PROCEDURAL CHANGE IN MATHEMATICS:

TALES OF ADOPTION AND RESISTANCE

by

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THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the

Department of Mathematics

SIMON FRASER UNIVERSITY

August 2003

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Procedural Change in Mathematics: Tales of Adoption and Resistance

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ABSTRACT

In mathematics, as in many fields, there are often several ways to solve a particular problem. A student may know one valid approach, which will give a correct answer, but there may also be a different method which is in some way better. I have observed that many students do not adopt the methods presented in class if a previously known method can be applied.

In this study I focus on three topics discussed in a mathematics course designed for pre-service elementary teachers: greatest common factor and least common multiple, compound percentage change, and addition and subtraction of mixed numbers. The procedures used by the students before and after instruction were recorded and clinical interviews were conducted to discover what motivated the students to adopt a new method or caused them to resist a new approach.

A theory of conceptual change, proposed by Posner, Strike, Hewson and Gertzog (1982), is used as a theoretical framework for interpreting the results, which indicate that a variety of factors influence a student's choice of procedure. The theory is then adapted to include procedural change, making it applicable to the adoption of a new method for solving a mathematical problem when a valid method is already known. This theory of procedural change can give a greater insight into what must take place in the classroom in order to make adoption more likely and thereby giving students the ability to choose the most suitable method for each situation.

DEDICATION

I dedicate this work to the Lord, who called me to it and sustained me throughout, and to my husband Peter, with whom I am one.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. Rina Zazkis, for her help and guidance, encouraging me to develop in many directions. Throughout the course of the work she managed to balance patience with prodding!

I am grateful to Dr. Malgorzata Dubiel for her friendship and her many words of wisdom, and Dr. Harvey Gerber for his direction in the early stages of my studies. My thanks also go to the members of the SFU Mathematics Education community and to my friends for their encouragement and support.

I would like to express my appreciation for the participants in the study, who so honestly shared their thoughts, feelings, and fears about mathematics with me.

Finally, I would not have been able to complete this work without the constant support of my husband, Peter. Thank you for believing in me.

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CHAPTER 1

INTRODUCTION

The following dialogue could have taken place in a faculty lounge or staffroom . . .

Researcher : Imagine a classroom where, after instruction in a particular topic, every student can correctly answer every question on that topic.

Responder : Ah, surely this would be ideal! All students understanding the approaches and methods presented by the teacher.

Researcher : At first glance it may appear so, but let us look deeper. What does the ability to get the correct answer indicate? Perhaps the students have learned to apply the methods correctly, but do not fully understand them.

Responder : Even so, this can be regarded as a success, since the students have learned something. All the students must have understood the teacher to some extent.

Researcher : Not so, for there may be a number of approaches, all of which lead to a correct solution. The student who did not grasp the newly presented method could achieve success using a previously known method.

Responder : But then the teacher would know from the use of an old method that the student had not understood the new.

Researcher : You are forgetting that we all have freedom of choice! Perhaps the student understood the new method, but *chose* to use a different one.

Responder : I see the difficulty - the teacher would not know why the student did not use the new method. But I have to ask, if the student already knows a valid approach, why should he or she adopt the method presented by the teacher?

Researcher : The new method may encourage a richer understanding of the concept, or may be more efficient. It may be that the new approach permits the student to move into realms that would otherwise be unattainable.

Responder : Then the teacher must point out these benefits.

Researcher : I fear this is not sufficient. We need to understand what it is that motivates a student to adopt a new method or causes resistance to a new approach.

Responder : With this knowledge the teacher would be better equipped to help the student progress.

Researcher : That is my desire for the outcome of my work.

For several years I have worked as a teaching assistant and an instructor with students enrolled in the Math 190 course, Principles of Mathematics for Teachers, at Simon Fraser University. The course is designed to encourage students to examine their understanding of elementary mathematics as a whole and to explore different methods with which to approach a variety of problems in several content areas. Many of the students in this course have little confidence in their mathematical ability and many have not studied the subject for several years.

In the Spring of 2000 I was given the opportunity to teach the Math 190 course and became aware of a difference between the methods I was discussing in class and the methods used by many of the students after instruction had taken place. For example, it is not unusual for students to find the value of $4+7\frac{1}{3}$ by using improper fractions: $\frac{4}{1}+\frac{22}{3} = \frac{12}{3}+\frac{22}{3} = \frac{34}{3} = 11\frac{1}{3}$, even after discussion in class which highlights the simplicity of adding the wholes directly. I wished to know if this type of resistance was common. In 2001 I taught the course once more and again I observed that the students often did not use the methods or approaches I had demonstrated in class. They either did not want to, or for some reason could not, use the newly introduced methods and were continuing to use procedures they knew before the course began. This phenomenon is not limited to the students in this course, or to the specific material covered. For example, I have observed students studying linear algebra solving a problem such as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} a & 2c \\ b & 2d \end{bmatrix} \text{ and } \det A = 3. \text{ Find } \det(AB^T).$$

by multiplying out the matrices, finding an expression for the determinant, and then substituting $ad - bc = 3$, rather than by using the methods discussed in class which would give

$$\det(AB^T) = \det(A)\det(B^T) \text{ and } \det(B^T) = 2\det(A)$$

$$\text{so} \quad \det(AB^T) = 3\times(2\times3) = 18$$

Why would a student use a method which requires so much more effort, takes longer, even though time is often precious in examinations, and is more likely to lead to a mistake in computation?

Schoenfeld (2000) describes the two main purposes of mathematics education research as "to understand the nature of mathematical thinking, teaching and learning" and "to use such understandings to improve mathematics instruction" (p. 641). Under the first of these purposes, we may attempt to classify a difficulty in learning, locate its source, and construct a theoretical framework to analyse it. Without attempts to do this a difficulty may remain unnoticed or poorly understood (Sierpinska, Kilpatrick, Balacheff, Howson, Sfard & Steinbring, 1993). From the examples mentioned above, it is clear that many students experience difficulty in adopting some methods presented in mathematics courses. Selden and Selden (1993), in their overview of collegiate mathematics education research, raised several issues which remain to be investigated, including the question, "What aspects of teaching influence a student's *inclination* to use a skill or apply a technique? The inclination to carry out a technique is distinct from the ability to do so" (p. 443). Students who have demonstrated their ability to use a particular method when required to do so, or under certain circumstances, at other times choose not to use that method. The decision may be unconscious, but can be described as a choice because the student has knowledge of, and the ability to use, a variety of methods. What factors influence this choice to resist or adopt the methods presented during instruction?

Since a student must possess the relevant knowledge before any choice can be made about which method to use, in the next chapter I will review literature discussing the acquisition of knowledge and obstacles to learning. Research about conceptual and procedural understanding is also considered, since it is suggested that students may have different types of understanding of procedures. The theory of conceptual change proposed by Posner, Strike, Hewson and Gertzog (1982) will be examined in some detail. Occasional personal observations and comments will be used to illustrate some of the ideas presented in the literature and to relate it to the current study.

The topics of greatest common factor and least common multiple, compound percentage change, and addition and subtraction of mixed numbers were chosen for examination in this study. The majority of students in the course knew a procedure for solving problems in each of these areas, but frequently their approach was different from that presented in class. These specific topics have not yet received much attention from researchers, but studies investigating content within the same domains are discussed in chapter 3. Some of the difficulties experienced by students relate to their understanding of the symbolic representations, or their ability to visualise and attribute meaning to the procedures. Research in these areas, and a brief look at some aspects of problem solving will also provide useful background information for the issues raised in this study.

In chapter 4 I give a description of the participants in this study, the instruction they received and the methods used to collect the data. This data is presented in a variety of formats in chapter 5. My initial observations were of students using methods and procedures that were not those promoted in the course. This raised the question, “What causes students to adopt a new method or what prevents them from so doing?” Some of the motivations given by the students in the interviews seemed to fit with the theory of conceptual change proposed by Posner et al. (1982). Therefore I shall use this framework in chapter 6 when interpreting the data collected in this study.

The literature discussing conceptual change, or the overcoming of obstacles in general, assumes that the student's current conception or knowledge is incorrect, or no longer valid in the new situation. The development that I would like to propose is to those situations where a student knows a correct and valid procedure, but where there exists a 'better' method. Recognising one method as better than another depends on one's view point and so it is necessary to consider what constitutes 'better' for the student. When discussing conceptual or procedural change, or the overcoming of obstacles, researchers seem to agree that the learner must first be convinced that their current knowledge will produce incorrect responses in situations where the new knowledge will give rise to correct responses. However, the 'old' procedures used by students for the topics in this study will give correct solutions to all the problems. The students must be convinced that the old methods are 'inadequate', even though they produce the right answer.

Students come to a course with certain knowledge, but they must still be able to adopt new approaches to familiar topics, giving them the ability to choose the most suitable method for each situation. If this adoption is difficult, then we must examine how to make it more likely. If there is still resistance to change, then educators will have to reconsider what is taught to students which causes this lack of flexibility. The conceptual change theory was originally designed to describe conditions necessary to correct scientific misconceptions. I develop and adapt this theory to apply to the adoption of a new method for solving a mathematical problem when a valid method is already known. This will give a greater insight into what must take place in the classroom in order to encourage more flexibility in students.

CHAPTER 2

KNOWLEDGE, OBSTACLES AND CHANGE

It is widely believed that learning is not only dependent on what a student is taught, but also on the knowledge and understanding which the student brings to the situation. Support for this view will be demonstrated throughout this chapter. We will consider how a student acquires knowledge and some of the obstacles which hinder learning. If a student's current knowledge or understanding is in some way inadequate, then conceptual change may be required. A distinction can be drawn between conceptual knowledge and procedural knowledge, and this issue is examined along with the types of understanding which can be associated with the different forms of knowledge. As I review literature in these areas I will show how it relates to the current study and include personal observations and comments to illustrate some of the ideas.

Knowledge Acquisition

At one time it was believed that knowledge could be acquired by transmission. By presenting information in a clear verbal form, or through careful monitoring of the behaviour of the learner until it resembled the desired behaviour, knowledge could be transmitted from teacher to student. This belief is not generally supported today since cognitive studies have shown that even with simple rules and algorithms learners will adapt the presented information in an attempt to make sense of it. Thus, learners construct their own knowledge from what is presented to them. Some of the evidence in support of the constructivist point of view is that this constructed knowledge often contains errors. For example, many young children use an incorrect algorithm for the subtraction of multi-digit numbers, although they have never been taught the variations which they implement (Brown & Burton, 1978). That knowledge is constructed by the learner, rather than simply transmitted to the learner, is the first of five characterisations of long-term acquisition of knowledge given by Hatano (1996).

The second characterisation of the acquisition of knowledge, according to Hatano, is that the knowledge is restructured. As the learner acquires more knowledge, it is reorganised. There is an attempt to integrate the new information with prior knowledge, to understand and think about the new in terms of what is already known (Glaser, 1984). Some separate pieces of knowledge may be recognised as closely related, while others may take on a new significance. If a student is faced with a new concept (or piece of knowledge) which is closely related to concepts that are already present in the student's knowledge structure then it can simply be added to the new knowledge within the existing structure. However, when the new concept is radically different from the concepts held by the student, the current concepts must be replaced, or the student's knowledge must be restructured. Following Piaget, the processes of adding to the current structure or restructuring are referred to as assimilation or accommodation respectively.

The construction and restructuring of knowledge is constrained. This is Hatano's third characterisation of the acquisition of knowledge. The constraints can be both positive and negative influences and are individual and societal in nature. Personal prior knowledge in a particular domain has been shown to enhance the acquisition of new pieces of knowledge within that domain (Glaser, 1984; Kuhara-Kojima & Hatano, 1991). Cultural constraints are those which are shared by the majority of a given community. They include documented pieces of knowledge, physical tools, social institutions and much more, but they also include shared beliefs. Although these constraints are external to the individual, they become internalised as knowledge when the person is repeatedly exposed to them or practises their use. It is these constraints which allow a person to reduce the number of possible hypotheses or interpretations in a given situation to a manageable level in order to make reasonable choices quickly.

The fourth of Hatano's characterisations is that knowledge acquisition is domain specific. The domains are essentially self-contained knowledge systems within which comprehension and problem solving take place. New knowledge acquired during these activities is stored in the domain in which the activity took place. Kuhara-Kojima and Hatano (1991) found that students who knew many facts in a particular domain were able to make more meaningful connections for new knowledge relating to that domain. The advantage of domain specificity is that by storing knowledge within a specific domain we are able to retrieve relevant information more rapidly, since we are searching only a part of our total knowledge system. Research has shown (e.g., Chi, Glaser & Rees, 1982) that relevant domain-specific knowledge is a critical factor in problem solving and that knowledge acquired through such activities is stored in that domain only.

Furthermore, all knowledge is acquired within a context (Brown, Collins & Duguid, 1989), and it may then be tied, more or less tightly, to that context. How the knowledge was acquired and how it has been used, including the goal to which the activity was directed, are contextual features with which the knowledge is closely associated. Becoming an expert is partly dependent on decontextualising, or desituating, the knowledge, allowing it to be retrieved and used in different contexts (Hatano & Inagaki, 1992). The fifth and final characterisation of knowledge acquisition given by Hatano is therefore that it is situated in contexts.

There are some problems inherent in the way we acquire knowledge as described by these characterisations. If the knowledge is constructed rather than transmitted, then there will always be the possibility that learners will construct their own knowledge which differs from that which was presented. This is the case when students create an incorrect version of an algorithm, such as the variations of the multi-digit subtraction algorithm mentioned earlier (Brown & Burton, 1978), or make an inappropriate assumption from what is presented, for example, conclude that when dividing, the quotient is always smaller than the dividend (Tirosh & Graeber, 1989).

Our prior knowledge can be seen as a constraint to our acquisition of new knowledge. On the one hand, existing conceptions guide the understanding and interpretation of new information and so our prior knowledge is essential to the acquisition of new pieces of knowledge. On the other hand, this prior knowledge may also be an obstacle to the acceptance of the new knowledge. A student who has a concept built into a stable structure may be reluctant to adapt or give up previously successful thinking when faced with a new concept which cannot be assimilated into that structure (Booker, 1996). In other words, the prior knowledge may actually prevent the more radical restructuring of accommodation from taking place. In this situation, the student must be more strongly convinced that a change is necessary. Glaser (1984) found that change occurs when there are challenges and contradictions to one’s prior knowledge, but Booker (1996) stated that it is not always easy to provide situations which encourage this type of change, especially if the concept has previously been successful, or if it has been practised for some time. Even if a student is willing to change, the choice of a new central concept will be influenced by the student's current concepts if more than one alternative view is presented (Posner, Strike, Hewson & Gertzog, 1982).

The constraints which can help us to retrieve more rapidly the relevant knowledge can also limit us and prevent us from making new connections. Social constraints, such as a method or algorithm presented and practised repeatedly, can lead to a belief that this is the only option when a particular situation is faced. Many beliefs may develop in the mathematics classroom and these beliefs can influence us in a variety of ways, they may prevent us from even considering a new concept. One such belief is that 'there is a formula for everything'. The students believe that questions presented to them in the mathematics classroom can be answered by the application of the formula given to them while studying the particular topic. For example, I have observed the work of Math 190 students who were presented with the following sequence : 2, 14, 98, 686, . . . Having established that this is a geometric sequence with first term, a = 2, and common ratio, r = 7, the students were asked, "If possible, find the sum to infinity. [If not possible, explain why.]" In spite of the hint in the question, many students did not appear to consider that it may not be possible to find the sum, and simply used the formula:

$$S_\infty = \frac{a}{(1-r)}, \qquad \text{giving } -\frac{1}{3} \text{ as the answer.}$$

Very few seemed troubled by the implausible value for the sum of the sequence, and some who recognised that it was not sensible commented that it must be correct because they had used the formula.

The situation of knowledge in contexts may give rise to a variety of difficulties for students. A clear occurrence of this was found in the street-vendor children of Brazil who could successfully carry out many arithmetic calculations relating to the cost of multiple items at the market (Carraher, Carraher & Schliemann, 1985). These calculations were performed mentally, using a variety of informal procedures. When these same calculations were represented mathematically, and pencil and paper were available, the children attempted to use the formal procedures taught in schools. However, they obtained very few correct answers because the school-taught procedures were incorrectly remembered. Sometimes learners may attempt to change the conditions surrounding a problem rather than face the conflict with previous concepts raised by the problem (Booker, 1996). For some students this avoidance of conflict takes the form of suspending common sense while in the mathematics classroom. For example, students are often taught to spot the key words in a question, such as the phrase 'all together' signifying 'add'. The students in the Math 190 course were given the question, "A water main for a street is being laid using a particular kind of pipe that comes in either 18-foot sections or 20-foot sections. The designer has determined that the water main would require 14 fewer sections of 20-foot pipe than if 18-foot pipe were used. Find the total length of the water main." (Musser, Burger & Peterson, 2001, p14). Several students struggled with understanding what this problem was describing and reverted to their belief that '14 fewer' meant 'subtract 14'. They were not sure whether to subtract the 14 from 20 or 18, but most gave 6 or 4 feet as the answer. The presence of certain words may encourage a student to use a certain algorithm, even if this is inappropriate to the problem, but also the association of certain words or types of question with a specific method or algorithm may prevent the student from using that method if those words are not present.

Obstacles

As mentioned earlier, if new knowledge is integrated into a learner's existing cognitive structure, it is termed assimilation. However, if the new concept cannot be assimilated, then a more radical restructuring must take place, termed accommodation. The new knowledge may be in conflict with previously held notions and may eventually replace them, but existing cognitive structures are often difficult to change significantly. The prior knowledge then becomes an obstacle to the construction of the new structures (Herscovics, 1989). In their research regarding the role of prior knowledge, Merenluoto and Lehtinen (2000) concluded that it is easier for a student if a concept need only be assimilated into the student's existing knowledge structure. However, prior knowledge which allows for the assimilation of a new piece of knowledge can be an obstacle to the student undergoing the more radical reorganisation of accommodation. In this situation the student may not see the need for the restructuring, that it would give a more advanced understanding. They note that the concepts which call for a radical revision of prior knowledge are quite resistant to traditional teaching.

For many years, researchers have asked questions about obstacles to learning, examining their nature and causes, and ways to overcome them. Working in the experimental sciences, Bachelard (1938, in Brousseau, 1997) formed a list of potential obstacles, which included such things as first experience, general knowledge, improper use of familiar images, and quantitative knowledge. Duroux (1982, in Brousseau, 1997) examined the nature of an obstacle and the conditions necessary for an obstacle to exist. He describes an obstacle as “a piece of knowledge or a conception . . . [which] produces responses which are appropriate within a particular, frequently experienced, context. But it generates false responses outside this context. A correct, universal response requires a notably different point of view” (p. 99). Duroux goes on to state that “this piece of knowledge withstands both occasional contradictions and the establishment of a better piece of knowledge. Possession of a better piece of knowledge is not sufficient for the preceding one to disappear” (pp. 99-100). Even “after its inaccuracy has been recognized, it continues to crop up in an untimely, persistent way” (p. 100). It is because the knowledge has been satisfactory so far that it is firmly anchored in the mind and now becomes an obstacle (Cornu 1983, as cited in Tall, 1991). Brousseau suggests that obstacles are made apparent by errors. These errors are persistent and reproduceable, not random or careless, and there is a reason behind the error.

The origins of various obstacles have been categorised by Brousseau (1997) as being ontogenic, didactical and epistemological. Ontogenic obstacles are those which arise because of the particular stage of development reached by a student. Didactical obstacles depend on the choices made in the educational system as to how, and in what order, to present concepts. There are also obstacles which are epistemological in origin. These obstacles are necessary “because of their formative rôle in the knowledge being sought” (Brousseau, 1997, p. 87) and they should not, indeed cannot, be avoided.

According to Sierpinska (1992), epistemological obstacles seem to belong to the meaning of the concepts themselves and are not just results of particular ways of teaching the concepts. They are not something that occurs in just one or two people, but are common in the frame of some culture, whether past or present. For example, the dictionary definition of vector is, “a quantity that has magnitude and direction and that is commonly represented by a directed line segment whose length represents the magnitude and whose orientation in space represents the direction” (Merriam-Webster, 1993, p. 1304). However, when studying linear algebra, the concept of a vector as an object having 'magnitude and direction' would be very limiting and would constitute an epistemological obstacle to a deeper understanding of the concept of vector. Sierpinska describes three areas from which epistemological obstacles can arise. Firstly there are our attitudes and beliefs, our 'world view', and secondly our schemes of thought, including how we approach problems, interpret situations, and the things we have learned by practice or imitation in the course of our socialisation and education. Thirdly, there is our technical knowledge, and the application of this knowledge is influenced by the contents of the other areas. Since many of our beliefs and schemes of thought are unconscious, they may function as obstacles to our thinking in the technical area. An obstacle of this nature is overcome only if we are able to step back and consider other points of view. It is worth remembering that if an obstacle is a piece of knowledge, then it cannot exist before that knowledge is acquired. Sierpinska (1994) found that in some cases younger children were more successful in their learning than older children because they did not have to overcome epistemological obstacles, since these had not yet been constructed.

Having acknowledged the existence of these cognitive obstacles, it is natural that researchers should turn their attention to matters of overcoming them. According to Sierpinska (1987, 1992), an epistemological obstacle is linked with some kind of conviction. In order to understand better, or see different aspects of the things we are considering, we have to become aware of our attitudes, beliefs and schemes of thought. She claims that to overcome an obstacle we should not simply replace one conviction by an opposite one, instead we should rise above these convictions. We must consider how we solve problems, recognising the reasons behind our choices, and become aware of other possibilities. Brousseau (1997) adds that it is not enough to identify where the obstacle fails, but its successes must be recognised.

While studying scientific misconceptions, researchers (e.g., Burbules & Linn, 1988; Viennot, 1979) found that misconceptions held by students prior to instruction are often highly robust and typically outlive teaching, or experimental results, which contradicts them. Brousseau (1997) explains that since an obstacle has the same nature as knowledge, it will resist rejection. Students must be provided with many situations where the knowledge is inadequate or wrong, to convince them to consider something else. To overcome the obstacle the students must carry out work of the same kind as when applying knowledge; it must be numerous, important to the student, and sufficiently different to require the acceptance of the new knowledge.

Several researchers talk of the necessity of mental conflict for overcoming obstacles (e.g., Sierpinska, 1987). The argumentation and justification which is involved in negotiating the meaning in certain mathematical situations is often referred to as cognitive conflict, but Booker (1996) points out that this term is more appropriately applied to a situation where an inadequate or inappropriate construction is ingrained in the learner. Tall (1991) found that a student can hold conflicting views, with each being evoked at different times, and not be aware of the conflict until they are evoked simultaneously. This can occur when the new ideas are not satisfactorily accommodated. When obstacles arise from deeply held convictions about mathematics they are difficult to remove. The beliefs may be suppressed in order for the student to apply the new knowledge successfully, but they will remain and may show themselves in feelings of uneasiness in certain situations. The use of an 'exposing event' to make students aware of their own pre-conceptions was suggested by Nussbaum & Novick (1982). The students should be invited to describe their current conceptions verbally or pictorially, and by stating their ideas clearly and concisely, they become aware of the elements in their own 'alternative framework'. This awareness can be increased by debating the pros and cons of their own preconceptions and noticing the differences in the ideas held by others.

