MODULATIONAL STABILITY OF OSCILLATORY PULSE SOLUTIONS OF THE PARAMETRICALLY-FORCED NONLINEAR SCHRÖDINGER EQUATION

by

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Modulational Stability of Oscillatory Pulse Solutions of the Parametrically-Forced Nonlinear Schrödinger Equation

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ABSTRACT

We employ a global quasi-stationary manifold to rigorously reduce the parametrically forced nonlinear Schrödinger equation (PNLS) to a finite-dimensional flow. While this manifold is not invariant, the long-time evolution of the full system is captured as a flow on the manifold through a renormalization group method. An explicit ODE for the flow is derived. Using this ODE, we show that the stationary pulse solution of the PNLS undergoes a Hopf bifurcation in a certain parameter regime, and that there exists a stable oscillatory limit cycle beyond criticality. In particular, we show that the Hopf bifurcation is supercritical.

DEDICATION

I dedicate this tome to Kiki for motivating me to work hard.

ACKNOWLEDGMENTS

I would like to thank Keith for his tremendous patience and guidance. I would also like to thank Ralf for his helpful comments and meticulous attention to detail. And I would like to thank all my committee members for reading this thesis.

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CHAPTER 1.

INTRODUCTION

We study the parametrically forced nonlinear Schrödinger equation (PNLS):

$$i\phi_{,t} + \frac{1}{2}\phi_{,xx} + |\phi|^2\phi + (i-a)\phi - \gamma\overline{\phi} = 0, \tag{1.1}$$

where γ and a are the forcing and detuning parameters respectively. With this scaling, the dissipationless case corresponds to $a \to \infty$. This equation describes a wide variety of physical phenomena, including the optical parametric oscillator in the large pump-detuning limit, Faraday resonance in water, spin waves and magnetic solitons in ferromagnets, and phase-sensitive parametric amplification of solitons in optical fibres [2, 5, 11]. For sufficiently strong parametric excitation, $\gamma > 1$, the system can produce and sustain the solitonic waves

$$\phi_{\pm} = \left(\sqrt{2\nu_{\pm}^{-1}}\,\mathrm{sech}\sqrt{2\nu_{\pm}^{-1}}x\right)e^{i\theta_{\pm}}, \tag{1.2}$$

where

$$\nu_{\pm} \equiv \frac{1}{a \pm \sqrt{\gamma^2-1}}, \tag{1.3}$$

$$e^{-2i\theta_{\pm}} = \frac{i \pm \sqrt{\gamma^2-1}}{\gamma}. \tag{1.4}$$

The lower branch solution ϕ_- exists only when $\gamma \in \left(1, \sqrt{1+a^2}\right)$, and it is always unstable due to the presence of point spectrum in the right-half complex plane [3]. We therefore disregard ϕ_- and study only the local behaviour and stability properties of the upper branch solution ϕ_+. Direct computer simulations were previously done in the stable [3] and unstable [5] regions in the parameter plane which revealed that the existence of two internal oscillation modes of ϕ_+. For $a > a_2 \approx 2.645$, the system undergoes a Hopf bifurcation as γ increases beyond the critical value $\gamma_c(a)$, and these modes resonate to produce a stable oscillatory solution for

$\gamma \in \left(\gamma_c(a), \sqrt{1+a^2}\right)$. This is accompanied by a complex conjugate pair of eigenvalues of the associated linearized operator crossing the imaginary axis into the right-half complex plane [6].

Previous work was done in [2] in which the supercritical dynamics of ϕ_+ were described analytically. The oscillatory solution was expressed as a perturbation expansion about ϕ_+, and reduced amplitude equations governing the nonlinear evolutions were obtained. The goals of this thesis are to extend this body of work as follows. First, we analyze the stability and long-time behaviour of the oscillatory solution itself. In doing so, we also obtain an analytic description of the behaviour of nearby solutions. The conditions under which these descriptions are valid are also investigated. Lastly, we explicitly show that the Hopf bifurcation is supercritical. Our results are summarized in Theorem 19. We note, however, that our investigations are valid for the fully dissipative case only (finite a), whereas the authors in [2] also investigated the stability problem for the dissipationless case ($a \to \infty$). In the latter case, the mechanism of soliton instability is due to the oscillatory-instability bifurcation, which is characterized by the collision of two pure imaginary eigenvalues of the associated linearized operator, one detaching from the essential spectrum and the other originating from the broken $U(1)$ gauge invariance [3].

The PNLS also possesss the trivial solution $\phi = 0$. It is stable when the forcing parameter is small, $\gamma \in \left(0, \sqrt{1+a^2}\right)$, and unstable against essential spectrum perturbations when the forcing parameter is large, $\gamma \in \left(\sqrt{1+a^2}, \infty\right)$. As with ϕ_-, we also disregard this solution.

In [6], the Hopf bifurcation curve $\gamma = \gamma_c(a)$ was computed by constructing Dirichlet expansions on the stable manifold of the eigenvalue problem associated with the linearization of the PNLS about ϕ_+. This expansion was then used to construct the Evans function, a

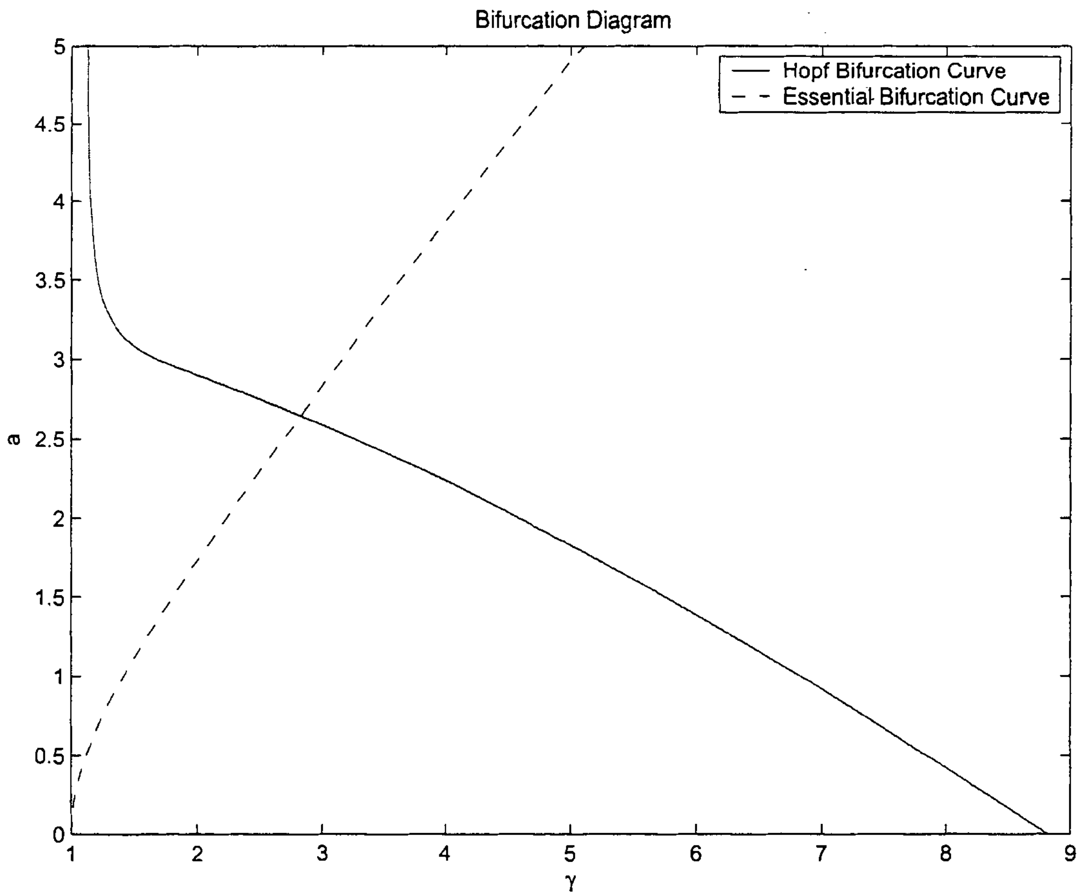

Figure 1.1: Bifurcation Diagram for the PNLS.

Wronskian-like analytic function whose zeros coincide with the eigenvalues of the linearized operator. The Evans function is particularly useful for detecting bifurcations because the order of the zero is equal to the algebraic multiplicity of the eigenvalue [1]. The technique used in [6] was also used in [13] to analyze the polarizational mode instability in birefringent fiber optics. Using the Evans function for the PNLS, the following stability diagram was obtained.

As γ increases from 1, the Hopf eigenvalues travel in the complex plane as follows [6]. To describe the eigenvalue trajectories which accompany the Hopf bifurcation, it is convenient to rescale the eigenvalue λ of the linearized operator as $\xi = \lambda / \nu_+$. The scaled eigenvalues enjoy four-fold symmetry with respect to the point -1 in the sense that, if ξ is a scaled eigenvalue, then so are $\bar{\xi}$, $-2-\xi$, and $\overline{-2-\xi}$. The linearized operator possesses the scaled eigenvalues $\xi = 0, -2$ and the scaled essential spectrum

$$\sigma_{e,\text{ scaled}}(L) = \Big\{ \xi = x + iy \Big| x \in \left(-1-\sqrt{\gamma^2-a^2}, -1+\sqrt{\gamma^2-a^2}\right),\ y=0 \\ \text{or } x=-1,\ y \in \left(-\infty, -\sqrt{a^2-\gamma^2}\right) \cup \left(\sqrt{a^2-\gamma^2}, \infty\right) \Big\}. \tag{1.5}$$

Consider first the case $a > a_1 \approx 1.132$. As γ increases from 1, the scaled eigenvalue ξ_H leaves the origin along the real axis towards -1, while the scaled eigenvalue ξ_E bifurcates out from the essential spectrum through the point $-1 + i\sqrt{a^2-1}$ along the line $-1+iy$ towards -1. Although the scaled essential spectrum also expands along $-1+iy$ towards -1, its edge always trails behind ξ_E. As γ further increases, ξ_H collides with its symmetrical counterpart $-2-\xi_H$ at -1. It then makes a right angle turn and moves along $-1+iy$ towards ξ_E. ξ_H then collides with ξ_E at some critical value of γ before moving off towards the imaginary axis. ξ_H and its symmetrical counterpart $\overline{\xi_H}$ then cross the imaginary axis as γ increases through $\gamma_c(a)$, and the Hopf bifurcation results.

Meanwhile, as γ increases, the scaled essential spectrum forms a "cross" centered at -1, and it eventually crosses the origin along the real axis. An essential bifurcation then results. For $a \in (a_1, a_2)$, the essential bifurcation occurs before the Hopf bifurcation, so ϕ_+ remains stable for its domain of existence. For $a \in (a_2, \infty)$, the Hopf bifurcation occurs first, at which point ϕ_+ becomes unstable and an oscillatory wave solution emerges.

The case $a < a_1$ is similar to that of $a > a_1$ except that the eigenvalues bifurcate off of the real axis rather than along the line $-1+iy$.

In the next chapter, we illustrate our methods by applying them to a toy example. We then proceed in the subsequent chapter to analyze the PNLS.

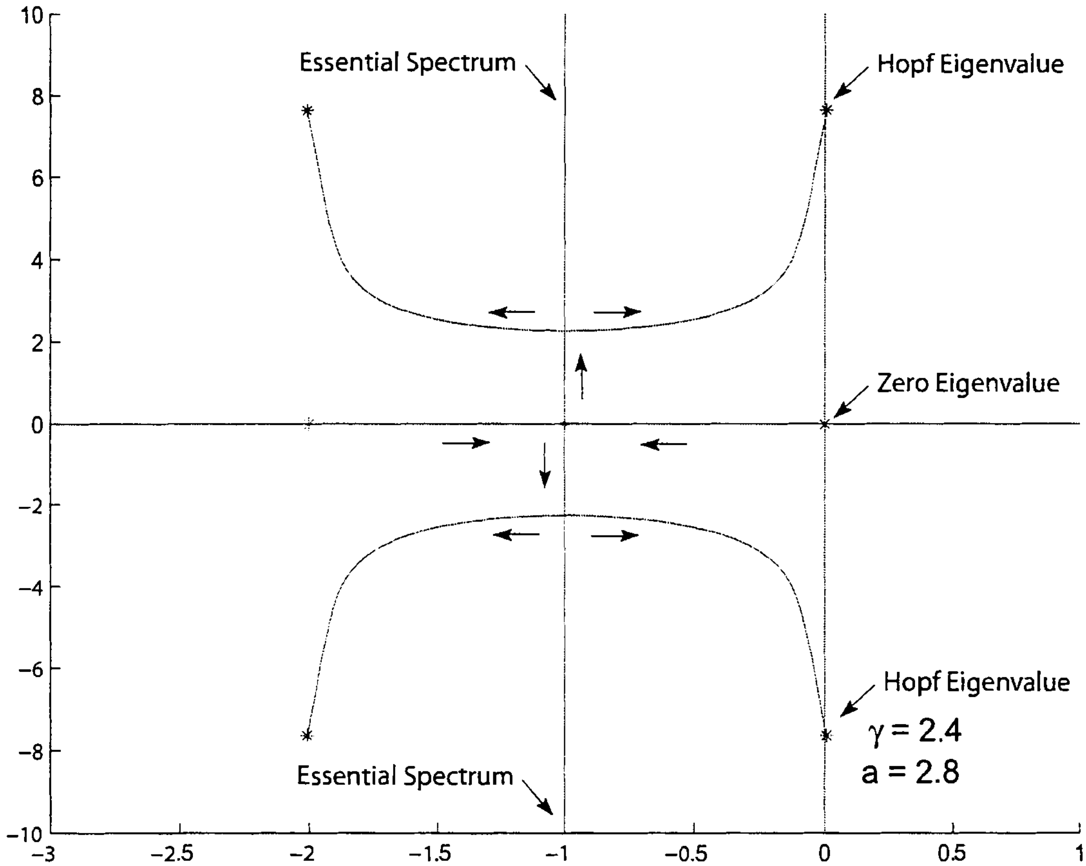

Figure 1.2: Scaled eigenvalue trajectory for increasing γ and fixed $a = 2.8$.

CHAPTER 2.

THE TOY EXAMPLE SYSTEM

The problems and solutions presented in this chapter is a simplified version of the ideas presented in [17]. In the next chapter wherein we will analyze the PNLS, those methods will be a slight generalization.

The purpose of this chapter is to illustrate the methods we will use to analyze the PNLS. We do so by applying these methods to an *ODE* problem. Our goal is to determine the long-time behaviour and stability properties of a quasi-stationary solution of an ODE

$$Z_{,t} = F(Z). \tag{2.1}$$

We do so by constructing a (not necessarily invariant) manifold parametrized by $\mathbf{q}$ which contains the quasi-stationary solutions $Q_{\mathbf{q}}$ which we study. We then reduce the flow onto $\widehat{\mathcal{M}}$ through the decomposition

$$Z = Q_{\mathbf{q}} + W \tag{2.2}$$

where W is the residual term. Our problem thus becomes one of identifying the flow on the manifold and obtaining estimates on W. The decay estimates which characterize these manifolds are exponential in nature, and are often obtained from the semigroup generated by the linearization $L_{\mathbf{q}} = F'(Q_{\mathbf{q}})$.

We describe the flow on the manifold using a series of local coordinate systems tied to the manifold itself. These coordinate systems are not chosen *a priori* however, but rather are selected to adapt to the flow on the manifold as the flow evolves. A key condition which characterizes our problem is that Z evolves slowly along the direction of the manifold. This enables us to control and remove the secularity through a slow modulation of the parameters and renormalization.

To explain our problem more precisely, let us discuss the properties of the linearization which characterize our problem. We then present the toy example and give an overview of our activities in this chapter. Our main results are summarized in Theorem 5.

2.1 Description of the ODE Problem

Let

$$Z_{,t} = \widehat{F}(Z), \ \widehat{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ Z \in \mathbb{R}^n \tag{2.3}$$

be an ODE which possesses an attractive manifold $\widehat{\mathcal{M}}$ of *stationary* solutions $Q_{\mathbf{q}}$ parametrized by $\mathbf{q} \in \mathbb{R}^{\widehat{N}}$, $\widehat{N} < n$:

$$\widehat{\mathcal{M}} \equiv \left\{ Q_{\mathbf{q}} \middle| \widehat{F}(Q_{\mathbf{q}}) = 0, \ \mathbf{q} \in \mathbb{R}^{\widehat{N}}, \ \widehat{N} < n \right\}. \tag{2.4}$$

Suppose we perturb this ODE by shifting a bifurcation parameter or adding small terms to $\widehat{F}$, say, thereby transforming (2.3) to

$$Z_{,t} = F(Z), \ F : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ Z \in \mathbb{R}^n. \tag{2.5}$$

In particular, this perturbation is such that $\widehat{\mathcal{M}}$ becomes a quasi-stationary manifold with respect to (2.5) in the sense that

$$\|F(Q_{\mathbf{q}})\| = O(\delta) \tag{2.6}$$

for some small parameter δ. What then remains of $\widehat{\mathcal{M}}$? What are the new dynamics near $\widehat{\mathcal{M}}$?

To further characterize our problem, let us expand solutions of (2.5) as

$$Z(t) = Q_{\mathbf{q}(t)} + W(t), \tag{2.7}$$

where $Q_{\mathbf{q}}$ is the quasi-stationary solution which shadows Z on $\widehat{\mathcal{M}}$ and W is a small residual term. Substituting this decomposition into (2.5) and linearizing F about $Q_{\mathbf{q}}$ yields

$$\widehat{\Upsilon}_{\mathbf{q}} \mathbf{q}_{,t} + W_{,t} = F(Q_{\mathbf{q}}) + L_{\mathbf{q}} W + \mathcal{N}_{\mathbf{q}}(W), \tag{2.8}$$

where

$$\widehat{\Upsilon}_{\mathbf{q}} \equiv \left(\partial_{q_1} Q_{\mathbf{q}}, \partial_{q_2} Q_{\mathbf{q}}, \ldots, \partial_{q_{\widehat{N}}} Q_{\mathbf{q}}\right) \tag{2.9}$$

is the n by $\widehat{N}$ matrix containing the partial derivatives of $Q_{\mathbf{q}}$ with respect to the components of $\mathbf{q}$,

$$L_{\mathbf{q}} \equiv F'\left(Q_{\mathbf{q}}\right) \tag{2.10}$$

is the linearization of F about $Q_{\mathbf{q}}$, and

$$\mathcal{N}_{\mathbf{q}}\left(\cdot\right) \equiv F\left(Q_{\mathbf{q}} + \cdot\right) - F\left(Q_{\mathbf{q}}\right) - L_{\mathbf{q}}\left(\cdot\right) \tag{2.11}$$

contains the higher-order nonlinear terms in $\cdot$.

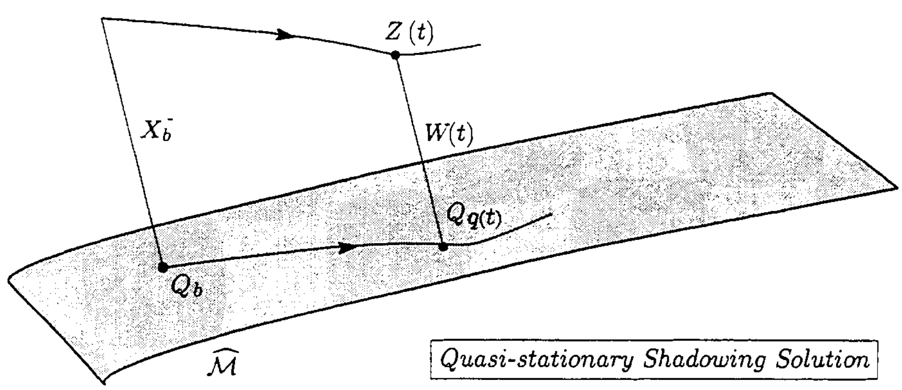

Figure 2.1: Schematic picture of the decomposition (2.7).

The following conditions characterize our problem.

Condition 1 (Normal Hyperbolicity) *The spectrum of each operator $L_{\mathbf{q}}$ may be decomposed into a stable part $\sigma_{\mathbf{q}}^-$ strictly contained in the left-half complex plane and an active part $\sigma_{\mathbf{q}}$ comprised of a fixed number of eigenvalues with small real part. In particular,*

$$\sigma\left(L_{\mathbf{q}}\right) = \sigma_{\mathbf{q}}^- \cup \sigma_{\mathbf{q}}, \tag{2.12}$$

where $\sigma_{\mathbf{q}}^-$ is contained in $\{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \leq -k\}$ for some $k > 0$ and $\sigma_{\mathbf{q}}$, which consists of $\widehat{N}$ eigenvalues including multiplicity, is contained in $\{z \in \mathbb{C} \,||\mathrm{Re}\, z| \leq \delta\}$ for some positive $\delta < k$. Both k and $\widehat{N}$ do not depend on $\mathbf{q}$.

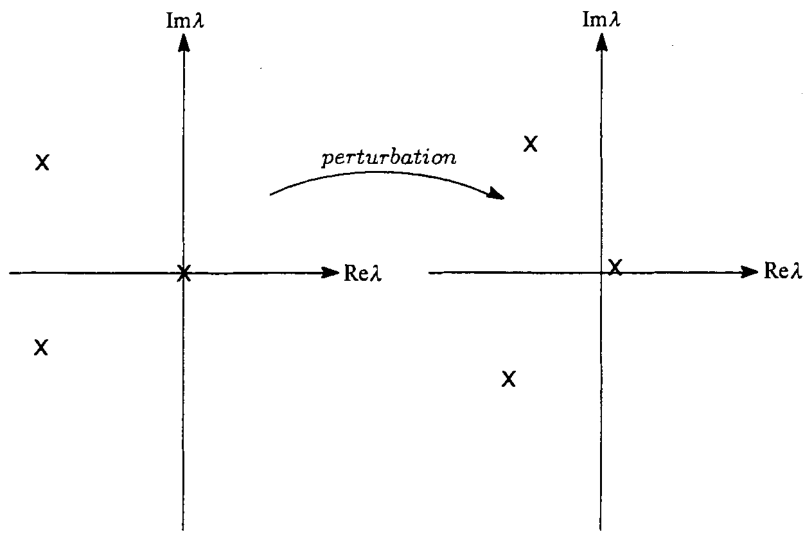

Figure 2.2: Schematic picture of the spectrum of the perturbed and unperturbed linearized operator, ODE case.

The $\widehat{N}$-dimensional $L_{\mathbf{q}}$-invariant subspace associated with $\sigma_{\mathbf{q}}$ is denoted by $X_{\mathbf{q}}$ and is called the active space. It contains the salient dynamics of the ODE. The complementary subspace of dimension $n - \widehat{N}$ is denoted $X_{\mathbf{q}}^-$ and is called the stable space. Under the action of $L_{\mathbf{q}}$, members of $X_{\mathbf{q}}^-$ satisfy the following decay estimate.

Condition 2 (Semigroup) *Each operator $L_{\mathbf{q}}$ generates a semigroup $S_{\mathbf{q}}$ which satisfies*

$$\|S_{\mathbf{q}}(t)\, Z\| \leq c_S e^{-kt} \|Z\| \tag{2.13}$$

for some constant $c_S \geq 1$, for all $Z \in X_{\mathbf{q}}^-$, and for all $t \geq 0$. The constant c_S is chosen to be independent of $\mathbf{q}$.

The semigroup decay estimate (2.13) is uniform in $\mathbf{q}$ in the sense that the constant c_S does not depend on $\mathbf{q}$, but this estimate is only applicable for each *fixed* $\mathbf{q}$. We wish to exploit this estimate to bound the residual term W but the corresponding manifold parameter function $\mathbf{q} = \mathbf{q}(t)$ varies with time t. Fortunately, $\mathbf{q}$ is seen to evolve *slowly* for the class of problems which we consider, so given the initial condition

$$\mathbf{q}(t_b) = \mathbf{b} \tag{2.14}$$

at the initial time t_b, $L_{\mathbf{b}}$ will approximate $L_{\mathbf{q}}$ well for a long time after t_b. We are thus motivated to impose the condition

$$W \in X_{\mathbf{b}}^{-} \tag{2.15}$$

under which W will decay under the action of $L_{\mathbf{b}}$ in accordance with (2.13). Bounds on W can then be obtained by exploiting (2.13). These bounds will be obtained for the toy example of this chapter in Sections 2.4 and 2.5. Observe that (2.15) implies that both $W_{,t}$ and $L_{\mathbf{b}}W$ also lie in $X_{\mathbf{b}}^{-}$.

The price for imposing the condition (2.15) is the appearance of a secular term which grows as $\mathbf{q}$ evolves away from $\mathbf{b}$. To see this, we "anchor" all $\mathbf{q}$-dependent terms in (2.8) to $\mathbf{b}$ like

$$A_{\mathbf{q}} = A_{\mathbf{b}} + (A_{\mathbf{q}} - A_{\mathbf{b}}), \tag{2.16}$$

where $A_{\mathbf{q}}$ represents any $\mathbf{q}$-dependent term, which recasts (2.8) as

$$\widehat{\Upsilon}_{\mathbf{b}}\mathbf{q}_{,t} + W_{,t} = F(Q_{\mathbf{b}}) + L_{\mathbf{b}}W + \mathcal{N}_{\mathbf{b}}(W) + \mathcal{S}_{\mathbf{b}}(\mathbf{q}, W), \tag{2.17}$$

where

$$\begin{aligned} \mathcal{S}_{\mathbf{b}}(\mathbf{q}, W) \equiv{}& \left(F(Q_{\mathbf{q}}) + L_{\mathbf{q}}W + \mathcal{N}_{\mathbf{q}}(W) - \widehat{\Upsilon}_{\mathbf{q}}\mathbf{q}_{,t}\right) \\ & - \left(F(Q_{\mathbf{b}}) + L_{\mathbf{b}}W + \mathcal{N}_{\mathbf{b}}(W) - \widehat{\Upsilon}_{\mathbf{b}}\mathbf{q}_{,t}\right) \end{aligned} \tag{2.18}$$

is the secular term. All terms in (2.17) except for $\mathcal{S}_{\mathbf{b}}$ depend on $\mathbf{b}$ and not on $\mathbf{q}$ while $\mathcal{S}_{\mathbf{b}}$ is at most $O(|\mathbf{q}-\mathbf{b}|)$. Since $\mathbf{q}$ evolves slowly, $\mathcal{S}_{\mathbf{b}}$ is guaranteed to be small compared to the other quantities in (2.17) for a long time after t_b. In essence, then, $\mathcal{S}_{\mathbf{b}}$ may be ignored until it grows to a size comparable to the other quantities in (2.17). This eventual happenstance and its resolution is addressed for the toy example of this chapter in Section 2.5.

The time t_b and the point $\mathbf{b}$ are hereby called the anchor time and anchor point respectively.

The eigenvectors of $L_{\mathbf{q}}$ which span $X_{\mathbf{q}}$ are denoted by $\Psi_{\mathbf{q}}^{(j)}$, $j = 1, ..., \widehat{N}$, and the corresponding adjoint eigenvectors by $\Psi_{\mathbf{q}}^{(j)\dagger}$. The corresponding eigenvalues are denoted by λ_j. The spectral projection operators corresponding to $X_{\mathbf{q}}$ and $X_{\mathbf{q}}^{-}$ are denoted by $\pi_{\mathbf{q}}$ and $\pi_{\mathbf{q}}^{-}$ respectively.

The evolution equation for $\mathbf{q}$ is obtained as follows. Taking the inner product of each side of (2.17) with $\Psi_{\mathbf{b}}^{(j)\dagger}$ yields

$$\sum_{k=1}^{\widehat{N}} \left[\widehat{\Pi}_{\mathbf{b}}\right]_{jk} q_{k,t} = \left\langle F\left(Q_{\mathbf{b}}\right) + \mathcal{N}_{\mathbf{b}}\left(W\right) + \mathcal{S}_{\mathbf{b}}\left(\mathbf{q}, W\right) \middle| \Psi_{\mathbf{b}}^{(j)\dagger} \right\rangle, \tag{2.19}$$

where we have used the fact that $W_{,t}, L_{\mathbf{b}}W \in X_{\mathbf{b}}^{-}$, and where we have introduced the matrix

$$\left[\widehat{\Pi}_{\mathbf{q}}\right]_{jk} \equiv \left\langle \partial_{q_k} Q_{\mathbf{q}} \middle| \Psi_{\mathbf{q}}^{(j)\dagger} \right\rangle. \tag{2.20}$$

The following condition guarantees the solvability of (2.19) for $\mathbf{q}_{,t}$.

Condition 3 (Compatibility) *The matrix $\widehat{\Pi}_{\mathbf{q}}$ is uniformly boundedly invertible in $\mathbf{q}$. In particular, we require the manifold parameters and eigenvectors to be ordered in such a way that*

$$\widehat{\Pi}_{\mathbf{q}} = I + O(\varepsilon), \tag{2.21}$$

where I is the identity matrix and ε is a small parameter.

