A FRAMEWORK FOR ENQUIRY INTO MATHEMATICAL KNOWLEDGE MANAGEMENT

by

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Abstract

The dynamics of knowledge creation, representation, and sharing by members of a community are intertwined with the nature of the community itself and are affected by such diverse influences as notions of authority, intellectual property, epistemology, and technological advancement. While influenced by informal and formal decision-making by individuals and groups from both within and outside the community, the development of domain specific knowledge management environments displays some of the characteristics of *self-organizing systems*. This thesis seeks to examine the *mathematical knowledge management* environment and the formative influences on its systems and their protocols and artifacts.

In all domains, knowledge management environments are currently bridging a divide defined by well-established typographic paradigms on one side and emerging digital paradigms on the other. In traversing this boundary, mathematical knowledge management systems are incorporating digital technology in knowledge creation, representation, sharing, and archiving. The emergent nature of the digital era presents the possibility that decisions made today concerning the allocation of mathematical knowledge management resources will have longstanding effects. At stake are tested and valued systems of knowledge verification and distribution as well as the economic models and community structures that support them. A central objective of this thesis will be the presentation of a framework for the discussion and analyses of trends in mathematical knowledge management.

Each society has its regime of truth, its "general politics" of truth: that is, the types of discourse which it accepts and makes function as true; the mechanisms and instances which enable one to distinguish true and false statements, the means by which each is sanctioned; the techniques and procedures accorded value in the acquisition of truth; the status of those who are charged with saying what counts as truth.

Michel Foucault

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Mathematical Knowledge

1.1 Introduction

Henri Lebesgue once remarked that "a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy".[1] (He went on to add that this was "an opinion, moreover, which has been expressed by many philosophers".[2]) The idea that mathematicians can *do* mathematics without a precise philosophical understanding of what they are doing is by observation, mercifully true. Unfortunately, while a neglect of philosophical issues does not impede mathematical discussion, discussion *about* mathematics tends to quickly become embroiled in philosophy. This is certainly the case with any discussion about Mathematical Knowledge Management (henceforth MKM) which can't get too far without confronting the question of exactly what is being managed. This question, the question of what is mathematical knowledge, has been taken up by each of the various schools of mathematical philosophy. With a view to establishing philosophical trends and tensions that resonate within MKM itself, this chapter briefly examines the major schools of the nineteenth and twentieth centuries and looks at the emerging state of the field in the early part of the twenty-first century. The classification presented is derived from that of Hersh and, at it's coarsest level, divides the field according to absolutist *vs.* humanist perspectives.[3]

[1] Freeman Dyson, "Mathematics in the Physical Sciences", *Scientific American* 211, no. 9 (1964): 130.
[2] ibid.
[3] Reuben Hersh, *What is Mathematics Really?*, (New York: Oxford University Press, 1997), 137-181.

1.2 Absolutism

In the pure mathematics we contemplate absolute truths which existed in the divine mind before the morning stars sang together, and which will continue to exist there when the last of their radiant host shall have fallen from heaven.[4]
Edward Everett (1794 - 1865)

A distinguished nineteenth century American statesman, clergyman, and one time president of Harvard, Edward Everett, to whom the above quote is attributed, was not a mathematician. His words however, capture perfectly a fundamental opinion that many people, and indeed many mathematicians, hold of mathematics. The idea of mathematics as being an investigation of absolute mathematical truth motivates most of the commonly acknowledged schools of mathematical philosophy and is characterized by a view of mathematical knowledge as being fixed, immutable, and most appropriately expressed in the form of deductive proof. Based on the general approach towards mathematical proof that they embrace, absolutist schools can be divided into two categories: the *Foundationist* and the *Platonist.*

1.2.1 Foundationism

Foundationism encompasses the schools of logicism, formalism, and constructivism and is distinguished by an attempt to establish mathematical knowledge in a secure deductive framework. The framework defines a system from which mathematical truths can be generated and against which mathematical truths can be tested.

Logicism

Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build my arithmetic.[5]
Gottlob Frege (1848 - 1925)

[4] Furman University *Mathematical Quotations Server* [online], 2003.

[5] Gottlob Frege, "Letter to Russell, 1902." in *From Frege to Gödel: A Sourcebook in Mathematical Logic, 1879-1931*, ed. Jean van Heijenoort, (Cambridge: Harvard University Press, 1967) 127.

Most famously associated with the work of Gottlob Frege, Bertrand Russell, and Alfred North Whitehead, logicism represents an attempt to provide a foundation for mathematics in classical propositional logic. Logicism was motivated by a recognition of the fundamental similarities between set theory and logic. For example, the logical statement $A \Rightarrow B \Rightarrow C$ becomes, in a set theoretic context, $A \subset B \subset C$. The work of von Neumann and Herbrand, among others, had previously established that the fields of geometry and analysis could be modelled upon arithmetic. If in turn, arithmetic could be derived from set theory, then all known branches of mathematics could ultimately be formulated as models of set theory and consequently, as models of propositional logic. Frege successfully derived arithmetic from set theory; the problem that would ultimately defeat logicism arose with set theory itself and its corresponding predicate logic.

The quote which opens this section is from a letter that Gottlob Fege received from Bertrand Russell in 1902 on the eve of publication of Frege's *Grundgesetze der Arithmetik*.[6] The letter described what is presently referred to as the "Russell Paradox", a paradox that severely weakened the Fregian system. Underlining the "I am a liar" type of circular contradiction which can occur in self-referencing predicates, this paradox asks whether or not the set of all sets that do not contain themselves is a subset of itself. The necessary conclusion that, in Russell's words, "there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves" seriously undermined Frege's attempt to model arithmetic on a predicate logic corresponding to set theory.[7] Attempts to resolve the inconsistencies proved fruitless and the foundationist movement suffered its first casuality.

Formalism

> *...I pursue a significant goal, for I should like to eliminate once and for all the questions regarding the foundations of mathematics, in the form in which they are now posed, by turning every mathematical proposition into a formula that can be concretely exhibited and strictly derived, thus recasting mathematical definitions and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole of science. I believe I can attain this goal*

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[6]Gottlob Frege, "Letter to Russell, 1902.", 127

[7]Bertrand Russell, "Letter to Frege, 1902." in *From Frege to Gödel: A Sourcebook in Mathematical Logic, 1879-1931*, ed. Jean van Heijenoort, (Cambridge: Harvard University Press, 1967) 125.

> *completely with my proof theory.*[8] ***David Hilbert (1862 - 1943)***

During the 1920s, David Hilbert proposed that mathematics could be divided into a *real* part and an *ideal* part. The real part includes all mathematical ideas that could be fully described without resorting to uncountably infinite entities. Number theory and logic are contained within the real part of mathematics. The ideal part includes everything that is not real; geometry and real analysis are contained within the ideal part. *Hilbert's program* was introduced as the proposition that:

1. all branches of mathematics could be formalized and ...
2. the resulting formalization could be proved consistent using only real methods.

The assumption of the existence of consistent formalizations would also fall victim to the Incompleteness Theorem and the formalist attempt at an absolute foundation for mathematics would be abandoned.

Constructivism:

Constructive mathematics is characterized by its rejection of "indirect proof", proof that in its logical formulation depends on the "Law of the Excluded Middle". Tracing its intellectual heritage to the work of Kronecker and others, modern constructivism is most closely associated with L.E.J. Brouwer's *intuitionism* and the work of Errett Bishop respectively.

Intuititionism

> *The subject for which I am asking your attention deals with the foundations of mathematics. To understand the development of the opposing theories existing in this field one must first gain a clear understanding of the concept "science"; for it is as a part of science that mathematics originally took its place in human thought.*[9] ***L. E. J. Brouwer (1881 - 1966)***

Brouwer based the intuitionist thesis on the idea that mathematics is comprised of human thought consructs that stem from an inner sense, or intuition, of the nature of time.

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[8]David Hilbert, "The Foundations of Mathematics, 1927." in *From Frege to Gödel: A Sourcebook in Mathematical Logic, 1879-1931*, ed. Jean van Heijenoort, (Cambridge: Harvard University Press, 1967) 464.

[9]L.E.J. Brouwer, "Intuitionism and Formalism", *Bulletin of the American Mathematical Society (New Series)*: 37, no. 9 (2000): 55.

> Mathematics arises when the subject of two-ness, which results from the passage of time, is abstracted from all special occurrences. The remaining empty form [the relation of n to n+1] of the common content of all these two-nesses becomes the original intuition of mathematics and repeated unlimitedly creates new mathematical subjects.[10]

In 1930, Brouwer's student and collaborator, Arend Heyting, published a set of axioms for intuitionistic logic. Intuitionism is distinguished from other forms of constructivism by the acceptance of "free choice sequences", sequences whose progression may not be encapsulated by a formula. The use of free choice sequences allows for the construction of real numbers whose place digits might, for example, be based on the possible, but unknown occurrence of a randomly chosen sequence of numbers in the decimal expansion of π. While the axiomatization of the intuitionistic approach expanded its exposure within the mathematical community beyond Brouwer's direct collaborators, the extra work that intuitionistic methods entailed, along with David Hilbert's high profile objections to its underlying assumptions, prevented intuitionism from gaining wide spread acceptance.

Bishop's Constructivism

> *The transcendence of mathematics demands that it should not be confined to computations that I can perform, or you can perform, or 100 men working 100 years with 100 digital computers can perform. Any computation that can be performed by a finite intelligence - any computation that has a finite number of steps - is permissable.*[11]
>
> ...
>
> *The only way to show that an object exists is to give a finite routine for finding it.*[12] ***Errett Bishop***

Bishop's constructivist approach is based on the idea that proofs should be constructed from finite routines and resonates strongly with the idea of computability. Constructivism

[10]Morris Kline, *Mathematical Thought from Ancient to Modern Times*, vol 3, (Oxford: Clarendon Press, 1972) 1199.

[11]Errett Bishop, and Donald Bridges, *Constructive Analysis, (Berlin: Springer-Verlag, 1985), 6.*

[12]Ibid., p. 11.

retains its radical and not-so-radical adherents in modern mathematics. While constructivist methods can be severely limiting, constructive proofs are valued by the mathematical community. In its orientation, Simon Fraser University's *Centre for Experimental and Constructive Mathematics* perhaps epitomizes a form of practical constructivism which embraces constructive methods where possible, aided by digital computation where necessary.

1.2.2 Platonism

> *I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations", are simply the notes of our observations.*[13]
> ***G.H. Hardy (1877 - 1947)***

Mathematical Platonism encompasses a range of perspectives that share the idea of a mathematical reality with properties and truths that can be discovered through ingenuity and insight. In its casual form, no assumptions are made regarding a unifying structure of the "mathematical reality". In its more developed form, it is postulated that the mathematical reality can be described by the language of set theory. Reuben Hersh quotes Kurt Gödel:

> Despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I do not see any reason why we should have any less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception.[14]

Mathematical Platonism is easily assailed on the grounds that it requires a huge leap in faith to believe in a reality that is only accessible by indirect observations and conjectures. Despite this achilles heel, Platonist sentiment is very common within the mathematical community. Platonism is accepting of any reasonable methodology and it places a minimal amount of responsibility on the shoulders of the mathematician. The Platonist perspective holds the mathematician responsible for observations and explanations, not creations.

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[13] G.H. Hardy, *A Mathematician's Apology.* (London: Cambridge University Press, 1967), 21.
[14] Hersh, p.10.

1.3 Experimental Mathematics

> *This new approach to mathematics - the utilization of advanced computing technology in mathematical research - is often called experimental mathematics. The computer provides the mathematician with a laboratory in which he or she can perform experiments: analyzing examples, testing out new ideas, or searching for patterns.*[15]
>
> ***David Bailey, Jonathan Borwein, and Roland Girgensohn***

Ideas from the developing field of *experimental mathematics* coalesce not so much into a philosophical perspective, an "ism" as it were, as they do into a body of thought concerning the judicious use of computational technology in mathematics. In *Experimentation in Mathematics*, Bailey, Borwein, and Girgensohn state that experimental mathematics comprehends a computational methodology which includes:

1. Gaining insight and intuition.
2. Discovering new patterns and relationships.
3. Using graphical displays to suggest underlying mathematical principles.
4. Testing and especially falsifying conjectures.
5. Exploring a possible result to see if it is worth formal proof.
6. Suggesting approaches for formal proof.
7. Replacing lengthy hand derivations with computer-based derivations.
8. Confirming analytically derived results.[16]

It is noteworthy that the methodology bears much in common with the *educational constructivist* perspective of mathematical pedagogy which, where appropriate, advocates techniques of learning by discovery. Both experimental mathematical research and constructivist mathematical pedagogy have benefited immensely from the development of *Computer Algebra Systems*, such as Maple™.[17] The following two pedagogical examples are aimed at

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[15]David Bailey and Jonathan Borwein, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, (New York: A.K. Peters Ltd, 2003), 2-3.

[16]Ibid, p. 3.

[17]*Maple is a trademark of Waterloo Maple Inc.*

the undergraduate level and illustrate symbolic calculations which can be carried out by hand and then verified with Maple or Mathematica™.[18]

$\pi \neq \frac{22}{7}$:

Math: In this case, a positive integral over $[0,1]$ evaluates to $\pi - \frac{22}{7}$, thus providing an integral "proof" that $\pi \neq \frac{22}{7}$. As an introduction to symbolic integration, students can be encouraged to first evaluate this integral by hand. (If the numerator of the integrand is expanded and the result divided by the denominator, the integrand can be expressed as the sum of its polynomial and rational parts; both of which integrate easily.) After convincing themselves of the result, they can execute the Maple integration command to verify that Maple is in agreement.

Maple code: This integral is expressible in a single line but is better displayed by the following three commands:

```
with(student):
A:=Int(x^4*(1-x)^4/(1+x^2),x=0..1):
A=value(A);
```

Execution of this code, yields the following output: $\int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = \frac{22}{7} - \pi$.

Methodology: This example falls neatly under item number 4 from the methodology, the testing and falsifying of conjectures. It also illustrates item number 7, the replacement of lengthy hand derivations by the use of digital tools for, along with its mathematical significance, it can motivate the use of symbolic integration by providing an example which is just enough work by hand for the student to appreciate the benefits of using the computer.

The sophomore's dream:

Math: In this case, Maple helps establish the following singular identity:

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^n}. \tag{1.1}$$

[18]Jonathan Borwein and Terry Stanway, *Numerical and Computational Mathematics at the Undergraduate Level* A presentation for the Conference on Technology in Mathematics Education at the Secondary and Tertiary Levels [online], 1999.

This depends on the series expansion: $x^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n (x\ln(x))^n}{n!}$. Integration produces: $\int_0^1 \frac{1}{x^x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n \int_0^1 (x\ln(x))^n dx}{n!}$. Then (1.1) may be proved by substituting $y = -\ln(x)$ and then $z = (n+1)y$.

Maple code: The series expansion for x^{-x} can be tested by the following command:

```
series(x^(-x),x=0,10);
```

The advantage to the student of using Maple for (1.1) is the ease with which it handles change of variables. To assign the substituted expression to a variable, the command is:

```
s0:=changevar(y=-ln(x),Int((x*ln(x))^n,x=0..1),y):
```

Methodology: This example also responds to both methodology items numbe 4 and number 7. As with the first integral, students can work it through by hand before trying it on the computer. (Students will need the series expansion of x^{-x} and perhaps the definition of the Γ-function.)

...

