# DATA REDUCTION FOR CONNECTED DOMINATING SET

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# Abstract

Computationally difficult problems are ubiquitous. Although, sometimes approximations come in handy, accuracy is often a must and hence complexity of optimization seems unavoidable. The classical viewpoint to the complexity considers the instant size as the only factor for computing its hardness, but while dealing with hard problems, many input instances consist of easy parts and other parts that form the hard core of the problem. Therefore, it seems reasonable that before starting a cost-intensive algorithm, a polynomial-time preprocessing phase is executed in order to shrink the instance to the hard core kernel. In fixed-parameter algorithms, this is known as data reduction to a problem kernel. In this thesis, we study data reduction for the connected dominating set problem. In particular, we introduce a set of data reduction rules for the connected dominating set problem and prove that the problem admits a linear-size kernel in planar graphs.

*To my parents*
*To my grandpa*

*"If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things."*

R. DESCARTES

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# Contents

**Approval** ii

**Abstract** iii

**Dedication** iv

**Quotation** v

**Acknowledgments** vi

**Contents** vii

**List of Figures** ix

**1 Introduction** 1

1.1 Parameterized algorithms . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
1.2 Graph domination . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
1.3 Related works . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
1.4 Contributions of this thesis . . . . . . . . . . . . . . . . . . . . . . . . . 4
1.5 Thesis outline . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5

**2 Preliminaries** 6

2.1 Parameterized algorithms . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
2.2 Graph domination . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

**3 Reduction Rules for Connected Domination** 12

**4 Linear-size Kernel** 25

**5 Extensions of reduction rules** **32**
5.1 Introduction . . . 32
5.2 A generalized rule . . . 33

**6 Conclusion** **35**
6.1 Summary of contributions . . . 35
6.2 Future work . . . 35

**Bibliography** **37**

# List of Figures


2.1 Examples of $N_i(v)$ and $N_i(v, w)$ $(i = 1, 2, 3)$. . . . . . . . . . . . . . . . . . . . . 9

2.2 (a) A region (b) A simple region of type 2 (c) A simple region of type 1. . . . 10

2.3 Example of a region decomposition for a plane graph. . . . . . . . . . . . . . 11

4.1 Example of bridges in $R(v, w)$ when $d_G(v, w) = 2$ and $d_G(v, w) = 3$. . . . . . . 26

4.2 a) Rules 2-4 apply b) $d_G(v, w) = 2$ but Rule 3 fails c) $d_G(v, w) = 3$ but Rule 4 fails. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 31

# Chapter 1

# Introduction

## 1.1 Parameterized algorithms

Computationally hard problems are ubiquitous and therefore it is critical to know how to best deal with them. Although, sometimes approximations come in handy, accuracy is often a must and hence complexity of optimization seems unavoidable.

A fixed-parameter algorithm computes an optimal solution to a discrete combinatorial problem. For an NP-hard problem, one can not hope for anything better than exponential running times. However, the fundamental idea is to restrict the corresponding, seemingly unavoidable, *combinatorial explosion* that causes the exponential growth in the running time of certain problem-specific parameters. It is hoped then that these parameters might take only relatively *small* values, resulting in an affordable exponential growth in which case, the fixed-parameter algorithm efficiently solves the given *parameterized problem* [36].

The field of parameterized algorithms continues to grow. Over twenty differently named techniques are known in the literature for designing parameterized algorithms. These include Bounded Search Trees [24], Data Reduction [36], Kernelization [24], The Extremal Method [26], The Algorithmic Method [39], Catalytic Vertices [26], Crown Reductions [17], Modeled Crown Reductions [21], Either/Or [40], Reduction to Independent Set Structure [41], Greedy Localization [21], Win/Win [25], Iterative Compression [21], Well-Quasi-Ordering [24], FPT through Treewidth [24], Search Trees [36], Bounded Integer Linear Programming [36], Color Coding [5], Method of Testsets [24], Interleaving [37]. Several survey articles [25, 21] and books [36, 24] have discussed the common themes like Bounded Search Trees, Kernelization and Win/Win. A full taxonomy of the mentioned techniques is provided in [43].

Although the classical viewpoint to the complexity usually considers the size of input as the only factor for computing its hardness, while dealing with computationally hard problems, many input instances consist of some parts that are relatively easy to deal with and other parts that form the real hard core of the problem. Therefore, it is reasonable that before starting a cost-intensive algorithm solving the difficult problem, a polynomial-time preprocessing phase is executed in order to shrink the input data to the hard core kernel. In the context of fixed-parameter algorithms, this basically comes down to what is known as *data reduction to a problem kernel.*

The work of Weihe [45, 46] demonstrates the ease and power of data reduction in tackling real life applications very well. Dealing with the problem of covering trains by stations they applied two simple data reduction rules to the equivalent NP-complete red/blue dominating set problem recursively until no further application was possible. Their result was later successfully tested on the data from German and European trains schedules. No matter the context in which the data reduction is applied, the main idea behind this approach is to shrink the problem instance by applying reduction rules into a very small instance (namely, *kernel*) for which a simple brute-force approach is sufficient to solve the computationally hard problems efficiently and optimally.

## 1.2 Graph domination

Given a graph $G$ with vertex set $V(G)$, the dominating set problem asks for a minimum subset $D \subseteq V(G)$ of vertices such that every vertex in $V(G) \backslash D$ has a neighbor in $D$. The dominating set problem is a classic NP-complete graph problem which belongs to a broader class of domination and covering problems on which hundreds of papers have been written. The dominating set problem and its variants are discussed in details in the book of Haynes, Hedetniemi, and Slater [31]. From applications' point of view, domination problems appear in numerous practical settings, ranging from strategic decisions such as locating radar stations or emergency services through computational biology to voting systems[4] and more than 200 research papers and more than 30 Ph.D. theses investigate the algorithmic complexity of domination and related problems (see Haynes et al. [31] for a survey). It is known that dominating set problem on arbitrary graphs is W[2]-complete [1] and hence is

[1] $W[2]$ is the class of decision problems that are fixed-parameter reducible to the Weighted Circuit-SAT with 2-unbounded fan-in along any path to the root.

not fixed-parameter tractable unless the parameterized complexity hierarchy collapses [24]. When restricted to planar graphs dominating set is fixed-paramater tractable [24]. The best known parameterized time complexity for the planar dominating set is $O(2^{11.98\sqrt{k}}) \cdot n^{O(1)}$ where $k$ is the domination number of the input graph [22].

Apart from the original dominating set problem, variations of dominations such as multiple domination, even/odd domination, distance domination, directed domination, independent domination and connected domination have found numerous applications and significant theoretical interests recently. While each of such problems may vary in terms of difficulty, most of them are proven to be NP-complete and hence it has also been a challenge to study the problems when restricted to a smaller class of graphs and check how the imposed restrictions affect tractability of the problems.

Out of these variants, the *connected dominating set* (abbreviated as CDS) has probably received the most attention, as apart from its theoretical significance, the connected dominating set lies at the heart of many practical settings. In this scenario, given a graph $G$ and a positive integer $k$, the question is whether a dominating set of size $k$ exists such that the induced graph on the set of dominating vertices is connected. It is known that the connected dominating set problem remains NP-complete even under the planarity restriction. In terms of parameterization results, the connected dominating set is not fixed-parameter tractable in arbitrary graphs but becomes tractable when restricted to planar graphs [23].

Probably the most prominent applications of connected dominating set appear in wireless ad hoc networks, where a wide range of network protocols use CDS as underlying structure for performing various communication functions. These include protocols for media access coordination [6]; unicast/routing [19, 18], multicast/broadcast [32, 33], location-based routing [20]; and energy conservation [14, 48]. For a survey on the algorithms and techniques for computing connected dominating sets in wireless ad hoc networks see [8].

## 1.3 Related works

Preprocessing of hard problems is not a new concept and indeed it can be traced back to the very beginning of algorithm research and hence it seems impossible to relate its origin to a particular piece of research. The concept of data reduction to a problem kernel was introduced by Downey and Fellows [24] for the first time in order to formalize reductions for parameterized complexity purposes. *Vertex cover* as a classical problem is probably one of the earliest problems studied in this line. A simple data reduction for the vertex cover is discussed by Buss and Goldsmith in [9]. Though much earlier and not in the context of data reduction, Nemhauser and Trotter [35] proved a fundamental result of $2k$ kernel for vertex cover. Cai et al. [11] proved that every fixed-parameter tractable problem is kernelizable. Mahajan and Raman [34] studied the parameterized and exact complexity of *MAX-SAT* and proved a quadratic-size problem kernel for the problem. Their work was later improved by Chen and Kanj [16]. Study of the NP-complete *MAX-2-SAT* problem is done in Gramm et al. [27]. They proved an upperbound of $L^{O(1)} \cdot 2^{K/5}$ for an input of size $L$ with $k$ 2-literal clauses. The kernelization of *cluster editing* is due to Gramm et al. [28]; while its more general form, correlation clustering had been studied before by Bansal et al. [7]. An exponential size problem kernel for *multicut in trees* was obtained by Guo and Niedermeier [29]. A problem kernel of cubic size for *3-hitting* set was shown by Niedermeier and Rossmanith [38]. A linear-size kernel of $335k$ for the *planar dominating set* problem was given in Alber et al. [4]. Chen et al. [15] improved this bound to $67k$ with better reduction rules and a more detailed analysis. They used an extra technique of marking vertices while applying rules in addition to the subsets of neighbors in Alber et al. [4]. On the experimental side, the work of Alber in [1] showed that their data reduction rules also perform well on non-planar sparse graphs.

## 1.4 Contributions of this thesis

In this thesis, we study the problem of the connected dominating set which has significant theoretical and practical importance. Although it is known that connected dominating set is fixed-paramater tractable when restricted to planar graphs, it has been open whether the problem in this setting admits a linear-size kernel. Here, we answer this question using data reduction techniques. In particular, having proposed a set of simple and easy-to-implement reduction rules for the connected dominating set, we prove that for planar graphs a linear-size problem kernel can be efficiently constructed. Considering the paramount importance of the problem in theory and practice, this brings us one step closer to efficient computation of many real life problems.

From algorithmic point of view, our linear kernel size result of $413k$ can be coupled with any of the previous algorithmic result to obtain a very efficient fixed-parameter algorithm for the connected dominating set problem on planar graphs. In particular, using the branch-decomposition based approach of [23], our result proposes a fixed-parameter algorithm of time $c^{\sqrt{k}} \cdot k + n^{O(1)}$ where $c$ is a constant and $k$ is the connected domination number.

## 1.5 Thesis outline

The organization of this thesis is as follows: In Chapter 2, we define the terminology and notation that we will be using throughout the thesis. In Chapter 3, we define a set of reduction rules for the connected domination in planar graphs. These rules are local in the sense that they use the neighborhood information of pairs of vertices only, for deciding whether the reduction can be applied. Later, we prove the correctness of the reduction rules and to assure the validity of reduction we prove polynomial computational time complexity for each rule. Chapter 4, deals with upperbounding the size of kernel resulted after applying reduction rules repetitively. In particular, we use a technique known as region decomposition to divide the reduced graph into a set of connected subgraphs called regions and prove that there can only exist a linear number of regions and that the size of each region is bounded by a constant and hence we obtain a linear upperbound for the size of kernel. In Chapter 5, we employ a similar idea as the one behind our previous reduction rules to obtain a master reduction rule. This rule is defined based on the SAT representation of the problem and introduces a general gadget on-the-fly based on the associated constraints. Finally, in Chapter 6, we conclude our work and suggest some possible future works in this line of research.

# Chapter 2

# Preliminaries

In this chapter, we set the foundation of definitions and terminology that is being used throughout the rest of the thesis.

## 2.1 Parameterized algorithms

**Definition.** Niedermeier [36] A *fixed-parameter algorithm* is the one that solves a problem with an input instance of size $n$ and a parameter $k$ in $f(k) \cdot n^{O(1)}$ time for some computable function $f$ depending solely on $k$. In other words for any fixed parameter value the algorithm gives a solution in polynomial time and the degree of polynomial is independent of $k$.

