

**USING COGNITIVE CONFLICT TO PROMOTE A
STRUCTURAL UNDERSTANDING OF GRADE 11
ALGEBRA**

by

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ABSTRACT

This study examines the effect of cognitive conflict interventions on high school students' understanding of algebra. The concepts of algebra can be conceived of as both processes to carry out, and objects that can be used in higher processes. I examine this double nature with reference to several epistemological frameworks and describe a conceptual divide between procedural and structural understanding. I formulated a system to analyze student understanding and developed cognitive conflict instruments intended to promote structural understanding. Both the system and instrument involve questions that contain 'procedural traps'. I carried out interventions and tested the students for any change in understanding. The majority of the students had a procedural understanding before the interventions, and a structural understanding afterwards. I concluded that cognitive conflict interventions could effect a significant advance to structural understanding on the part of a majority of the students, but were not effective with the weakest students.

Keywords: cognitive conflict; procedural trap; structural understanding; high school algebra

Subject Terms: Mathematics-study and teaching-research; Mathematics-study and teaching-secondary British Columbia; Mathematics-study and teaching-psychological aspects; cognition in children

DEDICATION

I dedicate this work to Phathi, Sara, and Vuli for their patience, love, and understanding.

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GLOSSARY

Algebra

Throughout this paper I will use algebra to mean high school algebra, which is the study of linear, quadratic, and polynomial equations and inequalities, and basic functions and their graphs. I do not use algebra in either of its more advanced meanings as the study of number systems or as a vector space.

When I refer to the ‘structures’ of algebra I am referring to these basic equations, functions, and graphs, not the topics of the more advanced algebra.

Cognitive conflict

Cognitive conflict is an individual’s awareness of contradictory pieces of information affecting a concept in that individual’s cognitive structure.

Procedure

I use procedure to mean a step-by-step algorithm.

Process

I use process to mean a particular course of action intended to achieve a result.

Procedural understanding

I will use the term *procedural understanding* for an understanding rooted in conceiving of an algebraic statement as a prescription for action.

Structural understanding

I will use the term *structural understanding* for an understanding characterized by the ability to conceive of an algebraic statement both as a prescription for action and as an object that can be used in further processes simultaneously.

INTRODUCTION

After eight years of teaching high school mathematics, I felt that the majority of my students did not have a level of comprehension that allowed them to use algebra to solve problems, but instead had a collection of memorized algorithms they could use on familiar questions. Kieran's (1992) article confirmed that the problem was widespread and much research had been done on it. This research revealed that for decades we have understood that there is a divide between conceiving of the structures of algebra as processes and conceiving of them as both processes and objects simultaneously. This divide is what must be overcome for the students to have the level of comprehension I considered necessary to use algebra. The search for a method of teaching to overcome this divide illuminated that it would not be a simple or straightforward matter. A number of teaching experiments over the years had produced conflicting results and no distinct method of teaching that would allow the students to cross the divide. The studies had narrowed the focus and provided a base to work from. It was not simply a matter of being able to conceive of the concepts of algebra as objects. They must be conceived of as processes and as objects simultaneously in order to master algebra. This sounds simple, but developing this ability in students is very difficult. The research on cognitive conflict offered methods that could develop this ability in students and engendered an intent to examine if using cognitive conflict could help students cross the divide from memorizing algorithms to comprehending algebra.

Thesis Organisation

I begin with a discussion and explanation of the theoretical underpinnings of why the students have this trouble with algebra. I then describe a theoretical framework of how students learn and the genesis of a teaching method that could reverse the typical outcome of the majority of the students not truly comprehending algebra.

In Chapter two I describe the specific research questions I narrowed my focus to, and explain the theory and practicalities behind the methods I used to test these research questions. I describe the origins of the methods and how and why I refined them.

Chapters three, four, five, and six display the results my methods produced. First the general results of the population are listed. These results are then examined for definitive behaviours that can be used to judge the efficacy of my teaching method. After presenting the results of the population, I present the results of individual students involved in the study with reference to the theory of learning I used. I then compare these results to the results of older, more proficient students who have not participated in the interventions.

In Chapter seven I summarize the results and present my conclusions about the research questions I asked, and in Chapter eight I examine the implications of these conclusions for the teaching of mathematics.

CHAPTER ONE: ON PROCEDURAL AND STRUCTURAL UNDERSTANDING

Algebra has a procedural and a structural nature

The structures of high school algebra can be perceived as two very different things at the same time. They can be seen both as processes to carry out, and as objects that can be used in other processes. In other words, algebraic concepts can be understood as prescriptions for action or actual objects that are symbols of generalized mathematical relationships. Booth (1989) explains that algebra has a *syntax*, the manipulation of terms and simplification of expressions, and a *semantic* aspect, the structural properties of mathematical operations and relations that govern the manipulation and simplification of expressions. According to Sfard (1991), a mathematical definition can define an abstract object and its properties, or it can define a computational process. Harper (1987) examined the relationship between the historical development of algebra and modern students' conceptual development. He states that the two major steps in the development of algebra were first, the introduction of the use of letters for unknown quantities by Diophantus and second, the use of letters for given quantities as well by Vieta, removing numbers from the equations altogether. He names Diophantine and Vietian solutions after Diophantus and Vieta. A Diophantine solution is procedural with specific numbers to be solved for, leaving it up to the reader to be able to generalize the concept. A Vietian

solution is a general relationship where symbolic forms can be used as objects. (See Harper (1987), or Sfard (1995) for more extensive discussions.)

What this dual character of algebra concepts means for students is that in order to understand algebra they must be able to conceive of the structures they are to learn as both processes and objects simultaneously. This innocuous statement is fraught with ominous portents for the learner of algebra. What the research into the learning of algebra has found is that students' understanding of algebra can be a facility with carrying out correct procedures, or it can be a more profound understanding of the mathematical properties involved. This superior understanding is a fusion of procedural proficiency combined with a vision of the structures of algebra as generalized mathematical relationships. Sfard uses "operational" to describe an understanding of the syntax (i.e. procedural proficiency) and uses "structural" understanding to describe an understanding of the semantics; the grasp of the mathematical properties of a concept so that the processes become objects that can themselves be used in higher level processes. Skemp (1976) calls these levels of understanding instrumental and relational respectively, and describes instrumental understanding as "rules without reasons" (p.2), simply the ability to use a procedure. Relational understanding he describes as knowing what to do and why to do it. I will use the term *procedural understanding* for an understanding rooted in conceiving of an algebraic statement as a prescription for action (Booth's (1989) understanding of syntax, Sfard's (1987) operational, Skemp's (1976) instrumental). I will use the term *structural understanding* to mean an understanding characterized by the ability to conceive of an algebraic statement both as a prescription for action and as an object that can be used in further processes simultaneously. I will elaborate later in this

chapter (in Oscillation) why I consider this ability to perceive of a concept both as an object and a process simultaneously as the defining characteristic of structural understanding.

Epistemological frameworks

Several researchers in the mathematical education field have created epistemological frameworks to describe how students learn mathematics, (see for example: Tall and Vinner, 1981; Biggs and Collis, 1982; Kaput, 1989; Dubinsky and McDonald, 2001; Sfard, 1987, 1991, 1995; Tall, 2004).

Pegg and Tall (2001, 2005) evaluated and compared these frameworks. They found that the researchers have different views on how a person progresses from a procedural to a structural understanding, but they all agree that the ultimate form of understanding is structural understanding. The researchers define the highest level of understanding as a type of relational understanding (after Skemp, 1976) where processes and objects are linked into an integrated structure. This structure is realized when a process has been converted (they use encapsulated or reified) to an object that can itself be used in higher processes. It is the ability to deal with the concept as a process and an object simultaneously that emerges as the foundation of structural understanding. Sfard's (1991) framework is based on the historical development of algebra and is predicated on the assertion that students initially learn the procedural prescription of an algebra statement and then progress to comprehending it as an object that can be used in higher operations. (see also Gray and Tall, 2001, Dubinsky and McDonald, 2001). Sfard sees the learning of algebra as a series of procedural to structural cycles the students must pass

through. She delineates each cycle into interiorization, condensation, and reification. During interiorization the students become more skilled at performing a process. During condensation they become more capable of thinking of the process as a whole. During reification the students pass through an ontological shift that solidifies the process into an object. Computational operations become permanent object-like entities, and these objects can then be used in higher processes. The process and the object can be conceived of simultaneously. Kieran (1991) summarizes that, "just as the historical development of algebra can be viewed as a cycle of procedural-structural evolution, the study of school algebra can be interpreted as a series of procedural – structural adjustments that must be made by students in coming to understand algebra" (p. 246).

The need for a structural understanding

A student can be reasonably successful in the high school mathematics classroom by being procedurally proficient. Indeed, prior to Vieta, mathematics developed for thousands of years purely procedurally. Today even in countries vaunted for the mathematical abilities of their students, mathematics educators suspect their students' high results on mathematical achievement tests are a result of better procedural efficiency rather than better understanding of the mathematical properties (see Fujii, 2003).

However, all the frameworks listed above predict that the lack of a structural understanding will eventually limit a student's mathematical development. Sfard (1991) states that with structural understanding "learning becomes more effective, more meaningful" (p. 28) and that "at certain stages of knowledge formation (or acquisition) the absence of a structural conception may hinder further development" (p. 29). Gray

and Tall (1994) feel without structural understanding the students "are actually doing a more difficult form of mathematics, causing a divergence in performance between success and failure" (p. 115) and the fact that many expressions can represent the same procedure means that, without the ability to conceive of the object "a student still at the procedure level might find these various expressions and their procedural meanings a considerable barrier to understanding" (Gray and Tall, 2001, p. 69). This means that a lack of structural understanding of a concept will prevent a student from understanding and working with the concept in higher processes and progressing to a higher level. Skemp (1976) says that until recently (i.e. the 1970's) he felt instrumental understanding, an understanding of only the syntax, should not even be called understanding. Booth (1989) argues that a mastery of the syntax must be built on an understanding of the semantics that underlies it, and that the majority of students learn the syntax without the semantics.

Sfard's (1991) framework is based on the concept that as a learner advances in mathematics they learn more and more processes and often have to perform a process on the result of another process. With a procedural understanding all of these processes are stored in memory in an unstructured sequential list. The sheer number of operations and procedures that have to be learned, even by grade school, will overwhelm the average person's working memory. But with a structural understanding the products of the processes learned are stored in memory as objects in a structured hierarchy, and information can be stored and retrieved faster and in exponentially higher volumes. Reification allows mathematical structures to be viewed as objects. This allows them to be compared for equivalency (of tremendous importance in algebra) and, more

importantly, permits a perception of the relationships between objects that is necessary to use them in a higher process. Tall, Gray, Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, Thomas, and Yusof (2001) found that "those who are (or who become) focused mainly on the procedural have a considerably greater burden to face in learning new mathematics than those who are able (in addition) to focus on the essential qualities of the symbolism as both process and concept." (p. 88). They were applying this to learners of mathematics from arithmetic to calculus. Sfard feels a delay in achieving a structural understanding of earlier concepts may cause "permanent harm" (Sfard, 1991, p. 33) to a student's ability to understand later concepts, and Sfard and Linchevski (1994) feel that once the chain of reification from procedural to structural understanding is broken "the process of learning is doomed to collapse" (p. 220). Gray and Tall (2001) give a spectrum of performance where structural understanding is needed to progress to the highest performance levels. At the highest level, "symbols act dually as process and concept, allowing the individual to think about relationships between the symbols in a manner which transcends process alone" (p. 69). Gray and Tall (1991) found this idea that a lack of structural understanding increases the burden on a student's mental faculties was exactly the case with arithmetic. They found that less able students had learned arithmetic concepts only as processes and to continue learning they had to continue adding processes to an increasingly long sequential list. They state that for these students, "arithmetic becomes increasingly more difficult" (p. 77) because any advancement makes for longer more complex procedures with higher possibility of error. The more able students, they found, had a structural understanding. They use 'proceptual understanding', which they define as:

"We characterize *proceptual thinking* as the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object. It is proceptual thinking that gives great power through the flexible, ambiguous use of symbolism that represents the duality of process and concept using the same notation".
(Gray and Tall, 1994, p. 7, italics theirs)

They also use 'process' as a specifically defined technical term they adopted from Davis (1983) (as cited in Gray and Tall, 2001) where "a *process* occurs when one or more procedures (having the same overall effect) are seen as a whole, without needing to refer to the individual steps" (Gray and Tall, 2001, p. 67). I use process in this paper to mean a particular course of action intended to achieve a result. This is similar to procedure, but I use procedure to mean a step-by-step algorithm. Gray and Tall (1991) found this structural understanding eased the burden on the students' memories and made arithmetic increasingly simple. Once the student could see a process as an object, they could compare it to other objects and use this object in higher processes in order to derive understanding of these. A procedural understanding means a process (not an object) is used in a higher process and perhaps again, creating levels of complexity that defeat the best memory. They feel:

"The less able child who is fixed in process can only solve problems at the next level up by coordinating sequential processes. This is, for them, an extremely difficult process. If they are faced with a problem two levels up, then the structure will almost certainly be too burdensome for them to support" (Gray and Tall, 1994, p. 21).

The lack of structural understanding they believed was a major contributory factor to failure in mathematics. "This lack of a developing proceptual structure becomes a major tragedy for the less able which we call *the proceptual divide*. We believe it to be a major contributory factor to widespread failure in mathematics" (Gray and Tall, 1994,

p. 18). I feel this portrayal of a divide between procedural and structural understanding is extremely important and will now examine this.

The difficulty of attaining structural understanding

The frameworks I've described delineate that a learner begins learning a mathematical concept as a process of action, becomes procedurally more and more efficient, and then moves to a point where they can conceive of the concept both as a process and as an object that can be used in other processes. The literature and my experience as a math teacher and researcher leads me to believe that this move is not a consequent and gentle evolution, but a mental 'leap' over a chasm, and a difficult one.

In developing structural understanding of a concept, procedural proficiency and conceiving of the process as an object are both needed. In the previous section *the need for structural understanding*, I explained why conceiving of a concept only as a process to be carried out prevents it being used as an object in a higher process, limiting further progress. Equally problematic is conceiving of a concept simply as an object without perceiving it is also a process that can be carried out. Linchevski and Sfard (1991) studied this and found that an ability to view a concept as an object without a complete grasp of the process this object also defines will lead to errors just as surely as a procedural understanding. They label this understanding of the structure of a concept without a grasp of the underlying process *pseudostructural understanding*. The problem this type of understanding creates is that without being able to view a concept as a process, the object is not attached to the process-object chain the learner has built. It is an object without meaning that is manipulated according to arbitrary rules. Without a grasp

of the process the concept defines, the learner cannot see the relationships that generate the rules of manipulation. They state:

"lacking operational underpinnings, this kind of conception would leave the new knowledge detached from the previously developed system of concepts, and the secondary processes would seem totally arbitrary. The student will still be able to perform these processes but his understanding will remain instrumental" (Linchevski and Sfard, 1991, p. 319).

Consequently, I believe the ability to conceive of a concept as both a process and an object simultaneously is necessary for a structural understanding, and Sfard (1991) asserts that both "*are prerequisite for each other*" (p. 31). In other words, reification comes from working with a concept as an object and a process at the same time. This is why developing a structural understanding is so difficult. The learning cycles are closed loops with a mutual dependence between the procedural and structural conception of the concept that can leave the learner without an immediate insight into a concept despite much hard work.

My experience as a math teacher led me to believe that the development from conceiving of a concept as a process to conceiving of it as a process and object concurrently is not a gentle, expected event, but a difficult leap over a divide, and the literature supported this. The next section, *Our students have a procedural understanding*, will substantiate the conclusion that for the majority of high school students' structural understanding cannot be considered a certain development. Correspondingly, the researchers developing the frameworks I've dealt with speak of the change from procedural to structural understanding as a difficult transition. Sfard (1991) states "*there is a deep ontological gap between operational and structural conceptions*"

(p. 4) and "reification is an instantaneous quantum leap" (p. 20) over this gap. She also posits that perhaps "reification is inherently so difficult that there may be students for whom the structural conception will remain practically out of reach" (Sfard 1989, p. 158). Davis (1975)(as cited in Sfard and Linchevski, 1994) characterizes it admirably:

"it is not entirely clear that this whole new point of view can be acquired by gentle accumulation of small increments. It may resemble the geological phenomenon of earthquake more than the phenomenon of erosion or dust deposit" (p. 29).

To summarize, neither a procedural nor a pseudostructural understanding provide a student with the level of understanding necessary to solve algebra problems rather than simply complete procedural exercises. It is structural understanding that confers this ability. Structural understanding is a progression from a procedural understanding, and is a more advanced stage of mathematical knowledge. Structural understanding means concepts can be conceived of both as processes and as objects simultaneously. These objects can then be compared or used in higher level processes. This allows the compact hierarchical information storage and retrieval, and the understanding of the relationships between these objects that are necessary for advancement in mathematics.

I feel a student's understanding cannot be characterized as structural without a high procedural proficiency that confers the ability to conceive of a concept as a process and an object simultaneously. This is why I have defined *structural understanding* as being distinguished by this ability, and will describe in detail (in Oscillation, under Themes) why I consider this mental flexibility as the foundation that demonstrates structural understanding. I also feel this structural understanding of a concept comes after

a student becomes procedurally proficient with the concept, but it is not an easy and consequent transition, it is a leap in understanding that is very difficult.

Our students have a procedural understanding

It is this 'dual nature' (Sfard's 1991 title) of algebra conceptions, being both processes and objects, that makes learning algebra so difficult for most people. The research into the teaching and learning of algebra over the past three decades has shown that the majority of high school students have a procedural and not a structural understanding of algebra (see for example: Collis, 1974, Davis, 1975, Matz, 1980, Kuchemann, 1981, Herscovics, 1989, Herscovics and Linchevski, 1994). Kieran (1992) and Nickson (2000) catalogue the many problems researchers have found that prevent students developing a deep understanding of algebra and describe the ongoing research to identify more. Kieran summarizes "The overall picture that emerges from an examination of the findings of algebra research is that the majority of students do not acquire any real sense of the structural aspects of algebra." (Kieran, 1991, p. 252). Sfard (1987), Linchevski and Sfard (1991), Kieran and Sfard (1999) feel that the majority of students can not 'see' the mathematical object that two processes are equivalent to, and so can not tell whether or not the processes are equivalent. Harper (1987) working with the development from a Diophantine to a Vietan understanding, felt that "few pupils actually achieve it." Maurer (1987) working with arithmetic and the modifications students make to their known procedures when faced with an unfamiliar question states, "in short, most student modifications are based on *syntactics*, the rules of symbol pushing, rather than *semantics*, the underlying meaning of the symbols." (p. 171). Over the decades this

assessment of students' knowledge has remained consistent. From Skemp (1979) who felt the majority of pupils did not achieve a structural understanding, Arcavi (1995) who says "in many classrooms today we see... Meaningless and syntactic symbol pushing is the major student activity in algebra" (p. 147), and Fujii (2003), who found that students had an 'instrumental' rather than a 'relational' understanding. Fujii (2003) and Kieran (1992) both felt that the majority of students have a procedural understanding because they feel comfortable with it, it works well for them to solve most of the questions they face in school, and they think it is better than a structural understanding.

What demonstrates the difference between a procedural and a structural understanding?

In this study I use special types of questions that demand structural understanding to determine if the students have a procedural or structural understanding. While observing students solve these types of questions it became clear that there were several behaviours students exhibited when they had a structural understanding. I found that students with structural understanding display it with a noticeable change in their vocabulary, their engagement with the material, their perseverance, and their desire to determine if their steps are legal manipulations.

A search of the literature supported this conclusion that students with structural understanding display several characteristic behaviours. I codified the behaviours into four discernible themes that demonstrate a student has a structural understanding. The four themes are: an ability to oscillate between a procedural and a structural representation of a concept rather than just perform a procedure, the language the student

uses, the justification of the transformations the student uses when faced with an unfamiliar problem, and evidence of a good control mechanism. I will now explain these themes in detail.

Oscillation

Sfard and Linchevski's (1994) concept of reification establishes that a student has to be able to oscillate between a procedural approach and a structural interpretation in order to have a structural understanding of algebra. They cite Moschkovich et al. (1992) (as cited in Sfard and Linchevski, 1994) as saying this flexibility of considering a concept as a process and as an object simultaneously is a hallmark of competence. As discussed previously, they discovered it is not enough just to be able to see the structure of a concept; that is, to have a pseudostructural understanding. The absence of this ability to see both the object and the process simultaneously means the procedures are detached from the structure of the question and seem random, even capricious, to the students. Mason in his work (1989, 1996) supports this and asserts that being able to shift between seeing an expression as a particular action and seeing it as the articulation of a manipulable object is what learning mathematics is. Gray and Tall (1991) considered this "duality between process and concept" (p. 72) in arithmetic and they felt this led to "a qualitatively different kind of mathematical thought between the more able and less able" (p. 72) that eventually determines success or failure in mathematics. The successful students were able to see a concept as a process and as the object created by the process simultaneously, and the students who saw the concept as just a process were doomed to find arithmetic increasingly more difficult until it became impossible. Tall et al (2001)

considered research ranging from arithmetic through algebra to calculus and felt the key to a structural understanding is when a "symbol can act as a *pivot*, switching from a focus on process to compute or manipulate, to a concept that may be *thought* about as a manipulable entity" (p. 84). The salient conclusion is that the ability to conceive of a concept as a process and as an object *simultaneously* and oscillate between the two views is critical for structural understanding. Consequently, I concluded that an exhibition of oscillation between a procedural and a structural outlook on the part of a student is the best demonstration of their structural understanding. It is this oscillation that demonstrates the ability to conceive of the process and the object simultaneously that is the fundamental requirement of structural understanding.

Language

According to Sfard (1991), a student with a procedural understanding will speak of a process and steps that need to be done, and provide a list of those steps as an explanation of why they produced an answer. A student with a structural understanding will describe the concept as an object, or compare the equivalency of objects, often supported by a visual image when asked to explain why they produced an answer. Sfard and Linchevski (1994) and Linchevski and Sfard (1991) also found that carefully listening to the language a student uses will indicate whether they have a procedural or a structural understanding. Fujii (1987) determined a student's level of understanding by analyzing the students' written opinions of a solution rather than their ability to solve a question. Further, Fujii (2003) probed students' structural understanding with interviews asking them to comment on another student's solution. The student's language can tell us

whether they have a procedural or a structural understanding. I concluded a student with a procedural understanding will list the steps of a procedure when asked to explain why they produced an answer, whereas, a student with a structural understanding will describe a concept as an object, explain how a process also represents an object, compare objects, or compare the equivalency of two structures, often supporting their explanation with a diagram.

Justification

In Cortes' (2003) study of the difference between experts solving a problem correctly and students solving it incorrectly, she found that when confronted with an unfamiliar problem the experts would analyze the relationships and then decide between various operations to perform. She found the experts "explicitly checked the validity of the transformations" (p. 258) they used with a quick example. Conversely, the students' solving methods resembled algorithms where they automatically performed transformations in a pattern used to solve a familiar problem without justifying these transformations. The students had a procedural understanding where they used an inappropriate algorithm or procedure to solve a problem, while the experts saw the structure of the problem and unravelled it. Because this justification of a transformation demands a connection between the mathematical properties of an object and the operation performed, (i.e. structural understanding) Cortes (2003) considered "mathematical justifications as fundamental" (p. 258) in the experts ability to properly solve unfamiliar problems. Sfard and Linchevski (1994) also considered this as a key indicator of structural understanding. They labelled the students' understanding as

pseudostructural if they "could not provide any sensible justification for the permissible operations" (p. 222), but instead saw solving algebra problems as performing an algorithm. Fujii (1987) considered structural understanding (he uses relational) to be characterized by the ability to justify how a question had been answered, and Kieran and Sfard(1999) argue "We would like students to understand what they are doing and be able to justify their decisions in ways that go beyond a mere recitation of the list of movements through which they went while transforming formal expressions." (p. 2). By 'we' they mean teachers and researchers, not just themselves. They consider the ability to justify the rules of algebra syntax as a meaningful understanding of algebra, and they found that students who could justify the algebraic rules they used, even partially, achieved considerably better than students who memorized a procedure. Maurer (1987) as well found that "students don't consciously test their generalizations" (p. 171) and it was these faulty generalizations that prevented a progression to a higher level of understanding. I concluded a student demonstrates structural understanding of a concept if, when faced with an unfamiliar problem involving the concept, they analyze the structure of the problem rather than automatically applying a familiar procedure, and justify their transformations as they solve it.