In defining the role of computational experimentation in mathematics, the experimental methodology accepts, as part of the experimental process, standards of certainty in mathematical knowledge which are more akin to the empirical sciences than they are to mathematics. As an experimental tool, the computer can provide strong, but not conclusive, evidence regarding the validity of an assertion. While with appropriate validity checking, confidence levels can in many cases be made arbitrarily high, it is notable that the concept of a 'confidence level' has traditionally been a property of statistically-oriented fields. It is important to note that the authors are not calling for a new standard of certainty in mathematical knowledge but rather the appropriate use of a methodology which may produce, as a product of its methods, definably uncertain transitional knowledge.

What the authors do advocate is closer attention to and acceptance of degrees of certainty in mathematical knowledge. This recommendation is made on the basis of three assertions:

1. almost certain mathematical knowledge is valid if treated appropriately

2. in some cases 'almost certain' is as good as it gets

3. in some cases an almost certain computationally derived assertion is at least as strong as a complex formal assertion.

The first assertion is addressed by the methodology. The second is a recognition of the limitations imposed by Gödel's Incompleteness Theorem. The third is more challenging for it addresses the idea that "certainty" is a function of the community's knowledge validation systems. The example used is Wiles' proof of Fermat's Last Theorem. In the authors' words:

> ...perhaps only 200 people alive can, given enough time, digest all of Andrew Wiles' extraordinarily sophisticated proof of Fermat's Last Theorem. If there is even a one percent chance that each has overlooked the same subtle error (and they may be psychologically predisposed so to do, given the numerous earlier results that Wiles' result relies on), then we must conclude that computational results are in many cases actually more secure than the proof of Fermat's Last Theorem.[19]

The idea that what is accepted as mathematical knowledge is, to some degree, dependent upon a community's methods of knowledge acceptance is an idea that is central to the *social constructivist* school of mathematical philosophy.

1.4 Social Constructivism

> *Social constructivism begins with the assumption that mathematical practices and institutions are a given; they are historically constituted and have a life of their own. In other words, the starting point is the existence of mathematical "forms of life," with their own participants, representations of knowledge, and so forth.*[20]
>
> ***Paul Ernest***

[19]Bailey, Borwein, and Girgensohn, p.10.

[20]Paul Ernest, *Social Constructivism As a Philosophy of Mathematics*, (Albany: State University of New York Press, 1998), 148.

Associated most notably with the writing of Paul Ernest, an English mathematician and Professor in the Philosophy of Mathematics Education, social constructivism seeks to define mathematical knowledge and epistemology through the social structure and interactions of the mathematical community. Ernest attributes social constructivism's intellectual antecedents to the writings of Imre Lakatos and Ludwig Wittgenstein.[21] From Lakatos, social constructivism derives its interpretation of the *methods of mathematics* and it absorbs:

> ... the account of the genesis of mathematical knowledge and proofs in Lakatos's LMD [Logic of Mathematical Discovery first developed in *Essays in the Logic of Mathematical Discovery*, his Cambridge Ph.D. thesis (1960)]. This describes the essentially social, dialectical process of mathematical knowledge creation and warranting.[22]

From Wittgenstein, social constructivism adopts its deeper epistemological foundation, locating:

> ... the basis for logical necessity and mathematical knowledge in linguistic rules (be they tacit or explicitly articulated) and practices (which are embedded in socially enacted forms of life).[23]

In *Proofs and Refutations*, Imre Lakatos famously imagines a dialogue in which a teacher and his students explore the nature of mathematical objects and mathematical knowledge. The challenge to absolutist notions of mathematical knowledge presented by both the methods of experimental mathematics, and the perspective of social constructivism bring this dialogue into the present.

1.5 Conclusion

> *Progress, far from consisting in change, depends on retentiveness. When change is absolute there remains no being to improve and no direction is set for possible*

[21] Ernest, p. 134.

[22] Ibid p. 135.

[23] Ibid p.135.

improvement... Those who cannot remember the past are condemned to repeat it.[24] ***George Santayana (1863 - 1952)***

In the early years of the 20^{th} century, the transformation of the socio-political landscape brought about by the First World War was complemented by transformations of the intellectual landscape brought about by the acceptance of new ideas, and rejection of old, in art, literature, and science. In art and literature the modernist perspective had informed such works as Picasso's early cubist piece, *Les Desmoiselles d'Avignon* and Santayana's *The Life of Reason.* In physics, Einstein had published *The Special Theory of Relativity*, challenging the determinism of Newtonian mechanics. Mathematics had not emerged unscathed. In his book, *What is Mathematics Really?*, Reuben Hersh describes the fractures that arose in the philosophy of mathematics after the widely accepted idea that all of mathematics could be ultimately derived from the principles of Euclidean geometry fell victim first to logically consistent non-euclidean geometries and second to geometrically counter-intuitive concepts such as space filling curves and such unavoidable consequences of analysis as continuous everywhere but nowhere differentiable curves. In Hersh's words:

> The situation was intolerable. Geometry served from the time of Plato as proof that certainty is possible in human knowledge - including religious certainty. Descartes and Spinoza followed the geometrical style in in establishing the existence of God. Loss of certainty in geometry threatened loss of all certainty.[25]

In the early days of the twenty-first century, at least the remnants of each of the twentieth century responses to this 'intolerable situation' have survived. The ideas and impulses that motivated foundationism are expressed in the field of automata theory and the efforts to develop automated theorem provers. The *QED* project provides an ambitious example of these efforts. The foundationist spirit of the project is represented eloquently by its mission statement, *The QED Manifesto*, which states in part:

> The QED system will conform to the highest standards of mathematical rigor, including the use of strict formality in the internal representation of knowledge

[24] George Santayana, *The Life of Reason: Reason in Common Sense* (New York: Scribner's, 1905-06), Volume 1, 284.

[25] Hersh, p. 137.

> and the use of mechanical methods to check proofs of the correctness of all entries in the system.[26]

With its broad, and some might argue shallow, acceptance of any mathematical method or statement which is consistent with the idea of mathematics as an act of discovery, mathematical Platonism emerges as the most encompassing perspective and, by consensus, the working perspective of most mathematicians. In *What is Mathematics Really?*, Hersh comments that "an inarticulate, half-conscious Platonism is nearly universal among mathematicians".[27]

The constructivist approach retains a high degree of acceptance amongst mathematicians and to furnish a constructive proof of a theorem, where previously only standard proofs were known, is considered to be an achievement. With its emphasis on defining algorithms, the constructivist approach resonates with the methodology of experimental mathematics.

In the effort to articulate a philosophical foundation for MKM, each of the aforementioned perspectives is incomplete for each fails to address the fact that "knowledge management" is ultimately a social activity. A philosophy of MKM must comprehend not only *what is being managed* but *how and for whom it is being managed.* In its consideration of knowledge as being constructed through social interaction, social constructivism does address these issues and, in so doing, appeals to the idea of "mathematical community". Writing about objective knowledge in mathematics, Paul Ernest states:

> the social constructivist philosophy of mathematics takes "objective knowledge" in mathematics to be that which is accepted as legitimately warranted by the mathematical community. Thus it is the mutually agreed upon, shared knowledge of that community, knowledge that satisfies its knowledge acceptance procedures and criteria, not something superhuman or absolute.[28]

Knowledge management systems are both defined by and help to define the structure of the mathematical community and its many subcommunities. In the concluding chapter, the question of how this structure might be analyzed is addressed. The following section presents a brief overview of the characteristics of mathematical communities.

[26] Anonymous, "The QED Manifesto", document in the public domain, 1997.

[27] Hersh, p. 11.

[28] Ernest, p. 148.

1.5.1 The Mathematical Community

The notion of community is loosely defined and can be used to refer to a lot of quite different types of social groupings. Intentionally, a broad definition of the mathematics community will be adopted to include all those involved with advancing the understanding of mathematics; either at its frontiers, the primary occupation of researchers, or within the existing body of mathematical knowledge such as teachers and students. The boundary is a porous one and relatively few would claim full time membership. Many others are interlopers, jumping in and out as the need arises or circumstances dictate. The following discussion briefly considers four interrelated factors that help to bind the community: the *language* of the community, the *purposes* of the community, the *methods* of the community, and the *meeting places* of the community.

The Language of the Community

> *A man is necessarily talking error unless his words can claim membership in a collective body of thought.*[29]
> ***Kenneth Burke***

It is tempting, but tautological, to state that the language of the community is the language of mathematics and it only extends the tautology to state that anyone who claims membership in the community knows what this statement means. In reality, it may be argued that the paradigm for mathematical discourse is the language of the published research paper. All other discourse approximates the paradigm by degree according to what level of rigor is appropriate to the situation and audience.

The special symbols of mathematics present a particular challenge to expressing mathematics in mechanically type set or digital forms. An individual claiming membership in the mathematical community can generally be assumed to have some understanding of how to overcome those challenges. In this area, it can be argued that digital methods have created fractures within the community that did not exist previously. While LaTeX is in wide use in research communities, it is almost unheard of in lower level mathematics education communities.

[29]Wayne C. Booth, *Modern Dogma and the Rhetoric of Assent* (Chicago: University of Chicago Press, 1974) 86.

The Purposes of the Community

> *If intellectual curiosity, professional pride, and ambition are the dominant incentives to research, then assuredly, no one has a fairer chance of gratifying them than a mathematician.*[30] ***G. H. Hardy***

In the sense it is used here, 'purpose' does not refer to the overriding *raison d'être* of the community; that has already been defined to be an interest in the advancement of mathematics. Rather, purpose here refers to what motivates an individual to seek membership in the community; and there are many. There is a professional motive which expresses itself by the simple statement that "I am involved with mathematics because this is how I earn my living". There is an egotistical motive which is expressed in the statement that "I am involved with mathematics because I take pleasure from proving to myself and others that I can overcome the challenges that the field affords". There is a social motive which is evident in the statement that "I am involved with mathematics because I benefit from the company of others who are involved with mathematics". And, finally, there is an aesthetic motive that is reflected in statements like "I am involved with mathematics because I wish to help unlock the beauty of mathematics".

With respect to mathematical knowledge management, an individual's reasons for being involved with mathematics strongly affects the individual's role in the community. This sense of purpose in the community in turn helps to determine the individual's information management needs.

The Methods of the Community

The *methods of the community* encompass how mathematicians do what they do and the tools that they use. Traditionally, mathematics has been one of the most purely cerebral of the sciences, depending, for its practice, on little more than pencil and paper. This austerity is tightly associated with underlying philosophical assumptions about the nature of mathematics. The foundational shifts of the last century and developments in computer technology paved the way to the present situation which finds mathematicians lining up with theoretical physicists, molecular biologists, and others to claim time on the world's most powerful super computers. An important consideration regarding the question of how

[30] G. H. Hardy, *A Mathematician's Apology*, (London: Cambridge University Press, 1967), 80.

mathematicians do mathematics is the question of how and to whom to mathematicians *express* their mathematics. The *meeting places of the community* of the community warrant special treatment and are addressed breifly here and again in chapter 5.

The Meeting Places of the Community

> *J.J. Sylvester sent a paper to the London Mathematical Society. His covering letter explained, as usual, that this was the most important result in the subject for 20 years. The secretary replied that he agreed entirely with Sylvester's opinion of the paper; but Sylvester had actually published the results in the L.M.S. five years before.*[31] ***J. E. Littlewood***

Tightly associated with the methods of the community, is the notion of the *meeting places of the community*. These are the venues in which mathematics is presented and discussed. Not only the offices, classrooms, seminar rooms, labs, and conference halls, but also the notes, postcards, letters, journals, and, in this electronic age, their digital equivalents.

...

In the early part of the last century it could be still be argued that mathematicians could claim membership in an *invisible college* which, while dispersed geographically, was connected by a common experience with a standard undergraduate mathematics curriculum. The rapid development of mathematics over the last century and the increasing application of mathematics in other domains has reduced this mathematical canon to a few courses which are still commonly taught in first and second year. As the community grows and becomes more permeable, the connective influence of the invisible college is being replaced by other shared experiences and, in many cases, those are the shared experiences of knowledge management methods.

[31] J. E. Littlewood, *Littlewood's Miscellany*, ed. Bella Bollobás, (London: Cambridge University Press, 1986) 148.

Chapter 2

A Overview of the Technological History of MKM

The history of knowledge management has been very much bound up in the history of information technology. Significant developments in information technology have catalyzed major shifts in knowledge management practice. In a period in which members of the extended mathematical community are wrestling with the challenge of how best to employ networked digital technology in the service of MKM, it is useful to examine the history of MKM for any lessons that might be derived from precedent. While the relationship of technology to MKM shares much in common with the relationship of technology to knowledge management in any particular domain, issues related to *mathematical knowledge representation* play a mediating role between the two. This chapter presents a survey of the technological history of western MKM from antiquity to the typographical era with an emphasis on how an understanding of the transition between the scribal era and the typographical era might inform our understanding of the transition between the typographical and digital eras.

2.1 MKM in Antiquity

Presently the object of a massive reconstruction effort, the library at Alexandrina was originally constructed in the third century BC in response to a decree issued by Ptolemy I. The library was the largest of the ancient libraries and eventually was home to an estimated 700,000 papyri, scrolls, and, in its later years, codices, the precursors of modern books.

Part of the *Mouseion*, which also included what would today be considered a research institute, the library contributed to Alexandria's ascendancy as, along with Athens, one of the two great intellectual centres of the period. In several respects, the preoccupations of the Alexandrian librarians remain important considerations in archival knowledge management. In his mid third-century BC tenure as chief librarian, the poet Callimachus developed a cataloguing system that arranged each work by subject sorted alphabetically by author. The catalogue entry included a bibliographical element and a brief critical account of the author's writing.[1] He applied this system to the 400,000 works in the library's collection at the time consuming 120 scrolls worth of entries.[2] Callimachus was immediately followed by the mathematician and geographer, Eratosthenes, who designed and implemented a shelving system for the growing collection. In an October 1999 speech at the University of Lièges, Birdie Maclennan, an Associate Professor of Library Sciences and Librarian at the University of Vermont, draws attention to the legacy of the librarianship work of Callimachus and Eratosthenes:

> Les bibliothécaires alexandrins ont passé un système d'organisation de la collection (c'est-à-dire le catalogage et la classification de documents) aux futurs générations de bibliothècaires. On peut se demander si les Pinakes et le "tetagmanos" de Callimachus et d'Eratosthenes étaient les prédécesseurs du système de Melville Dewey qui est employé aujourd'hui dans de nombreuses bibliothéques?[3]

Dr. Maclennan goes on to highlight an important parallel between the information environment of the Alexandrians and the environment of today: then, as now, there was a significant shift in the nature of publishing. While presently it is the shift from paper based publishing to digital publishing, in the Alexandrian period, it was a shift from the scroll to the codex.[4] Due to its compactness and the ease with which it afforded readers access to text, the codex was a such a vast improvement over the scroll that Martial, a poet writing at the end of the first century, was moved to proclaim, "assign your book-boxes to the great, this copy of me one hand can grasp".[5] It is interesting to note that portability

[1] Sameh M. Arab, "Bibliotheca Alexandrina", Arab World Books [online], 2000.

[2] Birdie Maclennan, "To be a librarian, today and tomorrow: Reflections on library education and practice in a changing world", U.D. Walthère Spring, Université de Liège [online], 2000.

[3] Ibid.

[4] Ibid.

[5] James Grout, "Scroll and Codex", *Encyclopaedia Romana* [online], 2003.

and ease of use alone were not enough to immediately guarantee wide spread acceptance of the codex. The replacement of scrolls by codices was only ensured after the early Christian church made a "head office" decision to record scripture and religious writings in codices as a visible means of distinguishing Christian works from that of Jewish and Pagan writings. This is an early example of a recurring theme in the development of knowledge management systems: superior technology alone is not enough to ensure the adoption of new systems. External societal factors frequently play an important role. In the case of codices, their adoption as the standard form of publication for longer works coincided with the spread of Christianity and codices had fully replaced scrolls by the end of the third century.

