HEDGING TRANCHES OF COLLATERALIZED DEBT OBLIGATIONS

by

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ABSTRACT

Collateralized debt obligations (CDO) are a recent development in credit derivatives market. Credit risk of the collaterals is securitized through issuing tranches. The values of those tranches depend on the default risk characteristics of the pool of collaterals. In this paper, a reduced form model of default is considered. The hazard rates of collaterals follow square-root diffusion processes that can be correlated. The question of hedging the values of tranches against default risk and uncertain movements of hazard rates is analyzed and a feasible hedging strategy using credit default swaps is suggested. The model avoids the static nature of copula models. Sensitivity of results to various parameters of the model is examined.

Keywords: CDO; CDS; Default intensity; Diffusion; Hedging; Tranche

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1. Introduction

A credit derivative is a financial instrument whose payoff depends on the creditworthiness of some companies or countries. Companies can use credit derivatives to trade credit risks in the same way that they trade market risks. The most exciting developments in derivatives markets have been in the credit derivatives area since the late 1990s. The total notional principal of outstanding credit derivatives contracts was around $800 billion in 2002. That grew to over $3 trillion by 2003. As a result, modelling credit risk has become an active and important field. However, modelling credit is very difficult since default is a rare event. There are not enough observations to test models and extract meaningful statistics. There are several approaches to modelling credit. They can be divided into two broad categories of structural and reduced-form models. Structural models try to relate default events to the internal structure of the issuer of the debt. Reduced-form models do not explain why default occurs. Instead, they model the probability of default using the notion of default intensity. This paper focuses on reduced-form models.

One of the most important advances in the securitization of credit risk is the issuance of collateralized debt obligations (CDOs). The underlying collateral is usually a portfolio of bonds or bank loans. The cash flow structure of a CDO allocates payments from the collateral pool to a prioritized collection of securities called tranches. That is the description of a cash CDO. There are also synthetic CDOs which use a pool of CDSs (described below) instead of a pool of bonds.

The default dependency of the debt instruments in the underlying collateral is crucial to pricing and hedging of the tranches. The key element in models for pricing and hedging of these products is the mechanism that generates default dependency. Credit default swaps can be used to hedge the tranches of a CDO. Credit default swaps (CDSs) are the most popular credit derivatives. They provide insurance against the default of a particular company. The protection buyer pays premium over time to the protection seller. In case of default by the company, the protection seller buys the bonds of that company for their face value. The market for CDSs has grown since the late 1990s. CDSs account for around 70% of all credit derivatives.

There are three major approaches to modelling default dependency within the category of reduced-form models. The first one is models that have dependent default intensities but conditionally independent default times such as Duffie and Garleanu (2001). The second one is factor copula models such as Li (2001), Laurent and Gregory (2005), Rogge and Schonbucher (2003), and Hull and White (2004). The dependence structure of default times is described by the copula function. The models in this approach are usually presented and used in a static fashion. This makes it hard to have intuition for dynamic aspects of the model. The third one is models with direct interaction between default intensities such as Jarrow and Yu (2001), Davis and Lo (2001), Giesecke and Weber (2006), and Yu (2005). In this approach, default intensities jointly jump upwards by a discrete amount when a credit event occurs (default contagion). The

disadvantage of this approach is that calibration is very difficult since estimation of the jump factor is nontrivial.

The risk management of CDO tranches has become very important for investors in credit markets. Tranches should be hedged against default risk of companies in the collateral and uncertain movements of default intensities (spread risk). Hedging is possible by buying or selling protection using single-name CDSs since the value of CDSs depends on the spreads. The standard approach for determining the sensitivity to spread changes (spread-deltas) is pricing the tranches using Gaussian copula and then varying the spreads to see how much the prices change. It assumes that all other parameters stay constant as one varies the intensities. Hedge ratios protecting tranches against the default of firms (default-deltas) can be calculated similarly as described by Neugebauer (2006). However, that method has deficiencies which are related to the weakness of copula models which is studied in Schonbucher (2001). Dynamic hedging of credit risky securities is studied by Laurent (2007) and Frey (2006). They analyze the hedging of CDO tranches with CDSs in a Markov-chain model with default contagion. However, they focus mainly on the case without spread risk. On the other hand, Industry practice tends to ignore hedging against default risk and hedges the spread risk only.

In this paper hedging tranches of a simple cash CDO against both spread risk and default risk is considered. The collateral consists of two bonds. A square-root diffusion model for default intensities is assumed which avoids the discretization of Markov chains and the static nature of copula models. The

diffusion of spreads of different firms can be correlated and intensities of different firms can be different. Hedge ratios for hedging against both the spread risk and default risk using CDSs were calculated using Monte Carlo simulations. It was shown that full hedge is possible if two CDSs with different maturities are traded in the market. One often finds that the relative accuracy of estimating sensitivities can be hundreds of times lower than the relative price precision. Variance reduction techniques were employed to improve the efficiency of estimating sensitivities. The model can be extended to include more bonds and also default contagion.

2. Model

This section explains the model, CDSs, CDOs, and the hedging strategy.

2.1. Intensity Modeling

The reduced-form model of default assumes that the probability of default over the next small interval of time is $\lambda\Delta t$ given the condition that the firm has survived up to that time. λ is called the hazard rate or default intensity which is a non-negative number. The default means the default of the firm under consideration. All the outstanding bonds issued by the firm default in case of bankruptcy of the firm. With deterministic hazard rate, the probability of survival from time t to T with the condition that it has survived up to time t is:

$$p(t,T) = e^{-\int_t^T \lambda(s)ds} \tag{1}$$

With stochastic hazard rate, the equation becomes:

$$p(t,T) = E_t[e^{-\int_t^T \lambda(s)ds}] \tag{2}$$

Processes in which the hazard rate is stochastic are called the Cox processes. The model used in this paper is the influential model of short-term interest rate r called CIR process. The interest rate r is replaced with the hazard rate λ:

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dz(t) \tag{3}$$

z is a standard Brownian motion. This process is also called a square-root

diffusion process. θ is the long-run mean of λ which means that $E_t[\lambda(s)]$

converges to θ as $s \to \infty$. κ is the rate of reversion to the long-term mean. It

keeps λ from getting too far from the long-term mean. σ is a volatility coefficient.

