

**EXPOSING PRE-SERVICE SECONDARY MATHEMATICS
TEACHERS' KNOWLEDGE THROUGH NEW RESEARCH
DESIGNED METHODOLOGY**

by

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ABSTRACT

This study is an extension to the ongoing research on secondary mathematics teachers' knowledge. This study focused on the concepts of logarithms and logarithmic functions. Several research studies have confirmed that high-school and undergraduate students have a very poor knowledge of logarithms and logarithmic functions. One of the possible reasons for students' difficulties could be an insufficient teachers' knowledge of this subject domain. As of yet, there has not been research into teachers' knowledge of logarithms. This study was an attempt to fill this gap.

The deeper understanding of teachers' knowledge, particularly subject matter knowledge and related pedagogical skills, leads towards improvement of instructional approaches for more effective teacher training. The questions posed in this study are: What do the designed tasks reveal about the nature of teachers' knowledge? What can be seen as the relationship between pre-service secondary mathematics teachers' subject matter knowledge and pedagogical content knowledge? To what extent are these tasks effective and useful as data collection tools for research in mathematics education?

This study identified that pre-service teachers are aware of possible difficulties of teaching or learning the concepts of logarithms and logarithmic functions. However, their insufficient subject matter knowledge disallowed participants to explain why the situations prompted by their own questions were indeed problematic and important. On the whole, the pre-service teachers' displayed a relatively weak content knowledge of

logarithms and logarithmic functions, exemplified by weak subject matter knowledge and related pedagogical content knowledge.

Another goal of this research was to determine the effectiveness of the research methodology developed and used in this study. I designed a unique research task, called the Job Interview, and utilized another research task, known as the Math Play. These activities allowed me to investigate pre-service teachers' knowledge from many different sources that yielded very diverse information about the participants' knowledge. Also, these tasks proved to be important learning activities. They allowed pre-service teachers to re-examine high school mathematics content and reflect on their practice, while keeping in focus students' meaningful learning.

Keywords: mathematics education, teachers' knowledge, logarithms, pre-service secondary teachers' training

DEDICATION

To my husband and my children

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EXPOSING PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE THROUGH NEW RESEARCH DESIGNED METHODOLOGY

Most mathematics educators are involved in the practice of teacher education at some level. Indeed, the field of mathematics education is predicated on the assumption that someone has to be educated to teach mathematics in our schools. That raises the question of what it means to be educated in order to become a teacher of mathematics. What kinds of knowledge do teachers need to become effective teachers of mathematics? What sorts of experiences are needed for teachers to acquire this knowledge? A fundamental question for the mathematics teacher educator is how the field of teacher education can be conceptualized so the programs and activities can be created to assist in the acquisition of this knowledge. Given the high visibility of standards developed by the National Council of Teachers of Mathematics (NTCM, 1989a, 1991, 1995), a question of interest to many is, What does it take to develop teachers who can move the field towards realizing these standards? (Cooney, 1994, p.608)

Chapter 1: Introduction

Over three decades ago Alba Thompson (1984) highlighted an assumption that teachers' knowledge affects their practice. In recent years, a concern with the quality of teachers' knowledge has resurfaced in mathematics education research. The renewed interest in teachers' mathematical knowledge, learning mathematics for teaching, and the demands of teaching mathematics has become a focus of several recent research studies (Ball, D. L., Hill, H.C, & Bass, H. 2005; Rowland, Thwaites, & Huckstep, 2003; Liljedahl, in press; Leikin, 2006).

The above quote from Cooney (1994) is intended to serve as an inspirational prologue for this research study. In answer to this quotation about the kinds of knowledge teachers need to teach mathematics, my research has focused on two necessary types of required knowledge: pedagogical and subject matter. The issue of teachers' experiences essential for the development of these types of knowledge is also anchored in my study. I have designed an original research task and utilized another research task, that have served as valuable experiences for both the participating pre-service teachers and myself. These tasks, it is hoped, may in the future go beyond the data collection and analysis undertaken here. Perhaps they can contribute to the "acquisition of knowledge" in the field of mathematics education. This study aims to contribute to the swelling of research and interest in the development of teachers who will be teaching mathematics to the generations to come.

In the existing body of mathematics education research, many studies attempt to explain the nature of the mathematics teachers' knowledge required for teaching, and the relationship between different components of that knowledge (Rowland, Martyn, Barber, & Heal, 2001; Tirosh, Fischbein, Graeber & Wilson, 1998; Hill, Rowan & Ball, 2005; Leikin, 2006). As mathematics educators, we must know what types of knowledge the novice teachers need in order to be successful in their profession. Also, we must investigate the relationship between the components of teachers' knowledge, what roles they play and how these roles differ as teachers differ in the knowledge they possess. To conduct such research studies, we have to develop methodologies that will enlighten our thinking in this area, and make diverse information about teachers' knowledge attainable for analysis.

There are research studies that explored whether and how teachers' mathematical knowledge contributes to students' achievement. Hill et al. (2005) have concluded that teachers' mathematical knowledge was significantly related to student achievement. This result caught my attention and inspired me to expand my previous study on high-school students' understanding of logarithms (Berezovski, 2004).

The interest in students' misunderstanding of logarithms and logarithmic functions has been identified as being on the rise in recent mathematics education research (Berezovski & Zazkis 2006, Kenney 2004, Weber 2002). All of these studies confirm that high school and undergraduate students have a very poor knowledge of logarithms and logarithmic functions.

One of the possible reasons for students' misconceptions related to logarithms might be an insufficient teachers' knowledge. And yet, there has not been much research into teachers' understanding of logarithms. It is time to give this problem the attention it deserves.

1.1 Personal Motivation

In my teaching career, I have had an opportunity to work with students learning mathematics from different perspectives and at varied levels. I taught mathematics to high school students, undergraduate students, and pre-service and in-service teachers. Each and every meeting I have with my students is a learning step for me in coming to an understanding of what mathematics really is about, and why it causes so many difficulties for others to learn and understand.

As the years passed, I began to notice how the learning of mathematics took place for learners and for me. While teaching Mathematics content courses at high school and the undergraduate level, I gained a deeper understanding of what mathematics is; however, only when working with teachers, did I begin to appreciate my job, and to realize the immense challenge of teaching mathematics. I began to reflect on the traditional learning methodologies I was exposed to as a student myself, and I questioned their validity when implementing them into my classrooms teaching. This inquiry became a serious project that resulted in my Master's thesis, where I examined in detail high school students' understanding of logarithms and logarithmic functions. That study provided a system of interpretive frameworks which were used to model the students' understanding of logarithms. The results describe students' difficulties with logarithms, and also suggested possible sources of these difficulties. Students' difficulties were viewed through the lens of the conceptual-epistemological obstacles. One of the major findings of this research was: high school students' knowledge is very limited and insufficient, and many students struggled to understand mathematics, particularly logarithms. Even though the research concluded with some learning activities that could be implemented by teachers for improvement of students' understanding of logarithms, I felt it was important to continue my journey, and explore pre-service teachers' knowledge of this content.

As mentioned earlier, some mathematics educators believe that students' learning is influenced by the teachers' knowledge of mathematics and relevant pedagogy. It is also my personal belief that a teacher is a guide to the student's success in mathematics learning. Therefore, I am interested not only in teachers' content knowledge of

logarithms, but also in their knowledge of related pedagogy, and aspects of students' learning of this particular mathematical domain.

1.2 Rationale for the study

Even though many studies in mathematics education research investigated the nature of the mathematics teachers' knowledge required for teaching, and the relationship between different components of that knowledge, very few of them involve pre-service secondary mathematics teachers, and none of them, to my knowledge, focused on the knowledge of logarithms. Therefore, this study is a significant contribution to the body of research on pre-service mathematics teachers' knowledge.

1.3 Purpose of the study and research questions

The primary purpose of this study is to investigate pre-service secondary mathematics teachers' knowledge in the context of logarithms and logarithmic functions. Particularly, it targets the subject matter knowledge and pedagogical content knowledge of pre-service teachers. Concurrent with those efforts, the study also focuses on the development of the research methodology for the purpose of the collection and analysis of data.

The following issues are the cornerstones of the present study. Firstly, the study provides an account of pre-service secondary school teachers' subject matter knowledge and pedagogical content knowledge of logarithms and logarithmic functions. Secondly, it explores how and why pre-service teachers envision applying their subject matter and pedagogical content knowledge of logarithms and logarithmic functions in designed

simulated activities. Finally, it describes the relationships between pre-service secondary school mathematics teachers' subject matter and pedagogical content knowledge.

In summary, there are research questions addressed in this study:

- a) What do the designed tasks reveal about the nature of teachers' knowledge?
- b) What can be seen as the relationship between pre-service secondary mathematics teachers' subject matter knowledge and pedagogical content knowledge?
- c) To what extent are these tasks effective and useful as data collection tools for research in mathematics education?

1.4 Thesis organization

Chapter 2 reviews the available literature from the existing body of research in mathematics education, which focuses on the notion of knowledge in mathematics. It then continues with a discussion of teachers' knowledge. Thirdly, it describes major research findings with regards to knowledge acquisition and understanding of the concepts of logarithms and logarithmic functions. The chapter concludes with a genesis of the logarithms and logarithmic functions.

In Chapter 3, a detailed explanation of the research methodologies utilized in this study is presented. The reader is also provided with the rare opportunity to learn of the research ideas and issues that guided the design of the Job Interview and the Math Play

tasks, followed by the rationale for considering each task as an appropriate instrument for the study.

The main focus of Chapter 4 is the setting of the study. This chapter describes the course, the methodology used for gathering data, and the participants in this research. Here, the reader will also find an extensive explanation of the purpose of each task, including a description of every particular source of information collected for the analysis.

Chapter 5 is the largest chapter of my dissertation. It is designated to present the results and analysis of participants' responses. It consists of three parts. The first part focuses on the analysis of data gathered from the Job Interview task. The research data is analyzed from using several tools: interview questions, questioning techniques, subject matter knowledge and pedagogical content knowledge. Pre-service teachers' conceptions of the significance of teaching logarithms and logarithmic functions are also discussed. The second part presents the analysis of pre-service teachers' knowledge from the Math Plays. Throughout this chapter, the connections between subject matter knowledge and pedagogical content knowledge exhibited by the pre-service teachers are established. Furthermore, the reader is presented with a detailed analysis, highlighting what the data has revealed about the subject matter and pedagogical content knowledge of the participants. At last, the chapter concludes with a discussion on what pre-service teachers identify as important to know for teaching, as well as how this reflects, or does not reflect, in their experiences.

Finally, chapter 6 summarizes the findings and the major outcomes of this research. The chapter presents the implications and contributions of this research study.

Firstly, the study provides a better understanding of pre-service teachers' knowledge of specific mathematical content such as logarithms and logarithmic functions. Secondly, it introduces a new methodology for investigating participants' knowledge of mathematics. As a by-product, the study also presents pedagogical tools for engaging pre-service or in-service secondary mathematics teachers in professional development. Lastly, the reader is introduced to the limitations of this study. The chapter finishes by commenting on the possible directions of future research.

Chapter 2: Theoretical Perspectives

This chapter begins with a presentation of the available literature from the existing body of research in mathematics education, which focuses on the notion of knowledge in mathematics. It then continues with a discussion on teachers' knowledge. Thirdly, it presents major research findings with regards to knowledge acquisition and understanding of the concepts of logarithms and logarithmic functions. The chapter concludes with a genesis of the logarithms and the logarithmic functions.

2.1 Frameworks for knowledge in mathematics

The notions of knowledge and understanding are multidimensional. In mathematics education literature, several forms and kinds of knowledge are described; as well, different kinds of understanding are used to characterize knowledge in mathematics. Among those are instrumental, relational, and logical (Skemp, 1978); procedural and conceptual (Hiebert & Lefevre, 1986); intuitive, algorithmic, and formal (Fischbein, 1993); and many other identified forms. It is important to note that aforementioned models of knowledge are not completely distinct. Frequently, they are used to portray analogous themes (Even & Tirosh, 2002).

2.1.1 Procedural and conceptual knowledge

Hiebert and Lefevre (1986) have offered a useful conceptualization of subject matter knowledge in mathematics through their description of two categories, which they refer to as procedural knowledge and conceptual knowledge. Procedural knowledge means knowledge of procedures - procedures that relate to the usage of mathematical symbols and procedures that describe how to complete mathematical tasks. The presence

of procedural knowledge does not always entail any knowledge of the meaning of symbols or procedures at hand. This knowledge of meaning would only exist if one has acquired conceptual knowledge. Conceptual knowledge consists of the knowledge of how underlying concepts support and connect the previously described procedures. Hiebert and Lefevre (1986) characterize this form of knowledge as a cohesive network. “In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information” (Hiebert & Lefevre, 1986, p.4).

2.1.2. Instrumental, relational, and logical understanding

Skemp (1978, 1987) described subject matter knowledge using three levels of understanding. Instrumental understanding is characterized by the ability to remember rules and procedures in order to derive the solution. It necessitates a memorization of a new algorithm or procedure for each new idea or task, even if it does not differ much from those encountered previously. According to Skemp, instrumental understanding entails “rules without reasons” (Skemp, 1978, p.9)

Relational understanding, like conceptual knowledge, is distinguished by the presence of general conceptual structures or relationships from which specific rules and procedures can be deduced. Relational understanding is described as knowing what to do and why. Having this kind of understanding eliminates the need for memorization of lengthy procedures and tedious algorithms.

Finally, Skemp’s model offers a notion of logical understanding, which is a type of conceptual knowledge but of a higher order. It is “closely related to the difference between being convinced oneself, for which relational understanding is sufficient, and

being able to convince other people” (Skemp, 1987, p.170). This type of understanding is characterized by the ability to demonstrate in some logical fashion that a mathematical idea is understood in a relational manner.

2.1.3. Intuitive, algorithmic, and formal

Fischbein’s theory of mathematical knowledge is guided by the principle that the mathematical knowledge of learners is embedded in a set of connections amongst algorithmic, formal, and intuitive dimensions of knowledge.

According to Fischbein (1999, 1993) any mathematical activity requires the use of these three dimensions. Fischbein believes that “...for the teaching of mathematics, it is very important that the teacher understands the interaction between the intuitive, the formal and the procedural aspects in the processes of understanding” (Fischbein, 1999, p.28).

The algorithmic dimension consists of knowledge of formal rules and procedures with regard to a particular domain of mathematics. “This dimension includes students’ capability to explain the successive steps included in various standard procedural operations” (Tirosh et al, 1998). Algorithmic dimension is procedural in nature.

The formal dimension of knowledge is expressed in the logical, deductive structure of mathematics, in the forms of axioms, definitions, theorems, and proofs. This dimension of knowledge is represented by definitions of the mathematical concepts, operations, theorems relevant to the specific subject matter knowledge and their proofs.

The intuitive dimension “refers to the degree of subjective, direct interpretation of mathematical concepts or statements” (Fischbein, 1999, p.32). It contains the ideas and beliefs one has about mathematics. It includes the mental constructs one uses to represent

mathematical concepts. Intuitive knowledge is described as a “direct, self-evident kind of knowledge” (Fischbein, 1999, p.29). It is a type of knowledge one tends to accept directly and confidently as being obvious, with a feeling that it needs no proof. This form of knowledge is the most powerful, because a person, who develops it, is inclined to abolish alternative representations, interpretations, or solutions.

2.2 Theoretical Background: teachers’ knowledge for teaching

That teachers of mathematics, or any subject for that matter, might require additional or special knowledge of their subject is not a new idea. In the past, John Dewey had made the following observation:

Every study or subject thus has two aspects: one for the scientist as a scientist; the other for the teacher as a teacher. These two aspects are in no sense opposed or contradicting. But neither are they immediately identical (Dewey, 1902, p.29)

What do teachers need to know?... It is clear to us that teachers need more than a personal understanding of the subject matter they are expected to teach. They must also possess a specialized understanding of subject matter, one that permits them to foster understanding in most of their students (Wilson, Shulman, Richert, 1987, p.104)

In recent years, the work of Shulman has played a major role in refocusing attention on a specialized knowledge of the subject matter unique to the needs of teaching that subject matter. He names specialized knowledge as pedagogical content knowledge. It is a “special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding” (Shulman, 1987, p.8). Though the pendulum of research on knowledge for teaching has swung from one end to the other within the century, it seems that at present, Shulman’s work on the importance

of subject matter knowledge has led the way to a reformation of teachers' education in order to enhance the quality of teaching and learning in schools.

2.2.1. Subject matter knowledge, pedagogical content knowledge, and curricular knowledge.

Lee Shulman (1987) presents his argument regarding the content, character, and sources, for a knowledge base of teaching. He identifies seven categories of knowledge comprising the 'knowledge base' for teachers. As to the dynamics of their developments, they can be grouped into two following subcategories: generic knowledge and content knowledge. In his previously published work, Shulman (1986) identifies three kinds of content knowledge: (a) subject matter knowledge, (b) pedagogical content knowledge and (c) curriculum knowledge.

Shulman refers to the *subject matter knowledge* (SMK) of teachers as the variety of ways in which the basic concepts and principles of the discipline are organized and used to establish validity or invalidity, of which truth or falsehood are held within the subject. Shulman elaborates:

Teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and practice. (Shulman, 1986, p.9)

Another kind of teachers' knowledge that "goes beyond knowledge of subject matter" is *pedagogical content knowledge* (PCK), which embodies the aspects of content most relevant to its teachability. At the heart of pedagogical content knowledge lies the transformation of teacher's subject matter knowledge into a form that supports their

students' attempts to gain an understanding of that content. Pedagogical content knowledge involves the blending of content and pedagogy in order to produce,

for the most regularly taught topics in one's subject area, the most useful forms of representations of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations..., the ways of representing and formulating the subject that make it comprehensible to others. (Shulman, 1986, p.9)

Pedagogical content knowledge also involves insights into the difficulties and misconceptions that students might have with knowledge acquisition of a particular content along with the capacity to call on useful strategies for guiding students through, and out of, these difficulties and misconceptions.

Curricular knowledge (CK) is the third kind of the content knowledge. It underlines the teacher's ability to situate the content of a specific topic not only within the discipline areas, but also within other subjects; and, includes the teacher's "understanding about the curricular alternatives available for instructions", such as texts, software, visual materials, etc. (Shulman, 1986, p.9)

To be effective at teaching, the teacher should first and foremost comprehend the subject matter knowledge with a degree of flexibility and adaptability that enables the teacher to transform that knowledge into "forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented to the students" (Shulman, 1987, p.15). Pre-service teachers have to develop the aforementioned competencies as well. However, the development of such expertise is difficult and is not a straightforward event. Pre-service teachers' development "from students to teachers, from the state of expertise as learners through novitiate as teachers, exposes and highlights the complex

bodies of knowledge and skills needed to function effectively as a teacher” (Shulman, 1987, p.4).

Pre-service teachers need to understand the content they want to teach. But they also need to understand how to unpack and present the content so that students can learn with understanding (Kilpatrick, et al., 2001). Teachers’ knowledge is dynamic, and is dialecticized by content knowledge, knowledge of pedagogy, and knowledge of students’ cognition.

The implication of teachers’ content knowledge on students’ mathematical achievements is that the more connected and broad the content knowledge of teacher, the richer the learning environment facilitated by the teacher can be. “The important factor in a positive relationship between content knowledge and classroom instruction appears to be mental organization of the knowledge that the teacher possesses” (Fennema & Franke, 1992, p.153). Expert teachers have a better-established relationship between mathematical content and pedagogy. Therefore, teachers’ content knowledge affects the teachers’ actions, and thus impacts students’ opportunity to learn. Opportunity to learn is believed to be the most important variable in students’ success in mathematics (Kilpatrick, et al., 2001; Ball, et al. 2005). The following statement from Fennema and Franke (1992) provides a useful perspective with respect to an investigation of teachers’ knowledge:

The transformation in action is understandably complex. Little research is available that explains the relationship between components of knowledge as new knowledge develops in teaching nor is information available regarding the parameters of knowledge being transformed through teacher implementation. Here all aspects of teacher knowledge and beliefs come together and all must be considered to understand the whole. The challenge is to develop methodologies and systematic studies that will provide information to enlighten our thinking in this area. The

future lies in understanding the dynamic interaction between components of teacher knowledge and beliefs, the role they play, and how the roles differ as teachers differ in the knowledge and beliefs they possess. (p.163).

2.2.2. Mathematics Teachers' Knowledge Base

Building on Shulman's work described in the previous section of this chapter, Harel (1993) suggested that three interrelated critical components define a teacher's knowledge base: (a) knowledge of mathematical content, (b) knowledge of student epistemology, and (c) knowledge of pedagogy. In this framework, knowledge of mathematical content refers to mathematical knowledge possessed by a teacher, and is "the cornerstone of teaching for it affects both what the teachers teach and how they teach it" (Harel & Lim, 2004). Knowledge of student epistemology consists of a teacher's understanding of fundamental psychological principles of learning, including the knowledge about the student's construction of understanding of new concepts. The knowledge of pedagogy is based on an understanding of how to assess students' knowledge, how to utilize assessments to pose problems, how to stimulate students' learning, and how to help students to firmly implant the knowledge they have constructed.

This framework was used by the researchers to model Bud's (a practicing secondary mathematics teacher) knowledge base, his teaching personality and rationale for his teaching actions. The findings of this research indicated that the components of the teacher's knowledge base were inseparable from each other. "One's ways of understanding and ways of thinking of mathematical concepts seem to dictate the nature of other components of knowledge" (Harel & Lim, 2004, p.31). Researchers suggested

that “integrated curricula, where the three components of knowledge base are addressed in a synergetic manner,” (Harel & Lim, 2004, p.31) could help teachers to tackle challenging mathematical content, reflect on their own learning, reflect on students’ learning, and subsequently appreciate epistemological and pedagogical issues.

2.2.3. Teachers’ Learning

Teachers’ knowledge and beliefs determine their decision making when planning, performing or reflecting. Teachers’ knowledge and beliefs have very complex structures. Leikin (2006) suggested using a 3D model of teachers’ knowledge, which described its complexity. The three main axes represent: kinds of teachers’ knowledge, sources of teachers’ knowledge, and forms and conditions of knowledge.