**Definition.** Niedermeier [36] Let $\mathcal{L}$ be a parameterized problem, that is, $\mathcal{L}$ consists of *input pairs* $(I, k)$ where $I$ is the input instance and $k$ is the parameter for $I$. Then, *kernelization or reduction to a problem kernel* means to replace instance $(I, k)$ by a reduced instance $(I', k')$ called problem *kernel* such that

$$k \leq k', |I'| \leq g(k)$$

for some function $g$ only depending on $k$, and

$$(I, k) \in \mathcal{L} \texttt{ iff } (I', k') \in \mathcal{L}$$

and this reduction must be computable in a polynomial time $T(|I|, k)$. Here, $g(k)$ is called the *kernel size.*

## 2.2 Graph domination

Throughout this thesis whenever we refer to $G$, we mean a graph that is simple and undirected. We denote by $V(G)$ the vertex set and $E(G)$ the edge set of a graph $G$ respectively. Readers may refer to any textbook on graph theory for the very elementary definitions that are skipped here.

For a vertex $v$ of a graph $G$, The open and closed neighborhoods of $v$ are defined as $N(v) = \{u | \{u, v\} \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ respectively.

Given a subset $U \subseteq V(G)$, let $N(U) = \{u | v \in U, \{u, v\} \in E(G)\}$ and $G[U]$ denote the subgraph induced by the vertices of $U$.
A vertex $u$ is *dominated* by a vertex $v$ if either $u = v$ or $\{u, v\} \in E(G)$. Similarly, A vertex $v$ is dominated by a vertex set $U$ if the $v$ is dominated by a vertex of $U$.
Consider a set of vertices $v_1, v_2, ..., v_l \in V$ we use the notation $v_1 - v_2 - . - . - v_{l-1} - v_l$ to denote a path between $v_1$ and $v_l$ passing the vertices $v_2, \cdots, v_{l-1}$ in order.
The distance between two vertices $v$ and $w$ in $G$, denoted by $d_G(v, w)$ is the length of the shortest path between $v$ and $w$ in $G$.

Given a graph $G = (V, E)$ and a nonnegative integer $k$, the *dominating set* problem asks whether there exists a subset $D$ of $V$ with at most $k$ vertices such that for every vertex $v \in V$, there is a vertex $u \in N[v]$ with $u \in D$.

A *connected dominating set* (CDS) of $G$ is a subset $D \subseteq V(G)$ such that $D$ is a dominating set of $G$ and the subgraph $G[D]$ induced by $D$ is connected. The minimum CDS problem is to find a CDS $D$ of $G$ with the minimum cardinality. Denoting the size of the minimum CDS of $G$ by $\gamma_c(G)$, the decision version of the CDS problem is to decide, given a graph $G$ and a positive integer $k$, whether $\gamma_c(G) \leq k$.

Alber et al. [4] partition the neighborhood of a vertex (pair of vertices) into three subsets based on the notion of domination. We adopt the same definitions from their work in order to introduce the domination concept in our reduction rules as it becomes clear later on. For a vertex $v \in V(G)$, we can define the following subsets of $N(v)$ (See Figure 2.1.$a$).

$$\begin{aligned} N_1(v) &= \{u|u \in N(v), N(u) \setminus N[v] \neq \emptyset\}, \\ N_2(v) &= \{u|u \in N(v) \setminus N_1(v), N(u) \cap N_1(v) \neq \emptyset\}, \\ N_3(v) &= N(v) \setminus (N_1(v) \cup N_2(v)). \end{aligned}$$

Also, for a pair of vertices $v, w \in V(G)$, let $N(v,w) = N(v) \cup N(w) \setminus \{v, w\}$ and $N[v,w] = N[v] \cup N[w]$. The neighborhood $N(v,w)$ is partitioned similarly into the following subsets(See Figure 2.1.$b$):

$$\begin{aligned} N_1(v,w) &= \{u|u \in N(v,w), N(u) \setminus N[v,w] \neq \emptyset\}, \\ N_2(v,w) &= \{u|u \in N(v,w) \setminus N_1(v,w), N(u) \cap N_1(v,w) \neq \emptyset\}, \\ N_3(v,w) &= N(v,w) \setminus (N_1(v,w) \cup N_2(v,w)). \end{aligned}$$

A *plane graph* is a planar graph drawn in the plane without edge crossings.

Let $G = (V, E)$ be a plane graph. A *region* $R(v,w)$ between two vertices $v, w$ is a closed subset of the plane with the following criteria:

- The *boundary* of $R(v,w)$ denoted by $\partial R$ is formed by two simple paths $P_1$ and $P_2$ in $V$ that connect $v$ and $w$, and the length of each path is at most three, and
- All vertices that are strictly inside the region $R(v,w)$ are from $N(v,w)$.

Figure 2.2.$a$ gives an example of a region.

For a region $R = R(v,w)$, let $V(R)$ denote the vertices belonging to $R$, that is,

$$V(R) := \{u \in V | u \text{ sits inside } R \text{ or on } \partial R\}$$

A region $R(v,w)$ between two vertices $v, w \in D$ is called *simple* if all vertices contained in $R(v,w)$ except for $v, w$ are common neighbors of both $v$ and $w$, that is, if $(V(R(v,w)) \setminus \{v,w\}) \subseteq N(v) \cup N(w)$. (See Figure 2.2.$b$)

A simple region is of *type* $i$ if $i, (i = 1, 2)$ of the vertices on its boundary except for $v, w$ have at least a neighbor outside the region. Examples of regions of type 2 and 1 are depicted

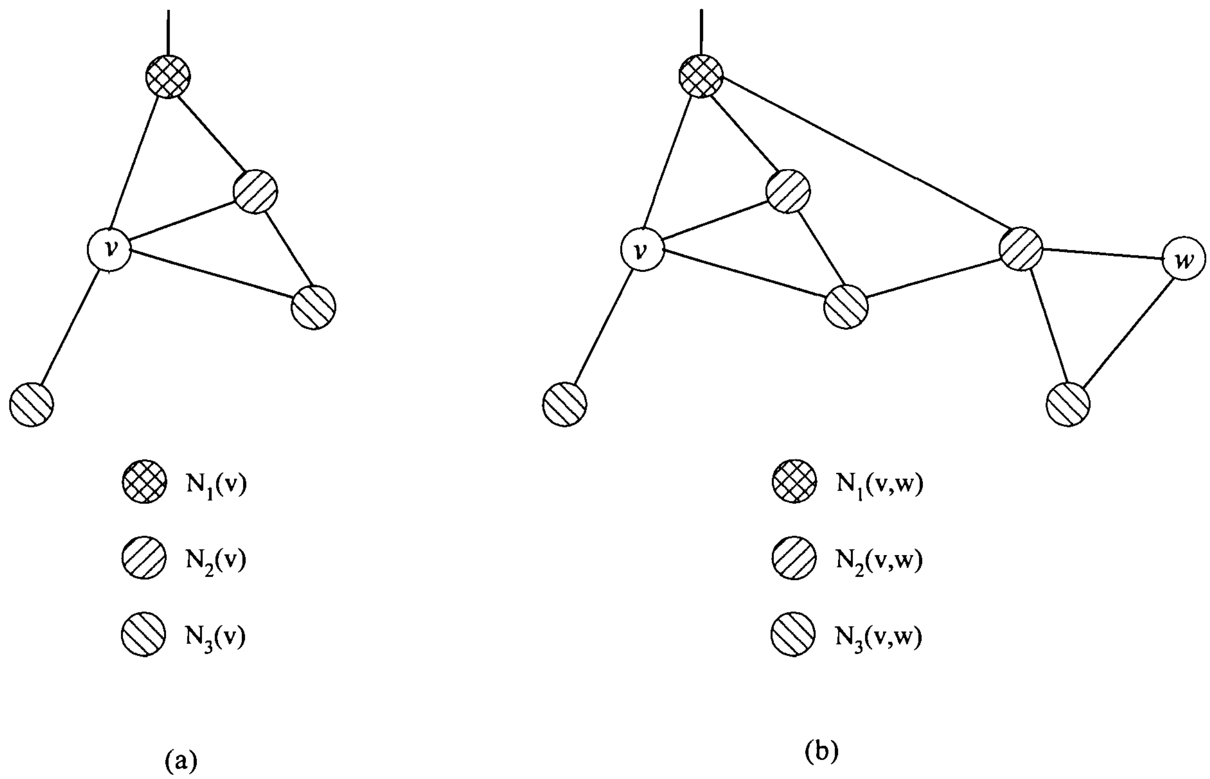


Figure 2.1: Examples of $N_i(v)$ and $N_i(v, w)$ $(i = 1, 2, 3)$.

in Figure 2.2.*b* and *c*.

Given the definition of the region, for a plane graph $G$, one can envision a decomposition of the graph into a set of non-overlapping regions. This notion is formalized in the next definition.

**Definition.** Given a plane graph $G$ and a subset $D \subseteq V(G)$, a *$D$-region decomposition* of $G$ is a set $\mathcal{R}$ of regions between pairs of vertices of $D$ such that

1. for $R(v, w)$, no vertex of $D \setminus \{v, w\}$ is in $V(R(v, w))$ and

2. for two regions $R_1, R_2 \in \mathcal{R}$, $(R_1 \cap R_2) \subseteq (\partial R_1 \cup \partial R_2)$.

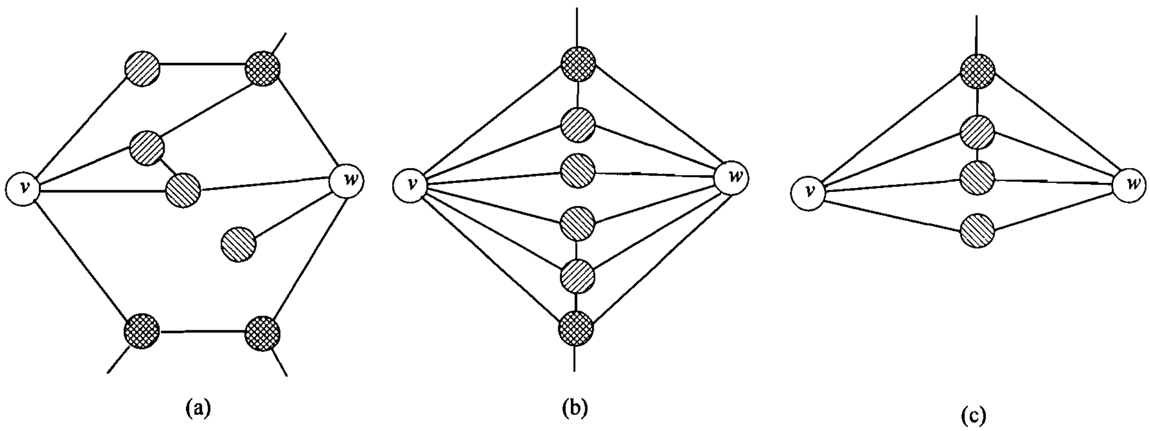


Figure 2.2: (a) A region (b) A simple region of type 2 (c) A simple region of type 1.

Figure 2.3 demonstrates an example of a region decomposition. With a little abuse of notation, we use $V(\mathcal{R})$ to denote all vertices in or on the boundary of a region in $\mathcal{R}$; i.e. $V(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} V(R)$.

The notion of region decomposition as defined before is not exact in the sense that it only specifies a set of criteria on the set of regions. In practice, there can be several different $D$-region decompositions for $G$ many of them even not covering major parts of the graph. Indeed, it is not hard to see that $\mathcal{R} = \emptyset$ is a valid decomposition. Therefore, in order to be more accurate we need to include the concept of maximality in our statement.

A $D$-region decomposition $\mathcal{R}$ is called *maximal* if no region $R$ can be added to $\mathcal{R}$ such that the resulting decomposition stays valid and more vertices are covered. In other words, $R \cup \mathcal{R}, R \notin \mathcal{R}$ is not a region decomposition if $V(\mathcal{R}) \subset V(\mathcal{R} \cup R)$.