Provided that this condition is satisfied, we may multiply both sides of (2.19) by $\widehat{\Pi}_{\mathbf{b}}^{-1}$ to obtain the equation

$$\mathbf{q}_{,t} = \omega_{\mathbf{b}}(W) + \widetilde{\mathbf{s}}_{\mathbf{b}}(\mathbf{q}, W) \tag{2.22}$$

which determines the evolution of $\mathbf{q}$, where

$$\begin{aligned} [\omega_{\mathbf{b}}(W)]_j &\equiv \sum_{k=1}^{\widehat{N}} \left[\widehat{\Pi}_{\mathbf{b}}^{-1}\right]_{jk} \left\langle F(Q_{\mathbf{b}}) + \mathcal{N}_{\mathbf{b}}(W) \middle| \Psi_{\mathbf{b}}^{(k)\dagger} \right\rangle, & (2.23) \\ [\widetilde{\mathbf{s}}_{\mathbf{b}}(\mathbf{q}, W)]_j &\equiv \sum_{k=1}^{\widehat{N}} \left[\widehat{\Pi}_{\mathbf{b}}^{-1}\right]_{jk} \left\langle \mathcal{S}_{\mathbf{b}}(\mathbf{q}, W) \middle| \Psi_{\mathbf{b}}^{(k)\dagger} \right\rangle. & (2.24) \end{aligned}$$

The evolution equation for W is obtained by projecting (2.17) onto $X_{\mathbf{b}}^{-}$:

$$W_{,t} = L_{\mathbf{b}} W + \pi_{\mathbf{b}}^{-} \left(F(Q_{\mathbf{b}}) + \mathcal{N}_{\mathbf{b}}(W) + \mathcal{S}_{\mathbf{b}}(\mathbf{q}, W) - \widehat{\Upsilon}_{\mathbf{b}} \mathbf{q}_{,t} \right). \tag{2.25}$$

Substituting $\mathbf{q}_{,t}$ (2.22), we rewrite (2.25) as

$$W_{,t} = L_{\mathbf{b}} W + \Omega_{\mathbf{b}}(W) + \widetilde{\mathcal{S}}_{\mathbf{b}}(\mathbf{q}, W), \tag{2.26}$$

where

$$\begin{aligned} \Omega_{\mathbf{b}}(W) &\equiv \pi_{\mathbf{b}}^{-} \left(F(Q_{\mathbf{b}}) + \mathcal{N}_{\mathbf{b}}(W) - \widehat{\Upsilon}_{\mathbf{b}} \omega_{\mathbf{b}}(W) \right), & (2.27) \\ \widetilde{\mathcal{S}}_{\mathbf{b}}(\mathbf{q}, W) &\equiv \pi_{\mathbf{b}}^{-} \left(\mathcal{S}_{\mathbf{b}}(\mathbf{q}, W) - \widehat{\Upsilon}_{\mathbf{b}} \widetilde{\mathbf{s}}_{\mathbf{b}}(\mathbf{q}, W) \right). & (2.28) \end{aligned}$$

Lastly, we substitute $\mathbf{q}_{,t}$ (2.22) into $\mathcal{S}_{\mathbf{b}}$ (2.18) to obtain

$$\begin{aligned} \mathcal{S}_{\mathbf{b}}(\mathbf{q}, W) = {} & (F(Q_{\mathbf{q}}) - F(Q_{\mathbf{b}})) + (L_{\mathbf{q}} W - L_{\mathbf{b}} W) + (\mathcal{N}_{\mathbf{q}}(W) - \mathcal{N}_{\mathbf{b}}(W)) \\ & - \left(\widehat{\Upsilon}_{\mathbf{q}} - \widehat{\Upsilon}_{\mathbf{b}} \right) \omega_{\mathbf{b}}(W) - \left(\widehat{\Upsilon}_{\mathbf{q}} - \widehat{\Upsilon}_{\mathbf{b}} \right) \widetilde{\mathbf{s}}_{\mathbf{b}}(\mathbf{q}, W) \end{aligned} \tag{2.29}$$

which implicitly defines $\mathcal{S}_{\mathbf{b}}$.

2.2 The Toy Example System

We now illustrate our methods by applying them to a toy example.

Notation The following notation is employed throughout this chapter only. Transposition is denoted by the superscript ^{t}. The components of a vector quantity A are denoted by $A = (a_1, a_2)^t$. The inner product $\langle \cdot | \cdot \rangle$ of two real vectors A and B is defined as

$$\langle A | B \rangle \equiv a_1 b_1 + a_2 b_2. \tag{2.30}$$

This inner product induces the norm $\|\cdot\|$ which in turn induces the operator norm $\|\cdot\|_*$. The adjoint of an operator L with respect to $\langle \cdot | \cdot \rangle$ is denoted by $L^\dagger$. The differentiation operator with respect to the variable x is denoted by ∂_x, and its action on a function f is denoted by $\partial_x f = f_{,x}$. Quantities will be enumerated with the superscript (j), where $j = 0, 1, 2, \ldots$. Constants are denoted by c_{text}, where "text" is an abbreviated description of the constant.

Description of the TES The unperturbed ODE which we study in this section is

$$Z_{,t} = \widehat{F}(Z) \equiv \begin{pmatrix} 0 \\ f(z_1) - z_2 \end{pmatrix}, \tag{2.31}$$

where $f \in C^2(\mathbb{R}, \mathbb{R})$, and it and its derivatives are uniformly bounded:

$$\max_z \left\{ f(z), f'(z), f''(z) \right\} \le c_f, \tag{2.32}$$

for some constant c_f. Examples of f include $f(z) = \sin z$, $f(z) = \operatorname{sech} z$, or $f(z) = \left(1 + z^2\right)^{-1}$ for instance. (2.31) possesses the manifold

$$\widehat{\mathcal{M}} \equiv \left\{ Q_q = (q, f(q))^t \, | q \in \mathbb{R} \right\} \tag{2.33}$$

of stationary solutions Q_q which coincides with the curve $z_2 = f(z_1)$ in the $z_1 z_2$-plane. Solutions of (2.31) near $\widehat{\mathcal{M}}$ are driven onto $\widehat{\mathcal{M}}$ along trajectories parallel to the z_2-axis.

We perturb (2.31) by adding the following terms. In the following, it is assumed that δ is a sufficiently small positive constant such that

$$\delta c_f < c_d \ll 1 \tag{2.34}$$

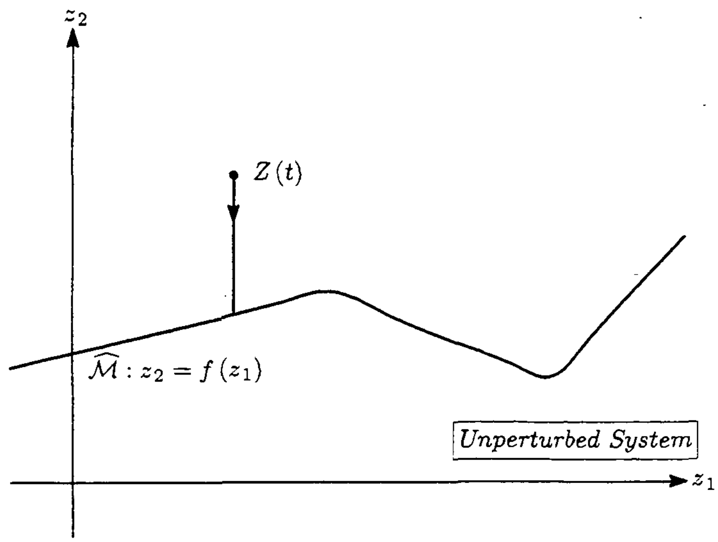

Figure 2.3: Solution of the unperturbed ODE.

for some positive constant c_d. Imagining that Z represents the position of a virtual particle, the term $(\delta, 0)^t$ represents "wind" which pushes the virtual particle along $z_2 = f(z_1)$, while $(-\delta f'(z_1), 0)^t$ represents "gravity" which tends to push the particle forward if it is on a downslope, and backwards if on an upslope. The nonlinear term $\left((f(z_1) - z_2)^2, 0\right)^t$ takes into account that the "wind" is stronger the further the particle is away from $z_2 = f(z_1)$. Adding these terms to (2.31) yields the perturbed ODE which we call the Toy Example System (TES):

$$Z_{,t} = F(Z) \equiv \begin{pmatrix} 0 \\ f(z_1) - z_2 \end{pmatrix} + \begin{pmatrix} \delta(1 - f'(z_1)) \\ 0 \end{pmatrix} + \begin{pmatrix} (f(z_1) - z_2)^2 \\ 0 \end{pmatrix}. \tag{2.35}$$

With respect to (2.35), $\widehat{\mathcal{M}}$ is now a quasi-stationary manifold in the sense that $\|F(Q_q)\| = O(\delta)$.

Our goal is to analyze the long-time evolution of TES solutions and their stability properties. We also wish to obtain the initial conditions and class of TES problems (i.e. bounds on δ) under which our analysis holds. To describe the evolution of solutions Z near

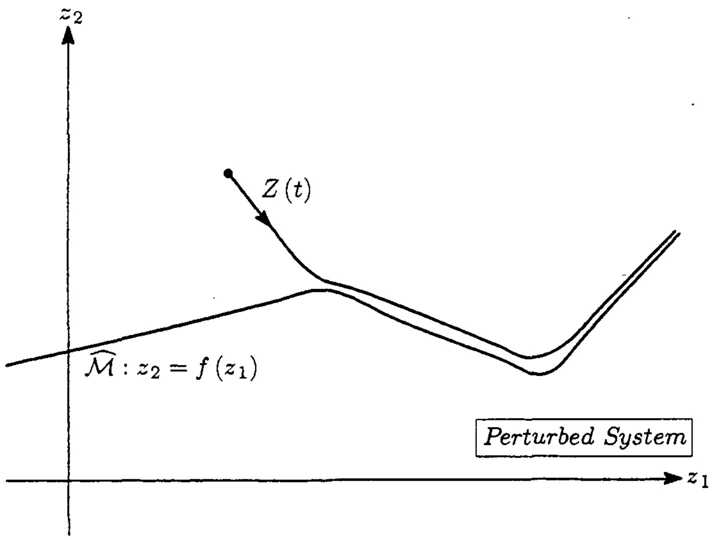

Figure 2.4: Solution of the perturbed ODE.

$\widehat{\mathcal{M}}$, we decompose Z as

$$Z(t) = Q_{q(t)} + W(t), \tag{2.36}$$

and substitute into the flow F (2.35) and linearize about the quasi-stationary solution Q_q (2.33). We then anchor q to an anchor point b, and the secularity in the system becomes apparent via the appearance of the term $\mathcal{S}_b(q, W)$ (2.18). By projecting the resulting evolution equation onto the active space X_b and the stable space X_b^-, the evolution equations for q and W are obtained. We then solve for the mild solution (2.74) for W from which we obtain the decay estimate (2.123) for W valid for the current anchor point b. This estimate shows that control of W is lost after a finite time period, and this is caused by secular growth. We then remove this secularity by rechoosing our anchor point to b^* in such a way that $W \in X_{b^*}^-$. This is done by Theorem 4. Lastly, we show that by appropriately choosing the fixed time length Δt in which each anchor point is used (see 2.144), the TES solution will approach and remain near the manifold under suitable initial conditions.

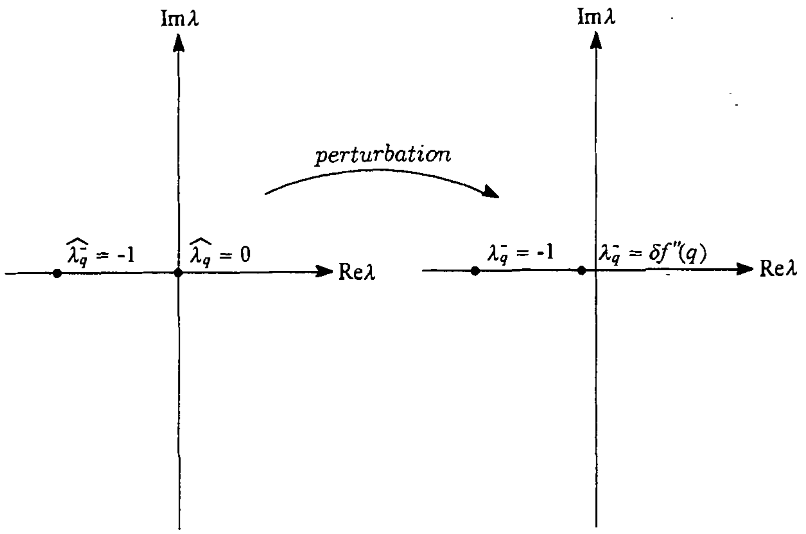

Figure 2.5: Spectrum of the unperturbed and perturbed linearized operator.

The Linearized Operator In this section, we calculate the linearized operator L_q, its adjoint $L_q^\dagger$, and their eigenvalues and eigenvectors. L_q possesses a small eigenvalue $\lambda_q = -\delta f''(q)$ and a fixed eigenvalue $\lambda_q^- = -1$, and its stable space X_q^- is spanned by a fixed vector $\Psi_q^- = (0,1)^t$. While these properties are nongeneric in the sense that Ψ_q^- does not depend on q, it does not diminish the generality of our methods. Lastly, we determine the semigroup decay estimate (2.50) which will be used to bound the residual term W.

For economy of notation, denote

$$n_q \equiv \frac{f'(q)}{1-\delta f''(q)}. \tag{2.37}$$

Under the assumptions (2.32,2.34), n_q is $O(1)$ and its denominator is uniformly bounded away from zero.

The linearized operator

$$L_q \equiv F'(Q_q) = \begin{pmatrix} -\delta f''(q) & 0 \\ f'(q) & -1 \end{pmatrix} \tag{2.38}$$

possesses the eigenvalue-eigenvector pair

$$\lambda_q = -\delta f''(q), \tag{2.39}$$

$$\Psi_q = \begin{pmatrix} 1 \\ n_q \end{pmatrix} \tag{2.40}$$

associated with the active space X_q. It also possesses the eigenvalue-eigenvector pair

$$\lambda_q^- = -1, \tag{2.41}$$

$$\Psi_q^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.42}$$

associated with the stable space X_q^-. Some remarks are in order. First, X_q^- does not depend on q since both λ_q^- and Ψ_q^- do not. Second, X_q^- is infinite-dimensional for the PDE case, so no PDE analogues of λ_q^- and Ψ_q^- exist. Lastly, the expressions (2.39,2.41) for λ_q and λ_q^- show that the Normal Hyperbolicity condition is satisfied.

The adjoint linearized operator

$$L_q^\dagger = \begin{pmatrix} -\delta f''(q) & f'(q) \\ 0 & -1 \end{pmatrix} \tag{2.43}$$

possesses the eigenvectors

$$\Psi_q^\dagger = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.44}$$

$$\Psi_q^{-\dagger} = \begin{pmatrix} -n_q \\ 1 \end{pmatrix} \tag{2.45}$$

which correspond to the eigenvalues λ_q and λ_q^- respectively. Both adjoint eigenvectors have been normalized so that $\langle A \,|A^\dagger \rangle = 1$, where $A = \Psi_q, \Psi_q^-$.

The operator which spectrally projects onto the active space X_q is given by

$$\pi_q(\cdot) = \left\langle \cdot \middle| \Psi_q^\dagger \right\rangle \Psi_q, \tag{2.46}$$

$$= \begin{pmatrix} 1 & 0 \\ n_q & 0 \end{pmatrix} (\cdot). \tag{2.47}$$

Its complementary operator which spectrally projects onto the stable space X_q^- is given by

$$\pi_q^-(\cdot) = (I - \pi_q)(\cdot), \tag{2.48}$$

$$= \begin{pmatrix} 0 & 0 \\ -n_q & 1 \end{pmatrix} (\cdot), \tag{2.49}$$

where I is the identity operator.

The relation $\langle L_q Z \,|\, Z \rangle = -\|Z\|^2$ holds for all $Z \in X_q^-$. The restriction of L_q to X_q^- therefore generates a strongly continuous semigroup of contractions S_q which satisfies the estimate

$$\|S_q(t)\, Z\| \leq e^{-t} \|Z\| \tag{2.50}$$

for all $Z \in X_q^-$ and $t \geq 0$. This shows that the Semigroup condition is satisfied with both c_S and k equal to 1.

2.3 The Evolution Equations

In this section, we compute explicit expressions for the evolution equations (2.22,2.26) for q and W. This primarily involves computing explicit expressions for $\omega_b(W)$ (2.23) and $\Omega_b(W)$ (2.27). We then obtain the mild solution (2.74) for W.

For the TES, $\omega_b(W)$ (2.23) is given by

$$\omega_b(W) = \widehat{\Pi}_b^{-1} \left\langle F(Q_b) + \mathcal{N}_b(W) \middle| \Psi_b^\dagger \right\rangle. \tag{2.51}$$

The terms comprising $\omega_b(W)$ are computed as follows. First, $\widehat{\Pi}_q$ is obtained by differentiating Q_q (2.33):

$$\widehat{\Upsilon}_q = Q_{q,q}, \tag{2.52}$$

$$= \begin{pmatrix} 1 \\ f'(q) \end{pmatrix}, \tag{2.53}$$

and then taking the inner product with $\Psi_q^\dagger$ (2.44):

$$\widehat{\Pi}_q = \left\langle \widehat{\Upsilon}_q \middle| \Psi_q^\dagger \right\rangle = 1. \tag{2.54}$$

The Compatibility condition is thus satisfied. Next, direct substitution of the solution Q_q (2.33) into the flow F (2.35) and subsequent dotting with $\Psi_q^\dagger$ (2.44) yields

$$\left\langle F(Q_q) \middle| \Psi_q^\dagger \right\rangle = \left\langle \begin{pmatrix} \delta(1 - f'(q)) \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \tag{2.55}$$

$$= \delta\left(1 - f'(q)\right). \tag{2.56}$$

Finally, since $\Psi_b^- = (0,1)^t$ (2.42) spans X_b^- and since $W \in X_b^-$, we may write $W = w\Psi_b^- = (0,w)^t$ where $w \equiv \|W\|$. Therefore, it follows by direct calculation that

$$\left\langle \mathcal{N}_b(W) \middle| \Psi_b^\dagger \right\rangle = \left\langle F(Q_q + W) - F(Q_q) - L_q W \middle| \Psi_b^\dagger \right\rangle, \tag{2.57}$$

$$= \left\langle \begin{pmatrix} w^2 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \tag{2.58}$$

$$= w^2. \tag{2.59}$$

With these expressions in hand,

$$\omega_b(W) = \delta\left(1 - f'(b)\right) + w^2. \tag{2.60}$$

Also, for the TES, $\widetilde{s}_b(q,W)$ (2.24) is given by

$$\widetilde{s}_b(q,W) = \left\langle \mathcal{S}_b(q,W) \middle| \Psi_b^\dagger \right\rangle. \tag{2.61}$$

The evolution equation (2.22) for q is thus explicitly given by

$$q_{,t} = \delta\left(1 - f'(b)\right) + w^2 + \left\langle \mathcal{S}_b(q, W) \middle| \Psi_b^\dagger \right\rangle . \tag{2.62}$$

For the TES, $\Omega_b(W)$ (2.27) is given by

$$\Omega_b(W) = \pi_b^-\left(F(Q_b) + \mathcal{N}_b(W) - \widehat{\Upsilon}_b\omega_b(W)\right). \tag{2.63}$$

Using the operator π_b^- (2.49) to spectrally project $F(Q_b)$, $\mathcal{N}_b(W)$, and $\widehat{\Upsilon}_b$ onto X_b^- yields

$$\pi_b^- F(Q_b) = \delta \begin{pmatrix} 0 \\ -n_b(1 - f'(b)) \end{pmatrix}, \tag{2.64}$$

$$\pi_b^- \mathcal{N}(W) = w^2 \begin{pmatrix} 0 \\ -n_b \end{pmatrix}, \tag{2.65}$$

$$\pi_b^- \widehat{\Upsilon}_b = \delta \begin{pmatrix} 0 \\ -n_b f''(b) \end{pmatrix}. \tag{2.66}$$

With these expressions in hand,

$$\Omega_b(W) = \delta\left(M_b^{(1)} + \delta M_b^{(2)}\right) + w^2\left(M_b^{(3)} + \delta M_b^{(4)}\right), \tag{2.67}$$

where

$$M_b^{(1)} \equiv \begin{pmatrix} 0 \\ -n_b(1 - f'(b)) \end{pmatrix}, \tag{2.68}$$

$$M_b^{(2)} \equiv \begin{pmatrix} 0 \\ n_b(1 - f'(b)) f''(b) \end{pmatrix}, \tag{2.69}$$

$$M_b^{(3)} \equiv \begin{pmatrix} 0 \\ -n_b \end{pmatrix}, \tag{2.70}$$

$$M_b^{(4)} \equiv \begin{pmatrix} 0 \\ n_b f''(b) \end{pmatrix}, \tag{2.71}$$

are $O(1)$ quantities which depend only on b. Also, for the TES, $\widetilde{\mathcal{S}}_b(q,W)$ (2.24) is given by

$$\widetilde{\mathcal{S}}_b(q,W) = \pi_b^-\left(\mathcal{S}_b(q,W) - \widehat{\Upsilon}_b\left\langle\mathcal{S}_b(q,W)\middle|\Psi_b^\dagger\right\rangle\right). \tag{2.72}$$

The evolution equation (2.26) for W is thus explicitly given by

$$W_{,t} = L_bW + \Omega_b(W) + \widetilde{\mathcal{S}}_b(q,W), \tag{2.73}$$

where $\Omega_b(W)$ and $\widetilde{\mathcal{S}}_b(q,W)$ are given as above, and it possesses the mild solution

$$W(t) = S(t-t_b)W_b + \int_{t_b}^{t} S(t-\tau)G_b(\tau)\,d\tau, \tag{2.74}$$

where the "forcing term" G_b is given by

$$G_b(\tau) \equiv \Omega_b(W(\tau)) + \widetilde{\mathcal{S}}_b(q(\tau), W(\tau)) \tag{2.75}$$

and $W_b \equiv W(t_b)$ is the initial residual with respect to the anchor point b.

2.4 Bounds on the Residual Term

From the mild solution (2.74) for W, we obtain the estimate (2.123) for W valid for the current anchor point b. This estimate shows that control of W is lost after a finite time period, and this is caused by the secular growth in $\widetilde{\mathcal{S}}_b$ (2.72). We remove this secularity in the following section by rechoosing our anchor point as b^* in such a way that $W \in X_{b^*}^-$. This is done by Theorem 4. Lastly, we show that by appropriately choosing the fixed time length Δt in which each anchor point is used (see 2.144), the TES solution will approach and remain near the manifold under suitable initial conditions.

Introduction Fix the anchor point at b on the time interval $[t_b, t_d)$ and denote the final value of q on $[t_b, t_d]$ by d. Also denote $w \equiv \|W\|$, and the initial and final values of w on $[t_b, t_d]$ by w_b and w_d. The following equations then hold:

$$q(t_b) = b,\ q(t_d) = d,\ w(t_b) = w_b,\ w(t_d) = w_d. \tag{2.76}$$

The following control quantities play prominent roles in our analysis. The quantity

$$\Delta t \equiv t_d - t_b \tag{2.77}$$

is the total length of time in which the anchor point b has been in use. The quantity

$$T^{(q)} \equiv \sup_{\theta \in [t_b, t_d)} |q(\theta) - b| \tag{2.78}$$

controls the distance between the manifold position parameter q and the anchor point b, and the quantity

$$T^{(w)} \equiv \sup_{\theta \in [t_b, t_d)} e^{\theta - t_b} w(\theta) \tag{2.79}$$

controls the size of W. W will decay exponentially in accordance with the semigroup decay estimate (2.50), so the purpose of the exponential factor $e^{\theta - t_b}$ in (2.79) is to compensate for this decay.

Apply the triangle inequality and the semigroup decay estimate (2.50) to the mild solution (2.74) for W to obtain

$$w(t) \leq e^{-(t-t_b)} w_b + \int_{t_b}^{t} e^{-(t-\tau)} g_b(\tau)\, d\tau, \tag{2.80}$$

where $g_b \equiv \|G_b\|$. A bound on G_b is obtained in the next section, which will then be used to bound W.

<u>Bound on the Forcing Term</u> The goal in this section is to obtain a bound on the "forcing term" G_b (2.75).

A bound on $\Omega_b(W)$ is obtained as follows. By the assumptions (2.32,2.34), n_b (2.37) satisfies

$$|n_b| \leq \frac{c_f}{1 - c_d}. \tag{2.81}$$

In conjunction with (2.32), we thus see that $M_b^{(j)}$ (2.68 thru 2.71) are uniformly bounded in b:

$$\left\| M_b^{(j)} \right\| \le c_M, \tag{2.82}$$

where

$$c_M \equiv \max \left\{ \frac{c_f (1+c_f)}{1-c_d}, \frac{c_f^2 (1+c_f)}{1-c_d}, \frac{c_f}{1-c_d}, \frac{c_f^2}{1-c_d} \right\}, \tag{2.83}$$

and where each of the arguments in the definition of c_M are bounds on $M_b^{(j)}$, $j = 1$ thru 4, respectively. Applying these estimates for $M_b^{(j)}$ to $\Omega_b(W)$ (2.67) then yields

$$\|\Omega_b(W)\| \le (c_M + \delta c_M)\left(\delta + w^2\right). \tag{2.84}$$

Bounds on $q_{,t}$ and $\mathcal{S}_b(q, W)$ are obtained as follows. Applying the triangle inequality to (2.18) yields

$$\begin{aligned} \|\mathcal{S}_b(q,W)\| &\le \|F(Q_q) - F(Q_b)\| + \|L_q W - L_b W\| \\ &\quad + \|\mathcal{N}_q(W) - \mathcal{N}_b(W)\| + \left\| \left(\widehat{\Upsilon}_q - \widehat{\Upsilon}_b\right) q_{,t} \right\|. \end{aligned} \tag{2.85}$$

Let us estimate each of the terms appearing in the right-hand side. Direct substitution of F (2.35) and $\widehat{\Upsilon}_q$ (2.53) yields

$$\|F(Q_q) - F(Q_b)\| = \left|f'(q) - f'(b)\right| \delta, \tag{2.86}$$

$$\left\| \left(\widehat{\Upsilon}_q - \widehat{\Upsilon}_b\right) q_{,t} \right\| = \left|f'(q) - f'(b)\right| |q_{,t}|. \tag{2.87}$$

Thus, by applying the Mean Value Theorem on f' and subsequently the uniform boundedness assumption (2.32) on f'',

$$\|F(Q_q) - F(Q_b)\| \le c_f T^{(q)} \delta, \tag{2.88}$$

$$\left\| \left(\widehat{\Upsilon}_q - \widehat{\Upsilon}_b\right) q_{,t} \right\| \le c_f T^{(q)} |q_{,t}|. \tag{2.89}$$

Next, since $\Psi_b^- = (0,1)^t$ (2.42) spans X_b^- and since $W \in X_b^-$, we may write $W = w\Psi_b^- = (0,w)^t$. By direct calculation then, $L_q W = (0,-w)^t$ and $\mathcal{N}_q(W) = (w^2,0)^t$, and so

$$L_q W - L_b W = 0, \tag{2.90}$$

$$\mathcal{N}_q(W) - \mathcal{N}_b(W) = 0. \tag{2.91}$$

Applying the above estimates to $\mathcal{S}_b$ (2.85) and $q_{,t}$ (2.62) then yields

$$\|\mathcal{S}_b(q,W)\| \leq c_f T^{(q)}\delta + c_f T^{(q)} |q_{,t}|, \tag{2.92}$$

$$|q_{,t}| \leq (1+c_f)\delta + w^2 + \|\mathcal{S}_b(q,W)\|, \tag{2.93}$$

which we combine as

$$\|\mathcal{S}_b(q,W)\| \leq \frac{c_f T^{(q)}}{1-c_f T^{(q)}}\left((2+c_f)\delta + w^2\right), \tag{2.94}$$

$$|q_{,t}| \leq \frac{1}{1-c_f T^{(q)}}\left(\left(1+c_f+c_f T^{(q)}\right)\delta + w^2\right). \tag{2.95}$$

Control of $\mathcal{S}_b$ and $q_{,t}$ by the estimates (2.94,2.95) is lost when $T^{(q)}$ grows too large. We therefore impose the constraint

$$T^{(q)} \leq \frac{1}{2c_f} \tag{2.96}$$

which restricts the possible size of Δt, thereby bounding $\mathcal{S}_b$ and $q_{,t}$ as

$$\|\mathcal{S}_b(q,W)\| \leq c_{qs}\left(\delta + w^2\right) T^{(q)}, \tag{2.97}$$

$$|q_{,t}| \leq c_{qs}\left(\delta + w^2\right), \tag{2.98}$$

where $c_{qs} \equiv \max\left\{2c_f(2+c_f), 2c_f, 2\left(\frac{3}{2}+c_f\right), 2\right\}$. We further impose the condition

$$w < c_\delta\sqrt{\delta} \tag{2.99}$$

which restricts the initial size of the residual to obtain

$$\|\mathcal{S}_b(q,W)\| \leq c_{qs}\left(1+c_\delta^2\right)\delta T^{(q)}, \tag{2.100}$$

$$|q_{,t}| \leq c_{qs}\left(1+c_\delta^2\right)\delta. \tag{2.101}$$

$T^{(q)}$ by its definition (2.78) then satisfies the estimate,

$$T^{(q)} \le \int_{t_b}^{t_d} |q_{,t}(\tau)|\, d\tau, \tag{2.102}$$

$$\le c_{qs}\left(1+c_\delta^2\right)\delta\Delta t, \tag{2.103}$$

which may be applied to (2.100) to obtain

$$\|\mathcal{S}_b(q,W)\| \le c_{qs}^2\left(1+c_\delta^2\right)^2\delta^2\Delta t. \tag{2.104}$$

Lastly, a bound on $\widetilde{\mathcal{S}}_b(q,W)$ is obtained as follows. Applying the triangle inequality to (2.72) yields

$$\left\|\widetilde{\mathcal{S}}_b(q,W)\right\| \le \left\|\pi_b^-\right\|_* \left(\|\mathcal{S}_b(q,W)\| + \left\|\widehat{\Upsilon}_b\right\| \left|\left\langle \mathcal{S}_b(q,W) \middle| \Psi_b^\dagger \right\rangle\right|\right). \tag{2.105}$$

By direct computation, both π_b^- (2.49) and $\widehat{\Upsilon}_b$ (2.53) have norms less than $\sqrt{1+c_f^2}$, and so

$$\left\|\widetilde{\mathcal{S}}_b(q,W)\right\| \le c_{\widetilde{\mathcal{S}}}\delta^2\Delta t, \tag{2.106}$$

where $c_{\widetilde{\mathcal{S}}} \equiv \sqrt{1+c_f^2}\left(1+\sqrt{1+c_f^2}\right)c_{qs}^2\left(1+c_\delta^2\right)^2$.