Control

Schoenfeld (1985) in his examination of mathematical problem solving found that, among other important characteristics, expert problem solvers initially analyze the problem and then constantly monitor and assess their solutions as they progress. They assess if each step they take is valid and if it will help them progress to a solution; "They

argue with themselves as they work" (p. 141). He found that inexpert problem solvers lose track of the overall goal, demonstrate a rigidity in procedural execution, and show an inability to anticipate or monitor the consequences of their actions during problem solving. Schoenfeld (1985) feels "selecting and pursuing the right approaches, recovering from inappropriate choices, and in general monitoring and overseeing the entire problem-solving process" (p. 98) is "a major determinant of problem solving success or failure" (p. 143). He uses the term *control* for "resource allocation during problem solving" (p. 143) and examines good and bad control behaviours and strategies. I will say a student has a good *control mechanism* or displays good *control behaviours* if that student displays control processes similar to an expert problem solver. Matz (1980) found that one of the main factors causing errors came from students applying a familiar procedure to an unfamiliar problem in an algorithmic way without recognizing the structure of the problem. This happens when a student has a procedural understanding, and can not conceive of the concept involved in the question as an object. They will use a familiar procedure that does not 'fit' the structure of the problem. A student with a structural understanding can see a concept as both a process and an object. Gray and Tall (1991, 2001), Sfard and Linchevski,(1994), and Cortes (2003) found that this vision gives concepts a structure that the more able and expert solvers use, sometimes unconsciously, in a continuous review mechanism that allows them to determine if they have made an error as they complete a question. Because they view the concepts simultaneously as processes and objects it becomes quickly evident to them if an operation or transformation they use is not legal (if the procedure does not 'fit'). This is because they can compare the consecutive steps in arithmetic or algebra as objects that must be

equivalent. This review mechanism alerts a student to an illegal transformation they have made and causes them to persevere until they find an answer. A student with procedural understanding is not alerted by illegal transformations they have made and is satisfied with an incorrect answer. Linchevski and Sfard (1991) state that it is evidence of pseudostructural understanding if a student "uses the criterion of transformation automatically and never returns to the underlying processes and abstract objects in order to verify his conclusions" (p. 321). Without structural understanding the procedures become uninformed memorized steps and the students follow the steps blindly and accept results that are illogical or do not answer the original question. If a student does not have a structural understanding they will use a familiar procedure without an analysis of the structure of the problem. They will perform transformations that are not permitted and will not realize they have created a contradiction, showing non-equivalent objects as equivalent or vice-versa. With a structural understanding this comparison of equivalent objects is a very fast and simple control mechanism. Gray and Tall (1991) found this makes mathematics increasingly easy for the more able students and increasingly difficult for the less able. If a student with a structural understanding comes to a point where the objects are not equivalent they will look for a structural clue and they will persevere and try to determine why the object they have ended up with is not equivalent. With procedural understanding, the operations and transformations are steps in an algorithm to be followed; there is no way to check the correctness except with a formal checking method like substituting the eventual answer back into the original question. As a student who has a procedural understanding completes a process they do not know if they have made an error. This means they will be satisfied with an answer that is patently wrong to

someone with a structural understanding, or does not solve the original problem, or if unsatisfied with the answer will not know what direction to take. Gray and Tall (2001) found a structural understanding "introduces the possibility of alternative methods allowing checking for possible errors in execution, even to an underlying unconscious feeling that something is wrong when an error is made." (p. 69). Sfard and Linchevski (1994) found that students with a procedural understanding believe the instruction to 'solve' an equation means to carry out a specific procedure. They found that to these students "solving equations and inequalities was tantamount to performing a certain algorithm" (p. 222). Students who are unable to see the object a concept represents will rigidly carry out a procedure until ' $x = 7$ ' or ' $x > 7$ ' act as a "halting signal" (Sfard and Linchevski, 1994, p. 222), so the students will leave an answer that is wrong or incomplete when they reach this stage. They found in their interviews that if a student could not see the object as well as the process that a concept represented (i.e. have a structural understanding) then in a situation where correct transformations cause the variable to disappear they become lost and will not be able to continue to the answer.

To sum up, Structural understanding is demonstrated by a student having a good control mechanism. This means they initially analyze a problem and constantly monitor their work to determine if a step is valid and will help to solve the problem. This results in them realizing their error if they perform an illegal transformation, or produce a valid answer that does not solve the original problem. This control mechanism is predicated on the ability to see a concept as a process and an object at the same time. A student with structural understanding will persevere at the point where an anomalous result is produced and compare the objects or test procedures in order to find the correct answer.

A student with procedural understanding will not demonstrate this good control mechanism. They will automatically begin with a procedure, lose track of the overall goal, rigidly pursue procedures that do not 'fit' the problem, and will not monitor their actions. This is demonstrated when they remain unaware of an anomaly they create and leave an improper answer, trusting the procedure is correct.

The four themes

We can see four strong themes emerge from this examination of structural understanding and I found these themes particularly important in evaluating a student's structural understanding, or lack thereof. I codified these themes as follows: A student demonstrates a structural understanding of a concept when they demonstrate the ability to oscillate between a procedural and a structural representation of a concept; when they verbally describe the concept as an object or compare objects for equivalency, often referring to, or producing a diagram; when they justify the transformations they use when faced with an unfamiliar problem; and when they demonstrate a good control mechanism that causes them to monitor and review their steps if they reach a point where they have created a contradiction (i.e. they have non-equivalent objects equated). A procedural understanding is demonstrated by the converse of these themes: a student automatically applying a procedure they are familiar with to a problem without reference to the structure of the problem, or grasping another familiar procedure when they reach a point where oscillation between a procedural and a structural view is needed; verbally describing a concept as a process or a set of steps; completing transformations without justifying their validity; not realizing they have made an error when two steps in their

procedure are not equivalent, and not knowing what to do if the variable disappears during a procedure, or treating 'x= 7' as a halting signal. It is these themes I will use (in addition to the ability to solve questions that demand structural understanding) in order to analyze whether or not my students have a structural understanding.

The change from procedural to structural understanding

I have outlined that most students master the syntax of algebra but not the semantic aspects of it; they have a procedural understanding. Yet, to be an "expert", to be able to use algebra, the students must have an understanding of the semantics; a structural understanding. I have also described the shift from procedural to structural understanding as a difficult leap. If structural understanding is so important and our students, in the main, have a procedural understanding, and the transition is difficult, how do we move them from a procedural to a structural understanding? We have seen from Sfard's (1991) framework that the students start with a procedural understanding and (in the best cases) move to a structural understanding through reification, but this reification is a difficult process; "The reification, which brings relational understanding, is difficult to achieve, it requires much effort, and it may come when least expected" (Sfard, 1991, p. 33).

Reification comes from working with a concept as a process and an object at the same time, but both of these are prerequisites of the other, so this is rather challenging to achieve. My goal was to find a method that would reliably help all the students, not just the best of them, make the leap to a structural understanding of algebra.

Cognitive Conflict

The goal of this study was to determine if there was a method of teaching that would promote the students from conceiving of a concept as a process to conceiving of it as both a process and an object simultaneously. The method I decided to employ to bring about this reification was predicated on this need for the students to conceive of a concept as both a process and an object at the same time, integrated with an appreciation of Piaget's work on cognitive development. Sfard (1989) details that simply teaching the concepts structurally "would be of little or no avail" (p. 152), the research tells us that the moving from conceiving of a concept as a step by step sequence to conceiving of it as a manipulatable entity is difficult, (see Pegg and Tall, 2001, for a good summary), and students are satisfied with a procedural understanding. All this means that generating this change in understanding is not an easy matter.

According to Piaget (1985), cognitive development is the result of three mental processes: assimilation, accommodation, and equilibration. Assimilation is absorbing new information and integrating it into existing cognitive structures. Accommodation is the modification of these existing cognitive structures in order to be able to integrate new information. Equilibration is a mental process that regulates the interplay between assimilation and accommodation and leads to the development of more complex cognitive structures. Accommodation causes disequilibria, instances where there is a conflict between new knowledge and existing cognitive structures. An individual must then determine to what extent new knowledge will modify existing cognitive structures, or conversely to what extent existing structures will modify the acceptance of new

knowledge. Piaget says, "In sum, the various forms of equilibration appear to constitute the fundamental factor in cognitive development" (p. 15).

This interplay of assimilation and accommodation was what I needed to manage since I intended to use new knowledge to cause a major change of the existing cognitive structures of my students (move them from their present procedural understanding, that they were satisfied with, to structural understanding). The essence of equilibration is that disequilibria provide an essential motivational force. "Disequilibria alone force the subject to go beyond his current state and strike out in new directions" (Piaget, 1985, p. 11). But disequilibria do not always lead to progress. They do so only when they give rise to developments that surpass what had previously existed. Disequilibria play a triggering role, but "progress is produced by reequilibration that leads to new forms that are better than previous ones" (Piaget, 1985, p. 11). The mechanism of equilibration after a disequilibrium is a combination of regulation and compensations. Regulation is positive or negative feedback. Positive feedback reinforces. It can reinforce an error, or it can reinforce a success. Negative feedback causes a compensation, a change designed to cancel the negative feedback.

Disequilibria, then, can trigger equilibration, a change in existing cognitive structures. Disequilibria are caused by perturbations or conflicts, but conflict can only be caused at certain developmental levels. Conflict is caused in proportion to the degree to which the structure being formed has been acquired:

No acquisition of structure. No conflict caused by new knowledge.	Partial acquisition of structure. Conflict caused by new knowledge.	Full acquisition of structure. No conflict caused by new knowledge.
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The significant assertions for my research were; "the most influential factor in acquiring new knowledge structures is perturbation or conflict" and "when conflictual situations like this are utilized in a systematic way, perturbations are overcome through new construction" (Piaget, 1985, p. 32 & 33).

Mischel (1971) distills a large portion of Piaget's work and synthesises that an individual's cognitive development advances through that individual striking a balance between construing new knowledge in terms of their existing cognitive structures or modifying these structures to accommodate this knowledge. It is cognitive conflict that motivates this striking of a balance, so that "the cognitive conflicts which the child engenders in trying to cope with his world, are then what motivates his cognitive development" (p. 332).

This led to an examination of the research to find the optimum methods to bring about the cognitive conflict and manage the regulation and compensation that would promote students from a procedural understanding to a structural understanding of algebra.

This concept of disequilibria or cognitive conflict is rather broad and has gone through much refinement over the intervening three decades since Piaget and Mischel. Lee and Kwon (2001) give us an excellent review of various researchers' definitions of the concept. These definitions stem from Piaget's work, but reflect the research concerns of the individual researchers. What the definitions all share is that cognitive conflict is an

awareness of contradictory information involving one's conceptions. The definition of cognitive conflict I will use to support my investigation is: Cognitive conflict is an individual's awareness of contradictory pieces of information affecting a concept in that individual's cognitive structure.

The literature dealing with cognitive conflict in science and math education is divided, with some research finding cognitive conflict effective at advancing a subject's cognitive structure, and other research finding that cognitive conflict did not lead to change. The findings demonstrate that cognitive conflict can have constructive, destructive, or meaningless potential. A considerable number of researchers (Posner, Strike, Hewson, and Gertzog, 1982, Hewson and Hewson, 1984, Tirosh and Graeber, 1990, Adey and Shayer, 1993, Bell, 1993, Niaz, 1995, Druyan, 1997, Fujii, 1987, Kwon, Lee, and Beeth, 2000, Lee and Kwon, 2001, Watson, 2002) found cognitive conflict an effective method of changing students' existing conceptions, while others (Dreyfus, Jungwirth, and Eliovitch, 1990, Tirosh, Stavey, and Cohen, 1998) found it was not consistently effective at modifying students' existing conceptions. There were key details in the methods of the two groups that attest that the strategy used to manage the cognitive conflict and the regulation and compensation is extremely important. After extensive examinations of the literature for and against cognitive conflict, several studies (Adey and Shayer, 1993, Fujii, 1987, Niaz, 1995, Watson, 2002) developed methods of managing the cognitive conflict that brought about cognitive development. The common features of these methods were:

1. An introduction of the relationships and the context of the concept.
2. Presentation of a problem that will induce cognitive conflict.
3. After having generated a conflict it is essential to provide an environment that will facilitate the proper resolution of the conflict.

The introduction to the concept came in the form of asking the subjects to solve a problem and discussing how they solved it, as there is a "necessity for students to make their conceptions explicit so they are available for change" (Watson, 2002, p. 228). The cognitive conflict was generated by giving a counter example, or two conflicting examples, where the subject's familiar method of solving the problem fails. The environment after the conflict varied, but it was necessary to provide the subjects with an alternate conception that they could understand. This was usually done with a sharing of ideas about *why* a method failed or defending *why* one method is superior to another.

The research that found cognitive conflict did not improve subjects' cognitive structures also lends support to the importance of these steps. For example Dreyfus, Jungwirth, and Elovitch (1990) found two main problems; in some cases they were not able to bring their students to a state of meaningful conflict, and in some cases the conflict was not successful in causing the construction of the desired knowledge. They attribute this to the fact that students must have enough of a structure that they find new information causes a conflict, and, "If a solution is proposed at a level which is beyond that of the students, it will remain meaningless to them and the effect of the conflict will be lost" (p. 568). They also found that unsuccessful students did not benefit from

cognitive conflict interventions because of their general attitudes towards schoolwork and high levels of anxiety generated by the conflict. Dreyfus, Jungwirth, and Eliovitch's work reinforces the importance of making the concept explicit and providing a problem that will induce conflict, and that it is essential to provide the students with a new conception they find understandable after the conflict. Adey and Shayer (1993) agree that "less able students often appear unaware of a conflict or at least are not bothered by it" and Niaz (1995) felt it would be extremely difficult to design a strategy to provide meaningful conflict for all students in a classroom. Another example is Tirosh et al (1998), who found a conflict-based intervention had only limited success in changing their students' conceptions. Their method was to provide the students with two contradictory statements and ask them to determine which was correct and justify their answer. They ascribe the lack of success to the fact that intuitive rules are very resistant to change. Both Dreyfus and Tirosh were working to replace students' intuitive, experience-based concepts with the correct scientific or mathematical concept.

A possible reason for disagreement among the researchers as to the effectiveness of cognitive conflict is that there are different types of cognitive conflict and there may be a significant difference in the level of conflict required to properly resolve these different types of conflict. Lee and Kwon (2001) describe sub-types of cognitive conflict researchers have categorised, for example: conflict between two internal concepts, conflict between two external sources of information, or conflict between an internal concept and an external source of information. Lee and Kwon (2001) and Kwon, Lee, and Beeth (2000) found that the different types of cognitive conflict demand different levels of conflict to resolve and it was the level of conflict that was most highly

correlated with cognitive change. The type of cognitive conflict I was interested in was conflict between an internal concept and an external source of information. This is Piaget's (1985) disequilibrium. Several researchers found this type of conflict to be effective in changing subjects' conceptions, and laid the groundwork for ensuring this type of conflict takes place. Fuji (1987) found cognitive conflict didn't develop all powers of understanding, but was useful in changing students' understanding from procedural to structural (he uses instrumental to relational). In research previous to Tirosh et al's (1998) negative findings, Tirosh and Graeber (1990) had found cognitive conflict effective in changing teachers' procedural knowledge of division to a structural conception. These results were encouraging as I intended to employ cognitive conflict in order to move my students from a procedural understanding to a structural understanding. Fujii (1987) dealt explicitly with the variable levels of understanding found in a class of students, and found that it was crucial that the problems used cause three types of conflict, 'behaviourized how', "verbalized how", and "justification of how". He found that a variation in the types of conflict experienced by the students caused a variation in the eventual levels of structural understanding reached by the students. These three types of conflict will activate students' thought at various levels and if all are invoked they can effect a change from procedural to structural understanding.

An important instrument I needed in order to use cognitive conflict was a method of determining whether or not my students had actually experienced the necessary cognitive conflict. Lee and Kwon (2001) reviewed many cognitive conflict studies and compiled a list of observable signs that indicate a student is experiencing cognitive conflict. The signs of cognitive conflict are: uncertainty and perplexity when a student

recognizes an anomaly that contradicts their expectations; hesitancy and reappraisal of the situation to try and resolve the conflict; curiosity arousal and a heightened interest in the situation; and tension, frustration and anxiety at finding a question more difficult to solve than expected. These reactions were what I used to judge if a student was experiencing cognitive conflict.

A final important consideration is that cognitive conflict can be meaningless, or even destructive, if the student feels frustrated or threatened by it (Dreyfus, Jungwirth, and Eliovitch 1990, Lee and Kwon, 2001). This dramatically reinforces the importance of the environment after the conflict. The methods used to present a new conception must be non-threatening.

My conclusion was that cognitive conflict, when the proper methodology is used, offers a valuable means of effecting the type of change in a student's cognitive structure I was interested in bringing about. The proper methodology included ensuring the students have enough of a structure that conflict will be induced, inducing conflict in a non-threatening environment, and providing students with a new conception they can understand after the conflict. In the next chapter I describe the system I designed in order to investigate how effective cognitive conflict would be in changing my students' understanding of algebra from a procedural level to a structural level.

CHAPTER TWO: RESEARCH QUESTIONS AND METHODOLOGY

Research questions

The overriding aim of this study was to determine if there was a method of teaching that would promote students' understanding to the point where they could use algebra to solve problems rather than just complete memorized exercises. This is a broad and multifaceted aim, and the research examined in Chapter one helped narrow the focus considerably. The literature indicated that it was probable the students had a procedural understanding, they would need to progress to a structural understanding, and cognitive conflict offered a possible method of helping them develop structural understanding. Accordingly, the main focus of this study was refined to: what effect would cognitive conflict interventions have on Grade 11 students' understanding of algebra?

In developing a methodology to examine this question, several ancillary questions arose. Firstly, in order to determine the effect cognitive conflict would have on understanding, I needed some way of testing the students' level of understanding and whether or not it had changed. I needed a test to administer before and after the application of my cognitive conflict technique that would tell me whether a student had a procedural or a structural understanding. Since the students can successfully answer most Grade 11 material with a procedural understanding, an important question became: what demonstrates a structural understanding on the part of the student?

Another pertinent concern that arose was the fact, as explained in Chapter one, that the research literature is ambiguous about the effectiveness of cognitive conflict interventions. Accordingly, another question to examine was: what are the elements that constitute an effective cognitive conflict intervention?

A final aspect to consider, also highlighted by the literature, was the many levels of proficiency present in a typical class of students. The contention of the frameworks, that a student progresses procedurally and then experiences reification, provoked a consideration of whether or not the technique would be effective with all ability levels. And so, a final question to examine was: what role would the individual differences of the students play in the effectiveness of the cognitive conflict interventions?

Methodology

In the rest of this chapter I will describe the participants and the setting of the study and then describe the methods I employed in order to answer my research questions. I will describe the genesis of the system I used to determine if the students had a structural understanding and then describe the design of the cognitive conflict interventions.

Participants and setting

The participants of this study were 54 Grade 11 students (26 female, 28 male) attending high school in British Columbia, Canada. I taught all these students in conventional Grade 11 mathematics classes from September to June. Also participating in the study were 18 Grade 12 students (10 male, 8 female) from the same school that I

had taught the previous year. Their participation was in order to compare the results of students who had participated in cognitive conflict interventions with students who had not, but had had the same teacher.

All of the students were made well aware when completing questionnaires, interviews, and the cognitive conflict interventions that it was for research purposes and had no bearing on their classroom results.

Fifty-four Grade 11 students were given a pre-questionnaire and 12 of these were also interviewed to determine the level of their understanding before the interventions. They then participated in four cognitive conflict lessons. Forty-four of these students (10 were absent either during the conflict sessions or the post-questionnaire) wrote a post-questionnaire and 12 were again interviewed in order to determine if there had been any change in their level of understanding. The 18 Grade 12 students were all given the same post-interview questions, and 5 were also interviewed in order to compare their level of understanding to the Grade 11's. The principle here was to compare the structural understanding of the Grade 12's, who had had the same teacher for Grade 11 mathematics, but had not participated in the cognitive conflict interventions, to that of the Grade 11's who had participated in the cognitive conflict interventions.

Before the interventions I appraised the Grade 11 students' general level of understanding of mathematics according to their regular class work and their performance on the pre-questionnaire and divided them into three groups: those students I felt had a weak understanding, those I concluded had a moderate understanding, and those I felt had a very good understanding of mathematics at their grade level. I will refer

to the three groups as foundation, moderate, and proficient, respectively, only in reference to their understanding of Grade 11 mathematics without any prejudice to their overall ability. I assessed 17 as foundation, 20 as moderate, and 17 as proficient students. I used this distinction into three groups for three purposes; to ensure I interviewed students of all ability levels, to inform my examination of the development of individual students detailed in Chapter five, and to help answer my research question: would the cognitive conflict technique promote students of all levels to a structural understanding? The students were never aware of my ranking of them, and this ranking is only a reflection of my appraisal of them *before* the cognitive conflict interventions.

The Grade 11 students who were interviewed were selected on the principle of a fair representation of the various levels of ability in the class. Four students from each of the three groups were selected and asked to participate in an interview. The students were also somewhat self-selected as they had to be willing to be interviewed outside of school time. This did not affect the balance of the three groups. There were three students who participated in the pre-interview and could not participate in the post-interview, so three other students of very similar ability were asked to participate in the post-interview instead. One post-interview of a student did not record properly and I did not have a chance to repeat it, so I discarded their pre-interview as well. The Grade 12s were all self-selected. The Grade 12s all had a moderate or a very good understanding of mathematics at their grade level, as weak students do not continue in Grade 12 mathematics.

For the Grade 11 students the pre-questionnaire and interviews took place in January and February after they had been introduced to quadratics, systems of equations, and inequalities. The first cognitive conflict session took place in February as a practice in order to help refine the technique, and the rest took place in April and May. The post-questionnaire and interviews took place in June. The Grade 12 post-questionnaire and interviews took place in June.

A dependable test of structural understanding

How can we determine a student has a structural understanding?

Since both the research (Sfard, 1995, Fuji, 2003, Mason, 1996) and my experience as a typical high school math teacher indicate that students can become very proficient with only a procedural understanding, it became necessary to establish what a test for structural understanding required. I determined that in order to differentiate between a procedural and a structural understanding I would need to ask the students questions at the Grade 11 level that demand a structural understanding in order to complete correctly, and I would, at the same time, have to watch for the behaviours described in the four themes detailed in Chapter one that a student demonstrates when they have a structural understanding of a concept. Recall that the four themes were: a student can oscillate between a procedural and a structural view, the language a student uses, the justification of transformations, and the presence of a good control mechanism.

A test of structural understanding

In order to decide whether or not the cognitive conflict interventions had moved any of the students from a procedural to a structural understanding, a dependable test to differentiate between the two was necessary. It needed to be consistent, not too onerous to use, and able to demonstrate clearly whether a student had a structural understanding or simply a procedural. The work of Sfard and Linchevski (1994) and Fuji (2003) provided types of questions at the eleventh grade level that require the student to have a structural knowledge in order to complete them successfully. The types of questions are: quadratic inequalities that have no solution or an infinite number of solutions, equations and inequalities or systems of equations and inequalities that have an infinite number of solutions, and parametric equations. Examples of these three respectively are:

$$\text{Solve } x^2 + x + 1 > 0 \text{ for } x.$$

$$\text{Solve } 2(3 - x) = -2x + 6 \text{ for } x.$$

Solve for b so that this system has no solution

$$\begin{aligned} y &= 2x + b \\ y &= (x - 1)^2 + 2 \end{aligned}$$

In addition, Linchevski and Sfard (1991) provide an excellent means of probing a student's understanding to differentiate between structural and pseudostructural understanding. Their method capitalizes on the predilection of students with procedural understanding to automatically apply procedures and not return to consider the underlying process or structure of a question. They gave students pairs of equations and inequalities that in some cases were equivalent and not transformable procedurally one to

the other, and in other cases were not equivalent but are very tempting for students to transform one into the other. Examples of these two types respectively are:

Do these two equations have the same solution set?

$$(x - 3)^2 = 0$$

$$5x - 4 = 2x + 5$$

Do these two inequalities have the same solution set?

$$4x^2 > 9$$

$$2x > 3$$

All of these 5 types of questions are effective in testing for structural understanding because they each contain what I call a 'procedural trap'. This is a point during a procedural solution where the procedures the students are taught produce an anomalous result. These anomalous results can only be resolved if the student can conceive of the structures as processes and objects simultaneously, and so the student's ability to oscillate between a structural and a procedural view is tested. There is no memorized procedure or answer that can be called upon at this point. The students are forced to go beyond their standard procedures and must relate the anomalous result to the underlying structure of the problem. It is also at this point in the question that the good control mechanism that comes with structural understanding is exposed. If the student has a structural understanding they will be able to compare processes and objects and recognize when the procedure produces an incompatible result, leading them to review their work with reference to the structure of the question. To someone with a structural understanding these traps may not seem to pose much of a problem, because they can oscillate views and unconsciously avoid them. Students with a procedural understanding find the traps impossible to resolve properly. An example is when the variable disappears in an equation with infinite solutions. When a student with a procedural understanding

reaches a point like $2x + 4 = 2x + 4$ their procedures produce $x = x$ or $0 = 0$. There is no way to produce the expected form of ' $x = 7$ ' that is the "halting signal" (Sfard and Linchevski, 1994, p. 222). Without the ability to oscillate to examine the structure of the problem, the student is left to search in vain for a procedure or invent an improper manipulation to produce the $x = 7$ halting signal. I will highlight the procedural trap in each of the questions I use for my tests. Tall (1998) uses the term procedural trap in a different context and the fields of computer science and law use the term in considerably different contexts.

Lastly the students were asked to explain why they were performing a procedure, either in writing on the questionnaires, or verbally during the interviews. This was intended to give a little more insight into the student's structural knowledge, but proved to be the most effective way to differentiate between Skemp's (1976) "knowing what to do" (procedural understanding) and "knowing what to do and why" (structural understanding).

Themes

The research described in Chapter one details the characteristics a student demonstrates when they have a structural understanding. The two strongest indicators were: an ability to oscillate between a procedural and a structural understanding, and language that indicates they are seeing the concept both as a process and as an object simultaneously. Harder to observe were the students justification of the transformations they use in a problem, and the presence of a good control mechanism, throughout their solving, that alerts them to errors.