Therefore, the instantaneous standard deviation as a fraction of level is $\sigma/\sqrt{\lambda(t)}$.

$\lambda(t)$ is always non-negative in this model which is required since hazard rate is

non-negative by definition. The analytical solution is:

$$p(t,T) = A(t,T)e^{-B(t,T)\lambda(t)} \tag{4}$$

where

$$A(t,T) = [\frac{2\gamma e^{(\kappa+\gamma)(T-t)/2}}{(\kappa+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}]^{2\kappa\theta/\sigma^2} \tag{5}$$

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\kappa+\gamma)(e^{\gamma(T-t)}-1)+2\gamma} \tag{6}$$

γ is defined as:

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2} \tag{7}$$

Now one can get the probability density function for default time τ :

$$f(t) = \lim_{\Delta t \to 0} \frac{P(t \leq \tau \leq t+\Delta t \mid \tau > 0)}{\Delta t} = -\frac{\partial p(0,t)}{\partial t} \tag{8}$$

In this paper it is assumed that short-term default-free interest rate $r(t)$ is

constant. To price a defaultable zero-coupon bond with unit face value and

maturity T we use:

$$B(0,T) = E_0[e^{-\int_0^T r(s)ds}1_{\{\tau>T\}}] = e^{-rT}E_0[1_{\{\tau>T\}}] = e^{-rT}E_0[e^{-\int_0^T \lambda(s)ds}] = e^{-rT}p(0,T) \tag{9}$$

The above formula uses the risk-neutral valuation principle which states that the

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value of a security is equal to its expected discounted cash flows under the risk-neutral probabilities. Therefore, the probability measure used in the model is the risk-neutral measure. Note that the real-world probability measure can be different.

There are two firms in this model. The hazard rate of each firm follows a CIR process. The two diffusion processes can be correlated:

$$d\lambda_1(t) = \kappa(\theta - \lambda_1(t))dt + \sigma\sqrt{\lambda_1(t)}dz_1(t) \tag{10}$$

$$d\lambda_2(t) = \kappa(\theta - \lambda_2(t))dt + \sigma\sqrt{\lambda_2(t)}dz_2(t) \tag{11}$$

The two diffusion processes are correlated:

$$Cov[dz_1(t), dz_2(t)] = \rho dt \tag{12}$$

No default contagion is assumed. That means if one firm defaults, the intensity of other firm does not jump. Moreover, conditional on the intensity paths, the default times of the two firms are independent. The only source of default dependence is the correlated diffusion of intensities. The parameters of the CIR process are the same for both firms, but the spot intensities can be different.

One can use the survival indicator variable to define a measure for default correlation by time T. The survival indicator d_i is zero if the firm i defaults by time T and it is one if it does not default. To measure the default correlation between firms i and j, covariance of d_i and d_j can be used:

$$E[d_i] = p_i = P(\tau_i > T) \tag{13}$$

$$Var[d_i] = E[d_i^2] - (E[d_i])^2 = p_i - p_i^2 = p_i(1 - p_i) \tag{14}$$

$$Cov[d_i, d_j] = E[d_i d_j] - E[d_i]E[d_j] = p_{ij} - p_i p_j \tag{15}$$

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where p_{ij} is the probability of both firms surviving. Now the binomial measure of default correlation can be defined as:

$$\rho_{d_i d_j} = \frac{Cov[d_i, d_j]}{\sqrt{Var[d_i]Var[d_j]}} = \frac{p_{ij} - p_i p_j}{\sqrt{p_i p_j (1 - p_i)(1 - p_j)}} \qquad (16)$$

2.2. Credit Default Swaps

Credit default swaps are the most important instruments in the credit derivatives market. They provide insurance against the default of an issuer (the reference entity). The buyer pays a premium until either the maturity of the contract or default on the reference entity, whichever comes first. If the default occurs, the seller of protection compensates the buyer by paying the face value of the bond in exchange for the defaulted bond (physical settlement). Zero recovery is assumed here in case of default. It means that the value of the bond drops to zero when default happens.

CDS has two legs: the premium leg and the protection leg. Protection buyer pays a premium until the default occurs. It is assumed that the premium is paid continuously. The protection seller pays the protection buyer the face value of the bond in case of default. Value of the premium leg is:

$$V_{prem} = c \int_0^T e^{-rt} p(t) dt \qquad (17)$$

where c is the CDS spread and $p(t)$ is the probability of survival until time t. Value of the protection leg is:

$$V_{prot} = \int_0^T e^{-rt} f(t) dt \qquad (18)$$

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where $f(t)$ is the probability density of default. The value of a CDS contract to the protection buyer is:

$$V_{CDS} = V_{prot} - V_{prem} = \int_0^T e^{-rt} f(t)\,dt - c \int_0^T e^{-rt} p(t)\,dt \qquad (19)$$

When the contract is signed at time zero, the CDS spread is determined so that the net value of the CDS is zero (break-even condition):

$$c = \frac{\int_0^T e^{-rt} f(t)\,dt}{\int_0^T e^{-rt} p(t)\,dt} \qquad (20)$$

The CDS spread is then locked and cannot be changed. That means that the CDS can gain nonzero value over time as the hazard rate changes. A change in the hazard rate would change the probability distribution function of default times, but leaves the CDS spread unchanged. Therefore, $f(t)$ and $p(t)$ change as initial intensity changes, but c stays the same. The partial derivative with respect to initial intensity is:

$$\frac{\partial V_{CDS}}{\partial \lambda} = \lim_{\Delta\lambda \to 0}$$

$$\{ [\int_0^T e^{-rt} f(\lambda + \Delta\lambda, t)\,dt - c \int_0^T e^{-rt} p(\lambda + \Delta\lambda, t)\,dt] - \qquad (21)$$

$$[\int_0^T e^{-rt} f(\lambda, t)\,dt - c \int_0^T e^{-rt} p(\lambda, t)\,dt] \} / \Delta\lambda$$

where $f(\lambda,t)$ and $p(\lambda,t)$ are functions of both time and the initial intensity.