The library at Alexandria is the first systematic attempt at the archiving and organization of knowledge for which there is a historical record. The library's demise was in no part an internal failure but rather a stark example of the influence of societal factors on knowledge management systems. The library's end coincided with the decline of Greek religion and the spread of Christianity. Succumbing to sectarian conflict, the last of the library's buildings was burned during a riot in 392 AD. The last known head librarian was the mathematician, Theon of Alexandria. His daughter, Hypatia, was also a mathematician whose adherence to Pagan religion brought about her premature death; she was torn limb from limb by a Christian mob in the second decade of the fifth century.[6]

2.2 Medieval Knowledge Management

In Medieval Europe, learning and the production of manuscripts came to be centred in universities and monasteries. During this period, there was progress in the way mathematics was practised, and consolidation and refinement in the ways in which mathematical texts were produced and archived.

The first significant technological advance illustrates the fundamental relationship between MKM and mathematical knowledge representation and concerns the method by which numbers are represented. Near the end of the first millenium, hindu-arabic numerals make their first, tentative appearance in European manuscripts. While the place-value properties of hindu-arabic numerals afford simpler computational algorithms than those of the Roman and Greek systems, adoption of the new system is a process which takes place over many

[6]Michael Deakin, "Hypatia and her Mathematics", *The American Mathematical Monthly* 101, no. 3, (1994): pp. 239-241.

centuries. In his famous work *Liber Abaci* written in 1202, Fibonacci describes his discovery of the "Indians' nine symbols":

> When my father, who had been appointed by his country as public notary in the customs at Bugia acting for the Pisan merchants going there, was in charge, he summoned me to him while I was still a child, and having an eye to usefulness and future convenience, desired me to stay there and receive instruction in the school of accounting. There, when I had been introduced to the art of the Indians' nine symbols through remarkable teaching, knowledge of the art very soon pleased me above all else and I came to understand it, for whatever was studied by the art in Egypt, Syria, Greece, Sicily and Provence, in all its various forms.[7]

The idea of advances in MKM being bound to advances in mathematics s also evident in the development and teaching of classical algebra. Prompted first by Latin translations of Al-jabr wa'l-muqabala by the Arab mathematician, al-Khwarizmi, geometrically inspired algebraic methods were introduced in Europe beginning in the twelfth century. Again, Fibonacci was instrumental. In her paper *The Art of Algebra from Al-Khwarizmi to Viète: A Study in the Natural Selection of Ideas*, Karen Hunger Parshall describes his contribution as follows:

> ... Fibonacci fashioned his mathematical environment by seeking out texts from which, and people from whom, he could learn more of the intricacies of arithmetic and algebra. One result of these studies, his most influential book, entitled Liber Abaci (1202, revised 1228), attested to his mastery not only of the Hindu-Arabic techniques of practical calculation but also of the theory of quadratic equations as found in the works of al-Khwarizmi, Abu-Kamil, and al-Karaji.[8]

The historical development of algebra offers several examples of how progress in mathematical thought is interconnected with progress in knowledge representation and consequently influences MKM. The late pre-typographic period also saw the rediscovery of the

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[7] J.J. O'Connor and E.F. Robertson, "The Arabic numeral system", The MacTutor History of Mathematics archive [online], 2001.

[8] Karen Hunger Parshall, "The Art of Algebra from Al-Khwarizmi to Viète: A Study in the Natural Selection of Ideas", *History of Science* 26 (1988): 133.

work of the Greek mathematician, Diophantus, whose Babylonian inspired algebra introduced the use of symbols as abstractions of numbers and sets.

While the Middle Ages saw progress in knowledge representation aspects of MKM, the systems of mathematical knowledge and archiving remained static and associated exclusively in monasteries, cathedrals, and universities. Developments did take place, however, in knowledge representation at the artifact level. The contemporary use of metadata for the classification of knowledge has an interesting antecedent in medieval scholarship. In his book *The Medieval Theory of Authorship*, A.J. Minnis describes the use of formal prologues found at the beginning of manuscripts. Here, he describes the so-called 'type C' prologue:

> In the systematisation of knowledge which is characteristic of the twelfth century, the 'type C' prologue appeared at the beginning of commentaries on textbooks os all disciplines: the arts, medicine, Roman law, canon law, and theology. Its standard headings, refined by generations of scholars and to some extent modified through the influence of other types of prologue, may be outlined as follows:*Titulus*, the title of the work...*Nomen auctoris*, the name of the author...*Intentio Auctoris*, the intention of the author...*Materia Libri*, the subject matter of the work...*Modus agendi*, the method of didactic procedure...*Ordo libri*, the order of the book...*Utilitas*, utility...*Cui parti philosophiae supponitur*, the branch of learning to which the work belonged.[9]

Cast as medieval metadata, these prologues indicate some effort on the part of medieval scholars to protect the integrity of scholarly works in a distributed publishing environment. Of relevence to present MKM concerns, the addition of structured elements to manuscripts provided one means of ensuring stability and protecting author's rights, such as they were, in an environment with multiple nodes of document reproduction.

2.3 The Typographic Period

> *The difference between the man of print and the man of scribal culture is nearly as great as between the non-literate and the literate. The components of Gutenberg technology were not new. But when brought together in the fifteenth*

[9] Minnis, A.J. *Medieval Theory of Authorship: Scholastic literary attitudes in the later Middle Ages*, (London: Scolar Press, 1984), 19.

> *century there was an acceleration of social and personal action tantamount to "take off" in the sense that W.W. Rostow develops this concept in* The Stages of Economic Growth *"that decisive interval in the history of a society in which growth becomes its normal condition."*[10]
>
> ***Marshall McLuhan (1911 - 1980)***

In *The Gutenberg Galaxy*, Marshall McLuhan describes the change in cultural orientations, expectations, and assumptions that occurred with the wide spread adoption of typography in the fifteenth and early sixteenth centuries. The rediscovery of Diophantus' number theoretic approach to algebra, coincided approximately with the invention of the printing press by Johannes Gutenberg in 1452. The first known printed mathematical work was the Campanus translation of Euclid's *E*lements executed at Venice in 1482 by the German printer, Erhard Ratdolt, from his shop.[11] Ratdolt's reputation in the history of printing is partly based on his pioneering work in the field of scientific and technical publishing. In an 1863 address to the *Bibliographical Society*, Gilbert Redgrave states:

> In the course of the year 1482, Ratdolt issued several of his most remarkable productions; among them the foremost place is due to the Euclid, with its beautiful border and elaborate diagrams. It constitutes the first attempt to illustrate the text of this author with wood-cuts of the problems, and it must ever be memorable for the skill and enterprise it displays in the accomplishment of what must have at that time been a most difficult task.[12]

The development of printing had a profound effect on scholarship, education, and civics. It led to a redefinition of what constitutes authority and notions of intellectual property which were foreign to the pre-typographic intellect. The Ratdolt edition of the *The Elements* provides a good example of another recurring theme in knowledge management: the preservation of old media artifacts in the early stages of new media. Ratdolt's work looks very much like a finely executed manuscript. Figure 2.1 is an image of the first page of *Book 1*[13]:

[10] *McLuhan, Marshall The Gutenberg Galaxy, (Toronto: University of Toronto Press, 1962), 90.*

[11] SunSITE DigitalMathArchive, *Erhard Ratdolt - first publisher of Euclid* [online], 2003.

[12] Ibid.

[13] By permission: Dr. William Casselman, The University of British Columbia. Original images obtained by Dr. Casselman with the cooperation of *The Fisher Rare Book Library* at The University of Toronto.

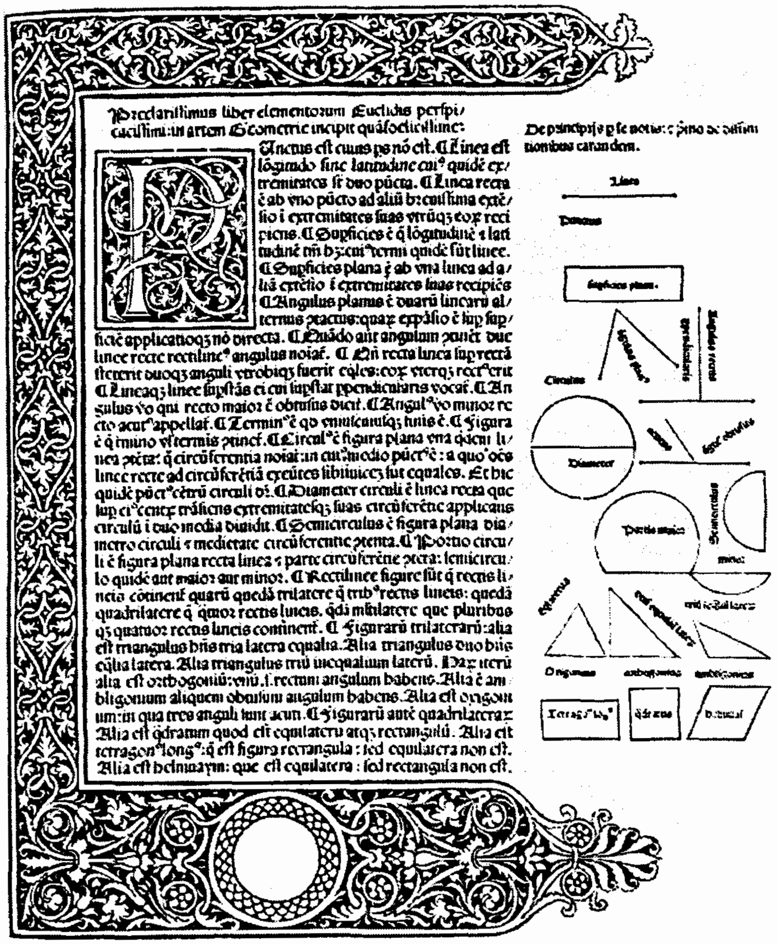

Figure 2.1: Book 1, Page 1 of The Elements (1482)

The overall effects on society of mechanized print has been the subject of much scholarly research and much debate. With a particular emphasis on effects which bear upon knowledge management, a representative list includes:

1. an increase in literacy rates
2. the commoditization of knowledge
3. the stabilization of knowledge
4. the reorganization of academic communities, including
   (a) the strengthening of communities around subject specialization, and
   (b) the reorganization of elites
5. the strengthening of notions of intellectual property and authorship.

In *Communication at a Distance: The Influence of Print on Sociocultural Oranization and Change*, David Kaufer and Kathleen Carley refer to the idea that print technology was a singular force behind the observed changes in the knowledge environment as the *Strong Print Hypothesis.*[14] They refute this hypothesis by developing an analysis which suggests that print was but one factor, albeit a catalyzing factor, in a process of change that was determined by a variety of socio-cultural variables. The history of scholarly publishing provides a good example of the gradual nature of the changes in knowledge management that took place after the advent of print.

At the time Henry Oldenburg was appointed as Secretary of *The Royal Society* in 1663, personal correspondence was the only means of sharing research within the scientific and mathematical communities. In fact, one of the secretary's main occupations was receiving research-oriented correspondence and communicating salient aspects of that correspondence to others in the field. Realizing that there was a need to ensure that important research was as widely disseminated as possible, in 1655 Oldenburg began publishing edited versions of the correspondence he received as the *Transactions of the Royal Society.* This publication rapidly adopted an informal system of peer review which would formalize into a system which has dominated 300 years of academic publishing. Two points are noteworthy:

[14]David S. Kaufer and Kathleen Carley, *Communication at a Distance: The Influence of Print on Socio-cultural Organization and Change*, (Hillsdale: Lawrence Erlbaum Associates, Inc., 1993), 254.

1. the first serialized scholarly publication did not appear until more than 150 years after the advent of type and

2. the publication began as a "repackaging" of a knowledge management system that predated type.

The *Transactions of the Royal Society* and other broadly focussed journals so dominated European academic publishing that the first academic journal dedicated to mathematics would not be published until 1795 at Leipzig. The *Archiv der reine und angewandte Mathematik* ceased publication after 10 years however it was followed in 1810 by Joseph Gergonne's *Annales de Mathématiques Pures et Appliquées.* This publication lasted until 1831 but, by this time, several other mathematical journals had entered into publication.[15] It is notable that the subject specialization that was at least partly the result of print did not appear in knowledge management at the journal level until the end of the eighteenth century.

2.3.1 Authorship and Intellectual Property

Discussing the impact of the printing press, McLuhan argues that, while pre-typographic culture was characterized by localized production and limited distribution of production - most abbeys would have at least one scribe but a single scribe can only produce so many manuscripts - typographic culture would come to be characterized by centralized production and mass distribution; a limited number of publishing houses producing and distributing many copies of individual texts. The mass production of identical copies of a text introduced notions of authority, authorship, and intellectual property that were completely unknown in scribal culture. He cites E.P. Goldschmidt, a scholar in medieval studies:

> One thing is immediately obvious: before 1500 or thereabouts, people did not attach the same importance to ascertaining the precise identity of the author of a book they were reading or quoting as we do now. We very rarely find them discussing these points...Not only were users of manuscripts, writes Goldschmidt, mostly indifferent to the chronology of authorship and to the "identity and personality of the author of the book he was reading, or in the exact period

[15] Laurent Rollet and Philippe Nabonnand, "Une bibliographie mathématique idéale? Le Répertoire bibliographique des sciences mathématiques", *Gazette des mathématiciens* 92 (2002): 12.

> at which this particular piece of information was written down, equally little, did he expect his future readers to be interested in himself."[16]

In 1710, England would pass the *Statute of Anne* which recognized for the first time in law an author's entitlement to a fixed term of ownership over his or her works. (Notably, the statute also guaranteed the deposition of one copy of any of any published work at each of nine selected public libraries.) That there would elapse two and a half centuries between the advent of mechanical typesetting and the first copyright law is another indication of the gradual nature of change in the knowledge management environment.

2.4 Conclusion

The connection between technology and knowledge management is manifest at many levels. It is present at the level of atomic mathematical entities such as numbers and formulas and it is present at the level of community wide systems of knowledge dissemination and archiving such as journals and libraries. At the atomic level, mathematics has benefited from a remarkable degree of universality and stability in knowledge representation. The monographs of Euler and Cauchy remains entirely readable today and are still cited. At the macro level, methods of knowledge representation have undergone significant shifts resulting from developments in information technology. An examination of the influence of print suggests that the initial response to the new medium may be a repackaging of artifacts that are characteristic of the old medium and that the development of new systems may be gradual. External societal factors may also influence the shape and pace of change.

In *The Rise of the Reading Public*, Elizabeth Eisenstein discusses the effect of print on social structures:

> Even while communal solidarity was diminished, vicarious participation in more distant events was also enhanced; and even while local ties were loosened, links to larger collective units were being forged. Printed materials encouraged silent adherence to causes whose advocates could not be found in any one parish and who addressed an invisible public from afar. New forms of group identity began

[16] McLuhan, p. 131.

to compete with an older, more localized nexus of loyalties.[17]

In mathematics, the "new form of group identity" was based around a shared literature. For the first time, those professing an interest in mathematics had access not only to the same authors, but also to the same version of the same texts. For the first time, the notion of a "mathematics community" made sense.

[17]Elizabeth Eisenstein, "The Rise of the Reading Public", *Communication in History: Technology, Culture, Society 2nd ed.* ed. D. Crowley and P.Heyer, (Cambridge: Cambridge University Press 1995), 112.

Chapter 3

Doing Mathematics in the Digital Age

This chapter is the revised version of an unpublished paper co-authored with Dr. Jonathan Borwein and presented to the 1999 meeting of the *Canadian Mathematics Education Study Group*. The focus of this meeting was information technology in the service of mathematics education.