Kinds of teachers’ knowledge were founded upon Shulman’s components of knowledge discussed earlier. Sources of teachers’ knowledge were rooted in Kennedy’s (2002) classification based on the practical classroom experiences of teachers. This dimension consisted of craft knowledge (mainly associated with orderly task flow and students’ willingness to participate in classroom activities), systematic knowledge (acquired through systematic studies of mathematics and pedagogy in universities, through reading research literature), prescriptive knowledge (knowledge of institutional policies, accountability systems, and texts of diverse nature), and finally, forms and conditions of knowledge, comprised of findings of several studies, such as teachers’ intuitive knowledge (impossible to premeditate teachers’ actions) and formal knowledge (planned actions), and teachers’ beliefs (expressed in teachers’ conceptions of teaching).

The three dimensional model of teachers’ knowledge was utilized to outline the relationships between the different dimensions in the process of the development of

knowledge through teaching. Results of this research indicated that mathematical intuition was transformed into formal mathematical knowledge, whereas pedagogical intuitions were transformed into beliefs, which was treated as evidence that teachers who participated in this research learned through their teaching.

2.3 On Logarithms and Logarithmic Functions in Mathematics Education

In the past decades, the understanding of how students acquire mathematical concepts has increased immeasurably. In this section, the reader is presented with an overview of the literature search on a knowledge acquisition of concepts of logarithms and logarithmic functions. The mathematics education research related to the topic of logarithms and logarithmic functions elicits two main themes: students' understanding of these mathematical concepts and the validity of the experimental instructional designs for the teaching of logarithms.

2.3.1. Revisiting some findings from using “frameworks of knowledge”

As is shown in the research literature, the notion of knowledge is multidimensional. Several frameworks are used to characterize knowledge. Some of these forms are not completely distinct; they could be used to illustrate similar ideas.

As an example of a procedural knowledge of logarithms, one might include the knowledge of how to simplify $\log 5 + \log 2$. To illustrate, the conceptual knowledge of logarithms could be exhibited when explaining how the product law of logarithms warrants the proper simplification of the same expression $\log 5 + \log 2$.

Skemp's model allows for the distinguishing of three levels in the understanding of logarithms. The instrumental understanding, characterized by memorization of rules, would permit the completion of the following task: present as a single logarithm $\log_2 + \log_3 - \log_5$. The conceptual understanding might be exhibited when completing and explaining how the final result was obtained. These two types of understanding resonate the findings offered by the conceptual/procedural framework. However, the third, most sophisticated level of understanding, called logical, would be characterized by the ability to provide several different logarithmic representations of the number 3. Such as $3 = \log_2 8 = \log_4 64 = \log_3 9 + \log_7 7$.

Fischbein's theory allows differentiating three dimensions of mathematical knowledge: algorithmic, formal, and intuitive. Fischbein's theory requires the learner to use all three dimensions for any mathematical activity. The algorithmic dimension is procedural in nature. For example, a student's ability to justify the successive steps in simplifying $\log(0.1) = \log 10^{-1} = -\log 10 = -1$ would reveal the presence of this particular dimension. If the same task is solved by using the definition of the logarithm, let's say the student formulated and answered the question: to what exponent should I raise 10 to get 0.1, then it can be viewed as an evidence of the formal knowledge. The degree of subjectivity and direct interpretation of logarithms would be measured by the presence of the intuitive dimension of knowledge. The student who operates in this dimension might answer the question about different logarithmic representations as "There are infinitely many representations possible, and any real number can be presented in the logarithmic form."

The frameworks offered by the research on knowledge allow differentiation and better interpretation of the various types of knowledge possessed by the participants in the study.

2.3.2. On students' understanding of Logarithmic Function notation

The main aim of the study reported by Kenney (2005) was to investigate students' understanding of logarithms through the lens of their interpretations of logarithmic notation. The participants of this study were college students enrolled in pre-calculus courses. The framework used to conduct the research is the procept theory of Gray and Tall (1994). This theory proposes that the usage of mathematical symbols enables students to consider mathematical concepts to be both a process and an object at the same time. It also endows learners with the ability to deal with this dual nature of function notation. Kenney believed that "understanding of logarithmic functions relies on being able to interpret the notion and symbols involved" (Kenney, 2005, p.2)

The results of this study indicated that the students did not, in general, have a proceptual understanding of logarithms. The major reason for this, as identified by Kenney, was that students could not make "meaningful connections to name logarithm and the notation used to present it" (Kenney, 2005, p.7). She also expressed the belief that for proceptual thinking to exist, the concepts of logarithms and exponents should be as closely linked as addition and subtraction. However no instructional advice of how it could be possibly done was presented.

2.3.3. On students' understanding of logarithms

The study conducted by Berezovski & Zazkis (2004, 2006) addressed not only the understanding of logarithms, but also common difficulties which high school students encounter as they study this topic. The research tools focused on different tasks: some standard and others non-standard, which involve logarithmic expressions or require the use of logarithms in a solution or explanation. The study systemized the knowledge about this content domain into the following three areas of specialized subject matter. To assess whether someone understands logarithms, means to validate their knowledge of:

- *Logarithms and Logarithmic Expressions as Numbers*
- *Operational Meaning of Logarithms*
- *Logarithms as Functions.*

The aforementioned interpretive system was used to model the high school students' understanding of logarithms. The results of the study indicated students' disposition towards a procedural approach and reliance on rules, rather than on the meaning of concepts. It presented a description of students' difficulties with logarithms, and also suggested possible explanations of the sources of these difficulties as they can be attributed to the conceptual-epistemological obstacle.

2.3.4. Searching for effective instructional designs

The problem with students' learning and understanding of logarithms became evident not only in the regular classrooms; the mathematics education community had also identified it. Some mathematics educators have gone beyond the search into cognitive developments, to the point where they have actually proposed instructional techniques to supplement or replace traditional pedagogy of logarithms (Confrey &

Smith, 1995, Berezovski, 2004). However, the effectiveness of these techniques has not been assessed.

An instructional design to teach students exponents and logarithms was proposed by Weber (2002). It was influenced by the learning theories of action/process (Dubinsky, 1991, Tall & Dubinsky, 1991) and those of operational/structural understanding (Sfard, 1991, 2000). The researcher postulated a set of mental constructs that students could make to understand exponents and logarithms. The instructional tools were designed to lead students to make these constructions. Two learning activities were implemented: MAPLE based program script writing, and paper-and-pencil worksheets.

The results of this study were encouraging, “students who completed the designed instructional activities outperformed students who received traditional instruction” (Weber, 2002, p.1026). Weber adds that the most significant achievement of the experimental learning was that “students can use their deep understanding of these topics [exponents and logarithms] to reconstruct forgotten symbolic knowledge” (Weber, 2002, p.1026).

2.4 Genesis of Logarithms and Logarithmic Functions

The development of a concept by an individual does not necessarily follow the same path as the historical development. However, there is much to be gained from the knowledge of the historical development of a mathematical concept. In particular, in the study of subject matter knowledge, an understanding of historical development of mathematical ideas provides us with another perspective on learners' activities. Commonalities that occur in the way a pre-service teachers' knowledge of a mathematical concept develops, and the way it developed historically are, according to Siepinska

(1994) citing Piaget and Garcia (1989) and Skarga (1989), attributable to commonalities in mechanisms of development and to preservation of the historical meaning of terminology.

Originally, nobody was thinking of logarithms as functions or even as exponents. Unlike the modern way of introducing this phenomenon at the high school level, the development of logarithms took an independent path from the concept of exponents. In the following section the historical genesis of logarithms and logarithmic functions is discussed. Special emphasis is placed on the mathematical knowledge that contributed to epistemology of logarithms and logarithmic function, and influenced their contemporary definition.

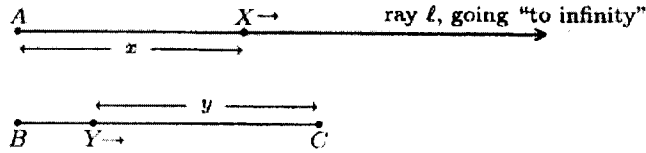
The power and importance of the logarithm lie in fact that it converts a product into a sum and a multiplication problem into an addition problem; and similarly, a quotient into a difference and a division problem into a subtraction problem. This obviously has computational significance. It was known during John Napier's time how a problem of multiplication could be changed into one of addition. For example, the following formulae convert products into sums:

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$
$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$
$$\sin x \cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$$

Figure 1: Multiplication to Addition Formulas

It seems likely that such facts would have helped Napier hit upon the idea of logarithms (Shirali, 2002). The following geometric model was used by Napier to juxtapose

arithmetically changing distance x , and geometrically changing distance y , and to interpret the nature of this relationship through logarithms.



Genesis of the Napierian logarithm

Figure 2: Napierian Logarithm

Consider a ray l , with A as the endpoint, and a line segment BC of unit length. Let X and Y start from A and B , and move along l and BC respectively, starting with the same initial speed; let X move at a constant speed, and let Y move at a speed proportional to the distance YC . This means that the speed of Y decreases steadily as it approaches C , and it takes infinitely long to reach its destination. Let x represent the distance AX , and y represent YC , then the relationship between x and y is a "Napierian Logarithm":

$$x = \text{Naplog } y$$

When x is 0, y is 1, and as x increases to infinity, y is decreases to 0. Napier coined the word *logarithm* from the Greek words *logos*, meaning *ratio*, and *arithmos*, meaning *number*. His original definition differs from our present interpretation of a logarithm. The relationship of the Napierian Logarithmic function to one of the functions we know today is: $\text{Naplog } y = -\ln y$.

Napier publicized his invention in 1614, in a book titled *A Description of the Wonderful Law of Logarithms*. (Shirali, 2002, p.36) This contained a table of Napierian logarithms. The significance of the new invention was quickly seen. In 1543, Copernicus

published his theory of the solar system. To proceed in his work, he needed to perform thousands of complicated and lengthy calculations. A similar difficulty faced Johannes Kepler (1571-1630), who completed an enormous number of arithmetical computations to obtain his famous laws of planetary motions. Napier's logarithms helped to solve this problem. The discovery of logarithms opened a whole era of discoveries in astronomy.

The miraculous powers of modern calculation are due to three inventions: the Arabic notation, Decimal Fractions and Logarithms. The invention of logarithms in the first quarter of the seventeenth century was admirably timed, for Kepler was then examining planetary orbits, and Galileo had just turned the telescope to the stars. During the Renaissance German mathematicians had constructed trigonometric tables with great accuracy, but its greater precision enormously increased the work of the calculator. It is no exaggeration to say that the invention of logarithms "by shortening the labors doubled the life of the astronomer".

Logarithms were invented by John Napier (1550-1617), Baron of Merchiston, in Scotland. It is one of the great curiosities of the history of science that Napier constructed logarithms before exponents were used. To be sure, Stifel and Stevin had made attempts to denote powers by indices, but this notation was not generally known - not even to T. Harriot, whose ALGEBRA appeared long after Napier's death. That logarithms flow naturally from the exponential symbol was not observed until much later (Cajori, 1919, cited in Shirali, 2002, p.37).

Even though, Napier's discovery was important to the development of mathematics and science, it is necessary to mention that it was John Briggs (1561-1631) who proposed to Napier, when they met in Scotland, that the definition of logarithm be modified. They agreed that the logarithm of 1 would be 0, and the logarithm of 10 would be 1. Thus was born the common logarithm (Boyer, 1968). After Napier's death, Briggs completed *Logarithmorum Chilias Prima* and *Arithmetica Logarithmica*, which contain the essence of the first table of common logarithms. Nevertheless, their views on logarithms were far from the contemporary one.

Strong evidence exists that exponents have long history in mathematics. Operations with exponents were apparent in works by al-Samawal's in mid-1100s, used by Nicole Oresme in the late 1300s. In 1484, Nicholas Chuquet made an explicit connection between multiplying terms and adding exponents (Smith, 2000). Among many mathematicians who worked with exponents was Michael Stifel. In 1544 he wrote the *Arithmetica Integra*, where he called powers *exponents*, and described the relationship between arithmetic and geometric sequences, discussing the process "when you multiply the numbers, you add the exponents."

By the middle of the seventeenth century A.A. de Sarasa made the connection between the natural logarithm and the number x that could be interpreted as $\int_1^x \frac{1}{t} dt$. The Danish mathematician Nicolaus Mercator discovered the formula for the expansion of natural logarithm into series. And with the invention of calculus by the end of the seventeenth century, Isaac Newton and Gottfried Leibniz both formalized the important relationship between logarithms and the area bounded by the x -axis, the vertical lines $x=1$ and $x=a$, and the hyperbolic curve $y = 1/x$. Logarithm came to play an important role in the development of new mathematics of calculus and computational sciences.

By the late 1700s, Leonard Euler formulated the relationship between exponential functions and logarithmic functions, when he defined $\log_a b = c$ to be true if and only if $a^c = b$; therefore $y=a^x$ was equivalent to the statement $x = \log_a y$. This definition was a long way from Napier's vision of a logarithm, which involved the distances traveled by two moving points.

Until quite recently, the students who study mathematics and sciences were given significant exposure to the computational methods involving logarithms. The major

mechanical device used to simplify computations was a slide rule. With the invention of the mechanical calculators the computational power of logarithms started to diminish. However, the meaning of logarithms secured its significance for many centuries to come. Logarithms are necessary to solve implicit differential equations that are used in mathematical modeling. There are many areas in science, sociology, economics, etc., that require knowledge of logarithms. Among such are compound interest, exponential growth and decay, pH, depreciation, measurement of the magnitude of volume, of earthquakes, and of sound.

With the development of new branches of mathematics such as fractal geometry, Benoit Mandelbrot, a 20th-century mathematician, needed to use the knowledge of logarithms. Algorithms involving logarithms proved to be efficient when working with fractional dimensions for fractals.

An overview of the historical development of the logarithms, allows one to identify the several meanings (logarithms as exponents, as areas, as series, or as inverse function) that one can possess about the particular concept. Consequently this brings up the possibility that one might teach the concept according to his or her knowledge of what it is.

2.5 Conclusion

The review of the literature revealed that very little has been undertaken in investigating the pre-service teachers' knowledge of logarithms and logarithmic functions. The question that was not addressed in the relevant literature is: how does teachers' knowledge inform their teaching practices, specifically at the high school level?

Based on the review of the literature on knowledge and on teachers' knowledge, when considering the specific case of teaching mathematics, there are three major areas where the research on pre-service secondary mathematics teachers' knowledge focuses: subject matter knowledge, pedagogical content knowledge, and the understanding of students' knowledge acquisition.

The next chapter focuses on the design and considerations of the research tasks for this study. Specifically, it presents a detailed explanation of relevant findings in the literature and my personal impetus.

Chapter 3: Designing and Considering the Research Tasks

“...research methodology is not merely a matter of choosing methods and research design, ... methodology is about the underlying basis for the choices that are being made...”
(Goodchild & English, 2003, p.xii)

In contemporary mathematics education, one encounters different ideas, methodologies, and various approaches to investigate research questions. For example, clinical interviews and questionnaires are the most commonly used instruments for collecting data. Some others are: journaling (Liljedahl, [in press]; Flückiger, 2005), error activities (Borasi, 1996), technology based tasks (Dubinsky, 1991; Weber, 2002), and example generation tasks (Bogomonly, 2006; Rowland, Thwaites & Huckstep 2003; Zazkis & Laikin, 2007). A detailed account of a variety of research methods can be found in Goodchild & English (2003). The research in mathematics education confirms that different methodologies and approaches allow for the creation of situations that enable researchers to collect more diverse data.

In this chapter, I introduce the research tasks used in this study, and present the reader with the discussions of the research ideas from an existing body of educational research that proved to be valuable in designing, understanding, and analyzing these research tasks.

3.1 Introduction of the Tasks

Both tasks used in this study represent simulations of teachers' life situations. I designed the first task, while the idea behind the second task belongs to Dr. Zazkis and

Dr. Liljedahl. However, I carried sole responsibility for its detailed descriptions, interpretations, and implementations.

In what follows I present the tasks and exemplify how they could be utilized.

Task 1: *the Job Interview*

Pre-service teachers were invited to assume the roles of personalities in a fictional mathematical interaction between the head of a mathematics department and an applicant for the position of substitute teacher, who would cover for a 3 week leave. Topics for teaching included logarithms and logarithmic functions.

The interviewer had to do his/her best to verify and evaluate the candidate's knowledge and understanding of the mathematical content required to be covered. The interviewer's questions were to reveal the main essence of the topic. The candidate had to do his/her best to answer the questions and to demonstrate his/her competence.

Data sources associated with this task comprised the following:

- a. Interviewer's rationale for the choice of the interview questions;
- b. Interviewer's anticipated answers for his/her interview questions;
- c. Transcript of the interview;
- d. Interviewer's evaluation of the candidate.

The purpose of each data source is elaborated upon in 4.2.

Task 2: *the Math Play*

Pre-service teachers were provided with an episode of a fictional mathematical interaction between a student and a teacher that presented a problematic situation in

which the student had developed a misunderstanding about logarithms. This situation was presented as Act 1, Scene 2.

Act 1, Scene 2

There is a conversation between a teacher and a student (there are 30 students in a class):

T: Why do you say that $\log_3 7$ is less than $\log_5 7$?

S: Because 3 is less than 5.

Pre-service teachers were asked to diagnose the fictional student's misunderstanding, formulate a plan for the remediation of the misunderstanding, and write out the balance of the interaction(s) in the form of a *math play*. They also were asked to provide their version of Act 1, Scene 1, as their personal view of what could lead to the presented situation, and where and how the problematic situation could take place. The remediation could be presented in the form of a play, or Act 2, Scene 1, and pre-service teachers were to create a teachable moment, whereby they would orchestrate the events, tasks, and conversations, to lead the student out of the problematic situation.

Data sources associated with this task consist of the following writings:

- a. Pre-service teachers' diagnosis(es) of the student's misconception;
- b. Their personal view on where and how the given situation could take place, in a form Act 1, Scene 1;
- c. The remediation part, where the pre-service teachers had to organize a situation to guide a student's learning, in the form of Act 2, Scene 1.

Similarly, the purpose of each data source is explained in Section 4.2.

3.2 Theoretical Considerations which Contributed to the Design of the Research Tasks

... in the majority of articles in journals and chapter books, a description is provided of “how” the research was done but rarely is an analysis given of “why” and, more particularly, out of all the methods that could have been used, what influenced the researcher to choose to do the research in the manner described. It comes as a refreshing change when one reads an author’s reflections on what impact such choices might have had on the research outcomes. (Burton, L. 2005, p.1)

In the following section, I present the discussions of several research studies that successfully implemented non-traditional instructional and research methodologies in various educational settings. Each study is discussed separately, highlighting the specific contribution it had on the design of the research tasks for this study.

3.2.1 Learning from Students’ Mathematical Errors

Throughout the years, the positive role of errors in the process of learning has been acknowledged. From my early childhood, I remember the encouraging words of the popular motto: “You learn from your mistakes”. Practice shows that errors as a source of learning have become recognized in certain areas of mathematics education research. Students’ mistakes receive constant attention from educators. In many studies they are collected, classified, and explored in terms of their roots, etc. Focusing on learners’ mistakes resulted in the intense assessment of students’ achievement, and became heavily dependent on the number of right or wrong solutions made on high stake examinations. Other errors that have attracted mathematics educators’ interest are those encountered in the epistemological development of mathematical concepts. The researchers who work in

this area try to investigate how the development of the entire mathematical enterprise became possible ‘by fixing the errors’ that prompted many to search for alternative ways of dealing with problems. The third and the most recent area of attention to errors recognized in the studies, focused on the integration of errors into teaching practices for the purpose of creating inquiry learning activities. This particular area of the research is important to my study; therefore, it requires a more detailed elaboration.

One of the most comprehensive works on this topic is done by Raffaella Borasi, and is presented in *Reconceiving Mathematics Instruction: A focus on Errors* (1996). Her entire study centered on mathematical errors. She believes that they play an important role in learning and teaching mathematics.

... an explicit focus on errors could be beneficial to mathematics students as it could contribute to developing some of the metacognitive skills identified as necessary to become an independent and efficient problem solver. First of all, such a focus could enable students to become familiar with specific strategies to critically review and check their mathematical work, at the same time developing the expectations that identifying and correcting errors is mainly the learners’ responsibility rather than teachers...(Borasi, 1996, p.32)

Her beliefs were based on a detailed analysis of several teaching experiments involving *error activities*. According to Borasi, *error activities* are “instructional activities designed so as to capitalize on the potential of ‘errors’ to initiate and support inquiry” (1996, p.30). Taking the constructivist approach of learner-based inquiry, she uses errors as an opportunity “...to generate doubt and questions that, in turn, can lead to valuable explorations and learning” (Borasi, 1996, p.285). Borasi identifies several types of error that can be successfully employed in *error activities*. The great majority of errors used in the case studies were students’ mathematical mistakes. These errors helped to generate

conflicts that, in turn, exposed and challenged the students' limited knowledge about mathematics. Borasi identifies three kinds of error activities: planned (previously selected and introduced by teacher), expected (made spontaneously by a student but expected by the teacher), and unexpected (made unexpectedly for the teacher by a student and pursued on the spot).

Also, this study shows that various error activities offer diverse learning opportunities. Among such opportunities are the prospects of experiencing constructive doubt and conflict regarding mathematical issues, pursuing mathematical exploration, engagements in challenging mathematical problem solving, experiencing the need for justification of the mathematical work, and taking an initiative and an ownership in the learning of mathematics. (Borasi, 1996).

The most important objective of Borasi's study is to regard mathematics instruction as supporting students' own inquiries, by "using errors as springboards for inquiry" that contributes to students' mathematical learning and growth in more than one way (Borasi, 1996, p.143). In this research, *error activities* are identified as alternatives to traditional methods of mathematics instruction. The inquiry approach focusing on using errors is introduced for the benefit of learners, including students taking mathematics courses and mathematics instructors, their understanding of mathematical content, and the nature of mathematics as a human endeavor. In that which follows, I discuss how Borasi's ideas contribute to the design of my research activities: how they can be extended to the learners of teaching of mathematics, and how they can be employed for the greater benefit of mathematics education researchers.

