INTERSECTING CONVEX SETS BY RAYS

by

Radoslav Fulek
M.Sc., Comenius University, Slovakia, 2005

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the School
of
Computing Science

SIMON FRASER UNIVERSITY
Summer 2008

APPROVAL

Name: Radoslav Fulek

Degree: Master of Science

Title of thesis: Intersecting convex sets by rays

Examining Committee: Dr. Bojan Mohar
Chair

Dr. Gábor Tardos, Senior Supervisor

Dr. Thomas C. Shermer, Supervisor

Dr. Binay Bhattacharya, SFU Examiner

Date Approved: June 23, 2008

Abstract

Intersecting convex sets by rays

What is the smallest number $\tau = \tau_d(n)$ such that for any collection $\mathcal{C}$ of n pairwise disjoint compact convex sets in $\mathbb{R}^d$, there is a point such that any ray (half-line) emanating from it meets at most τ sets of the collection? In this thesis we show an upper and several lower bounds on the value $\tau_d(n)$, and thereby we completely answer the above question for $\mathbb{R}^2$, and partially for higher dimensions. We show the order of magnitude for an analog of $\tau_2(n)$ for collections of fat sets with bounded diameter. We conclude the thesis with some algorithmic solutions for finding a point $\mathbf{p}$ that minimizes the maximum number of sets in $\mathcal{C}$ we are able to intersect by a ray emanating from $\mathbf{p}$ in the plane, and for finding a point that basically witnesses our upper bound on $\tau_d(n)$ in any dimension. However, the latter works only for restricted sets of objects.

Keywords: convex set, regression depth, Centerpoint Theorem, Helly Theorem

"To my parents"

Acknowledgements

I would like to thank my supervisor Gábor Tardos for introducing me the problem, fruitful discussions, and for his innumerable advices and remarks about the Thesis.

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Chapter 1

Introduction

Suppose we have an environment with a bunch of objects distributed in it, which we consider as the obstacles. Our goal is to find a position $\mathbf{p}$ in the environment such that the visibility between any location and $\mathbf{p}$ is not blocked by too many obstacles, i.e. the straight line connecting a location with $\mathbf{p}$ does not intersect too many obstacles. In the real world situation we assume that we want to set up in our environment a mobile wireless sensor network. The heart of our network is the base station, which has to be able to communicate with the sensor nodes, regardless of their position in the environment. According to a model of wireless positioning service recently patented by Liu and Hung [14], the signal sent by a sensor can penetrate only at most a certain number, t, of obstacles and will not be received by the base station if there is more than t obstacles between the sensor and the base station. They call this predetermined threshold *the obstacle number* of the network.

Based on this model we will investigate the following question asked by Jorge Urrutia [20]: What is the smallest number $\tau_d(n)$ such that for any set of n pairwise disjoint convex compact sets in the d-dimensional Euclidean space there is always a point $\mathbf{p}$, for which every ray emanating from it intersects at most $\tau_d(n)$ of these sets. So, basically we investigate the worst case scenario with respect to the obstacle number. Formally, we can express $\tau_d(n)$ as follows.

Let $\mathcal{C}$ be a collection of n convex sets in d-dimensional Euclidean space. For any point $\mathbf{p}$ we denote by $r(\mathbf{p})$ the set of all rays emanating from $\mathbf{p}$. Let $\tau(r, \mathcal{C})$ denote the number of sets in $\mathcal{C}$ intersected by a ray r, i.e. the number of sets C in $\mathcal{C}$ such that $r \cap C \neq \emptyset$. Then $\tau(\mathbf{p}, \mathcal{C}) = \max_{r \in r(\mathbf{p})} \tau(r, \mathcal{C})$ is the maximal number of sets from $\mathcal{C}$ we can intersect by a ray emanating from $\mathbf{p}$. The obstacle number of a given collection of sets is expressed as follows $\tau(\mathcal{C}) = \min_{\mathbf{p}\in\mathbb{R}^d} \tau(\mathbf{p},\mathcal{C})$. Finally, let $\tau_d(n) = \max_{\mathcal{C}} \tau(\mathcal{C})$, where $\mathcal{C}$ varies over all collections of n pairwise disjoint compact convex sets in d-dimensional Euclidean space. In our model the abstraction of an obstacle is a convex set living in the Euclidean space, and the base station and sensor nodes are represented by points in that space.

In this thesis, we study the asymptotic growth of the functions $\tau_d(n)$. We start, in Chapter 2, with presentation of some related problems, along with results that were obtained recently. In Chapter 3, we show an upper bound on $\tau_d(n)$. In Chapter 4, we provide some constructions attaining (or almost attaining) this bound in $\mathbb{R}^2$ and we give a general construction that works in any dimension, but leaves a large gap between the lower bound it provides and the general upper bound. Moreover, in Chapter 5 we study a restricted version of our problem in $\mathbb{R}^2$, in which we allow only fat convex sets with bounded diameter to be in our collection. We conclude this thesis with the presentation of some algorithms in Chapter 6 that for a collection of n pairwise disjoint compact convex sets $\mathcal{C}$ in $\mathbb{R}^2$ return a point $\mathbf{p}$ minimizing $\tau(\mathbf{p},\mathcal{C})$, or for a collection $\mathcal{C}$ in $\mathbb{R}^d$, for $d > 1$, return a point that has $\tau(\mathbf{p},\mathcal{C})$ not much higher than our upper bound on $\tau_d(n)$. However, in the latter case we have only algorithms that works for collections of sets of some restricted kind. The results in this thesis, most of which are from author's joint work [11] with Andreas Holmsen and János Pach, are as follows.

Theorem 1.1 ([11]). $\tau_d(n) \le \left\lfloor \frac{dn+1}{d+1} \right\rfloor$

Theorem 1.2. $\tau_d(n) \ge \left\lceil \frac{n-d+1}{2} \right\rceil$

Theorem 1.3 ([11]). *For any $k \in \mathbb{N}$, $k > 0$, there exists a collection of $3k$ pairwise disjoint discs in the Euclidean plane such that from any point there is a ray that intersects at least $2k-2$ of them.*

Theorem 1.4 ([11]). *For any $k \in \mathbb{N}$, $k > 0$, there exists a collection of $3k$ pairwise disjoint equal length segments in the Euclidean plane such that from any point there is a ray that intersects at least $2k-1$ of them.*

Theorems 1.3 and 1.4 show that our upper bound from Theorem 1.1 is tight (up to an additive constant) in two dimensions, and by Theorem 1.2 it is also tight in $\mathbb{R}$. However, the problem in $\mathbb{R}$ is much more simpler than in higher dimensions, as any collection of pairwise disjoint closed intervals gives us matching lower bound.

Moreover, for an analog $\tau_2'(n)$ of $\tau_2(n)$ for fat sets with bounded diameter we prove the following theorem.

Theorem 1.5 ([11]). $\tau_2'(n) \in \Theta(\sqrt{n \log n})$

Finally, we show that the computational problem of finding a point $\mathbf{p}$ witnessing $\tau(\mathcal{C})$, for a given set $\mathcal{C}$ of pairwise disjoint bounded complexity polygons in $\mathbb{R}^2$ having n sides in total, can be solved in deterministic polynomial time with respect to n.

Theorem 1.6. *There is an algorithm that computes a point $\mathbf{p}$ minimizing $\tau(\mathbf{p}, \mathcal{C})$ in $O(n^4 \log n)$ time using $O(n^4)$ space.*

1.1 Notation

In this section we summarize our notation.

We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real and natural numbers, respectively. $\mathbb{R}^d$ stands for the d-dimensional Euclidean space. The points (vectors) in $\mathbb{R}^d, d > 1$ are typeset in boldface. If $A \subseteq \mathbb{R}^d$, by cA, where $c \in \mathbb{R}$, we mean the set $\{c\mathbf{p}|\ \mathbf{p} \in A\}$. $\mathbf{0}$ denote the origin of $\mathbb{R}^d,\ d > 1$.

We call a *ray* from a point $\mathbf{p} \in \mathbb{R}^d$ the set of points $\{\mathbf{p}+t\mathbf{v}|\ t \in \mathbb{R}, t > 0\}$, for $\mathbf{v} \in \mathbb{R}^d, \mathbf{v} \neq \mathbf{0}$ (the direction). A ray r intersects a set S if $r \cap S \neq \emptyset$. By a *hyperplane* H in $\mathbb{R}^d$ we mean $(d-1)$-dimensional affine subspace, i.e. a set of points $\mathbf{x} = (x_1, \dots x_d) \in \mathbb{R}^d$ satisfying $b = \sum_{i=1}^{d} a_i x_i$, for some $b, a_i \in \mathbb{R},\ 1 \leq i \leq d$, and $\mathbf{a} = (a_1, \dots a_d) \neq \mathbf{0}$. We call $\mathbf{a}$ a normal vector of the hyperplane H. By a *vertical hyperplane* we mean a hyperplane having $a_d = 0$. We call a *closed half-space* defined by H the union of H with a connected component of its complement in $\mathbb{R}^d$.

By the point/hyperplane duality we will understand an injective mapping D that takes a point $\mathbf{a} = (a_1, \dots a_d) \in \mathbb{R}^d$ to a non-vertical hyperplane defined by the equation $x_d = \sum_{i=1}^{i=d-1} a_i x_i - a_d$. The hyperplane corresponding to a point $\mathbf{a}$ is $D(\mathbf{a})$, and the point corresponding to a hyperplane is $D^{-1}(\mathbf{a})$. Note that this correspondence preserves incidence.

The Euclidean distance between two points $\mathbf{p} = (p_1, \dots p_d)$, $\mathbf{q} = (q_1, \dots q_d)$ in $\mathbb{R}^d$ is denoted by $|\mathbf{pq}| = \sqrt{\sum_{i=1}^{d}(p_i - q_i)^2}$. A *ball* in $\mathbb{R}^d$ is a set of the form $\{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{xa}| \leq b\}$, for $\mathbf{a} \in \mathbb{R}^d$ (the centre) and $b \in \mathbb{R},\ b > 0$ (the radius). We call a two dimensional ball a *disc*. We call a *sphere* the boundary of a ball in $\mathbb{R}^d$, i.e a set $\{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{xa}| = b\}$, for $\mathbf{a} \in \mathbb{R}^d$ (the centre) and $b \in \mathbb{R},\ b > 0$ (the radius). We call a two dimensional sphere a *circle*. By *projection* of a set A in $\mathbb{R}^d$ on a sphere S with a centre $\mathbf{c}$ we mean the set of intersection points with S of the rays emanating from $\mathbf{c}$ and intersecting A. We denote by S^{d-1} the unit radius sphere in $\mathbb{R}^d$ with the centre at $\mathbf{0}$.

We call a *(simple) graph* $G(V, E)$ a pair of two finite sets, a set of *vertices* $V = V(G)$, and a set of *edges* $E = E(G)$, such that $E \subseteq \{\{u, v\}|\ u, v \in V,\ v \neq u\}$. We will refer to an edge $e = \{u, v\}$ shortly by uv. The edge $e = uv$ is *incident* to u and v.

The set $C \subseteq \mathbb{R}^d$ is *convex*, if for every $\mathbf{x}, \mathbf{y} \in C$ the line segment $\mathbf{xy}$ is contained in C. By the *convex hull* of a set $S \subseteq \mathbb{R}^d$, denoted by $\mathrm{conv}(S)$, we mean the smallest (with respect to inclusion) convex set containing S. By the *affine hull* of a set $S \subseteq \mathbb{R}^d$ we understand the smallest (with respect to inclusion) affine subspace of $\mathbb{R}^d$ containing S. By a *relative boundary* of a set S in $\mathbb{R}^d$ we mean the boundary of S with respect to its affine hull. A *compact* set is a set that is bounded and closed. Let A and B denote two compact subsets of $\mathbb{R}^d$. By *Hausdorff distance* between A and B we understand $\max\{\sup_{\mathbf{a}\in A}\inf_{\mathbf{b}\in B}|\mathbf{ab}|, \sup_{\mathbf{b}\in B}\inf_{\mathbf{a}\in A}|\mathbf{ab}|\}$.

We assume the reader to be familiar with the basic properties of the above notions, which could be found in many textbooks, e.g. [16, 23].

Chapter 2

Connections to regression depth

Recently robust statistics (see e.g. [17, 18]) became an active sub-area of the computational geometry due to the natural geometric formulation of many of its problems. Its motivation is to produce estimators that are not too much affected by small deviations from our model assumptions, i.e. estimators that are not much affected by outliers (in statistics: an observation distant from the rest of the data) presented in our data. The application of classical statistical methods (e.g. least square regression) turned out to be in some practical situation inappropriate, because their outcome was too much affected by these occasional exceptions.

The question we study is closely related to the statistical notion of regression depth of a hyperplane introduced by Rousseeuw and Hubert (1996) as a quality measure for robust linear regression. The robust linear regression in comparison with the least square regression, which assumes that error is normally distributed, allows some data to be affected by the completely arbitrary errors. Another robust estimators is e.g. slope selection [15, 8].

Let H be a non-vertical hyperplane in $\mathbb{R}^d$. Any vertical hyperplane (as defined in Section 1.1) H' in $\mathbb{R}^d$ together with H defines two pairs of opposite connected parts (double wedges) of the complement of $H \cup H'$ in $\mathbb{R}^d$. Let us denote by $R_1 = R_1(H, H')$ and $R_2 = R_2(H, H')$ the closure of the unions of these two pairs, respectively. Geometrically, the *regression depth* of a non-vertical hyperplane H with respect to a finite set of points $\mathcal{P}$ is $\min\{|\mathcal{P} \cap R_1|, |\mathcal{P} \cap R_2| : R_1 = R_1(H, H'),\ R_2 = R_2(H, H'),\ H' \text{ is vertical hyperplane}\}$. If H is a vertical hyperplane its *regression depth* with respect to $\mathcal{P}$ is $|H \cap \mathcal{P}|$.

In the dual settings of the hyperplane arrangement, this notion is equivalent to the notion of the undirected depth of a point in an arrangement of hyperplanes in $\mathbb{R}^d$. The *undirected* *depth* of a point $\mathbf{p} \in \mathbb{R}^d$ gives us the minimum number of hyperplanes in the arrangement any ray emanating from $\mathbf{p}$ intersects. Here we consider a hyperplane to be intersected by a ray that the hyperplane is parallel to. More formally, if $\mathcal{P}$ is a set of points in $\mathbb{R}^d$ then the regression depth of a hyperplane H with respect to $\mathcal{P}$ equals to the undirected depth of a point $D^{-1}(H)$ in $\{D(\mathbf{p})| \mathbf{p} \in \mathcal{P}\}$.

Given a collection of n hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$ notice that any ray r from a point $\mathbf{p}$ having undirected depth m in $\mathcal{H}$ intersects at most $n-m$ hyperplanes of $\mathcal{H}$ not containing $\mathbf{p}$ that are not parallel to r. To see this just observe that every ray is an opposite ray of some other ray starting at $\mathbf{p}$. Hence, a lower bound $l(n)$ on the maximal undirected depth for an arrangements of n hyperplanes $\mathcal{H}$ would give us the upper bound $n - l(n)$ for an analog of $\tau(n)$ for hyperplanes, where we consider rays to not contain their starting points. Thus, as we can convert any statement about regression depth into the statement about undirected depth (using the properties of the point/hyperplane duality), it is not hard to see that the following result gives us the same (up to an additive constant depending on d) upper bound as that of Theorem 1.1 for an analog of $\tau_d(n)$ for collections of flat sets in $\mathbb{R}^d$ and, in some sense, in "general position". What do we mean by general position in this case will be explained later. By a flat set we mean a set that is contained in a hyperplane. The proof of Theorem 2.1 from [3] relies on the same classical result as our proof of the general upper bound: Brouwer's fixed point theorem [5].