With the bounds (2.84,2.106) for $\Omega_b(W)$ and $\widetilde{\mathcal{S}}_b(q,W)$ in hand, the "forcing term" G_b (2.75) satisfies the bound

$$g_b(\tau) \le c_w\left(\delta + w^2(\tau) + \delta^2\Delta t\right), \tag{2.107}$$

where $c_w = \max\left\{c_M + \delta c_M, c_{\widetilde{\mathcal{S}}}\right\}$.

So long as the constraints $T^{(q)} \le \frac{1}{2c_f}$ (2.96) and $w < c_\delta\sqrt{\delta}$ (2.99) hold, then the estimates $|q_{,t}| \le c_{qs}\left(1+c_\delta^2\right)\delta$ (2.101) and $T^{(q)} \le c_{qs}\left(1+c_\delta^2\right)\delta\Delta t$ (2.103) also hold. On the other hand, if the estimate (2.103) holds, then the constraint (2.96) also holds if

$$c_{qs}\left(1+c_\delta^2\right)\delta\Delta t \le \frac{1}{2c_f}. \tag{2.108}$$

It is thus *self consistent* to replace the constraint (2.96) with the constraint (2.108), and we do so. Lastly, we rewrite (2.108) as

$$\Delta t \le c_t\delta^{-1}, \tag{2.109}$$

where $c_t \equiv \left(2c_f c_{qs}\left(1+c_\delta^2\right)\right)^{-1}$.

Residual Decay Estimates We assume that the constraints (2.99,2.109) hold, and we apply the bound (2.107) for G_b to the estimate (2.80) for W to obtain

$$w(t) \leq e^{-(t-t_b)}w_b + c_w \int_{t_b}^{t} e^{-(t-\tau)}\left(w^2(\tau)+\delta\left(1+\delta\Delta t\right)\right)d\tau, \tag{2.110}$$

Replacing t with θ, multiplying by $e^{\theta-t_b}$, and taking the supremum over $\theta \in [t_b, t_d)$, (2.110) becomes

$$\begin{aligned} T^{(w)} &\leq w_b + c_w \sup_{\theta\in[t_b,t_d)} e^{\theta-t_b}\int_{t_b}^{\theta} e^{-(\theta-\tau)}\left(w^2(\tau)+\delta\left(1+\delta\Delta t\right)\right)d\tau, && (2.111)\\ &\leq w_b + c_w \int_{t_b}^{t_d} e^{\tau-t_b}\left(w^2(\tau)+\delta\left(1+\delta\Delta t\right)\right)d\tau. && (2.112)\end{aligned}$$

Since

$$\begin{aligned} \int_{t_b}^{t_d} e^{(\tau-t_b)}w^2(\tau)\,d\tau &= \int_{t_b}^{t_d} e^{-(\tau-t_b)}\left(e^{(\tau-t_b)}w(\tau)\right)^2 d\tau, && (2.113)\\ &\leq \left(\int_{t_b}^{t} e^{-(\tau-t_b)}d\tau\right)\left(T^{(w)}\right)^2, && (2.114)\\ &= \left(1-e^{-\Delta t}\right)\left(T^{(w)}\right)^2, && (2.115)\end{aligned}$$

and

$$\int_{t_b}^{t_d} e^{\tau-t_b}d\tau = e^{\Delta t}-1, \tag{2.116}$$

(2.112) may be rewritten as

$$T^{(w)} \leq w_b + c_w\left(y_1(\Delta t)\left(T^{(w)}\right)^2 + \delta y_2(\Delta t)\right), \tag{2.117}$$

where

$$\begin{aligned} y_1(t) &\equiv 1-e^{-t}, && (2.118)\\ y_2(t) &\equiv \left(e^t-1\right)\left(1+\delta t\right). && (2.119)\end{aligned}$$

Note that y_2 is a strictly increasing function. The inequality (2.117) implies that either $T^{(w)} < z_1$ or $T^{(w)} > z_2$, where $z_1 < z_2$ are the two roots of the quadratic equation

$$c_w y_1 (\Delta t) z^2 - z + (w_b + c_w \delta y_2 (\Delta t)) = 0. \tag{2.120}$$

This quadratic equation possesses real solutions so long as its discriminant is positive, and this is always true for w_b and $\delta y_2 (\Delta t)$ sufficiently small. Since $T^{(w)}$ is continuous, $T^{(w)} (\Delta t = 0) = w_b$, and $z_2 (\Delta t = 0^+) = \infty$, it follows that initially $T^{(w)} (\Delta t = 0) \leq z_1 (\Delta t = 0)$ and hence, by continuity of $T^{(w)}$, z_1, and z_2 with respect to Δt, the inequality $T^{(w)} \leq z_1$ holds for all Δt. That is, with z_1 obtained via the quadratic formula, the inequality

$$T^{(w)} \leq \frac{1 - \sqrt{1 - 4 c_w y_1 (\Delta t) (w_b + c_w \delta y_2 (\Delta t))}}{2 c_w} \tag{2.121}$$

holds. By the approximation $\sqrt{1-x} \approx 1 - \frac{1}{2}x$ then,

$$T^{(w)} \leq c_\eta y_1 (\Delta t) (w_b + c_w \delta y_2 (\Delta t)), \tag{2.122}$$

where c_η is a constant slightly greater than 1. By further substituting $T^{(w)}$ (2.79) and the inequality $y_1 (\Delta t) \leq 1$, we obtain the residual decay estimate

$$w_d \leq c_\eta e^{-\Delta t} (w_b + c_w \delta y_2 (\Delta t)). \tag{2.123}$$

Moreover, by replacing Δt with $t - t_b$ and substituting y_2 (2.119), we obtain the residual decay estimate for general time t:

$$\begin{aligned} w(t) &\leq c_\eta e^{-(t-t_b)} (w_b + c_w \delta y_2 (t - t_b)), && (2.124) \\ &= c_\eta e^{-(t-t_b)} \left[w_b - c_w \delta - c_w \delta^2 (t - t_b)\right] + c_\eta c_w \left[\delta + \delta^2 (t - t_b)\right]. && (2.125) \end{aligned}$$

2.5 The Reanchor Method

The evolution equations (2.62,2.73) for q and W remain valid so long as the secular term $\mathcal{S}_b$ (2.18) remains sufficiently small. When this is no longer true, we rechoose the anchor point to remove this secular growth. Moreover, this new anchor point b^* is chosen so that the new residual lies in $X_{b^*}^-$. Theorem 4 will show that such a choice exists and is unique, provided that q is sufficiently close to the old anchor point b and provided that W is sufficiently small. The price of reanchoring is jump discontinuities in both q and W wherein W could in principle *increase*, yet we will show that the residual decay estimate (2.123) controls this possible growth provided that Δt, the length of time in which each anchor point is used, is suitably long. The residual decay estimate (2.123) is the key estimate which allows us to determine such a Δt so that secular growth is controlled and removed. Lastly, we will show that after an initial transient stage wherein the residual decays, the residual will remain small for all time and so our solution remains close to the manifold for all time.

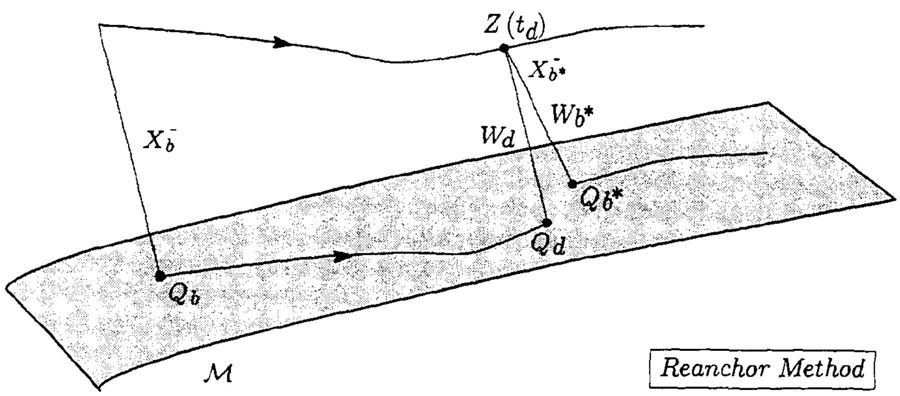

Figure 2.6: The reanchor method, ODE Case.

<u>The Reanchor Method</u> Again, fix the anchor point at b on the time interval $[t_b, t_d)$ and denote the final value of q on $[t_b, t_d]$ by d. When reanchoring, Z may be decomposed as either $Z = Q_d + W_d$ with respect to the old anchor point b, where $W_d \in X_b^-$, or as $Z = Q_{b^*} + W_{b^*}$ with respect to the new anchor point b^*, where b^* is to be determined so that $W_{b^*} \in X_{b^*}^-$. Reanchoring introduces a jump discontinuity in both q and W wherein q jumps from d to b^*

and W jumps from W_d to W_{b^*}. Equating these two decompositions and solving for W_{b^*}, we obtain

$$W_{b^*} = W_d + Q_d - Q_{b^*}. \tag{2.126}$$

Since $W_{b^*} \in X_{b^*}^-$, then

$$\left\langle W_d + Q_d - Q_{b^*} \middle| \Psi_{b^*}^\dagger \right\rangle = 0. \tag{2.127}$$

Given d and W_d, the following theorem shows that there exists a unique b^* such that (2.127) is satisfied. In addition, this theorem gives an estimate on the jump discontinuity in q when reanchoring.

Theorem 4 *Express $W_d = w_d \Xi_b$ for some scalar $w_d \geq 0$ and vector $\Xi_b \in X_b^-$ satisfying $\|\Xi_b\| = 1$. For w_d sufficiently small, there exists a unique smooth function $\mathcal{H} : \mathbb{R}^+ \to \mathbb{R}$ such that, by choosing $b^* = d + \mathcal{H}(w_d)$, (2.127) is satisfied. Moreover, the estimate*

$$|d - b^*| \leq c_r w_d |d - b| \tag{2.128}$$

holds for some constant c_r.

Proof. The equation (2.127) is equivalent to $\Gamma = 0$, where

$$\Gamma(w_d, b^*) \equiv \left\langle w_d \Xi_b + Q_d - Q_{b^*} \middle| \Psi_{b^*}^\dagger \right\rangle. \tag{2.129}$$

As $\left\langle -Q_{q,q} \middle| \Psi_q^\dagger \right\rangle = -1$ and $\left\langle \Xi_q \middle| \Psi_q^\dagger \right\rangle = 0$, the partial derivatives of Γ are

$$\begin{aligned}
\Gamma_{,b^*}(w_d, b^*) &= \left\langle -Q_{b^*,b^*} \middle| \Psi_{b^*}^\dagger \right\rangle + \left\langle w_d \Xi_b + Q_d - Q_{b^*} \middle| \Psi_{b^*,b^*}^\dagger \right\rangle, && (2.130)\\
&= -1 + \left\langle w_d \Xi_b + Q_d - Q_{b^*} \middle| \Psi_{b^*,b^*}^\dagger \right\rangle, && (2.131)\\
\Gamma_{,w_d}(w_d, b^*) &= \left\langle \Xi_b \middle| \Psi_{b^*}^\dagger \right\rangle, && (2.132)\\
&= \left\langle \Xi_b \middle| \Psi_{b^*}^\dagger - \Psi_b^\dagger \right\rangle. && (2.133)
\end{aligned}$$

Γ has a root at $(w_d, b^*) = (0, d)$ and $\Gamma_{,b^*}(0, d) = -1$ which is $O(1)$, so the implicit function theorem guarantees the existence of a smooth function $\mathcal{H}$ such that $b^* = d + \mathcal{H}(w_d)$. Moreover, since $\Gamma_{,w_d}(0, d)$ is $O(d-b)$, the implicit function theorem implies that $|\mathcal{H}'(0)|$ is $O(d-b)$ also. The estimate (2.128) then follows from the Mean Value Theorem. ∎

The reanchor method for the TES was presented in greater generality than was needed for pedagogical reasons. In particular, the unique choice of b^* is actually d because X_b^- is b-independent.

Estimates on the Reanchor Jump Discontinuities Apply the triangle inequality and substitute Q_q (2.33) to the identity (2.126) to obtain

$$\begin{aligned} w_{b^*} &\leq w_d + \|Q_d - Q_{b^*}\|, && (2.134)\\ &= w_d + \sqrt{|d-b^*|^2 + |f(d) - f(b^*)|^2}. && (2.135)\end{aligned}$$

Application of the Mean Value Theorem on f and the uniform boundedness assumption (2.32) on f' then yields

$$w_{b^*} \leq w_d + \sqrt{1 + c_f^2}\,|d - b^*|. \tag{2.136}$$

Inserting the estimate (2.128) for $|d - b^*|$ and subsequently (2.103) for $|d - b|$, we then obtain

$$\begin{aligned} w_{b^*} &\leq \left(1 + c_r\sqrt{1 + c_f^2}\,|d-b|\right) w_d, && (2.137)\\ &\leq (1 + \delta c_J \Delta t)\, w_d, && (2.138)\end{aligned}$$

where $c_J \equiv c_r\sqrt{1 + c_f^2}c_{qs}\left(1 + c_\delta^2\right)$. This estimate shows that w could in principle *increase* when reanchoring, yet we will show in the next section that the residual decay estimate (2.123) controls this possible growth provided that Δt, the length of time in which each anchor point is used, is suitably long.

The Iterations We now investigate two states in which, for some m to be determined, either $w_b \in \left(m\delta, c_\delta\sqrt{\delta}\right)$ or $w_b \in [0, m\delta]$ respectively. These states are called the initial transient and asymptotic states. We will show that w decreases on the whole in the initial transient state in the sense that $w_{b^*} < w_b$, while w remains small in the asymptotic state. Moreover, we will show that we can take $\Delta t = \ln(1 + 2c_w)$ and $m = y_2(\Delta t)$.

Inserting the residual decay estimate (2.123) into the reanchor jump estimate (2.138) yields

$$w_{b^*} \leq c_\eta e^{-\Delta t}\left(1 + \delta c_J \Delta t\right)\left(w_b + c_w \delta y_2(\Delta t)\right). \tag{2.139}$$

This inequality holds so long as the two constraints $w_b < c_\delta\sqrt{\delta}$ (2.99) and $\Delta t \leq c_t \delta^{-1}$ (2.109) holds. To continue using (2.139), we must rechoose our anchor point before either of these constraints fail.

Initial Transient State In the initial transient state wherein $w_b \in \left(m\delta, c_\delta\sqrt{\delta}\right)$, we further impose the constraint

$$\delta y_2(\Delta t) \leq w_b \tag{2.140}$$

on Δt and apply it to (2.139) to obtain

$$w_{b^*} \leq h(\Delta t)\, w_b, \tag{2.141}$$

where

$$h(x) \equiv c_\eta (1 + c_w)(1 + \delta c_J x) e^{-x}. \tag{2.142}$$

We now *choose* the fixed length of time Δt in which the current anchor point b is used such that $h(\Delta t) \approx \frac{2}{3}$. (The choice of $\frac{2}{3}$ is somewhat arbitrary. As long as $h(\Delta t) < 1$, our analysis can proceed forward.) By demanding that δ and c_δ are sufficiently small such that

$$\delta c_J = \delta c_r \sqrt{1 + c_f^2} c_{qs}\left(1 + c_\delta^2\right) < 1, \tag{2.143}$$

the choice

$$\Delta t = \ln(1 + 2c_w) \tag{2.144}$$

yields

$$h_m \equiv h(\Delta t) = \frac{c_\eta (1 + c_w)(1 + \delta c_J \ln(1 + 2c_w))}{1 + 2c_w} \leq \frac{2}{3} \tag{2.145}$$

and (2.141) thus becomes

$$w_{b^*} \leq h_m w_b \leq \frac{2}{3} w_b. \tag{2.146}$$

We further choose

$$\begin{aligned} m &= y_2(\Delta t), && (2.147) \\ &= 2c_w (1 + \delta \ln(1 + 2c_w)), && (2.148) \end{aligned}$$

so that the constraint (2.140) is automatically satisfied by virtue of the fact that $w_b \in \left(m\delta, c_\delta\sqrt{\delta}\right)$. It remains to show that our choice of Δt also satisfies the constraint (2.109), but this is easily achieved if we demand that δ satisfy

$$\delta \leq \frac{c_t}{\Delta t} = \frac{c_t}{\ln(1 + 2c_w)}. \tag{2.149}$$

We have thus proven that, if $w_b \in \left(m\delta, c_\delta\sqrt{\delta}\right)$ and $\Delta t = \ln(1 + 2c_w)$, the residual decays on the whole in the sense that $w_{b^*} \leq h_m w_b$. Moreover, we have determined the appropriate initial conditions and the class of TES problems under which our analysis holds. In particular, the solution must be close enough to the manifold such that the residual satisfies $w(0) < c_\delta\sqrt{\delta}$ (2.99) where c_δ is determined by (2.143). Also, the class of TES problems which we consider are constrained to satisfy (2.149).

The residual will continue to decay until $w_b \in [0, m\delta]$ at which point the system enters the asymptotic state.

Asymptotic State In the asymptotic state wherein $w_b \in [0, m\delta]$, we choose Δt as in (2.144):

$$\Delta t = \ln(1 + 2c_w). \tag{2.150}$$

Since $y_2(\Delta t) = m$ and $w_b \leq m\delta$, (2.139) then becomes

$$w_{b^*} \leq h(\Delta t)(m\delta), \tag{2.151}$$

where h is given by (2.142) as before. Since $h(\Delta t) = h_m \leq \frac{2}{3}$, we have

$$w_{b^*} \leq m\delta. \tag{2.152}$$

We have thus proven that, if $w_b \in [0, m\delta]$ and $\Delta t = \ln(1 + 2c_w)$, then $w_{b^*} \in [0, m\delta]$ also. In conjunction with the decay estimate (2.123), this shows that

$$w(t) \leq c_\eta (1 + c_w)(m\delta) \tag{2.153}$$

for all time t in the asymptotic state.

2.6 Conclusion

We conclude this chapter with the following theorem which summarizes our results.

Theorem 5 *Consider the Toy Example System (TES)*

$$Z_{,t} = \begin{pmatrix} \delta(1 - f'(z_1)) + (f(z_1) - z_2)^2 \\ f(z_1) - z_2 \end{pmatrix}, \tag{2.154}$$

where $\delta > 0$ and f and its derivatives are uniformly bounded. For δ sufficiently small and $w < c\sqrt{\delta}$, the TES posesses the solution

$$Z = (q, f(q))^t + W \tag{2.155}$$

for each fixed anchor point b, where q satisfies the evolution equation (in the asymptotic state)

$$q_{,t} = \delta(1 - f'(b)) + O(\delta^2), \; q(t_b) = b, \tag{2.156}$$

and W satisfies the condition

$$W \in X_b^- \tag{2.157}$$

and the bound

$$w(t) \le ce^{-(t-t_b)}\left(w_b + \delta y_2\left(t - t_b\right)\right) \tag{2.158}$$

with $y_2(t) \equiv \left(e^t - 1\right)(1 + \delta t)$. Moreover, one can use each anchor point for an $O(1)$ time period and rechoose the anchor point thereafter according to Theorem 4 such that, in the initial transient state, $w_{b^*} < h_m w_b$ for some fixed constant $h_m < 1$ and, in the asymptotic state, $w \le m\delta$ for some $O(1)$ constant m.

CHAPTER 3.

THE PNLS

We now generalize the methods from the previous chapter to analyze the PNLS. Our goal remains to analyze the stability and long-time behaviour of the oscillatory solution itself. In doing so, we also obtain an analytic description of the behaviour of nearby solutions. The conditions under which these descriptions are valid are also investigated. Lastly, we explicitly show that the Hopf bifurcation is supercritical. Our results are summarized in Theorem 19.

We construct a manifold consisting of the upper branch solutions ϕ_+ and the eigenfunctions corresponding to the Hopf eigenvalues of the linearization. The manifold parameters are

$$\mathbf{p} = (p_0, p_1, p_2) = (q, r_1, r_2)\,, \tag{3.1}$$

where q describes the position and $|r|$ describes the oscillation amplitude. The angular frequency can be obtained from the evolution equation for r. We reduce the flow onto the manifold and linearize the PNLS about ϕ_+ to obtain a general evolution equation. The evolution of $\mathbf{p}$ and W are then determined by projecting the general evolution equation onto active space X_b (which corresponds to the zero and Hopf eigenvalues) and the complementary space X_b^- (which corresponds to essential spectrum strictly contained in the left-half complex plane).

As before, we describe the flow on the manifold using a series of local coordinate systems tied to the manifold itself. These coordinate systems are not chosen *a priori* however, but rather are selected to adapt to the flow on the manifold as the flow evolves. Our key modification of this method from [17] is that, not only do we adapt the local coordinate systems, but we also adapt the manifold itself to the flow. In some sense, we are adapting the manifold to capture higher order modes which also resonate via the Hopf bifurcation (or perhaps some other mechanism) - higher order modes that are not adequately captureable by the unmodified manifold. We speculate that this adaptation also provides a method of constructing an invariant manifold for the PNLS, but we shall not show this. Only once we adapt the manifold to the flow are we able to exhibit and classify the Hopf bifurcation.

To explain our problem more precisely, let us discuss the properties of the linearization which characterize our problem. We then reintroduce the PNLS and give an overview of our activities in this chapter. We note that the notation for the PDE case is consistent with that for the ODE case.

3.1 Description of the PDE Problem

Let

$$Z_{,t} = \widehat{F}(Z) \tag{3.2}$$

be a PDE which possesses an attractive manifold of *stationary* solutions. Denote this manifold by $\widehat{\mathcal{M}}$, the stationary solutions by $Q_{\mathbf{q}}$, and the parameters which parametrize this manifold by $\mathbf{q} \in \mathbb{R}^{\widehat{N}}$:

$$\widehat{\mathcal{M}} \equiv \left\{ Q_{\mathbf{q}} \middle| \widehat{F}(Q_{\mathbf{q}}) = 0,\ \mathbf{q} \in \mathbb{R}^{\widehat{N}} \right\}. \tag{3.3}$$

Suppose we perturb this PDE by, say, shifting a bifurcation parameter or adding small terms to $\widehat{F}$, thereby transforming (3.2) to

$$Z_{,t} = F(Z). \tag{3.4}$$

In particular, we consider those perturbations which induce bifurcations in the system, thereby inducing new dynamical behaviour not adequately captureable as a reduced flow on $\widehat{\mathcal{M}}$. To capture this new behaviour then, we enlarge $\widehat{\mathcal{M}}$ as

$$\mathcal{M} \equiv \left\{ \Phi_{\mathbf{p}} = Q_{\mathbf{q}} + R_{\mathbf{p}} \middle| \|F(\Phi_{\mathbf{p}})\| = O(\delta),\ \mathbf{p} = (\mathbf{q}, \mathbf{r}) \in \mathbb{R}^{\widehat{N}} \times \mathbb{C}^{N-\widehat{N}} \right\}, \tag{3.5}$$

where δ is some small parameter. The N parameters $\mathbf{p}$ which parametrize $\mathcal{M}$ consist of the slowly-evolving parameters $\mathbf{q}$ and the small complex parameters $\mathbf{r}$.

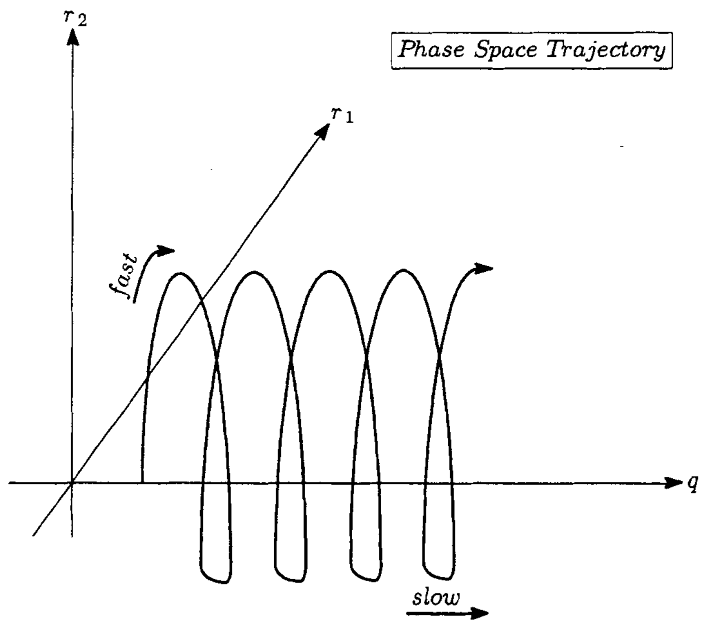

Figure 3.1: Schematic picture of the trajectory $\mathbf{p} = \mathbf{p}\,(t)$ in phase space.

To further characterize our problem, let us expand solutions of (3.4) as

$$Z\,(t) = \Phi_{\mathbf{p}(t)} + W\,(t)\,, \tag{3.6}$$

where $\Phi_{\mathbf{p}}$ is the quasi-stationary solution which shadows Z on $\mathcal{M}$ and W is a small residual term. Substituting this decomposition into (3.4) yields

$$\Upsilon_{\mathbf{p}}\mathbf{p}_{,t} + W_{,t} = F\,(\Phi_{\mathbf{p}} + W)\,, \tag{3.7}$$

where

$$\Upsilon_{\mathbf{p}} \equiv (\partial_{p_1}\Phi_{\mathbf{p}}, \partial_{p_2}\Phi_{\mathbf{p}}, \ldots, \partial_{p_N}\Phi_{\mathbf{p}}) \tag{3.8}$$

is the vector of partial derivatives of $\Phi_{\mathbf{p}}$ with respect to the components of $\mathbf{p}$. Because the Normal Hyperbolicity and Semigroup conditions pertain to the linearization of F about $Q_{\mathbf{q}}$, we linearize F about $Q_{\mathbf{q}}$ (as opposed to $\Phi_{\mathbf{p}}$) to obtain

$$\Upsilon_{\mathbf{p}}\mathbf{p}_{,t} + W_{,t} = F(Q_{\mathbf{q}}) + L_{\mathbf{q}}(R_{\mathbf{p}} + W) + \mathcal{N}_{\mathbf{q}}(R_{\mathbf{p}} + W), \tag{3.9}$$

where

$$L_{\mathbf{q}} \equiv F'(Q_{\mathbf{q}}) \tag{3.10}$$

is the linearization of F about $Q_{\mathbf{q}}$ and

$$\mathcal{N}_{\mathbf{q}}(\cdot) \equiv F(Q_{\mathbf{q}} + \cdot) - F(Q_{\mathbf{q}}) - L_{\mathbf{q}}(\cdot) \tag{3.11}$$

contains the higher-order nonlinear terms in $\cdot$.

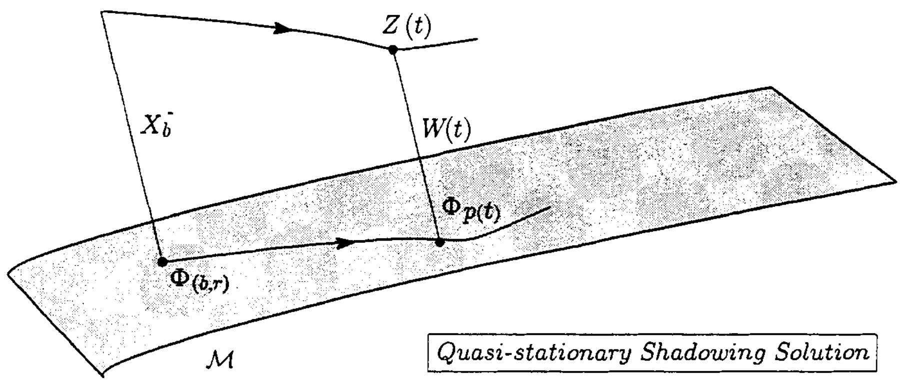

Figure 3.2: Schematic picture of the decomposition (3.6).

The following conditions characterize our problem.

Condition 6 (Normal Hyperbolicity) *The spectrum of each operator $L_{\mathbf{q}}$ may be decomposed into a stable part $\sigma_{\mathbf{q}}^-$ strictly contained in the left-half complex plane and an active part $\sigma_{\mathbf{q}}$ comprised of a fixed number of eigenvalues with small real part. In particular,*

$$\sigma(L_{\mathbf{q}}) = \sigma_{\mathbf{q}}^- \cup \sigma_{\mathbf{q}}, \tag{3.12}$$

where $\sigma_{\mathbf{q}}^-$ is contained in $\{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \leq -k\}$ for some $k > 0$ and $\sigma_{\mathbf{q}}$, which consists of N

eigenvalues including multiplicity, is contained in $\{z \in \mathbb{C}\,| |\mathrm{Re}\, z| \leq \delta\}$ *for some positive* $\delta < k$. *Both* k *and* N *do not depend on* $\mathbf{q}$.