We have seen that obstacles are pieces of knowledge, sometimes related to beliefs, and often unavoidable in the learning process. Even though new situations can show the knowledge to be false, it can be very resilient. Now let us consider a slightly different situation, one in which a student's current knowledge is appropriate within a particular context, but does not necessarily generate false responses outside this context. This can be illustrated by the example of finding the solution to a system of linear equations. When first introduced, the method of substitution is usually presented to the students and only two equations are involved. However, for larger systems this method becomes impractical and students encounter the method of linear combinations. It would seem reasonable that students adopt the new method, at least for larger systems, but some continue to use the method of substitution, frequently making computational mistakes in the process. Another example, from the work done by students involved in the current study, is that of finding a term in an arithmetic sequence. Given the first three terms of an arithmetic sequence the students are asked to find the 6th term and most simply find each term up to the 6th. However, when asked to find the 17th term it would seem sensible to use the formula for the nth term, namely $t_n = t_1 + (n-1)d$. Many students who have mastered the use of this formula still find every term up to the 17^{th}. Is the knowledge of how to find the 17^{th} term by finding all earlier terms of the sequence an obstacle to the students integrating the nth term formula into their network of knowledge from which the most appropriate method is chosen for any given problem?

One could question whether the examples given above are illustrations of true obstacles as Duroux (1982, in Brousseau, 1997) and others use the word. Listing every term of a sequence will always give the correct value of any given term and the method of substitution will always give the correct solution to the system of linear equations, provided that no careless computational errors are made. This does not seem to fit with the requirement for the knowledge or conception to generate false responses in certain contexts, and its inaccuracy cannot be demonstrated. However, these examples do satisfy the other criteria for an obstacle, especially that 'possession of a better piece of knowledge is not sufficient' and 'it continues to crop up in an untimely, persistent way'. The fact that calculation errors are much more likely to occur if a cumbersome method is used may be enough to elevate this type of situation to the status of obstacle. The characteristic of this situation is that the knowledge here is of a procedure, rather than a concept.

Procedural and Conceptual Knowledge

Hiebert, Lefevre and Wearne describe the distinction between conceptual and procedural knowledge (Hiebert & Lefevre, 1986; Hiebert & Wearne, 1986). According to these researchers, conceptual knowledge is rich in relationships and is gained by constructing relationships between pieces of information, which leads to understanding, whereas procedural knowledge is knowledge of symbols, or formal language, and the algorithms or rules for completing tasks. Knowledge of the meaning of the symbols or understanding of the algorithms is not necessary for successful application of procedures, and it is possible to learn procedures by rote. In contrast, conceptual knowledge must be learned meaningfully, in fact cannot be learned by rote, since meaning is generated as relationships between pieces of knowledge are formed. Procedures which are learned with meaning are linked to conceptual knowledge, and it is those procedures which make the conceptual knowledge observable. Both types of knowledge are necessary, and they must be related within the learner, together giving meaning and aiding the memory. Conceptual knowledge should be used in the selection of an appropriate procedure and to check for the reasonableness of the outcomes to procedures; procedural knowledge allows for efficient completion of tasks. Conceptual knowledge is gradually transformed into set routines with repeated use, thus becoming procedural, and new procedures can trigger the development of concepts.

Hiebert & Wearne (1986) claim that mathematical competence is characterized by connections between conceptual and procedural knowledge. They found that students often look at only surface features of problems and apply memorised symbol manipulation rules, but often this produces mathematically unreasonable answers. Three sites were identified where links between conceptual and procedural knowledge, or the lack of those links, are particularly significant. Site 1 is the interpretation of the problem, and it is here that the symbols in the presented problem are given some meaning. Site 2 is where procedures are selected and implemented, and Site 3 is where the solution is checked for reasonableness. Unless connections between conceptual and procedural knowledge are made at Site 1, it will be impossible to establish connections at the remaining sites. Although rules and algorithms are motivated by conceptual considerations, the absence of links between these procedures and the concepts involved may not hinder successful performance. It is not always clear whether a student who can use a rule appropriately actually understands why it works.

The absence of connections to conceptual knowledge can become obvious at Site 3, where the reasonableness of a solution is considered. Conceptual features of a problem do not generate exact answers, but they can provide information about the solution. Unfortunately, many students do not check for reasonableness of their answers either because they do not think to check, or because they have not made the necessary links at Site 1 and therefore do not know what the symbols mean. Their only means of checking is then whether the answer fulfils the rules of syntax of the symbols.

The process of selecting a procedure is passed over by Hiebert & Wearne who simply say, “After the problem has been interpreted, procedures are selected and applied to solve the problem. The execution of procedures is the domain of Site 2” (p. 7). They go on to suggest that procedures need not be linked to conceptual understanding, as long as they are recalled and applied correctly, but that the presence of such links would contribute to genuine competence. I believe that the selection process can not be taken for granted and that a lack of conceptual understanding will result in an inability to make appropriate choices when more than one procedure is applicable. Evidence supporting this belief was found during this study, and will be discussed in later chapters.

Hiebert & Lefevre (1986) noted that pre-school children seem to develop conceptual and procedural knowledge side by side, but the effect of schooling is to separate them. Written symbols are introduced whose meaning has not been well established, and then an emphasis is placed on the use of procedures involving these symbols. Throughout their school years, students can succeed at levels far beyond their conceptual understanding. Furthermore, it may be more difficult for students to connect symbols with referents once the symbol manipulation rules have been routinised (Hiebert, 1988).

Silver (1986) found that some students were not troubled by getting different answers to the same calculation when using manipulatives and symbolic paper and pencil techniques. This divorcing of the mathematics learnt in the classroom from the reality of the objects being represented in the procedures can be reduced if conceptual knowledge is adequately developed at the concrete level and relationships between physical procedures and conceptual knowledge are firmly established (Carpenter, 1986). Baroody & Ginsburg (1986) suggest that older children believe that they are supposed to calculate when presented with an arithmetical problem rather than look for patterns or relationships to simplify the problem. Some students go so far as to believe that finding an answer by a method other than calculation is cheating.

Understanding

It was stated above that conceptual knowledge, rich in relationships, leads to understanding, whereas procedural knowledge can be memorised without understanding. However, understanding means different things to different people and Skemp (1976) explores the possibility that there are in fact two types of understanding. Instrumental understanding is limited to knowing what to do in a given situation, whereas relational understanding is not only knowing what to do, but also why. Many of us may not regard instrumental understanding as actual understanding, but rather as memorisation of rules, or rote learning. However, Skemp suggests that for many students, and even teachers, the knowledge of a rule and the ability to use it is what they mean by understanding in mathematics. Many text books have been guilty of giving a rule or procedure without an explanation of why or how it works. Instrumental mathematics is easier to understand, and success is more immediate, but relational mathematics has the advantages of being adaptable to new tasks, easier to remember and more deeply satisfying.

Skemp's instrumental understanding is similar to what Zazkis & Campbell (1996a) term procedural understanding. They contrast this with conceptual understanding which relates to the construction of a mental object. The use of a procedure, or evidence of procedural understanding, does not necessarily mean a lack of conceptual understanding, but Zazkis & Campbell suggest that if a student spends time using a procedure which is not necessary for the solution of the problem, then this indicates a lack of conceptual understanding.

Harel & Tall (1991) demonstrate how it may be initially difficult to determine the type of understanding a student possesses. Consider some students who can each successfully use the method of linear combinations to solve simultaneous equations in two unknowns. They then meet a generalisation of this method for solving arrays of m x n equations by row operations on the matrix of coefficients. A student who has relational understanding can see the initial case as a particular example of the new. Harel & Tall refer to this as expansive generalisation, in which the range of applicability of an existing schema is expanded, without reconstructing it. An intermediate stage might be called reconstructive generalisation, where the existing schema is reconstructed in order to expand its applicability. Here the student has instrumental understanding of the method, but when given the new situation, begins to see the underlying meaning of the solution process and the initial case is seen as a particular example. Alternatively, a student who has instrumental understanding may simply memorise another procedure when given the new situation, without relating it to the original. This is disjunctive generalisation, where a new, disjoint schema is constructed to deal with the new situation. However, disjunctive generalisation is not in fact true generalisation, although the students may appear to have generalised, since they can perform the required procedures. Ideally, students should be led to a meaningful understanding from which expansive generalisation would take place. In reality, many students will have an instrumental understanding and so conditions must be provided in which reconstructive generalisation is likely to take place.

Pirie & Kieren (1994) have developed a theory of growth of mathematical understanding, describing levels through which a person may progress. They talk of 'don't need' boundaries, beyond which the learner is able to work without reference to the previous forms of understanding which gave rise to the current understanding. For example, if one has grasped a formal mathematical idea then one no longer needs a mental image of the idea. However, in order to extend current understanding, the learner must be able to return to earlier levels. Some of the difficulties experienced by students in developing their understanding may arise from forgetting how to form the images they have not used for a long time. A student who knows a rule can work at the formalising level, but if the student has no image, then the previous understanding cannot be recaptured and used to develop understanding when faced with future unsolvable problems.

Acts of understanding can be described in terms of qualitative changes in knowledge, or as jumps from old ways of knowing to new ways of knowing (Sierpinska 1990, 1992). If we look back at our old ways of knowing, we see things that prevented us from knowing in a new way, some of which may be epistemological obstacles, but if we look at what we have jumped to, we tend to speak of understanding. When considering epistemological obstacles we are focusing attention on what was wrong or insufficient in our ways of knowing. Our new way of knowing may, in turn, start functioning as an epistemological obstacle in a different situation. Hence, some acts of understanding are acts of overcoming epistemological obstacles.

We have seen that our prior knowledge can act as an obstacle to acquiring new knowledge, and that, in order for understanding to develop, the obstacles to the new knowledge must be overcome. The prior knowledge may be conceptual or procedural in nature, but both can be very resistant to change. We must discover what will motivate a student to consider a new way of knowing and what factors cause the resistance to change. Sierpinska (1987) suggests that examining the reasons for a student changing approach might be helpful in understanding the processes of overcoming obstacles. It is hoped that the current study will contribute to the research in this area. It may also be fruitful to expand the scope of the term obstacle to include the type of knowledge described above, and then to examine research in the area of obstacles in order to deepen our understanding of the choices made by students. Whether 'obstacle' is understood in a broader or narrower sense, overcoming an obstacle requires change in the knowledge structure of the learner, and it is to this process that we now turn.

Conceptual Change

Taking the constructivist view, and having recognised that knowledge construction takes place under certain constraints, within a domain and in a context, we now look in more detail at the conditions necessary for a student to restructure his or her knowledge system. Posner, Strike, Hewson and Gertzog (1982) asked the question, "Under what conditions does one central concept come to be replaced by another?" (p. 213).

The research of Posner et al. (1982) is in the scientific domain and it discusses the situation where a new experience cannot be explained by the student's current conception. For example, if a student has an incorrect concept of the motion of an object, it can be directly challenged by the presentation of experimental evidence. The student is then convinced that the current knowledge or conception is incorrect. This may well be similar to the evidence which can be presented to convince a student that some informal mathematical concepts brought into the classroom are not correct in all situations, or that a concept which has been generalised inappropriately should be limited to its original domain.

According to Posner et al. (1982), "Central concepts are likely to be rejected when they have generated a class of problems which they appear to lack the capacity to solve. A competing view will be accepted when it appears to have the potential to solve these problems" (p. 213). This can be seen to happen within mathematics when, for example, a child who has the concept of whole numbers meets subtraction. At first the child's concepts and knowledge are adequate, but then a class of problems is encountered for which a larger number must be subtracted from a smaller number. A new concept of number, which includes the negative integers, must be accepted in order for these problems to be solved. The accommodation which must take place does not involve a rejection of the original concept, but rather an expansion of it, so that the term 'number' now has a broader meaning.

Under what conditions is the accommodation of a new concept likely to occur? Posner et al. (1982) propose a set of four conditions which are common to most cases of the accommodation of a scientific concept. I will return to these conditions in chapter 6 where they will provide the structure for my exploration of students' motivations to adopt new mathematical procedures and their reasons for resisting some of the new approaches. The conditions are as follows:

1. *There must be dissatisfaction with existing conceptions.* A student must believe that his or her current concepts will not suffice, that there are anomalies which cannot be explained or problems which cannot be solved using the existing conceptions, and that these anomalies and inconsistencies must be reconciled with current beliefs. Attempts at assimilation into current knowledge structure must be seen to fail before accommodation will take place.

2. *A new conception must be intelligible.* A learner must be able to grasp how experience can be structured by the new concept sufficiently to explore further. At a superficial level, the meaning of the words and symbols must be known, but also the learner must construct or identify a coherent representation of what the new concept is portraying.

3. *A new conception must appear initially plausible.* The new concept must at least appear to have the capacity to solve the problems generated by previously held conceptions. It must be consistent with other knowledge previously constructed by the learner and with past experience. Also, the new conception must be compatible with one's beliefs and fundamental assumptions.

4. *A new concept should suggest the possibility of a fruitful research program.* It should have the potential to be extended, to open up new areas of inquiry.

It may be that this set of conditions for conceptual change accurately describes the conditions necessary for conceptual change in any field, or there may be some slight adaptations which will allow them to be applied to the accommodation of, for example, a mathematical concept or piece of knowledge.

It should be noted that the accommodation of a new concept is likely to be gradual, so that at any given time, some aspects will have been accommodated, but others may not.

Research Questions and Objectives

In the current study, I examine the methods used by students for solving particular mathematical problems, and the effect of their exposure to methods that are more efficient or elegant in certain circumstances. For some students the accommodation of a new concept may be necessary, but for others simply a change in procedure may be all that is required in order to adopt the new approaches.

The research of Baroody & Ginsburg (1986), examining addition procedures used by young children, shows that a procedure will often be chosen if it reduces effort, but then in simpler cases the new procedure may be discarded in favour of an old procedure. Whether this was because the old procedure had related conceptual understanding, or perhaps was just ingrained, they were unable to say. They did state that children often use informal procedures that make sense in terms of their informal concepts, rather than adopt procedures taught in school for which they do not have an adequate conceptual understanding. Of course, we must bear in mind that as we judge what a student knows or understands, and attribute choice to the student's resistance to a particular approach, it may simply be a temporary inability to recall the new knowledge (Asiala, Brown, Devries, Dubinsky, Mathews & Thomas, 1996).

My intention in this study is to reveal some of the factors which motivate students to adopt new procedures and also to highlight some causes of resistance to change. This will be achieved by pursuing the following objectives:

- to determine the methods used by students before instruction
- to observe the influence of the instructional intervention on the choice of method
- to examine the motivations for adoption or reasons for resistance of the presented methods
- to consider the applicability of the conceptual change theory to this situation
- to develop a theory of procedural change

The fourth and fifth objectives raise the two questions, "Does procedural change require the same conditions as those described for conceptual change?" and, "Can the four conditions provided by Posner et al. (1982) be expanded or modified to aid our understanding of what allows a student to adopt an alternative approach to solving a problem when some, perhaps inadequate but correct, method is already known?"

Based on the four conditions of conceptual change proposed by Posner et al., my specific research questions were:

- What causes dissatisfaction with a method?
- What affects the intelligibility of a mathematical procedure?

- In what ways might the new method fail to be initially plausible?
- What is the equivalent of 'fruitful research' for a mathematical procedure?

Before answering these questions, it will be helpful to consider the three content areas used in this study and to examine the findings of research carried out in those domains. In the next chapter I will also discuss literature on representations and visualisation, and problem solving strategies, to the extent that it is relevant to the findings of this study.

CHAPTER 3

PROCEDURES, SYMBOLS AND MEANING

The three content areas featured in the current study are multiples and factors, percentages and fraction arithmetic. Each of these has been explored to a greater or lesser extent within the mathematics education community. However, there appears to have been little research concerned with the specific topics under consideration here: greatest common factor and least common multiple, compound percentage change, and addition and subtraction of mixed numbers. A review of the literature most relevant to this study is presented in this chapter.

For each topic, certain procedures are taught to students. The meaning we attribute to the symbols used in algorithms has a great effect on our ability to understand the concepts underlying the procedures. Literature on symbolic representation, and the mental image evoked by these symbols, or by other forms of representation, is discussed and related to the current study. Finally, literature on the motivations to learn, and the strategies used by students when problem solving, is examined for its contribution to our knowledge of the choices made by students.

Content Areas

The understanding which students have of the different methods to find the greatest common factor (GCF) and least common multiple (LCM) has not been documented and little research is available on this topic. Brown, Thomas & Tolias (2002) examined conceptions of divisibility and observed that students often deal with number theoretic tasks without consciously using their knowledge of multiplicative structure. Students frequently choose to perform computations when reasoning about computations will suffice and they lack the understanding that the divisors of a number can be obtained directly from its prime factorisation. They identified three approaches to finding the LCM: 'set intersection', where multiples of each number are listed, 'create a multiple and divide', where multiples of one number are checked for divisibility by the other, and 'prime-factorization', where the powers of primes are compared and extracted from the prime factorisation of the numbers. Many students, they say, find it difficult to see that the prime factorisation method actually produces the LCM. They attribute the successful adoption of this method by students to the fact that it can be memorised and applied easily. Few of their students could explain why the procedure worked and from this they concluded that, “applying the algorithm correctly requires only a manipulation of surface features, not an understanding of its rationale” (p. 108).

Similarly, not much is known about the understanding of percentage change. Jabon & Tolias (2003) observed that college students taking a quantitative reasoning course had difficulties with compound percentage change questions, finding that even after instruction, 57% did not answer correctly. They called for further study of adults' understanding of percentages, suggesting that the use of percentages in everyday life does not correspond to what is taught in elementary schools.

Many studies have been conducted in the area of fractions, although little of this relates specifically to the notion of mixed numbers. Much of the work discusses the algorithms used for calculations involving fractions and the rote memorisation often associated with such procedures. Mack (1990) claims that many students' understanding of fractions is characterised by a knowledge of rote procedures, rather than by the concepts underlying the procedures. Some feel that the concept of the 'whole' is not sufficiently promoted as an essential idea in understanding fractions and that teachers' visual representations of fractions are incomplete and unsatisfactory (Linchevski & Vinner, 1989). Lamon (1999) discusses the confusion created by questions which do not indicate clearly the intended whole, which leads students to believe that it is either unimportant or a matter of personal choice. She goes on to describe several different interpretations of the fraction notation and points out that focusing on the part-whole comparison leads to an inadequate understanding and lays a poor foundation for the rational number system and all that builds on it. Tzur (1999) found that the part-whole emphasis when teaching fractions caused children to resist the concept of improper fraction, with the total amount being regarded as a new whole.

When the concept of fraction is introduced to children, a reasonable amount of time is spent relating the symbols to physical representations, but when arithmetic involving fractions is encountered, the algorithms are often introduced with great haste. Carpenter (1986) found that the procedures were not clearly connected to the conceptual knowledge which the children had. Poor foundations in fractions lead to rote learning of the algorithms (Booker, 1996). When students meet mixed numbers and improper fractions, less attention is given to the development of a sound conceptual base. Booker found that the change from a part-whole conception of fraction to mixed numbers causes confusion and Rees (1987) claims that the algorithm to change mixed numbers to improper fractions removes meaning and is confusing to students. She found, for example, that few students equate $^8/_3$ with $2^2/_3$.

Woodward (1998) used the term 'fraction sense' to refer to the ability to understand the meaning of fractions, to reason qualitatively and to make judgements about the reasonableness of calculations. She commented that little time is spent in schools developing this fraction sense and that students who could succeed by memorising the algorithms were frequently unable to say whether their answers were reasonable. After instruction, the students demonstrated good progress, but the results showed that they would abandon what they had learned when confronted with unfamiliar situations. It was concluded that fraction sense must be thoroughly developed and mastered before algorithms are introduced.

Others have also found that knowledge of procedures which have been memorised without understanding can interfere with, or even prevent, the adoption of a more meaningful approach (Hiebert & Wearne, 1988; Mack, 1990). If students carry out computations with the symbols linked to referents which are meaningful to them, then this understanding may inform the selection of an appropriate syntactic rule. However, Hiebert and Wearne suggest that most students rely on syntactic rules recalled without considering the meaning. Mack found that students rejected answers obtained by using their informal knowledge of fractions, trusting in memorised procedures, even when these were faulty. Overcoming their reliance on these procedures required a great deal of time and practice in taking real world situations and modelling them symbolically. These findings add weight to the argument that concepts should be taught thoroughly before algorithms are introduced. However, for students who already know the procedures, especially those who have memorised them correctly, the task of building meaning for the procedures might be much harder.

Representations and Visualisation

The dangers of rote memorisation of procedures have been mentioned several times in the preceding section, contrasting this form of knowledge with conceptual understanding. According to Harel and Kaput (1991), notations can be substituted for conceptual entities, but these notations do not refer to any mental content other than the physical structure of the notation itself. The inventors of the notations were looking for a way to express their own conceptions, but in schools we often teach the manipulation of the notations before the conceptions have been established. Zazkis and Campbell (1996b) confirmed that many students regard the form of the notation as an integral part of the meaning. Different representations can draw our attention to different properties of a number, but many students do not take advantage of this. For example, in the work of Zazkis & Gadowsky (2001), students were given $M = 3^3 \times 5^2 \times 7$ and asked if *M* is divisible by 7. Several students chose to carry out the multiplication, determining the value of *M*, then divided by 7 to see if a whole number resulted. They seemed to consider the prime factorisation as an instruction rather than as a representation of a number. I have observed that '18% of 534', for example, is often seen as an instruction to calculate rather than as a representation of a quantity, and similarly, some students appear to regard the mixed number notation as an instruction to calculate the equivalent improper fraction. Zazkis & Gadowsky suggest that students do not "think" before carrying out calculations, using sometimes extensive computation when it could be avoided. They described this strategy as 'not elegant' and attributed students' perception of number to their prior school practices, where more emphasis had been put on calculations than on attention to number structure. This may be the case for mixed number calculations, where converting to an improper fraction allows students to proceed with algorithms for multiplication or division as well as addition or subtraction. The conversion method is therefore a more general approach, requiring less thought before calculation commences, but the calculation involved is more complex than is necessary for addition or subtraction, thereby falling into the category of 'not elegant'.

The ability to capitalise on the strengths of a given representation is an important component of understanding mathematical ideas (Lesh, Behr & Post, 1987), but students need to develop meaning for the symbols before this is possible. Goldin (1998) suggests that we should help students to construct powerful, internal systems of representation rather than concentrating on the manipulation of formal notational systems. Herscovics (1996) suggested that students may even develop their own interpretations for notations when algorithms are learned without an understanding of the underlying concepts.

Symbols are manipulated during the application of procedures, but as argued above, the underlying meaning of the symbols, the concepts they represent, must not be lost. Hiebert (1988) proposed a theory of developing genuine competence with written symbols consisting of four stages: (1) connect symbols with referents, (2) develop symbol manipulation procedures, (3) elaborate and routinise the procedures, (4) build more abstract symbol systems using the symbols and rules as referents. He claims that many deficiencies of students can be explained by their not following this sequence. Stage 1 involves much repetition and must lead to the students being able to call up a mental image of the referent and reason directly about it to solve a problem presented symbolically. The numeric symbols must be well connected to the referents before the operation symbols can connect with the actions on those referents. Then the students can monitor their own actions on the referents and detect errors and modify their procedures (stage 2). However, this mental recollection of the referent must be stopped for further competence to develop, or the cognitive load would be too great. “The power of mathematics comes, not from the connections of symbols with referents, but from the fact that symbols can be manipulated without regard to their referents” (p. 341). At stage 3, routinising (executing automatically, with little conscious thought) separates rules from referents. Problems arise here if insufficient attention has been paid to the first two stages so that the meaning of the symbols cannot be accessed when needed. Further development at stage 4 then becomes meaningless rote memorisation. It may be more difficult to go back to rebuild the initial processes if they have not been established when the symbol system is first encountered. Hiebert found that instruction designed specially to promote stages 1 and 2 was less effective with students who had already routinised symbol manipulation rules. After instruction most of these students returned to using their prior approach, even when their procedure was flawed. Moving too quickly through the stages, leads to the commonly held view that mathematics consists of symbols on a page which have rules for manipulation and has little to do with intuitive thought or 'real' problems.