...

Technology has repeatedly promised to transform mathematics pedagogically. More recently it has made similar promises to the research community. That said, mathematics at the beginning of the twenty-first century looks a lot more like mathematics in 1939 than is the case with any of its sister sciences.

That this is about to change is inarguable. The confluence of ubiquitous compute power with new networking and collaborative environments will push the teaching and discovering of mathematics in conflicting directions often resistant to control. The burgeoning role of corporate edu-packages is hardly likely to diminish. Nor are battles over curriculum and its delivery about to stop. This chapter surveys and illustrates some of the ways in which twenty-first century mathematics will be changed by these new technologies. An attempt will be made to distinguish issues of ownership of technology from those of control over content and to discuss how mathematical educators might best prepare for the coming storms.

3.1 Introduction

In both the realms of research and education, technology has long been put forth as holding much promise to mathematicians. Experience, however, frequently leads to a confrontation with technology's pitfalls to such an extent that many in the pedagogical and research communities willingly choose to remain mathematical luddites. The sections that follow, include a discussion of the promises and pitfalls, both intellectual and technical, and an examination some of the intellectual property and commercial issues that influence the development and distribution of mathematical software. While the emphasis is on mathematical education, many of the arguments apply equally to mathematical research and indeed to the other sciences.

3.2 Intellectual Promises...

> *Today, after more than a century of electric technology, we have extended our central nervous system itself in a global embrace, abolishing both space and time as far as our planet is concerned.*[1]
> ***Marshall McLuhan***

Thirty-five years ago, Marshall McLuhan's pronouncements were viewed by many as, at best, encoded messages that required careful deciphering, or at worst, extreme examples of hyperbole. Today, the characteristics of McLuhan's electronic universe are part of everyday experience. The central promise of digital technology to mathematics is the extension of mathematical senses and faculties. Since ENIAC, computers have extended the human capacity for numerical calculations. With developments in the field of computer graphics, mathematical software promised to bring mathematics into colour. Today, it is largely soon to be resolved band-width problems that prevent the immersive manipulation of three dimensional surfaces in real time as they are generated by a colleague on the other side of the globe. Technology has also promised to extend the human capacity for inductive and deductive reasoning. Symbolics, an abstraction of the computer's traditional domain of numerics, were for a long time handled poorly by computers. Increase in computing power and advances in programming have resulted in applications such as Maple™which permit

[1] Marshall McLuhan, *Understanding Media*, (New York: Signet Press, 1964), 19.

background pattern checking and inverse calculation; the computer extends the capacity for abstraction.[2] Another promise is that of contracting frames of reference for time and space. As a means of processing mathematics, digital technology contracts time; as a means of communicating mathematics, it contracts space. This dual contraction permits rapid insight and entails a corresponding demand for rapid reinforcement via micro parallelism.

Advocates for computer technology have made specific promises to those involved in math education. As early as 1965, Seymour Papert conceived of a programming language which would support "constructivist" teaching and allow students to examine lively and realistic mathematics. In collaboration with Marvin Minsky and others, he developed the *Logo* programming language which, through many reincarnations, continues to thrive. Another promise lies in the development of "learner-centered" curricula. The adaptability of digital environments has suggested the possibility of expert learning systems that incorporate intelligent scaffolding to adapt to the learner's pace and learning style. Responding to the educational philosophy of "social constructivism", computer math environments promise students a richer means of representing and presenting the fruits of their mathematical imagination. Finally, by making difficult concepts accessible, digital mathematics has helped to promote the vision of unifying research and teaching, theory and practice. Even if students are not able to understand all of the mathematical complexity of a real world optimization problem, with a coarse-grained understanding of the mathematics and the appropriate software, they may still be able to gain insight.

3.3 Intellectual Pitfalls ...

> *It is generally the way with progress that it looks much greater than it really is.*[3] ***Ludwig Wittgenstein (1889 - 1951)***

If the promise of digital mathematics may occasionally lead to increasing optimism about its potential impact on research and education, its pitfalls can rapidly lead to a more realistic assessment. One of the major pitfalls is inappropriate use which often manifests itself as employing technology because it can be done rather than because it should be done. Nowhere is this tendency more evident than in the recent plethora of Java

[2] *Maple* is a trademark of Waterloo Maple Inc.

[3] *Evelyn Toynton, "The Wittgenstein Controversy", The Atlantic Monthly 279, no. 5 (1997): 39.*

applets which model poorly in a digital environment a concept which is done nicely by hand. In this vein, one encounters Java implementations of Newton's method which accomplish nothing more than what can be accomplished by any first year calculus student with a pencil and straight edge. By contrast, the full Java version of Euclid's elements at <http://aleph0.clarku.edu/~djoyce/java/elements/elements.html>, provides a wonderful example of how this technology can be used effectively. Another pitfall is loss of focus. In *The Aims of Education* a 1929 essay directed to educators, Alfred North Whitehead implored "do not teach too many subjects" and "what you teach, teach thoroughly", as his two "educational commandments"[4]. Digital technology puts a vast source of potential curriculum topics at the disposal of educators. It is necessary to make intelligent choices; it would be impossible to teach all of them thoroughly and it is therefore essential to distinguish educationally sound topics from those that are merely superficially attractive. Corresponding to the promise of learner-centred curricula, there is a pitfall resulting from loss of control over what mathematics the student is learning and at what pace. This pitfall also has strong implications for the teacher. Is the role of the teacher merely altered or is it eroded?

Another pitfall is the degradation of long-lived robust mathematical knowledge. Mathematics has advanced largely through the careful aggregation of a mathematical literature whose reliability has been established by a slow but thorough process of formal and informal scrutiny. Unlike the other sciences, mathematical works do not need to be recent to be pertinent. The democratization of the web and the instant publishing forum that it ensues, can make it difficult to distinguish reliable sources from unreliable ones. Another pitfall is the tendency to look for a total solution resulting in a growing reliance on effectively closed architecture software sold by such large firms as I.B.M., Sun, and PeopleSoft. The complete educational authoring environments that these firms offer, place much stronger limits on the flexibility of mathematical courseware than are encountered if a variety of mathematics software is selected. Within both the educational and research communities, computer technology has the tendency to amplify disparity between the haves and have-nots. Class-based disparity can driven purely by the financial challenge that computer systems impose. In Vancouver, a West-side school holding a silent auction to raise funds for technology can fare much better than an East-side school undertaking similar fundraising events for the same

[4]Alfred North Whitehead, *The Aims of Education*, (New York: The Free Press, 1957), 2.

purpose. Along with economic differences, disparity can also arise from prejudicial assumptions about class and attitudes based on gender and race. Finally, the perfusion software which encourages degeneration to machine-based rote learning is troubling. Media reports have heralded the adoption by a number of Ontario and British Columbia school districts of a computer-based mathematics instruction system that attempts to deliver the full grades nine through eleven curricula. The highly process-oriented approach to math education that these systems typically adopt merits caution; their reliance on rote learning and patterning does little to teach students about how to think mathematically, albeit being much easier to produce.

3.4 Technical Promises...

> *'I don't really start', he said, 'until I get my proofs back from the printer. Then I can begin serious writing.'*[5]
> *Sir Alec Cairncross quoting* ***John Maynard Keynes (1883 - 1946)***

While underlining the privileges that accompany high public esteem, few quotes also show how much publishing has changed since the first half of this century; proofs are now created and edited instantly. Digital technology has promised a similar revolution in the way mathematics is conducted. From the beginning, computers have been called upon to perform big numerical computations, long sortings, and large searches; these are things that computers do really well where humans labour. Computers excel at tasks such as these that can be described nicely by algorithms. Human cognition is not strongly rooted in this type of descriptive approach and therefore an attempt should be made to take full advantage of the computer whenever an algorithm can be identified as solving a particular problem. Related to student centred learning, another promise is that computers have the potential to help facilitate the teacher's ability to meet students' individual demands. Another promise is access to global databases; it is important to note that this access must be construed as *free access to information* not *access to free information*. The development of expert systems promises to support mathematical research and education by providing intelligent querying to these databases and to make use of advances in the understanding of learning to incorporate software that is genuinely able to take account of learners' performance and

[5] Sir Alec Cairncross "Keynes the man", *The Economist*, 339, no. 1962 (1996): 76.

personal history. Finally, mathematicians, working with programmers, are on the verge of developing seamless digital mathematical workspaces which marry text and computation in the same document.

3.5 Technical Pitfalls...

'When Gladstone was British Prime Minister he visited Faraday's laboratory and asked if some esoteric substance called 'Electricity' would ever have practical significance. "One day, sir, you will tax it." was the answer.' (Science, 1994)

The cost of the electricity consumed to keep computers running is minor compared the other costs that they entail. Each of these costs is related to a number of important technical pitfalls; some deriving from the way technology is acquired and others to the way we use it. One of the most pervasive is that of legacy software and hardware. When technology is purchased, an investment is made in the level of technology that is available to us at the time of purchase. A further investment in time and effort is then made in learning how to adapt the individual's way of doing mathematics to the new technology. Upgrades in hardware and software and upgrades in personal technical proficiency notwithstanding, it is the initial purchase and the initial effort that is put into adapting behaviour, that will most strongly determine the ability to incorporate new hardware and software as it becomes available. It would be pointless to try and run the latest version of Mathematica™ on a 486 class machine. Another pitfall is that in any computer-based math system, the weakest link will determine its value.[6] Examples of this abound; Maple™ running on a *Pentium IV*™ will bog down on large calculations if there is a shortage of RAM.[7] Yet another technical pitfall lies in the tendency of advocates to promise unrealistic payoffs and underestimate the effort required to achieve even modest advances. The technology component of the reform calculus initiative provides a good example. In this case, a lot of effort was put into developing

technology based exercises which were supposed to emphasize real world applications of calculus. In practice, instructors found it difficult to make time available to actually use these exercises with their classes. Another pitfall is the infinite time-sink that developing good mathematics courseware can entail. This is especially true for higher level courses;

[6] *Mathematica* is a trademark of Wolfram Research Inc.

[7] *Pentium IV* is a trademark of Intel Corporation.

effectively programming representations of high level mathematics requires high level skills and programming effort. There is also less of a chance that a program developed to help in one course will be able to be used successfully by another instructor teaching the same topic as different instructors inevitably take quite different approaches to the material presented. Finally, there is the growing (and unavoidable) reliance on commercial software. To be feasible, the amount of effort required to produce good quality mathematical software will in most cases require a commercial return. There is no guarantee that what is in the best interest of commerce is in the best interest of mathematics.

3.6 Intellectual Property and Commercial Issues

> *Today, with the arrival of automation, the ultimate extension of the electromagnetic form to the organization of production, we are trying to cope with such new organic production as if it were mechanical mass production.*[8]
> ***Marshall McLuhan***

The transition from scribal culture to typographic culture represented a shift from loose notions of authorship with distributed loci of publication and limited distribution to firm notions of authorship with centralized loci of publication and mass distribution, the transition to electronic culture turns the equation inside out, presenting the possibility of distributed authorship via mass collaboration and multiple nodes of production with various forms of near instantaneous mass publication. The transformation that occurred in the foundations of mathematics, from a quest for a unified perspective to a modernist acceptance of a plurality of perspectives, finds resonance in the media environment in which the mathematical community exists and has the potential to affect the language, purposes, methods, and meeting places of the community. In a speech entitled *The Medieval Future of Intellectual Culture: Scholars and Librarians in the Age of the Electron*, professor Stanley Chodorow states:

> In the not-so-distant future, our own intellectual culture will begin to look something like the medieval one. Our scholarly and information environment will have territories dominated by content, rather than by distinct individual

[8] McLuhan, *The Gutenberg Galaxy*, p. 130.

> contributions. The current geography of information was the product of the seventeenth-century doctrine of copyright. We are all worrying about how the electronic medium is undermining that doctrine. In the long run, the problem of authorship in the new medium will be at least as important as the problem of ownership of information.
>
> ...
>
> Works of scholarship produced in and through the electronic medium will have the same fluidity - the same seamless growth and alteration and the same de-emphasis of authorship - as medieval works had. The harbingers of this form of scholarship are the listservs and bulletin boards of the current electronic environment. In these forums, scholarly exchange is becoming instantaneous and acquiring a vigor that even the great scholarly battlers of old - the legendary footnote fulminators - would admire. Scholars don't just work side by side in the vineyard; they work together on common projects[9]

Applied to mathematics, Chodorow's ideas suggest the possibility that the community's elites, long having been composed of those individuals who demonstrate a particular "*individual* vision and brilliance", may undergo a process of reconstruction, resulting in elites whose members are those who have learned how to start with good ideas and develop them by using the Internet to harness the intellectual power of the community. In an age of massively parallel mathematical computation, the potential exists for massively parallel mathematical collaboration. Perhaps the best idea of what a fully digital mathematical scholarship and teaching environment *might* look like can be gleaned from the "hacker culture" of the open source programming community. The meeting places of this community are primarily email, threaded bulletin boards, and implementations of the Concurrent Version System. Those who identify themselves as members, speak of the community's "gift culture" which rewards the most talented and generous of members with status in the community meritocracy. In the opening section of *The Cathedral and the Bazaar*, Eric S. Raymond describes hacker culture:

> Many people (especially those who politically distrust free markets) would expect a culture of self-directed egoists to be fragmented, territorial, wasteful, secretive,

[9]Stanley Chodorow, "The Medieval Future of Intellectual Culture: Scholars and Librarians in the Age of the Electron", *Association of Research Libraries Proceedings* 189, [online]. 1996.

> and hostile. But this expectation is clearly falsified by (to give just one example) the stunning variety, quality and depth of Linux documentation. It is a hallowed given that programmers hate documenting; how is it, then, that Linux hackers generate so much of it? Evidently Linux's free market in egoboo [coined by the author for 'ego boost'] works better to produce virtuous, other-directed behavior than the massively-funded documentation shops of commercial software producers.[10]

He goes on to invoke the idea of a "community of interest":

> I think the future of open-source software will increasingly belong to people who know how to play Linus's game, people who leave behind the cathedral and embrace the bazaar. This is not to say that individual vision and brilliance will no longer matter; rather, I think that the cutting edge of open-source software will belong to people who start from individual vision and brilliance, then amplify it through the effective construction of voluntary communities of interest.[11]

If, indeed, *doing mathematics* in the digital age were to develop in a similar fashion to the way that *doing software development* has in the open source community, then the mathematics community must prepare itself for the loss of fixed notions of authorship and ownership and the accountability and economic models that those notions sustain.

Another intellectual property issue concerns the relationship between supervisor and student. As job security disappears more students see intellectual property as their future: to wit, there is the case of Ma vs Phong and Stein at Columbia University in which a grad student is suing his supervisor and a collaborator of his supervisor claiming that they stole parts of his work for articles that they submitted for publishing. While this suit is almost certainly groundless, this type of conflict is likely to become more frequent in the future. Finally, there is the researcher as CEO: conflicts of interest are inevitable. They must be declared, however, they are rarely resolved. Increasingly, researchers find themselves in the position of not only needing to capitalize on their intellectual property for their personal benefit, but also to help fund their research. Research may well get done or not done based primarily on intellectual property and disclosure issues.

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[10] Eric S. Raymond, *The Cathedral and the Bazaar*, (published by Eric S. Raymond under Open Publication License, 2000), [online].

[11] Raymond, *The Cathedral and the Bazaar*.

The extent to which mathematical software is adopted is strongly influenced by intellectual property issues. Different stakeholders can have very different views about what constitutes fair consideration for intellectual property in a particular setting. On the user side, one might expect the views of supervisors and teachers to differ from those of students and parents. On the producer side, different commercial entities will adopt different approaches to acquiring and protecting intellectual property.