An *isomorphism* between two graphs $G$ and $G'$ is a bijective map $M$ from the vertices of $G$ to the vertices of $G'$ such that there exists an edge from vertex $v$ to vertex $w$ in $G$ iff there is an edge from $M(v)$ to $M(w)$ in $G'$.

Assume a parameterized problem $\mathcal{L}$ with input instances of the type graph $G = (V, E)$ and a set of data reduction rules $\Phi$ for shrinking the size of instances. A rule $\phi$ is applied

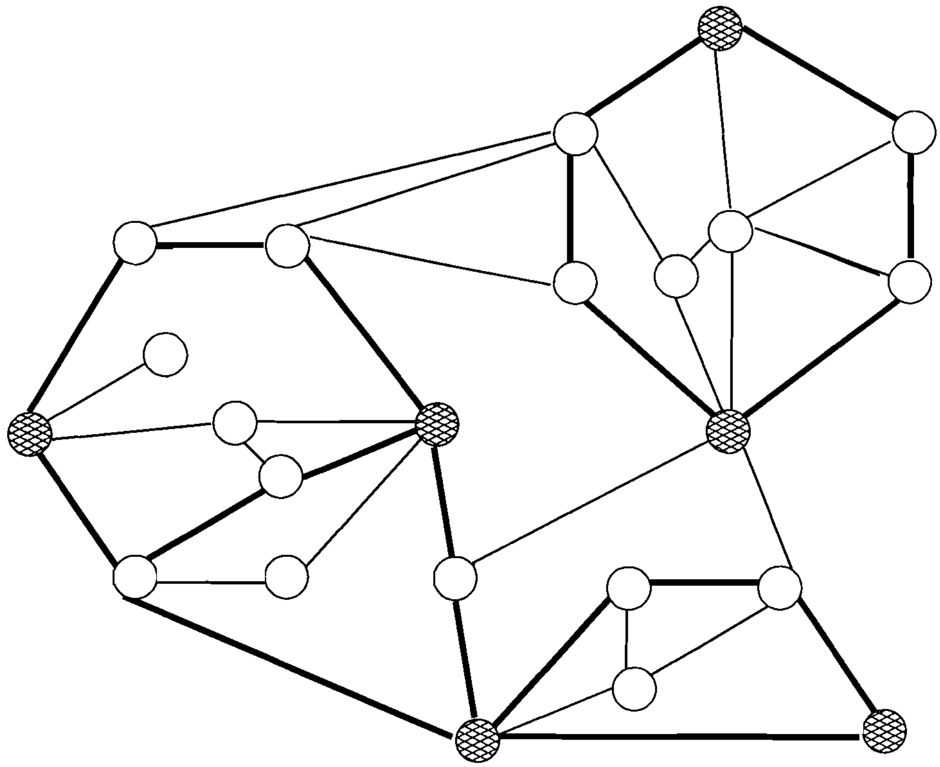

Figure 2.3: Example of a region decomposition for a plane graph.

*successfully* to a subset of vertices of $G$ if the graph obtained after applying rule $\phi$ is not isomorphic to $G$. Similarly, a graph $G$ is called *reduced* if applying rules $\Phi$ to $G$ results in a graph isomorphic to $G$; i.e. none of the rules in $\Phi$ can be applied successfully to any subset of $V$.

# Chapter 3

# Reduction Rules for Connected Domination

In this chapter, using the inherent properties of the connected domination, we introduce five simple rules for reducing the graph. Furthermore, in order to assure the validity of our fixed-parameter reduction, for each presented rule we prove correctness and polynomial time complexity.

The intuition behind the first and simplest reduction rule is that the vertices in $N_3(v)$ cannot be dominated by vertices from $N_1(v)$. Vertex $v$ is a good candidate for dominating $N_3(v)$ and the vertices in $N_2(v)$ and $N_3(v)$ can be removed.

**Rule 1** For $v \in V(G)$, if $N_3(v) \neq \emptyset$ then remove $N_2(v)$ and $N_3(v)$ from $G$ and add a new gadget vertex $v'$ with edge $\{v, v'\}$ to $G$.

**Lemma 3.0.1** *Given a graph $G$, let $G'$ be the graph obtained by applying Rule 1 to some $v \in V(G)$. Then $\gamma_c(G) = \gamma_c(G')$.*

**Proof:** We prove the lemma by showing that a minimum CDS $D'$ of $G'$ is a also a minimum CDS of $G$. Obviously, $D'$ contains vertex $v$ but not the gadget vertex $v'$. From $N_2(v) \cup N_3(v) \subseteq N(v)$, $D'$ is a CDS of $G$ as well. Let $D$ be a minimum CDS of $G$. Then $|D| \leq |D'|$. Notice that the vertices in $N_3(v)$ can only be dominated by either $v$ or some vertices from $N_2(v) \cup N_3(v)$.

Assume that $D$ contains $v$. Let $D_1 = D \setminus (N_2(v) \cup N_3(v))$; then $|D| \geq |D_1|$. From $N(N_2(v) \cup N_3(v)) \subseteq N[v]$, it is inferred that $D_1$ is a CDS of $G$. Since $D_1$ does not contain

any vertex from $N_2(v) \cup N_3(v)$, $D_1$ is also a CDS of $G'$. Therefore, $|D| \geq |D_1| \geq |D'|$. Thus, $|D| = |D'|$ and $D'$ is a minimum CDS of $G$.
Assume that $D$ does not contain $v$. Then, $D$ must contain at least one vertex from $N_2(v) \cup N_3(v)$. Let $D_1 = (D \setminus (N_2(v) \cup N_3(v))) \cup \{v\}$; then $D_1$ is a CDS of $G$ and $|D| \geq |D_1|$. Similarly, $D_1$ is a CDS of $G'$ and $D'$ is a minimum CDS of $G$. □

Our second reduction rule is applied to the vertices on an edge $\{v, w\}$ of $G$. The intuition behind introducing this rule is that a vertex of $N_3(v, w)$ can not be dominated by vertices of $N_1(v, w)$. Good candidates for dominating $N_3(v, w)$ are $\{v, w\}$ and therefore, some vertices in $N_2(v, w)$ and $N_3(v, w)$ can be removed.

**Rule 2** For $\{v, w\} \in E(G)$, assume that $|N_3(v, w)| \geq 2$ and $N_3(v, w)$ can not be dominated by a single vertex from $N_2(v, w) \cup N_3(v, w)$.

**Case 1:** $N_3(v, w)$ can be dominated by a single vertex from $\{v, w\}$.

- (1.1) If $N_3(v, w) \subseteq N(v)$ and $N_3(v, w) \subseteq N(w)$ then remove $N_3(v, w)$ and $N_2(v, w) \cap N(v) \cap N(w)$ from $G$ and add a new gadget vertex $z$ with edges $\{v, z\}$ and $\{w, z\}$ to $G$.
- (1.2) If $N_3(v, w) \subseteq N(v)$ but $N_3(v, w) \not\subseteq N(w)$ then remove $N_3(v, w)$ and $N_2(v, w) \cap N(v)$ from $G$ and add a new gadget vertex $v'$ with edge $\{v, v'\}$ to $G$.
- (1.3) If $N_3(v, w) \subseteq N(w)$ but $N_3(v, w) \not\subseteq N(v)$ then remove $N_3(v, w)$ and $N_2(v, w) \cap N(w)$ from $G$ and add a new gadget vertex $w'$ with edge $\{w, w'\}$ to $G$.

**Case 2:** If $N_3(v, w)$ can not be dominated by a single vertex from $\{v, w\}$ then remove $N_2(v, w)$ and $N_3(v, w)$ from $G$ and add new gadget vertices $v'$ and $w'$ with edges $\{v, v'\}$ and $\{w, w'\}$ to $G$.

**Lemma 3.0.2** *Given a graph $G$, let $G'$ be the graph obtained by applying Rule 2 to some edge $\{v, w\} \in E(G)$. Then $\gamma_c(G) = \gamma_c(G')$.*

**Proof:** Let $D$ be a minimum CDS of $G$ and $D'$ be a minimum CDS of $G'$. To prove the lemma, we show that $D'$ is also a minimum CDS of $G$, that is, $|D| = |D'|$. We first show that $D'$ is a CDS of $G$ and get $|D| \leq |D'|$. Later we show $|D| \geq |D'|$ to finally obtain $|D| = |D'|$. Let $X$ be the set of vertices removed by Rule 2 and $D_1 = (D \setminus X) \cup \{v, w\}$. Since $N(X) \subseteq N[v, w]$ and $\{v, w\} \in E(G)$, $D_1$ is a CDS of $G$. Since $D_1$ does not contain any vertex from $X$, $D_1$ is a CDS of $G'$ as well and thus, $|D_1| \geq |D'|$. Next, we show that $|D| \geq |D_1|$ to get $|D| \geq |D'|$. Notice that the vertices in $N_3(v, w)$ can only be dominated by vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$, and no single vertex from $N_2(v, w) \cup N_3(v, w)$ can dominate $N_3(v, w)$.

**Case 1:** $N_3(v, w)$ can be dominated by a single vertex from $\{v, w\}$.

- (1.1) $N_3(v, w) \subseteq N(v)$ and $N_3(v, w) \subseteq N(w)$.

  We claim that $D'$ does not contain the gadget vertex $z$. For the sake contradiction, assume that $z \in D'$. Since $z$ is connected only to $v$ and $w$, and $D'$ is a CDS of $G'$, $D'$ must contain at least one vertex from $\{v, w\}$. Because $\{v, w\}$ is an edge of $G$, we get $N(z) \subseteq N[v]$ and $N(z) \subseteq N[w]$. Therefore, $D'' = D' \setminus \{z\}$ is a CDS of $G'$ and $|D''| < |D'|$, a contradiction with $D'$ a minimum CDS of $G$. So $z \notin D'$. Since $z$ can only be dominated by a vertex from $\{v, w\}$, $D'$ contains either of $v$ or $w$, let's say it contains $v$. Since $X \subseteq N(v)$, $D'$ is a CDS of $G$ and $|D| \leq |D'|$. Next, we show that $|D| \geq |D'|$ to obtain $|D| = |D'|$. If $D$ contains at least two vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ then $|D| \geq |D_1| \geq |D'|$. Assume that $D$ contains one vertex from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$. Because no single vertex from $N_2(v) \cup N_3(v)$ can dominate $N_3(v)$, $D$ contains either of $v$ or $w$. Since $D$ does not contain any vertex from $X$, $D$ is also a CDS of $G'$ and $|D| \geq |D'|$.

- (1.2) $N_3(v, w) \subseteq N(v)$ but $N_3(v, w) \nsubseteq N(w)$.

  The proof of this case is similar to that for (1.1). Obviously, $D'$ does not contain the gadget vertex $v'$ and contains the vertex $v$. Since $X \subseteq N(v)$, $D'$ is a CDS of $G$ as well and $|D| \leq |D'|$.

  If $D$ contains at least two vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ then $|D| \geq |D_1| \geq |D'|$. Therefore, we assume that $D$ contains one vertex from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$. Because no single vertex from $\{w\} \cup N_2(v) \cup N_3(v)$ can dominate $N_3(v)$, $D$ contains $v$. Since $D$ does not contain any vertex from $X$, $D$ is also a CDS of $G'$ and $|D| \geq |D'|$.

- (1.3) If $N_3(v, w) \subseteq N(w)$ but $N_3(v, w) \nsubseteq N(v)$.

  The proof is symmetric to that for (1.2).

**Case 2:** $N_3(v, w)$ can not be dominated by a single vertex from $\{v, w\}$.

Obviously $D'$ does not contain any gadget vertex from $\{v', w'\}$ and contains vertices $v$ and $w$. Since $X \subseteq N(v, w)$, $D'$ is a CDS of $G$ as well and $|D| \leq |D'|$. Since $N_3(v, w)$ can not be dominated by a single vertex from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$, $D$ contains at least two vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$. Therefore, $|D| \geq |D_1| \geq |D'|$.

