i$;
  otherwise $\mathfrak{M}, x \models \Box(\alpha_1, \ldots, \alpha_n)$.
- $\mathfrak{M}, x \models \Diamond(\alpha_1, \ldots, \alpha_n)$ if $\exists y_1, \ldots, y_n : Rxy_1 \cdots y_n$ & $\forall i,\ \mathfrak{M}, y_i \models \alpha_i$;
  otherwise $\mathfrak{M}, x \not\models \Diamond(\alpha_1, \ldots, \alpha_n)$.

In particular, truth conditions for the $\Box$ and $\Diamond$ of $\mathcal{L}_1$ are the following:

- $\mathfrak{M}, x \models \Box\alpha$ if $\forall y,\ Rxy \implies \mathfrak{M}, y \models \alpha$; otherwise $\mathfrak{M}, x \not\models \Box\alpha$.
- $\mathfrak{M}, x \models \Diamond\alpha$ if $\exists y : Rxy$ & $\mathfrak{M}, y \models \alpha$; otherwise $\mathfrak{M}, x \not\models \Diamond\alpha$.

Truth conditions for the $\Box$ and $\Diamond$ of $\mathcal{L}_2$ are the following:

- $\mathfrak{M}, x \models \Box(\alpha, \beta)$ if $\forall y, z,\ Rxyz \implies (\mathfrak{M}, y \models \alpha$ or $\mathfrak{M}, z \models \beta)$; otherwise $\mathfrak{M}, x \not\models \Box(\alpha, \beta)$.
- $\mathfrak{M}, x \models \Diamond(\alpha, \beta)$ if $\exists y, z : Rxyz$ & $\mathfrak{M}, y \models \alpha$ & $\mathfrak{M}, z \models \beta$; otherwise $\mathfrak{M}, x \not\models \Diamond(\alpha, \beta)$. $\dashv$

We have defined validity and semantic entailment in the context of propositional models. These notions equally apply to relational models, which may be considered as augmentations of propositional models in the same way as modal languages are extensions of propositional languages. However the introduction of relations into the models allows us to consider validity not merely with respect to classes of models, but to classes of "frames" as well.

Definition 1.3.4 (Relational frames)**.** A *relational frame* $\mathfrak{F}$ of type $\tau = \langle \zeta, \rho\rangle$ is a pair $\langle U, \mathcal{R}\rangle$ where U is a non-empty set of points, and $\mathcal{R}$ a set of relations R_ξ's such that $\xi < \zeta$ and R_ξ is a $(\rho(\xi)+1)$-ary relation on U. $\dashv$

A relational model $\mathfrak{M} = \langle U, \mathcal{R}, V\rangle$ for a language of type τ can be considered as a frame $\mathfrak{F} = \langle U, \mathcal{R}\rangle$ of the same type supplemented with the valuation V. We also say that $\mathfrak{M}$ is a model on $\mathfrak{F}$.

Definition 1.3.5 (Validity on frames)**.** Let $\mathfrak{F}$ be a relational frame of type τ and α a formula of a modal language of the same type. α is said to be *valid* on $\mathfrak{F}$ (notation: $\mathfrak{F} \models \alpha$) if α holds in every model on $\mathfrak{F}$. $\dashv$

Definition 1.3.6 (Validity in classes of frames)**.** Let $\mathbb{C}$ be a class of relational frames of type τ and α a formula of a modal language of the same type. α is said to be *valid* in $\mathbb{C}$ (notation: $\models_{\mathbb{C}} \alpha$) if α is valid on every frame in $\mathbb{C}$. If $\mathbb{C}$ is the class of all relational frames, we simply say α is valid and write $\models \alpha$. ⊣

Semantic entailment can be defined with respect to classes of frames instead of classes of models. Note that preservation of truth is still local, i.e. Σ entails α in a class $\mathbb{C}$ of frames if and only if at any point in any model on any frame belonging to $\mathbb{C}$, truth of all formulas of Σ implies truth of α.

1.4 Neighbourhood semantics for modal languages

In the 1960's, neighbourhood semantics (for monadic modal languages) was developed independently by Montague and Scott (see Section 8, Chapter 1 of Segerberg (1971)). However, the most detailed development of the semantics and its application to the study of modal logic is perhaps Segerberg (1971). However, the generalization of neighbourhood models to interpret polyadic modal languages seems not to have been investigated in the literature (as far as the author knows).

Definition 1.4.1 (Neighbourhood models)**.** Let $\tau = \langle \zeta, \rho \rangle$ be a modal type and P a set of atoms. A *neighbourhood model* $\mathfrak{M}$ for the language $\mathcal{L}_\tau(P)$ is a triple $\langle U, \mathcal{N}, V \rangle$ where U and V are as in Definition 1.2.1, and $\mathcal{N}$ is a set of neighbourhood functions N_ξ's such that $N_\xi : U \to \mathscr{P}((\mathscr{P}(U))^{\rho(\xi)})$ for each $\xi < \zeta$. In other words, for each operator $\Box_\xi$, we have a neighbourhood function N_ξ mapping each element of U to a collection of $\rho(\xi)$-tuples of sets of points of U. N_ξ is said to be of type $\rho(\xi)$. ⊣

Definition 1.4.2 (Truth in neighbourhood models)**.** Let $\mathfrak{M} = \langle U, \mathcal{N}, V \rangle$ be a neighbourhood model of a modal language $\mathcal{L}_\tau(P)$ where $\tau = \langle \zeta, \rho \rangle$ is a modal type and P is a set of atoms. Truth conditions for $\mathcal{L}_\tau(P)$-formulas are those of Definition 1.2.2 plus the following:

- $\mathfrak{M}, x \models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$ if $\langle \|\alpha_1\|^{\mathfrak{M}}, \ldots, \|\alpha_{\rho(\xi)}\|^{\mathfrak{M}} \rangle \in N_\xi(x)$; otherwise $\mathfrak{M}, x \not\models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$. ⊣

Truth condition for $\Diamond_\xi$, the dual of $\Box_\xi$, can easily be derived as follows:

- $\mathfrak{M}, x \models \Diamond_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$ if $\langle \|\neg\alpha_1\|^{\mathfrak{M}}, \ldots, \|\neg\alpha_{\rho(\xi)}\|^{\mathfrak{M}} \rangle \notin N_\xi(x)$; otherwise $\mathfrak{M}, x \not\models \Diamond_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$.

Definition 1.4.3 (Neighbourhood frames)**.** A *neighbourhood frame* $\mathfrak{F}$ of type $\tau = \langle \zeta, \rho \rangle$ is a pair $\langle U, \mathcal{N} \rangle$ where U is a non-empty set of points, and $\mathcal{N}$ is a set of neighbourhood functions as in Definition 1.4.1. ⊣

Validity on a neighbourhood frame are defined as for validity in the relational idiom. We provide below, as examples, neighbourhood models for unimodal languages.

Example 1.4.4 (Neighbourhood models for the monadic unimodal language)**.** Recall that $\mathcal{L}_1$ is the modal language that has a single monadic modal operator $\Box$. A neighbourhood model $\mathfrak{M}$ for $\mathcal{L}_1$ is a triple $\langle U, N, V \rangle$ where

- U is a non-empty set of points,
- $N : U \to \mathscr{P}(\mathscr{P}(U))$, i.e. N assigns to each point a collection of sets of points, and
- V is a valuation, i.e. V assigns to each atom a set of points.

Truth conditions for $\Box$ and $\Diamond$ are stated thus:

- $\mathfrak{M}, x \models \Box\alpha$ if $\|\alpha\|^{\mathfrak{M}} \in N(x)$; otherwise $\mathfrak{M}, x \not\models \Box\alpha$.
- $\mathfrak{M}, x \models \Diamond\alpha$ if $\|\neg\alpha\|^{\mathfrak{M}} \notin N(x)$; otherwise $\mathfrak{M}, x \not\models \Diamond\alpha$. ⊣

Example 1.4.5 (Neighbourhood models for polyadic unimodal languages)**.** Recall that $\mathcal{L}_n$ is the modal language with a single modal operator $\Box$ of rank n ($n \geq 1$). A neighbourhood model $\mathfrak{M}$ for $\mathcal{L}_n$ is a triple $\langle U, N, V \rangle$ where

- U is a non-empty set of points,
- $N : U \to \mathscr{P}((\mathscr{P}(U))^n)$, i.e. N assigns to each point a collection of n-tuples of sets of points, and
- V is a valuation, i.e. V assigns to each atom a set of points.

Truth conditions for $\Box$ and $\Diamond$ are as follows:

- $\mathfrak{M}, x \models \Box(\alpha_1, \ldots, \alpha_n)$ if $\langle \|\alpha_1\|^{\mathfrak{M}}, \ldots, \|\alpha_n\|^{\mathfrak{M}} \rangle \in N(x)$;
  otherwise $\mathfrak{M}, x \not\models \Box(\alpha_1, \ldots, \alpha_n)$.
- $\mathfrak{M}, x \models \Diamond(\alpha_1, \ldots, \alpha_n)$ if $\langle \|\neg\alpha_1\|^{\mathfrak{M}}, \ldots, \|\neg\alpha_n\|^{\mathfrak{M}} \rangle \notin N(x)$;
  otherwise $\mathfrak{M}, x \not\models \Diamond(\alpha_1, \ldots, \alpha_n)$. ⊣

1.5 Hybrids of relational and neighbourhood semantics

In this section, we introduce two types of models (called prenormal and non-normal) for analyzing modal languages. They are characterized as hybrids of relational and neighbourhood semantics introduced in Sections 1.3 and 1.4. The universe of a prenormal model is divided into normal points and non-normal (or queer) points, with a relational component for the normal points and a neighbourhood component for the non-normal points. While at normal points modal formulas are evaluated as in a relational model, at non-normal points they are evaluated as in a neighbourhood model. A non-normal model is simply a prenormal model with the output of its neighbourhood function always being the empty set. Moreover, relational models can be considered as special cases of non-normal models, viz. those with normal points only. Although prenormal and non-normal models do not play any significant role in this dissertation (except Section 8.3), we include them here for general interest.

The notion of prenormal models is based on the semantics used in Chellas and Segerberg (1996) to study what the authors call “prenormal logics” (which are monadic modal systems). Chellas and Segerberg’s semantics is a recast and generalization of that developed in Cresswell (1972) for the study of Lewis system S1. The use of non-normal models (for monadic modal languages) can be traced to Kripke (1965), which deals with Lewis systems S2 and S3 and Lemmon’s E2 and E3. (We note here that prenormal semantics for monadic modal languages is used in Leung and Jennings (2005) to study weak modal systems in the vicinity of S1.)

Definition 1.5.1 (Prenormal models)**.** Let $\tau = \langle \zeta, \rho \rangle$ be a modal type and P a set of atoms. A *prenormal model* $\mathfrak{M}$ for the language $\mathcal{L}_\tau(P)$ is a quintuple $\langle U, Q, \mathcal{R}, \mathcal{N}, V \rangle$ where

- U is a non-empty set of points;
- Q is a subset of U (the elements of which are called non-normal or queer points);
- $\mathcal{R}$ is a collection of relations R_ξ’s where $\xi < \zeta$ such that $R_\xi \subseteq (U - Q) \times U^{\rho(\xi)}$;
- $\mathcal{N}$ is a collection of neighbourhood functions N_ξ’s where $\xi < \zeta$ such that $N_\xi : Q \to \mathscr{P}((\mathscr{P}(U))^{\rho(\xi)})$ satisfying the condition that $\langle a_1, \ldots, a_{i-1}, U, a_{i+1}, \ldots, a_n \rangle \notin N_\xi(x)$ for every $i \leq n$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \subseteq U$ and $x \in Q$;
- V is a valuation assigning to each atom a set of points. ⊣

Definition 1.5.2 (Truth in prenormal models)**.** Let $\mathfrak{M} = \langle U, Q, \mathcal{R}, \mathcal{N}, V\rangle$ be a prenormal model for a modal language $\mathcal{L}_\tau(P)$. Truth conditions for $\mathcal{L}_\tau(P)$-formulas are those of Definition 1.2.2 plus the following, where $\Box_\xi$ is an operator belonging to type τ and $\rho(\xi)$ is its rank:

- For $x \notin Q$: $\mathfrak{M}, x \models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$ if $\forall y_1, \ldots, y_{\rho(\xi)} \in U,\ R_\xi x y_1 \cdots y_{\rho(\xi)} \implies \exists i : \mathfrak{M}, y_i \models \alpha_i$; otherwise $\mathfrak{M}, x \not\models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$.
- For $x \in Q$: $\mathfrak{M}, x \models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$ if $\langle \|\alpha_1\|^{\mathfrak{M}}, \ldots, \|\alpha_{\rho(\xi)}\|^{\mathfrak{M}}\rangle \in N_\xi(x)$; otherwise $\mathfrak{M}, x \not\models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$. ⊣

Definition 1.5.3 (Non-normal models)**.** Let $\tau = \langle \zeta, \rho\rangle$ be a modal type and P a set of atoms. A *non-normal model* $\mathfrak{M}$ for the language $\mathcal{L}_\tau(P)$ is a quadruple $\langle U, Q, \mathcal{R}, V\rangle$ where U, Q, $\mathcal{R}$ and V are as in Definition 1.5.1. ⊣

Definition 1.5.4 (Truth in non-normal models)**.** Let $\mathfrak{M} = \langle U, Q, \mathcal{R}, V\rangle$ be a non-normal model of a modal language $\mathcal{L}_\tau(P)$. Truth conditions for $\mathcal{L}_\tau(P)$-formulas are those of Definition 1.2.2 plus the following, where $\Box_\xi$ is an operator belonging to type τ and $\rho(\xi)$ is its rank:

- For $x \notin Q$: $\mathfrak{M}, x \models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$ if $\forall y_1, \ldots, y_{\rho(\xi)} \in U,\ R_\xi x y_1 \cdots y_{\rho(\xi)} \implies \exists i : \mathfrak{M}, y_i \models \alpha_i$; otherwise $\mathfrak{M}, x \not\models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$.
- For $x \in Q$: $\mathfrak{M}, x \not\models \Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)})$. ⊣

Chapter 2

From Propositional Logic to Normal Modal Systems

Our study of polyadic modal logic begins with the so-called normal systems. As we shall see, these systems are closely related to the relational semantics introduced in Section 1.3. The organization of this chapter is as follows. We begin with some remarks on deductive systems and semantic idioms, followed by a discussion of the classical propositional logic, which serves as the base of our modal systems. Since languages with unary modal operators are the simplest modal languages, we study monadic normal systems first, then generalize them to systems in polyadic modal languages. Only the smallest of these polyadic systems, which we call K_n, is presented in this chapter, while extensions of K_n will be examined in the next two chapters. The systems that appear in this chapter are well-known in the literature. Therefore in most cases proofs of meta-theorems are omitted. (Standard references in this area are Hughes and Cresswell (1996) and Chellas (1980). For more recent exposition of the subject, see Chagrov and Zakharyaschev (1997) and Blackburn et al. (2001).)

2.1 Logics: syntax and semantics

2.1.1 Logics as deductive systems

In logical enquiry we are interested in finding out what sentences (conclusions) follow from a given set of sentences (hypotheses or assumptions). In arriving at the conclusions, we allow ourselves to make use of, beside the hypotheses, some sentences (axioms or postulates) which we accept unconditionally, and some rules of deduction which, like the axioms, are accepted as being correct without substantiation. The primitive axioms and rules constitute a system, on the basis of which we define the notion of deduction.

Definition 2.1.1 (Formal systems)**.** A *formal system* S in some object language $\mathcal{L}$ consists of a decidable set of $\mathcal{L}$-formulas, called the axioms of the system, and a set of reasonable rules, each of which specifies a formula as the output for a set of formulas. $\dashv$

Note that the set of axioms must be decidable, and the rules must be reasonable. We will not give precise definitions of decidability and reasonableness here, but roughly speaking a set of formulas is decidable if there is an algorithm that provides us the correct answer, in finite time, to the question as to whether a given formula belongs to the set or not. Similarly a rule is reasonable if we have an algorithm to check, in finite time, whether a given formula follows or not from a given set of formulas according to the rule. Note that a formal system is always defined in the context of some object language. It would be tedious, however, to repeat this fact every time we say something about a formal system. So henceforward we shall be silent in the matter of the object language.

Definition 2.1.2 (Deducibility)**.** A formula α is said to be *deducible* in S from a set Σ of formulas (notation: $\Sigma \vdash_{\mathrm{S}} \alpha$) if there exists a finite sequence of formulas $\beta_1, \ldots, \beta_n$ with the last member β_n being α and each β_i (where $1 \leq i \leq n$) satisfying one of the following conditions:

(1) β_i belongs to Σ.

(2) β_i is an axiom of S.

(3) β_i is the output of a rule of S for some previous formula(s) in the sequence. $\dashv$

An alternative term for "deduction" is "proof": if α is deducible in S from Σ, we also say that α is provable in S from Σ, and call the (finite) sequence of formulas in the deduction a *proof*. Sometimes the term "logical consequence" is used: α is a logical consequence of Σ in S if α is deducible from Σ in S. We also adopt the more compact expressions such as S-*deducible* etc. If the system is understood, we simply say that α is deducible from Σ and write $\Sigma \vdash \alpha$. The next definition introduces the class of theorems, those formulas that can be deduced without assumption.

Definition 2.1.3 (Theoremhood)**.** Let S be a formal system. A formula α is said to be a *theorem* of S or simply an S-theorem if it is deducible in S from the empty set, i.e. $\emptyset \vdash_{\mathrm{S}} \alpha$. If α is an S-theorem, we also write $\vdash_{\mathrm{S}} \alpha$. ⊣

The term "logic", when applied to particular logics rather than the study of such entities, is often used interchangeably in the literature with the term "system". However we distinguish between these two terms in this dissertation. Whereas formal systems have already been defined in Definition 2.1.1, logics are defined below.

Definition 2.1.4 (Logics)**.** A *logic* Λ is a set of formulas that is closed under a collection of rules. In other words, if α is the output for $\beta_1, \ldots, \beta_n$ according to one of the rules, and $\beta_1, \ldots, \beta_n$ are in Λ, then α is also in Λ. ⊣

Obviously the set of theorems of a system is closed under its rules. Accordingly a formal system determines a logic. However the reverse need not hold. If we could form a system by taking all the formulas belonging to a logic as axioms and all the rules of the logic as its primitive rules, then trivially the theorems of the resulting system would coincide with the logic. However there is no guarantee that the set of formulas that is the logic is decidable. In other words, there are logics that cannot be identified with the set of theorems of any system. We describe such logics as *unaxiomatizable*. Extending the notion of logic to include unaxiomatizable sets has the benefit of bringing them into the purview of logical enquiry.

If two systems yield the same set of theorems, they are called *equivalent axiomatizations* of the same logic. In fact we often treat them as if they were the same object. A system is said to *provide* a formula or a rule if the formula is among its axioms or theorems, or the rule is primitive or derivable in the system. The opposite is that a system *lacks* the formula or the rule. If a system S_2 provides all the axioms and rules of another system S_1, we say that S_2 is an *extension* of S_1, or, more concisely, S_2 is an S_1-system. On the other hand, if the set of theorems of S_1 is included in the set of theorems of S_2, i.e. if every S_1-theorem is an S_2-theorem, then S_1 is said to be *included* in S_2. Notice that if S_2 extends S_1, then S_1 is included in S_2. However the mere inclusion of one system in another system is insufficient for the latter to be an extension of the former. The reason is that the latter system may lack some of the rules of the former system.

2.1.2 Logics and semantic idioms

Recall that an object language is interpreted in an idiom, which comprises a class of models and a collection of truth conditions. Apparently we want the systems or logics we define to be correct with respect to the interpretation we intend for the object language. Put it another way, we require the theorems of a system to be valid in the intended idiom, and, more generally, any deduction in the system to be truth-preserving in the idiom we have chosen for the language. However we want something more than that from our systems or logics. Not only should they provide *only* valid theorems and truth-preserving deductions, but they should also give us *all* of the valid theorems and truth-preserving deductions. These considerations give rise to the following notions of soundness and completeness.

Definition 2.1.5 (Soundness)**.** A system S is *strongly sound* with respect to an idiom $\mathfrak{I}$ if for every set of formulas Σ and every formula α,

$$\Sigma \vdash_{\mathrm{S}} \alpha \implies \Sigma \models_{\mathfrak{I}} \alpha.$$

S is *weakly sound* with respect to $\mathfrak{I}$ if for every formula α,

$$\vdash_{\mathrm{S}} \alpha \implies \models_{\mathfrak{I}} \alpha. \qquad \dashv$$

Definition 2.1.6 (Completeness)**.** A system S is *strongly complete* with respect to an idiom $\mathfrak{I}$ if for every set of formulas Σ and every formula α,

$$\Sigma \models_{\mathfrak{I}} \alpha \implies \Sigma \vdash_{\mathrm{S}} \alpha.$$

S is *weakly complete* with respect to $\mathfrak{I}$ if for every formula α,

$$\models_{\mathfrak{I}} \alpha \implies \vdash_{\mathrm{S}} \alpha. \qquad \dashv$$

Definition 2.1.7 (Determination)**.** A system S is *strongly determined* by an idiom $\mathfrak{I}$ if it is both strongly sound and strongly complete with respect to $\mathfrak{I}$, i.e. for every set of formulas Σ and every formula α,

$$\Sigma \vdash_{\mathrm{S}} \alpha \iff \Sigma \models_{\mathfrak{I}} \alpha.$$

S is *weakly determined* by $\mathfrak{I}$ if it is both weakly sound and weakly complete with respect to $\mathfrak{I}$, i.e. for every formula α,

$$\vdash_{\mathrm{S}} \alpha \iff \models_{\mathfrak{I}} \alpha. \qquad \dashv$$

In this dissertation, we prove strong soundness and completeness (hence strong determination) of the systems we consider. Henceforth, "soundness", "completeness" and "determination" mean the strong versions of the respective notions. We mention here that if a system S satisfies a deduction theorem (e.g. PL and K_n, see Theorems 2.2.2 and 2.4.4), strong soundness and weak soundness collapse.

As we have stated earlier, the set of truth conditions for a given class of models or frames are typically fixed. Hence we need only mention the class of models or frames when referring to an idiom. This follows the usual practice of defining soundness and completeness with reference to a class of models or frames rather than to an idiom. In other words, instead of saying that a system is sound (or complete) with respect to an idiom, we simply say that it is sound (or complete) with respect to a class of frames or models, assuming that the reader already knows what the truth conditions are. For instance, strong determination of Definition 2.1.7 can be rephrased as follows: a system S is strongly determined by a class $\mathbb{C}$ of models or frames if for every set of formulas Σ and every formula α,

$$\Sigma \vdash_{\mathrm{S}} \alpha \iff \Sigma \models_{\mathbb{C}} \alpha.$$

2.2 Propositional Logic and its extensions

The modal systems we are going to study in this dissertation are extensions of classical propositional logic (PL), the system that axiomatizes the set of propositional formulas valid in the class of all propositional models. The set of valid propositional formulas coincides with the set of tautologies, formulas that are true on any assignment of truth-values to their atoms. For simplicity we take the set of tautologies for the set of axioms in our following definition of PL.

Definition 2.2.1. *Propositional Logic* (PL) in a propositional language $\mathcal{L}(P)$ has all of the tautologies as its axioms and the following two rules, known as *modus ponens* and *uniform substitution*:

$$[\mathrm{MP}] \quad \frac{\alpha, \alpha \to \beta}{\alpha}$$

$$[\mathrm{US}] \quad \frac{\vdash \alpha}{\vdash \alpha[p_i/\beta]}$$

where $\alpha[p_i/\beta]$ is the formula that results from substituting β for every occurrence of p_i in α. $\dashv$

Theorem 2.2.2. *Deducibility in* PL *has the following properties.*

(1) *(Finiteness) If* $\Sigma \vdash_{\mathrm{PL}} \alpha$*, then there is a finite subset* Σ' *of* Σ *such that* $\Sigma' \vdash_{\mathrm{S}} \alpha$*.*

(2) *(Monotonicity)* $\Sigma \vdash_{\mathrm{PL}} \alpha$*, then for any set of formulas* Σ'*,* $\Sigma \cup \Sigma' \vdash_{\mathrm{PL}} \alpha$*.*

(3) *(The deduction theorem,* [DT]*) If* $\Sigma \cup \{\alpha\} \vdash_{\mathrm{PL}} \beta$*, then* $\Sigma \vdash_{\mathrm{PL}} \alpha \to \beta$*.*

(4) *(The rule of replacement of (provable) equivalents,* [RRE]*) If* $\vdash_{\mathrm{PL}} \alpha \leftrightarrow \beta$ *and* $\vdash_{\mathrm{PL}} \gamma$*, then* $\vdash_{\mathrm{PL}} \gamma'$ *where* γ' *is the formula resulting from replacing some (possibly zero) occurrence of* α *in* γ *with an occurrence of* β*.*

Finiteness and monotonicity follow from the definition of deducibility, and so hold for deducibility in any formal system, not just PL. Note that the deduction theorem ([DT]) and the rule of replacement of equivalents ([RRE]) need not hold for every extension of PL. Although the modal systems we are going to study have these properties as well, they are results to be established separately. The following notions of consistency and maximal consistency are general and apply to any PL-system, including PL itself.

Definition 2.2.3. Let S be a PL-system.

(1) A set of formulas Σ is S-*consistent* if $\Sigma \nvdash_{\mathrm{S}} \bot$. Otherwise, Σ is S-*inconsistent*.

(2) A set of formulas Σ is *maximal* S-*consistent* if it is S-consistent, and, for any formula $\alpha \notin \Sigma$, $\Sigma \cup \{\alpha\}$ is S-inconsistent.

(3) The S-*proof set* of a formula α (notation: $|\alpha|_{\mathrm{S}}$) is the set of all the maximal S-consistent sets of formulas containing α.

(4) $\Box^{-}(\Sigma)$ is the set $\{\alpha | \Box\alpha \in \Sigma\}$ where Σ is a set of formulas. $\dashv$

Theorem 2.2.4. *Let* S *be a* PL*-system.*

(1) *If* Σ *is S-inconsistent, then for any formula* α*, we have* $\Sigma \vdash_{\mathrm{S}} \alpha$*.*

(2) *(The Extension Theorem or Lindenbaum's Lemma) If* Σ *is* S*-consistent, then there exists a set* Σ' *of formulas such that* Σ' *is maximal* S*-consistent and* $\Sigma \subseteq \Sigma'$*.*

(3) *(Deductive Closure) If* Σ *is maximal* S*-consistent and* $\Sigma \vdash_{\mathrm{S}} \alpha$*, then* $\alpha \in \Sigma$*.*

(4) *If* Σ *is maximal* S*-consistent, then the set of* S*-theorems is a subset of* Σ*.*

(5) *If* α *is not a theorem of* S*, then there is some maximal* S*-consistent set of which* α *is not an element.*

(6) *If* Σ *is maximal* S*-consistent, then exactly one of* α *and* $\neg\alpha$ *is an element of* Σ*.*

(7) *If* Σ *is maximal* S*-consistent, then* $\alpha \to \beta \in \Sigma$ *iff* $\alpha \notin \Sigma$ *or* $\beta \in \Sigma$*.*

2.3 Normal monadic systems

We begin our study of modal logic with what have commonly been called normal systems. Monadic languages are the simplest among the polyadic modal languages. So we start with normal monadic systems, from which we generalize to normal polyadic systems in the next section. Some formulas and rules pertaining to monadic systems are listed below.

[RE] $\dfrac{\vdash \alpha \leftrightarrow \beta}{\vdash \Box\alpha \leftrightarrow \Box\beta}$ [RE◊] $\dfrac{\vdash \alpha \leftrightarrow \beta}{\vdash \Diamond\alpha \leftrightarrow \Diamond\beta}$

[RM] $\dfrac{\vdash \alpha \to \beta}{\vdash \Box\alpha \to \Box\beta}$ [RM◊] $\dfrac{\vdash \alpha \to \beta}{\vdash \Diamond\alpha \to \Diamond\beta}$

[RR] $\dfrac{\vdash \alpha \wedge \beta \to \gamma}{\vdash \Box\alpha \wedge \Box\beta \to \Box\gamma}$ [RR◊] $\dfrac{\vdash \alpha \to \beta \vee \gamma}{\vdash \Diamond\alpha \to \Diamond\beta \vee \Diamond\gamma}$

[RK] $\dfrac{\vdash \alpha_1 \wedge \cdots \wedge \alpha_m \to \beta}{\vdash \Box\alpha_1 \wedge \cdots \wedge \Box\alpha_m \to \Box\beta}$ $(m \geq 0)$ [RK◊] $\dfrac{\vdash \alpha \to \beta_1 \vee \cdots \vee \beta_m}{\vdash \Diamond\alpha \to \Diamond\beta_1 \vee \cdots \vee \Diamond\beta_m}$ $(m \geq 0)$

[RN] $\dfrac{\vdash \alpha}{\vdash \Box\alpha}$ [RN◊] $\dfrac{\vdash \neg\alpha}{\vdash \neg\Diamond\alpha}$

[M] $\Box(p \wedge q) \to \Box p \wedge \Box q$ [M◊] $\Diamond p \vee \Diamond q \to \Diamond(p \vee q)$

[C] $\Box p \wedge \Box q \to \Box(p \wedge q)$ [C◊] $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$

[R] $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$ [R◊] $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$

[K] $\Box(p \to q) \to (\Box p \to \Box q)$ [K◊] $\neg\Diamond p \wedge \Diamond q \to \Diamond(\neg p \wedge q)$

[N] $\Box\top$ [N$\Diamond$] $\neg\Diamond\bot$

Definition 2.3.1 (Normal monadic systems)**.** A system in the monadic modal language $\mathfrak{L}_1$ is called *normal* if it provides, in addition to PL, rules [RM], [RN], and axiom [C]. ⊣

Definition 2.3.2. The weakest normal system is called K (after Kripke). It consists of the following axioms and rules.

K : PL, [RM], [RN], [C] ⊣

Other ways to characterize normal systems are as below:

- PL, [RN] and [K].
- PL and [RK].

Note that every normal monadic system has the formulas and rules listed earlier in this section.

2.4 Normal polyadic systems

In this section, we generalize normal monadic systems to normal n-adic systems where n is a positive integer. To simplify presentation of polyadic modal rules and principles, we adopt shorthands as follows.

Notation 2.4.1. Where n is a positive integer and $1 \leq i, k \leq n$,

- Instead of the longer $1 \leq i \leq n$, we write simply i.
- In formulas such as $\Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \to \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n)$, the formula β occurs at the i-th place as α_i does.
- $\vec{p}$ stands for the n-termed sequence $p_1, p_2, \ldots, p_n$.
- $\top^k$ stands for a k-termed sequence of $\top$'s. Similarly for $\bot^k$. ⊣

Polyadic modal rules and formulas pertaining to normal polyadic systems are listed below.

[RE_n^i] $$\frac{\vdash \alpha \leftrightarrow \beta}{\vdash \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \leftrightarrow \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n)}$$

[RM_n^i] $$\frac{\vdash \alpha_i \to \beta}{\vdash \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \to \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n)}$$

[RR_n^i] $$\frac{\vdash \alpha_i \wedge \beta \to \gamma}{\vdash \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \wedge \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n) \to \Box(\alpha_1, \ldots, \gamma, \ldots, \alpha_n)}$$

[RK_n^i] $$\frac{\vdash \alpha_i^1 \wedge \cdots \wedge \alpha_i^m \to \beta}{\vdash \Box(\alpha_1, \ldots, \alpha_i^1, \ldots, \alpha_n) \wedge \cdots \wedge \Box(\alpha_1, \ldots, \alpha_i^m, \ldots, \alpha_n) \to \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n)}$$

$(m \geq 0)$

[RN_n^i] $$\frac{\vdash \alpha_i}{\vdash \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)}$$

[M_n^i] $\Box(p_1, \ldots, p_i \wedge q, \ldots, p_n) \to \Box(p_1, \ldots, p_i, \ldots, p_n) \wedge \Box(p_1, \ldots, q, \ldots, p_n)$

[C_n^i] $\Box(p_1, \ldots, p_i, \ldots, p_n) \wedge \Box(p_1, \ldots, q, \ldots, p_n) \to \Box(p_1, \ldots, p_i \wedge q, \ldots, p_n)$

[R_n^i] $\Box(p_1, \ldots, p_i \wedge q, \ldots, p_n) \leftrightarrow \Box(p_1, \ldots, p_i, \ldots, p_n) \wedge \Box(p_1, \ldots, q, \ldots, p_n)$

[K_n^i] $\Box(p_1, \ldots, p_i \to q, \ldots, p_n) \to (\Box(p_1, \ldots, p_i, \ldots, p_n) \to \Box(p_1, \ldots, q, \ldots, p_n))$

[N_n^i] $\Box(p_1, \ldots, \top, \ldots, p_n)$

We list below dual forms of the above rules and formulas for reference.

[$\mathrm{RE}\Diamond_n^i$] $$\frac{\vdash \alpha_i \leftrightarrow \beta}{\vdash \Diamond(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \leftrightarrow \Diamond(\alpha_1, \ldots, \beta, \ldots, \alpha_n)}$$

[$\mathrm{RM}\Diamond_n^i$] $$\frac{\vdash \alpha_i \to \beta}{\vdash \Diamond(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \to \Diamond(\alpha_1, \ldots, \beta, \ldots, \alpha_n)}$$

[$\mathrm{RR}\Diamond_n^i$] $$\frac{\vdash \alpha_i \to \beta \vee \gamma}{\vdash \Diamond(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \to \Diamond(\alpha_1, \ldots, \beta, \ldots, \alpha_n) \vee \Diamond(\alpha_1, \ldots, \gamma, \ldots, \alpha_n)}$$

[$\mathrm{RK}\Diamond_n^i$] $$\frac{\vdash \alpha_i \to \beta_i^1 \vee \cdots \vee \beta_i^m}{\vdash \Diamond(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \to \Diamond(\alpha_1, \ldots, \beta_i^1, \ldots, \alpha_n) \vee \cdots \vee \Diamond(\alpha_1, \ldots, \beta_i^m, \ldots, \alpha_n)}$$

$(m \geq 0)$

$$[\mathrm{RN}\Diamond_n^i] \quad \frac{\vdash \neg\alpha_i}{\vdash \neg\Diamond(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)}$$

$$[\mathrm{M}\Diamond_n^i] \quad \Diamond(p_1, \ldots, p_i, \ldots, p_n) \vee \Diamond(p_1, \ldots, q, \ldots, p_n) \to \Diamond(p_1, \ldots, p_i \vee q, \ldots, p_n)$$

$$[\mathrm{C}\Diamond_n^i] \quad \Diamond(p_1, \ldots, p_i \vee q, \ldots, p_n) \to \Diamond(p_1, \ldots, p_i, \ldots, p_n) \vee \Diamond(p_1, \ldots, q, \ldots, p_n)$$

$$[\mathrm{R}\Diamond_n^i] \quad \Diamond(p_1, \ldots, p_i \vee q, \ldots, p_n) \leftrightarrow \Diamond(p_1, \ldots, p_i, \ldots, p_n) \vee \Diamond(p_1, \ldots, q, \ldots, p_n)$$

$$[\mathrm{K}\Diamond_n^i] \quad \neg\Diamond(p_1, \ldots, p_i, \ldots, p_n) \to (\Diamond(p_1, \ldots, q, \ldots, p_n) \to \Diamond(p_1, \ldots, \neg p_i \wedge q, \ldots, p_n))$$

$$[\mathrm{N}\Diamond_n^i] \quad \neg\Diamond(\alpha_1, \ldots, \bot, \ldots, \alpha_n)$$

Note that the above rules and formulas are specified in the form of schemas. For instance $[\mathrm{RM}_n^i]$, where $1 \leq i \leq n$, consists of n instances of the given schematic form. We refer to the instances collectively by $[\mathrm{RM}_n]$. Parallel nomenclature is used for other schemas of rules and formulas.

Definition 2.4.2 (Normal n-adic systems)**.** A system in the n-adic modal language $\mathcal{L}_n$ is called *normal* if it provides, in addition to PL, rules $[\mathrm{RM}_n]$, $[\mathrm{RN}_n]$ and axioms $[\mathrm{C}_n]$. ⊣

Definition 2.4.3. The weakest normal n-adic system is called K_n. It consists of the following axioms and rules.

$$\mathrm{K}_n : \mathrm{PL}, \quad [\mathrm{RM}_n], \quad [\mathrm{RN}_n], \quad [\mathrm{C}_n] \qquad \dashv$$

As in the case of monadic systems, there are other ways to characterize normal n-adic systems:

- PL, $[\mathrm{RN}_n]$ and $[\mathrm{K}_n]$.
- PL and $[\mathrm{RK}_n]$.

We have called the weakest normal n-adic system K_n. Note that K_1 is just K. Naming of the weakest normal n-adic system is not universally agreed. Bell (1996) calls it in the same way as we do here, whereas Blackburn et al. (2001) call it K_τ where τ is a modal similarity type. Other names have also been used: $E^{[n]}$ (E for entailment) in Gabbay (1976), and G_n in Johnston (1976). (Johnston names the system after Goldblatt for his introducing what amounts to G_2 in an unpublished paper “Temporal Betweenness”.)

Theorem 2.4.4 (Deduction Theorem for K_n)**.** *Let Σ be a set of $\mathfrak{L}_n$-formulas, and let α, β be $\mathfrak{L}_n$-formulas. If $\Sigma \cup \{\alpha\} \vdash_{\mathrm{K}_n} \beta$, then $\Sigma \vdash_{\mathrm{K}_n} \alpha \to \beta$.*

Proof. The proof is along the same lines of the proof of the deduction theorem for PL. We assume that $\Sigma \cup \{\alpha\} \vdash_{\mathrm{K}_n} \beta$, i.e. there is a K_n-proof of β from $\Sigma \cup \{\alpha\}$ consisting of a sequence of formulas $\gamma_1, \ldots, \gamma_k, \ldots, \gamma_m$ such that γ_m is β, and show by induction on k that there is a K_n-proof of $\alpha \to \gamma_m$ from Σ.

For the basis of the induction, we consider the following possibilities: γ_1 is an axiom of K_n, a member of Σ, or α itself. The cases common with PL are omitted here. For the case of γ_1 being $[\mathrm{C}_n^i]$, we note that the following is a K_n-proof of $\alpha \to \gamma_1$ from $\emptyset$ (so *a fortiori* a K_n-proof of $\alpha \to \gamma_1$ from Σ). (Note that the proof is the same as in the case of PL-axioms.)

1.	γ_1	$[\mathrm{C}_n^i]$
2.	$\gamma_1 \to (\alpha \to \gamma_1)$	PL
3.	$\alpha \to \gamma_1$	$1, 2, [\mathrm{MP}]$

For the inductive step, we assume $\Sigma \vdash_{\mathrm{K}_n} \alpha \to \gamma_g$ for every $g < k$ (the I.H.) and show that $\Sigma \vdash_{\mathrm{K}_n} \alpha \to \gamma_k$. The formula γ_k is either an axiom of K_n, a member of Σ, α itself, or the output of some earlier formula(s) of the sequence by a rule of K_n. We omit here the cases common with PL. The case for γ_k being $[\mathrm{C}_n^i]$ is the same as above. The remaining cases are those in which γ_k is obtained from an earlier formula γ_g by applying $[\mathrm{RN}_n^i]$ or $[\mathrm{RM}_n^i]$. We show the case for $[\mathrm{RN}_n^i]$ only (the case for $[\mathrm{RM}_n^i]$ is similar). Note that the use of $[\mathrm{RN}_n^i]$ requires γ_g be a K_n-theorem, i.e. there is a K_n-proof of γ_g from $\emptyset$. Then such a proof (say of m' lines) followed by the lines below is a K_n-proof of $\alpha \to \gamma_k$ from $\emptyset$ and *a fortiori* a K_n-proof of $\alpha \to \gamma_k$ from Σ:

$m' + 1.$	γ_k	$m', [\mathrm{RN}_n^i]$
$m' + 2.$	$\gamma_k \to (\alpha \to \gamma_k)$	PL
$m' + 3.$	$\alpha \to \gamma_k$	$m' + 1,\ m' + 2,\ [\mathrm{MP}]$

Note that the I.H. is not required in proving the cases for $[\mathrm{RN}_n^i]$ and $[\mathrm{RM}_n^i]$. But it is required for the case of [MP]. ⊣

2.5 Determination for K_n

In this section, we demonstrate the soundness and completeness of K_n, the weakest normal n-adic system, with respect to the class of all $(n + 1)$-ary relational frames.

Theorem 2.5.1 (Soundness of K_n)**.** *The weakest normal n-adic system, K_n, is sound with respect to the class of all $(n+1)$-ary relational frames.*

Proof. It is straightforward to show that $[\mathrm{RM}_n]$ and $[\mathrm{RN}_n]$ preserve validity, and $[\mathrm{C}_n]$ is valid in the class of all $(n+1)$-ary relational frames. ⊣

Our strategy of proving the completeness of a normal n-adic system S with respect to a class $\mathbb{C}$ of $(n+1)$-ary relational frames is to show that every set of $\mathfrak{L}_n$-formulas consistent in S has a model on a frame in $\mathbb{C}$. In fact, for any normal modal system, there exists a model that satisfies any consistent set of formulas. We call this model the canonical model of the system, and the corresponding frame its canonical frame. Given this result, all that remains to prove the completeness of S with respect to $\mathbb{C}$ is to show that the canonical frame of S belongs to $\mathbb{C}$.

In the following, we first define the canonical model of a normal n-adic system. Before showing that the canonical model is indeed a model for any consistent set of formulas, we prove an existence lemma and a truth lemma pertaining to such a system and its canonical model. (The proof for what we call the existence lemma here is based on Gabbay (1976). Another proof is found in Johnston (1976). For a more recent version, readers are advised to check Blackburn et al. (2001) pp. 200-201).

Definition 2.5.2 (Canonical frames and models)**.** Let S be a normal system in the modal language $\mathfrak{L}_n$. The S-canonical model, denoted $\mathfrak{M}_\mathrm{S}$, is a triple $\langle U_\mathrm{S}, R_\mathrm{S}, V_\mathrm{S}\rangle$ where:

- U_S is the set of all maximal S-consistent sets of $\mathfrak{L}_n$-formulas.
- For every $x, y_1, \ldots, y_n \in U_\mathrm{S}$, we have $R_\mathrm{S}xy_1 \cdots y_n$ iff for any $\mathfrak{L}_n$-formulas $\alpha_1, \ldots, \alpha_n$,

$$\Box(\alpha_1, \ldots, \alpha_n) \in x \implies \exists i : \alpha_i \in y_i.$$

- For every p_i, $V_\mathrm{S}(p_i)$ is the set $\{x \in U_\mathrm{S} \mid p_i \in x\}$.

We call the pair $\langle U_\mathrm{S}, R_\mathrm{S}\rangle$ the canonical frame of S. ⊣

Lemma 2.5.3 (Existence Lemma for normal n-adic systems)**.** *Let $\mathfrak{M}_\mathrm{S} = \langle U_\mathrm{S}, R_\mathrm{S}, V_\mathrm{S}\rangle$ be the canonical model of a normal n-adic system* S*. For any point $x \in U_\mathrm{S}$ and $\mathfrak{L}_n$-formulas $\alpha_1, \ldots, \alpha_n$, if $\neg\Box(\alpha_1, \ldots, \alpha_n) \in x$, then there exist $y_1, \ldots, y_n \in U_\mathrm{S}$ such that $\neg\alpha_1 \in y_1$, …, and $\neg\alpha_n \in y_n$, and $R_\mathrm{S}xy_1 \cdots y_n$.*

Proof. Assume $\neg\Box(\alpha_1, \dots, \alpha_n) \in x$. We show, by induction, that there exist $y_1, \dots, y_n \in U_{\text{S}}$ such that each y_i $(1 \leq i \leq n)$ satisfies both of the following requirements.

(E1) $\neg\alpha_i \in y_i$.

(E2) For any formulas $\gamma_1, \dots, \gamma_{i-1}, \beta$, if $\neg\gamma_1 \in y_1, \dots, \neg\gamma_{i-1} \in y_{i-1}$, and $\Box(\gamma_1, \dots, \gamma_{i-1}, \beta, \alpha_{i+1}, \dots, \alpha_n) \in x$, then $\beta \in y_i$.

For the existence of y_1, we first show that y_1^0 defined by letting

$$y_1^0 = \{\neg\alpha_1\} \cup \{\beta | \Box(\beta, \alpha_2, \dots, \alpha_n) \in x\}$$

is S-consistent. Assume, for reductio, y_1^0 is not S-consistent. Then, for some $\beta_1, \dots, \beta_m \in \{\beta | \Box(\beta, \alpha_2, \dots, \alpha_n) \in x\}$, the following hold.

$$\begin{array}{ll}
\{\beta_1, \dots, \beta_m, \neg\alpha_1\} \vdash_{\text{S}} \bot & \\
\vdash_{\text{S}} \beta_1 \wedge \dots \wedge \beta_m \to \alpha_1 & \\
\vdash_{\text{S}} \Box(\beta_1 \wedge \dots \wedge \beta_m \to \alpha_1, \alpha_2, \dots, \alpha_n) & ([\text{RN}_n]) \\
\vdash_{\text{S}} \Box(\beta_1 \wedge \dots \wedge \beta_m, \alpha_2, \dots, \alpha_n) \to \Box(\alpha_1, \alpha_2, \dots, \alpha_n) & ([\text{K}_n]) \\
\vdash_{\text{S}} \bigwedge_{j=1}^{m} \Box(\beta_j, \alpha_2, \dots, \alpha_n) \to \Box(\alpha_1, \alpha_2, \dots, \alpha_n) & ([\text{C}_n])
\end{array}$$

Since both $\Box(\beta_j, \alpha_2, \dots, \alpha_n) \in x$ for every j and x is maximal S-consistent, we have $\Box(\alpha_1, \alpha_2, \dots, \alpha_n) \in x$. But this is impossible, for by assumption $\neg\Box(\alpha_1, \alpha_2, \dots, \alpha_n) \in x$. Thus, by reductio, y_1^0 is S-consistent and so has a maximal S-consistent extension y_1 (by Lindenbaum's Lemma). It is straightforward to see that y_1 satisfies both requirements (E1) and (E2) (for $i = 1$).

To demonstrate the existence of the other members of the series, viz., $y_2, \dots, y_n$, assume that we already have $y_1, \dots, y_k \in U_{\text{S}}$ which satisfy (E1) and (E2) in place (where $k < n$). As in the case of y_1, we define an initial set y_{k+1}^0 that can be shown to have a maximal S-consistent extension y_{k+1} satisfying both (E1) and (E2). So let

$$\begin{aligned}
y_{k+1}^0 = \{\neg\alpha_{k+1}\} \cup \{\beta | \exists\gamma_1, \dots, \gamma_k : \neg\gamma_1 \in y_1, \dots, \neg\gamma_k \in y_k \ \& \\
\Box(\gamma_1, \dots, \gamma_k, \beta, \alpha_{k+2}, \dots, \alpha_n) \in x\}.
\end{aligned}$$

To show that y_{k+1}^0 is S-consistent, we assume otherwise. Then, for some $\beta_1, \dots, \beta_m \in y_{k+1}^0 - \{\neg\alpha_{k+1}\}$, the following hold.

$$\begin{array}{l}
\{\beta_1, \dots, \beta_m, \neg\alpha_{k+1}\} \vdash_{\text{S}} \bot \\
\vdash_{\text{S}} \beta_1 \wedge \dots \wedge \beta_m \to \alpha_{k+1}
\end{array}$$

For each β_j $(1 \leq j \leq m)$, there exist $\neg\gamma_{j.1} \in y_1, \ldots, \neg\gamma_{j.k} \in y_k$ such that

$$\Box(\gamma_{j.1}, \ldots, \gamma_{j.k}, \beta_j, \alpha_{k+2}, \ldots, \alpha_n) \in x.$$

Then by [RM$_n$] and [K$_n$] we get

$$\Box(\bigvee_{j=1}^{m} \gamma_{j.1}, \ldots, \bigvee_{j=1}^{m} \gamma_{j.k}, \bigwedge_{j=1}^{m} \beta_j, \alpha_{k+2}, \ldots, \alpha_n) \in x.$$

Since $\beta_1 \wedge \cdots \wedge \beta_m \to \alpha_{k+1} \in x$, we also have the following by [RN$_n$]

$$\Box(\bigvee_{j=1}^{m} \gamma_{j.1}, \ldots, \bigvee_{j=1}^{m} \gamma_{j.k}, \bigwedge_{j=1}^{m} \beta_j \to \alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n) \in x.$$

Thus by [K$_n$] we have

$$\Box(\bigvee_{j=1}^{m} \gamma_{j.1}, \ldots, \bigvee_{j=1}^{m} \gamma_{j.k}, \alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n) \in x.$$

Note that $\neg \bigvee_{j=1}^{m} \gamma_{j.1} \in y_1$, since $\neg\gamma_{j.1} \in y_1$ for all j $(1 \leq j \leq m)$, and y_1 is maximal S-consistent. Similarly, $\neg \bigvee_{j=2}^{m} \gamma_{j.2} \in y_2$, ..., and $\neg \bigvee_{j=1}^{m} \gamma_{j.k} \in y_k$. But $\bigvee_{j=1}^{m} \gamma_{j.k} \in y_k$, since y_k complies with our requirement (E2). Hence we derive a contradiction. By reductio y^0_{k+1} is S-consistent, and so has a maximal S-consistent extension y_{k+1}. It is straightforward to check that y_{k+1} satisfies requirements (E1) and (E2) (for $i = k + 1$).

We have now demonstrated the existence of $y_1, \ldots, y_n \in U_{\mathrm{S}}$ all of which satisfy requirements (E1) and (E2). It remains to show that $R_{\mathrm{S}}xy_1 \cdots y_n$. Assume that for any $\beta_1, \ldots, \beta_n$, $\Box(\beta_1, \ldots, \beta_n) \in x$, $\beta_1 \notin y_1$, ..., $\beta_{n-1} \notin y_{n-1}$. Then $\neg\beta_1 \in y_1$, ..., $\neg\beta_{n-1} \in y_{n-1}$. Since y_n satisfies (E2), we have $\beta_n \in y_n$. Thus $R_{\mathrm{S}}xy_1 \cdots y_n$ according to the definition of R_{S}. This completes our proof of the Existence Lemma. ⊣

Lemma 2.5.4 (Truth lemma for normal n-adic systems)**.** *Let* $\mathfrak{M}_{\mathrm{S}} = \langle U_{\mathrm{S}}, R_{\mathrm{S}}, V_{\mathrm{S}} \rangle$ *be the canonical model of a normal n-adic system* S. *For any* $\mathcal{L}_n$*-formula* α*, we have*

$$\forall x \in U_{\mathrm{S}},\ \mathfrak{M}_{\mathrm{S}}, x \models \alpha \iff \alpha \in x.$$

Proof. The proof is by induction on α. In the following we show the modal case of the inductive step only. Let α be $\Box(\alpha_1, \ldots, \alpha_n)$, and show that for an arbitrary $x \in U_{\mathrm{S}}$,

$$\mathfrak{M}_{\mathrm{S}}, x \models \Box(\alpha_1, \ldots, \alpha_n) \iff \Box(\alpha_1, \ldots, \alpha_n) \in x$$

by assuming the inductive hypothesis that the theorem holds for α_1, ..., and α_n.

For the direction $\Longrightarrow$, assume $\Box(\alpha_1, \ldots, \alpha_n) \notin x$, which is equivalent to $\neg\Box(\alpha_1, \ldots, \alpha_n) \in x$. Then, by the existence lemma, there exist $y_1, \ldots, y_n \in U_{\mathrm{S}}$ such that $\neg\alpha_1 \in y_1$, ..., and $\neg\alpha_n \in y_n$, and $R_{\mathrm{S}}xy_1 \cdots y_n$. Then for each i such that $1 \leq i \leq n$, $\alpha_i \notin y_i$ and by the inductive hypothesis $\mathfrak{M}_{\mathrm{S}}, y_i \not\models \alpha_i$. Thus $\mathfrak{M}_{\mathrm{S}}, x \not\models \Box(\alpha_1, \ldots, \alpha_n)$, as desired.

For the direction $\Longleftarrow$, assume $\Box(\alpha_1, \ldots, \alpha_n) \in x$. To show that $\mathfrak{M}_{\mathrm{S}}, x \models \Box(\alpha_1, \ldots, \alpha_n)$, we consider arbitrary $y_1, \ldots, y_n \in U$ such that $R_{\mathrm{S}}xy_1 \cdots y_n$. Then by the definition of R_{S}, $\alpha_i \in y_i$ for some i where $1 \leq i \leq n$. It follows from the inductive hypothesis that $\mathfrak{M}, y_i \models \alpha_i$, whence we conclude that $\mathfrak{M}_{\mathrm{S}}, x \models \Box(\alpha_1, \ldots, \alpha_n)$. ⊣

Corollary 2.5.5. *Let* S *be a normal n-adic system. Then any* S*-consistent set of formulas* Σ *is satisfiable in the* S*-canonical model* $\mathfrak{M}_{\mathrm{S}}$.

Proof. By Lindenbaum's Lemma, Σ can be extended to a maximal S-consistent set x of formulas. But every formula of Σ is true at x in $\mathfrak{M}_{\mathrm{S}}$ according to the truth lemma. Therefore, Σ is satisfiable in $\mathfrak{M}_{\mathrm{S}}$. ⊣

Theorem 2.5.6 (Completeness of K_n)**.** *The weakest n-adic normal system* K_n *is complete with respect to the class of all* $(n+1)$*-ary relational frames.*

Proof. It is enough to note that the canonical model of K_n is an $(n+1)$-ary relational model. ⊣

Chapter 3

Normal Systems from K_n to $\mathrm{S5}_n$

Whereas monadic systems extending K with axioms [P], [D], [T], [B], [4] and [5] have been studied in detail by modal logicians, polyadic normal systems (other than its weakest member K_n) seem to have been given little attention by many practitioners of modal logic. In this chapter we embellish modal logic by proposing n-adic counterparts of the aforementioned monadic axioms, and extending K_n with these n-adic axioms (Section 3.2). The classes of frames for the defined normal polyadic systems, as well as their completeness, are demonstrated in Sections 3.3 and 3.4. We also investigate the first-order relational properties corresponding to our n-adic modal axioms, culminating in the study of multi-ary equivalence relations (Section 3.5). But first we present in Section 3.1 results for the normal monadic systems P, D, T, B, S4 and S5. While the polyadic systems we propose in this chapter are new, their monadic cousins have been examined in standard textbooks such as Chellas (1980) and Hughes and Cresswell (1996).

It is worth mentioning that the n-adic axioms [P_n], [D_n], [T_n], [B_n], [4_n] and [5_n] we are going to present are Sahlqvist formulas. So their correspondences with first-order properties are expected, and the proofs are straightforward. Our aims here, however, are primarily studying these n-adic axioms and the resulting systems that can be said to generalize their monadic members, as well as investigating the corresponding conditions of multi-ary relations which, like the n-adic axioms and systems, are generalizations of their binary counterparts.

3.1 The normal monadic systems P, D, T, B, S4 and S5

In monadic modal logic, various axioms have been put forward to extend K. The following axioms (and their duals) have been studied for their theoretical and applicational interests.

[P]	$\Diamond\top$	[P$\Box$]	$\neg\Box\bot$
[D]	$\Box p \to \Diamond p$	[D$\Diamond$]	$\Box p \to \Diamond p$
[T]	$\Box p \to p$	[T$\Diamond$]	$p \to \Diamond p$
[B]	$p \to \Box\Diamond p$	[B$\Diamond$]	$\Diamond\Box p \to p$
[4]	$\Box p \to \Box\Box p$	[4$\Diamond$]	$\Diamond\Diamond p \to \Diamond p$
[5]	$\Diamond p \to \Box\Diamond\, p$	[5$\Diamond$]	$\Diamond\Box p \to \Box p$

Part of the theoretical significance of the above axioms is due to their correspondence to some simple first-order properties of binary relations, viz. seriality, reflexivity, symmetry, transitivity and euclideanness.

[P] :	[ser]	$(\forall x)(\exists y)Rxy$
[D] :	[ser]	$(\forall x)(\exists y)Rxy$
[T] :	[refl]	$(\forall x)Rxx$
[B] :	[sym]	$(\forall x)(\forall y)(Rxy \to Ryx)$
[4] :	[trans]	$(\forall x)(\forall y)(\forall z)(Rxy \wedge Ryz \to Rxz)$
[5] :	[eucl]	$(\forall x)(\forall y)(\forall z)(Rxy \wedge Rxz \to Ryz)$

By adding one or more of the above axioms to K, we obtain various systems (with some of them being equivalent systems). The following ones are important both historically and theoretically. (Alternative names are given in parentheses.)

KP	(P)	: K,	[P]	
KD	(D)	: K,	[D]	
KT	(T)	: K,	[T]	
KTB	(B)	: K,	[T],	[B]
KT4	(S4)	: K,	[T],	[4]
KT5	(S5)	: K,	[T],	[5]

From the correspondence results, the classes of frames for the systems P, D, T, B, S4 and S5 are as indicated below.

KP	(P)	: Serial frames
KD	(D)	: Serial frames
KT	(T)	: Reflexive frames
KTB	(B)	: Reflexive and symmetric frames
KT4	(S4)	: Reflexive and transitive frames
KT5	(S5)	: Equivalence frames

Moreover the listed systems are complete with respect to their classes of frames. In the following sections, we generalize the above results pertaining to extensions of the monadic K to extensions of the n-adic K_n.

3.2 The normal polyadic systems P_n, D_n, T_n, B_n, $\mathrm{S4}_n$ and $\mathrm{S5}_n$

The following formulas generalize the monadic [P], [D], [T], [B], [4] and [5]. (Notation: To improve readability, we write $\bot^n$ for an n-termed sequence of $\bot$'s, $\vec{p}$ for $p_1, \ldots, p_n$. If the i-th member of an n-termed sequence of $\bot$'s is replaced by p, we write simply $\bot, \ldots, p, \ldots, \bot$. Similarly instead of the longer $p_1, \ldots, p_{i-1}, \alpha, p_{i+1}, \ldots, p_n$, we use the shorter $p_1, \ldots, \alpha, \ldots, p_n$. Occasionally the above conventions are suspended in order to highlight syntactic features.)

$[\mathrm{P}_n]$	$\Diamond\top^n$
$[\mathrm{D}_n]$	$\Box\vec{p} \to \bigvee_i \Diamond(\top, \ldots, p_i, \ldots, \top)$
$[\mathrm{T}_n]$	$\Box\vec{p} \to \bigvee_i p_i$
$[\mathrm{B}_n^i]$	$p_i \to \Box(\neg p_1, \ldots, \Diamond\vec{p}, \ldots, \neg p_n)$
$[4_n^i]$	$\Box\vec{p} \to \Box(p_1, \ldots, \Box(\bot, \ldots, p_i, \ldots, \bot), \ldots, p_n)$
$[5_n^i]$	$\Diamond(\top, \ldots, p_i, \ldots, \top) \to \Box(\neg p_1, \ldots, \Diamond\vec{p}, \ldots, \neg p_n)$

The dual forms of the above formulas are as follows.

$[\mathrm{P}\Box_n]$	$\neg\Box\bot^n$
$[\mathrm{D}\Diamond_n]$	$\bigwedge_i \Box(\bot, \ldots, p_i, \ldots, \bot) \to \Diamond\vec{p}$
$[\mathrm{T}\Diamond_n]$	$\bigwedge_i p_i \to \Diamond\vec{p}$
$[\mathrm{B}\Diamond_n^i]$	$\Diamond(\neg p_1, \ldots, \Box\vec{p}, \ldots, \neg p_n) \to p_i$
$[4\Diamond_n^i]$	$\Diamond(p_1, \ldots, \Diamond(\top, \ldots, p_i, \ldots, \top), \ldots, p_n) \to \Diamond\vec{p}$

$$[5\Diamond_n^i] \quad \Diamond(\neg p_1, \ldots, \Box\vec{p}, \ldots, \neg p_n) \to \Box(\bot, \ldots, p_i, \ldots, \bot)$$

By adding one or more of the above axioms to K_n, we obtain n-adic counterparts of the monadic systems P, D, T, B, S4 and S5.

Definition 3.2.1. The following are extensions of K_n. Recall that K_n is the smallest system that provide PL, $[\mathrm{RM}_n]$, $[\mathrm{RN}_n]$ and $[\mathrm{C}_n]$. (Alternative names of the systems are given in parentheses.)

$\mathrm{K}_n\mathrm{P}_n$	(P_n)	:	K_n,	$[\mathrm{P}_n]$	
$\mathrm{K}_n\mathrm{D}_n$	(D_n)	:	K_n,	$[\mathrm{D}_n]$	
$\mathrm{K}_n\mathrm{T}_n$	(T_n)	:	K_n,	$[\mathrm{T}_n]$	
$\mathrm{K}_n\mathrm{T}_n\mathrm{B}_n$	(B_n)	:	K_n,	$[\mathrm{T}_n]$,	$[\mathrm{B}_n]$
$\mathrm{K}_n\mathrm{T}_n4_n$	$(\mathrm{S4}_n)$	:	K_n,	$[\mathrm{T}_n]$,	$[4_n]$
$\mathrm{K}_n\mathrm{T}_n5_n$	$(\mathrm{S5}_n)$	:	K_n,	$[\mathrm{T}_n]$,	$[5_n]$

⊣

Note that there are other ways to generalize monadic axioms. Some of them are listed below, followed by their dual forms. In order to distinguish them from the earlier set of axioms, we prefix their names with † (dagger).

$$\begin{array}{ll}
[\dagger\mathrm{D}_n^i] & \Box(\bot, \ldots, p, \ldots, \bot) \to \Diamond(\top, \ldots, p, \ldots, \top) \\
[\dagger\mathrm{T}_n^i] & \Box(\bot, \ldots, p, \ldots, \bot) \to p \\
[\dagger\mathrm{B}_n^i] & \bigwedge_i p_i \to \Box(\bot, \ldots, \Diamond\vec{p}, \ldots, \bot) \\
[\dagger 4_n^i] & \Box\vec{p} \to \Box(\bot, \ldots, \Box\vec{p}, \ldots, \bot) \\
[\dagger 5_n^i] & \Diamond\vec{p} \to \Box(\bot, \ldots, \Diamond\vec{p}, \ldots, \bot)
\end{array}$$

$$\begin{array}{ll}
[\dagger\mathrm{D}\Diamond_n^i] & \Box(\bot, \ldots, p, \ldots, \bot) \to \Diamond(\top, \ldots, p, \ldots, \top) \\
[\dagger\mathrm{T}\Diamond_n^i] & p \to \Diamond(\top, \ldots, p, \ldots, \top) \\
[\dagger\mathrm{B}\Diamond_n^i] & \Diamond(\top, \ldots, \Box\vec{p}, \ldots, \top) \to \bigvee_i p_i \\
[\dagger 4\Diamond_n^i] & \Diamond(\top, \ldots, \Diamond\vec{p}, \ldots, \top) \to \Diamond\vec{p} \\
[\dagger 5\Diamond_n^i] & \Diamond(\top, \ldots, \Box\vec{p}, \ldots, \top) \to \Box\vec{p}
\end{array}$$

The following theorems illustrate the deductive relations among the formulas listed so far. In order to highlight the rules and axioms used in deduction, we require the base logic to provide PL and rule $[\mathrm{RE}_n]$ only. (Such a system is called "classical". We shall study classical systems in Chapter 5.) Remarks about the deductive relations follow the proofs of the theorems.

Theorem 3.2.2. *Let* S *be a* PL*-system providing* $[\mathrm{RE}_n]$.

(1) $[\mathrm{D}_n] \to [\mathrm{P}_n]$ *is provable in* S *if it has* $[\mathrm{RN}_n]$.

(2) $[\mathrm{P}_n] \to [\mathrm{D}_n]$ *is provable in* S *if it has* $[\mathrm{RM}_n]$ *and* $[\mathrm{C}_n]$.

(3) $[\dagger\mathrm{D}_n] \to [\mathrm{P}_n]$ *is provable in* S *if it has* $[\mathrm{RN}_n]$.

(4) $[\mathrm{P}_n] \to [\dagger\mathrm{D}_n]$ *is provable in* S *if it has* $[\mathrm{C}_n]$.

(5) $[\mathrm{D}_n] \to [\dagger\mathrm{D}_n]$ *is provable in* S *if it has* $[\mathrm{RN}_n]$.

(6) $[\dagger\mathrm{D}_n] \to [\mathrm{D}_n]$ *is provable in* S *if it has* $[\mathrm{RM}_n]$, $[\mathrm{RN}_n]$ *and* $[\mathrm{C}_n]$.

Proof. For (1). We show that if S provides $[\mathrm{RN}_n]$, then $[\mathrm{D}_n] \vdash_{\mathrm{S}} [\mathrm{P}_n]$.

1.	$\top$	PL
2.	$\Box\top^n$	$1, [\mathrm{RN}_n]$
3.	$\Box\top^n \to \Diamond\top^n$	$[\mathrm{D}_n], \mathrm{PL}$
4.	$\Diamond\top^n$	$2, 3, [\mathrm{MP}]$

For (2). It suffices to show that if S provides $[\mathrm{RM}_n]$ and $[\mathrm{C}_n]$, then the following holds:

$$\{\Box\vec{p}, \Box(\neg p_1, \bot, \ldots, \bot), \Box(\bot, \neg p_2, \bot, \ldots, \bot), \ldots, \Box(\bot, \ldots, \bot, \neg p_n)\} \vdash_{\mathrm{S}} \Box\bot^n$$

since from the above we have $\vdash_{\mathrm{S}} \Box\vec{p} \wedge \bigwedge_i \Box(\bot, \ldots, \neg p_i, \ldots, \bot) \to \Box\bot^n$ (by [DT]) and so $\vdash_{\mathrm{S}} [\mathrm{P}_n] \to [\mathrm{D}_n]$ (by contraposition).

1.	$\Box(\neg p_1, \bot, \ldots, \bot)$	assumption
2.	$\Box(\neg p_1, \bot, \ldots, \bot) \to \Box(\neg p_1, p_2, \ldots, p_n)$	$\mathrm{PL}, [\mathrm{RM}_n]$
3.	$\Box(\neg p_1, p_2, \ldots, p_n)$	$1, 2, [\mathrm{MP}]$
4.	$\Box(p_1, p_2, \ldots, p_n)$	assumption
5.	$\Box(p_1 \wedge \neg p_1, p_2, \ldots, p_n)$	$3, 4, [\mathrm{C}_n], [\mathrm{MP}]$
6.	$\Box(p_1 \wedge \neg p_1, p_2, \ldots, p_n) \leftrightarrow \Box(\bot, p_2, \ldots, p_n)$	$\mathrm{PL}, [\mathrm{RE}_n]$
7.	$\Box(\bot, p_2, \ldots, p_n)$	$5, 6, [\mathrm{MP}]$
8.	$\Box(\bot, \neg p_2, \bot, \ldots, \bot)$	assumption

9.	$\Box(\bot, \neg p_2, p_3, \ldots, p_n)$	PL, $[\mathrm{RM}_n]$
10.	$\Box(\bot, p_2 \wedge \neg p_2, p_3, \ldots, p_n)$	$7, 9, [\mathrm{C}_n], [\mathrm{MP}]$
11.	$\Box(\bot, p_2 \wedge \neg p_2, p_3, \ldots, p_n) \leftrightarrow \Box(\bot, \bot, p_3, \ldots, p_n)$	PL, $[\mathrm{RE}_n]$
12.	$\Box(\bot, \bot, p_3, \ldots, p_n)$	$10, 11, [\mathrm{MP}]$

Using the rest of the assumptions, we eventually arrive at the formula $\Box(\bot, \ldots, \bot)$ as desired.

For (3). We show that if S provides $[\mathrm{RN}_n]$, then $[\dagger\mathrm{D}^i_n] \vdash_{\mathrm{S}} [\mathrm{P}_n]$.

1.	$\top$	PL
2.	$\Box(\bot^{i-1}, \top, \bot^{n-i})$	$1, [\mathrm{RN}_n]$
3.	$\Box(\bot^{i-1}, \top, \bot^{n-i}) \to \Diamond\top^n$	$[\dagger\mathrm{D}^i_n], [\mathrm{US}]$
4.	$\Diamond\top^n$	$2, 3, [\mathrm{MP}]$

For (4). We show that if S provides $[\mathrm{C}^i_n]$, then $\vdash_{\mathrm{S}} [\mathrm{P}_n] \to [\dagger\mathrm{D}^i_n]$.

1.	$\Box(\bot^{i-1}, p, \bot^{n-i}) \wedge \Box(\bot^{i-1}, \neg p, \bot^{n-i}) \to \Box(\bot^{i-1}, p \wedge \neg p, \bot^{n-i})$	$[\mathrm{C}^i_n], [\mathrm{US}]$
2.	$\Box(\bot^{i-1}, p \wedge \neg p, \bot^{n-i}) \leftrightarrow \Box(\bot^{i-1}, \bot, \bot^{n-i})$	PL, $[\mathrm{RE}_n]$
3.	$\Box(\bot^{i-1}, p, \bot^{n-i}) \wedge \Box(\bot^{i-1}, \neg p, \bot^{n-i}) \to \Box\bot^n$	$1, 2, [\mathrm{MP}]$
4.	$\neg\Box\bot^n \to \neg\Box(\bot^{i-1}, p, \bot^{n-i}) \vee \neg\Box(\bot^{i-1}, \neg p, \bot^{n-i})$	3, PL
5.	$\neg\Box\bot^n \to \neg\Box(\bot^{i-1}, p, \bot^{n-i}) \vee \Diamond(\top^{i-1}, p, \top^{n-i})$	4, PL, $[\mathrm{Df}\Diamond]$
6.	$\neg\Box\bot^n \to (\Box(\bot^{i-1}, p, \bot^{n-i}) \to \Diamond(\top^{i-1}, p, \top^{n-i}))$	5, PL

For (5). We show that if S provides $[\mathrm{RN}_n]$, then $[\mathrm{D}_n] \vdash_{\mathrm{S}} [\dagger\mathrm{D}_n]$.

1.	$\Box(\bot, \ldots, p_i, \ldots, \bot) \to \Diamond(\bot, \top, \ldots, \top) \vee \cdots \vee$ $\Box(\top, \ldots, p_i, \ldots, \top) \vee \cdots \vee \Box(\top, \ldots, \top, \bot)$	$[\mathrm{D}_n], [\mathrm{US}]$
2.	$\Box(\bot^{j-1}, \top, \bot^{n-j})$ where $1 \leq j \neq i \leq n$	$[\mathrm{RN}_n]$
3.	$\neg\Diamond(\top^{j-1}, \bot, \top^{n-j})$	$2, [\mathrm{Df}\Diamond]$
4.	$\Box(\bot, \ldots, p_i, \ldots, \bot) \to \Diamond(\top, \ldots, p_i, \ldots, \top)$	1, 3, PL

For (6). It follows from (3) and (2) that if S provides $[\mathrm{RM}_n]$, $[\mathrm{RN}_n]$ and $[\mathrm{C}_n]$, then $[\dagger\mathrm{D}_n] \to [\mathrm{D}_n]$ is provable. $\dashv$

Remark 3.2.3. $[\mathrm{D}_n]$ and $[\mathrm{P}_n]$ are provable equivalents in normal systems, but not so in systems that are weaker than K_n. In some interpretations of the modality $\Box$ (for instance, a deontic reading of $\Box$ as "it is obligatory that") we may wish to distinguish between these two axioms and so prefer a logic weaker than K_n. The same applies to [$\dagger\mathrm{D}_n$] and [P_n]. Note that [D_n] and [$\dagger\mathrm{D}_n$] are provable equivalents in normal systems. However in PL-systems providing [RM_n] and [RN_n] only, [D_n] is deductively stronger than [$\dagger\mathrm{D}_n$].

Theorem 3.2.4. *Let* S *be a* PL-*system providing* [RE_n].

(1) [T_n] $\to$ [D_n] *is provable in* S.

(2) [T_n] $\to$ [$\dagger\mathrm{T}_n$] *is provable in* S.

(3) [$\dagger\mathrm{T}_n$] $\to$ [P_n] *is provable in* S.

(4) [$\dagger\mathrm{T}_n$] $\to$ [$\dagger\mathrm{D}_n$] *is provable in* S.

Proof. For (1). We show that [T_n] $\vdash_{\mathrm{S}}$ [D_n].

1.	$\Box\vec{p} \to \bigvee_i p_i$	[T_n]
2.	$p_i \to \Diamond(\top^{i-1}, p_i, \top^{n-i})$	[$\mathrm{T}\Diamond_n$], [US]
3.	$\Box\vec{p} \to \bigvee_i \Diamond(\top^{i-1}, p_i, \top^{n-i})$	1, 2, PL

For (2). We show that [T_n] $\vdash_{\mathrm{S}}$ [$\dagger\mathrm{T}^i_n$].