2.3. Collateralized Debt Obligations

A collateralized debt obligation creates securities with different risk characteristics from a pool of debt instruments. Those securities are called

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tranches of the CDO. Cash flows from the pool of debt instruments are allocated to tranches in a prioritized way. There are typically three categories of tranches, senior, mezzanine, and equity tranches. Senior has priority over mezzanine, and mezzanine has priority over equity. That is the description of a cash CDO. There are also synthetic CDOs which use a pool of CDSs instead of a pool of bonds. Only cash CDOs are considered in this paper. Relative values of tranches depend on default dependencies of the collateral pool. For example, if all the bonds in the pool default together, all tranches have the same value. If defaults of different bonds are independent, the equity tranche becomes much riskier than the senior tranche.

In this paper, a portfolio of two zero-coupon bonds of identical maturity T is considered. The face value of each bond is $1. The two bonds are issued by two different firms. There are two tranches, a senior tranche and an equity tranche. The payoff structure of the tranches at maturity is assumed as the following:

- With no defaults, both tranches get $1 at maturity.
- With one default, the equity tranche gets zero and the senior gets $1.
- Both tranches get zero if both firms default.

The payoff structure makes the equity tranche responsible for the first 50% of loss and the senior tranche responsible for the second 50%. The following notation is used:

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$$p_i = P(\tau_i > T) \tag{22}$$

This means that p_i is the survival probability of firm i until time T which is the

maturity of the CDO. Let V_{eq} and V_{sn} be the values of the equity and senior

tranches at time zero. It is assumed that the default intensity of each firm follows

a CIR process according to Eq. (10), (11), and (12). The values of tranches at

time zero depend on the parameters of the model and the initial intensities of

firms:

$$V_{eq} = V_{eq}(\lambda_1, \lambda_2) \tag{23}$$

$$V_{sn} = V_{sn}(\lambda_1, \lambda_2) \tag{24}$$

where the fixed parameters are omitted since they are constant over time. One

can derive analytical solutions for prices of tranches when defaults are

independent:

$$V_{eq} = e^{-rT} p_1(\lambda_1) p_2(\lambda_2) \tag{25}$$

$$V_{sn} = e^{-rT}(1 - [1 - p_1(\lambda_1)][1 - p_2(\lambda_2)]) \tag{26}$$

where $p_1(\lambda_1)$ and $p_2(\lambda_2)$ are survival probabilities which depend on initial

intensities.

2.4. The Hedging Problem

The value of a hedge portfolio changes deterministically over time interval

Δt. That means that the hedge instruments should cancel the effects of sources

of uncertainty in the value of the hedged asset. Hedging is done for events with

probabilities that are $O(\Delta t)$. Events with probabilities that are $o(\Delta t)$ are not

considered. The reason is that as only the limit $\Delta t \to 0$ is considered for dynamic hedging. As time goes on, the hedge position should be adjusted dynamically to maintain the deterministic change in the hedged portfolio's value.

Eq. (23) and (24) are conditional on the survival to date of both firms. Given that condition, the two tranches should be hedged against fluctuations in both intensities. That means there are two sources of uncertainty. However, the defaults of the two firms can impact the values of the two tranches too. That means that there are four sources of uncertainty: changes in λ_1, changes in λ_2, default of firm 1, and the default of firm 2. This implies that one needs four hedge instruments to hedge each tranche. A possibility is using two CDSs with different maturities for each firm as explained below.

For hedging, two CDSs on each firm with maturities s and l ($s < l$) are used. s and l stand for short and long. That means that each tranche holder buys or sells CDSs on firms 1 and 2. The values of hedge portfolios for the equity and senior tranches are:

$$V_{eq}^{hedge} = V_{eq} + \alpha_{eq}^{1s} V_{CDS}^{1s} + \alpha_{eq}^{1l} V_{CDS}^{1l} + \alpha_{eq}^{2s} V_{CDS}^{2s} + \alpha_{eq}^{2l} V_{CDS}^{2l} \tag{27}$$

$$V_{sn}^{hedge} = V_{sn} + \alpha_{sn}^{1s} V_{CDS}^{1s} + \alpha_{sn}^{1l} V_{CDS}^{1l} + \alpha_{sn}^{2s} V_{CDS}^{2s} + \alpha_{sn}^{2l} V_{CDS}^{2l} \tag{28}$$

where α's are hedge ratios for different CDSs. For example, α_{eq}^{1s} is the number of CDSs with value V_{CDS}^{1s} and maturity s on firm 1 bought to hedge the equity tranche. The probability of both firms defaulting within Δt is $o(\Delta t)$ and not considered. If one firm defaults, the value of equity tranche drops to zero and the value of senior tranche becomes equal the price of the remaining bond. Hedging

against changes in intensities can be done by setting the partial derivatives equal to zero. Therefore, for hedging the equity tranche we should have:

$$V_{eq} = \alpha_{eq}^{1s} + \alpha_{eq}^{1l} \tag{29}$$

$$V_{eq} = \alpha_{eq}^{2s} + \alpha_{eq}^{2l} \tag{30}$$

$$-\frac{\partial V_{eq}}{\partial \lambda_1} = \alpha_{eq}^{1s}\frac{\partial V_{CDS}^{1s}}{\partial \lambda_1} + \alpha_{eq}^{1l}\frac{\partial V_{CDS}^{1l}}{\partial \lambda_1} \tag{31}$$

$$-\frac{\partial V_{eq}}{\partial \lambda_2} = \alpha_{eq}^{2s}\frac{\partial V_{CDS}^{2s}}{\partial \lambda_2} + \alpha_{eq}^{2l}\frac{\partial V_{CDS}^{2l}}{\partial \lambda_2} \tag{32}$$