A definition of a concept may be learned, but it may not always be understood. When a word is said, a mental image is invoked which may be a picture or remembered experiences or even some impressions, but it is rarely a definition. This mental image can be called the ‘concept image’ (Vinner , 1991). One description of understanding, suggested by Vinner, is that one has formed a concept image for the concept. Ideally, the concept image and the definition interact and both are consulted, but in reality, we usually rely on the concept image alone. This may be helpful, or may be misleading. For example, by definition, $|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$, but often we think of it as "the number part, without the sign". This concept image is adequate in some situations, but often leads to mistakes in others. A definition may help to form a concept image, but once the image is formed the definition is sometimes put aside, and may even be forgotten.

According to Vinner, educators often assume that a student will refer to the concept definition alone, or perhaps in conjunction with the concept image, but in reality, most refer only to the concept image. In order to encourage students to refer to the definition, problems for which the concept image is incomplete must be presented and this, in turn, will encourage the development of a more complete image. The fact that students rarely refer to the concept definition could explain why they are not troubled by some of the inconsistencies present in their work.

There is much debate over what is meant by internal representations or imaging or visualisation and these terms are used in different ways by different researchers. Zimmerman & Cunningham (1991), for example, define visualisation as the ability to represent a mathematical concept or problem by an appropriate diagram and to use the diagram to achieve understanding, and as an aid in problem solving. Dawe (1993) speaks of imaging, in which parts of the problem are visualised and brought together to form a whole, in order to understand a mathematical task. Goldin & Kaput (1996) state that an internal imagistic representation is essential to understanding and that to construct these internal representational systems we need to be able to interpret the formal rule-based procedural systems through representational acts. When faced with a task, the first step is to understand the problem and we do this by constructing a problem representation. How good this representation is determines the next step in our thinking (Glaser, 1984). If students see a question as invoking a rote procedure, but do not have a deeper understanding of the situation, they will not pause to think about a meaningful or different way to solve the problem.

Students discussed in Zazkis & Dubinsky (1996) showed a lack of ability to connect a diagram with its symbolic representation, but it was suggested that both visual and analytic thinking must be present and integrated in order to construct rich understandings of mathematical concepts. Students may be reluctant to visualise, or may not have been encouraged to do so, because of the assumptions present within the educational system that the use of visual strategies is connected with weak mathematical ability. However, weaker students exist among those who have a preference for analytic thinking as well as those preferring visual thinking. The difference may be that students who prefer analytic approaches can be more successful in school because of the emphasis on imitation of algorithms. At higher levels, where a deeper understanding is necessary, these students may no longer be able to succeed. According to Eisenberg & Dreyfus (1991), there is a prevalent belief that mathematics is non-visual among mathematicians and teachers, and this is passed to students, even when a visual representation is at the base of an idea. Mathematicians often use visualisation in their own work, but in their teaching, it is relegated to an illustrative role if it is used at all. They suggest that some learning problems might be eradicated by stressing a visual approach to mathematical concepts, but also claim that visual problem solving is cognitively more demanding than analytic, and it is harder to teach.

Problem Solving Strategies

It has been observed that children do not always choose to use the most superior strategy they know (Siegler & Shipley, 1995). Even when taught a strategy which improved their performance, few continued to use the new strategy when given a choice. Siegler & Shipley found that if the children were experienced in a particular domain, their strategy choice seemed to be a relatively automatic, hard-to-change process. While examining the development of problem solving strategies in children, Kuhn and Phelps (1982) found that knowledge of an effective strategy was not sufficient for students to abandon inadequate ones. Even after more advanced strategies had been successfully applied by the students, much time and practice was needed before their use became consistent. Using new methods is often not the safest or fastest way to achieve a correct answer and many students believe it is safer to rely on their old procedures. Hatano and Inagaki (1992) list as one of the conditions for comprehension activity that people must be "free from urgent external need" (p. 128), such as the pressures of time or being assessed on performance.

According to Schoenfeld (1985), what you know makes little difference if your beliefs will not let you even consider a particular approach. This is supported by the work of Cramer & Lesh (1988) in which pre-service teachers who solved story problems using their informal knowledge did not think to use their informal knowledge to solve problems presented in symbolic form. In fact they did not represent the story problem symbolically at all, but for symbolic problems they used only procedural methods. Many believe that mathematics is to do with following established procedures and rules, and our beliefs influence our decisions. There is also an affective component, such as what feels familiar. These beliefs and feelings are involved as we decide which are acceptable procedures to follow (Goldin, 1998).

The willingness to adopt new approaches also depends on what might be termed the personality of the learner. Nolen (1988) discusses different types of learner, or rather the different goals in learning they hope to achieve, which may cause students to choose, or persist with, a particular study strategy. These goals of learning relate to beliefs about causes of success in school. Different types of learner may use different learning strategies, which may in turn influence the nature of what is learned. The learning strategies described by Nolen could also apply to problem solving, and are presented here with slight modifications to fit with terminology appropriate to strategies for solving mathematical problems.

'Task oriented' students value learning for its own sake, they strive to understand and the more they learn, the more competent they feel. They believe that success depends on hard work, interest, co-operation and understanding rather than memorisation. These students are therefore more likely to use and value methods that facilitate understanding or are more elegant, even if they require more effort to learn. 'Ego oriented' students seek to do better than others, and for them learning and understanding is a means to an end. They believe success depends on competitiveness and impressing the teacher. They will change method only if they perceive that they are more likely to get the right answer, or if the teacher requires it. For 'work avoidance' students the amount of effort required is critical. They will adopt a new method only if they are convinced it will save them enough effort to be worth the effort involved in learning it. Nolen concludes that the competition for grades usually present in the education system may not be the best way to encourage meaningful learning and we need to do more to encourage students to value learning for its own sake.

In this chapter we have examined research in the domains of the mathematical topics used in this study. It has been suggested that many students, successful in the school system, carry out algorithms with little or no reasoning and do not relate the procedures to the concepts underlying them. We have seen how the symbols themselves can contribute to this removal from reality, encouraging rote memorisation when their meaning has not been firmly established. The beliefs and personality of the student are also factors in determining whether the learning is with understanding. These and other issues influence what students learn and what they choose to use in particular situations.

CHAPTER 4

METHOD

The Course

Math 190, 'Principles of Mathematics for Teachers', is a course offered at Simon Fraser University. Students who wish to enter the program for a teaching certificate in elementary education are required to take a mathematics course as part of their degree. The Math 190 course is recommended to these students and is designed to address their needs. A few students take this course who do not intend to be elementary teachers, but who have an interest in education because of, for example, involvement in private tutoring. Below I describe the structure of the course offering in which the study took place. There are variations in the specifics of the offerings, depending on the instructor, although these differences are minor.

The course was conducted through two 2-hour class meetings per week, for 13 weeks. Homework was assigned each week and a drop-in workshop was available to the students, where they could meet to work in groups and obtain help from experienced Teaching Assistants. Assessment of the students was based on two 1-hour midterm examinations, a 3-hour final examination, an extended investigation, and a small selection of the weekly homework problems. Throughout the course students were encouraged to work together, with much discussion and peer-teaching taking place. The use of calculators was permitted in every part of the course, although it is acknowledged that this reduces the necessity for students to choose procedures which lead to simpler calculations. However, because of the widespread acceptance and use of calculators in society, it was felt that to deny students this assistance would be unreasonable, creating an unrealistic environment in which to work.

The content of the course includes arithmetic, geometry, number theory, an introduction to problem solving techniques, and a small amount of statistics and probability. The majority of the content consists of topics which are typically met in the elementary school curriculum, although the approach and language used may be more advanced than that used with children. Three of the topics encountered in the course were selected for attention in this study and they will be discussed in detail at a later stage.

The weekly homework questions gave an opportunity for the students to examine, practise and discuss the material covered in class. Credit was given for any correct method used to solve problems and answer questions, although the techniques discussed in class were usually the most efficient, or the most explanatory, and these methods were used for the published solutions.

Students were also required to carry out an open-ended investigation. There was a choice of two topics, for each of which an initial simple problem was given. In investigations of this type, the students must develop the topic by asking, and attempting to answer, their own related questions which require the use of a higher level of mathematics. An emphasis is placed on clarity of communication and the reasoning used in solving the problems.

The Participants

The students taking the Math 190 course come from diverse ethnic backgrounds and although the majority is female, there is a significant minority of men each semester. There is a wide variety of ages within the group, from recent high school graduates to those in their forties. Consequently, many have not studied mathematics for several years. The pre-requisite for the course is grade 11 mathematics with a grade C, or an equivalent, or higher, qualification. A large number of students taking this course express a fear of the subject, or 'math anxiety', and often claim that they had 'bad experiences' during their school years.

To allow the homework, examinations, and observations made in class or during discussions to be used as data for this study, students in the Spring 2001 course offering were given the opportunity, during two of the class sessions, to sign a consent form (see Appendix A). Of the 95 students enrolled in the course, 81 gave their permission, however only 78 of these students completed all three of the written items used for data collection. All the students were then invited to participate in individual, clinical interviews. Out of the 23 volunteers, 20 were interviewed and 3 were not available during the times arranged for the interviews. Code names have been used for all students, although the gender indicated by the name remains unchanged.

Data Sources

The data is drawn from the following sources, the details of which are provided below:

- first homework
- second midterm examination
- clinical interviews
- final examination

At the beginning of the course, the students were asked to answer some questions on the three topics used in this study with the aim of establishing the methods they used before any instruction was given in the course. This took the form of the first homework and the students were informed that credit would be given to those who showed clearly the methods they used, rather than for the correctness of their answers. The written responses of those students who had signed consent forms were analysed.

Other sources of written response used as data for analysis were the second midterm examination and the final examination. Instruction on the three topics used for this study was given after the first midterm examination, during weeks 4 to 8. This timing allowed the students to be familiar with the style and requirements of the course before meeting the methods being introduced for these topics. The second midterm examination was at the beginning of week 10, which allowed time for the students to practise using the new methods after their introduction in class. There was a small amount of time during the final week of classes (week 13) for review of topics and this included some discussion of the three topics used for this study. The final examination was at the end of week 15.

After the second midterm, but before the final examination, clinical interviews were conducted with 20 volunteers from the course. A list of interview questions is provided in Appendix E. As outlined in Zazkis and Hazzan (1999), the questions asked initially in the interview were 'performance' questions, that is, questions of a type frequently found in mathematics lessons and text books. The interest from the point of view of the interviewer was the students' choice of method for answering each question rather than their success or failure in achieving a solution. After they had answered the questions, students were asked to reflect on their choice of method. They were encouraged to discuss why they had used their chosen procedures for the questions asked during the interview itself and to compare them with those they had used in the midterm and with the methods presented in class.

Throughout the course there were informal observations of students' work and discussions. These were made during class sessions, in the workshop where students were working together on the homework and investigations, and in my office when students came for help. Although these observations were not formally recorded, they contributed to my knowledge of the approaches used by the students, their motivations and the difficulties they experienced.

Mathematical Content

The three topics chosen for this study were greatest common factor and least common multiple, compound percentage change, and addition and subtraction of mixed numbers. Of the topics covered in the course, these appeared to be the areas where many students were familiar with a method which was different from that presented in class. For other topics the students were, in general, either reviewing familiar methods in order to gain a deeper understanding, or they did not know, or remember, any method to solve the problems.

Greatest Common Factor and Least Common Multiple

For this discussion we will consider the greatest common factor (GCF) or least common multiple (LCM) of two whole numbers, although the same methods can be applied for more than two numbers. The majority of students were familiar with the method of 'listing'. By this I mean the factors of each number are listed in order and then the largest common entry is selected to give the GCF. Some students modify this method in order to reduce the amount of work. They considered the factors of one number, from largest to smallest in the most efficient cases, and check, by division, whether each is a factor of the other number. Similarly, to find the LCM, the first few multiples of each number are listed and the first common entry is selected, the lists being extended if necessary. This is modified by some to checking each multiple of the larger number for divisibility by the smaller.

In the course, the method of listing was acknowledged as a useful introduction to the concept of GCF and LCM. It was then shown to be a very inefficient method to use in many situations. The students were subsequently taught the method of selecting appropriate powers of the prime factors from the prime factorisation of the numbers. A few students claimed to be familiar with this method.

Examples:

1. Find the GCF and LCM of 8 and 12.

By listing -

Factors of 8 = {1, 2, <u>4</u>, 8}
Factors of 12 = {1, 2, 3, <u>4</u>, 6, 12}
Multiples of 8 = {8, 16, <u>24</u>, 32, 40, . . .}
Multiples of 12 = {12, <u>24</u>, 36, 48, 60, . . .}
So GCF(8,12) = 4 and LCM(8,12) = 24

By modified listing -

Largest factor of 8 is 8, but 8 does not divide 12
Next largest factor of 8 is 4, and 12 ÷ 4 = 3
So GCF(8,12) = 4
First multiple of 12 is 12, but 8 does not divide 12
Second multiple of 12 is 24, and 24 ÷ 8 = 3
So LCM(8,12) = 24

By prime factorisation -

$8 = 2^3$ and $12 = 2^2 \times 3$

So $GCF(8,12) = 2^2 = 4$ and $LCM(8,12) = 2^3 \times 3 = 24$

It may be argued that the amount of work required to find the prime factorisation of the numbers means that there is not a significant saving of effort over that required for the listing methods. However, consider the following example.

2. Find the GCF and LCM of 360 and 378.

By listing -

Factors of 360 = {1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, <u>18</u>, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360}

Factors of 378 = {1, 2, 3, 6, 7, 9, 14, <u>18</u>, 21, 27, 42, 54, 63, 126, 189, 378}

Multiples of 360 = {360, 720, 1080, 1440, 1800, 2160, 2520, 2880, 3240, 3600, 3960, 4320, 4680, 5040, 5400, 5760, 6120, 6480, 6840, 7200, <u>7560</u>, . . .}

Multiples of 378 = {378, 756, 1134, 1512, 1890, 2268, 2646, 3024, 3402, 3780, 4158, 4536, 4914, 5292, 5670, 6048, 6426, 6804, 7182, <u>7560</u>, . . .}

So GCF(360,378) = 18 and LCM(360,378) = 7560

By modified listing -

The large factors of 360 are not easy to spot, without considering the smaller factors and obtaining the larger by dividing 360 by 2, 3, etc. Using this method, all the factors of 360 would be found before the 18 was identified as the GCF.

To find the LCM, the multiples of 378 would be listed in order, with each being checked for divisibility by 360.

By prime factorisation -

$360 = 2^3 \times 3^2 \times 5$ and $378 = 2 \times 3^3 \times 7$

So GCF(360,378) $= 2 \times 3^2 = 18$

and LCM(360,378) $= 2^3 \times 3^3 \times 5 \times 7 = 7560$

Clearly, the prime factorisation method is much quicker in this situation, and both the GCF and LCM can be found with little effort after the work done to find the prime factorisations.

Compound Percentage Change

The method most familiar to the participants for finding a quantity after a percentage increase or decrease was first to calculate the actual increase or decrease. This amount was then added to or subtract from the original value. I will refer to this method as the 'two-step' method. Some students were familiar with multiplying the original by a single number to find an increased value (here termed the 'one-step' method), but few used this method for a decrease.

When percentage increases and decreases were compounded the majority of students used the one-step or two-step methods to calculate the value after each change. The value after one change was used as the original amount for the next change. This is referred to as the 'repeated' one-step or two-step method. In class it was demonstrated how the one-step method could be applied to compounded percentage changes, thus allowing the complete calculation to be written as one string of multiplications ('compound one-step' method).

Examples:

1. Last year a house was valued at $379,000. Prices of houses of that type have fallen by 8%. What is the current value of the house?

By the two-step method - 8% of 379,000 = 0.08 x 379,000 = 30,320
new value = 379,000 - 30,320 = $348,680

By the one-step method - 92% of 379,000 = 0.92 x 379,000 = \$348,680

The initial representation of an '8% decrease' as '92% remaining' is regarded as simple enough to be found mentally, and is therefore not counted as a 'step' in the method. Similarly, it is assumed that the students are familiar with writing percentages as decimals.

2. A mutual fund has the following five-year history : 16% loss, 7% gain, 2% loss, 12% gain, 9% gain. What would be the overall percentage loss or gain on an investment made five years ago and cashed in today?

By the repeated two-step method - almost all students assume an arbitrary value for the investment, say \$1000 :

16% of 1,000 = 0.16 x 1,000 = 160
value after one year = 1000 - 160 = \$840
7% of 840 = 0.07 x 840 = 58.8
value after two years = 840 + 58.8 = \$898.80
2% of 898.80 = 0.02 x 898.80 = 17.976 = \$17.98
value after three years = 898.80 - 17.98 = \$880.82
12% of 880.82 = 0.12 x 880.82 = 105.6984 = \$105.70
value after four years = 880.82 + 105.70 = \$986.52
9% of 986.52 = 0.09 x 986.52 = 88.7868 = \$88.79
value after five years = 986.52 + 88.79 = \$1075.31
change in investment value = 1075.31 - 1000 = \$75.31
overall gain = 75.31/1000 x 100 = 7.531%

By the repeated one-step method - again, almost all students assume an arbitrary value for the investment, say \$1000:

84% of 1,000 = 0.84 x 1,000 = \$840
107% of 840 = 1.07 x 840 = \$898.80
98% of 898.80 = 0.98 x 898.80 = 880.824 = \$880.82
112% of 880.82 = 1.12 x 880.82 = 986.5184 = \$986.52

109% of 986.52 = 1.09 x 986.52 = 1075.3068 = $1075.31

overall change = 1075.31/1000 x 100 = 107.531%

overall gain = 7.531%

By the compound one-step method -

new value = 0.84 x 1.07 x 0.98 x 1.12 x 1.09 x old value

= 1.0753099392 x old value

overall gain = 7.531%

[The exact value of the product was noted because students obtained this answer when using calculators. There was then a discussion about choosing an appropriate number of significant figures for each stage in a calculation.]

The compound one-step method is obviously more efficient and, for those who understand the relationship between decimal fractions and percentages, the calculation is much clearer.

Addition and Subtraction of Mixed Numbers

Almost exclusively, the method used by the students was to convert the mixed numbers to improper fractions, find a common denominator, add or subtract and then, in most cases, convert back to a mixed number. This method, which I will refer to as the 'conversion' method, was used even when one of the numbers was a whole.

In class, we discussed the 'wholes-first' method in which the whole numbers and the fractional parts are added or subtracted separately. For addition, the resulting fractional part is then simplified, and converted to one whole and a fractional part, if appropriate. For subtraction, if the second fractional part is larger than the first then one whole from the resulting whole number, along with the first fractional part, could be converted to an improper fraction and the subtraction carried out by conversion. A slightly easier method is to subtract the second fractional part from one of the resulting wholes and then add the remainder to the first fractional part. I will refer to this as the 'wholes-first-one' method. This method was promoted in class, but it appeared that no students had encountered the wholes-first-one method before the course.

Examples:

1. Find the value of $7+8\frac{5}{6}$.

By conversion - $7+8\frac{5}{6} = \frac{7}{1}+\frac{53}{6} = \frac{42}{6}+\frac{53}{6} = \frac{95}{6} = 15\frac{5}{6}$

By wholes-first - $7+8\frac{5}{6} = 15\frac{5}{6}$

2. Find the value of $12\frac{1}{3}-9\frac{4}{5}$.

By conversion - $12\frac{1}{3}-9\frac{4}{5} = \frac{37}{3}-\frac{49}{5} = \frac{185}{15}-\frac{147}{15} = \frac{38}{15} = 2\frac{8}{15}$

By wholes-first - $12\frac{1}{3}-9\frac{4}{5} = 3\frac{1}{3}-\frac{4}{5} = 2\frac{4}{3}-\frac{4}{5} = 2\frac{20}{15}-\frac{12}{15} = 2\frac{8}{15}$

By wholes-first-one - $12\frac{1}{3}-9\frac{4}{5} = 3\frac{1}{3}-\frac{4}{5} = 2\frac{1}{3}+1-\frac{4}{5} = 2\frac{1}{3}+\frac{1}{5}=2\frac{5}{15}+\frac{3}{15}=2\frac{8}{15}$

One advantage of the wholes-first method is that calculations are carried out using smaller numbers, thereby reducing the likelihood of errors. Another advantage is that the whole number after the initial addition or subtraction is a reasonable estimate of the size of the final answer. The wholes-first-one method can also simplify the numbers involved in the calculation, with subtraction from one being a very easy operation and resulting in the addition of two small fractions. Also, the wholes-first-one method (perhaps even more than the wholes-first method) is more closely related to the actions carried out in physical situations.

Instruction

Greatest Common Factor and Least Common Multiple

Four hours of class time were spent discussing factors and multiples. The instruction took place in weeks 4 and 5 of the course. After a discussion of the definition of a factor, how to find all the factors of a number, and the definition of prime and composite numbers, the Fundamental Theorem of Arithmetic was introduced. Implications of this theorem were discussed and the students practised finding the prime factorisation of numbers. The meaning of the term 'greatest common factor of two numbers' (GCF(a,b)) was introduced by listing the factors of the numbers. The students were then asked if they could find a quicker way to find the GCF. Relating the GCF, found by listing, to the prime factorisation of the numbers, the students were able to see that the product of the primes common to the factorisations gave the GCF. This was discussed further and illustrated as the product of elements in the intersection of the sets of prime factors (multiple powers being repeated entries in the sets) on a Venn diagram. An example is shown in Figure 4.1. This method was then shown to be a much more efficient method than that of listing, by giving examples where the numbers have many factors.

$18 = 2 \cdot 3 \cdot 3$ and $30 = 2 \cdot 3 \cdot 5$ so $GCF(18,30) = 2 \cdot 3$

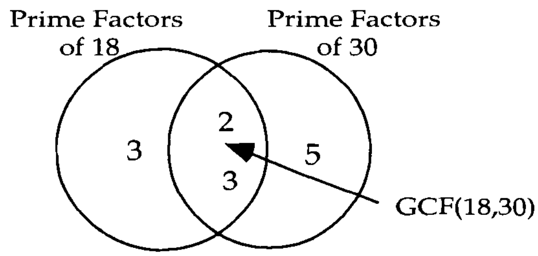

Figure 4.1 Visual Representation of GCF(a,b) from Prime Factors of a and b

The definition of multiple was then reviewed, and the concept of least common multiple (LCM) was illustrated by listing multiples of two numbers and finding the smallest value common to both lists. Again, the students were asked to find a quicker way to find the LCM. This time they found that the LCM is the product of elements in the union of the sets of prime factors (remembering to show multiple powers by repeated entries) and this was discussed further. This also was demonstrated to be much more efficient when the LCM is a large multiple of each number.

The students went on to practise these techniques and to use them to find the GCF and LCM of more than two numbers. The homework questions included direct questions such as "Find GCF(117, 195)" and word problems such as "Three neighbourhood dogs barked consistently last night. Spot, Patches and Lady began with a simultaneous bark at 11p.m. Then Spot barked every 4 minutes, Patches every 3 minutes and Lady every 5 minutes. Why did Mr. Jones suddenly awaken at midnight?" (Musser, Burger & Peterson, 2001, p. 201, 202).

Compound Percentage Change

After studying fractions and decimal fractions, one two-hour class session was spent discussing percentages in week 8 of the course. Percentages were introduced as a special fraction where the denominator is always 100. The similarity of the representations of fractions as decimals and as percentages was considered. After finding a percentage of a quantity using

$$x\%\ of\ Q = A \quad \Rightarrow \quad x\% \times Q = A \quad \text{(with } x\% \text{ written as a decimal)}$$

the concept of percentage increase or decrease was introduced. Emphasised in the discussion was the importance of identifying which quantity is the whole, or 100%, to which the increase or decrease is applied.