The pre and post-questionnaires and interviews were then created using these concepts. See Appendix B for the actual tests I used. I will describe the intention behind each question.

The pre-questionnaire and interview

Pre-questionnaire:

- 1.) Solve for x : $4x + 8 = 5x + 2$
- 2.) Solve for x : $b = vxt$
- 3.) Solve for x : $6x + 12 = 3(2x + 4)$
- 4.) Show using algebra that the sum of two consecutive numbers is always an odd number

Pre-interview:

- 1.) Solve for x : $6x + 12 = 3(2x + 4)$
- 2.) Find k so that this function will have 2 different real roots: $f(x) = x^2 + kx - 4$

The pre-questionnaire and interview consisted of questions that demanded a structural understanding and allowed me to observe the four themes outlined in Chapter one to evaluate if a student had a procedural or a structural understanding. The pre-questionnaire was designed not to intimidate or discourage any of the students, so it began with two questions that should have been quite familiar to them, and finished with two questions involving concepts they would have been exposed to previously in Grade 11. Question 1 was designed to test if the students could solve an equation they would have been expected to solve up to two years previously. This would give an indication of the students who were not yet procedurally proficient, and the language used in their explanation would give a good indication of whether or not they had a structural understanding. Question 2 was a straightforward parametric equation. Again, they would

have been exposed to this type of question for two years, and the language of their explanation would indicate whether they had a procedural or a structural understanding. Questions 3 and 4 were of the type that could only be properly solved if the student had a structural understanding. In question 3 the procedural trap is the variable disappearing, providing a point that exposes all the themes that differentiate between a procedural and a structural understanding. Question 4 differentiated between a procedural and a structural understanding because there is no procedure to solve it, and 'solving' an equation for a student with procedural understanding means completing a specific algorithm until a halting signal is reached. Typically, on this type of question, the students produce an equation similar to $(x) + (x + 1) = 2x + 1$. Both sides of the equation are processes, yet they cannot be carried out in order to find an answer. This question requires a suspension of calculation and requires the students to explain *why* the process represents an odd number rather than to complete a procedure. This is the procedural trap of this question; only with an oscillation between conceiving of the result as a process and as an object can a student explain this problem properly; automatically performing a procedure will not produce any useful result.

Question 1 of the pre-interview is a repeat of the question where the variable disappears. It required the students to be able to oscillate between a procedural view and a structural view, and was intended to elicit comments that would demonstrate if the student had a structural understanding. Question 2 of the pre-interview is a parametric equation that has a procedural trap. At this point in the year the students were adept at finding the roots of a given quadratic, and familiar with the discriminant of the quadratic formula, but this question necessitated a structural understanding because the procedure

they use to solve this type of question does not give an answer to what k is. Instead, their procedure produces a discriminant of $k^2 + 16 = 0$ or $k^2 + 16 > 0$ and, although the solution may seem obvious to us, this is the procedural trap. An oscillation to a structural view produces the realization that k can be any number. A student without a structural understanding continues the procedure to produce $k^2 = -16$ or $k^2 > -16$, decides the next step is to find the square root of both sides and answers 'no solution' because they have focussed on finding the square root of a negative number. This question elegantly displays whether or not a student has a good control mechanism because $k^2 + 16 > 0$ or $k^2 > -16$ should provoke the realization the any number squared will be greater than a negative number, but a student rigidly completing a procedure without monitoring their steps concentrates on the fact that finding a square root of a negative number is not possible and cannot find a halting signal.

The post-questionnaire and interview

Post-questionnaire:

- 1.) Solve for x : $x^2 + x > -1$
- 2.) Solve for x : $3(7x + 5) = 21x + 15$
- 3.) Show using algebra that the sum of an odd number and an even number is always an odd number.
- 4.) For what values of k will this system have a solution?

$$2x + y = k$$

$$y = 3 + k$$
- 5.) Do these two equations have the same solution set?

$$4x^2 > 9$$

$$2x > 3$$
- 6.) Solve for x : $x^3 - 3x^2 + 3x + 7 > 0$

Post-interview:

1.) Do these two equations have the same solution set?

$$(x - 3)^2 = 0$$

$$5x - 4 = 2x + 5$$

2.) For what values of b will there be no solution to this system of equations?

$$y = 2x + b$$

$$y = -2(x - 1)^2 + 2$$

The post-questionnaire and interview were designed very similarly to the previous questionnaire and interview as tools that would test the students' structural understanding and allow me to observe the four themes to decide if the students had a procedural or a structural understanding. Questions 1, 2, and 3, of the post-questionnaire are types that demand a structural understanding and that the students worked with during the cognitive conflict interventions. Questions 2 and 3 are similar to questions 3 and 4 of the pre-questionnaire in order to make the comparison of results between the pre and post-questionnaires as valid as possible. Questions 4, 5 and 6 were unfamiliar types so that I could determine if the students had progressed to a structural understanding rather than just become familiar with the type of questions involved in testing their understanding. In Question 4 one variable, k , disappears, but a value for x can be found. This is a very devious trap. A student without a structural understanding will be satisfied with this halting signal or will not know how to find k . Question 5 was designed to help differentiate between a structural and a pseudostructural understanding. We had worked with equations similar to this during the interventions, but the procedural trap is that inequalities written in this form tempt the students to disregard one of the roots of the quadratic. A student with procedural understanding answers incorrectly that the two inequalities have the same solution because they do not see the structure and

automatically apply a procedure (finding a square root) that establishes the two are equivalent when they are not, or the student may not be able to answer at all because they have never learned a procedure for this type of problem. Question 6 was designed to determine if the cognitive conflict interventions would only move the students from a procedural to a structural understanding of the specific types of questions studied during the interventions, or if the interventions would produce a structural understanding the students could use to solve unfamiliar problems.

Questions 1, 5, and 6 exploit the findings of Bazzini and Tsamir (2004), that students at this level approach inequalities with procedures, often borrowed from solving equations, that do not work on non-routine problems. This approach comes because inequalities are taught as subordinate to equations and "dealt with in an algorithmic manner ... resulting in a sequence of routine procedures" (p. 140). If questions 1, 5, and 6 had been equations the students would have easily solved them procedurally. By making them inequalities with none or only one root the students were forced to oscillate between a procedural and a structural understanding on a question that they were not familiar with. At the point where they found no roots they would have to switch to a structural outlook because the procedure we normally used to solve inequalities would not work properly at this point. The procedural trap is produced by the negative discriminant or no roots. In equations the students interpret this as no solution, but in these questions the negative discriminant must be assessed with reference to the structure of the inequality.

These multiple intentions made the post- questionnaire significantly longer and more difficult than the pre-questionnaire.

Question 1 of the post- interview may seem too simple a question to determine structural understanding (or it may not!), but I felt it was similar to the questions the students would work on during the cognitive conflict interventions but not exactly the same. This is the type of question from Linchevski and Sfard (1991) that would reveal a lack of structural understanding if the student automatically attacked it with a familiar procedure without a consideration of the actual objects. This question has a trap because the students use different procedures for solving quadratics and linear equations and so would see the two equations as having different solutions. It would reveal a structural understanding if the student compared the equations as objects. Question 2 of the post-interview was an extremely difficult question that the students would not be familiar with from class or from our cognitive conflict interventions. Because it is a parametric question, it is not solvable by the procedures we learn in class. Since it is presented as a system, it is very tempting for the students to use substitution, set the equations equal to each other and isolate b . At this point b will equal a quadratic and it is very tempting to use a familiar procedure to solve the quadratic (this would give the values of x when $b = 0$). This is the trap for a student with a procedural understanding. Without an oscillation between procedural and structural views at this point, they would not be able to solve this question. This question would definitely expose the students' ability to oscillate, their justification of transformations, and their comparison of objects. I felt it would be a litmus test to determine if the students had developed a structural understanding and that their comments would be very revealing.

The cognitive conflict interventions

The cognitive conflict interventions were designed using the accumulation of knowledge gained from the cognitive conflict studies related in Chapter one. Recall that in order for the cognitive conflict interventions to be effective: the students had to be at a level where conflict would be induced; they had to be presented with a problem that induced conflict; this had to take place in an environment that was not threatening; and they had to be provided an alternate conception they could understand.

I designed my interventions to incorporate these requisite factors and to be deliverable in a typical classroom setting. I created a method that was possible to use with entire classes at once, rather than the small numbers in interview settings that most researchers had worked with in the past. I created the worksheets found in Appendix C to lead the students through the phases of conflict and resolution listed in the previous paragraph. While the students were completing the worksheets I moved about the room to assess their progress and ensure the students were experiencing conflict and resolving it properly. The worksheets all had an identical look and organization with different concepts being addressed so that the interventions were all structurally similar and followed a similar pattern of implementation. The worksheets began by asking the students to solve a problem that demanded a structural understanding, and asking them to check and explain their answer. These problems all had a procedural trap where the student was forced to work with process and object simultaneously. This phase was meant to force the students to make their conceptions explicit so that they were available for change (after Watson, 2002), and because, as Sfard (1991) identifies, there is no reason to conceive of a process as an object if you are not required to use it as an object.

During this phase I moved around the room asking the students to check their answers with values that would trigger cognitive conflict. In order to determine if conflict had been induced in a student, I looked for evidence of the verbal and non-verbal signs that are indicators of cognitive conflict described in Chapter one. Recall that they are: a recognition of an anomalous result, interest in this result, anxiety, and reappraisal of the situation. If I felt conflict had not been induced, I worked to ensure that the student's understanding of the concept was enough to provoke meaningful conflict, and again asked a question that would induce cognitive conflict.

In the second phase the students were then presented with two solutions purportedly obtained by fictional students of previous years. They were given an incorrect procedural solution where the fictional student had fallen into the procedural trap and a correct solution that involved the oscillation between a procedural and a structural conception. It was made clear that the students were fictional, but the solutions typical. This was patterned after Watson (2002) and Fujii (2003). The students were then directed to decide which answer was correct and to choose a partner who had decided the opposite and justify why the answer they chose was correct. This phase had several objectives. The discordant results were designed to bring about conflict if the original problem had not, in an attempt to ensure the weaker students were aware of the conflict. The conflicting answers were intended to make the student examine both the process and the object, to become dissatisfied with their procedure, and to promote the oscillation between a procedural and a structural view. These discordant results were also designed to increase the level of conflict the stronger students felt, as Kwon, Lee, and Beeth (2000) had found very clearly that "students who expressed higher levels of conflict showed

higher rates of conceptual change" (p. 20). The discordant results were presented as previous students' solutions in order to increase conflict, but prevent the intimidation of the weaker students that Lee and Kwon (2001) and Dreyfuss, Jungwirth and Elovitch (1990) found. The solutions were fictional and given in written form, rather than using video of actual students, or drawing the solutions out of the students as other researchers (Watson, 2001, Fujii, 2003, Adey and Shayer, 1993, Bell, 1993) had done. This allowed me to make the responses optimal in structure and detail in order to make them conceptions the students could understand, and gave each student the solutions at hand to review while they struggled to determine the correct conceptions. This meant the method lent itself better to working with entire classes at once, rather than small numbers in interview settings. Asking the students, in pairs, to discuss and justify which response was correct was designed to ensure they were operating at Fujii's (1987) "justification of how" level and to immerse the students in the experts' method of solving algebra problems that Cortes (2003) found. When the solution is not evident and experts are confronted with a choice of methods, they explicitly check the validity of a method and justify the transformations used. Cortes considers "mathematical justification as fundamental" (p. 258) to the experts' solving methods. Even if the students chose to justify the procedural solution they were engaged in the experts' method. Requiring the students to converse with a partner and defend one of the answers was designed to make their conceptions explicit and bring about the proper resolution of the conflict.

Schoenfeld (1985) also feels,

"It seems reasonable that involvement in cooperative problem solving – where one is forced to examine one's ideas when challenged by others, and in turn keep an eye out for possible mistakes that are made by one's

collaborators – is an environment in which one could begin to develop the skills that, internalized, are the essence of good control" (p. 143).

The intention was to gently amass the negative feedback for the incorrect procedural solution and positive feedback for the correct solution. They were required to work with only a partner and to defend a fictional student's answer to prevent the intimidation created by defending one's own thoughts to a group. I endeavoured, in this way and by emphasizing that the interventions would not have any affect their classroom results, to provide as non-threatening an environment as possible. However, there is a pervasive power relationship between teacher and student that may have had an effect on the students' experience despite my effort to mitigate any apprehension. During this phase I moved around the class to ensure that the students were involved in justifying one of the answers and not just accepting one of them. I also ensured that justifying the procedural solution would demonstrate why it was incorrect.

In the final phase the students were asked to explain the concept involved in the original question as a generalized mathematical relationship, and to illustrate the concept with an example. This was combined with me moving about the class asking the students how they were thinking about the concept, in terms of a specific procedure, or in terms of a mathematical property. This was designed to elicit the consideration of the process and the object at the same time and again to encourage the oscillation from a procedural to a structural outlook. A classroom discussion of the mathematical properties of the original problem, and why these caused the procedural method to fail, followed this. This was a sharing of ideas and defending of them designed to provide the students with an alternate conception to their procedural understanding that they could understand.

The overarching goal was to make the students dissatisfied with their procedural method of solving and then effect a compensation that resulted in the students gaining a structural understanding. The dissatisfaction was brought about by demonstrating that their answer was wrong or that their method was similar to that of the fictitious student who was wrong. They were then delegated to discuss and justify which previous student was correct in order to engender some negative feedback for using just the procedural method and positive feedback for considering the structure at the same time as the procedure. The final discussion, defence of the answer they chose and creation of examples were to ensure their compensations were correct.

CHAPTER THREE: RESULTS

The questionnaires and interviews I conducted produced unexpectedly rich and abundant results that I realized could be analysed according to many criteria. For a list of future research ideas inferred please see the end of Chapter Eight: Pedagogical Implications. I will analyse the data only in terms of the specific intent of this study and examine and discuss the observations only in relation to what they show about the effect of the cognitive conflict interventions on the students' structural understanding of algebra. In this chapter I will first remark on my observations of the cognitive conflict interventions, and then list the numbers of students correctly answering the questions from the various instruments and give a quick summary of the types of their responses. In the next chapter I will perform an in-depth examination of the responses of the students as a population using the four themes developed in Chapter one. In each theme I will compare the Grade 11's results from before the cognitive conflict interventions to their results after the interventions to determine whether the population as a whole has demonstrated any growth from procedural to structural understanding. In the following chapter I will examine the changes demonstrated by specific individuals from each of the three groups (foundation, moderate, and proficient) that I had distinguished before the interventions. After this I will then compare the Grade 11 results to the Grade 12 results in order to compare the numbers of students with structural understanding between a group of students that participated in the cognitive conflict interventions and a group that

didn't, but had the same teacher for their Grade 11 year. All of the interviewed students also participated in the pre- and post-questionnaires, and their results are included in the population's. The interviews provide a group of results in addition to their performance on the questionnaires that enables a comprehensive analysis according to the four themes. The student names I use when I present quotes are all fictitious pseudonyms. Gender has been preserved, but I made no attempt to preserve the ethnic origins of names. I was the interviewer in every case.

The Cognitive Conflict Interventions

After each cognitive conflict intervention I recorded a point-form summary of the observations I had made of the students' behaviour, remarks, vocabulary, questions, interactions and engagement with the material during the intervention. During the interventions I endeavoured to ensure all the students were experiencing conflict, but not being threatened, and resolving the conflict properly. I used the indicators of conflict described in Methodology to decide if a student was experiencing conflict. I then used their answers to the final two parts of the worksheets (their defence of a fictional student's work, and could they explain the concept in general terms and create a further example) to decide if they had resolved it properly.

In each intervention I felt I had been successful in producing these outcomes. The questions did produce conflict according to the indicators I was evaluating for. In each intervention I found the students were intrigued when a value I gave them produced an anomalous result, or when they were presented with two answers, one of which was correct. Having the students discuss and justify which previous student was correct

worked extremely well in a non-threatening way. This provided them with an alternate conception that they understood, and defending someone else's conclusion (and discarding it if it is wrong) is much less threatening than being attacked on your own work. Explaining the concepts in general terms and producing a further example reinforced the new structural conception.

The Pre-questionnaire

Questions:

- 1.) Solve for x: $4x + 8 = 5x + 2$
- 2.) Solve for x: $b = vxt$
- 3.) Solve for x: $6x + 12 = 3(2x + 4)$
- 4.) Show using algebra that the sum of two consecutive numbers is always an odd number

The pre-questionnaire succeeded in not intimidating the students. When asked to write on the end of the sheet how hard they found the questions, the responses were all a form of either, "It was easy" or "The questions were mostly easy, but 4 was tougher".

Questions 1 and 2 of the pre-questionnaire did not demand a structural understanding and were mostly correctly answered. As expected, 53 of the 54 students correctly answered Question 1. Unexpectedly, only 46 students correctly answered Question 2. The written explanations accompanying their work offer evidence of a procedural understanding that I will examine in the next chapter. Questions 3 and 4 demanded structural understanding and so the results were reversed. Five students correctly answered question 3 and nine students correctly answered question 4.

Question 2

The eight students who answered incorrectly either left this question blank or subtracted the variables from both sides of the equation rather than dividing them. This was an unexpectedly low procedural ability on this type of question demonstrated by these eight.

Question 3

The students who answered this question incorrectly all had a form of the following three responses. They would perform well procedurally until they reached $x = x$ or $1 = 1$ or $0 = 0$. They would leave this as the answer or answer "x = 0" or "x is undefined".

Question 4

Many students did not know where to start this question and actually asked what procedure to use while they were writing the questionnaire. Most, with a little prompting, produced this: " $x + (x + 1) = \underline{\hspace{2cm}}$." 17 students at this point left a question mark or wrote, "I don't know how to show an odd number." 13 students substituted in various numbers as examples. Five students set up the equation $x + x + 1 = 0$ and solved to find a result of $x = -1/2$.

The Pre-interviews

Questions:

- 1.) Solve for x: $6x + 12 = 3(2x + 4)$
- 2.) Find k so that this function will have 2 different real roots. $f(x) = x^2 + kx - 4$

The pre-interviews reinforced the above results. Of eleven students, two of the students answered question one correctly and two of the students answered question two correctly. Only one of the students answered both questions correctly. The rest of the students' answers were incorrect.

Question 1

Two students answered this question correctly while the rest reached the $x = x$ or $0 = 0$ stage and could not proceed properly.

Question 2

Two of the students answered this question correctly, the rest of the students performed illegal manipulations or searched for a procedure when they reached $k^2 + 16 > 0$.

The Post-questionnaire

Questions:

- 1.) Solve for x : $x^2 + x > -1$
- 2.) Solve for x : $3(7x + 5) = 21x + 15$
- 3.) Show using algebra that the sum of an odd number and an even number is always an odd number.
- 4.) For what values of k will this system have a solution?
$$2x + y = k$$
$$y = 3 + k$$
- 5.) Do these two equations have the same solution set?
$$4x^2 > 9$$
$$2x > 3$$
- 6.) Solve for x : $x^3 - 3x^2 + 3x + 7 > 0$

The post-questionnaire showed that the majority of the students could now correctly answer questions that demand a structural understanding and explain their answers with ‘why’ instead of ‘what they did’. Forty-four students answered the post-questionnaire. Seven students clearly still had a procedural understanding, and another six could not answer questions 4, 5, and 6, but had excellent answers for 1, 2, and 3. This meant 31 students demonstrated impressive structural understanding. Of these, six students had one of questions 4 or 5 incorrect, and three had questions 5 and 6 incorrect, while 22 of the students answered all the questions correctly.

Table 1 Grade 11 results on post-questionnaire

Numbers of Grade 11 students able to answer questions of post-questionnaire correctly n = 44			
All questions incorrect	Correct on questions 1, 2, 3 only	Mistake on questions 4, 5, 6	All questions correct
7 16%	6 14%	9 20%	22 50%

Question 1

Two students wrote only $x^2 + x + 1 > 0$. Three students converted this to $x^2 + x + 1 = 0$, used the quadratic formula to solve, and when they found a discriminant of -3 explained there was no solution. The rest of the students answered this question correctly with an explanation that showed structural understanding. Interestingly, these students had a number of different and diverse explanations. I will examine this when I discuss the results according to the four themes.

Question 2

Seven students reached a solution of $x = x$ or $0 = 0$ and left this, or explained there was no solution. The rest of the students answered correctly with a form of these two explanations: "They are the same line so all values of x will work", "It is the same equation on both sides so x will give the same thing".

Question 3

Seven students did not answer this or used actual number examples, rather than algebra. The rest of the students gave some form these two explanations: They had written either $2x + 1$ or $4x + 1$ and explained *why* this represented an odd number.

Question 4

The students found this question very difficult. Eighteen students either left this question blank or solved a system of equations algebraically and reached $k - k = 2x + 3$ (the point where k disappears) and could not draw a conclusion or wrote "no solution" at this point. The rest of the students solved this question with many different explanations, recognizing that these were two lines with different slopes or that the k 's were eliminated and concluding k could have any value.

Question 5

Seventeen students answered incorrectly or not at all. Six of these answered all the other questions correctly. The 27 students who answered correctly had explanations or drew graphs involving the structure of the parabola and the line. I will examine these results in detail in the next chapter: Themes.

Question 6

Four students did not answer this question. Four answered with $x = -1$ rather than $x > -1$. Eight had correct workings, but no explanation after finding that the polynomial only had one root. The rest found that there was only one root to the polynomial and answered x had to be greater than -1 . Most of these then explained their algebra with a graph, or their graph with a comment.

The Post-interviews

Questions:

- 1.) Do these two equations have the same solution set?

$$(x - 3)^2 = 0$$

$$5x - 4 = 2x + 5$$

- 2.) For what values of b will there be no solution to this system of equations?

$$y = 2x + b$$

$$y = -2(x - 1)^2 + 2$$

The post-interviews showed that two of the students are still working with a procedural understanding, but the rest have gained a structural understanding.

Question 1

All the students were somewhat unfamiliar with the idea of a solution set because the British Columbia curriculum does not contain any work on sets, but I had briefly explained what I wanted in terms of a solution set during the cognitive conflict interventions. Nine of the interviewed students answered question 1 correctly. These students solved this question so quickly that I was surprised and asked further questions to check for a structural understanding. I will expand on this in the themes section below. Two students answered incorrectly.

Question 2

Two students needed quite a bit of prompting and did not demonstrate a structural understanding. Two students needed some prompting to correctly answer this question. The rest solved it quickly and much more easily than I expected. I pointed out arithmetic mistakes to two of the students.

My examination of the numbers of students who were able to solve these questions that demand structural understanding generates the conclusion that the students began the study with a procedural level of understanding. After the interventions, a small group ($n = 7$) are still at a procedural level, a small group ($n = 6$) have developed a structural understanding of the basic questions, and the majority ($n = 31$) have developed a broad structural understanding of Grade 11 concepts. These levels are borne out and refined in the next chapter when I examine the results according to the four themes.

The Grade 12's

Questions same as Grade 11 post interview.

Ten of the 18 Grade 12 students correctly answered question 1 of the post interview and 8 incorrectly. One student correctly answered question 2 and 17 answered incorrectly. In Chapter six I will compare their answers to the Grade 11's answers to show that the Grade 12's results are significantly more alike the Grade 11's results before the interventions than after the interventions.

Table 2 A summary of the Grade 11 results

The numbers of Grade 11 students answering correctly and incorrectly on the questionnaires and interviews								
	Pre-questionnaire n = 54		Pre-interview ¹ n = 11		Post-questionnaire n = 44		Post-interview ² n = 11	
	Correctly	Incorrectly	Correctly	Incorrectly	Correctly	Incorrectly	Correctly	Incorrectly
Question1	53 98%	1 2%	2 18%	9 82%	39 89%	5 11%	9 82%	2 18%
Question2	46 85%	8 15%	2 18%	9 82%	37 84%	7 16%	9 82%	2 18%
Question3	5 9%	49 91%			37 84%	7 16%		
Question4	9 17%	45 83%			26 59%	18 41%		
Question5					27 61%	17 39%		
Question6					36 81%	8 19%		

1. All interviewed students participated in pre-questionnaire and are included in those results.
2. All interviewed students participated in post-questionnaire and are included in those results.

Table 3**A comparison of the Grade 11 post-intervention results to the Grade 12 non-intervention results**

	Number of Students Exhibiting Procedural or Structural Understanding	
	Procedural	Structural
<i>Grade 11 post-questionnaire</i> n = 44	7 16%	37 84%
<i>Grade 11 post-interview</i> ¹ n = 11	2 18%	9 82%
<i>Grade 12 post-questionnaire</i> n = 18	17 94%	1 6%
<i>Grade 12 post-interview</i> ² n = 5	5 100%	0 0%

1. All interviewed students participated in post-questionnaire and are included in those results.
2. All interviewed students participated in post-questionnaire and are included in those results.

CHAPTER FOUR: THE FOUR THEMES

If we examine these results with reference to the four themes investigated in Chapter one as indicating structural understanding, we can make a solid evaluation of the level of structural understanding demonstrated by the students. Recall that the literature showed that a structural understanding is reflected in the student's language, their ability to oscillate between a procedural and a structural view of a problem, their explicit justification of a method they use when faced with an unfamiliar problem, and the exhibition of a good control mechanism to ensure an answer is correct.