For hedging the senior tranche we have:

$$V_{sn} - B_2 = \alpha_{sn}^{1s} + \alpha_{sn}^{1l} \tag{33}$$

$$V_{sn} - B_1 = \alpha_{sn}^{2s} + \alpha_{sn}^{2l} \tag{34}$$

$$-\frac{\partial V_{sn}}{\partial \lambda_1} = \alpha_{sn}^{1s}\frac{\partial V_{CDS}^{1s}}{\partial \lambda_1} + \alpha_{sn}^{1l}\frac{\partial V_{CDS}^{1l}}{\partial \lambda_1} \tag{35}$$

$$-\frac{\partial V_{sn}}{\partial \lambda_2} = \alpha_{sn}^{2s}\frac{\partial V_{CDS}^{2s}}{\partial \lambda_2} + \alpha_{sn}^{2l}\frac{\partial V_{CDS}^{2l}}{\partial \lambda_2} \tag{36}$$

Eq. (29) to (36) can be put in a compact form using matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \dfrac{\partial V_{CDS}^{1s}}{\partial \lambda_1} & \dfrac{\partial V_{CDS}^{1l}}{\partial \lambda_1} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \dfrac{\partial V_{CDS}^{2s}}{\partial \lambda_2} & \dfrac{\partial V_{CDS}^{2l}}{\partial \lambda_2} \end{bmatrix} \tag{37}$$

For equity tranche we get:

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$$
A\begin{bmatrix} \alpha_{eq}^{1s} \\ \alpha_{eq}^{1l} \\ \alpha_{eq}^{2s} \\ \alpha_{eq}^{2l} \end{bmatrix} = \begin{bmatrix} V_{eq} \\ -\dfrac{\partial V_{eq}}{\partial \lambda_1} \\ V_{eq} \\ -\dfrac{\partial V_{eq}}{\partial \lambda_2} \end{bmatrix}
\tag{38}
$$

And for senior tranche we get:

$$
A\begin{bmatrix} \alpha_{sn}^{1s} \\ \alpha_{sn}^{1l} \\ \alpha_{sn}^{2s} \\ \alpha_{sn}^{2l} \end{bmatrix} = \begin{bmatrix} V_{sn} - B_2 \\ -\dfrac{\partial V_{sn}}{\partial \lambda_1} \\ V_{sn} - B_1 \\ -\dfrac{\partial V_{sn}}{\partial \lambda_2} \end{bmatrix}
\tag{39}
$$

As long as matrix A is invertible, Eq. (38) and (39) can be solved and hedge ratios can be calculated. Matrix A is block-diagonal and is invertible if and only if the following equations hold:

$$
\frac{\partial V_{CDS}^{1s}}{\partial \lambda_1} \neq \frac{\partial V_{CDS}^{1l}}{\partial \lambda_1}
\tag{40}
$$

$$
\frac{\partial V_{CDS}^{2s}}{\partial \lambda_2} \neq \frac{\partial V_{CDS}^{2l}}{\partial \lambda_2}
\tag{41}
$$

Eq. (40) and (41) hold if the sensitivity of the value of a CDS to initial intensity depends on its maturity.

3. Simulation Methodology

This section explains the simulation and variance reduction methods used to estimate prices and sensitivities.

3.1. Estimating Prices

To analyze the case of correlated defaults, one has to resort to simulation since analytical solutions are often available only for the marginal distributions. In order to simulate the CIR process of Eq. (3), the following discrete-time approximation can be used:

$$\lambda(t + \Delta t) - \lambda(t) \approx \kappa(\theta - \lambda(t))\Delta t + \sigma\sqrt{\lambda(t)}\sqrt{\Delta t}\varepsilon(t) \tag{42}$$

where $\varepsilon(t)$ is drawn from a standard normal distribution. Negative outcomes for $\lambda(t + \Delta t)$ should be replaced with zero since hazard rate is non-negative. The draws from the standard normal are iid for different times.

It is usually easy to simulate a random variable X if the inverse of the cumulative distribution function $F^{-1}(x)$ is known. The inverse-transform method sets:

$$X = F^{-1}(U) \tag{43}$$

where U is a uniform random variable on [0,1]. In the case the CIR model used in this paper, the inverse of the CDF is not known analytically although one can use numerical methods like Newton's method to solve for time to default using

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the expression for the CDF. For the direct simulation of time to default, the following algorithm (count-down method) is used:

1. Simulate a uniform random variable $0 \le U \le 1$.

2. Generate a default intensity path up to time T.

3. Find τ' such that $\int_0^{\tau'} \lambda(s)\,ds = -\ln(U)$.

One has to show that the algorithm produces the true distribution of default time. That means showing τ (default time) and τ' have the same distribution. The following definition is used:

$$\gamma(t) = e^{-\int_0^t \lambda(s)\,ds} \tag{44}$$

Note that the distribution of τ can be found as:

$$P(\tau \ge T) = E_0[e^{-\int_0^T \lambda(s)\,ds}] = E[\gamma(T)] \tag{45}$$

Now let U be an independent uniform random variable on $[0,1]$ and denote by τ' the time at which $\gamma(\tau') = U$. The probability distribution of τ' is given by:

$$P(\tau' \ge T) = P(\gamma(T) \ge U) = E[1_{\gamma(T) \ge U}] = E[E[1_{\gamma(T) \ge U} \mid \gamma(T)]] = E[\gamma(T)] = P(\tau \ge T) \tag{46}$$

where the law of iterated expectation is used. The equation above shows that the distribution of τ' (time simulated in the algorithm) and τ have the same distribution.

The integration was done numerically using the trapezoidal method:

$$\int_0^{t_i} \lambda(s)\,ds \approx \sum_{j=1}^{t_i/\Delta t} \frac{1}{2}[\lambda(j\Delta t) + \lambda((j-1)\Delta t)]\Delta t \tag{47}$$

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where t_i is one of the points on the time grid:

$$0 = t_0 < t_1 < ... < t_n = T \tag{48}$$

The trapezoidal rule is based on the idea of approximating the function in each sub-interval by a straight line so that the shape of the area in the sub-interval is trapezoidal. As the number of sub-intervals used increases, the straight line will approximate the function more closely. τ' was found by minimizing

$| \int_0^{t_i} \lambda(s)\,ds + \ln(U) |$ over $t_i's$. The minimization works because $\int_0^t \lambda(s)\,ds$

is monotonically increasing in t. If the minimization returns $\tau = T$, it means that default has not occurred by time T.