Theorem 2.1. *For any d-dimensional set of n points $\mathcal{P}$ there exists a hyperplane having regression depth at least $\left\lceil \frac{n}{d+1} \right\rceil$.*

Although the proof of the above theorem is quite involved, the main argument can be formulated quite easily in terms of the projective transformation. First we embed Euclidean space containing our set of points $\mathcal{P}$ into the projective space as a hyperplane H_1 (see Figure 2.1) in $\mathbb{R}^{d+1}$ avoiding the origin and thereby enriching it by the points at infinity and the hyperplane at infinity.

Theorem 2.2 (Centerpoint Theorem). *Let $\mathcal{P}$ be the finite set of points in $\mathbb{R}^d$. Then there exists a point $\mathbf{p} \in \mathbb{R}^d$ (centerpoint) such that for all hyperplanes H, such that $\mathbf{p} \in H$, any closed half-space defined by H contains at least $\left\lceil \frac{n}{d+1} \right\rceil$ points in $\mathcal{P}$.*

Using the above Centerpoint Theorem (for the proof see e.g. [9]) observe that all we need to do is to prove that for any $\mathcal{P}$ there exists a projective transformation t of H_1 taking

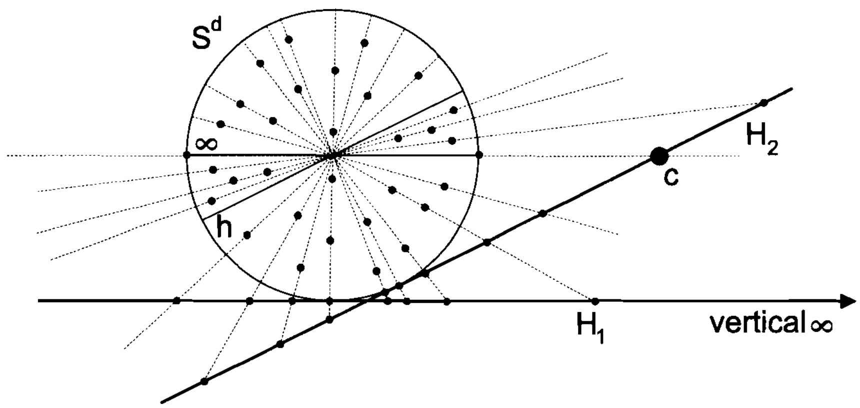

Figure 2.1: Projective transformation t

a point at the vertical infinity to the centerpoint $\mathbf{c}$ of the transformed set $t(\mathcal{P})$. The space containing the trasformed set $t(\mathcal{P})$ of points $\mathcal{P}$ is embeded into the projective space as a hyperplane H_2 (see Figure 2.1) in $\mathbb{R}^{d+1}$. As the inverse of t takes the hyperplane at infinity to a deep hyperplane h' (on Figure 2.1 their common corresponding projective hyperplane is denoted by h), the deep hyperplane exists. To see that h' has a regression depth at least $\left\lceil \frac{n}{d+1} \right\rceil$ it is enough to observe that a vertical hyperplane in H_1 corresponds to the same projective hyperplane as some hyperplane through $\mathbf{c}$ in H_2.

We show how to obtain a similar upper bound as that in Theorem 1.1 for an analog of $\tau_d(n)$ for collections of flat sets in $\mathbb{R}^d$.

If $\mathcal{C}$ is a collection of n flat sets in $\mathbb{R}^d$, by $\mathcal{H}(\mathcal{C})$ we denote one of the smallest collections of hyperplanes each of which contains a set from $\mathcal{C}$, and covering all sets in $\mathcal{C}$. Let $\mathcal{C}$ be a collection of n pairwise disjoint flat compact convex sets, so that we can choose $\mathcal{H}(\mathcal{C})$ such that none of the hyperplanes in $\mathcal{H}(\mathcal{C})$ is vertical (we can achieve this by choosing an appropriate coordinate system), and no k hyperplanes in $\mathcal{H}(\mathcal{C})$ have non-empty intersection. If $\mathcal{H}(\mathcal{C})$ can be chosen such that $k = d+1$, we say that $\mathcal{C}$ is in *general position.*

We apply the dual version of Theorem 2.1 on $\mathcal{H}(\mathcal{C})$ thereby obtaining a point $\mathbf{p} \in \mathbb{R}^d$. Let $\mathbf{p}' \in \mathbb{R}^d$ be $\mathbf{p}$, if $\mathbf{p}$ does not belong to any hyperplane in $\mathcal{H}(\mathcal{C})$. Otherwise let $\mathbf{p}'$ be a point very close to $\mathbf{p}$ that does not belong to any hyperplane in $\mathcal{H}(\mathcal{C})$, i.e. a point so that for all $H \in \mathcal{H}(\mathcal{C})$, such that $\mathbf{p} \notin H$, $\mathbf{p}'$ and $\mathbf{p}$ belong to the same half-space defined by H. For any ray r' let $\mathbf{p}(r')$ denote its translation starting at $\mathbf{p}$. We show that $\mathbf{p}'$ is the point having $\tau(\mathbf{p}, \mathcal{C}) \leq \left\lfloor \frac{dn}{d+1} \right\rfloor + k - 1$. If there were a ray r' from $\mathbf{p}'$ intersecting more than $\left\lfloor \frac{dn}{d+1} \right\rfloor + k - 1$ sets in $\mathcal{C}$, the opposite ray of $r = \mathbf{p}(r')$ would not intersect enough hyperplanes and thereby contradicts Theorem 2.1. Indeed, as $\mathbf{p}'$ does not belong to any hyperplane from $\mathcal{H}(\mathcal{C})$, r' must intersect more than $\left\lfloor \frac{dn}{d+1} \right\rfloor + k - 1$ hyperplanes not parallel to r'. Hence, r must intersect more than $\left\lfloor \frac{dn}{d+1} \right\rfloor \geq \left\lfloor \frac{|\mathcal{H}(\mathcal{C})|d}{d+1} \right\rfloor$ hyperplanes in $H \in \mathcal{H}(\mathcal{C})$ not containing $\mathbf{p}$ that are not parallel to r. Thus, the opposite ray of r must intersect less than $\left\lceil \frac{|\mathcal{H}(\mathcal{C})|}{d+1} \right\rceil$ hyperplanes and thereby yielding contradiction.

On the other hand, if we allow to have in our collection only d-dimensional balls, almost the same upper bound, i.e. $\left\lfloor \frac{dn}{d+1} \right\rfloor + 1$, as that in Theorem 1.1 is obtained by a simple application of the Centerpoint Theorem (Theorem 2.2). The argument is explained in the proof of Theorem 6.13.

It is interesting to note that we have basically the same upper bound for $\tau_d(n)$ as for its analog for collections of flat sets (not necessarily bounded) that are allowed to intersect, if we consider rays to not contain their starting points (otherwise the trivial upper bound n would be tight). In Chapter 4, we explain that this is due to the fact that we can place the objects in a way, so that they behave almost like hyperplanes with respect to the lines going through one point, i.e. any line through that point intersect almost all of them. Since each construction attaining the best known bound is based on the above idea, there is a strong reason to think that it is not possible to avoid some esoteric properties, such as exponential growth of the sizes of objects or the distances among them, in the constructions approaching the upper bound. On the other hand, simple examples show that there are less artificial configurations with τ_d still linear in the number of sets, whilst the situation is considerably different, if we consider just fat objects of bounded diameter (see Chapter 5).

The question solved by Theorem 2.1 was raised as a conjecture by Rousseeuw and Hubert in [19], where they observed that this bound is basically tight as witnessed by a set of n points on the moment curve, $\gamma = \{(t, t^2, \ldots t^d) |\ t \in \mathbb{R}\}$.

Theorem 2.3. *Every hyperplane H has regression depth at most $\left\lfloor \frac{n-d}{d+1} \right\rfloor + d$ with respect to any set of n points on the moment curve γ.*

Proof. Let $\mathcal{P}$ denote a set of n points on γ. To see that every hyperplane H has regression depth at most $m = m(n,d) = \left\lfloor \frac{n-d}{d+1} \right\rfloor + d$ with respect to $\mathcal{P}$, it is enough to introduce for each non-vertical hyperplane H a vertical hyperplane H', such that for $R_1 = R_1(H, H')$ and $R_2 = R_2(H, H')$, $\min\{|\mathcal{P} \cap R_1|, |\mathcal{P} \cap R_2|\} \leq m$ (R_1, R_2 are defined as above).

First assume that H divides γ into less than $d+1$ continuous parts by intersecting it with closed half-spaces defined by H ignoring the parts that are single points. As a vertical hyperplane can be determined by any $d-1$ points on γ, it is not very hard to see that we can choose H' such that either all points in $\mathcal{P}$ not belonging to H belongs to R_1 or all these points belong to R_2. We simply choose H' such that it passes through all endpoints of the considered connected parts of γ, and if we have less than $d-1$ of these endpoints, none of the other possible intersections of H' with γ lie between a pair in $\mathcal{P}$ with respect to γ, nor coincides with a point in $\mathcal{P}$.

Otherwise, we can obtain $d+1$ subsets of $\mathcal{P}$ (some of which might be empty) whose union is $\mathcal{P}$, such that each subset belongs to the same continuous part of γ in a closed half-space defined by H. Note that now none of the parts could be a single point as otherwise we would end up in the previous case. One of these subsets $\mathcal{S}$ has to consist of at most $\left\lfloor \frac{n-i}{d+1} \right\rfloor + 2$ points, where i is the number of points from $\mathcal{P}$ contained in H. Similarly as in the previous case, now, we can choose a vertical hyperplane H' such that either R_1 or R_2 contains at most $|\mathcal{S}| - 2 + i = \left\lfloor \frac{n-i}{d+1} \right\rfloor + i$ points. Clearly, this expression is maximized, if $i = d$. We choose H' such that it passes through all endpoints of the considered continuous parts of γ except one part that contains $\mathcal{S}$. Moreover, if we have $d-1$ of these endpoints, one more possible intersection of H' with γ does not lie between a pair in $\mathcal{P}$ with respect to γ, nor coincides with a point in $\mathcal{P}$.

□

Seeing all these similarities between regression depth and τ_d function, not to mention The Centerpoint Theorem, one would be quite surprised, if the upper bound in Theorem 2.1 were not tight. So far, we do not even have a proof that $\tau_d(n)$ is monotone with respect to the dimension. As we have indicated in the introduction, for our general case, we have only configurations that match the upper bound of $\tau_d(n)$ in less than 3 dimensions.

Chapter 3

General upper bound

In this section we give the proof of the general upper bound of Theorem 1.1. The main source of inspiration for the following proof is Chakerian's topological proof of Helly's Theorem from [6] using two classical results: Brouwer's Fixed Point [5] Theorem and Carathéodory Theorem (see e.g. [16]). Let $\mu(A)$, for $A \subseteq \mathbb{R}^d$, denote the Lebesgue measure (see e.g. [22]). Let us define the center of mass $m(A)$, for $A \subseteq \mathbb{R}^d$ such that $\mu(A) > 0$, as the center of mass with respect to the Lebesgue measure μ, i.e. $m(A) = \frac{1}{\mu(A)} \int_{\mathbf{a} \in A} \mathbf{a} d\mu$.

Proof. **(Theorem 1.1)** Let $\mathcal{C}$ be a collection of n convex compact pairwise disjoint sets in $\mathbb{R}^d$. Without loss of generality we can assume that all sets in $\mathcal{C}$ are properly contained in a unit ball B^d in $\mathbb{R}^d$ centred at $\mathbf{0}$.

In the following we define a continuous mapping f from $2B^d$ (our choice of domain for f will be explained later) to itself that depends on the positioning of the sets in $\mathcal{C}$. We proceed in several stages.

For each $S \subseteq \mathbb{R}^d$, let $K_S(\mathbf{p})$ be the convex hull of $S \cup \{\mathbf{p}\}$. Let $K_{\mathcal{C}'}(\mathbf{p})$ denote $\bigcap_{S \in \mathcal{C}'} K_S(\mathbf{p})$, for any $\emptyset \neq \mathcal{C}' \subseteq \mathcal{C}$. We define an auxiliary function $f_{\mathcal{C}'}$ for any $\emptyset \neq \mathcal{C}' \subseteq \mathcal{C}$ from $2B^d$ to $\mathbb{R}^d$:

$$f_{\mathcal{C}'}(\mathbf{p}) = \begin{cases} \mu(K_{\mathcal{C}'}(\mathbf{p}))(m_{\mathcal{C}'}(\mathbf{p}) - \mathbf{p}) & \mu(K_{\mathcal{C}'}(\mathbf{p})) > 0 \\ \mathbf{0} & otherwise \end{cases}$$

where $m_{\mathcal{C}'}(\mathbf{p}) = m(K_{\mathcal{C}'}(\mathbf{p}))$.

In what follows we show that $f_{\mathcal{C}'}$ is continuous for any $\emptyset \neq \mathcal{C}' \subseteq \mathcal{C}$.

Let $\{\mathbf{p_i}\}_{i=0}^{\infty}$ be the sequence of points in $\mathbb{R}^d$ converging to $\mathbf{p}$ (in Euclidean distance metric). To prove that $f_{\mathcal{C}'}$ is continuous, it is enough to show that

$$\lim_{i \to \infty} f_{\mathcal{C}'}(\mathbf{p_i}) = f_{\mathcal{C}'}(\mathbf{p}) \tag{3.1}$$

To demonstrate this we first prove, that for $S \in \mathcal{C}'$

$$\lim_{i \to \infty} K_S(\mathbf{p_i}) = K_S(\mathbf{p}) \tag{3.2}$$

in Hausdorff metric.

Let $|\mathbf{pp_j}| < \epsilon$, $\epsilon \in \mathbb{R}$, $j \in \mathbb{N}$. For every point $\mathbf{q} \in K_S(\mathbf{p})$, $\mathbf{q} = \alpha\mathbf{p} + (1-\alpha)\mathbf{r}$, where $\alpha \in \mathbb{R}, 0 < \alpha < 1$ and $\mathbf{r} \in S$, we can choose the point $\mathbf{q}' \in K_S(\mathbf{p_j})$ where $\mathbf{q}' = \alpha\mathbf{p_j} + (1-\alpha)\mathbf{r}$. It is easy to see that $|\mathbf{qq}'| < \epsilon$. By reversing the role of $\mathbf{p}$ and $\mathbf{p_j}$ in the above argument we show that Hausdorff distance between $K_S(\mathbf{p})$ and $K_S(\mathbf{p_j})$ is at most ϵ. That completes the proof of (3.2).

If $\mu(K_{\mathcal{C}'}(\mathbf{p})) > 0$, using (3.2) and convexity of $K_S(\mathbf{p_i})$, for every $i \in \mathbb{N}$ and $S \in \mathcal{C}$, one can show that $\lim\limits_{i \to \infty} K_{\mathcal{C}'}(\mathbf{p_i}) = K_{\mathcal{C}'}(\mathbf{p})$, in Hausdorff metric, as $K_{\mathcal{C}'}(\mathbf{p_i}) = \bigcap_{S \in \mathcal{C}'} K_S(\mathbf{p_i})$ for every $i \in \mathbb{N}$. We skip rather lengthy proof of this fact, as to prove it one can proceed by a straightforward application of some standard techniques.