The pre- questionnaire and pre-interview have shown that before the cognitive conflict interventions the majority of the students have a procedural understanding. Their language strongly indicates they are following a procedure and not looking at a question structurally. Several of the students can be prompted to look at a question structurally, but there is not an oscillation between a procedural and a structural view when they encounter the procedural trap, were it is needed. They are ready for a structural understanding, but they don't have it. At the point where an oscillation between views is needed they can find the correct solution if the structure is pointed out to them, but they do not do this on their own. The justification of methods and good control mechanisms are not present. The students rigidly complete procedures, they don't see contradictions their transformations create and are satisfied with an answer a procedure has given them

even if it looks inappropriate to them. They demonstrate perceiving 'x =' as a halting signal and being lost if the variable disappears.

When I examined the post-questionnaires according to these same themes it reinforced and clarified the results I displayed in the last chapter. I found the majority of the students had a structural understanding. Several (n=7) still had a procedural understanding and several (n=6) showed structural understanding only on the questions we specifically dealt with during the interventions, but were still at a procedural level for the more advanced concepts. The post-interviews demonstrate a similar pattern; two of the eleven students still have a procedural understanding, two have a minimal structural understanding, and the rest demonstrate a broad structural understanding. The majority of the students now use references to objects in their explanations rather than lists of steps. They show oscillation between a procedural and a structural view at the point where these questions demand it. When working with unfamiliar questions they justify their methods and show a good control mechanism that prevents them falling into the traps these questions set for those with a procedural understanding.

The ability to oscillate between a procedural and a structural view is the most critical test of structural understanding, but it is the students' language that provides the most accessible evidence, so I have begun with an examination of the students' language. In a number of the interview excerpts I have used a bold font in order to highlight the particular theme being examined and refer back to a specific point; it does not represent shouting or any emphasis on the part of the student or interviewer.

Language

In order to determine whether a student had a procedural or a structural understanding, I analyzed their language for the following key features. A student with a procedural understanding will list the steps of a procedure or an algorithm, and explain *how* they reached an answer when asked to explain *why* they produced an answer. They will demonstrate that a question asking them to 'solve' directs them to complete a procedure. Conversely, a student with a structural understanding will describe a concept as an object or explain how a process also represents an object. They will compare objects, or the equivalency of two structures. They will also support their explanation with a diagram. A student with a structural understanding will not provide a list of steps as an explanation; instead they will comment how the anomalous result produced by a procedural trap relates to the structure of the question.

Language before the interventions

Before the interventions the language of the students reveals their procedural understanding. Their talk is about the syntax rather than the semantics of the problems. The students just list the steps of their procedure as if it were an algorithm. They do not talk of the concepts as objects, and do not support their explanations with visual aids that describe their objects. Even when they answer a question correctly, their language can reveal that they don't have a structural understanding. When asked to explain *why* the answer is correct, they explain *what* they did. This is Skemp's (1976) "knowing what to do" not "knowing what to do and why". These results were consistent across the ability

levels of the class. Questions 3 and 4 of the pre-questionnaire were most often without comment, but I've supplied some examples that were present.

Questions 1 and 2 of the pre-questionnaire did not require structural understanding to solve, but the written explanations accompanying the solutions demonstrate the students' procedural knowledge. Some typical examples of question 1 (*Solve for x: $4x + 8 = 5x + 2$*) are:

*Mildred: First I moved the x's to one side and the regular numbers to the other side.
Then I divided by one so I could get x by itself.
I also had to divide the other side by 1.*

Joshua: Bring x's over to one side bring numerals over to other

Brian: Bring the x's over to one side of the equation and the numbers to the opposite side

Tom: You move the variable to the other side ($5x - 4x$) and the rest to the other

When asked to explain why they performed a step, the students wrote what they had done.

The following quotes from question 2 (*solve for x: $b = vxt$*) are exemplars of the population that demonstrate the students are listing the steps of a procedure:

Darren: you have to isolate the x's to one side. So put them all to the left than [sic] the non x's to the right. Then solve.

Georgia: Get x alone.

Laura: I divided by vt to get x by itself. I did this step on both sides.

Question 2 was especially interesting as five students used a correct procedure but showed they didn't understand why to use it. For example:

Eunice: $b/t/v = x$ *find the answer. ☹ I don't get it*

Katherine: $b/t + v = x$ *no numbers! I'm confused!*

without a halting signal ($x = 7$) the students feel they have not solved the problem.

Again in question 3 (*solve for x: $6x + 12 = 3(2x + 4)$*) the listing of steps is visible with no mention of the structure.

Walter: *Distribute $3(2x + 4)$. x can only be on one side of the equal sign.*

John: *Solve brackets first, expand then isolate x again*

Brin: *I multiplied 3 by everything in the bracket. The product was equal to the left side, thus, the two values cancelled out. Easy.*

We can see there is no indication they see the concepts as objects. This question was the most telling; there was no discussion of lines despite the fact that the students had studied the equations of lines the two previous years, we had spent a month on the intersections of lines this year, and if I had asked "where do the lines $6x+12$ and $3(2x+4)$ intersect" the students would find it an easy question.

On question 4 (*Show using algebra that the sum of two consecutive numbers is always an odd number*) as well, the students didn't show any indication they could conceive of the concepts as objects:

Jason: $x + (x + 1) = y$ so y is an odd number [no other explanation]

Marie: If you add an odd # then the answer must add up odd therefore I used #1 to add to x . x and $x+1$ are consecutive #'s. I don't think it worked!

Many came up with a correct expression $(2x+1)$ but saw this as a process that needed further transformation, not as an object that was the answer, so they gave number examples, or said they could not solve it.

Pre-interviews

On the pre-interviews the students displayed the same procedural understanding as on the pre-questionnaire. Here are the responses of two students solving question 1 (*Solve for x : $6x + 12 = 3(2x + 4)$*) that are typical of the nine students who could not solve this question:

Bella: First I'm going to put the three into the brackets

Interviewer: Distribute it?

Bella: Ya

Interviewer: how do you know that works?

Bella: Just from doing it

Interviewer: ok

Bella: and then um, I'm going to bring everything over to one side

Interviewer: ok, and why did you do that

Bella: um, cause that's just how I've done it forever

Bella: and then, um, I'd put it all together, the like terms

Bella: and I get zero = zero

and:

Mori: alright, so ah, multiply the three into brackets and get six x plus twelve, and the other side being six x plus twelve.

Interviewer: And how do you know to multiply it like that?

Mori: Well, you do it because that makes the equation easier.

Interviewer: Ok, now what do you do?

Mori: [laughs] umm... you move the six x and the twelve over to the other side, [mumbles works on paper], and you get zero equals zero.

The traits of procedural understanding are evident. They treat 'solve' as a direction to complete a procedure. They are listing steps of a familiar procedure. The answers they gave when I asked them to explain why they did a step are revealing; "just from doing it", "that's just how I've done it forever". They don't compare the objects or the relevance of their solution to the original question, and so when they reach $0 = 0$ they are satisfied with that as an answer.

Here are typical examples from question 2 (*Find k so that this function will have 2 different real roots. $f(x) = x^2 + kx - 4$*):

Nelson has reached ($k^2 + 16 > 0$) and said "k is greater than negative 4" and I've asked him to check a lower value.

Nelson: [types in] Hmm. What'd I do wrong?

Interviewer: Can you square root a negative number?

Nelson: No.

Interviewer: Can you just look at it here [$k^2 + 16 > \text{zero}$] and just decide what k could possibly be?

Nelson: Oh ya, you can't [find the square root of] a negative number so it's no solution.

Interviewer: Ok, what about the calculator, we just saw some.

Nelson: [(laughs) Oh ya.

...

Nelson: So I don't know how to do it, can you show me?

...

Interviewer: [Laughs] Try some values in the calculator.

Nelson: [(types away, tries a couple) They all worked.

Interviewer: What does that tell you?

*Nelson: **I don't ...there are some, but I don't know the way to get them.***

Another student:

Bella: ummm....right, right, right, a positive number gives two roots

Interviewer: ok

Bella: ok so k equals four
 ...
Interviewer: why don't you check it
Bella: ok k is four [does the math in head...very pleased] ... ok equals zero.
 ...
Interviewer: so when k is 4 we get two different real roots?
Bella: yep!
Interviewer: if you put one in instead of k what do you get?
Bella: um... ah ok, right, ah a positive number
Interviewer: so can you say what k should be?
Bella: ahhh....no...
(More prompting but no solution)

The students are using familiar procedures, neither mentions the structure of $k^2 + 16 > 0$, but the most telling point is when Nelson says "there are some, but I don't know how to get them". After I point them out, he realizes there are values of k that would solve the problem, but can not think of a procedure to find them. These students represent the majority of students before the cognitive conflict interventions.

All of the responses I have just quoted are summed up well by what the students wrote at the bottom of their questionnaire when I asked them to tell me how hard they had found it. Thirty-eight of the students wrote they had found it easy or extremely easy and the rest said moderately hard or left it blank. This showed that they did not have the structural understanding to see the contradictions they had created or to be alerted to the procedural traps of these questions.

The language of the students before the interventions compelled me to conclude that the vast majority of the students had a procedural understanding. Their explanations are *what* they did, not *why* an answer is so. They listed the steps of their procedures and did not mention the structure of the question even if we had worked with the structure

extensively. They did not draw any graphs or diagrams to aid their explanations. Solve, to them, means carry out a set of steps. They were satisfied with the incorrect answers their procedures produced and asked what procedure to use when anomalous results were pointed out to them.

Two of the students did demonstrate structural understanding on the pre-interview, but it was because they experienced the cognitive conflict during the interview and resolved it. I say this because their pre-questionnaires show they have a procedural understanding, and during the interview they exhibited the signs of cognitive conflict. I will explain with their results. On the pre-questionnaire, Dorin solved question 1 with a list of steps as an explanation, had only a question mark for question 2, went from $6x = 6x$ to solve question 3 as $x = 0$, and had $(x) + (x + 1) =$ with a large question mark on question 4. Cadmar solved questions 1 and 2 with explanations that were lists of steps, could not get beyond $6x + 12 = 6x + 12$ on question 3 and on question 4 wrote, " $(x) + (x + 1) =$ this is where I'm stuck, how do I show an odd number?" For both students, these are the traits that demonstrate procedural understanding. Their language was a listing of steps, they did not oscillate between a procedural and a structural view, and they did not justify their transformations.

Here are excerpts of the two students completing the pre-interviews. Recall that the indicators of cognitive conflict are: recognition of an anomalous result, interest in this result, anxiety, and a reappraisal of the situation. Dorin is completing question 1 ($6x + 12 = 3(2x + 4)$) and Cadmar is completing question 2 (*Find k so that this function will have 2 different real roots. $f(x) = x^2 + kx - 4$*):

Dorin: Ahh, first I'd move everything to one side, want me to show you?

Interviewer: Ya, yep.

Dorin: Ok, [works on paper] first I'll solve this bracket.

Interviewer: And how do you know how to do that?

Dorin: It just means three times two and three times 4

Interviewer: Ok

Dorin: Then it's just a matter of moving it all over, oh wait, ya sorry, that was a mistake. Ya yep, you want to make it all on the left side, so, you're going to minus six x to get there, so you're going to get no x's... it's ah, I don't know, it's equal....

Interviewer: Alright, what does x equal?

Dorin: Ah, zero, one, two, dammit this is weird, what the hell... it can equal anything!

Interviewer: Can it?

Dorin: Ya.

Interviewer: Ok, how'd you figure that out?

Dorin: Well, when you look at it, like, it's basically whatever you do to that side you do to that side so it's going to be the same no matter what, 'cause the 12's are going to be the same no matter what so it's basically whatever you times six by, you times six by there and it can be any number.

Cadmar:

Cadmar: Ok, ya, so greater than zero, so...[works]...so k...so k...[mumbles...looks at interviewer] plus 24...

Interviewer: 4 times 4...

Cadmar: Oh ya, plus 16...

Interviewer: You don't need the root sign when its just the discriminant.

Cadmar: Oh ya, so plus 16, so k squared should be greater than negative sixteen, na that won't work, [does the calculation several times on paper] so you're subtracting sixteen... [looks at interviewer...mumbles]

Interviewer: So what are you doing. I'm just saying this for the recording,

Cadmar: What am I doing? well...

Interviewer: You've got k squared greater than negative 16 and it looks like you're square rooting both sides...[says this for the recording]

Cadmar: Well I'm, I'm looking at this what I've established in the context of the whole discriminant... that's what killed me on the exam... [looks at it a long time] ...

Interviewer: Well what are we looking for, what are we trying to find?

Cadmar: A value of k , so something squared plus 16 should be greater than zero, and if it were negative... Wellll, I think any value of k works because anything squared is a positive, that's why you can't have a negative under the square root sign...[mumbles a lot]

Interviewer: So what's your solution for k ?

Cadmar: Solution for k well.... k should be greater than.....[looks at it a long time]

Cadmar: In the context of this, this, this, square root, and if k squared plus sixteen has to be greater than zero, anything's going to be greater than zero when it's squared so k can be any number.

Interviewer: Great, nice work.

Cadmar: Those are thinkers.

It was more evident in real life, but their anxiety, hesitation, and moments of reappraisal are palpable in these excerpts. Both students find the questions difficult, the opposite of their reactions to the pre-questionnaire. Both students demonstrated a similar leap to structural understanding on the other question of the interview as well.

Language after the interventions

The contrast in language from the pre-questionnaire to the post-questionnaire is overwhelmingly apparent when the students' explanations from the after the cognitive conflict interventions are compared to their explanations from before the interventions. The question papers had an entirely different look. On the pre-questionnaire the comments the students wrote were usually a list of sentences aligned with the step of the solution they restated. There were no visual aids used. On the post-questionnaire the comments were short explanations at the end of the solution, or at the point where they reached the procedural trap, that detailed or compared objects. 20 students used visual aids (graphs and charts) to help describe the object represented by their answer to the question.

I present here some typical written explanations that accompanied the questions. There are more examples than I supplied for the pre-questionnaire because I had many different explanations to choose from. On the pre-questionnaire the explanations were uniformly lists of steps, and one or two examples sufficed to represent them all. The students can now explain *why* they reached an answer and they do so. I have chosen examples from across the ability range of the students. After presenting them I will compare these to the explanations from the pre-questionnaire and interview.

These examples of question 1 (*Solve for x : $x^2 + x > -1$*) typify the students' explanations of *why* they reached an answer:

Cobert: [solved using quadratic formula, found negative discriminant] but all values of x work b/c the graph has no negative y values.

Elvina: No roots. All values of x work b/c the graph has no negative values.

Boris: x can equal any number because there are no zeros but it is a positive parabola so therefore always above the line.

Verna: If x is positive the equation will result in the addition of two positive numbers, which will always be greater than -1 . If the number is negative, it will become positive once it is squared, and, added to itself will only ever approach -1 , but will never reach it.

Colin: This parabolic function [$x^2 + x + 1 > 0$] never intersects or is equal to or less than zero.

The students demonstrate that they are analyzing the structure and explain in their own words. This is distinctly unlike the pre-questionnaire, where 'solve' meant to carry out steps and the negative discriminant meant 'no solution'. After the interventions the students analyze the structure of the problem and interpret the negative discriminant

correctly with reference to the structure rather than grasping the familiar "no solution".

They do not list any steps!

These exemplars from question 2 (*Solve for x: $3(7x + 5) = 21x + 15$*) highlight the change from the pre-questionnaire where there was no mention of the structure of the lines.

Jill: They are equal because they share a line on the graph and because you can work it out algebraically so x is any number.

Eugene: All values of x work because the lines are the same and thus they intersect at every point.

Elba: We can see immediately that the equations are the same, so that ANY value will always make the 2 equations equal.

The language shows us that the students have *not* been deceived by the procedural trap of the x disappearing. They explain *why* all values work for x with reference to the structure of the lines or by treating processes as objects.

Here are exemplars of question 3 (*Show using algebra that the sum of an odd number and an even number is always an odd number*) where the students' structural understanding is demonstrated by their description of why their solution will produce an odd number.

Brenda: This is because when you multiply any number by two it becomes an even number and when you add 1 to any even number, the sum is always an odd number.

Seamus: because '2x' represents an even number so '2x + 1' represents an even number plus one, which is odd.

Malcolm: Anything multiplied by 2 equals an even number and adding 1 will make it odd always.

The students produced an equation of $(x) + (x + 1) = 2x + 1$. On the pre-questionnaire they saw an equation like this only as a prescription for action (only as a process) and tried to solve for x or gave number examples. On the post-questionnaire there were 7 students who tried to solve it, but the rest did not. Without any procedural work, they explained why the process $2x + 1$ represented an odd number (an object). This question shows that the majority of the students are now conceiving of $2x + 1$ or $4x + 1$ as objects rather than as processes and so they can now explain why it represents an odd number. The different character of their answers compared to the language the students used on question 4 of the pre-questionnaire is phenomenal. The students have gone from using an invalid procedure explained by a listing of steps to an explanation of the object represented. We had not worked on this question in the intervening time. This analysis of the problem rather than automatically starting a procedure, and the suspension of calculation and switch to an examination of an object's properties rather than completing the procedure, I feel, requires the oscillation between conceiving of the structure as a process and an object concurrently that demonstrates structural understanding.

Question 4 (*For what values of k will this system have a solution? $2x + y = k$ $y = 3 + k$*) was especially powerful at demonstrating the students' gain in structural understanding because of its very tempting procedural trap.

Arax: k can be anything because k does not differ the shape of the line.

Knut: No matter what it [arrow points to $y = 3 + k$] will be a horizontal line, has no slope. k only changes the y intercept on [arrow points to $2x + y = k$] and because it has a slope it must intercept = lines are not parallel.

James: No matter what k is they would intersect and would have a solution.

There is no mention of the steps of a procedure, and all are correct interpretations of the structure of the lines, although these structures are not readily apparent in the question. This question was the most difficult for the students and the best at revealing their leap to structural understanding. The students had learned to solve systems by graphing, substitution, and elimination, and indeed many began with substitution or elimination and then crossed this out and answered by referring to the two lines having different slopes.

In question 5 (*Do these two inequalities have the same solution set? $4x^2 > 9$ $2x > 3$*) the students' language demonstrates that they were not lured by the seeming equivalence of the inequalities.

Hector: NO because thier lines don't over lap [sic]

Joyce: because this [an arrow points to the square root of x^2 and $9/4$] could also be the $(-x)$ then no they do not have the same solution set.

Henry: [$2x > 3$ and $4x^2 > 9$ solved with arrows pointing to correct graphs of the inequalities] No they r different. [sic]

The students use the structure to explain the deceptive nature of this question.

Again, on question 6 (*Solve this inequality: $x^3 - 3x^2 + 3x + 7 > 0$*) their language shows that the students were not deceived by the procedural trap; this time, the negative discriminant:

- Fritz: [arrow pointing to $x^2 - 4x + 7$] no roots so it is always greater than zero [along with correct answer]*
- Sean: (A solution using a factor table method that didn't work crossed out) cannot solve/no solution for equation $x^2 - 4x + 7$.
 $x > -1$ solution because $x^3 - 3x^2 + 3x + 7 > 0$ only intersects x-intercept once.*
- Gail: Since there is only one factor and it intersects at only one point [arrow pointing to a correct graph] must be $x > -1$*

The students' language on this question was a correct interpretation of the structure of the question when they reached the procedural trap of the quadratic having no roots. Before the interventions the lack of roots or negative discriminant would have prompted them to answer 'no solution' or use a familiar procedure inappropriately.

The quotes from all these questions are exemplars of what I found on the post-questionnaires. The difference from the pre-questionnaires is considerable. The students are no longer listing the steps in a procedure; instead they speak of the structure and explain the objects that appear. When the procedural traps appear the students explain anomalous results with reference to the structure of a question. Often if they had trouble expressing themselves they drew a correct diagram. These examples of the students' language combined with the evidence of oscillation, justification and control I will present next are testimony that the majority of them have made the leap to a structural understanding.

Post-interviews

On question 1 of the post-interview (*Do these two equations have the same solution set? $(x - 3)^2 = 0$ $5x - 4 = 2x + 5$*) nine of the students answered correctly so

quickly, with so little writing, that I asked additional questions to try to determine if they had a structural understanding. All of the students who answered correctly related the solution to the structure of the quadratic rather than showing or listing the procedure they had done. After years of teaching Grade 11 mathematics, I was surprised by the perception of their explanations. Here are two quotes that exemplify the majority of the students:

- Cadmar: Ok, [works for awhile] Well, in both $x = 3$ and so ... yes.*
Interviewer: Ok, but one's a quadratic and ones a straight line, so how many solutions do you expect for a quadratic?
Cadmar: Well, two, but in this one, one of the loops is above the $x -$ axis.
Interviewer: What do you mean, one of the loops is above the $x -$ axis?
Cadmar: [picks up the graphing calculator to type it in]
Interviewer: Try to explain it to me without the graphing calculator.
Cadmar: Ok, what I'm saying is that... it only touches 3 once, because it's right on there [sketches a correct graph], if we simplified it and got the discriminant, it would equal zero.

And:

- Interviewer: Ok, solve this first one and tell me what you're doing.*
Nelson: The bottom one is negative three, just because you bring it over, no, positive 3 because its $3x$ equals 9. So the top one is...is positive 3 as well. The top one is also positive three. So they do have the same solution set.
Interviewer: They do even though ones a quadratic and ones linear?
Nelson: Yep.
Interviewer: How can that be?
Nelson: The quadratic has two solutions, but they're the same solution, the vertex of the parabola only intersects the x axis at just one point, ah the vertex is the intersection point.

The students compare the structure of the two objects; they don't list the steps of the procedure they used. It's especially noticeable that their language reveals they are unconsciously dealing with the procedures and the objects simultaneously. The first student explains the structure while drawing a graph, and then explains procedurally, "if

we simplified it and got the discriminant, it would equal zero." The second student explains the procedural solutions and then describes the structure, "the vertex is the intersection point." They have reached the point where they can conceive of the two equations as both process and object and can compare both the procedural results and the structures.

Two of the students were still clearly procedural, but two showed what I felt was minimal structural understanding. For example, here is an interview excerpt of one of these students completing the same question that demonstrates the student has made some progress:

Mitch: Ok, if x minus 3 squared equals zero I FOIL it out so...[expands it] ohh, [disappointed] that doesn't help me, [sees he didn't have to expand] so x must equal 3. So for this one [collects it and solves for x] and x equals 3, so the answer is yes they have the same solution set.

Interviewer: Ok, but one's a, one's a quadratic and one's a straight line though.

Mitch: Yes, but they have the same solution set, because only 3, is the only x value that will give you zero for both of them.

Interviewer: Even though that's a quadratic? ...

Mitch: It doesn't matter, because its x minus 3 squared, if it was x minus 3 times x plus 3 it would be positive and negative, but because its squared then its just positive.

The student has solved this correctly, but has started with a familiar procedure (expanding the square) that was unnecessary. He does not mention the graphs when I prompt him, although he has a nice explanation of *why* the quadratic has only one solution rather than a list of steps.

Another example of the leap to structural understanding illustrated by the students' language is the number of different ways they expressed the same explanation.

On the pre-questionnaire the explanations were uniform and were step-by-step instructions. The post-questionnaire explanations were varied as the students explained the equivalency (or not) of the objects in their own words.

Oscillation

Another aspect I used to determine whether a student had a procedural or a structural understanding was an observation of their behaviour as they reached the procedural trap of a question. I analyzed their ability to oscillate between conceiving of the concept as a process and as an object. The behaviours that demonstrate a procedural understanding are: reaching the procedural trap and completing an illegal transformation or grasping a familiar procedure or answer from a recent class to produce a halting signal; or leaving an anomalous result as an answer, either because they are satisfied it answers the question or are unable to proceed without a procedure to apply. The behaviours that demonstrate structural understanding are: not being deceived by the procedural trap of a question, but, instead, switching to examining the structure of the question, or the objects presented when an anomalous result appears and providing a correct solution. A student with structural understanding may also begin by analyzing the structure of a question, and then performing a procedure to find an answer they apply to the structure to properly solve the question.

Oscillation before the interventions

The pre-questionnaire and interview have shown that before the interventions the students began to solve a problem with a procedure, performed the manipulations

admirably well, but when they reached a point where an oscillation to a structural outlook was demanded they didn't make the switch. Question 3 of the pre-questionnaire and question 1 of the pre-interview (which were both the same question: *solve* $6x + 12 = 3(2x + 4)$) were both excellent at showing the students' lack of oscillation. On the pre-questionnaire the students easily got to the stage: $6x + 12 = 6x + 12$, where a switch to a structural outlook will solve it neatly. Instead, 49 of the 54 students had a form of the following responses. They would perform valid algebraic transformations and reach $x = x$ or $1 = 1$ or $0 = 0$. They would then often write "there [!] both sides are equal". Or, upon reaching $0 = 0$, would finish with a question mark or by writing, "I don't think this is right". Showing they were unsatisfied with the answer, but are lost about how to continue. Several finished with " $x = 0$ " or " x is undefined". These I will examine in the justification theme below. The students have the previous knowledge necessary to solve this question. They had been working with this level of algebra for 2 years and, indeed, found the manipulations easy. They also have a solid enough understanding of the equation of a line in the form $y = mx + b$, that if I asked how the lines $6x + 12$ and $3(2x + 4)$ are related, they would say, "they are the same line", yet when they come to the point where viewing the statements as objects would give an immediate solution they either are lost, or blithely continue with a procedure.