In order to price the senior and equity tranches, two intensity paths should be simulated together according to Eq. (10), (11), and (12). Therefore, two correlated standard normal random variables ε_1 and ε_2 should be generated. The variance-covariance matrix of ε_1 and ε_2 is:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{49}$$

Cholesky decomposition of a positive-definite matrix can be used since Σ is positive-definite for $|\rho| < 1$. The matrix C that satisfies:

$$C^T C = \Sigma \tag{50}$$

is called the Cholesky decomposition of Σ. Now $C^T Z$ would have the desirable variance-covariance matrix where Z is a vector of independent standard normal variables:

$$E[(C^T Z)(C^T Z)^T] = E[C^T Z Z^T C] = C^T E[Z Z^T] C = C^T I C = \Sigma \tag{51}$$

Given the algorithms above one can simulate the two intensity paths for the two firms. The two uniform random variable U_1 and U_2 which are used to simulate τ_1 and τ_2 are independent because default times are independent conditional on the realization of intensities. Given the simulated default times, the discounted payoffs of senior and equity tranches can be calculated to price the tranches.

3.2. Estimating Sensitivities

Sensitivities have to be estimated in order to get the hedge ratios. Assume that one is interested in estimating the derivative of $\alpha(\lambda)$ where $\alpha(\lambda)$ is any function of the parameter λ. Also assume that:

$$\alpha(\lambda) = E[Y(\lambda)] \tag{52}$$

where the random variable $Y(\lambda)$ depends on the parameter λ. Central-difference method was used to get $\dfrac{d\alpha(\lambda)}{d\lambda}$:

$$\frac{d\alpha(\lambda)}{d\lambda} \approx \frac{\alpha(\lambda+h) - \alpha(\lambda-h)}{2h} \tag{53}$$

Variance of $Y(\lambda+h) - Y(\lambda-h)$ depends on how the simulation is done. Pathwise derivative estimates were used to estimate sensitivities. This method estimates derivatives directly. It uses common random numbers to simulate $Y(\lambda+h)$ and $Y(\lambda-h)$. That method is legitimate as long as:

$$E[\frac{dY(\lambda)}{d\lambda}] = \frac{d}{d\lambda} E[Y(\lambda)] \tag{54}$$

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Eq. (54) is equivalent to saying that the interchange of differentiation and expectation is justified:

$$\frac{d\alpha(\lambda)}{d\lambda} \approx \frac{\alpha(\lambda+h)-\alpha(\lambda-h)}{2h} = E[\frac{Y(\lambda+h)-Y(\lambda-h)}{2h}] \tag{55}$$

In most cases, variance of $[Y(\lambda+h)-Y(\lambda-h)]/(2h)$ blows up as $h \to 0$. On the other hand, the estimator becomes unbiased as $h \to 0$. Therefore, one has to worry about the trade-off between bias and variance when deciding what h to use.

3.3. Variance Reduction

Usually a large number of simulations are required to achieve the desirable accuracy. That is very expensive in terms of computation time. There are several variance reduction techniques available which are studied in Glasserman (2004). Their usefulness depends on the problem under consideration. Two variance reduction techniques were employed in the simulation: antithetic variates and control variates. The use of control variate was possible since there is an analytical solution for the price of a bond in the CIR model.

With the antithetic variable technique, two default times are simulated with each intensity path using U and $1-U$. It is based on the observation that if U is uniformly distributed over $[0,1]$, then $1-U$ is too. The price simulations from U and $1-U$ are then averaged. This method attempts to reduce variance by introducing negative dependence between pairs of replications. It also saves time

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in this model since only one intensity path is required to generate two default times.

The control variate technique exploits information about the errors in estimates of known quantities to reduce the error in an estimate of an known quantity. Assume you want to estimate $E[Y]$. Suppose on each replication of Y another output X is calculated along. Assume that $E[X]$ is known. Then for any fixed b we calculate:

$$Y(b) = Y - b(X - E[X])$$ (56)

Now it is clear that:

$$E[Y(b)] = E[Y] - b(E[X] - E[X]) = E[Y]$$ (57)

$$Var[Y(b)] = \sigma_Y^2 - 2b\sigma_X\sigma_Y\rho_{XY} + b^2\sigma_X^2$$ (58)

The optimal coefficient b^* minimizes the variance:

$$b^* = \frac{\sigma_Y}{\sigma_X}\rho_{XY} = \frac{Cov[X,Y]}{Var[X]}$$ (59)

The covariance and variance in Eq. (59) are not typically known, but they can be estimated using the simulations. The control variate in this model is based on the analytical solution for the price of a bond which is the expected value of discounted payoffs. Therefore, for estimating derivatives, the derivative of the price of a bond can be used as the control variate. For example, to estimate $\partial V_{eq} / \partial \lambda_1$, $\partial B_1 / \partial \lambda_1$ is used as the control variate.

4. Results and Discussion

Simulations were done $N = 10^5$ times. The time step chosen was $\Delta t = 0.01$. The unit of time is one year. It should be noted that there might be some bias because of time discretization. On the other hand, computation time increases as Δt is decreased. Δt should be small enough so that the discretization bias is much smaller than the Monte Carlo noise. Analytical solution is available for the zero correlation case. The tranche prices calculated by Monte Carlo can be compared with the analytical result to see if the bias is within the accuracy of the Monte Carlo result. That was the case for $\Delta t = 0.01$ and $N = 10^5$ which suggested that $\Delta t = 0.01$ is small enough. MATLAB code is attached in the appendix.