Questions such as "Company A made a profit of 10%, and Company B made a 60% profit. Which company is the more successful?" were considered in class. The answer to this question depends on how we define success. We also need to know how the percentage profit has been calculated. Suppose the profit is given as a percentage of the costs incurred by the company and that Company A had costs of $10,000, whereas Company B had costs of only $500. Then Company A made a profit of 10% of $10,000, which is $1000, and Company B made a profit of 60% of $500, which is $300.

Next, percentage change was discussed. We may know the initial size of a quantity and be given the increase, or decrease, as a percentage. We then want to know the new size of the quantity. This may arise in situations discussing population changes, loans and investments, and many others. In class we considered the new value as a percentage of the original value. For example,

for a 15% increase, new value = 115% of original value= 1.15 x original value
for a 7% decrease, new value = 93% of original value = 0.93 x original value

Compound percentage change problems were approached with a reference to 'real life' situations. The above method of calculation was shown to be very efficient when several changes were involved. For example, an increase of 20%, followed by a decrease of 6%, then an increase of 4% and finally a decrease of 15% can be represented by one calculation:

$$\begin{aligned} \text{new value} &= 1.20 \times 0.94 \times 1.04 \times 0.85 \times \text{original value} \\ &= 0.997152 \times \text{original value} \\ &\approx 99.7\% \text{ of original value} \end{aligned}$$

which is equivalent to an overall decrease of 0.3%

This method also has the advantage of not requiring knowledge of the original value, which is needed in the two-step method in which the actual increases and decreases are calculated.

The homework questions included compound problems such as "A bookstore had a spring sale. All items were reduced by 20%. After the sale, prices were marked up at 20% over the sale price. How do prices after the sale differ from prices before the sale?" (Musser, et al., 2001, p. 298)

Addition and Subtraction of Mixed Numbers

A total of three two-hour class sessions were spent discussing fractions and mixed numbers. The instruction took place in weeks 6 and 7 of the course. The essential concepts of the topic of fractions were introduced through a practical demonstration using two chocolate cakes. The session had a story theme running throughout which was based upon the story I had used when teaching this topic to 11 year-olds in a secondary school in England. I hoped that the story and the visual aids would help the students remember the concepts we discussed and also it was a demonstration of a style of teaching which had proved successful in teaching this topic to children.

When discussing calculations with mixed numbers I also used a visual aid and a story. Beginning with 12 chocolate bars, I explained to the students that my husband had eaten $\frac{4}{5}$ of a bar on the way to work that morning, so I had only $\frac{1}{5}$ of that bar, leaving a total of $11\frac{1}{5}$ bars now. Each chocolate bar consisted of 10 small pieces. "A student in the class wants 3 full bars and 7 small pieces to share with friends. How can this transaction be represented in the form of a calculation?" The students gave the expected response: $11\frac{1}{5}-3\frac{7}{10}$. I asked how I could carry out the calculation, and then suggested an answer to that question, based on the 'conversion method' commonly used by the students. My strategy would therefore be to open all the wrappers and break the bars into double pieces (to get 56 pieces of size one fifth of a bar). Then I would work out that 3 full bars and 7 small pieces gives 37 small pieces (of size one tenth of a bar) and so I needed to break up all the double pieces to get 112 small pieces. Then I could give 37 of these to the student and gather the remaining small pieces into groups of ten to put back into the wrappers and tape them up. Any small pieces left over I would place together in one wrapper, folding it up to keep them together. As I described my strategy, the students accused me of being ridiculous, and said that they would never do that! So I asked what they would do and was told first to give away the three full bars ($11\frac{1}{5}-3 = 8\frac{1}{5}$). Then there was some disagreement as to what to do next. Some suggested I give away the two small pieces in my part bar and open another bar and give away 5 of its 10 small pieces ($1\frac{1}{5}-\frac{7}{10} = \frac{12}{10}-\frac{7}{10} = \frac{5}{10}$), whereas others said it would be easier to take all 7 small pieces from a full bar and then put the two part bars together ($1-\frac{7}{10} = \frac{3}{10}$ then $\frac{1}{5}+\frac{3}{10} = \frac{5}{10}$).

Summarising the calculations representing the different strategies, we have: by conversion -

$$11\frac{1}{5}-3\frac{7}{10} = \frac{(11\times 5+1)}{5}-\frac{(3\times 10+7)}{10} = \frac{56}{5}-\frac{37}{10} = \frac{112}{10}-\frac{37}{10} = \frac{75}{10} = 7\frac{5}{10} = 7\frac{1}{2}$$

by wholes-first -

$$11\frac{1}{5}-3\frac{7}{10} = 8\frac{1}{5}-\frac{7}{10} = 7\frac{6}{5}-\frac{7}{10} = 7\frac{12}{10}-\frac{7}{10} = 7\frac{5}{10} = 7\frac{1}{2}$$

by wholes-first-one -

$$11\frac{1}{5}-3\frac{7}{10} = 8\frac{1}{5}-\frac{7}{10} = 7\frac{1}{5}+1-\frac{7}{10} = 7\frac{2}{10}+\frac{3}{10} = 7\frac{5}{10} = 7\frac{1}{2}$$

Since the 'wholes-first-one' method appeared to be unfamiliar to many of the students, we did several examples in class and found the method to be particularly efficient and elegant for calculations involving large numbers, for example:

$$732\frac{3}{8}-308\frac{11}{12} = 424\frac{3}{8}-\frac{11}{12} = 423\frac{3}{8}+1-\frac{11}{12} = 423\frac{9}{24}+\frac{2}{24} = 423\frac{11}{24}$$

rather than:

$$732\frac{3}{8}-308\frac{11}{12} = \frac{(732\times 8+3)}{8}-\frac{(308\times 12+11)}{12} = \frac{5859}{8}-\frac{3707}{12} = \frac{17577}{24}-\frac{7414}{24} = \frac{10163}{24} = 423\frac{11}{24}$$

In order to give further practice, the homework questions included questions such as "Calculate the following and express as mixed numbers in simplest form. a. $11\frac{3}{5}-9\frac{8}{9}$ b. $7\frac{5}{8}+13\frac{2}{3}$ c. $11\frac{3}{5}+9\frac{8}{9}$ d. $13\frac{2}{3}-7\frac{5}{8}$ " and word problems such as "A man measures a room for a wallpaper border and finds he needs lengths of 10 ft. $6\frac{3}{8}$ in., 14 ft. $9\frac{3}{4}$ in., 6 ft. $5\frac{1}{2}$ in., and 3 ft. $2\frac{7}{8}$ in. What total length of wallpaper border does he need to purchase? (Ignore amount needed for matching and overlap.)" (Musser, et al., 2001, p. 230, 231)

Pilot Study

In the fall of 2000 I was not teaching the Math 190 course, but I invited the students of that course to join a support group to which I offered my experience as a tutor. The students were asked to be committed to the group and to attend most of the weekly sessions throughout the semester. Nine students regularly attended. During these sessions any problems which arose from the material covered in class were discussed. The focus was not on the homework questions, but rather on their general understanding of the topics covered. Towards the end of the semester I requested that students allow me to interview them and an audio recording was made of each interview. Seven students volunteered for these interviews, during which I asked them to find the GCF and LCM of some numbers, to calculate some percentage changes in the context of real-life situations, and to carry out some additions and subtractions with mixed numbers. A full list of the questions used in these interviews can be found in Appendix B.

These interviews were then transcribed and studied to examine the methods used by the students to answer the questions, and to see whether they were those methods discussed in the support group sessions. Questions on the final examination for the course were also used to gather extra information on which methods the students chose to use.

The pilot interviews were a good source of information as to which questions were useful in determining the students' choice of method and which appeared to be perhaps confusing, causing the students to search for something irrelevant to the purposes of this study. Using this knowledge I was able to develop a more appropriate set of questions for the interviews in the main study. The following is a description of my reflections on the questions used in the pilot and how they could be improved.

In considering the questions on mixed numbers, I was concerned that by starting with simple fraction calculations, the students might be more inclined to convert the mixed numbers to improper fractions, in order to make the calculation of the same form as those they had just answered. There seemed to be few difficulties with adding and subtracting simple fractions, so I decided to remove these questions. Similarly, asking the students to explain the meaning of a mixed number, immediately before calculating with mixed numbers, could be an influence towards the wholes-first method, since many of the students drew pictures to help with the explanation. These thoughts led to a reduced number of questions in this section of the interview for the main study.

The students were familiar with basic percentage calculations, such as “Find 18% of 374”, and my main interest was in how they responded to percentage increase or decrease, so I removed these introductory questions for the main study. In the pilot interviews I found that the students were recognising 20% as a percentage which could be calculated mentally in a variety of ways, such as dividing by 10 to find 10% and then doubling that quantity. This seemed to distract the students from the task of finding the price after the 20% reduction. In the main study I avoided such 'special' percentages.

From observations of students in class and at the workshop, those who were inclined to use the one-step method seemed more likely to use it for an increase than for a decrease. In the main study I started with a simple increase question, rather than the questions about sale prices, in the hope that students would consider this method. This was followed by a simple decrease question to indicate whether there really was a greater reluctance to use the one-step method for decreases.

When compounding the same percentage change, the students in the pilot were tempted to multiply the single change by the number of changes. For the main study I avoided this problem by using different numerical values for the percentage changes to be compounded, and also interwove increases and decreases. The fact that two different percentage decreases had given the same actual decrease, in the used car question in the pilot, had been a source of confusion, so this was avoided in the main study.

The physical set-up of the interviews was found to be without major problems. The audio recordings were mostly clear, with little background noise. Some students were initially nervous, but relaxed after a few minutes, and were able to talk freely about their work and any difficulties they were experiencing. Each student had access to pen and paper, and a calculator, but was free to decide how much to use these.

Main Study

In the Spring of 2001 I had the privilege of teaching the Math 190 course for the second time. During the first week of the course the students were given a set of questions designed to determine how each student would approach the topics used in the study before any instruction had taken place. This was presented as the first homework, and a full listing of these questions can be found in Appendix C. The questions were based on those used in the pilot interviews. Fundamental understanding of percentage was tested first and then compounded percentages were introduced in the form of the interest on a GIC and price increases and decreases for an item in a shop. An example and explanation of greatest common factor and least common multiple was given, before students were asked to find the GCF and LCM of two numbers, in order to remind the students of the meaning of these terms. Finally, questions asking students to explain what fractions and mixed numbers are, by means of examples, and to carry out simple fraction calculations were included in order to establish the students' familiarity with this topic. A variety of addition and subtraction questions involving mixed numbers was then given.

When discussing both the greatest common factor and least common multiple, and percentages topics I reviewed the methods with which most students were familiar (those of listing the factors or multiples, and calculating the increase or decrease, followed by adding or subtracting, to find the new quantity) before developing the methods which I was promoting in the course. For the discussions of mixed numbers I used chocolate bars with ten segments to illustrate the methods for addition and subtraction.

After instruction in class, and after opportunities to practise the approaches used in the course, students were given a midterm examination which included questions designed to allow any reasonable approach. A full listing of these questions can be found in Appendix D. The students were again asked to add and subtract mixed numbers and to find the greatest common factor and least common multiple of two numbers. For the percentage questions, the students were asked to find a simple decrease, and were then given a compound question based on the gains and losses of a mutual fund over four years.

The students were then invited to participate in interviews in which they were first given questions similar to those on the midterm examination and then asked to comment on their chosen approaches in the interview and in the midterm. Interviews with 20 students were audio taped and later transcribed. A more detailed discussion of the interviews follows below.

Once again, students were asked questions of a similar type in the final examination for the course, which took place after the interviews. This provided additional data and was also used to check for any changes in the choice of methods used by the students who had, by participation in the interviews, been encouraged to consider further the benefits of the methods discussed in class. Since almost all of the students had used the prime factorisation method for finding the greatest common factor and least common multiple in the midterm examination, I did not include this topic on the final. Only one mixed number calculation was included, which was a subtraction with the second fractional part larger than the first. The compound percentages question was again based on price increases and decreases of an item in a shop. A full listing of these questions can be found in Appendix F.

The Interviews

The interviews were conducted in a private office and the students had access to pencil and paper, and also a calculator. They were encouraged to explain what they were doing as they answered the questions, but this was not required. After answering all questions the students were engaged in a discussion about their choice of method and were asked about their knowledge of methods prior to the commencement of the course. They were asked if they could explain why they used the methods they did, why they had changed, or not changed, from their pre-course methods and whether they could suggest what did, or would have, encouraged or enabled them to adopt the methods discussed in class. The students were assured that there were no right or wrong reasons, but that an accurate description of their thoughts would be most helpful. All written work produced by the students was collected at the end of the interview to aid the interpretation of the transcripts of their verbal work.

The interview questions can be found in Appendix E. The questions fell into two distinct sections: the first dealt with the addition and subtraction of mixed numbers, the second with percentages. Since the majority of the students had used the prime decomposition method for finding the GCF and LCM in the midterm examination, I did not ask them to answer questions of this type. Instead, I asked them if prime decomposition was a method with which they were familiar before the course, and if not, why they had decided to use it in the examination.

The first question, "Find the value of $8\frac{3}{4}-5\frac{2}{5}$ ", was a type of question very familiar to the students and was designed both to help them relax and also to show which method they would use for a straight forward calculation.

The second question, "Find the value of $6+2\frac{2}{7}$ ", was chosen to explore whether the student would recognise the simplicity of the wholes-first method in this situation.

The third question, "Find the value of $5-3\frac{2}{9}$ ", was included to discover if subtraction causes an approach different from addition when there is no fractional part in the first number from which to subtract the fractional part of the second number.

The fourth question, "Find the value of $42\frac{1}{3}-30\frac{3}{4}$ ", was chosen to both increase the size of the numbers and to discover how the students dealt with the subtracted fractional part being larger than the fractional part of the first mixed number.

Finally, the fifth question, "Find the value of $324\frac{7}{10}+213\frac{4}{5}$ ", was included to investigate the effect of large numbers on the choice of method.

The second section began with two simple, one-stage problems using percentages. They were included to examine the students' understanding of both increases and decreases.

1. House prices rose by 12% during 1994. If an average house cost $180,000 at the beginning of the year, how much would it cost at the end of the year?
2. House prices fell by 8% during 1998. How much would a house costing $250,000 at the beginning of the year be worth at the end of the year?

The third question was deliberately long to see if students who had used two-step methods for the first questions would use the one-step method when faced with multiple calculations:

3. The records of the Keep Fit gym show that in 1995 there were 600 members. The membership rose by 15% in 1996 and rose again, by 9%, in 1997. 1998 was a bad year and the number of members fell by 17%, and this was followed by another small decrease (of 4%) in 1999. 2000 was a better year and the membership rose by 19%. How many members did the gym have in 2000?

The students seemed more confident in these interviews than the students participating in the pilot interviews had been. One possible explanation of this was the change in the first questions. Most of the pilot students had been confused over the meaning of the words factor and multiple, either reversing them, or giving the same definition for both. In the main study the first questions were mixed number calculations, which the students were able to do correctly and confidently. In both studies some students were a little nervous to begin with, but soon seemed to relax, and the students appeared to be happy to talk about their work and their thoughts. No one seemed intimidated by the tape recorder or the general situation of the interview.

Throughout the interviews, an atmosphere of relaxed chatting was maintained in order to reduce the students' anxiety. Words of encouragement were given to help build their confidence. The students were also encouraged to talk freely about anything that occurred to them as they worked.

As they worked on the questions, occasional prompts were given by the interviewer. One intention of these interventions was to correct, or direct the student to correct, any simple calculation errors or any mistakes due to misreading a question. These were errors considered to be of little significance to the choice of method used by the student, but potentially distracting and confusing. Many students in the pilot interviews simply added the percentages when attempting to find the compounded changes. This was not helpful in determining whether they would use the two-step or one-step method when they remembered that it was incorrect to simply add or subtract, so in the main study a brief review of compound changes was explored with any students who were taking that path.

The interviewer was free to rephrase the questions and to answer any requests for clarification as to their meaning. Since the focus of this study is to examine the choice of method when solving problems, it was felt necessary that the students understood fully what they were being asked to do. In the pilot interviews the students were primarily describing what they were actually doing when solving the problems. In the main study the students were encouraged to talk much more about their choice of method, why they did what they did, and even to describe how they felt about their work.

In the presentation of excerpts from the interviews '. . .' is used to indicate that there was a pause, or that dialogue which is not relevant to the discussion has been omitted. Words such as um, err, OK, and repetitions of single words or parts of phrases, have been edited out where it was felt that they distracted from the clarity and ease of reading and did not add to what was being communicated.

Summary

This study uses data gathered from the work of students in the Math 190 course, 'Principles of Mathematics for Teachers', taught at Simon Fraser University in the Spring 2001 semester. The results of a pilot study conducted in the Fall 2000 semester served to inform the design of the research instruments for the main study.

Three mathematical topics were chosen for examination in this study: greatest common factor and least common multiple, compound percentage change, and addition and subtraction of mixed numbers. For these topics almost all students were familiar with a procedure which would give the correct answer to questions, but which was not the method promoted in class.

Written data, showing the methods used by the students, was gathered from questions in the first homework, the second midterm and the final examination. Clinical interviews were conducted with 20 students in which they were invited to explain their choice of method for answering the questions. This data is presented in the next chapter in a variety of formats.

CHAPTER 5

RESULTS

Introduction

Here I present the data collected from the first homework, the midterm examination, the interviews and the final examination. A full list of the questions used in this study and presented to the students on each of these occasions is given in Appendices C to F. I will give examples of the work of some individual students and will also summarise the number of students within the group using particular methods. The analysis of and commentary upon these data can be found in the next chapter. The privacy of the students has been protected by the use of code names, which do, however, maintain gender.

Students who adopted the approach presented in class for a particular topic, who had previously used a different method, are referred to as 'adopters' (A). Those who sometimes used the new approach, but sometimes reverted to a method they had used before instruction, are described as 'in process'(P). Some students continued to use a method which was not recommended in class, but which they knew before the beginning of the course. These students are referred to as ‘resisters’(R). Students who already used the recommended approach, before instruction in class, are described as 'users' (U).

First we will consider how each student responded to the instruction and whether he or she adopted or resisted the recommended methods. Later in this chapter we will look at each of the three content areas separately.

Change by Student

A full set of written data was collected from 78 students, 66 of whom adopted the prime factorisation method for finding the greatest common factor (GCF) and least common multiple (LCM), demonstrating a willingness and ability to learn new methods. In what follows I will consider how these 66 students responded to the methods presented in the course for the percentages and mixed numbers topics.

Table 5.1 shows that only one of these students already used the prescribed method for both percentages and mixed numbers. Four other students already used the prescribed method for mixed numbers, but showed no evidence of adopting the prescribed method for percentages. The new method was resisted for both topics by 15 students. All these 20 students, indicated by the cross-hatching in the table, demonstrated no change in method for either topic.

The 31 students represented in the right-hatched cells of Table 5.1 showed some change in method for one of the two topics. Of these, 14 students adopted, or showed progress in, the method used for percentages while resisting the wholes-first method for mixed numbers, but only 8 students adopted, or were in process of adopting, the wholes-first method while resisting the prescribed method for percentages. The remaining 9 students adopted, or were in process of adopting, the prescribed method for percentages, but already used the prescribed method for mixed numbers.

The white cells in Table 5.1 show that 7 students made progress in both topics, adopting for one topic and being in process for the other, or being in process for both topics, and 8 students fully adopted the prescribed methods for both topics.

		Mixed Numbers			
		U	R	P	A
Percentages	U	*1*	-	-	-
	R	*4*	*15*	*5*	*3*
	P	*3*	*4*	*1*	*3*
	A	*6*	*10*	*3*	*8*

U 'user': used prescribed method before instruction
R 'resister': used same method before and after instruction, or, in the case of mixed numbers, having used the wholes-first method for most questions in the first homework, did not always continue to use this method for subsequent work
P 'in process'
A 'adopter'

Table 5.1 Response of 66 Prime Factorisation Adopters to Methods for Percentages and Mixed Numbers

There were 12 students who did not fully adopt the prime factorisation method for the greatest common factor and least common multiple. If a student was familiar with, and chose to use, the prescribed methods before instruction in class took place, then it cannot be claimed that he or she adopted these methods. As can be seen from the columns of Table 5.2, seven students already used the prime factorisation method (U), three resisted this method (R), and two adopted the method for one of GCF and LCM (P). Only one student used the prescribed methods for all three topics before they were presented in class.

		GCF/LCM		
		U	R	P
Percentages / Mixed Numbers	UU	1	-	-
	RU	1	-	-
	RR	1	1	1
	PU	-	-	1
	PR	1	-	-
	UA	1	-	-
	AR	2	1	-
	AP	-	1	-

U 'user': used prescribed method before instruction

R 'resister': used same method before and after instruction, or, in the case of mixed numbers, having used the wholes-first method for most questions in the first homework, did not always continue to use this method for subsequent work

P 'in process'

A 'adopter'

Table 5.2 Response of 12 Prime Factorisation Non-Adopters to Methods for Percentages and Mixed Numbers

The cross-hatching in Table 5.2 indicates the students who showed no evidence of changing their approach, either because they resisted the methods presented in class, or because they already used them. Students who showed some progress, or fully adopted the prescribed method for one topic, while resisting or already using those prescribed for the other two topics, are identified by right-hatching. The two remaining students (white cells) made progress in, or adopted two topics, while resisting or using the third.

Four Illustrations

Since only one student resisted for all three topics and only 7 out of 78 students fully adopted the prescribed methods for all three topics, it is reasonable to conclude that it is not the case that a particular student can be described as an adopter or a resister in general terms. It appears that adoption or resistance to adoption depend in some way on the individual topic or specific method as well as on characteristics of the particular learner.

To illustrate the variety of factors influencing adoption or resistance, let us consider four students who responded quite differently to the methods taught in class.

The Story of Ray

Ray is an example of a student who adopted some methods, but resisted others. As did the majority of students, Ray adopted the prime factorisation method. He explained that prior to the course he felt he had no established method.

Ray: It would have been kind of hit and miss. I might have broken down the, like try to write out all the factors and look for the one that was common . . . the greatest common factor.

For the mixed numbers, Ray used improper fractions for every question in the first homework, but whole-heartedly adopted the wholes-first method after instruction, using it for every question in the midterm, interview and final. His explanation of his adoption of the new method contained a somewhat surprising comment.

Ray: It was so obviously a simpler, clearer, more sensible, more meaningful way.

Interviewer: So it was the impact with which it hit you, that it made more sense to do this.

Ray: Yeah, it was a tremendous relief actually.

Interviewer: Oh, OK (laugh).

Ray: Yeah, well I mean when you think about it, first of all, the lowest common denominator thing seems like a math classroom trick, right? Whereas, if you get the pizzas, you get the chocolate bars, you get the cases of beer, whatever they are, it's, you know after being around for a few years you don't, you just never would do that . . . right? And so this way, for one thing, it's easier to come up with a sensible answer, and the other part of it is that, even if you were making a mistake in the real world kind of thing, you'd be making a smaller one, you know . . .

Ray: Yeah and, you know, math has been a real challenge for me . . . I haven't understood it since I got out of grade 7. . . . And it just went downhill from there over the next few years.

His description of the feeling of relief shows that Ray is seeking meaning and understanding in his mathematics and wants to relate it to real life. He is a person who is influenced by his feelings towards the subject and his adoption of the wholes-first method for mixed numbers satisfied his desires. However, Ray did not adopt the compound one-step method for percentages. In the first homework, he used one step for increases and two steps for decreases, and continued to use repeated one-step or two-step methods on all occasions. When asked whether he would have used the repeated one-step method before the course, his response was again quite surprising.