□

Our next three rules are designed to be applied to a pair of vertices $v$ and $w$ of $G$ with $2 \leq d_G(v, w) \leq 3$. The intuition behind these rules is similar to that for Rule 2 but to remove some vertices from $N_2(v, w)$ and $N_3(v, w)$, we may need to keep some vertices which form a path between $v$ and $w$ to guarantee the connectivity of the graph induced by the dominating set while we also need to assure that there cannot be any other connected path of shorter length in $N_2(v, w)$ and $N_3(v, w)$ dominating $N_3(v, w)$. This makes the rules more complex than Rule 2 because there are different cases for keeping such vertices. We first introduce some notation.

A vertex $x \in N_3(v, w)$ is called a *bridge* if $x$ is dominated by a vertex from $N_2(v, w)$, $v$, and $w$, that is, $x \in N(N_2(v, w)) \cap N(v) \cap N(w)$. We denote by $B(v, w)$ the set of bridges for $v$ and $w$. Intuitively, a bridge is a good candidate for forming a path between $v$ and $w$ while since a bridge is connected to a vertex from $N_2(v, w)$ it can be on a candidate connected path of shorter length than the one connecting $v$ and $w$. A vertex $x \in N_2(v, w)$ is called a *key-neighbor* of $v$ w.r.t. $w$ if $x$ is dominated by $v$ and a vertex from $N_3(v, w) \cap N(w)$, that is, $x \in N(v) \cap N_2(v, w) \cap N(N(w))$. We denote by $K_w(v)$ the set of key-neighbors of $v$ w.r.t. $w$. We define similarly a key-neighbor of $w$ w.r.t. $v$ and $K_v(w)$. Intuitively, a key-neighbor and its respective bridge might be good candidates for a shorter connected path dominating $N_3(v, w)$.

**Rule 3** For $v, w \in V(G)$ with $d_G(v, w) = 2$, assume that $|N_3(v, w)| \geq 2$. We remove some vertices from $N_3(v, w)$ but keep a path $v - p - w$ in $G$.

**Case 1:** $N_3(v, w)$ can be dominated by a single vertex of $\{v, w\}$.

- (1.1) $N_3(v, w) \subseteq N(v)$ and $N_3(v, w) \subseteq N(w)$. If $N_3(v, w)$ can not be dominated by a subset $U$ of $N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 2$ and $G[U]$ connected then:
  - If $B(v, w) \neq \emptyset$ then select a vertex of $B(v, w)$ as $p$, otherwise select a vertex from $N_3(v, w)$ as $p$.

- Remove $N_3(v, w) \setminus (B(v, w) \cup \{p\})$.
- Add gadget vertex $z$ with edges $\{v, z\}$ and $\{z, w\}$ to $G$.

- (1.2) $N_3(v, w) \subseteq N(v)$ but $N_3(v, w) \nsubseteq N(w)$. If $N_3(v, w)$ can not be dominated by a subset $U$ of $\{w\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 2$ and $G[U]$ connected then:
  - If $K_v(w) \neq \emptyset$ then for each $x_i \in K_v(w)$, if $B(v, w)$ dominates $x_i$ then select a $y_i \in B(v, w)$ dominating $x_i$, let $Y$ be the set of such $y_i$'s, and select a vertex from $Y$ as $p$; otherwise,
    select a vertex from $N(v) \cap N(w)$ as $p$.
  - Remove $N_3(v, w) \setminus (Y \cup \{p\})$.
  - Add a gadget vertex $v'$ with edge $\{v, v'\}$ to $G$.

- (1.3) $N_3(v, w) \subseteq N(w)$ but $N_3(v, w) \nsubseteq N(v)$. If $N_3(v, w)$ can not be dominated by a subset $U$ of $\{v\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 2$ and $G[U]$ connected then:
  - If $K_w(v) \neq \emptyset$ then for each $x_i \in K_w(v)$, if $B(v, w)$ dominates $x_i$ then select a $y_i \in B(v, w)$ dominating $x_i$,
    let $Y$ be the set of such $y_i$'s. Select a vertex from $Y$ as $p$; otherwise, select a vertex from $N(v) \cap N(w)$ as $p$.
  - Remove $N_3(v, w) \setminus (Y \cup \{p\})$.
  - Add a gadget vertex $v'$ with edge $\{w, w'\}$ to $G$.

**Case 2:** $N_3(v, w)$ can not be dominated by a single vertex from $\{v, w\}$. If $N_3(v, w)$ can not be dominated by a subset $U$ of $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 2$ and $G[U]$ connected then: select a vertex from $N(v) \cap N(w)$ as $p$, remove $(N_2(v, w) \cup N_3(v, w)) \setminus \{p\}$, and add gadget vertex $v'$ and $w'$ with edges $\{v, v'\}, \{w, w'\}$ to $G$.

**Lemma 3.0.3** *Given a graph $G$, let $G'$ be the graph obtained by applying Rule 3 to some pair of vertices $v, w \in V(G)$ with $d_G(v, w) = 2$. Then $\gamma_c(G) = \gamma_c(G')$.*

**Proof:** To prove the lemma, we first show that there is a minimum CDS $D'$ of $G'$ which does not contain any gadget vertex. Then we prove that $D'$ is also a minimum CDS of $G$. Let $D''$ be a minimum CDS of $G'$. If $D''$ does not have any gadget vertex then the first statement is true by taking $D' = D''$. Assume that $D''$ contains some gadget vertex. If $D''$ contains $z$ (Case (1.1)) then $D''$ must contain $v$ or $w$ because $z$ is only connected to $v$ and

$w$, and $D''$ is a CDS of $G'$. Let $D' = (D'' \setminus \{z\}) \cup \{p\}$, then $D'$ is a CDS of $G'$, $|D'| \leq |D''|$ and $D'$ does not contain any gadget vertex. Since $D''$ is a minimum CDS of $G'$, $D'$ is a minimum CDS of $G'$. If $D''$ contains any gadget vertex from $\{v', w'\}$ (Cases (1.2), (1.3), and (2)) then $D' = D'' \setminus \{v', w'\}$ is a minimum CDS of $G'$ that does not contain any gadget vertex.

Next, we prove that $D'$ is also a minimum CDS of $G$. Similar to the proof for Lemma 3.0.2, let $D$ be a minimum CDS of $G$. We prove that $|D| = |D'|$ by showing that $D'$ is a CDS of $G$ to get $|D| \leq |D'|$ and then proving $|D| \geq |D'|$. Let $X$ be the set of vertices removed by Rule 3 and $D_1 = (D \setminus X) \cup \{v, w, p\}$. Since $D$ is a CDS of $G$, $N(X) \subseteq N[v, w]$, and $\{v, p\}, \{p, w\} \in E(G)$, $D_1$ is a CDS of $G$. Because $D_1$ does not contain any vertex from $X$, therefore, $D_1$ is a CDS of $G'$ as well and thus, $|D_1| \geq |D'|$. Next, we show that $|D| \geq |D_1|$ in order to get $|D| \geq |D'|$. Notice that the vertices in $N_3(v, w)$ can only be dominated by vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$.

**Case 1:** $N_3(v, w)$ can be dominated by one vertex from $\{v, w\}$.

- (1.1) Since $D'$ is a CDS of $G'$, $D'$ contains $v$ or $w$, say $v$. Since $X \subseteq N(v)$, $D'$ is a CDS of $G$ as well and $|D| \leq |D'|$. If $D$ contains at least three vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ then $|D| \geq |D_1| \geq |D'|$.

  Assume that $D$ contains at most two vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$. Since $N_3(v, w)$ can not be dominated by a subset $U$ of $N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 2$ and $G[U]$ connected, either (a) $D$ contains $v$ or $w$, or (b) $D$ has a subset $U$ of $N_2(v, w) \cup N_3(v, w)$ with $|U| = 2$ and $G[U]$ not connected.

  For Case (a), assume that $D$ contains $v$. If $D$ does not contain any vertex from $X$ then $D$ is a CDS of $G'$ and $|D| \geq |D'|$. Assume that $D$ contains one vertex $x$ of $X$. Since $x$ is not a bridge vertex and $x \in N(v) \cap N(w)$, $N(x) \subseteq N_3(v, w) \cup \{w\}$. From this and the fact that $N_3(v, w) \subseteq N(v)$ and $p$ dominates $w$, $N[x] \cup N[v] \subseteq N[p] \cup N[v]$. We replace $x$ by $p$ in $D$ to get $D_2$. Then $D_2$ is a CDS of $G$ and does not have any vertex from $X$. From this, $D_2$ is a CDS of $G'$ as well and $|D_2| \geq |D'|$. Since $|D| = |D_2|$, $|D| \geq |D'|$.

  For Case (b), let $D_2 = (D \setminus U) \cup \{v, w\}$; then $|D_2| = |D|$. Since $N(U) \subseteq N[v, w]$ and $G[U]$ not connected, if $D$ is a CDS of $G$ then $D_2$ is a CDS of $G$. Since $D_2$ does not contain any vertex from $X$, $D_2$ is a CDS of $G'$ as well. But $|D| = |D_2|$, therefore we can conclude $|D| \geq |D'|$.

- (1.2) Since $X \subseteq N(v)$, $D'$ is a CDS of $G$ as well and $|D| \leq |D'|$. If $D$ contains at least three vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ then $|D| \geq |D_1| \geq |D'|$.

  Assume that $D$ contains at most two vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$. Since $N_3(v, w)$ can not be dominated by a subset $U$ of $\{w\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 2$ and $G[U]$ connected, either (a) $D$ contains $v$ or (b) $D$ has a subset $U$ of $\{w\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| = 2$ and $G[U]$ not connected.

  For Case (a), if $D$ does not contain any vertex from $X$ then $D$ is a CDS of $G'$ and $|D| \geq |D'|$. Assume that $D$ contains a vertex $x$ of $X$. If $x$ does not dominates $w$ or any vertex of $N_2(v, w) \cap N(w)$ then we take $D_2 = D \setminus \{x\}$. Since $N[x] \subseteq N[v]$, $D_2$ is a CDS of $G$ and $|D| > |D_2|$, a contradiction to the fact that $D$ is a minimum CDS of $G$. So we assume that $x$ dominates $w$ or a vertex from $N_2(v, w) \cap N(w)$. Because $x$ is not a bridge dominating any key-neighbor of $w$, $x$ does not dominate both $w$ and a vertex of $N_2(v, w) \cap N(w)$. If $x$ dominates $w$ then we replace $x$ with $p$ to get $D_2$ with $|D| = |D_2|$. With a similar argument as in Case (a) of (1.1), we get $|D| = |D_2| \geq |D'|$. If $x$ dominates a vertex from $N_2(v, w) \cap N(w)$ then we replace $x$ by $w$ to get $D_2$ with $|D| = |D_2|$. Since $N[v] \cup N[x] \subseteq N[v, w]$, $D_2$ is a dominating set of $G$. Because $x \in D_3(v, w)$, $D$ has at most two vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$, and $v \in D$, $N(x) \cap D = \{v\}$. Therefore, $G[D]$ is connected implies that $G[D \setminus \{x\}]$ is connected. From this and the fact that $w$ is dominated by a vertex from $D$, we can conclude that $G[D_2]$ is connected and $D_2$ is CDS of $G$. Since $D_2$ does not have any vertex from $X$, $D_2$ is a CDS of $G'$ and $|D| = |D_2| \geq |D'|$.

  For Case (b), let $D_2 = (D \setminus U) \cup \{w\}$. Then $|D| \geq |D_2|$. Since $N(U) \subseteq N[v, w]$ and $G[U]$ not connected, if $D$ is a CDS of $G$ then $D_2$ is a CDS of $G$. Since $D_2$ does not contain any vertex from $X$, $D_2$ is a CDS of $G'$ as well. From this and $|D| \geq |D_2|$, we get $|D| \geq |D'|$.

- (1.3)

  The proof is symmetric to that for Case (1.2).

Case 2: $N_3(v, w)$ can not be dominated by one vertex from $\{v, w\}$.

Obviously $D'$ contains both $v$ and $w$. Since $X \subseteq N(v, w)$, $D'$ is a CDS of $G$ as well and $|D| \leq |D'|$. If $D$ contains at least three vertices from $N_2(v, w) \cup N_3(v, w) \cup \{v, w\}$ then $|D| \geq |D_1| \geq |D'|$.