Otherwise, i.e. if $\mu(K_{\mathcal{C}'}(\mathbf{p})) = 0$, we will show that the limit is the same. To see this it is enough to show that for any $\epsilon > 0$ we can choose a $\delta > 0$ so that if $|\mathbf{pq}| < \delta$, for some $\mathbf{q} \in \mathbb{R}^d$, then $\mu(K_{\mathcal{C}'}(\mathbf{q})) < \epsilon$.

Clearly, we have

$$\mu(K_{\mathcal{C}'}(\mathbf{q})) = \mu(\bigcap_{S \in \mathcal{C}'} K_S(\mathbf{q})) \leq \mu(\bigcup_{S \in \mathcal{C}'} K_S(\mathbf{q}) \setminus K_S(\mathbf{p})) \leq \sum_{S \in \mathcal{C}'} \mu(K_S(\mathbf{q}) \setminus K_S(\mathbf{p}))$$

On the other hand by (3.2) for any $S \in \mathcal{C}'$ and $\epsilon/|\mathcal{C}'|$ we can choose δ_S such that if $|\mathbf{pq}'| < \delta_S$, for some $\mathbf{q}' \in \mathbb{R}^d$, then by boundedness of S $\mu(K_S(\mathbf{q}') \setminus K_S(\mathbf{p})) < \epsilon/|\mathcal{C}'|$. We let $\delta = \min\limits_{S \in \mathcal{C}} \delta_S$. Hence, $\lim\limits_{i \to \infty} \mu(K_{\mathcal{C}'}(\mathbf{p_i})) = \mu(K_{\mathcal{C}'}(\mathbf{p}))$, for all $\mathbf{p} \in \mathbb{R}^d$.

It is a routine to show that $\lim\limits_{i \to \infty} m_{\mathcal{C}'}(\mathbf{p_i}) = m_{\mathcal{C}'}(\mathbf{p})$, whenever $\mu(K_{\mathcal{C}'}(\mathbf{p})) > 0$. Again we omit the proof, because it is just too technical. We finish the proof of (3.1) by considering separately the cases when $\mu(K_{\mathcal{C}'}(\mathbf{p})) > 0$ and $\mu(K_{\mathcal{C}'}(\mathbf{p})) = 0$. In the former case we use the fact that the sequences $\{\mu(K_{\mathcal{C}'}(\mathbf{p_i}))\}_{i=0}^{\infty}$, $\{\mathbf{p_i}\}_{i=0}^{\infty}$ and $\{m_{\mathcal{C}'}(\mathbf{p_i})\}_{i=0}^{\infty}$ converge to $\mu(K_{\mathcal{C}'}(\mathbf{p}))$, $\mathbf{p}$ and $m_{\mathcal{C}'}(\mathbf{p})$, respectively, if $\{\mathbf{p_i}\}_{i=0}^{\infty}$ converges to $\mathbf{p}$. The latter case is resolved by observing that we have $\mathbf{i_0} \in \mathbb{N}$ such that $m_{\mathcal{C}'}(\mathbf{p_i})$ is inside of $2B^d$ for any $\mathbf{i} > \mathbf{i_0}$. Moreover, as now $\{\mu(K_{\mathcal{C}'}(\mathbf{p_i}))\}_{i=0}^{\infty}$ converge to 0, we have $\lim\limits_{i \to \infty} f_{\mathcal{C}'}(\mathbf{p_i}) = \mathbf{0} = f_{\mathcal{C}'}(\mathbf{p})$, and that concludes the proof of the fact that $f_{\mathcal{C}'}$ is continuous for any $\emptyset \neq \mathcal{C}' \subseteq \mathcal{C}$

Before we define the function f we need to slightly enhance our collection of sets as follows. We inflate each set in $\mathcal{C}$ by a small ϵ obtaining a new collection $\mathcal{C}_\epsilon$, in which every pair of sets is still disjoint. By inflating by ϵ we mean conversion of a set S into the set $\text{infl}_\epsilon(S) = \{\mathbf{p} \in \mathbb{R}^d | \ \exists \mathbf{q} \in S \ |\mathbf{pq}| \leq \epsilon\}$. We can do that, because by compactness of our sets there exists $\epsilon > 0$ such that the minimal distance between the points of every pair of sets in $\mathcal{C}$ is more than 2ϵ. Clearly, inflating preserves the compactness, and by the following proposition we know that inflating preserves convexity as well.

Proposition 3.1. *If $S \subseteq \mathbb{R}^d$ is convex then $\text{infl}_\epsilon(S)$ is also convex.*

Proof. Let $\alpha \in [0,1]$. If $\mathbf{p}' \in S$ is ϵ-close to $\mathbf{p} \in \text{infl}_\epsilon(S)$, and $\mathbf{q}' \in S$ is ϵ-close to $\mathbf{q} \in \text{infl}_\epsilon(S)$, then $(\alpha\mathbf{p}' + (1-\alpha)\mathbf{q}') \in S$ is ϵ-close to $(\alpha\mathbf{p} + (1-\alpha)\mathbf{q})$. So, $(\alpha\mathbf{p} + (1-\alpha)\mathbf{q}) \in \text{infl}_\epsilon(S)$. □

We will use, that by inflating the sets in $\mathcal{C}$ we make every non-empty cone $K_{\mathcal{C}'}(\mathbf{p})$, $\mathcal{C}' \subseteq \mathcal{C}$, have Lebesgue measure greater than 0. It is also clear that a ray from any point intersects at least as many sets of $\mathcal{C}_\epsilon$ as of $\mathcal{C}$. Hence, we do not lose any generality by this enhancement.

Let τ be the greatest number such that for any point $\mathbf{p}$ there is a ray emanating from $\mathbf{p}$ that intersects τ sets in $\mathcal{C}$. Now, we define the function f' from $2B^d$ to $\mathbb{R}^d$ using the previously defined functions $f_{\mathcal{C}'}$. We have $f'(\mathbf{p}) = \sum_{\mathcal{C}' \subseteq \mathcal{C}_\epsilon, \ |\mathcal{C}'| = \tau} f_{\mathcal{C}'}(\mathbf{p})$.

Finally, we define f from $2B^d$ as follows:

$$f(\mathbf{p}) = \mathbf{p} + \frac{f'(\mathbf{p})}{|f'(\mathbf{p})| + 1}$$

It is easy to see that the image of f belongs to $2B^d$, as the ray emanating from $\mathbf{p} \in 2B^d$ having direction $f'(\mathbf{p}) \neq \mathbf{0}$ intersects B^d, and the length of a vector by which $\mathbf{p}$ is shifted by f is less than 1. Since f is continuous and maps $2B^d$ to $2B^d$, we can apply the Brouwer's Fixed Point Theorem to f to obtain a fixed point of f, $\mathbf{p}$, i.e. $f(\mathbf{p}) = \mathbf{p}$. Let $\mathcal{F}' = \{\mathcal{C}' \subseteq \mathcal{C}_\epsilon | \ f_{\mathcal{C}'}(\mathbf{p}) \neq \mathbf{0}, \ |\mathcal{C}'| = \tau\}$. Since τ is the obstacle number of $\mathcal{C}$, at least one non-empty cone $K_{\mathcal{C}'}(\mathbf{p})$, $\mathcal{C}' \subseteq \mathcal{C}$, $|\mathcal{C}'| = \tau$, exists for $\mathbf{p}$. This cone cannot have Lebesgue measure 0 after inflating. Thus, $\mathcal{F}'$ is non-empty. Clearly, $m_{\mathcal{C}'}(\mathbf{p}) \neq \mathbf{p}$, if $\tau > 1$, as $\mathbf{p}$ has to be on the boundary of $K_{\mathcal{C}'}(\mathbf{p})$, because the sets in $\mathcal{C}_\epsilon$ are disjoint. Moreover, we know that $\mathbf{p}$ is contained in a convex hull of the set $\{m_{\mathcal{C}'}(\mathbf{p}) \mid \mathcal{C}' \in \mathcal{F}'\}$. This can be derived as follows. Let

$a_{\mathcal{C}''}(\mathbf{p}) = \mu(K_{\mathcal{C}''}(\mathbf{p}))/\sum_{\mathcal{C}'\in\mathcal{F}',\ |\mathcal{C}'|=\tau}\mu(K_{\mathcal{C}'}(\mathbf{p}))$, $\mathcal{C}'' \in \mathcal{F}'$. Thus, $\sum_{\mathcal{C}'\in\mathcal{F}',\ |\mathcal{C}'|=\tau} a_{\mathcal{C}'}(\mathbf{p}) = 1$.

$$\begin{aligned} \mathbf{0} &= f'(\mathbf{p}) \\ \sum_{\mathcal{C}'\in\mathcal{F}',\ |\mathcal{C}'|=\tau} \mu(K_{\mathcal{C}'}(\mathbf{p}))\mathbf{p} &= \sum_{\mathcal{C}'\in\mathcal{F}',\ |\mathcal{C}'|=\tau} \mu(K_{\mathcal{C}'}(\mathbf{p}))m_{\mathcal{C}'}(\mathbf{p}) \\ \mathbf{p} &= \sum_{\mathcal{C}'\in\mathcal{F}',\ |\mathcal{C}'|=\tau} a_{\mathcal{C}'}(\mathbf{p})m_{\mathcal{C}'}(\mathbf{p}) \end{aligned}$$

Using Carathéodory Theorem we can choose a subset $\mathcal{F}$ of $\mathcal{F}'$ having size at most $d+1$ such that $\mathbf{p}$ belongs to the convex hull of the set $\{m_{\mathcal{C}'}(\mathbf{p}) \mid \mathcal{C}' \in \mathcal{F}\}$. Let us denote $R_{\mathcal{F}}$ the set of rays from $\mathbf{p}$ having the direction $m_{\mathcal{C}'}(\mathbf{p}) - \mathbf{p}$, for some $\mathcal{C}' \in \mathcal{F}$. It is easy to see that a ray in $R_{\mathcal{F}}$ corresponding to $\mathcal{C}' \in \mathcal{F}$ intersects all sets in $\mathcal{C}'$.

We finish the proof using double counting argument to count the number of pairs (r, S), where $r \in R_{\mathcal{F}}$, $S \in \mathcal{C}'$, and r has the direction $m_{\mathcal{C}'}(\mathbf{p}) - \mathbf{p}$.

Since a set in $\mathcal{C}_\epsilon$ cannot be intersected by all rays in $R_{\mathcal{F}}$, unless $\mathbf{p}$ is contained in it, and the sets in $\mathcal{C}_\epsilon$ are pairwise disjoint, we can have at most one set in $\mathcal{C}_\epsilon$ intersected by all rays in $R_{\mathcal{F}}$. Thus, we have

$$\tau(d+1) \leq \sum_{\mathcal{C}'\in\mathcal{F}} |\mathcal{C}'| \leq (n-1)d + d + 1$$

The above inequality concludes the proof. $\square$

Chapter 4

Lower bounds

A common feature of all collections of pairwise disjoint compact convex sets presented in this chapter is that all lines through certain points intersect almost all sets in it or almost all sets in a sub-collection. Therefore we introduce a notion of centre with respect to the collection of sets in $\mathbb{R}^d$. By the *centre* of a collection of sets in $\mathbb{R}^d$ we understand a point in $\mathbb{R}^d$ such that any line through it intersects all but at most $d-1$ sets in the collection.

4.1 General bound

We present a simple general construction providing relatively good lower bound on $\tau_d(n)$. The interesting fact about this construction is the presence of a variant of it as a building block in both known configuration that match the upper bound in $\mathbb{R}^2$. Therefore one might suspect that it is unavoidable in such constructions.

Let $\mathcal{H} = \{H_1, \dots H_n\}$ be the set of n hyperplanes in $\mathbb{R}^d$ containing $\mathbf{0}$ in general position. We note that $\mathbf{0}$ will be the centre of our construction.

Let $\epsilon \in \mathbb{R}$ be the minimum of the function $f: S^{d-1} \to \mathbb{R}$ that for a given point returns its (Euclidean) distance from dth closest hyperplane in $\mathcal{H}$. As each d-tuple of the hyperplanes in $\mathcal{H}$ has only $\mathbf{0}$ in its intersection, $f(\mathbf{p}) > 0$ for all $\mathbf{p} \in S^{d-1}$. Hence, by the compactness of S^{d-1} we have $\epsilon > 0$.

Our construction consists of n pairwise disjoint $(d-1)$-dimensional balls living in the hyperplanes parallel to the hyperplanes in $\mathcal{H}$. Let $\mathcal{B} = \{B_1, \dots B_n\}$ denote the collection of these balls. We construct the balls in $\mathcal{B}$ one by one as follows. Having constructed $\mathcal{B}_i = \{B_1, \dots B_{i-1}\}$, $i < n$, we choose a hyperplane H_i' parallel to H_i such that it is disjoint from $\bigcup_{B\in\mathcal{B}_i} B$. Then we choose $B_i \subseteq H_i'$ so that its projection P_i on S^{d-1} together with $-P_i$ cover all points $\mathbf{p} \in S^{d-1}$ having distance from H_i equal to ϵ or more than ϵ. An appropriate B_i always exists as we can obtain it e.g. by lifting a spherical cap on S^{d-1} defined by a hyperplane parallel to H_i at distance $\epsilon/2$ from H_i to H_i'. By lifting to H_i' we mean the mapping that maps a point $\mathbf{r} \in S^{d-1}$ to the intersection with H_i' of the line through $\mathbf{0}$ and $\mathbf{r}$. The property of $\mathcal{B}$ we are interested in is expressed by the following observation.

Observation 4.1. *From any point $p \in \mathbb{R}^d$ there is a ray that intersects at least $\lceil \frac{n-d+1}{2} \rceil$ elements of $\mathcal{B}$.*

Proof. Let L be the line through $\mathbf{0}$ and $\mathbf{p} \in \mathbb{R}^{\mathbf{d}}$, if $\mathbf{0} \neq \mathbf{p}$, and any line through $\mathbf{0}$, otherwise. If an intersection point $\mathbf{q}$ of L with S^{d-1} were disjoint from $P_i \cup -P_i$ for more than $(d-1)$ balls B_i, $1 \le i \le n$, we would have $f(\mathbf{q}) < \epsilon$. Hence, L intersects all elements of $\mathcal{B}$ but at most $(d-1)$. Thus, one of the rays from $\mathbf{p}$ contained in L intersects at least $\lceil \frac{n-d+1}{2} \rceil$ sets in $\mathcal{B}$. $\square$

The above observation proves Theorem 1.2.

Note that our general construction can easily suit any type of objects, such as d-dimensional balls, simplices etc. We only require that the projections on S^{d-1} of objects in a construction almost cover their corresponding hemispheres of S^{d-1}.

4.2 Constructions in 2 dimensions

As we have said previously our best constructions in lower dimensions use the idea of the general construction in Section 4.1. However, to prove $\tau(\mathcal{C}) > n/2$, that construction, because of its generality and freedom in positioning the sets, is not good as is as a building block. Therefore we need to make the construction more deterministic.

Let us introduce some new terminology needed in the sequel. By a *wedge* with the apex $\mathbf{p}$ we mean a convex hull of two non-colinear rays emanating from $\mathbf{p}$. By a *tangent* to a wedge W we mean a disc D inside W such that both rays defining W are tangents to D.

First, we present the construction in $\mathbb{R}^2$ that consists of pairwise disjoint discs, but can be modified to suit other shapes such as line segments, triangles etc.