In contrast with the ODE case, $\sigma_{\mathbf{q}}$ consists of N eigenvalues rather than $\widehat{N}$ eigenvalues.

Condition 7 (Semigroup) *Each operator* $L_{\mathbf{q}}$ *generates a* C_0 *semigroup* $S_{\mathbf{q}}$ *which satisfies*

$$\|S_{\mathbf{q}}(t)\, Z\| \leq c_S e^{-kt}\, \|Z\| \tag{3.13}$$

for some constant $c_S \geq 1$, *for all* $Z \in X_{\mathbf{q}}^{-}$, *and for all* $t \geq 0$. *The constant* c_S *is chosen to be independent of* $\mathbf{q}$.

The semigroup decay estimate (3.13) is uniform in $\mathbf{q}$ in the sense that the constant c_S does not depend on $\mathbf{q}$, but this estimate is only applicable for each *fixed* $\mathbf{q}$. We wish to exploit this estimate to bound the residual term W but the corresponding manifold parameter function $\mathbf{q} = \mathbf{q}(t)$ varies with time t. Fortunately, though, $\mathbf{q}$ is seen to evolve *slowly* for the class of problems which we consider. So, given the initial condition

$$\mathbf{q}(t_b) = \mathbf{b} \tag{3.14}$$

at the initial time t_b, $L_{\mathbf{b}}$ will approximate $L_{\mathbf{q}}$ well for a long time after t_b. We are thus motivated to impose the condition

$$W \in X_{\mathbf{b}}^{-} \tag{3.15}$$

under which W will decay under the action of $L_{\mathbf{b}}$ in accordance with (3.13). Bounds on W can then be obtained by exploiting (3.13). These bounds will be obtained for the PNLS in Sections 3.5 and 3.6. Observe that (3.15) implies that both $W_{,t}$ and $L_{\mathbf{b}}W$ also lie in $X_{\mathbf{b}}^{-}$.

We stress that only the slowly-evolving parameters $\mathbf{q}$ are anchored; the small complex parameters $\mathbf{r}$ are not.

The price for imposing the condition (3.15) is the appearance of a secular term which grows as $\mathbf{q}$ evolves away from $\mathbf{b}$. To see this, we "anchor" all $\mathbf{q}$-dependent terms in (3.9) to $\mathbf{b}$ like

$$A_{\mathbf{q}} = A_{\mathbf{b}} + (A_{\mathbf{q}} - A_{\mathbf{b}}), \tag{3.16}$$

$$A_{(\mathbf{q},\mathbf{r})} = A_{(\mathbf{b},\mathbf{r})} + \left(A_{(\mathbf{q},\mathbf{r})} - A_{(\mathbf{b},\mathbf{r})}\right), \tag{3.17}$$

where $A_{\mathbf{q}}$ and $A_{\mathbf{p}} = A_{(\mathbf{q},\mathbf{r})}$ represent any $\mathbf{q}$ and $\mathbf{p}$ dependent term respectively, which recasts (3.9) as

$$\Upsilon_{(\mathbf{b},\mathbf{r})}\mathbf{p}_{,t} + W_{,t} = F(Q_{\mathbf{b}}) + L_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})} + W\right) + \mathcal{N}_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})} + W\right) + \mathcal{S}_{\mathbf{b}}(\mathbf{p}, W), \tag{3.18}$$

where

$$\begin{aligned}\mathcal{S}_{\mathbf{b}}(\mathbf{p}, W) \equiv{}& \left(F(Q_{\mathbf{q}}) + L_{\mathbf{q}}\left(R_{(\mathbf{q},\mathbf{r})} + W\right) + \mathcal{N}_{\mathbf{q}}\left(R_{(\mathbf{q},\mathbf{r})} + W\right) - \Upsilon_{(\mathbf{q},\mathbf{r})}\mathbf{p}_{,t}\right) \\ & - \left(F(Q_{\mathbf{b}}) + L_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})} + W\right) + \mathcal{N}_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})} + W\right) - \Upsilon_{(\mathbf{b},\mathbf{r})}\mathbf{p}_{,t}\right)\end{aligned} \tag{3.19}$$

is the secular term. All terms in (3.18) except for $\mathcal{S}_{\mathbf{b}}$ depend on $\mathbf{b}$ and not on $\mathbf{q}$ while $\mathcal{S}_{\mathbf{b}}$ is at most $O(|\mathbf{q} - \mathbf{b}|)$. Since $\mathbf{q}$ evolves slowly, $\mathcal{S}_{\mathbf{b}}$ is guaranteed to be small compared to the other quantities in (3.18) for a long time after t_b. In essence, then, $\mathcal{S}_{\mathbf{b}}$ may be ignored until it grows to a size comparable to the other quantities in (3.18). This eventual happenstance and its resolution is addressed for the PNLS in Section 3.6.

The evolution equation for $\mathbf{q}$ is obtained as follows. Taking the inner product of each side of (3.18) with $\Psi_{\mathbf{b}}^{(j)\dagger}$, we then obtain

$$\sum_{k=1}^{N} \left[\Pi_{(\mathbf{b},\mathbf{r})}\right]_{jk} p_{k,t} = \left\langle F(Q_{\mathbf{b}}) + L_{\mathbf{b}} R_{(\mathbf{b},\mathbf{r})} + \mathcal{N}_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})} + W\right) + \mathcal{S}_{\mathbf{b}}(\mathbf{p}, W) \middle| \Psi_{\mathbf{b}}^{(j)\dagger}\right\rangle, \tag{3.20}$$

where

$$[\Pi_{\mathbf{p}}]_{jk} \equiv \left\langle \partial_{p_k}\Phi_{\mathbf{p}} \middle| \Psi_{\mathbf{q}}^{(j)\dagger}\right\rangle. \tag{3.21}$$

The following condition guarantees the solvability of (3.26) for $\mathbf{p}_{,t}$.

Condition 8 (Compatibility) *The matrix $\Pi_{\mathbf{p}}$ is uniformly boundedly invertible in $\mathbf{p}$. In particular, we require the manifold parameters and eigenfunctions to be ordered in such a way that*

$$\Pi_{\mathbf{p}} = I + O(\varepsilon), \tag{3.22}$$

where I is the identity matrix and ε is a small parameter.

It will be shown that $\varepsilon = |\mathbf{r}|$ for the PNLS. Provided that this condition is satisfied, we may multiply both sides of (3.20) by $\Pi^{-1}_{(\mathbf{b},\mathbf{r})}$ to obtain the equation

$$\mathbf{p}_{,t} = \omega_{\mathbf{b}}(\mathbf{r}, W) + \widetilde{\mathbf{s}}_{\mathbf{b}}(\mathbf{p}, W) \tag{3.23}$$

which determines the evolution of $\mathbf{p}$, where

$$[\omega_{\mathbf{b}}(\mathbf{r}, W)]_j \equiv \sum_{k=1}^{N} \left[\Pi^{-1}_{(\mathbf{b},\mathbf{r})}\right]_{jk} \left\langle F(Q_{\mathbf{b}}) + L_{\mathbf{b}} R_{(\mathbf{b},\mathbf{r})} + \mathcal{N}_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})} + W\right) \middle| \Psi^{(k)\dagger}_{\mathbf{b}} \right\rangle, \tag{3.24}$$

$$[\widetilde{\mathbf{s}}_{\mathbf{b}}(\mathbf{p}, W)]_j \equiv \sum_{k=1}^{N} \left[\Pi^{-1}_{(\mathbf{b},\mathbf{r})}\right]_{jk} \left\langle \mathcal{S}_{\mathbf{b}}(\mathbf{p}, W) \middle| \Psi^{(k)\dagger}_{\mathbf{b}} \right\rangle. \tag{3.25}$$

The evolution equation for W is obtained by projecting (3.18) onto $X^{-}_{\mathbf{b}}$:

$$W_{,t} = L_{\mathbf{b}} W + \pi^{-}_{\mathbf{b}}\left(F(Q_{\mathbf{b}}) + L_{\mathbf{b}} R_{(\mathbf{b},\mathbf{r})} + \mathcal{N}_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})} + W\right) + \mathcal{S}_{\mathbf{b}}(\mathbf{p}, W) - \Upsilon_{(\mathbf{b},\mathbf{r})}\mathbf{p}_{,t}\right). \tag{3.26}$$

Substituting the equation (3.23) for $\mathbf{p}_{,t}$, we rewrite (3.26) as

$$W_{,t} = L_{\mathbf{b}} W + \Omega_{\mathbf{b}}(\mathbf{r}, W) + \widetilde{\mathcal{S}}_{\mathbf{b}}(\mathbf{p}, W), \tag{3.27}$$

which determines the evolution of W, where

$$\Omega_{\mathbf{b}}(\mathbf{r}, W) \equiv \pi^{-}_{\mathbf{b}}\left(F(Q_{\mathbf{b}}) + L_{\mathbf{b}} R_{(\mathbf{b},\mathbf{r})} + \mathcal{N}_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})} + W\right) - \Upsilon_{(\mathbf{b},\mathbf{r})}\omega_{\mathbf{b}}(\mathbf{r}, W)\right), \tag{3.28}$$

$$\widetilde{\mathcal{S}}_{\mathbf{b}}(\mathbf{p}, W) \equiv \pi^{-}_{\mathbf{b}}\left(\mathcal{S}_{\mathbf{b}}(\mathbf{p}, W) - \Upsilon_{(\mathbf{b},\mathbf{r})}\widetilde{\mathbf{s}}_{\mathbf{b}}(\mathbf{p}, W)\right). \tag{3.29}$$

Lastly, we may substitute the equation (3.23) for $\mathbf{p}_{,t}$ into $\mathcal{S}_{\mathbf{b}}$ (3.19) to obtain

$$\begin{aligned}\mathcal{S}_{\mathbf{b}}(\mathbf{q},W) &= \left(F(Q_{\mathbf{q}})-F(Q_{\mathbf{b}})\right)+\left(L_{\mathbf{q}}R_{(\mathbf{q},\mathbf{r})}-L_{\mathbf{b}}R_{(\mathbf{b},\mathbf{r})}\right)\\ &\quad+\left(L_{\mathbf{q}}W-L_{\mathbf{b}}W\right)+\left(\mathcal{N}_{\mathbf{q}}\left(R_{(\mathbf{q},\mathbf{r})}+W\right)-\mathcal{N}_{\mathbf{b}}\left(R_{(\mathbf{b},\mathbf{r})}+W\right)\right)\\ &\quad-\left(\Upsilon_{(\mathbf{q},\mathbf{r})}-\Upsilon_{(\mathbf{b},\mathbf{r})}\right)\omega_{\mathbf{b}}(\mathbf{r},W)-\left(\Upsilon_{(\mathbf{q},\mathbf{r})}-\Upsilon_{(\mathbf{b},\mathbf{r})}\right)\widetilde{\mathbf{s}}_{\mathbf{b}}(\mathbf{p},W)\end{aligned}\tag{3.30}$$

which implicitly defines $\mathcal{S}_{\mathbf{b}}$.

To have $\mathcal{M}$ capture the dynamics of nearby solutions well, we require that W decays until it remains small. For instance, we require that W decay to $O\left(r^4\right)$ for the PNLS so that the reduced flow on $\mathcal{M}$ and its associated ODE can demonstrate the Hopf bifurcation and the existence of a stable oscillatory limit cycle beyond criticality. Because W is the same order as $\Omega_{\mathbf{b}}(\mathbf{r},W)$ in the asymptotic state, this requirement is equivalent to the following condition on $\Omega_{\mathbf{b}}(\mathbf{r},W)$.

Condition 9 (Quasi-Invariant Manifold) *For each anchor point b, the modified manifold must be chosen so that*

$$\Omega_{\mathbf{b}}(\mathbf{r},0)=O(\varsigma)\tag{3.31}$$

for some small parameter ς and all time t.

3.2 The PNLS

Notation The following notation is employed throughout this chapter only. Transposition is denoted by the superscript ^{t}. The components of a vector quantity A are denoted by $A=(a_1,a_2)^t$. The L^2 inner product $\langle\cdot\,|\cdot\rangle$ of two complex vectors A and B is defined as

$$\langle A\,|B\rangle \equiv \int_{-\infty}^{\infty} a_1(x)\,\overline{b_1(x)}+a_2(x)\,\overline{b_2(x)}dx.\tag{3.32}$$

This inner product induces the L^2 norm $\|\cdot\|$ and H^s norm $\|\cdot\|_{H^s}$ which in turn induce the operator norms $\|\cdot\|_*$ and $\|\cdot\|_{*,H^s}$ respectively. The orthogonal complement in L^2 is denoted by $^\perp$. Given an operator A, its adjoint with respect to $\langle\cdot|\cdot\rangle$ is denoted by $A^\dagger$, and its spectrum and resolvent sets by $\sigma(A)$ and $\rho(A)$. Quantities associated with a complex variable z include its complex conjugate $\overline{z}$, its magnitude $|z|$ and argument $\arg z$, and its real and imaginary parts $\operatorname{Re} z$ and $\operatorname{Im} z$. The differentiation operator with respect to the variable x is denoted by ∂_x, and its action on a function f is denoted by $\partial_x f = f_{,x}$. Quantities will be enumerated with the superscript (j), where $j = 0, 1, 2, \ldots$. Constants are denoted by c_{text}, where "text" is an abbreviated description of the constant. Lastly, we will use the (slightly bad) notation $\Psi_{b,b}^{(j)} \equiv \left(\Psi_{q,q}^{(j)}\right)_{q=b}$.

<u>Description of the PNLS</u> Denote

$$\nu \equiv \nu_+. \tag{3.33}$$

We rescale the dependent and independent variables of the PNLS (1.1) as

$$\widetilde{\phi}\left(\widetilde{x},\widetilde{t}\right) = \sqrt{\nu}e^{-i\theta}\phi\left(x,t\right),\ \widetilde{x} \equiv \sqrt{2\nu^{-1}}x,\ \widetilde{t} = \nu^{-1}t, \tag{3.34}$$

and drop the tilde notation to recast (1.1) as

$$i\phi_{,t} + \phi_{,xx} + |\phi|^2\phi + \nu_+\left(i-a\right)\phi - \nu_+\gamma e^{-2i\theta}\overline{\phi} = 0. \tag{3.35}$$

Introducing

$$Z = \left(z_1, z_2\right)^t = \left(\operatorname{Re}\phi, \operatorname{Im}\phi\right)^t \tag{3.36}$$

with $|Z| = \sqrt{z_1\overline{z_1} + z_2\overline{z_2}}$, we vectorize (3.35) to obtain the vectorized PNLS

$$Z_{,t} = F\left(Z\right), \tag{3.37}$$

where

$$F(Z) \equiv \begin{pmatrix} 0 & -\left(\partial_x^2 - \mu + |Z|^2\right) \\ \partial_x^2 - 1 + |Z|^2 & -2\nu \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{3.38}$$

and

$$\mu \equiv \nu_+/\nu_-. \tag{3.39}$$

For $a \in (0, \infty)$ and $\gamma \in \left(1, \sqrt{1+a^2}\right)$, the PNLS possesses the manifold

$$\widehat{\mathcal{M}} \equiv \left\{Q_q \equiv (s_q, 0)^t \,|q \in \mathbb{R}\right\} \tag{3.40}$$

of stationary pulse solutions Q_q, where

$$s_q(x) \equiv \sqrt{2}\,\text{sech}\,(x-q). \tag{3.41}$$

Note that s_q is simply the solitary wave solution ϕ_+ with the scalings (3.34) applied.

As discussed in the introduction, the PNLS undergoes a Hopf bifurcation as γ is increased beyond $\gamma_c(a)$ for fixed $a > a_c \approx 2.645$. To show this and to capture the resulting oscillatory pulse solution, we must enlarge $\widehat{\mathcal{M}}$ by adding new parameters and new terms. But how should this be done? Because the PNLS possesses two simple Hopf eigenvalues at criticality, let us add two small parameters r_1 and r_2:

$$\mathbf{p} = (p_0, p_1, p_2) = (q, r_1, r_2). \tag{3.42}$$

Note that the index in $\mathbf{p}$ starts at 0, not 1. We then expand the correction term $R_{\mathbf{p}}$ in powers of r_1 and r_2 as

$$R_{\mathbf{p}} = \sum_{j=1}^{3} R_{\mathbf{p}}^{(j)}, \tag{3.43}$$

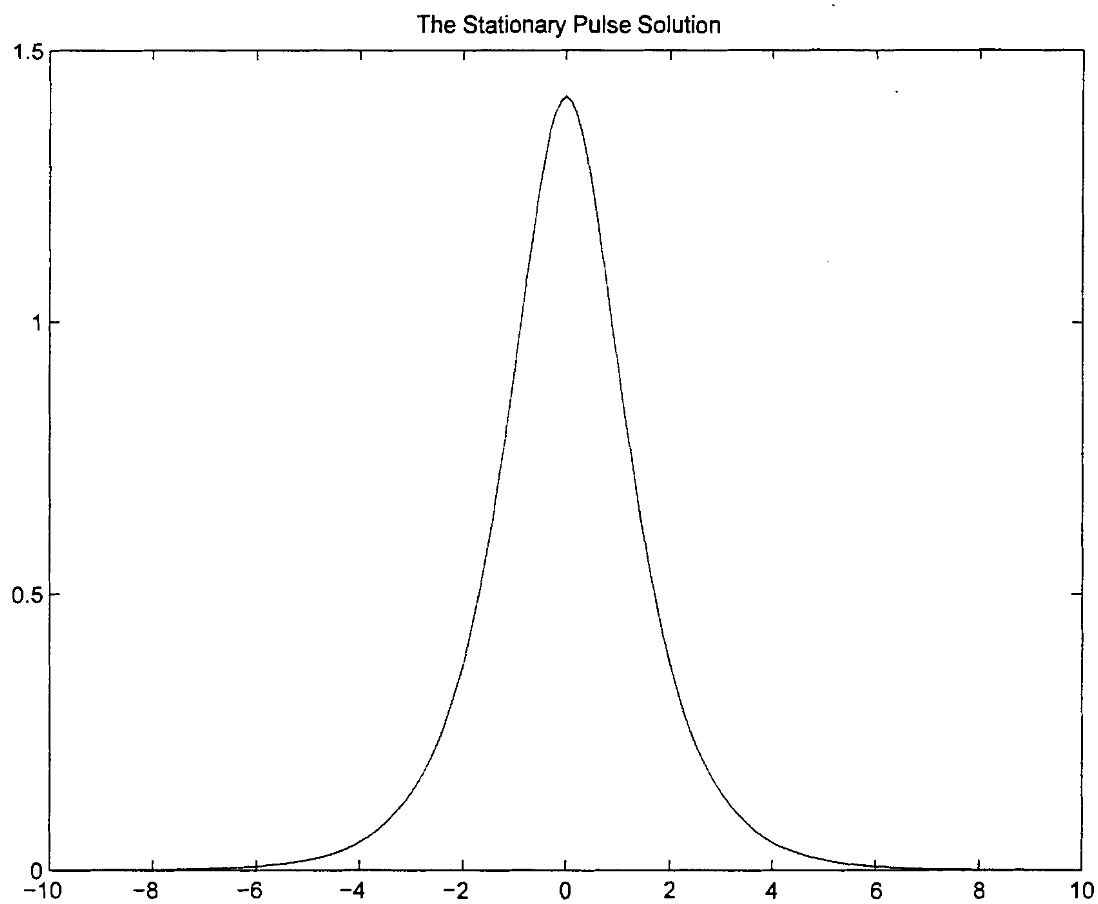

Figure 3.3: The stationary pulse solution of the PNLS.

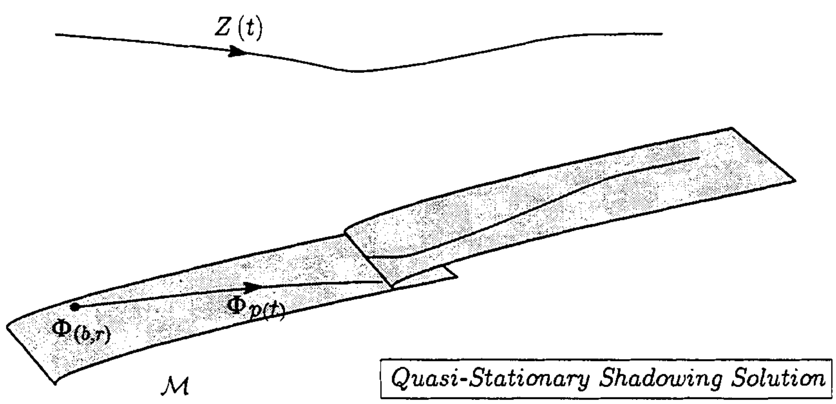

Figure 3.4: Conceptual picture of the solution of the PNLS.

where

$$R_{\mathbf{p}}^{(1)} = \sum_{j=1}^{2} r_j R_q^{(j)}, \tag{3.44}$$

$$R_{\mathbf{p}}^{(2)} = \sum_{j,k=1}^{2} r_j r_k R_q^{(jk)}, \tag{3.45}$$

$$R_{\mathbf{p}}^{(3)} = \sum_{j,k,l=1}^{2} r_j r_k r_l R_q^{(jkl)}, \tag{3.46}$$

and $R_q^{(j)}$, $R_q^{(jk)}$, and $R_q^{(jkl)}$ are $O(1)$ terms. Higher order correction terms need not be considered, as will be shown later. We now select $R_q^{(j)}$ so that the Compatibility condition is satisfied, while we select $R_q^{(jk)}$ and $R_q^{(jkl)}$ so that the Quasi-Invariant Manifold condition is satisfied.

We satisfy the Compatibility condition as follows. The modified manifold $\mathcal{M}$ consists of

$$\Phi_{\mathbf{p}} = Q_q + \sum_{j=1}^{2} r_j R_q^{(j)} + \sum_{j,k=1}^{2} r_j r_k R_q^{(jk)} + \cdots \tag{3.47}$$

with tangent plane spanned by

$$\partial_q \Phi_{\mathbf{p}} = Q_{q,q} + \sum_{j=1}^{2} r_j R_{q,q}^{(j)} + \sum_{j,k=1}^{2} r_j r_k R_{q,q}^{(jk)} + \cdots, \tag{3.48}$$

$$\partial_{r_j} \Phi_{\mathbf{p}} = R_q^{(j)} + 2\sum_{k=1}^{2} r_k R_q^{(jk)} + \cdots, \tag{3.49}$$

where we have introduced the symmetry condition $R_q^{(jk)} = R_q^{(kj)}$. To satisfy the Compatibility condition $\left\langle \partial_{p_k} \Phi_{\mathbf{p}} \middle| \Psi_q^{(j)\dagger} \right\rangle = \delta_{jk} + O(\varepsilon)$, where δ_{jk} are the components of the Kronecker delta, we choose

$$R_q^{(j)} = \Psi_q^{(j)}, \tag{3.50}$$

where $\Psi_q^{(j)}$ are the eigenfunctions of the linearized operator L_q corresponding to the Hopf eigenvalues λ_j, $j = 1, 2$. (It will later be shown that $Q_{q,q} = \Psi_q^{(0)}$.) Moreover, $\Phi_{\mathbf{p}}$ must be real because it is the vectorized form of a complex quantity (i.e. 3.36), so we choose

$$r_2 = \overline{r_1} \tag{3.51}$$

because $\Psi_q^{(2)} = \overline{\Psi_q^{(1)}}$. Henceforth, we shall denote

$$\lambda \equiv \lambda_1, \ r \equiv r_1. \tag{3.52}$$

Lemma 10 *By choosing $R_q^{(j)} = \Psi_q^{(j)}$, the Compatibility condition is satisfied.*

The physical significance of the parameters in $\mathbf{p}$ are that q describes the position and $|r|$ describes the oscillation amplitude. The angular frequency can be obtained from the evolution equation for r.

We choose $R_q^{(jk)}$ and $R_q^{(jkl)}$ so that the Quasi-Invariant Manifold condition is satisfied with $\varsigma = r^4$:

$$\Omega_b(\mathbf{r}, 0) = O\left(r^4\right), \tag{3.53}$$

so that the Hopf bifurcation for the PNLS can be analytically exhibited. This is because the number $\eta = \eta(a)$ which determines whether the bifurcation is supercritical or subcritical is obtained from the Poincaré normal form of the evolution equation for r, and this normal form requires that the evolution equation has all terms up to $O\left(r^3\right)$ explicitly given. The bifurcation is supercritical if $\operatorname{Re}\eta > 0$ and subcritical if $\operatorname{Re}\eta < 0$. The computation of η is performed in Section 3.4.

We sketch how the Quasi-Invariant Manifold condition motivates our choice of $R_q^{(jk)}$ and $R_q^{(jkl)}$ as follows. The details are presented in Section 3.3. By substituting $F(Q_q) = 0$ and $W = 0$ into $\Omega_b(\mathbf{r}, W)$ (3.28), we obtain

$$\Omega_b(\mathbf{r}, 0) = \pi_b^- \left(L_b R_{(b,\mathbf{r})} + \mathcal{N}_b\left(R_{(b,\mathbf{r})}\right) - \sum_{j=0}^{2} \left[\omega_b(\mathbf{r}, 0)\right]_j \left(\partial_{p_j} \Phi_{\mathbf{p}}\right)_{q=b} \right). \tag{3.54}$$

The Quasi-Invariant Manifold condition (3.53) is then equivalent to a system of equations for $R_b^{(jk)}$ and $R_b^{(jkl)}$ which we solve for. However, because

$$\partial_q \Phi_{\mathbf{p}} = \Psi_q^{(0)} + \sum_{j=1}^{2} r_j \Psi_{q,q}^{(j)} + \sum_{j,k=1}^{2} r_j r_k R_{q,q}^{(jk)} + \sum_{j,k,l=1}^{2} r_j r_k r_l R_{q,q}^{(jkl)} \tag{3.55}$$

contains q-derivatives of $R_q^{(jk)}$ and $R_q^{(jkl)}$, (3.53) is actually a system of *PDEs* for $R_q^{(jk)}$ and $R_q^{(jkl)}$ which are at best very difficult to solve! To circumvent this difficulty, we remove these q-derivatives by choosing $R_q^{(jk)}$ and $R_q^{(jkl)}$ such that they are *q-independent*:

$$R_q^{(jk)} = R_b^{(jk)}, \; R_q^{(jkl)} = R_b^{(jkl)}, \tag{3.56}$$

thereby recasting (3.53) as a system of equations for $R_q^{(jk)}$ and $R_q^{(jkl)}$ which now only depend on the anchor point b. The other condition which we choose is

$$R_b^{(jk)}, R_b^{(jkl)} \in X_b^-, \tag{3.57}$$

which is consistent with the condition $W \in X_b^-$ (3.15). This is a natural condition to impose since the correction terms $R_{\mathbf{p}}^{(2)}$ and $R_{\mathbf{p}}^{(3)}$ can be viewed as a resolution of the residual term W in the sense that $W \to R_{\mathbf{p}}^{(2)} + R_{\mathbf{p}}^{(3)} + W$.

As a consequence of (3.56), the modified manifold is anchor point dependent, and so the act of reanchoring becomes equivalent to rechoosing the modified manifold. This is the key modification of what was done in [17]. Not only do we adapt the local coordinate systems to the flow on the manifold (each of which is tied to the manifold itself), we also adapt the modified manifolds themselves to the flow!

Each modified manifold is given by

$$\mathcal{M}_b \equiv \left\{ \Phi_{\mathbf{p}} = Q_q + R_{\mathbf{p}} \,\middle|\, \mathbf{p} = (q, \mathbf{r}) \in \mathbb{R} \times \mathbb{C}^2 \right\} \tag{3.58}$$

with

$$R_{\mathbf{p}} \equiv \sum_{j=1}^{2} r_j \Psi_q^{(j)} + \sum_{j,k=1}^{2} r_j r_k R_b^{(jk)} + \sum_{j,k,l=1}^{2} r_j r_k r_l R_b^{(jkl)}. \tag{3.59}$$

A direct calculation (with details presented in Section 3.3) then reveals that

$$\Omega_b(\mathbf{r},0) = \sum_{j,k=1}^{2} r_j r_k \left(L_b R_b^{(jk)} - 2\lambda_j R_b^{(jk)} + U_b^{(jk)}\right) + \sum_{j,k,l=1}^{2} r_j r_k r_l \left(L_b R_b^{(jkl)} - 3\lambda_j R_b^{(jkl)} + U_b^{(jkl)}\right) + O\left(r^4\right), \quad (3.60)$$

where $U_b^{(jk)}$ and $U_b^{(jkl)}$ will be given by (3.128) and (3.129) respectively. Solving for $R_b^{(jk)}$ and $R_b^{(jkl)}$, and imposing the symmetry condition that $R_b^{(jk)}$ and $R_b^{(jkl)}$ are symmetric with respect to the interchange of any two indices, we obtain

$$R_b^{(jk)} = -\frac{1}{2}\left(L_b - \mu_{jk}\right)^{-1}\left(U_b^{(jk)} + U_b^{(kj)}\right), \quad (3.61)$$

$$R_b^{(jkl)} = -\frac{1}{3}\left(L_b - \mu_{jkl}\right)^{-1}\left(U_b^{(jkl)} + U_b^{(klj)} + U_b^{(ljk)}\right), \quad (3.62)$$

where $\mu_{jk} \equiv \lambda_j + \lambda_k$ and $\mu_{jkl} \equiv \lambda_j + \lambda_k + \lambda_l$. Thus, we have obtained the correction terms $R_b^{(jk)}$ and $R_b^{(jkl)}$ which will enable us to adapt the manifold to the flow to capture resonant behaviour in higher-order modes with "resonant frequencies" at μ_{jk} and μ_{jkl}.