Assume that $D$ contains at most two vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$. Then $D$ has a subset $U$ of $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| = 2$ and $G[U]$ not connected. Let $D_2 = (D \setminus U) \cup \{v, w\}$. Then $|D_2| = |D|$. Since $N(U) \subseteq N[v, w]$ and $G[U]$ is not connected, we know that if $D$ is a CDS of $G$ then $D_2$ is a CDS of $G$. Since $D_2$ does not contain any vertex from $X$, $D_2$ is a CDS of $G'$ as well. But since $|D| \geq |D_2|$, we obtain $|D| \geq |D'|$.

□

**Rule 4** For $v, w \in V(G)$ with $d_G(v, w) = 3$, assume that $|N_3(v, w)| \geq 2$. We remove some vertices from $N_3(v, w)$ but keep a path $v - p - q - w$ in $G$.

**Case 1:** $N_3(v, w)$ can be dominated by a single vertex of $\{v, w\}$.

- (1.1) $N_3(v, w) \subseteq N(v)$. If $N_3(v, w)$ can not be dominated by a subset $U$ of $\{w\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 3$ and $G[U]$ connected then:
  - If $K_v(w) \neq \emptyset$ then for each $x_i \in K_v(w)$, select a $y_i \in N(v) \cap N_3(v, w)$ dominating $x_i$, let $Y$ be the set of such $y_i$'s, select a vertex $y_i \in Y$ as $p$, and select $x_i$ dominated by $y_i$ as $q$; otherwise, select any two vertices $p$ and $q$ such that $v - p - q - w$ is a path of $G$.
  - Remove $N_3(v, w) \setminus (Y \cup \{p, q\})$.
  - Add a gadget vertex $v'$ with edge $\{v, v'\}$ to $G$.
- (1.2) $N_3(v, w) \subseteq N(w)$. If $N_3(v, w)$ can not be dominated by a subset $U$ of $\{v\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 3$ and $G[U]$ connected then:
  - If $K_w(v) \neq \emptyset$ then for each $x_i \in K_w(v)$, select a $y_i \in N(w) \cap N_3(v, w)$ dominating $x_i$, let $Y$ be the set of such $y_i$'s, select a vertex $y_i \in Y$ as $q$, and select $x_i$ dominated by $y_i$ as $p$; otherwise, select any two vertices $p$ and $q$ such that $v - p - q - w$ is a path of $G$.
  - Remove $N_3(v, w) \setminus (Y \cup \{p, q\})$.
  - Add a gadget vertex $w'$ with edge $\{w, w'\}$ to $G$.

**Case 2:** $N_3(v, w)$ can not be dominated by a single vertex of $\{v, w\}$. If $N_3(v, w)$ can not be dominated by a subset $U$ of $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 3$ and $G[U]$ connected then select any two vertices $p$ and $q$ such that $v - p - q - w$ is path of $G$, remove $(N_2(v,w) \cup N_3(v,w)) \setminus \{p,q\}$ from $G$, and add gadget vertices $v'$ and $w'$ with edges $\{v,v'\}$ and $\{w,w'\}$ to $G$.

**Lemma 3.0.4** *Given a graph $G$, let $G'$ be the graph obtained by applying Rule 4 to some pair of vertices $v, w \in V(G)$ with $d_G(v,w) = 3$. Then $\gamma_c(G) = \gamma_c(G')$.*

**Proof:** Obviously, a minimum CDS $D'$ of $G'$ does not contain any gadget vertex. We prove that $D'$ is also a minimum CDS of $G$. Let $D$ be a minimum CDS of $G$. We prove that $|D| = |D'|$ by showing that $D'$ a CDS of $G$ to get $|D| \leq |D'|$ and then proving $|D| \geq |D'|$. Let $X$ be the set of vertices removed by Rule 3 and $D_1 = (D \setminus X) \cup \{v,w,p,q\}$. Since $N(X) \subseteq N[v,w]$, $D$ is a CDS of $G$, and $\{v,p\},\{p,q\},\{q,w\} \in E(G)$, $D_1$ is a CDS of $G$. Since $D_1$ does not contain any vertex from $X$, $D_1$ is a CDS of $G'$ as well and $|D_1| \geq |D'|$. Considering all possible cases, we prove that $|D| \geq |D_1|$ in order to obtain $|D| \geq |D'|$. Notice that the vertices in $N_3(v,w)$ can only be dominated by vertices from $\{v,w\} \cup N_2(v,w) \cup N_3(v,w)$.

**Case 1:** $N_3(v,w)$ can be dominated by one vertex from $\{v,w\}$.

- (1.1) $D'$ contains $v$. Since $X \subseteq N(v)$, $D'$ is a CDS of $G$ as well and $|D| \leq |D'|$. If $D$ contains at least four vertices from $\{v,w\} \cup N_2(v,w) \cup N_3(v,w)$ then $|D| \geq |D_1| \geq |D'|$.

  Assume that $D$ contains at most three vertices from $\{v,w\} \cup N_2(v,w) \cup N_3(v,w)$. Since $N_3(v,w)$ can not be dominated by a subset $U$ of $\{w\} \cup N_2(v,w) \cup N_3(v,w)$ with $|U| \leq 3$ and $G[U]$ connected, either (a) $D$ contains $v$ or (b) $D$ has a subset $U$ of $\{w\} \cup N_2(v,w) \cup N_3(v,w)$ with $2 \leq |U| \leq 3$ and $G[U]$ not connected.

  For Case (a), if $D$ does not contain any vertex from $X$ then $D$ is a CDS of $G'$ and $|D| \geq |D'|$. Assume that $D$ contains a vertex from $X$. There are two subcases: (a1) $D$ contains one vertex $x \in X$ and (a2) $D$ contains two vertices $x, x' \in X$. For Subcase (a1), if $x$ dominates a key-neighbor $x_i \in K_v(w)$ then we replace $x$ by $y_i \in Y$ which dominates $x_i$. Since $N[v] \cup N[x] \subseteq N[v] \cup N[y_i]$, $D_2$ is a CDS of $G$. Since $D_2$ does have any vertex from $X$, $D_2$ is also a CDS of $G'$ and $|D_2| \geq |D'|$. But since $|D| = |D_2|$, $|D| \geq |D'|$. Otherwise ($x$ does not dominate any vertex from $N_2(v,w) \cap N[w]$) we remove $x$ from $D$ to get $D_2$. Since $N(x) \subseteq N(v)$, $D_2$ is a CDS of $G$ and $|D| \geq |D_2|$. Because $D_2$ does not have any vertex from $X$,

$D_2$ is a CDS of $G'$ and $|D| \geq |D_2| \geq |D'|$. For Subcase (a2), we replace $x$ and $x'$ by $w$ to obtain $D_2$. Since $N(\{x, x'\}) \subseteq N[v] \cup N[w]$, $D_2$ is a dominating set of $G$. But as $\{x, x'\} \cap D \subseteq N(v)$, if $D$ is a CDS of $G$ then $D_2$ is also a CDS of $G$. Since $D_2$ does not contain any vertex from $X$, $D_2$ is a CDS of $G'$ as well. Since $|D| \geq |D_2|$ and $D_2| \geq |D'|$, we get $|D| \geq |D'|$.

For Case (b), let $D_2 = (D \setminus U) \cup \{v, w\}$. Since $N(U) \subseteq N[v, w]$ and $G[U]$ is not connected, we conclude that if $D$ is a CDS of $G$ then $D_2$ must be a CDS of $G$. Since $D_2$ does not contain any vertex from $X$, $D_2$ is a CDS of $G'$ and $|D_2| \geq |D'|$. From $|D| \geq |D_2|$, we get $|D| \geq |D'|$.

- (1.2) The proof is symmetric to that for (1.1).

**Case 2:** $D'$ contains both $v$ and $w$. Since $X \subseteq N[v, w]$, $D'$ is a CDS of $G$ and $|D| \leq |D'|$. If $D$ contains at least four vertices from $N_2(v, w) \cup N_3(v, w) \cup \{v, w\}$, then $|D| \geq |D_1| \geq |D'|$.

Assume that $D$ contains at most three vertices from $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$. Since $N_3(v, w)$ can not be dominated by a subset $U$ of $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ with $|U| \leq 3$ and $G[U]$ connected, $D$ has a subset $U$ of $\{v, w\} \cup N_2(v, w) \cup N_3(v, w)$ with $2 \leq |U| \leq 3$ and $G[U]$ not connected. Let $D_2 = (D \setminus U) \cup \{v, w\}$. But since $N(U) \subseteq N[v, w]$ and $G[U]$ is not connected, we conclude that if $D$ is a CDS of $G$ then $D_2$ must be a CDS of $G$. Since $D_2$ does not contain any vertex from $X$, $D_2$ is a CDS of $G'$ and $|D_2| \geq |D'|$. But since $|D| \geq |D_2|$, we get $|D| \geq |D'|$.

□

For $v, w \in V(G)$, let $\mathcal{R} = \mathcal{R}(v, w)$ be the union of all maximal simple regions $R(v, w)$.

**Rule 5** For $v, w \in V(G)$, assume that $(V(\mathcal{R}) \cap N_3(v, w)) \setminus B(v, w) \neq \emptyset$. If $B(v, w) \neq \emptyset$ then select a vertex from $B(v, w)$ as $p$, otherwise select a vertex from $V(\mathcal{R}) \cap N_3(v, w)$ as $p$. Remove $(V(\mathcal{R}) \cap N_3(v, w)) \setminus (B(v, w) \cup \{p\})$. If $B(v, w) \neq \emptyset$ then add a gadget vertex $z$ with edges $\{v, z\}, \{z, w\}$ to $G$.

**Lemma 3.0.5** *Given a graph $G$, let $G'$ be the graph obtained by applying Rule 5 to $v, w \in V(G)$. Then $\gamma_c(G) = \gamma_c(G')$.*

**Proof:** Let $D$ be a minimum CDS of $G$ and $D'$ be a minimum CDS of $G'$. Since $z$ is dominated only by $v$ or $w$, $D'$ contains $v$ or $w$. Beause $N(z) \subseteq N(p)$, we can assume that

$D'$ does not contain $z$, otherwise we can replace $z$ by $p$. Since $(V(\mathcal{R}) \cap N_3(v,w)) \subseteq N(v)$, and $(V(\mathcal{R}) \cap N_3(v,w)) \subseteq N(w)$, $D'$ is a CDS of $G$ and $|D| \leq |D'|$. To prove the lemma, we show that $|D| \geq |D'|$. Let $X$ be the set of vertices removed by Rule 5. If $D$ does not contain any vertex from $X$ then $D$ is a CDS of $G'$ and $|D| \geq |D'|$. Assume that $D$ contains a vertex $x \in X$. If $D$ contains $v$ or $w$ then we can place $x$ by $p$ to make $D$ a minimum CDS of $G'$ and get $|D| \geq |D'|$. Assume that $D$ does not have any of $v$ and $w$. Because $D$ is a CDS, $D$ must contain a bridge $y \in B(v,w)$ and a vertex $y' \in N_2(v,w) \cap N(y)$. We replace $x, y, y'$ by $v, w, y$ in $D$. From $N[x] \cup N[y] \cup N[y'] \subseteq N[v] \cup N[w]$ and $G[\{v,y,w\}]$ is connected, $D$ is a minimum CDS of $G$ and does not contain any vertex from $X$ after the replacement. Therefore, $D$ is also a minimum CDS of $G'$ and $|D| \geq |D'|$. □

In Rules 1-4, adding gadget vertices $v'$ and $w'$ with edges $\{v,v'\}$ and $\{w,w'\}$ does not change the planarity of $G$. In Rule 5 and Case (1.1) of Rule 3, a gadget vertex $z$ with edges $\{v,z\},\{z,w\}$ is added to $G$ when a vertex $x \in N_3(v,w) \cap N(v) \cap N(w)$ is removed. We can place the edges $\{v,z\},\{z,w\}$ at the locations of $\{v,x\},\{x,w\}$ to keep $G$ planar. So graph $G'$ obtained from applying any of Rules 1-5 to $G$ is planar.