We construct a collection $\mathcal{D}$ of $3k$ discs in $\mathbb{R}^2$, for some fixed $k \in \mathbb{N}$, $k > 0$. We partition $\mathcal{D}$ into two subsets: $\mathcal{D}_1$ and $\mathcal{D}_2$, consisting of $2k$ and k elements, respectively.

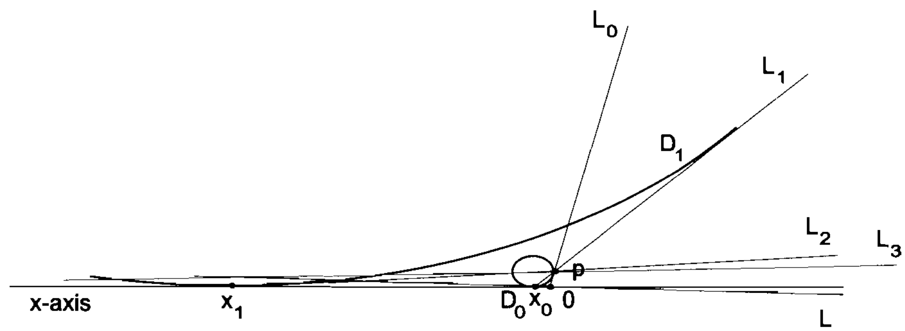

Figure 4.1: First two discs in $\mathcal{D}_1$

The collection $\mathcal{D}_1$ is constructed similarly as our general construction. Hence, we have a centre $\mathbf{p}$ of $\mathcal{D}_1$ and the set of $2k$ lines through $\mathbf{p}$ that divide a circle S' centred at $\mathbf{p}$ into $4k$ arcs. However, contrary to the general construction, where the projection of a ball in it covers all points in one hemisphere except those that were very close to its boundary, a projection of each disc in $\mathcal{D}_1$ on S' covers all $2k$ arcs belonging to one closed hemisphere except at most one arc having one end on its boundary.

$\mathcal{D}_2$ is the collection of discs whose elements are tangents to a certain wedge.

We start with the description of $\mathcal{D}_1 = \{D_0, \dots D_{2k-1}\}$ (Figure 4.1). Let $\mathbf{p} = (1, 1)$ and let $L_0 \subseteq \mathbb{R}^2$ be the line through $\mathbf{0}$ and $\mathbf{p}$. We place D_0 as a tangent to the wedge with the apex $\mathbf{0}$ having the negative part of x-axis as one boundary ray, and with another boundary ray inside L_0 containing the point $\mathbf{p}$. Moreover, we require that $\mathbf{p} \in D_0$. Having constructed $\mathcal{D}_i = \{D_0, \dots D_i\}$, for some $i < 2k - 1$, we put the next disc as a tangent to the wedge with the apex $\mathbf{x_i}$, where $\mathbf{x_i}$ is the point of D_i on x-axis. One boundary ray of this wedge is contained in the negative part of x-axis, and the other one contains $\mathbf{p}$. Since the union of the discs in $\mathcal{D}_i$ is bounded, we can always place D_{i+1} such that it is disjoint from all previous discs.

Let us call L_{i+1} the line through $\mathbf{x_i}$ and $\mathbf{p}$, for $0 \leq i < 2k$, and let L be a line through $\mathbf{0}$ passing below all intersections $D_i \cap L_{i+2}$, for $0 \leq i < 2k - 1$, and above the negative part of x-axis. The second sub-collection $\mathcal{D}_2$ consists of k discs that are tangents to the wedge with the apex $\mathbf{0}$, with one boundary ray being the negative part of x-axis and another boundary ray being the ray below x-axis belonging to L. Moreover, all discs in $\mathcal{D}_2$ touch x-axis in the points, whose x-coordinates are less than the x-coordinate of $\mathbf{x_{2k-1}}$. To bound $\tau(\mathcal{D})$ from

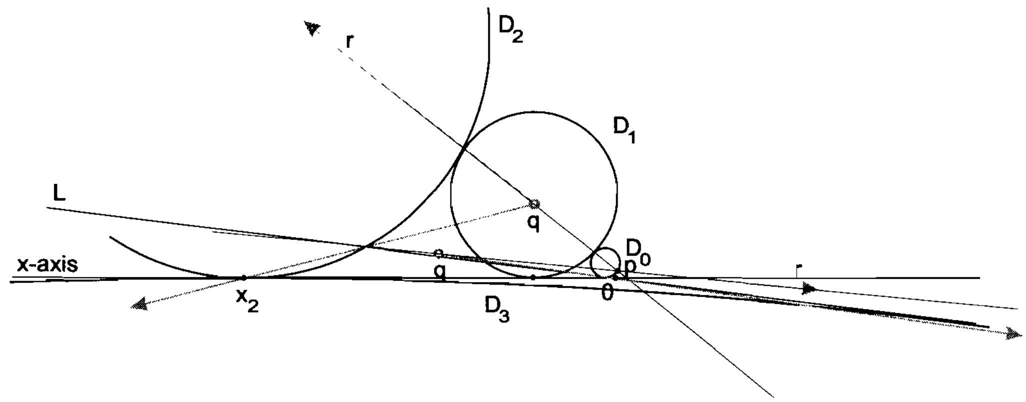

Figure 4.2: Illustration of the proof of Lemma 4.2

below we prove the following.

Lemma 4.2. *For every point* $\mathbf{q} \in \mathbb{R}^2$ *there is a ray starting at* $\mathbf{q}$ *that intersects at least* $2k-2$ *discs in* $\mathcal{D}$.

Proof. We do a simple case analysis according to the region, which $\mathbf{q}$ belongs to.

First suppose that $\mathbf{q}$ lies below or on the x-axis. In this case any ray through $\mathbf{p}$ intersects at least all sets in $\mathcal{D}_1$ but at most one. As $|\mathcal{D}_1| = 2k$ we are done with this case.

Otherwise, assume that $\mathbf{q}$ belongs to the wedge bounded by half of L from above and by the negative part of the x-axis from below. Similarly as in the previous case any ray r from $\mathbf{q}$ through $\mathbf{p}$ intersects all sets in $\mathcal{D}_1$ but at most two. To see this consider $m = \max\{j \in \mathbb{N} |\ \mathbf{q}$ lies above or on $L_j\}$. It is easy to see that r has to intersect all sets in $\mathcal{D}_1$ except D_m and D_{m-1}, if $m > 0$, and all sets in $\mathcal{D}_1$, otherwise.

The rest of the points will have a desired ray through either $\mathbf{0}$ or $\mathbf{x_{2k-1}}$, if its y coordinate is less than or equal to 1, and through either $\mathbf{0}$, $\mathbf{p}$ or $\mathbf{x_{2k-1}}$, if its y coordinate is more than 1 (Figure 4.2). Observe that now every ray from $\mathbf{q}$ through any point of the line segment $\mathbf{0x_{2k-1}}$ intersects all discs in $\mathcal{D}_2$.

In the former case a ray r from $\mathbf{q}$ through either $\mathbf{0}$ or $\mathbf{x_{2k-1}}$ intersects at least k discs in $\mathcal{D}_1$. As r intersects all discs in $\mathcal{D}_2$, we are done.

To see the latter case consider the line L' through $\mathbf{p}$ and $\mathbf{q}$. One of two rays from $\mathbf{q}$ contained in L' intersects at least k elements in $\mathcal{D}_1$. Let us call it r. If r does not contain $\mathbf{p}$, the ray r' from $\mathbf{q}$ through $\mathbf{x_{2k-1}}$ intersects all elements in $\mathcal{D}_2$ and at least k elements in $\mathcal{D}_1$.

Indeed, if we call $\mathbf{y}$ a point in the intersection of a disc D_i, for some $\mathbf{q} \notin D_i$, $0 \leq i < 2k$, in $\mathcal{D}_1$ with r then the line segment $\mathbf{yx_i}$, and therefore also D_i, is intersected by r'. This follows from the fact that in this case $\mathbf{yx_i}$ lies in the same closed half-plane defined by L' as $\mathbf{x_{2k-1}}$. If r contains $\mathbf{p}$, we choose as a desired ray r provided, that r intersect all elements in $\mathcal{D}_2$. Otherwise, r has to intersects x-axis in a point $(a, 0)$ with a less than x-coordinate of $\mathbf{x_{2k-1}}$ or $a > 0$. If $a > 0$ (the other case is treated analogously), we choose as a desired ray a ray r' through $\mathbf{0}$. The ray r' has to intersect every disc in $\mathcal{D}_1$ intersected by r by the same argument as we use above. Trivially, r' has to intersect all discs in $\mathcal{D}_2$, and that concludes the proof. □

The above Lemma proves Theorem 1.3.

We present another construction in $\mathbb{R}^2$ (see Figure 4.3 as an illustration). The objects in this construction are congruent but they are not fat. This complements the first construction where the objects were discs, but their sizes were different.

Intuitively, we take an equilateral triangle, extend all its sides in one direction preserving symmetry and lop off the small part from the beginning of each extended side in order to make them disjoint. It is easy to see that for each point in the plane there is a ray that meets at least two of these three adjusted sides of the triangle. Our construction is a blow up of this simple formation, in which we replace each segment by k almost parallel line segments that are very close to each other, and that are placed as in our general construction. Thus, our new construction $\mathcal{L}_\triangle$ can be partitioned into three disjoint collections of line segments $\mathcal{L}_{\triangle 1}, \mathcal{L}_{\triangle 2}$ and $\mathcal{L}_{\triangle 3}$ with equal number of elements, i.e. $|\mathcal{L}_{\triangle 1}| = |\mathcal{L}_{\triangle 2}| = |\mathcal{L}_{\triangle 3}| = k$.

We note that in the following construction the fact that the line segments are tangents to a circle is not essential. We chose this way of presentation just to make the argument more precise. Any collection of k line segments as in the general construction sufficiently close to each other would be equally fine.

First, we construct $\mathcal{L}_{\triangle 1}$. The line segments $L_{1j} \in \mathcal{L}_{\triangle 1}$ with endpoints $\mathbf{a_{1j}} \in S_1$ and $\mathbf{a'_{1j}}$, $j = 1, \ldots k$, are tangents to some unit circle S_1 with the centre $\mathbf{c_1}$ and their length is 2. All line segments in $\mathcal{L}_1$ leave the circle S_1 in the clockwise direction and the endpoints $\mathbf{a_{1j}}$, $j = 1, \ldots k$, on S_1 are ordered counterclockwise preserving the following exponential growth of the central angles defined by these points. We let $\mathbf{a'} = \mathbf{a_{11}}$, as this point will be a centre of $\mathcal{L}_{\triangle 1}$. Let $\measuredangle \mathbf{a_{11}c_1a_{1k}} = \alpha \in (0, \pi/100)$, then $\measuredangle \mathbf{a_{11}c_1a_{1l}} = \alpha/3^{k-l}$, $2 \leq l \leq k$. Let $\mathbf{a}$ denote a point on the line L_1 containing L_{11}, $\mathbf{a} \notin L_{11}$, such that $\measuredangle \mathbf{a_{1k}c_1a} = \alpha$. Next we

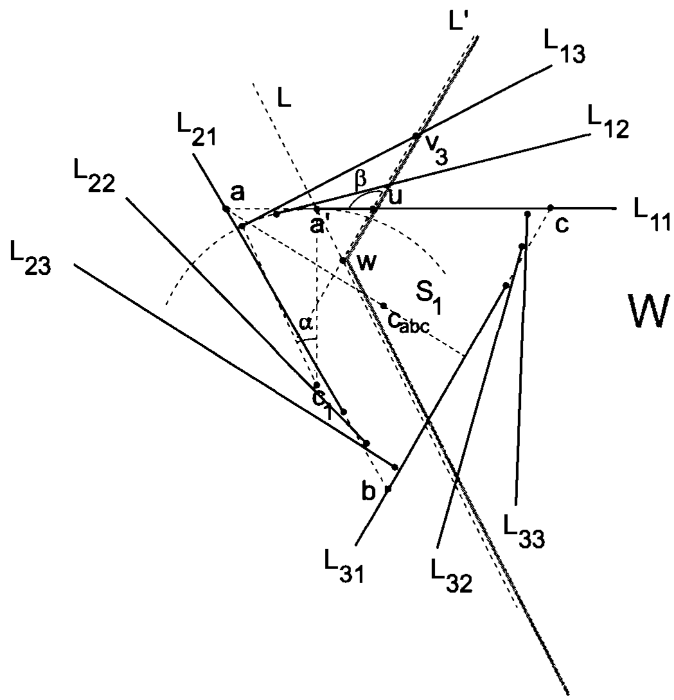

Figure 4.3: 3k line segments (k=3)

construct the equilateral triangle $\mathbf{abc}$, $\mathbf{c} \in L_{11}$, whose each side has unit length and that belongs to the same half plane defined by L_1 as the circle S_1. Having constructed $\mathcal{L}_1$ we can easily obtain $\mathcal{L}_i$, $i = 2, 3$, by the rotations r_i of $\mathcal{L}_1$ around the centre point $\mathbf{c_{abc}}$ of the triangle $\mathbf{abc}$ that send point $\mathbf{a}$ to $\mathbf{b}$, if $i = 2$, and to $\mathbf{c}$, if $i = 3$. Clearly, $\mathcal{L}$ is invariant under r_1 and r_2. We denote the line segments in $\mathcal{L}_{\triangle i}$, $i = 2, 3$ and their endpoints according to their preimages in r_i, i.e. $L_{1j}, \mathbf{a_{1j}}, \mathbf{a'_{1j}} \to L_{ij}, \mathbf{a_{ij}}, \mathbf{a'_{ij}}$. Analogously we label as L_2 and L_3 the lines containing L_{21} and L_{31}, respectively. The following Lemma proves Theorem 1.4.

Lemma 4.3. *We can choose $\alpha > 0$ such that for every point $\mathbf{p} \in \mathbb{R}^2$ there is a ray starting at $\mathbf{p}$ that intersects $2k - 1$ sets in $\mathcal{L}_\triangle$.*

Proof. We call L the line through $\mathbf{a'}$ and $\mathbf{a'_{2k}}$. As we found it convenient to make the construction depicted on Figure 4.3 look more comprehensible at a cost of loss in precision, the line L on it does not look like going through $\mathbf{a'_{2k}}$.

Clearly, L crosses all line segment in $\mathcal{L}_{\triangle 2}$.

Let δ be the smaller angle that is defined by L and L_{2k}. It is routine to show that $\alpha \to 0$ implies $\delta \to 0$. Let $\mathbf{u} \in \mathbf{a'c}$, such that $|\mathbf{a'u}| = 2\alpha$ and let L' be the line through $\mathbf{u}$ such that $\measuredangle \mathbf{a'uv_k} = 2\pi/3 - \alpha - \delta = \beta$, $\mathbf{v_i} = L' \cap L_{1i}$, $1 \le i \le k$. Clearly, L' intersects all segments in $\mathcal{L}_{\triangle 1}$. We denote by $\mathbf{w}$ the intersection $\mathbf{w} = L' \cap L$. Let W denote the wedge (on Figure 4.3 indicated by grey lines) with the apex $\mathbf{w}$, and with one boundary ray R_w containing the line segment $\mathbf{wu}$ and the other, inside L, not containing $\mathbf{a'_{2k}}$. Clearly, the angle defined by W has the size $2\pi/3$. Moreover, as with $\alpha \to 0$ also $\delta \to 0$ and $\mathbf{w} \to \mathbf{a}$, we can choose α, $0 < \alpha$ such that W contains $\mathbf{c_{abc}}$. From now on let $\alpha > 0$ be so that W contains $\mathbf{c_{abc}}$.