All of the students made similar mistakes, at first, on the pre-interview, but when prompted to examine $6x+12 = 6x + 12$ all except three could see the relationship and solve the question. One student came up with the correct answer with very little prompting, but two other students proved quite interesting. As these excerpts from their interviews show, they are quite capable of a structural understanding, but do not have it

at this point. Here are excerpts from their interviews from the point where the students reach $0 = 0$:

Mori: [laughs] umm... you move the six x and the twelve over to the other side, [mumbles works on paper], and you get zero equals zero. [indicates he's done]

Interviewer: Is that the solution?

Mori: [looks at it for a long time] That's a solution.

Interviewer: What, ah ... what do I mean, usually, when I say solve this?

Mori: You're generally looking for a value for x .

Interviewer: But we didn't get one here.

Mori: Right... [looks at it a long time]

Interviewer: So let's go back to here [$6x+12 = 6x + 12$] and look at that. Does that mean anything? Can you look at it and decide anything?

Mori: It ah ... in that ... both these equations are lines, and they are the same lines and so if we are looking for a point where they intersect, they don't have one point where they intersect, they intersect at an infinite number of points.

Interviewer: Ok, so if we are talking about solving what would the solution be?

Mori: Well.... x would equal... all numbers.

Here is another student who can see the structure when she is prompted, but does not oscillate:

Kristen: and I get zero = zero [indicates she's done]

Interviewer: is that the answer?

Kristen: [looks at it suspiciously] uh-huh

Interviewer: So let's go back to here [$6x+12 = 6x + 12$] and look at that. What do you think's going on?

Kristen: It's the same thing

Interviewer: so when I say solve what are we looking for?

Kristen: the x .

Interviewer: Can you find any x values?

Kristen: um, zero, um, well anything would work, because it's the same thing, but ...

Interviewer: so then what would your solution be?

Kristen: ahh... that x can be any number?

This next excerpt is very revealing. This student has always achieved excellent marks and is quite confident.

Kurt: ... and so zero = zero, so x is zero.

Interviewer: Can you check it?

Kurt: Ya I did in my head.

Interviewer: Can it be anything else?

Kurt: No.

Interviewer: No?

(more prodding by the interviewer but no change)

He is very confident of his procedural abilities and so is not stimulated to oscillate by the variable disappearing. The instruction *solve* directed these students to carry out a procedure, despite the fact that they say solve means find a value for x when I ask them what it means. This is evidence of their procedural understanding.

The interviews of the students who did not solve the problem despite my prompting were similarly informative. In the following two interviews the students show how firmly procedural their knowledge is even if I try to prompt them to see the relationships. The lack of ability to oscillate between a procedural and a structural view leaves them without a direction of attack when the procedures don't work and so they leave an answer that doesn't make sense:

Eunice: So then it just goes away? It can't just go away.

Interviewer: Think about it, it has a little trick.

Eunice: [Long pause]

Interviewer: You've got zero equals zero

Eunice: That's the answer!

Interviewer: No. What do I usually want when I say solve?

Eunice: You want us to solve for x.

Interviewer: Ok, and if you do it you get zero equals zero

Eunice: [laughs] I'm so not the right person to ask this.

Interviewer: [laughs] Don't worry this is what everyone does, these are tough. So just take a look at it, all your math is correct, but is there anything that could give you x.

Eunice: *What if I just solve it for both sides of the equals sign?*
 Interviewer: *Ok*
 Eunice: *Can I do that?*
 Interviewer: *Ok*
 Eunice: *[Solves $6x + 12$ and $3(2x + 4)$ separately], so x is 2, you can't do that though.*
 Interviewer: *Sorry, I misunderstood, no you can't just solve either side, I misunderstood what you meant.*
 Eunice: *[long staring at it]*
 Interviewer: *OK, your math is correct, your algebra is correct,*
 Eunice: *So x equals x .*
 Interviewer: *Ok, so x equals x , what does that give us?*
 Eunice: *[long pause] So that's right.*
 Interviewer: *That's the answer you'd put on a test?*
 Eunice: *Yep, that's right.*

Eunice realizes there is a problem with the x disappearing, but she looks for another procedure (even one she thinks is illegal), and finally backs up to $x = x$ so that she has a halting signal. Eunice also demonstrates that *solve*, to her, means carry out a procedure to find $x =$ to something. This is a hallmark of procedural understanding. Even with my prompting she does not look at the relationships of the objects.

Simon: *So there's no answer 'cause...[you can] do it many ways [and you always get] the same thing... [writes very quickly and solves a couple of ways, to get $x = x$, $12 = 12$].*
 Interviewer: *Can you just look at it back here [$6x + 12 = 6x + 12$] and see anything.*
 Simon: *Ya, see they're equal, ...[so you do can always do] the same thing to both sides and you never get an answer.*
 Interviewer: *When I say solve what do I mean?*
 Simon: *Find x [Smiles]*
 Interviewer: *Find a value for x ?*
 Simon: *Ya.*
 Interviewer: *So can you find a value that works?*
 Simon: *No, is it calculus or something that works?*
 Interviewer: *[laughs] No, this is just what we've been doing in class. So can you find a value for x ?*
 Simon: *[laughs] No, no solution.*

When I point out to Simon that he doesn't have a value for x , he assumes it is a procedure he doesn't know (calculus) and then falls back on a familiar answer, no solution. Both of these students have a procedural understanding that leaves them searching for a new procedure when they are faced with anomalous results.

This lack of ability to oscillate between a procedural and a structural outlook is shown well by question 4 (*Show using algebra that the sum of two consecutive numbers is always an odd number*) of the pre-questionnaire as well. Many students did not know where to start and actually asked what procedure to use during the questionnaire. Most, with a little prompting, correctly assigned x and $x + 1$ as consecutive numbers, but 17 students finished with a question mark or by writing "I don't know where to go", 13 students used number examples from this point. This is the Diophantine way; they are not able to generalize the concept and see it as an object, but can complete the procedure any number of times and leave it up to me to draw the conclusion. Six students wrote it out as an (improper) equation and actually solved for $x =$ a number. For example:

$$\begin{array}{l} \text{Brian:} \\ (x) + (x + 1) = (x - 1) \\ x = -2 \end{array}$$

Even when asked for a specific object (an odd number) the students want to use a procedure to solve the question. They create the proper object, but cannot switch to a structural view when it is required and so, either search in vain for a procedure or use a number example that provides a procedure with a clear end. Many ($n=30$) produced $x + x + 1$ or even $2x + 1$, but could not look at this as an object and could not explain why it represents an odd number. A typical example is:

$$\begin{array}{l} \text{Sheldon:} \\ (x) + (x + 1) = \text{This is where I'm stuck, how do I show an} \\ \text{odd number?} \end{array}$$

The students could not translate objects (consecutive numbers) into an expression, and when they had a correct answer ($2x+1$) they could not explain why this was an odd number. Here is an excellent example from a student who came up with the expressions for consecutive numbers without help:

Bashir: $x - a \text{ number}$
 $x + 1 - \text{the next number}$
 I don't know the steps to explain in algebra

This question showed the students created procedures, used numbers in order to have a procedure, or searched in vain for a procedure, when they needed to perceive the object.

Question 2 of the pre-interview (*Find k so that this function will have 2 different real roots. $f(x) = x^2 + kx - 4$*) showed that most students could not switch to a structural outlook on a harder question even when prompted. It is when the students reach the point $k^2 > -16$ that their procedural or structural knowledge is revealed. As I detailed above at the end of *Language before the Interventions*, two of the students produced the correct answer after pausing and thinking for a long time, and I believe they experienced the cognitive conflict and resolved it at that point. Their behaviour as they correctly answered this question would indicate a structural knowledge, not just because it demonstrates their ability to oscillate, but it also demonstrates the justification of their method and the good control mechanism that I examine next.

The rest of the students were familiar with the discriminant, but either found an illegal square root of -16 (we hadn't studied complex numbers) or searched in vain for a procedure to solve the question from this point.

Below are five interview excerpts that are typical of these students. They show that the students had a clear understanding that the discriminant had to be positive and were familiar with it from class, but wanted to use a procedure when they reached $k^2 + 16 > 0$. Despite my prompting them to check values with a graphing calculator (so that they realized that many values worked and had the visual prompt of the graph) they did not oscillate to a structural view, but fell back on familiar answers from class.

- Mori:* *Alright, so we can use the discriminant... which is k squared minus four times one times negative four so we get k squared plus 16... [long pause]...*
- Interviewer:* *What's the discriminant supposed to be for two different real roots?*
- Mori:* *[long pause]... Ah, it equals a positive number. Ok, you're not going to get a real answer here because k has to equal the square root of negative 16.*
- Interviewer:* *Ok, look at it back here [$k^2 + 16 > 0$], is there any k that would give me two different real roots?*
- Mori:* *No.*
- Interviewer:* *For sure no?*
- Mori:* *[Smiles] Pretty sure.*
- Interviewer:* *Pretty sure no?*
- Mori:* *Ahh...[does some calculations in head] ya there can't be... not to my mathematical knowledge.*

Mori understood the discriminant well, but could not oscillate to the structure of the question to interpret $k^2 + 16 > 0$ properly.

- Nelson:* *K squared! So therefore, K is greater than negative 4.*
- Interviewer:* *Can you square root a negative number?*
- Nelson:* *No.*
- Interviewer:* ***Can you just look at it here [$k^2 + 16 > 0$] and just decide what k could possibly be?***
- Nelson:* ***Oh ya, you can't [find the square root of] a negative number so it's no solution.***
- Interviewer:* ***Ok, what about the calculator, we just saw some.***
- Nelson:* *[laughs] Oh ya.*
- Interviewer:* *Can you look at it and come up with any possible values of k?*
- Nelson:* *No.*
- Interviewer:* *But we can put some in the calculator and they work.*

Nelson: Ya.
Interviewer: So?
Nelson: So I don't know how to do it, can you show me?
Interviewer: [laughs] You can figure it out, but there's, ah, no, ah, special method, you just have to look at the properties of the stuff that's there.
Nelson: Ok. [looks dubious]
Interviewer: [laughs] Try some values in the calculator.
Nelson: [types away, tries a couple] They all worked.
Interviewer: What does that tell you?
Nelson: I don't ...there are some, but I don't know the way to get them.
Interviewer: Can't just look at it and come up with anything?
Nelson: No.

Nelson could not oscillate to see the structure when presented with anomalous values, and felt he just didn't know the proper procedure. Again, like Eunice, Nelson demonstrates that *solve*, to him, means carry out a procedure to find 'x = a number'. Even though he has seen that there are values that work, he jumps to the familiar answer of 'no solution' when finding the square root of a negative number is encountered, rather than relating this information to the structure of the question.

Eunice: Oh ya, k is greater than negative four.
Interviewer: So if that's your answer is there any way of checking it?
Eunice: Ya, sub it in.
Interviewer: Ya, sub it in, so can you do it, let's make it easy for ourselves, let's make it zero.
Eunice: [Works it out] It works! [smiles, quite proud].
Interviewer: So let's test one lower than negative 4 then, just to be sure. Let's use the calculator. What's a number less than negative four?
Eunice: Oh, ok, [types into graphing calculator, sees two roots] Ah...that works too.
Interviewer: So can you just look at it back here [k squared + 16 greater than zero] and decide anything?
Eunice: Oh, ok so it's going to be anything between 4 and negative 4.
Interviewer: Anything else?
Eunice: No...oh and including 4 and negative four.

Interviewer: Anything about the k squared plus sixteen is greater than zero jump out at you? [emphasis on greater than]

Eunice: Just what I've said.

Eunice kept modifying the familiar answers from class rather than oscillating to look at the structure.

Mavis: So k squared equals greater than negative sixteen. Wait do I...so k is greater than negative 4

Interviewer: Is that your solution?

Mavis: Ya.

Interviewer: Is there any way of checking it?

Mavis: Ya.

*Interviewer: **Ok, let's do it...** [She tests several numbers on the graphing calculator that work]*

*Mavis: **Ohhh....so nothing works?***

Interviewer: No, can you just look at it back here [k squared + 16 greater than zero] and decide anything?

Mavis: [looks a long time] No.

Interviewer: So what solution would you give on a test?

Mavis: No real numbers.

Mavis found several values that worked, but rather than switching to a structural view, fell back on a familiar answer from class when a negative discriminant appeared - "no real numbers".

Bella: ummm....right, right, right, a positive number gives two roots

Interviewer: ok

Bella: ok so k equals four

...

Interviewer: why don't you check it

Bella: ok k is four [does the math in head, very pleased] ... ok equals zero.

...

Interviewer: so when k is 4 we get two different real roots?

Bella: yep!

Interviewer: if you put one in instead of k what do you get?

Bella: um... ah ok, right, ah a positive number

Interviewer: so can you say what k should be?

*Bella: ahhh....no...
(More prompting but no solution)*

Bella understood the discriminant, but also could not oscillate to see the structure when presented with anomalous values.

All of the students were well aware that k^2 always gives a positive result, but only two switched to a structural outlook at the $k^2 + 16 > 0$ stage and answered correctly that any value would solve the problem. Even when given the visual prompt of a graph on a graphing calculator only one other student switched to a structural outlook. Two students did not try to find the square root a negative number, but searched in vain for a procedure that would solve from this point. The other seven students were the same as those quoted above. They were shown that there are values that work, and saw the visual image of the graph on the calculator, but when asked to examine the statement, $k^2 + 16 > 0$, rather than seeing this as an object, saw only a process, and quickly grasped a familiar procedure from class. They either combined the fact that we can't find the square root of a negative number with the fact that the negative discriminant means no solution, to reach an answer of no solution, or when shown that many values work, grasped that k is greater than -4 and less than 4 , even if the values they tested were outside this range. When presented with evidence that their answer was wrong they quickly grasped another familiar answer without looking at the structure of the question.

Oscillation after the interventions

Question 1 of the post-questionnaire (*Solve for x: $x^2 + x > -1$*) demonstrated the progress the majority of the students had made to a structural understanding. On this question all but three students began to use the procedure we used in class to solve quadratic inequalities, but stopped at some point and explained that it was "a positive parabola that doesn't intersect the line" or drew a correct sketch. The procedure we learned in class to solve inequalities would result in a quadratic equation with a negative discriminant. I have already provided several examples from *before* the interventions where the response was 'no solution' whenever a negative discriminant was encountered, and indeed three students said exactly this. The rest of the students who found a negative discriminant had some form of the following, "no zeros but it is a positive parabola so all x's work". This beautifully demonstrates an oscillation to the structural view because the negative discriminant that meant 'no solution' before the interventions now means no roots and they interpret this correctly as the whole parabola is above the x-axis. *Solve* no longer means simply carry out a procedure; they oscillate between the procedure of finding no roots and analyzing what this means for the structure of the question. This is Cortes' (2003) expert's method. The majority of students used a procedure, and when it produced an unfamiliar result, switched to a structural view. This is the reverse of the results of the pre-questionnaire.

On question 2 of the post-questionnaire (*Solve for x: $3(7x + 5) = 21x + 15$*), 19 of the students only completed the procedure to the $21x + 15 = 21x + 15$ stage and answered correctly referring to the equality of the two sides of the equation, or that they are the same line. Seventeen students completed to the $x = x$ stage and then answered x can be

any number with several different correct explanations as to why, that all demonstrated an oscillation to the structural view. Here are two typical explanations after the students have reached the $x = x$ stage:

Natalie: x can equal any # because the lines intersect at every point

Boris: x can be any number because they are the same line and therefore at any x point they will be the same.

Seven students still completed the procedure to reach $x = x$ or $0 = 0$ and left this or had "no solution". This question was an excellent demonstration of the majority of the students' ability to oscillate between a procedural and a structural view at this point in the study. The students have begun or completed the same procedure as on the pre-questionnaire, but with radically different results this time. Their written explanations show they have switched to a structural view and compared the two sides of the equation as objects and determined they are equivalent objects or represent the same line.

Question 3 (*Show using algebra that the sum of an odd number and an even number is always an odd number*) also shows us that the majority of the students can now oscillate between a procedural and a structural view. They wrote $x + x + 1 = 2x + 1$ and did not carry out a procedure, but conceived of the process as an object and explained it as an odd number. They did not search for a procedure or try to solve beyond $2x+1$, but explained why this expression represented an odd number. This is in stark contrast to the results before the interventions where the students used number examples so they would have a procedure to complete, created procedures and solved them, or would not answer in the absence of a procedure.

Questions 4, 5, and 6 give the best evidence of the students improved ability to oscillate. Because the students had not seen these types of questions either in class or during the cognitive conflict interventions they would not be able to reproduce a remembered correct answer. The methods the students learned in class to solve similar types of questions all produce a point where the standard procedures don't make sense (the procedural trap) and a switch to a structural view is demanded. On the pre-questionnaire the students were happy to leave an answer that didn't make sense or they were lost at this point. On the post-questionnaire the majority of the students made the switch to a structural view and solved the questions complete with a good explanation and sometimes a diagram.

I felt question 4 (*For what values of k will this system have a solution? $2x + y = k$
 $y = 3 + k$*) most accurately distinguished the students with the ability to oscillate from a procedural to a structural view. If the students use their favoured procedures for solving (substitution or elimination) they reach a point similar to $2x + 3 + k = k$. This is a considerable trap because the k disappears and a fine solution appears ($x = -3/2$), complete with a halting signal. A student with procedural understanding will continue the procedure to the $x = -3/2$ stage and leave this without reference to the original question, or find this answer and be confused and not able to apply it to solve the original question. The students can only answer correctly if they oscillate to see the two equations as two lines with different slopes, or if they compare the two sides of the equation as objects, and neither of these is readily apparent. The 26 of 44 students who answered correctly wrote the question as $y = 2x + k$ above $y = 3 + k$ and wrote a good explanation involving the slopes of the lines, or they solved using substitution or elimination and finding k

disappear, provided a good explanation of the equivalency of the two equations, or they provided a good explanation without any procedural work, for example:

*Mandy: Diagonal [with arrow pointing to $2x + y = k$]
 Straight line horizontal [with arrow pointing to $y = 3 + k$]
 No matter what value for k both lines with [sic] have one
 solution*

All of this is very strong evidence that the majority of the students can comprehend and compare the two equations as objects and can oscillate to a structural view when a familiar procedure produces an unexpected result. This question is not solvable with the procedures the students know, but ‘fits’ deceptively well into their procedures and provides an enticing halting signal. Without the ability to oscillate between a procedural and a structural view the students easily fall into these traps and do not solve this question properly.

Question 5 (*Do these two equations have the same solution set? $4x^2 > 9$ $2x > 3$*) also exhibited the students’ ability to oscillate to a structural view. Again this question provides an alluring trap for those with a procedural understanding, especially since these students were unsure of what ‘solution set’ meant. It is very tempting to see the two inequalities as equivalent because you can simply find the square root of one of them. This may not seem readily believable to you if you have a structural understanding, but is exactly what I observed with the students with a procedural understanding and what Sfard and Linchevski (1994) found with a majority of students. An oscillation to view one of the equations as a parabola and one as a line is necessary to solve this question correctly. The 27 students who answered this question correctly began with a procedure and completed their answer with a graph or by explaining one is a quadratic and one a

straight line, or had just a correct explanation. Many, being unfamiliar with the wording 'solution set', cautiously hedged their answers by saying the inequalities shared half a solution set, but did not have the same solution set. Their explanations demonstrated they most definitely oscillated to the structural view.

Question 6 (*Solve this inequality: $x^3 - 3x^2 + 3x + 7 > 0$*) very well demonstrates the development in structural understanding of the students. The appearance of this question was familiar to the students from our class work, but the polynomial only has one root. This is the procedural trap of this question. If the students use the familiar procedure from class they will not be able to complete the question. They will encounter a quadratic factor that they cannot solve. A student with a procedural understanding at this point answers 'no solution' or is lost about how to proceed, as we saw in the pre-questionnaire and interview. Twenty-eight of the students reached the point where the negative discriminant appears and correctly answered this question. Many simply wrote the correct answer at this point, but there were 18 explanations and correct graphs. The explanations were all a form of "only one root so the equation intersects the x axis only once" demonstrating they had switched to a structural view at this point. I feel this clearly shows they have begun with a familiar procedure that does not involve a structural view and have switched to the structural view of the graph at the point where it is necessary. This confirms that the majority of the students now demonstrate this key requisite of structural understanding. They can oscillate between a procedural and a structural view on a concept that was not dealt with during the cognitive conflict interventions.

I found question 1 of the post-interview (*Do these two equations have the same solution set? $(x - 3)^2 = 0$ $5x - 4 = 2x + 5$*) very interesting because most of the students solved the linear equation first even though it was written below the quadratic equation, and then compared the two results. They just looked at the quadratic and found its vertex, then used pencil and paper to solve the linear equation. I feel this visualizing the quadratic's vertex without any procedure used to solve it, quickly solving the linear equation, and then comparing the two demonstrates a structural understanding. This is evidence that the students can see the structures as objects and compare the equivalency of the objects. When I asked the nine students who solved it so quickly (and correctly) how a quadratic could have the same solution as a linear equation, they explained why this is with reference to the structure of the two graphs. The two students who answered incorrectly could not refer to the structure. Here is a typical explanation from a student who answered correctly followed by the explanation of a student who answered incorrectly:

Correct student

Cadmar: Ok, what I'm saying is that... it only touches 3 once, because it's right on there [sketches a correct graph], if we simplified it and got the discriminant, it would equal zero.

Incorrect student

Eunice: No, they don't have the same solution set.

Interviewer: How come?

Eunice: Because in the first equation x is different than in the second equation.

Interviewer: Ok, what's x in the first equation?

Eunice: Its x squared.

Interviewer: Ok.

Eunice: And in the second one it's just x .

The first quote represents the nine students who answered the question correctly because they now have a structural understanding and can see the objects and the processes simultaneously. They explain their answer in terms of the structure of the graph *and* what the procedure would produce. The second quote represents the two students who still have a procedural understanding, so that, for them, solving an equation with x^2 in it is a different procedure than solving an equation with just x , without any reference to the structure of these specific equations.

Question 2 of the post-interview (*For what values of b will there be no solution to this system of equations? $y = 2x + b$ $y = -2(x - 1)^2 + 2$*) demanded an oscillation from a procedural to a structural outlook or it was not solvable. Here are excerpts from two students that are typical of the nine students who answered this question correctly:

Dorin: Ah, we will first...so we have to make it so they don't intersect each other, ok...I'm going to draw a graph...[sketches a rough graph] figure out where this intersects the y axis and we're doing great... so I figure out where it intersects the y axis. Then I do it so that $2x + b$ has to be greater than that point... Oh you can make em equal to each other, oh I guess I could do that, [does it] oh ya and just deal with the discriminant. [starts to work on it] b squared minus $4ac$, [solves it]

It is clear he has started with analyzing the structure and then switched to a procedure in order to find the exact answer. The next student oscillates in the opposite direction.

Nathan: Ahh, ya, umm... [looks at it a short time] you use this [writes out the discriminant].
Interviewer: The discriminant? [says this for the recording].

Nathan: Ya cause you'd use the discriminant and you'd somehow get those variables in the discriminant so that it becomes a negative number and you know there's nothing.

Interviewer: Beautiful, ok do it.

Nathan: So...[looks at it a little while] ok so...[sets them equal to each other and starts to solve, works awhile] ... b has to be more than a half.

Interviewer: beautiful, excellent, so what will happen if b is more than a half? [this is to determine if he can relate it to the structure or if he is procedurally very proficient].

Nathan: The discriminant will be a negative number and so no solution.

Interviewer: And what does that mean to the original question?

Nathan: Ah, so there is no intersection.

Interviewer: Ok, thanks a lot.

This student starts with a correct procedure and then shows he can use the structural view to correctly solve this unfamiliar question. I asked "And what does that mean to the original question?" in order to see if this student was just procedurally proficient and had guessed a proper procedure, and I feel his answer, "Ah, so there is no intersection", relating the answer to a visual image of the graph confirms a structural understanding.

The post-interviews confirm the evidence of the post-questionnaires that after the interventions the students were proficient at oscillating between a procedural and a structural view in order to solve unfamiliar problems. This is a strong verification that they had developed an impressive structural understanding of Grade 11 concepts. This conclusion will be markedly strengthened when I examine the Grade 12 results in the next chapter.

Justification

Recall from Chapter one that the experts explicitly test values to find a permissible transformation when faced with an unfamiliar problem. The behaviours I used to conclude that the students had a procedural understanding were: not explicitly justifying their methods when they used a familiar procedure that did not 'fit' (the wrong procedure for the concept) an unfamiliar question; performing transformations that resulted in expressions that were clearly not equivalent; performing illegal manipulations in order to create a halting signal when the variable disappeared; providing an answer of the form $0 = 0$ that did not answer the original question; or making an illegal manipulation so that the result looked similar to one we had recently had in class (we had been working on quadratic equations for several months). The behaviour I concluded demonstrated structural understanding was: pausing and considering when faced with an unfamiliar question or a procedural trap and explicitly testing a transformation to continue with.

Justification before the interventions

Before the cognitive conflict interventions the invented strategies of the students were much more in evidence than any justification of their methods. Although it is not possible to show a missing behaviour (the lack of justification), the students provided many examples of manipulations that demonstrated they must not have explicitly justified their methods while completing unfamiliar questions.