For instance, Borasi concludes that error activities are the type of instructional activities that would initiate and support learner-based inquiry, thereby exposing and challenging the students' limited mathematical knowledge. Even though error activities offer a certain freedom to the learner in terms of exploring the possibilities of fixing mathematical mistakes (which is beneficial for the students enrolled in a mathematical content course), the greater space for imagination is necessary when moving beyond correcting. To make imaginary students' cognition tangible to the pre-service teachers, the exploration into *when*, *why* and *how* such errors could occur should take place. In the task called the Math Play, pre-service teachers have to deal with a misconception developed by an imagined student. This misconception is presented as a student's erroneous answer to a posed mathematical problem. The pre-service teachers were asked to analyze the possible sources of this misconception, and accordingly situate them in the sequence of learning events, such as a lesson plan which contained a pre-existing dialogue between an imaginary student and a teacher. In addition to the diagnostic stage, pre-service teachers had to create a remediation part. The use of the erroneous examples in which pre-service teachers first considered student's conceptions and developed explanations, responses and remediation, promises to be a valuable activity for future teachers. For the researcher, this activity reveals both types of teacher's knowledge: subject matter and pedagogical. For more detailed explanation refer to Section 4.2.

3.2.2 Role-playing in Education

The multidisciplinary studies of aspects of the real world in the physical and social sciences over the past century has lead to the articulation of important new conceptual perspectives and methodologies that are of value to both researchers and

professionals in these fields. The simulation of real life problems has become one of the popular teaching methodologies in many subject areas. There are several studies that report on the effectiveness and importance of simulation activities in language education, science education, and in education in general (Blatner, 1995, 2002). For the purpose of this study, the following discussion will be focused on findings that view role-playing as a less technologically elaborate form of simulations activities, where participants personify somebody else for a particular reason.

In Webster's Dictionary, *Role-playing* has two definitions. First, it is defined as 'modification of one's behavior to accord with desired personal image, as to impress others or conform to particular environment'. And secondly, it is a 'method of psychotherapy aimed at changing attitudes and behavior, in which participants act out designated roles relevant to real life situation' (Webster, 1997, p.683)

The first definition resonates closely with the art of acting, when one can create a role, and improvise a performance. In fact, children do this all the time in their pretend play. The second meaning of role-playing is broadly utilized in education, because, 'like any good inquiry approach, role-playing transforms the content from information into experience' (Blatner, 2002). From a diagnostic perspective, the role-playing could expose how a person would act when placed in an imagined or pretend problematic situation. According to Blatner (1995), role-playing means learning, not just information, but, more notably, 'skills of interpersonal problem solving, communications and self-awareness.' Most significantly, he believes that it can be used for the training of professionals or in a classroom for the understanding of literature, history, and even science (Blatner, 2002). In his research on drama education, Blatner noted that when students exercise the

component skills of role playing, they are not only learning about the roles they're portraying; but of some greater importance, they are learning how to play with roles, and how to think a little like a playwright.

From the psychoanalytic point of view, role-playing generates a postmodern type of thinking. It involves interaction rather than position, and the shifting among several opinions rather than a reliance only on a one-person perspective. Role-playing represents the best way to learn how to understand what it's like to be in the other person's situation; thus, it is an effective method for developing the skills of understanding (Blatner, 2002). Role-playing develops a capacity for metacognition, the ability to think about the ways one thinks (Weinert & Kluwe, 1987).

Recognizing the importance of role-playing in education, brings to light another issue of creating or discovering real-life situations that are effective and complex enough so that it's not just a matter of cognitively "knowing a right answer" - any hypothesis must then be "sold" to partners or others involved in the play - and then as such must be tried out in order to see if it works. In this sense, role-playing is like a laboratory in which the various techniques of staging and bringing forth feelings and ideas are the equivalent elements to the scientific equipment (Kottler, 1994).

In the aforementioned studies, role-playing was shown to be a successful teaching methodology that can help students understand the more subtle aspects of education. It can help them become more interested and involved, in not only learning about the material, but also in learning to integrate the knowledge into action, by addressing problems, exploring alternatives, and seeking novel and creative solutions. Once again, in the upcoming chapter, I will show that role-playing can be utilized in pre-service teacher

education, and explore how the mathematics education researcher could possibly benefit from its utilization.

According to Blatner (2002), role-playing is also a good inquiry approach. It possesses two distinct properties: it transforms the content from information into experience, and it exposes how the person would act when placed in other person's situation (it could be either imagined or the act of pretending). Blatner claims that role-playing is an effective method for developing the ability to think about the ways one thinks: metacognition. It is also shown that role-playing is a powerful teaching methodology. This methodology helps students to understand the nature of education. Even though the research on role-playing was conducted with drama students, it seems that this approach can provide the pre-service secondary mathematics teachers with an opportunity, which was lacking in the previously mentioned error activity, to experience the understanding of the subject matter from someone else's perspective and position.

The role-playing approach was used in both research tasks for this study. In the first task, the Job Interview, two pre-service teachers were to play roles of the head of the mathematics department and the candidate for a position of a substitute teacher. They were to discuss a particular mathematical content, and the head of the mathematics department had a goal to assess and evaluate to candidate's content knowledge; whereas, the candidate was to convince the interviewer about the expertise he or she possessed. Being placed in the role of the department head, the pre-service teacher had to come up with questions; the answers to those would be the most revealing of the candidate's subject matter proficiency. The interviewer would have to react to the received answers immediately on the spot, and if necessary, adjust their line of inquiry accordingly. This

interview task allows pre-service teachers to explore and consequently expose their creativity and imagination. Their knowledge of the subject matter and pedagogy had to be verbalized and transformed into questions, prompts and activities in which the candidates were to engage.

The Math Play, as the second research task, was designed as an activity in which the pre-service teachers were to play roles of a classroom mathematics teacher and a student, simultaneously. In this setting, the pre-service teachers were to experience the metacognitive aspects that role-playing has to offer. When engaged in the interview task, the interviewer's behavior (ideally) depended on the candidate's response to the posed question; however, in the Math Play, the pre-service teachers would orchestrate the entire interaction for themselves and by themselves. The only constraint that remained would be the particular mathematical content. Though role-playing alone has great metacognitive potentials, a greater effectiveness can be reached when used specifically in combination with other approaches practiced in mathematics education research.

3.2.3 Questioning for Interviewing in Education and Mathematics education

We all ask questions, all the time. However, how we ask questions and how (we) reflect upon answers provided will determine what we say we 'know' and 'believe', will influence our relations with others, the world and our actions (Schostak, 2006, p.8).

In the recent years with the development of qualitative research, interviewing became one of the most frequently used methodologies in education. From one perspective, interviewing students has become a popular method of data-collection. The researcher's analysis of the responses collected in the interview creates the foundation of

many studies in mathematics education. From another perspective, interviewing is used in teaching practice for the purpose of testing.

Ginsburg (1997) identifies the interview method as a powerful technique for gaining insight into a child's thinking. Through a critical analysis, he argues, "traditional research and assessment research and assessment methods involving standardized administration are often inadequate for understanding the complexities and dynamics of the child's mind" (Ginsburg, 1997, p.30). He establishes the advantage interviewing has to offer, and demonstrates the effectiveness of the interviewing.

According to Ginsburg, the clinical interview is a form of social interaction "built around actors' (an adult interviewer, and a child interviewee) ideas", "considers participants' goals", and also "comprises acts". When focusing on the interviewer, the primary concern is the art of questioning (Ginsburg, 1997, p.75). Ginsburg suggests that the beginner interviewers have to be very conscientious when selecting and preparing tasks and questions for an interview. He offers advice on the use of theoretically meaningful tasks (beginning with easy, familiar ones, and following up with the more challenging), engaging to the child (making the tasks specific, not vague or unclear), the usage of various types of problems (variety of problems, variations on the same theme), and being open-ended (giving a child freedom to respond, and finally, the allowing of an expression of personal ways of thinking).

In mathematics education, the topic of questioning is considered to be very important, and receives a great deal of attention. In *Questions and Prompts for Mathematical Thinking* (1998), Anne Watson and John Mason have put together a collection of convincing and challenging questions which are designed to draw out

students' mathematical thinking. Besides a collection of questions, they present a framework for generating a wide range of mathematical questions and prompts which can be used by the teachers (for the purpose of development of their own approaches to mathematical content, and finding out more about their students' undertakings) and by the students (to make sense of mathematics and to question each other and their teachers). They illustrate how learning and teaching situations might be enhanced when using 'good' questions. The authors believe, "questions ...are intended as a source of inspiration and as an aid to change" (Watson & Mason, 1998, p.3). For Watson and Mason, questioning is a social and psychological activity, where a student's experience frames her/his view about the subject. Therefore, the authors intend to explore mathematical questions as "prompts and devices for prompting students to think mathematically, and thus becoming better at learning and doing mathematics" (Watson & Mason, 1998, p.4), to help students communicate mathematics.

An interesting approach to questioning is taken by Zazkis and Hazzan (1999). These authors looked at how researchers in mathematics education choose or design their questions for interviews. Reviewing different studies and reflecting on their own, allowed the authors to identify several types of questions most commonly used: performance questions, unexpected "why" questions, "twist" questions, construction tasks, reflection questions, and "give an example" tasks. However, their study did not stop at classification of the interview tasks and questions. Zazkis & Hazzan (1999) extended their investigation to elucidate the "whys". The authors sought an explanation of rationale for the design of interview tasks. Not surprisingly, "all our interviewees have admitted that their criteria for the choice of the interview questions were implicit and hard to

elicit” (Zazkis & Hazzan, 1999, p.435). However, the investigators were able to elude several interrelated themes from the collected data: theoretical analysis, subject-matter analysis, researcher’s practice and researcher’s personal mathematical understanding. The theoretical analysis-based design was structured upon a particular learning theory, where participants’ responses “serve as identifiers of different aspects or stages presented by a theory.” The second theme for collecting information is self-explanatory - investigations into the subject-matter knowledge. The “practice rooted” design was exemplified in the report as the design of questions for the purpose of isolation and determination of the sources of observed participants’ difficulties in a learning situation. And the last group of research designs centered on the researcher’s personal understanding of the concepts involved, and is guided by his/her view of the important features, components, and connectivity within related topics. Although these particular findings and their influence on the design of the tasks will be discussed once again in a later section of this chapter, I wish to emphasize that this work has helped me on a personal level, to think in terms of the choices pre-service teachers make when they prepare their questions for an interview. Taking this into consideration, I required as a condition for the completion of this task, that the interviewees submit a written explanation of the reasons behind their choices of interview questions.

To create a successful teaching situation in pre-service teacher education, one might consider an interview as a special type of role-playing activity. Indeed, the traditional interview setting includes an interviewer (the teacher), an interviewee (the student), and the content that is usually prepared by the interviewer and organized into questions and tasks that are aimed to prompt an interviewee’s knowledge of the content at

stake. For years, the major focus of an interview activity was on the interviewee's responses, when the teacher or the researcher tried to assess the student's understanding of a particular topic. Even now, contemporary educators (Ginsburg and others) identify the interview method as a powerful technique to gain insight into thinking of others. This also resonates with the metacognitive aspect of the role-playing.

Over the last decade, the interview, as a pedagogical and as a research instrument, has attracted the attention of many educators. This began with Ginsburg's (1997) study of a child's thinking, wherein the researcher described the advantage interviewing offers over the traditional administrative tests. Following this research, some mathematics educators looked closely at the component of questioning and task creation, for the purpose of teasing out students' mathematical thinking (Watson & Mason, 1998). Watson and Mason placed value on questioning, and broadened the merits of the effective interview as a pedagogical tool. Their extended work resulted in a detailed classification of questions and prompts. Finally, they illustrated how learning and teaching situations can be enhanced for the benefit of the learners.

In aforementioned studies, the interviewer and the interviewee, (the teacher and the student, or the researcher and the participant) would have different levels of expertise in the content involved. Thus, in one way or another, the pressure of giving a satisfactory answer could affect the interviewee's responses. Such pressure could be minimized if the interviewer and the interviewee were to have similar mathematical backgrounds, and could question each other. The peer-interview would be an appropriate alternative. Another possible situation would involve self-explanatory study, where the pre-service teacher would impersonate a mathematics teacher and a student. This situation would

allow self-reflection, and self-criticism. All these alterations to the reviewed, previously instructional/pedagogical practices should make learning situations of greater assistance to the learner, who is the pre-service secondary mathematics teacher, especially if the experiences are closely related to real-life ones.

Taking into account the previously discussed advantages that each of the aforementioned methodologies or approaches offered, the ideal research activity would have to consist of a combination of them. After giving a thoughtful consideration to the ideas and issues exposed in the discussed studies, the tasks finally emerged.

The Job Interview is a peer-interview activity, where two pre-service teachers pair off, and then each chooses to play the role of either the interviewer, as the head of a mathematics department, or the candidate (the interviewee), who is the applicant for the substitute teacher position. The innovative research approach introduced in this task focuses on the interviewers, rather than interviewees. Thus, the research data includes the interviewers' rationales for the choices of the interview questions, the interviewers' expected answers, transcripts of the interviews, and the interviewers' evaluation of the candidates.

The Math Play is a self-exploratory activity, which focused on the metacognitive aspects of mathematics teaching and learning. It is an example of an error activity that capitalizes on the imagined student's misconception. The pre-service teachers' investigations into the sources of such misconception can lead to valuable explorations into teaching and learning of a particular mathematical domain. In turn, such explorations might shed light on pre-service teachers' content knowledge. The detailed description of

the research tasks is provided in Section 4.2, but here the main intent was to focus on the design of these tasks.

3.3 Conclusion

When designing the tasks used in this study, I found myself in a dual-role position: as an instructor and as a researcher. As an instructor of the secondary mathematics methods course, *Designs for Learning Secondary Mathematics*, I hoped to create engaging activities and rich learning environments, where pre-service teachers would come into contact as closely as possible, with the real life situations of a mathematics teacher. These tasks would incorporate an implicit review of the mathematical content, while explicitly focusing on pedagogical implications. As a mathematics education researcher, I tried to construct methodologies that would reveal valuable insights about pre-service teachers' mathematical and pedagogical content knowledge, in particular related to logarithms and of logarithmic functions.

The possibility of employing pedagogical tools as research tools is not new in mathematics education. This idea was introduced and largely explored by Zazkis & Leikin (2006) in different teaching situations. The main goal of these researchers was to establish a framework where example generation tasks were used as a research lens through which learners' knowledge and understanding of a particular content could be examined. The researchers succeeded in achieving their goal, and showed that pedagogical tools can be used as research tools. I examined a similar idea, by employing the tasks designed for this study, for both instructional and research purposes.

Chapter 4: Research Setting

4.1 Setting of the Study

4.1.1. The Course

EDUC-415, *Designs for Learning: Secondary Mathematics*, is a standard one-semester, 4 credits, methods course at Simon Fraser University. This course is offered to prospective and practicing secondary school teachers who wish to explore the learning and teaching process as it applies to secondary school mathematics. It is a required course for students enrolled in the Professional Development Program and working towards their teacher certification, specializing in Secondary Mathematics Education. The course is conducted through one four-hour meeting per week, for 13 weeks. In addition to regularly scheduled classes, students are entitled and encouraged to attend office hours for assistance.

The objective of this course is to provide the participants with appropriate learning experiences so that they feel comfortable designing secondary school mathematics instructions, using appropriate instructional materials and methods. During the course, participants explore theoretical and practical aspects of mathematics teaching, their own learning, and their role as teachers. Different forms of mathematical learning and diverse instructional strategies are experienced and explored in class. The investigations into the theoretical and practical aspects of curriculum implementation, instructional materials (manipulatives, technology, etc.) and assessment in mathematics teaching and learning are offered.

The required textbook for this course is *Teaching Secondary Mathematics: Techniques and Enrichment Units* by Posamentier, A., Smith, B., and Stepelman, J. This

math methods textbook is written according to the NCTM standards, and reflects current curricular reforms. To give a better idea of the content included in the course and the chronological order in which it is taught, a summary of the syllabus is presented in Appendix A.

4.1.2 Population

Participants of this study were students enrolled in EDUC-415. The entire enrollment consisted of 47 pre-service secondary mathematics teachers, 15 males and 32 females. Twenty-one of them were mathematics majors, while the rest held a major in sciences. However, only ten pre-service secondary mathematics teachers participated in the study, completing the first task, the *Job Interview*. And only six [Greg, Natalia, Kurt, Mike, Nora and Kal] remained in the study and complete the second task, the *Math Play*. A detailed account of these events is described in the following section 4.1.3.

4.1.3 Data collection

The research data for this study was collected during the secondary mathematics methods course, *Designs for Learning Secondary Mathematics*, offered by the Faculty of Education at Simon Fraser University.

The data consists of the accumulated participants' responses gathered from their completion of the two tasks: peer-interviews conducted, transcribed and analyzed by participants; and, written responses in the form of *Math Play* scenarios. Both tasks were employed as ongoing learning activities during the method course. A detailed list of the collected resources is provided in the following section of this chapter.

The first task, called the *Job Interview*, a form of peer-interview, was assigned during the third session of the course. This session mainly focused on the role that questions play in mathematics classrooms. Students engaged in different activities, which centered on exploring the possibility of using questions for teaching and assessment purposes. Participants were invited to reflect on and discuss the following two readings:

Hazzan, O. and Zazkis, R. (1997). Constructing knowledge by constructing examples for mathematical concepts, *Proceedings of the 21st International Conference for the Psychology of Mathematics Education*, Vol. 4, pp. 299–306.

Mason, J.H. (2002). Minding Your Qs and Rs: effective questioning and responding in the mathematics classroom. In Haggerty, L. (ed.). *Aspects of Teaching Secondary Mathematics: perspectives on practice*, pp. 248-258. Routledge, London.

In the Job Interview task, pre-service teachers were invited to impersonate the head of a mathematics department and a candidate for the position of substitute teacher in a fictional job interview. Participants were given a choice of topics for the task. A total of 10 pre-service teachers chose to engage in the activity involving logarithms and logarithmic functions. It is important to mention that teachers could write the scripts of their interviews, long before the actual interviews would take place, and were to employ and consult any possible resources. They could complete this task on campus, or at the local library, or even at home. No constraints on the materials or the location were imposed. However, pre-service teachers had only 3 weeks to finish this assignment.

The second task, called the *Math Play*, was assigned during the eighth meeting of the course. This session was focused on assessment for understanding of mathematical content. One of the topics discussed in the class was the role of errors made by students in mathematics classrooms. Pre-service teachers were working on the activity of

assessing different “student made” erroneous solutions, and analyzed the possible sources of the mistakes that had occurred. As a follow up to such explorations, participants were asked to individually complete the Math Play task. Once again, several content choices were given to students; six students focused on logarithms and only those were analyzed for this study. A five-week period was provided for the completion of this task.

It should be mentioned that at the time of administering the tasks, pre-service teachers were not told that they undertook these activities for the purposes of the instructor’s research. Instead, they were asked to complete the tasks to the best of their knowledge, as a course requirement. After the discussion of the tasks’ results and the possibility of their valuable contribution to the planning of the next course, the pre-service teachers were given an option to either decline or accept the usage of their work in this study. If they were to decline their assignments would remain with them. If they were to accept, their work would be returned to the instructor, and used as data research in this study. No one withdrew participation. At the last meeting, the preliminary results were shared with the class.

It is essential to emphasize that both tasks were completed in out-of-class time. In the Job Interview task participants could, had they chosen to, plan together, revise or even stage the interview. They also were encouraged to consult and utilize any accessible resources from books, articles or the Internet. The main goal of such a setting was to present participants with the possibility to evaluate what was available, then select and consequently include in their assignments what they would consider the most important and relevant.

4.2 Detailed Description of each Task

I aimed to create situations that could expose the deepest insights into pre-service teachers' mathematical and pedagogical content knowledge, regarding logarithms and logarithmic functions. Influenced by the studies of Ball, Hill, & Bass (2005), Rowland, et al. (2003), and Leikin (2006), I believe it is important to mention that my personal predisposition towards the design of the tasks is based on the premise that teachers' knowledge is comprised of subject matter knowledge, knowledge of pedagogy, knowledge of students' cognition, and a given teacher's beliefs; and that they should all be situated in practice. However, for the purpose of the analysis in this study I decided to focus on knowledge.

Task1: The Job Interview

As mentioned earlier, pre-service teachers were invited to play the roles in a fictional mathematical interaction between the head of a mathematics department and an applicant for the position of substitute teacher, who would cover for 3-weeks leave. The mathematical content for the conversation was logarithms and logarithmic functions.

Each of the participants had their own objective to accomplish. For the interviewer, the goal was to do his/her best to verify and evaluate the candidate's knowledge and understanding of the mathematical content and pedagogy required to be covered. For the candidate, the goal was to do his/her best to answer the questions to demonstrate his/her competence.

The main purpose of this task was to analyze the interviewer, the person who played the role of the head of a math department. This decision was based on the premise that the participants' questions and intentions would reveal what they think is important

in teaching. Thus, participants were asked to provide a written explanation of their choices of interview questions, elaborating on the purpose and the type of each question. They were also asked to prepare anticipated answers for their interview questions, submit the transcript of the interview, and a detailed evaluation of the 'candidate'.

The interview task was designed in a way to produce further outcomes regarding pre-service teachers' knowledge that would allow me to substantiate or challenge my conjectures. By the phrase "further outcomes", I refer to the additional information revealed through the interviewer's reaction to the received answer, and his/her final evaluation of the candidate. For example, after analyzing the interviewer's intentions, questions, and anticipated answers, I could theorize that the pre-service teacher (interviewer) had strong mathematical knowledge. But later in the interview, if the interviewer received a very weak answer, containing some mathematical errors, to his/her question, and if the interviewer agreed to such an answer without further probing, it would indicate that the interviewer's knowledge was superficial. This limited his/her ability to justify the candidate's flaws or gaps in knowledge and indicated that my initial conjecture was rather premature and not valid.