Ray: Yeah, um, now let me think. (pause) Yeah, but I always felt I was cheating.

Interviewer: Oh? So what did you feel was expected of you?

Ray: Uh, to take, to figure out the 12% first and then add it on instead of doing it all in 'one swell foop' as they say.

Interviewer: OK. So when I came along and said, "Well actually this question we could do by just writing 5,000 times," [referring to an example used in class] and put a string of numbers, "1.12 x 1.13 x 0.84 x 1.08"; how do you feel about that? And, actually not even writing down these answers on the way?

Ray: And not even writing them down on the way, how would I feel about? Um, well it would depend on the purposes for which I was doing it. If I'm doing it for an exam I would like to show the work and the thinking. . . . If I were doing it to figure out something of sort of low consequence that I would never need to reflect on, I would probably do it that way.

Here Ray was influenced by his beliefs about the expectations of others, particularly his early teachers.

The Story of Paul

Paul is another student who adopted some methods, but resisted others. He adopted the prime factorisation method for finding the greatest common factor and least common multiple, and the compounded one-step method for percentages, immediately after instruction, but used the conversion method for mixed numbers throughout. When asked if he could explain why he did or did not adopt the methods presented in the course he gave the following reply.

Paul: Well I guess that being comfortable with these kinds of numbers in the first place, maybe I might be more willing to let go of the numbers that I've used. I guess for me fractions has always been a problem. . . . I mean I'm not a math student by any means (this is the only math course I've taken here at SFU), so when I see fractions, immediately I just say, "OK, well what have I always used?" . . . And I just, I use that method. But with, with these, with percentages and factors and multiples, I think that I feel comfortable with these concepts already. So in that sense, I might be more receptive to different ways of looking at them. . . . I guess, maybe I can't see fractions, or it's difficult for me to see fractions in ways other than what I've become accustomed to. . . . So for me, what I know is just what I use, . . . although obviously the methods that you're teaching are much quicker. . . . But I just don't feel comfortable looking at them in that way.

Paul was claiming that only if he had a good grasp of a concept was he able to adopt a new method for a procedure involving that concept.

The Story of Nicola

Nicola was a student who adopted all three new methods and yet gave different reasons for her adoption of each. When asked why she adopted the wholes-first approach to mixed numbers she responded that the wholes-first approach is "easier because then you're not dealing with as big numbers . . . and then you have less of a chance of not being able to put it in its lowest terms."

About using the compound one-step method for the third percentage question she indicated that the new method saved work.

Nicola: This was more than one year, so there was many. To me this is a lot of steps, where this was just finding the difference and then adding or subtracting to it.

Further, she associated the new method with a particular style of question.

Nicola: But now that there was more than one, you showed us this way and it just stayed in my head. . . . I didn't even think, I would have never even thought of doing it this way up here. I think because this is, typically, if I have to figure out a percentage, it's only one thing and then I'd either add or subtract, . . . in real life.

Interviewer: And yet you straight went to this when you knew that you had several steps to do, so you're obviously very comfortable with this, you knew exactly what to do. And yet you still wouldn't apply it to a simpler situation.

Nicola: It didn't even cross my mind to do it that way there. . . . This did stick in my head from the homework and from the review in class, this method, with more than one thing.

Nicola's adoption of the prime factorisation method for finding the greatest common factor and least common multiple was prompted by a need for speed.

Nicola: I've never, I don't even remember seeing a factor tree before. I would have, um, sat there and figured out every single [multiple and factor]. . . . I think I knew I had to [use the prime factorisation method], because you said we wouldn't have enough time in the exam if we did it the old way.

Nicola successfully adopted all three methods after instruction, using them in the midterm and final examinations. In the interview she also used the new methods, but only after a hesitant start. She talked about why she had used the conversion method for the first mixed number question, even though she then changed to the wholes-first for the next question. Partly it was a matter of confidence, Nicola: I think for the first one, I knew I could do it. And then once I felt like I could do it, I could then maybe try something I wasn't as sure of. But I knew that I would, by doing it that way, I would find the right answer.

but also familiarity.

Nicola: I haven't practised this method very much, so I think, . . . to do it first choice, . . . because I've only done the questions in the homeworks with it, where I've done the other way for . . . years (laugh).

To some extent she was still in the middle of the adoption process, with the old and new methods competing for supremacy.

Nicola: In the exam . . . I started doing it my way and then I got confused because I, your way was coming into my mind. So in the exam I had started doing my way and then went back to this other way. . . . And then I felt comfortable, because I double checked with the calculator, so I knew I had done it right.

Nicola used the two-step method for the simple percentage questions, but then used the compound one-step method for the third question. Her explanation for this also indicated familiarity or habit as the reason.

Nicola: Because that's, that was how I was taught and I've always done that. But then, in what I did at school, we would have only had to figure out one thing. . . . But now that there was more than one, you showed us this way and it just stayed in my head.

The Story of Diane

Of the 20 students interviewed, Diane was the most resistant to the methods taught in the course. She was one of only two students who did not use prime factorisation method for finding either of the greatest common factor and least common multiple on the midterm. She raised several issues in her explanation for her reluctance to change. About the mixed numbers and percentages she said she used her old methods "because it was an exam. I wanted to make sure I got it right." Diane seemed to be primarily concerned with certainty and familiarity.

Interviewer: In all these three cases in the exam, you've used, presumably, the method that you knew before, and you haven't used the method that I've taught in the course. . . . What is it that stopped you using the method that I taught?

Diane: Just how I feel comfortable. Like, I practised with your method, but when it comes to something that I have to get right, then I have to do what I feel comfortable.

Interviewer: OK, even though that takes longer?

Diane: Yeah.

Unlike Nicola, she did not feel the need for a faster method. She was able to succeed in the examinations because she had a good exam technique and was able to carry out calculations quickly enough.

Interviewer: OK, so you're kind of saying that because you've got good exam technique and because you can actually work quite quickly, you've got time to use these methods.

Diane: Yeah, yeah.

Interviewer: And you're more comfortable with these methods?

Diane: Yeah. If I knew that I had less time, I would have done the factor tree, . . . just because it's quicker. But I don't feel as confident with it, so I'd probably have to double check myself again using the old method.

Diane had two recommendations for encouraging herself to adopt the new methods. One was simply to practise more, to increase her comfort level: "I think that I needed to work way more with the factor tree to be more confident with it."

The other suggestion was: “Probably a quiz in class and you have to use that method.”

This brief examination of Ray, Paul, Nicola and Diane has shown that the reasons for a student adopting or resisting a new method for the solving of mathematical problems are numerous and varied. Although some of these reasons are dependent on the abilities and personalities of the learners and their prior knowledge, some influencing factors relate to a particular method or to the mathematics involved. We now turn to the three topics used in the study: greatest common factor and least common multiple, compound percentage change, and addition and subtraction of mixed numbers.

Greatest Common Factor and Least Common Multiple

Listing the factors and multiples of the numbers involved is a clear method to use to find the greatest common factor (GCF) and the least common multiple (LCM) of two or more numbers. Sometimes the students modified the listing method to reduce the amount of work. These methods were used by the majority of students in the first homework. However, during the course, the method of prime factorisation was discussed along with some of the benefits and drawbacks of using this method.

Almost all of the 20 students who volunteered for interview used the listing or modified listing method, or even just made a guess and tested it, to find both the GCF and the LCM in the first homework, but adopted the prime decomposition method for the midterm examination. Two students used some form of prime factorisation in the first homework. One of these, Laura, found the prime factors of 42 and 154, but then chose the largest of these as the GCF, rather than taking the product of the common prime factors. In the midterm she correctly used the prime factorisation method for both the GCF and the LCM.

In both the first homework and the midterm examination, Kevin correctly used the prime factorisation method for the LCM. When finding the GCF he first listed all the factors of the smaller number. In the first homework he also listed the factors of the larger number, up to the GCF, but did not indicate how he knew that he had reached the desired factor. In the midterm he used the modified listing method of testing, starting with the largest factor of the smaller number, until a factor of the larger number is found. Only one student, Diane, continued to use the listing method in the midterm examination, although she factored ten out of the 280 and 300 when finding the GCF, then listed the factors of 28 and 30.

Summary of the Whole Group

Data was collected from 78 members of the class. Of these, 73 used the prime decomposition method for finding both the GCF and the LCM in the midterm examination. Table 5.3 shows the methods which these students used in the first homework.

From Table 5.3 we can see that only 14 of the 73 students used prime decomposition before instruction, with only 7 of these using it for both GCF and LCM . Clearly the majority of students adopted the method presented in class.

		Method for LCM				
		X	G	L	M	P
Method for GCF	X	1	-	-	-	-
	G	-	7	2	10	-
	L	1	4	15	10	1
	M	-	-	-	9	-
	P	-	1	3	2	7

X had an incorrect understanding of the question
G appeared to simply guess until an answer was found
L used the listing method
M used the modified listing method
P found the prime decomposition of the numbers

Table 5.3 First Homework Methods for GCF/LCM of 73 Students Using Prime Factorisation in Midterm

Only 5 of the 78 students did not use the prime factorisation method for both the GCF and the LCM in the midterm examination. Two of these students used the prime factorisation method for finding the GCF in the midterm, but continued with their previous method for finding the LCM (one guessed, one listed). Kevin, mentioned above, used the prime factorisation method for the LCM, but continued to list factors to find the GCF. One student, who had simply guessed for the first homework, used the method of listing in the midterm, and Diane, also mentioned above, used the listing method throughout.

Percentages

The method most commonly used by students before the course began, for finding an amount after a percentage increase or decrease, was to find the actual increase or decrease and add this to, or subtract this from, the original amount. This is referred to as the two-step method. In class, the method of calculating the new amount in one step was discussed. It was shown how this could be used to simplify the calculation for situations where several percentage changes occurred consecutively. This method is referred to as the compound one-step method. If a student used either the one-step or the two-step method for each percentage change within the compounded problems, the term 'repeated' was used to describe the method.

Paul – An Adopter

The majority of students used the repeated two-step method for the percentage questions in the first homework. Paul was one such student. He showed each step in each calculation. However, in the midterm examination, after this topic was covered in class, he used the one-step method for the simple percentage decrease, and the compound one-step method for the question involving a mixture of compounded increases and decreases. His work was very clear and direct, indicating no hesitation to use this method.

During the interview, he temporarily reverted to the two-step method for the first simple increase: "I would want to figure out 12% of this, this cost here, and then add it to the original cost." However, in the very next question he realised that he could apply the one-step method.

Paul: OK, for this one (I guess you could use the same method for this [the previous question] as well) um, take this figure here, multiply it by .92 because you're losing 8% of that price. . . . So multiplying that by .92 would save you the trouble of having to add it to that cost there.

For the compounded question he immediately described the compound one-step method. Paul confirmed that the one-step method was not something that he had met before this course. In the final examination, Paul once more demonstrated a clear use of the compound one-step method.

Ray – In Process

Some students demonstrated that they had adopted part of the method taught in class, but had not adopted the full compound one-step method. For example, Ray used the repeated one-step method for question 2 of the first homework, where the compounded increase was the same each year, but used the two-step method for the decrease and one of the increases in question 3, when the percentages varied. By the midterm examination, Ray was using the one-step method for decreases, as well as increases, and used the repeated one-step method for compounding different percentage changes.

During the interview Ray used the one-step method for both the increase and decrease questions, as is shown in the following excerpt.

Ray: How much would a house cost? (pause) OK, alright, so if they fell by 8%, then it was worth 100% minus the 8%, so it would be worth 92%. . . . So 250,000 x 0.92 = $230,000.

However, when it came to the compound changes in the third question, he was a little confused.

Ray: 15% increase in '96, 9% in '97, so two years of increase was 24% increase, and there were 600 members. . . . So I'm multiplying by 1.24, and so at the end of that year there were 744 members. Now '98 was a bad year, so we're going down. It fell by 17%, followed by another small decrease of 4%. So because they're consecutive years, I can add those two together, I believe. So if it went down 21%, it was then 79% of what it was. So that times .79.

As I paused to consider what to say about his method of adding or subtracting the percentages for compounding, Ray reflected on his work:

Ray: Now I might just review that, having gone through it. . . . Now I wonder, actually now I have to think about this a bit. OK, it went up 15% in '96. No, you know I should have separated them, (pause) because I should be multiplying the product of these two numbers, of the 15% by the 9% I think. I don't think I can add, no I'm not sure. If I were doing this on an exam, or if I wanted to feel that I was accurate, I would try it using the separate numbers, and then add the two of them together, and see if I got the same answer.

Ray proceeded to check his work by using the repeated one-step method.

Ray: 690 are the number of people that there were in, uh, working in the Keep Fit Gym by the end of 1996, because the number of members had risen by 15%. And I now want to find out what it will be at the end of 1997, when it's gone up an additional 9%. So I would want to multiply this by 1.09. (pause) Yeah, that makes more sense, 752.10 which is quite different from this. . . . And I don't know if I'm seeing this very clearly or not, but I want to separate them, because if I add the two together, I'm kind of short-changing myself the opportunity to multiply the higher number by a certain amount of the percentage.

He then attempted the question again.

Ray: All right, well I will now keep them all as separate years. Yeah OK, that's what I will do. So 699 in '95 (I think I'll forget the years, just call it one) times 1.15, whoops, 690. The

> second year it went up by an additional 9%: 752.1 (I think I'll just leave that number in there, um, the decimal point that is). The bad year it went down by 17%. 17%, so 83 - that doesn't sound familiar, (pause). OK, (laugh) this half partial person is becoming stranger (pause). So that's the third year, the fourth year it fell another 4% (pause). This is really weird, (pause).

Ray was having difficulty knowing what to multiply by for a decrease of 4%. I asked him if he could explain why he found that a problem.

Ray: What's left after I take away the 4%, how many parts out of the 100 are left? . . . It's hard to describe, it's like a blur in my head . . . because I was trying to do two things at once . . . and one was shift from having 1'point'. Like this, this to me is fairly easy to take, to subtract, what I believe was correctly. 17% would leave me with 83% . . . and what I'm doing mentally is just subtracting that from that. . . . For some reason, going to taking four parts away on that side of the decimal it's, I don't know, it's just not as clear to me.

So Ray was attempting to adopt the methods used in class, but had not yet achieved the proficiency required to use them with confidence. Clearly he did not have a good conceptual understanding, and this contributed to his inability to fully adopt the one-step method. In the final examination he repeated his mistake of adding and subtracting the percentages for compounding. He then also considered an 'imaginary coat' costing $100. For this coat he calculated (correctly) the price after each change and found the amount saved by a customer, and correctly expressed that as a percentage. Faced with two different answers to the question, he was unsure what to do and wrote the comment, "Sorry to say – still not clear on this!"

Hazel – A Resister

Only a few students continued to use the repeated two-step method for the compound percentage questions after instruction in the one-step method. One such student was Hazel, in fact she was the only student of the 20 interviewed who persisted with this method through to the final examination.

In her first homework Hazel found the value of the investment after four years (question 2) by showing the amount of interest gained each year and then used the new value of the investment for the calculation of the interest for the following year. Also, for the price of the jacket in the third question, she found each increase or decrease and then the new price after each change. Similarly, in the midterm, she showed the amount of each gain or loss for the investment and then the value at the end of each year.

Hazel continued to use this method in the interview, for example, when finding the value of a house after prices had fallen by 8%, she found 8% of the price then subtracted it from the original price. As she began question 3, she described her thinking about a compound change which consisted of an original amount of 600 increasing twice, decreasing by 17% and 4% and then increasing again.

Hazel: Well I'd probably figure them out all individually, but I know that you taught us how to do the [inaudible comment]. So what do we start with? 600 (pause), this is right. This right here, is it 1.7? I have to do it individually.

Hazel clearly knew that a different method had been presented during the course, but had chosen to continue to use the repeated two-step method.

In the final examination Hazel once more used the repeated two-step method after assuming a price of $100 to enable her to calculate the actual increases and decreases. She correctly interpreted her final answer to give the percentage saved by the customer.

Summary

The vast majority (17 out of 20) of the interviewees used the two-step and repeated two-step methods in the first homework, although six students incorrectly calculated simple interest for question 2b. One student used the standard compound interest formula $A = A_o(1+x)^t$ for question 2, but used the repeated two-step method for question 3. Only one student added and subtracted the percentages for compounding. Nine of these 17 students fully adopted the compound one-step method after instruction and used this method in the midterm examination, the interview and the final. Of the other eight, five progressed to using the compound one-step method at least by the final examination, one attempted this method for the midterm and interview, but reverted to the repeated one-step method for the final, and one used the one-step method for increases, but continued to use the two-step method for decreases. Only one student used the repeated two-step method throughout the course.

Three students used the one-step method at some point in the first homework and I would like to consider each of these separately. Ray, as described above, initially used the one-step method for increases, but the two-step method for decreases. After instruction he used the repeated one-step method for both types of change, but did not progress to the compound one-step method. Laura certainly used the one-step method in the first homework, but did not clearly show her method for all calculations, making it impossible to know whether she used the one-step or two-step method for decreases. In the midterm she carried out the simple decrease in two steps, but used the repeated one-step method for compounding (including the decrease). However, during the interview she reverted to two steps for all decreases. In the final examination she compounded by adding and subtracting the percentages! In the first homework, Cathy used the two-step methods for all changes, except one of the increases, and in the midterm she used only the two-step methods. In the interview, Cathy used the two-step methods for decreases, but the one-step method for all increases. By the final examination, Cathy was able to use the compound one-step method, although some work had been erased indicating that she had used a different method first.

I consider there to be a progression in the sophistication of the methods used by the students, with the repeated two-step method (R2) being the most basic. Some students are able to calculate increased amounts in a single step, but have difficulty seeing the equivalent factor for decreases (1i2d). The next development is to calculate decreases, as well as increases, in one step, which leads to the use of the repeated one-step method (R1). It is interesting to note that no one used one step for decreases while using two steps for increases, which supports my ordering. Use of the compound one-step method (C1) requires the ability to express increases and decreases as single factors and so I regard this as the highest level of sophistication.

A summary for the whole class is given in Table 5.4, where the lay-out reflects the progression in sophistication. The 'method before instruction' is taken to be the method used in the first homework. For the 'method after instruction', the most advanced method demonstrated by the student in either examination is taken. An incorrect method of compounding (e.g. adding and subtracting the percentages) is denoted by 'X'.

		Method after instruction				
		X	R2	1i2d	R1	C1
Method before instruction	X	2	2	-	-	3
	R2	2	19	2	5	25
	1i2d	-	2	-	4	2
	R1	-	-	-	6	1
	C1	-	-	-	1	2

X incorrect understanding of compounding
R2 used repeated two-step method
1i2d used one-step for increases and two-step for decreases
R1 used repeated one-step method
C1 used compound one-step method

Table 5.4 Methods Used Before and After Instruction for Compound Percentage Change

The cells on the diagonal of Table 5.4 (right-hatching) indicate that 29 of the 78 students continued to use their original method. The cells below the diagonal (cross-hatching) show that five students actually used a less efficient method in the examinations than in the first homework. Of the remaining 44 students, appearing above the diagonal, 13 made some progress in the sophistication of their chosen method (left-hatching) and 31 adopted the compound one-step method (white cells).

Mixed Numbers

In class, the wholes-first method, in which the calculation is simplified by dealing with the whole numbers before the fractional parts, was discussed and students were encouraged to use this method when adding or subtracting mixed numbers. Several students were familiar with this method before instruction took place, but the majority chose to change the mixed numbers to improper fractions before adding or subtracting (the conversion method). The wholes-first-one method (see chapter 4) was also presented in class as an alternative approach to subtraction.

Ray – An Adopter

Ray was a student who used the conversion method before instruction, but chose to use the wholes-first method for the midterm examination. At the beginning of the course, he described the meaning of a mixed number by referring to the whole and fractional parts. Ray's explanation continued with, "It can alternatively be written as an improper fraction." He used the conversion method for each mixed number calculation in the first homework. By the time of the midterm he had adopted the wholes-first approach and answered every question using this method. However, he did not go so far as to use the wholes-first-one method for the subtraction questions.

During the interview Ray used the wholes-first method with confidence, making the following remarks.

Ray: I would separate them. I never would have done it this way before. . . . I would have taken all the chocolate bars and broken them up into 10ths (laughter) then handed them out. . . . I would have found the denominator . . . and turned them all into . . . improper fractions.

Jill - In Process

Many of the students were not as confident as Ray about using the new method. They saw benefits in the wholes-first approach, but often reverted to their previous conversion method. Jill is one such student. At the beginning of the course, her description of the meaning of a mixed number included the explanation that a whole number is a fraction in which the numerator and denominator are the same, and she showed how to convert the mixed number to an improper fraction. She used the conversion method for each mixed number calculation in the first homework.

After instruction in class, she chose to use the wholes-first method in the midterm examination, however, for the two questions which involved a whole number without a fractional part, she changed one whole into a fraction before proceeding. Her solution for the addition was:

$$8+3\tfrac{1}{4} = 7\tfrac{8}{8}+3\tfrac{1}{4} = 10\tfrac{8}{8}+\tfrac{1}{4} = 10\tfrac{8}{8}+\tfrac{2}{8} = 10\tfrac{10}{8} = 11\tfrac{2}{8} = 11\tfrac{1}{4}$$

It is interesting to note that in both these questions, the denominator she chose for the whole was the original number of wholes, and not the denominator of the fractional part of the other number. This may indicate a confusion in her understanding of the part-whole concept of fractions, or may relate to an attempt to apply the algorithm to convert mixed numbers to improper fractions.

During the interview Jill confirmed that her adoption of the wholes-first method was only partially complete. For the first mixed number question in the interview she immediately used the conversion method, but added a comment.

Jill: I know this isn't how you taught it, and now I'm remembering how you taught it, (pause). Yeah, like I said, this isn't how you taught it, but (pause) I'll do the next one like how you taught it, or the next next next one.

On consideration of the other questions, Jill decided that she would use the wholes-first method for questions 4 and 5, which involved large numbers, but admitted that she still was not confident with this method. When challenged with the second question ($6+2\frac{2}{7}$) she responded:

Jill: I think I would still use my old method. . . . $^{16}/_{7}$ (pause), it's just, um (pause). Wouldn't it just be $8^{2}/_{7}$? . . . (laugh) Yeah OK, I don't know why that one fumbles me. . . . But now that I think about it, yes it's 6 wholes, and then there's . . . 2 wholes, and it's just a fraction added on.

Her explanation of her use of the wholes-first method in the midterm was:

Jill: I knew for your test that that's how we were taught, it was still relatively fresh, and it made sense, and I knew that that's how you wanted us to do it.

Paul – A Resister

Some students who used only the conversion method for the questions on the first homework continued to use only that method. Paul gave a good description of the meaning of a mixed number, with no reference to improper fractions: “A mixed number consists of a whole number in addition to a fraction. If we have 3 whole pies, and two pieces of a pie that had been divided into 7 equal slices, then we would have $3^2/_7$ of pie.” He used the conversion method for all the first homework questions, however three out of six answers were wrong because of mistakes in converting the improper fractions back to mixed numbers.

In the midterm examination Paul again used the conversion method for all questions, but this time he was accurate in all his calculations. In the interview he did not hesitate to use the conversion method, although, as in the first homework, he made a mistake:

$$6+2\tfrac{2}{7} = \tfrac{6}{1}+\tfrac{2\times7+2}{7} = \tfrac{6}{1}+\tfrac{16}{7} = \tfrac{7}{7}+\tfrac{16}{7} = \tfrac{23}{7} = 3\tfrac{2}{7}$$

When asked about his choice not to use the method discussed in class he made the following comments:

Paul: Well fractions is always something that I’ve had problems with. . . . I guess maybe I can’t see fractions or it’s difficult for me to see fractions in ways other than what I’ve become accustomed to. . . . So for me, what I know is just what I use.