Commercial issues also strongly influence the type of mathematical software available and how it is adopted. The rules of the marketplace dictate that neither can you make what you can't sell nor can you sell what you can't make. As far as innovation goes, the edu-software business is characterized by a strong conservatism which is reinforced by the lack of a working model for research and development. Traditional publishing houses, for example, tend to view software like books offering royalties that are too small to justify the development required and shying away from anything but "work-for-hire" when contracting for software. Another issue is the commoditization of mathematical knowledge associated with the use of a standard set of commercial authoring tools. Software produced by these tools ends up having the same look and feel. Of greater pedagogical concern is that parts of the curriculum that can not be represented in the chosen authoring environment are jettisoned. Finally, there is the question of whether the latest and greatest software is necessary to meet educational (or research) objectives?

3.7 Suggestions and Conclusions ...

The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.[12]
Jacques Hadamard 1865 - 1963

I have no satisfaction in formulas unless I feel their numerical magnitude.[13]
Lord Kelvin 1824 - 1907

The above quote serves as a reminder of what one strives for with the appropriate use of technology in mathematical research and education. Some suggestions and conclusions

[12] G. Polya, *Mathematical discovery: On understanding, learning, and teaching problem solving (Combined Edition)*, (New York: John Wiley and Sons, 1981), vol. 2, 127.

[13] Furman University *Mathematical Quotations Server* [online], 2003.

follow. First, in order to choose the appropriate technology, it is important to clearly identify expectations and equally important to be realistic about the learning curve. Advanced software such as *Mathematica*™or *Maple*™, if used wisely, can yield great educational rewards however time must be alloted in the curriculum for the students to learn how to use it. Another recommendation is in order to avoid some of the pitfalls associated with the use of proprietary software, endeavour to use open architecture software whenever possible. Furthermore, if when participating in software development, it is advisable to consider the formation of not for profit and 'pre-competitive' consortia in order to share expertise and access to markets and to bolster the ability to compete with larger companies.

For mathematicians, mathematical software development provides the opportunity to recapture computing from mathematics' sister sciences. From the point of view of pedagogy and human-computer interface issues, it is now possible to take advantage of recent advances not only in software design but also in the field of cognitive neuroscience in which researchers have begun to investigate the neurophysiology of mathematical thought. The *Malthusian principle* that "expectations outstrip performance", dictates that good technology will never be cheap. This does not prevent us from hoping that in the not too distant future, appropriate technology will be accessible to all students and research communities. In conclusion, a final quote from Alfred North Whitehead the spirit of which serves as a reminder to consider the challenges and potential that computers present to mathematics educators.

> *...so long as we conceive intellectual education as merely consisting in the acquirement of mechanical mental aptitudes, and of formulated statements of useful truths, there can be no progress; although there will be much activity, amid aimless rearrangement of syllabuses, in the fruitless endeavour to dodge the inevitable lack of time.*[14]

[14] Alfred North Whitehead, "The Rythmic Claims of Freedom and Discipline", *The Aims of Education*, (New York: The Free Press, 1957), 31.

Chapter 4

Managing Digital Mathematical Discourse

> *According to the social constructivist account, new mathematical knowledge claims are constructed by individual persons, or groups of individuals working jointly, within the context of a mathematical practice or tradition. They are formulated linguistically (understood broadly to include diagrams, mathematical symbols, and other forms of representation) with reference to traditions of their form and content (composing part of the tacit knowledge of mathematics).*[1]
> ***Paul Ernest***

"Conversation and Rhetoric" is perhaps an unusual chapter title for a text on the philosophy of mathematics; however, in *Social Constructivism as a Philosophy of Mathematics*, this is the title that Paul Ernest choses to introduce his discussion of the problems associated with the exchange of mathematical knowledge.[2] Stating that "the identity and equivalence of linguistic forms and expressions that are admitted vary according to time, community, and context", Ernest cites three factors that have combined to distinguish mathematical knowledge exchange from the exchange of knowledge in other disciplines.[3] The first is the "explosive growth" of mathematics; he cites Davis and Hersh who have estimated that by the end of the twentieth century, mathematics was comprised of approximately thirty-four

[1] Ernest, p. 173.
[2] Ibid p. 162.
[3] Ibid. p. 197.

hundred subspecialisms. The second is the growing but robust set of mathematical symbols and icons. The third is the relative invariance of mathematical rule-based transformations which help preserve semantics across differences in time of writing, mathematical subspecialism, and even natural language. It is notable that much of eighteenth century mathematics remains readable today and it was not too long ago that Euclidean geometry was taught from the books of Euclid.

In the digital context, both the nature of mathematical knowledge and its increasingly high volume exchange present a serious challenge to knowledge management systems. The technical response to this challenge has been an attempt to fix ontologies typically in the form of XML based metadata standards; the OpenMath and MathML projects providing the two most notable examples. This chapter considers the use of flexible ontologies for the purpose of managing the "mathematical conversation" in its varied forms and is adapted from a paper presented at *The Second International Conference on Mathematical Knowledge Management (MKM 2003).*[4]

4.1 MKM's Intellectual Pedigree

Involving research mathematicians as well as specialists from such diverse fields as librarianship, education, cognitive science, and computer science, there are currently a wide range of initiatives and projects that may be considered as belonging to the field of Mathematical Knowledge Management (MKM). A perusal of the *Proceedings of the First International Workshop on Mathematical Knowledge Management* reveals that presentations with a focus on best practice in the exchange mathematical documents in digital environments, such as a presentation on the recommendations of the International Mathematics Union's *Committee on Electronic Information and Communication*, shared time with presentations focussed on foundational concepts, such as a presentation on the underlying logic and language of the *Theorema* theorem proving system.[5] The juxtaposition of topics represented by these two

[4] Jonathan Borwein and Terry Stanway, "Managing Digital Mathematical Discourse", *Lecture Notes in Computer Science no. 2594: Mathematical Knowledge Management: proceedings of The Second International Conference*, Andrea Asperti, Bruno Buchberger, and James C. Davenport eds., (Berlin: Springer-Verlag, 2003) 45. **(Portions of this chapter have been reprinted by permission of Springer-Verlag.)**

[5] The former presentation was by Dr. Jonathan Borwein and the latter by Dr. Bruno Buchberger. (Unpublished proceedings of the *First International Workshop on Mathematical Knowledge Management.*)

presentations represents a fundamental duality of focus in the field of MKM, as the intellectual foundations of the two presentations are distinct: those of the former stretching back to the libraries of antiquity and those of the latter, while more recent, reaching back at least as far as Leibniz' seventeenth century call for a *calculus philosophicus*.[6]

From this perspective, the pre-history of MKM is the dual histories of mathematical librarianship and mathematical logic. While both of these are 'meta-fields' in the sense that both are *about* mathematics, it would not have been immediately obvious to a pre-digital intellect that they share anything else in common. That emerging computer and network related technologies have redefined these fields in such a way that there are now good reasons to consider them as aspects of a single field, is an example of how a shift in media can lead to a shift in perspective.[7] The benefit of analysing MKM's intellectual pedigree is that not only does it help bring into focus the field's central preoccupations but it also helps identify some underlying tensions. From the librarianship side, MKM has inherited a concern for preservation, metadata, cataloguing, and issues related to intellectual property and accessibility. From the mathematical logic side, MKM has inherited a concern for foundations and issues related to automated or guided proof generation. Both traditions have bequeathed a concern for authentication of knowledge, albeit in different contexts. One task that is a concern in both of these two founding fields but is treated differently in each is the question of how to establish the underlying semantics that any exercise in information sharing requires. From a knowledge management perspective, this is the question of *ontology definition* and in the following section, we examine some of the problems presented by ontology definition in MKM.

4.2 Ontology Definition

The philosophical concept of *domain ontology* has important implications for MKM. In their *Scientific American* article, *The Semantic Web*, Berners-Lee, Hendler, and Lassila define "ontology" in the context of artificial intelligence and web-based applications:

> ...an ontology is a document or file that formally defines the relations among

[6] Gottlob Frege, "Begriffsschrift", *From Frege to Gödel: A Sourcebook in Mathematical Logic, 1879-1931*, Jean van Heijenoort ed., (Cambridge: Harvard University Press, 1967) 6.

[7] The conference description for the *First International Workshop on Mathematical Knowledge Management* describes MKM as an "exciting new field in the intersection of mathematics and computer science".

> terms. The most typical kind of ontology for the Web has a taxonomy and a set of inference rules.[8]

For our purposes, we will bear in mind this broad definition, but seek out a more precise description in order to address ontology problems in MKM. In particular, our focus will be on the *discourse* of mathematical communities as opposed to their literature and we will consider management problems arising from the informal exchange of information conducted in the shared vocabularies of communities that make up the broader mathematical community.

In *The Acquisition of Strategic Knowledge*, Thomas R. Gruber describes five overlapping stages of knowledge acquisition. These are: identification, conceptualization, formalization, implementation, and testing.[9] While originally conceived for the build-up of knowledge in expert systems, these five stages provide a useful framework for the description of MKM knowledge management tasks. In particular, the description of the conceptualization stage draws from the terminology of *ontological analysis*, specifying three distinct aspects of the broad ontology: static ontology, dynamic ontology, and epistemic ontology. Gruber describes this stage as follows:

> Conceptualization results in descriptions of categories or classes of the domain objects and how objects are related (the static ontology), the operators, functions, and processes that operate on domain objects (the dynamic ontology), and how all this knowledge can be used to solve the application task (the epistemic ontology).[10]

By way of an example, consider the application of the language of conceptualization stage ontological analysis to a typical MKM knowledge retrieval task: the discovery of publications which mention the *Bartle-Graves theorem* in their title, abstract, or keywords. In this case, the static ontology includes a definition of the 'publication', 'title', 'abstract', and 'keywords' entities as well as a definition of the entities to be searched. The dynamic ontology includes a definition of the protocols and processes involved in information access and retrieval; this

[8]Tim Berners-Lee, James Hendler, and Ora Lassila "The Semantic Web", *Scientific American*, 284 no. 5 (2001): 39.

[9]Thomas Gruber, *The Acquisition of Strategic Knowledge*, (New York: Academic Press, 1989), 128.

[10]Gruber, p. 128.

could be as simple as a SQL search on a fixed database or a more complex specification such as that of an agent-based query of a remote database. The epistemic ontology defines the interface level entities, both style related and logic related which will be invoked to determine the manners in which the information request and results interfaces may be presented to the user. According to a defined set of criteria, if the knowledge acquisition cycle is effective, the user is presented with output that, in some useful manner, lists publications that are related to the *Bartle-Graves theorem* along with relevant background information regarding these publications. A more sophisticated epistemic ontology may present related information based on an inference concerning what type of information might be of use to a particular user.

The dual inheritance of MKM is reflected in the definition of static ontologies. Applications that draw more strongly from the librarianship tradition, admit degrees of flexibility and ambiguity in their ontologies. Both Math-Net's *MPRESS* and the NSF funded *arXiv* mathematical document servers admit weakly defined elements in their metadata sets, asking submitting authors to make subjective assignments of topic descriptors.[11] Applications that draw from mathematical logic depend on highly fixed ontologies. The static ontology of *Theorema* is encoded at the implementation level as highly structured 'Theorema Formal Text', an implementation of high order predicate logic.[12]

Negotiating differences in ontologies is part of human communication. It is therefore not surprising that applications that have evolved from the highly human-centred discipline of librarianship have inherited a tolerance for subjectivity and a degree of ambiguity. It is similarly not surprising that applications that have evolved out of the field of mathematical logic depend upon fixed and highly defined ontologies. Certainly, ontology resolution is a problem that must be addressed in any effort to interconnect MKM applications. While much work has been done in this area resulting in significant progress, notably, by the *OpenMath* and *OMDoc* research groups, much work remains. Consider, for example, the task of determining whether the proof of a given proposition either exists in the literature or can be automatically generated. Significant refinements of current technology are required before this determination can be reliably accomplished by a purely agent-based query of

[11]In the case of MathNet, these descriptors are referred to as 'keywords' and in the case of arXiv, they are referred to as 'Mathematical Categories'.

[12]Bruno Buchberger, "Mathematical Knowledge Management in Theorema", proceedings of *The First International Workshop on Mathematical Knowledge Management*, ed. Olga Caprotti, unpublished proceedings (2001), 3

proof repositories or theorem proving systems. The task is made more difficult if the proposition is originally expressed using a human-centred application which accepts input, such as LaTeX or *Presentation MathML*, that maps directly to standard mathematical text. In this case, it is easy to imagine that some form of challenge and response interaction may be necessary in order to determine the semantic content of the query.

While the problems associated with ontology negotiation in MKM have received considerable attention, equally germane is the fundamental question of ontology construction. Motivated by the desire to build applications from a solid foundation in predicate logic, explicit attention to ontology construction has historically been a characteristic of research in artificial intelligence. This research has resulted in a number of languages and methods for building ontologies such as *Knowledge Interchange Format.*[13] Of note, in the domain of MKM, is the work done by Fürst, Leclère, and Trichet in developing a description of projective geometry based on the *Conceptual Graphs* model of knowledge representation.[14] Ontology definition is made more difficult if it is impossible to describe *a priori* aspects of the knowledge domain. This is precisely the case with *grey literature* in which elements of the static, dynamic, and epistemic ontologies will inevitably need to be extended as new fields and forms of knowledge are defined. In the next section, we examine the need for a flexible and extendable ontology in managing digital mathematical grey literature and mathematical discourse in general.

4.3 The Digital Discourse

As mathematical activity is increasingly conducted with the support of digital network helper technologies, it is becoming increasingly possible to address the questions of to what extent, and for what purposes, this activity can be captured and archived. The types of mathematical activity that are conducted with the aid of digital networks cover a spectrum with highly informal activities such as queries to search engines and mathematical databases at one end and, at the other end, activities of a much more formal nature such as the publication of papers in online journals and preprint servers. As the majority of

[13]The Logic Group - Stanford University, *Knowledge Interchange Format* [online], 2003.

[14]Frédéric Fürst, Michel Leclère, and Francky Trichet, *Contribution of the Ontology Engineering to Mathematical Knowledge Management* proceedings of *The First International Workshop on Mathematical Knowledge Management*

online mathematical activity is informal, the challenge that it presents to MKM is akin to the challenge presented by "grey literature" to the field of librarianship. A simple Web search reveals that the question of what constitutes grey literature is very much open to interpretation. Definitions range from the restrictive "theses and pre-prints only" to more inclusive interpretations such as the following from the field of medical librarianship:

> In general, grey literature publications are non-conventional, fugitive, and sometimes ephemeral publications. They may include, but are not limited to the following types of materials: reports (pre-prints, preliminary progress and advanced reports, technical reports, statistical reports, memoranda, state-of-the art reports, market research reports, etc.), theses, conference proceedings, technical specifications and standards, non-commercial translations, bibliographies, technical and commercial documentation, and official documents not published commercially (primarily government reports and documents).[15]

This idea of the "fugitive and sometimes ephemeral" nature of grey literature is particularly apt when applied to expression conveyed via digital networks. For purposes which will be discussed presently, we choose to adopt a definition of digital mathematical discourse (DMD) which encompasses the full *grey to white spectrum*. A list of examples from this spectrum might contain such diverse mathematical entities as email exchanges, bulletin boards, threaded discussions, CAS worksheets, transcripts of electronic whiteboard collaborations, exam questions, and database query strings as well as preprints and published papers. The motivation for this open-ended definition is the potential that methods of archiving and retrieving DMD hold for both gaining insight into the nature of mathematical activity and facilitating productivity in mathematical activity. We turn now to the problem of specifying an appropriate ontology development framework.