The base case of $\theta = 0.03$ (low-quality investment grade), $\kappa = 0.25$, $\lambda_1 = \lambda_2 = 0.03$, $r = 0.05$, and $\sigma = 0.17$ was considered. σ was chosen so that volatility of $\sigma / \sqrt{\theta} = 100\%$ is implied. $\dfrac{\partial V_{CDS}}{\partial \lambda}$ in Eq. (21) can be calculated with numerical integration and central-difference method for taking the derivative. The reason is that there are analytical expressions for $f(\lambda,t)$ and $p(\lambda,t)$. The CDSs used have maturities $s = 2$ and $l = 5$. The result of numerical integration and differentiation for $\lambda_1 = \lambda_2 = 0.03$ was:

$$\frac{\partial V_{CDS}^{1s}}{\partial \lambda_1} = \frac{\partial V_{CDS}^{2s}}{\partial \lambda_2} = 1.4439 \tag{60}$$

$$\frac{\partial V_{CDS}^{1l}}{\partial \lambda_1} = \frac{\partial V_{CDS}^{2l}}{\partial \lambda_2} = 2.3227 \tag{61}$$

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Eq. (60) and (61) imply that matrix A in Eq. (37) is invertible. That means that the hedging problem has a unique solution. A realization of intensity paths for the base case with $\rho = 0.8$ is shown in Figure 1. Prices of tranches and their partial derivatives with respect to initial intensities ($\lambda_1 = \lambda_2 = 0.03$) for the base case with $\rho = 0.8$ are shown in Table 1. SD is the estimate of the standard deviation of the sample mean. The analytical results for the same parameters but zero correlation are shown in Table 2. Hedge ratios for the base-case parameters and $\rho = 0.8$ are shown in Table 3. The analytical results for the same parameters but zero correlation are shown in Table 4.
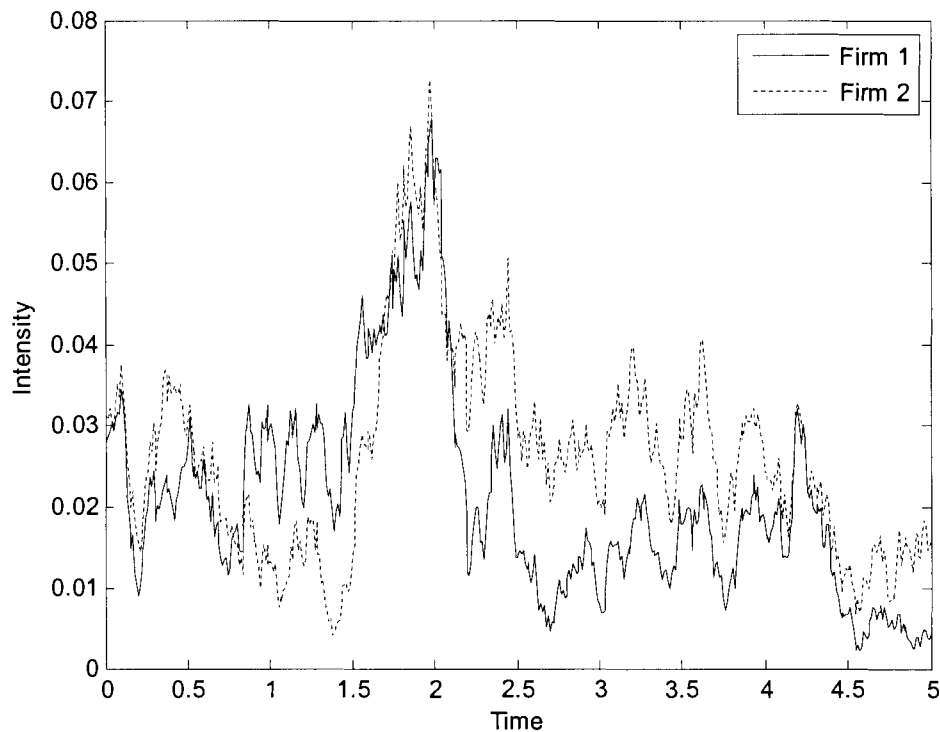

Figure 1 - Realization of intensity paths for the Base Case with $\rho = 0.8$

Table 1 - Prices of tranches and their derivatives for the Base Case with $\rho = 0.8$

	T=1		T=3		T=5		T=10	
	Estimate	SD	Estimate	SD	Estimate	SD	Estimate	SD
	Equity		Equity		Equity		Equity	
Price	0.8963	7.01E-04	0.723	9.98E-04	0.588	0.0011	0.3593	9.43E-04
Derivative	-0.7861	0.0025	-1.4504	0.0056	-1.5111	0.007	-1.0835	0.0066
	Senior		Senior		Senior		Senior	
Price	0.9504	8.97E-05	0.8523	2.68E-04	0.7586	3.91E-04	0.5603	5.09E-04
Derivative	-0.0273	0.0025	-0.1656	0.0056	-0.3004	0.007	-0.4082	0.0066

Table 2 - Prices of bonds, tranches, and their derivatives for the Base Case with $\rho = 0$

	T=1	T=3	T=5	T=10
	Equity	Equity	Equity	Equity
Price	0.896	0.7221	0.5854	0.3515
Derivative	-0.7895	-1.4801	-1.5703	-1.1353
	Senior	Senior	Senior	Senior
Price	0.9504	0.8546	0.765	0.572
Derivative	-0.0239	-0.1358	-0.241	-0.3561
	Bond	Bond	Bond	Bond
Price	0.9232	0.7884	0.6752	0.4617

Table 3 - Hedge Ratios for the Base Case with $\rho = 0.8$

	T=1		T=3		T=5		T=10	
CDS	Estimate	SD	Estimate	SD	Estimate	SD	Estimate	SD
	Equity		Equity		Equity		Equity	
1s	1.4744	0.0034	0.2605	0.0069	-0.1654	0.0085	-0.2833	0.0079
1l	-0.5781	0.0031	0.4625	0.0066	0.7534	0.0082	0.6426	0.0077
2s	1.4744	0.0034	0.2605	0.0069	-0.1654	0.0085	-0.2833	0.0079
2l	-0.5781	0.0031	0.4625	0.0066	0.7534	0.0082	0.6426	0.0077
	Senior		Senior		Senior		Senior	
1s	0.0408	0.0029	-0.0195	0.0064	-0.1214	0.008	-0.2039	0.0076
1l	-0.0136	0.0028	0.0834	0.0064	0.2048	0.008	0.3025	0.0076
2s	0.0408	0.0029	-0.0195	0.0064	-0.1214	0.008	-0.2039	0.0076
2l	-0.0136	0.0028	0.0834	0.0064	0.2048	0.008	0.3025	0.0076