Data sources associated with this task comprised the following writings:

- a. Interviewer's rationale for the choice of the interview questions:

Through these data I investigated interviewer's subject matter and pedagogical content knowledge. Subject matter knowledge was based on the conceptual aspects of the interview questions, what the interviewer would consider to be the most important to target, and why. The pedagogical content knowledge is exposed by the manner of questioning, and the types of questions employed.

b. Interviewer's anticipated answers for his/her interview questions:

Through anticipated answers, I planned to learn not only if the interviewer can solve his/her own problem, but also to verify whether there was a consistency between the prepared interview questions and answers. Basically, in combination with (a), I was able to assess if the interviewer did what he or she meant, and explored what the interviewer would do if the expected and received answers were different. The variety of the anticipated answers to a particular question could indicate the scope of the interviewer's subject matter knowledge and pedagogical content knowledge (the more the merrier).

c. Transcript of the interview:

The extent of the interviewer's pedagogical content knowledge was explored upon the interviewer's reaction to the candidate's response. For instance, if the response was satisfactory, would the interviewer prompt the candidate's knowledge of logarithm and logarithmic functions further? Or, if the answer contained an error, how would the interviewer reply to it? In this case, the ignorance of misconception might be the interviewer's misconception as well.

d. Interviewer's evaluation of the candidate:

The interviewer's evaluation of the candidate was the final, and one of the most important, pieces of information. If from (c) I was able to make an assumption about the interviewer's content knowledge, then in (d) the interviewer provided his/her own reflection on what happened in the interview. The details the interviewer included in the evaluation would be considered as the most important aspects of teaching and learning

logarithms and logarithmic function. In turn, this would support or contradict my evaluation of the interviewer's subject matter and pedagogical content knowledge.

Task 2: The Math Play

As mentioned in Section 4.1, the second task used in this study was the Math Play. The key difference between the job-interview and the math play is that in the first activity the interviewer's thinking is prompted by the interviewee's answers, whereas the play is a self-exploratory task. In the job interview, the pre-service teachers could investigate any aspect of teaching or learning of logarithms and logarithmic functions. In the math play, all the participants were prompted by the same task.

Each pre-service teacher was to analyse the erroneous situation that exposes a student's misunderstanding:

Act 1, Scene 2:

There is a conversation between a teacher and a student (there are 30 students in a class):

T: Why do you say that $\log_3 7$ is less than $\log_5 7$?

S: Because 3 is less than 5.

Pre-service teachers were asked to diagnose the student's misunderstanding, formulate a plan for remediation of the misunderstanding, and write out the balance of the interaction(s) in the form of a *math play*. The use of the word "diagnosis" in this situation, means that teachers were to establish how, when, why the misunderstanding could possibly occur. For this task, they were to write Act 1, Scene 1. From this, I anticipated an accumulation of some reliable assumptions about the participants' knowledge.

To confirm or refute my assumptions, I analysed Act 2, Scene 1 of the math play. In this remediation part of the activity, the pre-service teachers were to create a teachable

moment, when they would orchestrate the events, tasks, and conversations, to lead the student out of the problematic situation. It was expected that pre-service teachers would help the imaginary student realise the nature of his/her mistake, and in some way verify that the student corrected the mistake and demonstrated understanding of the provided explanation. The teachers' methods of such verification would expose their knowledge of mathematics and pedagogy that would allow me to confirm my initial evaluation, or on the contrary reject it, or perhaps simply modify it.

For example, in Act 1, Scene 1, a pre-service teacher may write that a misconception is a result of a student's misunderstanding of logarithmic notation. Then, in Act 2, Scene 1, the teacher would have to focus on the student's knowledge of logarithmic notation. There needs to be a consistency between the two acts. From the instruments used for remediation I learned about the extent of a pre-service teacher's subject matter and pedagogical knowledge. The absence of such consistency could be considered an indication of insufficiencies in the pre-service teacher's knowledge.

Yet again, data sources associated with this task included the following writings:

- a. Pre-service teachers' diagnosis(es) of the students misconception.

Through these particular data, I investigated how well pre-service teachers can assess a student's learning from a subject matter perspective. What exactly about the logarithms did the student not understand?

- b. Their personal encounter on where and how a given situation could take place, in the form Act 1, Scene 1.

In these materials, I was looking for the pre-service teacher's ability to situate the provided episode, Act 1, Scene 2, in a sequence of educational events.

- c. The remediation part, where the pre-service teachers had to organize a situation to guide student's learning, in the form of Act 2, Scene 1.

This was the most important part of the data collected in Task 2. Here, I was looking for the evidence of pre-service teachers working on the “fixing” of the student's misconception. Pre-service teachers' abilities to deal successfully with the given misconception served as an indication of their teaching proficiency. On the contrary, pre-service teachers' limited knowledge of the mathematical content and pedagogy resulted in their attempt to “re-teach” the concept.

4.3 Summary

The intention of this chapter was to introduce the reader to the participants of the study, the research setting, and the data collection processes. Furthermore, the chapter presented an extensive explanation of the purpose of each task designed and utilized in the study, including a description of every particular source of information collected for the analysis. The rationale for considering each task as an appropriate instrument for the study was offered. The next chapter presents the analysis of the data collected from each research task.

Chapter 5: Results and Analysis

In this chapter I introduce and analyze the data collected for this study along with a detailed report and explanation of it. The data was classified according to the common trends identified in the participants' responses. The materials produced by the pre-service secondary mathematics teachers functioned as the domain for the exploration. Only the facts related to subject matter knowledge and pedagogical content knowledge were analyzed in this research.

To reiterate, subject matter knowledge, for the purpose of this study, is the knowledge teachers possess with regards to logarithms and logarithmic functions. This knowledge was gained through the course of training at schools, colleges, and universities. To investigate the pre-service teachers' subject matter knowledge I focused on three major areas of knowledge: logarithms as numbers, the operational meaning of logarithms, and logarithmic functions (Berezovski, 2004, Berezovski & Zazkis, 2006).

Pedagogical content knowledge was consistent with the definition provided by Shulman (1986). Pedagogical content knowledge is knowledge of pedagogical instruments that enable a teacher to create situations of meaningful learning for students. It is considered a specific knowledge necessary for pre-service teachers to possess. It includes teachers' knowledge of the difficulties and misconceptions students experience when learning a particular content. The pedagogical content knowledge could be revealed through a variety of attributes, such as the quality of the questions asked, evidence of further prompting, and broad assortment of the assessment techniques, etc. Though pedagogical content knowledge is enhanced by teaching practice, it is my expectation

that pre-service teachers have acquired some of this knowledge in their prior courses and through student-teaching practicum.

Each of the research tasks designed for this study was very complex; the collected data was also multifaceted. While working through the data, I realized that teachers' knowledge was revealed not only by questions, prepared and used in the interviews, but also in the manners of how these questions were asked. Another important component influencing knowledge was in participants' conceptions of the importance of teaching logarithms and logarithmic functions. Therefore, before analyzing participants' subject matter knowledge and pedagogical content knowledge I focused on the interview questions, questioning techniques, and participants' explanations of the value of teaching logarithms.

In what follows I present the results organized according to each task. In section 5.1, the reader is introduced to the results obtained from the Job Interview task. In section 5.2, the discussion is centered on the participants' responses to the Math Play.

5.1 Analysis of teachers' knowledge from the Job Interview task

To reiterate, the main idea for the design of this particular task was to learn about pre-service secondary mathematics teachers' knowledge. The task was a peer interview type of activity, where two had to engage in a conversation about logarithms. The partners conducted a fictional job interview. One of them played the role of the head of a mathematics department, (referred to as an interviewer); the other impersonated a candidate for a teaching position, (referred to as an interviewee or a candidate). Unlike the usual interview analysis, where the attention is on the interviewee's responses, my

research interest focused on the interviewers, their choices of questions and tasks, assessment strategies, and evaluation criteria of the candidates. It is a novel methodology. I decided to try it, hoping to be able to delve into the multifaceted nature of teachers' knowledge. This decision was based on the premise that the participants' questions and intentions reveal what pre-service teachers think is important to know for successful teaching. For this matter, pre-service teachers were asked to provide: a) a written explanation of their choices of interview questions, while elaborating on the purpose and the type of each question, b) their own anticipated answer(s) for their interview questions, c) the transcriptions of their interviews, d) detailed evaluations of their peers - 'the candidates'.

5.1.1 Focusing on the interview questions

Various aspects of pre-service teachers' subject matter knowledge, manifested through interview questions, are presented in this section. First, I looked at the questions, asked by the interviewers, with an intention of identifying the interviewers' conceptual preferences towards logarithms and logarithmic functions. Then, I classified interviewers' questions according to the themes, and summarized my findings in Table 1. The common themes represent variety of questions, but they still fall within the interpretive system, discussed in Section 2.3.3. Thus, this system was used to analyze pre-service teachers' knowledge of logarithms and logarithmic functions.

Table 1: Types of Questions Asked In the Interview, focusing on the subject matter knowledge

Themes	Exponential equation/ expression	Logarithmic Laws/ Equations	Modeling questions	Logarithmic function	Application of logarithms	Definition of logarithm	Number of questions asked
Judi	XX	X	-	-		X	4
Sam		X		X	X	X	4
Nora		X				X	2
Natalia		X		X		X	3
Mike		X	X		X		3
Keara				X	X		2
Kal	X	X		X	X		4
Sophie			X		X	X	3
Kurt	X	X			X		3
Greg		XX	X		X		4
Total questions	4	9	3	4	7	5	32

For example, Natalia asked three questions:

1. Can the base of a logarithm be 1? Why?
2. What is the x-intercept of logarithmic function for any base?
3. Solve for x: $\log_2(x+2) + \log_2x = 3$

Question #1: “Can the base of a logarithm be 1? Why?” falls under the category ‘definition of a logarithm’. To answer this question, the candidate would have to possess a comprehensive knowledge of the definition of a logarithm. The response can be provided from analytic or graphical perspectives. The graphical consideration would make this question appropriate to the category ‘logarithmic function’. Perhaps, she could be finding out how the person chooses to respond. However, I believe if Natalia would have intended to receive graphical interpretation, she could have used more appropriate vocabulary, such as ‘graph’, ‘intercept’, or ‘zero’, etc. as in the following question.

Question #2: “What is the x-intercept of logarithmic function for any base?” has definite geometrical connotations and falls under the category ‘logarithmic functions’. It

seems that Natalia wanted to find out if the candidate understood the properties of the logarithmic functions. Moreover, the phrase “for any base” could be interpreted as her intention of exploring whether the candidate possessed general knowledge, rather than specific.

Question #3: “Solve for x : $\log_2(x+2) + \log_2 x = 3$ ” is a procedural type of question from category ‘logarithmic laws and equations’. There the candidate was asked to solve a logarithmic equation. It is most likely that Natalia expected to receive an algebraic solution, which would allow her to assess the candidate’s knowledge of logarithmic laws, and perhaps of definition. This question can also be solved graphically, with a help of a graphing calculator. However, if such a purpose were anticipated, the question would contain words like “solve graphically”, or “provide a graphical solution”. Nevertheless, it could be assumed that Natalia left this choice open to the candidate. In this case, her question could be expanded in different directions, allowing the interviewer to get a deeper sense of the candidate’s knowledge of logarithms and logarithmic functions.

Thus, when looking at the interview questions, one learns what Natalia considers being the most important aspects of the mathematics at hand, keeping in mind that her final evaluation should be based on the candidate’s responses to her questions. So, it could be concluded that Natalia believes that a person should know a definition of a logarithm and the main properties of logarithmic function, and be able to solve a logarithmic equation, in order to justify to her that this person knew logarithms and logarithmic functions.

The summary of cumulative findings was presented in Table 1. Ten interviewers asked a total of 32 interview questions. Questions were classified into seven categories

which should not be treated as completely isolated. As in the analysis of Natalia's question #1, the questions may have fallen into more than one category. In order to be consistent with the total number, I categorized most of them in themes indicated in the interviewers' analysis. Seven such themes were identified. Most of them are self-explanatory, except perhaps, 'modeling questions' and the 'application of logarithms'. Though the distinction between them is very vague, I tried to establish a difference. The 'application of logarithms' category contains the questions that intend to find out, in a general way, the areas of human endeavor where logarithms are used for modeling some kind of processes. This category also includes word problems that are supplied with the formulas, and were basically simplified into 'plug-in' logarithmic or exponential equations. For example,

Question 1: If a cell population doubles every three hours how long will it take 4 cells to reach a population of 16384?

Formula: $P(t) = P_0(2)^{kt}$, where $P(t)$ is the population after t hours
 P_0 is the original population
 k is the generation growth rate per hour
 t is the time

Question 2: Where is logarithm used in real life situations? Give an example about the application of logarithms.

By 'modeling questions' I mean questions that are not supplied with formulas. To solve such questions, the pre-service teachers were required to model first, in deriving an appropriate formula, prior to substituting the values and solving. For instance, the following question, "A radioactive substance has a half-life of 17 d. How long will it take for 300 g of this substance to decay to 95 g?" is a 'modeling question'. Though the distinction might seem to be insignificant, it is important to mention. In this particular

case, the question could belong to more than one category. Besides being a ‘modeling question’, it also could be interpreted as an ‘exponential equation’, because it is an example of an exponential decay.

All the interview questions had a mathematical connotation, ranging from basic ‘solve’ or ‘simplify’ questions, to conceptually advanced, modeling questions. In terms of knowledge of logarithms and logarithmic functions, 9 out of 32 were of a procedural nature, as defined by Hiebert and Lefevre (1986), focusing on the candidates’ knowledge of logarithmic laws. Eight out of 10 pre-service teachers decided that the ability to solve logarithmic equations was very important; therefore, they allocated from 25% (1 out of 4 questions in the interview) to 50% (2 out of 4 questions in the interview) of their interview time to the exploration of the candidates’ knowledge of this matter.

Three out of 32 questions were of the modeling type, which is less than 10%. Three out of 10 pre-service teachers included them in their interviews. Based upon this information, one can conclude that the pre-service teachers, who played interviewers, prioritized the procedural knowledge of solving logarithmic equations over that of conceptual knowledge and of modeling real life word problems. Though some questions point to the procedural orientation, conceptual orientation cannot be clearly identified.

For example, one of the questions asked by Mike was,

If the student was having difficulty understanding the product law of logarithms: $\log_a xy = \log_a x + \log_a y$, how could you prove this law by using the known law of exponents $a^m \cdot a^n = a^{m+n}$.

This question can be seen as inviting a ‘logging technique’ to both sides of the exponential equality, a view that points to a procedural intent. Alternatively, the intention of this question can be to establish the isomorphic relationship between products and

sums of numbers using logarithmic notations instead of the exponential notations. This intention would fit into the realm of conceptual knowledge. Therefore, the analysis of pre-service teachers' content knowledge cannot be solely based on the questions. Sometimes, different interviewers used the same question, but for different purposes. Some questions were asked with a student in mind, such as how to explain to a student or how to respond to a student's question. There were also questions directed strictly at the participant's subject matter knowledge. Then, it became important to focus on a precise formulation of the question, at questioning. Deeper understanding was to be gained, in consideration of the manner the interview questions were utilized by interviewers.

5.1.2 Categories of questioning techniques used in the interviews

To gain a better grasp of the meaning (participants placed into questions) of what pre-service teachers intended to ask, I focused on their questioning techniques, in the manner the questions were implemented. Several patterns emerged in the ways interviewers articulated their questions that seemed indicative of differences in questioning techniques. I categorized them as:

1. *Inventory questioning* – the method in which the interviewers proceed from one question to another in giving little regard to the candidates' answers, concerned only with the similarity to their own answers. These questions usually did not have follow-up questions, and the interviewers' responses were in the form of a spoken agreement (OK, good, right, etc.).
2. *Leading questioning* – the method in which the interviewers used the questions that included the words that had a direct impact on the interviewee's response. Very often these questions were rhetorical in nature, and required just

confirmation from the interviewees, thus limiting the possibility for their own thoughts.

3. *Skilled questioning* – the method in which the interviewers continued to prompt, using follow-up questions to ensure deeper investigation of the interviewees' knowledge. These interviewers typically employed different types of questions, such as: performance, twist, why, give an example, etc. For a full list of the types, see Section 3.2.3.

The inventory questioning technique was easily identified in the transcripts of the interviews. The interviewers simply read through the list of preplanned questions, obtained answers similar to the ones they expected, and moved on without any further probing of the candidates' thinking, as in the following example:

Mike: ...In terms of constructing proofs and whatnot, there are going to be students that have trouble relating knowledge they are gaining in terms of logarithms with knowledge they already should have of the exponential laws. So if you could relate the product law of logarithms: $\log_a xy = \log_a x + \log_a y$, and the sum law of exponents $a^m \cdot a^n = a^{m+n}$...

Candidate: This would be a very complicated way to get a kid to understand the relationship between logarithms and exponents, but here it is...(writes on paper):

$$\begin{aligned}
 &\underline{\text{Prove:}} \quad \log_a(xy) = \log_a x + \log_a y \\
 &\text{relating with } a^{(m+n)} = a^m \cdot a^n \\
 &\text{let } m = \log_a x \Rightarrow a^m = x \\
 &\quad n = \log_a y \Rightarrow a^n = y \\
 &\quad a^m \cdot a^n = a^{(m+n)} \\
 &\quad x \cdot y = a^{(m+n)} \\
 &\log_a(xy) = \log_a a^{m+n} \\
 &\log_a(xy) = m + n \\
 &\log_a(xy) = \log_a x + \log_a y
 \end{aligned}$$

Figure 3: Mike's Interviewee's Written Response

Mike: Thank you. That was a good job relating those two concepts.

In this episode, the candidate clearly expressed his or her concern with the requested answer, but the interviewer, Mike, completely disregarded it, and moved on, satisfied with a written response which matched his own. One of the possible explanations of Mike's behavior could be that by not knowing any other mathematical explanations, he felt anxious about the possibility that his limited knowledge could be exposed. The problem with this question was in its original wording. The interviewer had identified the problematic situation in students' learning of logarithms correctly. However, building a connection between two concepts from an algebraic point of view would become quite a challenge for an average student. Nevertheless, the candidate's concern, "This would be a very complicated way to get a kid to understand the relationship between logarithms and exponents" was left unattended by Mike. More students might understand the relationship as a preparation for abstract formulas that is done through explorations of the concrete examples.

Moreover, Mike led the candidate to the answer. He provided too many directions to the interviewee. His condition "relate the product law of logarithms: $\log_a xy = \log_a x + \log_a y$, and the sum law of exponents $a^m \cdot a^n = a^{m+n}$ ", provided the candidate with a hint of how to answer the question, limiting the interviewee's thinking. As a result, the interviewer found the candidate's response to be reasonable and satisfying of his expectations. Mike's own anticipated answer to the posed question, submitted as part of the assignment, is provided here for comparison: "The fundamental concepts in this proof are understanding that $\log_a x = m$ is equivalent to $a^m = x$ and understanding that when

switching between exponents and logarithms taking the log of both sides of the equation is needed.”

Another example of inventory questioning is illustrated below. According to the classification of questioning methods, this technique could be used to check whether the candidate does understand the meaning of a logarithm. But the interviewer’s frequent usage of the words ‘good’ and ‘yes’ influenced her chances for thorough assessment.

Sophie: ...what is a logarithm?

Candidate: Um, logarithm is a way of working backwards from an expression that uses exponents.

Sophie: Yes, like in $y = 3^x$?

Candidate: Yes, when we need to find x , we have to log both sides,

Sophie: Good.

Candidate: And then just solve for x .

Sophie: Right, that’s good...

Reading the transcript one may get the impression that Sophie’s words “good”, “right”, can be just culturally polite indicators of telling the interviewee “please proceed”. However, having listened to the recording it appears that in this dialogue, the interviewer, Sophie, is not only providing the answer to the candidate, as in the third line, but also cuts off the candidate’s chances for deeper thinking and explanations of the knowledge possessed. Though it may not have been the interviewer’s intention, it happened because she provided the candidate with signals of premature satisfaction (good, yes, right), indicating that the candidate is no longer required to continue his or her explanations.

Several pre-service teachers used questioning strategies that were identified as leading and in essence, attempted to guide and prompt the candidate to a correct answer.

Some questions were simply equipped with formulas; this reduced the assessment potential of the question. For instance,

Kal: Can you express $\log_2\left(\frac{15}{7}\right)$ in terms of x , y , and z , given that $x = \log_2 3$; $y = \log_2 5$, and $z = \log_2 7$, using the following formulas:

$$\log_b xy = \log_b x + \log_b y;$$
$$\log_b \frac{x}{y} = \log_b x - \log_b y .$$

Candidate: Sure. (writes solution on paper)

By providing too much information, the interviewer could only confirm that the candidate knew how to factor 15. However, the original question, without the formulas provided, could be used to target the candidate's knowledge of logarithmic properties.

Besides the inventory and leading questionings, there was another technique, identified as, or referred to, as skilled questioning. Skilled questioning usually included follow-up questions, sometimes preplanned, sometimes constructed on the spot, in reply to the candidate's answers. This technique allowed the interviewer to explore the depth of the interviewee's subject-matter knowledge. An interviewer's ability to elicit the deepest knowledge another person possessed could be considered as evidence of his or her personal knowledge necessary for successful teaching. However, considering how the task was presented and administered, the existing opportunity for planning and rethinking the prompts was taken advantage of only in several cases. In the following interview extract, the pre-service teacher probed the interviewee's correct response, by using that response as a part of the follow-up question:

Kurt: Can you solve an equation $(-2)^x = 64$?

Candidate: Yes, the answer is 6.

Kurt: Can you get the same answer by solving with logarithms?

This example of skilled questioning, illustrates how by attending to the responses, the interviewer, Kurt, can turn “the familiar at a glance” question into a novelty, when the introduction of logarithms was not previously utilized in a similar situation. The interviewee’s reaction to it would, indeed, provide insights about the candidate’s knowledge. However, the focus of this study is on the interviewers and their ability to form and use the questions and different method of questioning, which would serve as data about their personal knowledge of the subject, and their skills of assessing someone else’s knowledge.

The knowledgeable interviewers employed varied types of questioning. Their repertoire ranged from inventory to skilled methods. The following example is an illustration of how well-thought questioning can empower the interviewer’s ability to examine the candidate’s knowledge:

Judi: if you take the equation $(-2)^{x+1} = 16$,
and you take the log of both sides you get: $\log(-2)^{x+1} = \log 16$
then he says to just make inside the log positive and you get
 $(x + 1)\log(-2) = \log 16$
 $x + 1 = \log 16 / \log 2$
 $x + 1 = 4$
 $x = 3$
If you plug this back into the original equation you see that it gives the right
answer: $(-2)^{3+1} = (-2)^4 = 16$
Do you agree with the student’s reasoning? Why or why not?