**Lemma 3.0.6** *Given a planar graph $G$, Rule 1 can be performed in $O(n)$ time for all vertices of $G$, each of Rules 2-5 can be performed in $O(n^2)$ time for all pairs of vertices $v, w \in V(G)$.*

**Proof:** We first discuss Rule 1 as it is different from the rest of the rules in that it requires the calculation of neighborhood of a single vertex only. Considering the definition of the three neighborhood sets for a vertex $v$ of a planar graph $G$, $(N_1(v), N_2(v), N_3(v))$, in order to calculate these sets, it is sufficient to consider a subgraph of $G$ induced by the set of vertices in the distance at most 2 of $v$. To do so, we construct a breadth-first-search tree of depth two, rooted at $v$. We even may not need the complete tree as only some of the vertices at depth 2 are required. Calculating $N_1(v)$ vertices is very easy since it only requires us to explore all the vertices at distance one from $v$ and see if it has a neighbor in the second level of the tree. As soon as the first such neighbor is explored we can stop expanding the node's neighbors and proceeds with the next vertex in the first level. Denoting the degree of $v$ by $deg(v)$, the process takes time $O(deg(v))$ as there clearly are at most $2 \cdot deg(v)$ tree edges and since $G[N[v]]$ is planar, there can be only $O(deg(v))$ non-tree edges to be explored. To obtain $N_2(v)$, one has to go through all vertices from the first level of the tree that are not already marked as being in $N_1(v)$ but have at least one neighbor in $N_1(v)$. All this can be done in time $O(deg(v))$ within the planar graph induced by $N[v]$, using the already marked $N_1(v)$-vertices. Finally, $N_3(v)$ consists of vertices from the first level that are not marked yet. It is clear that removal of vertices and adding the gadget can also be performed in $O(deg(v)$. Since Rule 1 can be performed in $O(deg(v))$, applying Rule 1 for all vertices in $G$ takes $\sum_{v \in V(G)} O(deg(v))$ and as $G$ is planar this sum is bounded from above by $O(n)$.

Next, we consider Rules 2-5. All of these rules require the calculation of the joint neighborhood sets for a pair of vertices; therefore, we first focus on this section. To do this, we use a similar idea as the one employed for calculating neighborhood of one vertex. In particular, for computing the joint neighborhood sets $N_1(u,v), N_2(u,v), N_3(u,v)$, we construct the breadth-first tree up to 2 levels using a distance function defined as the minimum of the distance from $u$ and the distance from $v$. Having constructed the tree, we use the same process as before for calculating each of $N_1(u,v), N_2(u,v)$ and $N_3(u,v)$ sets. As before, the running time for calculating the neighborhood sets is determined by the size of the subgraph induced by $N[u,v]$ which has a size of $O(deg(u)+deg(v))$ for planar graphs. For enumerating bridge vertices we only need to test for each vertex of $N_3(u,v)$ whether it has an edge to a vertex from $N_2(u,v)$ as well as $u$ and $v$ which can obviously be performed in order of the size of $G[N[u,v]]$. Using a similar argument we can conclude that key-neighbors of $v$ ($u$) with respect to $u$ ($v$) can be calculated in the same time complexity. Next, we argue that the vertex removal and attachment of gadgets as prescribed by the rules can be performed within a linear order of the size of the respective subgraph. This is true since with a successful application of any of the rules there is always a constant number of vertices are being removed (added). Therefore, we need $\sum_{u,v \in V(G)} O(deg(u) + deg(v))$ for performing each of such rules on all of the vertex pairs which using the fact that $O(deg(v)) = O(n)$ this is upperbounded by $O(\sum_{u,v \in V(G)} deg(u) + \sum_{u,v \in V(G)} deg(v)) = O(n^2)$. □

We recall that a graph $G$ is called *reduced* if the graph obtained from applying any of Rules 1-5 to $G$ is isomorphic to $G$.

**Theorem 3.0.1** *A plane graph $G$ can be converted to a reduced plane graph $G'$ with $\gamma_c(G) = \gamma_c(G')$ by Rules 1-5 in $O(n^3)$ time.*

**Proof:** First of all notice that none of the 5 rules can increase the size of the graph. We recall that while introducing each of the reduction rules we proved that there exists a solution that does not contain any of the added gadgets. Therefore, without loss of generality we assume that reduction rules are never applied to any of the added gadgets. In practice, the gadget vertices can be marked so that (only) in the data reduction phase they are treated differently from original vertices of the graph. Please notice that the kernel includes the gadget vertices and while solving the kernel, whichever the technique is used, we need not distinguish between gadget and non-gadget vertices.

We claim that for a graph with $m$ edges, there can be at most $O(m)$ successful applications of reduction rules. This is true since after one application of Rules 1-5, which changes the graph, by definition, the resulting graph has at most the same number of vertices, but at least one edge less than before the application of the rule. Therefore, the graph can be reduced to a kernel in $O(m)O(n^2)$; but since graph is planar this is upperbounded by $O(n^3)$. $\square$

# Chapter 4

# Linear-size Kernel

In this chapter, we show that the reduced graph $G'$, obtained after repetitive application of reduction Rules 1-5, has $O(\gamma_c(G))$ vertices. The proof consists of three major parts. First, we obtain a maximal $D$-region decomposition $\mathcal{R}$ of $O(\gamma_c(G))$ regions for a plane graph $G$. Next, we show that having applied the reduction rules repetitively to the graph, each region in $\mathcal{R}$ can have $O(1)$ vertices only. Finally, we upperbound the number of vertices of $G$ not belonging to any region of $\mathcal{R}$ as $O(\gamma_c(G))$. The proof of the first part is almost the same as that in [4] and we only briefly recite some definitions and the main result here.

We recall that given a plane graph $G$ and a subset $D \subseteq V(G)$, a *$D$-region decomposition* of $G$ is a set $\mathcal{R}$ of regions between pairs of vertices of $D$ such that

1. for $R(v, w)$, no vertex of $D \setminus \{v, w\}$ is in $V(R(v, w))$ and
2. for two regions $R_1, R_2 \in \mathcal{R}$, $(R_1 \cap R_2) \subseteq (\partial R_1 \cup \partial R_2)$.

A $D$-decomposition $\mathcal{R}$ is called maximal if there is no region $R \notin \mathcal{R}$ such that $\mathcal{R}' = \mathcal{R} \cup \{R\}$ is a $D$-region decomposition and $V(\mathcal{R}) \subset V(\mathcal{R}')$.

**Lemma 4.0.7** *(Alber et al. [4]) Given a plane graph $G$ and a dominating set $D$ of $G$, a maximal $D$-decomposition $\mathcal{R}$ of at most $3|D|$ regions can be constructed.*

Using this result, by setting $D$ be to be a minimum CDS of $G$, we can have a maximal $D$-region decomposition $\mathcal{R}$ of at most $3\gamma_c(G)$ regions.

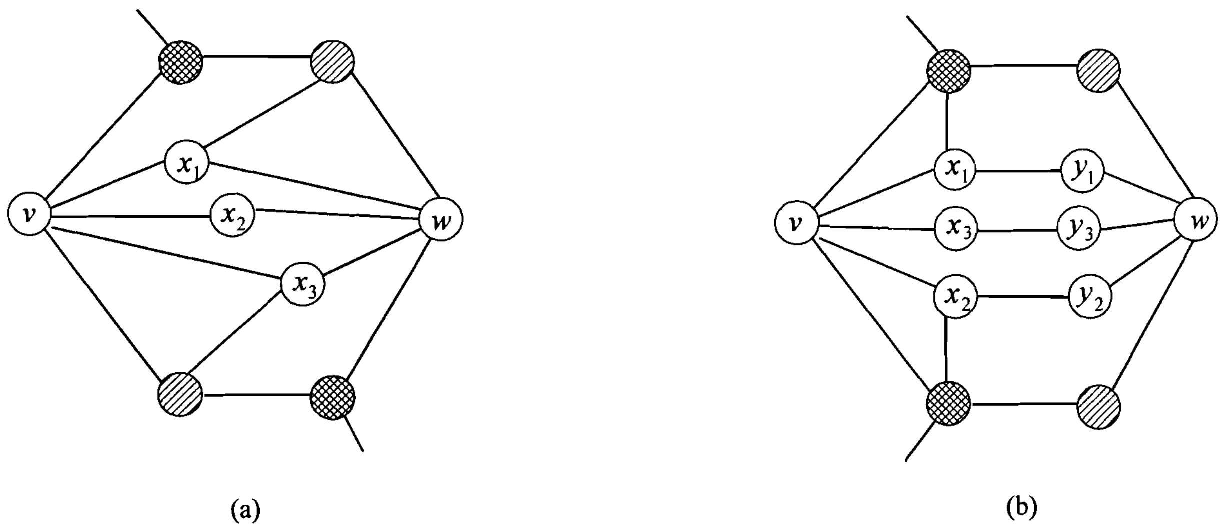


Figure 4.1: Example of bridges in $R(v, w)$ when $d_G(v, w) = 2$ and $d_G(v, w) = 3$.

Next, we calculate an upperbound on the number of vertices in a region.

**Proposition 4.0.1** *For any pair of vertices $v, w$ of a plane graph $G$, any region $R(v, w)$ has at most two bridges, that is, $|V(R) \cap B(v, w)| \leq 2$.*

**Proof:** Assume that there are at least three bridges $x_1, x_2, x_3$ in $R(v, w)$. Then one bridge, say $x_3$, must be strictly inside the region $R' = R'(v, w)$ formed by the paths $v - x_1 - w$ and $v - x_2 - w$ (see Figure 4.1 (a)). Since $R'$ is inside $R$ and $x_1, x_2 \in N_3(v, w)$, each vertex strictly inside $R'$ is not connected to any vertex from $N_2(v, w)$, a contradiction to the fact that $x_3$ is a bridge. Thus, there are at most two bridges in $R(v, w)$. □

**Proposition 4.0.2** *For any pair of vertices $v, w$ of a plane graph $G$, any region $R(v, w)$ has at most two key-neighbors of $v$ w.r.t. $w$ and at most two key-neighbors of $w$ w.r.t. $v$, that is, $|V(R) \cap K_w(v)| \leq 2$ and $|V(R) \cap K_v(w)| \leq 2$.*

**Proof:** Assume that there are at least three key-neighbors $x_1, x_2, x_3$ in $K_w(v)$. Let $y_i \in N(w) \cap N_3(v, w), i = 1, 2, 3$ be the neighbors of $x_i$, respectively. Then one vertex from $K_w(v)$, say $x_3$, must be strictly inside the region $R' = R'(v, w)$ formed by the paths $v - x_1 - y_1 - w$ and $v - x_2 - y_2 - w$ (see Figure 4.1 (b)). Since $R'$ is inside $R$ and $x_1, x_2 \in N_2(v, w)$ and

$y_1, y_2 \in N_3(v,w)$, each vertex strictly inside $R'$ is not connected to any vertex from $N_1(v,w)$, a contradiction to the fact that $x_3$ is a vertex of $N_2(v,w)$. Thus, there are at most two key-neighbors of $v$ w.r.t. $w$ in $R(v,w)$. Similarly, there are at most two key-neighbors of $w$ w.r.t. $v$ in $R(v,w)$. □

**Proposition 4.0.3** *Let $G'$ be the reduced graph obtained after repetitive application of Rules 1-5 to $G$. Then $G'$ has the following properties.*

1. *For every $v \in V(G')$, $N_3(v)$ does not have any vertex of $G$.*
2. *For every pair $v, w \in V(G')$, either*
   - *(a) there is a $U \subseteq N_2(v,w) \cup N_3(v,w)$ such that $|U| \leq 3$, $G[U]$ is connected, and $U$ dominates all vertices of $N_3(u,v)$, or*
   - *(b) for every region $R(v,w)$, $N_3(v,w) \cap V(R)$ has at most two vertices from $V(G)$.*

**Proof:** (1) follows from Rule 1. For (2), if no vertex is removed from $G$ by any of Rules 2-5 then by the definition of the rules, (a) holds, otherwise, by Propositions 4.0.1 and 4.0.2, (b) holds. □

Next, we use Proposition 4.0.3 to upperbound the size of simple regions. We recall that for $i = 1, 2$, a simple region $R(v,w)$ is called a type-$i$ region if $V(R)$ has $i$ vertices from $N_1(v,w)$.