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Table 4 - Hedge Ratios for the base case with $\rho = 0$

	T=1	T=3	T=5	T=10
CDS	Equity	Equity	Equity	Equity
1s	1.4699	0.2243	-0.2397	-0.363
1l	-0.5739	0.4978	0.8251	0.7144
2s	1.4699	0.2243	-0.2397	-0.363
2l	-0.5739	0.4978	0.8251	0.7144
CDS	Senior	Senior	Senior	Senior
1s	0.0446	0.0206	-0.0368	-0.1138
1l	-0.0174	0.0457	0.1266	0.2241
2s	0.0446	0.0206	-0.0368	-0.1138
2l	-0.0174	0.0457	0.1266	0.2241

To estimate derivatives, one has to choose a value for h in Eq. (55). Prices of senior and equity tranches are plotted in Figure 2 as a function of intensity of firm 2 for the base case parameters, zero correlation and $T = 3$. That can be done analytically since zero correlation is assumed. It is clear from Figure 2 that the price curves are very linear in the relevant range. That implies that choosing a large value for h is a safe choice. $h = 0.01$ was used for simulations.

Comparing the hedge ratios in Tables 3 and 4 reveals that hedge ratios are close for $\rho = 0.8$ and $\rho = 0$ when maturity T is short. As T increases, the difference between the hedge ratios becomes larger. It is also important to look at the estimated binomial measure of default correlation for the two bonds. The results are shown in Table 5. Those results show that default correlation increases as T increases. The default correlations in Table 5 are rather small. To

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test the sensitivity of that result to the parameters of the model, the default

correlation was estimated for different level of volatility and also for different

mean reversion rates. The results are shown in Table 6. The estimates in Table 6

show that it is possible to achieve higher default correlation with higher volatility

and lower mean reversion rate. Therefore, this model is capable of producing

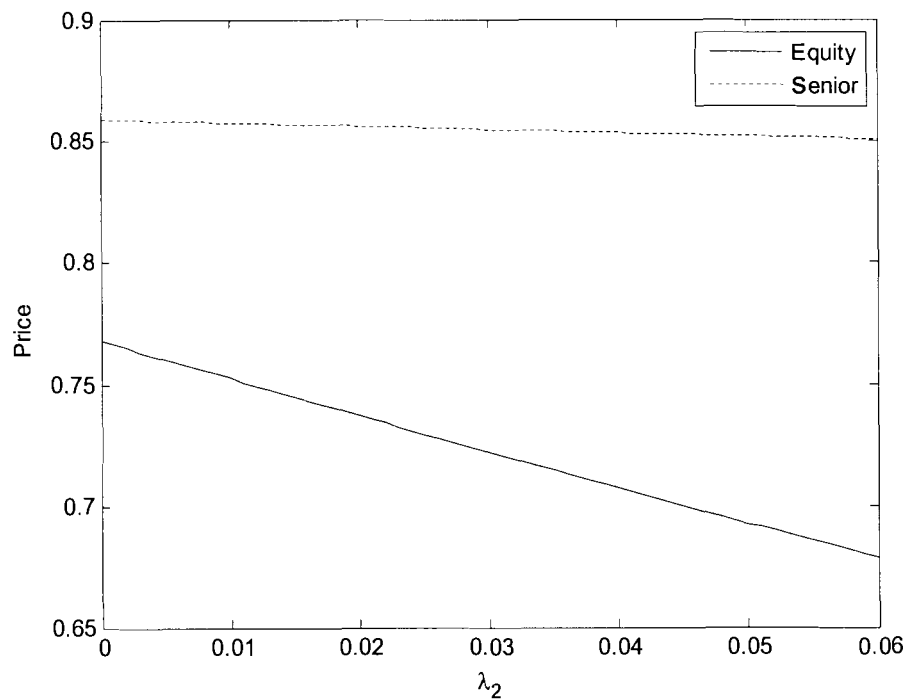reasonable default correlation for long maturities.

Figure 2 - Prices of tranches

Prices of tranches were calculated analytically for $\rho = 0$, $\lambda_1 = 0.03$, $T = 3$ and different values of λ_2. Other parameters were the base case parameters.

Table 5 - Estimates of default correlation for different maturities

Binomial measure of default correlation was estimated for the base case with $\rho = 0.8$ for different maturities.

T=1	T=3	T=5	T=10
0.0115	0.0336	0.0643	0.0961

Table 6 - Estimates of default correlation

Binomial measure of default correlation was estimated for $T = 10$ with different combinations of σ and κ. Other parameters are the base case parameters.

$\sigma = 0.17$			$\kappa = 0.25$		
$\kappa = 0.15$	$\kappa = 0.25$	$\kappa = 1.25$	$\sigma = 0.085$	$\sigma = 0.17$	$\sigma = 0.34$
0.139	0.0961	0.0102	0.0262	0.0961	0.1712

5. Conclusion

In this paper hedging tranches of a simple cash CDO was considered. The collateral consisted of two bonds. A square-root diffusion model for default intensities was assumed. The diffusion of spreads of different firms can be correlated and intensities of different firms can be different. The model avoids the discretization of Markov chain models and the static nature of copula models. It was shown that one can fully hedge the tranches using CDSs with different maturities. Hedge ratios for hedging against both the spread risk and default risk using CDSs were calculated using Monte Carlo simulations. Variance reduction techniques were employed to improve the efficiency of estimating sensitivities. Default correlations and hedge ratios were estimated for different parameters of the model. It was shown that the model is capable of producing reasonable default correlation for long maturity, high volatility, and low mean reversion rate. Even though the model has two bonds, it can be extended to include more bonds and also default contagion.