Our goal is to analyze the long-time evolution of the oscillatory solutions and their stability properties. We also wish to obtain the initial conditions under which our analysis holds. To describe the evolution of solutions Z near $\mathcal{M}$, we decompose Z as

$$Z(t) = \Phi_{\mathbf{p}(t)} + W(t). \quad (3.63)$$

and substitute into the flow F (3.38) and linearize about the quasi-stationary solution Q_q (2.33). We then anchor q to an anchor point b, and the secularity in the system becomes apparent via the appearance of the term $\mathcal{S}_b(\mathbf{p}, W)$ (3.19). By projecting the resulting evolution equation onto the active space X_b (associated with the zero and Hopf eigenvalues) and the stable space X_b^-, the evolution equations for $\mathbf{p}$ and W are obtained. We then solve for the mild solution (3.156) for W from which we obtain the decay estimate (3.234) for W valid for the current anchor point b. This estimate shows that control of W is lost after a finite time period, and this is caused by secular growth. We then remove this secularity by rechoosing our anchor point to b^* in such a way that $W \in X_{b^*}^-$. This is done by Theorem 4. In rechoosing our anchor point, not only to do we rechoose the local coordinate system but we also rechoose the modified manifold. However, except for a jump discontinuity in q, the values of the parameters $\mathbf{p}$ remain the same. Lastly, we show that by appropriately choosing the fixed time length Δt in which each anchor point is used (see 3.262), the oscillatory solution will approach and remain near the manifold under suitable initial conditions.

The remainder of the chapter is devoted to obtaining explicit descriptions on the evolution of $\mathbf{p}$ and rigorous bounds on W. In addition, the supercriticality of the Hopf bifurcation is to be demonstrated.

Due to the invariance of the PNLS under spatial translations, all quantities associated with the stationary pulse solution Q_q (with fixed q) may be obtained from their $q = 0$ counterparts by applying the translation $x \longrightarrow x - q$. Such quantities include the linearized operator L_q, its adjoint $L_q^\dagger$, and their eigenvalues and eigenfunctions. The spectrum of L_q remains invariant under such translations, however, as does the spectrum of its constituent operators C_q (3.65) and D_q (3.66).

The Linearized Operator In this section, we calculate the linearized operator L_q, its adjoint $L_q^\dagger$, and their eigenvalues and eigenfunctions. The eigenvalues and eigenfunctions of L_q are computed numerically using a Dirichlet expansion and the Evans function from [6], while various other quantities are derived analytically from these ones. Lemmas are presented which show that the normalization constants for the adjoint eigenfunctions are well defined in the sense that they are nonzero for the domain of existence of ϕ_-. While the zero eigenvalue is shown to be simple, we had to rely on numerical evidence to *infer* that the Hopf eigenvalues are simple. In particular, we plotted the Evans function $E = E(\xi)$ and observed that $\partial_\xi E(\xi) \neq 0$

at the scaled eigenvalue $\xi \equiv \lambda/\nu$. This suggests that these eigenvalues are simple because the order of the zero is equal to the algebraic multiplicity of the eigenvalue [1]. Lastly, we present the semigroup decay estimate (3.90) from [17] which will be used to bound the residual term W.

The linearized operator L_q is given by

$$L_q \equiv F'(Q_q) = \begin{pmatrix} 0 & D_q \\ -C_q & -2\nu \end{pmatrix}, \tag{3.64}$$

where C_q and D_q are the self-adjoint operators

$$C_q \equiv -\left(\partial_x^2 + 3s_q^2 - 1\right), \tag{3.65}$$

$$D_q \equiv -\left(\partial_x^2 + s_q^2 - \mu\right). \tag{3.66}$$

C_q has the eigenvalue-eigenfunction pairs $\{-3, s_q^2\}$, $\{0, s_{q,x}\}$ and essential spectra $[1,\infty)$, while D_q has the eigenvalue-eigenfunction pair $\{\mu - 1, s_q\}$ and essential spectra $z \in [\mu, \infty)$. In addition, D_q has a bounded self-adjoint inverse for $\gamma \in \left(1, \sqrt{1+a^2}\right)$. See [11] for more details.

Lemma 11 *For γ sufficiently close to $\gamma_c(a)$ and $a > 2.645$, the linearized operator L_q posesses an essential spectrum uniformly bounded away from the imaginary axis and three eigenvalues with small real part. Thus the Normal Hyperbolicity condition is satisfied.*

Of particular interest are the eigenvalue-eigenfunction pairs of L_q which correspond to the translational and oscillatory modes of the PNLS. The translational mode pair $\left\{\lambda_0, \Psi_q^{(0)}\right\}$ is given by

$$\lambda_0 = 0, \tag{3.67}$$

$$\Psi_q^{(0)} = \begin{pmatrix} s_{q,q} \\ 0 \end{pmatrix}, \tag{3.68}$$

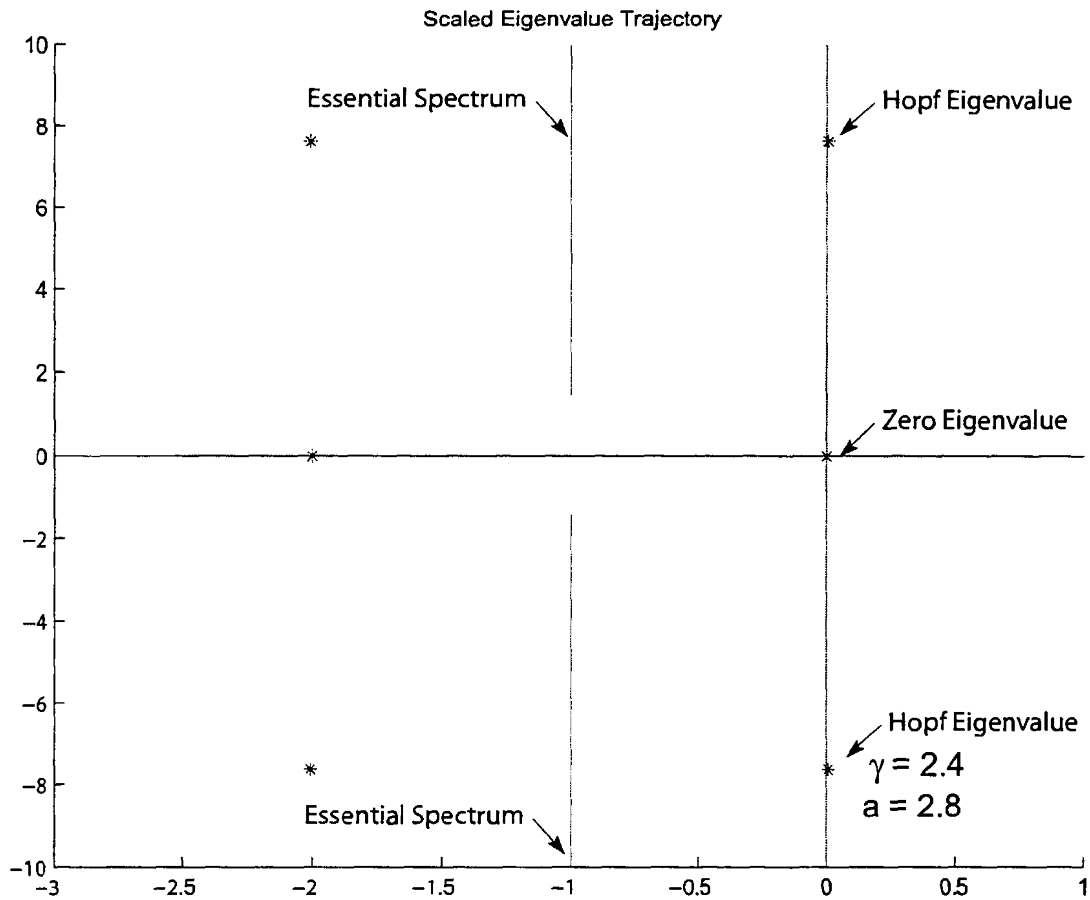

Figure 3.5: Spectrum of the linearized operator L_q for $\gamma = 2.4$ and $a = 2.8$.

while the oscillatory mode pairs $\left\{\lambda_1, \Psi_q^{(1)}\right\}$ and $\left\{\lambda_2, \Psi_q^{(2)}\right\}$ are numerically computed from the Evans function from [6]. The translational mode eigenfunction $\Psi_q^{(0)}$ is the q-derivative of the stationary pulse solution Q_q:

$$\Psi_q^{(0)} = Q_{q,q}, \tag{3.69}$$

because, since Q_q is a solution of the PNLS for each fixed q, then $F(Q_q) = 0$, and so $0 = \partial_q F(Q_q) = L_q Q_{q,q}$. The oscillatory mode eigenfunctions are complex conjugates of one another:

$$\Psi_q^{(2)} = \overline{\Psi_q^{(1)}}, \tag{3.70}$$

because L_q is real and $\lambda_2 = \overline{\lambda_1}$. For notational convenience, we further denote the components of $\Psi_q^{(1)}$ by

$$\Psi_q^{(1)} = \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix}. \tag{3.71}$$

The adjoint of L_q with respect to the L^2 inner product is given by

$$L_q^{\dagger} = \begin{pmatrix} 0 & -C_q \\ D_q & -2\nu \end{pmatrix}. \tag{3.72}$$

Since $s_{q,x}$ is the eigenfunction of C_q which corresponds to the zero eigenvalue of C_q, it follows that

$$\Psi_q^{(0)\dagger} = \frac{1}{n_0} \begin{pmatrix} 2\nu D_q^{-1} s_{q,x} \\ s_{q,x} \end{pmatrix} \tag{3.73}$$

is the adjoint eigenfunction which corresponds to the zero eigenvalue of L_q. Moreover, by writing out the components of the eigenvalue equation $L\Psi_q^{(1)} = \lambda_1 \Psi_q^{(1)}$:

$$-C_q \alpha_q = (\lambda_1 + 2\nu)\beta_q, \tag{3.74}$$

$$D_q \beta_q = \lambda_1 \alpha_q, \tag{3.75}$$

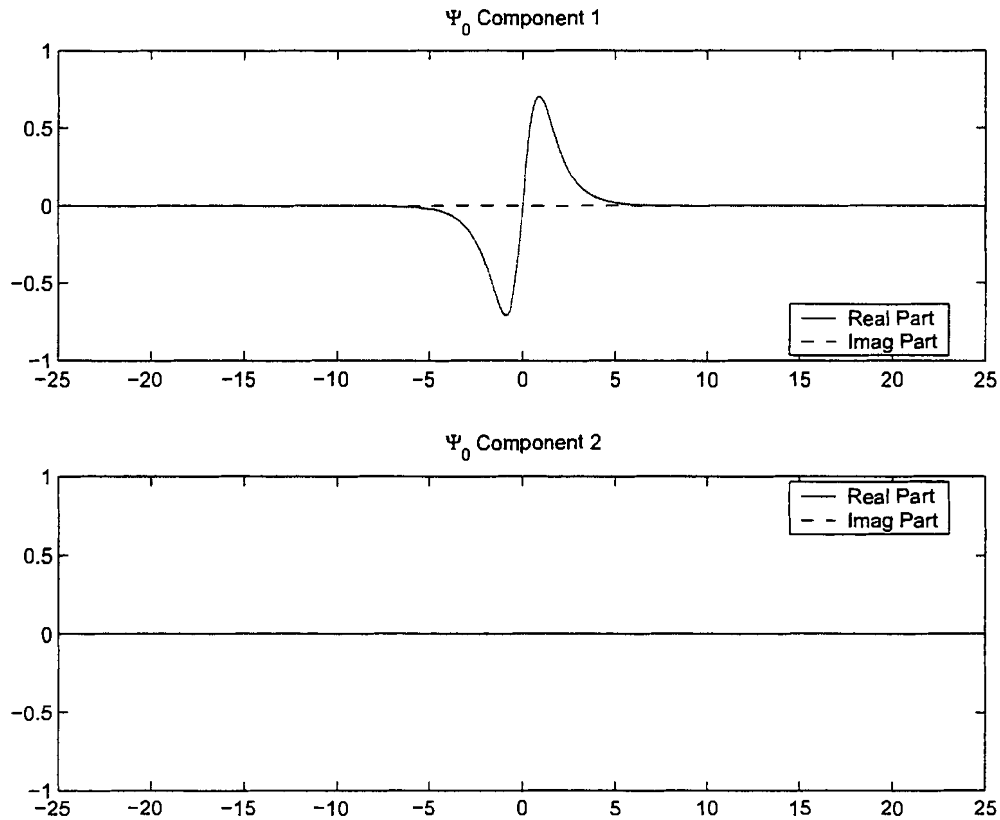

Figure 3.6: Graph of Ψ_0 for $\gamma = 2.4$ and $a = 2.8$.

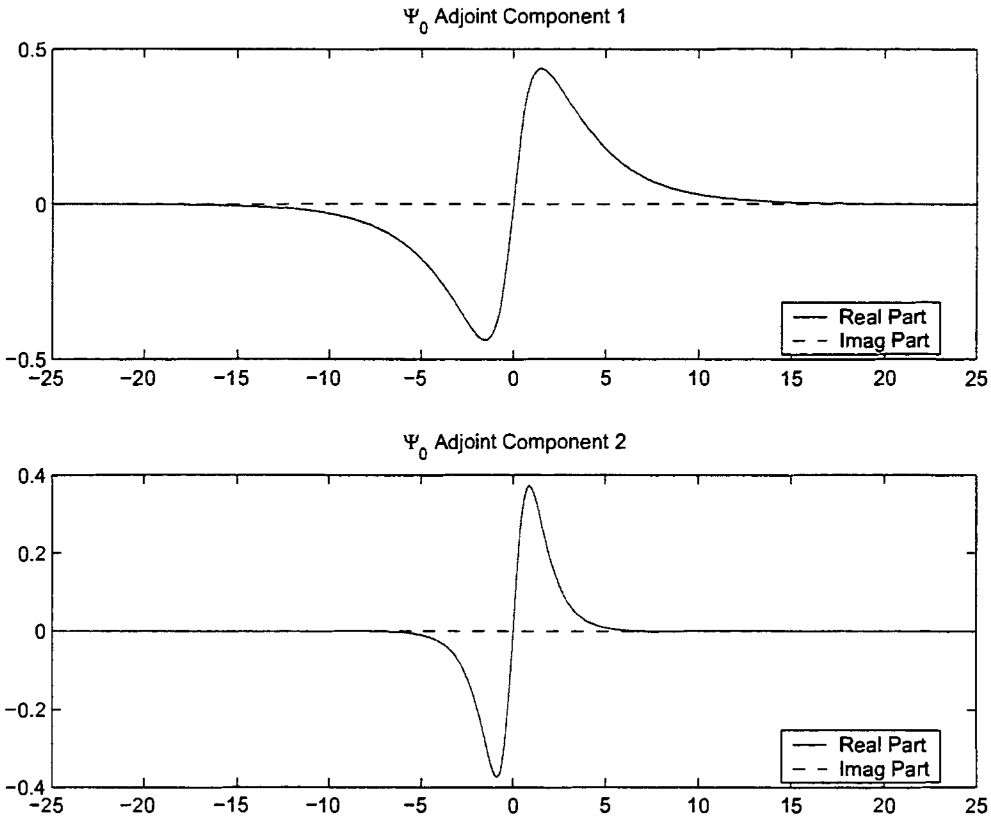

Figure 3.7: Graph of $\Psi_0^\dagger$ for $\gamma = 2.4$ and $a = 2.8$.

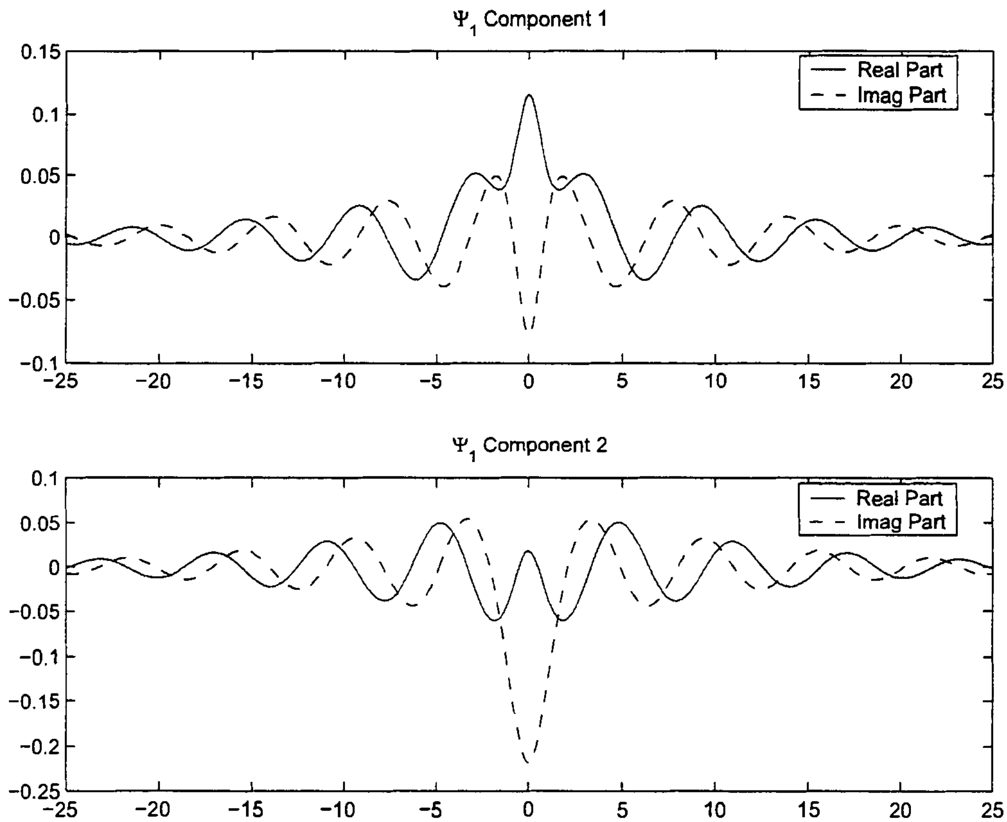

Figure 3.8: Graph of Ψ_1 for $\gamma = 2.4$ and $a = 2.8$.

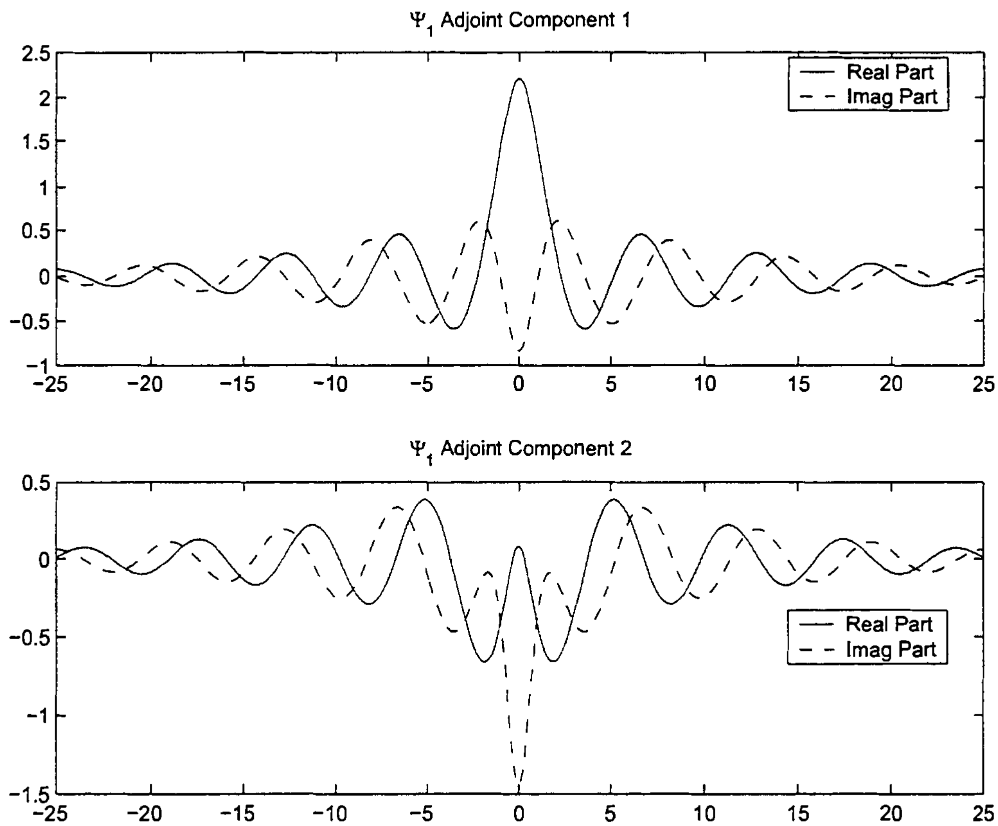

Figure 3.9: Graph of $\Psi_1^\dagger$ for $\gamma = 2.4$ and $a = 2.8$.

we see that the adjoint eigenfunctions which correspond to the eigenvalues λ_1 and λ_2 are given by

$$\Psi_q^{(1)\dagger} = \overline{\Psi_q^{(2)\dagger}}, \tag{3.76}$$

$$\Psi_q^{(2)\dagger} = \frac{1}{n_1}\begin{pmatrix} (\lambda_1 + 2\nu)\,\beta_q \\ \lambda_1 \alpha_q \end{pmatrix} \tag{3.77}$$

respectively. The normalization constants

$$n_0 \equiv -2\nu \left\langle D_q^{-1} s_{q,x} \,|\, s_{q,x} \right\rangle, \tag{3.78}$$

$$n_1 \equiv 2(\lambda_1 + \nu)\left\langle \alpha_q \,|\, \overline{\beta_q} \right\rangle, \tag{3.79}$$

have been selected so that the orthonormality condition

$$\left\langle \Psi_q^{(j)} \,\middle|\, \Psi_q^{(k)\dagger} \right\rangle = \delta_{jk} \tag{3.80}$$

holds, where δ_{jk} are the components of the Kronecker delta. These normalization constants remain invariant under spatial translations.

We now show that the adjoint eigenfunctions are well defined in the sense that their normalization constants n_j are nonzero within certain parameter regimes. We also show that the zero eigenvalue is simple, and we infer from some numerical evidence that the Hopf eigenvalues are simple.

Lemma 12 *If $\gamma \in \left(1, \sqrt{1+a^2}\right)$, then $\lambda_0 = 0$ is an eigenvalue of L_q with algebraic multiplicity 1. Furthermore, $n_0 < 0$.*

Proof. Since the kernel of C_q (3.65) is spanned by $s_{q,x}$ and since D_q (3.66) is invertible for $\gamma \in \left(1, \sqrt{1+a^2}\right)$, it follows that the kernel of L_q (3.64) is spanned by $\Psi_q^{(0)} = (-s_{q,x}, 0)^t$.

The component equations of the generalized eigenvalue equation $L_q Z = \Psi_q^{(0)}$ are

$$z_2 = -D_q^{-1} s_{q,x}, \tag{3.81}$$

$$C_q z_1 = 2\nu D_q^{-1} s_{q,x}. \tag{3.82}$$

By the Fredholm alternative and by the self-adjointness of C_q, (3.82) has a solution iff its RHS is in $(\ker C_q)^{\perp}$. That is, (3.82) has a solution iff $\langle D_q^{-1} s_{q,x} | s_{q,x} \rangle = 0$, or equivalently $n_0 = 0$. We show in the sequel that n_0 is negative, and so the generalized eigenvalue equation $L_q Z = \Psi_q^{(0)}$ has no solutions and the zero eigenvalue is simple.

Denote the space of all functions orthogonal to s_q by $s_q^{\perp}$, and denote the restriction of D_q to $s_q^{\perp}$ by $D_q^{s\perp}$. Since $D_q^{s\perp}$ is self-adjoint and $(-\infty, \mu) \subset \rho\left(D_q^{s\perp}\right)$, Theorem 2.6.6 of [14] implies that $\langle z | D_q^{s\perp} z \rangle \geq \mu \|z\|^2$ for all z in the domain of $D_q^{s\perp}$. In particular, this inequality applies for $z = s_{q,x}$ since $s_{q,x}$ is orthogonal to s_q. Hence,

$$\begin{aligned} -n_0 &= 2\nu \langle D_q^{-1} s_{q,x} | s_{q,x} \rangle, && (3.83)\\ &= 2\nu \left\langle \left(D_q^{s\perp}\right)^{-1} s_{q,x} \middle| s_{q,x} \right\rangle, && (3.84)\\ &= 2\nu \left\langle \left(D_q^{s\perp}\right)^{-1} s_{q,x} \middle| D_q^{s\perp} \left(D_q^{s\perp}\right)^{-1} s_{q,x} \right\rangle, && (3.85)\\ &\geq 2\mu\nu \left\| \left(D_q^{s\perp}\right)^{-1} s_{q,x} \right\|^2, && (3.86)\\ &> 0. && (3.87) \end{aligned}$$

The proof is complete. ■

Claim 13 *If* $\operatorname{Re}\lambda_1 \neq -\nu$ *and* $\operatorname{Im}\lambda_1 \neq 0$, *then* λ *is an eigenvalue of* L_q *with algebraic multiplicity 1.*

Numerical evidence suggests that $\partial_\xi E(\xi) \neq 0$ at the zero corresponding to the Hopf eigenvalues. A plot of the magnitude of the derivative of the Evans function along the Hopf

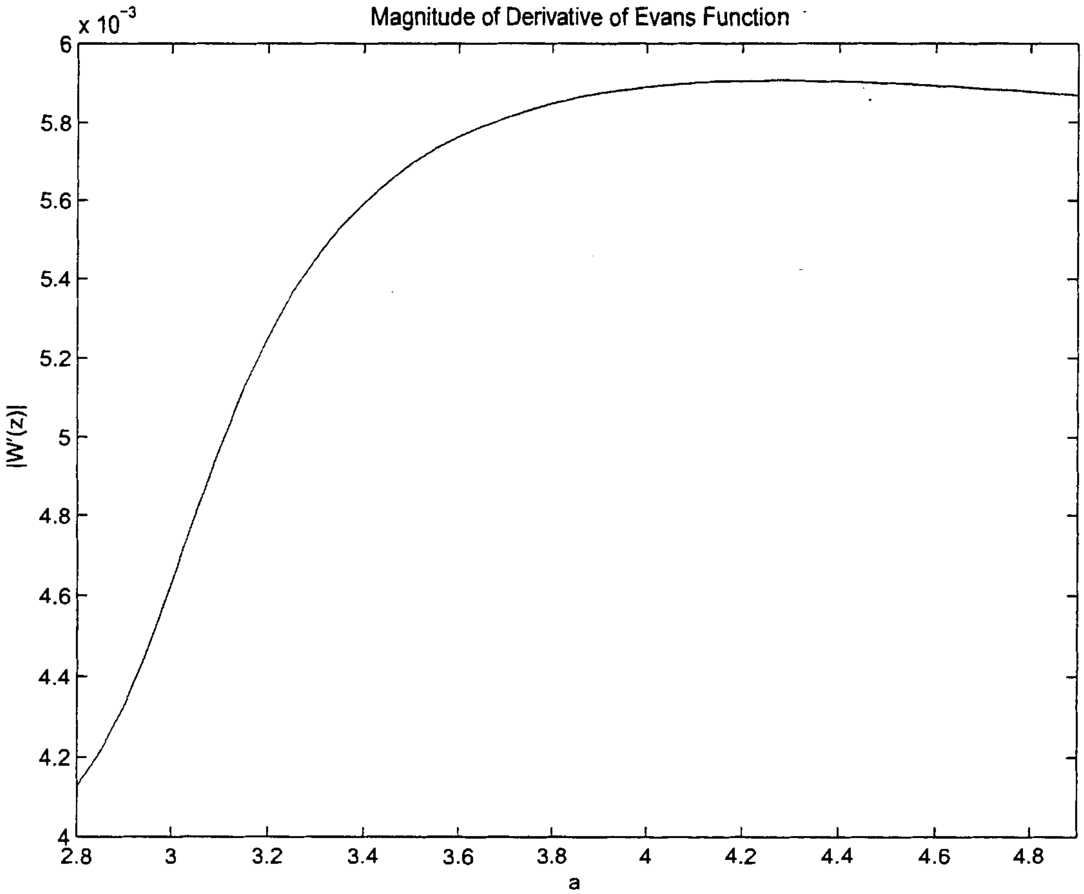

Figure 3.10: The magnitude of the derivative of the Evans function along the Hopf bifurcation curve.

bifurcation curve is shown above. Our claim then follows since the order of the zero is equal to the algebraic multiplicity of the eigenvalue [1]. Moreover, the normalization constant is always well defined by Theorem 1.1 of [9].

The operators which spectrally project onto the active space X_q and the stable space X_q^- are given by

$$\pi_q(\cdot) = \sum_{j=0}^{2} \left\langle \cdot \,\middle|\, \Psi_q^{(j)\dagger} \right\rangle \Psi_q^{(j)}, \tag{3.88}$$

$$\pi_q^-(\cdot) = (I - \pi_q)(\cdot), \tag{3.89}$$

respectively, where I is the identity operator.

Because the spectrum of the linearized operator is not contained in any sector of the complex plane, L_q generates only a C_0 semigroup $S_q(t)$. However, because $\sigma(L_q) \setminus \{\lambda_0, \lambda_1, \lambda_2\}$ is a strict subset of the left-half complex plane, the restriction of L_q to X_q^- enjoys the following estimate. See Proposition 4.1 of [11] for details.