**Proposition 4.0.4** *Given a reduced plane graph $G$ and a maximal $D$-region decomposition $\mathcal{R}$ for a CDS $D$ of $G$, a type-$i$ region $R(v,w)$ of $\mathcal{R}$ has at most $i$ vertices from $N_1(v,w)$, $i$ vertices from $N_2(v,w)$, and $i+1$ vertices from $N_3(v,w)$.*

**Proof:** For a simple region $R = R(v,w)$ in $\mathcal{R}$, only the vertices on the boundary can have a neighbor outside $R$. By the definition of simple region, $|N_1(v,w) \cap V(R)| \leq 2$. But since $G$ is planar, every vertex in $N_1(v,w) \cap V(R)$ can contribute at most one vertex to $N_2(v,w) \cap V(R)$. Hence, we get $|N_2(v,w) \cap V(R)| \leq |N_1(v,w) \cap V(R)|$. By Rule 5 and Proposition 4.0.1, $N_3(v,w) \cap V(R)$ has at most $|N_2(v,w) \cap V(R)|$ bridges and one gadget vertex. □

Next, we upperbound the number of vertices in a region $R(v, w)$ of $\mathcal{R}$. The key step in the proof is to decompose a worst case region $R(v, w)$ to a set of simple regions. From this upper bound and Propositions 4.0.3 and 4.0.4, we can get an upper bound on the number of vertices in $R(v, w)$.

**Lemma 4.0.8** *Given a reduced plane graph $G$ and a maximal $D$-region decomposition $\mathcal{R}$ for a CDS $D$ of $G$, every region $R = R(v, w)$ of $\mathcal{R}$ has at most 81 vertices.*

**Proof:** Let $P$ and $Q$ be the paths which form the boundary of $R$. Without loss of generality, we assume that both $P$ and $Q$ have length three (a shorter path will give a smaller number of vertices in $R$); let $P = v - p_1 - p_2 - w$ and $Q = v - q_1 - q_2 - w$. Since only the vertices on $P$ and $Q$ can be connected to vertices outside $R$, $|N_1(v, w) \cap V(R)| \leq 4$. The rest of the proof is divided into three cases: (1) One of Rules 2-4 has been successfully applied to $R$; (2) $d_G(v, w) = 2$ and the condition for applying Rule 3 is not satisfied; and (3) $d_G(v, w) = 3$ and the condition for applying Rule 4 is not satisfied.

**Case (1).** From Proposition 4.0.3 and the definition of Rules 2-4, $|N_3(v, w) \cap V(R)| \leq 3$. Since each vertex of $N_2(v, w)$ is dominated by a vertex of $N_1(v, w)$ and a vertex of $\{v, w\}$, the vertices of $N_2(v, w)$ are in simple regions between a vertex of $N_1(v, w)$ and a vertex of $\{v, w\}$. From the planarity of $G$, we conclude that there are at most six such regions (see Figure 4.2.$a$). In the worst case, 4 of the simple regions are type-1 and 2 are type-2. From Proposition 4.0.4, $|N_2(v, w) \cap V(R)| \leq 4 \cdot 4 + 2 \cdot 7 = 30$. Thus, $|V(R)| \leq |N_1(v, w) \cap V(R)| + |N_1(v, w) \cap V(R)| + |N_1(v, w) \cap V(R)| + |\{v, w\}| \leq 4 + 30 + 3 + 2 = 39$.

**Case (2).** From Proposition 4.0.3 and the definition of Rule 3, we know that there is a subset $U \subseteq N_3(v, w)$ such that $|U| \leq 2$, $G[U]$ is connected, and every vertex of $N_3(v, w)$ is dominated by $U$. Since each vertex of $N_3(v, w)$ is also dominated by a vertex from $\{v, w\}$, the vertices of $N_3(v, w)$ are in simple regions between a vertex of $U$ and a vertex from $\{v, w\}$. There are at most four such simple regions (see Figure 4.2.$b$). Therefore, $|N_3(v, w) \cap V(R)| \leq 4 \cdot 7 + 2 = 30$. Similar to Case (1), the vertices of $N_2(v, w)$ are in at most six simple regions between a vertex in $N_1(v, w)$ and a vertex from $\{v, w\}$. The total number of vertices in $V(R)$ is bounded by $4 + 30 + 30 + 2 = 66$.

**Case (3).** From Proposition 4.0.3 and the definition of Rule 4, there is a subset $U \subseteq N_3(v, w)$ such that $|U| \leq 3$, $G[U]$ is connected, and every vertex of $N_3(v, w)$ is dominated by $U$. Since each vertex of $N_3(v, w)$ is also dominated by a vertex of $\{v, w\}$, the vertices of

$N_3(v,w)$ are in simple regions between a vertex of $U$ and a vertex of $\{v,w\}$. There are at most six such simple regions (see Figure 4.2.c). Therefore, $|N_3(v,w) \cap V(R)| \leq 6 \cdot 7 + 3 = 45$. Similar to Case (1), the vertices of $N_2(v,w)$ are in at most six simple regions between a vertex of $N_1(v,w)$ and a vertex of $\{v,w\}$. The total number of vertices in $V(R)$ is bounded by $4 + 30 + 45 + 2 = 81$. □

Finally, we use the result of [4] to bound the number of vertices not in $V(\mathcal{R})$.

**Lemma 4.0.9** *(Alber et al. [4]) Given a plane reduced graph $G$ and a dominating set $D$ of $G$, if $\mathcal{R}$ is maximal $D$-region decomposition, then $|V(G) \setminus V(\mathcal{R})| \leq 2|D| + 56|\mathcal{R}|$.*

**Proof:** We recite the proof from Alber et al. [4] with the only difference that we use the result in Proposition 4.0.4 for bounding the size of simple regions. We claim that every vertex $u \in V \backslash V(\mathcal{R})$ is either a vertex in $D$ or belongs to a set $N_2(v) \cup N_3(v)$ for some $v \in D$. To see this, suppose that $u \notin D$. But since $D$ is a dominating set, we know that $u \in N(v)$ for some vertex $v \in D$. Since $\mathcal{R}$ is assumed to be maximal, by Lemma 6, we know that $N_1(v) \subseteq V(\mathcal{R})$. Thus, $u \in N_2(v) \cup N_3(v)$. For a vertex $v \in D$, let $N_2^*(v) = N_2(v) \backslash V(\mathcal{R})$, then $u \in N_3(v) \cup N_2^*(v), u \in D$. First, we bound the $N_3$ vertices. Using the fact that each vertex can have only one vertex in its $N_3$ neighborhood we get $|\bigcup_{v \in D} N_3(v)| \leq |D|$. Next, we upperbound the size of $N_2^*(v)$ for a given vertex $v \in D$. let's define $N_1^*(v)$ to be the subset of $N_1(v)$ that sit on the boundary of a region in $\mathcal{R}$ then by definition we know $N_2^*(v) \subseteq N(v) \cap N(N_1^*(v))$. Now for a vertex $v$ let $R(v,w_1), \ldots :, R(v,w_l)$ be all the regions between v and some other vertices $w_i \in D$, where $l = deg_{G_R}(v)$ is the degree of v in the induced region graph $G_R$. Then, every region $R(v,w_i)$ can contribute at most two vertices say $u_i^1$ and $u_i^2$ to $N_1^*(v)$, that is, $|N_1^*(v)| = 2deg_{G_R}(v)$. Next, we claim that there are a set of simple regions between vertices $v \in D$ and $N_1^*(v)$ such that they include all the vertices in $N_2^*(v)$. Formally, we claim that there exists a set $S_v$ of simple regions such that:

(1) every $S \in S_v$ is a simple region between $v$ and some vertex in $N_1^*(v)$,

(2) $N_2^*(v) \subseteq \bigcup_{S \in S_v} V(S)$ and

(3) $|S_v| \leq 2|N_1^*(v)|$.

The idea for the construction of the set $S_v$ is similar to the greedy-like construction of a maximal region decomposition as we used before. Starting with $S_v$ as empty set, one iteratively adds a simple region $S(v,x)$ between $v$ and some vertex $x \in N_1^*(v)$ to the set $S_v$ in such a way that (1) $S_v \cup \{S(v,x)\}$ contains more $N_2^*(v)$-vertices than $S_v$ , (2) $S(v,x)$ does not cross any region in $S_v$ and (3) $S(v,x)$ is maximal (in space) under all simple regions $S$ between $v$ and $x$ that do not cross any region in $S_v$ . The fact that we end up with at most $2|N_1^*(v)|$ many regions can be seen as follows: Consider the induced graph $GS_v$ , which has the set $\{v\} \cup N_1^*(v)$ as vertices and an edge between $v$ and a vertex $u \in N_1^*(v)$ if and only $S_v$ contains a simple region between $v$ and $u$. In other words, $G_{S_v}$ is a star with possible multiple edges. Since, by construction, all simple regions were chosen maximal in space, the graph $G_{S_v}$ is thin. It is not hard to see that a thin star on $n+1$ vertices can have at most $2n$ edges. In particular, this shows that $G_{S_v}$ has at most $2|N_1^*(v)|$ edges, that is, $|S_v| \leq 2|N_1^*(v)|$. Every simple region $S(v,x)$ with $x \in N_1^*(v)$ contains at most 7 vertices not counting $v$ and $x$, which clearly cannot be in $N_2^*(v)$,we conclude that:

$$|N_2^*(v)| \leq 7|S_v| = 14|N_1^*(v)| = 28deg_{G_R}(v).$$

Finally, adding up all the 3 sets we get:

$$|V \backslash V(\mathcal{R})| = |D| + |D| + \sum_{v \in D} |N_2^*(v)| \leq 2|D| + 28 \sum_{v \in D} deg_{G_R}(v) \leq 2|D| + 56|\mathcal{R}|$$

Taking a minimum CDS of $G$ as $D$, we get $|V(G) \setminus V(\mathcal{R})| \leq 2\gamma_c(G) + 56|\mathcal{R}|$.