1.	$\Box(\bot, \ldots, p_i, \ldots, \bot) \to \bot \vee \cdots \vee p_i \vee \cdots \vee \bot$	[T_n], [US]
2.	$\Box(\bot, \ldots, p_i, \ldots, \bot) \to p_i$	1, PL

For (3). We show that [$\dagger\mathrm{T}_n$] $\vdash_{\mathrm{S}}$ [P_n].

1.	$\Box\bot^n \to \bot$	[$\dagger\mathrm{T}_n$], [US]
2.	$\top \to \neg\Box\bot^n$	1, PL
3.	$\neg\Box\bot^n$	2, PL

For (4). We show that [$\dagger\mathrm{T}_n$] $\vdash_{\mathrm{S}}$ [$\dagger\mathrm{D}_n$].

1.	$\Box(\bot, \ldots, p, \ldots, \bot) \to \Diamond(\top, \ldots, p, \ldots, \top)$	[$\dagger\mathrm{T}_n$], [$\dagger\mathrm{T}\Diamond_n$], PL

$\dashv$

Remark 3.2.5. In PL-systems providing [RE_n] (and therefore in normal systems as well),

[D_n] is derivable from [T_n]. So are [P_n] and [$\dagger\mathrm{D}_n$] since both are derivable from [$\dagger\mathrm{T}_n$], which in turn is derivable from [T_n] in such systems.

Theorem 3.2.6. *Let* S *be a* PL*-system providing* [RE_n].

(1) [C^i_n] *is provable in* S *if it has* [RM^i_n] *and* [B^i_n].

(2) [N^i_n] *is provable in* S *if it has* [RM^i_n] *and* [B^i_n].

Proof. For (1). Let $\vec{p}$ be the sequence $p_1 \cdots p_i \cdots p_n$ and $\vec{q_i}$ the sequence $p_1, \ldots, q_i, \ldots, p_n$ (where q_i occurs at the ith-place as p_i does). The following sketches a proof of [C^i_n] in S.

1.	$\Diamond(\neg p_1, \ldots, \Box\vec{p}, \ldots, \neg p_n) \to p_i$	[$\mathrm{B}\Diamond^i_n$]
2.	$\Diamond(\neg p_1, \ldots, \Box\vec{q_i}, \ldots, \neg p_n) \to q_i$	[$\mathrm{B}\Diamond^i_n$]
3.	$\Diamond(\neg p_1, \ldots, \Box\vec{p}, \ldots, \neg p_n) \wedge \Diamond(\neg p_1, \ldots, \Box\vec{q_i}, \ldots, \neg p_n) \to p_i \wedge q_i$	1, 2, PL
4.	$\Diamond(\neg p_1, \ldots, \Box\vec{p} \wedge \Box\vec{q_i}, \ldots, \neg p_n) \to p_i \wedge q_i$	3, PL, [$\mathrm{RM}\Diamond^i_n$]
5.	$\Box(p_1, \ldots, \Diamond(\neg p_1, \ldots, \Box\vec{p} \wedge \Box\vec{q_i}, \ldots, \neg p_n), \ldots, p_n) \to$ $\Box(p_1, \ldots, p_i \wedge q_i, \ldots, p_n)$	4, [RM^i_n]
6.	$\Box\vec{p} \wedge \Box\vec{q_i} \to \Box(p_1, \ldots, \Diamond(\neg p_1, \ldots, \Box\vec{p} \wedge \Box\vec{q_i}, \ldots, \neg p_n), \ldots, p_n)$	[B^i_n], [US], [RE_n]
7.	$\Box\vec{p} \wedge \Box\vec{q_i} \to \Box(p_1, \ldots, p_i \wedge q_i, \ldots, p_n)$	5, 6, [MP]

For (2).

1.	$\top \to \Box(p_1, \ldots, \Diamond(\neg p_1, \ldots, \top, \ldots, \neg p_n), \ldots, p_n)$	[B^i_n], [US], [RE_n]
2.	$\Box(p_1, \ldots, \Diamond(\neg p_1, \ldots, \top, \ldots, \neg p_n), \ldots, p_n)$	1, PL
3.	$\Diamond(\neg p_1, \ldots, \top, \ldots, \neg p_n) \to \top$	PL
4.	$\Box(p_1, \ldots, \Diamond(\neg p_1, \ldots, \top, \ldots, \neg p_n), \ldots, p_n) \to$ $\Box(p_1, \ldots, \top, \ldots, p_n)$	3, [RM^i_n]
5.	$\Box(p_1, \ldots, \top, \ldots, p_n)$	2, 4, [MP]

$\dashv$

Remark 3.2.7. Both [C^i_n] and [RN^i_n] are provable if a S has, in addition to PL and [RE_n], both [RM^i_n] and [B^i_n]. The claim is a generalization to the n-ary $\Box$ of the result reported in Jennings (1981) for the unary $\Box$. An import of this is that a $\mathrm{K}_n\mathrm{T}_n\mathrm{B}_n$-system, also known as a Brouwersche system, can be characterized by a smaller set of modal axioms and rules, viz. [RM_n], [T_n] and [B_n] (in addition to PL).

3.3 Classes of frames for P_n, D_n, T_n, B_n, $\mathrm{S4}_n$ and $\mathrm{S5}_n$

A formula α is said to *correspond* to a frame property ϕ if the following holds: a frame $\mathfrak{F}$ validates α if and only if it has the property ϕ. In symbols,

$$\mathfrak{F} \models \alpha \iff \mathfrak{F} \models \phi.$$

The *class of frames* for a set Σ of formulas is the collection $\mathbb{C}$ of frames validating all the formulas belonging to Σ. In other words, for any frame $\mathfrak{F}$,

$$\mathfrak{F} \in \mathbb{C} \iff (\forall \sigma \in \Sigma,\ \mathfrak{F} \models \sigma).$$

We take the class of frames for a system S to be the class $\mathbb{C}$ of frames for the set of S-theorems. Thus, $\mathbb{C}$ comprises all the frames on which every theorem of S is valid. More formally, for any frame $\mathfrak{F}$,

$$\mathfrak{F} \in \mathbb{C} \iff (\forall \alpha,\ \vdash_{\mathrm{S}} \alpha \implies \mathfrak{F} \models \alpha).$$

Let $\mathbb{C}$ be the class of frames for a system S. If formulas $\mathrm{X}_1, \ldots, \mathrm{X}_n$ correspond respectively to frame properties $\phi_1, \ldots, \phi_m$, then the class of frames for the system $\mathrm{SX}_1 \cdots \mathrm{X}_\mathrm{m}$ (i.e. the extension of S with $\mathrm{X}_1, \ldots, \mathrm{X}_m$ as axioms) is the class $\mathbb{D} \subseteq \mathbb{C}$ of frames satisfying $\phi_1, \ldots, \phi_m$. For any frame $\mathfrak{F} \in \mathbb{D}$ validates all the rules and axioms of $\mathrm{SX}_1 \cdots \mathrm{X}_\mathrm{m}$, and any frame $\mathfrak{F} \notin \mathbb{D}$ invalidates some theorems of $\mathrm{SX}_1 \cdots \mathrm{X}_\mathrm{m}$.

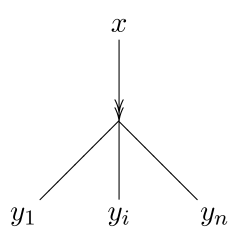
Note that the class of frames for K_n is the class of all $(n+1)$-ary relational frames since the axioms of K_n are valid, and the rules of K_n preserve validity in the class of all $(n+1)$-ary relational frames. In the following, we show that each of the principles $[\mathrm{P}_n]$, $[\mathrm{D}_n]$, $[\mathrm{T}_n]$, $[\mathrm{B}_n]$, $[4_n]$ and $[5_n]$ corresponds to a first-order property of $(n+1)$-ary relations. It follows from our earlier discussion that the classes of frames for $\mathrm{K}_n\mathrm{P}_n$, $\mathrm{K}_n\mathrm{D}_n$, $\mathrm{K}_n\mathrm{T}_n$, $\mathrm{K}_n\mathrm{T}_n\mathrm{B}_n$, $\mathrm{K}_n\mathrm{T}_n4_n$ and $\mathrm{K}_n\mathrm{T}_n5_n$ are precisely the classes of frames satisfying the relevant first-order conditions.

An $(n+1)$-ary relation R is said to be *serial*, *reflexive*, *symmetric*, *transitive* or *euclidean* at the i-th place (where appropriate) if it satisfies the following conditions, respectively.

$[\mathrm{ser}_{n+1}]$	$(\forall x)(\exists \vec{y})Rx\vec{y}$
$[\mathrm{refl}_{n+1}]$	$(\forall x)Rxx\cdots x$
$[\mathrm{sym}^i_{n+1}]$	$(\forall x)(\forall \vec{y})(Rx\vec{y} \to Ry_iy_1\cdots x\cdots y_n)$
$[\mathrm{trans}^i_{n+1}]$	$(\forall x)(\forall \vec{y})(\forall \vec{z})(Rx\vec{y} \wedge Ry_i\vec{z} \to Rxy_1\cdots z_i\cdots y_n)$
$[\mathrm{eucl}^i_{n+1}]$	$(\forall x)(\forall \vec{y})(\forall \vec{z})(Rx\vec{y} \wedge Rx\vec{z} \to Ry_iy_1\cdots z_i\cdots y_n)$

Note that there is only one instance of seriality and of reflexivity whereas there are n instances of each of symmetry, transitivity and euclideanness. If R is symmetric at every place, i.e. if R satisfies [sym_{n+1}] (which stands for the conjunction of all instances of [sym^i_{n+1}]), we simply call it symmetric. The same applies to transitivity and euclideanness.

Before demonstrating correspondence between modal axioms and relational properties, we mention here some heuristic devices which will be helpful in understanding the arguments. A binary relation is often characterized as a seeing relation, and shown in diagrams as a set of arrows. We propose similar devices here. Given an $(n+1)$-ary relation R, if $Rxy_1 \cdots y_n$, we say that x sees the n-tuple $y_1, \ldots, y_n$, i.e. x sees $y_1, \ldots, y_n$ and sees them in that order. In terms of the seeing relation, $(n+1)$-ary seriality is the property that every point sees at least one n-tuple; $(n+1)$-ary reflexivity is the property that every point sees the n-tuple consisting of itself only; and so on. Furthermore we represent the seeing relation in the form of arrows, each with n heads. If x sees the tuple $y_1, \ldots, y_i, \ldots, y_n$, i.e. if $Rxy_1 \cdots y_i \cdots y_n$, we draw the following picture.



Note that in the cases of symmetry, transitivity and euclideanness, new seeing arrangements arise from existing ones. We illustrate this by showing how points can be moved. For example, for symmetry at the i-th place, a new seeing arrangement results from swapping the positions of x and y_i. See Figures 3.1, 3.2 and 3.3 for symmetry, transitivity and euclideanness, respectively. (In each case, the diagram on the right is obtained from that on the left by moving point(s) as indicated with dotted arrow(s).)

We list below correspondences between modal formulas and frame properties. Proofs for them follow.

[P_n] : [ser_n]
[D_n] : [ser_n]
[T_n] : [refl_n]
[B^i_n] : [sym^i_n]
[4^i_n] : [trans^i_n]

$[5^i_n]$: $[\mathrm{eucl}^i_n]$

Theorem 3.3.1. $[\mathrm{P}_n]$ *corresponds to* $[\mathrm{ser}_{n+1}]$*, i.e. for any* $(n+1)$*-ary relational frame* $\mathfrak{F}$*,*

$$\mathfrak{F} \models [\mathrm{P}_n] \iff \mathfrak{F} \models [\mathrm{ser}_{n+1}].$$

Proof. For $\Longrightarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is not serial, i.e. there exists an x such that for all $y_1, \ldots, y_n$, $\neg Rxy_1 \cdots y_n$. Clearly for any model $\mathfrak{M}$ on $\mathfrak{F}$, we have $\mathfrak{M}, x \models \Box\bot^n$, i.e. $\mathfrak{M}, x \not\models \Diamond\top^n$. Thus $[\mathrm{P}_n]$ is invalid on $\mathfrak{F}$.

For $\Longleftarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is serial, i.e. every x is related to a tuple $y_1, \ldots, y_n$. Clearly $\mathfrak{M}, x \models \Diamond\top^n$. Thus $[\mathrm{P}_n]$ is valid on $\mathfrak{F}$. ⊣

Theorem 3.3.2. $[\mathrm{D}_n]$ *corresponds to* $[\mathrm{ser}_{n+1}]$*, i.e. for any* $(n+1)$*-ary relational frame* $\mathfrak{F}$*,*

$$\mathfrak{F} \models [\mathrm{D}_n] \iff \mathfrak{F} \models [\mathrm{ser}_{n+1}].$$

Proof. For $\Longrightarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is not serial, i.e. there exists an x such that for all $y_1, \ldots, y_n$, $\neg Rxy_1 \cdots y_n$. Clearly for any model $\mathfrak{M}$ on $\mathfrak{F}$, we have $\mathfrak{M}, x \models \Box\vec{p}$ but $\mathfrak{M}, x \not\models \Diamond(\top, \ldots, p_i, \ldots, \top)$ for every i. Thus $[\mathrm{D}_n]$ is invalid on $\mathfrak{F}$.

For $\Longleftarrow$. Consider a point x in a model $\mathfrak{M}$ on a serial frame $\mathfrak{F}$. Note that $Rxy_1 \cdots y_n$ for some points $y_1, \ldots, y_n$. Assume $\mathfrak{M}, x \models \Box\vec{p}$. Then for some i, $\mathfrak{M}, y_i \models p_i$ and so $\mathfrak{M}, x \models \Diamond(\bot, \ldots, p_i, \ldots, \bot)$. Thus $\mathfrak{M}, x \models [\mathrm{D}_n]$. Since $\mathfrak{M}$ and x are arbitrary, $[\mathrm{D}_n]$ is valid on $\mathfrak{F}$. ⊣

Theorem 3.3.3. $[\mathrm{T}_n]$ *corresponds to* $[\mathrm{refl}_{n+1}]$*, i.e. for any* $(n+1)$*-ary relational frame* $\mathfrak{F}$*,*

$$\mathfrak{F} \models [\mathrm{T}_n] \iff \mathfrak{F} \models [\mathrm{refl}_{n+1}].$$

Proof. For $\Longrightarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is not reflexive, i.e. there exists an x such that $\neg Rxx \cdots x$. Consider a model $\mathfrak{M}$ on $\mathfrak{F}$ such that for all i where $1 \leq i \leq n$,

$$V(p_i) = U - \{x\}.$$

Then $\mathfrak{M}, x \models \Box\vec{p}$ since for any $y_1, \ldots, y_n$ related to x, at least one of them, say y_j, is not x and so $\mathfrak{M}, y_j \models p_j$. But $\mathfrak{M}, i \not\models p_i$ for all i. Thus $\mathfrak{M}$ is a countermodel of $[\mathrm{T}_n]$.

For $\Longleftarrow$. Consider a point x in a model $\mathfrak{M}$ on a reflexive frame $\mathfrak{F}$. Note that $Rxx \cdots x$. Assume $\mathfrak{M}, x \models \vec{p}$. Then for some i, $\mathfrak{M}, x \models p_i$. In other words, $\mathfrak{M}, x \models [\mathrm{T}_n]$. But x and $\mathfrak{M}$ are arbitrary. So $[\mathrm{T}_n]$ is valid on $\mathfrak{F}$. ⊣

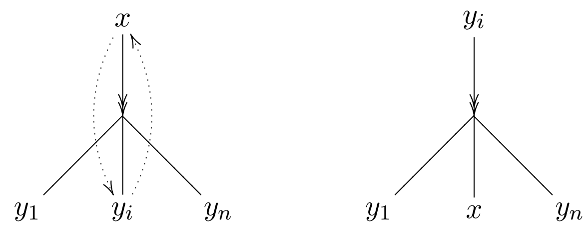


Figure 3.1: Symmetry at the i-th place

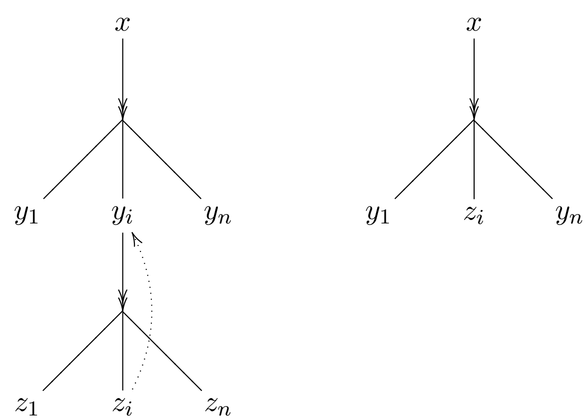


Figure 3.2: Transitivity at the i-th place

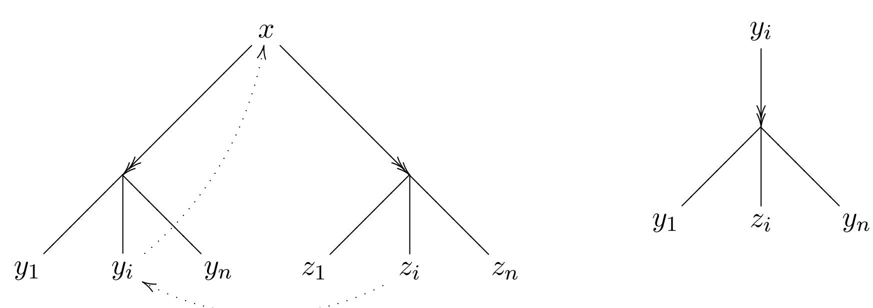


Figure 3.3: Euclideanness at the i-th place

Theorem 3.3.4. $[\mathrm{B}_n^i]$ *corresponds to* $[\mathrm{sym}_{n+1}^i]$, *i.e. for any* $(n+1)$*-ary relational frame* $\mathfrak{F}$,

$$\mathfrak{F} \models [\mathrm{B}_n^i] \iff \mathfrak{F} \models [\mathrm{symm}_{n+1}^i].$$

Proof. For $\Longrightarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is not symmetric at the i-th place, i.e. there exist x, $y_1, \dots, y_i, \dots, y_n$ such that $Rxy_1, \dots, y_i, \dots, y_n$ yet $\neg Ry_iy_1 \cdots x \cdots y_n$. Then any $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ satisfying the following will falsify $[\mathrm{B}_n^i]$ at x (where j ranges from 1 to n).

$$V(p_i) = \{x\}$$
$$V(p_j) = \{y_j\} \text{ for all } j \neq i$$

To prove that $\mathfrak{M}, x \not\models [\mathrm{B}_n^i]$, we first note that $\mathfrak{M}, x \models p_i$ and then show the following:

$$\mathfrak{M}, x \models \Diamond(p_1, \dots, \Box(\neg p_1, \dots, \neg p_n), \dots, p_n).$$

It is clear that $\mathfrak{M}, y_j \models p_j$ for all $j \neq i$. So it remains to show that $\mathfrak{M}, y_i \models \Box(\neg p_1, \dots, \neg p_n)$. Consider arbitrary $z_1, \dots, z_i, \dots, z_n$ such that $Ry_iz_1 \cdots z_i \cdots z_n$. Assume for all $j \neq i$ we have $\mathfrak{M}, z_j \models p_k$. Then $z_j = y_j$, from which it follows that $z_i \neq x$ (since by assumption $\neg Ry_iy_1 \cdots x \cdots y_n$). Thus $\mathfrak{M}, z_i \models \neg p_i$, whence we conclude $\mathfrak{M}, y_i \models \Box(\neg p_1, \dots, \neg p_n)$.

For $\Longleftarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is symmetric at the i-th place. Consider a point x in a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$. We show that $\mathfrak{M}, x \models [\mathrm{B}_n^i]$. So assume $\mathfrak{M}, x \models p_i$. For any $y_1, \dots, y_i, \dots, y_n$ such that $Rxy_1 \cdots y_i \cdots y_n$, if for all $j \neq i$ we have $\mathfrak{M}, x \models p_j$ then $\mathfrak{M}, y_i \models \Diamond(p_1, \dots, p_i, \dots, p_n)$ (since $Ry_iy_1 \cdots x \cdots y_n$ by $[\mathrm{sym}_{n+1}^i]$). Hence $\mathfrak{M}, x \models \Box(\neg p_1, \dots, \Diamond(p_1, \dots, p_i, \dots, p_n), \dots, \neg p_n)$. So $\mathfrak{M}, x \models [\mathrm{B}_n^i]$, whence we conclude that $\mathfrak{F} \models [\mathrm{B}_n^i]$. ⊣

Theorem 3.3.5. $[4_n^i]$ *corresponds to* $[\mathrm{trans}_{n+1}^i]$, *i.e. for any* $(n+1)$*-ary relational frame* $\mathfrak{F}$,

$$\mathfrak{F} \models [4_n^i] \iff \mathfrak{F} \models [\mathrm{trans}_{n+1}^i].$$

Proof. For $\Longrightarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is not transitive at the i-th place, i.e. there exist x, $\vec{y}$ and $\vec{z}$ such that $Rx\vec{y}$, $Ry_i\vec{z}$, yet $\sim Rxy_1 \cdots z_i \cdots y_n$. Consider a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ for which

$$V(p_i) = U - \{z_i\};$$
$$V(p_j) = U - \{y_j\}, \text{where } 1 \leq j \leq n \text{ and } j \neq i.$$

Then $\mathfrak{M}, x \models \Box\vec{p}$, since for any $\vec{w}$ such that $Rx\vec{w}$, if $\mathfrak{M}, w_j \not\models p_j$ for all $j \neq i$ then $w_j = y_j$ for all $j \neq i$ and so $w_i \neq z_i$, which implies $\mathfrak{M}, w_i \models p_i$. As well, $\mathfrak{M}, y_j \models \neg p_j$ for all $j \neq i$;

$\mathfrak{M}, y_i \models \Diamond(\top, \ldots, \neg p_i, \ldots, \top)$; hence $\mathfrak{M}, x \models \Diamond(\neg p_1, \ldots, \Diamond(\top, \ldots, \neg p_i, \ldots, \top), \ldots, \neg p_n)$ or equivalently $\mathfrak{M}, x \not\models \Box(p_1, \ldots, \Box(\bot, \ldots, p_i, \ldots, \bot), \ldots, p_n)$. In other words, $\mathfrak{M}, x \not\models [4^i_n]$.

For $\Longleftarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is transitive at the i-th place. Let x be an arbitrary point of U. To show $\mathfrak{M}, x \models [4^i_n]$, we assume $\mathfrak{M}, x \models \Box\vec{p}$ and show

$$\mathfrak{M}, x \models \Box(p_1, \ldots, \Box(\bot, \ldots, p_i, \ldots, \bot), \ldots, p_n).$$

Consider arbitrary $\vec{y}$ such that $Rx\vec{y}$ and assume $\mathfrak{M}, y_j \not\models p_j$ where $j \neq i$. For any $\vec{z}$ such that $Ry_i\vec{z}$ we have, by $[\mathrm{trans}^i_{n+1}]$, $Rxy_1 \cdots z_i \cdots y_n$. Consequently $\mathfrak{M}, z_i \models p_i$ since, by assumption, $\Box\vec{p}$ is true at x and p_j is false at y_j where $j \neq i$. Thus $\mathfrak{M}, y_j \models \Box(\bot, \ldots, p_i, \ldots, \bot)$ whence we conclude that $\mathfrak{M}, x \models \Box(p_1, \ldots, \Box(\bot, \ldots, p_i, \ldots, \bot), \ldots, p_n)$ as desired. $\dashv$

Theorem 3.3.6. $[5^i_n]$ *corresponds to* $[\mathrm{eucl}^i_{n+1}]$*, i.e. for any* $(n+1)$*-ary relational frame* $\mathfrak{F}$*,*

$$\mathfrak{F} \models [5^i_n] \iff \mathfrak{F} \models [\mathrm{eucl}^i_{n+1}].$$

Proof. For $\Longrightarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is not euclidean at the i-th place, i.e. there exist x, $\vec{y}$ and $\vec{z}$ such that $Rx\vec{y}$, $Rx\vec{z}$, yet $\sim Ry_iy_1 \cdots z_i \cdots y_n$. We show that the following is a countermodel for $[5^i_n]$. Let $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ be a model such that

$$V(p_i) = \{z_i\};$$
$$V(p_j) = \{y_j\}, \text{ where } 1 \leq j \leq n \text{ and } j \neq i.$$

Since $Rx\vec{z}$, we have $\mathfrak{M}, x \models \Diamond(\top, \ldots, p_i, \ldots, \top)$. It remains to show that

$$\mathfrak{M}, x \not\models \Box(\neg p_1, \ldots, \Diamond\vec{p}, \ldots, \neg p_n).$$

Note that $\mathfrak{M}, y_i \models \Box(\neg p_1, \ldots, \neg p_n)$ since for any $w_1, \ldots, w_n$ such that $Ry_iw_1 \cdots w_n$, if $\mathfrak{M}, w_j \models p_j$ for all $j \neq i$, then $w_j = y_j$ for all $j \neq i$, which implies $w_i \neq z_i$ and so $\mathfrak{M}, w_i \models \neg p_i$. Recall that $Rx\vec{y}$. Therefore $\mathfrak{M}, x \models \Diamond(p_1, \ldots, \Box(\neg p_1, \ldots, \neg p_n), \ldots, p_n)$. In other words, $\mathfrak{M}, x \not\models \Box(\neg p_1, \ldots, \Diamond\vec{p}, \ldots, \neg p_n)$ as desired.

For $\Longleftarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ is euclidean at the i-th place. Consider a point x in a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$. We show that $\mathfrak{M}, x \models [5^i_n]$. So assume $\mathfrak{M}, x \models \Diamond(\top, \ldots, p_i, \ldots, \top)$, i.e. there exist $y_1, \ldots, y_n$ such that $Rxy_1 \cdots y_n$ and $\mathfrak{M}, y_i \models p_i$. We need to show that $\mathfrak{M}, x \models \Box(\neg p_1, \ldots, \Diamond\vec{p}, \ldots, \neg p_n)$. So consider arbitrary $z_1, \ldots, z_n$ such that $Rxz_1, \ldots, z_n$ and $\mathfrak{M}, z_j \models p_j$, for all $j \neq i$, and show that $\mathfrak{M}, z_i \models \Diamond\vec{p}$. But this is obvious since $Rz_iz_1 \cdots y_i \cdots z_n$ (by $[\mathrm{eucl}^i_{n+1}]$). $\dashv$

It follows from the correspondence results that the class of frames for each of the normal systems defined in Definition 3.2.1 comprises those frames satisfying the first-order condition(s) that corresponds to the distinctive modal axiom(s) of that system.

Theorem 3.3.7. *The classes of frames for the following normal systems are as indicated.*

$\mathrm{K}_n\mathrm{P}_n$: *Serial frames*
$\mathrm{K}_n\mathrm{D}_n$: *Serial frames*
$\mathrm{K}_n\mathrm{T}_n$: *Reflexive frames*
$\mathrm{K}_n\mathrm{T}_n\mathrm{B}_n$: *Reflexive and symmetric frames*
$\mathrm{K}_n\mathrm{T}_n4_n$: *Reflexive and transitive frames*
$\mathrm{K}_n\mathrm{T}_n5_n$: *Reflexive and euclidean frames*

Modal axioms $[\dagger\mathrm{T}^i_n]$, $[\dagger\mathrm{B}^i_n]$, $[\dagger 4^i_n]$ and $[\dagger 5^i_n]$ correspond to notions of reflexivity, symmetry, transitivity and euclideanness which are different from the ones we defined at the beginning of this section. To distinguish these relational properties from the earlier ones, we add the prefix †, in the same way as we name the corresponding modal axioms.

$[\dagger\mathrm{refl}^i_{n+1}]$ $(\forall x)(\exists\vec{y})Rxy_1\cdots x\cdots y_n$
$[\dagger\mathrm{sym}^i_{n+1}]$ $(\forall x)(\forall\vec{y})(Rx\vec{y}\rightarrow Ry_ix\cdots x)$
$[\dagger\mathrm{trans}^i_{n+1}]$ $(\forall x)(\forall\vec{y})(\forall\vec{z})(Rx\vec{y}\wedge Ry_i\vec{z}\rightarrow Rx\vec{z})$
$[\dagger\mathrm{eucl}^i_{n+1}]$ $(\forall x)(\forall\vec{y})(\forall\vec{z})(Rx\vec{y}\wedge Rx\vec{z}\rightarrow Ry_i\vec{z})$

Correspondence theorem between the † axioms and first-order relational properties are given below. Note that $[\dagger\mathrm{D}^i_n]$ corresponds to the same notion of seriality as $[\mathrm{D}_n]$.

Theorem 3.3.8. *The following modal axioms correspond to the indicated first-order properties of* $(n+1)$*-ary relations.*

$[\dagger\mathrm{D}_n]$: $[\mathrm{ser}_{n+1}]$
$[\dagger\mathrm{T}_n]$: $[\dagger\mathrm{refl}_{n+1}]$
$[\dagger\mathrm{B}^i_n]$: $[\dagger\mathrm{sym}^i_{n+1}]$
$[\dagger 4^i_n]$: $[\dagger\mathrm{trans}^i_{n+1}]$
$[\dagger 5^i_n]$: $[\dagger\mathrm{eucl}^i_{n+1}]$

Proof. We leave the proof to the reader. ⊣

Figures 3.4, 3.5 and 3.6 give us a picture of these properties. As before, the diagrams on the right are obtained from those on the left by moving point(s) as indicated with dotted arrow(s). Note that the movement of points for †transitivity and for †euclideanness can be represented in a simpler way (see Figures 3.7 and 3.8). It is illuminating to compare these diagrams with those for symmetry, transitivity and euclideanness (Figures 3.1, 3.2 and 3.3).

Finally we note that for $n \geq 2$, the following $\mathfrak{L}_n$-formula

$$\Box\vec{p} \to \Diamond\vec{p}$$

does not have an $(n+1)$-ary relational frame, i.e. there is no $(n+1)$-ary relational frame on which the formula is valid. To see this, consider $\mathfrak{F} = \langle U, R\rangle$ where R is an $(n+1)$-ary relation on U. We let $\mathfrak{M}$ be the model $\langle \mathfrak{F}, V\rangle$ such that $V(p_1) = U$ and $V(p_j) = \emptyset$ for all $j \geq 2$. Then for any $x \in U$, $\mathfrak{M}, x \models \Box\vec{p}$ but $\mathfrak{M}, x \not\models \Diamond\vec{p}$. In other words, the formula $\Box\vec{p} \to \Diamond p$ is false at any point in such a model on $\mathfrak{F}$. It is thus invalid on $\mathfrak{F}$.

3.4 Determination for P_n, D_n, T_n, B_n, $\mathrm{S4}_n$ and $\mathrm{S5}_n$

It is straightforward to see that P_n, D_n, T_n, B_n, $\mathrm{S4}_n$ and $\mathrm{S5}_n$ are sound with respect to their respective classes of frames. It remains to show that they are also complete.

As discussed in Section 2.5, we prove completeness of a normal n-adic system with respect to a class of $(n+1)$-ary relational frames by establishing that its canonical model is on a frame belonging to that class.

Theorem 3.4.1. *The following normal n-adic systems are complete with respect to the indicated classes of $(n+1)$-ary relational frames:*

$\mathrm{K}_n\mathrm{P}_n$: *Serial frames*
$\mathrm{K}_n\mathrm{D}_n$: *Serial frames*
$\mathrm{K}_n\mathrm{T}_n$: *Reflexive frames*
$\mathrm{K}_n\mathrm{T}_n\mathrm{B}_n$: *Reflexive and symmetric frames*
$\mathrm{K}_n\mathrm{T}_n4_n$: *Reflexive and transitive frames*
$\mathrm{K}_n\mathrm{T}_n5_n$: *Reflexive and euclidean frames*

Proof. For $\mathrm{K}_n\mathrm{P}_n$. We show that the canonical model $\mathfrak{M}$ of any $\mathrm{K}_n\mathrm{P}_n$-system has a serial $(n+1)$-ary relation R. For any x of $\mathfrak{M}$, we have $\Diamond\top^n \in x$, Then, by the Truth Lemma for normal systems, $\mathfrak{M}, x \models \Diamond\top^n$. So there exist $y_1, \ldots, y_n$ of $\mathfrak{M}$ such that $Rxy_1 \cdots y_n$ and $\mathfrak{M}, y_i \models \top$. In other words, R is serial.

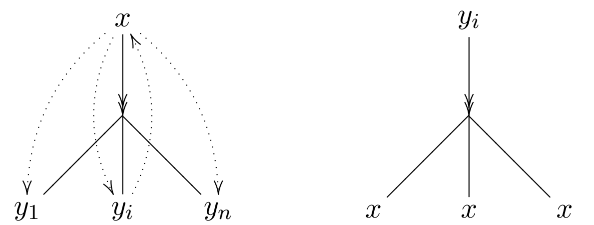


Figure 3.4: †Symmetry at the i-th place

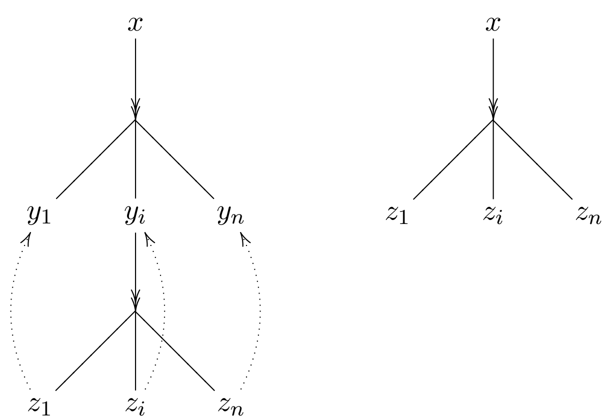


Figure 3.5: †Transitivity at the i-th place

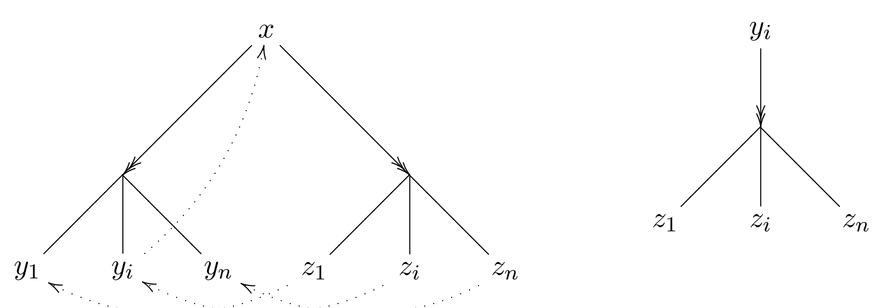


Figure 3.6: †Euclideanness at the i-th place

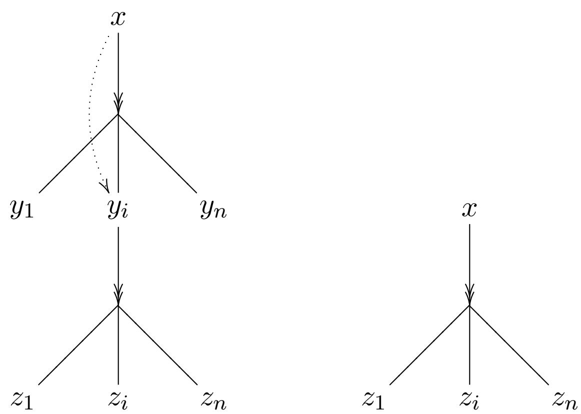


Figure 3.7: †Transitivity at the i-th place: an alternative picture

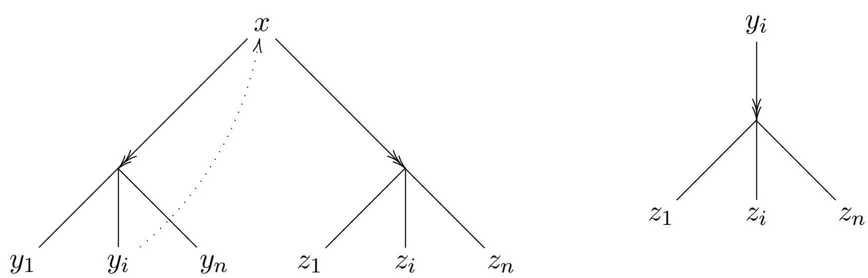


Figure 3.8: †Euclideanness at the i-th place: an alternative picture

For $\mathrm{K}_n\mathrm{D}_n$. We show that the canonical model $\mathfrak{M}$ of any $\mathrm{K}_n\mathrm{D}_n$-system has a serial $(n+1)$-ary relation R. For any x of $\mathfrak{M}$, we have, by substitution, $\Box\top^n \to \bigvee_i \Diamond(\top, \ldots, \top, \ldots, \top) \in x$, i.e. $\Box\top^n \to \Diamond\top^n \in x$. Since $\Box\top^n \in x$ (by $[\mathrm{RN}_n]$), we have, by deductive closure, $\Diamond\top^n \in x$. Hence R is serial, as already shown in the case of $\mathrm{K}_n\mathrm{P}_n$.

For $\mathrm{K}_n\mathrm{T}_n$. We show that the canonical model $\mathfrak{M}$ of any $\mathrm{K}_n\mathrm{T}_n$-system has a reflexive $(n+1)$-ary relation R. To demonstrate $Rxx\cdots x$ (where x is a point of $\mathfrak{M}$), we assume $\Box(\alpha_1, \ldots, \alpha_n) \in x$ and show $\alpha_i \in x$ for some $i \leq n$. From $\Box(\alpha_1, \ldots, \alpha_n) \to \bigvee_i \alpha_i \in x$, it follows that $\alpha_1 \vee \cdots \vee \alpha_n \in x$. Then, by the Truth Lemma, $\mathfrak{M}, x \models \alpha_1 \vee \cdots \vee \alpha_n$ and so, for some $i \leq n$, $\mathfrak{M}, x \models \alpha_i$, whence we conclude $\alpha_i \in x$ as desired.

For $\mathrm{K}_n\mathrm{T}_n\mathrm{B}_n$. We need only show that the canonical model $\mathfrak{M}$ of any $\mathrm{K}_n\mathrm{B}_n$-system has a symmetric relation R (since the canonical relation of any $\mathrm{K}_n\mathrm{T}_n$-system has already been shown to be reflexive). Assume $Rxy_1 \cdots y_n$ (for any x, $y_1, \ldots, y_n$ of $\mathfrak{M}$). To show that $Ry_i y_1 \cdots x \cdots y_n$ (where $i \leq n$), we assume $\Box(\alpha_1, \ldots, \alpha_n) \in y_i$ and demonstrate $\alpha_i \in x$. Based on the assumptions, we have $\mathfrak{M}, y_i \models \Box(\alpha_1, \ldots, \alpha_n)$ and so $\mathfrak{M}, x \models \Diamond(\neg\alpha_1, \ldots, \Box(\alpha_1, \ldots, \alpha_n), \ldots, \neg\alpha_n)$. Consequently $\Diamond(\neg\alpha_1, \ldots, \Box(\alpha_1, \ldots, \alpha_n), \ldots, \neg\alpha_n) \in x$ (by the Truth Lemma). But all substitutional instances of $[\mathrm{B}\Diamond^i_n]$ are in x. Thus $\alpha_i \in x$, which is what we want.

For $\mathrm{K}_n\mathrm{T}_n4_n$. We need only show that the canonical model $\mathfrak{M}$ of any K_n4_n-system has a transitive relation R (since the canonical relation of any $\mathrm{K}_n\mathrm{T}_n$-system has already been shown to be reflexive). Assume $Rxy_1 \cdots y_n$ and $Ry_i z_1 \cdots z_n$ (for any $x, y_1, \ldots, y_n, z_1, \ldots, z_n$). To show $Rxy_1 \cdots z_i \cdots y_n$ (for any $i \leq n$), we assume $\Box(\alpha_1, \ldots, \alpha_n) \in x$ and demonstrate that either $\alpha_k \in y_k$ (for some $k \neq i$) or $\alpha_i \in z_i$. So assume $\alpha_k \notin y_k$ (for all $k \neq i$). Then the following hold.

$\Box(\alpha_1, \ldots, \Box(\bot, \ldots, \alpha_i, \ldots, \bot), \ldots, \alpha_n) \in x$	($[4^i_n]$, deductive closure)
$\Box(\bot, \ldots, \alpha_i, \ldots, \bot) \in y_i$	($Rxy_1 \cdots y_n$, $\alpha_k \notin y_k$)
$\alpha_i \in z_i$	($Ry_i z_1 \cdots z_n$, $\bot \notin z_k$)

But this is what we want to demonstrate.

For $\mathrm{K}_n\mathrm{T}_n5_n$. We need only show that the canonical model $\mathfrak{M}$ of any K_n5_n-system has a euclidean relation R (since the canonical relation of any $\mathrm{K}_n\mathrm{T}_n$-system has already been shown to be reflexive). Assume $Rxy_1 \cdots y_n$ and $Rxz_1 \cdots z_n$ (for any $x, y_1, \ldots, y_n, z_1, \ldots, z_n$). To show $Ry_i y_1 \cdots z_i \cdots y_n$ (for any $i \leq n$), we assume $\Box(\alpha_1, \ldots, \alpha_n) \in y_i$ and demonstrate that either $\alpha_k \in y_k$ (for some $k \neq i$) or $\alpha_i \in z_i$. So assume $\alpha_k \notin y_k$ (for all $k \neq i$). Then following hold.

$\neg\alpha_k \in y_k$	($\alpha_k \notin y_k$ by assumption)
$\mathfrak{M}, y_k \models \neg\alpha_k$	(Truth Lemma)
$\mathfrak{M}, x \models \Diamond(\neg\alpha_1, \ldots, \Box(\alpha_1, \ldots, \alpha_n), \ldots, \neg\alpha_n)$	($Rxy_1 \cdots y_n$ by assumption)
$\Diamond(\neg\alpha_1, \ldots, \Box(\alpha_1, \ldots, \alpha_n), \ldots, \neg\alpha_n) \in x$	(Truth Lemma)
$\Box(\bot, \ldots, \alpha_i, \ldots, \bot) \in x$	($[5\Diamond^i_n]$, deductive closure)
$\alpha_i \in z_i$	($Rxz_1 \cdots z_n$ by assumption)

But this is what we want to demonstrate. $\dashv$

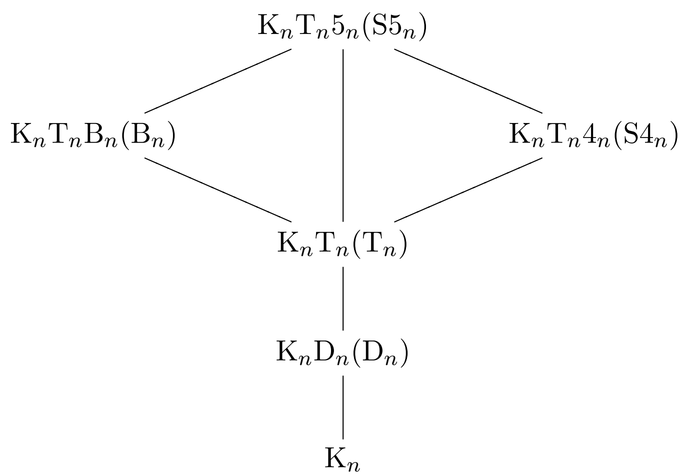


Figure 3.9: Inclusions among normal systems

Finally, inclusions among the normal systems studied in this paper are shown in Figure 3.9, where if two logics are linked by a line, the one at the top properly includes the one at the bottom.

3.5 First-order relational properties

In this section, the first-order relational properties which we have shown to be modally definable in Section 3.3 are studied for their own sake. We shall define a notion of equivalence relation on the basis of these properties.

3.5.1 Inter-derivability of relational properties

Theorem 3.5.1. *Let R be an $(n+1)$-ary relation.*

(1) *If R satisfies* $[\mathrm{refl}_{n+1}]$ *and* $[\mathrm{eucl}^i_{n+1}]$*, then it also satisfies* $[\mathrm{sym}^i_{n+1}]$ *and* $[\mathrm{trans}^i_{n+1}]$.

(2) *If R satisfies* $[\mathrm{sym}^i_{n+1}]$*, then* $[\mathrm{trans}^i_{n+1}]$ *is equivalent to* $[\mathrm{eucl}^i_{n+1}]$.

(3) *If R satisfies* $[\mathrm{sym}_{n+1}]$ *and* $[\mathrm{trans}_{n+1}]$*, then* $[\mathrm{ser}_{n+1}]$ *implies* $[\mathrm{refl}_{n+1}]$ *(whereas the converse holds generally).*

Proof. For (1). Assume that R satisfies $[\mathrm{refl}_{n+1}]$ and $[\mathrm{eucl}^i_{n+1}]$. If $Rxy_1 \cdots y_i \cdots y_n$, then $Ry_iy_1 \cdots x \cdots y_n$ by $[\mathrm{refl}_{n+1}]$ (which gives $Rx \cdots x$) and $[\mathrm{eucl}^i_{n+1}]$. In other words, R satisfies

[sym^i_{n+1}]. If $Rxy_1 \cdots y_i \cdots y_n$ and $Ry_iz_1 \cdots z_i \cdots z_n$, then $Ry_iy_1 \cdots x \cdots y_n$ (since R has already been shown to satisfy [sym^i_{n+1}]) and so $Ry_iy_1 \cdots z_i \cdots y_n$ by [eucl^i_{n+1}]. In other words, R satisfies [trans^i_{n+1}].

For (2). Assume R satisfies [sym^i_{n+1}]. To show that [trans^i_{n+1}] implies [eucl^i_{n+1}], assume R satisfies [trans^i_{n+1}]. If $Rxy_1 \cdots y_i \cdots y_n$ and $Rxz_1 \cdots z_i \cdots z_n$, then $Ry_iy_1 \cdots x \cdots y_n$ by [sym^i_{n+1}], and so $Ry_iy_1 \cdots z_i \cdots y_n$ by [trans^i_{n+1}]. In other words, R satisfies [eucl^i_{n+1}]. To show that [eucl^i_{n+1}] implies [trans^i_{n+1}], assume R satisfies [eucl^i_{n+1}]. If $Rxy_1 \cdots y_i \cdots y_n$ and $Ry_iz_1 \cdots z_i \cdots z_n$, then $Ry_iy_1 \cdots x \cdots y_n$ by [sym^i_{n+1}], and so $Rxy_1 \cdots z_i \cdots y_n$ by [eucl^i_{n+1}]. In other words, R satisfies [trans^i_{n+1}].

For (3). Assume R satisfies [sym_n] and [trans_{n+1}] (i.e. [sym^i_{n+1}] and [trans^i_{n+1}] for all $i \leq n$). To show that [ser_{n+1}] implies [refl_{n+1}], assume R satisfies [ser_{n+1}]. Then for any x, there exist $y_1, y_2, \ldots, y_n$ such that $Rxy_1y_2 \cdots y_n$. Then $Ry_1xy_2 \cdots y_n$ by [sym^1_{n+1}], and so $Rxxy_2 \cdots y_n$ by [trans^1_{n+1}]. By applying the same argument to $Rxxy_2 \cdots y_n$ using the conditions of symmetry and transitivity for the other places, we eventually arrive at $Rxx \cdots x$. In other words R satisfies [refl_{n+1}]. ⊣

An equivalence relation is often characterized as being reflexive, symmetric and transitive. The above shows that it can equally be characterized either as being reflexive and euclidean, or as being serial, symmetric and transitive.

Theorem 3.5.2. *Let* R *be an* $(n+1)$*-ary relation.*

(1) *If* R *satisfies* [sym^i_{n+1}], *then* [trans^i_{n+1}] *is equivalent to* [$\dagger\mathrm{trans}^i_{n+1}$].

(2) *If* R *satisfies* [sym^i_{n+1}], *then* [eucl^i_{n+1}] *is equivalent to* [$\dagger\mathrm{eucl}^i_{n+1}$].

(3) *If* R *satisfies* [sym_{n+1}] *and* [trans_{n+1}], *then* [refl_{n+1}] *implies* [$\dagger\mathrm{refl}_{n+1}$] *(whereas the converse holds generally).*

Proof. For (1). Assume R satisfies [sym^i_{n+1}]. To show that [trans^i_{n+1}] implies [$\dagger\mathrm{trans}^i_{n+1}$], we assume [$\mathrm{trans}^i_{n+1}$] holds for R. If $Rxy_1 \cdots y_i \cdots y_n$ and $Ry_iz_1 \cdots z_i \cdots z_n$, then both $Ry_iy_1 \cdots x \cdots y_n$ and $Rz_iz_1 \cdots y_i \cdots z_n$ (by [sym^i_{n+1}]) and so $Rz_iz_1 \cdots x \cdots z_n$ (by [trans^i_{n+1}]), from which we have $Rxz_1 \cdots z_i \cdots z_n$ (by [sym^i_{n+1}]). In other words, R satisfies [$\dagger\mathrm{trans}^i_{n+1}$]. To show that [$\dagger\mathrm{trans}^i_{n+1}$] implies [$\mathrm{trans}^i_{n+1}$], we assume that [$\dagger\mathrm{trans}^i_{n+1}$] holds for R. If $Rxy_1 \cdots y_i \cdots y_n$ and $Ry_iz_1 \cdots z_i \cdots z_n$, then both $Ry_iy_1 \cdots x \cdots y_n$ and $Rz_iz_1 \cdots y_i \cdots z_n$ (by [sym^i_{n+1}]) and so $Rz_iy_1 \cdots x \cdots y_n$ (by [$\dagger\mathrm{trans}^i_{n+1}$]), from which we have $Rxy_1 \cdots z_i \cdots y_n$ (by [sym^i_{n+1}]). In other words, R satisfies [trans^i_{n+1}].

For (2). Assume R satisfies $[\mathrm{sym}^i_{n+1}]$. To show that $[\mathrm{eucl}^i_{n+1}]$ implies $[\dagger\mathrm{eucl}^i_{n+1}]$, we assume $[\mathrm{eucl}^i_{n+1}]$ holds for R. If $Rxy_1 \cdots y_i \cdots y_n$ and $Rxz_1 \cdots z_i \cdots z_n$, then $Rz_iz_1 \cdots y_i \cdots z_n$ (by $[\mathrm{eucl}^i_{n+1}]$) and so $Ry_iz_1 \cdots z_i \cdots z_n$ (by $[\mathrm{sym}^i_{n+1}]$). In other words, R satisfies $[\dagger\mathrm{eucl}^i_{n+1}]$. To show that $[\dagger\mathrm{eucl}^i_{n+1}]$ implies $[\mathrm{eucl}^i_{n+1}]$, assume $[\dagger\mathrm{eucl}^i_{n+1}]$ holds for R. If $Rxy_1 \cdots y_i \cdots y_n$ and $Rxz_1 \cdots z_i \cdots z_n$, then $Rz_iy_1 \cdots y_i \cdots y_n$ (by $[\dagger\mathrm{eucl}^i_{n+1}]$) and so $Ry_iy_1 \cdots z_i \cdots y_n$ (by $[\mathrm{sym}^i_{n+1}]$). In other words, R satisfies $[\mathrm{eucl}^i_{n+1}]$.

For (3). Note that $[\dagger\mathrm{refl}_{n+1}]$ implies $[\mathrm{ser}_{n+1}]$. Furthermore, assuming $[\mathrm{sym}^i_{n+1}]$ and $[\mathrm{trans}^i_{n+1}]$, $[\mathrm{ser}_{n+1}]$ implies $[\mathrm{refl}_{n+1}]$ (item (3) of Theorem 3.5.1). Thus, on the same assumption, $[\dagger\mathrm{refl}_{n+1}]$ also implies $[\mathrm{refl}_{n+1}]$. ⊣

3.5.2 Symmetry and permutation

An $(n+1)$-ary relation R is said to satisfy *permutation* if $\langle x_0, x_1, \ldots, x_n\rangle \in R$ implies $\langle x_{\pi_0}, x_{\pi_1}, \ldots, x_{\pi_n}\rangle \in R$ for any permutation π of $0, 1, \ldots, n$. We also say that R is *permutational*. For example, a ternary relation R is permutational if $Rxyz$ implies $Rxzy$, $Ryxz$, $Ryzx$, $Rzxy$, and $Rzyx$. We demonstrate below that symmetry (at all places) is equivalent to permutation. It is trivial to show that permutation implies symmetry. So only the converse is of interest.

Theorem 3.5.3. *If an $(n+1)$-ary relation R satisfies $[\mathrm{sym}_{n+1}]$, then it is permutational. (Recall that $[\mathrm{sym}_{n+1}] = \bigwedge_i [\mathrm{sym}^i_{n+1}]$.)*

Proof. We prove the above by induction on n. For $n = 1$, i.e. for a binary relation R, if Rxy, then by symmetry Ryx. So R is permutational.

For the inductive step, we assume that any n-ary symmetric relation is permutational (I.H.), and show that any $(n+1)$-ary relation is also permutational. Let R be an $(n+1)$-ary relation, and let $\langle x_0, x_1, \ldots, x_{n-1}, x_n\rangle \in R$. Observe that any permutation of $0, 1, \ldots, n-1, n$ is of the following form:

$$\pi_0, \pi_1, \ldots, \pi_{i-1}, n, \pi_i, \ldots, \pi_{n-1}$$

where π is a permutation of $0, 1, \ldots, n-1$, and $0 \le i \le n-1$. Thus in order to show that R is permutational, we need to show that

$$\langle x_{\pi_0}, x_{\pi_1}, \ldots, x_{\pi_{i-1}}, x_n, x_{\pi_i}, \ldots, x_{\pi_{n-1}}\rangle \in R. \qquad (1)$$

We define an n-ary relation R_{x_n} as follows:

$$\langle y_0, \ldots, y_{n-1}\rangle \in R_{x_n} \iff \langle y_0, \ldots, y_n, x_n\rangle \in R.$$

It is easy to check that R_{x_n} is symmetric, given that R is symmetric. Thus R_{x_n} is permutational (by I.H.). Clearly $\langle x_0, x_1, \ldots, x_{n-1}\rangle \in R_{x_n}$. Hence $\langle x_{\pi_0}, x_{\pi_1}, \ldots, x_{\pi_{n-1}}\rangle \in R_{x_n}$. Then the following $(n+1)$-tuples are in R by the definition of R_{x_n} and the symmetry of R.

$$\begin{array}{l}
\langle x_{\pi_0}, x_{\pi_1}, \ldots, x_{\pi_{n-1}}, x_n\rangle \\
\langle x_n, x_{\pi_1}, \ldots, x_{\pi_{n-1}}, x_{\pi_0}\rangle \\
\langle x_{\pi_i}, x_{\pi_1}, \ldots, x_{\pi_{i-1}}, x_n, x_{\pi_{i+1}}, \ldots, x_{\pi_{n-1}}, x_{\pi_0}\rangle \\
\langle x_{\pi_{i+1}}, x_{\pi_1}, \ldots, x_{\pi_{i-1}}, x_n, x_{\pi_i}, \ldots, x_{\pi_{n-1}}, x_{\pi_0}\rangle \\
\qquad\vdots \\
\langle x_{\pi_{n-1}}, x_{\pi_1}, \ldots, x_{\pi_{i-1}}, x_n, x_{\pi_i}, \ldots, x_{\pi_{n-2}}, x_{\pi_0}\rangle \\
\langle x_{\pi_0}, x_{\pi_1}, \ldots, x_{\pi_{i-1}}, x_n, x_{\pi_i}, \ldots, x_{\pi_{n-2}}, x_{\pi_{n-1}}\rangle
\end{array}$$

In other words, we have shown (1). This concludes the inductive step. ⊣

3.5.3 Equivalence

An $(n+1)$-ary relation R is called an *equivalence relation* if is reflexive, symmetric and transitive at every co-ordinate, i.e. if R satisfies [refl_{n+1}], [sym^i_{n+1}] and [trans^i_{n+1}] for all i. Note that that any equivalence relation satisfies [eucl^i_{n+1}], [$\dagger\mathrm{trans}^i_{n+1}$], [$\dagger\mathrm{eucl}^i_{n+1}$] and permutation (see Theorems 3.5.1, 3.5.2 and 3.5.3). We shall make use of these conditions when proving properties of equivalence relations.

We show below that an $(n+1)$-ary equivalence relation R on a set U determines a partition of U, i.e. a collection of non-empty subsets of U such that each member of U belongs to one of the subsets and any two distinct subsets are disjoint. Moreover the partition has the following property: any $n+1$ members from the same subset (also called cell) of the partition are related under R, but members from different cells are not so related. First we define, for every x of U, the equivalence class of x modulo R as follows:

$$[x]_R = \{y \in U \mid \exists \vec{z} : Rx\vec{z} \;\&\; y = z_i \text{ for some } i\}.$$

Note that the subscript R is often dropped if it is clear from the context. The collection of all equivalence classes is called the quotient set of U by R (denoted by U/R, read "U

modulo R"). In other words,

$$U/R = \{[x] \mid x \in U\}.$$

Theorem 3.5.4. *Let R be an $(n+1)$-ary equivalence relation on U. Then the following holds for any $x, y \in U$:*

(1) $x \in [x]$.

(2) $[x] = [y]$ *iff* $\exists \vec{z} : Rx\vec{z}$ & $y = z_i$ *for some* i.

(3) *If* $[x] \neq [y]$ *then* $[x] \cap [y] = \emptyset$.

In other words, U/R is a partition of U.

Proof. (1) follows directly from reflexivity and the definition of equivalence class.

For the right-to-left direction of (2). Assume $[x] = [y]$. By (1), $y \in [y]$. Thus $y \in [x]$ and so, by the definition of $[x]$, we have $Rx\vec{z}$ for some $\vec{z}$ and $y = z_i$ for some i.

For the converse of (2), assume $Rx\vec{z}$ for some $\vec{z}$ and $y = z_i$ for some i. Consider arbitrary $w \in [x]$. Then, for some $\vec{v}$ and j, $Rx\vec{v}$ and $w = v_j$. Then by permutation, for some $\vec{v'}$, $Rx\vec{v'}$ and $w = v'_i$. Since R is euclidean, we thus have $Ryz_1 \cdots w \cdots z_n$ (where w occurs at the i-th place), from which it follows that $w \in [y]$. Hence $[x] \subseteq [y]$. It remains to show that $[y] \subseteq [x]$. Consider arbitrary $s \in [y]$. Then, for some $\vec{t}$ and k, $Ry\vec{t}$ and $s = t_k$. Then by permutation, for some $\vec{t'}$, $Ry\vec{t'}$ and $s = t'_i$. Since R is transitive, we thus have $Rxz_1 \cdots s \cdots z_n$ (where s occurs at the i-th place), from which it follows that $s \in [x]$. Hence $[y] \subseteq [x]$.

For (3). We proceed by contraposition. Assume $[x] \cap [y] \neq \emptyset$. It suffices to show the left hand side of (2), from which it follows that $[x] = [y]$. By assumption, there exists a w such that $w \in [x]$ and $w \in [y]$. In other words, both $Rx\vec{v}$ and $w = v_i$ (for some $\vec{v}$ and i) and $Ry\vec{t}$ and $w = t_j$ (for some $\vec{t}$ and j). Then by permutation, $Rw\vec{t'}$ and $y = t'_i$ (for some $\vec{t'}$). Then by transitivity, $Rxv_1 \cdots y \cdots v_n$ (where y occurs at the i-th place). Thus we have shown the left hand side of (2). ⊣

Theorem 3.5.5. *Let R be an $(n+1)$-ary equivalence relation on U. Then for any points $x_0, x_1, \ldots, x_n \in U$,*

$$Rx_0x_1 \cdots x_n \iff \exists C \in U/R : x_0, x_1, \ldots, x_n \in C.$$

Proof. For $\Longrightarrow$. Assume $Rx_0x_1\cdots x_n$. Then, by (2) of Theorem 3.5.4, we have $[x_0] = [x_1] = \cdots = [x_n]$. But $x_0 \in [x_0]$, $x_1 \in [x_1]$ and so on. Thus they belong to the same cell.

For $\Longleftarrow$. Assume that $x_0, x_1, \ldots, x_n$ belong to some cell, say $[y]$, of U/R. Then $Ryw_1\cdots x_0\cdots w_n$ with x_0 at the ith-place, and $Ryv_1\cdots x_1\cdots v_n$ with x_1 at the jth-place (for some $w_1, \ldots, v_1, \ldots$). By permutation, $Rx_0w_1\cdots y\cdots w_n$ and $Ryv_1\cdots x_1\cdots v_n$ with both y and x_1 occurring at the ith-place. Then by transitivity $Rx_0w_1\cdots x_1\cdots w_n$, from which we get $Rx_0x_1w_2\cdots w_n$ by permutation. Repeating the same argument, we eventually arrive at $Rx_0x_1\cdots x_n$. ⊣

An alternative to the account of equivalence relation given by Theorems 3.5.4 and 3.5.5 is as follows. An $(n+1)$-ary equivalence relation R (i.e. an $(n+1)$-ary relation that is reflexive, symmetric and transitive at all places) induces a binary relation R' where

$$R'xy \iff \exists\vec{z} : \exists i : Rx\vec{z} \;\&\; y = z_i.$$

Then R', which can be shown to be a binary equivalence relation, determines a partition of U having the following property: any $n+1$ members from the same cell of the partition are related under the $(n+1)$-ary relation R but members from different cells are not so related.

Chapter 4

Maximal Normal Systems

In this chapter we study two normal systems in the n-adic modal language—the Trivial system and the Verum system. They are the "extremes" for normal n-adic systems in the sense that every such system is included in either the Trivial system or the Verum system (or both). They are also maximal: adding any non-theorem to them would yield inconsistency. Thus they are like Propositional Logic (PL), which has no consistent extension. In fact, as we shall see, they can be translated to PL and so they are said to collapse into PL. Hughes and Cresswell (1996) has a clear exposition of the monadic Trivial system and Verum system. Here we generalize the results to the n-ary case.

4.1 The systems Triv_n and Ver_n

The Trivial system and the Verum system, denoted by Triv_n and Ver_n, are obtained from the smallest n-adic normal system K_n by adding, respectively, the schemas [Triv_n] and [Ver_n].

[Triv_n]	$\Box\vec{p} \leftrightarrow \bigvee_i p_i$
[$\mathrm{Triv}\Diamond_n$]	$\bigwedge_i p_i \leftrightarrow \Diamond\vec{p}$
[Ver_n]	$\Box\vec{p}$
[$\mathrm{Ver}\Diamond_n$]	$\neg\Diamond\vec{p}$

Definition 4.1.1. The n-adic Trivial system and the n-adic Verum system are the following extensions of K_n.

$\mathrm{Triv}_n : \mathrm{K}_n, \quad [\mathrm{Triv}_n]$

$\mathrm{Ver}_n \ : \mathrm{K}_n, \quad [\mathrm{Ver}_n]$

The axiom $[\mathrm{Triv}_n]$ combines $[\mathrm{T}_n]$ and its converse $\bigvee_i p_i \to \Box\vec{p}$. Note the the Trivial system Triv_n is deductively equivalent to the extension of $\mathrm{K}_n\mathrm{D}_n$ with $\bigvee_i p_i \to \Box\vec{p}$ as an axiom. The inclusion of the latter system in Triv_n follows directly from the deducibility of $[\mathrm{D}_n]$ from $[\mathrm{T}_n]$ in normal systems, while the inclusion of Triv_n in the latter system follows from the following theorem.

Theorem 4.1.2. *Let* S *be a* $\mathrm{K}_n\mathrm{D}_n$*-system.* $[\mathrm{T}_n]$ *is derivable if* S *has* $\bigvee_i p_i \to \Box\vec{p}$*.*

Proof. Assume S has $\bigvee_i p_i \to \Box\vec{p}$, or equivalently $\Diamond\vec{p} \to \bigwedge_i p_i$. Then

$\vdash_{\mathrm{S}} \Diamond(\top, \ldots, p_i, \ldots, \top) \to p_i$	by assumption and PL;
$\vdash_{\mathrm{S}} \Box\vec{p} \to \bigvee_i \Diamond(\top, \ldots, p_i, \ldots, \top)$	by $[\mathrm{D}_n]$;
$\vdash_{\mathrm{S}} \Box\vec{p} \to \bigvee_i p_i$	by [MP]. ⊣

$[\mathrm{Triv}_n]$ says that every sequence of $p_1, \ldots, p_n$ is necessary iff one of them is the case, whereas $[\mathrm{Ver}_n]$ says that every sequence of $p_1, \ldots, p_n$ is necessary. We next show that the Trivial system and the Verum system collapse into Propositional Logic in the following sense: every formula in the n-adic modal language ($\mathcal{L}_n$) is deductively equivalent to a formula in the language of propositional logic ($\mathcal{L}$).

Theorem 4.1.3. *Let t be a mapping of $\mathcal{L}_n$-formulas to $\mathcal{L}$-formulas for which α^t (called the $\mathcal{L}$-transform of α under t) is defined recursively as follows.*

$$p^t = p$$
$$\bot^t = \bot$$
$$(\neg\beta)^t = \neg(\beta^t)$$
$$(\beta \vee \gamma)^t = \beta^t \vee \gamma^t$$
$$(\Box(\beta_1, \ldots, \beta_n))^t = \bigvee_i {\beta_i}^t$$

Then for any $\mathcal{L}_n$-formula α, we have $\vdash_{\mathrm{Triv}_n} \alpha \leftrightarrow \alpha^t$.

Proof. The proof is by induction on the construction of α. The basis of the induction follows directly from the following theorems of PL (and so *a fortiori*, theorems of Triv_n): $p \leftrightarrow p$ and $\bot \leftrightarrow \bot$. For the inductive step, we argue as below:

(Case 1) $\alpha = \neg\beta$. By I.H. $\vdash_{\mathrm{Triv}_n} \beta \leftrightarrow \beta^t$. Thus by PL, $\vdash_{\mathrm{Triv}_n} \neg\beta \leftrightarrow \neg(\beta)^t$.

(Case 2) $\alpha = (\beta \vee \gamma)^t$. By I.H. both $\vdash_{\mathrm{Triv}_n} \beta \leftrightarrow \beta^t$ and $\vdash_{\mathrm{Triv}_n} \gamma \leftrightarrow \gamma^t$. Thus by PL, $\vdash_{\mathrm{Triv}_n} (\beta \vee \gamma) \leftrightarrow (\beta^t \vee \gamma^t)$

(Case 3) $\alpha = \Box(\beta_1, \ldots, \beta_n)$. By I.H. $\vdash_{\mathrm{Triv}_n} \beta_i \leftrightarrow {\beta_i}^t$ for all i. But by [Triv_n], $\vdash_{\mathrm{Triv}_n} \Box(\beta_1, \ldots, \beta_n) \leftrightarrow \bigvee_i \beta_i$. Thus by PL, $\vdash_{\mathrm{Triv}_n} \Box(\beta_1, \ldots, \beta_n) \leftrightarrow \bigvee_i (\beta_i)^t$. ⊣

Theorem 4.1.4. *Let t^* be a mapping of $\mathcal{L}_n$-formulas to $\mathcal{L}$-formulas for which $t^*(\alpha)$ (called the $\mathcal{L}$-transform of α under t^*) is defined recursively as follows.*

$$
\begin{aligned}
p^{t^*} &= p \\
(\neg\beta)^{t^*} &= \neg(\beta^{t^*}) \\
(\beta \vee \gamma)^{t^*} &= \beta^{t^*} \vee \gamma^{t^*} \\
(\Box(\beta_1, \ldots, \beta_n))^{t^*} &= \top
\end{aligned}
$$

Then for any $\mathcal{L}_n$-formula α, we have $\vdash_{\mathrm{Ver}_n} \alpha \leftrightarrow \alpha^{t^*}$.

Proof. The proof is by induction on α. The propositional cases are similar to those in the proof for Theorem 4.1.3. For the modal case, it suffices to note that $\vdash_{\mathrm{Ver}_n} \Box(\beta_1, \ldots, \beta_n) \leftrightarrow \top$ since $\vdash_{\mathrm{Ver}_n} \Box(\beta_1, \ldots, \beta_n)$. ⊣

We close this section by proving the following theorem which will be required later (see Proposition 4.3.3).

Theorem 4.1.5. *Let α be a constant formula (i.e. a formula constructed out of $\top$ and $\bot$ by truth-functional and modal connectives) and α^t its $\mathcal{L}$-transform under t. Then $\vdash_{\mathrm{K}_n\mathrm{D}_n} \alpha$ if α^t is* PL*-valid, and $\vdash_{\mathrm{K}_n\mathrm{D}_n} \neg\alpha$ otherwise.*

Proof. The proof is by induction on α. Note that α is a constant formula. So any subformula of α or its $\mathcal{L}$-transform is also a constant formula, i.e a formula that does not have any atoms. Furthermore, a constant formula is PL-valid iff it is satisfiable. Put it another way, a constant formula is PL-invalid iff it is unsatisfiable.

For the basis of induction, we let α be $\top$. The $\mathcal{L}$-transform of $\top$ is itself, which is both PL-valid and a theorem of $\mathrm{K}_n\mathrm{D}_n$.

For the induction step, we consider the following cases. In each case we show that (i) if α^t is PL-valid then α is a theorem of $\mathrm{K}_n\mathrm{D}_n$, and (ii) if α^t is PL-invalid then $\neg\alpha$ is a theorem of $\mathrm{K}_n\mathrm{D}_n$.

(Case 1) $\alpha = \neg\beta$. Then $\alpha^t = \neg\beta^t$.

(i) If $\neg\beta^t$ is PL-valid, then β^t is not PL-valid, and so by I.H. $\vdash_{\mathrm{K}_n\mathrm{D}_n} \neg\beta$.

(ii) If $\neg\beta^t$ is PL-invalid or equivalently $\neg\beta^t$ is unsatisfiable, then $\neg\neg\beta^t$ is PL-valid, and so by I.H. $\vdash_{\mathrm{K}_n\mathrm{D}_n} \neg\neg\beta$,

(Case 2) $\alpha = \beta \vee \gamma$. Then $\alpha^t = \beta^t \vee \gamma^t$.

(i) If $\beta^t \vee \gamma^t$ is PL-valid, then $\beta^t \vee \gamma^t$ is satisfiable, from which it follows that β^t or γ^t is satisfiable (hence PL-valid) and so, by I.H., β or γ is a theorem of $\mathrm{K}_n\mathrm{D}_n$. Therefore $\beta \vee \gamma$ is also a theorem of $\mathrm{K}_n\mathrm{D}_n$.

(ii) If $\beta^t \vee \gamma^t$ is PL-invalid, then $\beta^t \vee \gamma^t$ is unsatisfiable, from which it follows that both β^t and γ^t are unsatisfiable (hence PL-invalid) and so, by I.H., both $\neg\beta$ and $\neg\gamma$ are theorems of $\mathrm{K}_n\mathrm{D}_n$. Therefore $\neg(\beta \vee \gamma)$ is also a theorem of $\mathrm{K}_n\mathrm{D}_n$.

(Case 3) Let α be $\Box(\beta_1, \ldots, \beta_n)$. Then α^t is ${\beta_1}^t \vee \cdots \vee {\beta_n}^t$.

(i) If ${\beta_1}^t \vee \cdots \vee {\beta_n}^t$ is PL-valid and hence satisfiable, then one of ${\beta_i}^t$ is satisfiable and hence PL-valid, from which it follows by I.H. that one of β_i is a theorem of $\mathrm{K}_n\mathrm{D}_n$, implying that $\Box(\beta_1, \ldots, \beta_n)$ is also a theorem of $\mathrm{K}_n\mathrm{D}_n$ (by virtue of [RN_n]).

(ii) If ${\beta_1}^t \vee \cdots \vee {\beta_n}^t$ is PL-invalid and hence unsatisfiable, then all ${\beta_i}^t$ are unsatisfiable and hence PL-invalid, from which we argue as follows.

$\forall i,\ \vdash_{\mathrm{K}_n\mathrm{D}_n} \neg\beta_i$	(I.H.)
$\forall i,\ \vdash_{\mathrm{K}_n\mathrm{D}_n} \Box(\bot^{i-1}, \neg\beta_i, \bot^{n-i})$	([RN_n])
$\vdash_{\mathrm{K}_n\mathrm{D}_n} \bigwedge_i \Box(\bot^{i-1}, \neg\beta_i, \bot^{n-i})$	(PL)
$\vdash_{\mathrm{K}_n\mathrm{D}_n} \Diamond(\neg\beta_1, \ldots, \neg\beta_n)$	([$\mathrm{D}\Diamond_n$])
$\vdash_{\mathrm{K}_n\mathrm{D}_n} \neg\Box(\beta_1, \ldots, \beta_n)$	([Df$\Diamond$])

This completes the induction step of the proof. $\dashv$

4.2 Classes of frames and determination for Triv_n and Ver_n

We next show that [Triv_n] and [Ver_n] correspond to the following conditions of $(n+1)$-ary relations:

[triv_{n+1}]	$(\forall x)(\forall \vec{y})(Rx\vec{y} \leftrightarrow (\forall i)y_i = x)$
[ver_{n+1}]	$(\forall x)(\forall \vec{y})\neg Rx\vec{y}$

Note that an $(n+1)$-ary relation satisfying [triv_{n+1}] comprises all and only constant sequences $\langle x, \ldots, x\rangle$ of length $(n+1)$, and an $(n+1)$-ary relation satisfying [ver_{n+1}] is always empty.