4.4 Ontology Development for Digital Mathematical Discourse

There are two reasons that careful consideration of an ontology development framework for DMD is important. The first relates to the "Web Services" aspect of the Semantic Web specification and the potential for DMD oriented applications to both harvest and be

[15]The New York Academy of Medicine, *What is Grey Literature* [online], 2003.

harvested from, either manually or via agents. The second is more complicated and is related to the emerging state of metadata standards. Jokela, Turpeinen, and Sulonen have argued that if an application's main functions are content accumulation and content delivery, then a highly structured ontology definition and corresponding logic is unnecessary. In this case, ontologies may effectively be defined by way of formally specified metadata structures.[16]

One problem confronting DMD ontology development is that while it would be enticing to simply make the ontology implicit in an *application profile* combining, for example, the *Dublin Core* and the related *Mathematics Metadata Markup* metadata specifications, it is not clear that such a profile would be rich enough or offer fine enough granularity to meet the needs of a DMD specification. A second problem originates from the objective that DMD applications be acessable to the broader mathematical community, including individuals with possibly limited understanding of the metadata standards at their disposal. A lack of understanding of metadata standards introduces the possibility of inappropriate use of taxonomy and the unnecessary use of ad hoc taxonomy. For these reasons, for the definition of new forms of mathematical expression, an appropriate ontology based on a profile of existing metadata standards, must exhaust those standards and then encourage the intelligent use of ad hoc taxonomy to complete the definition to the desired degree of granularity. As the dominant metadata standards become more fully refined, the ad hoc component should be reconciled with the existing metadata profile. Figure 4.1 represents a component of the static ontology of the type of threaded discussion that might be motivated by an online class discussion; ad hoc elements are connected into the diagram by dotted lines. This ontology is based on an application profile that references the *Dublin Core* and *Educational Modeling Language* name spaces.

The extension of a metadata application profile by the addition of elements that are not defined in the schemata that make up the profile leads to potential management problems. These include incomplete object definition, in which not enough elements are defined to allow the desired degree of granularity, as well as contradictory or superfluous definition of elements. While it is certainly possible, depending on the size of the knowledge base, that these problems be resolved manually, the possibility of at least partially automating the management process merits attention.

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[16]Sami Jokela, Marko Turpeinen, and Reijo Sulonen, " Ontology Development for Flexible Content", *Proceedings of the Hawaii International Conference on System Science (IEEE)*, vol 6, (1998): 6506

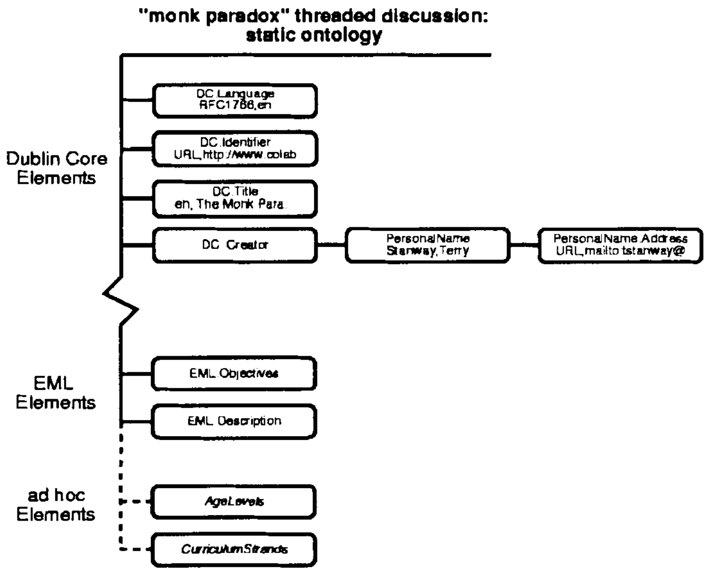

Figure 4.1: The Monk Paradox: Online Class Discussion

4.5 Conclusion and Future Work

Due largely to its diverse intellectual inheritance, the field of Mathematical Knowledge Management remains divided across a spectrum with applications whose main focus is derived from the field of mathematical librarianship at one end and applications whose main focus is derived from the field of mathematical logic at the other. In between, are applications such as Computer Algebra Systems, function libraries, and reverse look-up interfaces that interact with each other only with human intervention. Each of these applications are, in one respect or another, agents in the process of mathematical knowledge exchange. While the variety of MKM oriented applications reflects both the increasing diversity of the mathematics community and the needs of its various sub-communities, it presents a challenge to interaction between different MKM systems. While to date, much important work has focused on defining ontologies by way of metadata standards which are designed to facilitate low level data exchange between applications, it is not clear that fixed ontologies will provide the richness and variety needed by MKM systems designed to directly facilitate mathematical conversation between members of mathematical communities.

Dedicated to the investigation of advanced digital collaboration in mathematics research and education, the *CoLab* is part of Simon Fraser University's Centre for Experimental and Constructive Mathematics. Work is presently being undertaken on the design of MKM

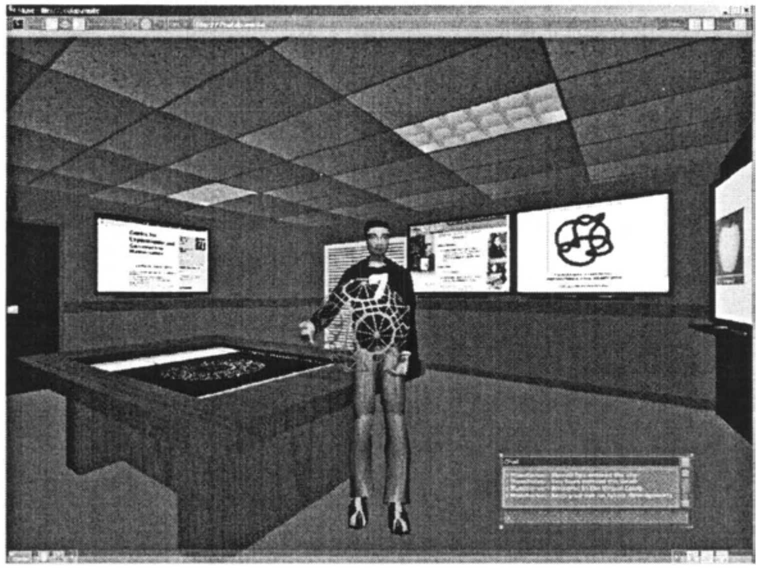

Figure 4.2: The (virtual) CoLab with avatar.[17]

interfaces which support dynamic ontologies; this work is described in more detail in the appendix.

Chapter 5

A Framework for the Analysis of MKM

> ...the invention of an ABC of man's thought was needed, and by putting together the letters of this ABC and by taking to bits the words made up by them, we would have an instrument for the discovery and testing of everything...if we had a body of signs that were right for the purpose of our talking about all our ideas as clearly and in as true and as detailed a way as numbers are talked about in Arithmetic or lines are talked about in the Geometry of Analysis, we would be able to do for every question, in so far as it is under the control of reasoning, all that one is able to do in Arithmetic and Geometry...[1]
>
> ***Gottfried Wilhelm von Leibniz (1646 - 1716)***

An examination of the recent history of MKM reveals a rich interplay of technological, philosophical, and economic influences as well as the influence of ideas from within the field of mathematics itself and its sister fields of logic and computer science. Depending upon their intellectual foundations, MKM initiatives may be roughly classified according to the degree to which they impose absolute *vs.* relativistic mathematical epistemologies, centralised *vs.* decentralised nodes of control, and fixed *vs.* flexible ontologies. Due to external societal factors, present trends in MKM are leading to increased diversity in the methods by which individuals and communities choose to represent, publish, and access mathematical

[1] Hersh p. 125.

knowledge. In a previous chapter, it has been argued that any attempt at MKM carries, either explicitly or implicitly, assumptions about the nature of mathematical knowledge and that it is through these assumptions that MKM defines epistemic communities within the broader mathematics community and thus gives shape to the community as a whole.

Uttered in his personal journal, the opening quote relates Leibniz' dream of an analytical approach to human reasoning. Almost 300 years further on, there is still no comprehensive "arithmetic of ideas" in sight however methods have been developed to aid decision making analysis in situations where reasonable assumptions can be made about human behaviour. This chapter is presented as a proposal regarding directions for the application of some of these methods to decision making in MKM.

The complex combination of factors affecting the development of MKM at this cusp between the typographic and digital eras encourages the point of view that MKM, like all forms of knowledge management, is a social experiment in progress. From this perspective, the development of MKM is an organic process under which its defining characteristics attain some form of equilibrium when, and if, stability comes to its various influencing factors. While this perspective respects the complex nature of MKM's development, it down plays the fact that the present state of MKM is the result of human decision making. Decisions made by individuals and groups concerning policy and the allocation of time and resources define the state of the MKM environment. These decisions range from the strictly personal, such as whether or not to prepare a mathematical document in LaTeX to the increasingly far-reaching, such as decisions by mathematical software firms concerning product development or decisions by educational policy makers concerning the use of information technology in mathematics education. At the global level, organizations such as the International Mathematics Union's *Committee on Electronic Information and Communication (CEIC)*, the *OpenMath Society*, and the *Math Working Group* of the World Wide Web Consortium enact decisions regarding standards and recommendations that are reported to the mathematics community as a whole. That all of these decisions necessarily will be made with incomplete knowledge, does not preclude the potential for the development of an analytical framework to support the decision making process. Indeed, the emergent nature of the digital era presents the possibility that decisions made today concerning the allocation of MKM resources will have longstanding effects. At stake are tested and valued systems of knowledge verification and distribution as well as the economic models and community structures that support them. With care, and perhaps some luck, the mathematics community may avoid QWERTY keyboard like legacies in its knowledge management systems.

Before proceeding with the discussion of an analytical framework, it is necessary to characterize good MKM decision making. This endeavour necessarily confronts the social nature of knowledge management and recognizes MKM artifacts, be they papers, Web pages, knowledge bases or other forms of represented knowledge, as rhetorical instruments designed to convey meaning to one or more communities within the mathematics community. In this respect, individuals make good MKM decisions when they choose to represent knowledge in a manner that is meaningful and useful to the intended audience. This criterion extends to group and community oriented policy making as 'good policy favours meaningful and useful forms of knowledge representation'. This rather obvious statement is more interesting when considered with respect to the variety of digital MKM systems which are currently being tested by the mathematics community. Here, the criterion must be interpreted to imply that effective systems support meaningful and useful knowledge representation within communities and may even encourage community formation.

When confronted with a decision regarding mathematical knowledge representation, individuals must consider the relative utility of the various options available to them. Decisions by groups concerning MKM policy or the implementation of an MKM system similarly will be based on the relative utility of the options. While it is possible that various utility criteria, such as financial incentives or access to technical expertise, may enter into the decision process, the present discussion considers decisions for which the sole criterion is the adoption of meaningful and useful forms of knowledge representation. In the following section, the MKM environment is characterized by a set of causal concepts which are then organized into a *Fuzzy Cognitive Map.*

5.1 A Complex Systems Approach to MKM Decision Making

Taking as its foundation the multi-valued set theory of Lotfi Zadeh, the concept of a *Fuzzy Dynamical System* has been developed by Kosko and others as a means of modelling systems in which statewise knowledge is either inherently uncertain or is affected by feedback processes to the extent that certainty is impractical.[2] In its simplest form, a *Fuzzy Cognitive Map* (henceforth, *FCM*) is a static construct which represents the causal concepts of a

[2]Bart Kosko, *Neural Networks and Fuzzy Systems*, (Englewood Cliffs: Prentice Hall, 1992), 152–170.

system as nodes of a digraph with edges taking values in $-1, 0, 1$. If concept C_i stimulates concept C_j (*i.e.* an increase in C_i causes an increase in C_j), then the corresponding edge e_{ij} is set to 1. Edge values of 0 and -1 represent neutral and detrimental relationships respectively.

The process of *FCM* model development has several stages. In the first stage, expert opinion is called upon for the definition of a set of causal concepts which describe the system. In the second stage, edge values are assigned to the relationship between concepts. These edge values may be arrived at through the analysis of relevant data, if such data is available, or by again calling on expert opinion. By way of example, the MKM environment is described by the following rubric of causal concepts:

1. *Perceived Need for Formal Knowledge Representation* - a measure of the desire for formal artifacts, such as journal style papers, amongst members of the target communities

2. *Perceived Need for Formal Validation* - a measure of the desire for formal methods of knowledge validation, such as peer review, amongst members of the target communities

3. *Perceived Need for Immediacy* - a measure of the desire for "instant gratification" amongst members of the target communities; to what extent must knowledge publication and retrieval be achievable with "the click of a mouse"

4. *Perceived Need for and Acceptance of Mass Publication* - a measure of the desire for and acceptance of broad, barrier-free dissemination of knowledge artifacts; the placement of papers on pre-print servers as opposed to their being restricted to subscribed journals

5. *Perceived Need for Maintenance of Community Structure* - a measure of the desire to maintain the roles which give the community its structure; one aspect of this factor is the desire to maintain community elites

6. *Perceived Need for and Acceptance of Collaboration* - a measure of the desire to support collaborative work and the acceptance of artifacts produced through collaboration

7. *Advances in MKM Media* - a measure of the anticipated influence of new hardware and software

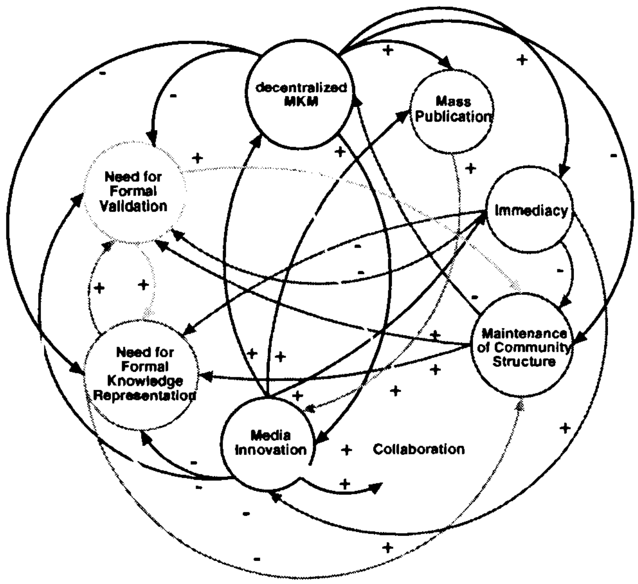

Figure 5.1: An FCM for MKM

8. *Acceptance of Decentralized MKM* - a measure of the willingness to accept and the desire for MKM systems characterized by flexible ontologies and decentralized nodes of control

Figure 5.1 depicts a possible *FCM* for MKM as described by these causal concepts. This *FCM* is encapsulated by the following causal connection edge matrix **K**:

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}$$

where K_{ij} is the effect of concept i on concept j.

The activation stage involves testing the model against an MKM decision. For a group, this decision might be framed along the lines of "given our present and anticipated resources and our understanding of trends and causal factors related to MKM, is our support of MKM initiative A a good idea". The test involves:

1. stating a concept vector, C_0, that corresponds to the MKM option being tested,
2. evaluating $\hat{C}_1 = K * C_0$,
3. evaluating $C_1 = f(\hat{C}_1)$, where f is an appropriate *threshold function*.

The process continues with $\hat{C}_i = K * C_{i-1}$ and $C_i = f(\hat{C}_i)$ until $C_i = C_{i-1}$ making C_{i-1} a fixed point of the *FCM* dynamical system.