In this situation, the candidate was asked to reflect on a solution produced by a imaginary student. It is important to recognize that here the interviewer, Judi, was able to prompt the candidate to respond not only to a challenging mathematical problem, but also to the *student’s* reasoning. This questioning technique is more likely to be used by the skilled interviewer, who possesses a solid knowledge of subject matter and also understands how students’ learning and the teaching of a particular content happens.

The interview questions collected for this study ranged from inadequately worded and poorly executed to well-thought and conceptually challenging. They varied in purpose, in nature, and in the ways they were executed during the interview. Interesting was the fact that eight out of ten pre-service teachers included varied questioning methods in their preplanned interviews. Yet, only six of the interviewers could productively implement such methods. The remaining two interviewers did not follow through with their initial intent. They neither prompted the received responses, nor did they ask the follow-up questions. In such cases, the interviewers' intentions, questions, and questioning methods were of a superficial nature.

Thus, the presented categorization allowed me to systematize the questioning techniques utilized by the interviewers. When analyzing the interviewing, based on the transcripts and rationales for the interview questions provided as parts of the assignment, many interesting facts about the interviewer's knowledge became tangible for further investigations. However, the ultimate goal of exploring the nature of the knowledge required for teaching, required additional considerations.

5.1.3 Pre-service Teachers' Conceptions of the Importance of Teaching Logarithms and Logarithmic Functions

In view of the recent findings from the body of research on teacher's knowledge in mathematics education literature, many studies have identified subject matter knowledge and pedagogical content knowledge as the key components of the knowledge necessary for teaching. This fact prompted the following investigation. When proceeding with further analysis in this section of the chapter, the study focuses on what knowledge pre-service teachers consider to be important for teaching a particular mathematical

content. The interviewers' rationale for the choice of interview questions as well as their anticipated answers were used for the analysis, shedding light on what knowledge pre-service teachers as interviewers identified as crucial in the teaching of logarithms and logarithmic functions.

To continue, this discussion allowed me to review the meaning of subject matter and pedagogical content knowledge, building on Shulman's (1986, 1987) components of knowledge for teaching. *Subject matter knowledge* refers to the breadth and depth of teachers' own knowledge of mathematics. *Pedagogical content knowledge* includes knowledge of how to teach, how to assess students' understanding, and how to create an educational setting for meaningful learning. This conceptual framework enabled me to describe pre-service secondary mathematics teachers' content knowledge and the rationale for their actions.

Once again, 10 pre-service secondary mathematics teachers' interview intentions, completed as part of their assignment, were analyzed. Seven out of ten pre-service secondary mathematics teachers explicitly indicated the importance of the practical aspects of mathematics, with an emphasis on the usefulness of mathematics in real-life applications and other subject areas, as a value and aim for teaching logarithms and logarithmic functions. For example, interviewer Greg wrote in his rationale, "...I think that the foundation of the teacher's knowledge should be from the purpose of the real world use..." Many other pre-service teachers shared his intention. Consequently, the pre-service teachers in the peer interviews frequently asked questions from the category of "application questions". For a detailed overview of different types of questions asked, consult Table 1 on page 59.

Some interviewers identified in their rationale the pedagogical importance of knowing how to use logarithms in real life. For example, Sophie justified her choice of the ‘application of logarithms’ type question in the following:

I feel it is important that mathematics teachers step back from theory more often than they do, to offer applications. The motivation to learn theory or concepts comes from the desire to apply that knowledge. A good candidate should be able to inspire students with the application of learned concept.

The aspect of motivation by the means of showing applicability was not the only one identified. Another teacher recognized one more importance of this particular knowledge. Greg’s view reveals the great conceptual significance of using logarithmic functions for the modeling of real life situations. He wrote, “I didn’t just want the candidate to know how to use the logarithmic function ... but also to be able to construct a mathematical problem from the real life situation presented”. These examples illustrate how knowing the value and aims of teaching logarithms and logarithmic functions is considered by pre-service secondary mathematics teachers as being an important part of teachers’ knowledge.

However, a deeper analysis was required to complete this line of judgment. The questions remaining on the research agenda were: do teachers possess this knowledge, do they know how and why applications of logarithms work and are indeed important? Do they mean what they asked? To answer these questions, the analysis of interviewers’ anticipated responses in juxtaposition with their candidate evaluations was undertaken, and presented in what follows.

One of the interviewers presented rather unique reasoning. In his rationale, Greg mentioned the importance of knowing logarithms for the purpose of modeling real life

situations. His intention was recognized as one of conceptual significance. However, in reality, the question was superficial in nature, revealing a shallow level of knowledge. This became evident from Greg's evaluation of the candidate's answer that was completed as part of the assignment. The following is an exact copy of Greg's question. It is a population problem that is provided with a formula for the exponential growth of population.

Interview task:

Cell population doubles over 3 hrs. How long would it take 4 cells to reach of population of 16 384?

Formula: $P(t) = P_0(2)^{kt}$, where $P(t)$ is the population after t hours
 P_0 is the original population
 k is the generation growth rate per hour
 t is the time

By providing the formula for the candidate, the interviewer disregarded a possible verification as to whether the candidate could model a presented, real life situation. Greg's deeds disagreed with his initial intention, or the way it was interpreted. The purpose of his task diminished to a very fundamental level of application, that of solving an exponential equation. Here is the candidate's answer:

$$2) P_t = P_0 = (2)^{kt}$$

$$P_0 = 4$$

$$P_t = 16,384$$

t = time in hours

$$k = 3 \text{ gen/hr}$$

$$16,384 = 4 = 2^{kt}$$

$$\frac{16,384}{4} = 2^{kt}$$

$$\log\left(\frac{16,384}{4}\right) = \log(2^{kt})$$

$$3.61 = t \log(2^k)$$

$$3.61 = t \log(2^3)$$

$$3.61 = t(0.100)$$

$$t \approx 36 \text{ hours}$$

Figure 4: Greg's Interviewee's Written Response

As evidenced from the response, the candidate applied logarithms only for the purpose of solving an exponential equation. From this solution, it is not clear if the candidate possessed any knowledge of the basic properties of logarithms. A calculator was available for use; therefore, the knowledge of logarithmic laws became irrelevant and unnecessary for obtaining a result. It still remained questionable if the candidate possessed such knowledge. It also remained unclear, if the candidate knew the definition of a logarithm. If the answer is yes, then the candidate could apply the knowledge of definition, and the third line of his or her solution would look like

$$kt = \log_2\left(\frac{16,384}{4}\right) \text{ where just substituting for } k \text{ would produce the answer. Though the}$$

correct solution was obtained, such an alternative approach would have provided an evidence for conceptual awareness rather than procedural implementation.

Disappointingly, none of these aspects were prompted by the interviewer; they were either missed or unrecognized. Surprisingly, in his final reflections, Greg expressed satisfaction with the candidate's response. With regard to the candidate's evaluation, he wrote, "I don't think there was any surprise here. I did give him the formula and the only part I thought there might be trouble was with the k value, but he [candidate] seemed to know exactly what it was. He also seemed very sure of himself in the use of the logs here..."

Every other pre-service teacher, who identified the application of logarithms as important knowledge for teaching and learning mathematics, expressed their expectations in a very narrow manner, by limiting the possibilities to some sort of checklist. Their intentions were satisfied as soon as the candidates followed their list. The most common items listed in the expectations were:

- ✓ Richter scale / intensity of earthquakes,
- ✓ Exponential growth/decay problems
- ✓ Intensity of Sound
- ✓ Intensity of Acidity

These facts are collected from the interviewers' anticipated answers, completed for the job interview task. Only seven entries of those interviewers, who were attempting to use questions about the application of logarithms, were considered.

In most cases, the provided answers included one or two items from the list. However, not one interviewer challenged the candidates by asking them to support their

responses with some concrete examples. This led me to conclude that even though pre-service teachers recognize the importance of real life applications that a particular mathematical phenomenon has, they are not confident in their knowledge; or perhaps they are not skilled in thinking of good questions, and/or their subject matter knowledge is so very limited, that they preferred to avoid any discussion in this regard.

The pre-service secondary mathematics teachers' common reasons for teaching logarithms and logarithmic functions, expressed in the interviewers' rationale for their choices of interview questions, were that these concepts are used in many fields of human endeavor, such as physics, biology, chemistry, and statistics. However, the findings were somewhat disappointing. Only six participants indicated that both the pedagogical and mathematical knowledge were of importance in teaching these concepts. None of the pre-service teachers expressed the thought that logarithms and logarithmic functions form a foundation or supporting structure for other mathematical ideas or further mathematical topics. One of the reasons behind the participants' shallow expectations of the candidate responses could be related to the inadequate mathematical and pedagogical content knowledge they possess, or their belief of no need for knowledge beyond the curriculum. In the following section, the study focuses specifically on the subject matter knowledge possessed by pre-service secondary mathematics teachers.

5.1.4 Pre-service teachers' subject matter knowledge of logarithms and logarithmic functions

In this section, I present an overview of relevant subject matter knowledge that pre-service teachers possess, by focusing on their ability to provide explanations of the meaning of logarithms and logarithmic functions. This knowledge was in evidence from

the interviewers' questions, and their rationales for the selection of these questions. It is also important to mention that the pre-service teachers did not review logarithms in the class during this course. They probably did individual or small group reviews when preparing for the interview task; however, it was done without any interference on the part of an instructor. Once again, I wish to remind the reader that all participants of this study majored in sciences or mathematics. They had successfully completed their degrees, and were in their final course at the finishing point of the teacher certification program.

The essence of logarithms and logarithmic functions can be explored in three major areas of knowledge:

- (area 1) logarithms as numbers
- (area 2) the operational meaning of logarithms
- (area 3) logarithms as functions (Berezovski & Zazkis, 2006, Berezovski, 2004).

At the heart of the first area of knowledge is the understanding that a logarithm is a real number, and that any real number can be presented in the form of a logarithm. An understanding of a definition of a logarithm is part of this area of knowledge. The second area focuses on the main properties of logarithms, and how they can be added or subtracted. This is otherwise known as the product and quotient laws of logarithms. And the third area encloses the knowledge of how the relationship between positive numbers and their logarithms becomes a function, and deals with the properties of logarithmic functions.

In this section of the chapter I analyzed pre-service secondary mathematics teachers' subject matter knowledge, exploring the three conceptual areas of knowledge mentioned above. Several situations are discussed in detail.

For example, the question, 'why can't we find the logarithm of a negative number?' would target the knowledge relevant to the first and/or the third areas of knowledge. Actually, this or similar questions were the most common: 8 out of 10 participants asked them in their interviews. This might indicate that interviewers identified it as problematic, or difficult for students' learning, but as important to understand. In the following episode, an interviewer, Nora, tries to assess if her candidate possessed such knowledge:

Nora: Okay good, Okay, now moving on. As a mathematics teacher, one has to ensure that students understand the reasoning behind mathematical ideas and not just memorizing them. If a student posed the question 'why are some logarithms undefined?' how would you explain it?

Candidate: I personally look to definitions as starting points, and would encourage students to examine what happens when constraints are violated. So I would have them start with a table of values of logs with negative numbers for bases, exponents, and arguments, and also examples of exponents with negative bases and exponents. And I would ideally use the question we just discussed¹ as a hook for a constructivist lesson, in which students would have to go back to the meaning of logs as they relate to exponents, and then come up with examples of different logs that are undefined, so they can see for themselves what's happening. I might even use this example to discuss continuous fractions and inverses, because it relates to that as well, or use it as an example of how to take observable relationships and translate them into a proof. So it's a very rich topic I think. I would have to structure the lessons so the students didn't get overwhelmed with too many different ideas though, uh but delving into the why's is definitely something I would want to encourage.

Nora: that is definitely true, thank you.

¹ Nora's previous question:

Solve for x : $\log_3(x-4) = 1 - \log_3(x-2)$ and one of your students,

Tom answered, $x = 5$ or $x = 1$. Is his answer correct, incorrect, or partially correct?

Please explain your reasoning.

Nora, as a department head, asked a very important question that had great potential to unravel the candidate's understanding of logarithms. It would be interesting to see how the candidate could handle it, what example would be chosen, and how the necessity of the existence of a logarithm would be established? But instead, the interviewer settled for less: a very blurry verbal description of some disorganized ideas. Did the interviewer really agree with such a response? The answer can be read from Nora's evaluation of the candidate, "Clara demonstrated she has content knowledge of logarithms and knew of the connection to exponential functions..." Another interesting question to consider might be why Nora agreed to this answer? To understand and explain it, I shall consult the interviewer's anticipated response. What did she expect to hear or to see, when the following question was asked?

...explaining to students why $\log_b n$ only exists for $n > 0$ will help students obtain understanding. It is also important to explain this idea step-by-step so that students understand the logic and don't just memorize this fact because they are frustrated by the explanation.

In this particular episode, the interviewer asked an important question that had the potential to target much deeper conceptual knowledge. Even though the answer was not well worded and revealed very little about the candidate's knowledge of a definition of a logarithm, Nora accepted it. It is questionable whether this indicated a polite response to the classmate or it meant that she possessed the same level of knowledge. To confirm, I compared it to Nora's expected answer, and confirmed that she indeed possessed a very limited knowledge of logarithms. She received a very unclear answer, and didn't ask any follow-up questions. Her personal response exposed the tendency to procedural learning. In this case, both Nora as the interviewer, and the interviewee possessed limited subject matter knowledge, particularly of logarithms as numbers (area 1).

In the following episode, the interviewer struggles to understand that there is more than one definition of logarithm in existence. This is a slightly different situation, but a very similar question,

Sam: If you could just maybe explain to me why they [logarithms] are undefined for negative values?

Candidate: ...Now by basic definition of a logarithm, when I say logarithm of a function x , it is an integral from 1 to x of 1 over t dt, *logarithm of x* is equal to $\int_1^x \frac{1}{t} dt$. So logarithm represents an area under the hyperbolic curve...(draw on paper)

Sam: Ok, is it possible to elaborate on that without using integrals?

Candidate: Without using integrals?

Sam: Yes

Candidate: Ok, if you put for the basic definition of logarithm, $\log 0$, it doesn't exist by itself, we call it undefined because $\log 1$ is actually 0. Beyond that, any value beyond $\log 1$ that will be, that one is zero by itself, If you want to do the integration or the area under the curve. Without integration aside, $\log 1 = 0$, any value that is greater than 1 will give you positive values. Any value less than 1 will give you negative values, but still you are limited between 0 and 1. Is that a question?

Sam: Ya, that's Ok. Let's say, you sort of touched on this response there, so we'll go with that to build on that, is there any value of x , so that $\log x = 0$?

Candidate: log of 1 equals 0.

Sam: If we switch them, and you let's say have log of a zero which is on the border, is there a definition at 0?

Candidate: Is the function defined at 0?

Sam: Yes, when $x = 0$

Candidate: when x approaches 0, the value of log approaches minus infinity, but actually it's an asymptote, ..., so we can't say it's defined exactly.

Sam: So then, using the explanations you have given here, if you have this basic question, $(-3)^x = 9$, can you use logarithms to find the value of x ?

Candidate: By doing the inspection, $3^x = 9$, so $x = 2$, if we say $(-3)^x$ equals 3^2 ... continues to write:

$$\begin{array}{l}
3^x = 9 \\
x = 2 \\
3^2 = 9
\end{array}
\qquad
\begin{array}{l}
(-3)^x = 9 \\
(-3)^2 = 9 \\
3^2 = (-3)^2 \qquad 3 = (-3) ? \\
\log_3 3^2 = \log_3 (-3)^x \\
2(\log_3 3) = x \log_3 (-3) \\
2 = x (\log_3 (-3)) = 1 ? \\
3^? = (-3)
\end{array}$$

Figure 5: Sam's Interviewee's Written Response

Candidate: (continues) This quantity equals to 1. Log of -3 base 3 what does it mean? Do I have a negative number, no I don't, so it doesn't fit with the definition so here we reach a contradiction with the following approach to solve it like that...a negative number doesn't belong in our domain, ok, it's not defined.

Sam: ok, next question...

By looking at this interview extract, it could be concluded that Sam, the interviewer, and his interviewee had different levels of mathematical knowledge. When the interviewer posed the question, the candidate provided an answer from the calculus perspective, which was unexpected to the interviewer. Several reasons could possibly contribute to Sam's decision to disregard the candidate's answer. Maybe it was Sam's poor interviewing technique, or perhaps he found this answer to be irrelevant to the current Mathematics 12 course curriculum. Also, it is possible that Sam didn't understand the candidate's response, and decided to ignore it, thus preventing the exposure of his lack of knowledge. From my familiarity with the student, I believe it was a latter option. This also explains why there were no follow-up questions, and Sam asked the candidate to take another try. Then, the candidate began to struggle, as his or her vocabulary was

limited, or probably was forgotten. The candidate's explanations were not clear, the candidate even tried to use his or her knowledge of calculus again.

Candidate: -3 to the power of x by the definition if we're doing the integration of $\log x$ the answer will be that if I'm going from integration of that function it will give me an absolute value. So basically a negative value doesn't exist by itself, it doesn't have a value...

Even though the candidate's answer was very vague, there were several moments where Sam could have interrupted to help. Especially at the beginning, when the candidate tried to use the definition of a logarithm as an area under the hyperbolic curve. Sam could have prompted the candidate to investigate whether he knew the properties of the function $y = (-3)^x$. However, the exponential functions with negative bases are beyond the school curriculum.

For the purpose of this research, I decided to stay focused on the behavior of the interviewer. Although this candidate's responses were unclear and not mathematically precise at the times, the interview kept on going, leaving the questionable answers unchallenged. This could have happened because the interviewer's mathematical knowledge and the candidate's subject matter knowledge did not coincide. The subject matter knowledge of the interviewer appeared to be limited. Sam was unable to comment or probably even comprehend the candidate's responses. That is how he commented on the received answer, in his evaluation of the candidate, "[Candidate] again referred to how logarithms are defined in terms of integrals, but failed to make a connection to how this problem could be solved..." The question that remained is, can this problem be solved? To remind the reader, here is Sam's question: "Can you use a logarithm to solve a question with a negative base, for example: $(-3)^x = 9$?" To clarify what Sam meant by saying "how this problem could be solved", I investigated his anticipated answer.

The question being asked is a special case, where the exponent in question is defined. In these special cases a student could treat the question as $3^x = 9$ and double-check their answer on the calculator. This would of course require that students be able to recognize these cases, and in doing so would imply they already know the answer from inspection so using $\log_3 9$ would not be beneficial to them.

In Sam's own response, I find it difficult to detect an answer to his question. Indeed, the particular equation $(-3)^x = 9$ can be called a special case, but the explanation of why this equation cannot be solved by using logarithms is not included in Sam's anticipated answer. It could be treated as evidence of Sam's limited knowledge of the definition of logarithm. This might also be a reason why Sam did not interfere in the candidate's responses. Perhaps, he did not know that a logarithm can be defined as an area under the hyperbolic curve, which can be treated as insufficient subject matter knowledge in area 1, previously mentioned on page 76. For detailed description of this interview episode, refer to page 79. As mentioned earlier, the reason for Sam's choice to ignore the candidate's geometric interpretation of logarithms could be that he considered this knowledge to be too sophisticated for high school students and therefore he sought alternatives. However, the examination of additional evidence – in Sam's anticipated answer and his analysis of the interview – suggests that this was not the case.

In the regular interview setting this situation would remain uncertain; however, the research task, designed for this study, allows one to get through the layers of the interviewer's knowledge, by clarifying some aspects through the interviewer's anticipated response, and the evaluation of the candidate. Prior to the interview, Sam wrote, "I expect that the candidate will make the connection to exponents and discuss how relating a logarithm as the inverse of exponential function does not permit negative values." Then, he concluded,

...I didn't feel the candidate's initial answer was on the fundamental level. By that I mean that logarithms and exponents are much more tightly linked than integrals and logarithms are, especially in the high school curriculum...I expected a response that was more in terms of the relationship rather than using integrals. I tried to redirect him in his explanations, though the candidate wasn't able to relate to logarithms without integrals...

In his own testimony, the interviewer could not relate to the explanations by the means of calculus; Sam remained true to the one definition he knew, probably from high school. Throughout the entire evaluation, I could identify that there were no alternatives that were accepted or acknowledged by the interviewer. This particular pre-service teacher, Sam, probably believed that, in order to teach, it was just enough to know what the students should know. This justified the conclusion that his subject matter knowledge is insufficient in the area 1, logarithms as numbers, previously mentioned on page 76. This includes the conceptual knowledge of the definitions of logarithms. It is important to note, that the aforementioned episode with Sam as an interviewer, was the only occasion where alternative definitions of logarithm were mentioned. This might be an indication that pre-service teachers (10 interviewers) do not know them, or believe that this particular knowledge is irrelevant when teaching high school mathematics.

Interestingly, historically, both definitions of logarithm through the concept of an inverse, and the concept of an area were developed independently of each other, and remained disconnected for decades. It is important that teachers know about it, and understand different representations of logarithms.

To explore pre-service teachers' subject matter knowledge in area 2, I analyzed the interviewers' knowledge of logarithmic laws. For that matter, I chose to focus on another interview that was inventive in its purpose. The interviewer, Kurt, was trying to assess if the candidate could explain why $\log(ab) = \log(a) + \log(b)$. A knowledgeable

teacher would try to establish that there exists an isomorphic connection between the product and the sum. When explaining this property to students, it would be important to mention and even illustrate the historical significance of the invention of logarithms, which allowed finding the product of numbers through addition. These are important conceptual aspects, the understanding of which could be evidence of specific subject matter knowledge.

Consider the following situation:

Kurt: So the question is, you have $\log(ab) = \log(a) + \log(b)$ How could I explain it in terms of

Candidate: I see some connection between $10^a \times 10^b = 10^{a+b}$ and $\log(ab) = \log(a) + \log(b)$

Kurt: What do you mean by connection?