Theorem 4.2.1. [Triv$_n$] *corresponds to* [triv$_{n+1}$], *i.e. for any* $(n+1)$*-ary relational frame* $\mathfrak{F}$,

$$\mathfrak{F} \models [\mathrm{Triv}_n] \iff \mathfrak{F} \models [\mathrm{triv}_{n+1}].$$

Proof. For $\Longrightarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ does not satisfy [triv$_{n+1}$]. In other words, either

(1) there exist an x and a $\vec{y}$ such that $Rx\vec{y}$ but at least one member of $\vec{y}$ is not x, or

(2) there exists an x such that $\neg Rx\cdots x$.

If (1), then define a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ where for all i ranging from 1 to n,

$$V(p_i) = \begin{cases} U - \{x\} & \text{if } y_i = x, \\ \{x\} & \text{if } y_i \neq x. \end{cases}$$

Then $\mathfrak{M}, x \models \Diamond(\neg p_1, \ldots, \neg p_n)$ or equivalent $\mathfrak{M}, x \not\models \Box(p_1, \ldots, p_n)$, but $\mathfrak{M}, x \models \bigvee_i p_i$ (since at least one of p_i is true at x in $\mathfrak{M}$). In other words, [Triv$_n$] fails at x in $\mathfrak{M}$. So it is invalid on $\mathfrak{F}$.

If (2), then define a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ where for all i ranging from 1 to n,

$$V(p_i) = U - \{x\}.$$

Then $\mathfrak{M}, x \models \Box(p_1, \ldots, p_n)$ since for any $\vec{y}$ such that $Rx\vec{y}$ at least one member y_i of $\vec{y}$ is not x and so p_i is true at y_i in $\mathfrak{M}$. (There may be no such $\vec{y}$, in which case $\Box(p_1, \ldots, p_n)$ is true trivially at x in $\mathfrak{M}$.) It is also clear that $\mathfrak{M}, x \not\models \bigvee_i p_i$ since each p_i is false at x in $\mathfrak{M}$. Thus [Triv$_n$] fails at x in $\mathfrak{M}$, from which it follows that it is invalid on $\mathfrak{F}$.

(1) and (2) are all the possible cases. So we conclude [Triv$_n$] is invalid on $\mathfrak{F}$.

For $\Longleftarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ satisfies [triv$_{n+1}$]. Consider an arbitrary point x in an arbitrary model $\mathfrak{M}$ on $\mathfrak{F}$. If $\mathfrak{M}, x \models \Box(p_1, \ldots, p_n)$, then at least one member of $\vec{p}$ is true at x in $\mathfrak{M}$ since x is related to the tuple $\langle x, \ldots, x\rangle$. If $\mathfrak{M}, x \not\models \Box(p_1, \ldots, p_n)$, then each member of $\vec{p}$ is false at x in $\mathfrak{M}$ since x is not related to any tuples other than $\langle x, \ldots, x\rangle$. In other words, $\mathfrak{M}, x \models$ [Triv$_n$], whence we conclude [Triv$_n$] is valid on $\mathfrak{F}$. ⊣

Theorem 4.2.2. [Ver$_n$] *corresponds to* [ver$_{n+1}$], *i.e. for any* $(n+1)$*-ary relational frame* $\mathfrak{F}$,

$$\mathfrak{F} \models [\mathrm{Ver}_n] \iff \mathfrak{F} \models [\mathrm{ver}_{n+1}].$$

Proof. For $\Longrightarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ does not satisfy [ver$_n$], i.e. $Rx\vec{y}$ for some $\vec{y}$. Then define a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ by letting for all i ranging from 1 to n,

$$V(p_i) = U - \{y_i\}.$$

Evidently for all i, $\mathfrak{M}, y_i \not\models p_i$. Hence $\mathfrak{M}, x \not\models \Box(p_1, \ldots, p_n)$ and so [Ver$_n$] is invalid on $\mathfrak{F}$.

For $\Longleftarrow$. Assume $\mathfrak{F} = \langle U, R\rangle$ satisfies [ver$_n$]. In other words, R is empty. So $\Box(p_1, \ldots, p_n)$ is true at any point in any model on $\mathfrak{F}$. Thus [Ver$_n$] is valid on $\mathfrak{F}$. ⊣

Theorem 4.2.3. *The classes of frames for the Trivial system (*Triv$_n$*) and the Verum system (*Ver$_n$*) are the classes of trivial and verum frames, respectively.*

Theorem 4.2.4. *The system* Triv$_n$ *is determined by its class of frames, viz. those frames satisfying* [triv$_{n+1}$].

Proof. It is easy to check that the axiom [Triv$_n$] is valid in the class of frames satisfying the condition [triv$_{n+1}$], and hence the system Triv$_n$ is sound with respect to this class of frames.

For the completeness of Triv$_n$, it suffices to show that its canonical model $\mathfrak{M}_{\mathrm{L}} = \langle U_{\mathrm{L}}, R_{\mathrm{L}}, V_{\mathrm{L}}\rangle$ (where L stands for Triv$_n$) belongs to the class of frames satisfying [triv$_{n+1}$]. In other words, we show that for arbitrary members x, $y_1, \ldots, y_n$ of U_{L},

$$R_{\mathrm{L}} x y_1 \cdots y_n \iff \forall y_i, y_i = x.$$

For $\Longrightarrow$, assume $R_{\mathrm{L}} x y_1 \cdots y_n$. Further assume, for reductio, $y_i \neq x$ for some i. Then there exists a formula α such that either (1) $\alpha \in y_i$ but $\alpha \notin x$ or (2) $\alpha \in x$ but $\alpha \notin y_i$. Suppose (1). Then we have the following.

$\neg\alpha \in x$	(x is maximal)
$\mathfrak{M}_{\mathrm{L}}, x \models \neg\alpha$	(Truth Lemma)
$\mathfrak{M}_{\mathrm{L}}, x \models \bot \vee \cdots \vee \neg\alpha \vee \ldots \bot$	(PL)
$\mathfrak{M}_{\mathrm{L}}, x \models \Box(\bot, \ldots, \neg\alpha, \ldots, \bot)$	([Triv$_n$])
$\Box(\bot, \ldots, \neg\alpha, \ldots, \bot) \in x$	(Truth Lemma)
$\neg\alpha \in y_i$	(Definition of R_{L})
$\alpha \notin y_i$	(y_i is maximal)

But this is absurd since by supposition $\alpha \in y_i$. Now suppose (2). Then $\alpha \in x$, and by an argument similar to the above we have $\alpha \in y_i$, which contradicts the supposition that

$\alpha \notin y_i$. Both (1) and (2) yield contradiction. So by reductio $y_i = x$ for all i, whence we conclude $R_{\mathrm{L}} x y_1, \ldots, y_n$.

For $\Longleftarrow$, what need to be shown is $R_{\mathrm{L}} x x \cdots x$. So we assume, for arbitrary $\alpha_1, \ldots, \alpha_n$, that $\Box(\alpha_1, \ldots, \alpha_n) \in x$ and check if there exists an i such that $\alpha_i \in x$.

$\Box(\alpha_1, \ldots, \alpha_n) \in x$	(Assumption)
$\alpha_1 \vee \cdots \vee \alpha_n \in x$	([Triv$_n$])
$\mathfrak{M}_{\mathrm{L}}, x \models \alpha_1 \vee \cdots \vee \alpha_n$	(Truth Lemma)
$\exists i : \mathfrak{M}_{\mathrm{L}}, x \models \alpha_i$	(PL)
$\exists i : \alpha_i \in x$	(Truth Lemma)

We have thus shown that R_{L} satisfies [triv$_{n+1}$]. It follows that system L, viz. Triv$_n$, is complete with respect to the class of frames satisfying [triv$_{n+1}$]. $\dashv$

Theorem 4.2.5. *The system* Ver$_n$ *is determined by its class of frames, viz. those frames satisfying* [ver$_{n+1}$].

Proof. For the soundness of the system Ver$_n$ with respect to the class of frames whose relation is empty, it suffices to show that the axiom [Ver$_{n+1}$] is valid in this class of frames, which will follow if any frame invalidating [Ver$_{n+1}$] has a non-empty relation. So suppose $\Box(p_1, \ldots, p_n)$ is invalid on a frame $\mathfrak{F} = \langle U, R \rangle$. Then there is a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ such that the formula $\Diamond(\neg p_1, \ldots, \neg p_n)$ is true at some $x \in U$, which implies that x is related to some $y_1, \ldots, y_n$, i.e. R is non-empty.

For the completeness of Ver$_n$, we show that its canonical model $\mathfrak{M}_{\mathrm{L}} = \langle U_{\mathrm{L}}, R_{\mathrm{L}}, V_{\mathrm{L}} \rangle$ (where L is Ver$_n$) has an empty relation. Assume, for reductio, R_{L} is non-empty, i.e. $R_{\mathrm{L}} x y_1 \cdots y_n$ for some elements x, $y_1, \ldots, y_n$ of U_{L}. But $\Box(\bot, \ldots, \bot) \in x$ by virtue of the axiom [Ver$_{n+1}$]. Then it follows from the definition of R_{L} that $\bot \in y_i$ for some i, which however is absurd. Thus by reductio the canonical relation R_{L} is empty. From this we conclude that Ver$_n$ is complete with respect to the class of frames satisfying the condition [ver$_{n+1}$]. $\dashv$

4.3 Maximality

In this section, we show that the Trivial system and the Verum system have the following properties:

(1) Triv_n is a consistent system, and so is Ver_n. However any system which has both $[\mathrm{Triv}_n]$ and $[\mathrm{Ver}_n]$ (in addition to PL) is inconsistent.

(2) Every n-adic normal system is included in Triv_n or in Ver_n (or in both). (A system S_1 is said to be included in another system S_2 if every S_1-theorem is an S_2-theorem.)

(3) Triv_n and Ver_n are maximal. (A system S is said to be maximal if it is consistent and adding any non-theorem to it as an axiom results in an inconsistent system.)

Note that (3) follows from (1) and (2). For if Triv_n were not maximal, i.e. if there existed a non-theorem of Triv_n (say α) such that adding it to Triv_n as an axiom would not result in an inconsistent system, then the resulting system (call it $\mathrm{Triv}_n\alpha$), being a normal system not included in Triv_n, would be included in Ver_n according to (2). In other words, if Triv_n were not maximal, then Ver_n would contain $[\mathrm{Triv}_n]$, which is absurd since by (1) Ver_n is consistent and any PL-system that contains both $[\mathrm{Triv}_n]$ and $[\mathrm{Ver}_n]$ is inconsistent. By a similar argument, we can show that Ver_n is maximal given (1) and (2).

Thus what need to be demonstrated are (1) and (2), from which (3) can be deduced. For (1) we argues as follows.

Proposition 4.3.1. Triv_n *and* Ver_n *are consistent systems, and any* PL*-system that contains both of the schemas* $[\mathrm{Triv}_n]$ *and* $[\mathrm{Ver}_n]$ *is inconsistent.*

Proof. Triv_n is sound with respect to the class of trivial frames, and Ver_n is sound with respect to the class of verum frames (and both classes of frames are non-empty). Therefore each of these systems is satisfiable and thus consistent. However any PL-system that has both of the schemas $[\mathrm{Triv}_n]$ and $[\mathrm{Ver}_n]$ is inconsistent because the falsum is provable in such a system. The following is a proof.

1.	$\Box(p,\ldots,p)$	$[\mathrm{Ver}_n], [\mathrm{US}]$
2.	$\Box(p,\ldots,p) \to p$	$[\mathrm{Triv}_n], \mathrm{PL}$
3.	p	$1, 2, [\mathrm{MP}]$
4.	$\bot$	$3, [\mathrm{US}]$

⊣

Observe that, as a consequence of (1), each of the systems Triv_n and Ver_n does not include the other. Next we show (2) by establishing the following two propositions:

- Every normal system which is not included in Ver_n contains $[\mathrm{D}_n]$, or equivalently $[\mathrm{P}_n]$.
- Every normal system which contains $[\mathrm{D}_n]$ is included in Triv_n.

Proposition 4.3.2. *Every consistent extension of* K_n *which is not included in* Ver_n *contains* $[\mathrm{P}_n]$ *(hence its equivalent* $[\mathrm{D}_n]$*).*

Proof. Let S be a consistent extension of K_n not included in Ver_n. It suffices to show that S has some theorem of the form $\Diamond(\delta_1, \ldots, \delta_n)$. For if so, then $\vdash_{\mathrm{S}} \Diamond(\top, \ldots, \top)$ by the tautology $\delta_i \to \top$ and rule $[\mathrm{RM}\Diamond_n]$.

Since S is not included in Ver_n, there exists a formula α such that

$$\vdash_{\mathrm{S}} \alpha \text{ and } \nvdash_{\mathrm{Ver}_n} \alpha.$$

We rewrite α in conjunctive normal form, then remove any negation before a modal formula by using PL-equivalences $\neg\Box(\alpha_1, \ldots, \alpha_n) \leftrightarrow \Diamond(\neg\alpha_1, \ldots, \neg\alpha_n)$ and $\neg\Diamond(\alpha_1, \ldots, \alpha_n) \leftrightarrow \Box(\neg\alpha_1, \ldots, \neg\alpha_n)$. The resulting formula α' is PL-equivalent (hence S-equivalent) to α. Thus we have,

$$\vdash_{\mathrm{S}} \alpha' \text{ and } \nvdash_{\mathrm{Ver}_n} \alpha'.$$

Moreover α' is of the form $C_1 \wedge \cdots \wedge C_k$ where each conjunct is either:

(a) a PL-formula, or

(b) a disjunction containing a disjunct of the form $\Box(\beta_1, \ldots, \beta_n)$, or

(c) a formula of the form $\Diamond(\beta_{11}, \ldots, \beta_{1n}) \vee \cdots \vee \Diamond(\beta_{m1}, \ldots, \beta_{mn})$, or

(d) a formula of the form $\beta \vee \Diamond(\gamma_{11}, \ldots, \gamma_{1n}) \vee \cdots \vee \Diamond(\gamma_{m1}, \ldots, \gamma_{mn})$ where β is a PL-formula.

But α' is an S-theorem and not a theorem of Ver_n. In other words,

$$\forall i, \vdash_{\mathrm{S}} C_i \text{ and } \exists j : \nvdash_{\mathrm{Ver}_n} C_j.$$

C_j could not be of type (a) since if C_j were a PL-formula, then given that C_j is provable in S we would have C_j provable in PL and so in Ver_n as well. Nor could C_j be of type (b) since $\Box(\beta_1, \ldots, \beta_n)$ is provable in Ver_n, which implies that a disjunction of type (b) is also provable in Ver_n. Therefore C_j is either of type (c) or type (d). We examine each of them below.

If C_j is of type (c), then for some $\delta_1, \ldots, \delta_n$, we have $\vdash_{\text{S}} \Diamond(\delta_1, \ldots, \delta_n)$ since the following is a theorem of K_n and hence also a theorem of S.

$$\Diamond(\beta_{11}, \ldots, \beta_{1n}) \vee \cdots \vee \Diamond(\beta_{m1}, \ldots, \beta_{mn}) \rightarrow \Diamond(\beta_{11} \vee \cdots \vee \beta_{m1}, \ldots, \beta_{1n} \vee \cdots \vee \beta_{mn})$$

If C_j is of type (d), then

$$\vdash_{\text{S}} \beta \vee \Diamond(\delta_1, \ldots, \delta_n)$$

for some formulas $\delta_1, \ldots, \delta_n$ since the following is a theorem of K_n and hence also a theorem of S.

$$\Diamond(\gamma_{11}, \ldots, \gamma_{1n}) \vee \cdots \vee \Diamond(\gamma_{m1}, \ldots, \gamma_{mn}) \rightarrow \Diamond(\gamma_{11} \vee \cdots \vee \gamma_{m1}, \ldots, \gamma_{1n} \vee \cdots \vee \gamma_{mn})$$

β must not be PL-valid because if it were then C_j would become PL-valid and so Ver_n-valid. Thus there is a substitutional instance β^* of β such that β^* is unsatisfiable and $\neg\beta^*$ is PL-valid. Therefore we have the following.

$$\vdash_{\text{S}} \beta^* \vee \Diamond(\delta_1, \ldots, \delta_n)$$
$$\vdash_{\text{S}} \neg\beta^* \rightarrow \Diamond(\delta_1, \ldots, \delta_n)$$
$$\vdash_{\text{S}} \Diamond(\delta_1, \ldots, \delta_n)$$

To summarize, C_j is either of type (c) or of type (d). Each of them implies that a $\Diamond$-formula is derivable in S, whence we conclude $\Diamond(\top, \ldots, \top)$ is also derivable in S. $\dashv$

Proposition 4.3.3. *Every consistent extension of* K_n *which contains* $[\text{D}_n]$ *is included in* Triv_n.

Proof. Let S be an extension of K_n containing $[\text{D}_n]$ (i.e. its axiom and rules include those of K_n, and $[\text{D}_n]$ is a theorem of it). We show that if there is a theorem of S which is not a theorem of Triv_n then S is inconsistent. So assume α is an S-theorem but not a Triv_n-theorem, i.e. assume $\vdash_{\text{S}} \alpha$ and $\nvdash_{\text{Triv}_n} \alpha$.

Given the assumption that $\nvdash_{\text{Triv}_n} \alpha$, the $\mathfrak{L}$-transform of α, denoted α^t, is not PL-valid. (For if α^t were PL-valid, we would have $\vdash_{\text{Triv}_n} \alpha^t$ and so $\vdash_{\text{Triv}_n} \alpha$ since $\vdash_{\text{Triv}_n} \alpha \leftrightarrow \alpha^t$.) Then there exists a substitutional instance α^* of α such that α^* is a constant formula whose PL-transform, viz. $(\alpha^*)^t$, is unsatisfiable. The claim we just made is substantiated by the following argument. Since α^t is not PL-valid, there is a truth-value assignment v that makes α^t false. Then substitute $\top$ for every atom p that occurs in α^t if $v(p) = 1$, and substitute $\bot$ for p otherwise. Note that the formula thus obtained from α^t is unsatisfiable. Apply the same substitution to α. The resulting formula is a constant formula whose $\mathfrak{L}$-transform is precisely the unsatisfiable formula we obtained earlier from α^t.

Since α^* is a constant formula whose $\mathfrak{L}$-transform is unsatisfiable, $\neg\alpha^*$ is a theorem of $\mathrm{K}_n\mathrm{D}_n$ by virtue of Theorem 4.1.5. It follows that $\neg\alpha^*$ is also a theorem of S. However by our original assumption α is a theorem of S, and so its substitutional instance α^* is also a theorem of S. This makes S inconsistent. $\dashv$

Chapter 5

Classical Systems of Modal Logic

The systems of polyadic modal logic we have studied so far are normal systems, which extend K_n, the smallest normal system, with one or more axioms. In this chapter, we investigate systems that are weaker than K_n: they have some but not necessarily all of the theorems and rules of K_n. We call them *classical* systems. While multi-relational frames are used to study polyadic normal systems, we use a more general type of structures for investigating polyadic classical systems, viz. the neighbourhood frames of finitary types discussed in Chapter 1. As in the case of normal systems, we present the simpler monadic classical systems (Section 5.1) before introducing the more general polyadic systems (Section 5.2). Whereas monadic classical systems are well documented (see Segerberg (1971) and Chellas (1980)), their polyadic counterparts appear not to have been studied in the literature. Thus we establish, in detail, the classes of frames and completeness for our polyadic classical systems. (Sections 5.3 to 5.5).

5.1 Classical monadic systems

Following Chellas, we define a series of monadic systems of increasing strength: classical systems, monotonic systems, regular systems, and normal systems. They extend Propositional Logic (PL) with one or several of the following rule and axioms. (These axioms and rules, together with their duals, already appear in Section 2.3. They are listed here again for easy reference.)

[RE] $\dfrac{\vdash \alpha \leftrightarrow \beta}{\vdash \Box\alpha \leftrightarrow \Box\beta}$ [RE$\Diamond$] $\dfrac{\vdash \alpha \leftrightarrow \beta}{\vdash \Diamond\alpha \leftrightarrow \Diamond\beta}$

[M]	$\Box(p \wedge q) \to \Box p \wedge \Box q$	[M$\Diamond$]	$\Diamond p \vee \Diamond q \to \Diamond(p \vee q)$
[C]	$\Box p \wedge \Box q \to \Box(p \wedge q)$	[C$\Diamond$]	$\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$
[N]	$\Box\top$	[N$\Diamond$]	$\neg\Diamond\bot$

Definition 5.1.1. Let S be a monadic system providing PL.

- S is classical if it provides [RE].
- S is monotonic if it is classical and provides [M].
- S is regular if it is monotonic and provides [C].
- S is normal if it is regular and provides [N].

Definition 5.1.2. The smallest classical, monotonic, regular and normal monadic systems are as follows. (Alternative names of the systems are enclosed in parentheses.)

E		:	PL,	[RE]		
EM	(M)	:	E,	[M]		
EMC	(R)	:	E,	[M],	[C]	
EMCN	(K)	:	E,	[M],	[C],	[N]

Each of the above systems can be extended by adding more axioms such as [P], [D], [T], [B], [4], and [5]. Note that the above list of classical systems does not exhaust all possibilities. There are other classical systems that are of interest, for example EMN. But to avoid a long chapter, we limit our attention to those listed above (and their n-adic counterparts in the following sections).

Classical systems weaker than K (in the sense that they lack some of the K-theorems) are incomplete with respect to the class of binary relational frames. A common semantic idiom for these systems is neighbourhood semantics (see Example 1.4.4 for details). The following determination results are standard.

Definition 5.1.3. Let N be a neighbourhood function (of type 1) on U and x a point. $N(x)$ is said to be *closed under supersets* and *closed under intersections*, and *contains the unit* if it satisfies the following conditions, respectively.

[sup]	$\forall a,\ \forall b,\ a \in N(x)\ \&\ a \subseteq b \implies b \in N(x)$.
[int]	$\forall a,\ \forall b,\ a \in N(x)\ \&\ b \in N(x) \implies a \cap b \in N(x)$.
[unit]	$U \in N(x)$.

$N(x)$ is a *quasi-filter* if it is closed under both supersets and intersections. A quasi-filter containing the unit is called a *filter*. (Equivalently a filter is a non-empty quasi-filter.) ⊣

The terminology defined above applies to neighbourhood functions, frames and models. For example, if $N(x)$ is closed under supersets for every x of U, we say that N, $\mathfrak{F} = \langle U, N\rangle$ and $\mathfrak{M} = \langle U, N, V\rangle$ are closed under supersets. Ditto for the other conditions of neighbourhood functions.

Theorem 5.1.4 (Determination of Classical Systems)**.** *The following classical systems are determined by the indicated classes of neighbourhood frames.*

E		:	*All frames*
EM	(M)	:	*Frames closed under supersets*
EMC	(R)	:	*Quasi-filters*
EMCN	(K)	:	*Filters*

5.2 Classical polyadic systems

In this section, we generalize the monadic systems of the previous section to n-adic systems, i.e. systems in the polyadic language $\mathfrak{L}_n$. The rules and axioms we need have already been stated in Section 2.4. They are listed here for quick reference.

[RE^i_n] $$\frac{\vdash \alpha_i \leftrightarrow \beta}{\vdash \Box(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n) \leftrightarrow \Box(\alpha_1,\ldots,\beta,\ldots,\alpha_n)}$$

[M^i_n] $\Box(p_1,\ldots,p_i \wedge q,\ldots,p_n) \rightarrow \Box(p_1,\ldots,p_i,\ldots,p_n) \wedge \Box(p_1,\ldots,q,\ldots,p_n)$

[C^i_n] $\Box(p_1,\ldots,p_i,\ldots,p_n) \wedge \Box(p_1,\ldots,q,\ldots,p_n) \rightarrow \Box(p_1,\ldots,p_i \wedge q,\ldots,p_n)$

[N^i_n] $\Box(p_1,\ldots,\top,\ldots,p_n)$

Duals of the above rules and axioms are as below.

[RE$\Diamond^i_n$] $$\frac{\vdash \alpha_i \leftrightarrow \beta}{\vdash \Diamond(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n) \leftrightarrow \Diamond(\alpha_1,\ldots,\beta,\ldots,\alpha_n)}$$

[M$\Diamond^i_n$] $\Diamond(p_1,\ldots,p_i,\ldots,p_n) \vee \Diamond(p_1,\ldots,q,\ldots,p_n) \rightarrow \Diamond(p_1,\ldots,p_i \vee q,\ldots,p_n)$

[C$\Diamond^i_n$] $\Diamond(p_1,\ldots,p_i \vee q,\ldots,p_n) \rightarrow \Diamond(p_1,\ldots,p_i,\ldots,p_n) \vee \Diamond(p_1,\ldots,q,\ldots,p_n)$

[N$\Diamond^i_n$] $\neg\Diamond(\alpha_1,\ldots,\bot,\ldots,\alpha_n)$

There are n instances of each of the above schemas of axioms and rules, and we refer to them collectively by $[\mathrm{RM}_n]$, $[\mathrm{M}_n]$, $[\mathrm{C}_n]$ and $[\mathrm{N}_n]$ (and similarly for their duals).

Definition 5.2.1. Let S be an n-adic system containing PL.

- S is classical if it provides $[\mathrm{RE}_n]$.
- S is monotonic if it is classical and contains $[\mathrm{M}_n]$.
- S is regular if it is monotonic and contains $[\mathrm{C}_n]$.
- S is normal if it is regular and contains $[\mathrm{N}_n]$.

Definition 5.2.2. The smallest classical, monotonic, regular and normal n-adic systems are listed below. (Alternative names of these systems are given in parentheses.)

$$
\begin{array}{llllll}
\mathrm{E}_n & & : \mathrm{PL}, & [\mathrm{RE}_n] & & \\
\mathrm{E}_n\mathrm{M}_n & (\mathrm{M}_n) & : \mathrm{E}_n, & [\mathrm{M}_n] & & \\
\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n & (\mathrm{R}_n) & : \mathrm{E}_n, & [\mathrm{M}_n], & [\mathrm{C}_n] & \\
\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n\mathrm{N}_n & (\mathrm{K}_n) & : \mathrm{E}_n, & [\mathrm{M}_n], & [\mathrm{C}_n], & [\mathrm{N}_n]
\end{array}
$$

Each of the above system can be extended by adjoining axioms such as $[\mathrm{P}_n]$, $[\mathrm{D}_n]$, $[\mathrm{T}_n]$, $[\mathrm{B}_n]$, $[4_n]$ and $[5_n]$. However we shall not study these extensions in this dissertation; our focus remains on the systems defined in Definitions 5.2.2.

We are not concerned here with proving formulas in classical systems. But the following two meta-theorems are of interest in our present study. The first one provides another characterization of monotonic, regular and normal systems, and the second one is the modal analogue of a correspondence result that we shall come across in the next section.

Theorem 5.2.3. *Let* S *be a classical system.*

(1) $[\mathrm{RM}_n^i]$ *is a rule of* S *iff* $[\mathrm{M}_n^i]$ *is provable in* S*.*

(2) $[\mathrm{RN}_n^i]$ *is a rule of* S *iff* $[\mathrm{N}_n^i]$ *is provable in* S*.*

Proof. For (1). Suppose S has the rule $[\mathrm{RM}_n^i]$. Since $p_i \wedge q \to p_i$ and $p_i \wedge q \to q$ are theorems of PL, they are theorems of S as well. Then by $[\mathrm{RM}_n^i]$, the following hold.

$$\vdash_{\mathrm{S}} \Box(p_1, \ldots, p_i \wedge q, \ldots, p_n) \to \Box(p_1, \ldots, p_i, \ldots, p_n)$$
$$\vdash_{\mathrm{S}} \Box(p_1, \ldots, p_i \wedge q, \ldots, p_n) \to \Box(p_i, \ldots, q, \ldots, p_n)$$

We thus have $[\mathrm{M}_n^i]$ as a theorem of S by PL.

Suppose S has $[\mathrm{M}_n^i]$ as a theorem. To show that S has the rule $[\mathrm{RM}_n^i]$, we assume $\alpha_i \to \beta$ is provable in S.

$$\begin{array}{ll} \vdash_{\mathrm{S}} \alpha_i \wedge \beta \leftrightarrow \alpha_i & \text{(Assumption, PL)} \\ \vdash_{\mathrm{S}} \Box(\alpha_1, \ldots, \alpha_i \wedge \beta, \ldots, \alpha_n) \leftrightarrow \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) & ([\mathrm{RE}_n^i]) \\ \vdash_{\mathrm{S}} \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \to \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n) & ([\mathrm{M}_n^i], \text{PL}) \end{array}$$

In other words, S has the rule $[\mathrm{RM}_n^i]$.

For (2). Suppose S has the rule $[\mathrm{RN}_n^i]$. Since $\top$ is an S-theorem (by virtue of PL), we have $\Box(\alpha_1, \ldots, \top, \ldots, \alpha_n)$ as a theorem of S by applying $[\mathrm{RN}_n^i]$.

Suppose S has $[\mathrm{N}_n^i]$ as a theorem. Assume α_i is provable in S. Then by PL, $\alpha_i \leftrightarrow \top$ is provable in S, and so, by $[\mathrm{RE}_n^i]$, $[\mathrm{N}_n^i]$ and PL, we have $\Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ provable in S as well. In other words, S has the rule $[\mathrm{RN}_n^i]$. $\dashv$

The import of the above theorem is that monotonic systems can be characterized as PL-systems providing $[\mathrm{RM}_n]$, regular systems as PL-systems providing $[\mathrm{RM}_n]$ and $[\mathrm{C}_n]$, and normal systems as PL-systems providing $[\mathrm{RM}_n]$, $[\mathrm{RN}_n]$ and $[\mathrm{C}_n]$. (This accords with the definition of normal systems in Chapter 2.)

Theorem 5.2.4. *Let* S *be a* PL-*system.*

(1) $[\mathrm{RM}_n]$ *are rules of* S *iff the following is a rule of* S.

$$[\mathrm{RM}_n^+] \quad \frac{\vdash \bigwedge_i (\alpha_i \to \beta_i)}{\vdash \Box(\alpha_1, \ldots, \alpha_n) \to \Box(\beta_1, \ldots, \beta_n)}$$

(2) $[\mathrm{M}_n]$ *are provable in* S *iff the following is provable in* S.

$$[\mathrm{M}_n^+] \quad \Box(p_1 \wedge q_1, \ldots, p_n \wedge q_n) \to \Box(p_1, \ldots, p_n) \wedge \Box(q_1, \ldots, q_n)$$

(Recall that $[\mathrm{RM}_n]$ *and* $[\mathrm{M}_n]$ *stand for the collections of all instances of* $[\mathrm{RM}_n^i]$ *and* $[\mathrm{M}_n^i]$, *respectively.)*

Proof. For (1). Given $[\mathrm{RM}_n^+]$, we get $[\mathrm{RM}_n^i]$ (where $1 \leq i \leq n$) simply by letting $\alpha_j = \beta_j$ for all $j \neq i$. For the converse, suppose S has $[\mathrm{RM}_n]$. Assuming S has $\bigwedge_i (\alpha_i \to \beta_i)$, we have the following by virtue of [RM$_n$].

$$\begin{aligned}
&\vdash_{\mathrm{S}} \Box(\alpha_1, \alpha_2, \ldots, \alpha_n) \to \Box(\beta_1, \alpha_2, \ldots, \alpha_n)\\
&\vdash_{\mathrm{S}} \Box(\beta_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \to \Box(\beta_1, \beta_2, \alpha_3, \ldots, \alpha_n)\\
&\vdots\\
&\vdash_{\mathrm{S}} \Box(\beta_1, \ldots, \beta_{n-1}, \alpha_n) \to \Box(\beta_1, \ldots, \beta_{n-1}, \beta_n)
\end{aligned}$$

It follows from the above that $\Box(\alpha_1, \ldots, \alpha_n) \to \Box(\beta_1, \ldots, \beta_n)$ is a theorem of S. In other words, S has [RM$_n^+$].

For (2). Given [M$_n^+$], we get [M$_n^i$] (where $1 \le i \le n$) by letting $p_j = q_j$ for all $j \neq i$. For the converse, suppose S has [M$_n$]. Then we have the following.

$$\begin{aligned}
&\vdash_{\mathrm{S}} \Box(p_1 \wedge q_1, p_2 \wedge q_2, \ldots, p_n \wedge q_n) \to \Box(p_1, p_2 \wedge q_2, \ldots, p_n \wedge q_n)\\
&\vdash_{\mathrm{S}} \Box(p_1, p_2 \wedge q_2, p_3 \wedge q_3, \ldots, p_n \wedge q_n) \to \Box(p_1, p_2, p_3 \wedge q_3, \ldots, p_n \wedge q_n)\\
&\vdots\\
&\vdash_{\mathrm{S}} \Box(p_1, \ldots, p_{n-1}, p_n \wedge q_n) \to \Box(p_1, \ldots, p_{n-1}, p_n)
\end{aligned}$$

From the above we have $\Box(p_1 \wedge q_1, \ldots, p_n \wedge q_n) \to \Box(p_1, \ldots, p_n)$ as an S-theorem. Similarly for $\Box(p_1 \wedge q_1, \ldots, p_n \wedge q_n) \to \Box(q_1, \ldots, q_n)$. Finally, by PL, [M$_n^+$] is a theorem of S. $\dashv$

Since all logics that appear in this chapter are classical systems (which are PL-systems), we shall freely make use of the equivalences stated in the above theorem. It is interesting to compare [C$_n$] (the collection of all instances of [C$_n^i$]) with the following formula.

$$[\mathrm{C}_n^+] \quad \Box(p_1, \ldots, p_n) \wedge \Box(q_1, \ldots, q_n) \to \Box(p_1 \wedge q_1, \ldots, p_n \wedge q_n)$$

It is easy to check that the above is not a theorem of K$_n$ (using relational semantics). So it is *not* equivalent to [C$_n$] in normal systems, let alone in PL-systems or classical systems. We shall return to this when discussing correspondence between modal formulas and properties of neighbourhood frames in the next section.

5.3 Properties of neighbourhood functions

The axioms [M$_n$], [C$_n$] and [N$_n$] are valid in the class of $(n+1)$-ary relational frames. Thus they correspond to the same class of relational frames, viz. the class of all such frames. Put another way, the relational idiom (or more particularly, the $(n+1)$-ary relational idiom) fails to distinguish between [M_n], [C_n] and [N_n]. As well, the systems E_n, M_n and R_n are incomplete with respect to the class of $(n+1)$-ary relational frames since they lack some of the theorems of K_n, which axiomatizes the set of validities in that class of frames. A more suitable semantics for studying classical logics (weaker that K_n) is the neigbhourhood idiom discussed in Section 1.4. In this section we define various properties of neighbourhood functions, and, in the next two, we show that [M_n], [C_n] and [N_n] correspond to these properties and E_n, M_n and R_n are complete with respect to their classes of neighbourhood frames.

5.3.1 Neighbourhoods from the algebraic perspective

A neighbourhood function N of type 1 on a set U of points assigns to each point x a collection $N(x)$ of sets of points. From the algebraic point of view, the collection of all sets of points of U ordered by set inclusion is a complemented distributive lattice, which we denote by $\langle \mathscr{P}(U), \subseteq \rangle$. Equivalently, it is a Boolean algebra, viz. $\langle \mathscr{P}(U), \cap, -, U \rangle$ for which Boolean meet and complementation are the set-theoretic operations of union and complementation, and the maximum (also called the unit element) is U. The algebraic perspective for neighbourhood functions of type 1 can be generalized to neighbourhood functions of arbitrary finite type n. Any such function N on U assigns to each point x a collection $N(x)$ of n-tuples of sets of points, i.e. $N : U \to \mathscr{P}((\mathscr{P}(U))^n)$. Note that the Cartesian product $(\mathscr{P}(U))^n$ is the collection of all n-tuples of sets of points. We define an ordering (denoted $\leq$ and called "less than") on $(\mathscr{P}(U))^n$ as below:

$$\langle a_1, \ldots, a_n \rangle \leq \langle b_1, \ldots, b_n \rangle \iff \forall i,\ a_i \subseteq b_i.$$

It is straightforward to check that $\langle (\mathscr{P}(U))^n, \leq \rangle$ is a complemented distributive lattice. Its corresponding Boolean algebra is $\langle (\mathscr{P}(U))^n, \wedge, -, 1 \rangle$ where the meet, complementation and maximum of the algebra are the following:

$$\langle a_1, \ldots, a_n \rangle \wedge \langle b_1, \ldots, b_n \rangle = \langle a_1 \cap b_1, \ldots, a_n \cap b_n \rangle;$$
$$-\langle a_1, \ldots, a_n \rangle = \langle -a_1, \ldots, -a_n \rangle;$$
$$1 = \langle U, \ldots, U \rangle.$$

Next we define the following properties of $N(x)$, which we call *closure under greater than* (or *upward closure*), *closure under (finite) meets*, and *presence of the maximum*, respectively.

(Recall that $\vec{a}$ and $\vec{b}$ stand for n-sequences or n-tuples of sets of points.)

$$\forall\vec{a},\ \forall\vec{b},\ \vec{a} \in N(x)\ \&\ \vec{a} \leq \vec{b} \implies \vec{b} \in N(x).$$
$$\forall\vec{a},\ \forall\vec{b},\ \vec{a} \in N(x)\ \&\ \vec{b} \in N(x) \implies \vec{a} \wedge \vec{b} \in N(x).$$
$$\langle U, \dots, U\rangle \in N(x).$$

$N(x)$ is called a *quasi-filter* if it is closed under both greater than and meets. It is called a *filter* if it is a quasi-filter and contains the maximum. Alternatively, a filter is a non-empty quasi-filter (since containing the maximum and being non-empty are the same thing if $N(x)$ is already closed under greater than). Given an $\vec{a}$, the collection of all $\vec{b}$ such that $\vec{a} \leq \vec{b}$ is a filter, also called the *principal filter* generated by $\vec{a}$.

5.3.2 Coordinate-wise properties

$N(x)$ where N is of type 1 can be considered as comprising 1-tuples of sets of points. The algebraic conditions of closure under greater than, closure under meets, and presence of the maximum defined in Section 5.3.1 can thus be said to generalize the properties of closure under supersets, closure under intersections, and presence of the unit which are ascribable to neighbourhood functions of type 1 in Section 5.1. However, there exist other ways to generalize the aforementioned properties of type 1 neighbourhood functions. In fact, from the perspective of our modal axioms [M^i_n], [C^i_n] and [N^i_n], a different set of conditions are more relevant. (As we shall see, although [M_n], the conjunction of all instances of [M^i_n], corresponds to closure under greater than, [C_n] does not correspond to closure under meets, nor does [N_n] correspond to presence of the maximum.). The following abbreviations will be used in defining the conditions corresponding to [M^i_n], [C^i_n] and [N^i_n].

$$\widetilde{a} = a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$$
$$a_1, \dots, b, \dots, a_n = a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n$$

The second abbreviation should be familiar to the reader by now. As for the first one, note that $\widetilde{a}$ is an $(n-1)$-sequence of sets of points obtained from $\vec{a}$ by the deletion of a_i. The reason for this unusual labelling will become clear below.

Definition 5.3.1. Let N be a neighbourhood function of type n on a set U of points, and let x be a point. $N(x)$ is said to be *closed under supersets at the i-th place*, to be *closed* *under intersections at the i-th place*, and to *contain the unit at the i-th place* if it satisfies the following conditions, respectively:

$[\mathrm{sup}_n^i]$ $\forall\widetilde{a},\ \forall a_i,\ \forall b,\ \langle a_1,\ldots,a_i,\ldots,a_n\rangle \in N(x)\ \&\ a_i \subseteq b \Longrightarrow \langle a_1,\ldots,b,\ldots,a_n\rangle \in N(x)$;

$[\mathrm{int}_n^i]$ $\forall\widetilde{a},\ \forall a_i,\ \forall b,\ \langle a_1,\ldots,a_i,\ldots,a_n\rangle \in N(x)\ \&\ \langle a_1,\ldots,b,\ldots,a_n\rangle \in N(x) \Longrightarrow \langle a_1,\ldots,a_i\cap b,\ldots,a_n\rangle \in N(x)$;

$[\mathrm{unit}_n^i]$ $\forall\widetilde{a},\ \langle a_1,\ldots,U,\ldots,a_n\rangle \in N(x)$.

If $N(x)$ satisfies $[\mathrm{sup}_n]$, $[\mathrm{int}_n]$ and $[\mathrm{unit}_n]$ (i.e. $[\mathrm{sup}_n^i]$, $[\mathrm{int}_n^i]$ and $[\mathrm{unit}_n^i]$ for every i), we call it simply *closed under supersets*, *closed under intersections*, and *containing the unit*, respectively. ⊣

The above terminology extends to the neighbourhood function N, the frame $\mathfrak{F} = \langle U, N\rangle$, and any model $\mathfrak{M} = \langle U, N, V\rangle$. For example, if every $N(x)$ is closed under supersets at the i-th place, we say that N, $\mathfrak{F}$ and $\mathfrak{M}$ are closed under supersets at the i-th place; similarly, if every $N(x)$ is closed under supersets, we say that N, $\mathfrak{F}$ and $\mathfrak{M}$ are closed under supersets.

Observe that the properties of closure under supersets, closure under meets and presence of the unit (both at the i-th place and at every i-th place) are defined partly with reference to sets of points occurring at some fixed position in the n-tuples. These properties may be described as "coordinate-wise" properties. In fact we could have called $[\mathrm{sup}_n^i]$ *coordinate-wise closure under supersets at the i-th place* and $[\mathrm{sup}_n]$ *coordinate-wise closure under supersets*, and similarly for the other properties. However for simplicity we omit "coordinate-wise". The coordinate-wise character of the properties can be made explicit by restatements deploying the following definition:

$$S_i(\widetilde{a},x) = \{b|\langle a_1,\ldots,b,\ldots,a_n\rangle \in N(x)\}$$

where i is a position, $\widetilde{a}$ is an $(n-1)$-sequence $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n$, all of which are sets of points, and x is a point. In other words, $S_i(\widetilde{a},x)$ consists of all those sets of points such that if any one of them is inserted between a_{i-1} and a_{i+1} in $\widetilde{a}$ then the resulting n-tuple is a member of $N(x)$. We can now restate the properties of neighbourhood functions in Definition 5.3.1 as follows:

$[\mathrm{sup}_n^i]$ $\forall\widetilde{a},\ \forall a_i,\ \forall b,\ a_i \in S_i(\widetilde{a},x)\ \&\ a_i \subseteq b \Longrightarrow b \in S_i(\widetilde{a},x)$;
$[\mathrm{int}_n^i]$ $\forall\widetilde{a},\ \forall a_i,\ \forall b,\ a_i \in S_i(\widetilde{a},x)\ \&\ b \in S_i(\widetilde{a},x) \Longrightarrow a_i \cap b \in S_i(\widetilde{a},x)$;
$[\mathrm{unit}_n^i]$ $\forall\widetilde{a},\ U \in S_i(\widetilde{a},x)$.

The above formulation of the coordinate-wise properties make clear that it is those sets $S_i(\widetilde{a}, x)$ that possess the coordinate-wise properties when we say that $N(x)$ has them. For example, to say that $N(x)$ is closed under supersets at the i-th place is to say that for any $\widetilde{a}$, $S_i(\widetilde{a}, x)$ is closed under supersets; to say that $N(x)$ is closed under supersets is to say that for any i and $\widetilde{a}$, $S_i(\widetilde{a}, x)$ is closed under supersets. Observe that $S_i(\widetilde{a}, x)$ defaults to $N(x)$ in the case of neighbourhood functions of type 1.

It is interesting to compare the coordinate-wise properties with closure under greater than, closure under meets, and presence of the maximum, which we discussed in Section 5.3.1. Closure under supersets (i.e. closure under supersets at every place) is equivalent to closure under greater than. However corresponding remarks cannot be made of the other two pairs of properties. On the one hand, closure under intersections does not imply closure under meets although the latter implies the former. On the other hand, presence of the unit implies presence of the maximum but the latter does not imply the former.

5.3.3 Quasi-filtroids and filtroids

Definition 5.3.2. Let N be a neighbourhood function of type n on a set U of points, and let x be a point. If $N(x)$ is closed under both supersets and intersections, then it is called a *quasi–filtroid*. If $N(x)$ is a quasi-filtroid and contains the unit, it is called a *filtroid*. $\dashv$

As before, given a neighbourhood function N on U of type n, we call the function N, the frame $\mathfrak{F} = \langle U, N \rangle$ or any model $\mathfrak{M} = \langle U, N, V \rangle$ a *(quasi-)filtroid* if every $N(x)$ is a (quasi-)filtroid.

To say that $N(x)$ is a filtroid is to say that for every i and $\widetilde{a}$, $S_i(\widetilde{a}, x)$ is a filter (in the lattice $\langle U, \subseteq \rangle$). Thus we could have called filtroids "coordinate-wise filters". For simplicity, we adopt the term "filtroids". That $N(x)$ is a filtroid does not imply that $N(x)$ is a filter (in the lattice $\langle (\mathscr{P}(U))^n, \leq \rangle$), and vice versa. Therefore the notions of filtroids and filters, despite some similarities, are independent of each other. What we have said of "filtroids" in this paragraph applies equally to quasi-filtroids. (The term "filtroid" comes from Bell (1996). In that paper, Bell proves among other things soundness and completeness of normal systems with respect to the class of filtroids, which coincide with our "coordinate-wise filters". Whereas the starting points of Bell are normal systems and filtroids, we build normal systems on the basis of classical systems, and develop the notion of filtroids from the more basic coordinate-wise properties.)

We have defined filtroids as quasi-filtroids containing the unit. An alternative characterization of filtroids is quasi-filtroids satisfying the following condition of coordinate-wise non-emptiness for every i.

Definition 5.3.3. Let N be a neighbourhood function of type n on a set U of points, and let x be a point. $N(x)$ is said to be *coordinate-wise non-empty at the ith-place* if the following holds:

$$[\mathrm{ne}_n^i] \quad \forall \widetilde{a},\ \exists b : \langle a_1, \ldots, b, \ldots, a_n \rangle \in N(x)$$

If $N(x)$ is coordinate-wise non-empty at every place, we say simply that it is *coordinate-wise non-empty.*

The properties of coordinate-wise non-emptiness extends to neighbourhood functions, frames and models as usual. Like the other coordinate-wise properties, the above can be reformulated as below:

$$[\mathrm{ne}_n^i] \quad \forall \widetilde{a},\ S_i(\widetilde{a}, x) \neq \emptyset.$$

The above makes clear what are non-empty when we say that $N(x)$ is coordinate-wise non-empty at the ith-place, viz. $S_i(\widetilde{a}, x)$ for any $\widetilde{a}$. Note that we use the term "coordinate-wise" in describing our property of non-emptiness (while we drop such a description for the other coordinate-wise properties). This avoids confusion with the condition that $N(x)$ is non-empty, i.e. $N(x) \neq \emptyset$. For neighbourhood functions of type 1, coordinate-wise non-emptiness coincides with non-emptiness. But for $n \geq 2$, while coordinate-wise non-emptiness implies that $N(x)$ is non-empty, the converse does not always hold. That some $\vec{b}$ is in $N(x)$ is insufficient to guarantee that for every i, for every $\widetilde{a}$, the set $S_i(\widetilde{a}, x)$ is non-empty; it guarantees only that for every i, for some $\widetilde{a}$, the set $S_i(\widetilde{a}, x)$ is non-empty (let $\widetilde{a}$ be $\vec{b}$ minus b_i).

That filtroids are precisely coordinate-wise non-empty quasi-filtroids follows from the following theorem.

Theorem 5.3.4. *For any neighbourhood function N of type n on a set U of points and a point x, if $N(x)$ satisfies $[\mathrm{sup}_n^i]$, then*

$$[\mathrm{unit}_n^i] \iff [\mathrm{ne}_n^i].$$

In other words, if $N(x)$ is closed under supersets at the i-th place, then it contains the unit at the ith-place just in case it is coordinate-wise non-empty at the same place.

We close this section with another property of $N(x)$ called augmentation.

Definition 5.3.5. Let N be a neighbourhood function of type n on a set U of points, and let x be a point. $N(x)$ is said to be *augmented at the i-th place* if it satisfies the following:

$$[\text{augm}_n^i] \quad \forall \widetilde{a},\ \forall b,\ S_i(\widetilde{a}, x) \neq \emptyset \ \&\ \bigcap S_i(\widetilde{a}, x) \subseteq b \implies b \in S_i(\widetilde{a}, x).$$

If $N(x)$ is augmented at every place, it is said to be *augmented*.

By extension, we say that N, $\mathfrak{F} = \langle U, N \rangle$ and any model $\mathfrak{M}$ on $\mathfrak{F}$ are augmented if $N(x)$ is augmented for every x. The following theorem provides another definition of augmentation at the i-th place.

Theorem 5.3.6. *For any neighbourhood function N of type n on a set U of points, and a point x, $N(x)$ satisfies $[\text{aug}_n^i]$ iff*

$$\forall \widetilde{a},\ S_i(\widetilde{a}, x) \neq \emptyset \implies S_i(\widetilde{a}, x) = F_d$$

where F_d is the filter generated by $d = \bigcap S_i(\widetilde{a}, x)$. In other words, $N(x)$ is augmented at the ith-place exactly when for any $\widetilde{a}$, either $S_i(\widetilde{a}, x)$ is empty or it is a principal filter.

Note that if $N(x)$ contains the unit and so is non-empty (for example when $N(x)$ is a filtroid), then $N(x)$ is augmented at the i-th place iff

$$\forall \widetilde{a},\ \forall b,\ \bigcap S_i(\widetilde{a}, x) \subseteq b \implies b \in S_i(\widetilde{a}, x),$$

or equivalently

$$\forall \widetilde{a},\ S_i(\widetilde{a}, x) = F_d$$

where F_d is the filter generated by $d = \bigcap S_i(\widetilde{a}, x)$.

5.4 Classes of frames for classical systems

All tautologies are valid in the class of neighbourhood frames, and the rules [MP], [US] and $[\text{RE}_n]$ preserve validity in the same class of frames. While we get PL and $[\text{RE}_n]$ for free in neighbourhood semantics, the same does not hold for $[\text{M}_n]$, $[\text{C}_n]$ and $[\text{N}_n]$. These modal axioms correspond to second-order formulas defining classes of neighbourhood frames we have studied in Section 5.3.2, viz. frames closed under supersets, closed under intersections, and containing the unit, respectively. Consequently the classes of frames for systems E_n, M_n, R_n and K_n are, respectively, the class of all (neighbourhood) frames, frames closed under supersets, quasi-filtroids and filtroids. In this section, we show that $[\mathrm{RE}_n]$ preserve validity in the class of all frames (while we leave the proof of validity-preservation by PL in the class of all frames to the reader). Correspondence results for $[\mathrm{M}_n]$, $[\mathrm{C}_n]$ and $[\mathrm{N}_n]$ then follow, leading to the theorem about the classes of frames for the weakest classical, monotonic, regular and normal systems aforementioned.

Theorem 5.4.1. $[\mathrm{RE}_n]$ *preserves validity in the class of all frames.*

Proof. Assume $\alpha_i \leftrightarrow \beta$ is valid in the class of all frames. To show that

$$\Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \leftrightarrow \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n)$$

is also valid in the class of all frames, we consider a point x of a model $\mathfrak{M}$ on a frame $\mathfrak{F} = \langle U, N \rangle$. By assumption, $\|\alpha_i\|^{\mathfrak{M}} = \|\beta\|^{\mathfrak{M}}$. Thus $\langle \|\alpha_1\|^{\mathfrak{M}}, \ldots, \|\alpha_i\|^{\mathfrak{M}}, \ldots, \|\alpha_n\|^{\mathfrak{M}} \rangle \in N(x)$ iff $\langle \|\alpha_1\|^{\mathfrak{M}}, \ldots, \|\beta\|^{\mathfrak{M}}, \ldots, \|\alpha_n\|^{\mathfrak{M}} \rangle \in N(x)$. In other words, $\mathfrak{M}, x \models \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ iff $\mathfrak{M}, x \models \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n)$, whence $\Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \leftrightarrow \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n)$ is true at x in $\mathfrak{M}$ and so valid on $\mathfrak{F}$ (since x and $\mathfrak{M}$ are arbitrary). We have thus established that $[\mathrm{RE}_n^i]$ preserves validity in the class of all frames. ⊣

Theorem 5.4.2. *Let* $\mathfrak{F} = \langle U, N \rangle$ *be a neighbourhood frame of type* n*.*

(1) $\mathfrak{F} \models [\mathrm{M}_n^i] \iff \mathfrak{F} \models [\sup_n^i]$, *for every* i*.*

(2) $\mathfrak{F} \models [\mathrm{M}_n] \iff \mathfrak{F} \models [\sup_n]$.

Proof. We prove (1) only, leaving to the reader the task of checking that (2) follows from (1). Let i be an arbitrary place.

($\Longrightarrow$) Assume $\mathfrak{F}$ is not closed under supersets at the i-th place, i.e. for some point x, some sequence $\widetilde{a}$ of sets of points, and some sets a_i and b of points, we have $\langle a_1, \ldots, a_i, \ldots, a_n \rangle \in N(x)$ and $a_i \subseteq b$ but $\langle a_1, \ldots, b, \ldots, a_n \rangle \notin N(x)$. Then define $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ by letting:

$$V(p_k) = a_k, \text{ where } 1 \le k \le n;$$
$$V(q) = b.$$

Clearly $\mathfrak{M}, x \models \Box(p_1, \ldots, p_i \wedge q, \ldots, p_n)$ since $\|p_i \wedge q\|^{\mathfrak{M}} = \|p_i\|^{\mathfrak{M}} \cap \|q\|^{\mathfrak{M}} = a_i \cap b = a_i$. However $\mathfrak{M}, x \not\models \Box(p_1, \ldots, q, \ldots, p_n)$ since $\langle \|p_1\|^{\mathfrak{M}}, \ldots, \|q\|^{\mathfrak{M}}, \ldots, \|p_n\|^{\mathfrak{M}} \rangle$ is $\langle a_1, \ldots, b, \ldots, a_n \rangle$, which by assumption is not in $N(x)$. Thus $[\mathrm{M}_n^i]$ is false at x in $\mathfrak{M}$ and so invalid on $\mathfrak{F}$.

($\Longleftarrow$) Assume $\mathfrak{F}$ is closed under supersets at the i-th place. Consider a point x of a model $\mathfrak{M}$ on $\mathfrak{F}$. Clearly $[\mathrm{M}^i_n]$ is true at x in $\mathfrak{M}$ since both $\|p_i\|^{\mathfrak{M}}$ and $\|q\|^{\mathfrak{M}}$ are supersets of $\|p_i \wedge q\|^{\mathfrak{M}}$, and $N(x)$ is closed under supersets at the i-th place. $\dashv$

Theorem 5.4.3. *Let $\mathfrak{F} = \langle U, N\rangle$ be a neighbourhood frame of type n.*

(1) $\mathfrak{F} \models [\mathrm{C}^i_n] \iff \mathfrak{F} \models [\mathrm{int}^i_n]$, *for every i.*

(2) $\mathfrak{F} \models [\mathrm{C}_n] \iff \mathfrak{F} \models [\mathrm{int}_n]$.

Proof. We prove (1), from which (2) follows straightforwardly. Let $i \leq n$ be an arbitrary place.

($\Longrightarrow$) Assume $\mathfrak{F}$ is not closed under intersections at the i-th place, i.e. for some point x, some sequence $\widetilde{a}$ of sets of points, and some sets a_i and b of points, we have both $\langle a_1, \ldots, a_i, \ldots, a_n\rangle \in N(x)$ and $\langle a_1, \ldots, b, \ldots, a_n\rangle \in N(x)$ but $\langle a_1, \ldots, a_i \cap b, \ldots, a_n\rangle \notin N(x)$. Then define $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ by letting:

$$V(p_k) = a_k, \text{ where } 1 \leq k \leq n;$$
$$V(q) = b.$$

Clearly both $\Box(p_1, \ldots, p_i, \ldots, p_n)$ and $\Box(p_1, \ldots, q, \ldots, p_n)$ are true at x in $\mathfrak{M}$. However $\Box(p_1, \ldots, p_i \wedge q, \ldots, p_n)$ are false at x in $\mathfrak{M}$ since $\|p_i \wedge q\|^{\mathfrak{M}}$ or equivalently $\|p_i\|^{\mathfrak{M}} \cap \|q\|^{\mathfrak{M}}$ is $a_i \cap b$.

($\Longleftarrow$) Assume $\mathfrak{F}$ is closed under intersections at the $i$the-place. Consider a point x of a model $\mathfrak{M}$ on $\mathfrak{F}$. Assume both $\Box(p_1, \ldots, p_i, \ldots, p_n)$ and $\Box(p_1, \ldots, q, \ldots, p_n)$ are true at x in $\mathfrak{M}$. Then so is $\Box(p_1, \ldots, p_i \wedge q, \ldots, p_n)$ since $\|p_i \wedge q\|^{\mathfrak{M}}$ is $\|p_i\|^{\mathfrak{M}} \cap \|q\|^{\mathfrak{M}}$ and $N(x)$ is closed under intersections at the i-th place. $\dashv$

Theorem 5.4.4. *Let $\mathfrak{F} = \langle U, N\rangle$ be a neighbourhood frame of type n.*

(1) $\mathfrak{F} \models [\mathrm{N}^i_n] \iff \mathfrak{F} \models [\mathrm{unit}^i_n]$, *for every i.*

(2) $\mathfrak{F} \models [\mathrm{N}_n] \iff \mathfrak{F} \models [\mathrm{unit}_n]$.

Proof. We show correspondence between $[\mathrm{N}^i_n]$ and $[\mathrm{unit}^i_n]$ only, from which correspondence between $[\mathrm{N}_n]$ and $[\mathrm{unit}_n]$ follows. Consider an arbitrary position i.

($\Longrightarrow$) Assume $N(x)$ does not contain the unit at the ith-place, i.e. there exists an $\widetilde{a}$ such that $\langle a_1, \ldots, U, \ldots, a_n\rangle \notin N(x)$. Then define $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ by letting:

$$V(p_k) = a_k, \text{ where } k \neq i;$$
$$V(p_i) = U.$$

Then $\Box(p_1, \ldots, \top, \ldots, p_n)$ is false at x in $\mathfrak{M}$ since $\|\top\|^{\mathfrak{M}}$ is U.

($\Longleftarrow$) Assume $\mathfrak{F}$ contains the unit at the i-th place. It is obvious that $[\mathrm{N}_n^i]$ is true at any point x of any model $\mathfrak{M}$ on $\mathfrak{F}$ since $\|\top\|^{\mathfrak{M}}$ is precisely U. ⊣

All the theorems of E_n are valid in the class of all neighbourhood frames (of type n). Consequently E_n has the class of all neighbourhood frames as its class of frames. Moreover, $[\mathrm{M}_n]$, $[\mathrm{C}_n]$ and $[\mathrm{N}_n]$ correspond to closure under supersets, closure under intersections, and presence of the unit, respectively. We therefore have the following result about the classes of frames for the smallest classical, monotonic, regular and normal systems.

Theorem 5.4.5. *The classes of neighbourhood frames of type n for the following classical systems are as indicated.*

E_n		:	*All frames*
$\mathrm{E}_n\mathrm{M}_n$	(M_n)	:	*Frames closed under supersets*
$\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n$	(R_n)	:	*Quasi-filtroids*
$\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n\mathrm{N}_n$	(K_n)	:	*Filtroids*

We close this section by the following remarks about the correspondence between modal formulas and the conditions of closure under greater than, closure under meets, and presence of the maximum.

Remark 5.4.6. (1) The following formula corresponds to closure under greater than.

$$\Box(p_1 \wedge q_1, \ldots, p_n \wedge q_n) \to \Box(p_1, \ldots, p_n) \wedge \Box(q_1, \ldots, q_n)$$

(2) The following formula corresponds to closure under meets.

$$\Box(p_1, \ldots, p_n) \wedge \Box(q_1, \ldots, q_n) \to \Box(p_1 \wedge q_1, \ldots, p_n \wedge q_n)$$

(3) The following formula corresponds to presence of the maximum.

$$\Box(\top, \ldots, \top)$$

5.5 General neighbourhood frames and completeness

We prove completeness of classical systems with respect to general neighbourhood frames. (Refer to Section 7.1 for the definitions of general neighbourhood frames and models.) We mention here that the monadic EM, EMC, EMCN can be shown to be complete with respect to their classes of "ordinary" neighbourhood frames by suitably supplementing their canonical neighbourhood functions (see Chapter 9 of Chellas (1980) for details). While similar supplementation still works for the n-adic $\mathrm{E}_n\mathrm{M}_n$, no such technique is forthcoming for $\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n$ and $\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n\mathrm{N}_n$ (for $n \geq 2$). By working with general neighbourhood frames, the proofs of completeness for classical systems become straightforward. This shows that general neighbourhood frames provide a more powerful tool for studying modal logic (which is a fragment of second-order logic) than ordinary neighbourhood frames do.

Definition 5.5.1 (Canonical models for classical systems)**.** Let S be a classical system. Its canonical (general neighbourhood) model $\mathfrak{M}_\mathrm{S}$ is the tuple $\langle U_\mathrm{S}, N_\mathrm{S}, A_\mathrm{S}, V_\mathrm{S}\rangle$ where

(1) U_S is the collection of all maximal S-consistent sets of formulas;

(2) N_S assigns to each maximal S-consistent set x a collection $N_\mathrm{S}(x)$ of n-tuples of sets of maximal S-consistent sets such that $\langle a_1, \ldots, a_n\rangle \in N_\mathrm{S}(x)$ iff

$$\exists \alpha_1, \ldots, \alpha_n : \Box(\alpha_1, \ldots, \alpha_n) \in x \ \&\ \forall i,\ a_i = |\alpha_i|_\mathrm{S}.$$

(3) A_S is the collection of all S-proof sets of formulas.

(4) V assigns to each propositional variable p the S-proof set of p, i.e. $|p|_\mathrm{S}$. ⊣

The model defined above is indeed a general neighbourhood model. The reason is as follows.

- U_S is non-empty, given that S is consistent.
- $N_\mathrm{S}(x)$ consists of n-tuples of proof sets, all of which are members of A_S.
- A_S contains $|\bot|_\mathrm{S}$, which is the empty set. Moreover it is closed under complementation, unions, and the operation l_{N_S} since $-|\alpha|_\mathrm{S} = |\neg\alpha|_\mathrm{S}$, $|\alpha|_\mathrm{S} \cup |\beta|_\mathrm{S} = |\alpha \vee \beta|_\mathrm{S}$ and $l_{\mathrm{N}_\mathrm{S}}(|\alpha_1|_\mathrm{S}, \ldots, |\alpha_n|_\mathrm{S}) = |\Box(\alpha_1, \ldots, \alpha_n)|_\mathrm{S}$.
- $V_\mathrm{S}(p)$, i.e. $|p|_\mathrm{S}$, is a member of A_S.

Theorem 5.5.2 (Truth lemma for classical systems)**.** *Let* $\mathfrak{M}_S = \langle U_S, N_S, V_S, A_S \rangle$ *be the canonical model of a classical system* S *in the* n*-adic language* $\mathfrak{L}_n$*. Then for every* $\mathfrak{L}_n$*-formula* α*, the following holds:*

$$\forall x \in U_S,\ \mathfrak{M}_S, x \models \alpha \iff \alpha \in x.$$

Proof. The proof is by induction on the construction of α. We show only the modal case of the inductive step:

$$\forall x \in U_S,\ \mathfrak{M}_S, x \models \Box(\beta_1, \ldots, \beta_n) \iff \Box(\beta_1, \ldots, \beta_n) \in x$$

on the inductive hypothesis that the theorem holds for every β_i with $1 \leq i \leq n$. Consider an arbitrary x in U_S.

For ($\Longrightarrow$), assume $\mathfrak{M}_S, x \models \Box(\beta_1, \ldots, \beta_n)$, i.e. $\langle \|\beta_1\|^{\mathfrak{M}_S}, \ldots, \|\beta_n\|^{\mathfrak{M}_S} \rangle \in N_S(x)$. Then $\langle |\beta_1|_S, \ldots, |\beta_n|_S \rangle \in N_S(x)$ by the inductive hypothesis. So for some formulas $\gamma_1, \ldots, \gamma_n$, we have $|\beta_1|_S = |\gamma_1|_S$, ..., $|\beta_n|_S = |\gamma_n|_S$, and $\Box(\gamma_1, \ldots, \gamma_n) \in x$. But $|\beta_i|_S = |\gamma_i|_S$ iff $\vdash_S \beta_i \leftrightarrow \gamma_i$, for all i from 1 to n; so by repeated application of [RE$_n$] we have $\vdash_S \Box(\beta_1, \ldots, \beta_n) \leftrightarrow \Box(\gamma_1, \ldots, \gamma_n)$. Since $\Box(\gamma_1, \ldots, \gamma_n) \in x$, we conclude $\Box(\beta_1, \ldots, \beta_n) \in x$.

For ($\Longleftarrow$), assume $\Box(\beta_1, \ldots, \beta_n) \in x$. Then $\langle |\beta_1|_S, \ldots, |\beta_n|_S \rangle \in N_S(x)$; so by the inductive hypothesis $\langle \|\beta_1\|^{\mathfrak{M}_S}, \ldots, \|\beta_n\|^{\mathfrak{M}_S} \rangle \in N_S(x)$. In other words, $\mathfrak{M}_S, x \models \Box(\beta_1, \ldots, \beta_n)$. ⊣

Theorem 5.5.3. *The following classical systems are complete with respect to the indicated classes of general neighbourhood frames of type n:*

E_n		:	*All frames*
E_nM_n	(M_n)	:	*Frames closed under supersets*
$E_nM_nC_n$	(R_n)	:	*Quasi-filtroids*
$E_nM_nC_nN_n$	(K_n)	:	*Filtroids*

Proof. Given the truth lemma for classical systems, we demonstrate completeness of the listed systems with respect to the indicated classes of frames by showing that their canonical frames $\mathfrak{M}_S = \langle U_S, N_S, A_S \rangle$ belong to the respective classes. For E_n, it suffices to note that its canonical model is a general neighbourhood model (see the explanation after Definition 5.5.1). For E_nM_n, $E_nM_nC_n$ and $E_nM_nC_nN_n$, we show in Theorems 5.5.4, 5.5.5 and 5.5.6 that their canonical models are superset-closed, a quasi-filtroid and a filtroid, respectively. (Note that in the context of general neighbourhood models and frames, the variables $a_1, \ldots, a_i, \ldots, a_n$ and b used in defining coordinate-wise properties (Definition 5.3.1) range over elements of A rather than elements of $\mathscr{P}(U)$.) ⊣

Theorem 5.5.4. *Let* S *be a monotonic system. Its canonical model* $\mathfrak{M}_S = \langle U_S, N_S, V_S, A_S \rangle$ *is closed under supersets, i.e. for every* x *of* U_S *and position* i*,* $N_S(x)$ *satisfies* $[\text{sup}^i_n]$.

Proof. Consider arbitrary $a_1, \ldots, a_i, \ldots, a_n$ and b, all of which are members of A_S. Assume $\langle a_1, \ldots, a_i, \ldots, a_n \rangle \in N_S$ and $a_i \subseteq b$. Then for some formulas $\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n$, we have $\Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \in x$ and $|\alpha_1|_S = a_1, \ldots, |\alpha_i|_S = a_i, \ldots, |\alpha_n|_S = a_n$. Given that b belongs to A_S, we have $b = |\beta|_S$ for some formula β. Then $|\alpha_i|_S \subseteq |\beta|_S$; so $\vdash_S \alpha_i \to \beta$. Since x is closed under $[\text{RM}^i_n]$, $\Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n) \in x$, and so $\langle a_1, \ldots, b, \ldots, a_n \rangle \in N_S(x)$. We thus conclude that $N_(x)$ is closed under supersets at the i-th place. ⊣

Theorem 5.5.5. *Let* S *be a regular system. Its canonical model* $\mathfrak{M}_S = \langle U_S, N_S, V_S, A_S \rangle$ *is a quasi-filtroid, i.e. for every* x *of* U_S *and position* i*,* $N_S(x)$ *satisfies both* $[\text{sup}^i_n]$ *and* $[\text{int}^i_n]$.

Proof. Given that a regular system is also monotonic, $N_S(x)$ already satisfies $[\text{sup}^i_n]$. It remains to show that $N_S(x)$ satisfies $[\text{int}^i_n]$ as well. Let $a_1, \ldots, a_i, \ldots, a_n$ and b be elements of A_S. Assume both $\langle a_1, \ldots, a_i, \ldots, a_n \rangle \in N_S(x)$ and $\langle a_1, \ldots, b, \ldots, a_n \rangle \in N_S(x)$. Then we have the following:

- for some formulas $\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n$, $|\alpha_1|_S = a_1, \ldots, |\alpha_i|_S = a_i, \ldots, |\alpha_n|_S = a_n$, and $\Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \in x$;
- for some formulas $\alpha'_1, \ldots, \beta, \ldots, \alpha'_n$, $|\alpha'_1|_S = a_1, \ldots, |\beta|_S = b, \ldots, |\alpha'_n|_S = a'_n$, and $\Box(\alpha'_1, \ldots, \beta, \ldots, \alpha'_n) \in x$.

But for all $j \neq i$, $|a_j|_S = |a'_j|_S$. Thus $\vdash_S \alpha_j \leftrightarrow \alpha'_j$. Since x is closed under $[\text{RE}_n]$, $\Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n) \in x$. Moreover x is closed under $[\text{C}^i_n]$. Therefore $\Box(\alpha_1, \ldots, \alpha_i \wedge \beta, \ldots, \alpha_n) \in x$; consequently $\langle a_1, \ldots, a_i \cap b, \ldots, a_n \rangle \in N_S(x)$ since $|\alpha_i \wedge \beta|_S = |\alpha_i|_S \cap |\beta|_S$. Thus we have shown that $N_S(x)$ is closed under intersections at the i-th place. ⊣

Theorem 5.5.6. *Let* S *be a normal system. Its canonical model* $\mathfrak{M}_S = \langle U_S, N_S, V_S, A_S \rangle$ *is a filtroid, i.e. for every* x *of* U_S *and position* i*,* $N_S(x)$ *satisfies all of* $[\text{sup}^i_n]$*,* $[\text{int}^i_n]$ *and* $[\text{unit}^i_n]$.

Proof. It is enough to show that $N_S(x)$ satisfies $[\text{unit}^i_n]$ since S is regular and so N_S already satisfies both $[\text{sup}^i_n]$ and $[\text{int}^i_n]$. Consider arbitrary elements $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ of A_S. There exist formulas $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n$ such that $a_1 = |\alpha_1|_S$ and so on. Since x contains $[\text{N}^i_n]$, $\Box(\alpha_1, \ldots, \top, \ldots, \alpha_n) \in x$ and so $\langle a_1, \ldots, a_{i-1}, U, a_{i+1}, \ldots, a_n \rangle \in N_S(x)$, whence we conclude that $N_S(x)$ contains the unit at the i-place ⊣

Remark 5.5.7. We have shown completeness of E_n, $\mathrm{E}_n\mathrm{M}_n$, $\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n$ and $\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n\mathrm{N}_n$ with respect to classes of general neighbourhood frames (Theorem 5.5.3). It can also be shown that these classes of frames are also the classes of general neighbourhood frames for the listed systems. The proof for Theorem 5.4.1 applies to general neighbourhood frames as it does to ordinary neighbourhood frames. The proofs for Theorems 5.4.2, 5.4.3 and 5.4.4 apply also to general neighbourhood frame $\mathfrak{F} = \langle U, N, A \rangle$: observe that the functions V defined in proving the direction $\Longrightarrow$ are valuations on $\mathfrak{F}$ (since they assign to each atom an elements of A), and, for the direction $\Longleftarrow$, all truth-sets of formulas in $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ are in A. In other words, E_n, $\mathrm{E}_n\mathrm{M}_n$, $\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n$ and $\mathrm{E}_n\mathrm{M}_n\mathrm{C}_n\mathrm{N}_n$ are determined by the their classes of general neighbourhood frames.