Even on question 2 of the pre-questionnaire (*solve for x: $b = vxt$*) which did not demand structural understanding, four students used the same procedure that they had used for question 1. For example:

Ethel: $b - v = xt$ *brought v over to the other side*
 $b - v - t = x$ *brought t over to other side. Solved.*

On question 3 of the pre-questionnaire (*solve for x: $6x + 12 = 3(2x + 4)$*), upon reaching $x = x$ or $1 = 1$ or $0 = 0$, most students wrote " $x = 0$ " or " x is undefined". Five rather creative students who reached $0 = 0$ or $1 = 1$, and not having a value for x , slipped a small x in to get $0x = 0$ or $1x = 1$ and solved as $x = 0$ or $x = 1$. Question 1 of the pre-interview (the same question) reinforced the evidence that the students are not justifying their transformations. Nine of the eleven incorrect students either stopped at the $0 = 0$ stage and, although they were not satisfied with this answer, did not test values for x , or they answered $x = 0$ after the $0 = 0$ stage; showing they did not test that this was a viable transformation.

On question 4 of the pre-questionnaire (*show using algebra that the sum of two consecutive numbers is always an odd number*) most students did not have an answer, but five students set up the equation $x + x + 1 = 0$ and solved this for an answer of $x = -1/2$. When I asked them why, they said that in class I had really stressed that to solve using the zero product rule, the equation must equal zero. This was, of course, during our recent work on *quadratic* equations. The look of $x + x + 1$ must have triggered the use of the procedure for $x^2 + x + 1 = 0$ without a justification of the transformation.

On question 2 of the pre-interview (*Find k so that this function will have 2 different real roots $f(x) = x^2 + kx - 4$*), six of eleven students interviewed found the square root of -16 and used it to create an answer; demonstrating they were not justifying their methods because they were well aware they could not find the square root of a negative number and we had not done any work on complex numbers.

The manipulations the students performed in questions 3 and 4 of the pre-questionnaire and question 2 of the pre-interview clearly showed the students were not justifying their methods. When the students reached the $0 = 0$ or $x = x$ stage of question 3 they were faced with an unfamiliar problem. It is evident the majority of the students chose transformations they had used recently and did not explicitly test the transformations they made. Their transformations of $0 = 0$ to $x = 0$ or x is undefined and $1 = 1$ to $x = 1$ show they must not have tested the transformation. In question 4 when they set $x + x + 1$ equal to 0 they were using a procedure we had recently used in class without testing the validity of its application. When asked "Can you find the square root of a negative number?", the students will answer "No" without hesitation, yet in question 2 of the interview when faced with a situation where the transformation is not familiar ($k^2 + 16 > 0$), six of the students found a square root of -16 and went on to find an (improper) answer. They automatically began with a familiar procedure we had used in class and used the procedure even though it involved a transformation they would recognize as illegal if given it on its own.

Justification after the interventions

Since the justifications of transformations are not written down, it is even harder to find evidence of them (other than the correct answers) in the post-questionnaires than the lack of them in the pre-questionnaires. The best evidence that the students were justifying their methods after the interventions was the way that they attacked question 2 of the post-interview (*For what values of b will there be no solution to this system of equations? $y = 2x + b$ $y = -2(x - 1)^2 + 2$*). If they had been given a system of a quadratic and a linear equation and asked to find the solution, they would quickly set the two equations equal and proficiently solve the resulting quadratic (on the final exam they did just this very well). But when faced with this unfamiliar parametric system, seven of the students tried a method and then justified their result to themselves. Here is the interview of a student that shows a style exactly alike the style of the experts that Cortes (2003) described; where, when faced with an unfamiliar problem, they explicitly checked the validity of the transformations they tried:

Henry: So, for what values of b will there be no solution to this...[long pause] ...ok, I'm just going to go one way I'm not sure it's going to work, ahh...so there'll be no solution to this... so the graph of the second equation that is... has a vertex at one comma two, and then goes downwards somehow.

Interviewer: ok.

Henry: If x is zero then the y intercept would be zero, and so something like that [he drew a correct graph]

Interviewer: Good.

Henry: And then the first equation is $2x$ plus b , you don't know what b is yet, y equals $2x$ goes like that, so if you want them to have no graph, they don't, they can't... ya it can't touch the graph basically, its got to be like apart, different.

Interviewer: Perfect.

Henry: So b 's got to be, I'm not sure what exact value, ...[long pause]... to find out the intercepts you go...[sets them

equal to each other] because they're both y equals the same thing.

Interviewer: Um, hm.

Henry: So $2x + b$ equals negative $2x - 1$ squared plus two.

(he sets up the substitution and solves)

When faced with an unfamiliar problem the student has not applied a procedure, but started with a visual image, justified the image to himself and then switched to a procedure and justified his steps as he progresses. Compare this to an equally accomplished student interviewed before the intervention:

Kurt: ... and so zero = zero, so x is zero.

Interviewer: Can you check it?

Kurt: Ya I did in my head.

Interviewer: Can it be anything else?

Kurt: No.

This student has accepted his procedural answer even though he has transformed $0 = 0$ to $x = 0$. He does not justify his method even after I prompt him.

Here is another student on question 2 of the post-interview:

Nathan: Ahh, ya, umm... [looks at it a short time] you use this [writes out the discriminant].

Interviewer: The discriminant? [says this for the recording].

Nathan: Ya cause you'd use the discriminant and you'd somehow get those variables in the discriminant so that it becomes a negative number and you know there's nothing.

Interviewer: Beautiful, ok do it.

Nathan: So...[looks at it a little while] ok so...[sets them equal to each other and starts to solve, works awhile] ... b has to be more than a half.

Interviewer: beautiful, excellent, so what will happen if b is more than a half? [this is to determine if he can relate it to the structure or if he is procedurally very proficient].

Nathan: The discriminant will be a negative number and so no solution.

Interviewer: And what does that mean to the original question?

Nathan: Ah, so there is no intersection.

Interviewer: Ok, thanks a lot.

It is harder to see because he solves it proficiently, but he also demonstrates the style of Cortes' (2003) experts. Recall the experts would analyze the relationships, then decide between various operations to perform, and check the validity of transformations. This student did not solve either of the questions on the pre-interview.

Control

Recall that students with a structural understanding demonstrate a good control mechanism. The behaviours that demonstrate procedural understanding are: the student loses track of the overall goal; is stymied by, or accepts as an answer, an anomalous result; demonstrates rigidity in procedural execution even if the procedure does not 'fit' the problem; and shows an inability to anticipate or monitor the consequences of their actions. The behaviours that demonstrate structural understanding are: the student initially analyzes a problem and monitors their work; is alerted that an anomalous result does not solve the original problem, or that they have performed an illegal transformation; and they use the structure of the question to determine a direction of action.

Control before the interventions

Before the interventions the students resoundingly demonstrate procedural understanding. They do not have the structural understanding that allows them to compare equivalent objects or alerts them when consecutive steps are not valid. This is demonstrated by their lack of a good control mechanism. I have several times previously

provided examples of the students reaching the $0 = 0$ stage and completing the question with one illegal manipulation. The students perform illegal transformations and are not alerted by any good control mechanism to the contradictions they create. Question 1 of the pre-interview provides more evidence of this. Two students answered correctly and nine students paused at the $0 = 0$ stage and were willing to leave this as an answer or were casting about for a procedure to use. Yet, when I asked them to review by saying, "When I say solve what do I mean?" or "Can you just look at it back here ($6x + 12 = 6x + 12$) and see anything" and induced a comparison of equivalent objects, all nine persevered and five made the switch to a structural outlook that allowed them to solve the question. Four of the students continued to look for a procedure and did not solve it. Here is an illustrative example of the five that solved after I had prompted them at the $0 = 0$ stage:

Interviewer: Good, so when I say solve this what am I looking for?

Cadmar: You're looking for a value of x .

Interviewer: So what are you going to do?

Cadmar: [mumbles] Could you divide this system by three?

Interviewer: Try it!

Cadmar: Cancels, cancels, $2x + 4$ equals $2x + 4$, same result!

Interviewer: [laughs], so...

Cadmar: [a lot of mumbling]

Interviewer: We're looking for a value for x .

Cadmar: A value for x ...ah... one'l work!

Interviewer: Ok, try that.

*Cadmar: [He does], if both sides equal each other... one works.
[long pause]*

Interviewer: Is one the solution?

Cadmar: X equals one is the solution.

Interviewer: Is it the only solution?

Cadmar: Well, ah, there could be other solutions... ahhhh ... ahhh, zero!

Interviewer: Ok

Cadmar: Negative one works, ah this doesn't really have a set of solutions does it?

Interviewer: [laughs] I want to see what you figure out, but you're doing awesome.

Cadmar: Ah, ok, ahhh, [works for quite a while] so the sides are equal, so I think, I think anything, anything can solve it right, because that means if the sides are equal any x value should have the same effect.

Interviewer: So x is...

Cadmar: x is all real numbers.

Cadmar is one of the two students I highlighted in Language as experiencing the cognitive conflict during the interview; this is also visible in this excerpt.

These results demonstrate that before the interventions the students do not have the structural understanding that provides a good control mechanism. The most telling comment comes from a pre-interview with a student who answered both the interview questions improperly at first. On question 1 he says "... ah, I kinda get lost" at the $0 = 0$ stage and answered question 2 with k greater than -4 and less than 4 . This is a familiar form of answer from solving quadratic inequalities. He answered both correctly after a little prompting. Here is the end of his interview:

Interviewer: Ok nicely done, thanks very much. You think you would have got those answers on, if it was just a test I gave you?

Nathan: I don't know. No, I wouldn't have thought about it as much.

Interviewer: And you probably wouldn't have checked it on a test.

Nathan: No, I would have just gone with it, and even if I had, it worked and so I wouldn't have tried outside.

He wouldn't have continued after the halting signal and he wouldn't have caught his mistakes.

At this point in the study a good control mechanism is not present in most of the students. Only a few have the structural understanding that allows them to compare

objects. They are not arrested by a transformation that creates a non-equivalent object, and they are satisfied with the answer a procedure has given them, despite it not answering the question. The absence of the ability to see both the object and the process simultaneously means the procedures are arbitrary to the students. They follow a procedure blindly and accept results that, although true, do not answer the original question. They treat ' $x = 7$ ' as a halting signal, even going to the extent of slipping in an x so that they can have an 'end' to the problem.

Control after the interventions

Although it was fairly easy to recognize the absence of a good control mechanism in the pre-questionnaire when the students would accept transformations that resulted in objects that were not equivalent, it is unwarranted to impute the presence of this in the post-questionnaire simply because the students correctly answered the questions.

Questions 1, 4, and 6 of the post-questionnaire provided good evidence that the students would begin a procedure, reach a point where their transformations produce an anomalous result, and would then review their work and obtain the correct answer. On question 1, eighteen students had completed a procedure and found a negative discriminant or no roots (sometimes this was crossed out). Beside this procedure these students had an explanation referring to the structure of the graph, and the correct answer. Rather than simply following a procedure and answering 'no solution' when no roots were found, these students exhibit a good control mechanism that makes them unsatisfied with the answer and causes them to review the structure of the question to obtain the correct answer. On 14 occasions on question 4 the students began with the familiar procedures

we used in class and found that the k disappeared. Again they often crossed out this work and answered beside this work with a correct explanation. Question 6 showed the students had a distinctly different type of understanding after the cognitive conflict interventions. Twenty-one students had begun to solve this question with a factor table (as we had done in class) and reached the point where the quadratic factor produces a negative discriminant and can't be factored. All of these students stopped at this point and drew a correct graph beside their procedural work or wrote an explanation of how the polynomial has only one root and answered the question correctly. On all these occasions the students did not fall into the traps these questions set for those with a procedural understanding. They did not answer 'no solution' when the variable disappeared or the discriminant was negative; they compared the objects while answering questions with these deceptive procedural traps and discovered the correct answer.

The post-interviews provide us with more examples of a good control mechanism. Question 1 was so quickly and competently completed by the students that there is no overt evidence of a good control mechanism. Question 2 was a much tougher question for the students and provides us with good evidence. Here are examples of question 2 (*For what values of b will there be no solution to this system of equations? $y = 2x + b$ $y = -2(x - 1)^2 + 2$*) from two students that display them working on a procedure and realizing they have made an improper manipulation:

Cadmar: Reads the question out loud [works for quite a while silently] Ah, I see what you're saying...

Interviewer: So what are you doing? [says this for the recording]

Cadmar: Well, so far I'm just simplifying both sides [he has set them equal to each other and expanded both sides. Works again, silently].

Interviewer: So what are you doing? [says this for the recording]

Cadmar: So that looks like a quadratic, [he has collected all on one side and set it equal to zero] but ... no solution would mean that...Uh I see, ok, I see.

Interviewer: So what, what are you going to do then? [says this for the recording]

Cadmar: I have to, it, the discriminant, it has to be um a negative number. [works for awhile, algebra is correct, he has made an arithmetic mistake and can see something is wrong] let me check this [picks up a graphing calculator]

Interviewer: Umm, something went wrong with the 36.

Cadmar: That's weird...so b is less than 4.5 but...[he can see from his sketch that this isn't good]

Interviewer: Ahh, you did everything perfect. That 6 was a problem from early on, but an arithmetic error.

Cadmar: Ok, what, ok, I see, [redoes his arithmetic] ah, sorry, that should be minus 2x, I even wrote minus 2x right there. [quickly redoes it correctly].

Interviewer: Ya, that's it, good, excellent.

The next student realizes in a similar way that he has made a mistake:

Dennis: [reads it out] So, ahh, so what I would do is FOIL this out and put it in the quadratic formula. [does this, trying to solve the system by elimination]

Interviewer: Ok.

Dennis: [gets half way and looks at it for awhile] Ah ok, so we have to substitute [starts again setting the two functions equal to each other, mumbles as he is doing it] And then, now you have to put it into the quadratic formula.

Interviewer: And why would you do that? Why would you put it into the quadratic formula?

Dennis: Because I know that I can, if, in the quadratic formula under the discriminant I get a negative number, I know that there is no possible solutions right? so that would be able to tell me. So I get b squared minus 4ac which is b, and then I get... and so...b is greater than point 5.

These two students both realized they had made a mistake part way through a procedure, stopped, and used a structural view in order to determine what to do.

Even though this was an unfamiliar question that we had not specifically covered in our conflict interventions, a very difficult question, and one that demanded a structural understanding, the students solved it adeptly.

All the post- interview examples above demonstrate the gain of a structural understanding the students experienced. The language shows the students are no longer simply applying an algorithm, but oscillating between a visual image and a procedure. While solving the procedure they justified their transformations and displayed evidence of a control mechanism that alerted them to illegal and non-productive steps. The students who made a mistake compared objects and discovered the problem.

This examination of the students' responses according to the four themes provides uncontroversial evidence that the population had a procedural understanding before the interventions and a structural understanding after the interventions. Their procedural understanding before the interventions is demonstrated by their language, the fact that they don't oscillate to a structural view, the lack of justification of their methods, and the lack of good control processes. The examples I provided show that the students explanations were lists of steps without reference to structures. When the students reached the procedural trap of a question they did not oscillate to a structural view, but instead searched for a procedure to use or performed illegal manipulations to produce a halting signal. They completed transformations without justifying their validity and they were not alerted by a good control mechanism when they made an illegal transformation.

The students' structural understanding after the interventions is demonstrated by how different their behaviour with respect to the four themes was. After the interventions

the students' language demonstrates they were analysing the structure of problems and comparing objects. As well, the procedural traps of the problems triggered an oscillation to a structural view that allowed them to complete and explain the problems correctly. There is evidence that they justified their methods when confronted by anomalous results, and evidence of the presence of good control mechanisms that alerted them to illegal manipulations.

CHAPTER FIVE: EXAMINATIONS OF INDIVIDUAL STUDENTS

This chapter will compliment the examination of the population according to the four themes I completed in the previous chapter. I will now examine individual students' results with respect to the four themes. I include this because an examination of the aggregate of an individual student's language, oscillation, justification, and control provides an excellent insight into the students' development that can be diluted by the analysis of the previous chapter.

When we examine the results of specific students it is clearly illuminated that students from every ability level made the leap to structural understanding, but this growth was not uniformly spread across the ability levels present in the population. Recall from Chapter two that after the pre-questionnaire I divided the students into three groups (foundation, moderate, and proficient) according to my judgement of their understanding of Grade 11 mathematics at that point. Because only five students demonstrated a structural understanding before the interventions, this was an assessment of their procedural ability. It is important to distinguish that, throughout this thesis, when I refer to a student as being a member of one of the three groups it reflects only my assessment of their ability *before* the interventions. I will examine students from each of the groups to demonstrate that it is possible for students of all ability levels to make the jump to structural understanding.

The results of the post-questionnaire showed seven students who clearly still demonstrated a procedural understanding. These seven were all students I had determined were in the foundation group before the interventions. I concluded they still had a procedural understanding because they were deceived by the procedural traps of all six questions and answered all incorrectly. They displayed all the traits of procedural understanding that the majority of students displayed before the interventions. There was no comparison of objects, all seven students left $x = x$ or $0 = 0$ as the solution to question 2, and they could not explain $2x + 1$ as an object, but used number examples instead, so no oscillation between a procedural and a structural view was evidenced. They answered 'no solution' when a negative discriminant appeared in questions 1 and 6. They were deceived by the procedural trap of question 4, leaving the halting signal $x = -3/2$ as their answer, and were also deceived by the tempting improper manipulation of question 5 and found the two inequalities equivalent. There were very few comments for questions 1, 2, and 3, but the comments present were lists of the steps of a procedure similar in form to the majority of the students on the pre-questionnaire. None of the seven had written comments for questions 4, 5, and 6 other than "no solution" on question 4.

Here is a comparison of the pre- and post-interviews of a student who exemplifies this group that demonstrated a procedural understanding both before and after the cognitive conflict interventions. The student made some gains in language, oscillation, and control, but has not gained a structural understanding. The following excerpts from the pre-interview demonstrate the student's procedural understanding before the interventions:

Here is the student completing question 1 of the pre-interview (*solve for x: $6x + 12 = 3(2x + 4)$*):

Katya: ok so I'm going to move that over till I get.... oh... you're going to get zero = zero ...

Interviewer: That's the problem with this one.

Katya: Why is it a problem? 'cause that's right eh? That makes sense.

Interviewer: Ok, um, what do I mean when I say solve? what do I want you to do?

Katya: solve for x?

Interviewer: ok, does it?

Katya: ah, ok, no, that's not good.... [laughs, waits]

...more prompting, but no switching to the structural view and she finishes with:

Katya: Well ... you can't...anything you do to one side you have to do to the other right? So if you're going to solve it for x on this side you have to solve it for x on that side as well.

Katya demonstrates a procedural understanding on this question (similar to the majority of students). To Katya, *solve* means carry out a procedure. Her language is a list of steps, she accepts the $0 = 0$ answer as correct until I point out it doesn't solve for x , and she does not demonstrate a control mechanism that would alert her to the fact that this does not answer the original question. She doesn't oscillate to the structural view even with prompting, but searches for a procedure and jumps to familiar answers.

Here is her question 2 (*Find k so that this function will have 2 different real roots.*

$f(x) = x^2 + kx - 4$) answer:

Interviewer: So I want this function to have two different real roots, what jumps out about that?

Katya: Umm, nothing! [laughs]

Interviewer: [laughs]) What tells you the number of roots of a function. [long pause] Remember the quadratic formula? What part of that helps here?

Katya: Ah ya, the discriminant, the part that's in the thing...
Interviewer: Perfect, ok what are you going to do here?
Katya: Plug in the A, B, and C?
Interviewer: Ok, check it out and see what happens.
Katya: Ok, [works through it] ... b squared is k squared.... minus four times one times negative four, ok k squared plus 16 ... now what?
Interviewer: Ok, you need something else, what is it that shows it has two different real roots? Do you remember?
Katya: About this part of it?
Interviewer: Um - hm
Katya: That shows it has two different real roots?
Interviewer: Um - hm
Katya: It has to be positive?
Interviewer: Great. Do you know how we show that it has to be positive?
Katya: A plus sign?
Interviewer: Remember Greater Than?
Katya: Oh, with the, like that? [draws a greater than sign]
Interviewer: Ok
Katya: Ok, so you get k squared equals, no, is greater than negative 16, ok, k squared, no k equals, no is greater than... ah can you do that? [referring to finding square root of negative 16]
Interviewer: Can you?
Katya: No, not so much, so it doesn't really help?
Interviewer: So what happens?
Katya: Oh ya, oh ya, no solutions!

...more prompting, but she stuck with no solution.

Once again, like her classmates before the interventions, she knows she cannot find the square root of a negative number, but jumps to a familiar answer rather than looking at the structure of the question. I have included such a long quote to demonstrate how weak she is procedurally. She needs a lot of my help to begin to attack the question with the quadratic formula, more help with understanding the relevance of the discriminant, and she mistakes how we would express a positive number in earlier grades for how we show an expression must be a positive number.

Katya's answers on the post- interview show some good development, but her procedural weakness limited the gains she made in structural understanding. Here she is on question 1 (*Do these two equations have the same solution set? $(x - 3)^2 = 0$ $5x - 4 = 2x + 5$*):

Katya: Ok, so I'm writing out the x minus 3 twice because there's a square root there [she means power of 2 and writes $(x - 3)(x - 3)$], and then for the second part I'm writing out x minus 4 and bringing the 2x over, and then doing the same, bringing it all over to one side, Umm, then I just realized I need the number on the right side, so it's 3x equal to 9 and I divided both by 3 and it's x is equal to 3. And I think that they both have the same solution set ... because both, all values of x are equal to 3.

Interviewer: Both? All?

Katya: Um hm.

Interviewer: Ok, this one's a quadratic and this one's a straight line, so how could they have the same solution set?

Katya: Because they're both the same [she points to the two factors $x - 3$].

Interviewer: So?

Katya: So they're both solved by only 3.

She does not fall into the trap of this question (that a quadratic and a linear equation must have different solutions), but the language is a list of steps. She does not mention the structures of the quadratic and the line, but defends her answer well explaining the equivalency.

Here is Katya answering question 2 of the post-interview (*For what values of b will there be no solution to this system of equations? $y = 2x + b$ $y = -2(x - 1)^2 + 2$*):

Katya: [Reads it to herself] Whew there's only two questions, I feel under pressure [laughs]. So should I find the vertex?

Interviewer: You could, what would that do for you?

Katya: Uh, I know that the vertex is 1, 2 so I know the way it would be if there was no solution [sketches a line missing a parabola].

Interviewer: Ah, good, nice, how are you going solve it?
Katya: Ah, I don't know...[laughs]
Interviewer: Ah, how do you ah, usually solve this kind of thing? What's the best way?
Katya: Graph it.
Interviewer: That could be, but in this case we've got that b...
Katya: Right! so...substitute! [Big smile]
...
I helped with every step of the quadratic formula and she finished with:

Katya: [Checks it] So we're looking for just a cut off – so if it was negative .25 ah, it would work.

This student is weak procedurally and has trouble with the quadratic formula and discriminant on both interviews (these are both topics we work on in Grade 11). This prevents her performing the algebra to solve question 2 despite the fact that she has a structural view and can sketch a correct solution to the system. This is a pseudostructural understanding. She has shown some development in her ability to compare equivalent objects, and demonstrates a good control mechanism on question 1, but begins with automatically applying a procedure that is not needed. She has a good grasp of the structure of question 2, but her low procedural proficiency prevents her gaining structural understanding. This student, and the group she exemplifies, demonstrates how a high procedural proficiency is necessary before the student can make the leap to structural understanding. Even with the ability to conceive of the object the concept represents, without the procedural proficiency that permits a student to conceive of the concepts as processes and objects simultaneously, structural understanding is out of their reach.

Foundation Student

Ten of the 17 students that I classified as foundation advanced to a structural understanding. I will examine the pre- and post-questionnaires of a student who I chose to highlight because she was among the weakest of all the students on the pre-questionnaire and then answered all 6 questions of the post-questionnaire correctly.

This student, Anna, answered question 1 of the pre-questionnaire as follows:

Anna: $6 = 1x$
 1. *Bring the x's over to one side of the equation and the other numbers to the opposite side.*
 2. *Divide x by other side if necessary.*

On question 2 (*solve for x: $b = vxt$*) she answered with:

Anna: $x = b - v - t$
 1. *Move everything to other side.*

showing that the parametric equation without numbers is a procedural trap for students with this level of understanding.

She answered question 3 ($6x + 12 = 3(2x + 4)$) with:

Anna: *3 goes in by xing [multiplying] it in.*
 Bring over numbers to isolate x.
 $x = 0$

Showing she was deceived by the trap. She answered question 4 (*Show using algebra that the sum of two consecutive numbers is always an odd number*) with only a question mark.

Anna commented that the questionnaire was "easy except #4."

We can see that her explanations were lists of the steps with no reference to the objects. She demonstrated an overwhelmingly procedural understanding, even to the point of using the same procedure on question 1 and 2; a clear indication she did not justify her transformations and didn't have a good control mechanism. Her comment that she found the questionnaire easy indicates she did not have the structural understanding that would alert her to the anomalous results.

Anna solved all 6 problems on the post questionnaire properly and showed the language, oscillation and good control that demonstrates structural understanding. For example on question 1 (*solve for x : $x^2 + x > -1$*):

Anna: [solved the quadratic $x^2 + x + 1 > 0$] All values of x would work because y values are positive.

On question 2 (*solve for x : $3(7x + 5) = 21x + 15$*):

Anna: They are equal because they share a line on the graph and because you can work it out algebraically [with an arrow pointing to the $21x + 15 = 21x + 15$ step].