Candidate: I mean both equalities have the same base 10, left sides of both are products, and the right sides are sums...but how can I get from one to another? Let's try to log both sides of the first equality...

$$\begin{aligned} \log(10^a \cdot 10^b) &= \log(10^{a+b}) \\ \log(10^a) + \log(10^b) &= \log(10^{a+b}) \\ a \log 10 + b \log 10 &= (a+b) \log 10 \\ a + b &= a + b \quad ? \end{aligned}$$

Figure 6: Kurt's Interviewee's Written Response 1

Candidate: Oops

Kurt: Maybe take a closer look at each expression

Candidate: They are inverses of each other, how would I connect them...I think I know, (writes on paper):

$$\begin{aligned} \text{Let } \log ab &= \log a + \log b \\ ab = 10^z & \quad a = 10^x \quad b = 10^y \\ 10^x \cdot 10^y &= 10^z \\ 10^{x+y} &= 10^z \\ x+y &= z \\ \log a + \log b &= \log ab \end{aligned}$$

Figure 7: Kurt's Interviewee's Written Response 2

Kurt: That's right, thank you...

From this episode, one could learn that the interviewer's subject matter knowledge about the product law is formal (Fischbein, 1999), or relational (Skemp, 1987). In a way, the interviewer tried to verify if the candidate could provide a formal proof of the product law. Kurt indicated this in his rationale for the following question, "...though this question is a performance question, it should reveal a higher level of understanding. The ability to prove the identity and communicate clearly how it is done will provide information about the candidate's advanced training in mathematics and the understanding of logarithms..." The candidate's response to the answer fell right into interviewer's expectations, except for one detail. To fulfill the expectations, the candidate should be able to "discuss how the product of the exponents with the same base is the sum of the exponents and logarithms are a way of bringing multiplication down to addition..." This quote allowed me to become more precise in my analysis. Even though the wording was awkward, it was possible to sense that the interviewer tried to reach to

deepest knowledge he could possibly explore. Nevertheless, Kurt settled for less. However, the intent and the interviewee's evaluation indicated that Kurt's knowledge of basic properties of logarithms went beyond instrumental and relational. It would border between relational and logical, according to Skemp (1987), or between algorithmic and formal, according to Fischbein (1999). Though the majority of pre-service teachers exhibited an understanding of logarithmic properties only at the procedural level, which indicated their limited knowledge in area 2, this particular episode exemplified an exceptional situation, where the participant moved beyond the procedural. Kurt exhibited thoughtful proficiency in the second area of knowledge.

Area 3, previously mentioned on page 76, encloses the knowledge of how the relationship between positive numbers and their logarithms becomes a function, and deals with the properties of logarithmic functions. Seven participants included questions related to logarithmic function in their interviews. The questions ranged from how to define this function, to applications of logarithmic functions. The reader is invited to look at the treatment of the fundamental knowledge of logarithmic function offered by one of the interviewees, called Kal.

Kal: What is a logarithmic function?

This question could be addressed to the teacher and to the student. On the student level, it would probably be enough to repeat the most popular definition from the school textbook, something like, it is an inverse of the exponential function, and draw a graph symmetric in the line $y = x$, to the exponential function. The description of the main properties such as, domain, range, points of intersection, symmetries, asymptotes, etc should follow it. On the teachers' level, it could be anticipated that the interviewer knew about different representations of logarithmic functions, and how they are connected.

This knowledge could be exhibited in the follow up questions or prompts that would lead the candidate to explain how logarithmic functions are used to model real life situations, and why they are appropriate for this matter. In this regard, the analysis of the entire interaction between the interviewer and the candidate seems to be most revealing.

Candidate: Well, just to give you the definition of logarithmic function. I would say that it's the inverse of exponential function. Umm... just to give you brief history behind it, this logarithm operation was invented simply um ... just to simplify long numerical operations to find the inverse of exponential functions. Can I give you an example?

Kal: Sure!

Candidate: ...(showing work on paper) if we have the logarithm of a number x in base b, let's say $\log_b k = n$ then its inverse is the exponential function of the base b raised to the power of n equal to the number x, $b^n = x$.

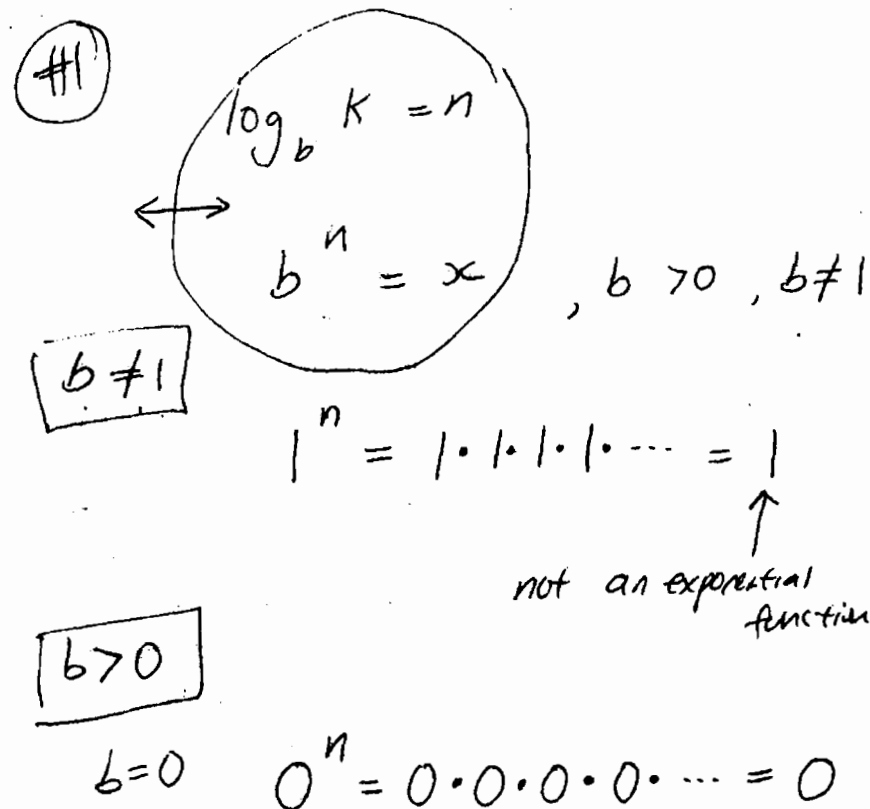


Figure 8: Kal's Interviewee's Written Response 1

Candidate: (continued)...the logarithm is the inverse of exponential function given that the base is positive, and doesn't equal 1. The base cannot be one, because this simply

means that what we are doing is, umm... if the base equals 1 that means 1 to the power of n, which means we are multiplying 1 by the number of power that we have and so on. This will always be 1. However, 1 is not an exponential function. Therefore, this condition must apply to this definition. The second condition is that the base has to be greater than 0. And if we assume that the base equals to 0, which means we have 0 to the power of n. This, in turn, means we are multiplying 0 by how much the number of power it is raised to. This will always be 0. Once again, it is not an exponential function... Let's see if b equals a fractional number, that means if we have a fraction of half raised to the power of n,

$$b < 0 \quad b = \text{fractional} \# \neq \text{negation}$$

$$\textcircled{1} \left(-\frac{1}{2}\right)^n = \text{positive} \quad \text{when pos} = n$$

$$\textcircled{2} \left(-\frac{1}{2}\right)^n = \text{neg.} \quad \text{when neg} = n$$

$$\textcircled{3} \left(-\frac{1}{2}\right)^n = \text{undefined.} \quad \text{when } n = \text{fract}$$

Figure 9: Kal's Interviewee's Written Response 2

Candidate: (continued) Oh mind you, this has to be a fractional number and also has to be negative, because $b < 0$. So, we have negative half raised to the power of n. If we look at an even power, that means I will have a positive value and when I have a negative n value, then I would have a negative value. Here's another case. That is in (1) and (2). (3) is that if my power is a fraction that means I cannot take the power of a negative fraction number. So this will be undefined. Given these three cases and plus the examples I've proved above, the conditions that the base must be greater than 0 and the base must not be 1 must meet in order for the definition of logarithm to be satisfied.

Kal: well, thanks for your answer...

It is evident that the candidate provided a very extensive overview of the different base exponents, and some were even incorrect, such as in what the candidate wrote (1),

when n is positive, for example, let $n=3$, the power is negative: $\left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$, which is

contradictory to the statement provided. Or in (2), when n is negative, for example, let

$n=-1$, the power is positive: $\left(-\frac{1}{2}\right)^{-2} = 4$. This contradicts the candidate's response. Even

the third statement provided is incorrect, for the counter example, let $n = \frac{1}{3}$, the power

will exist: $\left(-\frac{1}{2}\right)^{\frac{1}{3}} = -\left(\sqrt[3]{\frac{1}{2}}\right) = -\frac{1}{\sqrt[3]{2}}$. However, the interviewer noticed none of these. In

her evaluation she wrote, "...the candidate answered my question very well, and his answer is almost the same as my anticipated answer." What became apparent from this quote is that Kal has a very limited knowledge not only of exponents, but also of logarithms and functions. Her expectations were set even lower than the requirements in high school mathematics. There was no evidence that pre-service teachers possess the knowledge of how real numbers form logarithmic functional relation. None of the interviewers asked any questions that highlight this relation; for example, "locate on the graph and compare the values $\log_{1/2}3$ and $\log_{1/2}5$ ". If interviewers did not question about this particular knowledge, they either did not know it themselves, or did not consider it important for a candidate to possess. The only knowledge present in the interviewers' data was a commonly used relation between exponential and logarithmic function. Data indicated that two of the interviewers did not understand this relation. Possibly, they did not know it, because they did not understand exponents in the first place, as in the aforementioned analyzed episode.

In the setting of the clinical interview, Kal did not say much, and revealed very little of what she may have known. However, the task was designed in such a way that allowed for the discovery of a deeper insight into her' knowledge, evident through her self-evaluation, implicitly provided throughout her questions, answers and evaluation. The old saying comes to mind, "who knows you better than you do?"

In general, the peer-interview task provided a view of an individual's subject matter knowledge of logarithms and logarithmic functions. Through the rationales for the

selection of questions for the interview, interviewers' expected answers, and their evaluations of the candidates, I could investigate the level of the subject matter knowledge of the interviewers, who were in fact, pre-service secondary mathematics teachers, impersonating the heads of mathematics departments.

On one hand, by choosing good questions, participants revealed their awareness of possible difficulties when teaching or learning a particular content. On another, the subject matter knowledge exhibited by the pre-service teachers was generally insufficient, lacking understanding in all three areas: numerical, operational and functional meaning. This prevented them from further development of their interviews into thorough investigations of their candidates understanding and abilities.

5.1.5 Pre-service teachers' pedagogical content knowledge of logarithms and logarithmic functions.

In this section, I describe pre-service teachers' pedagogical content knowledge of logarithms and logarithmic functions, as well as illustrate how their subject matter knowledge is related to their pedagogical content knowledge. To reiterate, the meaning of pedagogical content knowledge is consistent with one defined in Shulman's (1986, 1987) components of knowledge for teaching. *Pedagogical content knowledge* includes knowledge of how to teach, how to assess students' understanding, and how to create an educational setting for meaningful learning. This conceptual framework enabled me to describe pre-service secondary mathematics teachers' pedagogical content knowledge.

There is a growing body of research that considers pedagogical content knowledge as an essential part of teaching and teacher training. Among these are Fenstermacher (1994), Ball, D. L., Hill, H.C, & Bass, H. (2005), Rowland, Thwaites &

Huckstep (2003), Harel & Lim (2004), Leikin (2006). The majority of these studies focused on basic mathematics and elementary teacher preparation. Some of them established that many participants possessed insufficient pedagogical content knowledge. Therefore, it is important to investigate pre-service secondary mathematics teachers' pedagogical content knowledge, and to learn about its relationship to the subject matter knowledge. I investigated the issues in the context of teaching and learning the concepts of logarithms and logarithmic functions; heretofore, no previous studies have examined pre-service teachers' pedagogical content knowledge in this conceptual domain.

Table 2: Types of Questions Asked In the Interview, focusing on the pedagogical content knowledge

Themes	Reference to student's solution	Request for justification	Provide an example
Number of questions asked in each category	7	11	4
Number of pre-service teachers who asked the questions	5	8	4
Example	Solve for x : $\log_3(x-4) = 1 - \log_3(x-2)$ and one of your students, Tom answered, $x = 5$ or $x = 1$. Is his answer correct, incorrect, or partially correct? Please explain your reasoning.	Why are some logarithms undefined? Explain.	Can you give me an example of where logarithmic scales are used in "real" life?

First, I consider the examples of how several pre-service teachers identified pedagogical content knowledge as a necessary component for successful teaching. These

thoughts are excerpted from the interviewers' remarks made before the actual interviews took place. Supporting quotes, from interviewers' explanations for their choices of the interview questions, were identified. Then, I describe pre-service teachers' questions, relating them to the anticipated and actual answers, and finally to the teachers' evaluations of the candidate. This multifaceted process is similar to the one that was employed in the previous section, when analyzing participants' subject matter knowledge.

The majority of the interviewers, 8 out of 10, set their minds to interview for pedagogical content knowledge. The summary of these questions is presented in Table 2. The questions were divided into three themes: reference to a student's misunderstanding or erroneous solutions; questions containing a request for justification; request to provide an example, searching for further clarification/justification. However, not many of the interviewers could successfully carry out their preplanned missions. When explaining the purpose of her interview, one interviewer, Keara, expressed the concern that she often finds students are spending the majority of class time working on 'theoretical' mathematics. In all probability, what she meant by 'theoretical' would be the proving of conjectures and theorems, or time spent on drilling exercises. Keara believed that by engagement in these activities, students lose 'an opportunity to explore the subject on a deeper level'. This judgment stemmed from the belief that a distinction between knowledge and understanding exists. For Keara, knowledge was "a set of facts that students obtain from a subject area; whereas, understanding requires the student to make sense of what the knowledge attained means." For clarification of her point of view, she provided an example, "...being able to solve for x an equation $2x + 3 = 9$ is the result of

knowledge that the student has attained. However, understanding occurs when the student understands why one takes the steps to obtain the answer $x = 3$ ". Further to this, she explained that a focus on understanding was the underlying principle for her interview questions, "It would be important to me to find a candidate who would teach towards students' understanding, especially since there is a provincial exam at the end of Mathematics 12 course".

The pedagogically intended questions were identified in the participants' questions. These questions were concerned with a student's difficulty and often prompted an interviewee to create a teaching situation that would help an imaginary student out of the problematic condition. The evidence of the pedagogical content knowledge was explicitly expressed by the pre-service teachers in their written explanations and justifications provided in the forms of rationales for interview questions, anticipated answers, and evaluations of the candidates. Further, it was clearly articulated during their actual interviews.

When reasoning his purpose for an interview question, an interviewer, Kurt, wrote, "For the next line of questioning, ...I wanted to ... see if he (the candidate) can recognize student behavior and diagnose where the student was going in his thinking for solving...I tried to shift his (the candidate's) focus from trying to obtain a response to how the response is obtained". From this extract it is evident that the interviewer's objective was to target the candidate's ability to diagnose the student's understanding. This could be categorized as an attempt to assess the interviewee's pedagogical knowledge. In another example, the interviewer, Greg, is more explicit about the intent of

his interview, “I posed my questions broadly enough to find out more about his (candidate’s) pedagogical abilities”.

As a means of targeting the candidate’s knowledge, another interviewer, Nora, chose the following question, “If the student posed the question ‘Why are some logarithms undefined,’ how would you explain this to him?” Nora’s explicit pedagogical intent was to discover the candidate’s explanation given to the imagined students. She wanted to evaluate the candidate’s accountability for, and knowledge of, students’ learning and understanding of a particular content. At first, Nora exhibited the behavior of the pedagogically knowledgeable interviewer; she shifted the attention from a direct answer to her, to providing an answer to some third person. The purpose in doing so could have been to reduce the level of anxiety in the candidate. Nora also explicitly advocated for the sustainability of using a ‘why’ question to accomplish the purpose. In the reasoning behind this decision, she stated, “... this question ties into the idea of having students attain understanding over just pure knowledge...Explaining to students’ why $\log_b n$ only exists for $n > 0$ will help students obtain understanding. It is also important to explain this idea step-by-step, so that students understand the logic and don’t just memorize this fact ... Hence the use of an unexpected ‘why’ question seemed appropriate here as this question gives an opportunity for the candidate to reflect on why certain mathematical ideas are the way they are...”

There seemed to be a strong connection between what Nora wanted to find out about the candidate’s pedagogical knowledge and how she decided to get it done. Then again, the research task was designed in a way that allowed me to continue this line of

inquiry further. By analyzing the interviewer's treatment of the candidate's response, I was able to conclude if a desirable objective was reached.

Nora: Okay good, Okay, now moving on. As a mathematics teacher, one has to ensure that students understand the reasoning behind mathematical ideas and not just memorizing them. If a student posed the question 'why are some logarithms undefined?' how would you explain it?

Candidate: I personally look to definitions as starting points, and would encourage students to examine what happens when constraints are violated. So I would have them start with a table of values of logs with negative numbers for bases, exponents, and arguments, and also examples of exponents with negative bases and exponents. And I would ideally use the question we just discussed as a hook for a constructivist lesson, in which students would have to go back to the meaning of logs as they relate to exponents, and then come up with examples of different logs that are undefined, so they can see for themselves what's happening. I might even use this example to discuss continuous fractions and inverses, because it relates to that as well, or use it as an example of how to take observable relationships and translate them into a proof. So it's a very rich topic I think. I would have to structure the lessons so the students didn't get overwhelmed with too many different ideas though, uh but delving into the why's is definitely something I would want to encourage.

Nora: that is definitely true, thank you.

In the response to her question, the interviewer received a narrative of implicit and vague ideas, from describing some possible tasks to structuring a lesson. The candidate did not demonstrate any attempt to answer the question, "why are some logarithms undefined". The candidate also used mathematical terms inappropriately at the wrong times. For instance, the candidate talks about 'the table of values of logs' with 'negative bases, exponents, and arguments'. It would be interesting to find out how the interviewee plans to distinguish exponents and logarithms in this table. Nevertheless, Nora decided not to prompt the candidate's response. Possibly she agreed to the response received. I cannot ignore the fact that Nora may not have wanted to criticize her classmate, the candidate. It could have been due to the specific way in which the assignment was set. The candidate and the interviewer had to switch their roles and

repeat the Job Interview task focusing on a different mathematical content. Therefore, being over critical towards the candidate's responses could have resulted in reciprocation once the candidate became the interviewer. Though, there is another possible explanation of her agreement to the candidate's response. This could mean that her initial intentions were superficial. Intuitively, Nora knew what was important for teachers to know, but there was no evidence that she knew why. Therefore, an insignificant, and even meaningless, response was treated as satisfying. In her evaluation of the candidate, Nora wrote, "I believe that Candidate would make a very capable mathematics teacher, and I would hire him for the position. The answers were given with confidence". Evidently, Nora exhibited rather limited pedagogical content knowledge.

I have purposely chosen this particular episode, which was analyzed in the preceding section for subject matter knowledge. Earlier, it was established that the same interviewer, Nora, possessed very limited subject matter knowledge. Indeed, her mathematical knowledge was shallowly presented and discussed, when the main focus was directed at the pedagogical issues. On the surface, Nora showed deep interest in the pedagogical competence of the candidate, but her limited subject matter knowledge restricted her from a possible meticulous investigation of pedagogical issues. Many pre-service teachers who possessed weak subject matter knowledge exhibited inadequate pedagogical content knowledge. This finding coincides with the research findings (Rowland, 2003; Ball, 2005, Leikin, 2006).

In the following section, I investigate the data collected from the second research task, called the Math Play. The key difference between the job interview and the math play, is that in the first activity the interviewer's thinking was prompted by the

interviewee's answers, whereas the play is a self-exploratory task, because it was a one person monologue authored by a pre-service teacher. In the job interview, the pre-service teachers investigated any aspect of teaching or learning of logarithms and logarithmic functions, of their choice. However, in the math play, all the participants were prompted by the same task, and were to respond to the same prompt.

5.2 Analysis of teachers' knowledge from the Math Play

Chronologically, the Math Play was an activity that followed the Job Interview. Pre-service teachers were given a choice of topics for this next assignment. Six students out of ten who had participated in the Job Interviews, decided to continue with logarithms activities. Four of the participants dropped the topic of logarithms and (they) changed their focus of study. The reasons for this 'change of heart' could be that pre-service teachers found this content very challenging; or on the contrary, they found it very easy and decided to challenge themselves with other concepts, or they just wanted variety.

To reiterate, in the Math Play, each pre-service teacher was to analyse an erroneous situation that would expose a student's misunderstanding:

Act 1, Scene2:

There is a conversation between a teacher and a student (there are 30 students in a class):

T: Why do you say that $\log_3 7$ is less than $\log_5 7$?

S: Because 3 is less than 5.

Pre-service teachers were asked to diagnose the student's misunderstanding, formulate a plan for remediation of the misunderstanding, and write out the balance of the interaction(s) in the form of a *math play*. The research data consisted of the pre-service teachers' written assignments. They were asked to submit their version of Act 1, Scene 1, detailing personal encounters of where and how the problematic situation could

take place. Pre-service teachers were also to write the remediation part. It could be presented in the form of a narrative, or Act 2, Scene 1. In Act 2, Scene 1, pre-service teachers were to create a teachable moment, whereby they would orchestrate the events, tasks, and conversations, to lead a student out of a problematic situation.

Data sources associated with this task consisted of the following writings:

- a. Pre-service teachers' diagnosis(es) of the students misconception;
- b. Their personal account on where and how the given situation could take place, in the form Act 1, Scene 1;
- c. The remediation part, where the participants had to organize a situation to guide a student's learning, in the form of Act 2, Scene 1.

5.2.1 Teachers' explanation(s) of the possible reasons why the given error occurred

This particular lens allowed me to measure the depth of the participants' subject matter knowledge. It is an open-ended task, as it provides the participants with the freedom to elaborate not only on the issues related to the knowledge of logarithms, but also on those of pedagogy. This task presents participants with an opportunity to reflect on possible pedagogical inconsistencies that turned out to be fatal to student understanding.