In the MKM example, to test a decision about whether or not to support an initiative that advocates a strongly decentralized approach to MKM, a C_0 concept vector is constructed with each element switched to 0 except that corresponding to distributed MKM which is switched to 1:

$$C_0 = (0, 0, 0, 0, 0, 0, 0, 1).$$

A typical threshold function for this test would be defined as follows:

$$f(\hat{C}_i^k) = \begin{cases} 1 & \text{if } k = 8 \\ 0 & \text{if } k \neq 8 \textit{ and } \hat{C}_i^k < \frac{1}{2} \\ 1 & \text{if } k \neq 8 \textit{ and } \hat{C}_i^k \geq \frac{1}{2} \end{cases}$$

With this definition, the "decentralized MKM" concept is *clamped* at 1 while each of the other concepts will be set to either 1 or 0 at each iteration. The iterations proceed as follows:

$$\begin{aligned} K * C_0 = \hat{C}_1 &= (0, 0, 0, 0, -1, 1, 1, 0) \\ f(\hat{C}_1) = C_1 &= (0, 0, 0, 0, 0, 1, 1, 1) \\ K * C_1 = \hat{C}_2 &= (0, 0, 1, 1, -1, 2, 2, 2) \\ f(\hat{C}_2) = C_2 &= (0, 0, 1, 1, 0, 1, 1, 1) \\ K * C_2 = \hat{C}_3 &= (0, 0, 1, 1, -1, 2, 4, 3) \\ f(\hat{C}_3) = C_3 &= (0, 0, 1, 1, 0, 1, 1, 1) \end{aligned}$$

$C_2 = C_3$ which implies that $(0, 0, 1, 1, 0, 1, 1, 1)$ is a fixed point. This simple analysis postulates that a commitment to a decentralized MKM initiative is consistent with an

MKM environment that favours immediacy, mass publication, collaboration, and innovation in media.

By the same process, clamping each of 'Formal Knowledge Representation', 'Formal Validation', and 'Maintenance of Community Structure' in separate executions leads to the steady state vector $(1,1,0,0,1,0,0,0)$ suggesting that these three factors of the MKM environment are mutually consistent.

FCM modelling provides a very simple method for representing the causal relationships that characterize a particular field. Experts in the field can quickly define sets of causal concepts and generate the corresponding FCM. With modern computational tools, calculating the steady state of a given input, if it exists, is not hard and cycles are easy to detect. Along with a consideration of the history and trends of MKM and knowledge management in general, the result of an FCM iteration is readily interpreted and provides another means of formulating decisions in MKM. The approach does, however, have weaknesses which, if unaccounted for, may lead to misleading statements.

A fundamental weakness of an FCM approach is described by Kosko:

> Yet an FCM equally encodes the expert's knowledge or ignorance, wisdom or prejudice. Worse different experts differ in how they assign causal strengths to edges and in which concepts they deem causally relevant. The FCM seems only to encode its designers biases and may not even encode them accurately.[3]

By way of a partial solution to this problem, Kosko proposes an algorithm for the combination of adjacency matrices generated by different experts. If k experts generate k adjacency matrices, $\{E_1, E_2, \ldots, E_k\}$, then a combined FCM adjacency matrix F is constructed by:

1. determining n, the total number of distinct concepts used by the experts
2. ordering the set of concepts as $C_1, C_2, \ldots, C_n$
3. for each expert's adjacency matrix, $E_i, i \epsilon \{1, 2, \ldots, k\}$, construct the corresponding $n \times n$ *augmented adjacency matrix*, F_i, by permutating the rows and columns of E_i to match the concept ordering in step 2 and adding 'zero' rows and columns where necessary

[3]Kosko p. 155.

4. calculating F as the normalized sum of the F_i:

$$F = \frac{1}{k}\sum_{i=1}^{k} F_i.$$

If each expert chooses adjacency matrix edge values in {-1,0,1}, the elements, f_{ij} of F will take values in $[-1, 1]$. As Kosko points out, given a sufficiently large representation of experts, it should be possible to state a confidence measure for how closely F matches an ideal population adjacency matrix. Even this approach, however, can not avoid the inherent uncertainties associated with ranking by consensus implied by Arrow's theorem.

There are limitations of the FCM model that go beyond the vagaries of capturing expert opinion. In *Fuzzy Logic for Business and Industry*, Cox outlines three of these limitations.[4] The first concerns the meta-analysis nature of the FCM approach. FCM models are based on statements about the causal relationships between factors affecting a system and not from any raw data that might be available through monitoring the system. For an MKM system, it is possible to imagine a situation where data concerning the frequency of particular types of interaction with a Web site might be germane. The FCM approach does not afford an easy way of incorporating this data directly into the model. A second limitation stems from the static nature of the FCM. While causality is indicated by arrow directions and edge weights, it is very difficult to separate out any but the most direct *lead-lag* relationships between concepts. A third limitation is related to computability. As the number of concepts, n, belonging to an FCM grows, the size of the adjacency matrix grows only as n^2. A problem arises with an exhaustive analysis of the system in which it might be desirable to clamp not only single concepts but all, or almost all, possible combinations of concepts. Unfortunately, the number of possible input concept vectors, C_0, grows as 2^n. The situation is even worse if it is desirable to look at a significant subset of the possible adjacency matrices. Assuming that the elements of the adjacency matrix take values in $\{-1, 0, 1\}$, the number of possible matrices grows as 3^{n^2}.

The usefulness of FCM's in analyzing MKM systems may potentially be improved by allowing the nodes of the concept map to be updated by learning algorithms as the execution stage passes through its iterations. Research in this area applies ideas from the theory of *neural networks* to the construction of dynamic FCM's. Seminal work by Kosko focuses

[4] Earl Cox, *Fuzzy Logic for Business and Industry*, (Rockland: Charles Media Inc., 1995), 360.

on *Differential Hebbian Learning* algorithms.[5] Named in honour of the Canadian cognitive psychologist Donald Hebb, Hebbian learning is perhaps best explained in Dr. Hebb's own words from his influential connectionist text, *The Organization of Behaviour*:

> When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that A's efficiency, as one of the cells firing B, is increased.[6]

For FCM's, this general rule implies that if two concepts are either augmented or diminished simultaneously at any given step in the iteration process then the weight of the edge adjoining them is increased. Variations of Hebbian learning may be useful in capturing some of the dynamics of MKM environments such as subsets of causal factors that tend to reinforce each other or pairs of causal factors that change relative to each other in a fashion consistent with competition. In contemplating the application of adaptive FCM's to the modelling of MKM environments, the challenge will be to determine the most appropriate learning algorithms. It is not clear that a single alogorithm will be appropriate for all concept pairs. In the MKM rubric described above, some concepts, such as the desire for formal knowledge validation, depend upon human attitudes while others, such as the anticipated influence of new media, are at least partially dependent upon more empirical factors, in this case, Moore's Law.

The FCM approach provides a means of describing an MKM environment in terms of concepts and the causal relationships between them. When considering the development or deployment of a collaborative MKM system the big question, "if we build it, will they come?", is addressed by constructing the appropriate input concept vector, stepping through the execution stage, and analyzing the steady state vector if it exists. Support for the initiative is indicated if the steady state vector is consistent with the characteristics of the desired or anticipated MKM environment. This form of analysis indirectly respects the idea that the development of MKM environments is the result of individual and collective decision-making with the perceptions that influence decisions bound up in the FCM's conceptual framework. An approach that directly addresses decision-making at the level of individuals and groups is provided by game theory. In this area, recent work by Pappus and others on

[5]Kosko pp. 152–165.

[6]Donald Hebb, *The Organization of Behaviour*, (New York: John Wiley and Sons Inc., 1949), 62.

the development of Open Source Software (henceforth *OSS*) as the *private provision of a public good* is instructive.[7]

5.2 Ideas from Game Theory: MKM as a Public Good

Including such diverse entities as roads, trash collection, and the regulation of broadcast bands, a *public good* is something that, once provided, is available at sustainable levels for the benefit of all. Global public goods are those whose benefits extend across international boundaries. In *Providing Global Public Goods: Managing Globalization* the editors focus on a "top ten" which includes such ideas as the respect for sovereignty between nations and the sustainable management of ecosystems. At number eight in their list, there is "the concerted management of knowledge, including global property rights".[8] Game theoretic models of OSS provision adopt a world view which perceives the potential contributors to an OSS project as a community of self-interested agents (in this case, programmers), each with privately understood costs and benefits of making a contribution to the project. Barring potential benefits related to the perceived gain in personal prestige that one might experience by making a contribution, the analysis proceeds along the lines of a *prisoner's dilemma*: the greatest utility would be to freeload as long as enough others contribute that the project reaches a useful state of completion. If too many freeload, then no one benefits.

The adaptation of these models to the context of a Web-based MKM system rests on the idea that there are real costs, possibly financial but at least in terms of time and effort, associated with making use of the system. In an era when professionals and academics of all stripes are confronted with so many demands on their time that the maxim *time is money* often enters into decision-making around the prioritization of even the most mundane tasks, the idea that an individual might conduct an informal cost-benefit analysis while contemplating the use of a particular MKM system is not far-fetched. There are, however, significant differences between common MKM activity and OSS development. Unlike an OSS project which, while typically ongoing, can be modelled as a single game which ends when a production level version of the software is attained, MKM systems tend to be

[7]Justin Pappas Johnson, "Economics of Open Source Software", (extension of chapter from M.I.T. Ph.D. dissertation. [online], (2001), 5-16

[8]Inge Kaul et al., The United Nations Development Programme, *Providing Global Public Goods: Managing Globalization* [online], (2003).

open-ended. In particular, if an MKM system sustains a stable user community, then the ongoing participation of community members needs to be effectively accounted for. This suggests that adapting the OSS model to MKM situations might involve its reinterpretation in the form of a "repeated game". While there is considerable literature from the field of mathematical economics regarding the sequential equilibria of repeated games, most of this literature involves *symmetric equilibria*, equilibria arising from games in which all participant's strategies are identical. It is not obvious that any given MKM system can be described in such a way that all classes of participants share the same participation strategy. Consequently, it may be difficult to elaborate a general game theoretic model for MKM systems.

5.3 Recent Trends in Modelling Socio-Economic Phenomena

Ultimately, the effort to model productive, collaborative MKM may be best served by research which combines game theory's respect for the role of individual decision makers with the utility of the dynamical systems approach. In a follow-up to a presentation of a relational algebra for FCM's, Chaib-draa examines the use of FCM's in modelling organizational decision-making and presents an example of hypothetical departmental decision-making at a university.[9] This analysis takes an agent-based approach in which the interests of different types of community members are represented as FCM's which are then synthesized into a single FCM by the application of a combination algorithm. Figure 5.2 adapts Chaib-draa's approach to the MKM-related context of a "secondary school mathematics teacher" agent.

Attempts to resolve the analytical approach of game theory and its inherent dependence on human players making rational (and optimal) precisely-defined decisions with approaches that admit multiple causal influences which may be weakly or fuzzily defined have originated from within the field of mathematical economics itself. In an introduction to an upcoming text on *Cognitive Economics*, the editors, Paul Bourgine and Jean-Pierre Nadal, cite two research programs that have been brought together under the emerging field:

> an epistemic program, grounded on individual beliefs and reasoning, develops a procedural individual rationality, now symbolized by a so-called 'homo cogitans

[9]Brahim Chaib-draa, "Causal Maps: Theory, Implementation and Practical Applications in Multiagent Environments", *IEEE Trans. on Knowledge and Data Engineering*, vol. 14 no. 6 (2002): 9.

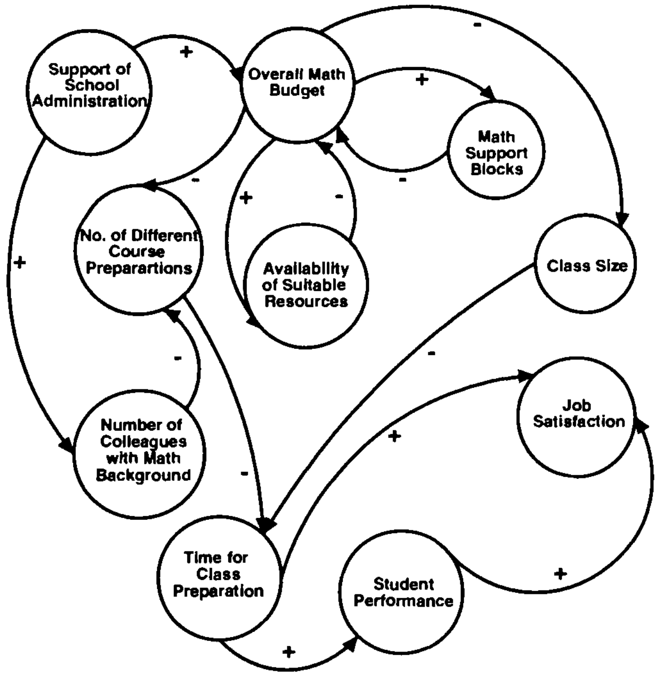

Figure 5.2: Agent-based Causal Map for a Secondary School Mathematics Teacher

> /adaptans'; [and] an evolutionist program, based on interaction networks and adaptation processes, which has developed the idea of self-organization of a system and which studies emergent structures.[10]

Drawing ideas and methods from such fields as cognitive psychology, statistical mechanics, and viability theory, cognitive economics seeks a much broader view of the factors that affect economic activity than the view afforded by classical economics. In particular the perspective of cognitive economics recognizes the importance of the formation and maintenance of networks and coalitions. In a chapter (as yet untranslated) tentatively entitled *Formation of Networks on the Internet*, the authors state their objective as follows:

> Cet article s'intéresse au problème de l'èchange d'informations sur Internet et aux propriétés d'auto-organisation que cet échange peut faire émerger, en termes de segmentation d'individus consommateurs.[11]

[10]Paul Bourgine and Jean-Pierre Nadal, "What is Cognitive Economics", *Towards Cognitive Economics* [online], 2001.

[11]N. Curien, E. Fauchart, G. Laffond, J. Laine, J. Lesourne, F. Moreau, "Forums de consommation sur Internet :un modèle évolutionniste", *Towards Cognitive Economics* [online], 2001.

While presented from a marketing perspective, the impetus to examine the processes by which individuals form groups whose interaction is mediated by the Internet is important to the understanding of MKM. The model presented draws its theoretical foundation from the game theory side of systems science but emphasizes strong descriptive capabilities. It incorporates, for example, parameters which represent consumer's trust (fidélité) in both the Internet itself and the information available in on-line consumer forums. As it time-steps towards a final state, these parameters are modified according to an evolutionary algorithm.[12]

A more theoretical treatment is presented in the proposed chapter on Viability Theory by Jean-Pierre Aubin.[13] Aubin introduces viability theory as relevant to the study of:

1. cognitive systems that must design learning processes allowing them to adapt and evolve in an environment defined by viability constraints in unexpected environmental circumstances, before reaching another subset regarded as a target if needed
2. of the architecture of a network described by connectionist tensors operating on a coalition of actors.[14]

The foundations of viability theory are strongly rooted in optimization however the approach imports ideas from the theory of neural networks. Adjacency matrices with elements defined by learning algorithms are advocated as one method of regulating a system as it evolves towards a desired target.[15] Insight into the system is gained through an understanding of the choice of initial conditions and the nature of the learning algorithms needed to attain a particluar target.

5.4 *Quo Vadis?*

The study of how mathematical knowledge is defined, validated, and shared within and between mathematical communities is a worthwhile undertaking in its own right. In situations

[12] Curien p.12.

[13] Jean-Pierre Aubin, "Elements of Viability Theory for the Regulation of the Evolution of the Architecture of Networks", *Towards Cognitive Economics* [online], 2001.