Chapter 6

Modal Algebras and General Relational Frames

Stone (1936) established that every Boolean algebra was isomorphic to a set algebra . The result was extended by Jónsson and Tarski (1951) to cover what they called Boolean algebras with operators: every such algebra was isomorphic to a set algebra or, more particularly, a subalgebra of the complex algebra of a relational structure. The connection between Boolean algebras and propositional logic had long been observed by logicians. (In fact, propositional logic as conceived by Boole in the 19th-century was algebraic in character: hence the name *Boolean algebra*.) However the relationship between modal logic and Boolean algebras with operators went unnoticed by philosophers for some time after the publication of Jónsson and Tarski's work even though relational semantics had become popular among modal logicians since Kripke (1959). (For example, Lemmon (1966a,b), writing on algebraic semantics for modal logics, made no reference to Jónsson and Tarski's work.) It was only in the 1970's that modal logicians started to incorporate the representation theorem of Jónsson and Tarski into the theory of relational structures. Goldblatt (1974) and Thomason (1975) showed that certain categories of binary relational structures were dually equivalent to categories of Boolean algebras with unary operators.

In Goldblatt (1974) it was proved that the category of descriptive (binary) relational frames and the category of normal modal algebras (or Boolean algebras with normal unary operations) were dually equivalent by two contravariant functors, which Goldblatt denoted by $^+$ and $_+$. We generalize the result to the n-ary case where n is a finite number. Specifically, we show that the category of descriptive relational frames (DRF) and the category of normal modal algebras (NMA) are dually equivalent by two contravariant functors, which we denote by $\sharp$ and $\flat$. Whereas $\sharp$ transforms DRF to NMA, $\flat$ goes back from NMA to DRF. They do so in such a way that composing them gives us an isomorphic copy of the original category. In technical terms, we have the result that, on the one hand, the composite $\flat \circ \sharp$ is naturally isomorphic to the identity functor on DRF, and, on the other hand, the composite $\sharp \circ \flat$ is naturally isomorphic to the identity functor on NMA. It is in this sense that the two categories are equivalent (dually since $\sharp$ and $\flat$ are contravariant functors).

This chapter is organized in the following way. We first define the categories of modal algebras and normal modal algebras (Section 6.1) as well as the category of descriptive relational frames (Section 6.2). Then we show in Section 6.3 that the function $\sharp$ is a contravariant functor from DRF to NMA. In Section 6.4 we do the same thing for the function $\flat$ that transforms NMA to DRF. Finally both categories are shown to be dually equivalent by these two functors (Section 6.5). Background information about Boolean algebras and category theory is given separately in Appendices A and B, respectively.

6.1 Modal algebras and normal modal algebras

In this section, we define the categories of modal algebras and normal modal algebras. These algebras extend Boolean algebras with n-ary operations. In the case of modal algebras, no conditions are imposed on these n-ary operations. However conditions are imposed on them in the case of normal modal algebras. Note that what we call normal modal algebras here are also known as Boolean algebras with operators (a name due to Jónsson and Tarski).

Definition 6.1.1 (Modal algebras)**.** A *modal algebra* $\mathfrak{A}$ is a tuple $\langle A, +, -, 0, l\rangle$ where $\langle A, +, -, 0\rangle$ is a Boolean algebra and l is an n-ary operation on A. $\dashv$

Boolean meet $\cdot$ and the unit element 1 are defined as for Boolean algebras. The dual of l, denoted m, is the operation

$$m(a_1, \dots, a_n) = -l(-a_1, \dots, -a_n)$$

where $a_1, \dots, a_n$ are elements of the carrier A of the algebra $\mathfrak{A}$.

We define validity of formulas on modal algebras as we do for validity of formulas on Boolean algebras. In other words, a formula α is said to be valid on a modal algebra $\mathfrak{A}$ if the equation $\alpha \approx \top$ holds in $\mathfrak{A}$ or, equivalently, if $V(\alpha) = 1$ for every valuation V on $\mathfrak{A}$. In symbols,

$$\begin{aligned} \mathfrak{A} \models \alpha &\iff \mathfrak{A} \models \alpha \approx \top \\ &\iff V(\alpha) = 1, \text{ for every } V \text{ on } \mathfrak{A}. \end{aligned}$$

(Note that we treat atomic formulas as algebraic variables and, more generally, formulas as terms.)

In what follows, we define various types of mappings between modal algebras that preserve algebraic operations. Note that we call these mappings "algebraic" in order to distinguish them from structure-preserving maps between other types of structures (for instance, relational frames). However when it is clear that we are talking about algebras, we usually drop the adjective "algebraic".

Definition 6.1.2 (Algebraic homomorphisms)**.** Let both $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ and $\mathfrak{A}' = \langle A', +, -, 0, l\rangle$ be modal algebras. A map $f : A \to A'$ is a *homomorphism* if it preserves all algebraic operations, i.e.

$$\begin{aligned} f(a + b) &= f(a) + f(b); \\ f(-a) &= -f(a); \\ f(0) &= 0; \\ f(l(a_1, \dots, a_n)) &= l(f(a_1), \dots, f(a_n)). \end{aligned}$$ ⊣

Note that in the above definition we use the same set of symbols for the operations of $\mathfrak{A}$ and $\mathfrak{A}'$. The context makes clear which algebraic operations we are talking about.

Definition 6.1.3 (Algebraic embeddings)**.** An *embedding* of $\mathfrak{A}$ in $\mathfrak{A}'$ is an injective homomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$. $\mathfrak{A}$ is *embeddable* in $\mathfrak{A}'$ if there is an embedding of $\mathfrak{A}$ in $\mathfrak{A}'$. ⊣

Definition 6.1.4 (Algebraic isomorphisms)**.** An *isomorphism* from $\mathfrak{A}$ to $\mathfrak{A}'$ is a surjective embedding of $\mathfrak{A}$ in $\mathfrak{A}'$ or, equivalently, a bijective homomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$. $\mathfrak{A}$ is *isomorphic* to $\mathfrak{A}'$ if there is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$. ⊣

If $\mathfrak{A}$ is isomorphic to $\mathfrak{A}'$ under f, then $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}$ under f^{-1}. Hence we often call $\mathfrak{A}$ and $\mathfrak{A}'$ isomorphic to each other ($\mathfrak{A} \cong \mathfrak{A}'$).

We say that a mapping f between modal algebras $\mathfrak{A}$ and $\mathfrak{A}'$ (and between any structured sets) *preserves* a property P ascribable to them if $\mathfrak{A}'$ has P whenever $\mathfrak{A}$ has it. If f preserves P in the other direction, i.e. if $\mathfrak{A}'$ has P only when $\mathfrak{A}$ has it, we say that f *respects* P. If P is preserved and respected by f, it is said to be *invariant* under f.

Modal algebras are used as interpretations of modal languages. Thus we are interested in knowing what types of mappings between modal algebras preserve or respect or both preserve and respect validity of formulas. We note here that validity is respected by embeddings, and invariant under isomorphisms.

- If $\mathfrak{A}$ is embeddable in $\mathfrak{A}'$, then for any formula α,

$$\mathfrak{A}' \models \alpha \implies \mathfrak{A} \models \alpha.$$

- If $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic, then for any formula α,

$$\mathfrak{A} \models \alpha \iff \mathfrak{A}' \models \alpha.$$

Definition 6.1.5 (Normal modal algebras)**.** A modal algebra $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ is *normal* if the operation l satisfies the following conditions of normality and multiplicativity, respectively. (Note that $\langle a_1, \ldots, 1, \ldots, a_n\rangle$ stands for $\langle a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n\rangle$, and the same applies to other similar cases.)

$$l(a_1, \ldots, 1, \ldots, a_n) = 1.$$
$$l(a_1, \ldots, a_i, \ldots, a_n) \cdot l(a_1, \ldots, b, \ldots, a_n) = l(a_1, \ldots, a_i \cdot b, \ldots, a_n). \qquad \dashv$$

Given our definition of the dual operation m, it can easily be checked that the following conditions of normality and additivity hold for normal modal algebras.

$$m(a_1, \ldots, 0, \ldots, a_n) = 0.$$
$$m(a_1, \ldots, a_i, \ldots, a_n) + m(a_1, \ldots, b, \ldots, a_n) = m(a_1, \ldots, a_i + b, \ldots, a_n).$$

Normal modal algebras are also known as Boolean algebras with operators (BAO). We shall use the terms interchangeably. In the following we define the categories of modal algebras and normal modal algebras.

Definition 6.1.6 (The category of modal algebras)**.** *The category of modal algebras*, MA, consists of all modal algebras as its objects and all homomorphisms between modal algebras as its arrows. The operations of domain, codomain, composition and identity are the usual ones for functions or maps. $\dashv$

Definition 6.1.7 (The category of normal modal algebras)**.** *The category of normal modal algebras*, NMA, consists of all normal modal algebras as its objects and all homomorphisms between normal modal algebras as its arrows. The operations of domain, codomain, composition and identity are the usual ones for functions or maps. ⊣

6.2 General relational frames

The notions of general relational frames and models extend that of relational frames and models with an additional element A which is a collection of sets of points of a frame or model subject to certain closure conditions. (The reason for calling this set A will be explained later.) If it is clear that we are talking about general relational frames or models rather than ordinary relational frames or models, we use the simpler description "relational frame" or "relational model". The even simpler term "frame" or "model" is used if the type of frames or models is obvious from the context.

Given an $(n+1)$-ary relation R on a set W, we let l_R be an n-ary operation on $\mathscr{P}(W)$ defined as follows (where $a_1, \ldots, a_n \subseteq W$):

$$l_R(a_1, \ldots, a_n) = \{x_0 \in W \mid \forall x_1, \ldots, x_n \in W, Rx_0x_1 \cdots x_n \implies \exists i \geq 1 : x_i \in a_i\}.$$

The dual of l_R, denoted m_R, is the operation

$$m_R(a_1, \ldots, a_n) = -l_R(-a_1, \ldots, -a_n).$$

It follows from the above that

$$m_R(a_1, \ldots, a_n) = \{x_0 \in W \mid \exists x_1, \ldots, x_n \in W : Rx_0x_1 \cdots x_n \ \& \ \forall i \geq 1, x_i \in a_i\}.$$

Recall that if $Rx_0x_1 \cdots x_n$ we say that x_0 sees the tuple $\langle x_1, \ldots, x_n \rangle$. Thus, $l_R(a_1, \ldots, a_n)$ consists of all points x_0 such that whatever tuple, say $\langle x_1, \ldots, x_n \rangle$, it sees has a member x_i in a_i. Using the same metaphor, x_0 is in $m_R(a_1, \ldots, a_n)$ iff x_0 sees a tuple, say $\langle x_1, \ldots, x_n \rangle$, such that each member x_i comes from a_i.

Definition 6.2.1 (General relational frames)**.** A *general* $(n+1)$*-ary relational frame* $\mathfrak{F}$ is a triple $\langle W, R, A \rangle$ of which:

(1) W is a non-empty set of points;

(2) R is an $(n+1)$-ary relation on W;

(3) $A \subseteq \mathscr{P}(W)$ contains $\emptyset$, and is closed under $\cup$, $-$ and l_R. ⊣

Definition 6.2.2 (General relational models)**.** Let $\mathfrak{F} = \langle W, R, A\rangle$ be a general relational frame. A *general relational model* $\mathfrak{M}$ on $\mathfrak{F}$ is a pair $\langle \mathfrak{F}, V\rangle$ or equivalently a quadruple $\langle W, R, A, V\rangle$ where V, called a valuation on $\mathfrak{F}$, assigns to each atom an element of A. ⊣

Truth of formulas in general relational models and validity of formulas on general relational frames are defined as in the cases of ordinary relational models and frames. Note that for any general relational frame $\mathfrak{F} = \langle W, R, A\rangle$, the set A contains all the truth-sets of formulas in any model $\mathfrak{M}$ on $\mathfrak{F}$. In other words, for any formula α and model $\mathfrak{M} = \langle W, R, A, V\rangle$, we have $\|\alpha\|^{\mathfrak{M}} \in A$.

Indeed A is the carrier of a modal algebra, viz. $\langle A, \cup, -, \emptyset, l_R\rangle$. This explains why we denote the set by the symbol A, where A stands for "algebra". Some authors use the symbol A in the sense of "admissible": the set A is a collection of admissible sets of points of W, and a valuation V on $\mathfrak{F}$ assigns to each atom an admissible set of points.

Observe that ordinary relational frames and models are special cases of general relational frames and models, viz. those with their components A being identical to the power set of W. In other words, the accompanying algebra of an ordinary relational frame $\mathfrak{F} = \langle W, R\rangle$ or model $\mathfrak{M} = \langle W, R, V\rangle$ is the power set algebra $\langle \mathscr{P}(W), \cup, -, \emptyset, l_R\rangle$.

A structure preserving map f from a relational structure $\langle W, R\rangle$ to another one $\langle W', R'\rangle$, as studied in first-order model theory, is usually of one of the following types.

- Homomorphisms: if $Rx_0x_1 \cdots x_n$ then $R'f(x_0)f(x_1)\cdots f(x_n)$.
- Strong homomorphisms: $Rx_0x_1 \cdots x_n$ iff $R'f(x_0)f(x_1)\cdots f(x_n)$.
- Embeddings: injective strong homomorphisms.
- Isomorphisms: surjective embeddings, or equivalently bijective homomorphisms.

While surjective strong homomorphism is sufficient for the preservation of validity of modal formulas, it is stronger than necessary. There is a weaker but more useful notion, which we call "general relational frame morphism" (or simply "frame morphism" if the type of frames is clear).

Definition 6.2.3 (General relational frame morphisms)**.** Let $\mathfrak{F} = \langle W, R, A\rangle$ and $\mathfrak{F}' = \langle W', R', A'\rangle$ be frames. A map $f : W \to W'$ is a *frame morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ if all of the following conditions hold. (Unless otherwise stated, $x_0, x_1, \ldots, x_n$ range over the elements of W, $y_1, \ldots, y_n$ over the elements of W', and b over the elements of A'.)

(R1) $Rx_0x_1 \cdots x_n \implies R'f(x_0)f(x_1) \cdots f(x_n)$.

(R2) $R'f(x_0)y_1 \cdots y_n \implies (\exists x_1, \ldots, x_n \in W : Rx_0x_1 \cdots x_n \ \&\ \forall i \geq 1, f(x_i) = y_i)$.

(A1) $f^{-1}[b] \in A$. ⊣

Note that if the algebraic component A is dropped, frame morphism is what has been known as p-morphism (for pseudo-epimorphism), bounded morphism or zig-zag morphism in the literature.

In the following definitions, let $\mathfrak{F} = \langle W, R, A \rangle$ and $\mathfrak{F}' = \langle W', R', A' \rangle$ be general relational frames. As in the case of general relational frame morphisms, the description "general relational" will be omitted wherever avoidable.

Definition 6.2.4 (General relational frame morphic images)**.** $\mathfrak{F}'$ is a *frame morphic image* of $\mathfrak{F}$ if there is a surjective frame morphism from $\mathfrak{F}$ to $\mathfrak{F}'$. ⊣

Definition 6.2.5 (General relational frame embeddings)**.** An *embedding* of $\mathfrak{F}$ in $\mathfrak{F}'$ is an injective frame morphism f from $\mathfrak{F}$ to $\mathfrak{F}'$ satisfying the following (where a ranges over the elements of A):

(A2) $f[a] = b \cap f[W]$, for some $b \in A'$.

$\mathfrak{F}$ is *embeddable* in $\mathfrak{F}'$ if there is an embedding of $\mathfrak{F}$ in $\mathfrak{F}'$. ⊣

Definition 6.2.6 (General relational frame isomorphisms)**.** An *isomorphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ is a surjective embedding of $\mathfrak{F}$ in $\mathfrak{F}'$. $\mathfrak{F}$ is *isomorphic* to $\mathfrak{F}'$ if there is an isomorphism from $\mathfrak{F}$ to $\mathfrak{F}'$. ⊣

If $\mathfrak{F}$ is isomorphic to $\mathfrak{F}'$ under f, then $\mathfrak{F}'$ is isomorphic to $\mathfrak{F}$ under f^{-1}. Thus when there is an isomorphism from $\mathfrak{F}$ to $\mathfrak{F}'$ we often say that $\mathfrak{F}$ and $\mathfrak{F}'$ are isomorphic to each other ($\mathfrak{F} \cong \mathfrak{F}'$). Note that $\mathfrak{F}$ is isomorphic to $\mathfrak{F}'$ under f iff f is a bijective frame morphism from $\mathfrak{F}$ to $\mathfrak{F}'$, and its inverse f^{-1} is a frame morphism from $\mathfrak{F}'$ to $\mathfrak{F}$. This provides another characterization of isomorphism.

Validity of modal formulas is preserved by taking frame morphic images. Moreover it is invariant under isomorphisms. In detail, we note the following.

- Let f be a frame morphism from $\mathfrak{F} = \langle W, R, A\rangle$ to $\mathfrak{F}' = \langle W', R', A'\rangle$, and let V' be a valuation on $\mathfrak{F}'$. Then V mapping each atom p to $f^{-1}[V'(p)]$ is a valuation on $\mathfrak{F}$. Moreover for any formula α,

  $$\|\alpha\|^{\mathfrak{M}} = f^{-1}[\|\alpha\|^{\mathfrak{M}'}]$$

  where $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ and $\mathfrak{M}' = \langle \mathfrak{F}', V'\rangle$.

- If $\mathfrak{F}'$ is a frame morphic image of $\mathfrak{F}$, then for any formula α,

  $$\mathfrak{F} \models \alpha \implies \mathfrak{F}' \models \alpha.$$

- If $\mathfrak{F}$ and $\mathfrak{F}'$ are isomorphic, then for any formula α,

  $$\mathfrak{F} \models \alpha \iff \mathfrak{F}' \models \alpha.$$

We have observed earlier that the set A of a general relational frame $\mathfrak{F} = \langle W, R, A\rangle$ comprises all the truth-sets of formulas in models definable on $\mathfrak{F}$. Recall that the truth-set of a formula in a model is the set of points (states or worlds) at which the formula is true in the model. These truth-sets are often held to be propositions expressed by the formulas of the object language. Thus viewed, the set A comprises all the propositions that can be expressed in the language, and the set of elements of A to which a point x belongs, viz. the following set

$$Ax = \{a \in A \mid x \in a\}$$

comprises all the propositions true in x. Following Goldblatt (1974), we call Ax the *truth-description* of x. Some properties of frames are intuitively plausible:

- Each state of affairs is uniquely determined by the propositions true in that state.
- A consistent and exhaustive selection from among all propositions defines a state of affairs.
- If all sequences of n propositions necessarily true of x are true of a sequence of n states $\langle y_1, \ldots, y_n\rangle$, then $Rxy_1 \cdots y_n$.

We call a frame satisfying the above properties "descriptive general relational frame" (or simply "descriptive relational frame" or more simply "descriptive frame" if no confusion would arise). A more formal definition is given below.

Definition 6.2.7 (Descriptive general relational frames)**.** A frame $\mathfrak{F} = \langle W, R, A\rangle$ is *descriptive* if it satisfies all of the following. (Unless otherwise stated, $x, y, y_1, \ldots, y_n$ range over the elements of W; $a_1, \ldots, a_n$ range over the elements of A; u ranges over the ultrafilters in $\langle A, \cup, -, \emptyset, l_R\rangle$.)

(D1) $Ax = Ay \implies x = y$.

(D2) $u = Ax$, for some $x \in W$.

(D3) $\big(x \in l_R(a_1, \ldots, a_n) \implies \exists i : y_i \in a_i\big) \implies Rxy_1 \cdots y_n$. ⊣

Remark 6.2.8. Note that converses of the above conditions hold for *all* frames.

(1) If $x = y$, then $Ax = Ay$.

(2) The set Ax is an ultrafilter in $\langle A, \cup, -, \emptyset, l_R\rangle$.

(3) If $Rxy_1 \cdots y_n$, then for every $a_1, \ldots, a_n$ such that $x \in l_R(a_1, \ldots, a_n)$ we have $y_i \in a_i$ for some i. ⊣

In the following, we define the category of descriptive general relational frames, or simply the category of descriptive relational frames.

Definition 6.2.9 (The category of descriptive general relational frames)**.** *The category of descriptive general relational frames*, DRF, comprises all descriptive frames as its objects and all frame morphisms between descriptive frames as its arrows. The operations of domain, codomain, composition and identity are the usual ones for functions or maps. ⊣

6.3 Transformation of DRF to NMA

In this section, we define a function (denoted $\sharp$ and read "sharp") that transforms descriptive frames to set algebras, and their frame morphisms to maps between these set algebras (but with the directions reversed). As we shall see, the set algebras we get by $\sharp$ are normal modal algebras, and the maps between these algebras we get by $\sharp$ are homomorphisms. Moreover the transformation preserves both composition of morphisms and the identity morphisms. Therefore, the function $\sharp$ is a contravariant functor from the category of descriptive frames (DRF) to the category of normal modal algebras (NMA). (Refer to Appendix B.5 for the definition of contravariant functors.)

Definition 6.3.1 (The function $\sharp$ for descriptive relational frames and their frame morphisms)**.** The function $\sharp$ (read "sharp") assigns to each descriptive frame $\mathfrak{F} = \langle W, R, A\rangle$ a set algebra $\mathfrak{F}^\sharp$, and to each frame morphism f from descriptive frame $\mathfrak{F}_1 = \langle W_1, R_1, A_1\rangle$ to descriptive frame $\mathfrak{F}_2 = \langle W_2, R_2, A_2\rangle$ a map $f^\sharp$ from the set algebra ${\mathfrak{F}_2}^\sharp$ to the set algebra ${\mathfrak{F}_1}^\sharp$ as follows.

- $\mathfrak{F}^\sharp = \langle A, \cup, -, \emptyset, l_R\rangle$.
- $f^\sharp : A_2 \to A_1$ is defined, for every $b \in A_2$, by

$$f^\sharp(b) = f^{-1}[b]. \qquad \dashv$$

Note that $f^\sharp$ is well defined since, by condition (A1) of frame morphism (Definition 6.2.3), $f^{-1}[b]$ is guaranteed to be in A_1. Also observe that the arrows are reversed: whereas f maps A_1 to A_2, $f^\sharp$ maps A_2 to A_1.

We next show that $\mathfrak{F}^\sharp$ is a normal modal algebra and $f^\sharp$ is a homomorphism. In addition, $\sharp$ preserves composition of morphisms as well as the identity morphisms. Note that in proving the above (and so the function $\sharp$ is a contravariant functor from DRF to NMA), we do not make use of (D1), (D2) and (D3) of Definition 6.2.7, which are the distinctive frame conditions for descriptive frames. However these conditions will be required when we show, in Section 6.5.1, that $\sharp$ is an equivalence from DRF to NMA.

Theorem 6.3.2. *For any frame $\mathfrak{F} = \langle W, R, A\rangle$, $\mathfrak{F}^\sharp = \langle A, \cup, -, \emptyset, l_R\rangle$ is a normal model algebra (called the full complex algebra of $\mathfrak{F}$).*

Proof. It follows directly from the definition of frames (Definition 6.2.1) that the set A contains $\emptyset$, and is closed under $\cup$, $-$ and l_R. Hence, according to Definition 6.1.1, $\mathfrak{F}^\sharp$ is a modal algebra. It remains to show that $\mathfrak{F}^\sharp$ is normal, i.e. l_R satisfies both the conditions of normality and multiplicativity (see Definition 6.1.5).

For normality, observe that for any x and $\vec{y}$, if $Rx\vec{y}$ then trivially $y_i \in W$ for all i. Thus, by the definition of l_R, we have

$$l_R(a_1, \ldots, W, \ldots, a_n) = W.$$

For the condition of multiplicativity, i.e.

$$l_R(a_1, \ldots, a_i, \ldots, a_n) \cap l_R(a_1, \ldots, b, \ldots, a_n) = l_R(a_1, \ldots, a_i \cap b, \ldots, a_n),$$

we argue as follows:

- Assume x is a member of both $l_R(a_1, \ldots, a_i, \ldots, a_n)$ and $l_R(a_1, \ldots, b, \ldots, a_n)$. Consider arbitrary $\vec{y}$ such that $Rx\vec{y}$. If $y_j \notin a_j$ for all $j \neq i$, then $y_i \in a_i$ and $y_i \in b$, i.e. $y_i \in a_i \cap b$. Hence x is a member of $l_R(a_1, \ldots, a_i \cap b, \ldots, a_n)$.

- Assume x is a member of $l_R(a_1, \ldots, a_i \cap b, \ldots, a_n)$. Consider arbitrary $\vec{y}$ such that $Rx\vec{y}$. Then either (i) $x \in a_i \cap b$, i.e. $x \in a_i$ and $x \in b$, or (ii) $x \in a_j$ for some $j \neq i$. In other words, *both* $x \in a_i$ or $x \in a_j$ for some $j \neq i$ *and* $x \in b$ or $x \in a_j$ for some $j \neq i$. Hence x is a member of both $l_R(a_1, \ldots, a_i, \ldots, a_n)$ and $l_R(a_1, \ldots, b, \ldots, a_n)$.

We have shown that l_R is both normal and multiplicative. Thus $\mathfrak{F}^\sharp$ is a normal modal algebra. $\dashv$

Theorem 6.3.3. *For any frame morphism f from frame $\mathfrak{F}_1 = \langle W_1, R_1, A_1 \rangle$ to frame $\mathfrak{F}_2 = \langle W_2, R_2, A_2 \rangle$, $f^\sharp$ is a homomorphism from ${\mathfrak{F}_2}^\sharp = \langle A_2, \cup, -, \emptyset, l_{R_2} \rangle$ to ${\mathfrak{F}_1}^\sharp = \langle A_1, \cup, -, \emptyset, l_{R_1} \rangle$.*

Proof. What needs to be shown is that $f^\sharp$ preserves the set-theoretic operations of ${\mathfrak{F}_2}^\sharp$. The following hold simply by virtue of the definition of inverse relations (where b, b_1, b_2, etc. are elements of A_2):

$$\begin{aligned} f^{-1}[b_1 \cup b_2] &= f^{-1}[b_1] \cup f^{-1}[b_2]; \\ f^{-1}[-b] &= -f^{-1}[b]; \\ f^{-1}[\emptyset] &= \emptyset. \end{aligned}$$

Hence $\cup$, $-$ and $\emptyset$ are preserved under $f^\sharp$, i.e.

$$\begin{aligned} f^\sharp(b_1 \cup b_2) &= f^\sharp(b_1) \cup f^\sharp(b_2); \\ f^\sharp(-b) &= -f^\sharp(b); \\ f^\sharp(\emptyset) &= \emptyset. \end{aligned}$$

For the preservation of the modal operation l_{R_2}, i.e.

$$f^\sharp(l_{R_2}(b_1, \ldots, b_n)) = l_{R_1}(f^\sharp(b_1), \ldots, f^\sharp(b_n)),$$

we show the following:

$$f^{-1}[l_{R_2}(b_1, \ldots, b_n)] = l_{R_1}(f^{-1}[b_1], \ldots, f^{-1}[b_n]).$$

First, consider arbitrary $x_0 \in f^{-1}[l_{R_2}(b_1, \ldots, b_n)]$. Suppose $R_1 x_0 x_1 \cdots x_n$. Then by condition (R1) of frame morphism (Definition 6.2.3), we have $R_2 f(x_0) f(x_1) \ldots f(x_n)$. But

$f(x_0) \in l_{R_2}(b_1, \ldots, b_n)$. Thus there exists an $i \geq 1$ such that $f(x_i) \in b_i$, i.e. $x_i \in f^{-1}[b_i]$. Consequently $x_0 \in l_{R_1}(f^{-1}[b_1], \ldots, f^{-1}[b_n])$.

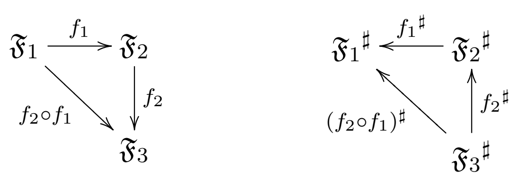
Secondly, consider arbitrary $x_0 \in l_{R_1}(f^{-1}[b_1], \ldots, f^{-1}[b_n])$. Suppose $R_2 f(x_0) y_1 \cdots y_n$. Then by condition (R2) of frame morphism (Definition 6.2.3), there exist $x_1, \ldots, x_n$ such that $R_1 x_0 x_1 \cdots x_n$, $f(x_1) = y_1$, ..., and $f(x_n) = y_n$. Given our initial assumption about x_0, we have for some $i \geq 1$, $x_i \in f^{-1}[b_i]$, i.e. $f(x_i) = y_i \in b_i$. Consequently $f(x_0) \in l_{R_2}(b_1, \ldots, b_n)$ or equivalently $x_0 \in f^{-1}[l_{R_2}(b_1, \ldots, b_n)]$. ⊣

Theorem 6.3.4. *The function ♯ preserves both composition and identity, i.e.*

(1) $(f_2 \circ f_1)^\sharp = {f_1}^\sharp \circ {f_2}^\sharp$, *for any frame morphisms* $f_1 : \mathfrak{F}_1 \to \mathfrak{F}_2$ *and* $f_2 : \mathfrak{F}_2 \to \mathfrak{F}_3$*;*

(2) ${\mathrm{id}_{\mathfrak{F}}}^\sharp = \mathrm{id}_{\mathfrak{F}^\sharp}$ *for any frame* $\mathfrak{F}$.

Proof. For (1). What needs to be shown is that the following diagram commutes, i.e. $(f_2 \circ f_1)^\sharp = {f_1}^\sharp \circ {f_2}^\sharp$.



It is straightforward to check the following, where c is an element of ${\mathfrak{F}_3}^\sharp$.

$$\begin{aligned}
(f_2 \circ f_1)^\sharp(c) &= (f_2 \circ f_1)^{-1}[c] && \text{(Definition of } \sharp) \\
&= {f_1}^{-1}[{f_2}^{-1}[c]] && \text{(Definition of inverse relations and compositions)} \\
&= {f_1}^\sharp({f_2}^\sharp(c)) && \text{(Definition of } \sharp) \\
&= ({f_1}^\sharp \circ {f_2}^\sharp)(c) && \text{(Definition of compositions).}
\end{aligned}$$

Note that the above $(f_2 \circ f_1)^{-1}[c] = {f_1}^{-1}[{f_2}^{-1}[c]]$ because of the definitions of inverse relations and composition of maps. The detail is as follows.

$$\begin{aligned}
x &\in (f_2 \circ f_1)^{-1}[c]. \\
(f_2 \circ f_1)(x) &\in c. \\
f_2(f_1(x)) &\in c. \\
f_1(x) &\in {f_2}^{-1}[c]. \\
x &\in {f_1}^{-1}[{f_2}^{-1}[c]].
\end{aligned}$$

For (2). We note that for any a of $\mathfrak{F}^\sharp$,

$$\mathrm{id}_{\mathfrak{F}}{}^\sharp(a) = \mathrm{id}_{\mathfrak{F}}{}^{-1}[a] = a = \mathrm{id}_{\mathfrak{F}^\sharp}(a). \qquad \dashv$$

Theorem 6.3.5. *The function $\sharp$ is a contravariant functor from the category* DRF *to the category* NMA*.*

Proof. The theorem follows immediately from Theorems 6.3.2, 6.3.3 and 6.3.4. $\dashv$

6.4 Transformation of NMA to DRF

We are going to define a function called $\flat$ (read "flat") that is the converse of the function $\sharp$: whereas $\sharp$ transforms descriptive frames and their frame morphisms to normal modal algebras and their homomorphisms, $\flat$ transforms normal modal algebras and their homomorphisms to descriptive frames and their frame morphisms. Similarly, while the function $\sharp$ is a contravariant functor from the category DRF to the category NMA, the function $\flat$ is a contravariant functor from NMA to DRF.

Let $\mathfrak{A}$ be a modal algebra. We denote the collection of all ultrafilters in $\mathfrak{A}$ by $\mathrm{Uf}\,\mathfrak{A}$. For every element a of $\mathfrak{A}$, Ua is the set of ultrafilters containing a. In other words,

$$Ua = \{u \in \mathrm{Uf}\,\mathfrak{A} \mid a \in u\}.$$

Definition 6.4.1 (The function $\flat$ for normal modal algebras and their homomorphisms)**.** The function $\flat$ (read "flat") assigns to each normal modal algebra $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ a relational structure $\mathfrak{A}^\flat$, and to each homomorphism f from normal modal algebra $\mathfrak{A}_1 = \langle A_1, +, -, 0, l\rangle$ to normal modal algebra $\mathfrak{A}_2 = \langle A_2, +, -, 0, l\rangle$ a map from ${\mathfrak{A}_2}^\flat$ to ${\mathfrak{A}_1}^\flat$ as follows.

- $\mathfrak{A}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}, R_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle$ where:
  (1) $\mathrm{Uf}\,\mathfrak{A}$ is the set of all ultrafilters in $\mathfrak{A}$;
  (2) $R_{\mathfrak{A}}$ is an $(n+1)$-ary relation on $\mathrm{Uf}\,\mathfrak{A}$ such that for any $u_0, u_1, \ldots, u_n \in \mathrm{Uf}\,\mathfrak{A}$, $R_{\mathfrak{A}}u_0u_1\cdots u_n$ iff $\forall a_1\cdots a_n \in A,\ l(a_1, \ldots, a_n) \in u_0 \implies \exists i \geq 1 : a_i \in u_i$;
  (3) $A_{\mathfrak{A}}$ is the set $\{Ua \mid a \in A\}$.
- $f^\flat : \mathrm{Uf}\,\mathfrak{A}_2 \to \mathrm{Uf}\,\mathfrak{A}_1$ is defined, for every $v \in \mathrm{Uf}\,\mathfrak{A}_2$, by

$$f^\flat(v) = f^{-1}[v]. \qquad \dashv$$

Note that $f^\flat$ is well defined because $f^{-1}[v]$ is an ultrafilter in $\mathfrak{A}_1$ (given that v is an ultrafilter in $\mathfrak{A}_2$ and f is a homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$). It is easy to check that $f^{-1}[v]$ is a filter (since it is non-empty, closed under taking meets, and is upward closed) and, for every $a \in A_1$, exactly one of a and $-a$ is in it.

The next two theorems show that for any normal modal algebra $\mathfrak{A}$, the relational structure $\mathfrak{A}^\flat$ is a frame (Theorem 6.4.2) and is descriptive (Theorem 6.4.3). We also call $\mathfrak{A}^\flat$ the ultrafilter frame of $\mathfrak{A}$.

Theorem 6.4.2. *For any normal modal algebra* $\mathfrak{A} = \langle A, +, -, 0, l \rangle$, $\mathfrak{A}^\flat = \langle \mathrm{Uf}\, \mathfrak{A}, R_{\mathfrak{A}}, A_{\mathfrak{A}} \rangle$ *is a frame.*

Proof. $\mathrm{Uf}\,\mathfrak{A}$ is non-empty and $R_{\mathfrak{A}}$ is an $(n+1)$-ary relation on $\mathrm{Uf}\,\mathfrak{A}$. It remains to show that $A_{\mathfrak{A}}$ contains $\emptyset$, and is closed under $\cup$, $-$ and $l_{R_{\mathfrak{A}}}$.

Since every element of $\mathfrak{A}_{\mathfrak{A}}$ is of the form Ua (for some $a \in A$), it is sufficient to note the following (where a, $a_1, \ldots, a_n$ and b are elements of $\mathfrak{A}$).

- $\emptyset = U0$ since no ultrafilters in $\mathfrak{A}$ contain the zero element.
- $Ua \cup Ub = U(a+b)$ since for any ultrafilter u in $\mathfrak{A}$, $u \in Ua \cup Ub$ iff $u \in Ua$ or $u \in Ub$ iff $a \in u$ or $b \in u$ iff $a+b \in u$ iff $u \in U(a+b)$. (The only interesting step is the inference that $a \in u$ or $b \in u$ iff $a+b \in u$, which follows from the proprieties of ultrafilters.)
- $-Ua = U(-a)$ since $u \in -Ua$ iff $a \notin u$ iff $-a \in u$ iff $u \in U(-a)$.
- $l_{R_{\mathfrak{A}}}(Ua_1, \ldots, Ua_n) = U(l(a_1, \ldots, a_n))$ since the following are equivalent, where $u_0 \in \mathrm{Uf}\,\mathfrak{A}$.

$$u_0 \in l_{R_{\mathfrak{A}}}(Ua_1, \ldots, Ua_n). \tag{1}$$

$$\forall u_1, \ldots, u_n \in \mathrm{Uf}\,\mathfrak{A},\ R_{\mathfrak{A}} u_0 u_1 \cdots u_n \implies \exists i \geq 1 : u_i \in Ua_i. \tag{2}$$

$$l(a_1, \ldots, a_n) \in u_0. \tag{3}$$

$$u_0 \in U(l(a_1, \ldots, a_n)). \tag{4}$$

For the last item, note that (1) $\iff$ (2) by the definition of $l_{R_{\mathfrak{A}}}$, (3) $\iff$ (4) by the definition of $U(l(a_1, \ldots, a_n))$, and (3) $\implies$ (2) by the definition of $R_{\mathfrak{A}}$ (bear in mind that $u_i \in Ua_i$ iff $a_i \in u_i$). The only interesting inference is that (2) $\implies$ (3), which we prove by contraposition. Assume

$$l(a_1, \ldots, a_n) \notin u_0,$$

and show

$$\exists u_1, \ldots, u_n \in \mathrm{Uf}\,\mathfrak{A} : R_{\mathfrak{A}} u_0 u_1 \cdots u_n \;\&\; \forall i \geq 1, -a_i \in u_i.$$

To show the above, it suffices to establish (by induction) that there exist a series of ultrafilters $u_1, \ldots, u_n$ of $\mathfrak{A}$, each of which satisfies the following conditions (where $1 \leq i \leq n$).

(i) $-a_i \in u_i$.

(ii) If $b_1 \notin u_1, \ldots, b_{i-1} \notin u_{i-1}$ and $l(b_1, \ldots, b_{i-1}, b_i, a_{i+1}, \ldots, a_n) \in u_0$, then $b_i \in u_i$ (for any $b_1, \ldots, b_{i-1}, b_i \in A$).

For if so then we have $-a_i \in u_i$ for all $i \geq 1$, and, for any $b_1, \ldots, b_n \in A$, $l(b_1, \ldots, b_n) \in u_0$ implies $b_i \in u_i$ for some $i \geq 1$ (hence $R_{\mathfrak{A}} u_0 u_1 \cdots u_n$).

(The basis) We show that the following subset s_1 of A has the finite intersection property (i.e. the meet of every finite subset of s_1 is not the zero element of $\mathfrak{A}$) and so can be extended to an ultrafilter in $\mathfrak{A}$.

$$s_1 = \{-a_1\} \cup \{c \in A \mid l(c, a_2, \ldots, a_n) \in u_0\}.$$

Suppose, for reductio, that s_1 does not have the finite intersection property, i.e. the meet of some finite subset of s_1 is the zero element of $\mathfrak{A}$. But $-a_1 \neq 0$ since l is normal and by assumption $l(a_1, \ldots, a_n) \notin u_0$. Hence there exist $c_1, \ldots, c_m \in s_1 - \{-a_1\}$ such that the following hold (where j ranges from 1 to m):

$$-a_1 \cdot \prod c_j = 0;$$
$$\prod c_j \leq a_1;$$
$$l(\prod c_j, a_2, \ldots, a_n) \leq l(a_1, a_2, \ldots, a_n).$$

But for all j, $l(c_j, a_2, \ldots, a_n) \in u_0$. We thus have

$$l(\prod c_j, a_2, \ldots, a_n) \in u_0,$$

since u_0 is closed under taking meets and l is multiplicative. In addition, u_0 is upward closed. Therefore,

$$l(a_1, a_2, \ldots, a_n) \in u_0,$$

which is contrary to the initial assumption that $l(a_1, a_2, \ldots, a_n) \notin u_0$. Hence, by reductio, s_1 has the finite intersection property. Accordingly it can be extended to an ultrafilter u_1 in $\mathfrak{A}$. Obviously $-a_1 \in u_1$ (since $-a_1 \in s_1 \subseteq u_1$). Moreover, for any $b \in A$, if $l(b, a_2, \ldots, a_n) \in u_0$ then $b \in u_1$ (since $b \in s_1 \subseteq u_1$). In other words, u_1 satisfies both (i) and (ii) (for the case of $i = 1$).

(The inductive step) The I.H. is that there already exist $u_1, \ldots, u_k \in \mathrm{Uf}\,\mathfrak{A}$ (where $1 \leq k < n$) satisfying both (i) and (ii). Consider the following subset of A.

$$s_{k+1} = \{-a_{k+1}\} \cup \{c \in A \mid \exists d_1, \ldots, d_k \in A : -d_1 \in u_1, \ldots, -d_k \in u_k \ \& \\ l(d_1, \ldots, d_k, c, a_{k+2}, \ldots, a_n) \in u_0\}.$$

We show, by reductio, that s_{k+1} has the finite intersection property. So assume not, i.e. there is a finite subset of s_{k+1} such that its meet is the zero element of $\mathfrak{A}$. But $-a_{k+1} \neq 0$ since l is normal and by assumption $l(a_1, \ldots, a_n) \notin u_0$. Thus for some $c_1, \ldots, c_m \in s_{k+1} - \{-a_{k+1}\}$, we have the following (where j ranges from 1 to m).

$$-a_{k+1} \cdot \prod c_j = 0.$$
$$\prod c_j \leq a_{k+1}.$$

For each j, there exist $-d_j^1 \in u_1, \ldots, -d_j^k \in u_k$ such that

$$l(d_j^1, \ldots, d_j^k, c_j, a_{k+2}, \ldots, a_n) \in u_0.$$

Then, by the upward closure of u_0, we have for each j

$$l(\sum d_j^1, \ldots, \sum d_j^k, c_j, a_{k+2}, \ldots, a_n) \in u_0.$$

Then, by the closure of u_0 under taking meets, and the multiplicativity of l,

$$l(\sum d_j^1, \ldots, \sum d_j^k, \prod c_j, a_{k+2}, \ldots, a_n) \in u_0,$$

from which it follows by the upward closure of u_0 that

$$l(\sum d_j^1, \ldots, \sum d_j^k, a_{k+1}, a_{k+2}, \ldots, a_n) \in u_0.$$

Note that for all j, $-d_j^1 \in u_1$. So $\prod(-d_n^1) \in u_1$, whence we derive $-\sum d_j^1 \in u_1$ and thus $\sum d_j^1 \notin u_1$. Similarly, we have $\sum d_j^2 \notin u_2$, $\ldots$, $\sum d_j^{k-1} \notin u_{k-1}$ and $\sum d_j^k \notin u_k$.

However by the I.H. u_k complies with (ii). Hence $\sum d_j^k \in u_k$. We thus arrive at a contradiction. Therefore, by reductio, s_{k+1} has the finite intersection property. Accordingly it can be extended to an ultrafilter u_{k+1} in $\mathfrak{A}$. Clearly $-a_{k+1} \in u_{k+1}$ (since $-a_{k+1} \in s_{k+1} \subseteq u_{k+1}$). Moreover if $b_1 \notin u_1, \ldots, b_k \notin u_k$ and $l(b_1, \ldots, b_k, b_{k+1}, a_{k+2}, \ldots, a_n) \in u_0$, then $b_{k+1} \in u_{k+1}$ (since $b_{k+1} \in s_{k+1} \subseteq u_{k+1}$). Therefore u_{k+1} satisfies (i) and (ii) (for the case of $i = k+1$). This concludes the inductive proof that there exist ultrafilters $u_1, \ldots, u_n$ in $\mathfrak{A}$ satisfying (i) and (ii), which is what is needed to show (2) $\Longrightarrow$ (3). $\dashv$

Theorem 6.4.3. *Let $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ be a normal modal algebra. The frame $\mathfrak{A}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}, R_\mathfrak{A}, A_\mathfrak{A}\rangle$ is descriptive.*

Proof. We show that $\mathfrak{A}^\flat$ satisfies conditions (D1), (D2) and (D3) of descriptive frames (see Definition 6.2.7).

To show (D1), i.e. $A_\mathfrak{A}u = A_\mathfrak{A}v \implies u = v$, we suppose $u \neq v$ and demonstrate $A_\mathfrak{A}u \neq A_\mathfrak{A}v$. By supposition, there exists an $a \in A$ such that both $a \notin u$ and $a \in v$ (or both $a \in u$ and $a \notin v$, in which case the following argument applies *mutatis mutandis*). Then $Ua \notin A_\mathfrak{A}u$ but $Ua \in A_\mathfrak{A}v$. Hence $A_\mathfrak{A}u \neq A_\mathfrak{A}v$.

(D2) stipulates that every ultrafilter μ in $A_\mathfrak{A}$ is of the form $A_\mathfrak{A}u$ where u is an ultrafilter in $\mathfrak{A}$. (Note that μ is a maximal collection of Ua's, where Ua is the set of ultrafilters in $A_\mathfrak{A}$ containing a.) To demonstrate this, it suffices to show that the set

$$v = \{a \in A \mid Ua \in \mu\}$$

is an ultrafilter in $\mathfrak{A}$, because if it is then $A_\mathfrak{A}v = \{Ub \mid v \in Ub\}$ is simply μ (to see this, note that $Ua \in \mu$ iff $a \in v$ iff $v \in Ua$ iff $Ua \in A_\mathfrak{A}v$). Indeed, v is an ultrafilter in $\mathfrak{A}$ because it is non-empty, closed under Boolean meet, upwardly closed, and, for each $a \in A$, exactly one of a and $-a$ is in v. Details are as follows:

- $1 \in v$ since $U1 = \mathrm{Uf}\,\mathfrak{A} \in \mu$.
- Suppose $a, b \in v$, i.e. $Ua, Ub \in \mu$. Then $Ua \cap Ub \in \mu$. But $Ua \cap Ub = U(a \cdot b)$. Thus $a \cdot b \in v$.
- Suppose $a \in v$ and $a \leq b$. Given the latter, $Ua \subseteq Ub$ (since if $u \in Ua$ or equivalently $a \in u$ then $b \in u$ or equivalently $u \in Ub$). Given that $a \in v$, we have $Ua \in \mu$ and so $Ub \in \mu$, i.e. $b \in v$.

- Suppose it is false that exactly one of a and $-a$ is in v, i.e. *either* (i) both a and $-a$ are in v *or* (ii) neither a nor $-a$ is in v. If (i) then $Ua, U(-a) \in \mu$, then $Ua \cap U(-a) = U(a \cdot -a) = U0 = \emptyset \in \mu$, which is absurd. If (ii) then $Ua, U(-a) \notin \mu$, then $-Ua, -U(-a) \in \mu$, then $U(-a), U(a) \in \mu$, which contradicts the earlier derivation that $Ua, U(-a) \notin \mu$. Thus, by reductio, exactly one of a and $-a$ is in v.

For (D3), we suppose that for any $u_0, u_1, \ldots, u_n \in \mathrm{Uf}\,\mathfrak{A}$ and $a_1, \ldots, a_n \in A$,

$$u_0 \in l_{R_\mathfrak{A}}(Ua_1, \ldots, Ua_n) \implies \exists i \geq 1 : u_i \in Ua_i, \text{ i.e. } a_i \in u_i,$$

and show that $R_\mathfrak{A} u_0 u_1 \cdots u_n$ or, equivalently, for any $a_1, \ldots, a_n$,

$$l(a_1, \ldots, a_n) \in u_0 \implies \exists i \geq 1 : a_i \in u_i.$$

So assume $l(a_1, \ldots, a_n) \in u_0$. Then by the definition of $R_\mathfrak{A}$ we have for any $u_1, \ldots, u_n \in \mathrm{Uf}\,\mathfrak{A}$,

$$Ru_0 u_1 \cdots u_n \implies \exists i \geq 1 : a_i \in u_i, \text{ i.e. } u_i \in Ua_i.$$

But this just means that $u_0 \in l_{R_\mathfrak{A}}(Ua_1, \ldots, Ua_n)$ (by the definition of $l_{R_\mathfrak{A}}$). Thus by supposition there exists an $i \geq 1$ such that $a_i \in u_i$ as desired.

We have shown that $\mathfrak{A}^\flat$ satisfies (D1), (D2) and (D3). It is thus a descriptive frame. ⊣

Theorem 6.4.4. *For any homomorphism f from modal algebra $\mathfrak{A}_1 = \langle A_1, +, -, 0, l\rangle$ to modal algebra $\mathfrak{A}_2 = \langle A_2, +, -, 0, l\rangle$, $f^\flat$ is a frame morphism from ${\mathfrak{A}_2}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}_2, R_{\mathfrak{A}_2}, A_{\mathfrak{A}_2}\rangle$ to ${\mathfrak{A}_1}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}_1, R_{\mathfrak{A}_1}, A_{\mathfrak{A}_1}\rangle$.*

Proof. We show that $f^\flat$ satisfies conditions (R1), (R2) and (A1) of frame morphisms (Definition 6.2.3). In the following, let $u_0, u_1, \ldots, u_n$ be ultrafilters in $\mathfrak{A}_1$ and let $v_0, v_1, \ldots, v_n$ be ultrafilters in $\mathfrak{A}_2$.

For (R1), assume $R_{\mathfrak{A}_2} v_0 v_1 \cdots v_n$ and show $R_{\mathfrak{A}_1} f^\flat(v_0) f^\flat(v_1) \cdots f^\flat(v_n)$, or equivalently $R_{\mathfrak{A}_1} f^{-1}[v_0] f^{-1}[v_1] \cdots f^{-1}[v_n]$, or equivalently if $l(a_1, \ldots, a_n) \in f^{-1}[v_0]$ then there exists an $i \geq 1$ such that $a_i \in f^{-1}[v_i]$ i.e. $f(a_i) \in v_i$. So suppose $l(a_1, \ldots, a_n) \in f^{-1}[v_0]$, i.e. $f(l(a_1, \ldots, a_n)) \in v_0$. Then $l(f(a_1), \ldots, f(a_n)) \in v_0$ since f is a homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$. But $R_{\mathfrak{A}_2} v_0 v_1 \cdots v_n$ by assumption. So $f(a_i) \in v_i$ for some $i \geq 1$, as desired.

For (A1), what needs to be shown is ${f^\flat}^{-1}[Ua] \in A_{\mathfrak{A}_2}$ for an arbitrary $a \in A_1$. It suffices to establish that ${f^\flat}^{-1}[Ua] = U(f(a))$ since $U(f(a))$ is a member of $A_{\mathfrak{A}_2}$. Consider a $v \in \mathrm{Uf}\,A_2$. Then,

$$
\begin{array}{rcll}
v \in {f^\flat}^{-1}[Ua] & \iff & f^\flat(v) \in Ua & \text{(definition of inverse relations)} \\
& \iff & f^{-1}[v] \in Ua & \text{(definition of } \flat\text{)} \\
& \iff & a \in f^{-1}[v] & \text{(definition of } Ua\text{, and } f^{-1}[v] \in \mathrm{Uf}\,\mathfrak{A}_1\text{)} \\
& \iff & f(a) \in v & \text{(definition of inverse relations)} \\
& \iff & v \in U(f(a)) & \text{(definition of } U(f(a))\text{).}
\end{array}
$$

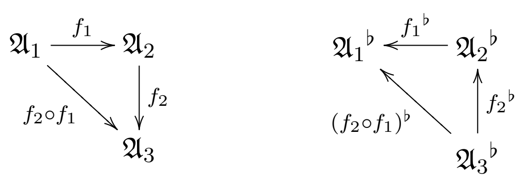
$f^\flat$ satisfies (R1), (R2) and (A1). It is thus a frame morphism. ⊣

Theorem 6.4.5. *The function $\flat$ preserves compositions of morphisms and the identity morphisms, i.e.*

(1) $(f_2 \circ f_1)^\flat = {f_1}^\flat \circ {f_2}^\flat$ *whenever f_2 is composable with f_1;*

(2) $(\mathrm{id}_{\mathfrak{A}})^\flat = \mathrm{id}_{\mathfrak{A}^\flat}$.

Proof. For (1). What needs to be shown is that the following diagram commutes, i.e. $(f_2 \circ f_1)^\flat = {f_1}^\flat \circ {f_2}^\flat$.



It is straightforward to check the following, where w is an element of ${\mathfrak{A}_3}^\flat$.

$$
\begin{array}{rcll}
(f_2 \circ f_1)^\flat(w) & = & (f_2 \circ f_1)^{-1}[w] & \text{(Definition of } \flat\text{)} \\
& = & {f_1}^{-1}[{f_2}^{-1}[w]] & \text{(Definition of inverse relations and compositions)} \\
& = & {f_1}^\flat({f_2}^\flat(w)) & \text{(Definition of } \flat\text{)} \\
& = & ({f_1}^\flat \circ {f_2}^\flat)(w) & \text{(Definition of compositions).}
\end{array}
$$

As noted above, $(f_2 \circ f_1)^{-1}[w] = {f_1}^{-1}[{f_2}^{-1}[w]]$ by virtue of the definitions of inverse relations and compositions of maps. The detail is as follows.

$$
\begin{array}{c}
x \in (f_2 \circ f_1)^{-1}[w]. \\
(f_2 \circ f_1)(x) \in w. \\
f_2(f_1(x)) \in w. \\
f_1(x) \in {f_2}^{-1}[w]. \\
x \in {f_1}^{-1}[{f_2}^{-1}[w]].
\end{array}
$$

For (2). We note that for any u of $\mathfrak{A}^\flat$,

$$\mathrm{id}_{\mathfrak{A}}{}^\flat(u) = \mathrm{id}_{\mathfrak{A}}{}^{-1}[u] = u = \mathrm{id}_{\mathfrak{F}^\flat}(u). \qquad \dashv$$

Theorem 6.4.6. *The function $\flat$ for normal modal algebras and their homomorphisms is a contravariant functor from the category* NMA *to the category* DRF.

Proof. The theorem follows immediately from Theorems 6.4.2, 6.4.3, 6.4.4 and 6.4.5. $\dashv$

6.5 Dual equivalence between DRF and NMA

In the previous two sections, we have established $\sharp$ and $\flat$ to be contravariant functors from DRF to NMA, and from NMA to DRF, respectively. We now show that they are also equivalences between the two categories.

Theorem 6.5.1. *The categories* DRF *and* NMA *are dually equivalent.*

Proof. We demonstrate the following regarding contravariant functor $\sharp$ (from DRF and NMA) and contravariant functor $\flat$ (from NMA to DRF).

- The composite functor $\flat \circ \sharp$ is naturally isomorphic to the identity functor on DRF (Theorem 6.5.4).
- The composite functor $\sharp \circ \flat$ is naturally isomorphic to the identity functor on NMA (Theorem 6.5.7).

Further details of the proof are given in Section 6.5.1 and 6.5.2. $\dashv$

Background information about natural transformation, equivalence and contravariance is provided in B.3, B.4 and B.5. The setup is technical but the underlying idea is simple. The most important thing we show is the following:

- Every descriptive frame $\mathfrak{F} = \langle W, R, A\rangle$ is isomorphic to $\mathfrak{F}^{\sharp^\flat}$ (the ultrafilter frame of the complex algebra of $\mathfrak{F}$) under the map $x \mapsto Ax$.
- Every normal modal algebra $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ is isomorphic to $\mathfrak{A}^{\flat^\sharp}$ (the complex algebra of the ultrafilter frame of $\mathfrak{A}$) under the map $a \mapsto Ua$.

6.5.1 Natural isomorphism between $\mathrm{Id}_{\mathsf{DRF}}$ and $\flat \circ \sharp$

Throughout this section, $\mathfrak{F}$, $\mathfrak{F}^\sharp$ and $\mathfrak{F}^{\sharp\flat}$ are as follows.

- $\mathfrak{F} = \langle W, R, A\rangle$ is a descriptive frame, i.e. frames satisfying (D1), (D2) and (D3). (See Definition 6.2.7.)
- $\mathfrak{F}^\sharp = \langle A, \cup, -, \emptyset, l_R\rangle$ is the normal modal algebra we get from $\mathfrak{A}$ by $\sharp$. Recall that l_R is the n-ary operation on A defined, for every $a_1, \ldots, a_n \in A$, by

$$l_R(a_1, \ldots, a_n) = \{x \in W \mid \forall y_1, \ldots, y_n,\ Rxy_1 \cdots y_n \implies \exists i : y_i \in a_i\}.$$

- $\mathfrak{F}^{\sharp\flat} = \langle \mathrm{Uf}\,\mathfrak{F}^\sharp, R_{\mathfrak{F}^\sharp}, A_{\mathfrak{F}^\sharp}\rangle$ is the ultrafilter frame we get from $\mathfrak{F}^\sharp$ by $\flat$. Note that
  - $\mathrm{Uf}\,\mathfrak{F}^\sharp$ is the collection of all ultrafilters in $\mathfrak{F}^\sharp$;
  - $R_{\mathfrak{F}^\sharp} u_0 u_1 \cdots u_n$ iff $l_R(a_1, \ldots, a_n) \in u_0 \implies \exists i \geq 1 : a_i \in u_i$;
  - $A_{\mathfrak{F}^\sharp} = \{Ua \mid a \in A\}$ where Ua is the set of ultrafilters in $\mathfrak{F}^\sharp$ containing a.

We let η be the function that assigns to each descriptive frame $\mathfrak{F} = \langle W, R, A\rangle$ the map $\eta_{\mathfrak{F}} : W \to \mathrm{Uf}\,\mathfrak{F}^\sharp$ defined, for every $x \in W$, by

$$\eta_{\mathfrak{F}}(x) = Ax.$$

The map $\eta_{\mathfrak{F}}$ is well defined since every Ax is an ultrafilter in $\mathfrak{F}^\sharp = \langle A, \cup, -, \emptyset, l_R\rangle$. See (2) of Remark 6.2.8.

Theorem 6.5.2. $\eta_{\mathfrak{F}} : W \to \mathrm{Uf}\,\mathfrak{F}^\sharp$ *is a frame morphism from* $\mathfrak{F}$ *to* $(\mathfrak{F}^\sharp)^\flat$.

Proof. We show that $\eta_{\mathfrak{F}}$ satisfies (R1), (R2) and (A1) of Definition 6.2.3. For (R1) and (R2), we establish the following equivalences first (where $x_0, x_1, \ldots, x_n \in W$).

$$R_{\mathfrak{F}^\sharp}(Ax_0)(Ax_1)\ldots(Ax_n). \tag{1}$$

$$\forall a_1, \ldots, a_n \in A,\ l_R(a_1, \ldots, a_n) \in Ax_0 \implies \exists i : a_i \in Ax_i. \tag{2}$$

$$\forall a_1, \ldots, a_n \in A,\ x_0 \in l_R(a_1, \ldots, a_n) \implies \exists i : x_i \in a_i. \tag{3}$$

$$Rx_0x_1 \cdots x_n. \tag{4}$$

In the above, (1) $\iff$ (2) by the definition of $R_{\mathfrak{F}^\sharp}$; (2) $\iff$ (3) by the definition of Ax_0 and Ax_i and the closure of A under l_R; (3) $\implies$ (4) by (D3) while (4) $\implies$ (3) by the definition of l_R.

For (R1), we assume $Rx_0x_1 \cdots x_n$. Then by the above $R_{\mathfrak{F}^\sharp}(Ax_0)(Ax_1)\dots(Ax_n)$. In other words, $R_{\mathfrak{F}^\sharp}\eta_{\mathfrak{F}}(x_0)\eta_{\mathfrak{F}}(x_0)\dots\eta_{\mathfrak{F}}(x_0)$.

For (R2), we assume, for arbitrary $x_0 \in W$ and $u_1, \dots, u_n \in \mathrm{Uf}\,\mathfrak{F}^\sharp$, $R_{\mathfrak{F}^\sharp}\eta_{\mathfrak{F}}(x_0)u_1\dots u_n$. But $\eta_{\mathfrak{F}}(x_0) = Ax_0$. Moreover, according to (D2), $u_1 = Ax_1$ for some $x_1 \in W$ and similarly for $u_2, \dots, u_n$. Thus for some $x_1, \dots, x_n \in W$, $R_{\mathfrak{F}^\sharp}(Ax_0)(Ax_1)\dots(Ax_n)$, from which it follows from the above equivalences that $Rx_0x_1 \cdots x_n$ where for all $i \geq 1$, $\eta_{\mathfrak{F}}(x_i) = Ax_i = u_i$.

(A1) stipulates that ${\eta_{\mathfrak{F}}}^{-1}[Ua] \in A$ for any $a \in A$. (Note that $A_{\mathfrak{F}^\sharp}$ is the set $\{Ua \mid a \in A\}$.) To show (A1) we establish that ${\eta_{\mathfrak{F}}}^{-1}[Ua] = a$ (and so ${\eta_{\mathfrak{F}}}^{-1}[Ua] \in A$ since $a \in A$). For any $x \in W$,

$$\begin{aligned} x \in {\eta_{\mathfrak{F}}}^{-1}[Ua] &\iff \eta_{\mathfrak{F}}(x) \in Ua \\ &\iff Ax \in Ua \\ &\iff a \in Ax \\ &\iff x \in a. \end{aligned}$$

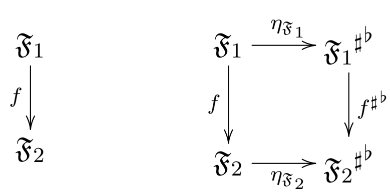
Thus ${\eta_{\mathfrak{F}}}^{-1}[Ua] = a$, as desired. ⊣

Theorem 6.5.3. *η is a natural transformation from* $\mathrm{Id}_{\mathsf{DRF}}$ *to* $\flat \circ \sharp$.

Proof. We have proved in Theorem 6.5.2 that every component $\eta_{\mathfrak{F}}$ of η is a frame morphism from $\mathfrak{F}$ to $\mathfrak{F}^{\sharp^\flat}$, i.e. from $\mathrm{Id}_{\mathsf{DRF}}(\mathfrak{F})$ to $(\flat \circ \sharp)(\mathfrak{F})$. It remains to show that the following holds for any frame morphism f from descriptive frame $\mathfrak{F}_1 = \langle W_1, R_1, A_1 \rangle$ to descriptive frame $\mathfrak{F}_2 = \langle W_2, R_2, A_2 \rangle$,

$$f^{\sharp^\flat} \circ \eta_{\mathfrak{F}_1} = \eta_{\mathfrak{F}_2} \circ f.$$

In other words, what needs to be shown is that the following diagram commutes.



We recall here that $f^\sharp : A_2 \to A_1$ and $f^{\sharp^\flat} : \mathrm{Uf}\,{\mathfrak{F}_1}^\sharp \to \mathrm{Uf}\,{\mathfrak{F}_2}^\sharp$ are defined by:

$$\forall b \in A_2,\ f^\sharp(b) = f^{-1}[b];$$

$$\forall u \in \mathrm{Uf}\,{\mathfrak{F}_1}^\sharp,\ f^{\sharp^\flat}(u) = {f^\sharp}^{-1}[u].$$

Observe that $f^{\sharp^\flat} \circ \eta_{\mathfrak{F}_1} = \eta_{\mathfrak{F}_2} \circ f$ iff for any $x \in W_1$,

$$(f^{\sharp^\flat} \circ \eta_{\mathfrak{F}_1})(x) = (\eta_{\mathfrak{F}_2} \circ f)(x)$$

or equivalently

$$f^{\sharp^\flat}(A_1 x) = A_2(f(x)).$$

To show the above identity, we consider arbitrary $b \in A_2$. The following are equivalent.

$$\begin{aligned}
b \in f^{\sharp^\flat}(A_1 x) &\iff b \in A_2(f(x)). \\
b \in f^{\sharp^{-1}}[A_1 x] &\iff f(x) \in b. \\
f^{\sharp}(b) \in A_1 x &\iff f(x) \in b. \\
f^{-1}[b] \in A_1 x &\iff f(x) \in b. \\
x \in f^{-1}[b] &\iff f(x) \in b. \\
f(x) \in b &\iff f(x) \in b.
\end{aligned}$$

But the last statement is obviously true. Thus we have shown that $f^{\sharp^\flat}(A_1 x) = A_2(f(x))$ for any $x \in W_1$, from which it follows that $f^{\sharp^\flat} \circ \eta_{\mathfrak{F}_1} = \eta_{\mathfrak{F}_2} \circ f$, as argued above. $\dashv$

Theorem 6.5.4. *η is a natural isomorphism from $\mathrm{Id}_{\mathsf{DRF}}$ to $\flat \circ \sharp$. Thus $\mathrm{Id}_{\mathsf{DRF}}$ is naturally isomorphic to $\flat \circ \sharp$.*

Proof. We already know that η is a natural transformation from $\mathrm{Id}_{\mathsf{DRF}}$ to $\flat \circ \sharp$ (Theorem 6.5.3). For η to be a natural isomorphism, every component $\eta_{\mathfrak{F}}$ of it must be a frame isomorphism. In other words, we need to show that for every frame morphism $\eta_{\mathfrak{F}}$ from $\mathfrak{F} = \langle W, R, A \rangle$ to $\mathfrak{F}^{\sharp^\flat} = \langle \mathrm{Uf}\,\mathfrak{F}^{\sharp}, R_{\mathfrak{F}^\sharp}, A_{\mathfrak{F}^\sharp} \rangle$, there exists a frame morphism $\theta_{\mathfrak{F}}$ from $\mathfrak{F}^{\sharp^\flat}$ to $\mathfrak{F}$ such that

$$\theta_{\mathfrak{F}} \circ \eta_{\mathfrak{F}} = \mathrm{id}_{\mathfrak{F}};$$
$$\eta_{\mathfrak{F}} \circ \theta_{\mathfrak{F}} = \mathrm{id}_{\mathfrak{F}^{\sharp^\flat}}.$$

Let $\theta_{\mathfrak{F}} : \mathrm{Uf}\,\mathfrak{F}^{\sharp} \to W$ be defined as follows: for every $u \in \mathrm{Uf}\,\mathfrak{F}^{\sharp}$

$$\theta_{\mathfrak{F}}(u) = x, \quad \text{whenever } u = Ax.$$

Note that $\theta_{\mathfrak{F}}$ is well-defined since

- by (D2) every ultrafilter u in $\mathfrak{F}^\sharp$ is of the form Ax for some $x \in W$ and so is assigned some member of W;
- by (D1) every ultrafilter u in $\mathfrak{F}^\sharp$ is assigned at most one member of W (for if $u = Ax$ and $u = Ay$, then $x = y$).

Moreover $\theta_{\mathfrak{F}}$ as defined earlier is a frame morphism from $\mathfrak{F}^{\sharp^\flat}$ to $\mathfrak{F}$ because it satisfies (R1), (R2) and (A1) of Definition 6.2.3. In detail, we have:

- if $R_{\mathfrak{F}^\sharp}(Ax_0)(Ax_1)\dots(Ax_n)$, then $Rx_0x_1\cdots x_n$ where for all $i \geq 0$, $x_i = \theta_{\mathfrak{F}}(Ax_i)$;
- if $R\theta_{\mathfrak{F}}(Ax_0)x_1\cdots x_n$, then $Rx_0x_1\cdots x_n$, then $R_{\mathfrak{F}^\sharp}(Ax_0)(Ax_1)\dots(Ax_n)$ where all $i \geq 1$, $Ax_i \in \mathrm{Uf}\,\mathfrak{F}^\sharp$ and $\theta_{\mathfrak{F}}(Ax_i) = x_i$.
- for all $a \in A$, ${\theta_{\mathfrak{F}}}^{-1}[a] \in A_{\mathfrak{F}^\sharp}$ because ${\theta_{\mathfrak{F}}}^{-1}[a] = Ua$. (To see the latter, assume $u \in \theta_{\mathfrak{F}}^{-1}[a]$. Then for some $x \in W$, $u = Ax$ and $x \in a$ or equivalently $a \in Ax$. Then $u \in Ua$. The argument can be reversed.)

Finally for any $x \in W$ and $Ax \in \mathrm{Uf}\,\mathfrak{F}^\sharp$,

$$\begin{aligned}(\theta_{\mathfrak{F}} \circ \eta_{\mathfrak{F}})(x) &= \theta_{\mathfrak{F}}(\eta_{\mathfrak{F}}(x)) = \theta_{\mathfrak{F}}(Ax) = x;\\ (\eta_{\mathfrak{F}} \circ \theta_{\mathfrak{F}})(Ax) &= \eta_{\mathfrak{F}}(\theta_{\mathfrak{F}}(Ax)) = \eta_{\mathfrak{F}}(x) = Ax.\end{aligned}$$

Thus, both $\theta_{\mathfrak{F}} \circ \eta_{\mathfrak{F}} = \mathrm{id}_{\mathfrak{F}}$ and $\eta_{\mathfrak{F}} \circ \theta_{\mathfrak{F}} = \mathrm{id}_{\mathfrak{F}^{\sharp\flat}}$. ⊣

6.5.2 Natural isomorphism between $\mathrm{Id}_{\mathsf{NMA}}$ and $\sharp \circ \flat$

Throughout this section, $\mathfrak{A}$, $\mathfrak{A}^\flat$ and $\mathfrak{A}^{\flat^\sharp}$ are as follows.

- $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ is a normal modal algebra. (See Definition 6.1.5.)
- $\mathfrak{A}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}, R_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle$ is the descriptive frame we get from $\mathfrak{A}$ under $\flat$ as defined in Definition 6.4.1. Recall that:
  - $\mathrm{Uf}\,\mathfrak{A}$ is the collection of all ultrafilters in $\mathfrak{A}$;
  - $R_{\mathfrak{A}}u_0u_1\cdots u_n$ iff
    $$\forall a_1, \dots, a_n \in A,\ l(a_1, \dots, a_n) \in u_0 \implies \exists i \geq 1 : a_i \in u_i;$$
  - $A_{\mathfrak{A}} = \{Ua \mid a \in A\}$ where Ua consists of all ultrafilters in $\mathfrak{A}$ containing a.

- $\mathfrak{A}^{\flat\sharp} = \langle A_{\mathfrak{A}}, \cup, -, \emptyset, l_{R_{\mathfrak{A}}} \rangle$ is the normal modal algebra we get from $\mathfrak{A}^\flat$ under $\sharp$ as defined in Definition 6.3.1. Note that $l_{R_{\mathfrak{A}}}(Ua_1, \ldots, Ua_n)$, which consists of ultrafilters u_0 in $\mathfrak{A}$ satisfying the condition

$$\forall u_1, \ldots, u_n \in \mathrm{Uf}\,\mathfrak{A},\ R_{\mathfrak{A}} u_0 u_1 \cdots u_n \implies \exists i \geq 1 : u_i \in a_i,$$

is simply $U(l(a_1, \ldots, a_n))$ (see the proof of Theorem 6.4.2).