The rotations r_2 and r_3 of W gives us two wedges that together with W cover the whole plane. As our construction is invariant under these rotations, to prove the lemma, it will be enough to show that for every point $\mathbf{p}$ in W the ray from $\mathbf{p}$ through $\mathbf{a'}$ meets at least $2k - 1$ members of $\mathcal{L}_\triangle$. Moreover, it suffices to consider only the case when $\mathbf{p} \in R_w$, as the ray from $\mathbf{p}$ in W through $\mathbf{a'}$ meets R_w. Since L and the line parallel to L' through $\mathbf{a'}$ meets all members of $\mathcal{L}_{\triangle 2}$, also a ray through $\mathbf{a'}$ from some point in W meets them. Thus, it is enough to show that the ray through $\mathbf{a'}$ from every point $\mathbf{p}$ on R_w meets all but at least one line segments in $\mathcal{L}_{\triangle 1}$. We proceed as follows.

If we draw a line L'' (Figure 4.4) through $\mathbf{a_{11}} = \mathbf{a'}$ and $\mathbf{a_{1(i+2)}}$, for some $1 \le i < k - 1$, L'' intersects $L_{1(i+1)}$ and L_{1i} in the points $\mathbf{d'_i}$ and $\mathbf{d_i}$, respectively. The existence of these intersections is guaranteed by an exponential growth of the sizes of the central angles defined

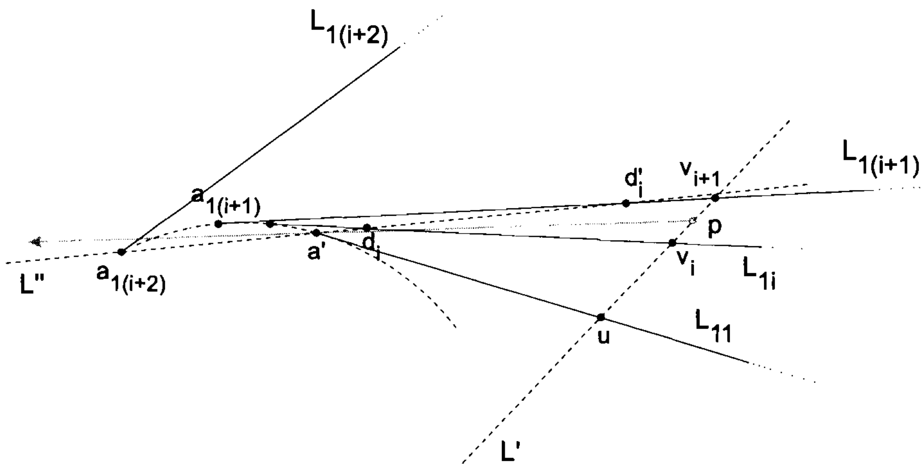

Figure 4.4: Illustration for the proof of Lemma 4.3

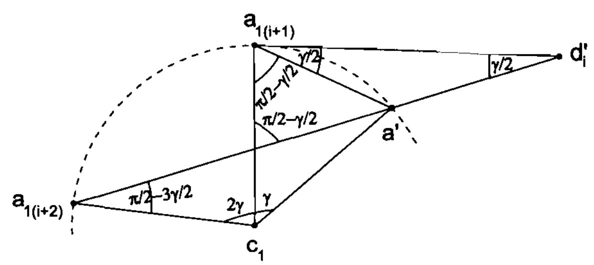

Figure 4.5: The triangle $a_{1(i+1)}a'd'_i$

by the touching points of the line segments. The straightforward geometric argument, based on the fact that the triangle $\mathbf{a'd'_ia_{1(i+1)}}$ is isosceles with the base $\mathbf{a_{1(i+1)}d'_i}$ and the base angle $1/2\measuredangle\mathbf{a'c_1a_{1(i+1)}}$ (Figure 4.5), shows that $|\mathbf{d'_ia_{1(i+1)}}|$ is less than $2\sin\alpha$. As we have $|\mathbf{d_ia_{1i}}| < |\mathbf{d'_ia_{1(i+1)}}| < 2\alpha = |\mathbf{a_{11}u}| < |\mathbf{a_{1i}v_i}| < |\mathbf{a_{1(i+1)}v_{i+1}}|$, the ray through $\mathbf{a'}$ starting at $\mathbf{p} \in \mathbf{v_iv_{i+1}}$, for any $1 \leq i < k-1$, intersects $L_{11}, \ldots L_{1i}$ on the way to $\mathbf{a'}$ and after it meets the circle S_1 for the second time, it intersects the rest of the line segments in $\mathcal{L}_{\triangle 1}$ with possible exception of not intersecting L_{i+1}. Trivially, a ray through $\mathbf{a'}$ from every point on R_w not between $\mathbf{v_1}$ and $\mathbf{v_k}$ meets all segments in $\mathcal{L}_{\triangle 1}$ and $\mathcal{L}_{\triangle 2}$, and that concludes the proof.

□

It is a natural idea to generalize the construction $\mathcal{L}_\triangle$ into higher dimensions. However, our attempts to do it failed, because in the higher dimensions we are unable to make sure that all points below some 'good' hyperplane with respect to the one of the sub-collections (think of an analog of L' with respect to $\mathcal{L}_{\triangle 1}$) have rays intersecting all objects in more than one other sub-collection. A better potential for generalization seems to have the construction $\mathcal{D}$, especially when we replace discs by flats, i.e. sets living in hyperplanes. So far, we were able to extend it to three dimensions, but the construction giving us the lower bound $2/3n - 4$, where n is the number of sets in it, is quite complicated and not very nice. Therefore we omitted its presentation. Nevertheless, there is still hope that nice and simple general construction proving lower bound roughly $2/3n$ can be obtained by a generalization of $\mathcal{D}$.

Finally, it is worth to mention that Theorems 1.1, 1.3, and 1.4 together with the treatment in Chapter 5 gives the complete characterization of the behaviour of $\tau_2(\mathcal{C})$ with respect to the fatness and boundedness of our objects.

We conclude this chapter with an open problem.

Open Problem 4.4. *Provide a lower bound on $\tau_d(n)$ higher than $\frac{n}{2} + O(1)$ ($\frac{2n}{3} + O(1)$) in more than 3 dimensions (in 3 dimensions), or an upper bound smaller than $\frac{nd}{d+1} + O(1)$ in more than 2 dimensions*

Chapter 5

Fat objects

Up to this point we have studied the worst case behaviour of an obstacle number for a collection $\mathcal{C}$ of pairwise disjoint compact convex sets without taking into account some realistic assumptions about sets in $\mathcal{C}$, under which it is more likely that $\mathcal{C}$ could directly model some real world situation, e.g. an environment in which we want to establish a wireless network. To address the above clearly reasonable objection, in this chapter, we study the worst case behaviour of the obstacle number for collections consisting of sets in $\mathbb{R}^2$ having, roughly speaking, similar size.

Formally, our restriction on convex sets is expressed by (a, b)-boundedness defined as follows. We call a convex set $A \in \mathbb{R}^d$, (a, b)-*bounded*, where $0 < a < b$, $a, b \in \mathbb{R}$, if it contains a d-dimensional ball of diameter a, and it is contained in a d-dimensional ball of diameter b.

From now on in this Chapter we consider all convex sets to belong to $\mathbb{R}^2$. We show that it is not a coincidence that none of the constructions attaining the upper bound from Theorem 1.1 in the previous section is a collection of (a, b)-bounded sets, because it turns out that the maximum value of $\tau(\mathcal{C})$, if $\mathcal{C}$ consists of n (a, b)-bounded convex sets, is $\Theta(\sqrt{n \log n})$ with the constants hidden in the upper bound of Θ-notation depending on a, b.

Both the upper and lower bound are simple corollaries of the work [2] by N.Alon, M.Katchalski and W.R.Pulleyblank. In that paper they studied the asymptotic behaviour of the function $f(n)$ that returns the minimum integer f such that for any family of n pairwise disjoint (a, b)-bounded convex sets, for fixed $0 < a < b$, there is a direction α such that any line having the direction α intersects at most f convex sets. Let us state their result by the subsequent theorem (follows from Theorem 1.1, Lemma 3.4, and concluding remarks in

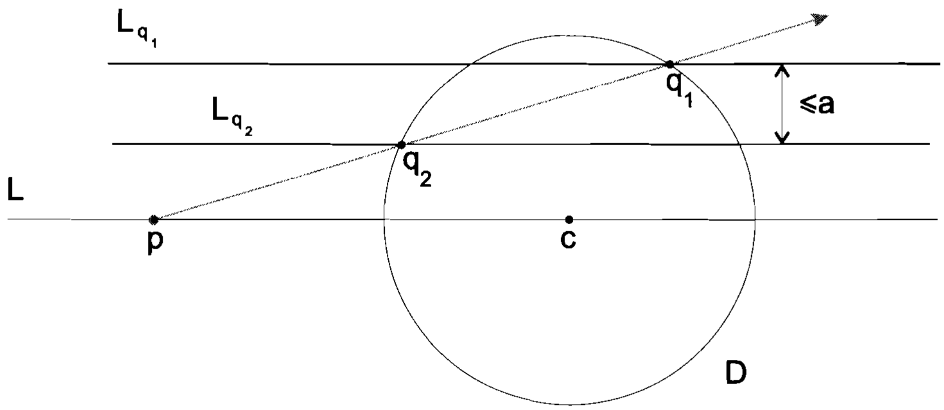

Figure 5.1: Illustration of the proof of Theorem 5.2

[2]).

Theorem 5.1. *There exist two positive constants d_1 and $d_2 = d_2(a, b)$ such that*

$$d_1\sqrt{n \log n} \leq f(n) \leq d_2\sqrt{n \log n}$$

for all $n > 0$.

The proof of the upper bound of $f(n)$ is by a simple counting argument, that could be applied to our case as well. However, in our argument we actually use only the existence of this bound. The construction for the lower bound, that consists of unit discs, is more involved and relies heavily on the famous construction of Besicovitch [4] from his solution to the Kakeya Needle Problem.

Let $\tau'_d(n) = \max_{\mathcal{C}} \tau(\mathcal{C})$, with $\mathcal{C}$ varying over all collections consisting of n pairwise disjoint convex (a, b)-bounded sets, for fixed $0 < a < b$, in $\mathbb{R}^d$. The aim of the following is to show that the order of magnitude of $\tau'_2(n)$ and $f(n)$ is the same, i.e. to prove Theorem 1.5. This result follows immediately from Theorems 5.2 and 5.3.

Theorem 5.2. *Let $\mathcal{C}$ be a collection of $n > 0$ pairwise disjoint (a, b)-bounded sets, for some fixed $0 < a < b$. Then there exists a point $\mathbf{p}$ in the plane such that any ray starting at $\mathbf{p}$ intersects $O(\sqrt{n \log n})$ sets in $\mathcal{C}$ with the constant in O-notation depending on a and b.*

Proof. We denote by D a disc with the centre $\mathbf{c}$ containing all elements of $\mathcal{C}$. Let α be the direction, whose existence is guaranteed by Theorem 5.1, such that any line having this

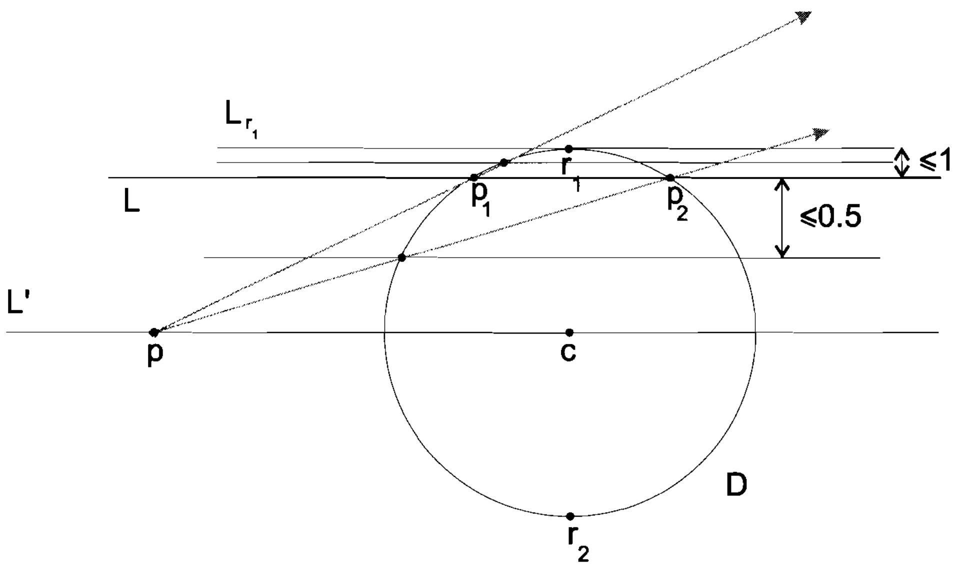

Figure 5.2: Illustration of the proof of Theorem 5.3

direction intersects $f(n) \in O(\sqrt{n \log n})$ sets in $\mathcal{C}$. We call L the line through $\mathbf{c}$ with the direction α. Let $\mathbf{p}$ denote a point such that $\mathbf{p} \in L$, and $\mathbf{p} \notin D$. Given a ray r from $\mathbf{p}$ intersecting the interior of D we denote $\mathbf{q_1}, \mathbf{q_2}$ its two intersections with the boundary of D. We call $L_{\mathbf{q_1}}$ and $L_{\mathbf{q_2}}$ the lines parallel to L through $\mathbf{q_1}$ and $\mathbf{q_2}$, respectively.

If $|\mathbf{pc}| \to \infty$ then $|L_{\mathbf{q_1}} L_{\mathbf{q_2}}| \to 0$. Hence, we can choose $\mathbf{p} \in L$ so that for any ray r from $\mathbf{p}$ intersecting the interior of D the distance between $L_{\mathbf{q_1}}$ and $L_{\mathbf{q_2}}$ is less than a.

Hence, r intersects only the sets intersected by $L_{\mathbf{q_1}}$ or $L_{\mathbf{q_2}}$, i.e. $O(\sqrt{n \log n})$ sets, and that concludes the proof. □

Theorem 5.3. *For each $n > 0$ there exists a collection $\mathcal{E}$ of n pairwise disjoint unit discs in the plane such that for every point $\mathbf{p}$ in the plane there exists a ray starting at $\mathbf{p}$ that intersects $\Omega(\sqrt{n \log n})$ discs in $\mathcal{E}$.*

Proof. Let $\mathcal{D}$ be the collection of $\lfloor n/2 \rfloor$ pairwise disjoint unit discs satisfying the lower bound in Theorem 5.1. We denote by D a disc with the centre $\mathbf{c}$ containing all elements of $\mathcal{D}$.

Let $\mathbf{p}$ denote a point in the plane such that $\mathbf{p} \notin D$. Let L' denote the line through $\mathbf{p}$ and $\mathbf{c}$. We call $\mathbf{r_1}, \mathbf{r_2}$ the two intersection points of the boundary of D with the line through $\mathbf{c}$ perpendicular to L'. Given a ray r from $\mathbf{p}$ intersecting D we denote by $\mathbf{q_1}, \mathbf{q_2}$ its intersections with the boundary of D. We consider $\mathbf{q_1} = \mathbf{q_2}$ if r is a tangent to D. Given a point $\mathbf{s}$ we denote by $L_{\mathbf{s}}$ the line parallel to L' through the point $\mathbf{s}$. It is easy to see that there exists $d \in \mathbb{R}$ such that if $|\mathbf{cp}| > d$, then holds the following. For any ray r from $\mathbf{p}$ that intersects the interior of D the distance between $L_{\mathbf{q_1}}$ and $L_{\mathbf{q_2}}$ is less than $1/2$, and for any ray r from $\mathbf{p}$ that is a tangent to D the minimum of the distances between $L_{\mathbf{q_i}}$ and $L_{\mathbf{r_j}}$, for $i, j \in \{1, 2\}$, is less than $1/2$. We label by D_1 a disc with the centre $\mathbf{c}$ and the diameter d.