Lemma 14 *Each operator L_q generates a C_0 semigroup S_q which satisfies*

$$\|S_q(t) Z\|_{H^1} \leq c_S e^{-kt} \|Z\|_{H^1} \tag{3.90}$$

for some constant $c_S \geq 1$, for all $Z \in X_q^-$, and for all $t \geq 0$. The constant c_S is chosen to be independent of q. Thus the Semigroup condition is satisfied.

We end this section with the following lemma.

Lemma 15 *If $\gamma \in \left(1, \sqrt{1+a^2}\right)$ and $\operatorname{Re}\lambda_1 \neq -\nu$ and $\operatorname{Im}\lambda_1 \neq 0$, then $\left\langle \Psi_{q,q}^{(j)} \middle| \Psi_q^{(k)\dagger} \right\rangle = 0$ for $j, k = 1, 2$.*

Proof. Because the PNLS is invariant under spatial translations, it suffices to show that $\left\langle \Psi_{q,q}^{(j)} \middle| \Psi_q^{(k)\dagger} \right\rangle = 0$ for $q = 0$. For notational convenience, denote $\Psi \equiv \left(\Psi_q^{(1)}\right)_{q=0}$.

Now, L (3.64) is an even operator in the sense that it is invariant under the reflection $x \longrightarrow -x$. So, by replacing x with $-x$ in the eigenvalue equation

$$L(x)\Psi(x) = \lambda\Psi(x), \tag{3.91}$$

we obtain

$$L(x)\Psi(-x) = \lambda\Psi(-x) \tag{3.92}$$

Adding and subtracting (3.91) and (3.92) yields

$$L(x)(\Psi(x) + \Psi(-x)) = \lambda(\Psi(x) + \Psi(-x)), \tag{3.93}$$

$$L(x)(\Psi(x) - \Psi(-x)) = \lambda(\Psi(x) - \Psi(-x)) \tag{3.94}$$

respectively. By Claim 13, λ has algebraic multiplicity 1 and so it is associated with only one linearly independent eigenfunction. Because $\Psi(x) - \Psi(-x)$ and $\Psi(x) + \Psi(-x)$ are linearly independent, one of these functions must be zero, and so Ψ is either even or odd. By the expressions (3.76,3.77) for the adjoint eigenfunctions, this is also true of $\Psi^{\dagger}$.

We remark that the numerical computations from [6] show that $\Psi_q^{(j)}$ for $q = 0$, $j = 1, 2$, is even.

Functions of opposite parity are mutually orthgonal since, if A is odd and B is even, we have

$$\begin{aligned}
\langle A(x) | B(x) \rangle &= \langle -A(-x) | B(-x) \rangle, && (3.95) \\
&= -\langle A(x) | B(x) \rangle. && (3.96)
\end{aligned}$$

To prove our lemma, then, it is sufficient to show that, for functions of definite parity, differentiation changes parity. If A is odd, then

$$A'(-x) = \frac{\partial A(-x)}{\partial(-x)} = \frac{\partial x}{\partial(-x)} \frac{\partial A(-x)}{\partial x} = \frac{\partial A(x)}{\partial x} = A'(x), \tag{3.97}$$

and so A' is even. Similarly, if B is even, then B' is odd. ∎

<u>The Nonlinear Term</u> We compute the nonlinear term $\mathcal{N}_q$ (3.11) by computing each of the terms on the right-hand side of

$$\mathcal{N}_q(Y) \equiv F(Q_q + Y) - F(Q_q) - L_q(Y) \tag{3.98}$$

as follows. Denoting the components of Y as $Y = (y_1, y_2)^t$, direct substitution of $Q_q + Y = (s_q + y_1, y_2)^t$ into F (3.38) yields

$$F(Q_q + Y) = \begin{pmatrix} -y_{2,xx} + \mu y_2 - (s_q + y_1)^2 y_2 - y_2^3 \\ (s_q + y_1)_{,xx} - (s_q + y_1) + (s_q + y_1)^3 + (s_q + y_1) y_2^2 - 2\nu y_2 \end{pmatrix}. \tag{3.99}$$

As Q_q is a stationary solution of the PNLS for all q, then

$$F(Q_q) = 0. \tag{3.100}$$

Finally, direct multiplication of L_q (3.64) by Y yields

$$L_q Y = \begin{pmatrix} 0 & -\partial_x^2 - s_q^2 + \mu \\ \partial_x^2 + 3s_q^2 - 1 & -2\nu \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{3.101}$$

$$= \begin{pmatrix} -y_{2,xx} - s_q^2 y_2 + \mu y_2 \\ y_{1,xx} + 3s_q^2 y_1 - y_1 - 2\nu y_2 \end{pmatrix}. \tag{3.102}$$

Substituting these expressions into (3.98), as well as applying the identity

$$s_{q,xx} - s_q + s_q^3 = 0 \tag{3.103}$$

which arises from (3.100), we obtain

$$\mathcal{N}_q(Y) = s_q N_2(Y, Y) + N_3(Y, Y, Y), \tag{3.104}$$

where

$$N_2(A, B) \equiv \begin{pmatrix} -2a_1 b_2 \\ 3a_1 b_1 + a_2 b_2 \end{pmatrix}, \tag{3.105}$$

$$N_3(A, B, C) \equiv \begin{pmatrix} -a_1 b_1 c_2 - a_2 b_2 c_2 \\ a_1 b_1 c_1 + a_1 b_2 c_2 \end{pmatrix}. \tag{3.106}$$

Observe that N_2 and N_3 are bilinear and trilinear in each of their arguments respectively. For instance, $N_2(A, B + C) = N_2(A, B) + N_2(A, C)$ and $N_2(A, cB) = cN_2(A, B)$.

In this section, we determine the correction terms

$$R_{\mathbf{p}}^{(2)} \equiv \sum_{j,k=1}^{2} r_j r_k R_b^{(jk)}, \tag{3.107}$$

$$R_{\mathbf{p}}^{(3)} \equiv \sum_{j,k,l=1}^{2} r_j r_k r_l R_b^{(jkl)} \tag{3.108}$$

so that the Quasi-Invariant Manifold condition is satisfied with $\varsigma = r^4$:

$$\Omega_b(\mathbf{r},0) = \pi_b^- \left(L_b R_{(b,\mathbf{r})} + \mathcal{N}_b\left(R_{(b,\mathbf{r})}\right) - \sum_{j=0}^{2} [\omega_b(\mathbf{r},0)]_j \left(\partial_{p_j}\Phi_{\mathbf{p}}\right)_{q=b} \right) = O\left(r^4\right). \tag{3.109}$$

This is necessary so that W will decay to $O\left(r^4\right)$ which in turn is necessary for transforming the evolution equation for r into Poincaré normal form in the asymptotic state. As noted earlier, we demand that $R_b^{(jk)}$ and $R_b^{(jkl)}$ depend only on the anchor point and not on q. Moreover, we impose the same condition

$$R_b^{(jk)}, R_b^{(jkl)} \in X_b^- \tag{3.110}$$

as was imposed on the residual term W.

We impose the following symmetry conditions on $R_b^{(jk)}$ and $R_b^{(jkl)}$ to ease our calculations.

(S1) $R_b^{(jk)}$, $R_b^{(jkl)}$ is symmetric with respect to the interchange of any two indices.

(S2) Denote $\bar{1} = 2$ and $\bar{2} = 1$. Then, $R_b^{(\bar{j}\bar{k})} = \overline{R_b^{(jk)}}$ and $R_b^{(\bar{j}\bar{k}\bar{l})} = \overline{R_b^{(jkl)}}$.

Note that, under these conditions, $R_b^{(12)}$ is real since $R_b^{(12)} = R_b^{(21)}$ and $R_b^{(21)} = \overline{R_b^{(12)}}$.

We calculate the terms comprising (3.109) as follows. Recall from (3.59) that

$$R_{\mathbf{p}} \equiv \sum_{j=1}^{2} r_j \Psi_q^{(j)} + \sum_{j,k=1}^{2} r_j r_k R_b^{(jk)} + \sum_{j,k,l=1}^{2} r_j r_k r_l R_b^{(jkl)}. \tag{3.111}$$

First, multiplying L_b (3.64) by $R_{(b,\mathbf{r})}$ (3.59) yields

$$L_b R_{(b,\mathbf{r})} = \sum_{j=1}^{2} \lambda_j r_j \Psi_b^{(j)} + \sum_{j,k=1}^{2} r_j r_k L_b R_b^{(jk)} + \sum_{j,k,l=1}^{2} r_j r_k r_l L_b R_b^{(jkl)}. \tag{3.112}$$

Next, by direct substitution of $R_{\mathbf{p}}$ (3.59) into $\mathcal{N}_b$ (3.104) and using the bilinearity and trilinearity of N_2 (3.105) and N_3 (3.106) respectively,

$$\mathcal{N}_b \left(R_{(b,\mathbf{r})}\right) = \sum_{j,k=1}^{2} r_j r_k H_b^{(jk)} + \sum_{j,k,l=1}^{2} r_j r_k r_l H_b^{(jkl)} + O\left(r^4\right), \tag{3.113}$$

where

$$\begin{aligned} H_q^{(jk)} &\equiv s_q N_2 \left(\Psi_q^{(j)}, \Psi_q^{(k)}\right), & (3.114) \\ H_q^{(jkl)} &\equiv s_q N_2 \left(\Psi_q^{(j)}, R_b^{(kl)}\right) + s_q N_2 \left(R_b^{(jk)}, \Psi_q^{(l)}\right) + N_3 \left(\Psi_q^{(j)}, \Psi_q^{(k)}, \Psi_q^{(l)}\right). & (3.115) \end{aligned}$$

Next, direct differentiation of (3.58) yields

$$\begin{aligned} \partial_q \Phi_{\mathbf{p}} &= \Psi_q^{(0)} + \sum_{j=1}^{2} r_j \Psi_{q,q}^{(j)}, & (3.116) \\ \partial_{r_j} \Phi_{\mathbf{p}} &= \Psi_q^{(j)} + 2\sum_{k=1}^{2} r_k R_b^{(jk)} + 3 \sum_{k,l=1}^{2} r_k r_l R_b^{(jkl)}, & (3.117) \end{aligned}$$

where we have used the identity $Q_{q,q} = \Psi_q^{(0)}$ (3.69) and applied the symmetry conditions (S1,S2). Substituting these expressions for $\partial_{p_j} \Phi_{\mathbf{p}}$ into $\Pi_{(b,\mathbf{r})}$ (3.21) then yields

$$\Pi_{(b,\mathbf{r})} = \begin{pmatrix} 1 + \sum_{j=1}^{2} r_j \left\langle \Psi_{b,b}^{(j)} \middle| \Psi_b^{(0)\dagger} \right\rangle & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.118}$$

where we have used the orthonormality condition (3.80) and Lemma 15. The inverse of $\Pi_{(b,\mathbf{r})}$ is then

$$\Pi_{(b,\mathbf{r})}^{-1} = \begin{pmatrix} \left(1 + \sum_{j=1}^{2} r_j \left\langle \Psi_{b,b}^{(j)} \middle| \Psi_b^{(0)\dagger} \right\rangle\right)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.119}$$

Finally, substituting $F(Q_q) = 0$ and $W = 0$ into $\omega_b(\mathbf{r}, W)$ (3.24) yields

$$[\omega_b(\mathbf{r}, 0)]_j \equiv \sum_{k=1}^{N} \left[\Pi_{(b,\mathbf{r})}^{-1}\right]_{jk} \left\langle \lambda_k R_{(b,\mathbf{r})} + \mathcal{N}_b\left(R_{(b,\mathbf{r})}\right) \middle| \Psi_b^{(k)\dagger} \right\rangle, \tag{3.120}$$

where we have used the relation

$$\left\langle L_b R_{(b,\mathbf{r})} \middle| \Psi_b^{(k)\dagger} \right\rangle = \left\langle R_{(b,\mathbf{r})} \middle| L_b^\dagger \Psi_b^{(k)\dagger} \right\rangle = \left\langle \lambda_k R_{(b,\mathbf{r})} \middle| \Psi_b^{(k)\dagger} \right\rangle. \tag{3.121}$$

Substituting (3.119) for $\Pi_{(b,\mathbf{r})}^{-1}$, (3.111) for $R_{(b,\mathbf{r})}$, and (3.113) for $\mathcal{N}_b\left(R_{(b,\mathbf{r})}\right)$ then yields

$$[\omega_b(\mathbf{r}, 0)]_0 = \left[\Pi_{(b,\mathbf{r})}^{-1}\right]_{00} \left(\sum_{k,l=1}^{2} \gamma_{kl0} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klm0} r_k r_l r_m + O\left(r^4\right) \right), \tag{3.122}$$

and, for $j = 1, 2$,

$$[\omega_b(\mathbf{r}, 0)]_j = \lambda_j r_j + \sum_{k,l=1}^{2} \gamma_{klj} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klmj} r_k r_l r_m + O\left(r^4\right), \tag{3.123}$$

where we have introduced the constants

$$\gamma_{jkl} \equiv \left\langle H_b^{(jk)} \middle| \Psi_b^{(l)\dagger} \right\rangle, \tag{3.124}$$

$$\gamma_{jklm} \equiv \left\langle H_b^{(jkl)} \middle| \Psi_b^{(m)\dagger} \right\rangle, \tag{3.125}$$

which are independent of b due to the invariance of the PNLS under spatial translations. Collecting all of the above expressions (3.112,3.113,3.116,3.117,3.122,3.123), we substitute into the Quasi-Invariant Manifold condition (3.109) and group the $O\left(r^2\right)$ and $O\left(r^3\right)$ terms to obtain

$$\sum_{j,k=1}^{2} r_j r_k \left(L_b R_b^{(jk)} - 2\lambda_j R_b^{(jk)} + U_b^{(jk)} \right) = 0, \tag{3.126}$$

$$\sum_{j,k,l=1}^{2} r_j r_k r_l \left(L_b R_b^{(jkl)} - 3\lambda_j R_b^{(jkl)} + U_b^{(jkl)} \right) = 0, \tag{3.127}$$

where

$$U_b^{(jk)} \equiv \pi_b^- H_b^{(jk)}, \tag{3.128}$$

$$U_b^{(jkl)} \equiv \pi_b^- H_b^{(jkl)} - \gamma_{kl0} \pi_b^- \Psi_{b,b}^{(j)} - \sum_{m=1}^{2} 2\gamma_{klm} R_b^{(mj)}. \tag{3.129}$$

Applying the symmetry conditions (S1,S2), we obtain

$$\left(L_b - \mu_{jk}\right) R_b^{(jk)} = -\frac{1}{2}\left(U_b^{(jk)} + U_b^{(kj)}\right), \tag{3.130}$$

$$\left(L_b - \mu_{jkl}\right) R_b^{(jkl)} = -\frac{1}{3}\left(U_b^{(jkl)} + U_b^{(klj)} + U_b^{(ljk)}\right), \tag{3.131}$$

where

$$\mu_{jk} \equiv \lambda_j + \lambda_k, \tag{3.132}$$

$$\mu_{jkl} \equiv \lambda_j + \lambda_k + \lambda_l. \tag{3.133}$$

Both the R's and U's lie in the decay space X_b^-, so the R's are solvable provided the operators $(L_b - \mu)$ are boundedly invertible on X_b^-. The restriction of L_b to X_b^- has the spectrum

$$\sigma\left(L_b|_{X_b^-}\right) = \sigma\left(L_b\right) \backslash \{\lambda_0, \lambda_1, \lambda_2\} \tag{3.134}$$

contained in $\{z \in \mathbb{C}\,|\mathrm{Re}\, z \leq -k\}$. Thus, by a slight generalization of Lemma 4.2 from [11], $(L_b - z)^{-1}$ exist and are uniformly bounded on $\{z \in \mathbb{C}\,|\mathrm{Re}\, z > -\delta > -k\}$. Because the resonances μ lie in $\{z \in \mathbb{C}\,|\mathrm{Re}\, z > -\delta > -k\}$, the R's are solvable.

Lemma 16 *By choosing the correction terms $R_{\mathbf{p}}^{(2)}$ and $R_{\mathbf{p}}^{(3)}$ as*

$$R_{\mathbf{p}}^{(2)} \equiv \sum_{j,k=1}^{2} r_j r_k R_b^{(jk)}, \tag{3.135}$$

$$R_{\mathbf{p}}^{(3)} \equiv \sum_{j,k,l=1}^{2} r_j r_k r_l R_b^{(jkl)}, \tag{3.136}$$

where

$$R_b^{(jk)} = -\frac{1}{2}\left(L_b - \mu_{jk}\right)^{-1}\left(U_b^{(jk)} + U_b^{(kj)}\right), \tag{3.137}$$

$$R_b^{(jkl)} = -\frac{1}{3}\left(L_b - \mu_{jkl}\right)^{-1}\left(U_b^{(jkl)} + U_b^{(klj)} + U_b^{(ljk)}\right), \tag{3.138}$$

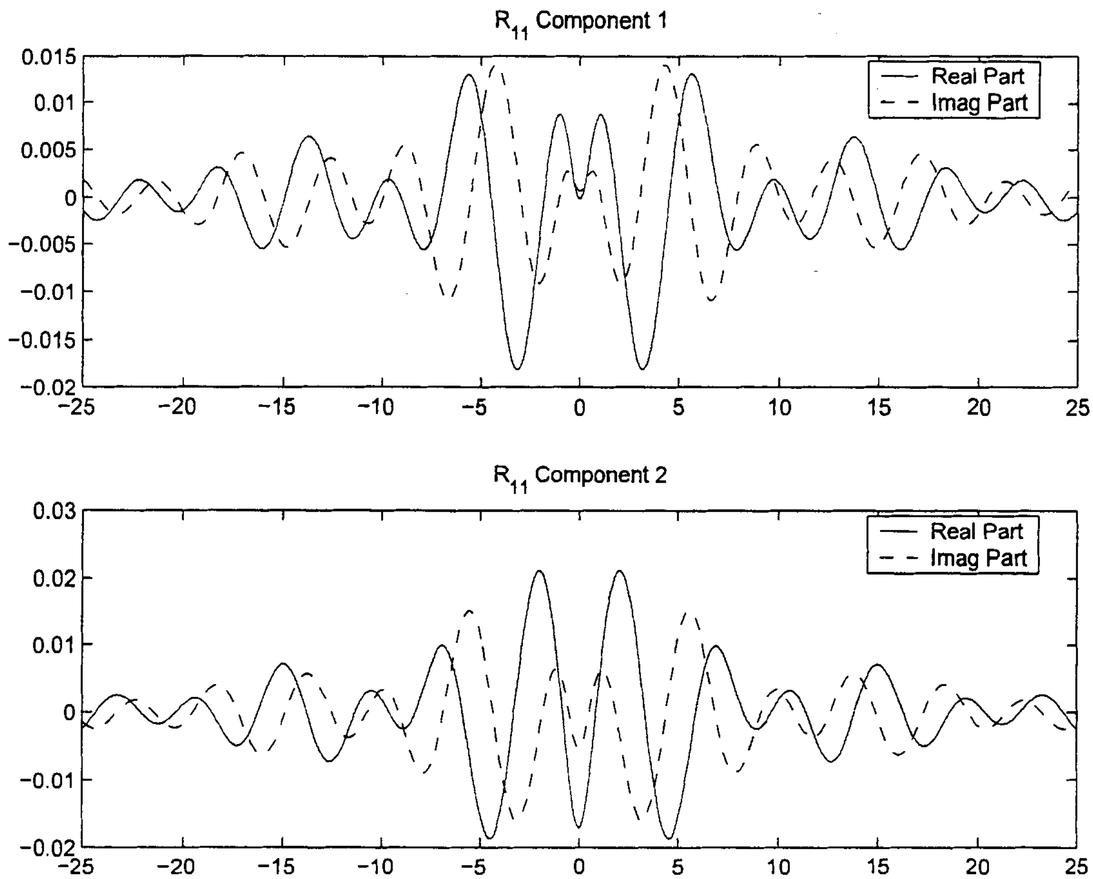

Figure 3.11: Graph of the manifold correction term R_{11} for $\gamma = 2.4$ and $a = 2.8$.

and

$$\mu_{jk} \equiv \lambda_j + \lambda_k, \tag{3.139}$$

$$\mu_{jkl} \equiv \lambda_j + \lambda_k + \lambda_l. \tag{3.140}$$

the Quasi-Invariant Manifold condition is satisfied.

3.4 The Evolution Equations

In this section, we compute explicit expressions for the evolution equations (3.23,3.27) for $\mathbf{p}$ and W. This primarily involves computing explicit expressions for $\omega_{\dot{b}}(W)$ (3.24) and

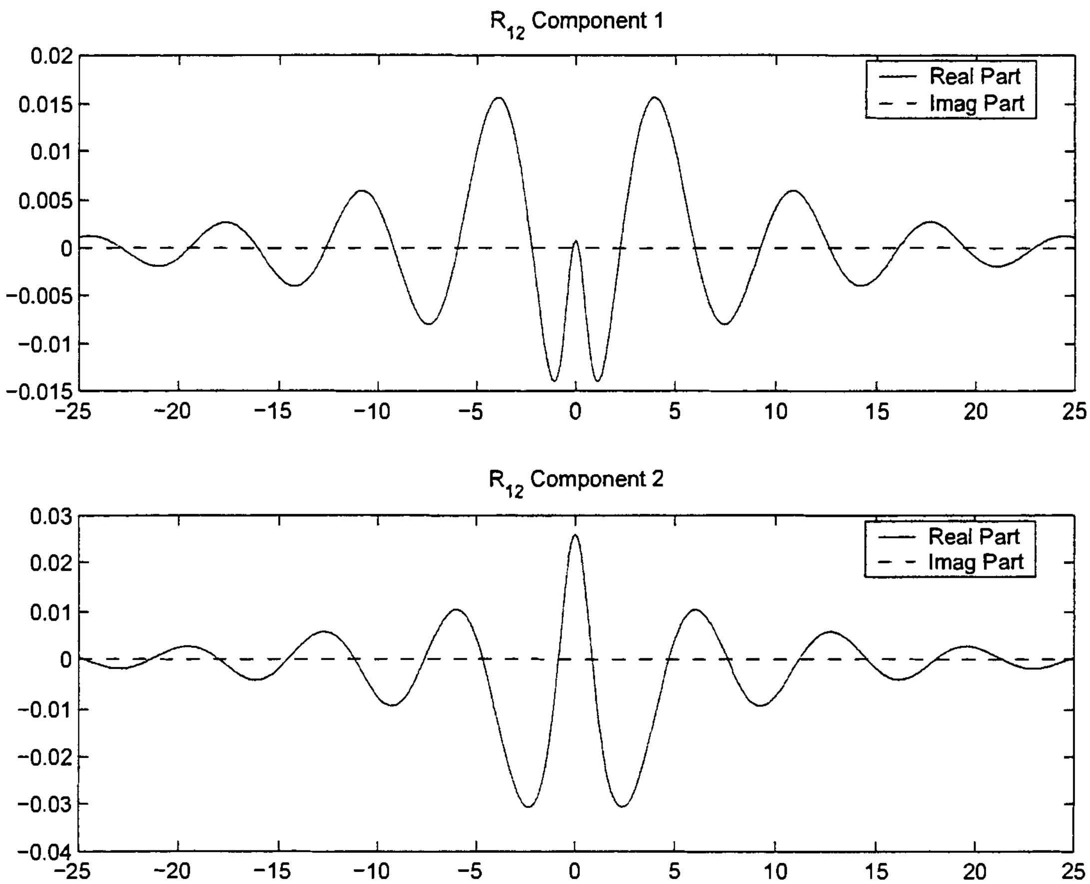

Figure 3.12: Graph of the manifold correction term R_{12} for $\gamma = 2.4$ and $a = 2.8$.

$\Omega_b(W)$ (3.28). We then obtain the mild solution (2.74) for W. Lastly, we transform the evolution equation for r into Poincaré normal form which analytically exhibits the Hopf bifurcation. In particular, we shall show that the coefficient η of the cubic term in the Poincaré normal form has positive real part at criticality, and this implies that the Hopf bifurcation is supercritical.

Most of the following calculations have already been performed in Section 3.3.

The Evolution Equations Recasting $\omega_b(\mathbf{r}, W)$ (3.24) as

$$\omega_b(\mathbf{r}, W) = \omega_b(\mathbf{r}, 0) + \widetilde{\omega}_b(\mathbf{r}, W), \tag{3.141}$$

where

$$\widetilde{\omega}_b(\mathbf{r}, W) \equiv \omega_b(\mathbf{r}, W) - \omega_b(\mathbf{r}, 0) \tag{3.142}$$

we substitute the expressions (3.122,3.123) for the components of $\omega_b(\mathbf{r}, 0)$ to obtain

$$\begin{aligned} [\omega_b(\mathbf{r}, W)]_0 &= \left[\Pi^{-1}_{(b,\mathbf{r})}\right]_{00} \left(\sum_{k,l=1}^{2} \gamma_{kl0} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klm0} r_k r_l r_m\right) \\ &\quad + \left[\Pi^{-1}_{(b,\mathbf{r})}\right]_{00} \left(O\left(r^4\right) + [\widetilde{\omega}_b(\mathbf{r}, W)]_0\right). \end{aligned} \tag{3.143}$$

and, for $j = 1, 2$,

$$\begin{aligned} [\omega_b(\mathbf{r}, W)]_j &= \lambda_j r_j + \sum_{k,l=1}^{2} \gamma_{klj} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klmj} r_k r_l r_m \\ &\quad + O\left(r^4\right) + [\widetilde{\omega}_b(\mathbf{r}, W)]_j, \end{aligned} \tag{3.144}$$

where

$$\left[\Pi^{-1}_{(b,\mathbf{r})}\right]_{00} = 1 + \sum_{j=1}^{2} r_j \left\langle \Psi^{(j)}_{b,b} \middle| \Psi^{(0)\dagger}_b \right\rangle. \tag{3.145}$$

Also, the components of $\widetilde{\mathbf{s}}_b(\mathbf{p}, W)$ (3.25) are given by

$$[\widetilde{\mathbf{s}}_b(\mathbf{p}, W)]_0 = \left[\Pi^{-1}_{(b,\mathbf{r})}\right]_{00} \left\langle \mathcal{S}_b(\mathbf{p}, W) \middle| \Psi^{(0)\dagger}_b \right\rangle, \tag{3.146}$$

$$[\widetilde{\mathbf{s}}_\mathbf{b}(\mathbf{p}, W)]_j = \left\langle \mathcal{S}_b(\mathbf{p}, W) \middle| \Psi^{(j)\dagger}_b \right\rangle. \tag{3.147}$$

The evolution equation (3.23) for $q_{,t}$ is thus explicitly given by (to leading order)

$$\begin{aligned} q_{,t} &= \left[\Pi^{-1}_{(b,\mathbf{r})}\right]_{00} \left(\sum_{k,l=1}^{2} \gamma_{kl0} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klm0} r_k r_l r_m \right); \\ &\quad + \left[\Pi^{-1}_{(b,\mathbf{r})}\right]_{00} \left(O\left(r^4\right) + [\widetilde{\omega}_b(\mathbf{r}, W)]_0 \right) + [\widetilde{\mathbf{s}}_b(\mathbf{p}, W)]_0 . \end{aligned} \tag{3.148}$$

and $r_{j,t}$ by

$$\begin{aligned} r_{j,t} &= \lambda_j r_j + \sum_{k,l=1}^{2} \gamma_{klj} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klmj} r_k r_l r_m \\ &\quad + O\left(r^4\right) + [\widetilde{\omega}_b(\mathbf{r}, W)]_j + [\widetilde{\mathbf{s}}_b(\mathbf{p}, W)]_j . \end{aligned} \tag{3.149}$$

Recasting $\Omega_b(\mathbf{r}, W)$ (3.28) as

$$\Omega_b(\mathbf{r}, W) = \Omega_b(\mathbf{r}, 0) + \widetilde{\Omega}_b(\mathbf{r}, W), \tag{3.150}$$

where

$$\begin{aligned} \widetilde{\Omega}_b(\mathbf{r}, W) &\equiv \Omega_b(\mathbf{r}, W) - \Omega_b(\mathbf{r}, 0), & (3.151) \\ &= \pi_b^- \widetilde{\mathcal{N}}_b(\mathbf{r}, W) - \pi_b^- \Upsilon_{(b,\mathbf{r})} \widetilde{\omega}_b(\mathbf{r}, W), & (3.152) \end{aligned}$$

and

$$\widetilde{\mathcal{N}}_q(\mathbf{r}, W) \equiv \mathcal{N}_q\left(R_{(q,\mathbf{r})} + W\right) - \mathcal{N}_q\left(R_{(q,\mathbf{r})}\right). \tag{3.153}$$

Also, $\widetilde{\mathcal{S}}_{\mathbf{b}}(\mathbf{p}, W)$ (3.29) is given by

$$\widetilde{\mathcal{S}}_b(\mathbf{p}, W) \equiv \pi_b^- \left(\mathcal{S}_b(\mathbf{p}, W) - \Upsilon_{(b,\mathbf{r})} \widetilde{\mathbf{s}}_b(\mathbf{p}, W)\right). \tag{3.154}$$

The evolution equation (3.27) for W is thus given by

$$W_{,t} = L_b W + \Omega_b(\mathbf{r}, 0) + \widetilde{\Omega}_b(\mathbf{r}, W) + \widetilde{\mathcal{S}}_b(\mathbf{p}, W), \tag{3.155}$$

which posesses the mild solution

$$W(t) = S(t - t_b) W_b + \int_{t_b}^{t} S(t - \tau) G_b(\tau) \, d\tau, \tag{3.156}$$

where the "forcing term" G_b is given by

$$G(\tau) \equiv \Omega_b(\mathbf{r}(\tau), 0) + \widetilde{\Omega}_b(\mathbf{r}(\tau), W(\tau)) + \widetilde{S}_b(\mathbf{p}(\tau), W(\tau)) \tag{3.157}$$

and $W_b \equiv W(t_b)$ is the initial residual with respect to the anchor point b.