[14] Aubin, p.1

[15] Aubin, p. 15

that require decisions regarding the allocation of MKM resources, mathematical modelling will never replace informed "common sense" combined with an understanding of MKM's past and the various factors that are shaping its future. Modelling techniques, however, do hold the potential to encourage clarity of thought. If, when queried, a well-reasoned model were to provide a counter-intuitive answer, then a further investigation of the model and its assumptions would be warranted. Amongst both educational and research-oriented mathematical communities, there are enough examples of inappropriate and discarded MKM systems to warrant careful, informed decision-making supported by all practical means. On the scale of mathematical communities, there is potentially more at stake than the effective use of time and finances.

In the theory of software development processes, *Conway's Law* is cited as a caveat regarding the tendency of a software project's logical design to take on the characteristics of the organizational structure of the work groups that create it. The full statement of the law describes a set of complementary forces in which software architecture informs organizational structure and vice versa. Ultimately, however, the two become aligned and it is therefore important in the early stages of a project to build as much flexibility as possible into both architecture and organizational structure.[16] While these ideas about the software development process were never intended to apply to an undertaking such as the development of MKM systems, the potential connection between organizational structure and system design is worth considering.

In an earlier chapter, a broad definition of *mathematical community* was adopted encompassing all "those involved with advancing the understanding of mathematics; either at its frontiers, the primary occupation of researchers, or within the existing body of mathematical knowledge such as teachers and students". This definition is at odds with the experience of most who might claim either full or part time membership in the community. If there is truth to the idea that an individual's experience of "community" is formed by the group of people with whom he or she exchanges ideas, then it can be argued that any undertaking that attempts to define systems of mathematical knowledge management affects the structure of the mathematical community.

It can reasonably be argued that many of the factors that have determined the current divisions within the broad mathematics community, such as domain specialization in

[16]James O Coplien, *A Development Process Generative Pattern Language, AT&T* [online], (1995): 19.

research and age group specialization in education, have their origin in the perspectives of typographic culture. If these divisions are to be breached, new MKM systems need to support the creation of "meeting places" that bring the broader community together. The schism that presently exists between school level and university level mathematics provides a good example. If properly designed, there is no reason that those interested in exchanging ideas regarding a topic in a high school mathematics curriculum could not visit the same electronic mathematical "community centre" as members of a particular research community. While members of one group may have little use for the specific resources of the other, the opportunities for interaction and consultation offer up the hope that each community, even if only accidentally, may gain a better understanding of the other's priorities and concerns.

Today, digital computation and networking technologies are exerting a complex influence on the forms of mathematical knowledge validated by mathematical communities and the means by which that knowledge is managed and exchanged. The close relationship between mathematics and computer science is illustrated in the fact that not only was numerical computation the first task of digital computers but, in 1969, the *Culler-Fried Interactive Mathematics Center* at the University of California at Santa Barbara became only the third node on the Arpanet. While the experimental approach presents a digital methodology for doing mathematics, at this still early stage in the development of network technologies for mathematical knowledge management, it is important to consider the effect that those technologies can have on the meeting places of the community and, among other things, which communities within the broader mathematical community are invited to those meeting places.

Appendix A

Some Software and Hardware Initiatives

The first [axiom] said that when one wrote to the other (they often preferred to exchange thoughts in writing instead of orally), it was completely indifferent whether what they said was right or wrong. As Hardy put it, otherwise they could not write completely as they pleased, but would have to feel a certain responsibility thereby. The second axiom was to the effect that, when one received a letter from the other, he was under no obligation whatsoever to read it, let alone answer it, - because, as they said, it might be that the recipient of the letter would prefer not to work at that particular time, or perhaps that he was just then interested in other problems....The third axiom was to the effect that, although it did not really matter if they both thought about the same detail, still, it was preferable that they should not do so. And, finally, the fourth, and perhaps most important axiom, stated that it was quite indifferent if one of them had not contributed the least bit to the contents of a paper under their common name; otherwise there would constantly arise quarrels and difficulties in that now one, and now the other, would oppose being named co-author.[1]

The "axioms" of the Hardy-Littlewood Collaboration as described by ***Harald Bohr (1887 - 1951).***

[1] Littlewood, pp. 10-11.

The above recounting by Harald Bohr describes mathematical knowledge management at the interpersonal level as designed by G.H. Hardy and John Littlewood. While entirely pre-digital, the nature of this collaboration illustrates the need to adapt and customize knowledge management systems. In this case the systems being adapted are nothing more sophisticated than the interpersonal protocols surrounding the handling of postcards via mail and the joint submission of papers to journals. Digital environments provide much greater potential for customization with respect to both hardware and software. This appendix describes two projects being undertaken at Simon Fraser Univerty's CoLab which are designed to facilitate research in MKM. The first, *emkara*, is software project which investigates the design of interfaces that support dynamic ontologies. The second, the *CoLabPad*, involves both hardware and software design and is intended to facilitate research in human-computer interaction issues related to mathematical education and collaboration.

A.1 Emkara

The purpose of the Emkara project is to provide an experimental environment for investigations concerning the archiving and retrieval of mathematical knowledge via web-based interfaces. At the design level, emkara is a document management system that provides for and encourages extensive use of metadata and allows for flexibility in the definition of the form of mathematical content to be archived. For each definition of a new form of mathematical content, Emkara generates default edit and view interfaces. These interfaces may be modified or replaced in order to provide flexibility in both the manner in which mathematical content is archived and the manner in which it is retrieved. The types of investigations that might be undertaken with the aid of emkara interfaces include:

- data mining: what forms of data can profitably be collected from user interaction with emkara based interfaces?
- grey literature: what forms of archiving are suitable for grey literature?
- interface design:
  - what form of interface is suitable for a particular device?
  - what interface are necessary to support commercial access to data?

- data encoding: is it more efficient to encode data in multiple formats or to provide translation? If translation, should it take place on the server or on the client?
- interaction with remote services: to what extent can an emkara system interact with remote services (i.e. a pre-print server) via SOAP/XMLP or the emerging Web Ontology Language protocols.
- use of metadata:
  - what forms of metadata are necessary for different communities?
  - to what extent can system "decisions" based on metadata be exposed to the user?

A major component of the project is the design and construction of a digital mathematical discourse management application based on open source technology. Corresponding respectively to the static, dynamic, and epistemic ontologies, this application has three fundamental components: an archiving component which respects the static ontology, an object and interface building component that respects the dynamic ontology, and a user interfaces component which respects the epistemic ontology. Caste as design objectives, these three components reflect the intent to provide:

- qualified user control over granularity of object classification
- qualified user control over functionality and interface design, and ...
- end user access to knowledge creation and retrieval interfaces.

Secondary design objectives include the ability to store all forms of mathematical content and the ability to translate structured mathematical text between LaTeX, MathML, and, ultimately, OMDoc formats.

At the implementation level, an Emkara system consists of a front-end CSS and MathML compatible web interface together with a MathML editor configured as a "helper application". The mid-level implementation consists of a SQL compatible relational database management system, functions which support the creation and management of user interfaces, functions which support user authentication and session management, and functions which support data transformations such as translation of structured text formats. The back-end consists of the database tables and a data directory tree. Used primarily for the storage of

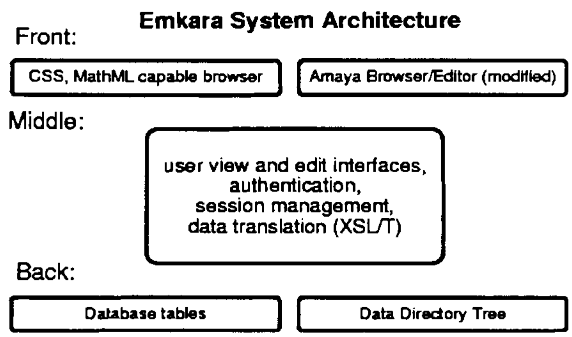

Figure A.1: Emkara System Architecture

metadata and internal system parameters, the tables also provide a convenient location to store data obtained from data mining processes. The directory tree is used for the storage of documents and large objects.[2] *Figure 2* illustrates the Emkara system architecture.

While the prototype is intended mainly as an experimental interface, a fully implemented Emkara system presents opportunities for research which respond to each of the fundamental ontologies. As alluded to in the last section, making the static ontology open and subject to extension by qualified users presents design questions concerning how to present an ontology editing environment along with the question of how to ensure that the ontology is being edited in a useful manner. That the static ontology is being represented as XML makes testing for "well-formedness" a simple test for consistency however, such a test makes no comment about superfluousness or lack of detail. The latter failings may only become apparent with use. It is a valid question, therefore, whether or not data regarding user interaction with the system can be processed in such a way as to reveal strengths and weaknesses in the static ontology. If so, then it is possible that the process of ontology management can be at least partially automated.[3]

A compelling question related to the introduction of unneeded vocabulary in the construction of static ontologies concerns the potential differences between the language that members of a particular community of interest use to describe their field and the vocabulary

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[2]The prototype is implemented using Postgresql™and PHP4™with a lightly modified Amaya™browser serving as the MathML editor helper application. The modified Amaya browser must be installed on the client along with a CSS and MathML compatible browser.

[3]The project is expected to make use of the considerable amount of work done by the members of the *Ontolingua* research team concerning ontology development environments and automated ontology analysis.

presented by the relevant schemata. The latter are typically developed through extensive committee work and are designed to meet a much broader range of needs than those of an individual communicating ideas to fellow members of a given community. It is important that methods be developed to support the presentation of schemata vocabularies in such a way that users can identify the elements they need to express their ideas and make the appropriate association with the language of their particular communities.

The dynamic ontology underlies the relationship between entities in the static ontology and the way they interact with each other. A "flash card" object may consist of only two main fields: question and answer. The dynamic ontology allows for the description of how these two fields interact in the context of the flash card object and directly reflects the functions and methods that determine object behaviour.[4] A valid question concerns the numbers and types of general object interactions that are necessary in order to construct the type of behaviour required by DMD. A related question is that of ontology representation: how many and what types of object access methods must be exposed in order to afford qualified "non-programmers" the flexibility they need to create the behaviour they desire. With respect to these questions, inspiration, guidance, and, possibly, code, can be derived from the work of the members of the *Protege-2000* research team who have created a visual ontology editor and undertaken research into its application in the creation of ontologies in several Semantic Web compatible knowledge interchange languages.[5]

Questions that can be addressed at the level of the epistemic ontology include any question related to the type of interface appropriate for a particular task and user device. If the task is helping students understand the material in a particular lesson, a threaded discussion might be part of the solution. The related question is what access and acquisition methods make sense for different devices?

Related to the epistemic ontology, a number of valid research objectives are derived from the fact that an Emkara system is necessarily a *stateful* system which must enforce authentication and a system of maintaining read and edit access privileges. From this follow questions related to how users interact with the system and the system's function as a facilitator of "on-line community". In particular, users are instances of a "user object" that

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[4]In the prototype, the dynamic ontology is bound up in PHP4 classes and scripts and therefore some knowledge of PHP is required to customize system behaviour.

[5]Natalya F. Noy and Michael Sintek et al., "Creating Semantic Web Contents with Protege-2000", *IEEE Intelligent Systems*, [online], March-April (2001), 60-71.

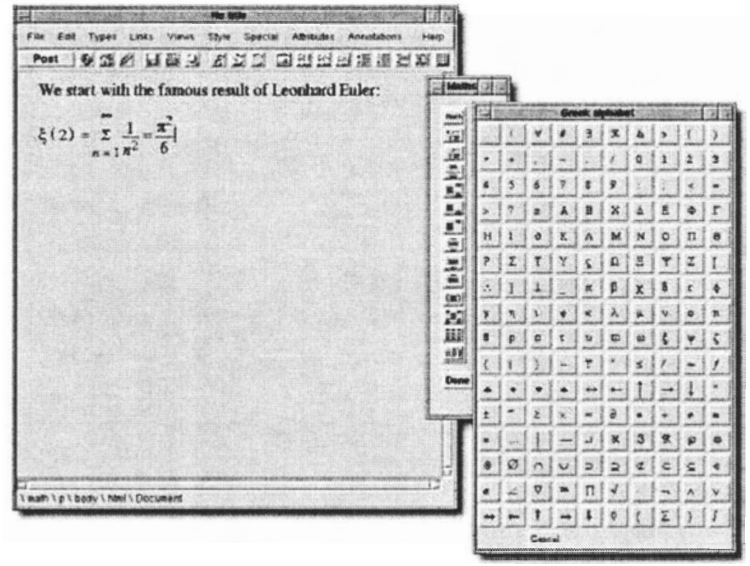

Figure A.2: The modified Amaya MathML editor

includes metadata that makes up a user profile. This permits the collection and processing of data regarding what types of users are accessing what types of data and using what kinds of interfaces.

One of the most compelling features of any system that attempts to provide management functions for digital mathematical discourse is the potential that the system has to interact with other MKM applications via the emerging Semantic Web interchange languages and Web Services protocols. While atomic in nature, search strings for pre-print servers and input to theorem provers are forms of grey literature. If a DMD application is able to provide interfaces for remote access to these systems coupled with organized methods of storing and retrieving the output, it is possible that a significant proportion of access to MKM applications would take place via interfaces that could provide meaningful information about the ways in which mathematics is being accessed and shared through the Web. In this situation, the payoff would be data concerning what types of users are accessing what types of documents in pre-print servers and what types of users are using what types of services provided by theorem provers.

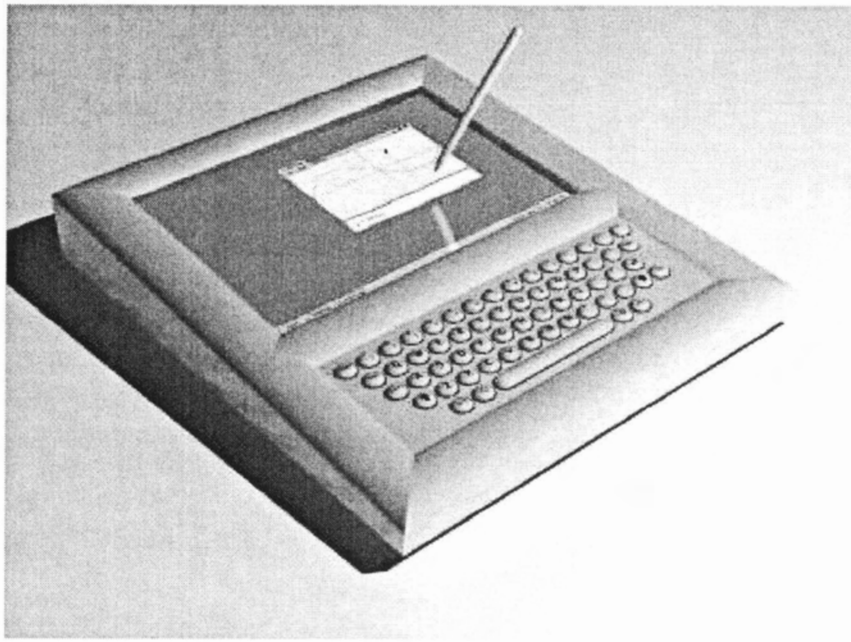

Figure A.3: An early rendition of the CoLabPad.

A.2 CoLabPad

The objective of this project is to build and test a portable wireless net-booting mathematics environment that affords instructors and researchers a high degree of control over the available mathematical tools in educationally or research oriented collaborative environments. The system will consist of a server, CoLabPads, mathematical applications, and the necessary operating system software and wireless hardware.

In this context, the "CoLabPad" is a terminal that provides students with a set of resources that can be defined by the instructor according to the educational or assessment objectives. Appliances such as these would hold a number of advantages over graphing calculators. Among them, better display capabilities, reconfigurability, and the ability to offer a level playing field by ensuring that all students have the same resources at their disposal. As well, such appliances would be less prone to theft as they would be close to useless when not connected to the server.

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