It holds that for every direction α there exists a line with direction α intersecting $\Omega(\sqrt{n \log n})$ discs. Let $\mathbf{p}$ be an arbitrary point in the $D_1^c = \mathbb{R}^2 \setminus D_1$. Using the above property of $\mathcal{D}$ we obtain a line L having the same direction as the line containing $\mathbf{p}$ and $\mathbf{c}$, and intersecting $\Omega(\sqrt{n \log n})$ discs in $\mathcal{D}$. Let $\mathbf{p_1}$ and $\mathbf{p_2}$ denote the two intersection points of L with the boundary of D. It can be easily checked (see Figure 5.2) that a ray from $\mathbf{p}$ either through $\mathbf{p_1}$ or $\mathbf{p_2}$ has to intersect at least half of the discs intersected by L. Indeed, every disc in $\mathcal{D}_1$ intersected by L is intersected by the ray from $\mathbf{p}$ through $\mathbf{p_1}$ or $\mathbf{p_2}$.

Next, we obtain the collection $\mathcal{D}'$ of $\lfloor n/2 \rfloor$ pairwise disjoint unit discs by a translation t of $\mathcal{D}$ by the distance of at least $diam(D_1)$ and the addition of one more arbitrary disc if n is odd. Now, using similar argument as for $\mathcal{D}$, we can show that every point in $t(D_1)^c = \mathbb{R}^2 \setminus t(D_1)$ has a ray emanating from it and intersecting enough elements of $\mathcal{D}'$. As $D_1^c \cup t(D_1)^c = \mathbb{R}^2$, setting $\mathcal{E} = \mathcal{D} \cup \mathcal{D}'$ concludes the proof.

Alternatively, one can obtain a better constant hidden in our bound by constructing $\mathcal{D}'$ as a collection of discs with the centres on one line through $\mathbf{c}$ far enough from $\mathbf{c}$. □

As for the higher dimensions d, $d > 2$, it is not hard, using the counting argument from [2], to impose on $\tau_d'(n)$ the upper bound $O(\sqrt{n \log n})$. However, the straightforward generalization of this argument is pretty loose, and hence, leaves room for an improvement. On the other hand, so far, we were able to put only trivial lower bound $\Omega(\sqrt[d]{n})$ on $\tau_d'(n)$, for $d > 2$. The construction $\mathcal{C}$ proving this bound can be obtained by cutting d-dimensional cube regularly with the hyperplanes parallel to the hyperplanes defining its sides. By regularly we mean that by this process we obtain a partition of the cube into many smaller congruent cubes. Thus, $\mathcal{C}$ is the set of these congruent cubes perturbed slightly so that they are pairwise disjoint.

Open Problem 5.4. *Find the right order of magnitude for $\tau_d'(n)$, for $d > 2$, or at least provide a lower bound better than $\Omega(\sqrt[d]{n})$ or an upper bound better than $O(\sqrt{n \log n})$.*

Chapter 6

Algorithms

In this chapter we address the following computational problem: Given a collection $\mathcal{C}$ of pairwise disjoint compact convex sets in $\mathbb{R}^d$, we want to find a point in $\mathbb{R}^d$ that either witnesses our general upper bound from Theorem 1.1, or witnesses $\tau(\mathcal{C})$, i.e. a point that minimizes the maximum number of sets intersected by a ray emanating from it.

For the description and analysis of most of the algorithms in this chapter we use a theoretical model, in which every basic algebraic operation (+,-,*,/) is assumed to be carried out in a constant time. Our algorithms are deterministic.

We present an algorithm that for a given collection $\mathcal{C}$ of pairwise disjoint convex polygons in $\mathbb{R}^2$ finds a point minimizing $\tau(\mathbf{p}, \mathcal{C})$ in a polynomial time. For any dimension we have two algorithms, but these require as an input restricted collections of objects. Both are the reductions to other problems. The first reduction works for collections of pairwise disjoint balls, and basically gives us a witnessing point for the upper bound from Theorem 1.1, i.e. a point $\mathbf{p}$ with $\tau(\mathbf{p}, \mathcal{C}) \leq \left\lfloor \frac{nd}{d+1} + 1 \right\rfloor$. The second reduction requires our sets to be flat and in general position (see Chapter 2) to give us a point having $\tau(\mathbf{p}, \mathcal{C})$ at most $\left\lfloor \frac{nd}{d+1} \right\rfloor + d$. In fact for this reduction we do not need to have sets in general position, but if we relax this constraint, the point $\mathbf{p}$ we obtain can have $\tau(\mathbf{p}, \mathcal{C})$ bigger than $\left\lfloor \frac{nd}{d+1} \right\rfloor + d$.

Let us introduce a standard notion of hyperplane arrangement, which will be needed in the sequel.

Let $\mathcal{H}$ be a collection of hyperplanes in $\mathbb{R}^d$. Let $\mathcal{H}' \subseteq \mathcal{H}$, and let i, $0 \leq i < d$, be the dimension of an affine subspace $A = \bigcap_{H \in \mathcal{H}'} H$. If $\mathcal{H}' = \emptyset$ we set $A = \mathbb{R}^d$. We call a *cell* C defined by $\mathcal{H}$ with $\mathcal{H}'$ a connected component of the complement of $A \cap \bigcup_{H \in \mathcal{H} \setminus \mathcal{H}'} H$ in

A. Then the *dimension* of C is i. Clearly, the cells defined by $\mathcal{H}$ (with any $\mathcal{H}' \subseteq \mathcal{H}$) form a partition of $\mathbb{R}^d$. We denote by $\mathcal{A}(\mathcal{H})$ the *arrangement of hyperplanes* in $\mathcal{H}$, that is a set that consists of all cells defined by $\mathcal{H}$. By the boundary of a cell C we will understand the relative boundary of C.

6.1 Algorithm in 2 dimensions

In this section we present an algorithm running in time $O(n^4 \log n)$ that returns a point $\mathbf{p} \in \mathbb{R}^2$ witnessing $\tau(\mathcal{C})$ for a collection $\mathcal{C}$ of pairwise disjoint convex polygons having n sides in total. In fact, we do not have to restrict ourselves to polygons, but we do it to avoid some technicalities, that would distract our presentation.

Let $S \subset \mathbb{R}^2$ denote a circle. We call a closed arc $I \subset S$ a *cyclic interval.* We denote two endpoints of I by $e_1(I)$ and $e_2(I)$ so that traversing S in the positive direction from $e_1(I)$ to $e_2(I)$ takes place in I. Let us define the ternary relation $\leq_c$ on S, such that $u \leq_c v \leq_c w$, iff v is contained in the cyclic interval with $e_1(I) = u$ and $e_2(I) = w$. We say that two n-tuples of arcs $\mathcal{I} = (I_1, \dots I_m)$ and $\mathcal{I}' = (I'_1, \dots I'_m)$, $I_i \subset S$, $I'_i \subset S'$, for $1 \leq i \leq m$, $S, S' \subset \mathbb{R}^2$ are the circles, have the same combinatorial arrangement, if the cyclic order along S of the endpoints $e_i(I_j)$, for $i \in \{1,2\}$ and $0 < j \leq m$, is the same as the cyclic order along S' of the corresponding endpoints $e_i(I'_j)$, for $i \in \{1,2\}$ and $0 < j \leq m$. It is easy to see that if $\mathcal{I}$ and $\mathcal{I}'$ have the same combinatorial arrangement, then $e_{j_1}(I_{i_1}) \leq_c e_{j_2}(I_{i_2}) \leq_c e_{j_3}(I_{i_3})$ iff $e_{j_1}(I'_{i_1}) \leq_c e_{j_2}(I'_{i_2}) \leq_c e_{j_3}(I'_{i_3})$, for $j_1, j_2, j_3 \in \{1,2\}$ and $i_1, i_2, i_3 \in \{1, \dots m\}$. Thus, a combinatorial arrangement of $\mathcal{I}$ is defined by the restriction of $\leq_c$ on the set of all endpoints of the intervals in $\mathcal{I}$. Let us denote by $\Pi(\mathcal{I})$ this restriction of $\leq_c$.

Let $\mathcal{C} = \{C_1, \dots C_m\}$ be a collection of pairwise disjoint compact convex sets in $\mathbb{R}^2$. Given a point $\mathbf{p} \in \mathbb{R}^2$, we denote by $\mathcal{I}_{\mathcal{C}}(\mathbf{p})$ the n-tuple of arcs $(I_{C_1}(\mathbf{p}), \dots I_{C_m}(\mathbf{p}))$ we get by projections of all elements in $\mathcal{C}$. I_{C_i} is the projection of C_i, for $1 \leq i \leq m$, on a unit circle S centred at $\mathbf{p}$, if $\mathbf{p} \notin C_i$, and by definition we set $I_{C_i} = \emptyset$, otherwise. Let $\mathcal{I}_{C,\mathbf{p}}$ be the set of cyclic intervals covering S, whose elements are bounded by two consecutive endpoints in the cyclic ordering of the endpoints of intervals in $\mathcal{I}_{\mathcal{C}}(\mathbf{p})$. Let $d(I)$, for $I \in \mathcal{I}_{C,\mathbf{p}}$, denote the number of intervals in $\mathcal{I}_{\mathcal{C}}(\mathbf{p})$ covering I.

Observation 6.1. *The values of $d(I)$, for $I \in \mathcal{I}_{C,\mathbf{p}}$, are determined by $\Pi(\mathcal{I}_{\mathcal{C}}(\mathbf{p}))$.*

Proof. Follows easily from the fact that $d(I) = |\{I' \in \mathcal{I}_{C,\mathbf{p}}|\ e_1(I') \leq_c e_1(I) \leq_c e_2(I) \leq_c e_2(I')\}|$ □

Observation 6.2. *$\tau(\mathbf{p}, \mathcal{C}) = \max_{I \in \mathcal{I}_{\mathcal{C},\mathbf{p}}} d(I)$, if $\mathbf{p} \notin C$, for all $C \in \mathcal{C}$, and $\tau(\mathbf{p}, \mathcal{C}) = \max_{I \in \mathcal{I}_{\mathcal{C},\mathbf{p}}} d(I) + 1$, otherwise.*

In the light of the above observations all we need to do in order to find a point $\mathbf{p}$ minimizing $\tau(\mathbf{p}, \mathcal{C})$ is to partition $\mathbb{R}^2$ into finitely many connected components such that within each region R combinatorial arrangements of $\mathcal{I}_{\mathcal{C}}(\mathbf{q})$, $\mathbf{q} \in R$, do not change, and then traverse those regions while computing $\max_{I \in \mathcal{I}_{\mathcal{C},\mathbf{q} \in R}} d(I)$, for each region R. It turns out that such partition can be obtained by cutting $\mathbb{R}^2$ with polynomially many lines with respect to the total number of sides of polygons in $\mathcal{C}$.

By a *tangent* to a convex set C we will understand a line L having a non-empty intersection with C such that C is contained in a closed half-plane determined by L. Let C_1 and C_2 denote two disjoint compact convex sets in $\mathbb{R}^2$. The following observation about common tangents of C_1 and C_2 in $\mathbb{R}^2$ is well-known.

Observation 6.3. *C_1 and C_2 have at most 4 common tangents.*

We have another simple observation telling us how to partition $\mathbb{R}^2$ such that the combinatorial arrangement of $\mathcal{I}_{\{C_1,C_2\}}(\mathbf{p})$ is invariant for all $\mathbf{p}$ within any part (see Figure 6.1 for an illustration). Let $\mathcal{L} = \mathcal{L}(C_1, C_2) = \{L_i | \ i \in \{0, \dots 3\}\}$ be the set of common tangents of C_1 and C_2. We denote by $\mathcal{A}'(\mathcal{L})$ the set of regions in $\mathbb{R}^2 \setminus (C_1 \cup C_2)$ we get by intersecting the cells in $\mathcal{A}(\mathcal{L})$ by $\mathbb{R}^2 \setminus (C_1 \cup C_2)$. Thus, $\mathcal{A}'(\mathcal{L})$ contains every cell in $\mathcal{A}(\mathcal{L})$ that is disjoint from C_1 and C_2, and the connected parts of the intersections of other cells in $\mathcal{A}(\mathcal{L})$ with $\mathbb{R}^2 \setminus (C_1 \cup C_2)$.

Observation 6.4. *Let R be a region in $\mathcal{A}'(\mathcal{L})$. Then for all $\mathbf{q} \in R$, $\mathcal{I}_{\{C_1,C_2\}}(\mathbf{q})$ have the same combinatorial arrangement.*

Proof. Let $\mathbf{p}$ and $\mathbf{r}$ denote two points in $\mathbb{R}^2$ such that $\mathcal{I}_{\{C_1,C_2\}}(\mathbf{p})$ and $\mathcal{I}_{\{C_1,C_2\}}(\mathbf{r})$ have different combinatorial arrangement. Thus, we have three endpoints $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of intervals in $\mathcal{I}_{\{C_1,C_2\}}(\mathbf{p})$ having different cyclic order as the order of their corresponding endpoints in $\mathcal{I}_{\{C_1,C_2\}}(\mathbf{r})$. For the sake of contradiction let us assume that $\mathbf{p}, \mathbf{r} \in R$.

As R is connected, we can get from $\mathbf{p}$ to $\mathbf{r}$ by a continuous motion, which takes place in R. Thus, during such motion we are not allowed to meet C_1, C_2 or their common tangents not containing $\mathbf{p}$ and $\mathbf{r}$. On the other hand, by a standard argument one can show that during such motion we have to visit a point $\mathbf{s}$, in which two of three endpoints of cyclic intervals in $\mathcal{I}_{\{C_1,C_2\}}(\mathbf{s})$ corresponding to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ become identical. Moreover, these two points are

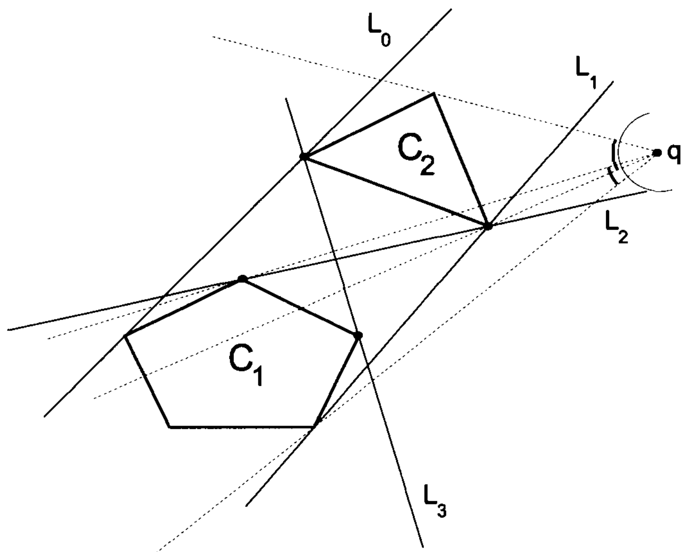

Figure 6.1: Observation 6.4

not the endpoints of the same cyclic interval in $\{C_1, C_2\}$. It is easy to see that $\mathbf{s}$ belongs to a common tangent of C_1 and C_2, that does not contain $\mathbf{p}$ and $\mathbf{r}$. Thus, we obtain a contradiction. $\square$

From now on let $\mathcal{C}$ be a collection of pairwise disjoint convex polygons in $\mathbb{R}^2$. Let n be the total number of sides of polygons in $\mathcal{C}$. Moreover, we suppose that the polygons in $\mathcal{C}$ are given by the vertices on their boundaries, and that for each polygon those vertices are sorted according to their appearance on the boundary.