We let η be the function that assigns to each $\mathfrak{A}$ the map $\eta_{\mathfrak{A}} : A \to A_{\mathfrak{A}}$ defined, for every $a \in A$, by

$$\eta_{\mathfrak{A}}(a) = Ua.$$

Theorem 6.5.5. *$\eta_{\mathfrak{A}} : A \to A_{\mathfrak{A}}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}^{\flat\sharp}$.*

Proof. We show that $\eta_{\mathfrak{A}}$ preserves the algebraic operations, i.e.

$$\begin{aligned} \eta_{\mathfrak{A}}(a+b) &= \eta_{\mathfrak{A}}(a) \cup \eta_{\mathfrak{A}}(b); \\ \eta_{\mathfrak{A}}(-a) &= -\eta_{\mathfrak{A}}(a); \\ \eta_{\mathfrak{A}}(0) &= \emptyset; \\ \eta_{\mathfrak{A}}(l(a_1, \ldots, a_n)) &= l_{R_{\mathfrak{A}}}(\eta_{\mathfrak{A}}(a_1), \ldots, \eta_{\mathfrak{A}}(a_n)). \end{aligned}$$

But the above is a consequence of the following, which we have already demonstrated when proving that the set $A_{\mathfrak{A}}$ is closed under $\cup$, $-$, $\emptyset$ and $l_{R_{\mathfrak{A}}}$ (see the proof of Theorem 6.4.2):

$$\begin{aligned} U(a+b) &= U(a) \cup U(b); \\ U(-a) &= -U(a); \\ U(0) &= \emptyset; \\ U(l(a_1, \ldots, a_n)) &= l_{R_{\mathfrak{A}}}(U(a_1), \ldots, U(a_n)). \end{aligned}$$

Thus $\eta_{\mathfrak{A}}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}^{\flat\sharp}$. ⊣

Theorem 6.5.6. *η is a natural transformation from $\mathrm{Id}_{\mathsf{NMA}}$ to $\sharp \circ \flat$.*

Proof. We have proved in Theorem 6.5.5 that every component $\eta_{\mathfrak{A}}$ of η is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}^{\flat\sharp}$, i.e. from $\mathrm{Id}_{\mathsf{NMA}}(\mathfrak{A})$ to $(\sharp \circ \flat)(\mathfrak{A})$. It remains to show that the following holds for any homomorphism f from $\mathfrak{A}_1 = \langle A_1, +, -, 0, l \rangle$ to $\mathfrak{A} = \langle A, +, -, 0, l \rangle 2$ (both are normal modal algebras),

$$f^{\flat\sharp} \circ \eta_{\mathfrak{A}_1} = \eta_{\mathfrak{A}_2} \circ f.$$

In other words, what needs to be shown is that the following diagram commutes.

$$\begin{array}{c} \mathfrak{A}_1 \\ {\scriptstyle f}\downarrow \\ \mathfrak{A}_2 \end{array} \qquad\qquad \begin{array}{ccc} \mathfrak{A}_1 & \xrightarrow{\eta_{\mathfrak{A}_1}} & {\mathfrak{A}_1}^{\flat\sharp} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\flat\sharp}} \\ \mathfrak{A}_2 & \xrightarrow[\eta_{\mathfrak{A}_2}]{} & {\mathfrak{A}_2}^{\flat\sharp} \end{array}$$

We recall here that $f^\flat : \mathrm{Uf}\,\mathfrak{A}_2 \to \mathrm{Uf}\,\mathfrak{A}_1$ and $f^{\flat\sharp} : A_{\mathfrak{A}_1} \to A_{\mathfrak{A}_2}$ are defined by:

$$\forall v \in \mathrm{Uf}\,\mathfrak{A}_2,\ f^\flat(v) = f^{-1}[v];$$
$$\forall a \in A_1,\ f^{\flat\sharp}(Ua) = {f^\flat}^{-1}[Ua].$$

Observe that $f^{\flat\sharp} \circ \eta_{\mathfrak{A}_1} = \eta_{\mathfrak{A}_2} \circ f$ iff for any $a \in A_1$,

$$(f^{\flat\sharp} \circ \eta_{\mathfrak{A}_1})(a) = (\eta_{\mathfrak{A}_2} \circ f)(a)$$

or equivalently

$$f^{\flat\sharp}(U_1 a) = U_2(f(a))$$

where $U_1 a$ consists of all ultrafilters in $\mathfrak{A}_1$ containing a, and $U_2(f(a))$ consists of all ultrafilters in $\mathfrak{A}_2$ containing $f(a)$. To show the above identity, we consider arbitrary $v \in \mathrm{Uf}\,\mathfrak{A}_2$. The following are equivalent.

$$\begin{array}{rcl} v \in f^{\flat\sharp}(U_1 a) & \iff & v \in U_2(f(a)). \\ v \in {f^\flat}^{-1}[U_1 a] & \iff & f(a) \in v. \\ f^\flat(v) \in U_1 a & \iff & f(a) \in v. \\ f^{-1}[v] \in U_1 a & \iff & f(a) \in v. \\ a \in f^{-1}[v] & \iff & f(a) \in v. \\ f(a) \in v & \iff & f(a) \in v. \end{array}$$

But the last statement is obviously true. Thus we have shown that $f^{\flat\sharp}(U_1 a) = U_2(f(a))$ for any $a \in A_1$, from which it follows that $f^{\flat\sharp} \circ \eta_{\mathfrak{A}_1} = \eta_{\mathfrak{A}_2} \circ f$, as argued above. ⊣

Theorem 6.5.7. *η is a natural isomorphism from* $\mathrm{Id}_{\mathsf{NMA}}$ *to* $\sharp \circ \flat$. *Thus* $\mathrm{Id}_{\mathsf{NMA}}$ *is naturally isomorphic to* $\sharp \circ \flat$.

Proof. We already know that η is a natural transformation from $\mathrm{Id}_{\mathsf{NMA}}$ to $\sharp \circ \flat$ (Theorem 6.5.6). For η to be a natural isomorphism, every component $\eta_{\mathfrak{A}}$ of it must be a isomorphism. In other words, we need to show that for every homomorphism $\eta_{\mathfrak{A}}$ from $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ to $\mathfrak{A}^{\flat\sharp}$, there exists a homomorphism $\theta_{\mathfrak{A}}$ from $\mathfrak{A}^{\flat\sharp}$ to $\mathfrak{A}$ such that

$$\theta_{\mathfrak{A}} \circ \eta_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}};$$
$$\eta_{\mathfrak{A}} \circ \theta_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}^{\flat\sharp}}.$$

Let $\theta_{\mathfrak{A}} : A_{\mathfrak{A}} \to A$ be defined as follows: for every $Ua \in A_{\mathfrak{A}}$,

$$\theta_{\mathfrak{A}}(Ua) = a.$$

$\theta_{\mathfrak{A}}$ as defined above is a homomorphism from $\mathfrak{A}^{\flat\sharp}$ to $\mathfrak{A}$ iff the following hold:

$$\begin{aligned}
\theta_{\mathfrak{A}}(Ua \cup Ub) &= \theta_{\mathfrak{A}}(Ua) + \theta_{\mathfrak{A}}(Ub),\\
\theta_{\mathfrak{A}}(-Ua) &= -\theta_{\mathfrak{A}}(Ua),\\
\theta_{\mathfrak{A}}(\emptyset) &= 0,\\
\theta_{\mathfrak{A}}(l_{R_{\mathfrak{A}}}(Ua_1, \ldots, Ua_n)) &= l(\theta_{\mathfrak{A}}(Ua_1), \ldots, \theta_{\mathfrak{A}}(Ua_n)),
\end{aligned}$$

or equivalently the following hold:

$$\begin{aligned}
\theta_{\mathfrak{A}}(U(a+b)) &= \theta_{\mathfrak{A}}(Ua) + \theta_{\mathfrak{A}}(Ub),\\
\theta_{\mathfrak{A}}(U(-a)) &= -\theta_{\mathfrak{A}}(Ua),\\
\theta_{\mathfrak{A}}(0) &= 0,\\
\theta_{\mathfrak{A}}(U(l(a_1, \ldots, a_n))) &= l(\theta_{\mathfrak{A}}(Ua_1), \ldots, \theta_{\mathfrak{A}}(Ua_n)).
\end{aligned}$$

But the last set of identities are obvious, given our definition of $\theta_{\mathfrak{A}}$.

Finally for any $a \in A$ and $Ua \in A_{\mathfrak{A}}$, we have

$$\begin{aligned}
(\theta_{\mathfrak{A}} \circ \eta_{\mathfrak{A}})(a) &= \theta_{\mathfrak{A}}(\eta_{\mathfrak{A}}(a)) &&= \theta_{\mathfrak{A}}(Aa) &&= a;\\
(\eta_{\mathfrak{A}} \circ \theta_{\mathfrak{A}})(Ua) &= \eta_{\mathfrak{A}}(\theta_{\mathfrak{A}}(Ua)) &&= \eta_{\mathfrak{A}}(a) &&= Ua.
\end{aligned}$$

Thus, both $\theta_{\mathfrak{A}} \circ \eta_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}}$ and $\eta_{\mathfrak{F}} \circ \theta_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}^{\sharp\flat}}$. ⊣

Chapter 7

Modal Algebras and General Neighbourhood Frames

We showed in the previous chapter that the categories of descriptive relational frames and normal modal algebras are dually equivalent. More general than the relational frames are the neighbourhood structures. Došen (1989) establishes dual equivalence between descriptive neighbourhood frames of type 1 and modal algebras with arbitrary unary operations. In this chapter, we generalize Došen's result to duality between descriptive neighbourhood frames of type n and modal algebras with arbitrary n-ary operations.

The plan of this chapter is similar to that of the previous chapter. We define the categories of descriptive neighbourhood frames (DNF) in Section 7.1. (Note that the category of modal algebras MA has already been defined in Section 6.1.) A function $\sharp$ is defined in Section 7.2 for descriptive neighbourhood frames and their frame morphisms. It transforms a frame to a set algebra, and a frame morphism to a homomorphism between set algebras. We then show that the function $\sharp$ is a contravariant functor from DNF to MA. We proceed similarly in Section 7.3 for the contravariant functor $\flat$, which transforms modal algebras and homomorphisms to descriptive neighbourhood frames and their frame morphisms. Finally, the categories of descriptive neighbourhood frames and modal algebras are demonstrated to be dually equivalent by the contravariant functors $\sharp$ and $\flat$ (Section 7.4).

7.1 General neighbourhood frames

Consider a neighbourhood function N of type n on a set W of points. Every point x is assigned a collection of n-tuples of sets of points (with the sets of points being called the neighbourhoods of x). In symbol, $N(x) \subseteq (\mathscr{P}(W))^n$. We let l_N be an n-ary operation on $\mathscr{P}(W)$ defined as follows (where $a_1, \ldots, a_n \subseteq W$):

$$l_N(a_1, \ldots, a_n) = \{x \in W \mid \langle a_1, \ldots, a_n \rangle \in N(x)\}.$$

The dual operation of l_N, denoted m_N, is thus:

$$m_N(a_1, \ldots, a_n) = -l_N(-a_1, \ldots, -a_n),$$

where $-$ is set-complementation (relative to W). It is easy to check that the following identity holds:

$$m_N(a_1, \ldots, a_n) = \{x \in W \mid \langle -a_1, \ldots, -a_n \rangle \notin N(x)\}.$$

Definition 7.1.1 (General neighbourhood frames)**.** A *general neighbourhood frame* $\mathfrak{F}$ is a triple $\langle W, N, A \rangle$ of which:

(1) W is a non-empty set of points;

(2) N is a neighbourhood function of type n on W, i.e. $N : W \to \mathscr{P}((\mathscr{P}(W))^n)$;

(3) $A \subseteq \mathscr{P}(W)$ contains $\emptyset$ as well as all neighbourhoods, and is closed under $-$, $\cup$ and l_N. (A neighbourhood a is a set of points such that for some point x and sets $b_1, \ldots, b_n$ of points, we have $\langle b_1, \ldots, b_n \rangle \in N(x)$ and a is one of $b_1, \ldots, b_n$.) $\dashv$

Definition 7.1.2 (General neighbourhood models)**.** Let $\mathfrak{F} = \langle W, N, A \rangle$ be a general neighbourhood frame, and V a function that assigns to each atom an element of A. $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, or equivalently $\mathfrak{M} = \langle W, N, A, V \rangle$, is called a *general neighbourhood model* on $\mathfrak{F}$. $\dashv$

Truth in general neighbourhood models and validity on general neighbourhood frames are defined in the same way as truth in ordinary neighbourhood models and validity on ordinary neighbourhood frames. Note that for any formula α and general neighbourhood model $\mathfrak{M} = \langle W, N, A, V \rangle$ we have $\|\alpha\|^{\mathfrak{M}} \in A$. As in the case of general relational frames and models, the set A is so named because it is the carrier of a modal algebra, viz. $\langle A, \cup, -, \emptyset, l_N \rangle$.

Another reason is that the members of A are sometimes called admissible sets and a valuation on $\mathfrak{F} = \langle W, N, A\rangle$ assigns to each atom an admissible set of points.

When the context makes clear we are talking about general neighbourhood frames and models, we use the expressions “neighbourhood frames” and “neighbourhood models”, or simply say “frames” and “models”. The same applies to morphisms between general neighbourhood frames that we are going to define.

Definition 7.1.3 (General neighbourhood frame morphisms)**.** Let $\mathfrak{F}_1 = \langle W_1, N_1, A_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, N_2, A_2\rangle$ be frames. A map $f : W_1 \to W_2$ is called a *frame morphism* from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ if all of the following conditions hold. (In the following, x ranges over the elements of W_1, and $b, b_1, \ldots, b_n$ range over the elements of A_2.)

(N1) $\langle f^{-1}[b_1], \ldots, f^{-1}[b_n]\rangle \in N_1(x) \iff \langle b_1, \ldots, b_n\rangle \in N_2(f(x))$.

(A1) $f^{-1}[b] \in A_1$. ⊣

In the following definitions, $\mathfrak{F}_1 = \langle W_1, N_1, A_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, N_2, A_2\rangle$ are general neighbourhood frames.

Definition 7.1.4 (General neighbourhood frame morphic images)**.** $\mathfrak{F}_2$ is a *frame morphic image* of $\mathfrak{F}_1$ if there is a surjective frame morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$. ⊣

Definition 7.1.5 (General neighbourhood frame embeddings)**.** A frame morphism f from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ is called an *embedding* of $\mathfrak{F}_1$ in $\mathfrak{F}_2$ if it is injective and satisfies the following condition (where a ranges over elements of A_1):

(A2) $f[a] = b \cap f[W_1]$, for some $b \in A_2$.

If there is an embedding of $\mathfrak{F}_1$ in $\mathfrak{F}_2$, $\mathfrak{F}_1$ is said to be *embeddable* in $\mathfrak{F}_2$ ⊣

Definition 7.1.6 (General neighbourhood frame isomorphisms)**.** A surjective embedding of $\mathfrak{F}_1$ in $\mathfrak{F}_2$ is called an *isomorphism* from $\mathfrak{F}_1$ to $\mathfrak{F}_2$. If there is an isomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, $\mathfrak{F}_1$ is said to be *isomorphic* to $\mathfrak{F}_2$ ⊣

If $\mathfrak{F}_1$ is isomorphic to $\mathfrak{F}_2$ under f, then $\mathfrak{F}_2$ is isomorphic to $\mathfrak{F}_1$ under f^{-1}. Thus when there is an isomorphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, we often say that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are isomorphic to each other ($\mathfrak{F}_1 \cong \mathfrak{F}_2$). $\mathfrak{F}_1$ is isomorphic to $\mathfrak{F}_2$ under f iff f is a bijective frame morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, and its inverse f^{-1} is a frame morphism from $\mathfrak{F}_2$ to $\mathfrak{F}_1$. This provides an alternative definition of isomorphism.

Validity of modal formulas is preserved by taking frame morphic images, and it is invariant under isomorphisms. In detail, we note the following.

- Let f be a frame morphism from $\mathfrak{F}_1 = \langle W_1, N_1, A_1 \rangle$ to $\mathfrak{F}_2 = \langle W_2, N_2, A_2 \rangle$, and let V_2 be an admissible valuation on $\mathfrak{F}_2$. Then V_1 assigning to each atom p the set $f^{-1}[V_2(p)]$ of points of W_1 is a valuation on $\mathfrak{F}_1$. Moreover for any formula α,

  $$\|\alpha\|^{\mathfrak{M}_1} = f^{-1}[\|\alpha\|^{\mathfrak{M}_2}]$$

  where $\mathfrak{M}_1 = \langle \mathfrak{F}_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle \mathfrak{F}_2, V_2 \rangle$.

- If $\mathfrak{F}_2$ is a frame morphic image of $\mathfrak{F}_1$, then for any formula α,

  $$\mathfrak{F}_1 \models \alpha \implies \mathfrak{F}_2 \models \alpha.$$

- If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are isomorphic, then for any formula α,

  $$\mathfrak{F}_1 \models \alpha \iff \mathfrak{F}_2 \models \alpha.$$

Analogous to descriptive relational frames, we define the following class of general neighbourhood frames characterizable as descriptive.

Definition 7.1.7 (Descriptive general neighbourhood frames)**.** A frame $\mathfrak{F} = \langle W, N, A \rangle$ is said to be *descriptive* if it satisfies all of the following conditions. (Unless otherwise stated, x and y range over the elements of W, and u ranges over the ultrafilters in $\langle A, \cup, -, \emptyset, l_N \rangle$.)

$$\text{(D1)} \quad Ax = Ay \implies x = y.$$
$$\text{(D2)} \quad u = Ax, \text{for some } x \in W.$$

(Recall that Ax is the set $\{a \in A \mid x \in a\}$.) ⊣

The converses of the above conditions hold generally. If $x = y$, then trivially $Ax = Ay$. Moreover every Ax can be shown to be an ultrafilter in $\langle A, \cup, -, \emptyset, l_N \rangle$.

Definition 7.1.8 (The category of descriptive general neighbourhood frames)**.** *The category of descriptive general neighbourhood frames* (DNF), comprises all descriptive frames as its objects and all frame morphisms between descriptive frames as its arrows. The operations of domain, codomain, composition and identity are the usual ones for functions or maps. ⊣

7.2 Transformation of DNF to MA

In the rest of this chapter, descriptive frames means descriptive general neighbourhood frames, and frame morphisms means general neighbourhood frame morphisms.

Definition 7.2.1 (The function $\sharp$ for descriptive neighbourhood frames and their frame morphisms)**.** The function $\sharp$ (read "sharp") assigns to each descriptive frame $\mathfrak{F} = \langle W, N, A\rangle$ a set algebra $\mathfrak{F}^\sharp$, and to each frame morphism f from descriptive frame $\mathfrak{F}_1 = \langle W_1, N_1, A_1\rangle$ to descriptive frame $\mathfrak{F}_2 = \langle W_2, N_2, A_2\rangle$ a map $f^\sharp$ from the set algebra ${\mathfrak{F}_2}^\sharp$ to the set algebra ${\mathfrak{F}_1}^\sharp$ as follows.

- $\mathfrak{F}^\sharp = \langle A, \cup, -, \emptyset, l_N\rangle$.
- $f^\sharp : A_2 \to A_1$ is defined, for every $b \in A_2$, by

$$f^\sharp(b) = f^{-1}[b]. \qquad \dashv$$

Note that $f^\sharp$ is well defined since, by condition (A1) of frame morphism (Definition 7.1.3), $f^{-1}[b]$ is guaranteed to be in A_1. Note that the arrows are reversed: whereas f maps A_1 to A_2, $f^\sharp$ maps A_2 to A_1.

We next show that $\mathfrak{F}^\sharp$ is a modal algebra (also called the full complex algebra of $\mathfrak{F}$) and $f^\sharp$ is a homomorphism. In addition, $\sharp$ preserves composition of morphisms as well as the identity morphisms. Note that in proving the above (and so the function $\sharp$ is a contravariant functor from DNF to MA), we do not make use of (D1) and (D2) of Definition 7.1.7, which are the distinctive frame conditions for descriptive frames. However these conditions will be required when we show, in Section 6.5.1, that $\sharp$ is an equivalence from DNF to MA.

Theorem 7.2.2. *For any frame $\mathfrak{F} = \langle W, N, A\rangle$, $\mathfrak{F}^\sharp = \langle A, \cup, -, \emptyset, l_N\rangle$ is a modal algebra.*

Proof. It follows directly from the definition of frames (Definition 7.1.1) that the set A contains $\emptyset$, and is closed under $\cup$, $-$ and l_N. Hence $\mathfrak{F}^\sharp$ is a modal algebra by Definition 6.1.1. $\dashv$

Theorem 7.2.3. *For any frame morphism f from frame $\mathfrak{F}_1 = \langle W_1, N_1, A_1\rangle$ to frame $\mathfrak{F}_2 = \langle W_2, N_2, A_2\rangle$, $f^\sharp$ is a homomorphism from ${\mathfrak{F}_2}^\sharp = \langle A_2, \cup, -, \emptyset, l_{N_2}\rangle$ to ${\mathfrak{F}_1}^\sharp = \langle A_1, \cup, -, \emptyset, l_{N_1}\rangle$.*

Proof. It can easily be shown that the following hold generally:

$$f^{-1}[b_1 \cup b_2] = f^{-1}[b_1] \cup f^{-1}[b_2];$$
$$f^{-1}[-b] = -f^{-1}[b];$$
$$f^{-1}[\emptyset] = \emptyset.$$

Hence $\cup$, $-$ and $\emptyset$ are preserved under $f^\sharp$, i.e.

$$f^\sharp(b_1 \cup b_2) = f^\sharp(b_1) \cup f^\sharp(b_2);$$
$$f^\sharp(-b) = -f^\sharp(b);$$
$$f^\sharp(\emptyset) = \emptyset.$$

For the preservation of the modal operation, i.e. $f^\sharp(l_{N_2}(b_1, \ldots, b_n)) = l_{N_1}(f^\sharp(b_1), \ldots, f^\sharp(b_n))$, we show that $f^{-1}[l_{N_2}(b_1, \ldots, b_n)] = l_{N_1}(f^{-1}[b_1], \ldots, f^{-1}[b_n])$, or equivalently for any $x \in W_1$,

$$x \in f^{-1}[l_{N_2}(b_1, \ldots, b_n)] \iff x \in l_{N_1}(f^{-1}[b_1], \ldots, f^{-1}[b_n]).$$

For $\Longrightarrow$, we argue as follows:

$x \in f^{-1}[l_{N_2}(b_1, \ldots, b_n)]$	by assumption;
$f(x) \in l_{N_2}(b_1, \ldots, b_n)$	by the definition of f^{-1};
$\langle b_1, \ldots, b_n \rangle \in N_2(f(x))$	by the definition of l_{N_2};
$\langle f^{-1}[b_1], \ldots, f^{-1}[b_n] \rangle \in N_1(x)$	by (N1) of frame morphisms;
$x \in l_{N_1}(f^{-1}[b_1], \ldots, f^{-1}[b_n])$	by the definition of l_{N_1}.

The $\Longleftarrow$ direction holds as well since the above argument can be reversed.

We thus have shown that $f^\sharp$ preserves all the relevant algebraic operations, and so $f^\sharp$ is a homomorphism from ${\mathfrak{F}_2}^\sharp$ to ${\mathfrak{F}_1}^\sharp$. ⊣

Theorem 7.2.4. *The function $\sharp$ preserves both composition and identity, i.e.*

(1) $(f_2 \circ f_1)^\sharp = {f_1}^\sharp \circ {f_2}^\sharp$*, for any frame morphisms* $f_1 : \mathfrak{F}_1 \to \mathfrak{F}_2$ *and* $f_2 : \mathfrak{F}_2 \to \mathfrak{F}_3$*, and*

(2) ${\mathrm{id}_{\mathfrak{F}}}^\sharp = \mathrm{id}_{\mathfrak{F}^\sharp}$ *for every frame* $\mathfrak{F}$*.*

Proof. The proof is the same as that for Theorem 6.3.4. ⊣

Theorem 7.2.5. *The function* $\sharp$ *is a contravariant functor from the category* DNF *to the category* MA*.*

Proof. The theorem follows immediately from Theorems 7.2.2, 7.2.3 7.2.4. $\dashv$

7.3 Transformation of MA to DNF

In the following, the collection of all ultrafilters in an algebra $\mathfrak{A}$ is denoted by $\mathrm{Uf}\,\mathfrak{A}$, and the set of all ultrafilters containing an element a of $\mathfrak{A}$ is denoted by Ua. In other words,

$$Ua = \{u \in \mathrm{Uf}\,\mathfrak{A} \mid a \in u\}.$$

Definition 7.3.1 (The functions $\flat$ for modal algebras and homomorphisms)**.** The function $\flat$ (read "flat") assigns to each modal algebra $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ a neighbourhood structure $\mathfrak{A}^\flat$, and to each homomorphism f from modal algebra $\mathfrak{A}_1 = \langle A_1, +, -, 0, l\rangle$ to modal algebra $\mathfrak{A}_2 = \langle A_2, +, -, 0, l\rangle$ a map from ${\mathfrak{A}_2}^\flat$ to ${\mathfrak{A}_1}^\flat$ as follows.

- $\mathfrak{A}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}, N_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle$ where:
  (1) $\mathrm{Uf}\,\mathfrak{A}$ consists of all ultrafilters in $\mathfrak{A}$;
  (2) $N_{\mathfrak{A}} : \mathrm{Uf}\,\mathfrak{A} \to \mathscr{P}((\mathscr{P}(\mathrm{Uf}\,\mathfrak{A}))^n)$ such that for each $u \in \mathrm{Uf}\,\mathfrak{A}$,
  $$N_{\mathfrak{A}}(u) = \{\langle Ua_1, \ldots, Ua_n\rangle \mid l(a_1, \ldots, a_n) \in u\};$$
  (3) $A_{\mathfrak{A}} = \{Ua \mid a \in A\}$.
- $f^\flat : \mathrm{Uf}\,\mathfrak{A}_2 \to \mathrm{Uf}\,\mathfrak{A}_1$ is defined, for every $v \in \mathrm{Uf}\,\mathfrak{A}_2$, by
  $$f^\flat(v) = f^{-1}[v]. \qquad \dashv$$

Theorem 7.3.2. *For any modal algebra* $\mathfrak{A} = \langle A, +, -, 0, l\rangle$*,* $\mathfrak{A}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}, N_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle$ *is a descriptive frame.*

Proof. We first show that $A_{\mathfrak{A}}$ contains the $\emptyset$ as well as all neighbourhoods, and is closed under $\cup$, $-$ and $l_{N_{\mathfrak{A}}}$ (hence $\mathfrak{A}^\flat$ is a frame, given that $\mathrm{Uf}\,\mathfrak{A}$ is non-empty and $N_{\mathfrak{A}}$ is a neighbourhood function of type n on $\mathrm{Uf}\,\mathfrak{A}$), and secondly show that conditions (D1) and (D2) of Definition 7.1.7 are satisfied (hence $\mathfrak{A}^\flat$ is descriptive).

For the first part, it suffices to check the following, where a, $a_1, \ldots, a_n$ and b range over elements of $\mathfrak{A}$.

- $\emptyset = U0$ since no ultrafilter in $\mathfrak{A}$ contains the zero element.
- All neighbourhoods are of the form Ua.
- $Ua \cup Ub = U(a+b)$ since for any ultrafilter v in $\mathfrak{A}$, $v \in Ua \cup Ub$ iff $v \in Ua$ or $v \in Ub$ iff $a \in v$ or $b \in v$ iff $a+b \in v$ iff $v \in U(a+b)$. (The only interesting step is the inference that $a \in v$ or $b \in v$ iff $a+b \in v$, which follows from the proprieties of ultrafilters.)
- $-Ua = U(-a)$ since for any ultrafilter v in $\mathfrak{A}$, $v \in -Ua$ iff $a \notin v$ iff $-a \in v$ iff $v \in U(-a)$. (The only interesting step is the inference that $a \notin v$ iff $-a \in v$, which follows from the proprieties of ultrafilters.)
- $l_{N_\mathfrak{A}}(Ua_1, \ldots, Ua_n) = Ul(a_1, \ldots, a_n)$ since the following (where u is an ultrafilter in $\mathfrak{A}$) are equivalent.

$$
\begin{array}{ll}
u \in l_{N_\mathfrak{A}}(Ua_1, \ldots, Ua_n) & \\
\langle Ua_1, \ldots, Ua_n \rangle \in N_\mathfrak{A}(u) & \text{(Definition of } l_{N_\mathfrak{A}}) \\
l_{N_\mathfrak{A}}(a_1, \ldots, a_n) \in u & \text{(Definition of } N_\mathfrak{A}) \\
u \in Ul(a_1, \ldots, a_n) & \text{(Definition of } Ul(a_1, \ldots, a_n))
\end{array}
$$

$\mathfrak{A}^\flat$ is thus a frame. We next show that it is descriptive, i.e. conditions (D1) and (D2) of descriptive frames are satisfied (see Definition 7.1.7).

To show (D1), i.e. $A_\mathfrak{A}u = A_\mathfrak{A}v \implies u = v$, we suppose $u \neq v$ and demonstrate $A_\mathfrak{A}u \neq A_\mathfrak{A}v$. Note that $A_\mathfrak{A}u$ and $A_\mathfrak{A}v$ consists of all the elements of $A_\mathfrak{A}$ containing u and v, respectively, and Ua consists of all ultrafilters containing a. Therefore:

$$
\begin{aligned}
A_\mathfrak{A}u &= \{Ua \mid u \in Ua\} = \{Ua \mid a \in u\}; \\
A_\mathfrak{A}v &= \{Ua \mid v \in Ua\} = \{Ua \mid a \in v\}.
\end{aligned}
$$

By supposition $u \neq v$. Thus there exists an $a \in A$ such that either (i) both $a \in u$ and $a \notin v$ or (ii) both $a \notin u$ and $a \in v$. If (i), then $Ua \in A_\mathfrak{A}u$ but $Ua \notin A_\mathfrak{A}v$, and so $A_\mathfrak{A}u \neq A_\mathfrak{A}v$. Similarly if (ii), we have $A_\mathfrak{A}u \neq A_\mathfrak{A}v$. In other words, we have shown that $\mathfrak{A}^\flat$ satisfies (D1).

(D2) stipulates that every ultrafilter μ in $\langle A_\mathfrak{A}, \cup, -, \emptyset, l_{N_\mathfrak{A}} \rangle$ is of the form $A_\mathfrak{A}u$ where u is an ultrafilter in $\mathfrak{A} = \langle A, +, -, 0, l \rangle$. (Note that μ is a maximal collection of Ua's, where Ua consists of ultrafilters in $\mathfrak{A}$ containing a.) To demonstrate (D2), it suffices to show that the set

$$v = \{a \in A \mid Ua \in \mu\}$$

is an ultrafilter in $\mathfrak{A}$, because if it is then $A_{\mathfrak{A}}v = \{Ub \mid v \in Ub\}$ is simply μ. (To see this, assume $v \in \mathrm{Uf}\,\mathfrak{A}$, then $Ua \in \mu$ iff $a \in v$ iff $v \in Ua$ iff $Ua \in A_{\mathfrak{A}}v$). Thus what remains to be shown is that v is an ultrafilter in $\mathfrak{A}$. Our argument is that v is non-empty, closed under Boolean meet, and upward closed (hence v is a filter) and for each $a \in A$, exactly one of a and $-a$ is in v (hence v is an ultrafilter). Details are as follows:

- $1 \in v$ since $U1 = \mathrm{Uf}\,\mathfrak{A} \in \mu$.
- Suppose $a, b \in v$, i.e. $Ua, Ub \in \mu$. Then $Ua \cap Ub \in \mu$. But $Ua \cap Ub = U(a \cdot b)$. Thus $a \cdot b \in v$.
- Suppose $a \in v$ and $a \leq b$. From $a \in v$, we have $Ua \in \mu$. From $a \leq b$, we have $Ua \subseteq Ub$ (since if $u \in Ua$ or equivalently $a \in u$ then $b \in u$ or equivalently $u \in Ub$). Thus, $Ub \in \mu$, from which it follows that $b \in v$.
- Suppose it is false that exactly one of a and $-a$ is in v, i.e. *either* (i) both a and $-a$ are in v *or* (ii) neither a nor $-a$ is in v. If (i) then $Ua, U(-a) \in \mu$, then $Ua \cap U(-a) = U(a \cdot -a) = U0 = \emptyset \in \mu$, which is absurd. If (ii) then $Ua, U(-a) \notin \mu$, then $-Ua, -U(-a) \in \mu$, then $U(-a), U(a) \in \mu$, which contradicts the earlier derivation that $Ua, U(-a) \notin \mu$. Hence, by reductio, exactly one of a and $-a$ is in v.

This concludes the proof that $\mathfrak{A}^\flat$ is a descriptive frame. ⊣

Theorem 7.3.3. *For any homomorphism f from modal algebra $\mathfrak{A}_1 = \langle A_1, +, -, 0, l\rangle$ to modal algebra $\mathfrak{A}_2 = \langle A_2, +, -, 0, l\rangle$, $f^\flat$ is a frame morphism from ${\mathfrak{A}_2}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}_2, N_{\mathfrak{A}_2}, A_{\mathfrak{A}_2}\rangle$ to ${\mathfrak{A}_1}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}_1, N_{\mathfrak{A}_1}, A_{\mathfrak{A}_1}\rangle$.*

Proof. We show that $f^\flat$ satisfies conditions (N1) and (A1) of frame morphisms (see Definition 7.1.3).

For (N1), we note that the following are equivalent, where $v \in \mathrm{Uf}\,\mathfrak{A}_2$ and $a_1, \ldots, a_n \in \mathfrak{A}_1$.

$$\begin{aligned}
\langle {f^\flat}^{-1}[Ua_1], \ldots, {f^\flat}^{-1}[Ua_n]\rangle &\in N_{\mathfrak{A}_2}(v) \\
\langle U(f(a_1)), \ldots, U(f(a_n))\rangle &\in N_{\mathfrak{A}_2}(v) \\
l_2(f(a_1), \ldots, f(a_n)) &\in v \\
f(l_1(a_1, \ldots, a_n)) &\in v \\
l_1(a_1, \ldots, a_n) &\in f^{-1}[v] \\
l_1(a_1, \ldots, a_n) &\in f^\flat(v) \\
\langle Ua_1, \ldots, Ua_n\rangle &\in N_{\mathfrak{A}_1}(f^\flat(v))
\end{aligned}$$

For (A1), what needs to be shown is ${f^\flat}^{-1}[Ua] \in A_{\mathfrak{A}_2}$ for an arbitrary $a \in A_1$. It suffices to establish that ${f^\flat}^{-1}[Ua] = U(f(a))$ since $U(f(a))$ is a member of $A_{\mathfrak{A}_2}$. Consider a $v \in \mathrm{Uf}\, A_2$. The following are equivalent.

$$\begin{aligned} v &\in {f^\flat}^{-1}[Ua] \\ f^\flat(v) &\in Ua \\ f^{-1}[v] &\in Ua \\ a &\in f^{-1}[v] \\ f(a) &\in v \\ v &\in U(f(a)) \end{aligned}$$

Note that $a \in f^{-1}[v]$ implies $f^{-1}[v] \in Ua$ because $f^{-1}[v]$ is an ultrafilter in $\mathfrak{A}_1$ (given that v is an ultrafilter in $\mathfrak{A}_2$ and f is a homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$). ⊣

Theorem 7.3.4. *The function* $\flat$ *preserves both composition and identity, i.e.*

(1) $(f_2 \circ f_1)^\flat = {f_1}^\flat \circ {f_2}^\flat$, *for any homomorphisms* $f_1 : \mathfrak{A}_1 \to \mathfrak{A}_2$ *and* $f_2 : \mathfrak{A}_2 \to \mathfrak{A}_3$, *and*

(2) ${\mathrm{id}_{\mathfrak{A}}}^\flat = \mathrm{id}_{\mathfrak{A}^\flat}$ *for every modal algebra* $\mathfrak{A}$.

Proof. The proof is the same as that for Theorem 6.4.5. ⊣

Theorem 7.3.5. *The function* $\flat$ *for modal algebras and homomorphisms is a contravariant functor from the category* MA *to the category* DNF.

Proof. The theorem follows immediately from Theorems 7.3.2, 7.3.3 and 7.3.4. ⊣

7.4 Dual equivalence between DNF and MA

In the previous two sections, we have established $\sharp$ and $\flat$ to be contravariant functors from DNF to MA, and from MA to DNF, respectively. We now show that they are also equivalences between the two categories.

Theorem 7.4.1. *The categories* DNF *and* MA *are dually equivalent.*

Proof. We demonstrate the following regarding contravariant functor $\sharp$ (from DNF and MA) and contravariant functor $\flat$ (from MA to DNF).

- The composite functor $\flat \circ \sharp$ is naturally isomorphic to the identity functor on DNF (Theorem 7.4.4).
- The composite functor $\sharp \circ \flat$ is naturally isomorphic to the identity functor on MA (Theorem 7.4.7).

Further details of the proof are given in Section 7.4.1 and 7.4.2. ⊣

As in duality between DRF and NMA, the basic idea is the following.

- Every descriptive frame $\mathfrak{F} = \langle W, N, A\rangle$ is isomorphic to $\mathfrak{F}^{\sharp\flat}$ (the ultrafilter frame of the complex algebra of $\mathfrak{F}$) under the map $x \mapsto Ax$.
- Every modal algebra $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ is isomorphic to $\mathfrak{A}^{\flat\sharp}$ (the complex algebra of the ultrafilter frame of $\mathfrak{A}$) under the map $a \mapsto Ua$.

7.4.1 Natural isomorphism between $\mathrm{Id}_{\mathsf{DNF}}$ and $\flat \circ \sharp$

Throughout this section, $\mathfrak{F}$, $\mathfrak{F}^{\sharp}$ and $\mathfrak{F}^{\sharp\flat}$ are as follows.

- $\mathfrak{F} = \langle W, N, A\rangle$ is a descriptive frame, i.e. frames satisfying (D1) and (D2). (See Definition 7.1.7.)
- $\mathfrak{F}^{\sharp} = \langle A, \cup, -, \emptyset, l_N\rangle$ is the normal modal algebra we get from $\mathfrak{A}$ by $\sharp$. Recall that l_N is the n-ary operation on A defined, for every $a_1, \ldots, a_n \in A$, by

$$l_N(a_1, \ldots, a_n) = \{x \in W \mid \langle a_1, \ldots, a_n\rangle \in N(x)\}.$$

- $\mathfrak{F}^{\sharp\flat} = \langle \mathrm{Uf}\,\mathfrak{F}^{\sharp}, N_{\mathfrak{F}^{\sharp}}, A_{\mathfrak{F}^{\sharp}}\rangle$ is the ultrafilter frame we get from $\mathfrak{F}^{\sharp}$ by $\flat$. Note that
  - $\mathrm{Uf}\,\mathfrak{F}^{\sharp}$ is the collection of all ultrafilters in $\mathfrak{F}^{\sharp}$;
  - $N_{\mathfrak{F}^{\sharp}}(u) = \{\langle Ua_1, \ldots, Ua_n\rangle \mid l_N(a_1, \ldots, a_n) \in u\}$, for every $u \in \mathrm{Uf}\,\mathfrak{F}^{\sharp}$;
  - $A_{\mathfrak{F}^{\sharp}} = \{Ua \mid a \in A\}$ where Ua is the set of ultrafilters in $\mathfrak{F}^{\sharp}$ containing a.

We let η be the function that assigns to each descriptive frame $\mathfrak{F}$ the map $\eta_{\mathfrak{F}} : W \to \mathrm{Uf}\,\mathfrak{F}^{\sharp}$ defined, for every $x \in W$, by

$$\eta_{\mathfrak{F}}(x) = Ax.$$

The map $\eta_{\mathfrak{F}}$ is well defined since every Ax is an ultrafilter in $\mathfrak{F}^{\sharp}$. See (2) of Remark 6.2.8.

Theorem 7.4.2. *$\eta_{\mathfrak{F}} : W \to \mathrm{Uf}\,\mathfrak{F}^{\sharp}$ is a frame morphism from $\mathfrak{F}$ to $\mathfrak{F}^{\sharp\flat}$.*

Proof. We show that $\eta_{\mathfrak{F}}$ satisfies (N1) and (A1) of Definition 7.1.3.

(N1) stipulates that for any $x \in W$, $a_1, \dots, a_n \in A$,

$$\langle {\eta_{\mathfrak{F}}}^{-1}[Ua_1], \dots, {\eta_{\mathfrak{F}}}^{-1}[Ua_n]\rangle \in N(x) \iff \langle Ua_1, \dots, Ua_n\rangle \in N_{\mathfrak{F}^{\sharp}}(\eta_{\mathfrak{F}}(x)),$$

which is equivalent to

$$\langle a_1, \dots, a_n\rangle \in N(x) \iff \langle Ua_1, \dots, Ua_n\rangle \in N_{\mathfrak{F}^{\sharp}}(Ax),$$

since $\eta_{\mathfrak{F}}(x) = Ax$ and, for every i from 1 to n, ${\eta_{\mathfrak{F}}}^{-1}[Ua_i] = a_i$. (To see the latter, consider arbitrary $x \in W$, then $x \in {\eta_{\mathfrak{F}}}^{-1}[Ua_i]$ iff $Ax \in Ua_i$ iff $a_i \in Ax$ iff $x \in a_i$.) But $\langle Ua_1, \dots, Ua_n\rangle \in N_{\mathfrak{F}^{\sharp}}(Ax)$ iff $l_N(a_1, \dots, a_n) \in Ax$ iff $x \in l_N(a_1, \dots, a_n)$ iff $\langle a_1, \dots, a_n\rangle \in N(x)$. Thus $\eta_{\mathfrak{F}}$ satisfies (N1).

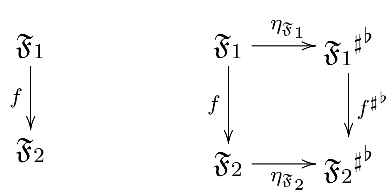
(A1) requires that for every $a \in A$, ${\eta_{\mathfrak{F}}}^{-1}[Ua] \in A$. But this is obvious since we already know that ${\eta_{\mathfrak{F}}}^{-1}[Ua] = a$. $\dashv$

Theorem 7.4.3. *η is a natural transformation from $\mathrm{Id}_{\mathsf{DNF}}$ to $\flat \circ \sharp$.*

Proof. We have proved in Theorem 7.4.2 that every component $\eta_{\mathfrak{F}}$ of η is a frame morphism from $\mathfrak{F}$ to $\mathfrak{F}^{\sharp\flat}$, i.e. from $\mathrm{Id}_{\mathsf{DNF}}(\mathfrak{F})$ to $(\flat \circ \sharp)(\mathfrak{F})$. It remains to show that the following holds for any frame morphism f from $\mathfrak{F}_2 = \langle W_2, N_2, A_2\rangle$ to $\mathfrak{F}_2 = \langle W_2, N_2, A_2\rangle$ (both are descriptive frames),

$$f^{\sharp\flat} \circ \eta_{\mathfrak{F}_1} = \eta_{\mathfrak{F}_2} \circ f.$$

In other words, what needs to be shown is that the following diagram commutes.



The proof is the same as that for descriptive relational frame, and is omitted here. $\dashv$

Theorem 7.4.4. *η is a natural isomorphism from $\mathrm{Id}_{\mathsf{DNF}}$ to $\flat \circ \sharp$. Thus $\mathrm{Id}_{\mathsf{DNF}}$ is naturally isomorphic to $\flat \circ \sharp$.*

Proof. We already know that η is a natural transformation from $\mathrm{Id}_{\mathsf{DNF}}$ to $\flat \circ \sharp$ (Theorem 7.4.3). For η to be a natural isomorphism, we need to show that every component $\eta_{\mathfrak{F}}$ of it is a frame isomorphism from $\mathfrak{F}$ to $\mathfrak{F}^{\sharp^\flat}$, i.e. there exists a frame morphism $\theta_{\mathfrak{F}}$ from $\mathfrak{F}^{\sharp^\flat}$ to $\mathfrak{F}$ such that

$$\theta_{\mathfrak{F}} \circ \eta_{\mathfrak{F}} = \mathrm{id}_{\mathfrak{F}};$$
$$\eta_{\mathfrak{F}} \circ \theta_{\mathfrak{F}} = \mathrm{id}_{\mathfrak{F}^{\sharp^\flat}}.$$

Let $\theta_{\mathfrak{F}} : \mathrm{Uf}\,\mathfrak{F}^\sharp \to W$ be defined as follows: for every $u \in \mathrm{Uf}\,\mathfrak{F}^\sharp$

$$\theta_{\mathfrak{F}}(u) = x, \quad \text{where } u = Ax.$$

Note that $\theta_{\mathfrak{F}}$ is well-defined since

- by (D2) every ultrafilter u in $\mathfrak{F}^\sharp$ is of the form Ax for some $x \in W$ and so is assigned some member of W;
- by (D1) every ultrafilter u in $\mathfrak{F}^\sharp$ is assigned at most one member of W (for if $u = Ax$ and $u = Ay$, then $x = y$).

Moreover $\theta_{\mathfrak{F}}$ as defined earlier is a frame morphism from $\mathfrak{F}^{\sharp^\flat}$ to $\mathfrak{F}$ because it satisfies (N1) and (A1) of Definition 7.1.3. The reasons are as follows.

- (N1) stipulates that for every $Ax \in \mathrm{Uf}\,\mathfrak{F}^\sharp$, $a_1, \ldots, a_n \in A$,

  $$\langle {\theta_{\mathfrak{F}}}^{-1}[a_1], \ldots, {\theta_{\mathfrak{F}}}^{-1}[a_n] \rangle \in N_{\mathfrak{F}^\sharp}(Ax) \iff \langle a_1, \ldots, a_n \rangle \in N(\theta_{\mathfrak{F}}(Ax)),$$

  which is equivalent to

  $$\langle Ua_1, \ldots, Ua_n \rangle \in N_{\mathfrak{F}^\sharp}(Ax) \iff \langle a_1, \ldots, a_n \rangle \in N(x),$$

  since for every i from 1 to n,

  $$\begin{aligned} {\theta_{\mathfrak{F}}}^{-1}[a_i] &= \{Ax \in \mathrm{Uf}\,\mathfrak{F}^\sharp \mid x \in a_i\} \\ &= \{Ax \in \mathrm{Uf}\,\mathfrak{F}^\sharp \mid a_i \in Ax\} \\ &= Ua_i. \end{aligned}$$

  But $\langle Ua_1, \ldots, Ua_n \rangle \in N_{\mathfrak{F}^\sharp}(Ax)$ iff $l_N(a_1, \ldots, a_n) \in Ax$ iff $x \in l_N(a_1, \ldots, a_n)$ iff $\langle a_1, \ldots, a_n \rangle \in N(x)$. Thus $\eta_{\mathfrak{F}}$ satisfies (N1).

- (A1) requires that for every $a \in A$, ${\theta_{\mathfrak{F}}}^{-1}[a] \in A_{\mathfrak{F}^\sharp}$. But this is obvious since we already know that ${\theta_{\mathfrak{F}}}^{-1}[a] = Ua$.

Finally for any $x \in W$ and $u = Ax \in \mathrm{Uf}\,\mathfrak{F}^\sharp$,

$$\begin{aligned}(\theta_{\mathfrak{F}} \circ \eta_{\mathfrak{F}})(x) &= \theta_{\mathfrak{F}}(\eta_{\mathfrak{F}}(x)) &= \theta_{\mathfrak{F}}(Ax) &= x;\\ (\eta_{\mathfrak{F}} \circ \theta_{\mathfrak{F}})(Ax) &= \eta_{\mathfrak{F}}(\theta_{\mathfrak{F}}(Ax)) &= \eta_{\mathfrak{F}}(x) &= Ax.\end{aligned}$$

Thus, both $\theta_{\mathfrak{F}} \circ \eta_{\mathfrak{F}} = \mathrm{id}_{\mathfrak{F}}$ and $\eta_{\mathfrak{F}} \circ \theta_{\mathfrak{F}} = \mathrm{id}_{\mathfrak{F}^{\sharp\flat}}$. ⊣

7.4.2 Natural isomorphism between $\mathrm{Id}_{\mathsf{MA}}$ and $\sharp \circ \flat$

Throughout this section, $\mathfrak{A}$, $\mathfrak{A}^\flat$ and $\mathfrak{A}^{\flat\sharp}$ are as follows.

- $\mathfrak{A} = \langle A, +, -, 0, l \rangle$ is a normal modal algebra. (See Definition 6.1.5.)
- $\mathfrak{A}^\flat = \langle \mathrm{Uf}\,\mathfrak{A}, R_{\mathfrak{A}}, A_{\mathfrak{A}} \rangle$ is the descriptive frame we get from $\mathfrak{A}$ under $\flat$ as defined in Definition 6.4.1. Recall that:
  - $\mathrm{Uf}\,\mathfrak{A}$ is the collection of all ultrafilters in $\mathfrak{A}$;
  - $R_{\mathfrak{A}} u_0 u_1 \cdots u_n$ iff

$$\forall a_1, \ldots, a_n \in A,\ l(a_1, \ldots, a_n) \in u_0 \implies \exists i \geq 1 : a_i \in u_i;$$

  - $A_{\mathfrak{A}} = \{Ua \mid a \in A\}$ where Ua consists of all ultrafilters in $\mathfrak{A}$ containing a.
- $\mathfrak{A}^{\flat\sharp} = \langle A_{\mathfrak{A}}, \cup, -, \emptyset, l_{R_{\mathfrak{A}}} \rangle$ is the normal modal algebra we get from $\mathfrak{A}^\flat$ under $\sharp$ as defined in Definition 6.3.1. Note that $l_{R_{\mathfrak{A}}}(Ua_1, \ldots, Ua_n)$, which consists of ultrafilters u_0 in $\mathfrak{A}$ satisfying the condition

$$\forall u_1, \ldots, u_n \in \mathrm{Uf}\,\mathfrak{A},\ R_{\mathfrak{A}} u_0 u_1 \cdots u_n \implies \exists i \geq 1 : u_i \in a_i,$$

  is simply $U(l(a_1, \ldots, a_n))$ (see the proof of Theorem 6.4.2).

We let η be the function that assigns to each $\mathfrak{A}$ the map $\eta_{\mathfrak{A}} : A \to A_{\mathfrak{A}}$ defined, for every $a \in A$, by

$$\eta_{\mathfrak{A}}(a) = Ua.$$

Theorem 7.4.5. *$\eta_{\mathfrak{A}} : A \to A_{\mathfrak{A}}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}^{\flat\sharp}$.*

Proof. We show that $\eta_{\mathfrak{A}}$ preserves the algebraic operations, i.e.

$$\begin{aligned}
\eta_{\mathfrak{A}}(a+b) &= \eta_{\mathfrak{A}}(a) \cup \eta_{\mathfrak{A}}(b);\\
\eta_{\mathfrak{A}}(-a) &= -\eta_{\mathfrak{A}}(a);\\
\eta_{\mathfrak{A}}(0) &= \emptyset;\\
\eta_{\mathfrak{A}}(l(a_1,\ldots,a_n)) &= l_{R_{\mathfrak{A}}}(\eta_{\mathfrak{A}}(a_1),\ldots,\eta_{\mathfrak{A}}(a_n)).
\end{aligned}$$

But the above is a consequence of the following, which we have already demonstrated when proving that the set $A_{\mathfrak{A}}$ is closed under $\cup$, $-$, $\emptyset$ and $l_{R_{\mathfrak{A}}}$ (see the proof of Theorem 6.4.2):

$$\begin{aligned}
U(a+b) &= U(a) \cup U(b);\\
U(-a) &= -U(a);\\
U(0) &= \emptyset;\\
U(l(a_1,\ldots,a_n)) &= l_{R_{\mathfrak{A}}}(U(a_1),\ldots,U(a_n)).
\end{aligned}$$

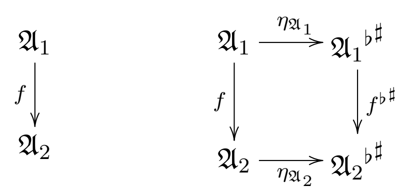
Thus $\eta_{\mathfrak{A}}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}^{\flat\sharp}$. ⊣

Theorem 7.4.6. *η is a natural transformation from $\mathrm{Id}_{\mathsf{MA}}$ to $\sharp \circ \flat$.*

Proof. We have proved in Theorem 6.5.5 that every component $\eta_{\mathfrak{A}}$ of η is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}^{\flat\sharp}$, i.e. from $\mathrm{Id}_{\mathsf{MA}}(\mathfrak{A})$ to $(\sharp \circ \flat)(\mathfrak{A})$. It remains to show that the following holds for any homomorphism f from $\mathfrak{A}_1 = \langle A_1, +, -, 0, l\rangle$ to $\mathfrak{A} = \langle A, +, -, 0, l\rangle 2$ (both are normal modal algebras),

$$f^{\flat\sharp} \circ \eta_{\mathfrak{A}_1} = \eta_{\mathfrak{A}_2} \circ f.$$

In other words, what needs to be shown is that the following diagram commutes.



We recall here that $f^{\flat} : \mathrm{Uf}\,\mathfrak{A}_2 \to \mathrm{Uf}\,\mathfrak{A}_1$ and $f^{\flat\sharp} : A_{\mathfrak{A}_1} \to A_{\mathfrak{A}_2}$ are defined by:

$$\forall v \in \mathrm{Uf}\,\mathfrak{A}_2,\ f^{\flat}(v) = f^{-1}[v];$$
$$\forall a \in A_1,\ f^{\flat\sharp}(Ua) = {f^{\flat}}^{-1}[Ua].$$

Observe that $f^{\flat\sharp} \circ \eta_{\mathfrak{A}_1} = \eta_{\mathfrak{A}_2} \circ f$ iff for any $a \in A_1$,

$$(f^{\flat\sharp} \circ \eta_{\mathfrak{A}_1})(a) = (\eta_{\mathfrak{A}_2} \circ f)(a)$$

or equivalently

$$f^{\flat\sharp}(U_1 a) = U_2(f(x))$$

where $U_1 a$ consists of all ultrafilters in $\mathfrak{A}_1$ containing a, and $U_2(f(a))$ consists of all ultrafilters in $\mathfrak{A}_2$ containing $f(a)$. To show the above identity, we consider arbitrary $v \in \mathrm{Uf}\,\mathfrak{A}_2$. The following are equivalent:

$$\begin{aligned}
v \in f^{\flat\sharp}(U_1 a) &\iff v \in U_2(f(a));\\
v \in f^{\flat^{-1}}[U_1 a] &\iff f(a) \in v;\\
f^{\flat}(v) \in U_1 a &\iff f(a) \in v;\\
f^{-1}[v] \in U_1 a &\iff f(a) \in v;\\
a \in f^{-1}[v] &\iff f(a) \in v;\\
f(a) \in v &\iff f(a) \in v.
\end{aligned}$$

But the last statement is obviously true. Thus we have shown that $f^{\flat\sharp}(U_1 a) = U_2(f(a))$ for any $a \in A_1$, from which it follows that $f^{\flat\sharp} \circ \eta_{\mathfrak{A}_1} = \eta_{\mathfrak{A}_2} \circ f$, as argued above. ⊣

Theorem 7.4.7. *η is a natural isomorphism from $\mathrm{Id}_{\mathsf{MA}}$ to $\sharp \circ \flat$. Thus $\mathrm{Id}_{\mathsf{MA}}$ is naturally isomorphic to $\sharp \circ \flat$.*

Proof. We already know that η is a natural transformation from $\mathrm{Id}_{\mathsf{MA}}$ to $\sharp\circ\flat$ (Theorem 7.4.6). For η to be a natural isomorphism, every component $\eta_{\mathfrak{A}}$ of it must be a isomorphism. In other words, we need to show that for every homomorphism $\eta_{\mathfrak{A}}$ from $\mathfrak{A} = \langle A, +, -, 0, l\rangle$ to $\mathfrak{A}^{\flat\sharp}$, there exists a homomorphism $\theta_{\mathfrak{A}}$ from $\mathfrak{A}^{\flat\sharp}$ to $\mathfrak{A}$ such that

$$\theta_{\mathfrak{A}} \circ \eta_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}};$$

$$\eta_{\mathfrak{A}} \circ \theta_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}^{\flat\sharp}}.$$

Let $\theta_{\mathfrak{A}} : A_{\mathfrak{A}} \to A$ be defined as follows: for every $Ua \in A_{\mathfrak{A}}$,

$$\theta_{\mathfrak{A}}(Ua) = a.$$

$\theta_{\mathfrak{A}}$ as defined above is a homomorphism from $\mathfrak{A}^{\flat\sharp}$ to $\mathfrak{A}$ iff the following hold:

$$\begin{aligned}
\theta_{\mathfrak{A}}(Ua \cup Ub) &= \theta_{\mathfrak{A}}(Ua) + \theta_{\mathfrak{A}}(Ub), \\
\theta_{\mathfrak{A}}(-Ua) &= -\theta_{\mathfrak{A}}(Ua), \\
\theta_{\mathfrak{A}}(\emptyset) &= 0, \\
\theta_{\mathfrak{A}}(l_{R_{\mathfrak{A}}}(Ua_1, \ldots, Ua_n)) &= l(\theta_{\mathfrak{A}}(Ua_1), \ldots, \theta_{\mathfrak{A}}(Ua_n)),
\end{aligned}$$

or equivalently the following hold:

$$\begin{aligned}
\theta_{\mathfrak{A}}(U(a + b)) &= \theta_{\mathfrak{A}}(Ua) + \theta_{\mathfrak{A}}(Ub), \\
\theta_{\mathfrak{A}}(U(-a)) &= -\theta_{\mathfrak{A}}(Ua), \\
\theta_{\mathfrak{A}}(0) &= 0, \\
\theta_{\mathfrak{A}}(U(l(a_1, \ldots, a_n))) &= l(\theta_{\mathfrak{A}}(Ua_1), \ldots, \theta_{\mathfrak{A}}(Ua_n)).
\end{aligned}$$

But the last set of identities are obvious, given our definition of $\theta_{\mathfrak{A}}$.

Finally for any $a \in A$ and $Ua \in A_{\mathfrak{A}}$, we have

$$\begin{aligned}
(\theta_{\mathfrak{A}} \circ \eta_{\mathfrak{A}})(a) &= \theta_{\mathfrak{A}}(\eta_{\mathfrak{A}}(a)) &&= \theta_{\mathfrak{A}}(Aa) &&= a; \\
(\eta_{\mathfrak{A}} \circ \theta_{\mathfrak{A}})(Ua) &= \eta_{\mathfrak{A}}(\theta_{\mathfrak{A}}(Ua)) &&= \eta_{\mathfrak{A}}(a) &&= Ua.
\end{aligned}$$

Thus, both $\theta_{\mathfrak{A}} \circ \eta_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}}$ and $\eta_{\mathfrak{F}} \circ \theta_{\mathfrak{A}} = \mathrm{id}_{\mathfrak{A}^{\sharp\flat}}$. ⊣

Chapter 8

Translation in Modal Logic

This chapter explores translation between various types of modal logic: between monadic and polyadic systems, and between normal and non-normal systems. We start with a discussion of the notions of translation schemes and translational equivalence.

8.1 Translation and translational equivalence

Following Pelletier and Urquhart (2003), we adopt the following definitions of translations between languages and translational equivalence between systems.

Definition 8.1.1 (Translation schemes and translations)**.** A *translation scheme* t from object language $\mathfrak{L}$ to object language $\mathfrak{L}'$ has the following form (where $\alpha_1, \ldots, \alpha_n$ are parameters or place-holders in formulas):

- Every atom p_i of $\mathfrak{L}$ is assigned a formula A_i of $\mathfrak{L}'$;
- For any n-ary connective f of $\mathfrak{L}$, the formula $f(\alpha_1, \ldots, \alpha_n)$ is assigned a formula B_f of $\mathfrak{L}'$ containing only parameters from $\alpha_1, \ldots, \alpha_n$.

The *translation* determined by a translation scheme t is the mapping ^{t} from formulas of $\mathfrak{L}$ to formulas of $\mathfrak{L}'$ as given by the following recursive definition:

(1) If p_i is an atom of $\mathfrak{L}$, then ${p_i}^t$ is A_i.

(2) If f is an n-ary connective of $\mathfrak{L}$, then $(f(A_1, \ldots, A_n))^t$ is $B_f[\alpha_1/{A_1}^t, \ldots, \alpha_n/{A_n}^t]$, which is the formula resulting from substituting ${A_1}^t$ for every occurrence of α_1 in B_f (and similarly for ${A_2}^t$ etc.). $\dashv$

Note that the translation schemes (and the corresponding translations) we considered in this chapter assign each atom to itself, and assign each formula constructed from a truth-functional connective to the formula itself. Thus in stating the translation schemes, we stipulate only the modal case.

Definition 8.1.2 (Sound and exact translations)**.** Let S and S′ be systems in the languages $\mathfrak{L}$ and $\mathfrak{L}'$, and α a formula in $\mathfrak{L}$. A translation ^{t} from $\mathfrak{L}$ to $\mathfrak{L}'$ is *sound* if α^t is a theorem of S′ whenever α is a theorem of S. The translation is *exact* if α^t is a theorem of S′ exactly when α is a theorem of S. ⊣

Definition 8.1.3 (Translational equivalence)**.** Two systems S and S′ (in the languages $\mathfrak{L}$ and $\mathfrak{L}'$ respectively) are *translationally equivalent* if there are translations t from $\mathfrak{L}$ to $\mathfrak{L}'$ and t' from $\mathfrak{L}'$ to $\mathfrak{L}$ such that

(1) Both t and t' are sound;

(2) For any formula α in $\mathfrak{L}$, $(\alpha^t)^{t'} \leftrightarrow \alpha$ is a theorem of S;

(3) For any formula α in $\mathfrak{L}'$, $(\alpha^{t'})^t \leftrightarrow \alpha$ is a theorem of S′. ⊣

8.2 Monadic fragments of polyadic systems

8.2.1 Diagonalization

Consider the following translation scheme from the monadic modal language $\mathfrak{L}_1$ to the n-adic modal language $\mathfrak{L}_n$:

$$\Box\alpha \longmapsto \Box(\alpha, \dots, \alpha)$$
$$\Diamond\alpha \longmapsto \Diamond(\alpha, \dots, \alpha)$$

The unary $\Box$ and $\Diamond$ can be described as the diagonalization of the n-ary $\Box$ and $\Diamond$: if we consider the set of all n-tuples of formulas as an n-dimensional matrix, then the subset of tuples whose coordinates are the same formula can be viewed as a diagonal across such a matrix. Hence we call the translation scheme d_n (for n-diagonalization). Similarly the unary $\Box$ and $\Diamond$ are called the n-diagonal $\Box$ and $\Diamond$, respectively. (We do not mention n if it is clear what n is.)

Given the translation scheme d_n and the usual interpretation of the n-adic modal language, it can readily be seen that the interpretation of the diagonal $\Box$ and $\Diamond$ in the class of $(n+1)$-ary relational frames is as follows:

- $\mathfrak{M}, x \models \Box\alpha$ iff $\forall y_1, \ldots, y_n,\ Rxy_1 \cdots y_n \implies \exists i (1 \leq i \leq n) : \mathfrak{M}, y_i \models \alpha$.
- $\mathfrak{M}, x \models \Diamond\alpha$ iff $\exists y_1, \ldots, y_n : Rxy_1 \cdots y_n \ \&\ \forall i (1 \leq i \leq n),\ \mathfrak{M}, y_i \models \alpha$.

We call the above idiom for the monadic modal language "the n-diagonal idiom". (Recall that an idiom is a class of frames with a truth-theory.) The set of monadic formulas valid in the n-diagonal idiom, which we refer to as the n-diagonal logic, is finitely axiomatizable. For details, see Jennings and Schotch (1984), Apostoli and Brown (1995), and Nicholson et al. (2000).

Definition 8.2.1 (n-diagonal logics)**.** $\mathrm{K}_n^{\mathrm{d}}$ has PL, [RM], [RN], and the following axiom.

$$[\wedge_n^{\mathrm{d}}] \quad \Box p_1 \wedge \cdots \wedge \Box p_{n+1} \to \Box \bigvee_{i=1}^{n+1} \bigvee_{j=i+1}^{n+1} (p_i \wedge p_j) \qquad \dashv$$

The characteristic axioms of the first two members of the series of diagonal logics are as follows:

$$[\wedge_1^{\mathrm{d}}] \quad \Box p \wedge \Box q \to \Box(p \wedge q)$$

$$[\wedge_2^{\mathrm{d}}] \quad \Box p \wedge \Box q \wedge \Box r \to \Box((p \wedge q) \vee (p \wedge r) \vee (q \wedge r))$$

Note that the axiom $[\wedge_1^{\mathrm{d}}]$ is the familiar [C], and the logic $\mathrm{K}_1^{\mathrm{d}}$ is just K. The n-diagonal logic $\mathrm{K}_n^{\mathrm{d}}$ (where $n > 1$) can be described as a "weakly aggregative modal logic" since its aggregation principle $[\wedge_n^{\mathrm{d}}]$ is a weakening of the following principle of complete aggregation, which is a theorem of K.

$$\Box p_1 \wedge \cdots \wedge \Box p_n \to \Box(p_1 \wedge \cdots \wedge p_n)$$

$\mathrm{K}_n^{\mathrm{d}}$ can be extended by adding the familiar monadic formulas [C], [P], [T], [B], [4], and [5], and the resulting systems are called, respectively, $\mathrm{K}_n^{\mathrm{d}}\mathrm{C}$, $\mathrm{K}_n^{\mathrm{d}}\mathrm{P}$, $\mathrm{K}_n^{\mathrm{d}}\mathrm{T}$, $\mathrm{K}_n^{\mathrm{d}}\mathrm{B}$, $\mathrm{K}_n^{\mathrm{d}}4$, and $\mathrm{K}_n^{\mathrm{d}}5$. Note that $\mathrm{K}_n^{\mathrm{d}}\mathrm{C}$ is just $\mathrm{K}_1^{\mathrm{d}}$ or K, since $[\wedge_n^{\mathrm{d}}]$ is derivable from $[\wedge_1^{\mathrm{d}}]$ or [C] (in the presence of PL and [RM]. In the following, we list correspondence and determination results for these formulas and logics (in the n-diagonal idiom).

Theorem 8.2.2. *The following monadic modal formulas correspond to the indicated first-order conditions on $(n+1)$-ary relations.*

[C]	:	$(\forall x)(\forall \vec{y})(Rx\vec{y} \rightarrow (\exists y_i \in \vec{y})Rxy_i \cdots y_i)$	(D-binarity)
[P]	:	$(\forall x)(\exists \vec{y})Rx\vec{y}$	(Seriality)
[T]	:	$(\forall x)Rxx \cdots x$	(Reflexivity)
[B]	:	$(\forall x)(\forall \vec{y})(Rx\vec{y} \rightarrow (\exists y_i \in \vec{y})Ry_ix \cdots x)$	(D-symmetry)
[4]	:	$(\forall x)(\forall \vec{y}, \vec{z}_1, \ldots, \vec{z}_n)(Rx\vec{y} \wedge Ry_1\vec{z}_1 \wedge \cdots \wedge Ry_n\vec{z}_n \rightarrow$ $(\exists \vec{w} \subseteq \vec{z}_1 \cup \cdots \cup \vec{z}_n)Rx\vec{w})$	(D-transitivity)
[5]	:	$(\forall x)(\forall \vec{y}, \vec{z})(Rx\vec{y} \wedge Rx\vec{z} \rightarrow (\exists y_i \in \vec{y})Ry_i\vec{z})$	(D-euclideanness)

Note: In the condition of d-transitivity, $\exists \vec{w} \subseteq \vec{z}_1 \cup \cdots \cup \vec{z}_n$ means the following: there exists a $\vec{w}$ such that every $w_k \in \vec{w}$ belongs to the set of $z_{i.j}$'s where $1 \leq i, j \leq n$.

Observe that the frame properties corresponding to [B], [4], and [5] are weaker than those corresponding to [B_n], [4_n], and [5_n], which is expected since the former formulas are derivable from the latter ones. We distinguish the weaker properties from the stronger ones by prefixing it with the letter D or d (for diagonal).

Theorem 8.2.3. *The following n-diagonal logics are determined by the indicated classes of $(n+1)$-ary relational frames.*

$\mathrm{K}_n^{\mathrm{d}}$: *All frames*
$\mathrm{K}_n^{\mathrm{d}}\mathrm{C}$: *D-binary frames*
$\mathrm{K}_n^{\mathrm{d}}\mathrm{P}$: *Serial frames*
$\mathrm{K}_n^{\mathrm{d}}\mathrm{T}$: *Reflexive frames*
$\mathrm{K}_n^{\mathrm{d}}\mathrm{B}$: *D-symmetric frames*
$\mathrm{K}_n^{\mathrm{d}}4$: *D-transitive frames*
$\mathrm{K}_n^{\mathrm{d}}5$: *D-euclidean frames*

Theorem 8.2.4. *The monadic n-diagonal system $\mathrm{K}_n^{\mathrm{d}}$ is exactly translatable into the n-adic normal system K_n under the translation d_n, i.e. for every $\mathcal{L}_1$-formula α,*

$$\vdash_{\mathrm{K}_n^{\mathrm{d}}} \alpha \iff \vdash_{\mathrm{K}_n} \alpha^{d_n}.$$

8.2.2 Furcation

In this section, we introduce another translation scheme from the monadic modal language $\mathfrak{L}_1$ to the n-adic modal language $\mathfrak{L}_n$.

$$\Box\alpha \longmapsto \Box(\alpha, \bot^{n-1}) \vee \Box(\bot, \alpha, \bot^{n-2}) \vee \cdots \vee \Box(\bot^{n-1}, \alpha)$$
$$\Diamond\alpha \longmapsto \Diamond(\alpha, \top^{n-1}) \wedge \Diamond(\top, \alpha, \top^{n-2}) \wedge \cdots \wedge \Diamond(\top^{n-1}, \alpha)$$

The above can be written in a more condensed but less readable form.

$$\Box\alpha \longmapsto \bigvee_{i=1}^{n} \Box(\bot^{i-1}, \alpha, \bot^{n-i})$$
$$\Diamond\alpha \longmapsto \bigwedge_{i=1}^{n} \Diamond(\top^{i-1}, \alpha, \top^{n-i})$$

For illustration, we present the case for the dyadic modal language.

$$\Box\alpha \longmapsto \Box(\alpha, \bot) \vee \Box(\bot, \alpha)$$
$$\Diamond\alpha \longmapsto \Diamond(\alpha, \top) \wedge \Diamond(\top, \alpha)$$

The translation "splits" the unary $\Box$ into n n-ary $\Box$'s. Hence we call it *n-furcation* (forking into n branches). If n is two, we have a case of bifurcation. We denote the translation by f_n. As in the case of diagonalization, we call the unary $\Box$ and $\Diamond$ n-furcate modal connectives. If the value of n is clear in the context, we will not mention it.

Truth conditions for the n-furcate $\Box$ and $\Diamond$ in $(n+1)$-ary relational frames can easily be derived as follows:

- $\mathfrak{M}, x \models \Box\alpha$ iff $\exists i (1 \leq i \leq n) :\ \forall \vec{y},\ Rx\vec{y} \implies \mathfrak{M}, y_i \models \alpha$;
- $\mathfrak{M}, x \models \Diamond\alpha$ iff $\forall i (1 \leq i \leq n),\ \exists \vec{y} :\ Rx\vec{y}\ \&\ \mathfrak{M}, y_i \models \alpha$.

We call the resulting idiom the *n-furcate idiom.* Note that the above interpretation of the unary $\Box$ and $\Diamond$ effectively treats an $(n+1)$-ary relation R as consisting of n binary relations. This is made more succinct by first defining the set of i-th relata of x under R:

$$R_i(x) = \{y | \exists \vec{y} : Rx\vec{y}\ \&\ y = y_i\},$$

then rewriting the truth conditions for the unary $\Box$ and $\Diamond$:

- $\mathfrak{M}, x \models \Box\alpha \iff \exists i (1 \leq i \leq n) :\ R_i(x) \subseteq \|\alpha\|^{\mathfrak{M}}$;

- $\mathfrak{M}, x \models \Diamond\alpha \iff \forall i(1 \leq i \leq n),\ R_i(x) \cup \|\alpha\|^{\mathfrak{M}} \neq \emptyset$.

The case of bifurcation is given below for illustration:

- $\mathfrak{M}, x \models \Box\alpha$ iff $R_1(x) \subseteq \|\alpha\|^{\mathfrak{M}}$ or $R_2(x) \subseteq \|\alpha\|^{\mathfrak{M}}$;
- $\mathfrak{M}, x \models \Diamond\alpha$ iff $R_1(x) \cup \|\alpha\|^{\mathfrak{M}} \neq \emptyset$ or $R_2(x) \cup \|\alpha\|^{\mathfrak{M}} \neq \emptyset$,

where $R_1(x) = \{y|\exists z : Rxyz\}$ and $R_2(x) = \{z|\exists y : Rxyz\}$.

In fact we could have translated the monadic (uni)modal language $\mathfrak{L}_1$ to a multi-modal language consisting of unary $\Box_1$, $\Box_2$, $\ldots$, $\Box_n$ (call the language $\mathfrak{L}_{1,\ldots,1}$). The translation scheme would look like the following.

$$\Box\alpha \longmapsto \Box_1\alpha \vee \Box_2\alpha \vee \cdots \vee \Box_n\alpha$$

$$\Diamond\alpha \longmapsto \Diamond_1\alpha \wedge \Diamond_2\alpha \wedge \cdots \wedge \Diamond_n\alpha$$

Truth conditions for the unary $\Box$ and $\Diamond$ in a multi-relational model $\mathfrak{M} = \langle U, R_1, \ldots, R_n, V\rangle$ would be:

- $\mathfrak{M}, x \models \Box\alpha$ iff $\exists i(1 \leq i \leq n) :\ \forall y,\ R_ixy \implies \mathfrak{M}, y \models \alpha$;
- $\mathfrak{M}, x \models \Diamond\alpha$ iff $\forall i(1 \leq i \leq n),\ \exists y :\ R_ixy\ \&\ \mathfrak{M}, y \models \alpha$.

While the earlier approach, i.e. translating $\mathfrak{L}_1$ to $\mathfrak{L}_n$, is the official one adopted here, it is straightforward to see what results would obtain if we were to follow the second approach, i.e. translating $\mathfrak{L}_1$ to $\mathfrak{L}_{1,\ldots,1}$.

The set of monadic formulas valid in the n-furcate idiom is axiomatized by the following system denoted $\mathrm{K}^{\mathrm{f}}_n$.