<u>The Hopf Bifurcation</u> The evolution equation for r_j is given by (3.149):

$$r_{j,t} = \lambda_j r_j + \sum_{k,l=1}^{2} \gamma_{klj} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klmj} r_k r_l r_m + O\left(r^4, |\widetilde{\omega}_b(\mathbf{r}, W)|, |\widetilde{\mathbf{s}}_b(\mathbf{p}, W)|\right), \tag{3.158}$$

where it will be shown in Section 3.5 that $\widetilde{\omega}_b(\mathbf{r}, W)$ is $O\left(|r|\, w, w^2\right)$ and $\widetilde{\mathbf{s}}_b(\mathbf{p}, W)$ is $O(|q-b|)$. It will also be shown in Section 3.5 that, in the asymptotic state wherein w is $O\left(r^4\right)$, $\widetilde{\omega}_b(\mathbf{r}, W)$ and $\widetilde{\mathbf{s}}_b(\mathbf{p}, W)$ are also $O\left(r^4\right)$. Thus, in the asymptotic state, all terms in (3.149) up to $O\left(r^3\right)$ are explicitly given. Defining

$$h_{20} = 2\gamma_{111}, \tag{3.159}$$

$$h_{11} = \gamma_{121} + \gamma_{211}, \tag{3.160}$$

$$h_{02} = 2\gamma_{221}, \tag{3.161}$$

$$h_{30} = 6\gamma_{1111}, \tag{3.162}$$

$$h_{21} = 2\left(\gamma_{1121} + \gamma_{1211} + \gamma_{2111}\right), \tag{3.163}$$

$$h_{12} = 2\left(\gamma_{1221} + \gamma_{2121} + \gamma_{2211}\right), \tag{3.164}$$

$$h_{03} = 6\gamma_{2221}, \tag{3.165}$$

we rewrite (3.149) as

$$r_{,t} = \lambda r + \sum_{2 \le k+l \le 3} \frac{h_{kl}}{k!l!} r^k \overline{r}^l + O\left(r^4\right). \tag{3.166}$$

We now apply Lemma 3.6 from [12], which we restate below, to transform the evolution equation (3.166) for r into the Poincaré normal form for the Hopf bifurcation.

Lemma 17 *The equation (3.166) with* $\lambda = \lambda(\gamma)$, $\operatorname{Re}\lambda(\gamma_c) = 0$, $\operatorname{Im}\lambda(\gamma_c) > 0$, *and* $h_{kl} = h_{kl}(\gamma)$, *can be transformed by the invertible parameter-dependent change of complex coordinate:*

$$r = v + \sum_{2 \le k+l \le 3} \frac{j_{kl}}{k!l!} v^k \overline{v}^l \tag{3.167}$$

with $j_{21} = 0$ *into an equation with only the resonant cubic term:*

$$v_{,t} = \lambda v + \eta |v|^2 v + O(v^4) \tag{3.168}$$

for all γ *such that* $|\gamma - \gamma_c|$ *is sufficiently small. Furthermore,*

$$\eta(\gamma) = \frac{h_{20}h_{11}(2\lambda + \overline{\lambda})}{2|\lambda|^2} + \frac{|h_{11}|^2}{\lambda} + \frac{|h_{02}|^2}{2(2\lambda - \overline{\lambda})} + \frac{h_{21}}{2} \tag{3.169}$$

which, at the critical bifurcation value γ_c, *reduces to*

$$\eta_c = \eta(\gamma_c) = \frac{i}{2\operatorname{Im}\lambda(\gamma_c)}\left(h_{20}h_{11} - 2|h_{11}|^2 - \frac{1}{3}|h_{02}|^2\right) + \frac{h_{21}}{2}. \tag{3.170}$$

If $\operatorname{Re}\eta < 0$, *the Hopf bifurcation is supercritical; otherwise, the Hopf bifurcation is subcritical.*

The coefficient η_c of the cubic term was numerically computed along the Hopf bifurcation curve for $a \in (2.8, 4.9)$ according to (3.170). These computations were performed using Matlab R13. The most challenging aspects of this computation were the numerical determination of the Hopf eigenfunction Ψ (3.71) and the correction terms $R_b^{(jk)}$ (3.137) and $R_b^{(jkl)}$ (3.138). Using results from [6], the Hopf eigenfunction was numerically computed as a linear combination of Dirichlet expansions on the stable manifold of the associated linearized eigenvalue problem. The corresponding Evans function plays a key role in this computation, yielding the Hopf eigenvalues as its zeros and the coefficients used in the linear combination. See [6] for details. On the other hand, the computation of the correction terms involved implicitly solving ODEs of the type

$$(L - \mu) R = U, \tag{3.171}$$

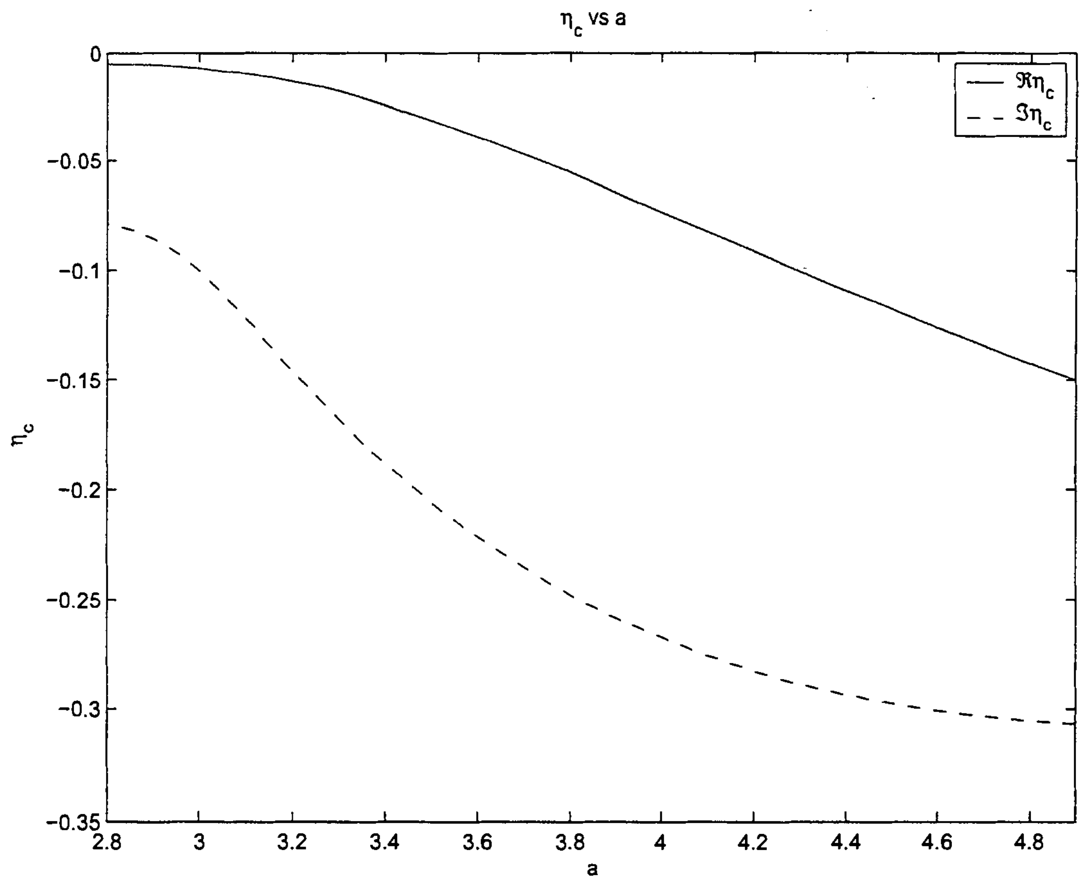

Figure 3.13: Hopf bifurcation constant η versus detuning parameter a along Hopf bifurcation curve.

where L is the linearized operator, μ is the resonance, R is a correction term, and U is some combination of the terms $U_b^{(jk)}$ (3.128) and/or $U_b^{(jkl)}$ (3.129). This was done using Matlab R13's two-point boundary value problem solver *bvp4c* with boundary conditions $R = 0$ and $R' = 0$ at $x = 100$ (i.e. infinity). The results are shown in the figure above.

Because $\mathrm{Re}\,\eta\,(\gamma_c\,(a)) < 0$, we conclude that the Hopf bifurcation is supercritical.

3.5 Bounds on the Residual Term

From the mild solution (3.156) for W, we obtain the estimate (3.234) for W valid for the current anchor point b. This estimate shows that control of W is lost after a finite time period, and this is caused by the secular growth in $\widetilde{S}_b$ (3.154). We remove this secularity in the following section by rechoosing our anchor point as b^* in such a way that $W \in X_{b^*}^-$. This is done by Theorem 18. Lastly, we show that by appropriately choosing the fixed time length Δt in which each anchor point is used (see 3.262), the oscillatory solution will approach and remain near the manifold under suitable initial conditions.

Introduction Fix the anchor point at b on the time interval $[t_b, t_d)$ and denote the final value of q on $[t_b, t_d]$ by d. Also denote $w \equiv \|W\|_{H^1}$, and the initial and final values of w on $[t_b, t_d]$ by w_b and w_d. The following equations then hold:

$$q(t_b) = b, q(t_d) = d, w(t_b) = w_b, w(t_d) = w_d. \tag{3.172}$$

The following control quantities play prominent roles in our analysis. The quantity

$$\Delta t \equiv t_d - t_b \tag{3.173}$$

is the total length of time in which the anchor point b has been in use. The quantity

$$T^{(q)} \equiv \sup_{\theta \in [t_b, t_d)} |q(\theta) - b| \tag{3.174}$$

controls the distance between the manifold position parameter q and the anchor point b, the quantity

$$T^{(r)} \equiv \sup_{\theta \in [0, t_d)} |r(\theta)| \tag{3.175}$$

controls the amplitude of the oscillating solution, and the quantity

$$T^{(w)} \equiv \sup_{\theta \in [t_b, t_d)} e^{k(\theta - t_b)} w(\theta) \tag{3.176}$$

controls the size of W. W will decay exponentially in accordance with the semigroup decay estimate (3.90), so the purpose of the exponential factor $e^{k(\theta - t_b)}$ in (3.176) is to compensate for this decay.

Apply the triangle inequality and the semigroup decay estimate (3.90) to the mild solution (3.156) for W to obtain

$$w(t) \leq c_S e^{-k(t-t_b)} w_b + c_S \int_{t_b}^{t} e^{-k(t-\tau)} g_b(\tau)\, d\tau \tag{3.177}$$

where $g_b \equiv \|G_b\|_{H^1}$. A bound on G_b is obtained in the next section, which will then be used to bound W.

Bound on the Forcing Term The goal in this section is to obtain a bound on the "forcing term" G_b (3.157). The constant

$$\begin{aligned} c_\Psi \;=\; & \max_{j,k,l} \left\{ \|Q_q\|_{H^1}, \|Q_{q,q}\|_{H^1}, \left\|\Psi_q^{(j)}\right\|_{H^1}, \left\|\Psi_{q,q}^{(j)}\right\|_{H^1}, \right. \\ & \left. , \left\|R_b^{(jk)}\right\|_{H^1}, \left\|R_b^{(jkl)}\right\|_{H^1}, \gamma_{jkl}, \gamma_{jklm} \right\} \end{aligned} \tag{3.178}$$

is chosen for convenience to facilitate our computations. We shall frequently make use of this fact about the $H^1 = H^1(\mathbb{R})$ norm:

$$\|z_1 z_2\|_{H^1} \leq \|z_1\|_{H^1} \|z_2\|_{H^1}, \tag{3.179}$$

and we shall frequently use inequalities like

$$r + r^2 + \cdots \leq 2r, \;\; r \text{ small} \tag{3.180}$$

to dismiss the higher order terms in r. In particular, take note of the factor 2 on the right-hand side of this inequality.

Bounds on $\Omega_b(\mathbf{r}, 0)$ and $\widetilde{\Omega}_b(\mathbf{r}, W)$ are obtained as follows. All anchor point dependent quantities such as the linearized eigenfunctions $\Psi_b^{(j)}$ and the correction terms $R_b^{(jk)}$ have norms which are anchor point *independent* because the PNLS is invariant under spatial translations. Moreover, the nonlinearity in the PNLS is polynomial in nature, which means that these nonlinearities satisfy some type of (bi- or tri-) linearity property. In particular, all the $O\left(r^4\right)$ terms appearing in Sections 3.3 and 3.4 arise from $\mathcal{N}_b\left(R_{\mathbf{p}}\right)$ and therefore satisfy bounds like $\|A\|_{H^1} \leq c\,|r|^4$ for some constant c which is anchor point *independent.* It therefore follows from the Quasi-Invariant Manifold condition that

$$\|\Omega_b\left(\mathbf{r},0\right)\|_{H^1} \leq c_\Omega \left(T^{(r)}\right)^4 \tag{3.181}$$

for some constant c_Ω *independent* of b. As for $\widetilde{\Omega}_b\left(\mathbf{r},W\right)$, let us estimate each of the terms appearing in the right-hand side of (3.152). First, applying the triangle and Cauchy-Schwarz inequalities to the spectral projection operator π_b^- (3.89) yields

$$\begin{aligned} \left\|\pi_b^-\left(A\right)\right\|_{H^1} &= \left\| A - \sum_{j=0}^{2} \left\langle A \,\middle|\, \Psi_q^{(j)\dagger} \right\rangle \Psi_q^{(j)} \right\|_{H^1}, & (3.182)\\ &\leq \left(1+3c_\Psi^2\right) \|A\|_{H^1}, & (3.183) \end{aligned}$$

for any A, and so

$$\left\|\pi_b^-\right\|_{*,H^1} \leq 1 + 3c_\Psi^2. \tag{3.184}$$

Also, by direct substitution of $\mathcal{N}_q$ (3.104) in $\widetilde{\mathcal{N}}_q$ (3.153) and the bilinearity and trilinearity of N_2 (3.105) and N_3 (3.106) respectively,

$$\begin{aligned} \left\|\widetilde{\mathcal{N}}_q\left(\mathbf{r},W\right)\right\|_{H^1} &\leq \|s_q\|_{H^1} \left\|N_2\left(R_{(q,\mathbf{r})}+W\right) - N_2\left(R_{(q,\mathbf{r})}\right)\right\|_{H^1} \\ &\quad + \left\|N_3\left(R_{(q,\mathbf{r})}+W\right) - N_3\left(R_{(q,\mathbf{r})}\right)\right\|_{H^1}, & (3.185)\\ &\leq 2c_\Psi\left(c_\Psi w T^{(r)} + w^2\right). & (3.186) \end{aligned}$$

Next, the components of $\pi_b^- \Upsilon_{(b,\mathbf{r})}$ are obtained by applying the spectral projection operator

π_b^- (3.89) on the derivatives of $\Phi_{\mathbf{p}}$ (3.116,3.117):

$$\pi_b^- \left(\partial_q \Phi_{\mathbf{p}}\right)_{q=b} = \sum_{j=1}^{2} r_j \pi_b^- \Psi_{b,b}^{(j)}, \tag{3.187}$$

$$\pi_b^- \left(\partial_{r_j} \Phi_{\mathbf{p}}\right)_{q=b} = 2\sum_{k=1}^{2} r_k R_b^{(jk)} + 3\sum_{k,l=1}^{2} r_k r_l R_b^{(jkl)}. \tag{3.188}$$

Finally, $\|\widetilde{\omega}_b(\mathbf{r},W)\|_{H^1}$ is obtained by substituting $\omega_b(\mathbf{r},W)$ (3.24) and subsequently $\Pi_{(b,\mathbf{r})}^{-1}$ (3.119) into $\widetilde{\omega}_b(\mathbf{r},W)$ (3.142), and then applying the estimate (3.186) for $\widetilde{\mathcal{N}}_q$:

$$\left|[\widetilde{\omega}_b(\mathbf{r},W)]_j\right| = 2\left|\left\langle \widetilde{\mathcal{N}}_b(\mathbf{r},W) \middle| \Psi_b^{(j)\dagger} \right\rangle\right|, \tag{3.189}$$

$$\leq 4c_\Psi^2 \left(c_\Psi w T^{(r)} + w^2\right). \tag{3.190}$$

Applying the above estimates to $\widetilde{\Omega}_b(\mathbf{r},W)$ (3.152), we obtain

$$\left\|\widetilde{\Omega}_b(\mathbf{r},W)\right\|_{H^1} = \left\|\pi_b^- \widetilde{\mathcal{N}}_b(\mathbf{r},W) - \pi_b^- \Upsilon_{(b,\mathbf{r})}\widetilde{\omega}_b(\mathbf{r},W)\right\|_{H^1}, \tag{3.191}$$

$$\leq c_{\widetilde{\Omega}} \left(w T^{(r)} + w^2\right), \tag{3.192}$$

where $c_{\widetilde{\Omega}} \equiv \max\left\{\left(1+3c_\Psi^2\right)2c_\Psi^2, \left(1+3c_\Psi^2\right)2c_\Psi, 4c_\Psi^3, 4c_\Psi^2\right\}$.

Bounds on $q_{,t}$ and $\mathcal{S}_b(\mathbf{p},W)$ are obtained as follows. Applying the triangle inequality to (3.19) yields

$$\begin{aligned}\|\mathcal{S}_b(\mathbf{p},W)\|_{H^1} \leq\ & \|F(Q_q) - F(Q_b)\|_{H^1} + \|L_q W - L_b W\|_{H^1} \\ & + \left\|\mathcal{N}_q\left(R_{(q,\mathbf{r})} + W\right) - \mathcal{N}_b\left(R_{(b,\mathbf{r})} + W\right)\right\|_{H^1} \\ & + \left\|\left(L_q R_{(q,\mathbf{r})} - \Upsilon_{(q,\mathbf{r})}\mathbf{p}_{,t}\right) - \left(L_b R_{(b,\mathbf{r})} - \Upsilon_{(b,\mathbf{r})}\mathbf{p}_{,t}\right)\right\|_{H^1}. \end{aligned} \tag{3.193}$$

Let us estimate each of the terms on the right-hand side. As Q_q is a stationary solution of the PNLS for all q, then $F(Q_q) = 0$ for all q, and so

$$F(Q_q) - F(Q_b) = 0. \tag{3.194}$$

Next, by the expression (3.64) for L_q, $L_q - L_b$ is a multiplicative operator satisfying the bound $\|L_q - L_b\|_{*,H^1} \leq c_\Psi^2 |q-b|$, and so

$$\begin{aligned} \|L_q W - L_b W\|_{H^1} &\leq \|L_q - L_b\|_{*,H^1} w, && (3.195)\\ &\leq c_\Psi^2 w T^{(q)}. && (3.196) \end{aligned}$$

Next, by direct substitution of $R_{(b,\mathbf{r})} + W$ into $\mathcal{N}_q$ (3.104) and the bilinearity and trilineary of N_2 (3.105) and N_3 (3.106) respectively,

$$\left\|\mathcal{N}_q\left(R_{(q,\mathbf{r})} + W\right) - \mathcal{N}_b\left(R_{(b,\mathbf{r})} + W\right)\right\|_{H^1} \leq 2c_\Psi^3 \left(T^{(r)} + w\right)^2 T^{(q)}. \qquad (3.197)$$

Finally, multiplying L_q (3.64) by $R_\mathbf{p}$ (3.111) and applying the eigenvalue equation $L_q \Psi_q^{(j)} = \lambda_j \Psi_q^{(j)}$,

$$L_q R_{(q,\mathbf{r})} = \sum_{j=1}^{2} \lambda_j r_j \Psi_q^{(j)} + L_q\left(R_\mathbf{p}^{(2)} + R_\mathbf{p}^{(3)}\right) \qquad (3.198)$$

while, by direct substitution of $\partial_q \Phi_\mathbf{p}$ (3.116), $\partial_{r_j} \Phi_\mathbf{p}$ (3.117), and $r_{j,t}$ (3.149),

$$\begin{aligned} \left(\partial_q \Phi_\mathbf{p}\right) q_{,t} &= \left(\Psi_q^{(0)} + \sum_{j=1}^{2} r_j \Psi_{q,q}^{(j)}\right) q_{,t}, && (3.199)\\ \left(\partial_{r_j} \Phi_\mathbf{p}\right) r_{j,t} &= \left(\Psi_q^{(j)} + \partial_{r_j}\left(R_\mathbf{p}^{(2)} + R_\mathbf{p}^{(3)}\right)\right)\left(\lambda_j r_j + \left(r_{j,t} - \lambda_j r_j\right)\right). && (3.200) \end{aligned}$$

Therefore the fourth term on the right-hand side of (3.193), which we denote by s_4, satisfies

$$\begin{aligned} s_4 &\equiv \left\|\left(L_q R_{(q,\mathbf{r})} - \Upsilon_{(q,\mathbf{r})}\mathbf{p}_{,t}\right) - \left(L_b R_{(b,\mathbf{r})} - \Upsilon_{(b,\mathbf{r})}\mathbf{p}_{,t}\right)\right\|_{H^1}, \\ &\leq 2c_\Psi^2\left(T^{(r)}\right)^2 T^{(q)} + 2c_\Psi T^{(q)} |q_{,t}| + 2c_\Psi T^{(q)} |r_{,t} - \lambda r|. && (3.201) \end{aligned}$$

Applying the above estimates to $\mathcal{S}_b$ (3.193), $q_{,t}$ (3.148), and $r_{,t}$ (3.149) then yields

$$\begin{aligned} \|\mathcal{S}_b(\mathbf{p}, W)\|_{H^1} &\leq c_\sigma\left(w + \left(T^{(r)}\right)^2 + |q_{,t}| + |r_{,t} - \lambda r|\right) T^{(q)}, && (3.202)\\ |q_{,t}| &\leq c_\sigma\left(\left(T^{(r)}\right)^2 + wT^{(r)} + w^2 + \|\mathcal{S}_b(\mathbf{p}, W)\|_{H^1}\right), && (3.203)\\ |r_{,t} - r\lambda| &\leq c_\sigma\left(\left(T^{(r)}\right)^2 + wT^{(r)} + w^2 + \|\mathcal{S}_b(\mathbf{p}, W)\|_{H^1}\right), && (3.204) \end{aligned}$$

where $c_\sigma \equiv \max\left\{2c_\Psi^3 + 2c_\Psi^2, 2c_\Psi, 8c_\Psi^3, 8c_\Psi^2\right\}$. We further combines these estimates as

$$\|\mathcal{S}_b(\mathbf{p}, W)\|_{H^1} \leq \frac{c_\sigma\left(2w + (1 + 2c_\sigma)\left(T^{(r)}\right)^2\right)T^{(q)}}{1 - 2c_\sigma^2 T^{(q)}}, \tag{3.205}$$

$$|q_{,t}| \leq \frac{c_\sigma\left(2\left(T^{(r)}\right)^2 + wT^{(r)} + w^2 + c_\sigma w T^{(q)}\right)}{1 - 2c_\sigma^2 T^{(q)}}, \tag{3.206}$$

where we have used the fact that $|q_{,t}|$ and $|r_{,t} - r\lambda|$ obey the same estimate.

Control of $\mathcal{S}_b$ and $q_{,t}$ by the estimates (3.205,3.206) is lost when $T^{(q)}$ grows too large. We therefore impose the constraint

$$T^{(q)} \leq \frac{1}{4c_\sigma^2} \tag{3.207}$$

which restricts the possible size of Δt, thereby bounding $\mathcal{S}_b$ and $q_{,t}$ as

$$\|\mathcal{S}_b(\mathbf{p}, W)\|_{H^1} \leq c_{qs}\left(\left(T^{(r)}\right)^2 + w\right)T^{(q)}, \tag{3.208}$$

$$|q_{,t}| \leq c_{qs}\left(\left(T^{(r)}\right)^2 + wT^{(r)} + w^2 + wT^{(q)}\right), \tag{3.209}$$

where $c_{qs} \equiv \max\left\{4c_\sigma, 2c_\sigma(1 + 2c_\sigma)\right\}$. We further impose the condition

$$w < c_\delta\left(T^{(r)}\right)^2 \tag{3.210}$$

which restricts the initial size of the residual to obtain

$$\|\mathcal{S}_b(\mathbf{p}, W)\|_{H^1} \leq c_{qs}(1 + c_\delta)\left(T^{(r)}\right)^2 T^{(q)}, \tag{3.211}$$

$$|q_{,t}| \leq 2c_{qs}\left(T^{(r)}\right)^2. \tag{3.212}$$

$T^{(q)}$ by its definition (3.174) then satisfies the estimate,

$$T^{(q)} \leq \int_{t_b}^{t_d} |q'(\tau)|\, d\tau, \tag{3.213}$$

$$\leq 2c_{qs}\left(T^{(r)}\right)^2 \Delta t, \tag{3.214}$$

which may be applied to (3.211) to obtain

$$\|\mathcal{S}_b(\mathbf{p}, W)\|_{H^1} \leq 2c_{qs}^2(1 + c_\delta)\left(T^{(r)}\right)^4 \Delta t. \tag{3.215}$$

Lastly, a bound on $\widetilde{\mathcal{S}}_b(\mathbf{p}, W)$ is obtained as follows. Applying the triangle inequality to (3.154) yields

$$\left\|\widetilde{\mathcal{S}}_b(\mathbf{p}, W)\right\|_{H^1} \leq \left\|\pi_b^-\right\|_{*,H^1}\left(\left\|\mathcal{S}_b(\mathbf{p}, W)\right\|_{H^1} + \left\|\Upsilon_{(b,\mathbf{r})}\widetilde{\mathbf{s}}_b(\mathbf{p}, W)\right\|_{H^1}\right) \tag{3.216}$$

and subsequent application of the above estimates yields

$$\left\|\widetilde{\mathcal{S}}_b(q, W)\right\| \leq c_{\widetilde{S}}\left(T^{(r)}\right)^4 \Delta t, \tag{3.217}$$

where $c_{\widetilde{S}} \equiv 2c_{qs}^2(1+c_\delta)\left(1+3c_\Psi^2\right)\max\left\{1, 2c_\Psi^2\right\}$. Moreover,

$$\begin{aligned} \left|[\widetilde{\mathbf{s}}_b(\mathbf{p}, W)]_j\right| &= \left|\left\langle \mathcal{S}_b(\mathbf{p}, W) \middle| \Psi_b^{(j)\dagger}\right\rangle\right|, && (3.218) \\ &\leq c_\Psi c_{\widetilde{S}}\left(T^{(r)}\right)^4 \Delta t. && (3.219) \end{aligned}$$

With the bounds (3.181,3.192,3.217) for $\Omega_b(W)$, $\widetilde{\Omega}_b(\mathbf{r}, W)$, $\widetilde{\mathcal{S}}_b(q, W)$ in hand, the "forcing term" G_b (3.157) satisfies the bound

$$g_b(\tau) \leq c_w\left(\left(T^{(r)}\right)^4 + w(\tau)T^{(r)} + \left(T^{(r)}\right)^4 \Delta t\right), \tag{3.220}$$

where $c_w = c_S \max\left\{c_\Omega, 2c_{\widetilde{\Omega}}, c_{\widetilde{S}}\right\}$.

So long as the constraints $T^{(q)} \leq \frac{1}{4c_\sigma^2}$ (3.207) and $w < c_\delta\left(T^{(r)}\right)^2$ (3.210) hold, then the estimates $|q_{,t}| \leq 2c_{qs}\left(T^{(r)}\right)^2$ (3.212) and $T^{(q)} \leq 2c_{qs}\left(T^{(r)}\right)^2 \Delta t$ (3.214) also hold. On the other hand, if the estimate (3.214) holds, then the constraint (3.207) also holds if

$$2c_{qs}\left(T^{(r)}\right)^2 \Delta t \leq \frac{1}{4c_\sigma^2}, \tag{3.221}$$

It is thus *self consistent* to replace the constraint (3.207) with the constraint (3.221), and we do so. Lastly, we rewrite (3.221) as

$$\Delta t \leq c_t\left(T^{(r)}\right)^{-2}, \tag{3.222}$$

where $c_t \equiv \left(8c_\sigma^2 c_{qs}\right)^{-1}$.