We denote by $\mathcal{L}_\mathcal{C}$ a set of common tangents of pairs of objects in $\mathcal{C}$ and the lines determining the boundaries of the polygons in $\mathcal{C}$. Note that by Observation 6.3 we know that the size of $\mathcal{L}_\mathcal{C}$ is $O(n + |C|^2) = O(n^2)$.

We present a lemma, which generalize Observation 6.4, and which our algorithm mostly depends on.

Lemma 6.5. *Let C be a cell in $\mathcal{A}(\mathcal{L}_\mathcal{C})$. Then all collections of cyclic intervals $\mathcal{I}_\mathcal{C}(\mathbf{p})$, for $\mathbf{p} \in C$, have the same combinatorial arrangement.*

Proof. Assume that we have two points $\mathbf{r}$ and $\mathbf{q}$ in C such that $\mathcal{I}_\mathcal{C}(\mathbf{r})$ and $\mathcal{I}_\mathcal{C}(\mathbf{q})$ have different combinatorial arrangements. By the proof of Observation 6.4, during a continuous motion that takes us from $\mathbf{r}$ to $\mathbf{q}$ we have to cross either a common tangent, not containing $\mathbf{r}$ and $\mathbf{q}$, of a pair of sets in $\mathcal{C}$, or a boundary segment of a set in $\mathcal{C}$. Hence, $\mathbf{r}$ and $\mathbf{q}$ cannot belong to the same cell of $\mathcal{A}(\mathcal{L}_\mathcal{C})$. Thus, we obtain a contradiction. $\square$

Note that in the above observation instead of cells in $\mathcal{A}(\mathcal{L}_\mathcal{C})$ one can consider a partition of the plane into smaller number of regions determined only by the common tangents of every pair of objects in $\mathcal{C}$ and the boundaries of the polygons in $\mathcal{C}$. However, the asymptotic complexity of this partition in the worst case would be the same. Therefore this inefficiency does not bother us.

We abuse our notation and define $\Pi(\mathcal{I}_\mathcal{C}(R)) = \Pi(\mathcal{I}_\mathcal{C}(\mathbf{p}))$, where R is a region in $\mathbb{R}^2$ such that, for all $\mathbf{p} \in R$, $\mathcal{I}_\mathcal{C}(\mathbf{p})$ have the same combinatorial arrangement.

By *cyclic binary search* we will understand an analog of the binary search for cyclically ordered array described as follows. Let $A[0], \dots A[m-1]$ denote elements in a cyclically ordered array A. Thus, elements in A are ordered according to the ternary relation $\leq_c$, i.e. $A[i] \leq_c A[(i+j) \mod m] \leq_c A[(i+k) \mod m]$, for all $i,j,k \in \{0,\dots m-1\}$, $0 < j < k$. Given an element e the position of which in A we want to determine at one step of a cyclic binary search we find out whether $A[1] \leq_c e \leq_c A[\lceil m/2 \rceil]$ or $A[1] \leq_c A[\lceil m/2 \rceil] \leq_c e$ holds, and thereby we decrease the length of A for the next recursive step at least by half.

Observe that $\Pi(\mathcal{I}_\mathcal{C}(\mathbf{p}))$ for some $\mathbf{p}$ can be represented (in a data structure) by a linear order of the endpoints of the intervals in $\mathcal{I}_\mathcal{C}(\mathbf{p})$ accompanied by an information indicating which of the endpoints coincide. Thus, by computing (determining) $\Pi(\mathcal{I}_\mathcal{C}(\mathbf{p}))$ we mean obtaining of the above representation of $\Pi(\mathcal{I}_\mathcal{C}(\mathbf{p}))$.

The following observation allows us to perform a cyclic binary search in $O(\log^2 n)$ to find a position of an endpoint in $\Pi(\mathcal{I}_\mathcal{C}(\mathbf{p}))$, for $\mathbf{p} \in \mathbb{R}^2$.

Observation 6.6. *Let P_1, P_2 and P_3 denote three polygons in $\mathcal{C}$. Given a point $\mathbf{p} \in \mathbb{R}^2$ we can determine $\Pi(\mathcal{I}_{\{P_1,P_2,P_3\}}(\mathbf{p}))$ in $O(\log n)$.*

Proof. Follows easily from the fact that we can compute the tangents to every polygon P_1, P_2 and P_3 through $\mathbf{p}$ in $O(\log n)$. $\square$

We call $G' = G'(V,E)$ a graph fully representing $\mathcal{A}(\mathcal{L}_\mathcal{C})$, i.e. the vertices in $V(G')$ represent the cells in $\mathcal{A}(\mathcal{L}_\mathcal{C})$, and each vertex in $V(G')$ corresponding to an i-dimensional cell of $\mathcal{A}(\mathcal{L}_\mathcal{C})$, $i \in \{1,2\}$, is joined with every vertex corresponding to an $(i-1)$-dimensional cell of $\mathcal{A}(\mathcal{L}_\mathcal{C})$ on its boundary.

In what follows we define a subgraph G of G', that represents $\mathcal{A}(\mathcal{L}_\mathcal{C})$, and that will play an important role in computing $\tau(\mathcal{C})$.

$G = G(V, E)$ is obtained from G' by removing for every $P \in \mathcal{C}$ all but one of the edges uv, such that exactly one of v and u corresponds to a cell contained in P. Thus, we make each subgraph induced by the vertices corresponding to the cells contained in $P \in \mathcal{C}$ joined with the rest of G by just one edge. Clearly, G is connected.

We define $\tau(A, \mathcal{C})$, $A \subseteq \mathbb{R}^d$, to be $\min_{\mathbf{p}\in A} \tau(\mathbf{p}, \mathcal{C})$.

Now, we are ready to prove the existence of a polynomial algorithm that computes a point $\mathbf{p}$ minimizing $\tau_2(\mathbf{p}, \mathcal{C})$.

Proof. (**Theorem 1.6**) Since $\tau(\mathbf{p}, \mathcal{C})$, $\mathbf{p} \in \mathbb{R}^2$, depends only on the combinatorial arrangement of $\mathcal{I}_\mathcal{C}(\mathbf{p})$, by Lemma 6.5, for finding a point witnessing $\tau(C)$ it is enough to compute $\tau(C, \mathcal{C})$ for each cell C in $\mathcal{A}(\mathcal{L}_\mathcal{C})$, and output a point in a cell C with the minimum value of $\tau(C, \mathcal{C})$. We proceed by traversing all vertices of G such that whenever we visit a vertex corresponding to a cell C we compute $\tau(C, \mathcal{C})$. The reason why we use G instead of G' is that traversing an edge for which exactly one of its incident vertices corresponds to a cell contained in some polygons from $\mathcal{C}$ might take (using our approach) much more time than traversing a typical edge, as we will see later. Thus, we want to limit the number of those edges. It is clear that a bottleneck of our algorithm is the computation and traversing of G. Therefore, most of our effort is put into technical details, which should convince the reader that the time complexity of our algorithm only slightly exceeds the time complexity of the algorithm that computes and traverses G.

First, our algorithm computes the lines in $\mathcal{L}_\mathcal{C}$. Since we are able to compute common tangents to each pair of disjoint polygons in $O(\log n)$ by the result from [12] (Theorem 5), this phase takes $O(n^2 \log n)$ time in total.

By the result from [10] we are able to compute $\mathcal{A}(\mathcal{L}_\mathcal{C})$ in $O(n^4)$ time represented by $G' = G'(V, E)$. It is very easy to see (e.g. [10]), that in $O(n^4)$ time we can obtain for all cells $C \in \mathcal{A}(\mathcal{L}_\mathcal{C})$ the points $p(C) \in C$ representing them. For each vertex in $V(G')$ we also store an information completely describing its corresponding cell (the lines defining its boundary, the supporting lines) in a data structure $D(V(G))$. Moreover, in $D(V(G))$ we store an information telling us, whether a cell belongs to a boundary of a polygon in $\mathcal{C}$ and if yes, which polygon in $\mathcal{C}$ does it belong to. Clearly, we do not spend more than $O(n^4)$ time computing $D(V(G))$. Using $D(V(G))$ we can compute $G = G(V, E)$ in $O(n^4)$, as we can decide for a given edge $e \in E(G')$, whether e is a candidate for removal, in $O(1)$.

In another data structure $D(\mathcal{L}_C)$ we will store for each common tangent L to a pair of polygons in $\mathcal{C}$ the order of appearance on L of its touching points with polygons in $\mathcal{C}$. Notice, that again one common tangent in $\mathcal{L}_C$ can be a common tangent to many ($>> 2$) polygons in $\mathcal{C}$. For each tangent $L \in \mathcal{L}_C$ we will store in $D(\mathcal{L}_C)$ two ordered lists of its touching points. Each of them stores touching points of L with the polygons living in one closed half-plane defined by L. Clearly, $D(\mathcal{L}_C)$ can be prepared in $O(n^2 n \log n)$ time.

Our algorithm uses a data structure DS that stores an ordered array, and allows updating (deleting, inserting) in $O(\log k)$ and searching in $O(c \log k)$ time per item, where $O(c)$ is a time needed to carry out a comparison, and k is the number of stored values (there are plenty of such data structures, see e.g. [1]).

To store the cyclic ordering defined by $\Pi(\mathcal{I}_C(C))$, $C \in \mathcal{A}(\mathcal{L}_C)$, our algorithm maintains one instance of DS, $D(\mathcal{I}_C)$, whose each stored value is dedicated to one endpoint. The representation of each endpoint in $D(\mathcal{I}_C)$ is also accompanied by an information indicating, whether it coincides with its predecessor and successor in the ordering. Hence, we have the full description of $\Pi(\mathcal{I}_C(C))$ stored in $D(\mathcal{I}_C)$. For each endpoint we also keep pointer to its representation in $D(\mathcal{I}_C)$. Our algorithm stores in another instance $D'(\mathcal{I}_C)$ of DS a representation of the set of cyclic intervals $\mathcal{I}_{C,\mathbf{p}}$ sorted according to $d(I)$. For each $\mathcal{I}_{C,\mathbf{p}}$ we also keep pointer to its representation in $D'(\mathcal{I}_C)$. By Observation 6.2 the maximal element of $\mathcal{I}_{C,\mathbf{p}}$ in $D'(\mathcal{I}_C)$ witnesses $\tau(C, \mathcal{C})$.

During its main phase our algorithm performs a depth-first search (DFS) in the graph G to compute $\tau(C, \mathcal{C})$ for each cell C in $\mathcal{A}(\mathcal{L}_C)$. In fact instead of DFS we could use any search that visit every vertex in G and traverse each edge at most twice. In what follows we describe behaviour of our algorithm according to the cell that corresponds to the currently visited vertex during DFS. For convenience by the relative position of a vertex in V with respect to the lines in $\mathcal{L}_C$ or a polygon in $\mathcal{C}$ we mean the relative position of its corresponding cell with respect to those objects.

Let C be a cell in $\mathcal{A}(\mathcal{L}_C)$ that corresponds to a currently visited vertex v of $V(G)$ during DFS. Let $\mathbf{p} = p(C) \in C$.

If C corresponds to the starting vertex of DFS our algorithm performs, we initialize $D(\mathcal{I}_C)$, and $D'(\mathcal{I}_C)$. Thus, we need to add to $D(\mathcal{I}_C)$ one by one representations of all endpoints of intervals in $\mathcal{I}_C(\mathbf{p})$, while updating $D'(\mathcal{I}_C)$. To find a right position in $D(\mathcal{I}_C)$ for the representation of an endpoint in $\mathcal{I}_C(\mathbf{p})$ we perform a cyclic binary search. By Observation 6.6, using $\mathbf{p}$ the initialization can be performed in $O(n \log^2 n)$, as one comparison in a cyclic

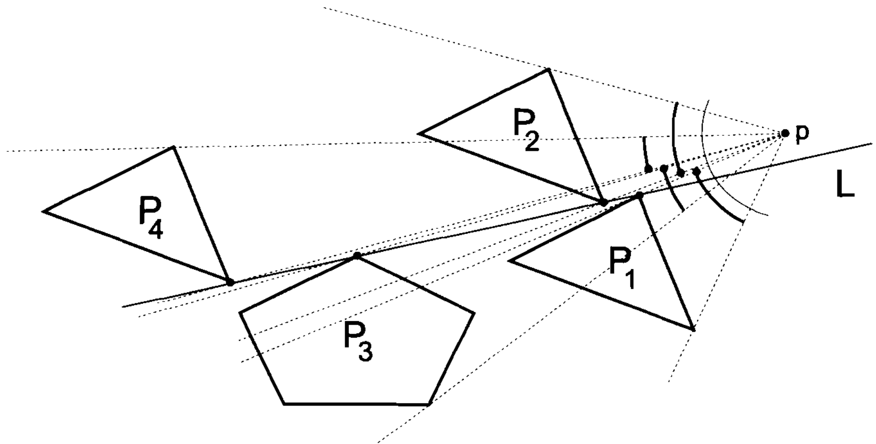

Figure 6.2: A common tangent L to the polygons P_1, P_2, P_3, P_4

binary search costs us $O(\log n)$. The initialization of $D'(\mathcal{I}_C)$ takes $O(n^2 \log n)$ time, as by an addition of new cyclic interval from $\mathcal{I}_C(\mathbf{p})$ we can affect $O(n)$ intervals in $\mathcal{I}_{C,\mathbf{p}}$.

Otherwise, C does not correspond to the starting vertex. Let e denote an edge in $E(G)$ we have traversed to get to the currently visited vertex v that corresponds to C. Let us assume that the number of lines we have changed the relative position to by traversing e is $k = k(e)$, if we do not count the lines that coincide with multiplicity, and if we count only the lines that are common tangents to a pair of polygons in $\mathcal{C}$. Let us denote these lines by $L_1, \dots L_k$. We denote by $i_j = i_j(e) \in \mathbb{N}$ the multiplicity of L_j, $1 \leq j \leq k$, in $\mathcal{L}_C$, i.e. a line that is a common tangent to m polygons is counted $\binom{m}{2}$. Let $k' = k'(e) = \sum_{1 \leq j \leq k} \sqrt{i_j}$. We update the representations of at most $O(k')$ endpoints in $D(\mathcal{I}_C)$ affected by the latest step in DFS, while updating $D'(\mathcal{I}_C)$ accordingly. Moreover, by taking the advantage of an information stored in $D(\mathcal{L}_C)$ we can update $D(\mathcal{I}_C)$ in $O(k')$ time. Indeed, in the case, that the affected endpoints do not coincide anymore, they can be divided into at most $2k$ parts, such that the endpoints in each of these parts follow consecutively one after another in the ordering defined by $\Pi(\mathcal{I}_C(\mathbf{p}))$ (see Figure 6.2). Moreover, their orders according to $\Pi(\mathcal{I}_C(\mathbf{p}))$ corresponds to the orders of the lists of its corresponding touching points representing L_j, for $1 \leq j \leq k$, in $D(\mathcal{L}_C)$. Thus, by taking into account the position of C with respect to the line L_j, for some $1 \leq j \leq k$, we can determine the relative order among the affected endpoints in $O(\sqrt{i_j})$ time, as we keep for each affected endpoint a pointer to its representation in $D(\mathcal{I}_C)$.