The variety of explanations provided by the participants as to why the particular mathematical error occurs would reveal the depth of pre-service teachers' knowledge. Participants' imagination, subject matter knowledge and understanding of envisioned students' performances are main factors that contribute to the assessment of teachers' preparedness to teach. While four out of six pre-service teachers provided only one

explanation of why a given misunderstanding took place, two participants, Mike and Greg presented multiple reasons.

The summary of sources for a student’s misconception provided by each of six pre-service teachers is presented in the following table.

Table 3: Sources of the Student's Misconception (from the Math Play)

Name	Root(s) of misconception		
Greg	Missing meaning – change into exponential form	Student’s attitude	Misunderstand multiplication
Natalia	Misunderstanding of Change of base law, common logarithm and definition		
Kurt	Definition of logarithm		
Mike	Misunderstanding how logs are related to the exponents $\log_5 7$ misread as 5^7 and $\log_3 7$ misread as 3^7	“misconceive solution $\log_5 7 = y_1$ and $\log_3 7 = y_2$ to comparison”	Relating logarithms to fractions $\log_3 7$ misread as $3/7$ and $\log_5 7$ misread as $5/7$
Nora	Change of base law		
Kal	Missing meaning – change into exponential form		

The data provided in the table presents the synopsis of the participants’ responses. The three columns to the right of the pre-service teachers’ names represent the information about the sources of the student’s possible misconceptions. Teachers indicated that these particular misconceptions led the student to a faulty response to the

given task. As is shown in the table, all six participants were concerned with mathematical content. For them incomplete or absent mathematical knowledge is one reason for the student's inability to respond correctly. There was only one pre-service teacher, Greg, who expressed his concerns not only with the content, but also with the student's attitude. For Greg, the student's personality could be a possible contributor to the mistaken answer. The detailed analysis of his responses is provided in the following sections of the chapter.

5.2.2 Procedural Subject Matter Knowledge and Workable Pedagogy

In the following section I present and discuss the situations where pre-service teachers with limited mathematical knowledge, but a workable knowledge of pedagogy, were able to transform some of their limited knowledge to trickle down to the students. Let's return to Greg. He provided three possible causes for the student's mistake. Here are his explanations:

1. "This could mean that the student understands what the meaning of logarithms is but that they just were too lazy and lacked the experience to know that their intuitive beliefs were wrong."
2. "The student may not understand exactly how to change it into exponential form, which means that they don't understand the logarithmic operator."
3. "The big problem is when the student knows how to convert it into the exponential form and still thinks that the $\log_3 7$ is less than $\log_5 7$. This would show a wider problem that involves the child's understanding of exponents and maybe even multiplication."

The first reason provided by Greg could be considered as the most obvious to him. Here the pre-service teacher tried to connect “the meaning of logarithm” as constructed by the student with his “intuitive beliefs”. According to Fischbein (1999), when a student understands something intuitively, this means that he believes that this is true without any doubts, questioning, or proof. So, Greg could have believed that the student possessed a mistaken meaning of logarithm, and then acted upon his intuitive belief. Greg also expressed his concerns about the student’s behavior, for being “too lazy”. For Greg, a student is not only a mathematics scholar, but also a human being who often acts upon his/her beliefs. Further analysis of the remediation scene might better inform the study regarding these issues.

The second explanation presented, is lacking proper wording. What did Greg mean by ‘the logarithmic operator’? Perhaps the student tried to establish a link, very algorithmic in nature, between the process of inverting logarithms into exponential form and some kind of self-invented, algebraic operator? It would be interesting to see where Greg could go with it, but this situation was not chosen by him for future remediation. It is possible that Greg himself was not comfortable discussing this issue any further, because he was lacking in subject matter knowledge.

The last reason goes beyond the mathematical content at stake. Greg expressed the belief that the misunderstanding might be caused by the missing prerequisite knowledge. Greg also implied that knowing “how to convert it into the exponential form” might not be enough to deal with the question in the proper manner. This diagnosis places Greg in a situation of endless persistent questioning, which would limit him from any

conclusiveness about the problem. The pre-service teacher chose not to continue along this path of inquiry.

The first response received an interesting continuation in Greg's remediation scene (Act 2, Scene 1). But before appealing to further analysis, let's look back. The pre-service teacher questioned the student's understanding of the meaning of logarithm. What could Greg possibly do about evaluating his assumption, and 'fixing' the problem? One of the possible solutions could be checking the student's knowledge of the different representations of logarithm, such as graphical, or algebraic. Hopefully, at least one of them would work, and if it did, then the next step could be building up the connection between graphical and algebraic.

Greg's means to remediation is described in the following: "I would have the student convert both logs to exponential form and then ask them again. If the student still doesn't see the problem I could simplify the problem by using $\log_2 16$ and $\log_4 16$. When this is put into exponential form then we get $2^x=16$ and $4^y=16$. Now I would show how the question is, which is at least x or y . Students quickly see that $x=4$ and $y=2$ so y is least, but that means $\log_4 16$ is smaller than $\log_2 16$. Once the student sees this they may understand, and if they don't I know that the problems is much bigger and should be dealt with individually."

It is evident, that Greg decided to illustrate how similar, but less complicated problems can be solved. The reduction of the complexity comes with utilization of perfect square 16 instead of a prime number 7 as a power². Sixteen is an easy number; it

² The term "power" refers both to the exponent and the exponential expression. For example, b^c is a power, and c is a power of b .

can be represented as a power. Drawing a comparison between two examples would provide the grounds for further discussion.

What can be learned from this episode with regards to Greg's knowledge? It was identified earlier that he possesses a procedural knowledge of logarithms, and is consistent throughout. Though his subject matter knowledge is limited, he exhibited a good choice of the appropriate pedagogical tool. This enabled Greg to transfer his knowledge on to the student. However, in Act 2, Scene 1 there is no information about what convinced the teacher that understanding indeed happened.

Greg's Act 2, Scene 1 in the narrative form,

I would introduce logarithms as the inverse function to the exponential function and then go through the definition of the different form and how to switch them back and forth. (Maybe do a few quick examples and then have them try a few to make sure they get it)

I would put them into pairs and each pair will graph the functions $y = 10^x$ and $y = \log_{10}x$ and then they would have to find the relationships between them.

I would also ask them to see what happens if we change the base 10 to different bases including negative ones and then what happens when we multiply or divide the function or x by a constant.

We would discuss as a class the results that we found and from there come up with the domain and range of the function.

It is in here that a student would say that $\log_3 7 < \log_5 7$

Because the class discussed the difference in bases they would have looked at this sort of thing so I would ask

"So when we increase the base then what does the graph do?" The students should notice that the graph is showing that if x is greater than 1 then the larger the base the smaller the values but for x is less than 1 the larger the base the greater the value.

Then I would ask, "why is this happening? Can we use the definition to show that this makes sense?"

The students will look at the definition and switch out forms to exponential form to show why this is happening.

A similar situation was witnessed in another episode. However, in this case, the example used by the pre-service teacher, Kal, to overcome the student's misconception, was more closely related to the given problem,

Act 2, Scene 1

Kal: Ok. Then consider this pair of logarithms: $\log_3 9$ and $\log_5 9$. Can you see why this is a similar problem?

Student: Yeah, both logarithms have the same number as the power but different bases.

Kal: And which of these two are larger?

At this point, the student could make the connection that $\log_3 9$ is equal to 2 since $3^2 = 9$. They would then notice that $5^2 = 25$ and hopefully make the connection that $\log_3 9 > \log_5 9$. For the sake of argument, let's assume the student makes the same error in reasoning:

Student: $\log_5 9$?

Kal: Well let's find out. Everyone write $\log_3 9$ and $\log_5 9$ in exponential form.

wait

Student: What do we get for $\log_3 9$? Tommy?

Tommy: $3^x = 9$

Kal: Right. And for $\log_5 9$, Rachel?

Rachel: $5^x = 9$

Kal: Well let's see if we can find out what x is. Any suggestions?

Student₂: $3^2 = 9$

Kal: Good. This is one that we can solve if we know some powers of 3. The second one is harder though, since 9 is not a power of 5. How can we find x for this one? Any suggestions?

Student₃: Guess and Test?

Kal: Ok. Could you explain that in more detail?

Student₃: We could test values for x using our calculators. Like, start with $x = 1.2$, because $x = 1$ gives 5 which is too low, and $x = 2$ gives 25 which is too high.

Kal: Very good. So try this method.

wait

Kal: What do you find for x?

Student₄: $x = 1.36$

Kal: Ok, so we have $3^2 = 9$ and $5^{1.36} = 9$. Convert these back to logarithmic form.

wait

Kal: What do we get? *Ask original student*

Student: $\log_3 9 = 2$ and $\log_5 9 = 1.36$

Kal: Good! Now looking back at the original question, can you tell me which is larger?

Student: $\log_3 9$

Kal: Right. So can you see where you made your mistake?

Student: I think so.

Kal: Well, go ahead and test yourself with the rest of the examples to see if you've got the concept. Don't be afraid to convert the logarithms to exponential form if you're unsure.

The teacher can check in with this student later when the rest of the class is working on something else. If the student is still struggling, the teacher can point out the area where the student is making their error in reasoning.

In the diagnosis, Kal wrote,

Since the powers were the same, the only other two numbers to compare were the bases... So, since 5 is larger than 3, the student reasons that $\log_5 7$ is greater than $\log_3 7$. Since this student's struggle was caused by their desire to find a 'quick-fix' solution method rather than draw on the information given to them in the lesson about the value of logs, my remediation plan involves having them re-visit the meaning of the logarithmic form. To do this, I plan to have them convert the log into exponential form, compute the value of the missing variable (by guess and check), then revert to log form and compare the value of the two logs.

From Kal's diagnosis, I learned that she identified the source of the student's misconception as the student's misunderstanding of "the meaning of the logarithmic form". The word "form" gives her explanation a procedural connotation. And indeed, her plan for the remediation supports this. "I plan to have them convert the log into exponential form," says Kal. For this pre-service teacher, the "meaning" is associated with an exchange into the exponential form. In Act 2, Scene 1, she is consistent with her explanation. Her 'fixing' plan is based on the substitution of one problem with another, similar in nature, but easier for the sake of the procedure. In the remediation stage, Kal commented, "At this point, the student could make the connection that $\log_3 9$ is equal to 2

since $3^2 = 9$. They would then notice that $5^2=25$ and hopefully make the connection that $\log_3 9 > \log_5 9$. For the sake of argument, let assume the student makes the same error in reasoning.”

The remediation stages of both aforementioned pre-service teachers, Greg and Kal, also were similar, because of similar pedagogical techniques. Both pre-service teachers worked out procedurally easier problems first, and then moved onto the given one.

It is also important to mention that Greg’s attention to the student’s behavior and attitude did not receive further development. In the remediation part of the task, Greg completely abandoned the idea he initially expressed about the student being too lazy. His concerns with mathematical content overpowered his explanations. Unfortunately, other participants never even considered the possibility that student’s personality could be a contributor to his or her decisions or actions. Participants chose to attend only to students’ knowledge of mathematics. This conditioned my analysis of the participants’ responses.

Witnessing this, neither Greg nor Kal identified logarithms as mathematical objects, real numbers. Their treatments were seen as well-related to the student’s knowledge and yet, there was a missed opportunity of comparing logarithms and comparing numbers. As such, the notion of logarithms as numbers was overlooked. For them, “log” simply means to convert into exponential form. However, a limited mathematical knowledge, with a workable pedagogy may allow for the transformation of this limited knowledge to be passed down to the students. The majority of the pre-service teachers’ responses, four out of six, would fit into this category.

5.2.3 Extensive Subject Matter Knowledge and Skilled Pedagogy

In the following example, I turn to a very original math play that was written by Mike. He was another pre-service teacher, who presented three justifications for the given misconception. All of them are concerned with the mathematical content. However, these justifications are diverse in their mathematical nature. When some of them looked very superficial, others were right to the point. Mike was trying to take an original approach to the problem. Here is an excerpt of his diagnosis:

1. I think that the misconception that $\log_3 7$ is less than $\log_5 7$ since 3 is less than 5 is what it is because the student does not have a full understanding of how logarithms are related to exponents. It is very likely that the student could be viewing the expression $\log_3 7$ as 3^7 and the expression $\log_5 7$ as 5^7 due to a misconception of the exponential relatedness of logarithms. In this case the student would be entirely correct in saying that 3^7 is less than 5^7 since 3 is less than 5.
2. Another possibility for the formation of this misconception is that the student just read the question too quickly and did not stop to consider that $\log_3 7$ is equivalent to $3^x = 7$ and $\log_5 7$ is equivalent to $5^y = 7$. If they rushed through these equivalencies then they could misconceive the solution to the comparison. They could also have gotten the comparison backwards or unwittingly switched the terms in their mind, verbalizing that $\log_3 7$ is less than $\log_5 7$ but actually thinking that $\log_5 7$ is less than $\log_3 7$, which is the correct answer.
3. The student could also be forming this misconception by relating logarithms to fractions. The student could be relating $\log_3 7$ to $3/7$ and $\log_5 7$ to $5/7$. In this case they would be correct in saying that $3/7$ is less than $5/7$ because 3 is less than 5. A student might make a leap to this relation because they might be comfortable with fractions and uncomfortable with logarithms and thus would lean toward what they are comfortable with. By viewing the concept of a logarithm as a mysterious entity the student might make a quick generalization, gravitating toward concepts that they have spent more time on and have practiced more.

Further, he summarized his pedagogical strategy, “Overall, now that this misconception has come about, as a teacher I need to think about how I can steer the student toward a fuller understanding of the concept so that they can recognize that their

statement was false. Merely telling them that their answer was wrong will do nothing for their understanding, so questioning is very important in this scenario to steer the student towards an understanding of the correct answer.” Mike’s pedagogical strategy was to use questioning. At first glance at Mike’s diagnosis, it seems that he had a strong background in mathematics, and was familiar with different pedagogical strategies. To verify this assumption about his ability, I refer to his actual play.

Act 1, Scene 1:

Mike: We can see where logarithmic values come from by looking at the general graph of the logarithmic function (draws the function). What would be the one point on the curve that would be the same for any base, Joseph?

Student: Well from your drawing it looks like that point is (1,0).

Mike: Could you explain why this might be the case, Irman? Think about exponents.

Student: Oh, I think that it is because anything to the power of 0 equals 1. Is that right?

Mike: Exactly right, I am glad to see that you were able to use your knowledge of exponents. Now, can you describe the behavior of the logarithmic function between x equals 0 and x equals 1, Jessica?

Student: I guess the y -value keeps decreasing forever, but the x – value never quite makes it to 0?

Mike: Good for you. What about the behavior from x equals 1 onward? Joey?

Student: Well, it looks like the y -value is increasing slowly.

Mike: Right, but, you know what, we need to be a bit more specific. If I am talking about log base 10, what would be an easy point to observe on this graph when x is greater than 0, Joey again?

Student: Oh, okay, when y equals 1, x will equal 10.

Mike: That’s it, now what about say base 2, how will this function compare to base 2, Rebecca?

Student: Umm, well for base 2 at y equals 1 x will only equal 2 so the function will be increasing faster than for base 10, is that what you were getting at when you asked about behaviour before?

Mike: Yes thanks for clarifying that so nicely for everyone. So now if we look at different bases (drawing graphs for different bases with (1,0) and (a,1) marked) we can

see that as the base increases the graph of the logarithmic function increases slower and slower, see how the graphs look flatter?

Student: (nodding, whispering to each other)

Act 1, Scene 2:

Mike: Well, you are all nodding so that means that you are experts now, right? Excellent. So if asked you to compare $\log_3 7$ to $\log_5 7$, which would you say is less, Emmanuel?

Student: Hmm, what? Oh um $\log_3 7$

Mike: Really? Why do you say that $\log_3 7$ is less than $\log_5 7$?

Student: Because 3 is less than 5.

Act 2, Scene 1:

Mike: Okay, I can see how you might have got that answer, but can you draw how that would look for me?

Student: Well I guess that you would draw the graphs of y equals log base 3 and log base 5. . . . Hmm, both graphs would pass through (1,0) and then log base 3 would also go through (3,1) and log base 5 would go through (5,1). I guess you would increase steeply to come up to (1,0) and then flatten out when moving toward both (3,1) and (5,1) so...

Mike: Sorry, could you stop right there for a second?

Student: Sure.

Mike: Which graph is going to be steeper?

Student: Uh, the log base 3 one I guess...

Mike: Great, so when you get to x equals 7 on the log base 3 graph, are you going to be at a higher or lower y -value than at x equals 7 on the log base 5 graph?

Student: I'm going to be higher... oh oops, how did I just mess that up? That sucks.

Mike: You know what? That's okay. Now you know that for the same x value, a graphical understanding tells you that a lower base will have a steeper graph so the y -value for the lower base will be higher than the y -value for the higher base. I have to say that you making that mistake was a good learning opportunity for all of us. Maybe you just jumped the gun and went for the easy solution, but look what you were able to do once you went through the process, you were spot on.

Student: Yeah I guess I do understand that whole graphical approach thing and the exponent connection, I just had to think about it. Guess I won't make that mistake on a test!

Mike: There you go. I think that you had to take a moment to think because its kind of a new concept and something that you are not used to dealing with, so it was good that

you went through it. Believe it or not there are probably others here that would have given the same answer. You just got to be centre stage and we all learned from you going through the process. Thank you (goes on to give some more sample examples and homework problems).

From a mathematical point of view, Mike was extremely consistent. He situated the problem within the graphical context, and remained true to his idea to the very end. It is evident that this pre-service teacher understood the piece of mathematics discussed. The choice of the graphical approach is not typically used to discuss an algebraically represented situation, which adds value to the teacher's subject matter knowledge.

Pedagogically wise, his commitment to the questioning strategy was evident throughout: "can you draw how that would look for me?" "Which graph is going to be steeper?" "are you going to be at a higher or lower y-value than at x equals 7 on the log base 5 graph?" Mike did not spoon-feed the answer to the student, but in small increments brought him or her to the realization of the misconception made.

In this episode, participant Mike did not state openly that the student's personality could be the reason for his or her failure to answer the teacher's question correctly. Nevertheless, Mike's concerns regarding the student's knowledge of mathematics were addressed at a level sufficient for a novice teacher. He was able to combine his subject matter knowledge and pedagogical knowledge into a coherent teaching experience, though an artificial one.

While the majority of the responses were simply inconsistent, and some pre-service teachers evidently exhibited very superficial subject matter knowledge, the examples like the one just discussed suggest that there are still good mathematics

teachers out there in the field. They possess a solid, balanced knowledge of mathematics and pedagogy, and obviously can employ it successfully in their day-to-day practice.

5.3 What pre-service teachers identify as important to know for teaching

As each of the pre-service teachers outlined their justifications for the choice of interview questions that were required as a part of the Job Interview task, they provided some evidence of the knowledge they considered to be important. Pre-service teachers planned ahead of time what to ask the candidate, and the majority of them were very explicit about the knowledge they prepared to target.

It was not a big surprise when most (9 out of 10) of the participants' intentions were to assess the subject matter knowledge of the candidate. For example, all three questions Nora prepared for the interview were mathematical knowledge oriented. In her rationale, she justified her goal: "All these questions were designed to examine the candidate's skills and knowledge of mathematics." Interviewer Sophie stated in her rationale: "I want to ask a conceptual question about logarithms. This way I could really tell if my interviewee knows what logarithms are..." Mike also explained the aim of his interview, "The original tasks for this interview are to verify and evaluate the candidate's knowledge and understanding of logarithms. I want to make sure that the candidate not only understands the reasons one teaches logarithms, but the significance of logarithms and their fundamental connection to other important mathematical concepts..."

The second important area of expertise, identified by the pre-service teachers as being essential to possess, was knowledge of pedagogy (6 out of 10). For example, Sophie indicated in her rationale that mathematics knowledge is important; however, she

also believes that it is extremely important in teaching mathematics to be able “to view the subject from the student’s eyes and teach in a language that can be understood...” Another aspect of knowledge the teacher must possess is an ability to assess a student’s mistake on the spot. The following idea was expressed by several participants: pre-service teachers were concerned as to whether a candidate could recognize a student’s behaviour and diagnose “...where the student was going.” Among other important characteristics of the pedagogical content knowledge was a teacher’s ability to motivate students, conduct an efficient assessment, and provide quick feedback. It was interesting to find that more than half of the group believed in the importance of possessing both pedagogical and subject matter knowledge.

5.4 From Intentions to Actions

From the strong responses of the pre-service teachers, when their intentions were consistent with their actions, the data exposed that those pre-service teachers who assessed their candidates for the knowledge of only basic facts and insignificant conceptual aspects, possessed limited subject matter knowledge. They achieved their goals, by utilizing very basic pedagogical tools and limited inventory questioning. These pre-service teachers were satisfied not only with primitive, but at the times, even incorrect responses. From the data collected from the Math Play, there was one pre-service teacher Mike, who exhibited subject matter knowledge beyond the school curriculum, and managed to remain consistent to his purpose, and eventually reached his goal. Mike also used a questioning method as his pedagogical technique. It allowed him to complete the assignment to his personal standards; thus, he achieved satisfaction.

The data collected from the above-described situations are quite revealing about pre-service teachers' knowledge, because of the consistencies between pre-service teachers' intentions and their actions. However, there were instances where collected data had to be examined more extensively. Several inconsistencies between pre-service teachers' intentions and actual questions were identified. In some cases, interviewers meant to seek out meaningful mathematical knowledge, but their questions were too shallow to accomplish that goal. These situations were analyzed for the possible causes of these phenomena, which turned out to be either a lack of the subject matter knowledge or a deficiency in pedagogical content knowledge.

Other inconsistencies were identified between pre-service teachers' intentions and their evaluations of the candidates. Interviewers had well prepared questions and answers to their questions prior to the interviews. However, at the time of the interviews, they received responses that were in conflict with their anticipated answers. Rather than probe further, they accepted these answers. These situations were analyzed as to possible causes of interviewers' actions. There were a number of reasons that were identified as explanatory to these episodes. The one most likely possibility was that the subject matter knowledge possessed by the teacher was insufficient to question the candidate's response. Another explanation was that interviewers may have had some personal doubts about their subject matter knowledge, and would feel insecure exposing it in a scenario with a more knowledgeable interviewee.