**Theorem 4.0.2** *For a planar graph $G$ which is reduced with respect to Rules 1 to 5,* $|V(G)| \leq 413\gamma_c(G)$

**Proof:** From Lemma 4.0.7, there are at most $3\gamma_c(G)$ regions in $\mathcal{R}$. From Lemma 4.0.8, each region has at most 81 vertices. From Lemma 4.0.9, $|V \backslash V(\mathcal{R})| \leq 2|D| + 56|\mathcal{R}|$. Therefore, $|V(G) \leq 2\gamma_c(G) + 56 \times 3\gamma_c(G) + 81 \times 3\gamma_c(G) = 413\gamma_c(G)$. □

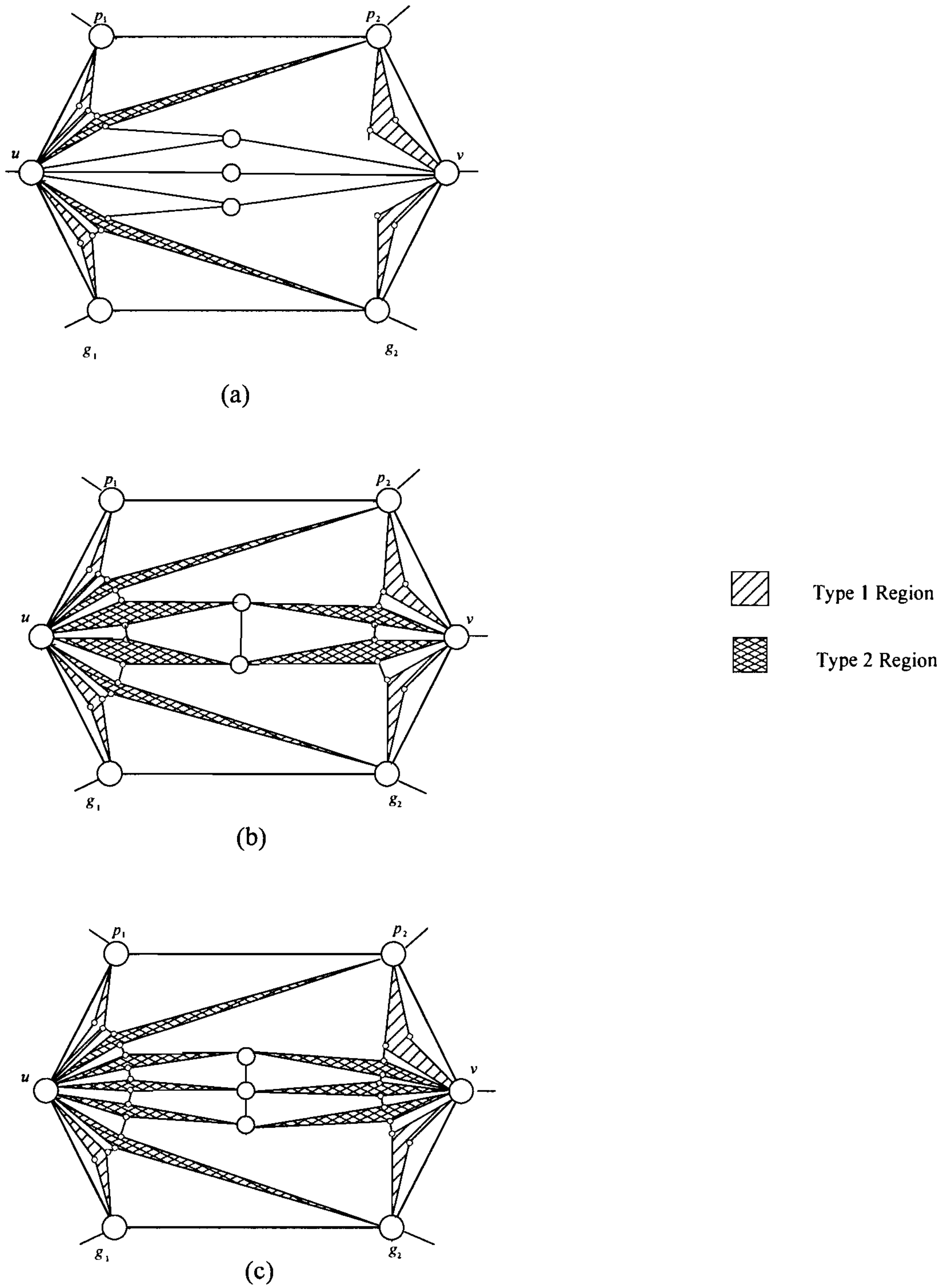


Figure 4.2: a) Rules 2-4 apply b) $d_G(v, w) = 2$ but Rule 3 fails c) $d_G(v, w) = 3$ but Rule 4 fails.

# Chapter 5

# Extensions of reduction rules

## 5.1 Introduction

The idea behind our proposed reduction rules in the previous chapters was to examine the graph locally and decide whether the subgraph can be replaced by a smaller graph while maintaining the problems propreties. In particular, the reduction rules use the information regarding a joint neighborhood of a pair of vertices as the criteria for choosing which subset of vertices can be preferred to another subset.

A straight forward extension to the proposed rules is using neighborhood information of 3 or more vertices in reduction rules. This can be done by generalizing the same notion of $N_1, N_2, N_3$ sets for a set $V_r$ of vertices in the following way:

$$
\begin{aligned}
N_1(V_r) &= \{u \in N(V_r) | N(u) \backslash N[V_r] \neq \emptyset\}, \\
N_2(V_r) &= \{u \in N(V_r) \backslash N_1(V_r) | N(u) \cap N_1(V_r) \neq \emptyset\}, \\
N_3(V_r) &= N(V_r) \backslash (N_1(V_r) \cup N_2(V_r))
\end{aligned}
$$

The main idea behind the generalized rule is to explore the joint neighborhood of $r$ distinct vertices for a given constant $r$. This more complex setting requires the introduction of a new gadget which generalizes the simple gadget used previously for a pair of vertices.

## 5.2 A generalized rule

As discussed in the previous chapters, data reduction rules identify sets of vertices that should not be included in the solution (need to be dominated) in comparison to more preferred sets. Figuratively speaking, this proposes a number of candidate sets for inclusion in the solution which can be formally specified as a set of constraints on the respective candidate sets. We use this notion to formalize the reduction rules as boolean constant formulas.

**SAT reduction.** Our general reduction rule will on the fly generate a boolean constraint formula for an optimal connected dominating set $D$ of the given graph considering this setting: We identify the vertices $V$ of a graph $G = (V, E)$ with boolean variables, where the meaning of a 1(0)-assignment is that the corresponding vertex will (not) belong to $D$. A constraint on the choice of vertices for an optimal connected dominating set then can be formalized as a boolean formula over the variables $V$.

**Definition.**[3] Let $\mathcal{W} \subseteq 2^V$ be a collection of subsets of $V$. The constraint associated with $\mathcal{W}$ is a boolean formula $F_{\mathcal{W}}$ in disjunctive normal form: $F_{\mathcal{W}} = \bigvee_{W \in \mathcal{W}} \bigwedge_{w \in W} w$.

**Definition.**[3] Let $G = (V, E)$ and let $F_{\mathcal{W}}$ be a constraint associated with some set system $\mathcal{W} = \{W_1, ..., W_s\} \subseteq 2^V$ of $r := |\bigcup_{i=1}^{s} W_i|$ vertices. An $F_{\mathcal{W}}$-gadget is a set of $p = \prod_{i=1}^{s} |W_i|$ new selector vertices $S = \{u_{(x_1,...,x_s)} | x_i \in \{1, ..., |W_i|\}\}$ and if $p < r$ another $(r - p)$ blocker vertices $B$ which are connected to $G$ by the following additional edges: For each $1 \leq i \leq s$ with $W_i = \{w_{i1}, ..., w_{i|W_i|}\}$ and each $1 \leq j \leq |W_i|$, we add edges between $w_{ij}$ and all selector vertices in $\{u_{(x_1,...,x_s)} \in S | x_i = j\}$ and between $w_{ij}$ and all blocker vertices in $B$. We denote the resulting graph by $G \oplus F_{\mathcal{W}}$.

The proposed definitions imply that a set $D \subseteq V$ fulfills constraint $F_{\mathcal{W}}$ if the assignment where each vertex in $D$ is set to 1 and each vertex in $V \backslash D$ is set to 0 satisfies $F_{\mathcal{W}}$.

A set system $\mathcal{W} \subseteq 2^V$ is said to be *compact* if for any pair of its elements $W, W' \in \mathcal{W}$ we have: $W \subseteq W' \Longrightarrow W = W'$.

Notice that any set system $\mathcal{W} \subseteq 2^V$ can be transformed into a *minimal compact subset* $\widehat{\mathcal{W}} \subseteq \mathcal{W}$ in a process called *compactification* such that $F_{\mathcal{W}}$ is logically equivalent to $F_{\widehat{\mathcal{W}}}$. Furthermore, $\widehat{\mathcal{W}}$ can be found in polynomial time [3].

**Definition.**[3] For two sets $\emptyset \neq W, W' \subseteq V$, we say that $W$ is better than $W'$ and write it as $W \leq W'$ if $|W| \leq |W'|$ and $N[W] \supseteq N[W']$.

**Generalized $r$-rule.** Using the setting put forth in the previous section, we propose a generalized $r$-rule in the following way:
Consider $r$ pairwise distinct vertices $V_r = \{v_1, ..., v_r\} \subseteq V$ and suppose $N_3(V_r) \neq \emptyset$. For $u, v \in NV_r$, let $P_{u,v}$ be the set of vertices on the shortest paths between $u$ and $v$ in $GNV_r$. Then for all vertex subsets $U$ of $V_r$ with $|U| \geq 2$, we can compute the set $\mathcal{P}_U = \bigcup_{u,v \in U} P_{u,v}$.

A $W \subseteq P_U$ is a *minimum candidate* if a) $G[W]$ is connected, b) $w$ dominates $N_3(V_r)$ and c) for any $W'$ satisfying a) and b) criteria, $|W| \leq |W'|$.

Now, we define $\mathcal{W} = \{W \subseteq \bigcup_{U \subseteq V_r} | W$ is a minimum candidate $\}$. Let $m = \min_{W \in \mathcal{W}} |W|$, we also define the set of all alternative connected paths to dominate $N_3(V_r)$ with less than $m$ vertices as $\mathcal{W}_{altern} = \{X \subseteq N[V_r] | G[X]$ is connected and $N_3(V_r) \subseteq N[X]$ and $|X| < m\}$. Next, we compute the compactifications $\widehat{\mathcal{W}}$ of $\mathcal{W}$ and $\widehat{\mathcal{W}}_{altern}$ of $\mathcal{W}_{altern}$.

If $(\forall W \in \widehat{\mathcal{W}}_{altern} \exists W' \in \widehat{\mathcal{W}} : W' \leq W)$, then $\widehat{\mathcal{W}}$ is considered a better candidate for connected domination and the $r$-rule applies as a result. We remove an independent subset of

$$R = \{v \in N_2(V_r) \cup N_3(V_r) \quad | \quad v \texttt{ is not a seperating set of } G[N[V_r]] \quad \texttt{and } N[v] \subseteq \bigcap_{W \in \widehat{\mathcal{W}}} N[W]\}$$

Put an $F_{\widehat{\mathcal{W}}}$-gadget to $G$ for the constraint associated with $\widehat{\mathcal{W}}$.

The proof of the correctness for the generalized rule is left as a future work.

# Chapter 6

# Conclusion

## 6.1 Summary of contributions

In this work, we studied data reduction for connected domination on planar graphs. Connected domination has a significant practical importance as it frequently appears in various fields of networking and communication. In mobile ad hoc networks (MANETs), connected dominating set is well-known to be the core or virtual backbone and hence has been found extremely useful in routing, message broadcast, and collision avoidance. This work addresses the problem of finding a linear size kernel for connected dominating set on planar graphs. Having proposed a set of simple and easy to implement reduction rules for connected dominating set, we proved that for planar graphs, a linear-size problem kernel of size $413k$ can be efficiently constructed using our set of reduction rules. Considering the paramount importance of the problem in the realm of graph theory is considered one step closer to solving the problem efficiently. To the best of authors knowledge this is the first study considering kernelization of connected dominating set in planar graphs.

## 6.2 Future work

Parameterization as a novel viewpoint to the complexity is still in its childhood. There are still a lot of untouched arenas that are yet to be explored. There can be various direction to take in order to extend this work or use it as a starting point for future studies.

One direction is to use similar data reduction techniques for attacking related graph problems and there are still many of them left untouched. The ideas introduced in this work are basically in two different categories and both can be applied to other problem. One is coding the characteristics of the problem in a set of effective reduction rules which can be easily modified to be suitable for other problems. The second and more significant category is the techniques employed to perform the analysis for calculating the size of kernel.

Another direction is to improve this work by further lowering the worst-case upper bound on the size of the problem kernel. There can be various ways to address this problem. This can be addressed using a set of more sophisticated reductions rules, for example, colored vertices as in Chen et al.[15] but it may require a lot work to analyze the kernel size. It might also be possible to capture more of the connectivity properties of the connected domination problem in the reduction rules in order to further improve the upper bound. One possible way is to consider the planar dual of the graph and look for a cycle of certain length say $m$ in its outer boundary as one such a cycle may correspond to an edge-cut of size $m$ in the original graph. This, can be used as a piece of information about the connectivity properties of the graph while designing reduction rules.

In terms of upperbounding the kernel size, the techniques used in this work, are using rough estimates in many cases, in order to keep analysis as simple and elegant as possible. A more detailed analysis may help upperbounding the kernel with a constant smaller than 413. It might also be possible to use a technique different from region decomposition for performing the analysis and obtain a better upperbound.

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