Definition 8.2.5 (n-furcate logics)**.** $\mathrm{K}^{\mathrm{f}}_n$ has PL, [RM], [RN], and the following axiom.

$$[\wedge^{\mathrm{f}}_n] \quad \Box p_1 \wedge \cdots \wedge \Box p_n \to \bigvee_{i=1}^{n+1} \bigvee_{j=i+1}^{n+1} \Box(p_i \wedge p_j)$$ ⊣

For illustration, we list the characteristic axioms $\mathrm{K}^{\mathrm{f}}_1$ and $\mathrm{K}^{\mathrm{f}}_2$:

$$[\wedge^{\mathrm{f}}_1] \quad \Box p \wedge \Box q \to \Box(p \wedge q)$$

$$[\wedge^{\mathrm{f}}_2] \quad \Box p \wedge \Box q \wedge \Box r \to \Box(p \wedge q) \vee \Box(p \wedge r) \vee \Box(q \wedge r)$$

Note that the axiom $[\wedge_1^{\mathrm{f}}]$ is the familiar [C], and the system $\mathrm{K}_1^{\mathrm{f}}$ is just K. Like $\mathrm{K}_n^{\mathrm{d}}$, the system $\mathrm{K}_n^{\mathrm{f}}$ (where $n > 1$) can be described as a "weakly aggregative modal logic" since its aggregation principle $[\wedge_n^{\mathrm{f}}]$ is a weakening of the following principle of complete aggregation, which is a theorem of K.

$$\Box p_1 \wedge \cdots \wedge \Box p_n \rightarrow \Box(p_1 \wedge \cdots \wedge p_n)$$

Any system that provides $\mathrm{K}_n^{\mathrm{f}}$ is called an n-furcate system or logic. Thus $\mathrm{K}_n^{\mathrm{f}}$ is the smallest n-furcate system. It can be extended by adding the familiar monadic formulas [C], [P], [T], [B], [4], and [5], and the resulting systems are called, respectively, $\mathrm{K}_n^{\mathrm{f}}\mathrm{C}$, $\mathrm{K}_n^{\mathrm{f}}\mathrm{P}$, $\mathrm{K}_n^{\mathrm{f}}\mathrm{T}$, $\mathrm{K}_n^{\mathrm{f}}\mathrm{B}$, $\mathrm{K}_n^{\mathrm{f}}4$, and $\mathrm{K}_n^{\mathrm{f}}5$. Note that $\mathrm{K}_n^{\mathrm{f}}\mathrm{C}$ is just $\mathrm{K}_1^{\mathrm{f}}$ or K, since $[\wedge_n^{\mathrm{f}}]$ is derivable from $[\wedge_1^{\mathrm{f}}]$ or [C] (in the presence of PL and [RM]. In the following, we list correspondence and determination results for these formulas and systems (in the n-furcate idiom).

Theorem 8.2.6. *The following monadic modal formulas correspond to the indicated first-order conditions on $(n+1)$-ary relations R.*

[C]	:	$\forall x, \forall i, \forall j, R_i(x) \subseteq R_j(x)$ *or* $R_j(x) \subseteq R_i(x)$	(F-binarity)
[P]	:	$\forall x, \forall i, R_i(x) \neq \emptyset$	(Seriality)
[T]	:	$\forall x, x \in R_i(x) \cap \cdots \cap R_n(x)$	(F-reflexivity)
[B]	:	$\forall x, \exists i : \forall y \in R_i(x), x \in R_1(y) \cap \cdots \cap R_n(y)$	(F-symmetry)
[4]	:	$\forall x, \forall i, \exists j : \forall y \in R_j(x), \exists k : R_k(y) \subseteq R_i(x)$	(F-transitivity)
[5]	:	$\forall x, \forall \vec{y}, \vec{z} \in R_1(x) \times \cdots \times R_n(x), \exists y \in \vec{y} : \forall i, \exists z \in \vec{z} : z \in R_i(y)$	(F-euclideanness)

Theorem 8.2.7. *The following n-furcate systems are determined by the indicated classes of $(n+1)$-ary relational frames.*

$\mathrm{K}_n^{\mathrm{f}}$: *All frames*
$\mathrm{K}_n^{\mathrm{f}}\mathrm{C}$: *F-binary frames*
$\mathrm{K}_n^{\mathrm{f}}\mathrm{P}$: *Serial frames*
$\mathrm{K}_n^{\mathrm{f}}\mathrm{T}$: *F-reflexive frames*
$\mathrm{K}_n^{\mathrm{f}}\mathrm{B}$: *F-symmetric frames*
$\mathrm{K}_n^{\mathrm{f}}4$: *F-transitive frames*
$\mathrm{K}_n^{\mathrm{f}}5$: *F-euclidean frames*

Theorem 8.2.8. *The monadic* n*-furcate system* $\mathrm{K}_n^{\mathrm{f}}$ *is exactly translatable into the* n*-adic normal system* K_n *under the translation* f_n*, i.e. for every* $\mathcal{L}_1$*-formula* α*,*

$$\vdash_{\mathrm{K}_n^{\mathrm{f}}} \alpha \iff \vdash_{\mathrm{K}_n} \alpha^{f_n}$$

8.3 Equivalence between non-normal systems and normal systems

In this section we consider translation between the n-adic regular system $\mathrm{R}_n\mathrm{P}_n$ and the n-adic normal system K_n:

$$\begin{array}{llllll} \mathrm{R}_n\mathrm{P}_n & : \mathrm{PL}, & [\mathrm{E}_n], & [\mathrm{M}_n], & [\mathrm{C}_n], & [\mathrm{P}_n] \\ \mathrm{K}_n & : \mathrm{PL}, & [\mathrm{RM}_n], & [\mathrm{RN}_n], & [\mathrm{C}_n] & \end{array}$$

Regular systems are defined in Section 5.2 and normal systems are defined in Section 2.4.

We show that the following translation scheme, which we call t_1 in this section, is a sound translation of $\mathrm{R}_n\mathrm{P}_n$ to K_n:

$$\Box_1(\alpha_1, \ldots, \alpha_n) \longmapsto \Box_2(\alpha_1, \ldots, \alpha_n) \wedge \Diamond_2 \top^n$$

where $\Box_1$ is the modal operator of $\mathrm{R}_n\mathrm{P}_n$ and $\Box_2$ is the modal operator of K_n. On the other hand, K_n can be soundly translated to $\mathrm{R}_n\mathrm{P}_n$ by the following translation scheme called t_2 here.

$$\Box_2(\alpha_1, \ldots, \alpha_n) \longmapsto \Box_1(\alpha_1, \ldots, \alpha_n) \vee \Diamond_1 \bot^n$$

Note that t_1 and t_2 map formulas of $\mathcal{L}_n$ to formulas of $\mathcal{L}_n$. Moreover both translations preserve propositional variables and truth-functional connectives. As we shall see, $\mathrm{R}_n\mathrm{P}_n$ and K_n are equivalent under the above translations. We proceed semantically, making use of the following determination results:

- $\mathrm{R}_n\mathrm{P}_n$ is sound and complete with respect to the class of serial non-normal $(n+1)$-ary relational frames. Non-normal semantics is discussed in Section 1.5. We shall not prove here the determination of $\mathrm{R}_n\mathrm{P}_n$ by the class of serial non-normal $(n+1)$-ary relational frames. A determination proof for the monadic RP can be found in Leung (2003). Generalization of the proof to the n-adic case is straightforward.

- K_n is sound and complete with respect to the class of $(n+1)$-ary relational frames respectively. (Refer to Section 2.5 for the determination of K_n.)

Theorem 8.3.1. *Every $(n+1)$-ary relational model $\mathfrak{M} = \langle U, R, V\rangle$ is simulated by a serial non-normal $(n+1)$-ary relational model $\mathfrak{M}' = \langle U', Q', R', V'\rangle$ with respect to the translation t_1. In other words, there is a one-to-one correspondence between the points of U and that of U' such that the following holds for every $\mathcal{L}_n$-formula α:*

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_1} \iff \mathfrak{M}', x' \models \alpha$$

where x' is the point in U' corresponding to x.

Proof. Given a relational model $\mathfrak{M} = \langle U, R, V\rangle$, we define its simulation model $\mathfrak{M}' = \langle U', Q', R', V'\rangle$ by letting $U' = U$, $R' = R$, $V' = V$ and

$$Q' = U - \{x \in U \mid \exists \vec{y} \in U^n : Rx\vec{y}\}.$$

Note that non-normal points of $\mathfrak{M}'$ are precisely those points of U that are not related to any tuple under R, and the normal points of $\mathfrak{M}'$ are precisely those points of U that are related to some tuple under R. Clearly $\mathfrak{M}'$ is a non-normal relational model since $R' \subseteq (U' - Q') \times U'^n$. Moreover R' is serial since every normal point is related to some tuple under R'.

We show that if each point x of U is mapped to itself, then for every $\mathcal{L}_n$-formula α,

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_1} \iff \mathfrak{M}', x \models \alpha.$$

The proof is by induction on α. We show the modal case only, i.e.

$$\forall x \in U,\ \mathfrak{M}, x \models (\Box_1(\alpha_1, \dots, \alpha_n))^{t_1} \iff \mathfrak{M}', x \models \Box_1(\alpha_1, \dots, \alpha_n),$$

which is equivalent to

$$\forall x \in U,\ \mathfrak{M}, x \models \Box_2({\alpha_1}^{t_1}, \dots, {\alpha_n}^{t_1}) \wedge \Diamond_2 \top^n \iff \mathfrak{M}', x \models \Box_1(\alpha_1, \dots, \alpha_n).$$

In the following, let x be a point of U.n

For $(\Longrightarrow)$. Assume $\mathfrak{M}, x \models \Box_2({\alpha_1}^{t_1}, \dots, {\alpha_n}^{t_1}) \wedge \Diamond_2 \top^n$. Then $x \in U' - Q'$ since x is R-related to some tuple. Consider arbitrary $\vec{y}$ such that $R'x\vec{y}$. Then $Rx\vec{y}$. Hence for some i, $\mathfrak{M}, y_i \models \alpha^{t_1}$ and so by I.H. $\mathfrak{M}', y_i \models \alpha_i$. But $\vec{y}$ is arbitrary. Thus $\mathfrak{M}', x \models \Box_1(\alpha_1, \dots, \alpha_n)$.

For $(\Longleftarrow)$. Assume $\mathfrak{M}', x \models \Box_1(\alpha_1, \dots, \alpha_n)$. Then $x \in U' - Q'$. Then x is R-related to some tuple, whence $\mathfrak{M}, x \models \Diamond_2 \top^n$. Moreover $\mathfrak{M}, x \models \Box_2({\alpha_1}^{t_1}, \dots, {\alpha_n}^{t_1})$ because of the following. Suppose $Rx\vec{y}$. Then $R'x\vec{y}$. Hence for some i, $\mathfrak{M}', y_i \models \alpha_i$ and so by I.H. $\mathfrak{M}, y_i \models \alpha_i^{t_1}$. ⊣

Theorem 8.3.2. *Every serial non-normal $(n+1)$-ary relational model $\mathfrak{M} = \langle U, Q, R, V\rangle$ is simulated by an $(n+1)$-ary relational model $\mathfrak{M}' = \langle U', R', V'\rangle$ with respect to the translation t_2. In other words, there is a one-to-one correspondence between the points of U and that of U' such that the following holds for every formula $\mathcal{L}_n$-formula α:*

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_2} \iff \mathfrak{M}', x' \models \alpha$$

where x' is the point in U' corresponding to x.

Proof. Given a serial non-normal relational model $\mathfrak{M} = \langle U, Q, R, V\rangle$, we define its simulation model $\mathfrak{M}' = \langle U', R', V'\rangle$ by letting $U' = U$, $R' = R$ and $V' = V$. Clearly $\mathfrak{M}'$ is a relational model.

We show that if each point x of U is mapped to itself, then for every $\mathcal{L}_n$-formula α,

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_2} \iff \mathfrak{M}', x \models \alpha.$$

The proof is by induction on α. We show the modal case only, i.e.

$$\forall x \in U,\ \mathfrak{M}, x \models (\Box_2(\alpha_1, \ldots, \alpha_n))^{t_2} \iff \mathfrak{M}', x \models \Box_2(\alpha_1, \ldots, \alpha_n),$$

which is equivalent to

$$\forall x \in U,\ \mathfrak{M}, x \models \Box_1({\alpha_1}^{t_2}, \ldots, {\alpha_n}^{t_2}) \vee \Diamond_1 \bot^n \iff \mathfrak{M}', x \models \Box_2(\alpha_1, \ldots, \alpha_n).$$

In the following, let x be a point of U.

For $(\Longrightarrow)$, assume that $\mathfrak{M}, x \models \Box_1({\alpha_1}^{t_2}, \ldots, {\alpha_n}^{t_2}) \vee \Diamond_1 \bot^n$. Then either (i) $\mathfrak{M}, x \models \Box_1({\alpha_1}^{t_2}, \ldots, {\alpha_n}^{t_2})$ or (ii) $\mathfrak{M}, x \models \Diamond_1 \bot^n$.

- Suppose (i) is the case. Then $x \in U - Q$. Consider arbitrary $\vec{y}$ such that $R'x\vec{y}$. Then $Rx\vec{y}$. Hence for some i, $\mathfrak{M}, y_i \models {\alpha_i}^{t_2}$ and so by I.H. $\mathfrak{M}', y_i \models \alpha_i$. But $\vec{y}$ is arbitrary. Thus $\mathfrak{M}', x \models \Box_2(\alpha_1, \ldots, \alpha_n)$.

- Suppose (ii) is the case. Then $x \in Q$. Then x is not R'-related to any tuple. So $\mathfrak{M}, x \models \Box_2(\alpha_1, \ldots, \alpha_n)$ trivially.

In either case, we have $\mathfrak{M}, x \models \Box_2(\alpha_1, \ldots, \alpha_n)$.

For $(\Longleftarrow)$, assume $\mathfrak{M}', x \models \Box_2(\alpha_1, \ldots, \alpha_n)$. Either (i) x is R'-related to some tuple or (ii) x is not R'-related to any tuple.

- If (i), then $x \in U - Q$. Consider arbitrary $\vec{y}$ such that $Rx\vec{y}$. Then $R'x\vec{y}$. Then for some i, $\mathfrak{M}', y_i \models \alpha_i$ and so by I.H. $\mathfrak{M}, y_i \models {\alpha_i}^{t_2}$. Since $\vec{y}$ is arbitrary, we have $\mathfrak{M}, x \models \Box_1({\alpha_1}^{t_2}, \ldots, {\alpha_n}^{t_2})$.

- If (ii), then $x \in Q$ since R is serial. Then $\mathfrak{M}, x \models \Diamond_1 \bot^n$ trivially.

In either case, we have $\mathfrak{M}, x \models \Box_1({\alpha_1}^{t_2}, \ldots, {\alpha_n}^{t_2}) \vee \Diamond_1 \bot^n$. ⊣

Theorem 8.3.3. *Both of the translations t_1 and t_2 are sound, i.e. for every $\mathfrak{L}_n$-formula α,*

(1) *If $\vdash_{\mathrm{R}_n\mathrm{P}_n} \alpha$, then $\vdash_{\mathrm{K}_n} \alpha^{t_1}$.*

(2) *If $\vdash_{\mathrm{K}_n} \alpha$, then $\vdash_{\mathrm{R}_n\mathrm{P}_n} \alpha^{t_2}$.*

Proof. For (1). Given the determination results for $\mathrm{R}_n\mathrm{P}_n$ and K_n, it suffices to note that if α^{t_1} fails in a relational model, then α fails in a serial non-normal relational model according to Theorem 8.3.1.

For (2). Given the determination results for $\mathrm{R}_n\mathrm{P}_n$ and K_n, it suffices to note that if α^{t_2} fails in a serial non-normal relational model, then α fails in a relational model according to Theorem 8.3.2. ⊣

Theorem 8.3.4. *For any $\mathfrak{L}_n$-formula α, $(\alpha^{t_1})^{t_2} \leftrightarrow \alpha$ is a theorem of $\mathrm{R}_n\mathrm{P}_n$.*

Proof. Given that $\mathrm{R}_n\mathrm{P}_n$ is determined by the class of serial non-normal $(n+1)$-ary relational frames, it needs to be shown that for any $\mathfrak{L}_n$-formula α, $\alpha^{t_1 t_2} \leftrightarrow \alpha$ is valid in the same class of frames. In other words, we demonstrate that for any serial non-normal $(n+1)$-ary relational model $\mathfrak{M} = \langle U, Q, R, V\rangle$, the following holds for any $\mathfrak{L}_n$-formula α:

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_1 t_2} \iff \mathfrak{M}, x \models \alpha.$$

The proof is by induction on α. Only the modal case of the induction step is of interest:

$$\forall x \in U,\ \mathfrak{M}, x \models (\Box(\beta_1, \ldots, \beta_n))^{t_1 t_2} \iff \mathfrak{M}, x \models \Box(\beta_1, \ldots, \beta_n).$$

Note that

$$\begin{aligned} (\Box(\beta_1, \ldots, \beta_n))^{t_1 t_2} &= (\Box({\beta_1}^{t_1}, \ldots, {\beta_n}^{t_1}) \wedge \Diamond \top^n)^{t_2} \\ &= (\Box({\beta_1}^{t_1}, \ldots, {\beta_n}^{t_1}))^{t_2} \wedge (\neg \Box \bot^n)^{t_2} \\ &= (\Box({\beta_1}^{t_1 t_2}, \ldots, {\beta_n}^{t_1 t_2}) \vee \Diamond \bot^n) \wedge (\Diamond \top^n \wedge \Box \top^n). \end{aligned}$$

In the following, let x be a point of U.

For ($\Longrightarrow$). Assume $\mathfrak{M}, x \models (\Box(\beta_1, \ldots, \beta_n))^{t_1 t_2}$. Then $x \in U - Q$ since $\mathfrak{M}, x \models \Box\top^n$. Consider arbitrary $\vec{y}$ such that $Rx\vec{y}$. But $\mathfrak{M}, x \models \Box({\beta_1}^{t_1 t_2}, \ldots, {\beta_n}^{t_1 t_2})$ (since $x \in U - Q$ and so $\mathfrak{M}, x \not\models \Diamond\bot^n$.) Thus for some i, $\mathfrak{M}, y_i \models {\beta_i}^{t_1 t_2}$, whence by I.H. $\mathfrak{M}, y_i \models \beta_i$. Given that $\vec{y}$ is arbitrary, we thus have $\mathfrak{M}, x \models \Box(\beta_1, \ldots, \beta_n)$.

For ($\Longleftarrow$). Assume $\mathfrak{M}, x \models \Box(\beta_1, \ldots, \beta_n)$. Then $x \in U - Q$. Trivially $\mathfrak{M}, x \models \Box\top^n$. Since R is serial, $\mathfrak{M}, x \models \Diamond\top^n$. It remains to show that $\mathfrak{M}, x \models \Box({\beta_1}^{t_1 t_2}, \ldots, {\beta_n}^{t_1 t_2})$. Consider arbitrary $\vec{y}$ such that $Rx\vec{y}$. By assumption, there exists an i such that $\mathfrak{M}, y_i \models \beta_i$. Then by I.H. $\mathfrak{M}, y_i \models {\beta_i}^{t_1 t_2}$. Given that $\vec{y}$ is arbitrary, we thus have $\mathfrak{M}, x \models \Box({\beta_1}^{t_1 t_2}, \ldots, {\beta_n}^{t_1 t_2})$, as desired. ⊣

Theorem 8.3.5. *For any $\mathfrak{L}_n$-formula α, $(\alpha^{t_2})^{t_1} \leftrightarrow \alpha$ is a theorem of* K_n.

Proof. Given that K_n is determined by the class of $(n+1)$-ary relational frames, it needs to be shown that for any $\mathfrak{L}_n$-formula α, $\alpha^{t_2 t_1} \leftrightarrow \alpha$ is valid in the same class of frames. In other words, we demonstrate that for any $(n+1)$-ary relational model $\mathfrak{M} = \langle U, R, V\rangle$, the following holds for any $\mathfrak{L}_n$-formula α:

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_2 t_1} \iff \mathfrak{M}, x \models \alpha.$$

The proof is by induction on α. Only the modal case of the induction step is of interest:

$$\forall x \in U,\ \mathfrak{M}, x \models (\Box(\beta_1, \ldots, \beta_n))^{t_2 t_1} \iff \mathfrak{M}, x \models \Box(\beta_1, \ldots, \beta_n).$$

Note that

$$\begin{aligned}(\Box(\beta_1, \ldots, \beta_n))^{t_2 t_1} &= (\Box({\beta_1}^{t_2}, \ldots, {\beta_n}^{t_2}) \vee \Diamond\bot^n)^{t_1} \\ &= (\Box({\beta_1}^{t_2}, \ldots, {\beta_n}^{t_2}))^{t_1} \vee (\neg\Box\top^n)^{t_1} \\ &= (\Box({\beta_1}^{t_2 t_1}, \ldots, {\beta_n}^{t_2 t_1}) \wedge \Diamond\top^n) \vee (\Diamond\bot^n \vee \Box\bot^n).\end{aligned}$$

In the following, let x be a point of U.

For ($\Longrightarrow$). Assume $\mathfrak{M}, x \models (\Box(\beta_1, \ldots, \beta_n))^{t_2 t_1}$. Since $\mathfrak{M}, x \not\models \Diamond\bot^n$, we have either

(1) $\mathfrak{M}, x \models \Box({\beta_1}^{t_2 t_1}, \ldots, {\beta_n}^{t_2 t_1}) \wedge \Diamond\top^n$, or

(2) $\mathfrak{M}, x \models \Box\bot^n$.

If (1), then $\mathfrak{M}, x \models \Box(\beta_1, \ldots, \beta_n)$ since, for any $\vec{y}$ such that $Rx\vec{y}$, we have, for some i, $\mathfrak{M}, y_i \models {\beta_i}^{t_2 t_1}$ and so $\mathfrak{M}, y_i \models \beta_i$ according to the I.H. If (2), then x is not related to any tuple under R, then trivially $\mathfrak{M}, x \models \Box(\beta_1, \ldots, \beta_n)$. Thus, in either case, $\mathfrak{M}, x \models \Box(\beta_1, \ldots, \beta_n)$.

For ($\Longleftarrow$). Assume $\mathfrak{M}, x \models \Box(\alpha_1, \ldots, \alpha_n)$. Either (i) x is R-related to some tuple or (ii) x is not R-related to any tuple. If (i), then $\mathfrak{M}, x \models \Diamond\top^n$ and $\mathfrak{M}, x \models \Box({\beta_1}^{t_2 t_1}, \ldots, {\beta_n}^{t_2 t_1})$. (Note that the latter holds generally by virtue of the assumption and the I.H.) If (ii), then trivially $\mathfrak{M}, x \models \Box\bot^n$. Thus, in either case, $\mathfrak{M}, x \models (\Box(\beta_1, \ldots, \beta_n))^{t_2 t_1}$. ⊣

Theorem 8.3.6. $\mathrm{R}_n\mathrm{P}_n$ *and* K_n *are translationally equivalent under* t_1 *and* t_2.

Proof. The theorem follows directly from Definition 8.1.3, Theorems 8.3.3, 8.3.4 and 8.3.5. ⊣

8.4 Equivalence between polyadic systems and monadic systems

The monadic system KP or equivalently KD (also called Standard Deontic Logic) is defined in Section 3.1, and the n-adic system $\mathrm{DR!}_n$ (also called the smallest system of strong deontic residuation) is defined in Section 10.2.3. They are the following systems:

KP	: PL,	[RM],	[RN],	[C]	[P]	
$\mathrm{DR!}_n$	: PL,	$[\mathrm{RM}_n]$,	$[\mathrm{RN}_n]$,	$[\mathrm{C}_n]$,	$[\mathrm{P}_n]$	$[\mathrm{Re!}_n]$

We show in this section that the following translation t_1 of $\mathfrak{L}_1$-formulas to $\mathfrak{L}_n$-formulas is a sound translation of KP to $\mathrm{DR!}_n$

$$\Box\alpha \longmapsto \Box(\alpha, \bot^{n-1}).$$

Furthermore, the following translation t_n of $\mathfrak{L}_n$-formulas to $\mathfrak{L}_1$-formulas is a sound translation of $\mathrm{DR!}_n$ to KP.

$$\Box(\alpha_1, \alpha_2, \ldots, \alpha_n) \longmapsto \Box(\alpha_1 \vee \Box(\alpha_2 \vee \cdots \vee \Box(\alpha_{n-1} \vee \Box\alpha_n) \cdots))$$

For example, the translations t_2 and t_3 are as follows.

$$\Box(\alpha_1, \alpha_2) \longmapsto \Box(\alpha_1 \vee \Box\alpha_2)$$
$$\Box(\alpha_1, \alpha_2, \alpha_3) \longmapsto \Box(\alpha_1 \vee \Box(\alpha_2 \vee \Box\alpha_3))$$

Finally, KP and $\mathrm{DR!}_n$ are translationally equivalent under t_1 and t_n.

As before we proceed semantically, making use of the following determination results.

- KP is determined by the class of serial binary relational frames. (Refer to Section 3.1.)
- $\mathrm{DR!}_n$ is determined by the class of serial and strongly semital $(n+1)$-ary relational frames. (Refer to Section 10.4.)

Theorem 8.4.1. *Every serial and strongly semital $(n+1)$-ary relational model $\mathfrak{M} = \langle U, R, V\rangle$ is simulated by a serial binary relational model $\mathfrak{M}' = \langle U', R', V'\rangle$ with respect to the translation t_1. In other words, there is a one-to-one correspondence between the points of U and that of U' such that the following holds for every $\mathcal{L}_1$-formula α:*

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_1} \iff \mathfrak{M}', x' \models \alpha$$

where x' is the point in U' corresponding to x.

Proof. Given a serial and strongly semital $(n+1)$-ary relational model $\mathfrak{M} = \langle U, R, V\rangle$, we define its simulation model $\mathfrak{M}' = \langle U', R', V'\rangle$ by letting $U' = U$, $V' = V$, and R' be as follows: for any $x_0, x_1 \in U'$,

$$R'x_0x_1 \iff \exists x_2, \ldots, x_n \in U : Rx_0x_1x_2 \cdots x_n.$$

Evidently $\mathfrak{M}'$ is a serial binary relational model.

We show that if each point x of U is mapped to itself, then for every $\mathcal{L}_1$-formula α,

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_1} \iff \mathfrak{M}', x \models \alpha.$$

The proof is by induction on α. We show the modal case only, i.e.

$$\forall x \in U,\ \mathfrak{M}, x \models (\Box\alpha)^{t_1} \iff \mathfrak{M}', x \models \Box\alpha,$$

which is equivalent to

$$\forall x \in U,\ \mathfrak{M}, x \models \Box(\alpha^{t_1}, \bot^{n-1}) \iff \mathfrak{M}', x \models \Box\alpha.$$

In the following, let x be a point of U.

For ($\Longrightarrow$). Assume $\mathfrak{M}, x \models \Box(\alpha^{t_1}, \bot^{n-1})$. Consider arbitrary y_1 such that $R'xy_1$. Then $Rxy_1y_2 \cdots y_n$ for some $y_2, \ldots, y_n$. Then $\mathfrak{M}, y_1 \models \alpha^{t_1}$ and so by I.H. $\mathfrak{M}', y_1 \models \alpha$. Since y_1 is arbitrary, we have $\mathfrak{M}', x \models \Box\alpha$.

For ($\Longleftarrow$). Assume $\mathfrak{M}', x \models \Box\alpha$. Consider arbitrary $\vec{y}$ such that $Rx\vec{y}$. Then $R'xy_1$. Then $\mathfrak{M}', y_1 \models \alpha$ and so by I.H. $\mathfrak{M}, y_1 \models \alpha^{t_1}$. But $\vec{y}$ is arbitrary. Thus $\mathfrak{M}, x \models \Box(\alpha^{t_1}, \bot^{n-1})$. ⊣

Theorem 8.4.2. *Every serial binary relational model $\mathfrak{M} = \langle U, R, V\rangle$ is simulated by a serial and strongly semital $(n+1)$-ary relational model $\mathfrak{M}' = \langle U', R', V'\rangle$ with respect to the translation t_n. In other words, there is a one-to-one correspondence between the points of U and that of U' such that the following holds for every $\mathfrak{L}_n$-formula α:*

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_n} \iff \mathfrak{M}', x' \models \alpha$$

where x' is the point in U' corresponding to x.

Proof. Given a serial binary relational model $\mathfrak{M} = \langle U, R, V\rangle$, we define its simulation model $\mathfrak{M}' = \langle U', R', V'\rangle$ by letting $U' = U$, $V' = V$, and R' be as follows: for every $x_0, x_1, \ldots, x_n$ in U',

$$R'x_0x_1 \cdots x_n \iff x_0Rx_1 \cdots x_{n-1}Rx_n$$

where $x_0Rx_1 \cdots x_{n-1}Rx_n$ stands for "Rx_0x_1, …, and $Rx_{n-1}x_n$." Note that given R is serial, R' is both serial and strongly semital.

We show that if each point x of U is mapped to itself, then for every $\mathfrak{L}_n$-formula α,

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_n} \iff \mathfrak{M}', x \models \alpha.$$

The proof is by induction on α. We show the modal case only, i.e.

$$\forall x \in U,\ \mathfrak{M}, x \models \Box(\alpha_1, \ldots, \alpha_n)^{t_n} \iff \mathfrak{M}', x \models \Box(\alpha_1, \ldots, \alpha_n),$$

which is equivalent to

$$\forall x \in U,\ \mathfrak{M}, x \models \Box({\alpha_1}^{t_n} \vee \Box({\alpha_2}^{t_n} \vee \cdots \vee \Box({\alpha_{n-1}}^{t_n} \vee \Box{\alpha_n}^{t_n}) \cdots)) \iff \mathfrak{M}', x \models \Box(\alpha_1, \ldots, \alpha_n).$$

For any $x \in U$, the following are equivalent.

$\mathfrak{M}, x \models \Box({\alpha_1}^{t_n} \vee \Box({\alpha_2}^{t_n} \vee \cdots \vee \Box({\alpha_{n-1}}^{t_n} \vee \Box{\alpha_n}^{t_n}) \cdots))$.
$\forall y_1, \ldots, y_n,\ xRy_1 \ldots y_nRy_n \implies \exists i : \mathfrak{M}, y_i \models {\alpha_i}^{t_n}$.
$\forall y_1, \ldots, y_n,\ R'xy_1 \cdots y_n \implies \exists i : \mathfrak{M}', y_i \models \alpha_i$.
$\mathfrak{M}', x \models \Box(\alpha_1, \ldots, \alpha_n)$.

We thus have shown the modal case of the inductive step. ⊣

Theorem 8.4.3. *Both of the translations t_1 and t_2 are sound. In other words:*

(1) *For every $\mathfrak{L}_1$-formula α, if $\vdash_{\mathrm{KP}} \alpha$, then $\vdash_{\mathrm{DR!}_n} \alpha^{t_1}$.*

(2) *For every $\mathfrak{L}_n$-formula α, if $\vdash_{\mathrm{DR!}_n} \alpha$, then $\vdash_{\mathrm{KP}} \alpha^{t_n}$.*

Proof. For (1). Given the determination results for KP and DR!$_n$, it suffices to note that if α^{t_1} fails in a serial and strongly semital $(n+1)$-ary relational model, then α fails in a serial binary relational model according to Theorem 8.4.1.

For (2). Given the determination results for KP and DR!$_n$, it suffices to note that if α^{t_n} fails in a serial binary relational model, then α fails in a serial and strongly semital $(n+1)$-ary relational model according to Theorem 8.4.2. ⊣

Theorem 8.4.4. *For any $\mathfrak{L}_1$-formula α, $\alpha^{t_1 t_n} \leftrightarrow \alpha$ is a theorem of* KP.

Proof. Given that KP is determined by the class of serial binary relational frames, it needs to be shown that for any $\mathfrak{L}_1$-formula α, $\alpha^{t_1 t_n} \leftrightarrow \alpha$ is valid in the same class of frames. In other words, we demonstrate that for any serial binary relational model $\mathfrak{M} = \langle U, R, V\rangle$, the following holds for any $\mathfrak{L}_1$-formula α:

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_1 t_n} \iff \mathfrak{M}, x \models \alpha.$$

The proof is by induction on α. Only the modal case of the induction step is of interest:

$$\forall x \in U,\ \mathfrak{M}, x \models (\Box\beta)^{t_1 t_n} \iff \mathfrak{M}, x \models \Box\beta.$$

Note that

$$\begin{aligned}(\Box\beta)^{t_1 t_n} &= (\Box(\beta^{t_1}, \bot^{n-1}))^{t_n}\\ &= \Box(\beta^{t_1 t_n} \vee \Box(\bot \vee \cdots \vee \Box(\bot \vee \Box\bot)\cdots)).\end{aligned}$$

In the following, let x be a point of U.

For ($\Longrightarrow$). Assume $\mathfrak{M}, x \models (\Box\beta)^{t_1 t_n}$. Consider arbitrary $y \in U$ such that Rxy. Since R is serial, we have, for any point z of U, $\mathfrak{M}, z \not\models \Box\bot$, $\mathfrak{M}, z \not\models \Box(\bot \vee \Box\bot)$, and so on. Thus by assumption $\mathfrak{M}, y \models \beta^{t_1 t_n}$, whence by I.H. $\mathfrak{M}, y \models \beta$. But y is arbitrary. Thus $\mathfrak{M}, x \models \Box\beta$.

For ($\Longleftarrow$). Assume $\mathfrak{M}, x \models \Box\beta$. Consider arbitrary $y \in U$ such that Rxy. Then by assumption $\mathfrak{M}, y \models \beta$. Then by I.H. $\mathfrak{M}, y \models \beta^{t_1 t_n}$, whence $\mathfrak{M}, y \models \beta^{t_1 t_n} \vee \Box(\bot \vee \cdots \vee \Box(\bot \vee \Box\bot)\cdots)$. Since y is arbitrary, we have $\mathfrak{M}, x \models (\Box\beta)^{t_1 t_n}$. ⊣

Theorem 8.4.5. *For any $\mathfrak{L}_n$-formula α, $\alpha^{t_n t_1} \leftrightarrow \alpha$ is a theorem of* DR!$_n$.

Proof. Given that DR!$_n$ is determined by the class of serial and strongly semital $(n+1)$-ary relational frames, it needs to be shown that for any $\mathfrak{L}_n$-formula α, $\alpha^{t_n t_1} \leftrightarrow \alpha$ is valid in the same class of frames. In other words, we demonstrate that for any serial and strongly semital $(n+1)$-ary relational model $\mathfrak{M} = \langle U, R, V \rangle$, the following holds for any $\mathfrak{L}_n$-formula α:

$$\forall x \in U,\ \mathfrak{M}, x \models \alpha^{t_n t_1} \iff \mathfrak{M}, x \models \alpha.$$

The proof is by induction on α. Only the modal case of the induction step is of interest:

$$\forall x \in U,\ \mathfrak{M}, x \models (\Box(\beta_1, \dots, \beta_n))^{t_n t_1} \iff \mathfrak{M}, x \models \Box(\beta_1, \dots, \beta_n).$$

Note that $(\Box(\beta_1, \dots, \beta_n))^{t_n t_1}$ is the following:

$$(\Box(\beta_1{}^{t_n} \vee \Box(\beta_2{}^{t_n} \vee \cdots \vee \Box(\beta_{n-1}{}^{t_n} \vee \Box\beta_n{}^{t_n}) \cdots)))^{t_1};$$
$$\Box(\beta_1{}^{t_n t_1} \vee \Box(\beta_2{}^{t_n t_1} \vee \cdots \vee \Box(\beta_{n-1}{}^{t_n t_1} \vee \Box(\beta_n{}^{t_n t_1}, \bot^{n-1}), \bot^{n-1}) \cdots, \bot^{n-1}), \bot^{n-1}).$$

In the following, let x be a point of U.

For $(\Longrightarrow)$. We proceed by contraposition. Assume $\mathfrak{M}, x \not\models \Box(\beta_1, \dots, \beta_n)$, i.e. $\mathfrak{M}, x \models \Diamond(\neg\beta_1, \dots, \neg\beta_n)$. Thus, there exist $y_1, \dots, y_n$ such that $Rxy_1 \cdots y_n$ and for all i, $\mathfrak{M}, y_i \models \neg\beta_i$ or equivalently $\mathfrak{M}, y_i \models \neg\beta_i{}^{t_n t_1}$ (by I.H.). Starting from $i = n$, we note that

$$\mathfrak{M}, y_{n-1} \models \Diamond(\neg\beta_n{}^{t_n t_1}, \top^{n-1})$$

since R is serial and strongly semital. (Details are as follows. By seriality, we have $Ry_n z_1 \cdots z_n$ for some $z_1, \dots, z_n$. But $Rxy_1 \cdots y_n$. So $Ry_{n-1} y_n z_1 \cdots z_{n-1}$ by the condition of strong semita. Finally note that $\mathfrak{M}, y_n \models \neg\beta_n{}^{t_n t_1}$.) Repeating the same argument for $i = n-1$ and so on, we establish that the following is true at x in $\mathfrak{M}$:

$$\Diamond(\neg\beta_1{}^{t_n t_1} \wedge \Diamond(\neg\beta_2{}^{t_n t_1} \wedge \cdots \wedge \Diamond(\neg\beta_{n-1}{}^{t_n t_1} \wedge \Diamond(\neg\beta_n{}^{t_n t_1}, \top^{n-1}), \top^{n-1}) \cdots, \top^{n-1}), \top^{n-1}),$$

which is equivalent to $\neg(\Box(\beta_1, \dots, \beta_n))^{t_n t_1}$. Thus, $\mathfrak{M}, x \not\models (\Box(\beta_1, \dots, \beta_n))^{t_n t_1}$.

For $(\Longleftarrow)$. Again we proceed by contraposition. Assume $\mathfrak{M}, x \not\models (\Box(\beta_1, \dots, \beta_n))^{t_n t_1}$. In other words, the following is true at x in $\mathfrak{M}$:

$$\Diamond(\neg\beta_1{}^{t_n t_1} \wedge \Diamond(\neg\beta_2{}^{t_n t_1} \wedge \cdots \wedge \Diamond(\neg\beta_{n-1}{}^{t_n t_1} \wedge \Diamond(\neg\beta_n{}^{t_n t_1}, \top^{n-1}), \top^{n-1}) \cdots, \top^{n-1}), \top^{n-1}).$$

Then we have the following (where $\vec{y}_1$ stands for the n-termed sequence $y_{1.1}, \dots, y_{1.n}$, and similarly for $\vec{y}_2$, etc.):

$$\begin{array}{ll}
\exists\vec{y}_1 & : Rx\vec{y}_1 \ \& \ \mathfrak{M}, y_{1.1} \models \neg{\beta_1}^{t_n t_1}; \\
\exists\vec{y}_2 & : Ry_{1.1}\vec{y}_2 \ \& \ \mathfrak{M}, y_{2.1} \models \neg{\beta_2}^{t_n t_1}; \\
 & \qquad\qquad \vdots \\
\exists\vec{y}_{n-1} & : Ry_{(n-2).1}\vec{y}_{n-1} \ \& \ \mathfrak{M}, y_{(n-1).1} \models \neg{\beta_{n-1}}^{t_n t_1}; \\
\exists\vec{y}_n & : Ry_{(n-1).1}\vec{y}_n \ \& \ \mathfrak{M}, y_{n.1} \models \neg{\beta_n}^{t_n t_1}.
\end{array}$$

Since R is semital, $Rxy_{1.1}y_{2.1}\ldots y_{(n-1).1}y_{n.1}$. Moreover by I.H. $\mathfrak{M}, y_{i.1} \models \neg\beta_i$ for all i from 1 to n. Thus $\mathfrak{M}, x \models \Diamond(\neg\beta_1, \ldots, \neg\beta_n)$, i.e. $\mathfrak{M}, x \not\models \Box(\beta_1, \ldots, \beta_n)$. ⊣

Theorem 8.4.6. KP *and* $\mathrm{DR}!_n$ *are translationally equivalent under* t_1 *and* t_n*.*

Proof. The theorem follows directly from Definition 8.1.3, Theorems 8.4.3, 8.4.4 and 8.4.5. ⊣

Chapter 9

Formal Representation of Deontic Reasoning

The root "deon" comes from the Greek term $\delta\acute{\epsilon}o\nu$, which means "that which is binding, duty". Deontic logic can thus be understood as the logic of obligation and other related notions such as permission, prohibition and supererogation. Obligations (and permissions etc.) provide us, *qua* agents, norms for action. The term "norm" (Latin "norma") is used here in a general sense: besides obligations, there are norms for belief (epistemic norms), norms for preference (norms for rational choice) and so on. In this dissertation our focus is restricted to the class of deontic concepts, which is only one species of normative concepts. We shall not go into philosophical analysis of the notion of obligation here. Questions about the nature of obligations and their obligatoriness, however significant they are, require an analysis deeper than any that can be offered in this dissertation. Our focus is on the more mundane task of studying the logical relations between deontic statements, and proposing a formal representation of deontic reasoning.

In this chapter we provide a survey of modern deontic logic, starting with the so-called Standard Deontic Logic and motivations for it (Section 9.1). The development of deontic logic has been driven by a set of core problems also known as "paradoxes", which we examine in Section 9.2. Finally we give a selective summary of contemporary approaches to deontic reasoning in Section 9.3. The following sources have been consulted when preparing this chapter: introductory chapters on deontic logic in various handbooks and guides, for example, Åqvist (2002), Carmo and Jones (2002), Hilpinen (2001) and McNamara (2006). More detailed information is found in anthologies dedicated to deontic logic such as Hilpinen (1971), Hilpinen (1981) and the workshop proceedings of DEON 1998 (McNamara and Prakken (1999)), DEON 2000 (Demolombe and Hilpinen (2000, 2001)), DEON 2004 (Lomuscio and Nute (2004)) and DEON 2006 (Goble and Meyer (2006)).

9.1 Modern deontic logic

Modern deontic logic is often said, rightly or wrongly, to begin with the publication of "Deontic logic" in *Mind* by von Wright (1951). The tribute is correct in so far as influences on later authors are concerned, for it is von Wright's paper that initiates a line of research which is still active today (and, admittedly, this dissertation is part of that research tradition). But it has also been pointed out that, before von Wright's paper, Mally (1926) had already put forward, in *Grundgesetze des Sollens, Elemente der Logik des Willens*, a deontic logic which is "modern" in every aspect in which von Wright's deontic logic of 1951 is. Therefore it would not be historically incorrect to say that modern deontic logic begins with Mally's pioneer work, although the impact among logicians of his *Grundgesetze* is less than that of von Wright's "Deontic logic". (We note here that Mally's system suffers a significant defect: the collapse of what ought to be into what is the case.)

The use of the term "modern" in describing our subject matter acknowledges the fact that deontic logic, as the formal study of obligation and other deontic notions, has its origin in much earlier periods. As far as Western philosophy is concerned, discussions of normative reasoning already appeared in Aristotle's writings (for example practical syllogism in *Nicomachean Ethics*). A logic of norms began to emerge in the works of medieval philosophers, and the formal study of norms continued well into the early modern period. (Secondary literature about "pre-modern" deontic logic is rather rare. Knuuttila (1981) still provides valuable information about the development of deontic logic in the 14th century.)

9.1.1 Analogies between deontic concepts and modal concepts

Since the early days of modern deontic logic (in fact since the medieval period), logicians have noticed similarities between deontic logic and (alethic) modal logic. For example, some deontic concepts are inter-definable in the same way as modal concepts are, and deontic statements are related logically in a pattern that is analogous to the logical relations between modal statements (and categorical statements). The deontic and modal squares of opposition (Figures 9.3 and 9.2), together with the traditional square of opposition (Figure 9.1), illustrate the analogy between deontic, modal and quantificational concepts.

From the deontic square of opposition, we can derive the following set of principles (where $\Box\alpha$ stands for "Obligatorily α", and $\Diamond\alpha$ stands for "Permissibly α").

[Df$\Diamond$]	$\Diamond\alpha \leftrightarrow \neg\Box\neg\alpha$
[Df$\Box$]	$\Box\alpha \leftrightarrow \neg\Diamond\neg\alpha$
[D]	$\Box\alpha \rightarrow \Diamond\alpha$

While [Df$\Diamond$], [Df$\Box$] and [D] can be deduced from the deontic square, we can proceed the other way round, viz. deriving the deontic square from the trio [Df$\Diamond$], [Df$\Box$] and [D] (with PL as the base logic). Thus the deontic square is tautologously equivalent to [Df$\Diamond$], [Df$\Box$] and [D]. Note that if the base logic is classical (i.e. if it provides PL and [RE]), then the deontic square is equivalent to the pair [Df$\Diamond$] and [D], or the pair [Df$\Box$] and [D] (since in classical systems, [Df$\Diamond$] and [Df$\Box$] are inter-derivable).

[Df$\Diamond$] and [Df$\Box$] are so called because they can be considered as definitions of permission (in terms of obligation) and of obligation (in terms of permission). Thus, according to these two principles, α is permissible if and only if its negation is non-obligatory, and α is obligatory if and only if its negation is impermissible. The principle [D]$\Box\alpha \rightarrow \Diamond\alpha$ asserts that what is obligatory is also permissible. In view of the interdefinability of obligation and permission, the principle can be stated thus:

[D] $\Box\alpha \rightarrow \neg\Box\neg\alpha$

In other words, it stipulates that obligations cannot conflict: if α is obligatory then its negation is not obligatory. The principle is called [D] (for deontic) because being free of conflicts is often held to be a defining characteristic of deontic necessity.

It is common in modern deontic logic to define other deontic notions in terms of the primitive notion of obligation (or permission).

It is forbidden that α	$=_{\text{def}} \Box\neg\alpha$	(or $\neg\Diamond\alpha$)
It is gratuitous that α	$=_{\text{def}} \neg\Box\alpha$	(or $\Diamond\neg\alpha$)
It is optional that α	$=_{\text{def}} \neg\Box\alpha \wedge \neg\Box\neg\alpha$	(or $\Diamond\alpha \wedge \Diamond\neg\alpha$)

Figure 9.1: The traditional square of opposition

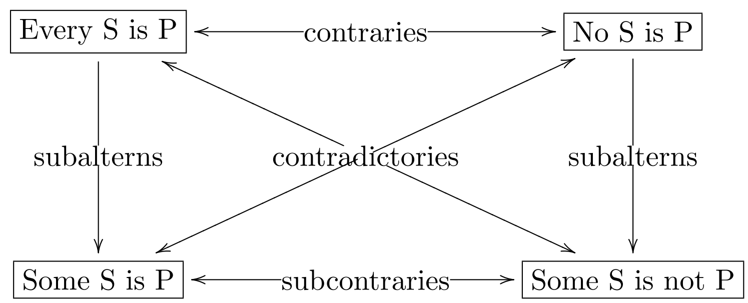


Figure 9.2: The modal square of opposition

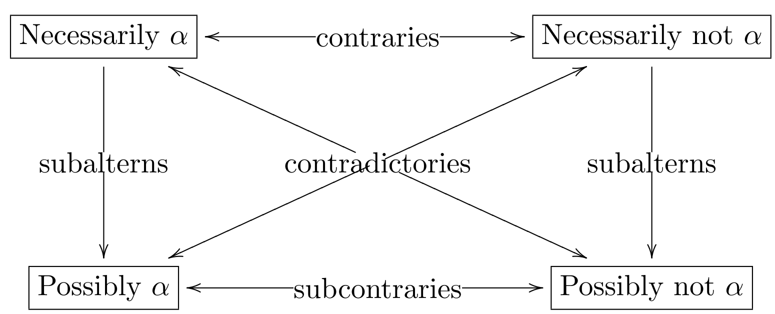
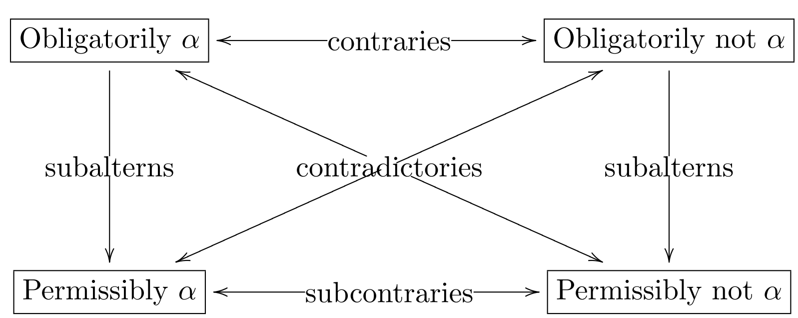


Figure 9.3: The deontic square of opposition



9.1.2 Standard Deontic Logic and its semantics

The so-called Standard Deontic Logic (SDL for short) is the system KD, which is obtained by adding the deontic principle [D] to K, the smallest normal system. (Recall that K is axiomatized by PL, [K], [RN], or alternatively by PL, [C], [RM], [RN].) We note here that in any normal system, [D] is equivalent to the following principle.

[P] $\neg\Box\bot$

(Following Chellas we call the above principle [P] where P stands for possibilitation). Since [D] and [P] are inter-derivable in any normal system of modal logic, SDL can be defined to be the system KP instead of KD. (More specifically, the inter-deducibility between [D] and [P] is as follows. [P] is derivable from [D] using PL and [RM], and [D] is derivable from [P] using PL and [C].)

According to the relational semantics, $\Box\alpha$ is true at a point (state or world) x iff α is true at all points to which x is related. It has been shown that SDL is determined by the class of relational frames satisfying seriality, the condition that every point is related to some point(s). The relata of x (the points to which x is related) are often said to be the ideal alternatives of x (i.e. states or worlds where things go as they should according to the norms of x). Thus viewed, the deontic accessibility or alternative relation is one for which every state or world has a non-empty set of ideal alternatives.

9.1.3 Reduction of SDL to alethic modal logic

In the 1950's, Anderson proposed a reduction of deontic logic to alethic logic. (See Anderson (1956, 1958).) Kanger, a contemporary of Anderson, had a similar proposal. In fact the idea of analyzing deontic modalities in terms of alethic ones can be found in Leibniz's works. (Hilpinen (2001) has more details on this.)

Let us consider a language which has two modal operators $\Box$ and $\boxdot$, and a propositional constant v (in addition to the usual propositional connectives and variables). Assume the logic for $\boxdot$ is the smallest alethic modal system KT and the following axioms hold.

[Viol] $\neg \boxdot v$

[Df$\Box$] $\Box p \leftrightarrow \boxdot(\neg p \to v)$

Then the logic for $\Box$ satisfies the theorems of SDL (i.e. KD). It is common to read the constant v as the proposition that a violation of some relevant norms has occurred.

9.2 Problem set for deontic logic

There has been an accumulation of core problems for modern deontic logic that are widely referred to as paradoxes (or puzzles). Although they are often directed at the so-called Standard Deontic Logic (SDL), these "paradoxes" are applicable to any formalization of deontic reasoning as they are to SDL. Thus in what follows, we discuss these problems, whenever possible, from the general perspective of deontic logic (rather than from the perspective of SDL). Nevertheless, the rules or axioms belonging to, or absent from, SDL will be used to illustrate the problems. (Not all the problems are peculiar to modern deontic logic. Some of them, for example, those relating to conditional obligations and closure of obligation under consequence, were discussed by medieval logicians (cf. Knuuttila (1981)).

9.2.1 Representation of norms

Ought-to-do and ought-to-be

Obligations and prohibitions are often expressed in sentences such as "A ought to do X" and "A ought not to do X" where the term A refers to some agent(s), definite or indefinite, and the term X an action or action type. For example, the ideal of fidelity is often promulgated by such principles as "you ought not to cheat on your partners". These sentences are often referred to as statements expressing *ought-to-do* since deontic concepts are applied to actions which are to be carried out or avoided by agents. (Instead of postulating a separate class of "ought-not-to-do" for prohibitions, we treat them as obligations to refrain from executing certain actions.)

The idea of deontic concepts as ought-to-do presents a problem to logicians. Since its early days, modern deontic logic has been developed as a branch of modal logic. It is a common practice in modal logic to apply modal terms to indicative sentences. In other words, modal concepts are ascribed to factual propositions or states of affairs. For example, "S must be P" is treated as having the same meaning as"It is necessarily the case that S is P" or "Necessarily S is P", which is symbolized by $\Box\alpha$ where $\Box$ is the alethic modality of necessity and α the sentence expressing the proposition that S is P. Following the practice of modal logic, normative statements in deontic logic are typically treated as indicative sentences with modal operators applied to them. So "A ought to do X" becomes "It ought to be the case that A does X" or "Obligatorily, A does X", and in symbol $\Box\alpha$ where $\Box$ denotes the notion of obligatoriness and α the proposition that A does X. Sentences of the form "It ought to be the case that A does X" are said to be statements expressing *ought-to-be*, in order to distinguish them from statements expressing ought-to-do, which is of the from "A ought to do X".

Given the distinction between ought-to-do and ought-to-be, one may ask whether the practice in deontic logic of representing ought-to-do by ought-to-be is adequate or not. There are in fact two questions involved. The first one is about the appropriateness of the translation of ought-to-do to ought-to-be we have discussed earlier, viz. treating "A ought to do X" as being equivalent to "It ought to be the case that A does X". However, even though the above translation may be shown to be faulty, there may still be other reductions that do the job better. So there is a more general question of whether every ought-to-do can be reduced to an ought-to-be.

Norms and truth

There is a philosophical tradition according to which norms are non-factual items and as such lack truth values. On the other hand, there is the view that logical relations exist among norms and as a result a logic of norms is possible. Let us call the first view "non-cognitivism" and the second view "logical approach to norms". These two views become incompatible if one also accepts that logical relations are dependent upon the possession of truth values by the items entering into those relations. Several responses are possible.

(1) One may simply reject non-cognitivism and endorse the logical approach to norms. However in accepting the logical approach, explanation has to be given as to how norms acquire truth values (despite the apparent differences between normative sentences and factual ones).

(2) One may adopt the non-cognitivist position and reject the logical approach to norms. If so, then the remaining problem is to explain away the appearance of a logic of norms.

(3) Instead of choosing between non-cognitivism and the logical approach to norms, one may accept both and reject the assumption that logical relations among norms require the ascription of truth values to norms. The challenge then is coming up with a theory of validity that is not based on the notion of truth.

9.2.2 Violability and fulfillability of norms

Some have argued that obligations must be violable, and so actions (events or states) can be obligatory only if they are avoidable. Violability could be applied in different contexts: we may ask whether it is psychologically, physically or conceptually possible for someone to default on his obligation. The case for violability looks contentious if what matters is psychological possibility. For instance it may well be true that parents protect their children from harm as a matter of psychological (or biological) fact. But it still makes sense (at least for some theorists) to say that parents ought to protect their children from harm. However the principle of violability appears more convincing in those cases involving physical possibility. For example, it is physically impossible for us to change events that have already happened; so no one is obligated to keep past events from being changed. In the following discussion, we refrain from entering into this debate by considering violability in a logical context. The specific type of considerations, whether it is psychological, physical or conceptual, can be incorporated into the underlying logic as domain-specific axioms.

The principle of violability (in a logical context) is usually specified by a rule to the effect that if α is a theorem of the logic, i.e. logically necessary, then α is not obligatory ($\vdash \alpha \implies \vdash \neg\Box\alpha$). In other words, no theorems are obligatory ($\neg\Box\top$). This principle obviously leads to contradiction in any logic that has [RN], or equivalently [N], both of which stipulate that every theorem is obligatory ($\Box\top$). But the problem is not restricted to [RN] or [N] only. Indeed in any system that provides [RM], the principle of violability entails that nothing should be obligatory if contradiction is to be avoided. For according to [RM], if any thing is obligatory at all then so is the verum.

While (it has been argued that) obligations should be violable, (it has also been argued that) obligations should be fulfillable. As in the case of violability, the notion of fulfillability can be applied in various contexts: psychological, physical, conceptual, etc. Thus if it is a fact of psychology that parents protect their children from harm, then no parents should be obligated to sacrifice their children willingly (though as in the case of violability this claim may be contested). For the same reason that no one ought to prevent past events from being changed, viz. the fact that history cannot be altered, no one ought to change past events either. To avoid controversy regarding the principle of fulfillability in different contexts, we consider logical fulfillability in our discussion.

In comparison with the principle of violability (in a logical context), the principle of fulfillability presents a lesser challenge to deontic logic. It can be represented either by a rule to the effect that nothing logically impossible should be obligatory ($\vdash \neg\alpha \implies \vdash \neg\Box\alpha$), or by the principle that the falsum is not obligatory ([P]$\neg\Box\bot$). Unlike the principle of violability, the principle of fulfillability does not lead to logical inconsistency in systems that has [RN] or [RM]. However the principle of fulfillability has a problem of its own: if (unrestricted) aggregation of obligations is permitted, then the principle of fulfillability excludes cases of conflicting obligations, situations which, according to some theorists, are plausible. (We shall discuss more of this in the next section.)

In a language that has both deontic modality ($\Box$) and alethic modality ($\boxdot$), the principle of violability cum fulfillability (i.e. $\neg\Box\top \wedge \neg\Box\bot$) can be formalized by the formula $\Box\alpha \rightarrow \neg \boxdot \alpha \wedge \neg \boxdot \neg\alpha$, which says α is obligatory only if α is contingent.

9.2.3 Normative conflicts

Not all cases of normative conflicts are irresolvable. For in some cases the appearance of conflicts between obligations can be removed by balancing the reasons that support each of the obligations. However some philosophers and logicians maintain that not all apparent conflicts can be resolved by deliberation (for example when the reasons for the incompatible obligations are equally strong) and so there are genuine cases of normative conflicts. The following two examples, taken from Plato and Sartre (see Lemmon 1962) illustrate the above.

Case 1. Plato in the *Republic* describes the following scenario. A man demands his friend to return weapons as promised. But the man is now in a rage and intends unjustly to kill someone with the weapon. While it is obligatory for his friend to keep his promise, it is also obligatory for him to save innocent life. Apparently he cannot fulfil both obligations.

Case 2. A character in Sartre's essay has to choose between joining the resistance to revenge his brother's death and fight the Nazi occupation, and staying at home to aid his ailing mother. It seems that he is obligated to do both, even though performing one means neglecting the other.

The first case, many will say, is not irresolvable, for the reason to save life outweighs the reason to keep promise. But the second case presents a greater challenge since the reason for joining the resistance is as strong as the reason to stay at home. The existence of these two types of conflicts—those that are resolvable by balancing the strength of reasons and those that are (or at least seem to be) irresolvable by such calculation—suggests a distinction between prima facie obligations and all-things-considered obligations. While it is generally accepted that prima facie oughts may conflict with each other, there is no such consensus on the question of whether there are genuine conflicts among all-things-considered oughts. For example, while some ethicists regard Case 2 as an instance of irresolvable normative conflicts, some consider it as a case in which one has a disjunctive obligation (i.e. an obligation to do either one of the two options) and not a case in which one has two obligations (i.e. an obligation to do one option and another obligation to do the other option).

The rejection of conflicting obligations (let us assume they are all-things-considered oughts) can be represented by the so-called deontic consistency principle [D] $\Box\alpha \rightarrow \neg\Box\neg\alpha$. (More generally, if the logic has rule [RM], then [D] is equivalent to $\Box\alpha \rightarrow \neg\Box\beta$ where $\alpha \wedge \beta \rightarrow \perp$ is a theorem of the logic.) This principle, however, should be distinguished from the principle of fulfillability discussed earlier (viz. [P] $\neg\Box\perp$), which says that there are no logically impossible obligations. One may accept [P] while denying [D]. In other words, one may reject the existence of logically impossible obligations but accept the existence of conflicting obligations. But the distinction between [D] and [P], which seems compelling, is destroyed if the logic endorses aggregation of obligations, usually formalized by [C] $\Box\alpha \wedge \Box\beta \rightarrow \Box(\alpha \wedge \beta)$. For it is obvious that in such a logic [D] and [P] are provable equivalents.

Rejecting the aggregation principle, which collapses conflicts into impossible obligations, is important for logicians who wish to allow for normative conflicts (while maintaining fulfillability of obligations). But a total rejection of aggregation would appear too drastic, for aggregation seems desirable when no conflict would arise (see van Fraassen, 1973). While there are systems and semantics designed for distinguishing [P] from [D], devising a logic that endorses aggregation of compatible obligations remains a challenge. (The following solutions have been proposed: a logic with the axiom $\neg \boxdot \neg(p \wedge q) \wedge \Box p \wedge \Box q \rightarrow \Box(p \wedge q)$, and defeasible deontic logic.)

9.2.4 Closure of obligation under consequence

The intuition that the logical consequence of what is obligatory is also obligatory is usually formalized by the rule [RM] (from $\alpha \rightarrow \beta$ infer $\Box\alpha \rightarrow \Box\beta$). The problem of logical inconsistency caused by this rule in the presence of the principle of violability has been discussed in Section 9.2.2 on page 156. Perhaps the most obvious puzzle brought about by this rule is the so-called the Good Samaritan Paradox. For instance if we ought to relieve the suffering of the poor, then the poor ought to suffer, by virtue of [RM] and the fact that relief works presupposes that the poor suffer. A similar puzzle arises in the case of what may be called epistemic obligation. Suppose there is a fire in the town. Then the fire chief ought to know that there is a fire in the town. But this knowledge entails there is a fire; so by [RM] there ought to be a fire in the town.

Another puzzle in connection with the closure principle is Ross's paradox: A duty of posting some letter implies a disjunctive obligation of posting it or burning it, since the proposition that the letter is posted or the letter is burned is a logical consequence of the proposition that the letter is posted. But it seems odd that a duty of posting a letter begets another one which can be fulfilled by burning it.

9.2.5 Commitments or derived obligations

Representing conditional obligations generates a range of problems for deontic logic, some of which have counterparts in other modal notions, while others are peculiar to deontic notions. The so-called paradox of commitment or derived obligation (discussed in this section) belongs to the first category, and the paradox of contrary-to-duty (discussed in the next section) belongs to the second category.

Commitments have the general form "A's action X commits him to do Y". Let α be the proposition "A did X", and β the proposition "A does Y". At first glance, we can formalize commitment in one of the following two ways (where $\rightarrow$ is the material conditional).

(1) $\alpha \rightarrow \Box\beta$

(2) $\Box(\alpha \rightarrow \beta)$

However each of the above approach has problems of its own.

Suppose commitment is represented by (1). We have the following by virtue of propositional logic (for any α and β).

- $\alpha \rightarrow (\neg\alpha \rightarrow \Box\beta)$
- $\Box\beta \rightarrow (\alpha \rightarrow \Box\beta)$

In other words, the negation of a true proposition commits one to everything, and any proposition commits one to an existing obligation. These consequences, although not causing any logical contradiction, would seem odd. Readers may notice that this is similar to the paradox of material implication.

Suppose we represent commitment by (2). Analogous to the paradox of strict conditional, we have the following by virtue of propositional logic and [RM].

- $\Box\alpha \rightarrow \Box(\neg\alpha \rightarrow \beta)$
- $\Box\beta \rightarrow \Box(\alpha \rightarrow \beta)$

What the above says is that violating an existing obligation commits one to everything, and anything commits one to whatever is already obligatory. These results look strange, if not totally outrageous.

9.2.6 Contrary-to-duty obligations

The original Chisholm paradox

In "Contrary-to-duty imperatives and deontic logic" (1963) Chisholm argues that contrary-to-duty imperatives (imperatives telling us what we ought to do if we neglect certain of our duties) cannot be given an adequate representation in the deontic systems proposed by Mally, von Wright, Prior, and Anderson. (We can substitute Standard Deontic Logic SDL for the target of Chisholm's criticism.)

Chisholm observes that CTD imperatives cannot be represented in the form of an obligatory conditional: "It is obligatory that if a then b". His reason is that given "It is obligatory that not a", we can derive "It is obligatory that if a then b", for any b. (If one should refrain from performing the act of doing a, then one should refrain from performing the joint act of doing a and not doing b, no matter what b may be.) But apparently this is not what we intend when using CTD imperatives. For example, breaking a promise requires remedial action, but the misdeed does not give us license to do anything we want. (In SDL, $\Box\neg\alpha$ entails $\Box(\alpha \rightarrow \beta)$ for any β by virtue of propositional logic and [RM].)

If CTD imperatives cannot be represented as obligatory conditionals, then, Chisholm points out, they must be represented in the form of a conditional with an obligatory consequent ("If a, then it is obligatory that b"). But unfortunately this leads to logical contradiction in the presence of [K] and [D]):

[K] $\Box p \wedge \Box(p \rightarrow q) \rightarrow \Box q$

[D] $\neg(\Box p \wedge \Box\neg p)$

Chisholm illustrates the problem with an example which consists of the following four sentences:

(1) It ought to be that a certain man goes to the assistance of his neighbours.

(2) It ought to be that if he does go he tells them he is coming.

(3) If he does not go, then he ought not to tell them he is coming.

(4) He does not go.

Or in symbolic form:

(1) $\Box go$.

(2) $\Box(go \rightarrow tell)$.

(3) $\neg go \rightarrow \Box\neg tell$.

(4) $\neg go$.

From the first two sentences, we can derive $\Box tell$ using [K]. From the last two sentences, we can derive $\Box\neg tell$ using modus ponens. It then follows that $\Box tell \wedge \Box\neg tell$, which contradicts [D]. (Deriving $\Box tell$ from the first two sentences is sometimes called deontic detachment, and deriving $\Box\neg tell$ from the other two sentences is sometimes called factual detachment.) In conclusion, the above four sentences, which can be generalized to describe most of the situations in which CTD obligations arise, are mutually inconsistent (in the presence of principles [K] and [D].)

One may wonder whether the second sentence of the Chisholm set can be formalized as "$go \rightarrow \Box tell$" (thus $\Box tell$ can no longer be derived from (1) and (2) by deontic detachment). But if (2) is represented as a conditional with an obligatory consequent, then it becomes deducible from (4) simply by virtue of PL. Independence is likewise lost if we represent (3) as "$\Box(\neg go \rightarrow \neg tell)$" (thereby avoiding the derivation of $\Box\neg tell$ from (3) and (4) by factual detachment). The reason is that if (3) is so represented, it will be derivable from (1) by using PL and [RM]. Therefore the problem of representing CTD obligations is how to describe situations in which such imperatives arise by a set sentences or formulas whose members are independent of the others and logically consistent when taken together.

Note that even if [D] is dropped and so logical contradiction is avoided, the derivation of a pair of contradictory obligations (which represents a situation of practical conflict) is problematic, since intuitively the Chisholm set does not present a dilemma at all. The person, in Chisholm's example, should not tell his neighbours he is coming because he does not go to help.

Another version of the Chisholm paradox

In the original Chisholm paradox, the action described in the antecedent of the obligatory conditional ($\Box(go \rightarrow tell)$) takes place (ideally) after the action described in its consequent. In other words, helping one's neighbours should happen after telling them that one is coming. The same temporal ordering can be said of the antecedent action and the consequent action of the CTD obligation ($\neg go \rightarrow \Box\neg tell$). We can reverse this temporal ordering as in the following version of the paradox.

(1) It ought to be the case that John does not impregnate Suzy Mae.

(2) It ought to be the case that if John does not impregnate Suzy Mae, then he does not marry her.

(3) If John impregnates Suzy, then it ought to be the case that he marries her.

(4) John impregnates Suzy.

Timeless and actionless CTD examples

The CTD scenarios considered so far involve some kind of temporality of actions. However there are examples of CTD not depending on any temporal ordering of actions at all. The following are two examples.

Example 1:

(1) There ought to be no dog.

(2) If there is no dog, there ought to be no warning sign.

(3) If there is a dog, there ought to be a warning sign.

(4) There is a dog.

Example 2:

(1) There must be no fence.

(2) -

(3) If there is a fence, then it must be a white fence.

(4) There is a fence.

The Gentle Murderer Paradox

Some paradoxes discussed in the literature have a similar structure to the Chisholm paradox. They describe scenarios in which some obligations arise in less than ideal situations. However, under [RM] (closure of obligations under logical consequence), such obligations imply other obligations that are highly problematic. We describe one such case (the gentle murderer paradox) here, and another (the good Samaritan paradox) in the next section.

(1) Smith ought not to kill his mother.

(2) If Smith kills his mother, he ought to kill her gently.

(3) Smith kills his mother.

The last two sentences entails that Smith ought to kill his mother gently. But killing gently implies killing. So one may conclude that Smith ought to kill his mother, which is (deontically) inconsistent with the first sentence.

The Good Samaritan Paradox

A good Samaritan gives help to people in trouble (for example, the victim of a robbery). But her good deed implies the existence of misery of someone. So, given that obligations are closed under logical consequence (rule [RM]), someone ought to suffer. Like the CTD examples above, the Good Samaritan paradox involves some less than ideal situation. Although the agent (the good Samaritan) has not violated any primary obligation, we can present the paradox in the standard CTD format as follows:

(1) John ought not to be robbed.

(2) If John has been robbed, Mary ought to help him.

(3) John has been robbed.

Mary ought to help John, who has been robbed (from the last two sentences). Since helping the victim of a robbery means that the person in question has been robbed, we arrive at the conclusion that John ought to be robbed (by applying [RM]). This contradicts the first sentence (in the presence of D, the deontic consistency principle).

Responses to the CTD "paradoxes"

In "Deontic logic and contrary-to-duties" (2002), Carmo and Jones state the following requirements that an adequate formalization of the Chisholm set (and other CTD scenarios) should meet:

(1) The set should be consistent.

(2) The sentences in the set should be logically independent.

(3) The formalization should be applicable to timeless and actionless CTD examples.

(4) The assignment of logical form to each of the norms in the set should be independent of the other norms in it.

(5) We should be able to derive actual obligations.

(6) We should be able to derive ideal obligations.

(7) Pragmatic oddity should be avoided.

In the following we outline some proposals that address the problem of representing CTD.

9.3 After SDL: new approaches to deontic logic

We list below some of the contemporary approaches to deontic logic. Although a broad range of deontic logics are covered, the examples we give represent only a small subset of the different theories available. Note that the approaches are not exclusive of each other. Quite often the same deontic logic may incorporate elements from several approaches. Our discussion here is brief. So interested readers are advised to check the references we provide below.

9.3.1 Temporal approaches

In traditional deontic logics such as SDL, obligation statements are evaluated at a state or world. But obligation changes as the state or world evolves over time, and in the traditional approaches such changes of obligation cannot be represented easily. For example, in the

Chisholm paradox (Section 9.2.6), before it is settled that the person does not go to help his neighbours, he has an obligation to tell them that he is coming . However, once the matter is settled, the original obligation is replaced by a contrary one: the obligation of not telling them that he is coming.

In order to deal with the Chisholm paradox or other situations in which temporality plays an important role, a formal language that can represent time is desirable, and deontic logic becomes an extension of temporal logic. There are two families of deontic temporal logics: the indexed or the non-indexed.

(1) In indexed temporal deontic logics (e.g. van Eck (1982)) , the object language typically has the following symbols:

- time terms t_1, t_2, etc.
- time-indexed propositional variables p_{t_1}, p_{t_2}, etc.
- time-indexed necessity operator $\boxdot_{t_1}$, $\boxdot_{t_2}$, etc.
- time-indexed deontic operator $\Box_{t_1}$, $\Box_{t_2}$, etc.

(2) In non-indexed temporal deontic logics, the base tense logic has temporal operators such as F, G, P, H (for "it will be the case that", "it is always going to be the case that", "it was the case that", and "it has always been the case that", respectively). In addition to the temporal operators, the object language has a necessity operator $\boxdot$ and a deontic operator $\Box$. See, for example, Chellas (1980), Thomason (2002, 1981).

A model for temporal deontic logic usually consists of a set of histories, which are instantaneous world-states ordered temporally. For each history h at a moment of time t, there is a set of histories h' that is called the deontic alternatives of h at t (subject to the constraint that h' shares the same past as h at t). Thus worlds or states in the model for SDL are replaced by histories in the model for deontic temporal logic, and the relation of deontic alternativeness is no longer between worlds or states, but between histories and relativized to time.

9.3.2 Action-based approaches

The distinction between ought-to-be and ought-to-do has been discussed in Section 9.2.1. In traditional deontic logics, the obligation operator is applied to sentences expressing states of affairs rather than to terms denoting actions. There are contemporary deontic logics that allow us to represent actions explicitly.

(1) In Casteñeda's deontic logic, a distinction is made between propositions and practitions, for example, conditional obligations can be expressed by formulas in the form of $\Box_s(p \rightarrow q*)$ where s is a particular sense of obligation, p is the circumstance or condition of a deontic judgement, and $q*$ is an action practically considered. See Tomberlin (1983b, 1986a) for a discussions of Casteñeda's logic.

(2) The dynamic deontic logics of Meyer (1988) and Meyer et al. (1998) are based on propositional dynamic logic, which has terms for both actions and propositions. For example, the formula $[\phi]\alpha$ means that execution of the action ϕ leads to some state where the proposition α holds. The object language has a propositional constant v denoting violation. Formulas of the form $[-\phi]v$ thus states that the negated action $-\phi$ leads to a state of violation, or, put it another way, the action ϕ is obligatory.