Summary

Of the 20 students who volunteered for interview, 11 used only the conversion method in their first homework. Five of these used the wholes-first method for the midterm examination and can therefore be regarded as being willing to change their method. This may not be a complete change, as was shown in Jill's interview above, however, these students have demonstrated the ability to use the new method when they choose. Three students did not use the wholes-first method at any stage (midterm, interview or final), but continued to use the conversion method which they knew prior to the course. The remaining three students showed a gradual change as the course progressed, increasingly using the wholes-first method.

Of the nine students who were obviously familiar with the wholes-first method before the instruction during the course, three chose to use predominantly the conversion method for the examinations (midterm and final) although they used the wholes-first method in the first homework and the interview. Five students used the wholes-first method for almost all questions at all times, but one used this method for only some of the questions on each occasion.

Data from 78 students was collected, showing the methods which they used in the first homework, the midterm examination and the final examination. Tables 5.5 to 5.7 display this information. For the first homework there were six mixed number calculations. In the category 'Mostly Wholes-First' shown in Table 5.5, in which only one or two questions were answered using the conversion method, 13 out of the 15 students used the conversion method for the second question ($6-2\frac{5}{7}$). Students who answered only the first ($4+7\frac{1}{3}$), or first and second questions, by the wholes-first method ('Mostly Conversion') have been considered separately from those who answered two or three of the other questions by that method ('Mostly Wholes-First'), since these questions involve a whole without a fractional part.

Method Used in First Homework	Number of Students
Conversion Only (All 6 questions)	39
Mostly Conversion (First or first two only by Wholes-First)	5
Some by Wholes-First (2 or 3 from remaining four)	4
Mostly Wholes-First (4 or 5 questions)	15
Wholes-First (All 6 questions)	15

Table 5.5 Method Used in First Homework for Mixed Number Calculations

Method Used in Midterm	Number of Students
Conversion Only (All 5 questions)	31
Mostly Conversion (3 or 4 questions)	13
Mostly Wholes-First (3 or 4 questions)	11
Wholes-First (All 5 questions)	23

Table 5.6 Method Used in Midterm Examination for Mixed Number Calculations

A summary of the methods used in the midterm examination are shown in Table 5.6. Of the 13 students in the 'Mostly Conversion' category, 9 used the wholes-first approach for question (ii) $8+3\frac{1}{4}$. In the 'Mostly Wholes-First' category, all students used the wholes-first method for question (v) $57\frac{8}{11}+211\frac{3}{7}$, the conversion method being used for one or two of the other questions.

In the final examination the students were asked to, "Find the value of $3\frac{1}{4}-1\frac{7}{8}$ and draw a diagram to illustrate the calculation". The method used was taken to be that of any written calculation, regardless of what any diagrams showed. If a student demonstrated both methods of calculation, he or she was counted as 'wholes-first', since an ability and willingness to use that method had been demonstrated. Some students subtracted the '1', but then faced with $2\frac{1}{4}-\frac{7}{8}$ they resorted to the conversion method.

Method Used in Final	Number of Students
Conversion Only	36
Subtract the whole, then conversion	4
Wholes-First	25
Diagram only (all divided into 8^{ths})	8
Diagram only (indicating wholes-first)	5

Table 5.7 Method Used in Final Examination for Mixed Number Calculations

As is shown in Table 5.7, 13 students showed diagrams without a clearly written calculation, and of these, 5 clearly described the wholes-first approach.

For the other 8 it was not possible to determine their approach, but each of these students showed all the wholes divided into eight parts.

Table 5.8 shows how the method used by the students changed from before to after instruction. Some students (indicated by cross-hatching) actually decreased their usage of the wholes-first method. The right-hatching indicates students who exhibited no significant change and the left-hatched cells show those who increased their usage of the wholes-first method only slightly. There are 74 students represented in this table, but only 14 significantly increased their usage of the wholes-first method (white cells).

		Method after instruction			
		CO	MC	MW	WO
Method before instruction	CO	*20*	*6*	*4*	*7*
	MC	-	*2*	*1*	*2*
	SW	*2*	*2*	-	-
	MW	*4*	*1*	*5*	*5*
	WO	*2*	-	*4*	*7*

CO conversion method only
MC mostly conversion method
SW some by wholes-first method
MW mostly wholes-first method
WO wholes-first method only

Table 5.8 Methods Used by 74 Students Before and After Instruction for Mixed Number Calculations

The summary given in Table 5.8 does not include 4 of the 78 students. These students did not fit the common pattern of change and are worth considering individually.

- One student used the conversion method for all but one question in the first homework, midterm and final, but in the interview used the wholes-first approach to all questions.
- One student used the conversion method for all questions in the first homework, the wholes-first method for all the midterm questions, but then the conversion method for the final examination.
- One student who used the wholes-first method for the first homework and midterm examination used the conversion method for the final examination.
- One student used the wholes-first method for all the first homework questions, then the conversion method for all the midterm examination questions, but returned to using the wholes-first method for the final.

It was not simple to categorise students as users, resisters, in process and adopters for the mixed numbers topic. Most students had some knowledge of the wholes-first method before instruction in the course, but few were consistent in their choice of method. To be considered a 'user' the student must have used the wholes-first method for all questions, or all but one of questions 3 to 6, of the first homework. Those who were not considered to be 'users' and who used the wholes-first method for all questions, or all but one of questions 3 to 5 of the midterm, were considered to be 'adopters'. Those who decreased, or did not increase, their usage of the wholes-first method were considered to be 'resisters', as were those whose only usage of the wholes-first method was for one of questions 1 and 2 in the midterm. The remaining students showed some significant increase in their usage of the wholes-first method and were deemed to be 'in process'.

Where The Results Lead

We have seen that students are not by nature adopters or resisters, and that a particular method will not be adopted or resisted by all students. In the next chapter I will present more of the wide variety of factors which the students indicated influenced their ability or desire to adopt the methods presented in the course. These factors have been grouped into reasons to adopt and reasons to resist using the prescribed methods. By considering all the influences, I hope to expand the theory of Posner, Strike, Hewson and Gertzog (1982), which refers to the accommodation of a scientific concept, to include procedural change in mathematics.

The theory was originally applied to situations where the students can be shown that their current concept cannot explain a new experience or some new data. Here, the students know that all methods, if applied correctly, will solve the problem correctly. The question, "Under what conditions is the accommodation of a new concept likely to occur?", attended to in the conceptual change theory of Posner et al., can now be modified to ask, "Under what conditions is the adoption of a method or procedure likely to occur?".

CHAPTER 6

ANALYSIS

Introduction

The results in the previous chapter support my initial observation that frequently students do not adopt the methods presented to them if they already know a method which will give a correct answer. This prior knowledge of a method which can be successfully applied in a given situation was an obstacle to their adoption of a new method which would be better in some way, and in some situations, than the method they currently use. What is it that motivates students to overcome this obstacle? Clearly their satisfaction with their prior method was different for the different content areas.

The conceptual change theory presented by Posner, Strike, Hewson and Gertzog (1982), and discussed in detail earlier, responds to the question, "Under what conditions is the accommodation of a new concept likely to occur?", by providing a set of four conditions which they found common to most cases. Their theory was developed from research carried out in the scientific domain and related to situations where a new experience cannot be explained by the student's current conception. The student is presented with experimental evidence showing the inadequacy of his or her present knowledge and this must be resolved in some way. The student may not always achieve the goal of accommodation of the new concept, but may simply store the new concept as a separate piece of knowledge. The new knowledge may be retrieved in appropriate situations, but the prior concept may continue to be recalled and used inappropriately. Frequently the two concepts can exist side by side without the student being aware of a conflict.

This theory, I believe, can be applied to the learning of mathematical concepts, as students develop from initial understandings to more complex situations. For example, many students believe, from early experience, that division always leads to a quotient smaller than the dividend (Tirosh & Graeber, 1989). Later, when division by fractions is encountered, this belief is challenged. Evidence is presented which shows that the student's current conception is inadequate and the student may respond by expanding his or her concept of division, or may learn the outcome of division by fractions and store this as a separate piece of knowledge. In the future, either concept may be recalled and applied appropriately, or sometimes inappropriately.

In this study we consider a different scenario: the students have learnt a valid procedure, appropriate perhaps to their mathematical knowledge at the time, or useful in the development of understanding of a particular concept. At a later date, the students are presented with a different procedure which is in some way better than the original in at least some circumstances. The students are not asked to reject their prior methods, but rather to add to their repertoire these new methods which they can then call upon when appropriate. For some this may involve a change in their conceptions or beliefs, but for others, it is simply a case of accepting that the new method is worth learning. The theory of conceptual change can be applied to this situation; the same four conditions hold true. What must change is our interpretation of, for example, causes of dissatisfaction, or 'fruitfulness'. It may be true that each problem presented to the students can be solved using their current method, but at the same time their procedure 'will not suffice' in some way. While examining the interview transcriptions I focused on the following issues raised by the conceptual change theory of Posner et al.:

- What causes dissatisfaction with a method?
- What affects the intelligibility of a mathematical procedure?
- In what ways might the new method fail to be initially plausible?
- What is the equivalent of 'fruitful research' for a mathematical procedure?

Applying the Conceptual Change Theory

Many reasons were suggested by the students in this study for their adoption of, or resistance to, the methods and procedures presented in class. By considering these reasons, grouped under the four conditions given by Posner, Strike, Hewson and Gertzog (1982), I will show how this theory can be expanded to help our understanding of the students' choice of methods. Ultimately it is hoped that this understanding could help in the development of teaching strategies to encourage students to adopt new methods. The examples given to illustrate each motivation are taken from the interviews during which students were encouraged to examine why they adopted or resisted a particular method and also to discuss differences in response to different topics. The abbreviations GCF/LCM, %C and M# will be used to indicate when a student was referring to greatest common factor or least common multiple, percentage change and mixed numbers respectively.

There Must Be Dissatisfaction With the Existing Method

The first condition given in the theory of conceptual change is that there must be dissatisfaction with what the student currently uses. We will examine what it was that students felt encouraged them to adopt the new procedures. The following factors were raised by students in the interviews.

- The new method is easier or requires less effort – the prior method 'fails in complexity of execution'.

GCF/LCM: **Priscilla**: Because it's easier and it takes less time, I think.

Laura: Because I hate writing that much stuff out. [Referring to listing]

%C: **Kevin**: . . . because this seems like such a shorthand method, I really found this really efficient, and so I was like, oh that's awesome.

Sue: Um, well because you've showed me an easier way, it's much faster and it's easier. You can't, it's not as many steps.

M#: **Cathy**: (laugh) Because it was so revolutionary to me, it was like wow! . . . This is so much easier than trying to make them the top-heavy fractions.

Elise: It kind of lets you skip out all the stuff from the middle.

- The new method is faster – the dissatisfaction comes from the amount of time needed.

GCF/LCM: **Tanya**: Because actually what I was used to, I didn't really like doing that either, because it took a long time, . . . so I didn't really like that method.

Sue: Well I think also, like this was, when I looked at these lowest common multiple, the factor trees, and that was just, that was that wow!, you know, it saves you so much time, it's so much easier, that's so great. This [wholes-first], the steps, I mean you save a little bit of time, but not enough time to be so significant that that's all you want to do is that.

Nicola: I think I knew I had to, because you said we wouldn't have enough time in the exam if we did it the old way.

%C: **Mandy**: Because it's faster.

Sue: Um, well because you've showed me an easier way, it's much faster and it's easier, . . . it's not as many steps.

M#: **Jill**: And I mean this is a lot faster than making that into a fraction . . . then finding the common denominator for it and adding it or subtracting whatever it is out. And then reducing it even more.

- The new method is less error prone – the dissatisfaction comes from the lack of reliability of their method.

GCF/LCM: **Tanya**: I think, well definitely greatest common factor one, I always hated doing those. Because sometimes even after you, like so much work, sometimes if you, I don't know, if you have a mental lapse or something and you're not, and you miss a multiple or something, then you get, your answer is completely wrong, and I hated it. . . . Because it took so long, and even if you took a long time doing it, but this is, this way it was more reliable.

Fran: It gets to the right number every time.

%C: **Isabel**: However, in doing it in a hurry, it's like writing an exam, you want to get it right the first time.

M#: **Elise**: Well this is a lot simpler in a lot of ways, because you don't end up dealing with bigger numbers. Like it seems like there's less margin for error.

- They did not previously have what they would describe as a method (it seems they did not consider the method of listing to be a genuine method).

GCF/LCM: **Ray**: It would have been kind of hit and miss, I might have broken down the, like try to write out all the factors and look for the one that was common, . . . the greatest common factor.

Mandy: I didn't know an efficient method for finding greatest common factor, least common multiple before any ways.

Olga: I can't remember, I had to relearn the greatest. I knew, when we talked about it, I knew what they were, and I knew that I should know it, I knew that I had done it, but I couldn't remember how.

%C: **Mandy**: I don't know if I even really learned percents. If I did, it was a small lesson. I think I just know it from like working and, I don't know, . . . it's easier to give up on this method and take a new one.

- They were dissatisfied with the conversion method for mixed numbers because they had simply memorised it, without any understanding.

M#: **Elise**: Oh yeah, well I think, well at least in my experience when I was in elementary school . . . I was never really encouraged to look at things that way very much at all. . . . It would be like, OK a fourth is . . . this thing here, but we'd never really be shown how to relate that to doing addition, subtraction. . . . Like especially not with mixed numbers, but I don't think even really with simpler fractions. . . . It was really just like, OK this is what you're supposed to do, and do it. . . . That's why I really like this class, because it's so much stuff that I feel like, OK I basically know, but to really understand why it works is something that you don't really get taught in school.

Olga: I think that I didn't really get what I was doing with them. . . . They just seemed like a whole bunch of unruly numbers that were all over the place, and just learning a bunch of rules for adding them and subtracting them and multiplying them and things like that.

Priscilla: I think because we've constantly just had numbers given to us and we just had to do it. So I think it was the process, how do you find the answer, the fastest, instead of just thinking it, I don't think of it as being pieces of anything. . . . And it seems weird to take this class and think about what it actually means. . . . I was really, really good in math . . . all through school. And I don't know if it was because I could memorise formulas or something. . . . But

I've never really been taught that this is why you're doing something - 'here's a formula and do it'.

- The presence of large numbers in the mixed number questions encouraged them to use the wholes-first method.

M#: **Isabel**: Well as I got down the page here, and I noticed these numbers were too big to do that, like it would just be ridiculous numbers I'd be dealing with.

Kevin: And I know it kind of gets messy when you're dealing with big numbers and stuff, so I kind of explore the method of how I'd do it.

Nicola: But the other way is easier because then you're not dealing with as big numbers, . . . and then you have less of a chance of not being able to put it in its lowest terms.

This can be contrasted with the reasons given by students who did not adopt the methods taught in class.

- Their prior method was an ingrained habit - rote learning is robust.

%C: **Tanya**: It's automatic for me to do it that way. . . . I know I should try to learn to do the quicker way.

Cathy: It comes from doing that so much. That's the way I've done it for years and years.

M#: **Mandy**: I'm just like programmed to do it this way. So many, like so many years of practising doing it the same way.

Tanya: It's almost automatic to do mixed fractions this way for me. . . . When I see fractions, I usually immediately just do what I, I don't know, for this I just immediately went to what I knew, what I was used to.

Jill: I've been, I don't know, conditioned to do it that way ever since like elementary school.

Olga: Just forcing myself to do it the new way I think, because the old way is just because that's the way I learned to do it, it's just a habit.

➢ They found their prior method fast enough.

GCF/LCM: **Diane**: [Diane had good exam technique and worked quickly] If I knew that I had less time I would have done the factor tree.

M#: **Mandy**: I can use my calculator quickly.
Interviewer: Whereas these other two methods, you knew it would save enough time to be worth learning it, . . . is that it?
Mandy: Yeah, yeah.

Hazel: You taught us the way, but I'm slower at that way I think, a little bit more. Like it makes sense when you taught us how to draw it out, but I can't draw them myself and visualize it that way.

➢ They were generally comfortable with, or confident in, their old method.

GCF/LCM: **Diane**: Just how I feel comfortable. Like I practised with your method, but when it comes to something that I have to get right, then I have to do what I feel comfortable. . . . I needed to work way more with the factor tree to be more confident with it.

M#: **Paul**: See that, that's the thing: I've just always been more comfortable with converting it.

Sue: I think if I was just more comfortable with it. I think part of the problem is that I'm rushing to get my homework done, so I'm doing it the way I know best instead of practising the way that I was shown. And so I just didn't have enough practice, I didn't feel comfortable enough to use it on the exam.

➢ They knew they would get the right answer with their old method.

%C: **Diane**: Yeah, because it was an exam, I wanted to make sure I got it right.

M#: **Jill**: And with math it's just, it's scary for me, it's really scary. So if I find a formula, if I find a way that I know that is right, . . .

Laura: Not making them top-heavy, that's the quick way to do it. . . . I didn't realise that I was going to be so pressed for time, so I decided to take the time and make sure that they are right, because they're easy marks.

A New Method Must Be Intelligible

The second condition for the adoption of a new method is that it must be intelligible. The students talked about the benefits to understanding of the methods presented in class.

- The new method helped them to understand the concepts involved or they understood how the method worked – their prior method 'failed because divorced from understanding'.

GCF/LCM: **Hazel**: I always confused them before. And so that's one thing you went over, so then I studied the method in the textbook, and that you offered, really trying to nail it.

Fran: I can understand how like the numbers can simplify down into these numbers. Like I can understand this is equal to that, and therefore like I can visualize how it works.

%C: **Gwen**: I just think it was because every problem we did using this, I just really understood it. So that just made me remember it.

M#: **Ray**: It was so obviously a simpler, clearer, more sensible, more meaningful way.

Jill: Only this class has taught me how to, OK, it's really 6 of these and 2 of those, and then just add them together. . . . I'm not very confident with math, with um manipulating . . . numbers . . . and being confident that I know what I'm doing.

- The new method made sense to the students.

GCF/LCM: **Olga**: . . . and it made sense to me.

Jill: It made sense with the prime numbers. . . . The way that the numbers, knowing that it's prime, and you're just trying to, . . . these are the factors for this number, 280, and then these are the factors for 300, and then there's, there's a commonality within them. So then once you line them all up and multiply them, because we're trying to find the least common multiple, then it would be all of these numbers . . . and all of these numbers, but you don't need to repeat the ones that have, like, exponents.

%C: **Priscilla**: I only really remember with percentages and stuff, doing cross-multiplication to find the missing thing that you have. But this makes more sense to me now.

M#: **Elise**: Yeah, yeah, I really like this idea of like, yeah you take it away, that totally makes sense to me.

Priscilla: That makes sense when you go through it like that.

- A few said that their good prior understanding of the concepts allowed them to explore new ways of looking at procedures involving those concepts and therefore to use different methods.

GCF/LCM: **Mandy**: No I've known, I knew how to do factor trees, but I didn't know this.

Bev: Yeah, because this [prime factorisation] was very similar . . . to something I've met before.

%C: **Interviewer**: So are you saying that you feel you understand what percentages are about, and so you can adapt to a new method?
Paul: Yeah, yeah.

M#: **Laura**: I wasn't taught that way, but . . . we had 2 hours of math at the very beginning of every single day, so I became very, very comfortable with numbers and fractions and just like easy arithmetic. . . . So I can do a lot of the stuff in my head. So even though I hadn't been taught to deal with the whole numbers first, I did it in my mind probably.

- Some talked about understanding what the symbols represented, or the relationship between different symbolic representations – again, the old 'failed because divorced from understanding'.

GCF/LCM: **Fran**: Well I can understand how like the numbers can simplify down into these numbers. Like I can understand this is equal to that, and therefore like I can visualize how it works.

Alice: That one, (pause) I think it's something that I learned, but I, it's pretty surprising, because this course has, I'm understanding a lot more things now. Like I think I have done that process, but I never really understood everything. . . . And now I understand a lot better, but I think previously what I did do was like list all of them and then match up the greatest common factor, the lowest common multiple, I never got the factor tree.

%C: **Sue**: I didn't know that a 12% increase was the same as 1.12, timesing it by 1.12.

M#: **Kevin**: If you were to actually physically draw, you know, 8 pies, and then you have plus 3 more pies and 1 quarter of a pie left, how many do you have? I mean it's so easy that way.
Interviewer: Right, so it's just constantly reinforcing the picture of what is represented side by side with the symbols?
Kevin: Um hmm, definitely.

Sue: Well the language, when you use language, when you write it out in English, so I can see how mathematical language works with the English language, so they're not separate any more, they actually can be combined, that really helps me.

- One student related the method presented in class to his real-life experiences, and contrasted that with his lack of understanding of his previous method – once more, the old method 'failed because divorced from understanding'.

M#: **Ray**: Yeah, well I mean when you think about it, first of all, the lowest common denominator thing seems like a math classroom trick, right? Whereas, if you get the pizzas, you get the chocolate bars, you get the cases of beer, whatever they are, it's, you know after being around for a few years you don't, you just never would do that . . . right? And so this way, for one thing, it's easier to come up with a sensible answer, and the other part of it is that, even if you were making a mistake, in the real world kind of thing, you'd be making a smaller one, you know.

Only a few students mentioned not understanding the methods as reasons for not adopting what was presented in class. However, some other comments were made which may relate to the intelligibility of the method to the student.

- Several couldn't remember the procedures well enough to implement the methods.

GCF/LCM: **Kevin**: That's probably something I don't remember. [Prime factorisation method]

%C: **Tanya**: I don't even remember. I mean, I think in my review I didn't even, I think I skipped over that part.

Hazel: Like I know we had two homework questions on that, but I never can remember how to do it for the subtraction, so I always figure it out individually and then keep going from there.

M#: **Elise**: [$12^1/_3$ - $^3/_4$] Yeah, hmm, what am I going to do now? Gees, I totally feel like I'm forgetting something doing all of these. Oh I know that I did it differently before, but I just, it's not coming to me right now (laugh). Um, OK, so I'm thinking I'm going to do what I did over there, even though I don't really want to, but I can't really remember what else to do.

Hazel: It's just trying to remember all these different things [%C and M#]. There's so many things, like you've taught us so many different things. The fractions I can remember when I'm doing it, and the homework I do really well, but when I have to do it without my, looking at my examples, I don't feel confident to try it. . . . And so for me to try to memorise all the examples you've given us, and then the formulas, and then coming in and writing it on a test, I can't remember. So I always go back to the way that I know is going to work. It takes longer but I'm more confident in my answer.

- One student had missed the class session in which one of the new methods was presented.

%C: **Laura**: I learned the 1.12, yeah, I learned that in wherever, elementary school or wherever it was. And I hadn't learned 0.92, and then I wasn't there for that lesson. . . . So, and then I read over my friend's notes, but I was, I was, you know, I was trying to quick study and . . .

- Some students did admit that they just hadn't understood the methods presented in class, and were therefore unwilling to use them.

%C: **Tanya**: Um, I don't really understand it. I still have to go over it, actually. I'm not quite clear on how to do it that way.

Diane: [Asked why she wrote 0.17] Because it's a decrease, um, I'm not really sure. I just know that you had to write the 0.

- A few students said that they had not understood part of the method and therefore used the old method in certain situations.

M#: **Fran**: Well you can't, you have to have a larger fraction to take away [from], and this obviously doesn't work.