Residual Decay Estimates We assume that the constraints (3.210,3.222) hold, and we apply the bound (3.220) for G_b to the estimate (3.177) for W to obtain

$$w(t) \leq c_S e^{-k(t-t_b)} w_b + c_w \int_{t_b}^{t} e^{-k(t-\tau)} \left(w(\tau) T^{(r)} + \left(T^{(r)}\right)^4 (1+\Delta t) \right) d\tau. \tag{3.223}$$

Replacing t with θ, multiplying by $e^{k(\theta-t_b)}$, and taking the supremum over $\theta \in [t_b, t_d)$, (3.223) becomes

$$\begin{aligned} T^{(w)} &\leq c_S w_b \\ &\quad + c_w \sup_{\theta\in[t_b,t_d)} e^{k(\theta-t_b)} \int_{t_b}^{\theta} e^{-k(\theta-\tau)} \left(w(\tau) T^{(r)} + \left(T^{(r)}\right)^4 (1+\Delta t) \right) d\tau, && (3.224) \\ &\leq c_S w_b + c_w \int_{t_b}^{t_d} e^{k(\tau-t_b)} \left(w(\tau) T^{(r)} + \left(T^{(r)}\right)^4 (1+\Delta t) \right) d\tau. && (3.225) \end{aligned}$$

Since

$$\begin{aligned} \int_{t_b}^{t_d} e^{k(\tau-t_b)} w(\tau)\, d\tau &\leq \int_{t_b}^{t_d} T^{(w)} d\tau, && (3.226) \\ &\leq T^{(w)} \Delta t, && (3.227) \end{aligned}$$

and

$$\int_{t_b}^{t_d} e^{k(\tau-t_b)} d\tau = k^{-1} \left(e^{k\Delta t} - 1 \right), \tag{3.228}$$

(3.225) may be rewritten as

$$T^{(w)} \leq c_S w_b + c_w \left(T^{(r)} T^{(w)} \Delta t + \left(T^{(r)}\right)^4 y_2(\Delta t) \right), \tag{3.229}$$

where

$$y_2(t) \equiv k^{-1} \left(e^{kt} - 1 \right) (1+t). \tag{3.230}$$

Note that y_2 is a strictly increasing function. Upon solving for $T^{(w)}$, we obtain

$$T^{(w)} \leq \frac{c_S w_b + c_w \left(T^{(r)}\right)^4 y_2(\Delta t)}{1 - c_w T^{(r)} \Delta t} \tag{3.231}$$

which, by inserting $T^{(w)}$ (3.176), yields the decay estimate

$$w_d \leq \frac{e^{-k\Delta t}\left(c_S w_b + c_w \left(T^{(r)}\right)^4 y_2\left(\Delta t\right)\right)}{1 - c_w T^{(r)} \Delta t}. \tag{3.232}$$

Because control of w_d by (3.232) is lost when $\Delta t = c_w^{-1}\left(T^{(r)}\right)^{-1}$, we impose the additional constraint

$$\Delta t \leq \frac{1}{2}\left(c_w T^{(r)}\right)^{-1} \tag{3.233}$$

so that the denominator in (3.232) is uniformly bounded away from zero. It is clear that this constraint is always more exigent than $\Delta t \leq c_t \left(T^{(r)}\right)^{-2}$ (3.222) for $T^{(r)}$ sufficiently small, and so we hereby replace that constraint with this one. With this constraint, (3.232) becomes

$$w_d \leq 2e^{-k\Delta t}\left(c_S w_b + c_w \left(T^{(r)}\right)^4 y_2\left(\Delta t\right)\right). \tag{3.234}$$

Moreover, by replacing Δt with $t - t_b$ and substituting y_2 (3.230), we obtain the residual decay estimate for general time t:

$$\begin{aligned} w\left(t\right) &\leq 2e^{-k(t-t_b)}\left(c_S w_b + c_w \left(T^{(r)}\right)^4 y_2\left(t - t_b\right)\right), & (3.235)\\ &= 2e^{-k(t-t_b)}\left(c_S w_b - c_w k^{-1}\left(T^{(r)}\right)^4 - c_w k^{-1}\left(T^{(r)}\right)^4 (t - t_b)\right) \\ &\quad +2c_w k^{-1}\left[\left(T^{(r)}\right)^4 + \left(T^{(r)}\right)^4 (t - t_b)\right]. & (3.236)\end{aligned}$$

3.6 The Reanchor Method

The evolution equations (3.148,3.149,2.73) for $\mathbf{p}$ and W remain valid so long as the secular term $\mathcal{S}_b$ (3.19) remains sufficiently small. When this is no longer true, we rechoose the anchor point to remove this secular growth. Moreover, this new anchor point b^* is chosen so that the new residual lies in $X_{b^*}^{-}$. Theorem 18 will show that such a choice exists and is unique, provided that q is sufficiently close to the old anchor point b and provided that W is sufficiently small. The price of reanchoring is jump discontinuities in both q and W wherein W

could in principle *increase*, yet we will show that the residual decay estimate (3.234) controls this possible growth provided that Δt, the length of time in which each anchor point is used, is suitably long. The residual decay estimate (3.234) is the key estimate which allows us to determine such a Δt so that secular growth is controlled and removed. Lastly, we will show that after an initial transient stage wherein the residual decays, the residual will remain small for all time and so our solution remains close to the manifold for all time.

The Reanchor Method Again, fix the anchor point at b on the time interval $[t_b, t_d)$ and denote the final value of q on $[t_b, t_d]$ by d. Let us introduce the term

$$\widetilde{Q}_{(q,\mathbf{r})} = Q_q + R^{(1)}_{(q,\mathbf{r})} \tag{3.237}$$

and rewrite the decomposition (3.63) for Z as

$$Z = \widetilde{Q}_{(q,\mathbf{r})} + \widetilde{W}. \tag{3.238}$$

When reanchoring, Z may be decomposed as either $Z = \widetilde{Q}_{(d,\mathbf{r})} + \widetilde{W}_d$ with respect to the old anchor point b, where $\widetilde{W}_d \in X_b^-$, or as $Z = \widetilde{Q}_{(b^*,\mathbf{r})} + \widetilde{W}_{b^*}$ with respect to the new anchor point b^*, where b^* is to be determined so that $\widetilde{W}_{b^*} \in X_{b^*}^-$. In particular, note that $\widetilde{W}_d \in X_b^{(0)\perp}$ and $\widetilde{W}_{b^*} \in X_{b^*}^{(0)\perp}$. Reanchoring introduces a jump discontinuity in both q and W wherein q jumps from d to b^* and $\widetilde{W}$ jumps from $\widetilde{W}_d$ to $\widetilde{W}_{b^*}$. Equating these two decompositions and solving for $\widetilde{W}_{b^*}$, we obtain

$$\widetilde{W}_{b^*} = \widetilde{W}_d + \widetilde{Q}_{(d,\mathbf{r})} - \widetilde{Q}_{(b^*,\mathbf{r})}. \tag{3.239}$$

Since $\widetilde{W}_{b^*} \in X_{b^*}^{(0)\perp}$, then

$$\left\langle \widetilde{W}_d + \widetilde{Q}_{(d,\mathbf{r})} - \widetilde{Q}_{(b^*,\mathbf{r})} \middle| \Psi_{b^*}^{(0)\dagger} \right\rangle = 0. \tag{3.240}$$

Given d and $\widetilde{W}_d$, the following theorem shows that there exists a unique b^* such that (3.240) is satisfied. In addition, this theorem gives an estimate on the jump discontinuity in q when reanchoring.

Theorem 18 *Express* $\widetilde{W}_d = \widetilde{w}_d \Xi_b$ *for some scalar* $\widetilde{w}_d \geq 0$ *and function* $\Xi_b \in X_b^{(0)\perp}$ *satisfying* $\|\Xi_b\|_{H^1} = 1$. *For* $\widetilde{w}_d$ *sufficiently small, there exists a unique smooth function* $\mathcal{H} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that, by choosing* $b^* = d + \mathcal{H}(\widetilde{w}_d)$, *(3.240) is satisfied. Moreover, the estimate*

$$|d - b^*| \leq c_r \widetilde{w}_d |d - b| \tag{3.241}$$

holds for some constant c_r.

Proof. The equation (3.240) is equivalent to $\Gamma = 0$, where

$$\Gamma(\widetilde{w}_d, b^*) \equiv \left\langle \widetilde{w}_d \Xi_b + \widetilde{Q}_{(d,\mathrm{r})} - \widetilde{Q}_{(b^*,\mathrm{r})} \middle| \Psi_{b^*}^{(0)\dagger} \right\rangle. \tag{3.242}$$

As $\partial_{b^*} \widetilde{Q}_{(b^*,\mathrm{r})} = \Psi_{b^*}^{(0)} + \sum_{j=1}^{2} r_j \Psi_{b^*,b^*}^{(j)}$ and $\left\langle \Xi_b \middle| \Psi_b^{(0)\dagger} \right\rangle = 0$, the partial derivatives of Γ are

$$\begin{aligned}
\Gamma_{,b^*}(\widetilde{w}_d, b^*) &= \left\langle -\partial_{b^*} \widetilde{Q}_{(b^*,\mathrm{r})} \middle| \Psi_{b^*}^{(0)\dagger} \right\rangle + \left\langle \widetilde{w}_d \Xi_b + \widetilde{Q}_{(d,\mathrm{r})} - \widetilde{Q}_{(b^*,\mathrm{r})} \middle| \Psi_{b^*,b^*}^{(0)\dagger} \right\rangle, & (3.243)\\
&= -1 - \sum_{j=1}^{2} r_j \left\langle \Psi_{b^*,b^*}^{(j)} \middle| \Psi_{b^*}^{(0)\dagger} \right\rangle + \left\langle \widetilde{w}_d \Xi_b + \widetilde{Q}_{(d,\mathrm{r})} - \widetilde{Q}_{(b^*,\mathrm{r})} \middle| \Psi_{b^*,b^*}^{(0)\dagger} \right\rangle & (3.244)\\
\Gamma_{,\widetilde{w}_d}(\widetilde{w}_d, b^*) &= \left\langle \Xi_b \middle| \Psi_{b^*}^{(0)\dagger} \right\rangle, & (3.245)\\
&= \left\langle \Xi_b \middle| \Psi_{b^*}^{(0)\dagger} - \Psi_b^{(0)\dagger} \right\rangle. & (3.246)
\end{aligned}$$

Γ has a root at $(\widetilde{w}_d, b^*) = (0, d)$ and $\Gamma_{,b^*}(0, d) = -1 + O(r)$ which is $O(1)$, so the implicit function theorem guarantees the existence of a smooth function $\mathcal{H}$ such that $b^* = d + \mathcal{H}(\widetilde{w}_d)$. Moreover, since $\Gamma_{,\widetilde{w}_d}(0, d)$ is $O(d-b)$, the implicit function theorem implies that $|\mathcal{H}'(0)|$ is $O(d-b)$ also. The estimate (3.241) then follows from the Mean Value Theorem. ■

Estimates on the Reanchor Jump Discontinuities Since N_2 (3.105) and N_3 (3.106) are bilinear and trilinear with respect to their arguments respectively, they are also continuous maps from $H^1 \times H^1$ and $H^1 \times H^1 \times H^1$ to H^1 respectively. Moreover, L_b (3.64) and $\Psi_b^{(k)}$ (3.68,3.71) are continuous with respect to b, and so it follows that $R_b^{(jk)}$ and $R_b^{(jkl)}$ are continuous with respect to b as well. This continuity property plays a crucial role in the computation of the reanchor jump estimate, as we shall see forthwith.

When reanchoring, Z may be decomposed as either

$$Z = Q_d + R^{(1)}_{(d,\mathbf{r})} + R^{(2)}_{(b,\mathbf{r})} + R^{(3)}_{(b,\mathbf{r})} + W_d \tag{3.247}$$

with respect to the old anchor point b, where $W_d \in X_b^-$, or as

$$Z = Q_{b^*} + R^{(1)}_{(b^*,\mathbf{r})} + R^{(2)}_{(b^*,\mathbf{r})} + R^{(3)}_{(b^*,\mathbf{r})} + W_{b^*} \tag{3.248}$$

with respect to the new anchor point b^*, where $W_{b^*} \in X_{b^*}^-$. Equating these two decompositions and solving for W_{b^*} yields

$$W_{b^*} = W_d + \left(Q_d - Q_{b^*} + R^{(1)}_{(d,\mathbf{r})} - R^{(1)}_{(b^*,\mathbf{r})}\right) + \left(R^{(2)}_{(b,\mathbf{r})} - R^{(2)}_{(b^*,\mathbf{r})} + R^{(3)}_{(b,\mathbf{r})} - R^{(3)}_{(b^*,\mathbf{r})}\right). \tag{3.249}$$

Apply the triangle inequality to obtain

$$\begin{aligned} w_{b^*} \;\; \leq \;\; & w_d + \|Q_d - Q_{b^*}\|_{H^1} + \left\|R^{(1)}_{(d,\mathbf{r})} - R^{(1)}_{(b^*,\mathbf{r})}\right\|_{H^1} \\ & + \left\|R^{(2)}_{(b,\mathbf{r})} - R^{(2)}_{(b^*,\mathbf{r})}\right\|_{H^1} + \left\|R^{(3)}_{(b,\mathbf{r})} - R^{(3)}_{(b^*,\mathbf{r})}\right\|_{H^1}. \end{aligned} \tag{3.250}$$

Since Q_q, $R^{(1)}_{(q,\mathbf{r})}$, $R^{(2)}_{(b,\mathbf{r})}$, $R^{(3)}_{(b,\mathbf{r})}$ are continuous with respect to their arguments,

$$w_{b^*} \leq w_d + 2c_\Psi \left|d - b^*\right| + 2c_\Psi \left|r\right|^2 \left|b - b^*\right|, \tag{3.251}$$

and since $|b - b^*| = |b - d + d - b^*| \leq |d - b| + |d - b^*|$,

$$w_{b^*} \leq w_d + 4c_\Psi \left|d - b^*\right| + 2c_\Psi \left|r\right|^2 \left|d - b\right|. \tag{3.252}$$

Finally, inserting the estimates (3.241) for $|d - b^*|$ yields

$$w_{b^*} \leq w_d + \left(4c_\Psi c_\tau \widetilde{w}_d + 2c_\Psi \left|r\right|^2\right) \left|d - b\right| \tag{3.253}$$

which, by inserting the estimate (3.214) for $|d-b|$ and using the fact that $\widetilde{w}_d = \left\| R_{\mathbf{p}}^{(2)} + R_{\mathbf{p}}^{(3)} \right\|_{H^1}$, becomes

$$w_{b^*} \le w_d + c_J \left(T^{(r)}\right)^4 \Delta t, \tag{3.254}$$

where $c_J \equiv \left(8c_\Psi^2 c_r + 2c_\Psi\right) 2c_{qs}$. This estimate shows that w could in principle *increase* when reanchoring, yet we will show in the next section that the residual decay estimate (3.234) controls this possible growth provided that Δt, the length of time in which each anchor point is used, is suitably long.

<u>The Iterations</u> We now investigate two states in which, for some m to be determined, either $w_b \in \left(m\left(T^{(r)}\right)^4, c_\delta \left(T^{(r)}\right)^2\right)$ or $w_b \in \left[0, m\left(T^{(r)}\right)^4\right]$ respectively. These states are called the initial transient and asymptotic states. We will show that w decreases on the whole in the initial transient state in the sense that $w_{b^*} < w_b$, while w remains small in the asymptotic state. Moreover, we will show that we can take $\Delta t = k^{-1} \ln\left(h_m^{-1}\left(4c_S + 4c_w + 2kc_J\right) + 1\right)$ and $m = y_2(\Delta t)$ for some $h_m < 1$.

Inserting the residual decay estimate (3.234) into the reanchor jump estimate (3.254) yields

$$w_{b^*} \le 2e^{-k\Delta t}\left(c_S w_b + c_w \left(T^{(r)}\right)^4 y_2(\Delta t)\right) + c_J \left(T^{(r)}\right)^4 \Delta t \tag{3.255}$$

This inequality holds so long as the two constraints $w_b < c_\delta \left(T^{(r)}\right)^2$ (3.210) and $\Delta t \le \frac{1}{2}\left(c_w T^{(r)}\right)^{-1}$ (3.233) holds. To continue using (3.255), we must rechoose our anchor point before either of these constraints fail.

<u>Initial Transient State</u> In the initial transient state wherein $w_b \in \left(m\left(T^{(r)}\right)^4, c_\delta\left(T^{(r)}\right)^2\right)$, we further impose the constraint

$$\left(T^{(r)}\right)^4 y_2(\Delta t) \le w_b \tag{3.256}$$

on Δt and apply it and substitute the expression (3.230)

$$y_2(\Delta t) \equiv k^{-1}\left(e^{k\Delta t} - 1\right)(1 + \Delta t) \tag{3.257}$$

into (3.255) to obtain

$$\begin{aligned} w_{b^*} &= 2e^{-k\Delta t}\left(c_S w_b + c_w \left(T^{(r)}\right)^4 y_2(\Delta t)\right) + \frac{c_J \left(T^{(r)}\right)^4 \Delta t}{w_b} w_b, && (3.258)\\ &\leq 2e^{-k\Delta t}(c_S + c_w) w_b + \frac{c_J \Delta t}{y_2(\Delta t)} w_b, && (3.259)\\ &= h(\Delta t) w_b, && (3.260) \end{aligned}$$

where

$$h(x) \equiv \frac{2(c_S + c_w)}{e^{kx}} + \frac{k c_J x}{(e^{kx} - 1)(1 + x)}. \tag{3.261}$$

We now *choose* the fixed length of time Δt in which the current anchor point b is used such that $h(\Delta t) \approx h_m$ for some $h_m < 1$. The choice

$$\Delta t = k^{-1} \ln\left(h_m^{-1}(4c_S + 4c_w + 2kc_J) + 1\right) \tag{3.262}$$

yields

$$\begin{aligned} h_m &\equiv h(\Delta t), && (3.263)\\ &= \frac{2(c_S + c_w) h_m}{4(c_S + c_w) + 2kc_J + h_m} + \frac{k c_J h_m}{4(c_S + c_w) + 2kc_J}\left(\frac{\Delta t}{1 + \Delta t}\right), && (3.264)\\ &< \frac{h_m}{2} + \frac{h_m}{2}, && (3.265)\\ &= h_m, && (3.266) \end{aligned}$$

and (3.260) thus becomes

$$w_{b^*} \leq h_m w_b. \tag{3.267}$$

We further choose

$$m = y_2(\Delta t) \tag{3.268}$$

so that the constraint (3.256) is automatically satisfied by virtue of the fact that $w_b \in \left(m\left(T^{(r)}\right)^4, c_\delta\left(T^{(r)}\right)^2\right)$. It remains to show that our choice of Δt also satisfies the constraint (3.233), but this is easily satisfied if we demand that $T^{(r)}$ satisfy

$$T^{(r)} \leq \frac{1}{2c_w \Delta t}. \tag{3.269}$$

For what initial conditions on r and w does (3.269) hold? We again transform the evolution equation (3.166) for r into Poincaré normal form, but this time we retain the higher order terms $\widetilde{\omega}_b + \widetilde{\mathrm{s}}_b$ since we're working in the initial transient state:

$$v_{,t} = \lambda v + \eta\left|v\right|^2 v + \widetilde{\omega}_b + \widetilde{\mathrm{s}}_b + O\left(v^4\right). \tag{3.270}$$

Express $v = |v|\, e^{i\theta}$ and gather real parts.

$$|v|_{,t} = (\mathrm{Re}\,\lambda)\,|v| + (\mathrm{Re}\,\eta)\,|v|^3 + \mathrm{Re}\,\widetilde{\omega}_b + \mathrm{Re}\,\widetilde{\mathrm{s}}_b + O\left(|v|^4\right). \tag{3.271}$$

Applying the bounds (3.142) and (3.219) for $\widetilde{\omega}_b(\mathbf{r}, W)$ and $\widetilde{\mathrm{s}}_b(\mathbf{p}, W)$ then yields

$$|v|_{,t} = (\mathrm{Re}\,\lambda)\,|v| + (\mathrm{Re}\,\eta)\,|v|^3 + 4c_\Psi^3 w T^{(r)} + 4c_\Psi^2 w^2 + c_\Psi c_{\widetilde{S}}\left(T^{(r)}\right)^4 \Delta t + O\left(|v|^4\right). \tag{3.272}$$

Finally, applying the constraints $w < c_\delta\left(T^{(r)}\right)^2$ (3.210) and the choice (3.222) for Δt, we obtain

$$|v|_{,t} = (\mathrm{Re}\,\lambda)\,|v| + (\mathrm{Re}\,\eta)\,|v|^3 + 4c_\delta c_\Psi^3\left(T^{(r)}\right)^3 + O\left(|v|^4\right). \tag{3.273}$$

Choose c_δ is sufficiently small such that $\mathrm{Re}\,\eta + 4c_\delta c_\Psi^3 < 0$. Then $|r|$ will always satisfy the bound

$$|r| \leq \sqrt{-\frac{\mathrm{Re}\,\lambda}{\mathrm{Re}\,\eta + 4c_\delta c_\Psi^3}} \tag{3.274}$$

if $|r|$ is initially small enough, while $|r|$ will eventually satisfy this bound if $|r|$ is initially large provided the constraint (3.269) initially holds. Note that, since w decays, $|r|$ actually approaches $\sqrt{-\operatorname{Re}\lambda/\operatorname{Re}\eta}$. In any case, for $\operatorname{Re}\lambda$ and c_δ sufficiently small, the constraint (3.269) always holds.

We have thus proven that, if $w_b \in \left(m\left(T^{(r)}\right)^4, c_\delta\left(T^{(r)}\right)^2\right)$ and Δt is chosen as in (3.262), the residual decays on the whole in the sense that $w_{b^*} \leq h_m w_b$. Moreover, we have determined the appropriate initial conditions under which our analysis holds. In particular, the solution must be close enough to the manifold and the oscillation amplitude must be small enough such that the residual satisfies $w(0) < c_\delta\left(T^{(r)}\right)^2$ (3.210) and such that the condition (3.269) holds.

The residual will continue to decay until $w_b \in \left[0, m\left(T^{(r)}\right)^4\right]$ at which point the system enters the asymptotic state.

<u>Asymptotic State</u> In the asymptotic state wherein $w_b \in \left[0, m\left(T^{(r)}\right)^4\right]$, we choose Δt as in (3.262):

$$\Delta t = k^{-1}\ln\left(h_m^{-1}\left(4c_{\mathrm{S}} + 4c_w + 2kc_{\mathrm{J}}\right) + 1\right). \tag{3.275}$$

Since $y_2(\Delta t) = m$ and $w_b \leq m\left(T^{(r)}\right)^4$, (3.255) then becomes

$$\begin{aligned} w_{b^*} &\leq 2e^{-k\Delta t}\left(c_{\mathrm{S}} + c_w\right) m\left(T^{(r)}\right)^4 + \frac{c_{\mathrm{J}}\Delta t}{m} m\left(T^{(r)}\right)^4, & (3.276)\\ &= h(\Delta t)\left(m\left(T^{(r)}\right)^4\right), & (3.277)\end{aligned}$$

where h is given by (3.261) as before. Since $h(\Delta t) = h_m$, we have

$$w_{b^*} \leq m\left(T^{(r)}\right)^4. \tag{3.278}$$

We have thus proven that, if $w_b \in \left[0, m\left(T^{(r)}\right)^4\right]$ and Δt is chosen as in (3.262), then

$w_{b^*} \in \left[0, m\left(T^{(r)}\right)^4\right]$ also. In conjunction with the decay estimate (3.234), this shows that

$$w(t) \le 2(c_S + c_w)\left(m\left(T^{(r)}\right)^4\right) \tag{3.279}$$

for all time t in the asymptotic state.

3.7 Conclusion

We conclude our thesis with the following theorem which summarizes our results.

Theorem 19 *Consider the vectorized parametrically-forced nonlinear Schrödinger equation (PNLS):*

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_{,t} = \begin{pmatrix} 0 & -\partial_x^2 + \mu - |Z|^2 \\ \partial_x^2 - 1 + |Z|^2 & -2\nu \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.280}$$

Provided that $\operatorname{Re}\lambda$ *and* w *are sufficiently small, the PNLS posesses the solution*

$$Z(t) = \Phi_{\mathbf{p}(t)} + W(t) \tag{3.281}$$

for each fixed anchor point b*, where*

$$\Phi_{\mathbf{p}} = Q_q + \sum_{j=1}^{2} r_j \Psi_q^{(j)} + \sum_{j,k=1}^{2} r_j r_k R_b^{(jk)} + \sum_{j,k,l=1}^{2} r_j r_k r_l R_b^{(jkl)}. \tag{3.282}$$

Here, Q_q *is the stationary pulse solution,* $\Psi_q^{(j)}$ *are the linearized eigenfunctions, and* $R_b^{(jk)}$ *and* $R_b^{(jkl)}$ *are the anchor-point dependent correction terms. The manifold parameters satisfy the evolution equations (in the asymptotic state)*

$$q_{,t} = \sum_{k,l=1}^{2} \gamma_{kl0} r_k r_l + O\left(r^3\right), \tag{3.283}$$

$$r_{j,t} = \lambda_j r_j + \sum_{k,l=1}^{2} \gamma_{klj} r_k r_l + \sum_{k,l,m=1}^{2} \gamma_{klmj} r_k r_l r_m + O\left(r^4\right), \tag{3.284}$$

where λ_j *are the Hopf eigenvalues, and* γ_{jkl} *and* γ_{jklm} *are anchor-point independent numbers. The residual term* W *satisfies the condition*

$$W \in X_b^- \tag{3.285}$$

and the bound

$$w(t) \leq 2e^{-k(t-t_b)} \left(c_S w_b + c_w \left(T^{(r)}\right)^4 y_2(t - t_b)\right) \tag{3.286}$$

with $y_2(t) \equiv k^{-1}\left(e^{kt} - 1\right)(1 + t)$. *Moreover, one can use each anchor point for an* $O(1)$ *time period and rechoose the anchor point thereafter according to Theorem 18 such that, in the initial transient state,* $w_{b^*} < h_m w_b$ *for some fixed constant* $h_m < 1$ *and, in the asymptotic state,* $w \leq m\delta$ *for some* $O(1)$ *constant* m.

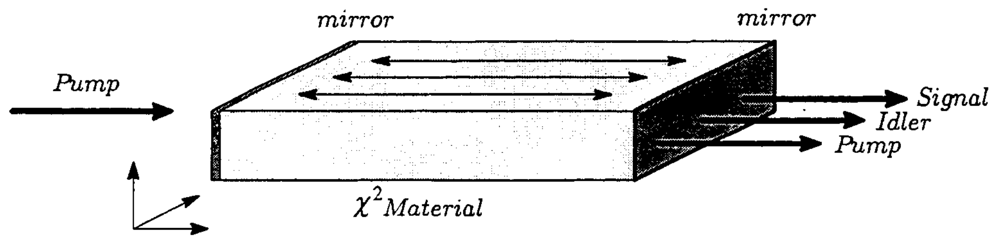

Figure 3.14: Schematic diagram of an OPO (Promislow).

APPENDIX

The Optical Parametric Oscillator An optical parametric oscillator (OPO) is a nonlinear optical device consisting of a non-centrosymmetric medium and a Fabry Perot cavity. The device converts sufficiently powerful pump waves into two tuneable coherent waves named the signal and idler waves. Like conventional lasers, gain is produced at the signal and idler frequencies, and this gain is combined with feedback to produce coherent radiation. Unlike conventional lasers, no population inversion takes place, so OPOs are capable of producing frequencies not easily producible by conventional lasers. In particular, OPOs can be tuned to hit band-gap frequencies with high precision. First demonstrated by Giordmaine & Miller in 1965, OPOs became outdated due to the lack of efficient resistant crystals and the development of dye lasers. In recent years, however, improvement and availability of high quality crystals have renewed interest in OPOs as the phase-matching characteristics and broad wavelength coverage of these new crystals have given the OPO a much wider frequency tuning range. Moreover, no degradation in the active medium is exhibited by the new OPOs, raising the possibility of maintenance-free operation.

In this section, we derive the PNLS as the good cavity, large pump detuning limit of the OPO. This derivation is from [11] and is summarized here for completeness and convenience.

We consider the degenerate OPO near the single-mode resonances at the fundamental and second harmonic of a planar cavity in the good cavity limit. The mean-field model for dimensionless signal and pump field envelopes are respectively denoted by u and v and are governed by the equations

$$u_{,t} = \frac{1}{2} i u_{,xx} + \overline{u} v - (1 + i\Delta_1) u, \tag{3.287}$$

$$v_{,t} = \frac{1}{2} i \rho v_{,xx} - u^2 - (\alpha + i\Delta_2) v + s, \tag{3.288}$$

where Δ_1 and Δ_2 are the cavity detuning parameters, ρ is the diffraction ratio between the signal and pump fields, α is the pump-to-signal loss ratio, and s represents the pumping term. See [18, 19]. The OPO exhibits subcritical bistable behaviour for both front and solitary wave solutions, but we investigate this behaviour for only solitary wave solutions in the large pump-detuning regime for which $|\Delta_2| \gg 1$. Rewriting (3.288) as

$$\Delta_2^{-1} \left(v_{,t} - \frac{1}{2} i \rho v_{,xx} + \alpha v \right) = \Delta_2^{-1} \left(s - u^2 \right) - i v, \tag{3.289}$$

we assume $s = O(|\Delta_2|)$, $u = O\left(\sqrt{|\Delta_2|}\right)$, and $v = O(1)$, as well as neglect the left-hand side terms to obtain

$$i \Delta_2 v = s - u^2. \tag{3.290}$$

For consistency with (3.287,3.288), we also require that $|\Delta_1| \gg \alpha / |\Delta_2|$. Inserting (3.290) into (3.287) yields the PNLS equation

$$i u_{,t} + \frac{1}{2} u_{,xx} + \Delta_2^{-1} |u|^2 u + (i - \Delta_1) u - \Delta_2^{-1} s \overline{u} = 0. \tag{3.291}$$

Rescaling as $u = \sqrt{|\Delta_2|} \phi$, $a = \Delta_1 > 0$, and $\gamma = \Delta_2^{-1} s$, we obtain the focusing PNLS

$$i \phi_{,t} + \frac{1}{2} \phi_{,xx} + |\phi|^2 \phi + (i - a) \phi - \gamma \overline{\phi} = 0, \tag{3.292}$$

where γ is the pump strength and a is the detuning parameter.

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