(3) The deontic logic of Horty (2001) represents actions with the help of an operator called "cstit" ("stit" for "see to it that" and "c" for Chellas). The statement that an agent A sees to it that a state α is the case is formalized by $[A : \text{cstit}\, \alpha]$. Ought-to-do and conditional ought-to-do are represented as follows:

- $\Box[A : \text{cstit}\, \alpha]$: A ought to see to it that α.
- $\Box([A : \text{cstit}\, \alpha]/\beta)$: A ought to see to it that α under the condition β.

9.3.3 Preference-based approaches

In the model for SDL, each state is assigned a collection of states, called its deontic alternatives (sometimes called ideal states or better permissible states). A formula α is said to be obligatory at a state x if α holds at every deontic alternative of x. This type of model is too crude, as critics point out, to represent all of the deontic notions we are interested in. An example is the contrary-to-duty obligations. These duties arise when some other duties are violated. However, in the model for SDL, no duties that are applicable of a state x go unsatisfied in the deontic alternatives of x. In order to model CTD scenarios, we need some kind of grading of states. For instance, in the Gentle Murderer Paradox, those states in which Smith kills his mother gently are better than those in which he kills but not gently although in either case he has violated his obligation of not killing his mother.

Lewis (1974) discusses four types of semantics. They postulate some kinds of value structures on the basis of which worlds or states are compared.

- Hansson (1969)
- Føllesdal and Hilpinen (1971)
- van Fraassen (1972, 1973) (Appendix C.1)
- Lewis (1973)

For more recent preference-based deontic logics, see Jennings (2001) and Goble (2000, 2003, 2004) (Appendix C.2). Horty (2001) is a combination of temporal, action and preference based approaches.

9.3.4 Rule-based approaches

The traditional way of doing deontic logic (and modal logic in general) has been to define a class of models and determine the logic it validates (or vice versa, that is, to define an axiom system and find a class of models that validates the theorems of the system). In this approach, a normative statement is considered as expressing a proposition just like any non-modal statement is. But this methodology has been questioned for various reasons. One concerns the truth-aptness of norms, which we have discussed in Section 9.2.1. Another criticism is directed at its failure to represent our normative reasoning, for example, the application of aggregation when doing so would not cause any problem (Section 9.2.3. This type of reasoning is difficult to be represented in a traditional modal system (see van Fraassen (1973)). The last mentioned criticism has led Horty (1997) to treat norms as default rules, thus showing that van Fraassen's semantics can better be captured by treating a set of norms as a default theory. Yet another criticism of the traditional approach is that it is difficult to formulate a defeasible theory. Note that the classical consequence relation $\models$ is monotonic (and so is the classical derivability relation $\vdash$). In order to get a nonmonotonic deontic logic, a more radical approach than the traditional one becomes necessary. (The same situation is also found in default reasoning, which adopts formalisms such as circumscription and default logic.) In the following, we list some approaches that treat norms, not as ordinary modal formulas, but as rules.

- Horty's nonmonotonic deontic logic: Horty (1997, 2003) (Appendix C.3)

- Nute's defeasible deontic logic: Nute (1997a, 1999) (Appendix C.4)
- Makinson and van der Torre's input/output logics: Makinson and van der Torre (2000, 2001, 2003) (Appendix C.5)

Chapter 10

The Logic of Deontic Residuation

In this chapter, we present a class of normal polyadic systems that are interpreted as logics of deontic residuation, the concept that a principal obligation passes to another obligation, for example, through neglect of the agent or change of circumstances. The second obligation could also pass to other obligations, and the process, or residuation as we call it, may continue for some further steps. Deontic residuation is contrasted with the idea that a single sanction attends every omission of obligation, a thesis introduced by Anderson in his reduction of deontic logic to alethic modal logic. Making certain assumptions about the polyadic operators, we show that our deontic logics can be strengthened and the resulting systems can be embedded into the so-called Standard Deontic Logic (SDL). But even in these reductions Anderson's notion of there being a single sanction following different transgressions is avoided.

The idea of deontic residuation introduced here generalizes that of contrary-to-duty obligation (CTD): whereas the notion of CTD involves a primary obligation and a secondary obligation (hence it is dyadic), our notion of deontic residuation allows for a finite sequence of obligations starting from a principal obligation and going through successive residual obligations (hence it is polyadic). The adoption of polyadic language permits us to represent the change of obligation more effectively than using monadic or dyadic language.

We begin in Section 10.1 with a discussion of the shortcoming of SDL in representing the consequences of moral transgressions, which is exposed by Anderson's famous reduction of SDL to alethic modal logic. Systems of deontic residuation and strong residuation are then presented in Section 10.2, followed by their classes of frames in Section 10.3. These systems are demonstrated to be complete with respect to their classes of frame (Section 10.4). We show that the systems of strong deontic residuation can be embedded in SDL (Section 10.5). An interpretation of the deontic rules and axioms is provided in Section 10.6, where we also illustrate how normative conflicts are dealt with. (Section 10.1 is contributed by Ray Jennings, who has also suggested the names “semita” and “deontic residuation” to the author.)

10.1 From Anderson’s sanction to deontic residuation

So many authors have brought serious charges against each of the principles of SDL, that one must almost reject the system KD from the role as a standard or reconstrue the role. This much is true. KD has become the customary point of departure for pure research into formal models of deontic language, and that perhaps, *post hoc*, justifies the title. It is obtained, following Kripke’s recipe, by replacing the alethic principle [T] by the so-called deontic law, [D]. But both that alethic and that deontic logic represent a set of necessities as a classical theory, that is, as a (classically) deductively closed set, which is therefore either consistent or the whole language. This has seemed to some to misrepresent deontic discourse both by collapsing deontically significant distinctions and by multiplying obligations beyond moral capacity. On the first score, Jennings and Schotch, severally and jointly have explored systems that preserve the distinction between

[D] $\Box p \rightarrow \Diamond p$,

which seems to preclude moral conflict and

[P$\Box$] $\neg\Box\bot$,

which merely rejects obligatory contradictions. See Schotch and Jennings (1980, 1981), Jennings and Schotch (1981) and Jennings (2001). That distinction requires the rejection of

[C] $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$.

In the second matter, various authors, supposing themselves to be honouring von Wright’s principle that only contingencies can be obligatory, have rejected the principle

[RN] $\dfrac{\vdash \alpha}{\vdash \Box\alpha}$, which makes all tautologies obligatory. In fact, however, von Wright's contingency principle would require the rule

$$[\text{Anti-RN}]\ \frac{\vdash \alpha}{\vdash \neg\Box\alpha},$$

the adoption of which would, in its turn, require the rejection of the unrestricted monotonicity principle

$$[\text{RM}]\ \frac{\vdash \alpha \to \beta}{\vdash \Box\alpha \to \Box\beta}.$$

When all of these authors have made their excisions, only [P□] remains as a bedrock deontic principle. On the other hand, the remaining, restricted monotonicity principle, which holds for provable consequents other than tautologies, would seem to require a strengthened underlying logic such as S5 to distinguish contingent consequents from those universally verified in a model.

At a somewhat more foundational level, no argument has been advanced to justify the system KT, or latterly K, as the most natural starting point for modal logic, and that system has at least one competitor that is at at least as intuitively compelling. Algebraically, one might insist, it is plausible to assume that $\bot$ is not necessary, that $\top$ is necessary and that anything above a necessity is a necessity. On this intuition, a reasonable starting common point of departure for both alethic and deontic systems would be the system axiomatized by [P□], [N] ($\Box\top$), and [RM].

The deontic path would retain [P□], replace [N] by [Anti-RN] and [RM] by a rule yielding only contingent obligations. Again, the approach would seem to require some such system as S5 rather than PL as its foundation, though this would confer another benefit in that it would let us explicitly represent the Kantian law that ought implies can.

The problem with these surgery-cum-prosthesis (SCP) approaches is that in ordinary deontic discourse, we do, upon occasion, feel compelled to infer obligations from obligations using something like [RM], even if we do not infer tautologous ones, and we do from time to time aggregate obligations, even if we do not aggregate those that conflict. Who is to say that it is not unweakened [RM] that we are "using" or not [C]? In the nature of things, the whole of humanity in the whole history of its deontic deliberations will not have used even every acceptable instance of either principle. Moreover, even if every acceptably attributed obligation lies strictly between the verum and the falsum, the practical principle that ensures this is that we don't expect from one another even the physically, or psychologically impossible, and we don't represent conditions already achieved as present obligations. (Perhaps the real deontic correspondent of [T] is [Anti-T] $\Box p \to \neg p$.) So there may be some point in trying to separate inferential discourse from its logic. The former is the set of all historical, correct inferences; the latter is the smallest closed set of conditionals capable of expressing them. A similar attitude would dissipate some of the gloom from the study of relevant logic.

A second and more illuminating alternative to SCP approaches would take up some advice offered by Max Cresswell [1]: Don't re-axiomatize; define. The idea, as it applies to deontic logic, would be to exploit the expressive power of the language of SDL or even K to embed systems closer to the heart's desire. So, for example, a connective $\Box_1$ defined by

$$\Box_1 \alpha = \Box \alpha \wedge \Diamond \alpha$$

would admit [D] and defines a non-normal deontic sublogic of K, just as the connective $\Box_2$ defined by

$$\Box_2 \alpha = \Box \alpha \wedge \alpha$$

yields the alethic logic T as a sublogic of K. Such a strategy has the theoretical advantage of introducing a new question, namely, how is the $\Box$ of the parent system to be understood in the definition?

Now a characteristic of deontic necessity is its adventitiousness even in the time scale of day-to-day living. Obligations can arise accidentally as the outcome of morally indifferent events. A complete stranger, by falling ill in one's presence, creates an obligation. And obligations are created dynamically by one's responses. Without self-loathing, one does not cower in one's room knowing of a heart attack behind the next door. The loathing is hardly diminished if one takes on the task of phoning the wife while delegating the duty of care. Even the notification seems to require an offer of transport.

Now it may be simply a matter of time-scale that distinguishes the adventitiousness of deontic necessity from that of, say, physical necessity. We do not know what accidents early in the evolution of the physical universe created its present nomological profile. Nor, presumably, can we answer corresponding questions set in an even larger multidimensional-scale for mathematical necessity. However, in the realm of the deontic, our social lives are

[1] In conversation with Ray Jennings, 1973.

daily shaped by necessities born of accidence, whether chance spatio-temporal coincidence, or failures on our own or on others' parts to meet social expectations.

It is evident that not only moral failures have morally significant consequences. But some have thought that among human doables, it is precisely the anticipatable untoward moral significance of failure that distinguishes the obligatory from the non-obligatory. Of course, simply as a matter of logic, failures of obligations commit one to every act, as negations of alethic necessities strictly imply every sentence. Alan Ross Anderson's *reduction* of deontic logic to alethic modal logic can be understood as an attempt to refine this purely formal mark of the obligatory by more closely specifying a *morally* significant outcome of failure (Anderson (1956, 1958)).

One might rather have said that his was a study of a deontic system restricted to those obligations for which penalties invariably attend defaults. The reduction averages over any distinctions among sanctions that might correspond to differences among their triggering transgressions. We need not interpret this as the stern moralism of an uncompromising religionist

> Ye have heard that it was said by them of old time, Thou shalt not kill; and whosoever shall kill shall be in danger of the judgment. But I say unto you . . . whosoever shall say, Thou fool, shall be in danger of hell fire. *Matthew* 5: 21, 22.

for if we average the sanctions we will not be hanged even for a sheep, let alone for a lamb. Anderson himself offered "All Hell breaks loose" as a reading for his constant S, but we can understand this as merely a moral hell, which, like Mr. Bennett, we get through pretty well.

> It was not till the afternoon, when he joined them at tea, that Elizabeth ventured to introduce the subject [of Lydia's elopement]; and then, on her briefly expressing her sorrow for what he must have endured, he replied, "Say nothing of that. Who would suffer but myself? It has been my own doing, and I ought to feel it."
>
> "You must not be too severe upon yourself," replied Elizabeth.
>
> "You may well warn me against such an evil. Human nature is so prone to fall into it! No, Lizzy, let me once in my life feel how much I have been to blame. I am not afraid of being overpowered by the impression. It will pass away soon enough." (Austen, 1813)

Again, in ordinary life, an undertaking, as often as not, has a shelf-life: some sought-after benefit may be gained by its timely fulfilment, but once its "best-before" date has passed, we soldier on without the benefit or we seek another. Grant application deadlines come readily to mind. If one is missed, there will be another, and in the meantime other sources of support for destitute research students may present themselves. Virtue ethics makes us complacent. It is a virtue to feed the starving, but the starving we have with us always: it will be just as virtuous to feed next year's batch after this year's batch have succumbed.

The idea that every failure to fulfill one's obligations is a grave matter is an idol of moral vanity. As Kipling remarked: All men count, but none too much. Our obligations, as often as not, present themselves with, or even as the consequences of failure, and also with means of mitigation. The general notion of behaving well is one of behaving in such a way as to minimize duties of mitigation for ourselves and for others. In popular expression it is to create or leave as few pieces as possible for ourselves and others to pick up and reassemble. But in general we live our lives on the avails of restitution.

This general feature of mitigation creates another source of difficulty for SDL. Since its introduction by Chisholm, it has been discussed under the heading of contrary-to-duty-imperatives (see Section 9.2.6 for details). Solving the problem requires the recognition that in real life obligations of any moment residuate. Obligations of the absolute non-residuating variety cannot have much moment, precisely because no further obligations arise when we fail to fulfill them. This chapter is an attempt to chart a representation of moral residuation in polyadic modal logics and SDL.

10.2 Normal deontic systems and their residuating extensions

Recall that an n-adic system (in the language $\mathcal{L}_n$) is said to be normal if it includes PL and provides the following rules of inference and axioms. (In what follows, $1 \leq i \leq n$, and β, q, and $p_i \wedge q$ occur in the ith place of $\Box$ as α_i and p_i do.)

$$[\mathrm{RM}^i_n] \quad \frac{\vdash \alpha_i \to \beta}{\vdash \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \to \Box(\alpha_1, \ldots, \beta, \ldots, \alpha_n)}$$

$$[\mathrm{RN}^i_n] \quad \frac{\vdash \alpha_i}{\vdash \Box(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)}$$

$$[\mathrm{C}_n^i] \qquad \Box(p_1,\ldots,p_i,\ldots,p_n) \wedge \Box(p_1,\ldots,q,\ldots,p_n) \rightarrow \Box(p_1,\ldots,p_i \wedge q,\ldots,p_n)$$
$$\Box(p_1,\ldots,p_i \wedge q,\ldots,p_n)$$

The weakest normal n-adic modal logic, called K_n, axiomatizes the set of formulas valid in the class of $(n+1)$-ary relational frames (see Section 2.5). In the following, we extends K_n to normal deontic logics, which are further extended to logics of deontic residuation and strong deontic residuation.

10.2.1 Normal deontic systems

Definition 10.2.1 (Normal deontic systems)**.** A normal n-adic system is said to be *deontic* if it provides the following axiom.

$$[\mathrm{P}\Box_n] \quad \neg\Box(\bot,\ldots,\bot)$$

The weakest normal n-adic deontic system is called D_n. ⊣

$[\mathrm{P}\Box_n]$ is the dual of the possibilitation principle $[\mathrm{P}_n]$ $\Diamond_n(\top,\ldots,\top)$. (Note that $[\mathrm{P}\Box_n]$ is logically equivalent to $[\mathrm{P}_n]$.) We might refer to the formula $\neg\Box_n(\bot,\ldots,\bot)$ as the deontic principle and notate it as $[\mathrm{D}_n]$. However it is now common practice to use the symbol "D" (for "deontic") to designate another principle, viz. $\Box p \rightarrow \Diamond p$. Thus, in naming the axioms of our normal deontic logics, we adapt the nomenclature of Chellas, who calls the formula $\Diamond\top$ "P" and the formula $\neg\Box\bot$ "P□".

By way of illustration, we list the inferential rules and axioms of D_1 and D_2 below. Note that an alternative axiomatization of D_1 is obtained by adding to the weakest normal system K_1 the axiom [D] $\Box\alpha \rightarrow \Diamond\alpha$ instead of our $[\mathrm{P}\Box_1]$. D_1 is also known as "Standard Deontic Logic" (SDL) in the literature.

Example 10.2.2. D_1 (in the language $\mathcal{L}_1$) consists of PL and the following rules of inference and axioms.

$$[\mathrm{RM}] \quad \frac{\vdash \alpha \rightarrow \beta}{\vdash \Box\alpha \rightarrow \Box\beta}$$

$$[\mathrm{RN}] \quad \frac{\vdash \alpha}{\vdash \Box\alpha}$$

$$[\mathrm{C}] \quad \Box p \wedge \Box q \rightarrow \Box(p \wedge q)$$

$$[\mathrm{P}\Box] \quad \neg\Box\bot$$

Example 10.2.3. D_2 (in the language $\mathcal{L}_2$) consists of PL and the following rules of inference and axioms.

$$[\mathrm{RM}_2] \quad \frac{\vdash \alpha \to \beta}{\vdash \Box(\alpha,\gamma) \to \Box(\beta,\gamma)} \qquad \frac{\vdash \alpha \to \beta}{\vdash \Box(\gamma,\alpha) \to \Box(\gamma,\beta)}$$

$$[\mathrm{RN}_2] \quad \frac{\vdash \alpha}{\vdash \Box(\alpha,\beta)} \qquad \frac{\vdash \alpha}{\vdash \Box(\beta,\alpha)}$$

$$[\mathrm{C}_2] \quad \Box(p,r) \wedge \Box(q,r) \to \Box(p \wedge q, r)$$

$$\Box(r,p) \wedge \Box(r,q) \to \Box(r, p \wedge q)$$

$$[\mathrm{P}\Box_2] \quad \neg\Box(\bot,\bot)$$

10.2.2 Systems of deontic residuation

A normal deontic system can be extended by adding what we call "residuation principles", resulting in a system of deontic residuation.

Definition 10.2.4 (Systems of deontic residuation)**.** A normal n-adic deontic system is said to be a *system of deontic residuation* if it provides the following axioms of residuation (where $1 \leq i \leq n$ and $\bot^k$ is a k-tuple of $\bot$'s).

$$[\mathrm{Re}_n^i] \quad \Box(p_1,\ldots,p_n) \to \Box(p_1,\ldots,p_{i-1},p_i \vee \Box(p_{i+1},\ldots,p_n,\bot^i),\bot^{n-i})$$

The weakest n-adic system of deontic residuation is called DR_n. ⊣

Observe that there are n instances of $[\mathrm{Re}_n^i]$. We list the first two below.

$$[\mathrm{Re}_n^1] \quad \Box(p_1,\ldots,p_n) \to \Box(p_1 \vee \Box(p_2,\ldots,p_n,\bot),\bot^{n-1})$$

$$[\mathrm{Re}_n^2] \quad \Box(p_1,\ldots,p_n) \to \Box(p_1,p_2 \vee \Box(p_3,\ldots,p_n,\bot,\bot),\bot^{n-2})$$

The last instance, $[\mathrm{Re}_n^n]$, is the tautology $\Box(p_1,\ldots,p_n) \to \Box(p_1,\ldots,p_n)$. As in the case of other rules and axioms, we use $[\mathrm{Re}_n]$ to denote the collection of the instances of $[\mathrm{Re}_n^i]$. DR_1 is just D_1, and our real interest in the logic of deontic residuation begins with DR_2, the axioms and rules of inference of which are given below.

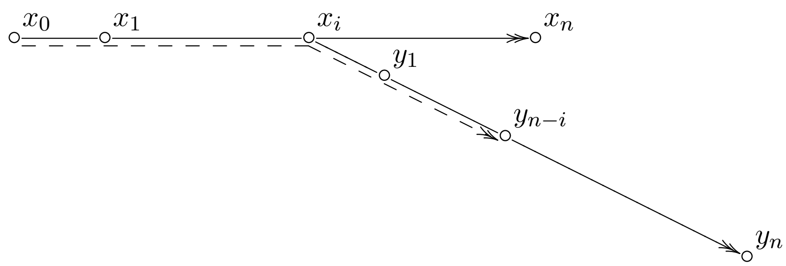


Figure 10.1: Semita at the i-th place

Example 10.2.5. DR_2 (in the language $\mathcal{L}_2$) consists of PL, [RM_2], [RN_2], [C_2], [$\mathrm{P}\Box_2$], and the following axiom.

$$[\mathrm{Re}_2] \quad \Box(p, q) \to \Box(p \vee \Box(q, \bot), \bot)$$

Note that we omit the second instance of [Re_2], which is the tautology $\Box(p, q) \to \Box(p, q)$.

We show in Theorem 10.3.3 that [Re_n^i] (with $1 \leq i \leq n$) corresponds to the following property of an $(n+1)$-ary relation R: for any $x_0, x_1, \ldots, x_n$, and $y_1, \ldots, y_n$,

$$Rx_0x_1 \cdots x_n \ \&\ Rx_iy_1 \cdots y_n \implies Rx_0x_1 \cdots x_iy_1 \cdots y_{n-i}.$$

One way to read the above condition is to treat R as consisting of paths, each with $(n+1)$ nodes, i.e. each tuple $\langle x_0, \ldots, x_n \rangle$ of R is a path originating at x_0 and passing through successively $x_1, \ldots, x_{n-1}$ before ending at x_n. What the condition says is thus the following: if there is a path $\langle x_0, \ldots, x_n \rangle$ and the path branches at x_i, i.e. there is another path $\langle x_i, y_1, \ldots, y_n \rangle$, then there is a path $\langle x_0, x_1, \ldots, x_i, y_1, \ldots, y_{n-i} \rangle$. Representing an $(n+1)$-tuple or path as an extended arrow passing through $(n+1)$ nodes, we get the picture as shown in Figure 10.1.

We call an $(n+1)$-ary relational frame *semital* if it satisfies the above condition for every $1 \leq i \leq n$. ("Semital" is based on the Latin word "semita", which means path.) The system DR_n is both sound and complete with respect to the class of $(n+1)$-ary relational frames that are both serial and semital (see Section 10.4).

10.2.3 Systems of strong deontic residuation

In this section, we strengthen systems of deontic residuation to what are described as systems of strong deontic residuation.

Definition 10.2.6 (Systems of strong deontic residuation)**.** An n-adic deontic system is said to be a *system of strong deontic residuation* if it provides the following axioms of strong residuation. (In the following, $2 \leq j \leq n$, and $\bot^k$ stands for an k-tuple of $\bot$'s.)

$$[\mathrm{Re!}^1_n] \quad \Box(p_1, p_2, \ldots, p_n) \to \Box(p_1 \vee \Box(p_2, \ldots, p_n, \bot), \bot^{n-1})$$
$$[\mathrm{Re!}^j_n] \quad \Box(\bot^{j-2}, \neg\Box(p_1, \ldots, p_n), p_1 \vee \Box(p_2, \ldots, p_n, \bot), \bot^{n-j})$$

The weakest n-adic system of strong deontic residuation is called $\mathrm{DR!}_n$. ⊣

Note that $[\mathrm{Re!}^1_n]$ is the same formula as $[\mathrm{Re}^1_n]$. Indeed, we can derive all of the instances of $[\mathrm{Re}^i_n]$ from $[\mathrm{Re!}^i_n]$, given PL, $[\mathrm{RM}_n]$ and $[\mathrm{C}_n]$ (see Theorem 10.2.8). This justifies our calling the axioms $[\mathrm{Re!}^i_n]$ "strong principles of deontic residuation", and the resulting systems "systems of strong deontic residuation". As in the case of DR_n's, the system $\mathrm{DR!}_1$ is a degenerative case: it is just D_1 (or SDL). The system $\mathrm{DR!}_2$ is given below as an example.

Example 10.2.7. $\mathrm{DR!}_2$ (in the language $\mathfrak{L}_2$) consists of PL, $[\mathrm{RM}_2]$, $[\mathrm{RN}_2]$, $[\mathrm{C}_2]$, $[\mathrm{P}\Box_2]$, and the following axioms.

$$[\mathrm{Re!}_2] \quad \Box(p, q) \to \Box(p \vee \Box(q, \bot), \bot)$$
$$\Box(\neg\Box(p, q), p \vee \Box(q, \bot))$$

The axiom $[\mathrm{Re!}^i_n]$ (with $1 \leq i \leq n$) corresponds to the following property of an $(n+1)$-ary relation R: for any $x_0, x_1, \ldots, x_n$, and $y_1, \ldots, y_n$,

$$Rx_0x_1 \cdots x_n \ \&\ Rx_iy_1 \cdots y_n \implies Rx_{i-1}x_iy_1 \cdots y_{n-1}.$$

Using arrows to represent R, we get the dotted path from the two solid paths in Figure 10.2. An $(n+1)$-ary relation satisfying the above condition for every i from 1 to n is said to be *strongly semital*. The class of $(n+1)$-ary relational frames that are both serial and strongly semital determines the system $\mathrm{DR!}_n$. (See Theorem 10.3.4 for the correspondence result and Section 10.4 for the determination result.)

Theorem 10.2.8. *The principles of residuation* $[\mathrm{Re}^i_n]$ *is provable in* $\mathrm{DR!}_n$*. Hence the n-adic system of deontic residuation* DR_n *is included in the n-adic system of strong deontic residuation* $\mathrm{DR!}_n$*.*

Proof. The proof is by induction on i. The base case is obvious since $[\mathrm{Re}^1_n]$ is just $[\mathrm{Re!}^1_n]$. For the inductive case, assume

$$[\mathrm{Re}^i_n] \quad \Box(p_1, \ldots, p_n) \to$$
$$\Box(p_1, \ldots, p_{i-1}, p_i \vee \Box(p_{i+1}, \ldots, p_n, \bot^i), \bot^{n-i})$$

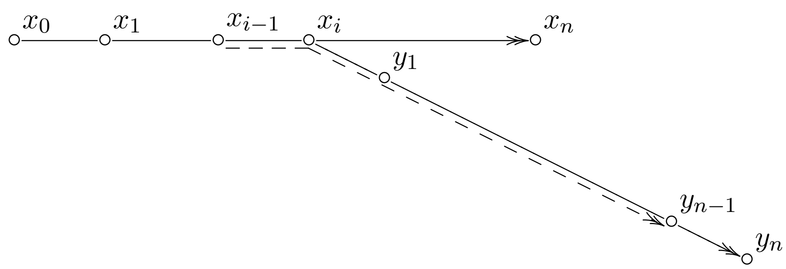


Figure 10.2: Strong semita at the i-th place

is provable in $\mathrm{DR!}_n$ (the inductive hypothesis), and show that

$$[\mathrm{Re}_n^{i+1}]\quad \Box(p_1,\ldots,p_n) \to \Box(p_1,\ldots,p_{i-1},p_i,p_{i+1}\vee\Box(p_{i+2},\ldots,p_n,\bot^{i+1}),\bot^{n-i-1})$$

is provable in $\mathrm{DR!}_n$. Given the inductive hypothesis, it suffices to show that the following is provable in $\mathrm{DR!}_n$.

$$\Box(p_1,\ldots,p_{i-1},p_i\vee\Box(p_{i+1},\ldots,p_n,\bot^i),\bot^{n-i}) \to \Box(p_1,\ldots,p_{i-1},p_i,p_{i+1}\vee\Box(p_{i+2},\ldots,p_n,\bot^{i+1}),\bot^{n-i-1}) \quad (\dagger)$$

Applying suitable uniform substitutions to $[\mathrm{Re!}_n^{i+1}]$, we obtain

$$\Box(\bot^{i-1},\neg\Box(p_{i+1},\ldots,p_n,\bot^i),p_{i+1}\vee\Box(p_{i+2},\ldots,p_n,\bot^{i+1}),\bot^{n-i-1})$$

from which by $[\mathrm{RM}_n]$ we derive the following theorem of $\mathrm{DR!}_n$.

$$\Box(p_1,\ldots,p_{i-1},\neg\Box(p_{i+1},\ldots,p_n,\bot^i),p_{i+1}\vee\Box(p_{i+2},\ldots,p_n,\bot^{i+1}),\bot^{n-i-1})$$

On the other hand, the following is derivable in $\mathrm{DR!}_n$ by using PL and $[\mathrm{RM}_n]$.

$$\Box(p_1,\ldots,p_{i-1},p_i\vee\Box(p_{i+1},\ldots,p_n,\bot^i),\bot^{n-i}) \to \Box(p_1,\ldots,p_{i-1},p_i\vee\Box(p_{i+1},\ldots,p_n,\bot^i),p_{i+1}\vee\Box(p_{i+2},\ldots,p_n,\bot^{i+1}),\bot^{n-i-1})$$

From the last two displayed formulas, we obtain by PL the following theorem of $\mathrm{DR!}_n$.

$$\Box(p_1,\ldots,p_{i-1},p_i\vee\Box(p_{i+1},\ldots,p_n,\bot^i),\bot^{n-i}) \to \Box(p_1,\ldots,p_{i-1},p_i\vee\Box(p_{i+1},\ldots,p_n,\bot^i),p_{i+1}\vee\Box(p_{i+2},\ldots,p_n,\bot^{i+1}),\bot^{n-i-1})\wedge \Box(p_1,\ldots,p_{i-1},\neg\Box(p_{i+1},\ldots,p_n,\bot^i),p_{i+1}\vee\Box(p_{i+2},\ldots,p_n,\bot^{i+1}),\bot^{n-i-1})$$

Finally by [C_n], [RM_n], and the following PL-valid formula

$$(p_i \vee \Box(p_{i+1}, \dots, p_n, \bot^i)) \wedge \neg\Box(p_{i+1}, \dots, p_n, \bot^i) \rightarrow p_i$$

we get the desired result (†). ⊣

Another sense in which $DR!_n$ is a strong system is that it can be embedded in D_1 by the following translation scheme * mapping formulas of $\mathcal{L}_n$ to those of $\mathcal{L}_1$ (see Section 10.5).

$$\Box(\alpha_1, \dots, \alpha_n)^* = \Box(\alpha_1{}^* \vee \Box(\alpha_2{}^* \vee \dots \vee \Box(\alpha_{n-1}{}^* \vee \Box\alpha_n{}^*) \dots)).$$

In other words, the n-ary modal operator $\Box$ of $DR!_n$ can be represented by the unary $\Box$ of D_1 or equivalently the so-called Standard Deontic Logic SDL.

10.3 Classes of frames for D_n, DR_n and $DR!_n$

The class of frames for a system is the class of frames that validates every theorem of the system. We show that the classes of $(n+1)$-ary relational frames for DR_n and $DR!_n$ are, respectively, the class of serial and semital frames, and the class of serial and strongly semital frames. Given that PL and [C_n] are valid, and [RM_n] and [RN_n] preserve validity in the general class of $(n+1)$-ary relational frames, it is sufficient to show that the classes of frames validating the remaining axioms of these systems, viz. [$P\Box_n$], [Re_n], and [$Re!_n$], are the classes of serial, semital, and strongly semital frames, respectively. In other words, we show that each of the axioms (or axiom schema) just mentioned is valid on an $(n+1)$-ary relational frame if and only if the frame is in the indicated class of frames.

Definition 10.3.1. Let $n \geq 1$. An $(n+1)$-ary relational frame $\mathfrak{F} = \langle U, R \rangle$ is said to be *serial* if R satisfies the following condition.

$$[\text{Seriality}_{n+1}] \quad (\forall x)(\exists y_1, \dots, y_n) R x y_1 \cdots y_n$$

$\mathfrak{F}$ is said to be *semital* if R satisfies the following condition for all i with $1 \leq i \leq n$.

$$[\text{Semita}^i_{n+1}] \quad (\forall x_0, x_1, \dots, x_n, y_1, \dots, y_n)(R x_0 x_1 \cdots x_n \wedge R x_i y_1 \cdots y_n \rightarrow R x_0 x_1 \cdots x_i y_1 \cdots y_{n-i})$$

$\mathfrak{F}$ is said to be *strongly semital* if R satisfies the following condition for all i with $1 \leq i \leq n$.

$$[\text{Semita!}^i_{n+1}] \quad (\forall x_0, x_1, \dots, x_n, y_1, \dots, y_n)(R x_0 x_1 \cdots x_n \wedge R x_i y_1 \cdots y_n \rightarrow R x_{i-1} x_i y_1 \cdots y_{n-1})$$

⊣

As we explained in Section 10.2.2, one way to read the conditions of semita and strong semita is to treat the $(n+1)$-ary relation R as consisting of paths, each of $(n+1)$ nodes. See also Figures 10.1 and 10.2.

Theorem 10.3.2. [P$\Box_n$] *corresponds to* [Seriality$_{n+1}$]*, i.e. for any* $(n+1)$*-ary relational* $\mathfrak{F} = \langle U, R\rangle$*,*

$$\mathfrak{F} \models [\mathrm{P}\Box_n] \iff \mathfrak{F} \models [\mathrm{Seriality}_{n+1}].$$

Proof. For $\Longrightarrow$, assume $\mathfrak{F}$ is not serial, i.e. there exists an x such that for all $y_1, \ldots, y_n$, we have $\neg Rxy_1, \ldots, y_n$. It follows directly from the truth condition for $\Box$ that $\Box(\bot, \ldots, \bot)$ is true at x in any model on $\mathfrak{F}$. Thus $\mathfrak{F} \not\models [\mathrm{P}\Box_n]$.

For $\Longleftarrow$, assume $\mathfrak{F}$ is serial. It is straightforward to see that for any x in any $\mathfrak{M}$ on $\mathfrak{F}$, we have $\mathfrak{M}, x \models \Diamond_n(\top, \ldots, \top)$, i.e. $\mathfrak{M}, x \models \neg\Box(\bot, \ldots, \bot)$. Hence $\mathfrak{F} \models [\mathrm{P}\Box_n]$. ⊣

Theorem 10.3.3. [Re^i_n] *corresponds to* [Semita$^i_{n+1}$] *(where* $1 \leq i \leq n$*), i.e. for every* $(n+1)$*-ary relational frame* $\mathfrak{F} = \langle U, R\rangle$*,*

$$\mathfrak{F} \models [\mathrm{Re}^i_n] \iff \mathfrak{F} \models [\mathrm{Semita}^i_{n+1}].$$

Proof. For $\Longrightarrow$, assume $\mathfrak{F}$ does not satisfy [Semita$^i_{n+1}$], i.e. there exist x_0, $x_1, \ldots, x_i, \ldots, x_n$, $y_1, \ldots, y_n$ such that $Rx_0x_1 \cdots x_i \cdots x_n$ and $Rx_iy_1 \cdots y_n$ but $\neg Rx_0x_1 \cdots x_iy_1 \cdots y_{n-i}$. Consider a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ where the valuation V satisfies the following conditions.

$$\begin{aligned}
V(p_1) &= U - \{x_1\}\\
V(p_2) &= U - \{x_2\}\\
&\vdots\\
V(p_i) &= U - \{x_i\}\\
V(p_{i+1}) &= U - \{y_1\}\\
&\vdots\\
V(p_n) &= U - \{y_{n-i}\}
\end{aligned}$$

It is not difficult to see that $\mathfrak{M}, x_0 \models \Box(p_1, p_2, \ldots, p_n)$. (Observe that if $Rx_0z_1z_2 \cdots z_n$ for some arbitrary $z_1, z_2, \ldots, z_n$, then at least one of the following identities does not hold: $z_1 = x_1$, $z_2 = x_2$, $\ldots$, $z_i = x_i$, $z_{i+1} = y_1$, $\ldots$, $z_n = y_{n-i}$ since $\neg Rx_0x_1 \cdots x_iy_1 \cdots y_{n-i}$.)

Furthermore, we have $\mathfrak{M}, x_1 \models \neg p_1$, $\mathfrak{M}, x_2 \models \neg p_2$, $\ldots$, $\mathfrak{M}, x_i \models \neg p_i$, and $\mathfrak{M}, x_i \models \Diamond_n(\neg p_{i+1}, \ldots, \neg p_n, \top, \ldots, \top)$ (note that $Rx_i y_1 \cdots y_n$). Since $Rx_0 x_1 \cdots x_n$, the following holds.

$$\mathfrak{M}, x_0 \not\models \Box(p_1, \ldots, p_{i-1}, p_i \vee \Box(p_{i+1}, \ldots, p_n, \bot, \ldots, \bot), \bot, \ldots, \bot)$$

Thus $\mathfrak{F} \not\models [\mathrm{Re}^i_n]$.

For $\Longleftarrow$, assume $\mathfrak{F}$ satisfies $[\mathrm{Semita}^i_n]$. It is straightforward to verify that for any point x in any $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ we have $\mathfrak{M}, x \models [\mathrm{Re}^i_n]$. ⊣

Theorem 10.3.4. $[\mathrm{Re!}^i_n]$ *corresponds to* $[\mathrm{Semita!}^i_{n+1}]$ *(where* $1 \leq i \leq n$*), i.e. for every* $(n+1)$*-ary relational frame* $\mathfrak{F} = \langle U, R \rangle$*,*

(1) $\mathfrak{F} \models [\mathrm{Re!}^1_n] \iff \mathfrak{F} \models [\mathrm{Semita!}^1_{n+1}]$*, and*

(2) $\mathfrak{F} \models [\mathrm{Re!}^j_n] \iff \mathfrak{F} \models [\mathrm{Semita!}^j_{n+1}]$*, where* $2 \leq j \leq n$*.*

Proof. Case (1). $[\mathrm{Re!}^1_n]$ is just $[\mathrm{Re}^1_n]$, and $[\mathrm{Semita!}^1_n]$ just $[\mathrm{Semita}^1_n]$. Thus case (1) follows directly from Theorem 10.3.3

Case (2). For $\Longrightarrow$, assume $\mathfrak{F}$ does not satisfy $[\mathrm{Semita!}^j_{n+1}]$ or equivalently there exist x_0, $x_1, \ldots, x_j, \ldots, x_n$, $y_1, \ldots, y_n$ such that both $Rx_0 x_1 \cdots x_j \cdots x_n$ and $Rx_j y_1 \cdots y_n$ but $\neg Rx_{j-1} x_j y_1 \cdots y_{n-1}$. Consider a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ satisfying the following conditions.

$$
\begin{aligned}
V(p_1) &= U - \{x_j\} \\
V(p_2) &= U - \{y_1\} \\
V(p_3) &= U - \{y_2\} \\
&\vdots \\
V(p_n) &= U - \{y_{n-1}\}
\end{aligned}
$$

Again it is not difficult to show that $\mathfrak{M}, x_{j-1} \models \Box(p_1, p_2, \ldots, p_n)$. (Observe that if we have $Rx_{j-1} z_1 z_2 \cdots z_n$ for some arbitrary $z_1, z_2, \ldots, z_n$, then at least one of the following does not hold: $z_1 = x_j$, $z_2 = y_1$, $z_3 = x_2$, $\ldots$, $z_n = y_{n-1}$, since $\neg Rx_{j-1} x_j y_1 \cdots y_{n-1}$.) Furthermore, $\mathfrak{M}, x_j \models \neg p_1$ and $\mathfrak{M}, x_j \models \Diamond_n(\neg p_2, \ldots, \neg p_n, \top)$ (note that $Rx_j y_1 \cdots y_n$). So $\mathfrak{M}, x_j \models \neg(p_1 \vee \Box(p_1, \ldots, p_n, \bot))$. Given that $Rx_0 x_1 \cdots x_j \cdots x_n$, we thus have $\mathfrak{F} \not\models [\mathrm{Re!}^j_n]$.

For $\Longleftarrow$, assume $\mathfrak{F}$ satisfies $[\mathrm{Re!}^j_n]$. It is straightforward to verify that for any point x in any $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ we have $\mathfrak{M}, x \models [\mathrm{Re!}^j_n]$. ⊣

Theorem 10.3.5. *The classes of* $(n+1)$*-ary relational frames for the following normal* n*-adic systems are as indicated:*

D_n : *Serial*
DR_n : *Serial and semital*
$\mathrm{DR!}_n$: *Serial and strongly semital*

Proof. The theorem follows directly from Theorems 10.3.2, 10.3.3, and 10.3.4. ⊣

10.4 Determination for D_n, DR_n and $\mathrm{DR!}_n$

Soundness of D_n, DR_n, and $\mathrm{DR!}_n$ with respect to their classes of frames follows immediately from Theorem 10.3.5. In the following, we demonstrate that they are also complete by showing that every D_n-consistent set of formulas is satisfiable on a serial $(n+1)$-ary relational frame, every DR_n-consistent set of formulas is satisfiable on a serial and semital $(n+1)$-ary relational frame, and every $\mathrm{DR!}_n$-consistent set of formulas is satisfiable on a serial and strongly semital $(n+1)$-ary relational frame. Given that these systems are normal systems, we make use of the result that the canonical model of any normal system is a model for every set of formulas consistent in that system. (Refer to Section 2.5.) What remains to be shown for completeness of our systems is thus the following: the canonical model of D_n is serial, that of DR_n is serial and semital, and that of $\mathrm{DR!}_n$ is serial and strongly semital. We show the above after describing the canonical model of a normal system.

We recall here that the canonical model of a normal n-adic system S, denoted $\mathfrak{M}_{\mathrm{S}}$, is the triple $\langle U_{\mathrm{S}}, R_{\mathrm{S}}, V_{\mathrm{S}}\rangle$ where:

- U_{S} is the set of all maximal S-consistent set of $\mathfrak{L}_n$-formulas.
- For every $x, y_1, \ldots, y_n \in U_{\mathrm{S}}$, $R_{\mathrm{S}}xy_1 \cdots y_n$ iff the following condition holds for any $\mathfrak{L}_n$-formulas $\alpha_1, \ldots, \alpha_n$:

$$\Box(\alpha_1, \ldots, \alpha_n) \in x \implies \exists i : \alpha_i \in y_i.$$

- For every propositional variable p_i and $x \in U_{\mathrm{S}}$, $x \in V_{\mathrm{S}}(p_i)$ iff $p_i \in x$.

In the ensuing proofs, we make use of, often silently, the following properties of canonical models and maximal consistent sets of formulas (for any normal n-adic system S):

- Every $\mathfrak{L}_n$-formula α is true at a point x in $\mathfrak{M}_{\mathrm{S}}$ if and only if α belongs to x.
- Every maximal S-consistent sets of formulas contains all the theorems of S and is closed under logical consequence.

Theorem 10.4.1. *Let* S *be a normal deontic n-adic system, and* $\mathfrak{M}_{\mathrm{S}} = \langle U_{\mathrm{S}}, R_{\mathrm{S}}, V_{\mathrm{S}}\rangle$ *its canonical model. Then* R_{S} *is serial.*

Proof. To show that R_{S} is serial, we consider an arbitrary x in U_{S}. Since $\Diamond_n(\top, \ldots, \top) \in x$, we have $\mathfrak{M}_{\mathrm{S}}, x \models \Diamond_n(\top, \ldots, \top)$. Then there exist $y_1, \ldots, y_n$ such that $R_{\mathrm{S}} x y_1 \cdots y_n$. In other words, R_{S} is serial. ⊣

Theorem 10.4.2. *Let* S *be an n-adic system of deontic residuation, and* $\mathfrak{M}_{\mathrm{S}} = \langle U_{\mathrm{S}}, R_{\mathrm{S}}, V_{\mathrm{S}}\rangle$ *its canonical model. Then* R_{S} *is both serial and semital.*

Proof. Given that S is also a normal deontic system, we know that R_{S} is serial from Theorem 10.4.1. To show that R_{S} is semital, i.e. it satisfies the condition [Semita$^i_{n+1}$] for all i from 1 to n, we assume $R_{\mathrm{S}} x_0 \cdots x_i \cdots x_n$ and $R_{\mathrm{S}} x_i y_1 \cdots y_n$ (for arbitrary $x_0, \ldots, x_i, \ldots, x_n$, $y_1, \ldots, y_n$, and $1 \le i \le n$), and show that $R_{\mathrm{S}} x_0 \cdots x_i y_1 \cdots y_{n-i}$. In other words, we show that if $\Box(\alpha_1, \ldots, \alpha_n) \in x_0$ (for arbitrary $\alpha_1, \ldots, \alpha_n$) then at least one of the following holds: $\alpha_1 \in x_1$, $\ldots$, $\alpha_i \in x_i$, $\alpha_{i+1} \in y_1$, $\ldots$, $\alpha_n \in y_{n-i}$. So assume $\Box(\alpha_1, \ldots, \alpha_n) \in x_0$. Since [Re$^i_n$] $\in x_0$, we have:

$$\Box(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \vee \Box(\alpha_{i+1}, \ldots, \alpha_n, \bot^i), \bot^{n-i}) \in x_0;$$
$$\mathfrak{M}_{\mathrm{S}}, x_0 \models \Box(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \vee \Box(\alpha_{i+1}, \ldots, \alpha_n, \bot^i), \bot^{n-i});$$
$$\alpha_1 \in x_1 \text{ or } \cdots \text{ or } \alpha_{i-1} \in x_{i-1} \text{ or } \alpha_i \vee \Box(\alpha_{i+1}, \ldots, \alpha_n, \bot^i) \in x_i.$$

So if $\alpha_1 \notin x_1$, $\ldots$, and $\alpha_{i-1} \notin x_{i-1}$, then $\alpha_i \vee \Box(\alpha_{i+1}, \ldots, \alpha_n, \bot^i) \in x_i$. If in addition $\alpha_i \notin x_i$, then

$$\Box(\alpha_{i+1}, \ldots, \alpha_n, \bot^i) \in x_i;$$
$$\mathfrak{M}_{\mathrm{S}}, x_i \models \Box(\alpha_{i+1}, \ldots, \alpha_n, \bot^i);$$
$$\alpha_{i+1} \in y_1 \text{ or } \cdots \text{ or } \alpha_n \in y_{n-i}$$

which is what we want. ⊣

Theorem 10.4.3. *Let* S *be an n-adic system of strong deontic residuation, and* $\mathfrak{M}_{\mathrm{S}} = \langle U_{\mathrm{S}}, R_{\mathrm{S}}, V_{\mathrm{S}}\rangle$ *its canonical model. Then* R_{S} *is both serial and strongly semital.*

Proof. Seriality follows from Theorem 10.4.1. For [Semita$!^i_n$] where $1 \leq i \leq n$, note that the case of $i = 1$ has already been shown in Theorem 10.4.2 since [Semita$!^1_n$] is the same as [Semita^1_n], and [Re$!^1_n$] the same as [Re^1_n]. It remains to show R_S satisfies [Semita$!^j_n$] where $2 \leq j \leq n$.

Assume $R_\mathrm{S}x_0x_1\cdots x_j\cdots x_n$ and $R_\mathrm{S}x_jy_1\cdots y_n$ (for arbitrary $x_0,\ldots,x_n$, $y_1,\ldots,y_n$ and $1 \leq j \leq n$), and show that $R_\mathrm{L}x_{j-1}x_jy_1\cdots y_{n-1}$ or equivalently if $\Box(\alpha_1,\ldots,\alpha_n) \in x_{j-1}$, then $\alpha_1 \in x_j$ or $\alpha_k \in y_{k-1}$ for some k such that $2 \leq k \leq n$. So assume $\Box(\alpha_1,\ldots,\alpha_n) \in x_{j-1}$. Since [Re$!^j_n$] is in x_0, we argue as follows:

$\Box(\bot^{j-2}, \neg\Box(\alpha_1,\ldots,\alpha_n), \alpha_1 \vee \Box(\alpha_2,\ldots,\alpha_n,\bot), \bot^{n-j}) \in x_0$;
$\mathfrak{M}_\mathrm{S}, x_0 \models \Box(\bot^{j-2}, \neg\Box(\alpha_1,\ldots,\alpha_n), \alpha_1 \vee \Box(\alpha_2,\ldots,\alpha_n,\bot), \bot^{n-j})$;
$\neg\Box(\alpha_1,\ldots,\alpha_n) \in x_{j-1}$ or $\alpha_1 \vee \Box(\alpha_2,\ldots,\alpha_n,\bot) \in x_j$;
$\alpha_1 \vee \Box(\alpha_2,\ldots,\alpha_n,\bot) \in x_j$;
$\mathfrak{M}_\mathrm{S}, x_j \models \alpha_1 \vee \Box(\alpha_2,\ldots,\alpha_n,\bot)$;
$\mathfrak{M}_\mathrm{S}, x_j \models \alpha_1$ or $\mathfrak{M}_\mathrm{S}, x_j \models \Box(\alpha_2,\ldots,\alpha_n,\bot)$;
$\mathfrak{M}_\mathrm{S}, x_j \models \alpha_1$ or $\mathfrak{M}_\mathrm{S}, y_1 \models \alpha_2$ or $\cdots$ or $\mathfrak{M}_\mathrm{S}, y_{n-1} \models \alpha_n$;
$\alpha_1 \in x_j$ or $\alpha_2 \in y_1$ or $\cdots$ or $\alpha_n \in y_{n-1}$

which is what we want. ⊣

Theorem 10.4.4. *The following normal n-adic systems are both sound and complete with respect to the indicated classes of $(n+1)$-ary relational frames:*

D_n : *Serial*
DR_n : *Serial and semital*
$\mathrm{DR!}_n$: *Serial and strongly semital*

Proof. Soundness follows directly from Theorem 10.3.5. Completeness follows from Theorems 10.4.1, 10.4.2, and 10.4.3. ⊣

10.5 Embedding of $\mathrm{DR!}_n$ in D_1

A system S in a language $\mathfrak{L}$ is said to be embeddable in another system S′ in another language $\mathfrak{L}'$ if there is a translation ^{t} from $\mathfrak{L}$ to $\mathfrak{L}'$ such that for every $\mathfrak{L}$-formula α,

$$\vdash_\mathrm{S} \alpha \iff \vdash_{\mathrm{S}'} \alpha^t.$$

In this section, we show that $\mathrm{DR!}_n$ can be embedded in D_1 (or SDL) under the following translation.

Definition 10.5.1. The translation * maps formulas of the n-adic modal language $\mathfrak{L}_n$ to formulas of the monadic modal language $\mathfrak{L}_1$ according to the following condition for $\Box$:

$$\Box(\alpha_1, \ldots, \alpha_n)^* = \Box({\alpha_1}^* \vee \Box({\alpha_2}^* \vee \cdots \vee \Box({\alpha_{n-1}}^* \vee \Box{\alpha_n}^*) \cdots))$$

while propositional variables and truth-functional connectives are preserved under the translation. ⊣

(The notion of embedding defined here is weaker than the notion of translational equivalence defined in Section 8.1. So the embedding of $\mathrm{DR!}_n$ in D_1 can be derived as a corollary of Theorem 8.4.6. Nonetheless we include this section in order to provide a direct proof of the embedding result.)

We already know that D_1 is determined by the class of serial binary relational frames and $\mathrm{DR!}_n$ by the class of serial and strongly semital $(n+1)$-ary relational frames. Thus, showing that $\mathrm{DR!}_n$ is embedded in D_1 under * is equivalent to showing that every $\mathfrak{L}_n$-formula α is valid in the class of serial and strongly semital $(n+1)$-ary frames if and only if its translation α^* is valid in the class of serial binary frames. That this is the case follows from the next two theorems, which show that a serial binary frame can be simulated by a serial and strongly semital $(n+1)$-ary frame, and vice versa.

Theorem 10.5.2. *Every serial binary relational model $\mathfrak{M} = \langle U, R, V \rangle$ is pointwise equivalent to a serial and strongly semital $(n+1)$-ary relational model $\mathfrak{M}' = \langle U, R', V \rangle$ with respect to the translation *, i.e. for every $\mathfrak{L}_n$-formula α and every x in U,*

$$\mathfrak{M}, x \models \alpha^* \iff \mathfrak{M}', x \models \alpha.$$

Proof. We define R' according to the following condition: for every x_0, $x_1, \ldots, x_n$ in U,

$$R' x_0 x_1 \cdots x_n \iff x_0 R x_1 \cdots x_{n-1} R x_n$$

where $x_0 R x_1 \cdots x_{n-1} R x_n$ stands for "$R x_0 x_1$, …, and $R x_{n-1} x_n$." The proof is by induction on the formation of α. The cases for atomic formulas and truth-functional connectives are trivial, and are omitted here. For the modal case, we consider a sub-formula of α in the form of $\Box_n(\beta_1, \ldots, \beta_n)$ and argue first that if its translation is true at an arbitrary point x_0 in $\mathfrak{M}$ then it is true at the same point in $\mathfrak{M}'$. Details are as follows:

$$\begin{aligned}
&\mathfrak{M}, x_0 \models \Box(\beta_1{}^* \vee \Box(\beta_2{}^* \cdots \vee \Box(\beta_{n-1}{}^* \vee \Box\beta_n{}^*) \cdots));\\
&\forall x_1, \ldots, x_n,\ x_0Rx_1 \cdots x_{n-1}Rx_n \Longrightarrow \mathfrak{M}, x_1 \models \beta_1{}^* \text{ or } \cdots \text{ or } \mathfrak{M}, x_n \models \beta_n{}^*;\\
&\forall x_1, \ldots, x_n,\ R'x_0x_1 \cdots x_n \Longrightarrow \mathfrak{M}', x_1 \models \beta_1 \text{ or } \cdots \text{ or } \mathfrak{M}', x_n \models \beta_n;\\
&\mathfrak{M}', x_0 \models \Box_n(\beta_1, \ldots, \beta_n).
\end{aligned}$$

Moreover the above steps can be reversed; so we have proved the modal case. ⊣

Theorem 10.5.3. *Every serial and strongly semital $(n+1)$-ary relational model $\mathfrak{M} = \langle U, R, V\rangle$ is pointwise equivalent to a binary relational model $\mathfrak{M}' = \langle U, R', V\rangle$ with respect to the translation *, i.e. for every $\mathfrak{L}_n$-formula α and every x in U,*

$$\mathfrak{M}, x \models \alpha \iff \mathfrak{M}', x \models \alpha^*.$$

Proof. We define R' as follows: for any x_0, x_1 in U,

$$R'x_0x_1 \iff \exists x_2, \ldots, x_n : Rx_0x_1x_2 \cdots x_n.$$

The proof is by induction on the formation of α. We omit the cases of atomic formulas and truth-functional connectives. For the modal case, we consider a subformula of α in the form of $\Box_n(\beta_1, \ldots, \beta_n)$, and argue first that if it is true at an arbitrary point x_0 in $\mathfrak{M}$ then its translation is also true at the same point in $\mathfrak{M}'$ (and vice versa). Details are as follows:

$$\begin{aligned}
&\mathfrak{M}, x_0 \models \Box_n(\beta_1, \ldots, \beta_n);\\
&\forall x_1, \ldots, x_n,\ Rx_0x_1 \cdots x_n \Longrightarrow \exists i(1 \leq i \leq n) : \mathfrak{M}, x_i \models \beta_i;\\
&\forall x_1, \ldots, x_n,\ x_0R'x_1 \cdots x_{n-1}R'x_n \Longrightarrow \exists i(1 \leq i \leq n) : \mathfrak{M}', x_i \models \beta_i{}^*;\\
&\mathfrak{M}', x_0 \models \Box(\beta_1{}^* \vee \Box(\beta_2{}^* \cdots \vee \Box(\beta_{n-1}{}^* \vee \Box\beta_n{}^*) \cdots)).
\end{aligned}$$

Moreover the above steps can be reversed. For the reasoning from line two to line three above (and the reverse direction), observe that given that R is serial and strongly semital, we have

$$Rx_0x_1 \cdots x_n \iff x_0R'x_1 \cdots x_{n-1}R'x_n.$$

For the left-to-right direction, assume that $Rx_0 \cdots x_{i-1}x_i \cdots x_n$. Since R is serial, we have $Rx_iy_1 \cdots y_n$ (for some $y_1, \ldots, y_n$). But R is strongly semital. Thus $Rx_{i-1}x_iy_1 \cdots y_{n-1}$, from which it follows that $R'x_{i-1}x_i$. For the right-to-left direction, we start from $R'x_{n-1}x_n$. By the definition of R', $Rx_{n-1}x_ny_1\cdots y_{n-1}$ for some $y_1, \ldots, y_{n-1}$, and $Rx_{n-2}x_{n-1}z_1\cdots z_{n-1}$ for some $z_1, \ldots, z_{n-1}$. But R is strongly semital. Thus $Rx_{n-2}x_{n-1}x_ny_1\cdots y_{n-2}$. By repeating the same argument, we eventually get $Rx_0x_1\cdots x_n$. $\dashv$

Theorem 10.5.4. *The n-adic modal system* $\mathrm{DR!}_n$ *is embedded in the monadic modal system* D_1 *under the translation* *. *In other words, for every $\mathcal{L}_n$-formulas,*

$$\vdash_{\mathrm{DR!}_n} \alpha \iff \vdash_{\mathrm{D}_1} \alpha^*.$$

Proof. For $\Longrightarrow$, assume $\nvdash_{\mathrm{D}_1} \alpha^*$. Then α^* does not hold a serial binary relational model. Then, by Theorem 10.5.2, α does not hold in a serial and strongly semital $(n+1)$-ary relational model. In other words, $\nvdash_{\mathrm{DR!}_n}$. Argument for the $\Longleftarrow$ direction is similar but makes use of Theorem 10.5.3. $\dashv$

10.6 The logic of deontic residuation: an interpretation

10.6.1 Principal and residual obligations

In the introduction, we remarked that different sanctions might attend different omissions of obligation, and analyzing deontic necessity in terms of a single sanction simply ignores this subtlety of our deontic discourse. Let us consider the following scenario. Suppose you ought to help your neighbour, because of a previous promise, for example. Unexpected circumstances might prevent you from fulfilling your obligation. In that case, you incur an obligation to apologize to your neighbour. But should you fail to do that either, you ought to avoid your neighbour lest you might embarrass yourself. We say that you have a principal obligation to help you neighbour, and, defaulting on that, you incur a residual obligation to apologize to your neighbour. The residuation of "previous" obligations could go on for some further steps, as our example above illustrated.

In our systems of deontic residuation, we formalize the residuation of deontic necessity by a series of polyadic modal operators $\Box$'s. The formula $\Box(\alpha, \beta)$ means that default on α makes β obligatory, or, in our terminology, an obligation of α residuates into an obligation of β. We also call α the principal, and β the residuum, of the residuating obligation $\Box(\alpha, \beta)$. In the general case of $\Box(\alpha_1, \alpha_2, \ldots, \alpha_n)$, the sequence $\langle \alpha_2, \ldots, \alpha_n \rangle$ is said to the residua of the principal α_1, with α_{i+1} $(1 \leq i < n)$ being called the i-th residuum. In the rest of this section, we limit our consideration to the dyadic system of deontic residuation for simplicity. See Example 10.2.5 for the axioms and rules of DR_2. An interpretation of them is offered below.

- [RM_2]: Obligations, principal or residual, are closed under logical consequence.
- [RN_2]: Any logical truth is an absolute obligation, principal or residual. (An obligation is said to be absolute if, in the case of a principal obligation, every formula is a residue of it, or, in the case of a residual obligation, it is a residue of every formula. Note that given the rule [RM_2], the above amounts to saying that a principal obligation of, say α, is absolute if $\Box(\alpha, \bot)$ holds, and a residual obligation of, say β, is absolute if $\Box(\bot, \beta)$ holds.) The sense of an absolute obligation is that one cannot shirk it. Since an obligation of logical truth is trivially fulfilled, the rule [RN_2] is pretty harmless, even for those who dislike the idea of logic being able to impose obligation by itself.
- [C_2]: If two principal obligations shares the same residue, they aggregate. If two residual obligations come from the same principal, they aggregate.
- [$P\Box_2$]: The axiom excludes the unwelcome situation of having the false both as a principal obligation and as a residue (i.e. the situation in which one ought to do the logically impossible, or failing that, which is bound to happen, one is still obligated to do it). Put it another way, the false is not a persistent obligation. (An obligation of α is said to be persistent if $\Box(\alpha, \alpha)$ holds, i.e. if it has itself as a residue.)
- [Re_2]: A principal obligation of α with a residuum of β implies an absolute principal obligation of $\alpha \vee \Box(\beta, \bot)$. Whereas all the rules and axioms discussed so far, viz. [RM_2], [RN_2], [C_2], and [$P\Box_2$], deal with obligations in the same place of $\Box$, the principle of residuation [Re_2] shows how an obligation in the second place gives rise to an obligation in the first place. Facing a residuating obligation with α as the principal and β as the residuum, one is obligated to make a choice (which is unavoidable) between realizing α and bringing upon oneself an obligation of realizing β. Note that the second obligation is "unshirkable", reflecting the limitation of deontic residuation to one residuum in the case of the dyadic $\Box$.

10.6.2 Representation of normative conflicts

Two obligations (say the obligation to realize α and the obligation to realize β) are said to be in conflict if $\{\alpha,\ \beta\}$ is logically inconsistent, or in the formal representation, the false can be derived from $\{\alpha,\ \beta\}$. Plausibly, there are genuine cases of conflicting obligations, as for example, when one ought to help one's neighbour and ought not to help him. A more subtle (and more usual) form of deontic conflict, however, involves impracticability rather than logical inconsistency, as when one inadvertently commits to helping one's neighbour and to visiting one's parents but it is practically impossible to do both. We can represent such impracticabilities by augmenting our system of deontic residuation (DR_2) with domain-specific axioms, for instance, the axiom that if you help your neighbour, then you do not visit your parents:

$$HelpNeighbour \rightarrow \neg VisitParents.$$

In such an augmented system, which we might call DR_2^+, the duties of being an attentive neighbour and being a dutiful offspring become logically inconsistent.

We note that two conflicting *absolute* principal obligations yield logical inconsistency in our system DR_2^+. For if $HelpNeighbour \rightarrow \neg VisitParents$ is part of our augmented system, then we can derive from $\Box(HelpNeighbour, \bot)$ and $\Box(VisitParents, \bot)$, by applying $[C_2]$ and $[RM_2]$, the conclusion that $\Box(\bot, \bot)$, which contradicts $[P\Box_2]$. This intolerance of conflicts among absolute principal obligations mirrors similar intolerance of conflicting obligations in the monadic DR_1, which is of course the same system as D_1 or SDL.

By contrast, DR_2 allows non-absolute principal obligations to conflict. To continue the example above, suppose that the only residue of the principal obligation of helping the neighbour is to apologize to him, and the only residue of the principal obligation of visiting one's parents is to call them. In other words, we have the following:

$$\Box(HelpNeighbour, ApologizeToNeighbour);$$
$$\Box(VisitParents, CallParents).$$

Even though $HelpNeighbour$ and $VisitParents$ are inconsistent in our augmented system DR_2^+, we cannot derive $\Box(\bot, \bot)$ from the above two obligations with residues. Instead we derive, by using $[C_2]$ and $[RM_2]$, the formula $\Box(\bot, ApologizeToNeighbour \vee CallParents)$.

More interestingly, if we also use [Re_2], we arrive at the following absolute obligation:

$$\Box\Big(\big((HelpNeighbour \wedge \Box(CallParents, \bot)\big) \vee \\ \big(VisitParents \wedge \Box(ApologizeToNeighbour, \bot)\big) \vee \\ \big(\Box(ApologizeToNeighbour, \bot) \wedge \Box(CallParents, \bot)\big)\big), \bot\Big).$$

The above obligation gives all the normative situations one possibly faces: (a) one helps the neighbour but incurs an absolute obligation to call one's parents; (b) one visits one's parents but incurs an absolute obligation to apologize to the neighbour; (c) one does neither and incurs an absolute obligation to apologize to the neighbour and another to call one's parents.

Appendix A

Algebraic Systems and Boolean Algebras

In this appendix, we present background material in the area of algebraic logic connected with our study of polyadic modal logic. We start with some universal algebraic notions, then move to Boolean algebras, which is then extended to modal algebras (or Boolean algebras with operators). Algebraic logic is a big topic in both algebra and logic. We mention here some of the references consulted when preparing this appendix. For an introduction to universal algebra, see Burris and Sankappanavar (1981), Denecke and L. (2002). An algebraic account of propositional logic can be found in Bell and Slomson (1971). Goldblatt (2000) provides a useful survey on the application of algebraic ideas to modal logic but a more detailed treatment is given in modern textbooks on modal logic such as Chagrov and Zakharyaschev (1997) and Blackburn et al. (2001).

A.1 Algebras

An algebra $\mathfrak{A}$ consists of a non-empty set A of objects $a, b, c, \ldots$ together with a collection of finitary operations $\phi, \psi, \chi \ldots$ on A (these operations, which may be infinitely many, are called the basic operations of $\mathfrak{A}$). By an n-ary operation ϕ on A, we means an n-ary function from the n-th Cartesian power of A to A (i.e. $\phi : A^n \to A$). The number n is called the rank or arity of ϕ. Note that we require n be a finite number. In this dissertation we use ordinals to index the operations of an algebra. Thus the collection of operations of $\mathfrak{A}$ comprises $\phi_0, \phi_1, \ldots, \phi_\xi, \ldots$ where $\xi < \zeta$ for some ordinal ζ. This setup facilitates comparison of algebras, which may have different symbols to denote their operations. Two algebras can now be said to be similar (or belong to the same type) if there is a one-to-one correspondence between their collections of operations and corresponding operations have the same rank. We give below formal definitions of abstract types and algebras. (The types are called abstract because they are independent of the symbols used to denote the operations of particular types of algebras).

Definition A.1.1 (Abstract type)**.** An *abstract type* (or simply a *type*) is a pair $\tau = \langle \zeta, \rho \rangle$ where ζ is an ordinal, and ρ (called the rank function of τ) maps ζ into ω, i.e. for any ordinal $\xi < \zeta$, $\rho(\xi)$ is a natural number n (called the rank or arity of ξ). ⊣

Definition A.1.2 (Algebras)**.** Let $\tau = \langle \zeta, \rho \rangle$ be a type. An *algebra* $\mathfrak{A}$ of type τ is a pair $\langle A, O \rangle$ where A is a non-empty set called the carrier of the algebra, and O is a collection of operations $\phi_0, \phi_1, \ldots, \phi_\xi, \ldots$ such that $\xi < \zeta$ and each ϕ_ξ is a $\rho(\xi)$-ary operation on A. (We often denote $\mathfrak{A}$ by $\langle A, \phi_\xi \rangle_{\xi<\zeta}$ in order to display the operations of $\mathfrak{A}$.) ⊣

Our definition of an algebra allows for nullary operations, the output of each of which is a fixed member of the algebra. As a matter of convention, we call a nullary operation by the name of the object it picks out from the carrier of the algebra. Two algebras $\mathfrak{A} = \langle A, \phi_\xi \rangle_{\xi<\zeta}$ and $\mathfrak{B} = \langle B, \psi_\xi \rangle_{\xi<\eta}$ are said to be *similar* if they are of the same type, i.e. $\zeta = \eta$, and for every $\xi < \zeta$ the rank of ϕ_ξ is the same as that of ψ_ξ.

When a type $\tau = \langle k, \rho \rangle$ is finite, i.e. when k is a finite ordinal, we represent τ by the sequence $\rho(0), \rho(1), \ldots, \rho(k-1)$. An algebra $\mathfrak{A}$ of finite type $\tau = \langle k, \rho \rangle$ can thus be written as a tuple $\langle A, \phi_0, \phi_1, \ldots, \phi_{k-1} \rangle$, and it is said to be of type $\rho(0), \rho(1), \ldots, \rho(k-1)$. Whereas in our definition of algebras the operations are called ϕ_ξ's, particular types of algebras studied in mathematics often denote their operations by specific symbols, as the following examples illustrate.

Example A.1.3 (Groups and Abelian groups)**.** A *group* is an algebra $\langle G, \cdot, ^{-1}, 1 \rangle$ of type $2, 1, 0$ satisfying the following axioms.

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \tag{G1}$$

$$a \cdot 1 = 1 \cdot a = a \tag{G2}$$

$$a \cdot a^{-1} = a^{-1} \cdot a = 1 \tag{G3}$$

G is an *Abelian* group (or *commutative group*) if it satisfies, in addition to (G1), (G2) and (G3), the following axiom.

$$a \cdot b = b \cdot a \tag{G4}$$

Note that when the binary operation on G is denoted by $\cdot$, the group is said to be written multiplicatively, in which case the unary operation is denoted by $^{-1}$ (with a^{-1} called the inverse of a), and the nullary operation is denoted by 1 (called the identity element of the group). Sometimes a group is presented additively: its binary operation is denoted by $+$. In such a case, the inverse of a is usually denoted by $-a$ (also called the negative of a), and the identity element is denoted by 0 (also called the zero element). In other words, a group, when presented additively, is an algebra $\langle G, +, -, 0\rangle$ of type 2, 1, 0 satisfying the following conditions (G1'), (G2') and (G3'). An Abelian group, when presented additively, satisfies (G4') as well.

$$a + (b + c) = (a + b) + c \tag{G1'}$$

$$a + 0 = 0 + a \quad = a \tag{G2'}$$

$$a + (-a) = (-a) + a \quad = 0 \tag{G3'}$$

$$a + b = b + a \tag{G4'}$$

Example A.1.4 (Semigroups and monoids)**.** A *semigroup* is an algebra $\langle S, \cdot\rangle$ of type 2 satisfying (G1). A *monoid* is an algebra $\langle M, \cdot, 1\rangle$ of type $2, 0$ satisfying (G1) and (G2). ⊣

Example A.1.5 (Rings)**.** A *ring* is an algebra $\langle R, +, \cdot, -, 0\rangle$ of type $2, 2, 1, 0$ satisfying the following.

$$\langle R, +, -, 0\rangle \text{ is an Abelian group.} \tag{R1}$$

$$\langle R, \cdot\rangle \text{ is a semigroup.} \tag{R2}$$

$$\begin{aligned} x \cdot (y + z) &= (x \cdot y) + (x \cdot z) \\ (x + y) \cdot z &= (x \cdot z) + (y \cdot z) \end{aligned} \tag{R3}$$

A *ring with identity* is an algebra $\langle R, +, \cdot, -, 0, 1\rangle$ of type $2, 2, 1, 0, 0$ such that (R1), (R2), (R3) and (G2) hold. In other words, $\langle R, +, \cdot, -, 0, 1\rangle$ is a ring with identity iff $\langle R, +, \cdot, -, 0\rangle$ is a ring and $\langle R, \cdot, 1\rangle$ is a monoid. ⊣

A.2 Operations on algebras

In the following, we describes several ways of constructing new algebras from given ones, viz, the constructions of homomorphic images, subalgebras and product algebras, which lead us to the important notion of a variety of algebras. Note that the new algebras are of the same type as the original ones.

A.2.1 Homomorphisms

Definition A.2.1 (Homomorphisms)**.** Let $\mathfrak{A} = \langle A, \phi_\xi \rangle_{\xi<\zeta}$ and $\mathfrak{B} = \langle B, \psi_\xi \rangle_{\xi<\eta}$ be two algebras of the same type. A map $f : A \to B$ is a *homomorphism* if for any ϕ_ξ and $a_1, \ldots, a_n \in A$ (where n is the rank of ϕ_ξ),

$$f(\phi_\xi(a_1, \ldots, a_n)) = \psi_\xi(f(a_1), \ldots, f(a_n)). \qquad \dashv$$

For any two similar algebras $\mathfrak{A}$ and $\mathfrak{B}$, we say that $\mathfrak{B}$ is a *homomorphic image* of $\mathfrak{A}$ if there is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. Given a class $\mathbb{C}$ of algebras, $\mathsf{H}\mathbb{C}$ denotes the class of homomorphic images of the algebras in $\mathbb{C}$.

A bijective homomorphism, or equivalently a homomorphism whose inverse is also a homomorphism, is called an *isomorphism*. If there is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* (or they are *isomorphic images* of each other) and write $\mathfrak{A} \cong \mathfrak{B}$. In general we do not distinguish between isomorphic algebras.

A.2.2 Subalgebras

Definition A.2.2 (Subalgebras)**.** Let $\mathfrak{A} = \langle A, \phi_\xi \rangle_{\xi<\zeta}$ be an algebra, and B a subset of A closed under every operation ϕ_ξ. Then we call $\mathfrak{B} = \langle B, \psi_\xi \rangle_{\xi<\zeta}$ a *subalgebras* of $\mathfrak{A}$ if ψ_ξ is $\phi_\xi \restriction B$, i.e. the restriction of ϕ_ξ to B. $\dashv$

An algebra $\mathfrak{B}$ is said to be *embeddable* in another algebra $\mathfrak{A}$ if $\mathfrak{B}$ is isomorphic to a subalgebra of $\mathfrak{A}$, and the isomorphism is called an *embedding*. $\mathsf{S}\mathbb{C}$ denotes the class of isomorphic images of subalgebras of algebras in class $\mathbb{C}$.