A New Method Must Appear Initially Plausible

The third condition given by Posner et al. is that the new method must appear initially plausible.

- That the new method solved some of the problems of the old seemed to arise mostly in the discussions of causes of dissatisfaction with their prior method, but the following is an example of how some adopters felt.

GCF/LCM: **Fran**: Yeah but this is much more, I've got this, I find this very useful.

- Students who adopted a new method talked of needing to check that it gave the same answer as their old method.

%C: **Fran**: Well maybe if I, I checked it, like I did it both ways to make sure.
Interviewer: So just do a few examples to convince yourself it gave the same answer?
Fran: Yeah, that's probably it, because it just seems sometimes too good to be true. . . . You have to see how it worked out.

- Two students who did not get the correct answer using the method presented in class reverted to their prior method, although others continued trying to learn to use the new methods correctly and eventually succeeded.

M#: **Mandy**: Like I know the other method for sort of estimating, . . . but I was really bad at it. . . . Like I practised it in the homework, but I was, I always got the wrong answer.

Sue: You see I messed up on this one, and that's what scared me. . . . I did it that way and it's like, oh I got the wrong answer, so I'm going to just resort to the way I know.

- A few students rejected the prescribed methods because they were incompatible with their beliefs about what they were expected to do in a mathematical problem.

%C: **Ray**: Yeah, um, now let me think, (pause). Yeah, but I always felt I was cheating. [Using one-step method]
Interviewer: So what did you feel was expected of you?
Ray: To take, to figure out the 12% first, and then add it on instead of doing it all in 'one swell foop', as they say.

M#: **Tanya**: [Looking at $7^3/_5$] Well if it's in an exam, I just, it's an equation to do.

Elise: I must say I'm thinking of it [$2 - {}^2/_9$] in terms of numbers, I'm not really thinking of it in terms of things. Like if I thought about that as a picture, then I'd be able to solve it without doing math.

- Some were uncomfortable with the concepts involved in the methods and so had memorised procedures to obtain correct answers. When faced with different approaches, these students could not relate the new approaches to their prior knowledge and so rejected them.

M#: **Paul**: Um, well I guess that being comfortable with these kinds of numbers in the first place, maybe I might be more willing to let go of the numbers that I've used. I guess for me fractions has always been a problem, . . . so when I see fractions, immediately I just say, "OK, well what have I always used?" . . . And I just, I use that method. But with, with these, with percentages and factors and multiples, I think that I feel comfortable with these concepts already. So in that sense, I might be more receptive to different ways of looking at them. . . . I guess maybe I can't see fractions, or it's difficult for me to see fractions, in ways other than what I've become accustomed to. . . . So for me, what I know is just what I use. . . . Although, obviously the methods that you're teaching are much quicker, . . . but I just don't feel comfortable looking at them in that way.

Kevin: [$8 + 3^1/_4$] Yeah, it's because I don't see anything here [after the 8]. So I think to myself, "Well then, I have to find a common denominator for this." And it's, there's nothing there though, right? . . . So you just keep the $^1/_4$. But I don't, I guess I haven't practised it enough to realise that when nothing is there, the $^1/_4$ just remains, and it's 8 + 3.

A New Method Should Suggest the Possibility of Fruitfulness

The last condition suggested by Posner et al. is that the new method should be fruitful, or there should be the possibility of application to new areas. This is perhaps the hardest to relate to the learning of mathematical methods. Some of the comments made by students do, however, relate to this condition, if it can be considered in a broader context.

- One student valued the prime factorisation method because both the GCF and the LCM could be obtained from the one procedure.

GCF/LCM: **Sue**: It's much more simple. You don't, can't mess up in your multiplying all the time, and it's a lot faster. Those numbers would take a long time. . . . And also, you can use them both for the same purposes, like once you've, whereas you'd have to do two different steps.

- In contrast, one student saw the compound one-step method as being useful for only one particular style of question, and not as a more generally applicable procedure.

%C: **Nicola**: It didn't even cross my mind to do it that way there [For a single change]. . . . Because that's, that was how I was taught and I've always done that, but then in what I did at school, we would have only had to figure out one thing. . . . But now that there was more than one, you showed us this way and it just stayed in my head.

- A more common reference to the fruitfulness of a method was that it was expected or required by the instructor.

GCF/LCM: **Gwen**: I think because when I was doing the homework and I had to do it that way . . .

Tanya: I know that's the method you taught, so then I thought that's what you wanted to see, I think, too.

M#: **Jill**: I knew for your test that that's how we were taught. It was still relatively fresh, and it made sense, and I knew that that's how you wanted us to do it.

Priscilla: Maybe that's why I did do that in the exam, because it's not always about just finding the final answer. . . . And that might be why, but when, when there was, when I was in high school and stuff, it was more, you would just have to find the answer, really.

- Another interpretation of fruitfulness could be that it would allow the student to get more marks.

GCF/LCM: **Hazel**: So as soon as you say that no one ever gets it, then I'm like, "Oh I've got to learn how to do that, or I'm going to lose all these marks."

All topics: **Diane**: Um, probably a quiz in class and you have to use that method.
Interviewer: OK, so actually forcing you by the assessment, saying you'll get it wrong if you don't use this method?
Diane: Yeah, yeah.

Topic Dependence

Some reasons given by the students applied to all three methods under examination in this study, but other reasons were mentioned only with regard to a specific topic. In what follows I would like to consider the motivations for adoption of, or resistance to, the specific procedures presented in the course. What aspect of the prime factorisation method caused so many students to adopt it, or what were the factors which contributed to their resistance to the wholes-first method? Was there something about the specific topic or method that made it easier, or harder, for students to change?

Greatest Common Factor and Least Common Multiple

1. There must be dissatisfaction with the existing method

Several students were dissatisfied with the amount of time or effort required by the method of listing, and some mentioned that it is easy to make an error when listing out the factors or multiples. Another compelling reason for adopting the new method was that many students did not regard listing as a genuine method, saying that they did not have a method prior to the course. Perhaps this is related to the observation that some students seem to give up if the lists becomes longer than about ten entries.

2. A new method must be intelligible

In general, the students felt that they understood the method and the symbols and how the method is applied to the problems. They were also able to recall the method at appropriate times.

3. A new method must appear initially plausible

There were no comments about a lack of plausibility of the new method. Clearly the prime factorisation method overcame the inadequacies of the method of listing felt by the students, as was commented on under the first condition. There was no mention of conflicts with the learners' prior knowledge, past experiences, or beliefs.

4. A new method should suggest the possibility of fruitfulness

Some students were conscious that the instructor expected this method to be used during the course, and one referred to the advantage that both the GCF and the LCM could be obtained easily, once the prime factorisations of the numbers were found.

Percentages

1. <u>There must be dissatisfaction with the existing method</u>

Several students found the compound one-step method easier and faster than the repeated two-step method for compounded percentage changes, but only two said that they had no method to solve these problems before the course. In contrast, some felt that their prior method was fast enough, or were confident that they could get the correct answer using it. Some felt that they used their old method out of habit.

2. <u>A new method must be intelligible</u>

Some students talked of understanding the compound one-step method, or said that it made sense and gave them a clearer picture of what the percentage change meant. For some, the understanding came during the interview and they were able to use the compound one-step method in the final examination. However, just as many students commented on not understanding the method and several struggled to remember it while in the examination.

3. <u>A new method must appear initially plausible</u>

Most students accepted that the compound one-step method was more efficient than the repeated two-step method, but some students needed reassurance that it would give the same answers. Once this had been confirmed, they were able to accept the new method. Some students indicated that they had not spent enough time working with the new method to be convinced of its value. One student felt that it was 'cheating' to not show the intermediate steps in the calculation of a percentage change.

4. A new method should suggest the possibility of fruitfulness

Once again, few comments were made relating to this condition, with only one student talking about the expectations of the instructor. One student used the compound one-step method for questions of a particular style only and did not recognise that the method could be applied to any percentage change situation.

Mixed Numbers

1. There must be dissatisfaction with the existing method

No students felt that they did not have a method for adding and subtracting mixed numbers before the course, although several recognised the wholes-first method as easier, faster and less error prone, especially when large numbers were involved. However, several students were satisfied that the conversion method gave the correct answer and they were confident in its use. Many felt that the conversion method was such an ingrained habit that they used it without thinking, but only a few were dissatisfied enough with their rote learning of the procedure to adopt the wholes-first method.

2. A new method must be intelligible

Several students talked of understanding the wholes-first method in a way that they had not understood the conversion method. They were able to relate the symbols to physical objects in the wholes-first method, and for some that allowed them to relate the mathematics to their real-life experiences. As with percentages, a few students struggled to remember the method, and two students did not understand how to apply the method in certain situations.

3. A new method must appear initially plausible

A few students had been discouraged in their attempts to use the wholes-first method by initially getting the wrong answer. Since they knew they could get the correct answer by using improper fractions, they quickly gave up on the new method. Several talked about feeling uncomfortable with fractions and therefore wanting to stick to the method they had learnt by rote. Some even felt that to use the wholes-first method was not 'doing math', or that the mixed number is an instruction to convert to an improper fraction.

4. A new method should suggest the possibility of fruitfulness

Relating to the fruitfulness of using the wholes-first method, a few students referred to the expectations of the instructor, but no other comments were made.

One additional issue arose in two of the interviews regarding the method used by the students for mixed number questions. It seems that their memories were not entirely reliable. In the first homework, Alice used the wholes-first method in addition to the conversion method for most of the questions. However, in the interview, when asked if she was familiar with the wholes-first method before the course, she said it was new to her, but

> "I worked on it and I figured it out and it, yeah, it made more sense. . . . Because on the first homework that you gave us, I did change everything into like top-heavy fractions, and then figured it out, and then brought it back to a mixed."

Cathy also used the wholes-first method in the first homework, but only for the addition questions. When asked, in the interview, if the wholes-first method was a new idea for her, she said,

> "Yeah, it was totally new to me, because it was so revolutionary to me. It was like wow, this is so much easier than trying to make them the top-heavy fractions. . . . This is one of those things that once you showed it to us, it just totally hit me how much easier . . ."

A possible explanation for their perception that they had not known the wholes-first method before the course is that they may not have understood the method, or what the symbols actually represented, until receiving instruction during the course. Also, they may not have previously thought about the different methods available to them, and the advantages of each, and were therefore struck by the benefits of the wholes-first method. Perhaps they saw the method in such a new way that they forgot they had actually used it before.

Motivations

Some motivations to adopt the methods presented in class were common to a number of students, whereas some were suggested by only one. Similarly, the reasons for resisting the new methods were sometimes mentioned by several students. Table 6.1 shows a summary of the number of distinct students who mentioned a particular motivation in relation to a particular method. The motivations have been grouped under the four conditions of the conceptual change theory.

Reason for adopting	GCF /LCM	%C	M #	Reasons for resisting	GCF /LCM	%C	M #
1. Dissatisfaction with existing method							
new easier/less effort	5	6	5				
new faster	6	4	4	old fast enough	1	2	-
new less error prone	3	1	3	old gives right answer	-	4	5
new easier for large numbers	-	-	6				
no acceptable existing method	7	2	-	comfortable or confident with old	1	-	5
old just rote	-	-	3	old ingrained habit	-	4	10
2. Intelligibility							
already confident with topic	2	1	1	didn't remember new	1	4	3
understood what symbols represent	2	1	3	missed class	-	1	-
new helped understanding	2	1	5	didn't understand new	-	6	2
new relates to experience	-	-	1				
new makes sense	2	3	5				
3. Plausibility							
solves problems of old	1	-	-	not compatible with beliefs	-	1	2
checked new against old	-	2	-	didn't get right answer	-	-	2
				no representation for symbols	-	-	5
4. Fruitfulness							
applicable to more than one situation	1	-	-	used in specific context only	-	1	-
gets more marks	1	-	-				
expected by instructor	2	1	2				

Table 6.1 Number of Distinct Students Mentioning Particular Motivations (by topic and conceptual change theory condition)

The findings presented in the previous chapter showed that almost all of the students adopted the prime factorisation method for finding the greatest common factor and least common multiple, whilst only a few used this method before instruction. Very few students used the compound one-step method, or even the one-step method for single percentage decreases, in the first homework. However, over half of the non-users showed progress in using the one-step method for percentage changes, with about two fifths of the students adopting the compound one-step method. The wholes-first method for mixed numbers was used in at least some of the non-trivial questions by a little under half of the students in the first homework, but only about one-third of the remaining students adopted the wholes-first approach to this extent. This information is given in detail in Table 6.2.

	Some Use in First Homework		Adoption to this level	
	No. of students	% of students	No. of students	% of remaining students
Prime factorisation for at least one of GCF/LCM	14	18	62	97
One-step for percentage decreases (as well as increases)	10	13	39	57
Compound one-step	3	4	31	41
Wholes-first for more than simplest mixed number questions	34	44	15	34

Table 6.2 Number of Users and Adopters to a Given Level by Topic

The theory can be used to explain the differences in the number of students who adopted the methods presented in the course. By considering the motivations suggested by the students, which were summarised by topic above, we can recognise which conditions were responsible for the differences and which features of the methods contributed most significantly.

The prior method with which the students were most dissatisfied was that of listing for finding the greatest common factor and least common multiple. This, I believe, was the main reason why so many students adopted the prime factorisation method. They found that the new method was much faster and required less effort, and in fact, several students did not regard listing as a genuine method. The students mentioned no problems with the intelligibility or plausibility of the prime factorisation method and some mentioned that the expectations of the instructor were important to them.

In contrast, the conversion method for mixed numbers was an ingrained habit which the students found difficult to break. Many found that, with the use of a calculator, improper fractions were easy to deal with accurately and quickly enough. Although several indicated that they were able to understand the wholes-first method, whereas they had simply memorised the conversion method, a few struggled with the application of wholes-first to some of the subtraction questions. Many students were unwilling to take the time, or put in the effort, to learn to use the new method because they were sufficiently confident in their ability to use their old method, and they believed it was adequate for their purposes.

For some students the belief that a mixed number is an instruction to calculate the improper fraction prevented them from considering the wholes-first approach. A quote from an interview with one of the students in the pilot study shows how extreme this belief can be. When asked the value of $4 + 7^1/_3$ she responded:

> "OK, well I know I can't do 7 and 4, 'cos this is, like, a whole thing right here [referring to the 7 and $^1/_3$ as inseparable]. And I think I have to put this [4] over 1. . . . I can't add 4. . . . I could say this is 11 and $^1/_3$, but that wouldn't be right. . . . I just don't think I can add this [4] to this [$7^1/_3$], . . . 'cos it's, like I feel like this [$7^1/_3$] is a whole thing, like, by itself, that I have to break down before I can add it to 4."

She explained, as did several students taking the course, that the whole number and the mixed number must be converted into the same type of number, i.e. an improper fraction, before they can be added or subtracted.

The adoption rate of the compound one-step method for percentages lay between that of the prime factorisation and the wholes-first methods. It was recognised by many as faster or requiring less effort than their prior method, but others were confident in their use of the repeated one or two-step methods. Some students found that studying the one-step method increased their understanding of percentage change, but many students said they did not understand or remember how to apply the compound one-step method. It may be that those students who struggled to remember the one-step method did so because of a lack of understanding. Others were not convinced that it would give them the correct answer. The reasons suggested for adopting and resisting the new method were more balanced for this topic.

Summary

It is not sufficient that a method presented is 'better' than a student's current method. It is clear that only if there is great dissatisfaction with the current method for solving a problem, will a student invest the time to learn a new method. The main cause of dissatisfaction which has arisen in this study is the amount of time or effort needed to implement the method. This was countered by the old method being so practised that it had become an ingrained habit. A second source of dissatisfaction was the student's lack of confidence that the correct answer could reliably be obtained using the current method.

The level of dissatisfaction with a current method will determine how much effort a student is willing to make in order to comprehend and remember the new method. To be intelligible, the student must certainly recognise the symbols and know how to apply the method, and in what circumstances it should be used. Unfortunately, it is not necessary for the student to understand how the method works, although this knowledge will encourage the adoption of the new method and aid the memory of the procedure.

The student must be convinced that the new method gives the same answer as the old method and agree that the new method truly fulfils the shortcomings of the old method by being significantly faster or easier or less error prone. It must also be compatible with the student's beliefs about what mathematics is or about the requirements of external authorities. The fruitfulness of a method did not seem to be an important issue for the students, other than with respect to getting more marks or fulfilling the requirements of the instructor.

Additional Comments

The above analysis is based on what students said when giving their reasons for using the different procedures. Before closing this chapter, I would like to speculate about some factors which may, perhaps unconsciously, have influenced the students. My speculations are derived from informal observations of the students and the conversations which took place throughout the course.

In chapter 3 there was discussion of the importance of students forming strong connections between the concept and the symbolic representation of the concept. Students must be able to access the meaning of the symbols in order to develop their understanding. If this is lacking, then their only option is rote memorisation. Frequently students do not regard certain combinations of symbols as representing numbers, but rather as being instructions to calculate a value which will be the number. For example, $8^{1/3}$ must be re-written as 2 before it is considered a number, and $2 + 3$ is seen as an instruction to be carried out, not as one of many representations of the number 5. I believe that many students do not think of mixed numbers as numbers. In the current study, students were asked to explain what a mixed number is, or what it means, using $3\frac{2}{7}$ as an example. Some students appeared to regard the mixed number notation as an instruction to calculate the equivalent improper fraction, giving answers such as "It's 7 x 3 + 2, over 7". Several students responded with, "A mixed number is a fraction where the top number (numerator) is greater than the bottom number (denominator)" or similar descriptions, and one student even added, "the original number was $\frac{23}{7}$ ". When students come to multiply and divide with mixed numbers, they are taught to convert the mixed number to an improper fraction as the first step. This approach also works for addition and subtraction. If the students have lost sight of the meaning of the symbols, then it may be that they choose, or are encouraged to use, this approach for all mixed number calculations in order to reduce the number of procedures they must memorise.

From observation of students in the current study, I would suggest that the emphasis on changing mixed numbers to improper fractions can prevent students from seeing that the wholes do not need to be split into fractions with denominator determined by the fractional part of the mixed number. For example, when explaining the meaning of $3\frac{2}{7}$ there were responses such as the following "You'd have 3 complete . . . numbers that had been divided into slices . . . and then the fraction would only have 2 out of 7. So what it's saying is that you have $\frac{7}{7}+\frac{7}{7}+\frac{7}{7}$ (so that would be the three whole things) and then $^{2}/_{7}$." This lack of understanding of the meaning of mixed numbers may have contributed to the students' resistance to the wholes-first approach.

As discussed in chapter 2, learners will assimilate new knowledge into their existing structures whenever possible, but the accommodation of a concept requires a much more radical restructuring. Students with a strong belief that mixed numbers must be converted into improper fractions would need to accommodate a new concept for mixed numbers in order to adopt the wholes-first method. The difficulty of accommodation may help to explain the high proportion of resisters for this topic. In contrast, a student who knows the one-step method for percentage increases may be able to assimilate the one-step method for decreases. In this study, 6 out of 8 students who initially used the one-step method for increases, but not for decreases, adopted the method for decreases. However, only 55% of students who did not initially use the one-step method adopted this method for decreases.

Returning to the discussion of different representations of numbers, a percentage of a quantity is another form of number. However, I have observed that '18% of 534', for example, is often seen as an instruction to calculate rather than as a representation of a quantity. This perspective would hinder the acceptance that the new amount can be found after a percentage change by the one-step method, with students feeling that they must follow the instruction to calculate the actual value of the change first.

The prime factorisation is also a representation of number. Why is it that a lack of understanding of this did not interfere with students' use of the prime factorisation method to find the greatest common factor or least common multiple of two numbers? Perhaps this understanding is less important when applying this method. A more likely explanation is that the effort required to memorise this method by rote was outweighed by the students' dissatisfaction with the method of listing and so many used the method without understanding the concepts underlying it. For the mixed number calculations and, to a lesser extent for the percentage change questions, the students were not sufficiently dissatisfied with their prior methods to put in the effort to memorise the new methods by rote.

Many students seemed uncomfortable with mixed numbers and fractions, preferring decimal notation. One contribution to this unease may be the increased use of calculators and computers. Typing fractions, and especially mixed numbers, is more difficult than using decimals, and this has led to mixed numbers and fractions becoming less common in every day life. Percentages are used in abundance in the media and other areas of life, but frequently they are misunderstood and used incorrectly, especially with percentages greater than 100%. There is often a confusion between 'percentage change' and 'percentage of original'. For example, a 50% increase, leading to the new amount being 150% of the original, is often referred to as a 150% increase. These two factors could encourage students to be comfortable with the percentage notation, but lead them to a poor understanding of its meaning, which may account for some choosing to 'stick with what they know'. However, greatest common factor and least common multiple are concepts very rarely explicitly expressed in real life. It may be that this is another encouragement for rote memorisation, as these concepts are regarded as useful only in the mathematics classroom and not related to life outside.

The method of listing is clearly useful in helping students to understand the concept of greatest common factor or least common multiple. This method is transparent, being closely connected to the meaning of the concepts. Similarly, the two-step method for finding the amount after a percentage change is also very easy to understand and follow. These were the methods used by the majority of students at the beginning of the course. The prime factorisation method for finding the greatest common factor and least common multiple and the compound one-step method for percentage changes were presented in class as being more efficient, but it is acknowledged that it is less obvious that these procedures lead to the desired outcome. Conversely, the wholes-first method for mixed number addition and subtraction is more transparent than the conversion method, being more closely related to the physical actions which would normally be used to carry out the operations. However, almost all students adopted the prime factorisation method, the majority adopted, or progressed towards adopting, the compound one-step method, but a smaller proportion of students adopted the wholes-first method. This could indicate that intelligibility of a method is not a high priority for many of the students when choosing a method. If a student is content with rote memorisation, then intelligibility is not an important consideration. In contrast, it can be seen from comments made in the interviews that for some, the understanding brought by the wholes-first approach was a strong motivation for adoption of this method.

Next we will consider how the analysis presented in this chapter gives rise to a theory of procedural change, growing out of the conceptual change theory of Posner, Strike, Hewson and Gertzog (1982). The implications of this theory for developing teaching strategies which encourage students to adopt new procedures are also examined. Finally, areas for further research are suggested which will stimulate the development of this theory of procedural change.

CHAPTER 7

DEVELOPMENT OF A THEORY

A Theory of Procedural Change

In chapter 5 we looked at how students responded to procedures for solving problems in topics for which they already knew a valid approach, considering whether they adopted or resisted these new methods. The results suggested that the outcome depended upon both the particular topic and the individual student. The data collected from the interviews pointed towards a variety of motivations for adopting, and reasons for resisting, a new approach. In chapter 6 it was shown that organising this data along the lines of the conceptual change theory of Posner, Strike, Hewson and Gertzog (1982) allowed a better understanding of the students' motivations. Continuing to follow the ideas of grounded theory (Glaser & Strauss, 1967), I now propose a theory of procedural change.

By broadening Duroux's description of an obstacle (as cited in Brousseau, 1997) from "A piece of knowledge or a conception . . . [which] produces responses which are appropriate within a particular, frequently experienced, context, . . . [but] generates false responses outside this context" (p. 99) to include 'a piece of knowledge which, although correct, is inhibiting the acceptance of another piece of knowledge which is applicable, and more appropriate, in certain situations', we can apply the discussions on overcoming obstacles to this type of obstacle. For example, Sierpinska (1987) speaks of overcoming an obstacle by considering the means one uses to solve problems, recognising the reasons behind the choices, and becoming aware of other possibilities. We must recognise the successes of the obstacle as well as its failures (Brousseau, 1997). Misconceptions were found to be highly robust, typically outliving teaching which contradicts them. Similarly, these procedural obstacles were highly robust in some students and outlived the teaching designed to encourage the students to adopt the new approaches. Brousseau (1997) said that students must be provided with many situations where the knowledge is inadequate, to convince them to consider something else. Overcoming the obstacle demands work of the same kind as applying the knowledge; problems must be numerous, important to the student, and sufficiently different from those previously encountered to require the leap to the acceptance of the new knowledge. The same can be said of the adoption of new procedures.