The analysis of the collected data suggests that out of ten participants, only one pre-service teacher, Mike, possessed sufficient mathematical and pedagogical knowledge. A vast majority of the pre-service teachers exhibited limited subject matter knowledge,

which prevented them from constructing effective pedagogical strategies. By “limited,” I mean that the knowledge exposed in the tasks was mainly procedurally oriented, and restricted by the school curriculum, despite the opportunities to go beyond. It can be concluded that it is necessary for a secondary mathematics teacher to have solid subject matter knowledge in order to develop the pedagogical content tools in the context of this study. However, knowing the mathematical content does not automatically guarantee successful development of pedagogy. The case of Kal, see section 5.1.3, supports this statement. At first glance, the pre-service teacher appeared to have a strong mathematical background; she posed a good mathematical problem, but struggled with pedagogical implementations, providing too much information for a solution. A detailed account of this episode is provided on page 67. However, when analyzing this particular person, I could not collect any hard evidence to confirm my assumption. Perhaps another research methodology, such as a clinical interview, could provide data in this regard.

5.5 Summary

This chapter presented the results and analysis of the research data. When focusing on the concepts of logarithms and logarithmic functions, a systematic approach was used in analyzing pre-service teachers’ understanding of these particular topics. In the procedure for analyzing the data, I looked for common trends, and clustered it around common themes. Teachers’ pedagogical knowledge (at the times they are considered to be inseparable) was explored through the lens of the teachers’ prepared questions and examples, and used in the designed instructional activities.

It was found that the subject matter knowledge of pre-service secondary mathematics teachers within the content domain of logarithms and logarithmic functions is very limited. It was also found that the pedagogical content knowledge teachers hold is related to their subject matter knowledge (subject matter knowledge). It was especially evident in the participants' responses in the Job Interview task. Even though participants were able to identify and utilize the possible difficulties in teaching and learning of logarithms and logarithmic functions, their limited content knowledge prevented them from pursuing their inquiry further.

The Math Play showed that most participants were able to compensate their limited content matter knowledge with a workable level of pedagogical knowledge. This allowed them to situate learning around the mathematics they were comfortable with.

Chapter 6: Conclusion

This study is an extension to the ongoing research on secondary mathematics teachers' knowledge. As the focus for this study, the concepts of logarithms and logarithmic functions were used. Several research studies have confirmed that high-school and undergraduate students have a very poor knowledge of logarithms and logarithmic functions. One of the possible reasons for students' difficulties could be an insufficient teachers' knowledge of this subject domain. As of yet, there has not been research into teachers' knowledge of logarithms. This study was an attempt to fill this gap. The deeper understanding of teachers' knowledge, particularly subject matter knowledge and related pedagogical skills, leads towards improvement of instructional approaches for more effective teacher training.

Three research questions were posed in this study:

- a) What do the designed tasks reveal about the nature of teachers' knowledge?
- b) What can be seen as the relationship between pre-service secondary mathematics teachers' subject matter knowledge and pedagogical content knowledge?
- c) To what extent are these tasks effective and useful as data collection tools for research in mathematics education?

In what follows, I summarize the findings by addressing each of the above questions. I describe the contributions of my research to the field of mathematics

education. I present the limitations and implications of this study to high school teaching and learning, pre-service teacher education, and future research.

6.1 Main findings of this study

One of the goals of this research was to investigate the pre-service teachers' knowledge of logarithms and logarithmic functions. My study has identified that pre-service teachers are aware of possible difficulties of teaching or learning the concepts of logarithms and logarithmic functions. An example of this is the case of the interviewer Nora, who asked one of the crucial questions: "Why are some logarithms undefined?" This question could have led to a deep, meaningful investigation of the candidate's understanding of logarithms. However, this opportunity was abandoned, because Nora's own knowledge of logarithms was insufficient to further pursue the received response. This episode is detailed in chapter 5, section 5.1.4.

On the whole, the pre-service teachers' subject matter knowledge was insufficient for meaningful engagement in learning about logarithms and logarithmic functions. This prevented them from developing thorough investigations of their peers' understanding and abilities. In the Job Interview task, the majority of participants, who were interviewers, simply could not explain why the situations prompted by their own questions were indeed problematic and important.

Many pre-service teachers who possessed weak subject matter knowledge exhibited inadequate pedagogical content knowledge. The majority of the interviewers, eight out of ten, chose to interview their candidates for pedagogical content knowledge, as discussed in Section 5.1.5. However, not many of them could successfully carry out

their intentions. For example, when analyzing Nora's interview, I recognized that her pedagogical content knowledge was also very limited.

In all but one case the data collected from the Math Play confirmed the findings from the Job Interview task. The exception was the case of the participant Mike. His results from the interview task were consistent with all the other participants, in that he too, exhibited insufficient subject matter knowledge and limited pedagogical content knowledge. However, in his Math Play assignment, he showed exceptional knowledge of both logarithmic functions and pedagogical methodology. A detailed account of this can be found in Section 5.2.3. This finding does not contradict the connection of pedagogical content knowledge to the subject matter knowledge; moreover, it illustrates how tight this connection is. Poor mathematical content knowledge is related to inadequate pedagogy, while proficient subject matter knowledge connected to the relevant pedagogy. Other research alerted to this as well. The relationship between the two types of teachers' knowledge was mentioned in the research findings of Ball, et al (2005) and Rowland, et al (2004).

Another goal of this research was to determine the effectiveness of the research methodology developed and used in my study. I designed a unique research task, called the Job Interview, and utilized another research task, known as the Math Play. These activities allowed me to investigate pre-service teachers' knowledge from many different sources that yielded very diverse information about the participants' knowledge. The detailed account of the collected materials from each research task is presented in section 4.2 of chapter 4. Both tasks proved to be valuable data collection tools and were used for

the purposes of analysis. Furthermore, they created meaningful experiences for both the participating pre-service teachers and myself.

To summarize this section, the participants in my study displayed a relatively weak content knowledge of logarithms and logarithmic functions, exemplified by weak subject matter knowledge and related pedagogical content knowledge. The study showed that subject matter knowledge and pedagogical content knowledge are strongly related to each other. The two non-traditional research tasks designed and utilized in this study served as effective data collection tools, and proved to be useful for the purposes of the analysis.

6.2 Contributions of the study

There are several contributions of this study to the field of teachers' mathematics education. Firstly, focusing on methodology, it introduces effective data collection tools to investigate pre-service teachers' knowledge of mathematical concepts. The tasks developed in this study provide researchers with useful tools to investigate the scope of teachers' understanding. Secondly, focusing on specific mathematical content, it provides a finer and deeper analysis of pre-service teachers' knowledge of logarithms and logarithmic functions.

In chapter 2, I referred to the limited amount of research in the subject area of teachers' knowledge and understanding of the concepts of logarithms and logarithmic functions. This study expands on the growth of research and interest in the development of teachers' knowledge necessary for teaching of this particular mathematical content. It

provides information with regards to the nature of pre-service secondary mathematics teachers' knowledge of logarithms and logarithmic functions.

The methodological contribution that this study brings to the field of mathematics education consists of the methods used for gathering the field data. To collect data for investigating the pre-service secondary mathematics teachers' knowledge of logarithms and logarithmic functions, I designed the peer-interview task and utilized the writing of a script for a play activity. Using both tasks in this research study allowed me to collect different types of data that better informed my investigations. It is interesting to note that mainly, the findings from one task supported the findings from the other.

Moreover, in focusing on pedagogy, the study enhances the teaching of pre-service mathematics teachers by highlighting their learning through utilization of the research tasks for instructional purposes. As was pointed out in Chapter 4, one of the reasons for pre-service teachers' difficulties in teaching mathematics is the lack of pedagogical practices and simulation of real teaching situations, that allow them to experience firsthand, their preparedness for the teaching of secondary mathematics. Therefore, both research tasks are a valuable addition to teacher training in mathematics education, since they serve not only as an assessment tool but also as an instructional tool that provides learners with an opportunity to engage in meaningful learning.

6.3 Pedagogical implications

Pedagogical implications of this research were dual in nature: as an assessment tool and an instructional tool. Through the purpose of this research study, I tried to examine the factors that influence pre-service teachers' knowledge for teaching. It is an

important topic within the conceptual framework that deals with research on teaching (Shulman, 1986, 1987). The assessment of the pre-service teachers' mathematical and pedagogical content knowledge was accomplished by the two specifically designed and utilized tasks: the Job interview, and the Math play. Firstly, I find that these particular activities required pre-service teachers to review the mathematical content at hand. Secondly, they provided pre-service teachers with an opportunity to experience the complexity of real teaching situations that required knowledge of mathematics, pedagogy, and the students' learning. Pre-service teachers, who participated and completed these tasks, exposed deeper insights into their subject matter and pedagogical content knowledge, regarding logarithms and logarithmic functions.

Two pre-service teachers, the interviewer and the interviewee produced the information gathered through the first task. Ideally, this task could be considered more objective because of another person's involvement. Whereas in the second task, one pre-service teacher could entirely orchestrate the data gathered, allowing for the exposure of the knowledge of which he or she is sure, and avoiding the disclosure of what the pre-service teacher does not know. The peer-interview task contains an element of unexpected surprise. Pre-service teachers' reactions to such surprises are very revealing in terms of their knowledge, as seen in the interview conducted by Sam (page 76). His candidate used a calculus approach to define a logarithm as an area under the hyperbolic curve. Sam remained inattentive to these explanations, most likely because he was inexperienced with the material his candidate was presenting.

The pedagogical tools of utilizing student-to-student interactions were an important component of these explorations. The research tasks used in this study enable

me, as the instructor of the course, to orchestrate classroom discourse, create a learning environment that emphasizes students' meaningful learning, communication and reasoning. At the same time, it allowed participants to reflect on their actions, focusing on students' learning; and consequently, the instructional decision-making.

From the feedback given by the participants of this study, I found that they related positive comments to me about their experiences. They generally felt that they experienced for themselves the importance of teaching knowledge. The pre-service teacher, Nora wrote,

This interview educated me in many ways. It gave me a chance to interact with my fellow student as a head of the math department in a secondary school. I felt happy to ask my partner ... some questions. I was receiving honest answers from my partner. The interviewing process made me feel comfortable and knowledgeable in my attempt to become a teacher.

This quote is Nora's testimony to the fact that there still exist a number of students, trying to pursue a career in teaching of mathematics, who not only possess a limited subject matter knowledge, but are also completely unaware of it. Perhaps, the methods course is the place where meaningful experiences could bring such matters to light. Only after acknowledging the problem, can it be dealt with appropriately.

6.4 Further Considerations and Implications

6.4.1 Undergraduate and high school mathematics teaching/learning

During the follow-up discussion of their responses to both tasks, the pre-service teachers voiced concerns about the manner in which they were taught in high school. They argued the concepts they learned were not connected; therefore, they displayed difficulties exhibiting and utilizing a meaningful understanding of logarithms and logarithmic functions. In their university level mathematics courses, the emphasis was always placed on their procedural knowledge, rather than conceptual knowledge. These all contributed to the lack of conceptual knowledge evidenced through the activities they completed. As to the college and high school teachers, my study advocates for sufficient stress on the meaning of the definitions, basic properties, and finally, functional relationships of the logarithms.

6.4.2 Pre-service teacher education

As became evident from this research, there is a lack of mathematical knowledge on the part of the pre-service teachers. The reasons for this might be a lack of the reviews of the high school curriculum in the undergraduate courses, which were completed by the participants. The tasks designed and used in this research are important activities for reexamining high school mathematics content, from what Usiskin, et al., (2003) refer to as an advanced perspective. They allow pre-service teachers to reflect on their practice, while keeping in focus students' meaningful learning.

Teachers' experiences in every day teaching are very complex. To simulate such experiences for teacher training is very challenging. However, the designed research

tasks create such an opportunity. The main idea behind my design was to integrate different research methodologies, as in Borasi (1996), Blatner (2002), Zazkis and Hazzan (1999), Watson and Mason (1998), into one, creating a multifaceted activity. For a full description of these tasks refer to sections 3.1 and 4.2.

This study has provided some compelling anecdotal evidence, which implies that professional mathematics educators, mathematics teachers, and pre-service teacher educators can all productively engage in these activities, and benefit from them in a variety of ways. Possibly, these strategies could even be employed in elementary school teachers' development programs. As a result, the novel approach to interviews advocated in this study seems both valuable and feasible for mathematics educators. It can also be implemented in different contexts, serving diverse purposes from both research and instructional perspectives.

6.4.3 Future research and limitations

This study introduces new research methodology. The novel task, called *Job Interview* was designed and administered for the first time. The participants conducted this task on their own, in out-of-class time. Even though they had an opportunity to rethink or even stage the interview, their lack of interviewing experience and limited interview skills might have influenced the collected data.

Some limitations of the study included the assessment of the pre-service secondary mathematics teachers' pedagogical knowledge. It was not situated in actual practice, and thus, may not have reflected the participants' full understanding of pedagogy of the concepts. Also, there were only ten participants involved in the study.

This may not be grounds for generalization. However, the cumulative evidence extends beyond this particular research.

Another limitation comes from using course assignments as research data, because the quality of students' work on these assignments may be influenced by outside factors, and may not give a true reflection of one's knowledge. As mentioned previously an inconsistency was established when comparing results from the analysis of Mike's interview and his math play. When analysing his manner of questioning, it was concluded that he possessed insufficient pedagogical content knowledge, and it was assumed that his subject matter knowledge was also limited. Nevertheless, his performance in the math play was outstanding.

I also cannot disregard the fact that I was familiar with the participants of this study prior to the research. They were all enrolled in the course, which I instructed. This may have influenced my conclusions.

The results of this study provide insight into pre-service teachers' knowledge for teaching school mathematics. As expected with every research study, the articulation of these recent findings has given rise to new possibilities that call for further research.

Several possible extensions of this study could be explored. One can investigate pre-service secondary teachers' knowledge of logarithms and logarithmic functions more deeply, but restructure the investigation to account for growth; it can be done through an intervention into the methods course. The main goal of this intervention would be an increase in pre-service teachers' knowledge.

Another possible extension could be focused on the investigation of the pedagogical values of the designed research tasks. In the present research it was not explored, therefore it calls for future inquiry.

The research tasks designed and utilized in this study served a double purpose. They functioned as research methodologies for collecting and analyzing data. They also were successfully used as instructional activities in the pre-service teachers' training course, the Designs for Learning Secondary Mathematics. There possibly exists an interplay between instructional and research tools, and the relationship between them both could be symbiotic in nature. This warrants further investigation in future research as well.

6.5 On the crossroads

Mathematics as a discipline has existed for thousands of years. People worked and developed new knowledge that allowed our civilization to advance. Society has placed great emphasis on the necessity of possessing such knowledge, with the consequence of the learning of mathematics in early childhood. Many advanced mathematical courses are required for the successful completion of degrees in various disciplines and fields of study. All of this makes the teaching of mathematics extremely important, and places teacher education under a magnifying glass of high expectation. How successful are we as a civilization in the transferring of mathematical knowledge? Is it enough to know formal mathematics for the teaching of mathematics?

From my personal experiences, as a high school mathematics teacher of 10 years, the number of students enjoying the study of mathematics is on the decline. Why are we

losing these students? Despite of the variety of curricular reforms, the conventional treatment of logarithms and logarithmic functions in the local curriculum did not change. We work on improving our pedagogical skills and methodologies, so what is the problem?

In looking back, mathematics education research has focused on the teacher, on the content, and on the student, but the problem still exists. Perhaps we have to investigate all the components as a complex phenomenon. Or possibly, there exist some components, essential to teaching and learning mathematics, which we have thus far been unable to detect. What can it be? The answer may lie in the teachers' lack of ability in the flexible transformation of formal mathematical knowledge into teachable mathematics. What are the characteristics or features teachers have to possess and develop in order to warrant such flexibility? How can pre-service teachers be prepared for this? Should we focus on technology, or the human factor?

This study has brought me to a crossroads. It leads me to many possible avenues for future investigations. They may vary from experimental work on the development of new methodologies for pre-service teachers' training, to completely theoretical work on the components of teachers' knowledge necessary for the teaching of mathematics.

Appendix A

Education 415

Designs for Learning: Secondary Mathematics

COURSE DESCRIPTION

This course is designed for prospective and practicing secondary school teachers who wish to explore the learning and teaching process as it applies to secondary school mathematics.

The objective of the course is to provide the participants with appropriate learning experiences so that they feel confident designing secondary school mathematics instructions within a consistent framework using appropriate instructional materials and methods.

Participants will explore theoretical and practical aspects of mathematics teaching, their own learning, and their role as teachers. Different forms of mathematics learning and different instructional strategies will be experienced and explored in class, and theoretical and practical aspects of curriculum implementation, instructional materials (calculator, manipulatives, internet, etc.) and assessment and evaluation in mathematics teaching and learning will be investigated.

Dates and Times

May 9	5:30pm – 9:20pm	June 27	5:30pm – 9:20pm
May 16	5:30pm – 9:20pm	July 4	5:30pm – 9:20pm
May 23	5:30pm – 9:20pm	July 11	5:30pm – 9:20pm
May 30	5:30pm – 9:20pm	July 18	cancelled
June 6	cancelled	July 25	5:30pm – 9:20pm
June 13	5:30pm – 9:20pm	August 1	5:30pm – 9:20pm
June 20	5:30pm – 9:20pm		

Text: Posamentier, A.S., Smith, B., Stepelman, J. (2006). *Teaching Secondary Mathematics: Techniques and Enrichment Units (7th edition)*.

Graphing Calculator: Texas Instruments TI-83 Plus

Schedule of Topics and Readings

Topic	Reading	Assignment
Let's do some math. Thinking about being a math teacher	<i>Thinking about Being a Mathematics Teacher</i> , reflective problems	<i>Math Journal:</i> The Treasure of Captain Bird <i>Writing Journal:</i> What is Math? Why do we teach it? What does it mean to teach/learn mathematics?
What is Mathematics? Why teach mathematics? Theory and Practice of Learning Mathematics	Wheeler, D. <i>Humanizing Mathematics Education</i> . Ernest, P. <i>Why Teach Mathematics?</i> Chapter 1, pp.1-14	<i>Math Journal:</i> Corner to corner problem <i>Writing Journal:</i> Respond to the readings. Compare your response from today to your response from the last class.
Role of questions in mathematics classroom: Questioning as an instructional tool; Questioning as an assessment tool	Zazkis, R., Hazan, O. <i>Interviewing in Mathematics Education Research: Choosing the questions</i> . Mason, J. <i>Minding Your Qs and Rs: effective questioning and responding in the mathematics classroom</i> Chapter 3, pp. 74-83 The Job Interview Assignment is due in three weeks.	First two math problems due. <i>Math Journal:</i> Calendar problem <i>Writing Journal:</i> Respond to the readings, provide examples of different types questions.
Problem Solving	Chapter 4, pp. 109-135	<i>Math Journal:</i> Toss a ball problem <i>Writing Journal:</i> What do you think makes a good problem? Why is that? What is the role of problem solving in teaching/learning mathematics?
Using Technology: Geometer's Sketchpad and Graphing Calculator	Chapter 5, pp.135-165	<i>Math Journal:</i> Using GS create a stable rhombus, in two or more distinct ways Create a short activity using GC <i>Writing Journal:</i> What is the role of technology in teaching?

Assessment Probability and Statistics	STATISTICS CANADA Learning Resources Guskey, T.R. <i>Computerized Gradebooks and the Myth of Objectivity</i> Chapter 6, pp.166-196	<i>Math Journal:</i> Four pocket round table problem <i>Writing Journal:</i> Pick three assessment strategies that you see yourself using in the future and comment on their applicability, feasibility and accountability The Job Interview assignment is Due
Planning in Teaching. NCTM standards	<u>The Open-Ended Approach: A New Proposal for Teaching Mathematics</u> , pp.113-115, 153-155 Chapter 2, pp.14-62 Detailed Lesson plan with the Math Play assignment is due in two weeks	<i>Math Journal:</i> Tessellation project <i>Writing Journal:</i> What are the main aspects of the effective planning? Respond to the readings.
Enrichment	Chapter 7, pp.197 - 216	<i>Math Journal/ Writing Journal:</i> Create an open-ended “enrichment” problem and find a possible solution (s). Reflect upon your thinking. How do you think this problem can help in enhancing students’ learning?
Extracurricular activities in Mathematics Math and Arts, Math and Music	Chapter 8, pp.217-230	<i>Writing Journal:</i> What is a purpose of the extracurricular activities? Respond to the reading Detailed Lesson plan with the Math Play assignment is due.
History of Mathematics	Burton, D. <u>The History of Mathematics. An Introduction</u> , selected topics	Prepare a poster-presentation on the topic of your choice. Due last class.
Group Presentations	Extracurricular mini lessons	Math Journal is due
Math Resources: Manipulatives, Internet sources, literature		Writing Journal is Due

Course Requirements

1. Mathematical Journal (20%)

Throughout the course you will be given several mathematics-based problems to work on. Your task is to keep a journal of your attempts to solve it. Evaluation will take into account your analysis of your attempts, not only your "solution". The journal is due on June 27th.

2. Writing Journal (20%)

Throughout the course you will be asked to reflect on assigned readings and in-class discussions.

The journal is due on August 1st.

3. Individual Assignment (20%)

An important aspect of understanding and teaching mathematics is learning about its origins, discoverers, and explorers. You will prepare a poster researching a notable mathematician and outline the relevance of the chosen topic to the high school curriculum. You will present it in class.

This is due on July 4th.

4. Peer Interview (20%)

You will interview your classmate and will be interviewed by your classmate on the topic of high school curriculum as if conducting a job interview for a substitute teacher position. You will submit the transcript of the interview, and the evaluation according to the criteria discussed in class.

This is due on June 13th

5. Detailed Lesson Plan (20%)

This assignment will focus on a detailed lesson plan that incorporates a *Math Play*: For this the assignment you will be given an episode of a fictional mathematical interaction between a student and a teacher. The interaction may be in the form of a conversation, an in-class lesson, an in-class question and answer session, a teacher reading a student's work (homework, project, or test) and it will present to you a problematic situation in which a student has developed a misunderstanding about something. You are to diagnose the misunderstanding, formulate a plan for the remediation of the misunderstanding, and write out the balance of the interaction in the form of a *play*.

This is due on July 25th.

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