If by moving to C we leave a polygon P from $\mathcal{C}$, we insert to $D(\mathcal{I_C})$ the representations of two endpoints of a cyclic interval, which corresponds to P in $\mathcal{I_C}(\mathbf{p})$. That can be done in $O(\log^2 n)$, by the same argument as we used in the initialization step. If, on the other hand, by moving to C we enter a polygon in $\mathcal{C}$, we delete the representations of its endpoints in $D(\mathcal{I_C})$ in $O(\log n)$ time. Notice that both of these events can be detected in $O(1)$ time by querying $D(V(G))$.

In the case that we do not leave or enter a polygon from $\mathcal{C}$ the updating of $D'(\mathcal{I_C})$ takes $O(k' \log n)$ time (we delete and re-insert $O(k')$ items), since we stored for each affected interval a pointer to its representation in $D'(\mathcal{I_C})$. The changed values $d(I)$, $I \in \mathcal{I}_{\mathcal{C},\mathbf{p}}$, in $D'(\mathcal{I_C})$ are adjusted in a straightforward way in $O(k')$ time. Otherwise, the update of $D'(\mathcal{I_C})$ costs $O(n \log n)$, because $O(n)$ cyclic intervals in $\mathcal{I}_{\mathcal{C},\mathbf{p}}$ could be affected.

Since during DFS we traverse each edge of $E(G)$ at most twice and only $O(n)$ edges cause addition or deletion of a representation of a cyclic interval to $D(\mathcal{I_C})$, $O(n^2 \log n + \log n \sum_{e \in E(G)}(k'(e)+1))$ is, clearly, its worst case time complexity. The claimed time complexity of our algorithm easily follows if the lines in $\mathcal{L_C}$ are in general position, as $k'(e) = O(1)$, for all $e \in E$. However, we show that always we have $\sum_{e \in E(G)}(k'(e)) = O(n^4)$.

Let $E_0(G) \subseteq E(G)$ denote the set of all edges that are incident to a vertex corresponding to a 0-dimensional cell in $\mathcal{A}(\mathcal{L_C})$. For $e \in E_0(G)$, the corresponding 0-dimensional cell of its incident vertex contributes to the decrease in the number of cells of $\mathcal{A}(\mathcal{L_C})$ by $\Omega((\sum_{j=1}^{k(e)} i_j(e))^2 - \sum_{j=1}^{k(e)} i_j^2(e)))$ in comparison with the case when there are no degeneracies in $\mathcal{A}(\mathcal{L_C})$, i.e. when $\mathcal{A}(\mathcal{L_C})$ consists of $O(n^4)$ cells. Let v be a vertex in G. We denote by $deg(v)$ the number of edges in G incident to v. Let e' be the edge having the maximal $k'(e')$ among the edges containing v. As for each vertex v that corresponds to a 0-dimensional cell we have $\sum_{e=vw, w \in V} k'(e) \leq deg(v)k'(e')$ and $k(e') = (deg(v)-2)/2$, then $\sum_{e=vw,w \in V}(k'(e)) = O((\sum_{j=1}^{k(e')} i_j(e'))^2 - \sum_{j=1}^{k(e')} i_j^2(e'))$. Hence $\sum_{e \in E_0(G)}(k'(e)) = O(n^4)$. It is easy to see that we can put the same bound on $\sum_{e \in E(G) \setminus E_0(G)} k'(e) = O(n^4)$. Therefore, a total time needed for traversing whole G is $O(n^4 \log n)$.

Notice that each of the finitely many above described stages of the algorithm can be carried out in $O(n^4 \log n)$ time. Thus, a total running time of our algorithm is $O(n^4 \log n)$. □

Note that the algorithm in Theorem 1.6 would work for any collection $\mathcal{C}'$ of pairwise disjoint convex sets in $\mathbb{R}^2$, such that we can effectively compute a common tangent to a pair in $\mathcal{C}'$, and for a given line L and $C \in \mathcal{C}'$ we can effectively compute the intersection points of L with the boundary of C.

Because our solution of the problem of finding one of the points witnessing an obstacle number for a collection $\mathcal{C}$ (of pairwise disjoint polygons with bounded complexity) seems to waste computational resources, it is highly likely that a more effective algorithm for this problem exists. Despite some effort so far we were not able to devise any specific geometric properties of $\mathcal{A}(\mathcal{L}_{\mathcal{C}})$ to reduce the order of magnitude of the algorithm. However, it should not be hard to get rid of the logarithmic factor in our bound by some preprocessing, or by performing clever updating of the data structures as we traverse the arrangement. On the other hand the worst case running time of our algorithm might be still close to an optimal one in the case, when one wants to determine all regions containing the points witnessing the obstacle number. It is easy to see that the union of these regions (cells in $\mathcal{A}(\mathcal{L}_{\mathcal{C}})$) does not have to be connected. Moreover, this union could consist of $\Theta(n^4)$ connected components. In order to see that we consider the case, when $\mathcal{C}$ is the set of n points in the plane in general position.

Our approach seems to be not so easily extensible to the higher dimensions, because the analogous partition of a higher dimensional space, such that $\tau(\mathbf{p}, \mathcal{C}')$ is invariant within each part for an input collection $\mathcal{C}'$, gives us the regions that do not have to have hyperplanes defining its boundaries, but rather some more complicated surfaces.

We conclude this section with some open problems.

Open Problem 6.7. *Provide an algorithm with the running time in $o(n^4 \log n)$ that finds a point $\mathbf{p}$ minimizing $\tau(\mathbf{p}, \mathcal{C})$ for a given collection $\mathcal{C}$.*

Open Problem 6.8. *Provide an algorithm that finds a point a point $\mathbf{p}$ minimizing $\tau(\mathbf{p}, \mathcal{C}')$ for a given collection of pairwise disjoint compact convex sets $\mathcal{C}'$ in $\mathbb{R}^d$, for $d > 2$.*

6.2 Reductions to other problems

Our task in this section is to provide an algorithm that for a given collection $\mathcal{C}$ in $\mathbb{R}^d$, for any $d > 1$, of n pairwise disjoint sets of some restricted kind finds a point $\mathbf{p}$ having $\tau(\mathbf{p}, \mathcal{C})$ not much higher than the bound on $\tau(n)$ guaranteed by Theorem 1.1.

When $\mathcal{C}$ consists only of n flat convex sets living in $\mathbb{R}^d$, we are able to reduce our problem, the same way as we have described it in Chapter 2, to the problem of computing a point having the maximum undirected depth (see Chapter 2) with respect to some finite set of hyperplanes. Thus, for each set C in $\mathcal{C}$ we compute a hyperplane, which contains C. Then it is enough to compute a point having the maximum undirected depth with respect to obtained hyperplanes. In $\mathbb{R}^2$, this approach outperforms the running time of the above algorithm. However, the downside of this reduction, besides the fact that it works only for flats, is that it does not necessarily give us a point $\mathbf{p}$ having $\tau(\mathbf{p}, \mathcal{C})$ at least 'reasonably' close to $\tau(\mathcal{C})$, since the upper bound from Theorem 1.1 could be very far from an obstacle number of $\mathcal{C}$.

Moreover, by this approach to have a guarantee that the obtained point witnesses at least some bound that is reasonable smaller than the trivial one (i.e. n), the sets in $\mathcal{C}$ must be in general position (as defined in Chapter 2), and the number of sets in $\mathcal{C}$ must be more than $(d+1)^2$.

Recently, it was shown in [13] that there is an optimal $O(n \log n)$ running time algorithm for finding a line that has the maximum regression depth with respect to a given set of n points in $\mathbb{R}^2$. Thus, the point/hyperplane duality gives us $O(n \log n)$ running time algorithm for computing a point having the maximum undirected depth with respect to n hyperplanes in $\mathbb{R}^2$.

In the higher dimensions we compute the undirected depth for a collection $\mathcal{H}$ of n hyperplanes in $\mathbb{R}^d$ as follows. We say that two d-dimensional cells in $\mathcal{A}(\mathcal{H})$ are neighbouring, if their boundaries contain a common $(d-1)$-dimensional cell from $\mathcal{A}(\mathcal{H})$. It is easy to see that all points contained in one cell of the arrangement $\mathcal{A}(\mathcal{H})$ have the same undirected depth with respect to $\mathcal{H}$. Hence, by an undirected depth of a cell we can understand the undirected depth of any point belonging to it. Thus, by the following theorem from [21] we have an algorithm for computing maximal undirected depth in $\mathbb{R}^d$ with the running time $O(n^d)$.

Theorem 6.9. *For a set $\mathcal{H}$ of n hyperplanes in $\mathbb{R}^d$, the undirected depth of each cell in $\mathcal{A}(\mathcal{H})$ can be computed in $O(n^d)$ time by building the arrangement and traversing the graph of adjacent cells.*

Proof. The algorithm uses the same idea as that in Theorem 1.6, i.e. we traverse a graph defined by the arrangement of hyperplanes $\mathcal{A}(\mathcal{H})$. However, in this case the graph we traverse is defined in a way such that its vertices correspond only to d-dimensional cells.

Let $G(V, E)$ be the graph whose vertices correspond to the d-dimensional cells in $\mathcal{A}(\mathcal{H})$, and the edges join the vertices corresponding to neighbouring cells. Moreover, we add to V one more vertex v, and we join it with all vertices that correspond to unbounded cells.

Let us $d(C)$ denote an undirected depth of a cell in $\mathcal{A}(\mathcal{H})$.

Observation 6.10. *Directions of the rays witnessing $d(C)$ of a d-dimensional cell C are the directions of the rays witnessing $d(C')$ of its neighbouring cells C' having $d(C') = d(C) - 1$.*

Proof. If r is a witnessing ray for $d(C')$, r does not cross a hyperplane separating C and C'. Hence, we can translate it, so that it starts in C, and thereby obtain a ray with one more intersection with hyperplanes in $\mathcal{H}$, that is a witnessing ray for $d(C)$.

On the other hand, given a witnessing ray for $d(C)$, we can translate it so that it starts in some neighbouring cell C' of C, and witnesses $d(C') \leq d(C) - 1$. However, $d(C')$ is not less than $d(C) - 1$ as that would contradict the undirected depth of C. □

By the above observation to obtain $d(C)$ for every d-dimensional cell $C \in \mathcal{A}(\mathcal{H})$ it is enough to do a breadth-first search from v in G during which we assign undirected depth to every d-dimensional cell, which is equal to the minimum distance between its corresponding vertex in G and a vertex that corresponds to an unbounded d-dimensional cell.

The undirected depth for the lower dimensional cells is assigned easily according to the following observation.

Observation 6.11. *Let C be a cell in $\mathcal{A}(\mathcal{H})$ incident to k hyperplanes in $\mathcal{H}$, and let C' be a d-dimensional cell in $\mathcal{A}(\mathcal{H})$ with minimal $d = d(C')$ among the cells having C on its boundary. Then we have $d(C) = d + k$.*

Proof. Clearly, $d(C) \leq d + k$, as a witnessing ray for $d(C')$ can be translated so that it starts in C. On the other hand, a witnessing ray for $d(C)$, if it is not contained in a hyperplane from $\mathcal{H}$, can be made shorter so that it starts in some d-dimensional cell C'' having C on the boundary. Hence, $d(C) - k \geq d(C'') \geq d$. If a witnessing ray for $d(C)$ is contained in a hyperplane from $\mathcal{H}$, we can rotate it slightly, such that it no longer belongs to a hyperplane from $\mathcal{H}$ and use the previous argument. □

The time complexity of our algorithm follows similarly as the time complexity in Theorem 1.6 from the result in [10] stating that we can construct a graph representing $\mathcal{A}(\mathcal{H})$ in $O(n^d)$ time. □

It is a challenging open problem to introduce more efficient algorithm for computing maximal regression depth in dimensions higher than 2. So far, the only improvement was made in by Kreveld et.al. in [21], where the space used in the former algorithm was reduced by linear factor to $O(n^{d-1})$ using standard ϵ-cutting method, see e.g. [16].

Given a collection of n pairwise disjoint convex compact flat sets $\mathcal{C}$ in $\mathbb{R}^d$ such that we can compute $\mathcal{H}(\mathcal{C})$, as defined in Chapter 2 so that no $k+1$ hyperplanes in $\mathcal{H}(\mathcal{C})$ has non-empty intersection, in $f(n)$ deterministic time. By the previously mentioned algorithms from [13, 21] we have.

Theorem 6.12. *(i) If $d=2$ then there exists an algorithm with the running time $O(f(n)+n\log n)$, that computes a point $\mathbf{p}$ such that any ray starting at $\mathbf{p}$ intersects at most $\left\lfloor \frac{2n}{3} \right\rfloor + k - 1$ elements in $\mathcal{C}$.*

(ii) If $d=2$ then there exists an algorithm with the running time $O(f(n)+n^d)$ using $O(f(n)+n^{d-1})$ space, that computes a point $\mathbf{p}$ such that any ray starting at $\mathbf{p}$ intersects at most $\left\lfloor \frac{dn}{d+1} \right\rfloor + k - 1$ elements in $\mathcal{C}$.

Other simple reduction pops up, when one wants to find a point basically witnessing the general upper bound (Theorem 1.1) in the case, that our collection $\mathcal{C}$ of pairwise disjoint compact convex sets consists only of balls.

Given a finite set of points $\mathcal{P}$ in $\mathbb{R}^d$, *Tukey median* is a point $\mathbf{p}$ in $\mathbb{R}^d$, which maximize the minimum number $d_t(\mathbf{p})$ of points of $\mathcal{P}$ belonging to a closed half-space defined by a hyperplane through $\mathbf{p}$. Formally,
$d_t(\mathbf{p}) = \min\{|P \cap \gamma| :$ where γ is a halfspace defined by a hyperplane through $\mathbf{p}\}$.

As we have a randomized algorithm due to Chan [7] that computes Tukey median in $O(n\log n)$ time for $\mathbb{R}^2$ and in $O(n^{d-1})$ time for $\mathbb{R}^d$, the proof of the following theorem is rather simple.

Theorem 6.13. *There exists a randomized algorithm running in $O(n\log n)$ $(O(n^{d-1}))$ time, that given a collection $\mathcal{C}$ of n pairwise disjoint balls in $\mathbb{R}^2$ $(\mathbb{R}^d, d>2)$, computes a point $\mathbf{p}$ such that any ray starting at $\mathbf{p}$ intersects at most $\lfloor 2/3n \rfloor + 1$ $(\lfloor dn/(d+1) \rfloor + 1)$ elements in $\mathcal{C}$.*

Proof. The Centerpoint Theorem (Theorem 2.2) guarantees that Tukey median $\mathbf{p}$ satisfies that $d_t(\mathbf{p})$ is at least $\lceil n/(d+1) \rceil$. Hence, given a hyperplane H perpendicular to a ray r emanating from $\mathbf{p}$, each of two half-spaces it defines contains at least $\lceil n/(d+1) \rceil$ centres of the balls in $\mathcal{C}$. To this end, it is enough to show that Tukey median $\mathbf{p}$, with respect to the set of centres of the balls in $\mathcal{C}$, is our witnessing point. Since r can either intersect at most one ball with the centre in one half-space defined by H or intersect at most one ball with the centre in another half-space defined by H, r cannot intersect more than $\lfloor dn/(d+1) \rfloor + 1$ balls of $\mathcal{C}$ in total. $\square$

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