The four conditions given by Posner et al. (1982), for the accommodation of a new concept when prior knowledge is an obstacle, can now be re-written to give a 'theory of procedural change'. Italics are used to highlight changes from the conceptual change theory. Each condition is followed by a brief explanation, which includes comments on any significant differences from the original theory.

1. <u>There must be dissatisfaction with existing *procedures*</u>

A student must believe that his or her current *procedure* will not suffice. *The student must be sufficiently dissatisfied to take the time, or put in the effort, to learn to use the new method, to comprehend it and remember it, and to overcome the habit and comfort of familiarity of the old method.*

We are not presenting students with evidence which contradicts their current beliefs. The dissatisfaction here comes from a variety of sources, such as the amount of time or effort required to implement the procedure or the likelihood of computational errors, rather than from problems that cannot be solved.

2. <u>A new *method* must be intelligible</u>

A learner must be able to grasp how *the new procedure can be applied to problems* sufficiently to explore further. At a superficial level, the *learner must know how to manipulate the symbols.*

Unfortunately, it is not necessary for a student to understand what the symbols mean or how a method works, since the steps can be memorised by rote, although understanding will aid the memory of the procedure and increase motivation to use the method. If the method itself helps understanding of a concept or if the student had a good prior understanding of the concepts involved, or if the student can relate the symbols or the procedure to physical objects or real life experiences, the method is more likely to be adopted.

3. <u>A new *procedure* must appear initially plausible</u>

The new *method* must at least appear to have the capacity to solve the problems generated by previously *used procedures*. It must be consistent with other knowledge previously constructed by the learner and with past experience. Also, the new *method* must be compatible with one's beliefs and fundamental assumptions.

These beliefs and assumptions include the expectations of external authorities and an understanding of the nature of mathematics. It may be impossible to adopt the new method if the prior method was rote memorised without understanding, since there is no foundation for understanding the new, or perhaps no belief that mathematics can be understood. Certainly, the new method must be seen to generate the same solutions as the old method. Sufficient time and practice is needed to become comfortable with the new procedure or to fully recognise its value.

4. <u>A new *procedure* should suggest the possibility of *fruitfulness*</u>

The new procedure should be seen to be applicable to many situations and should meet the requirements of any assessment.

This replaces the condition for a fruitful research program, with the potential for extension of the concept or of application to new areas of inquiry. Using the new method, students should be able to solve a variety of problems, not just one specific type of problem, and by fulfilling the assessment criteria, its use should allow the students to have greater success.

It is acknowledged that information gathered in interviews, asking questions such as why a student adopted or resisted a method, cannot lead to absolute conclusions. However, it is hoped that by examining this issue, the data collected can suggest an explanation of the observed behaviour and eventually lead to the development of teaching strategies which will help students to accept a wider variety of approaches to mathematical problems, increasing their repertoire of procedures and algorithms, and allowing them the freedom to choose the most appropriate and efficient method.

Implications for Teaching

Some students adopted the new methods, showing that change is possible, but it should not be taken for granted that adoption will always take place as a consequence of instruction. Clearly it is not sufficient that a method presented to the students is 'better' in some way, or for some problems, than their current method. That some students are resistant to change, and even when willing to adopt a new method can find it difficult to do so, is of great concern. It implies that what and how a student is first taught is of crucial importance, and yet usually we entrust teaching in the early years to non-specialists, who are often uncomfortable with the subject. If an elementary teacher considers mathematics to consist of rules and procedures to be memorised, then this belief will be passed on to the students. If it is conveyed that there is one correct way to solve a problem, then students will find it difficult to adopt new approaches in the future. If getting the right answer is all that counts, then students will not be concerned with understanding. Teachers often mimic how they were taught, and so we must be especially careful how we teach teachers (Grouws, 1992). Since we cannot turn the clock back to change how potential teachers were taught in elementary school, we must do all that we can to teach them to be receptive to new approaches and new understanding when participating in pre-service courses.

Nussbaum and Novick (1982) suggest a teaching strategy to encourage the accommodation of a scientific concept. This can be related to the instruction given to the students in this study.

1. *Expose students' alternative conceptions through their responses to an 'exposing event' and encourage them to describe their preconceptions verbally and pictorially.* This was the aim of the first homework.

2. *Make students aware of their own and other students' alternative conceptions through discussion and debate.* The different methods commonly used for each topic were discussed in class, with the benefits of each highlighted.

3. *Create conceptual conflict by students trying to explain a discrepant event.* Problems were discussed which showed that the students' prior methods were inefficient.

4. *Encourage and guide accommodation of the new cognitive concept.* The methods recommended in the course were discussed and their application to problems demonstrated.

This is a strategy commonly used when introducing new approaches to problems in mathematics. However, Nussbaum and Novick (1982), and Smith

(1983), report limited success when using this strategy, with most students reaching an intermediate conception between their initial conception and the accepted scientific conception. The data in the current study also supports the view that this teaching strategy is not entirely successful.

Can the theory of procedural change proposed above give rise to any further suggestions for a teaching strategy? Let us consider the implications of the four conditions in the theory.

1. <u>There must be dissatisfaction with existing procedures</u>

Examples and homework questions should involve situations in which the students' prior methods prove inadequate, for example, they require too much effort and time, such as many compounded percentage changes or large numbers. Understanding of how the procedure works can be encouraged by requiring the students to explain what they are doing at each stage, in some cases relating it to physical actions, or to the underlying concepts. If a student has memorised an algorithm without understanding, this will become apparent.

Only when the students have become dissatisfied with their current method should the new method be introduced. Examples used to illustrate a new method are typically chosen for simplicity and clarity, but perhaps these do not best illustrate the advantages of the new method over the old. A balance must be achieved between simplicity, to allow students to follow the application of the procedure, and complexity, to induce dissatisfaction with prior methods.

Students who use an old method out of habit can be asked to describe more than one approach and then explain which they think is most appropriate.

However, encouraging a student to use the new approach by too much repetitive practice may result in rote memorisation of that procedure without an understanding of the concepts on which it is based, an appreciation of its value, or the ability to discern when it is the most appropriate method to use.

2. <u>A new method must be intelligible</u>

Students are more likely to remember and use a method which makes sense to them. A clear connection should be established between the symbols in the procedure and the objects or concepts they represent, rather than teaching the manipulation of notational systems. This may involve revisiting the concepts underlying the procedure. It cannot be assumed that students have grasped these concepts at an earlier stage, they may simply have memorised algorithms. In some cases, the new method will facilitate understanding of the concepts involved.

Understanding how and why a procedure solves a problem must be valued by the educational community (the classroom, school or broader community) before students will value understanding over simply getting the right answer. The repeated application of the procedure, always accompanied by discussion of what is represents at each stage, will help to develop this understanding and reduce the likelihood of rote memorisation.

3. <u>A new procedure must appear initially plausible</u>

The examples and homework questions mentioned under condition 1, which require much effort and time when using the students' prior method, must then be seen to be solved easily using the new method. It is also important to consider appropriate use of current technology when developing questions which are intended to show the strengths of the new procedure. The same problems should be solved using both methods to convince the students that the same answer is obtained, and the students should be encouraged to propose advantages and disadvantages of each method. This should prevent a given method being associated with a particular style of question and will encourage the students to make informed choices for approach in the future.

It is to be expected that students will need time and practice in order to become sufficiently familiar with a method that they will use it in situations of consequence, such as examinations. However, confirming that the new method is not only acceptable, but desirable, will help to combat beliefs held by some students about the type of approaches and procedures that are acceptable to the mathematics community, and which may prevent the adoption of some methods.

Since the new method must be consistent with other knowledge previously constructed by the learner, it may be necessary to review key ideas before introducing the new procedure. This illustrates the importance of teaching for understanding at every stage in mathematics education, since a lack of understanding can prevent a student from learning at a later stage.

4. A new procedure should suggest the possibility of fruitfulness

The students should apply the new procedure to different situations, solving a variety of problems, not just one specific type of question. Perhaps other procedures based on the same approach could be demonstrated. Students should be encouraged to examine the procedure and seek out any new mathematical directions it may suggest. Assessment tools can be used to encourage the new approach either by requiring its use, or by giving more credit to those who use an elegant or efficient method.

Further Research

I would like to continue the work of this study by developing instruction, based on this teaching strategy, to promote the methods under discussion here. My instruction given in the course underestimated what was needed to make some students dissatisfied with the conversion method for mixed number calculations and the repeated two or one-step methods for percentage change. Problems given in class and for homework should have been more 'extreme'. The fact that different notations can represent a given quantity, without having to calculate the value and represent it in decimal notation, is knowledge that would be beneficial for all the topics and this should have been emphasised more. For some students the compound one-step method was difficult to understand and more time was needed for relating $(100 \pm x)\%$ to an increase or decrease of $x\%$. Students should be encouraged to explain what is represented by each step in the procedures and they should themselves describe the advantages and disadvantages of each method. Lastly, the students should be required to demonstrate the ability to use all the methods discussed in the course and to choose appropriately for particular questions. This is especially important for the students in this study, most of whom will become teachers and will need to be able to present more than one method to their own students.

The theory of conceptual change has been developed to give a theory of procedural change, but in this study I did not explore to what degree the procedural change was a result of conceptual change. Some comments made by students clearly indicated that they had adopted a method because of a new grasp of the concept, but others did not refer to their conceptual understanding. Similarly, it was not investigated to what extent resistance was due to a lack of conceptual understanding. Conducting research into the relationship between conceptual change or understanding and the adopting of or resistance to new procedures could be very productive. A further component of this work could be to discover the robustness of the adoption or resistance: whether students continued to use methods adopted during the course, or resisters eventually adopted the new procedures, months or years later.

The introduction of new approaches to problems that can be solved by a procedure already known to the students is not limited to the topics under discussion in this study, nor to the content of courses for pre-service elementary teachers. Examples can be taken from many areas and can be found in advanced, as well as elementary, mathematics. In fact, whenever there is more than one possible approach to solving a problem, students will usually meet the different methods sequentially rather than simultaneously, and must therefore be able to adopt a new approach when a correct method is already known.

In addition to the examples taken from linear algebra, discussed in an earlier chapter, there are several topics suitable for examination in calculus. Consider the following integral:

$$\int_0^1 4\,x\,(2\,x-1)^4\,dx$$

A student first learns to integrate polynomials and can find the value of this integral by expanding $4\,x\,(2\,x-1)^4 = 64\,x^5-128\,x^4+96\,x^3-32\,x^2+4\,x$, then

$$\begin{aligned}\int_0^1 64\,x^5-128\,x^4+96\,x^3-32\,x^2+4\,x\,dx &= \left[\tfrac{64}{6}\,x^6-\tfrac{128}{5}\,x^5+\tfrac{96}{4}\,x^4-\tfrac{32}{3}\,x^3+\tfrac{4}{2}\,x^2\right]_0^1 \\ &= \tfrac{64}{6}-\tfrac{128}{5}+\tfrac{96}{4}-\tfrac{32}{3}+\tfrac{4}{2} = 26-25\tfrac{3}{5} = \tfrac{2}{5}\end{aligned}$$

Later, the student meets integration by substitution and finds that a much more efficient way to solve the above integral is by substituting $u = 2\,x-1$, giving

$$\int_{-1}^{1} 2\,(u+1)\,u^4\cdot\tfrac{1}{2}\,du = \int_{-1}^{1}(u^5+u^4)\,du = \left[\tfrac{1}{6}\,u^6+\tfrac{1}{5}\,u^5\right]_{-1}^{1} = \left[\tfrac{1}{6}+\tfrac{1}{5}\right]-\left[\tfrac{1}{6}-\tfrac{1}{5}\right] = \tfrac{2}{5}$$

Is there a resistance to adopting the method of substitution when prior methods will suffice? What motivates students to adopt the new method in this situation?

Further research could be conducted in areas of advanced mathematics to discover if the adoption of the new methods is influenced by the same factors. The theory of procedural change should be tested against the findings of such research and adapted as appropriate. Improved teaching strategies can then be developed for greater success in encouraging students to be flexible in their approach, allowing them to choose the most efficient method for a given situation.

The list of possible topics in mathematics is almost endless and this situation is also encountered in other disciplines, for example, medicine. Some doctors are reluctant to move to new ways of healing because of a lack of time to learn the new ways and a belief that the risks involved with the current method are acceptable. In other words, they are not sufficiently dissatisfied with their current methods to adopt a new approach. Perhaps they do not fully comprehend the new approach, or perhaps they do not think it will fulfil the shortcomings of the old method, or it is incompatible with their beliefs about medicine - the new method is not initially plausible. If they do not consider it to have the potential to be fruitful, then perhaps they do not consider learning it a good investment of their time.

Summary

A desirable outcome of mathematics education is to equip students with the ability to choose an appropriate strategy for a given problem, from a range of procedures and algorithms. We want to discourage students from simply doing the first thing that they think of, out of habit, and we want them to understand the concepts behind the procedures. In this study I have examined the reasons why a student may adopt or resist a new approach to solving a problem. The student's prior knowledge, including beliefs, affects this often subconscious choice and simply pointing out the benefits of the new method is not always enough.

The theory of conceptual change proposed by Posner, Strike, Hewson and Gertzog (1982), provided a basis for considering what motivates students.

This theory was then adapted and extended to fit a broader range of circumstances, giving rise to a theory of procedural change. The new theory is applicable to situations where the prior knowledge of the student is not incorrect.

Some of the implications of the procedural change theory for teaching strategies were considered and the need for further research has been acknowledged. The theory should be tested using other mathematical topics, and even topics from other disciplines, in order to confirm that is able to explain the behaviour of students.

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APPENDIX A: ETHICAL APPROVAL

SIMON FRASER UNIVERSITY

OFFICE OF VICE-PRESIDENT, RESEARCH

BURNABY, BRITISH COLUMBIA
CANADA V5A 1S6
Telephone: (604) 291-4370
FAX: (604) 291-4860

February 15, 2001

Ms. Christine Stewart
Graduate Student
Department of Mathematics
Simon Fraser University

Dear Ms. Stewart:

Re: Knowledge as an Obstacle to Learning in Pre-Service Elementary School Teachers Studying Mathematics

I am pleased to inform you that the above referenced Request for Ethical Approval of Research has been approved on behalf of the University Research Ethics Review Committee. This approval is in effect for twenty-four months from the above date. Any changes in the procedures affecting interaction with human subjects should be reported to the University Research Ethics Review Committee. Significant changes will require the submission of a revised Request for Ethical Approval of Research. This approval is in effect only while you are a registered SFU student.

Best wishes for success in this research.

Sincerely,

Dr. James, R.P. Ogloff, Chair
University Research Ethics Review Committee

c: R. Zazkis, Supervisor

/bjr

FORM #2

SIMON FRASER UNIVERSITY

INFORMED CONSENT BY SUBJECTS TO PARTICIPATE IN A RESEARCH PROJECT

The University and those conducting this project subscribe to the ethical conduct of research and to the protection at all times of the interests, comfort, and safety of subjects. This form and the information it contains are given to you for your own protection and full understanding of the procedures. Your signature on this form will signify that you voluntarily agree to participate in the research project.

This research is essential in order to understand how preservice teachers learn mathematics and the effect of specific instructional approaches on students' learning.

Participants in this research will give their permission to use their work in the course MATH 190 - Principles of Mathematics for Teachers - as research data. This includes observation of students' work in class and in the lab, analysis of students' homework, projects and exams, and conducting interviews with individual students.

Materials collected in this research will be held in a secure location. Participants' identity will be protected in case of publication of the research results or conference presentations.

I understand that I may withdraw my participation in this research at any time.

I also understand that I may register any complaint I might have about the research with Christine Stewart (291-5754) or with Dr. Rina Zazkis (291-3662), Faculty of Education, at Simon Fraser University.

I may obtain copies of the results of this study, upon its completion, by contacting Christine Stewart.

I understand that my identity is protected in case of publication or presentation of research results.

I agree to participate by allowing observation of my work in class and in the lab, analysis of my homework, exams, and other assignments, and by being interviewed on the material related to the course. I agree that these interviews may be audiotaped.

I further understand that my involvement in this research or my refusal to participate has absolutely no relation to the grading of my assignments in MATH 190 and to the grading of the course.

NAME (please type or print legibly): ______________________________

ADDRESS: ______________________________

____

SIGNATURE: ____________________

DATE: ____________________

APPENDIX B: PILOT INTERVIEW QUESTIONS

Math190 Support Group Interviews, Fall 2000

SECTION A

1) What is a multiple?
2) What is a factor?
3) Give some multiples of 12.
4) Give some factors of 12.
5) What is the least common multiple (LCM) of some numbers?
6) What is the greatest common factor (GCF) of some numbers?
7) Find the LCM of 42 and 154.
8) Find the GCF of 42 and 154.
9) Find the LCM of 9 and 15.
10) Find the GCF of 9 and 15.

SECTION B

1) Explain the meaning of a fraction, for example $\frac{4}{5}$.
2) Find the value of $\frac{2}{3}+\frac{4}{5}$.
3) Find the value of $\frac{7}{8}-\frac{5}{6}$.
4) Explain the meaning of a mixed number, for example $3\frac{2}{7}$.
5) Find the value of $4+7\frac{1}{3}$.
6) Find the value of $6-2\frac{5}{7}$.
7) Find the value of $31\frac{3}{8}+6\frac{1}{3}$.
8) Find the value of $94\frac{4}{5}-11\frac{1}{4}$.
9) Find the value of $27\frac{5}{9}+8\frac{4}{5}$.
10) Find the value of $89\frac{1}{4}-34\frac{7}{8}$.

SECTION C

1) What is a percentage? For example, what does 18% mean?

2) What is a percentage of a quantity? For example, what is 18% of 374?

3)(a) An item is marked $30. It is then put in a '20% off' sale. What is the sale price of the item?

(b) An item is marked $54. It is then put in a '33% off' sale. What is the sale price of the item?

4) The population of a small town was 15,000 in 1997. The annual increase in the population of the town is 16%.

(a) What was the population in 1998?

(b) What was the population in the year 2000?

5) A car cost $32,000 when it was new. The first owner sold it for 20% less than he paid for it. The second owner sold it for 25% less than he paid for it. The third owner sold it for 10% less than he paid for it.

(a) How much did the third owner get for the car?

(b) Can you think of a quicker way to get the answer to part (a)?

APPENDIX C: FIRST HOMEWORK QUESTIONS

Math190 Homework 1, 9th January, 2001

A1) (a) Using 18% as an example, explain what a percentage is.
(b) Find 18% of 374.

A2) A GIC gives 4% interest per year. I have $500 which I can use to buy a GIC now. How much will I have
(a) 1 year from now? (b) 4 years from now?

A3) Five weeks before Christmas a crafty store manager raised all prices by 15%. In the week before Christmas she announced a sale and lowered the prices by 10%. On Boxing Day she put up a sign saying, "Final clearance, an extra 10% off all prices!"
A customer bought a ski-jacket in early November. She was so happy with it that she told a friend, who decided to wait until the Boxing Day sales. The jacket was originally marked $250.

(a) How much did the friend pay for the jacket?
(b) Would you have waited for the Boxing Day sale?
(Explain your reasoning.)

An assistant in the store told the manager that he had found a much easier way to calculate the final sale price. Since they had added 15%, taken off 10%, then another 10%, he could just take 5% off the original price.

(c) Should the manager let him do that?
(d) Is there a quick way to calculate the final sale price from the original price?

B1) (a) Give three multiples of 12. (b) Give three factors of 12.

The least common multiple of 9 and 15 [written LCM(9,15)] is 45, because this is the smallest number which is a multiple of 9 and is also a multiple of 15.
The greatest common factor of 9 and 15 [written GCF(9,15)] is 3, because this is the largest number which is a factor of 9 and is also a factor of 15.

B2) (a) Find LCM(42,154). (b) Find GCF(42,154).

C1) Using $\frac{4}{5}$ as an example, explain what a fraction is (you may draw a picture).

C2) Find the value of (a) $\frac{2}{3}+\frac{4}{5}$ (b) $\frac{7}{8}-\frac{5}{6}$

C3) Using $3\frac{2}{7}$ as an example, explain what a mixed number is.

C4) Find the value of (a) $4+7\frac{1}{3}$ (b) $6-2\frac{5}{7}$

C5) Find the value of (a) $31\frac{3}{8}+6\frac{1}{3}$ (b) $94\frac{4}{5}-11\frac{1}{4}$

C6) Find the value of (a) $27\frac{5}{9}+8\frac{4}{5}$ (b) $89\frac{1}{4}-34\frac{7}{8}$

APPENDIX D: MIDTERM EXAMINATION QUESTIONS

Math190 Midterm #2, 13th March, 2001

1.(c) Find the value of each of the following:
(show enough working to clearly indicate your method of calculation)

(i) $7\frac{3}{5} - 5\frac{1}{4}$

(ii) $8 + 3\frac{1}{4}$

(iii) $4 - 2\frac{5}{7}$

(iv) $24\frac{1}{3} - 18\frac{7}{9}$

(v) $57\frac{8}{11} + 211\frac{3}{7}$

2.(a) Find the greatest common factor <u>and</u> least common multiple of 280 and 300.

GCF(280,300) = ____________________

LCM(280,300) = ____________________

3.(a) The population of Castleton was 3251 in 1980 and 5179 in 1990.

(i) What was the percentage growth in the population between 1980 and 1990?

(ii) Between 1990 and 2000 the population of Castleton fell by 19%. What was the population in 2000?

3.(b) Investors in mutual funds are warned that the value of the fund may fall in some years. On 1st January 1997 Sheila invested $5000 in a mutual fund. Here is the performance of her fund:

1997 gained 12%
1998 gained 23%
1999 lost 16%
2000 gained 8%

What was the value of Sheila's investment on 1st January 2001?

APPENDIX E: INTERVIEW QUESTIONS

Math190 Interviews, Spring 2001

SECTION A

1) Find the value of $8\frac{3}{4} - 5\frac{2}{5}$.

2) Find the value of $6 + 2\frac{2}{7}$.

3) Find the value of $5 - 3\frac{2}{9}$.

4) Find the value of $42\frac{1}{3} - 30\frac{3}{4}$.

5) Find the value of $324\frac{7}{10} + 213\frac{4}{5}$.

SECTION B

1) House prices rose by 12% during 1994. If an average house cost $180,000 at the beginning of the year, how much would it cost at the end of the year?

2) House prices fell by 8% during 1998. How much would a house costing $250,000 at the beginning of the year be worth at the end of the year?

3) The records of the Keep Fit gym show that in 1995 there were 600 members. The membership rose by 15% in 1996 and rose again, by 9%, in 1997. 1998 was a bad year and the number of members fell by 17%, and this was followed by another small decrease (of 4%) in 1999. 2000 was a better year and the membership rose by 19%. How many members did the gym have in 2000?

APPENDIX F: FINAL EXAMINATION QUESTIONS

Math190 Final Examination, 20th April, 2001

2.(d) A store raised the price of its winter sports equipment by 16% in October.

In January there was a New Year sale:	25% off marked prices.
In March prices were reduced again:	30% off sale price.
In April a final clearance sale was announced:	40% off March prices.

What percentage of the original (September) price was saved by a customer making a purchase in April?

4.(c) Find the value of $3\frac{1}{4} - 1\frac{7}{8}$ and draw a diagram to illustrate the calculation.