She answered question 3 (*Show using algebra that the sum of an odd number and an even number is always an odd number.*) with:

Anna: $x + (x + 1) = 2x + 1$ If you sub in any even number for x , you will get an odd result, x and $x + 1$ are consecutive, so one has to be odd and the other even.

These answers demonstrate that she has gained a structural understanding. The procedural traps of these questions now stimulate an oscillation to the structural view and an analysis of the structure of the question. Her language includes explanations of the

structure and characteristics of objects and comparisons of the equivalency of objects. On question 4 (*For what values of k will this system have a solution? $2x + y = k$ $y = 3 + k$*) Anna solved until the k disappeared, drew a graph of a diagonal line crossing a horizontal line and commented "k can be any number. She answered just "no" on question 5, but with "parabola" written beside $4x^2 > 9$ and "straight line" beside $2x > 3$. She solved question 6 (*Solve this inequality: $x^3 - 3x^2 + 3x + 7 > 0$*) with the procedure from class, but didn't work past the point where the quadratic factor has no roots and answered correctly without an explanation. The procedural traps of these questions have triggered the necessary oscillation to a structural view and she has explained the anomalous results with reference to the structure of the questions

After the interventions Anna has not been deceived by the procedural traps, she has demonstrated an oscillation to the structural view on questions 1, 2, 4, and 6; an analysis of the structure of the question when confronted with anomalous results; language describing objects rather than steps in a procedure on questions 1, 2, and 3, and a good control mechanism on questions 2, 4, and 6. She has made the advance to a structural understanding.

The results of the foundation students demonstrated that seven are still predominantly procedural after the cognitive conflict interventions, and the rest have gained a structural understanding.

Moderate student

I will highlight the questionnaires and interviews of a moderate student, Nelson, because of the clarity of his results. I will use his answers from his pre- questionnaire as a demonstration of his procedural understanding before the interventions and his answers from his post-interview.

He answered $x = 0$ on question 3 (*solve for x: $6x + 12 = 3(2x + 4)$*) of the pre-questionnaire with the following list of steps:

*Nelson: Expand the brackets
 Move everything to one side
 Solve*

He answered question 4 with a number example.

The following exchange with the interviewer during question 2 of the pre-interview when he has reached the $k^2 > -16$ stage is an exemplar of the moderate students before the cognitive conflict interventions:

*Nelson: Ok, k is greater than or equal to negative sixteen.
Interviewer: K?
Nelson: K squared! So therefore, K is greater than negative 4.
Interviewer: That's your solution?
Nelson: Ya, that works.
Interviewer: Can you check it?
Nelson: Ya, you just slot it in.
Interviewer: Ok, check it out, just use the calculator, it's easier to graph
 it than do the algebra.
Nelson: [types it in] Works!
Interviewer: Will any other value work?
Nelson: Ya, but it's got to be greater than negative 4.
Interviewer: What if we try something less than negative four?
Nelson: It won't work.
Interviewer: Ok, try it.
Nelson: [types in] Hmm. What'd I do wrong?
Interviewer: Can you square root a negative number?*

Nelson: No.

Interviewer: Can you just look at it here [$k^2 + 16 > 0$] and just decide what k could possibly be?

Nelson: Oh ya, you can't [square root] a negative number so it's no solution.

Interviewer: Ok, what about the calculator, we just saw some.

Nelson: [laughs] Oh ya.

Interviewer: Can you look at it and come up with any possible values of k ?

Nelson: No.

Interviewer: But we can put some in the calculator and they work.

Nelson: Ya.

Interviewer: So?

Nelson: So I don't know how to do it, can you show me?

This is an archetypal example of a student with procedural understanding. His language is a list of steps. He was deceived by the procedural trap of question 3 and inappropriately produced the halting signal $x = 0$. He used number examples so that he would have a procedure on question 4. On question 2 of the interview, he used a familiar procedure without reference to the structure of the question, performed an illegal manipulation (finding a square root of -16), did not oscillate to a structural view when I demonstrated values that satisfied the inequality, and crucially, jumped to the familiar answer of 'no solution' despite just finding values that worked. He finished by asking for a procedure that works rather than oscillating to the structural view. Contrast these answers where he does not oscillate to a structural view even with prompting, and is deceived by the procedural traps, to his answers on the post-interview:

Question 1 *Do these two equations have the same solution set?*

$$(x - 3)^2 = 0$$

$$5x - 4 = 2x + 5$$

Nelson: The bottom one is negative three, just because you bring it over, no, positive 3 because its 3x equals 9. So the top one is...is positive 3 as well. The top one is also positive three. So they do have the same solution set.

Interviewer: They do even though ones a quadratic and ones linear?

Nelson: Yep.

Interviewer: How can that be?

Nelson: The quadratic has two solutions, but they're the same solution, the vertex of the parabola only intersects the x axis at just one point, ah the vertex is the intersection point.

Question 2 *For what values of b will there be no solution to this system of equations?*

$$y = 2x + b$$

$$y = -2(x - 1)^2 + 2$$

*Nelson: Ya so there can be no intesection between the ah, line and the ah, parabola. So, ya ok. **So all values of x are valid, well no, because you have to find b.** [works hard] Ah ya, there we go, I made it into one quadratic equation and then, ah which has b in it, and then you use the disriminant to find out what that can't equal, so root of b squared minus 4ac, umm, would be 16 minus 4 times negative two times whatever negative b would be so, no must be minus, so b is all values greater than 2 ...because b is negative in the quadratic so that at 2... the discriminant is zero and therefore they intersect once so all values greater than that they do not intersect because the discriminant is negative.*

Interviewer: Ah, ok can you check it?

Nelson: Owww...ok let me see, I probably made an error, oh yes I have, it should be ah ... a half.

The growth to a structural understanding here is recognizable. Before the cognitive conflict interventions the student demonstrates the characteristics of procedural understanding: grasping a familiar procedure at the point where oscillation between a procedural and a structural view is needed, even when it is pointed out; verbally describing a concept as a process or a set of steps; not realizing he has made an error

when his procedure produces a contradiction, and using 'x= 0' as a halting signal when the variable disappears during a procedure.

After the interventions Nelson begins with an analysis of the question, rather than automatically performing a procedure; he catches an important misstep unconsciously, "So all values of x are valid, well no, because you have to find b", showing evidence of good control; he oscillates between a procedural and a structural view, and his explanations describe the objects involved. These are all signature traits of structural understanding.

Proficient student

There were approximately 5 proficient students who displayed a structural understanding before the interventions. I will examine a student who exemplifies the proficient students who progressed from a procedural understanding to a structural understanding.

Here is the student answering question 1 (*solve: $6x + 12 = 3(2x + 4)$*) of the pre-interview:

Mori: alright, so ah, multiply the three into brackets and get six x plus twelve, and the other side being six x plus twelve.

Interviewer: And how do you know to multiply it like that?

Mori: Well, you do it because that makes the equation easier.

Interviewer: Ok, now what do you do?

Mori: [laughs] umm... you move the six x and the twelve over to the other side, [mumbles works on paper], and you get zero equals zero [indicates he's done].

Here he is answering question 2 (Find k so that this function will have 2 different real roots. $f(x) = x^2 + kx - 4$):

Mori: Alright, so we can use the discriminant... which is k squared minus four times one times negative four so we get k squared plus 16... [long pause].

Interviewer: I want two different real roots

Mori: smiles [long pause].

Interviewer: What's the discriminant supposed to be for two different real roots?

Mori: [long pause]... Ah, it equals a positive number. Ok, you're not going to get a real answer here because k has to equal the square root of negative 16.

Interviewer: Ok, is there any k that would give me two different real roots?

Mori: No.

After prompting by the interviewer he answered question 1 correctly, but stuck with 'no solution' on question 2. These excerpts show that the student used familiar procedures and did not oscillate to a structural view without prompting. His language was a list of steps. When his transformations created a contradiction he grasped a familiar answer (no real number), and he did not justify his methods.

After the cognitive conflict interventions, on the post-interview, Mori demonstrates structural understanding. He solved question 1 (*Do these two equations have the same solution set? $(x - 3)^2 = 0$ $5x - 4 = 2x + 5$*) so quickly with so little written work that I asked him to explain his answer since one was a quadratic and one a linear equation. His response was:

Mori: Because these two solutions are the same. [points to $(x - 3)(x - 3)$]

Interviewer: So what's that look like?

Mori: The vertex is the only intersection.

Here is his explanation as he solves question 2 (*For what values of b will there be no solution to this system of equations? $y = 2x + b$ $y = -2(x - 1)^2 + 2$*):

Mori: Oh ya, ok, so they don't intersect, so ... well, if you wanted to find the intersect you'd make them equal to each other... ya, ok [sets them equal to each other and works]. Ok, so why not bring them over, find out what b should be and then make b what it shouldn't be?

Interviewer: Sounds ok, can you do it?

Mori: Ummm, ya...oh ya, since its all a quadratic I can use the discriminant so b should be...[works it out] a half, no that's so they're equal, so negative discriminant, so b should be greater than a half.

On these two questions Mori has avoided the procedural traps, oscillated between a procedural and a structural view, compared the concepts as equivalent objects, and demonstrated a good control mechanism.

These examples provide evidence that there are students from all three ability levels who have progressed from a procedural understanding before the cognitive conflict interventions to a structural understanding afterwards. This gain in understanding is unequivocally demonstrated by their behaviour when they are confronted by the procedural traps of the questions. Before the interventions the students are deceived by the procedural traps of the questions, and they search for a procedure at this point without reference to the structure of the problem. They leave an incomplete answer or perform illegal manipulations without justifying their validity in order to have a halting signal. Their language shows they are performing the steps in a procedure rather than analysing the structure of the problem. After the interventions the procedural traps had the opposite effect. They alerted the students to anomalous results that the students reconciled by

oscillating to a structural view and justifying their transformations. Their language showed they were analysing the structures and comparing objects, and they showed evidence of good control mechanisms and justification of their methods.

A small minority of students demonstrate the behaviours associated with procedural understanding both before and after the interventions. Although these students can visualize the structures of the concepts, their lack of procedural proficiency prevents them from conceiving of the concepts as processes and objects simultaneously, preventing the rise to structural understanding.

CHAPTER SIX: THE GRADE 12'S

Ten of the eighteen grade 12's correctly answered question 1 (*Do these two equations have the same solution set? $(x - 3)^2 = 0$ $5x - 4 = 2x + 5$*) of the post-interview and one student correctly answered question 2 (*For what values of b will there be no solution to this system of equations? $y = 2x + b$ $y = -2(x - 1)^2 + 2$*). If we examine their responses according to the themes, it is clear that only one student could be assessed as having a minimal structural understanding of both of these questions. Eight students answered question 1 incorrectly, with answers that demonstrate a procedural understanding. Most expanded the $(x - 3)^2$ and then factored or used the quadratic formula to solve and then solved the linear equation. At some point they all made an illegal transformation and ended with an extra $x = 0$ or $x = -3$. Of the ten that answered question 1 correctly, four actually wrote "I don't know where to start" and then " $x = 3$ ", and answered yes. These, I felt, were all a correct guess, and the students confirmed this when I asked them. A typical response was, "I solved both and they seemed the same and so I said yes". Another four expanded the $(x - 3)^2$ to $x^2 - 6x + 9$, like those that were incorrect, and factored or used the quadratic formula to solve, and answered yes. These, I feel, were procedurally proficient. They demonstrated the automatic use of a familiar procedure even though it was not necessary and just returned them to the starting point. This use of an unnecessary procedure without a consideration of the structure, I feel, demonstrates a procedural understanding.

The language of all these students (correct and incorrect) was a listing of steps with no visual image or reference to structure. There were no graphs drawn or mention of the structures of the quadratic and the linear equation. There was no evidence of oscillation to a structural view, and no comparison of objects. When asked to justify their answer they justified their steps as correct. Here are two examples that are typical of these students. One is a written comment from the questionnaire and one is an interview excerpt.

Vanessa: [solved quadratic and linear properly] both have $x = -3$, but $(x - 3)^2$ has $x = 0$

And:

Ranjit: [again solves both correctly] Now I'm just going to factor this out [he means the top equation] umm, I mean FOIL. [expands and uses quadratic formula to solve it properly]. Then go $5x$ minus $2x$ is $3x$...then 5 plus 4 is 9 . . . $3x$ minus 9 equals 0 ... and so x is 3 and so the answer is no.

Interviewer: Ok, and how come you say that?

Ranjit: Well, this is a quadratic solution. And that's not.

Both of these show that the students are applying a familiar procedure and even though they are proficient and produce a correct result to the procedure, they do not oscillate to a structural view, or compare the two equations as objects.

Two of the students that answered correctly did not expand $(x - 3)^2$, and only solved the linear equation on paper. This is similar to the Grade 11's after the intervention. One said, "both have x intercepts at only $x = 3$ " and the other said, "3 is the zero [of the quadratic] there is only one intercept and $x = 3$ ". As with the Grade 11's, I feel this visualizing the quadratic's vertex without any procedure used to solve it, then

quickly solving the linear and comparing their equivalency, combined with the language comparing the two equations as objects, demonstrates a structural understanding.

Question 2 of the Grade 12 questionnaire and interview (*For what values of b will there be no solution to this system of equations? $y = 2x + b$ $y = -2(x - 1)^2 + 2$*) was very telling. Only 1 of 18 students demonstrated a structural understanding. Four students had no workings and an incorrect answer. Eleven students set the two systems equal to each other and solved for b (the procedure that is tempting). This results in b equalling a quadratic and is where this question demands a switch to a structural view. At this point they either solved the quadratic by factoring or using the quadratic formula as if the quadratic were equal to 0 or 2, and then used these answers for b (resulting in $b = 0, 1$ and $b < 0$), or they would find the square root of b and conclude either ‘no answer’ or ‘ $b > 0$ ’. These are the improper transformations that a procedural understanding produces (the trap that this question sets). These students have used a familiar procedure and not oscillated to a structural view when necessary, nor have they justified their transformations. They have not demonstrated a good control mechanism that would alert them to the fact that they have set the quadratic equal to b but solved it as equal to zero. The following explanation from a questionnaire and two interview excerpts are typical. They come after the students have set b equal to a quadratic and tried to solve:

Christina: $b = -2x(x - 1)$ b cannot equal 0, 1

Excerpts

*Menachem: b must be greater than zero because of the negative root.
[in the discriminant]*

Brandy: you will eventually have to square root b and so in order for there to be no solutions b must be less than zero.

The students are using a familiar procedure (the zero product rule) and a familiar answer (no solutions means a negative discriminant) without attending to the actual structure of the problem. These students were procedurally more proficient than the Grade 11's, but they demonstrated the automatic use of a familiar procedure when faced with an unfamiliar question, the lack of oscillation to a structural view and use of a visual image, the lack of justification of transformations and comparison of objects, and the absence of a good control mechanism that are the hallmarks of procedural understanding. We can compare this to the nine Grade 11 students who answered this question correctly. We have seen previously that the Grade 11's, after participating in the cognitive conflict interventions, demonstrated the oscillation between a procedural and a structural view, and the good control needed to solve this problem.

Three of the Grade 12 students sketched a graph of the parabola and the straight line, typed them into their graphing calculators and moved the line up and down by substituting numbers for b in order to find b . One of these students came up with approximately $\frac{1}{2}$ as an answer, one with 1 and one with 2. These last two students had been interviewed and I felt that although they had compared the two equations as objects, they did not show the oscillation between a procedural and a structural view needed to solve the problem. Here is a typical interview:

Martin: Can I just graph it.

Interviewer: Ok

Martin: So no intersections between the two equations...

Martin: So by 2 because that makes it so that the line won't intersect.

Interviewer: Can you check it?

Martin: Ya, ok. Ya it checks out.

(More prompting, no using substitution)

The students did not compare the two equations using substitution and find an accurate answer, despite prompting. This demonstrates they are not able to consider the object and the process at the same time. Their substituting numbers in for b was a trial and error mechanism, not evidence of a good control mechanism. I felt they had a pseudostructural understanding; they grasped the structure but could not see it as a procedure at the same time. The student who answered approximately $\frac{1}{2}$ (on a questionnaire) also solved the question using substitution and fell into the trap of setting b equal to a quadratic and solving as if b equalled zero, but she crossed all this work out and wrote "for all values of b greater than ~ 0.5 there will be no solution, just by graphing". Her answer of *approximately* $\frac{1}{2}$, I believe, is because she didn't solve it accurately procedurally. This student had correct workings for question 1 and found an answer of $x = 3$ for both equations, but had a written explanation of

*Lorna: No, because the squared one has a solution of \pm ____.
 Then again, - 3 doesn't work, so yes? I think I've failed*

I felt her questionnaire demonstrated evidence of an oscillation to a structural view on question 2 when the procedure didn't work, along with a good control mechanism on both, and should be construed as a structural understanding. These results demonstrate that two of the 18 Grade 12 students (none of whom had the cognitive conflict interventions) had a structural understanding of the concept of question 1, and one student had a minimal structural understanding on both questions, but none showed the solid structural understanding of the Grade 11's after the interventions.

A comparison of the Grade 11's responses to the Grade 12's responses on the post interview gives excellent support to the conclusion that the interventions promoted structural understanding. The Grade 12 results on question 1 demonstrate that they are procedurally more proficient than the Grade 11's but they have a procedural understanding of the material that is similar to the Grade 11's *before* they had the cognitive conflict interventions. On question 1 the similarity of the Grade 12's to the Grade 11's before the interventions is remarkable. Two of the Grade 12's show a structural understanding similar to the weakest of the Grade 11's after the interventions, but the rest demonstrate the same procedural understanding leading to the same mistakes, and use the same language as the Grade 11's before the interventions. The Grade 12's automatically use a procedure and do not oscillate to the structural view, contrary to the Grade 11's after the intervention, who oscillate between a procedural and a structural view and compare the equivalency of the two objects.

On question 2 both the Grade 11's and 12's were unsure of a direction to take because of the unfamiliarity of the problem. The Grade 11's analyzed the structure, oscillated between a procedural and a structural view, and justified their transformations. While using the procedure the Grade 11's demonstrated a good control mechanism that made them stop and consider at the $b = a$ quadratic stage, the procedural trap where an oscillation to a structural view is demanded. On the same question, all except one of the Grade 12's applied familiar procedures at this point, performed improper transformations and came up with an incorrect answer. The Grade 12's showed no evidence of oscillation to a structural view, justification of transformations, or of a good control mechanism that would have alerted them at the $b = a$ quadratic stage. Comparing the post-questionnaires

of the two very weak foundation students highlighted in *Foundation Students* above to the Grade 12's demonstrates that many of the weakest Grade 11's after the interventions have a broader, more effective structural understanding than the most proficient Grade 12's who had had no intervention.

CHAPTER SEVEN: SUMMARY AND CONCLUSIONS

In this chapter I will summarize the results related in the last four chapters and then explain the conclusions I've drawn from these results with respect to the research questions I listed in Chapter 2. I will discuss the gains in structural understanding the Grade 11's have shown, then discuss the differences in gains of structural understanding that are observable between the different groups of students, and then compare the Grade 11's results to the Grade 12's.

My examination of the results provides evidence that the majority of the students made the leap from a procedural to a structural understanding. The pre-questionnaire demonstrated that the overwhelming majority of the students had a procedural understanding. Forty-nine of fifty-four students could not answer the questions that demanded structural understanding. These 49 also displayed the traits of procedural understanding according to the four themes I've demarcated: language, oscillation between a procedural and a structural conception, justification of their transformations, and a good control mechanism. Their explanations were lists of the steps they had carried out, they did not refer to any objects, even at points like $6x + 12 = 6x + 12$ that seemed obvious to them after the interventions, and there was no evidence of an oscillation between a procedural and a structural view. At times, they could see the objects when prompted, but did not combine this with a simultaneous view of the

processes. They automatically applied procedures that did not 'fit' the structure of the question, and then rigidly completed these procedures with illegal transformations. They stopped at the $x = x$ or $0 = 0$ stage with an explanation that was a list of steps, or used $x = 0$ as a halting signal and so made illegal transformations without justifying them or realizing it. The pre-interviews demonstrate that at that point the students could see contradictions that their transformations created when they were pointed out to them, but they did not have a good control mechanism that would have alerted them to these contradictions as they made them.

The post-questionnaires and interviews demonstrate that the majority of the population, including students from each of the three groups (foundation, moderate, and proficient) have developed a structural understanding of this level of material. The majority can now correctly answer questions that demand structural understanding. The students' language demonstrates they are comparing objects rather than listing steps in a procedure. Their explanations are varied and supported by visual aids. The students demonstrate an oscillation from a procedural to a structural view where it is demanded rather than automatically using familiar procedures that do not 'fit' the problem. They do not fall into the procedural traps of unfamiliar questions because they justify their transformations and have good control mechanisms that alert them to anomalous results.

Language

Taking the questionnaires and interviews as a whole, the difference in the students' language before and after the interventions is exceptional and can be seen in the representative examples I used in the results. Before the interventions the explanations

are uniform and are lists of the steps of the procedure used. After the interventions the explanations are varied and are *why* objects are equivalent or not. They are cogent explanations, not lists of steps detailing *how* the answer was reached; they are explanations, in the students' words, of the equivalency of the objects. During the pre-interviews the students asked me for a procedure to use when I pointed out values other than their solutions that solved the questions. On the post-interviews the students avoided the procedural traps, and explained to me *why* there were unusual or unexpected values and how they apply to solving the questions. This is a strong confirmation that the interventions have moved the students to a structural understanding.

Oscillation

The results display very well that before the interventions the students were unable to oscillate between a procedural and a structural view. The students reached $6x + 12 = 6x + 12$ and could not see the relationship, so they continued with a procedure. They reached $2x + 1$ and could not explain it as an odd number, but wanted to solve it procedurally; or they reached $k^2 + 16 > 0$ and could not see the structure, but grasped familiar answers and procedures that produced answers that did not solve the problem.

After the interventions the students demonstrate an ability to oscillate to a structural view when faced with $21x + 15 = 21x + 15$, $x^2 + x + 1 > 0$, disappearing variables, and negative discriminants.

A persuasive example of the difference in ability to oscillate from a procedural to a structural view was the difference in answers between question 4 of the pre-

questionnaire (*show that the sum of two consecutive numbers is always odd*) and question 3 of the post (*show that the sum of an odd and an even number is always odd*). Before the intervention, even though many produced a correct equation, the students could not oscillate between a procedural and a structural view. They conceived of $2x + 1$ as a process to be completed and searched for a procedure or used number examples in order to produce a halting signal. After the intervention the majority explained *why* the *process* $2x+1$ produces an odd number (an object).

The strongest evidence that the cognitive conflict interventions improved the ability of the majority of the students to oscillate from a procedural to a structural understanding comes from the results of the questions that were unfamiliar to the students; questions 4, 5, and 6 of the post-questionnaire and the questions of the post-interview. The students began these questions using familiar procedures and most were not deceived by the traps the procedures create (e.g. the k disappearing, or b equalling a quadratic). The majority of students switched to a structural view and solved the problems. This is the opposite of the results from before the interventions where the students did not switch to a structural view even at junctions like $6x + 12 = 6x + 12$ or $k^2 + 16 > 0$. The Grade 12's and some of the foundation students demonstrated that conceiving of the question structurally is not enough. The students must be able to oscillate between a procedural and a structural view to solve these questions.

Justification

Before the interventions the students quickly grasped a familiar procedure or answer when they had reached an unfamiliar result. They would automatically apply a

procedure and rigidly complete it without reference to the structure of the question. There was no justification of the transformations. They would transform $x = x$ or $0 = 0$ to $x = 0$, and answer 'no solution' whenever a negative discriminant was found, without reference to what it meant to the question. After the interventions the students did not make these same errors. They analyzed the question and if a variable disappeared or a negative discriminant appeared they related this to the structure of the question and answered correctly. Most importantly, they justified their transformations on unfamiliar questions. The students referred back to the structure of the question as they worked and when they reached a confusing point, like $b = -2x^2 + 2x$, they tested their transformations.

Control

Before the cognitive conflict interventions the students performed many illegal transformations because they did not have a control mechanism that would alert them to contradictions they created or cause them to review a step if they performed an illegal transformation. They were satisfied with a procedural answer even if it did not answer the original question or involved an illegal transformation like finding the square root of a negative number. This made them vulnerable to falling into the procedural traps set by these questions that demand structural understanding. The students' behaviour after the interventions is distinctly different from their behaviour before. They began to solve the questions similarly to the way they did before the interventions, but they did not fall into the procedural traps. They referred to the structure of the question as they completed procedures and compared objects, so that they were alerted when their procedure

produced a contradiction. They then reviewed the question and discarded (even crossed out) their work and completed it with a structural explanation.

All four themes coalesce when we examine the difference in results between question 3 of the pre-questionnaire and question 4 of the post-questionnaire. Question 3 (*solve for x: $6x + 12 = 3(2x + 4)$*) presents the students with $6x + 12 = 6x + 12$. The students had worked with this level of algebra for three years, and the equations of lines in this form for two, but they did not oscillate to a structural view, did not compare the equivalency of the equations, and did not analyze the structure and test values. They searched in vain for a procedure, or used illegal transformations to provide a procedure with a halting signal. Question 4 (*For what values of k will this system have a solution? $2x + y = k$ $y = 3 + k$*) is the opposite end of the spectrum. Indeed, Sfard and Linchevski (1994), from whom I adopted the question, feel "the conceptual step that must be taken to reach this interpretation [two lines that intersect are represented] may be higher than even the most experienced teachers would guess" (p. 114). The students had not worked with this type of parametric equation before, it is not solvable procedurally, and it has a devious procedural trap, yet the majority of the students answered correctly; in the process displaying the behaviours of the four themes that demonstrate structural understanding. When the k disappeared and a satisfying halting signal appeared the students reviewed their work, analyzed the structure, and answered correctly either with reference to the two lines represented or with a comparison of the effect of k on the two equations.