A.2.3 Products of algebras

Definition A.2.3 (Products of algebras)**.** Let I be an index set, and $\{\mathfrak{A}_i\}_{i\in I}$ a collection of algebras of type $\tau = \langle \zeta, \rho \rangle$. In other words, for each $i \in I$, $\mathfrak{A}_i$ is $\langle A_i, \phi^i_\xi \rangle_{\xi<\zeta}$ and the rank of

ϕ^i_ξ is $\rho(\xi)$. The *product* of these algebras, denoted $\prod_{i\in I}\mathfrak{A}_i$, is the algebra $\langle\prod_{i\in I}A_i, \phi_\xi\rangle_{\xi<\zeta}$, where $\prod_{i\in I}A_i$ the Cartesian product of the carriers of A_i's, and the operation ϕ_ξ of rank $n=\rho(\xi)$ is defined as follows:

$$\phi_\xi(\langle a^i_1\rangle_{i\in I}, \ldots, \langle a^i_n\rangle_{i\in I}) = \langle\phi^i_\xi(a^i_1, \ldots, a^i_n)\rangle_{i\in I},$$

where for any $i \in I$, $a^i_1, \ldots, a^i_n \in A_i$. If $\mathfrak{A}_i$'s are the same algebra $\mathfrak{A}$, we write $\mathfrak{A}^I$ and call it a *power* of $\mathfrak{A}$.

Given a class $\mathbb{C}$ of algebras, $\mathsf{P}\mathbb{C}$ is the class of isomorphic copies of products of algebras in $\mathbb{C}$.

A.2.4 Varieties

A class $\mathbb{C}$ of algebras is called a *variety* if it is closed under taking homomorphic images, subalgebras, and products, i.e. $\mathsf{H}\mathbb{C} \subseteq \mathbb{C}$, $\mathsf{S}\mathbb{C} \subseteq \mathbb{C}$, and $\mathsf{P}\mathbb{C} \subseteq \mathbb{C}$. Where $\mathbb{C}$ is a class of algebras, $\mathsf{V}\mathbb{C}$ denotes the smallest variety including $\mathbb{C}$. We also say that it is the variety generated by $\mathbb{C}$.

Theorem A.2.4. *Let $\mathbb{C}$ be a class of algebras. Then we have $\mathsf{V}\mathbb{C} = \mathsf{HSP}\mathbb{C}$.*

An import of the above theorem is that we can obtain the variety generated by $\mathbb{C}$, by first taking products of algebras in $\mathbb{C}$, then taking subalgebras of $\mathsf{P}\mathbb{C}$, and finally forming homomorphic images of $\mathsf{SP}\mathbb{C}$. Further applications of these operations will not produce any new algebras.

A.3 Equational classes

A.3.1 Algebraic languages

In order to develop a metatheory of a class of similar algebras, say of type $\tau = \langle\zeta, \rho\rangle$, we require a formal language that is suitable for the purpose. First of all, the language should have a set S of operation symbols corresponding to the operations of the algebras. In other words, S consists of a sequence of operation symbols $s_0, s_1, \ldots, s_\xi, \ldots$, with $\xi < \zeta$, and the rank of s_ξ is $\rho(\xi)$. Besides the operation symbols, we need a set X of variables in order to talk about elements of the algebras. We call this language the algebraic language of type

τ over X, and denote it by $\mathfrak{L}_\tau(X)$. The terms of $\mathfrak{L}_\tau(X)$ is defined by the following rule in BNF:

$$t ::= x | s_\xi(\underbrace{t, \ldots, t}_{\rho(\xi)\text{times}}),$$

where x ranges over the set X of variables, and s_ξ over the set S of operation symbols. An equation is a pair of terms $\langle t_1, t_2 \rangle$, often written as $t_1 \approx t_2$.

A.3.2 Valuation and satisfaction

An algebra $\mathfrak{A} = \langle A, \phi_\xi \rangle_{\xi<\zeta}$ of type $\tau = \langle \zeta, \rho \rangle$ can be considered as an interpretation of the algebraic language $\mathfrak{L}_\tau(X)$ — the operation ϕ_ξ is simply the denotation of the operation symbol s_ξ. Each term of the language is assigned an element of the carrier A according to a valuation or assignment function V satisfying the following conditions:

- $V(x) \in A$, for every variable x;
- $V(s_\xi(t_1, \ldots, t_n)) = \phi_\xi(V(t_1), \ldots, V(t_n))$, for every operation symbol s_ξ of rank n, and terms $t_1, \ldots, t_n$.

Note that we could have defined V as a function mapping X to A, and extend it to another function V^+ that covers every term of the language. But for simplicity, we define V in such a way that it already covers all the terms of the language, including its variables.

An $\mathfrak{A}$ is said to satisfy an equation $t_1 \approx t_2$ if $V(t_1) = V(t_2)$ for every V on $\mathfrak{A}$. A class $\mathbb{C}$ of algebras is said to satisfy $t_1 \approx t_2$ if every algebra of $\mathbb{C}$ satisfies it. When $\mathfrak{A}$ (or $\mathbb{C}$) satisfies $t_1 \approx t_2$, we also say that the equation is true in $\mathfrak{A}$ (or $\mathbb{C}$) or holds in $\mathfrak{A}$ (or $\mathbb{C}$). More formally, we have the following definitions.

- $\mathfrak{A} \models t_1 \approx t_2$ if for all V on $\mathfrak{A}$, $V(t_1) = V(t_2)$; otherwise $\mathfrak{A} \not\models t_1 \approx t_2$.
- $\mathbb{C} \models t_1 \approx t_2$ if for all $\mathfrak{A} \in \mathbb{C}$, $\mathfrak{A} \models t_1 \approx t_2$; otherwise $\mathbb{C} \not\models t_1 \approx t_2$.

A.3.3 Equational classes

The notion of satisfaction for an equation can be extended to a set E of equations.

- $\mathfrak{A} \models E$ if for any equation $t_1 \approx t_2$ in E, $\mathfrak{A} \models t_1 \approx t_2$; otherwise, $\mathfrak{A} \not\models E$.

- $\mathbb{C} \models E$ if for any equation $t_1 \approx t_2$ in E, $\mathbb{C} \models t_1 \approx t_2$; otherwise, $\mathbb{C} \not\models E$.

A class $\mathbb{C}$ of algebras is said to be defined by a set E of equations if for every algebra $\mathfrak{A}$, $\mathfrak{A} \in \mathbb{C}$ iff $\mathbb{C} \models E$. We say that $\mathbb{C}$ is *equationally definable* or an *equational class* if it is defined by a class of equations. The following theorem due to Birkhoff provides an important link between the structure theoretic and the equational approaches in studying universal algebra.

Theorem A.3.1 (Birkhoff). *A class $\mathbb{C}$ of algebras is an equational class iff it is a variety.*

A.4 Algebraic semantics for propositional languages

Recall that a propositional language over a set P of variables contains connectives $\vee$, $\neg$, and $\bot$, which are binary, unary, and nullary connectives respectively. Such a language can be considered as an algebraic language of type $\langle 2, 1, 0\rangle$, and any algebra of the same type could be used to interpret it. However, we want the algebraic operations to follow certain rules that reflect the meaning we intend for the connectives. The equational class of Boolean algebras are commonly held to be suitable for interpreting propositional languages.

A.4.1 Boolean algebras

Definition A.4.1 (Boolean algebras). A *Boolean algebra* (BA) is a $2, 1, 0$-type algebra $\mathfrak{A} = \langle A, +, -, 0\rangle$ whose operations, called respectively joint, complementation, and the zero element, satisfy the following conditions for any $a, b, c \in A$.

$a + b = b + a$	$a \cdot b = b \cdot a$	(Commutative laws)
$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$	(Associative laws)
$a + 0 = a$	$a \cdot 1 = a$	(Identity laws)
$a + (-a) = 1$	$a \cdot (-a) = 0$	(Complement laws)
$a + (b \cdot c) = (a + b) \cdot (a + c)$	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	(Distributive laws)

Observe that we have used the following shorthands in stating the above conditions:

- $a \cdot b$, called the meet of a and b, abbreviates $-a + -b$.

- 1, called the unit element, abbreviates -0.

Finally, the class of all Boolean algebras is denoted by $\mathbb{BA}$. ⊣

A Boolean algebra $\mathfrak{A} = \langle A, +, -, 0 \rangle$ can be considered as a complemented distributive lattice $\langle A, \leq \rangle$ where $a \leq b$ iff $a + b = b$ or equivalently $a \cdot b = a$. Conversely a complemented distributive lattice $\langle A, \leq \rangle$ can be considered as a Boolean algebra $\mathfrak{A} = \langle A, +, -, 0 \rangle$ where $a + b$ is the supremum of $\{a, b\}$, $-a$ is the lattice complement of a, and 0 is the minimum of lattice.

An element a of a Boolean algebra is said to be less than, or below, another element b ($a \leq b$ in symbol) if $a \vee b = b$ or equivalently $a \wedge b = a$. Indeed, a Boolean algebra $\mathfrak{A} = \langle A, +, -, 0 \rangle$ can be considered as a complemented distributive lattice $\langle A, \leq \rangle$.

Given a set $\{a_i\}_{i \in I}$ of (possibly infinitely many) elements of a Boolean algebra, we define the joint and meet of the set as follows,

$$\sum_{i \in I} a_i = inf(\{a_i\}_{i \in I}) \qquad \prod_{i \in I} a_i = sup(\{a_i\}_{i \in I})$$

where for any set B of elements of $\mathfrak{A}$, $inf(B)$ is the infimum or the greatest lower bound of B in $\mathfrak{A}$, and $sup(B)$ is the supremum or the least upper bound of B in $\mathfrak{A}$. If $\{a_i\}_{i \in I}$ is finite, i.e. it consists of $a_1, \ldots, a_i, \ldots, a_n$ for some natural number n, its joint can be written as $a_1 + \cdots + a_i + \cdots + a_n$ and its meet as $a_1 \cdot \cdots \cdot a_i \cdot \cdots \cdot a_n$.

In the following we define filters and ultrafilters in Boolean algebras and list important theorems of them which we will require later.

Definition A.4.2 (Filters and ultrafilters)**.** Let $\mathfrak{A} = \langle A, +, -, 0 \rangle$ be a Boolean algebra, and a, b elements of the algebra.

(1) A subset F of A is a *filter* in $\mathfrak{A}$ if

  (i) it is non-empty;

  (ii) it is closed under taking meets, i.e. if $a, b \in F$, then $a \cdot b \in F$;

  (iii) it is upward closed, i.e. if $a \in F$ and $a \leq b$, then $b \in F$.

(2) A filter F in $\mathfrak{A}$ is *proper* if $F \neq A$.

(3) An *ultrafilter* is a proper filter which has no proper extensions which are also proper filters. The collection of all ultrafilters in $\mathfrak{A}$ is denoted by $\mathrm{Uf}(\mathfrak{A})$.

Theorem A.4.3. *The following hold for any Boolean algebra* $\mathfrak{A} = \langle A, +, -, 0 \rangle$.

(1) *A filter F in $\mathfrak{A}$ is an ultrafilter iff for each $a \in A$, either $a \in F$ or $-a \in F$ but not both.*

(2) *(The ultrafilter theorem) Every proper filter in $\mathfrak{A}$ can be extended to an ultrafilter.*

(3) *Each set of elements of $\mathfrak{A}$ having the finite intersection property can be extended to an ultrafilter. (A set B of elements of $\mathfrak{A}$ is said to have the finite intersection property if the meet of every finite subset of B is not identical with 0.)*

(4) *Each non-zero element of $\mathfrak{A}$ is contained in some ultrafilters.*

(5) *If a and b are distinct element of $\mathfrak{A}$, then there is an ultrafilter containing one but not the other.*

A.4.2 Interpretation of propositional languages In Boolean algebras

We remarked earlier that a propositional language $\mathcal{L}(P)$ (where P is a set of propositional variable), considered as an algebraic language of type $\langle 2, 1, 0\rangle$, can be interpreted in a Boolean algebra $\mathfrak{A} = \langle A, +, -, 0\rangle$. More specifically, the connectives $\vee$, $\neg$, and $\bot$ are read as respectively the binary operation $+$ (joint), the unary operation $-$ (complementation), and the nullary operation 0 (the zero element — recall that a nullary operation is named after its output). The defined connective $\wedge$ thus corresponds to the defined operation $\cdot$ (meet), and the constant $\top$ to 1 (the unit element). A propositional formula is treated as a term. So, given a valuation V on $\mathfrak{A}$, a formula denotes an element of the algebra according to the following rules:

- $V(p) \in A$, for every $p \in P$;
- $V(\phi \vee \psi) = V(\phi) + V(\psi)$;
- $V(\neg\phi) = -V(\phi)$;
- $V(\bot) = 0$.

We say that a formula ϕ is valid in $\mathfrak{A}$ if the equation $\phi \approx \top$ is valid in $\mathfrak{A}$, i.e. $\mathfrak{A} \models \phi \approx \top$ or, using more English, for every valuation V on $\mathfrak{A}$, we have $V(\phi)$ identical with 1. This is a rather sloppy piece of terminology since we define validity in an algebra for *equations* rather than for *terms* (which are here formulas). Continuing the same sloppiness, we say that ϕ is valid in a class $\mathbb{C}$ of algebras if the equation $\phi \approx \top$ is valid in every algebras of $\mathbb{C}$, i.e. $\mathbb{C} \models \phi \approx \top$.

A.4.3 Boolean algebras and propositional models

Example A.4.4 (The algebra of truth values)**.** *The algebra of truth values* is the tuple $\mathbf{2} = \langle 2, +, -, 0 \rangle$, where for any $a, b \in 2 = \{0, 1\}$,

$$a + b = \max(\{a, b\});$$
$$-a = 1 - a.$$

Example A.4.5 (Power set algebras and set algebras)**.** Given a set X, the *power set algebra* of X is the tuple $\mathfrak{P}(X) = \langle \wp(X), \cup, -, \emptyset \rangle$, where $\wp(X)$ is the collection of all subsets of X, $\cup$ is set union, $-$ is set complementation, and $\emptyset$ is the empty set. A subalgebra of a power set algebra is called a *set algebra*. The class of all set algebras is denoted by $\mathbb{SET}$.

Theorem A.4.6. *Every power set algebra is isomorphic to a power of the algebra of truth values* $\mathbf{2}$*, and vice versa. Hence every set algebra is isomorphic to a subalgebra of a power of* $\mathbf{2}$*, and vice versa.*

Proof. Given a power set algebra $\mathfrak{P}(X) = \langle \wp(X), \cup, -, \emptyset \rangle$, we show that it is isomorphic to $\mathbf{2}^X = \langle 2^X +, -, 0 \rangle$. Consider a map $f : \wp(X) \to 2^X$ defined as follows: for any $Y \subseteq X$, $f(Y) = \langle a_x \rangle_{x \in X}$ with $a_x = 1$ if $x \in Y$ and $a_x = 0$ otherwise. It is not difficult to check that f is a homomorphism, i.e. for any $X_1, X_2 \subseteq X$,

$$f(X_1 \cup X_2) = f(X_1) + f(X_2);$$
$$f(-X_1) = -f(X_1).$$

Moreover f is a bijective. Thus it is an isomorphism.

Conversely, given a power $\mathbf{2}^I = \langle 2^I +, -, 0 \rangle$ of the algebra of truth values, we show that it is isomorphic to the power set algebra $\mathfrak{P}(I) = \langle \wp(I), \cup, -, \emptyset \rangle$. Consider a map $\theta : 2^I \to \wp(I)$ defined by letting $\theta(\langle a_i \rangle_{i \in I})$ be the set of i's such that $a_i = 1$. Again it is straightforward to show the following:

$$\theta(\langle a_i \rangle_{i \in I} + \langle b_i \rangle_{i \in I}) = \theta(\langle a_i \rangle_{i \in I}) \cup \theta(\langle b_i \rangle_{i \in I});$$
$$\theta(-\langle a_i \rangle_{i \in I}) = -\theta(\langle a_i \rangle_{i \in I}).$$

Thus θ is a homomorphism. Since it is bijective, θ is an isomorphism. $\dashv$

Theorem A.4.7 (Stone's representation theorem)**.** *Every Boolean algebra is isomorphic to a set algebra, hence to a subalgebra of a power of* $\mathfrak{2}$*.*

Theorem A.4.8. *Validity of propositional formulas in the class of all propositional models is equivalent to that in the following algebra and classes of algebras:*

- *the algebra of truth values* $\mathfrak{2}$*;*
- *the class* $\mathbb{SET}$ *of all set algebras;*
- *the class* $\mathbb{BA}$ *of all Boolean algebras.*

A.5 Algebraic semantics for modal languages

A.5.1 Modal algebras

In the previous section, we use Boolean algebras to interpret propositional languages. Recall that modal languages are extensions of propositional languages with modal operators. Hence it is natural to supplement Boolean algebras with additional operations, which provide interpretations for modal operators. We call these structures "modal algebras".

Definition A.5.1 (Modal algebras)**.** Let $\tau = \langle \zeta, \rho \rangle$ be a modal type. A *modal algebra* (MA) of type τ is an algebra $\mathfrak{A} = \langle A, +, -, 0, l \rangle m \xi \zeta$ where $\langle A, +, -, 0 \rangle$ is a Boolean algebra and each operation m_ξ maps $A^{\rho(\xi)}$ into A. ⊣

For each operation m_ξ we define another operation l_ξ (called its dual) as follows (where $n = \rho(\xi)$).

$$l_\xi(a_1, \dots, a_n) = -m_\xi(-a_1, \dots, -a_n)$$

In our definition of modal algebras, the operations m_ξ's are completely general. Classes of modal algebras can be specified by imposing conditions on these operations. In the following, we define an important class which has been commonly called "Boolean algebras with operators". They are modal algebras that satisfy the conditions of normality and additivity (see below for details). A more precise description of them is "normal and additive modal algebras", but we follow the tradition of calling them Boolean algebras with operators or BAO in short. Note that some authors have called them "normal modal algebras" or simply "modal algebras".

Definition A.5.2 (Boolean algebras with operators)**.** Let $\tau = \langle \zeta, \rho \rangle$ be a modal type. A *Boolean algebra with operators* (BAO) of type τ is a modal algebra $\mathfrak{A} = \langle A, +, -, 0, m_\xi \rangle_{\xi<\zeta}$ where every m_ξ satisfies the following for any $i \leq n = \rho(\xi)$ and any $a_1, \ldots, a_i, \ldots, a_n$ and a'_i in A.

$$m_\xi(a_1, \ldots, 0, \ldots, a_n) = 0 \quad \text{(Normality)}$$

$$m_\xi(a_1, \ldots, a_i + a'_i, \ldots, a_n) = m_\xi(a_1, \ldots, a_i, \ldots, a_n) + m_\xi(a_1, \ldots, a'_i, \ldots, a_n) \quad \text{(Additivity)}$$

$\dashv$

Given our earlier definition of l_ξ, it is easy to check that the following identities hold for any BAO.

$$l_\xi(a_1, \ldots, 1, \ldots, a_n) = 1 \quad \text{(Normality)}$$

$$l_\xi(a_1, \ldots, a_i \cdot a'_i, \ldots, a_n) = l_\xi(a_1, \ldots, a_i, \ldots, a_n) \cdot l_\xi(a_1, \ldots, a'_i, \ldots, a_n) \quad \text{(Multiplicativity)}$$

A.5.2 Interpretation of modal languages in modal algebras

A modal language of type $\tau = \langle \zeta, \rho \rangle$ has, in addition to the truth-functional connectives $\wedge$, $\neg$, and $\bot$, a series of modal connectives $\Box_\xi$'s (where $1 \leq \xi < \zeta$) whose arities are determined by the function ρ. It is easy to see that a modal algebra of type $\langle \zeta, \rho \rangle$ can be used to interpret such a modal language: the denotation of the modal connective $\Box_\xi$ is simply the algebraic operation l_ξ, and that of $\Diamond_\xi$ is m_ξ. Thus given an assignment v on a modal algebra $\mathfrak{A} = \langle A, +, -, 0, l \rangle m\xi\zeta$, a formula of the modal language $\mathfrak{L}_\tau(P)$ is assigned a member of the algebra. The rules for the truth-functional connectives are as before and that for the modal connectives are as follows:

- $V(\Box_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)}) = l_\xi(V(\alpha_1), \ldots, V(\alpha_{\rho(\xi)}))$.
- $V(\Diamond_\xi(\alpha_1, \ldots, \alpha_{\rho(\xi)}) = m_\xi(V(\alpha_1), \ldots, V(\alpha_{\rho(\xi)}))$.

Validity of $\mathfrak{L}_\tau(P)$-formulas in modal algebras and in classes of modal algebras are as in the case of propositional formulas and Boolean algebras. Given that Boolean algebras with operators are modal algebras, the above applies to them as well.

A.5.3 Boolean algebras with operators and relational models

Definition A.5.3 (Full complex algebras and complex algebras)**.** Let $\mathfrak{F} = \langle U, R_\xi \rangle_{\xi<\alpha}$ be a relational frame of type $\tau = \langle \alpha, \rho \rangle$. The *full complex algebra* of $\mathfrak{F}$, denoted $\mathfrak{F}^\sharp$, is the algebra $\langle \wp(U), \cup, -, \emptyset, m_\xi \rangle_{\xi<\alpha}$ where $\langle \wp(U), \cup, -, \emptyset \rangle$ is the powerset algebra of U, and for every $X_1, \ldots, X_n \subseteq U$ with $n = \rho(\xi)$, $x \in m_\xi(X_1, \ldots, X_n)$ iff there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $R_\xi x x_1 \cdots x_n$. A *complex algebra* is a subalgebra of a full complex algebra. Where $\mathbb{C}$ is a class of relational frames, we denote the class of the full complex algebras of frames in $\mathbb{C}$ by $\mathbb{C}^\sharp$.

Proposition A.5.4. *Let $\mathfrak{F} = \langle U, R_\xi \rangle_{\xi<\alpha}$ be a relational frame of type $\tau = \langle \alpha, \rho \rangle$. Its full complex algebra $\mathfrak{F}^\sharp = \langle \mathfrak{P}(U), m_\xi \rangle_{\xi<\alpha}$ is a BAO.*

Proposition A.5.5 (Johnsson-Tarski theorem)**.** *Every BAO is isomorphic to a complex algebra.*

Theorem A.5.6. *Let $\mathfrak{L}_\tau(P)$ be a modal language. Validity of $\mathfrak{L}_\tau(P)$-formulas in the class $\mathbb{C}$ of all relational frames is equivalent to that in the following classes of algebras:*

- *the class $\mathbb{C}^\sharp$ of full complex algebras of frames in $\mathbb{C}$;*
- *the class $\mathbb{BAO}$ of all Boolean algebras with operators.*

A.6 Lindenbaum-Tarski algebras

Propositional language $\mathfrak{L}(P)$ can be treated as an algebraic language of the similarity type $\langle 2, 1, 0 \rangle$. We call the corresponding term algebra *formula algebra*.

Definition A.6.1 (Formula algebras)**.** Let P be a set of propositional variables, and $Form_{\mathfrak{L}}(P)$ the set of $\mathfrak{L}$-formulas over P. The *formula algebra* of $\mathfrak{L}$ over P is the tuple $\mathfrak{Form}_{\mathfrak{L}}(P) = \langle Form_{\mathfrak{L}}(P), +, -, \bot \rangle$ where for any formulas α, β in $Form_{\mathfrak{L}}(P)$,

$$\alpha + \beta = \alpha \vee \beta;$$
$$-\alpha = \neg\alpha. \qquad \dashv$$

Note that formula algebras are not Boolean algebras. Nonetheless, there is a class of Boolean algebras based on formula algebras, and they play an important role in the study of algebraic completeness of propositional logic PL.

Definition A.6.2 (Lindenbaum-Tarski algebra)**.** Let P be a set of propositional variables, and $Form(P)/\equiv_{\text{PL}}$ be the set of equivalence classes that $\equiv_{\text{PL}}$ induces on the set of formulas and $[\alpha]$ be the equivalence class containing α. Then the *Lindenbaum-Tarski algebra* for this language is the structure $\mathfrak{A}_{\text{PL}}(P) = \langle Form(P)/\equiv_{\text{PL}}, +, -, 0\rangle$, where $+$, $-$, and 0 are defined as follows:

$$\begin{aligned}
[\alpha] + [\beta] &= [\alpha \vee \beta];\\
-[\alpha] &= [\neg\alpha];\\
0 &= [\bot].
\end{aligned}$$

⊣

Appendix B

Basic Category Theory

In this appendix we review the basic notions of category theory used in Chapter 6. For a comprehensive treatment of the subject, seeMac Lane (1998). The following books provide accessible approach to category theory: Adámek et al. (1990), Bell (1988) and Goldblatt (1979).

B.1 Categories

Definition B.1.1 (Categories)**.** A *category* C is a tuple $\langle \mathrm{Obj}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \circ, \mathrm{id} \rangle$ where Obj is a class of objects (a, b, c, etc.), Arr is a class of arrows (f, g, h, etc. also called morphisms), and dom, cod, $\circ$ and id are the following operations.

(1) Each arrow f is assigned a pair of objects $\mathrm{dom}\, f$ and $\mathrm{cod}\, f$ (called its domain and codomain). If $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$, we say that f is an arrow from a to b, and write $f : a \to b$.

(2) Each pair of arrows f and g for which $\mathrm{cod}\, f = \mathrm{dom}\, g$ (such arrows are said to be composable) is assigned an arrow $g \circ f$ from $\mathrm{dom}\, f$ to $\mathrm{cod}\, g$, called the composition or composite of f and g.

(3) Each object a is assigned an arrow $\mathrm{id}_a : a \to a$ called the identity arrow on a.

In addition, the operations of composition and identity satisfy the following laws:

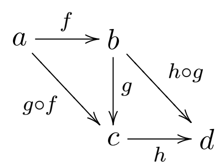
- (Associative law) For any arrows $f : a \to b$, $g : b \to c$ and $h : c \to d$, we have

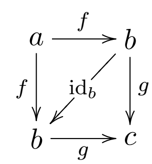
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (Identity law) For any arrows $f : a \to b$ and $g : b \to c$, we have

$$\mathrm{id}_b \circ f = f; \quad g \circ \mathrm{id}_b = g \qquad \dashv$$

The associative law and the identity law can be represented by the following two commutative diagrams.





Note that the identity arrow for each object b is uniquely determined by the identity law: for if $h : b \to b$ satisfies the identity law, i.e. if the law still holds after replacing id_b with h, then we have $h \circ \mathrm{id}_b = \mathrm{id}_b$ (from substituting h for id_b and id_b for f in the first part of the identity law) and $h \circ \mathrm{id}_b = h$ (from substituting h for g in the second part of the identity law), whence we conclude h is id_b. In other words, there is a bijective mapping between objects and identity arrows.

In our definition of category, we do not require the class of objects and the class of arrows to be sets. They can be proper classes, collections that are too "large" to be sets. A category is said to be *small* if its class of arrows is a set; otherwise it is called *large*. Observe that if a category is small, then its class of objects, like its class of arrows, is also a set since there is a one-one correspondence between objects and identity arrows. In the case of the large categories, i.e. those categories whose arrows do not form sets, it is still possible that the class of arrows from an object a to another object b—denoted by $\hom_{\mathsf{C}}(a, b)$ or simply $\hom(a, b)$, and called the *hom-class* of $\langle a, b\rangle$—is a set. If such is the case, we say that the category is *locally small*, and call $\hom(a, b)$ the *hom-set* of $\langle a, b\rangle$. Note that category C is small iff it is locally small and its class of objects is a set. Right-to-left follows from our previous remark, whereas left-to-right follows from the fact that the union $\bigcup \hom(a, b)$ where a and b range over Obj is a set given that Obj and all $\hom(a, b)$'s are sets. (Some authors incorporate the "local smallness" condition into the definition of categories, and under their definition a category is said to be small if its class of objects is a set.)

Examples of categories are given below. Quite often the objects are sets (structured or not) and the arrows are functions or maps between sets. For these types of categories, the domain and codomain of arrows are simply the domain and codomain of functions;

composition of arrows and identity arrows are the usual composition of functions and identity functions. So we omit them in our examples whenever no ambiguity would arise.

Example B.1.2. The category Set has the class of all sets as its class of objects and the class of all functions between sets as its class of arrows. ⊣

Although Set is a large category, it is locally small since the collection of functions between two sets X and Y (denoted by Y^X) is a set. So are the following two examples whose arrows are structure-preserving maps between structured sets.

Example B.1.3. The category $\mathsf{Alg}(\tau)$ (where τ is an algebraic type) consists of algebras of type τ as objects and homomorphisms between these algebras as arrows. ⊣

Example B.1.4. The category $\mathsf{Rel}(\tau)$ (where τ is a modal type) consists of relational structures of type τ as objects and homomorphisms between these structures as arrows. ⊣

Finally we give an example of categories whose objects, unlike the previous ones, are not always sets.

Example B.1.5. A pre-ordered class $\langle X, \leq\rangle$ is a category whose class of objects comprises members of X, and class of arrows comprises pairs $\langle a, b\rangle$ whenever $a \leq b$. The domain and codomain of $\langle a, b\rangle$ are a and b, respectively. The composite of $\langle a, b\rangle$ and $\langle b, c\rangle$ is $\langle a, c\rangle$, and the identity arrow on a is $\langle a, a\rangle$. (Note that the composite arrow and the identity arrow defined above exist since $\leq$ is transitive and reflexive. Moreover there is exactly one arrow from each object to itself and at most one arrow from one object to another.) ⊣

Definition B.1.6 (Isomorphisms between objects)**.** An arrow $f : a \to b$ is an *isomorphism* from a to b (written $f : a \cong b$) if there exists an arrow $g : b \to a$ such that

$$g \circ f = \mathrm{id}_a; \quad f \circ g = \mathrm{id}_b\,.$$

Such an arrow g is called an *inverse* of f. Object a is *isomorphic* to object b ($a \cong b$ in symbol) if there is an isomorphism from a to b. ⊣

It is easy to check the following.

- Each arrow f has at most one inverse g (for if g and g' are inverses of f, then $g = g \circ \mathrm{id}_b = g \circ (f \circ g') = (g \circ f) \circ g' = \mathrm{id}_a \circ g' = g'$). Since an inverse g of f, if it exists, is unique, we say that g is "the" inverse of f, and denote it by f^{-1}.

- For each object a, $a \cong a$ (since id_a is an isomorphism).
- If $a \cong b$, then $b \cong a$ (since if $f : a \cong b$, then $f^{-1} : b \cong a$).
- If $a \cong b$ and $b \cong c$, then $a \cong c$ (since if $f : a \cong b$ and $g : b \cong c$, then $g \circ f : a \cong c$).
- Given the last three points, the relation "is isomorphic to" is an equivalence relation on the class of objects. If a is isomorphic to b, we simply call them isomorphic, and treat them as "essentially" the same object.

B.2 Functors

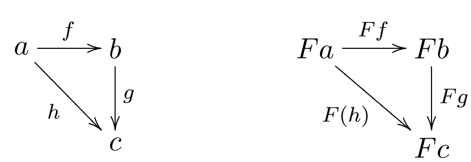
Categories themselves can be considered as objects with arrows definable between them. The arrows (between categories) we are going to define in this section are called functors, which, like the arrows of a category, are composable in such a way that the associative law and the identity law (for functors) hold for them. Isomorphic categories are defined similarly as isomorphic objects of a category.

Definition B.2.1 (Functors)**.** A *functor* F from category C to category D ($F : \mathsf{C} \to \mathsf{D}$ in symbol) is a function that assigns to each C-object a a D-object Fa and to each C-arrow $f : a \to b$ a D-arrow $Ff : Fa \to Fb$ subject to the following conditions.

(1) $F(g \circ f) = Fg \circ Ff$, whenever composition is defined;

(2) $F\,\mathrm{id}_a = \mathrm{id}_{Fa}$, for any C-object a. ⊣

Condition (1) of the above definition of functors can be restated as below: If the first diagram commutes in category C, then second diagram commutes in category D.



It is clear from the definition of functors that a functor $F : \mathsf{C} \to \mathsf{D}$ preserves the operations dom and cod since $F(\mathrm{dom}\, f) = \mathrm{dom}(Ff)$ and $F(\mathrm{cod}\, f) = \mathrm{cod}(Ff)$. Obviously F also preserves the operations of composition and identity since this is exactly what conditions (1) and (2) say. It is in this sense that F is said to provide a picture of C in D.

Functors may be composed in a way which is associative, and identity functors defined to act as identities for the composition.

Definition B.2.2 (Composition of functors)**.** For any functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$, their *composite* $G \circ F : \mathsf{C} \to \mathsf{E}$ is the function defined by letting

$$G \circ F(a) = G(Fa); \quad G \circ F(f) = G(Ff)$$

for any C-object a and C-arrow f. ⊣

Definition B.2.3 (Identity functors)**.** For any category C, the *identity functor* Id_{C} is the function that maps each object to itself and each arrow to itself. ⊣

Proposition B.2.4. *Composite functors and identity functors satisfy the following conditions.*

- *(Associative law) Whenever composition is defined,*

$$H \circ (G \circ F) = (H \circ G) \circ F.$$

- *(Identity law) For any functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$,*

$$\mathrm{Id}_{\mathsf{D}} \circ F = F; \quad G \circ \mathrm{Id}_{\mathsf{D}} = G.$$

The following are examples of functors. It can easily be checked that they preserve composition and identity as required in our definition of functors.

Example B.2.5. For any category C consisting of structured sets as objects and structure-preserving maps as arrows, we can define a forgetful functor (or underlying functor) $U : \mathsf{C} \to \mathsf{Set}$ which assigns to each structured set its underlying set and to each structure-preserving map the map itself. For instance we have the forgetful functors $U : \mathsf{Alg}(\tau) \to \mathsf{Set}$ and $U : \mathsf{Rel}(\tau) \to \mathsf{Set}$. ⊣

Example B.2.6. An order-preserving map f from a pre-ordered class P_1 to another pre-ordered class P_2 is a functor from P_1 to P_2, each of which is considered as a category. ⊣

Definition B.2.7 (Isomorphisms between categories)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is an *isomorphism* from C to D ($F : \mathsf{C} \cong \mathsf{D}$ in symbol) if there exists a functor $G : \mathsf{D} \to \mathsf{C}$ such that

$$G \circ F = \mathrm{Id}_{\mathsf{C}}; \quad F \circ G = \mathrm{Id}_{\mathsf{D}}.$$

Such a functor G is called an *inverse* of F. Category C is *isomorphic* to category D ($\mathsf{C} \cong \mathsf{D}$ in symbol) if there exists an isomorphism from C to D. ⊣

As in the case for isomorphism between objects, an inverse of functor F if exists is unique. We call it the inverse of F, and denote it by F^{-1}. The relation "is isomorphic to" is reflexive, symmetric and transitive. So it is an equivalence relation on the collection of all categories. Isomorphic categories are treated as essentially the same entity. However in the next section we shall consider a weaker but more useful notion of sameness between categories. In the following, we define some properties pertaining to functors and provide another characterization of isomorphism.

A functor $F : \mathsf{C} \to \mathsf{D}$ is called *injective* on arrows if F maps C-arrows one-one to D-arrows, and *surjective* if F maps C-arrows onto D-arrows. Similarly for objects. Given a functor $F : \mathsf{C} \to \mathsf{D}$ and a pair of C-objects $\langle a, b \rangle$, the term "hom-class restriction" means the restriction of the domain and codomain of F to $\hom_{\mathsf{C}}(a, b)$ and $\hom_{\mathsf{D}}(Fa, Fb)$, respectively.

Definition B.2.8. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor.

(1) F is *full* if all hom-class restrictions are surjective.

(2) F is *faithful* if all hom-class restrictions are injective.

(3) F is an *embedding* if it is injective on arrows, or equivalently if it is faithful, and injective on objects.

(4) F is *dense* if for any D-object b, there is a C-object a such that $Fa \cong b$. ⊣

Note that if F is an embedding then F is faithful. However the converse does not always hold. Consider a faithful $F : \mathsf{C} \to \mathsf{D}$ which maps two distinct C-objects a and a' to the same D-object (i.e. $a \neq a'$ and $Fa = Fa'$). Then F is not injective on arrows since $\mathrm{id}_a \neq \mathrm{id}_{a'}$ but $F\,\mathrm{id}_a = \mathrm{id}_{Fa} = \mathrm{id}_{Fa'} = F\,\mathrm{id}_{a'}$.

Proposition B.2.9. *A functor $F : \mathsf{C} \to \mathsf{D}$ is an isomorphism iff it is bijective on both objects and arrows, or equivalently iff it is full, faithful, and bijective on objects.*

B.3 Natural transformations

In the previous section, we mentioned that a functor from category C to category D can be thought of as providing a picture of C inside D. We are interested in knowing when two such functors F and G are "essentially" the same. With this aim in mind, we define arrows between these functors (treated as objects) so that a notion of isomorphism between functors becomes definable. The guiding ideas are that an arrow from F to G is a transformation of the F-picture of C inside D into the G-picture of C inside D, and F is isomorphic to G if the F-picture and the G-picture of C look the same inside D

Definition B.3.1 (Natural transformations)**.** Let F and G be functors from C to D. A *natural transformation* η from F to G (denoted by $\eta : F \to G$) is a function that assigns to each C-object a a D-arrow $\eta_a : Fa \to Ga$ in such a way that for every C-arrow $f : a \to b$,

$$Gf \circ \eta_a = \eta_b \circ Ff.$$

The arrows η_a, η_b, etc. are called the *components* of η. ⊣

The above "naturality" condition can be re-stated as the condition that the following diagram commutes.

$$\begin{array}{ccccc} a & \qquad & Fa & \xrightarrow{\eta_a} & Ga \\ {\scriptstyle f}\downarrow & & {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ b & & Fb & \xrightarrow[\eta_b]{} & Gb \end{array}$$

Intuitively, $\eta : F \to G$ uses the arrows of D to turn the F-picture of C inside D into a G-picture. Next we make precise the idea that the F-picture and the G-picture of C look the same inside D.

Definition B.3.2 (Natural isomorphisms)**.** Let F and G be functors from C to D. A natural transformation $\eta : F \to G$ is called a *natural isomorphism* from F to G (written $\eta : F \cong G$) if every component of it is an isomorphism, i.e. if $\eta_a : Fa \cong Ga$ for every C-object a. (Recall that $\eta_a : Fa \cong Ga$ iff there exists an arrow $\theta : Ga \to Fa$ such that $\theta \circ \eta_a = \mathrm{id}_{Fa}$ and $\eta_a \circ \theta = \mathrm{id}_{Ga}$.) F is said to be *naturally isomorphic* to G (written $F \cong G$) if there is a natural isomorphism from F to G. ⊣

Observe that the relation "is naturally isomorphic to" between functors from category C to category D is reflexive, symmetric and transitive. Hence it is an equivalence relation on the collection of all functors from C to D.

B.4 Equivalence of categories

Isomorphism between categories C and D requires the existence of functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that they are inverse of each other (i.e. $G \circ F = \mathrm{Id}_{\mathsf{C}}$ and $F \circ G = \mathrm{Id}_{\mathsf{D}}$). As mentioned earlier, this notion of isomorphism between categories is too strong—there are categories which we consider as being the same but fail to be isomorphic. In this section, we define the notion of equivalence (between categories) which is weaker than the notion of isomorphism. The idea is that for C and D to be equivalent, we require the existence of functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F$ is naturally isomorphic to Id_{C}, and $F \circ G$ is naturally isomorphic to Id_{D} (instead of requiring the composite functors to be identical to the respective identity functors). Naturally isomorphism, as we have already seen, is weaker than identity.

Definition B.4.1 (Equivalences between categories)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence* from C to D (written $F : \mathsf{C} \equiv \mathsf{D}$) if there exists a functor $G : \mathsf{D} \to \mathsf{C}$ such that

$$G \circ F \cong \mathrm{Id}_{\mathsf{C}}; \quad F \circ G \cong \mathrm{Id}_{\mathsf{D}} .$$

A category C is *equivalent* to another category D (written $\mathsf{C} \equiv \mathsf{D}$) if there exists an equivalence from C to D. ⊣

The relation "is equivalent to" between categories is reflexive, symmetric and transitive. So it is an equivalence relation on the collection of all categories.

The notion of equivalence defined here is weaker than the notion of isomorphism (between categories): if $F : \mathsf{C} \cong \mathsf{D}$, then $F : \mathsf{C} \equiv \mathsf{D}$ since both $F^{-1} \circ F = \mathrm{Id}_{\mathsf{C}}$ and $F \circ F^{-1} = \mathrm{Id}_{\mathsf{D}}$ (by isomorphism between categories), and both $\mathrm{Id}_{\mathsf{C}} \cong \mathrm{Id}_{\mathsf{C}}$ and $\mathrm{Id}_{\mathsf{D}} \cong \mathrm{Id}_{\mathsf{D}}$ (by reflexivity of natural isomorphisms between functors).

The following proposition provides another definition of equivalence, which is analogous to the alternative characterization given to the notion of isomorphism between categories. Note the stronger condition of bijection on objects for isomorphism is replaced by the weaker condition of denseness for equivalence.

Proposition B.4.2. *A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence from C to D if it is full, faithful and dense.*

B.5 Contravariance and opposites

Definition B.5.1 (Opposite or dual categories)**.** Let $\mathsf{C} = \langle \mathrm{Obj}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \circ, \mathrm{id} \rangle$ be a category. The *opposite* or *dual category* of C is the tuple $\mathsf{C}^{op} = \langle \mathrm{Obj}, \mathrm{Arr}, \mathrm{dom}^{op}, \mathrm{cod}^{op}, \circ^{op}, \mathrm{id} \rangle$ where

(1) $\mathrm{dom}^{op} f = \mathrm{cod}\, f$ and $\mathrm{cod}^{op} f = \mathrm{dom}\, f$.

(2) $f \circ^{op} g = g \circ f$. ⊣

Note that C^{op} has the same objects, arrows, and identities as C. The difference between them is that the "direction" of arrows is reversed. The tuple C^{op} is indeed a category since its operations of composition and identity observe the associative law and the identity law.

- For any C^{op}-arrows $f : a \to b$, $g : b \to c$ and $h : c \to d$, we have C-arrows $f : b \to a$, $g : c \to b$ and $h : d \to c$, and

$$h \circ^{op} (g \circ^{op} f) = (f \circ g) \circ h = f \circ (g \circ h) = (h \circ^{op} g) \circ^{op} f.$$

- For any C^{op}-arrows $f : a \to b$ and $g : b \to c$, we have C-arrows $f : b \to a$ and $g : c \to b$, and

$$\mathrm{id}_b \circ^{op} f = f \circ \mathrm{id}_b = f; \quad g \circ^{op} \mathrm{id}_b = \mathrm{id}_b \circ g = g.$$

Functors as defined previously are sometimes called covariant functors, in order to distinguish them from contravariant functors, which reverse the direction of corresponding arrows and hence the order of composition.

Definition B.5.2 (Contravariant functors)**.** A *contravariant functor* F from category C to category D is a function that assigns to each C-object a a D-object Fa and to each C-arrow $f : a \to b$ a D-arrow $Ff : Fb \to Fa$ subject to the following conditions.

(1) $F(g \circ f) = Ff \circ Fg$, whenever composition is defined;

(2) $F\,\mathrm{id}_a = \mathrm{id}_{Fa}$, for any C-object a. ⊣

A contravariant functor F from C to D can be represented as a (covariant) functor $F : \mathsf{C}^{op} \to \mathsf{D}$ since

$$F(f \circ^{op} g) = F(g \circ f) = Ff \circ Fg$$

whenever composition is defined. Equivalently F can be represented as a (covariant) functor $F : \mathsf{C} \to \mathsf{D}^{op}$ since

$$F(g \circ f) = Ff \circ Fg = Fg \circ^{op} Ff.$$

Definition B.5.3 (Dually equivalent categories)**.** Categories C and D are *dually equivalent* if $\mathsf{C}^{op} \equiv \mathsf{D}$ or equivalently $\mathsf{C} \equiv \mathsf{D}^{op}$. ⊣

The following proposition provides another characterization of dually equivalent categories which are sometimes more convenient for proving equivalence than the above definition.

Proposition B.5.4. *Categories* C *and* D *are dually equivalent iff there exist contravariant functors* F *from* C *to* D *and* G *from* D *to* C *such that*

$$G \circ F \cong \mathrm{Id}_{\mathsf{C}}; \quad F \circ G \cong \mathrm{Id}_{\mathsf{D}}\,.$$

Appendix C

Contemporary Deontic Logics

C.1 van Fraassen's deontic logics

Reference: van Fraassen (1972, 1973).

C.1.1 The axiological thesis

- X is what ought to be done because its being so would be good. (What ought to be is what is best or better than its alternatives.)
- A model $\mathfrak{M}$ is a tuple $\langle U, \mathcal{V}, >, f\rangle$ where
  - U is a set of alternative possibilities we are evaluating.
  - $\mathcal{V}$ is a set of values.
  - $>_x$ (for every $x \in U$) is a binary relation on $\mathcal{V}$ such that $>$ is asymmetric, transitive, and connected on its field (i.e., a linear ordering of values for x).
  - f_x (for every $x \in U$) is a function assigning to each $y \in U$ a set of values from the field of $>_x$ (i.e., $f_x(y) \subseteq \text{fld}(>_x)$).
- Evaluation of monadic ought statements:

$$\mathfrak{M}, x \models \Box\alpha \iff \exists y \in \|\alpha\|^{\mathfrak{M}} : \exists v \in f_x(y) : \forall z \in \|\neg\alpha\|^{\mathfrak{M}}, \forall w \in f_x(z), v >_x w.$$

  That is, it ought to be α exactly if some value attaching to some α-state is greater than any value attaching to any non α-state, or simply α is better $\neg\alpha$.

- Evaluation of dyadic ought statements:

$$\mathfrak{M}, x \models \Box(\alpha|\beta) \iff \exists y \in \|\alpha \wedge \beta\|^{\mathfrak{M}} : \exists v \in f_x(y) : \\ \forall z \in \|\neg\alpha \wedge \beta\|^{\mathfrak{M}}, \forall w \in f_x(z), v >_x w.$$

- The class of frames determines the (dyadic) logic CD, which includes a dyadic version of KD plus some other principles. Note that CD does not have the principle of augmentation (or strengthening of antecedent) $\Box(\alpha|\beta) \rightarrow \Box(\alpha|\beta \wedge \gamma)$.

- A problem though: moral conflicts ruled out by the axiological thesis.

C.1.2 Evaluation by imperatives

- Motivating problems:
  - Possibility of unresolvable normative conflicts (replace [D] with [P] while keeping [RM]). Note that the earlier logic CD motivated by the axiological thesis is no longer a candidate because it has [D].
  - Agglomeration of oughts when there are no conflicts.

- Let I_x be the set of imperatives in force at world x. The set of worlds (or states of affairs) at which an imperative i is fulfilled is denoted by i^+. Then,

$$\mathfrak{M}, x \models \Box\alpha \iff \exists i \in I_x : i^+ \subseteq \|\alpha\|^{\mathfrak{M}}.$$

- With the restriction that any imperative in forces can be fulfilled, we have PL, [RM], and [P]. (We also have [N] if every world has some imperative in force.)

C.1.3 Aggregation of oughts

- Motivating problem – If one's choice is between fulfilling two imperatives in force and fulfilling only one of them, one ought to do the first. An example (given by Stalnaker):

  (1) Honor thy father or thy mother!

  (2) Honor not thy mother!

  (3) Hence, thou shalt honor thy father.

Applying [RM] to the tautology $((f \vee m) \wedge \neg m) \rightarrow f$, we have that $\Box((f \vee m) \wedge \neg m) \rightarrow \Box f$, If we can aggregate the obligations from the first two premises, then $\Box f$ is derivable. Note that van Fraassen cannot not simply adopt [C] (the conjunction principle) since [D] is derivable from [C] and [P]. Put it in another way, unrestricted aggregation of oughts (for example, conflicting norms) may contradict [P]. Thus what van Fraassen would require is agglomeration of oughts when doing so causes no problems.

- Let I_x be the set of imperatives in force at world x. The set of worlds (or states of affairs) at which an imperative i is fulfilled is denoted by i^+. The score of a world y with respect to x is defined as follows.

  $$score_x(y) = \{i \in I_x | y \in i^+\}$$

  An ought-sentence Oα is true at a world x in a model $\mathfrak{M}$ iff

  $$\exists y \in \|\alpha\|^{\mathfrak{M}} : \forall z \notin \|\alpha\|^{\mathfrak{M}}, score_x(y) \not\subseteq score_x(z).$$

- Logic:
  - The schemata [N], [P], and the rule [RM] are validated yet [D] and [C] are not. (Hence normative conflicts would not lead to inconsistency.)
  - The semantics supports aggregation of imperatives if they are compatible with each other: for any $i_1, i_2 \in I_x$ and any formula α,

    $$(i_1^+ \cap i_2^+ \neq \emptyset \ \& \ i_1^+ \cap i_2^+ \subseteq \|\alpha\|^{\mathfrak{M}}) \Longrightarrow \mathfrak{M}, x \models \Box\alpha.$$

    But this condition cannot be expressed in a logic of ought-statements alone.

C.2 Goble's deontic logics

Reference: Goble (2000, 2003, 2004).

C.2.1 A preference-based semantics

According to Kripke-style relational semantics, $\Box\alpha$ is true (at a world x in a model $\mathfrak{M}$) just in case α is true (in $\mathfrak{M}$) at all the deontically perfect or ideal possible worlds (for x). Thus, conflicts of obligation are not allowed in the logic it determines (i.e., SDL). In a preference semantics, however, possible worlds are compared with each other instead of being classified either as ideal or not. This semantics determines a weaker logic called P, where normative conflicts could occur.

C.2.2 Simple preference frames

A simple preference frame is a duple $\mathfrak{F} = \langle U, P \rangle$ where P assigns to each point x of U a binary relation (called a preference relation) P_x on U. Evaluation of ought-statements in a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ is as follows.

$$\mathfrak{M}, x \models \Box\alpha \iff \exists y \in \mathrm{fld} P_x : y \in \|\alpha\|^{\mathfrak{M}} \ \&\ \forall z, P_x z y \Longrightarrow z \in \|\alpha\|^{\mathfrak{M}}.$$

Conditional oughts are evaluated according to the following rule:

$$\mathfrak{M}, x \models \Box(\alpha|\beta) \iff \exists y \in \mathrm{fld} P_x : y \in \|\alpha \wedge \beta\|^{\mathfrak{M}} \ \& \\ \forall z : (P_x z y \ \&\ z \in \|\beta\|^{\mathfrak{M}}) \Longrightarrow z \in \|\alpha\|^{\mathfrak{M}}.$$

Monadic Deontic Logics SDL and P

- SDL is determined by the class of all *standard* simple preference frames, i.e., those frames whose preference relations are transitive, connected and so reflexive (on their fields).
- The logic P (axiomatized by PL, [RM], [N], and [P]) is determined by the class of all simple preference frames.

Dyadic Deontic Logics DP and SDDL

- The logic SDDL (standard dyadic deontic logic) is determined by the class of all standard simple preference frames. SDDL is axiomatized by PL together with the following schemas and rules.

[RCE] $\dfrac{\vdash \beta \leftrightarrow \beta'}{\vdash \Box(\alpha|\beta) \leftrightarrow \Box(\alpha|\beta')}$

[RCM] $\dfrac{\vdash \alpha \leftrightarrow \alpha'}{\vdash \Box(\alpha|\beta) \leftrightarrow \Box(\alpha'|\beta)}$

[CK] $\Box(\alpha \rightarrow \beta|\gamma) \rightarrow (\Box(\alpha|\gamma) \rightarrow \Box(\beta|\gamma))$

[CD] $\Box(\alpha|\beta) \rightarrow \neg\Box(\neg\alpha|\beta)$

[CN] $\Box(\top|\top)$

[C$\Box\wedge$] $\Box(\alpha|\beta) \to \Box(\alpha \wedge \beta|\beta)$

[trans] $((\alpha \geq \beta) \wedge (\beta \geq \gamma)) \to (\alpha \geq \gamma)$

where $\alpha \geq \beta$ abbreviates $\neg\Box(\neg\alpha|\alpha \vee \beta)$. (Intuitively, $\alpha \geq \beta$, read "α is at least as good as β", represents a preference ordering of formulas in terms of conditional obligation.)

- The logic DP (dyadic P) is determined by the class of all reflexive, transitive simple preference frames. DP is axiomatized by PL together with the following schemas and rules.

[RCE] $\dfrac{\vdash \beta \leftrightarrow \beta'}{\vdash \Box(\alpha|\beta) \leftrightarrow \Box(\alpha|\beta')}$

[RCM] $\dfrac{\vdash \alpha \leftrightarrow \alpha'}{\vdash \Box(\alpha|\beta) \leftrightarrow \Box(\alpha'|\beta)}$

[CN] $\Box(\top|\top)$

[CP] $\neg\Box(\bot|\alpha)$

[C$\Box\wedge$] $\Box(\alpha|\beta) \to \Box(\alpha \wedge \beta|\beta)$

[trans] $((\alpha \geq \beta) \wedge (\beta \geq \gamma)) \to (\alpha \geq \gamma)$

[C$\Box\vee$] $\Box(\alpha|\beta \vee \gamma) \to (\Box(\alpha|\beta) \vee \Box(\alpha|\gamma))$

C.2.3 Multiple preference frames

A multiple preference frame (MP-frame) is a duple $\mathfrak{F} = \langle U, \mathcal{P}\rangle$ where $\mathcal{P}$ assigns each $x \in U$ a non-empty set $\mathcal{P}_x$ of preference relations P's on U. (We assume each P is non-empty.) We can define two modal operators $\Box_e$ and $\Box_a$ (corresponding to the indefinite sense and the core or definite sense of ought) as follows.

$$\mathfrak{M}, x \models \Box_e\alpha \iff \exists P \in \mathcal{P}_x : \exists y \in \text{fld}P : y \in \|\alpha\|^{\mathfrak{M}} \ \&\ \forall z, Pzy \Longrightarrow z \in \|\alpha\|^{\mathfrak{M}}.$$

$$\mathfrak{M}, x \models \Box_a\alpha \iff \forall P \in \mathcal{P}_x : \exists y \in \text{fld}P : y \in \|\alpha\|^{\mathfrak{M}} \ \&\ \forall z, Pzy \Longrightarrow z \in \|\alpha\|^{\mathfrak{M}}.$$

Monadic deontic logics SDL$_a$P$_e$ and P$_a$P$_e$

The (bimodal) logic SDL$_a$P$_e$ is determined by the class of all standard MP-frames. SDL$_a$P$_e$is axiomatized by PL together with the following rules and schemas.

[K_a], [D_a], [RN_a]	[K], [D], and [RN] with $\Box_a$ as the modal operator.
[RM_e], [N_e], [P_e]	[RM_e], [N_e], [P_e] with $\Box_e$ as the modal operator.
[K_{ae}]	$\Box_a(\alpha \to \beta) \to (\Box_e\alpha \to \Box_e\beta)$

The (bimodal) logic P_aP_e is determined by the class of all MP-frames (or all reflexive or transitive MP-frames. P_aP_e is axiomatized by PL together with the following rules and schemas.

[RM_a], [N_a], [P_a]	[RM_a], [N_a], [P_a] with $\Box_a$ as the modal operator.
[RM_e], [N_e], [P_e]	[RM_e], [N_e], [P_e] with $\Box_e$ as the modal operator.
[$\Box_a\Box_e$]	$\Box_a\alpha \to \Box_e\alpha$

C.2.4 Ranked multiple frames

A ranked multiple preference frame (MP$\leq$-frame) is a triple $\langle U, \mathcal{P}, \leq\rangle$ where $\leq$ assigns to each point x of U a binary relation $\leq_x$ (a ranking) on the set $\mathcal{P}_x$ of preference relations.

Besides $\Box_a$ and $\Box_e$ we also have $\preceq$ as a new dyadic operator. "$\alpha \preceq \beta$" could be read as saying that β is at least as obligatory as α. Truth evaluation is as follows.

$$\mathfrak{M}, x \models \alpha \preceq \beta \iff (\forall P \in \mathcal{P}_x,\ \mathfrak{M}, P \models \alpha \Longrightarrow \exists Q \in \mathcal{P}_x : \mathfrak{M}, Q \models \beta \ \&\ P \leq_x Q)$$

where $\mathfrak{M}, P \models \alpha$ means that

$$\exists y \in \mathrm{fld}P :\ \mathfrak{M}, y \models \alpha \ \&\ \forall z, Pzy \Longrightarrow \mathfrak{M}, z \models \alpha.$$

Monadic deontic logics $\mathbf{SDL}_a\mathbf{P}_e \leq$ and $\mathbf{P}_a\mathbf{P}_e \leq$

The logic $SDL_aP_e \leq$ is determined by the class of all standard MP$\leq$-frames. $SDL_aP_e \leq$ is axiomatized by SDL_aP_e together with the following schemas.

[$\Box_a \preceq$]	$\Box_a(\alpha \to \beta) \to (\alpha \preceq \beta)$
[$\neg\Box_e \preceq$]	$\neg\Box_e\alpha \to (\alpha \preceq \beta)$
[$\preceq \Box_e$]	$(\alpha \preceq \beta) \to (\Box_e\alpha \to \Box_e\beta)$
[$\preceq$-trans]	$((\alpha \preceq \beta) \wedge (\beta \preceq \gamma)) \to (\alpha \preceq \gamma)$

The logic $SDL_aP_e \leq_c$ is determined by the class of all standard MP$\leq$-frames whose $\leq_x$ (for every $x \in U$) is connected. ${}_aP_e \leq_c$ is $SDL_aP_e \leq$ plus the following schema.

[$\preceq$-connex] $(\alpha \preceq \beta) \vee (\beta \preceq \alpha)$

The logic determined by the class of all MP$\leq$-frames (or the class of all reflexive or transitive MP$\leq$-frames) is $\mathrm{P}_a\mathrm{P}_e \leq$, i.e., $\mathrm{P}_a\mathrm{P}_e$ plus the following.

$[\Box_a \preceq']$	$\Box_a(\alpha) \to (\beta \preceq \alpha)$
$[\neg\Box_e \preceq]$	$\neg\Box_e\alpha \to (\alpha \preceq \beta)$
$[\preceq \Box_e]$	$(\alpha \preceq \beta) \to (\Box_e\alpha \to \Box_e\beta)$
$[\preceq$-trans$]$	$((\alpha \preceq \beta) \land (\beta \preceq \gamma)) \to (\alpha \preceq \gamma)$
$[\mathrm{R} \preceq]$	$\dfrac{\vdash \alpha \to \beta}{\vdash \alpha \preceq \beta}$

If $\leq_x$ (for every x) is connected, we have $\mathrm{P}_a\mathrm{P}_e \leq_c$, i.e., $\mathrm{P}_a\mathrm{P}_e \leq$ plus [$\leq$-connex].

C.3 Horty's deontic logics

Reference: Horty (1997, 2003).

C.3.1 Nonmonotonic foundations for deontic logic

A nonmonotonic approach is better than standard model-theoretic approach, especially in two particular areas of normative reasoning.

- conflicting oughts.
- prima facie oughts (conditional oughts that can be overridden by other norms or by some facts).

C.3.2 Normative conflicts and van Fraassen's proposal

Let Γ be a set of ought statements $\Box\alpha$ etc., and $\mathfrak{M}$ a propositional model of an ought-free language (that is, an assignment of truth value to propositional letters). the score of $\mathfrak{M}$ with respect to Γ is defined as follows.

$$score_\Gamma(\mathfrak{M}) = \{\Box\alpha \in \Gamma \mid \mathfrak{M} \models \alpha\}$$

Van Fraassen's notion of deontic consequence ($\vdash_F$) is captured by the following.

$$\Gamma \vdash_F \Box\alpha \iff \exists\mathfrak{M}_1 \in \mathrm{Mod}\ \alpha : \forall\mathfrak{M}_2 \in \mathrm{Mod}\ \neg\alpha,\ score_\Gamma(\mathfrak{M}_1) \not\subseteq score_\Gamma(\mathfrak{M}_2).$$

(Mod α is the class of all models of the formula α. Similarly Mod Γ is the class of all models that satisfy the set Γ of formulas.) An equivalent definition of the consequence relation $\vdash_F$ is given below . (Let $\Box^-(\Gamma)$ be the set of formulas α's such that $\Box\alpha$ is in Γ.)

$$\Gamma \vdash_F \Box\alpha \iff \Sigma \vdash \alpha, \text{for some consistent subset } \Sigma \text{ of } \Box^-(\Gamma).$$

C.3.3 Oughts as defaults

A set Γ of ought statements induces a default theory $\langle \mathcal{W}, \mathcal{D} \rangle$ where

- $\mathcal{W} = \emptyset$, and
- $\mathcal{D} = \{\frac{:\alpha}{\alpha} \mid \Box\alpha \in \Gamma\}$.

The following shows that the consequence relation $\vdash_F$ can be seen as deduction in default logic, and justifies therefore the claim that oughts are default rules.

$$\Gamma \vdash_F \Box\alpha \iff \alpha \in \mathcal{E}, \text{ for some extension } \mathcal{E} \text{ of the default theory induced by } \Gamma.$$

As an example, let Γ be the set $\{\Box p, \Box\neg p\}$. Then the default theory induced by Γ is $\langle \emptyset, \{: p/p, : \neg p/\neg p\}$. It can readily be seen that any variable q distinct from p will not be in any extension of our default theory. Thus $\Gamma \not\vdash_F \Box q$, i.e., deontic explosion is avoided even we have competing obligations.

C.3.4 A skeptical reasoning strategy

The following are equivalent definitions.

- $\Gamma \vdash_S \Box\alpha \iff \alpha \in \mathcal{E}$, for each extension $\mathcal{E}$ of the default theory induced by Γ.
- $\Gamma \vdash_S \Box\alpha \iff \Sigma \vdash \alpha$, for each consistent subset Σ of $\Box^-(\Gamma)$.

C.3.5 The strategy of articulating the premise set

Given a set Γ of ought statements, the articulated set Γ^* is defined as the smallest superset of Γ that contains both $\Box(\ldots\alpha\ldots)$ and $\Box(\ldots\beta\ldots)$ whenever it contains one of the following:

- $\Box(\ldots(\alpha \wedge \beta)\ldots)$ with the occurrence of the conjunction positive;
- $\Box(\ldots(\alpha \vee \beta)\ldots)$ with the occurrence of the disjunction negative.

An articulated variant of $\vdash_F$ is as follows.

$$\Gamma \vdash_{FA} \Box\alpha \iff \Gamma^* \vdash_F \Box\alpha.$$

C.3.6 An articulated skeptical strategy

Combining the previous two strategies, we get:

$$\Gamma \vdash_{SA} \Box\alpha \iff \Gamma^* \vdash_S \alpha.$$

C.3.7 Conditional obligations

An ought context is a duple $\langle \mathcal{W}, \Gamma \rangle$, where $\mathcal{W}$ is a set of facts, and Γ a set of conditional ought statements in the form of $\Box(\alpha|\beta)$.

An ought statement $\Box(\alpha|\beta)$ is overridden in a context $\langle \mathcal{W}, \Gamma \rangle$ iff there exists another ought statement $\Box(\gamma|\delta) \in \Gamma$ such that all of the following hold.

(1) $\mathrm{Mod}\, \mathcal{W} \subseteq \mathrm{Mod}\, \delta$.

(2) $\mathrm{Mod}\, \delta \subset \mathrm{Mod}\, \beta$ (i.e., δ is more specific than β).

(3) $\mathcal{W} \cup \{\alpha, \gamma\}$ is inconsistent.

In other words, a conditional ought can be overridden (only) by a single opposing statement, which is both applicable in the context and more specific.

A set $\mathcal{E}$ of formulas is a conditioned extension of $\langle \mathcal{W}, \Gamma \rangle$ iff there exists a set $\mathcal{A}$ of formulas such that all of the following hold.

(1) $\alpha \in \mathcal{A}$ iff

  (i) $\Box(\alpha|\beta) \in \Gamma$,

  (ii) $\mathrm{Mod}\, \mathcal{W} \subseteq \mathrm{Mod}\, \beta$,

  (iii) $\Box(\alpha|\beta)$ is not overridden in $\langle \mathcal{W}, \Gamma \rangle$, and

  (iv) $\neg\alpha \notin \mathcal{E}$.

(2) $\mathcal{E} = \mathrm{Cn}(\mathcal{W} \cup \mathcal{A})$.

Note that a conditioned extension of an ought-context can be considered as a way to fulfil the oughts given the world knowledge. Moreover it can be shown that every ought context has a conditioned extension.

Finally we can define a consequence relation for conditional oughts as follows.

$$\langle \mathcal{W}, \Gamma \rangle \vdash_{CF} \Box(\alpha|\beta) \iff \alpha \in \mathcal{E}, \quad \text{for some conditioned extension } \mathcal{E} \text{ of } \langle \mathcal{W} \cup \{\beta\}, \Gamma \rangle.$$

In particular, if $\mathcal{W}$ is empty, we have that

$$\Gamma \vdash_{CF} \Box(\alpha|\beta) \iff \alpha \in \mathcal{E}, \text{for some conditioned extension } \mathcal{E} \text{ of } \langle \{\beta\}, \Gamma \rangle.$$

C.3.8 Outstanding problems

- The account, as it now stands, does not allow for *any* kind of transitivity of conditional oughts.

  $$\frac{\Box(\alpha|\beta), \Box(\beta|\gamma)}{\Box(\alpha|\gamma)}$$

  What we want is transitivity as a defeasible rule.

- The account does not allow reasoning with disjunction antecedents.

  $$\frac{\Box(\alpha|\beta), \Box(\alpha|\gamma)}{\Box(\alpha|\beta \vee \gamma)}$$

- The account allows overridden of a norm only by a single opposing norm, but not by a set of opposing norms.

- According to the present account, an overridden norm cannot be reinstated when the overriding norm is itself overridden.

C.4 Nute's defeasible deontic logic

Reference: Nute (1997a, 1999). We restrict ourselves to formulas that are either literals (i.e., p or $\neg p$ where p is a propositional letter) or modal formulas (i.e., $\Box\lambda$ or $\neg\Box\lambda$ where λ is a literal). Note that we do not have compound formulas that are disjunctions or conjunctions. Neither do we have iterated negations or embedded modalities.

There are three types of rules recognized in the logic. (In the following, A is a set of formulas and ϕ is a formula.)

- strict rule: $A \rightarrow \phi$ (e.g., Penguins are birds.)
- defeasible rule: $A \Rightarrow \phi$ (e.g., Birds fly.)
- undercutting defeaters: $A \rightsquigarrow \phi$ (e.g., A damp match might not burn.)

Norms are thus represented as rules rather than as formulas. (For example, "You ought not to lie" as "$\Rightarrow \Box\neg l$", "If lying saves lives, you ought to lie" as "$s \Rightarrow \Box l$".)

A deontically closed defeasible theory T is a tuple $\langle F, R, C, \prec\rangle$ where

(1) F is a set of formulas called facts.

(2) R is a set of rules.

(3) C is a set of conflict sets, i.e., finite sets of formulas satisfying all of the following conditions for any formula ϕ.

  (i) $\{\phi, \sim \phi\} \in C$.

  (ii) For every set $S \in C$ and every strict rule $A \rightarrow \phi \in R$, if $\phi \in S$, then $A \cup (S - \{\phi\}) \in C$.

  (iii) For every $\{\phi_1, \ldots, \phi_n\} \in C$, there exists $\{\psi_1, \ldots, \psi_n\} \in C$, such that for each $1 \leq i \leq n$, either

    i. ϕ_i is a literal and ψ_i is $\Box\phi_i$, or

    ii. ϕ_i is not a literal and ψ_i is ϕ_i.

(4) $\prec$ is an acyclic binary relation (precedence relation) on the non-strict rules in R.

A defeasible deontic proof (of a formula ϕ from a theory T) is an argument tree whose top node has the formula ϕ, and is constructed according to a series of rules. Without going into details of the rules here, we will illustrate how the logic works with a simple example. Consider the theory T whose components are as follows. (Let s be the statement that lying saves lives, l the statement that you lie.)

- The set F of facts is $\{s\}$.
- The set R of rules is $\{\Rightarrow \Box\neg l, s \Rightarrow \Box l\}$

- The set C of conflict set has the following members.

$$\{l, \neg l\}, \{s.\neg s\}, \{\Box l, \neg\Box l\}, \{\Box\neg l, \neg\Box\neg l\}, \{\Box s, \neg\Box s\}, \{\Box\neg s, \neg\Box\neg s\},$$
$$\{\Box l, \Box\neg l\}, \{\Box s, \Box\neg s\}.$$

- The set $\prec$ is $\{\Rightarrow \Box\neg l \prec s \Rightarrow \Box l\}$.

Given the fact s (that lying saves lives), we can derive $\Box l$ (It is obligatory to lie) by applying the rule $s \Rightarrow \Box l$ by deontic detachment. $\Box l$ is in the conflict set $\{\Box l, \Box\neg l\}$ but it is not defeated because the derivation of $\Box\neg l$ could be done only by a weaker rule $\Rightarrow \Box\neg l$. On the other hand, we cannot apply deontic detachment to the rule $\Rightarrow \Box\neg l$ since it is defeated (in the sense that its consequent $\Box\neg l$ is in the conflict set $\{\Box l, \Box\neg l\}$, and $\Box l$ is derivable by the stronger rule $s \Rightarrow \Box l$.

C.5 Makinson and van der Torre's Input/Output Logics

Reference: Makinson and van der Torre (2000, 2001, 2003).

C.5.1 Terminology

- A conditional norm is an ordered pair of propositions (a, x) where the body a represent a condition and x what is deemed desirable given the condition.
- A normative code G is a set of conditional goals or obligations (also called a generating set).
- An input A is a set of propositions (representing a situation).
- An output of G under A ($out(G, A)$) is a set of propositions (representing what are deemed desirable given the situation and the code).
- $x \in deriv(G, a)$ (or $(a, x) \in deriv(G)$) iff (a, x) is in the least set that includes G, contains the pairs (t, t) where t is a tautology, and is closed under a set of rules (to be given).
- $x \in deriv(G, A)$ (or $(A, x) \in deriv(G)$) iff $x \in deriv(G, a)$ where a is a finite conjunction of some elements of A.
- $x \in G(A)$ iff for some $(a, x) \in G$, $a \in A$.x

C.5.2 Output Operations

- Simple-minded output: $out_1(G, A) = Cn(G(Cn(A)))$
- Basic output: $out_2(G, A) = \cap\{Cn(G(V)) : A \subseteq V, V \text{complete}\}$
- Reusable simple-minded output: $out_3(G, A) = \cap\{Cn(G(B)) : A \subseteq B = Cn(B) \supseteq G(B)\}$
- Reusable basic output: $out_4(G, A) = \cap\{Cn(G(V)) : A \subseteq V \supseteq G(V), V \text{complete}\}$
- Simple-minded throughput: $out_1^+(G, A) = out_1(G \cup I, A)$ where I is the set of all pairs of formulas (a, a). (Similarly for other output operations)

C.5.3 Derivation rules

- SI(strengthening input): $(a, x) \implies (b, x)$ whenever $b \vdash a$
- AND(conjoining output): $(a, x), (a, y) \implies (a, x \wedge y)$
- WO(weakening output): $(a, x) \implies (a, y)$ whenever $x \vdash y$
- OR(disjoining input): $(a, x), (b, x) \implies (a \vee b, x)$
- CT(cumulative transitivity): $(a, x) \implies (a \wedge x, y)$
- ID(identity): $\implies (a, a)$

C.5.4 Derivability and Output Operations

- $out_1(G, A) = deriv_1(G, A)$ where $deriv_1$ has the rules SI, AND, WO.
- $out_2(G, A) = deriv_2(G, A)$ where $deriv_2$ has the rules SI, AND, WO, OR.
- $out_3(G, A) = deriv_3(G, A)$ where $deriv_1$ has the rules SI, AND, WO, CT.
- $out_4(G, A) = deriv_4(G, A)$ where $deriv_1$ has the rules SI, AND, WO, OR, CT.

C.5.5 Constrained Outputs

- $maxfamily(G, A, C)$: the family of all maximal $H \subseteq G$ such that $out(H, A)$ is consistent with C (constraint set).
  - $maxfamily(G, A, \emptyset)$
  - $maxfamily(G, A, A)$
- $outfamily(G, A, C)$: the family of all outputs under input A generated by elements of $maxfamily(G, A, C)$.
- $\cap outfamily(G, A, C)$: full meet constrained output.
- $\cup outfamily(G, A, C)$: full join constrained output.

C.5.6 Constrained Outputs and Reiter's Default Logic

The pair (G, A) can be considered as a normal Reiter's default theory by taking (a, x) to be $\frac{a;x}{x}$. The family of all extensions of (G, A) is denoted by $extfamily(G, A)$. In the following, the output operation is out_3^+ (reusable simple-minded throughput), and we assume A is consistent.

- $extfamily(G, A) \subseteq outfamily(G, A)$
- For every $X \in outfamily(G, A)$, there is an $E \in extfamily(G, A)$ with $X \subseteq E$.
- $extfamily(G, A)$ consists exactly the maximal elements of $outfamily(G, A)$.
- $\cup(extfamily(G, A)) = \cup(outfamily(G, A))$.

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