The examination of the students' results according to the four themes found that, undeniably, the proportion of the population with a structural understanding was inverted by the cognitive conflict interventions, from a majority with a procedural understanding before the interventions to a majority with a structural understanding afterwards.

Differences amongst the groups

The results show a leap to a structural understanding on the part of a considerable piece of the population, but they also show that a significant proportion of the foundation students may have benefited from the cognitive conflict interventions, but have not developed a structural understanding. This group has made gains in recognizing the structure of the problem, but their procedural proficiency limits their ability to oscillate between a procedural and a structural view. The rest of the foundation group, the moderate group, and the proficient group have shown great gains. They have gone from a procedural understanding before the interventions to a structural understanding, even on unfamiliar questions, afterwards. The proficient group had some members who displayed a structural understanding before the interventions, but also had many members who progressed from a procedural understanding to a very strong structural understanding shown by their ability to answer all 6 questions on the post-questionnaire correctly with very good explanations.

The results of those students who did not gain a structural understanding articulate that the cognitive conflict technique I used must be refined. Recall from the frameworks in Chapter one that Piaget's (1985) findings tell us that if a structure has not been acquired, no conflict will be caused by new knowledge. Similarly, Dreyfus,

Jungwirth, and Eliovitch (1990) tell us that if the proposed solutions provided in the interventions are beyond the ability of the students, no conflict will be generated. Also, Fujii (1987) found that a variation in experiencing the three types of conflict, "behaviourized how", "verbalized how", and "justification of how" can cause a variation in the students' understanding. And finally, Sfard's framework tells us that students must be at a level of procedural proficiency that allows the insight of seeing a process as an entire object in order to permit reification. The pre-questionnaires and interviews and the weak procedural performance on the post-interviews of these students who did not make the leap to structural understanding demonstrate that they did not have a strong enough procedural understanding at the beginning of the study in order to progress to a structural understanding. These students are still at Sfard's (1991) interiorization level and are becoming more proficient with the procedure. They have not reached a level of condensation where they can see the process as a whole that will support reification of these concepts. The foundation student (Katya), who's interviews I used to demonstrate that these students made advances but still have a procedural understanding, exemplifies that these students are at the behaviourized how and verbalized how level. They have increased their understanding of *how* a procedure works, but have not crossed the divide to understand *why* this is so. Notwithstanding these results with some of the foundation students, the interventions were effective with many very weak students and led them to progress to structural understanding. To make this technique effective for all the students, they must all be brought to a level of procedural proficiency that enables them to conceive of a concept as both a process and an object simultaneously so that conflict can be engendered.

The Grade 12's

The Grade 12's demonstrated a procedural understanding similar to what the Grade 11's showed before the cognitive conflict interventions, but slightly more proficient. They solved one question that demanded structural understanding better than the Grade 11's before the intervention, but could not solve a question that the Grade 11's could after the interventions. More conclusively, the Grade 12's did not demonstrate the language, the oscillation between a procedural and a structural view, the justification of transformations, or the good control mechanisms that confirm structural understanding, while the Grade 11's did. The Grade 12's language demonstrated that even though they could solve question 1 procedurally more proficiently, they did not see the equivalency of the objects. Their answers on question 2 showed they could not oscillate between a procedural and a structural view; leading them to perform the same type of invalid transformations that the Grade 11's performed before their interventions. The Grade 12's could not explain *why* they had reached an answer while the Grade 11's could. The similarity of the Grade 12's to the Grade 11's before the interventions despite more practice with higher level concepts corroborates that it was the interventions that moved the Grade 11's to a structural understanding.

An argument can be made that the cognitive conflict interventions had little effect and that the students had simply progressed to a structural understanding by the end of the year as can be expected, or that the work on these questions during the interventions had simply made the students more procedurally proficient. This is indeed what the frameworks elaborate; that progress to a structural understanding comes after much procedural work with a concept. The evidence supporting my assertion that the Grade 11

students developed a structural understanding of Grade 11 concepts due to the cognitive conflict interventions is solid. Some support for this contention is that the proportion of students who have demonstrated a structural understanding is much higher than the literature would lead us to expect, but it is the comparison of the students' abilities according to the four themes and a comparison of the Grade 11 results to the Grade 12 results that provide striking support. After the cognitive conflict interventions the change in the students' language, their ability to oscillate between a process and an object, their justification of transformations, and their good control all demonstrate a broad structural understanding, not a procedural proficiency. The Grade 12's results provide the evidence that the students wouldn't have developed this structural understanding without the interventions. Both the proportion of Grade 11's demonstrating structural understanding and the strength of their understanding is higher than the Grade 12's. Crucially, many of the weakest Grade 11's showed a structural understanding superior to the most proficient Grade 12's who had had the same teacher for Grade 11, along with an additional year of procedural practice. Because the Grade 12's had had all the instruction that the Grade 11's had, and an extra year of working at a higher level, but had not had the cognitive conflict interventions, I concluded the interventions had promoted the dramatic increase in structural understanding the Grade 11's displayed.

This assertion then engenders another counter-argument; that my students had recently worked with this material and so were masters at it, but they would not display the same mastery at the end of their Grade 12 year. That, perhaps, their results a year later would be similar to the Grade 12's I enlisted for this study. This concern is somewhat addressed by questions 4, 5, and 6 of the post-questionnaire and question 2 of

the post-interview. I intended these questions to be unfamiliar enough that it would not simply be the proximity to work during the interventions that would allow the students to answer them. But I feel a definitive answer waits on another research project that would assess the students' structural understanding directly after the interventions and then again a year later, and perhaps two years later as well.

Another argument that can be made is that the students gave more than their usual effort and thought while I was assessing them because they were aware it was for my research and were expecting the questions to be more difficult. As the results on the pre-questionnaire and interview show, and the Grade 12 results reinforce, students with a procedural understanding either can not discern the difficulty of a question, and confidently solve it incorrectly, or they are not able to solve it if aware of the difficulty. We have seen from the literature (Sfard, 1991, Linchevski and Sfard, 1991, Gray and Tall, 1991) that structural understanding is not just a matter of hard work, and these questions demanded structural understanding. Without an oscillation from a procedural to a structural view these questions cannot be solved. Even with my prompting on the pre-interviews the students with a procedural knowledge were often not able to oscillate to a structural view. After the interventions the students were able to answer questions that demanded structural understanding, and they displayed the behaviours that demonstrate structural understanding. The students were aware both before and after the interventions that the questionnaires were for my research and their expectations and effort were equal before and after. The significant events in the interval between the questionnaires were the interventions.

Conclusions

I will now present and explain the conclusions I drew from these results. Recall from Chapter two that my research questions were: what effect would cognitive conflict interventions have on Grade 11 students' understanding of algebra; what demonstrates a structural understanding on the part of the student; what are the elements that constitute an effective cognitive conflict intervention; and what role would the individual differences of the students play in the effectiveness of the cognitive conflict interventions?

I will begin with my conclusions to the question *what demonstrates a structural understanding on the part of the student*, because it is addressed by a theoretical examination while the other three questions are answered by the empirical results of the study. My review of the literature, combined with my experience during the study, led me to conclude that the types of questions I used in my questionnaires and interviews, coupled with an assessment of the student's solution in relation to the four themes delineated in Chapter one, gave a very dependable method of distinguishing whether a student had a procedural or a structural understanding. I concluded that problems that contain a procedural trap force a student to consider a process and an object simultaneously, and provide an elegant opportunity to test for structural understanding. If a student's solution to one of these questions is analyzed for the language the student uses, the ability to oscillate between a procedural and a structural conception, the justification of methods used, and the display of a good control mechanism, the result is a system that provides an excellent insight into the understanding of the student.

I'll now consider the question; *what are the elements that constitute an effective cognitive conflict intervention*. My observations during the interventions, (and during the interviews, see *Dorin and Cadmar in Pre- interviews*) led me to conclude that the questions with procedural traps instigated cognitive conflict in the students and succeeded in making them unsatisfied with their procedural methods of solving the problems. These questions demanded the students work with concepts as processes and objects simultaneously. Following this demand with the discussion of two previous students' answers presented the students with conceptions they could understand in a non-threatening environment. These elements in concert provided the environment needed to resolve the conflict properly.

My conclusions relating to these first two questions lead up to my conclusion of what effect cognitive conflict interventions have on Grade 11 students' understanding. The results I have summarized led me to conclude that the cognitive conflict brought about during the interventions resulted in a significant change in the students' understanding. I believe that the majority of the students began the study with a procedural understanding and that the interventions were the driving force behind the acquisition of structural understanding that the majority of the population of students demonstrated after the interventions. Achieving structural understanding is tremendously hard work for most students. I concluded that the cognitive conflict intervention technique I used in this study significantly diminished the typical disparity between the effort a student invests and the progress they make toward structural understanding.

The results demonstrated that, despite the gain in structural understanding the majority of the students displayed, a group of the foundation students did not gain a structural understanding. These results indicate that the cognitive conflict interventions did not promote structural understanding in students of all ability levels. I concluded that the cognitive conflict interventions as I designed them are not effective if a student does not have a strong enough procedural understanding to enable them to be impelled to consider a structure as a process and an object simultaneously when confronted with a conflict situation.

The definitive conclusion that the results reveal is that, after the interventions the majority of the students have an ability to solve questions that demand structural understanding that they did not have before. The demands for structural understanding of these questions combined with the fact that the students recognized the difficulty of these questions with inherent traps and were able to oscillate between a procedural and a structural conception when it was required, demonstrate the leap they made to a structural understanding. The students also exhibit the four characteristics I feel are distinguishing traits of structural understanding after the interventions while they did not before. I believe that these escalations in the students' ability to solve these questions and their displays of structural understanding can not be ascribed to the students spending more time with the concepts, becoming more procedurally proficient, or working harder after the interventions. I believe the progress came because the cognitive conflict interventions helped so many of the students cross the divide from procedural to structural understanding, and so, I concluded cognitive conflict interventions can effectively promote a structural understanding of Grade 11 algebra.

CHAPTER EIGHT: PEDAGOGICAL CONTRIBUTIONS AND IMPLICATIONS

I feel the results of this study offer compelling evidence that the cognitive conflict technique I designed promoted a stronger structural understanding in the majority of the students. This coupled with the fact that the interventions were designed to be used in a typical classroom setting, and significantly decreased the disparity between the amount of effort the students typically dedicate and their gain of a structural understanding, produces the conclusion that these interventions should be used regularly in the classroom. Indeed, this study has altered my own teaching considerably. The cognitive conflict interventions of the study have become standard lessons in my methodology, and I have begun to create lessons that use the cognitive conflict technique for all the concepts we study. My methodology exploits the discoveries I made during this study. I impel the students to become as procedurally proficient as possible and then use cognitive conflict in order to help the students make the leap to structural understanding.

I found that the students enjoyed the interventions and, although conflict was induced, they were not intimidated by the technique. Their engagement with the material was much higher with these questions that cause cognitive conflict, and they worked harder and thought more on these questions because they were intrigued. I found my students reinforced a conclusion of Skemp (1979), "We ourselves know that the achievements of relational understanding can be very pleasurable" (p. 46), and they were

very pleased when they realized they had a better understanding of the material. I now endeavour to use this pleasurable feeling that comes with structural understanding to motivate the students to work for understanding.

The crucial elements of the interventions are easily modifiable to produce structural understanding of diverse concepts other than Grade 11 algebra. The technique of using worksheets with questions that contain a procedural trap, providing a procedural and a structural answer from previous students, and discussing and justifying the answers with a partner, worked well in a standard classroom situation. It does not demand extra class time and was not onerous to prepare or operate. The only unusual task is finding (or creating) questions with a procedural trap that demand the students operate with a concept as a process and an object at the same time.

Because the students show the strongest gains in structural understanding on the questions that we specifically worked with during the cognitive conflict interventions, it behoves us as mathematics teachers to use more of these questions that demand structural understanding in class. However, the literature has shown us that just asking the students to solve these questions does little to promote understanding. My hardest working students and the most proficient, hardest working Grade 12's did not have a structural understanding without the interventions. These questions that demand structural understanding have to be used in conjunction with cognitive conflict interventions.

The questions I used on my questionnaires and interviews are much more likely to be found on math contests than the tests and exams generally written at the Grade 11 level (because the majority of Grade 11 students would answer them incorrectly!). This

means a simple, non-revolutionary way of using more of these questions is to prepare for, and write, math contests using the cognitive conflict intervention technique. Once this becomes commonplace these types of questions can be added to tests.

I concluded in the last chapter that in order to provide the best opportunity for all students to gain a structural understanding (which was my ambition) the intervention technique must be refined. In order to use this method and have the foundation students benefit from it, they must be brought up to a level of procedural understanding where the conflict is meaningful. This means an improvement of their procedural proficiency so that they will be able to view a concept as a process and an object at the same time when required to do so.

Areas of future research generated by the data

As stated at the beginning of the results chapter, the questionnaires and interviews presented an abundance of data that I had not expected and that generated the following topics it would need future research to illuminate.

I found that asking the students to explain their answers provided unexpected insight into their thinking and understanding. It would be valuable to investigate if my technique could be modified to add to the body of research studying if having students explain their answers promotes understanding. Two important issues my results generated that need further exploration are: whether or not it is possible to bring the weakest students to a structural understanding and what will be the long term effects of the gain in structural understanding the students experienced. Another consideration

arose while I was comparing the students' individual results. The partner the students chose to discuss and justify their answers with seemed to significantly affect their resolution of the conflict. It would be interesting to study if certain combinations of students working together promote better conflict resolution. A final concern that arose from the results of the weakest foundation students was whether it was the intimidation of these students throughout their school careers by material that is above their level of understanding that prevents them gaining a structural understanding.

I believe this study made three specific contributions to the mathematics education field.

I feel the system I designed to assess a student's understanding has contributed an effective tool for math teachers and researchers to use in testing for understanding. The system incorporates the independent work of many researchers, codifies these observations into four themes, and uses questions with procedural traps to reveal a student's understanding and allow analysis according to the themes. This system competently reveals whether a student has a procedural or a structural understanding.

I also feel the design of a cognitive conflict intervention that is practical to use provides a valuable tool to math teachers and researchers. The structure of using a question with a procedural trap to induce conflict, incorporated with the elements that are necessary to properly resolve the conflict, was very effective at instigating conflict and resolving it properly. This capability to instigate and resolve cognitive conflict can be used in numerous teaching and research situations.

The final contribution of this study is as an example of a successful implementation of the cognitive conflict interventions and the system of assessing students' understanding. This study used the analysis of the students' solutions according to the four themes, and it found the interventions significantly advanced the understanding of the majority of the students.

APPENDICES

APPENDIX A

PERMISSION SLIP

SIMON FRASER UNIVERSITY

Form 2- Informed Consent By Participants In a Research Study

The University and those conducting this research study subscribe to the ethical conduct of research and to the protection at all times of the interests, comfort, and safety of participants. This research is being conducted under permission of the Simon Fraser Research Ethics Board. The chief concern of the Board is for the health, safety and psychological well-being of research participants.

Should you wish to obtain information about your rights as a participant in research, or about the responsibilities of researchers, or if you have any questions, concerns or complaints about the manner in which you were treated in this study, please contact the Director, Office of Research Ethics by email at hweinber@sfu.ca or phone at 604-268-6593.

Your signature on this form will signify that you have received a document which describes the procedures, possible risks, and benefits of this research study, that you have received an adequate opportunity to consider the information in the documents describing the study, and that you voluntarily agree to participate in the study.

Any information that is obtained during this study will be kept confidential to the full extent permitted by the law. Knowledge of your identity is not required. You will not be required to write your name or any other identifying information on research materials. Materials will be maintained in a secure location.

You are being asked to participate in this study on a purely volunteer basis. You do not have to participate in this study, and not participating will have no effect whatsoever on your school work. The study is completely separate from school. Participating or not participating in this study will not contribute to or affect your mark in any way whatsoever. If you do decide to participate, you may quit at any time with no repercussions.

Title: **Using cognitive conflict as an instructional strategy to help students develop a more expert method of solving algebra problems**
Investigator Name: **Duncan Fraser**
Investigator Department: **Education**

Having been asked to participate in the research study named above, I certify that I have read the procedures specified in the Study Information Document describing the study. I understand the procedures to be used in this study and the personal risks to me in taking part in the study as described below:

Risks to the participant, third parties or society:
None

Benefits of study to the development of new knowledge:
The study will help find more effective methods of teaching algebra to high school

students.

Procedures:

Complete 2 paper and pencil worksheets, some subjects will be interviewed after completing the worksheets.

I understand that I may withdraw my participation at any time. I also understand that I may register any complaint with the Director of the Office of Research Ethics or the researcher named above or with the Chair, Director or Dean of the Department, School or Faculty as shown below.

Department, School or Faculty: Chair, Director or Dean:
Education Dr. Rina Zazkis

8888 University Way,
Simon Fraser University,
Burnaby, British Columbia, V5A 1S6, Canada

I may obtain copies of the results of this study, upon its completion by contacting:

**Duncan Fraser
1329 East 13th Ave
Vancouver, British Columbia
V5N 2B5
604 708 4143**

I have been informed that the research will be confidential.

I understand that my supervisor or employer may require me to obtain his or her permission prior to my participation in a study of this kind.

I understand the risks and contributions of my participation in this study and agree to participate:

The participant and witness shall fill in this area. Please print legibly	
Participant Last Name:	Participant First Name:
Participant Contact Information:	
Participant Signature:	Witness (if required by the Office of Research Ethics):
Date (use format MM/DD/YYYY):	

APPENDIX B QUESTIONNAIRES AND INTERVIEWS

The question sheets used during the questionnaires and interviews.

Pre-questionnaire.

Remember this questionnaire is for research and will have no effect on your class mark and will be kept completely confidential.

Please answer each of the following. Please write down each of your steps (even if you can do it in your head) and explain what you are doing at that step and why that step works.

1.) Solve for x:

$$4x + 8 = 5x + 2$$

2.) Solve for x:

$$b = vxt$$

3.) ¹Solve for x:

$$6x + 12 = 3(2x + 4)$$

4.) ²Show using algebra that the sum of two consecutive numbers is always an odd number

¹ After Fujii 2003

² From Lee and Wheeler in Kieran 1989

Pre-interview

Remember this questionnaire is for research and will have no effect on your class mark and will be kept completely confidential. Please tell me what you are doing and why as you do it for the camera.

¹Solve this for x:

$$6x + 12 = 3(2x + 4)$$

²Find k so that this function will have 2 different real roots

$$f(x) = x^2 + kx - 4$$

¹ After Fujii 2003

² After Sfard and Linchevski, 1994

Post-questionnaire

Remember this questionnaire is for research and will have no effect on your class mark and will be kept completely confidential.

Please answer each of the following. Please write down each of your steps (even if you can do it in your head) and explain what you are doing at that step and why that step works.

1.) ¹Solve for x: $x^2 + x > -1$

2.) ²Solve for x: $3(7x + 5) = 21x + 15$

3.) ³Show using algebra that the sum of an odd number and an even number is always an odd number.

¹ After Sfard and Linchevski, 1994

² After Fujii 2003

³ From Lee and Wheeler in Kieran 1989

4.) ¹For what values of k will this system have a solution?

$$2x + y = k$$

$$y = 3 + k$$

5.) ²Do these two inequalities have the same solution set?

$$4x^2 > 9$$

$$2x > 3$$

6.) Solve this inequality:

$$x^3 - 3x^2 + 3x + 7 > 0$$

¹ After Linchevski and Sfard, 1991

² After Sfard and Linchevski, 1994

Post-interview

Remember this questionnaire is for research and will have no effect on your class mark and will be kept completely confidential. Please tell me what you are doing and why as you do it for the camera.

1.) ¹Do these two equations have the same solution set?

$$(x - 3)^2 = 0$$

$$5x - 4 = 2x + 5$$

2.) For what values of b will there be no solution to this system of equations?

$$y = 2x + b$$

$$y = -2(x - 1)^2 + 2$$

¹ After Linchevski and Sfard, 1991

APPENDIX C INTERVENTION WORKSHEETS

The worksheets used during the cognitive conflict interventions

Principles of Math 11 Inequalities worksheet

Name _____

Can you solve this?

$$-8x > 16$$

Check your answer to see that it makes the inequality true.

Find someone who got a different answer than you and compare your operations with theirs. Can you explain why your operations are correct and theirs are wrong?

Last year Anastasia solved this same question and her answer was:

$$x > -2$$

She justified this answer by saying, "all you have to do is divide both sides by the same number. We learned that long ago".

Last year Bertrand solved this same question and his answer was:

$$x < -2$$

He justified this answer by saying, "I graphed the two lines and the graph shows that when x is less than negative two, $-8x$ is greater than 16.

Pick a partner. One of you take Anastasia's side and one of you take Bertrand's and argue who is correct.

Can you explain the concept we are dealing with in general terms?

Can you write another example that illustrates the same concept? Can you write an example that further extends this concept?

Can you solve:

$$-5x - 8 > 32$$

Principles of Math 11 Equations worksheet

Name _____

Can you solve:

$$2(x + 3) = 2x + 6$$

Check your answer to see that it makes the equation true.

Find someone who got a different answer than you and compare your operations with theirs. Can you explain why your operations are correct and theirs are wrong?

Last year Anastasia solved this same question and her answer was:

x can be any number

She said, "If you think of these two lines, then there are an infinite number of points that solve the equation."

Last year Bertrand solved this same question and his answer was:

$$0 = 0$$

So he said there was no solution to the equation.

He justified this answer by saying, "Whatever you do to one side you have to do to the other, and when you do that you never get x equals anything."

Pick a partner. One of you take Anastasia's side and one of you take Bertrand's and argue who is correct.

Can you explain the concept we are dealing with in general terms?

Can you write another example that illustrates the same concept? Can you write an example that further extends this concept?

Can you solve:

$$5x - 15 = 5(x - 3)$$

Principles of Math 11 Inequalities worksheet

Name _____

Can you solve:

$$x^2 + 2x + 4 > 0$$

Check your answer to see that it makes the equation true.

Find someone who got a different answer than you and compare your operations with theirs. Can you explain why your operations are correct and theirs are wrong.

Last year Anastasia solved this same question and her answer was:

x can be any number

She justified her answer by saying, "I thought of the graph of $x^2 + 2x + 4$ in my head, and decided x could be any number."

Last year Bertrand solved this same question and his answer was:

there is no solution

He said, "I solved $x^2 + 2x + 4$ using the quadratic formula and found

$$\frac{-2 \pm \sqrt{-12}}{2}$$

the -12 as a discriminant told me there are no solutions."

Pick a partner. One of you take Anastasia's side and one of you take Bertrand's and argue who is correct.

Can you explain the concept we are dealing with in general terms?

Can you write another example that illustrates the same concept? Can you write an example that further extends this concept?

Can you solve:

$$-2x^2 + 12x - 23 < 0$$

Principles of Math 11 Equations Worksheet

Name _____

Solve:

And solve:

a.) $(x + 2)(x - 2) = (x - 2)$

b.) $x^2 - 4 = x - 2$

Check your answer to see that it makes the equation true.

Check the original equations with 2 and -1.

Find someone who got a different answer than you and compare your operations with theirs. Can you explain why your operations are correct and theirs are wrong

Last year Anastasia solved these same questions and her answers were:

Last year Bertrand solved these same questions and his answers were:

a. $x = -1$

a. $x = 2, -1$

b. $x = 2, -1$

b. $x = 2, -1$

Pick a partner. One of you take Anastasia's side and one of you take Bertrand's and argue who is correct.

Can you explain the concept we are dealing with in general terms?

Can you write another example that illustrates the same concept? Can you write an example that further extends this concept?

Can you solve:

$$(x + 3)(x - 3) \geq x + 3$$

Principles of Math 11 Systems of equations worksheet

Name _____

Solve this system of equations:

$$\begin{aligned}2(3x - 5) &= 2 - y \\6x + y &= 12\end{aligned}$$

Check your answer to see that it makes both equations true.

Find someone who got a different answer than you and compare your operations with theirs. Can you explain why your operations are correct and theirs are wrong

Last year Anton solved this same question and his answer was:

There are no solutions to this system.

He justified this by saying, "I multiplied out to get

$$\begin{aligned}6x + y &= 12 \\6x + y &= 12\end{aligned}$$

Then subtracted and got

$$0 = 0$$

and I decided there are no solutions"

Last year Betty solved this same question and her answer was:

$$y = -6x + 12$$

Her justification was: "I multiplied out to get

$$\begin{aligned}6x + y &= 12 \\6x + y &= 12\end{aligned}$$

and I realized these describe the same line, and so any point on that line solves the system."

Pick a partner. One of you take Anton's side and one of you take Betty's and argue who is correct.

Can you explain the concept we are dealing with in general terms?

Can you write another example that illustrates the same concept? Can you write an example that further extends this concept?

Can you solve:

$$-2(x - 3)^2 + 7 = -2x^2 + 12x - 11$$

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