Appendix

MATLAB Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Pricing the equity and senior tranches using Monte Carlo.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

t0=clock;

T = 10;
N = 100000;
M = T*100;
dt = 1 / 100;

lambda1_0 = 0.03;
lambda2_0 = 0.03;

theta = 0.03;
kappa = 0.25;
r = 0.05;
rho_cop = 0;
rho_diff = 0.8;
sigma = 0.17;

C_cop = chol([1,rho_cop;rho_cop,1]);
C_diff = chol([1,rho_diff;rho_diff,1]);

lambda1 = zeros(M+1,1);
lambda2 = zeros(M+1,1);

area1 = zeros(M+1,1);
area2 = zeros(M+1,1);

senior = zeros(N,1);
equity = zeros(N,1);

default1 = zeros(N,1);
default2 = zeros(N,1);

lambda1(1)=lambda1_0;
lambda2(1)=lambda2_0;

for i=1:N
    Z = C_cop' * randn(2,1);
    P1 = -log(normcdf(Z(1)));
    P2 = -log(normcdf(Z(2)));

    for j=1:M
        W = C_diff' * randn(2,1);
```

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```
        lambda1(j+1) = lambda1(j) + kappa * (theta - lambda1(j)) * dt +
sigma * sqrt(lambda1(j)) * W(1) * sqrt(dt);
        lambda2(j+1) = lambda2(j) + kappa * (theta - lambda2(j)) * dt +
sigma * sqrt(lambda2(j)) * W(2) * sqrt(dt);

        if lambda1(j+1) < 0
            lambda1(j+1) = 0;
        end

        if lambda2(j+1) < 0
            lambda2(j+1) = 0;
        end

        area1(j+1) = area1(j) + dt * (lambda1(j+1) + lambda1(j))/2;
        area2(j+1) = area2(j) + dt * (lambda2(j+1) + lambda2(j))/2;
    end

    [a1,b1]=min(abs(P1-area1));
    [a2,b2]=min(abs(P2-area2));

    tau1 = (b1 - 1) * dt;
    tau2 = (b2 - 1) * dt;

    if tau1 < T
        default1(i) = 1;
    else
        default1(i) = 0;
    end

    if tau2 < T
        default2(i) = 1;
    else
        default2(i) = 0;
    end

    if (tau1 == T) & (tau2 == T)
        senior(i) = exp(-r*T);
        equity(i) = exp(-r*T);
    elseif (tau1 < T) & (tau2 < T)
        senior(i) = 0;
        equity(i) = 0;
    else
        senior(i) = exp(-r*T);
        equity(i) = 0;
    end
end

senior_price = mean(senior)
stdev_senior = std(senior)/sqrt(N)

equity_price = mean(equity)
stdev_equity = std(equity)/sqrt(N)
```

```
cov_default = cov(default1,default2);
default_corr = cov_default(1,2) / sqrt(cov_default(1,1) *
cov_default(2,2))


l2=etime(clock,t0)/60



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Calculates the partial derivatives of values of senior and equity
% tranches with respect to initial intensity of firm 1 using Monte
% Carlo. Partial derivatives with respect to initial intensity of
% firm 2 can be calculated similarly.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

t0=clock;


T = 10;
N = 100000;
M = T*100;
dt = 1 / 100;


lambda1_0 = 0.03;
lambda2_0 = 0.03;


theta = 0.03;
kappa = 0.25;
r = 0.05;
rho_cop = 0;
rho_diff = 0.8;
sigma = 0.17;


h = 0.02;


C_cop = chol([1,rho_cop;rho_cop,1]);
C_diff = chol([1,rho_diff;rho_diff,1]);


lambda1_up = zeros(M+1,1);
lambda1_down = zeros(M+1,1);


lambda2 = zeros(M+1,1);


area1_up = zeros(M+1,1);
area1_down = zeros(M+1,1);


area2 = zeros(M+1,1);


senior_up = zeros(N,1);
senior_down = zeros(N,1);


anti_senior_up = zeros(N,1);
anti_senior_down = zeros(N,1);


equity_up = zeros(N,1);
equity_down = zeros(N,1);
```

```matlab
anti_equity_up = zeros(N,1);
anti_equity_down = zeros(N,1);

bond1_up = zeros(N,1);
bond1_down = zeros(N,1);

anti_bond1_up = zeros(N,1);
anti_bond1_down = zeros(N,1);

lambda1_up(1)=lambda1_0 + h/2;
lambda1_down(1)=lambda1_0 - h/2;

lambda2(1)=lambda2_0;

for i=1:N
    Z = C_cop' * randn(2,1);

    P1 = -log(normcdf(Z(1)));
    P2 = -log(normcdf(Z(2)));

    anti_P1 = -log(normcdf(-Z(1)));
    anti_P2 = -log(normcdf(-Z(2)));

    for j=1:M
        W = C_diff' * randn(2,1);

        lambda1_up(j+1) = lambda1_up(j) + kappa * (theta -
lambda1_up(j)) * dt + sigma * sqrt(lambda1_up(j)) * W(1) * sqrt(dt);
        lambda1_down(j+1) = lambda1_down(j) + kappa * (theta -
lambda1_down(j)) * dt + sigma * sqrt(lambda1_down(j)) * W(1) * sqrt(dt);

        if lambda1_up(j+1) < 0
            lambda1_up(j+1) = 0;
        end

        if lambda1_down(j+1) < 0
            lambda1_down(j+1) = 0;
        end

        area1_up(j+1) = area1_up(j) + dt * (lambda1_up(j+1) +
lambda1_up(j))/2;
        area1_down(j+1) = area1_down(j) + dt * (lambda1_down(j+1) +
lambda1_down(j))/2;

        lambda2(j+1) = lambda2(j) + kappa * (theta - lambda2(j)) * dt +
sigma * sqrt(lambda2(j)) * W(2) * sqrt(dt);

        if lambda2(j+1) < 0
            lambda2(j+1) = 0;
        end

        area2(j+1) = area2(j) + dt * (lambda2(j+1) + lambda2(j))/2;
    end
```

```
[a1_up,b1_up]=min(abs(P1-area1_up));
[a1_down,b1_down]=min(abs(P1-area1_down));

[anti_a1_up,anti_b1_up]=min(abs(anti_P1-area1_up));
[anti_a1_down,anti_b1_down]=min(abs(anti_P1-area1_down));

[a2,b2]=min(abs(P2-area2));
[anti_a2,anti_b2]=min(abs(anti_P2-area2));

tau1_up = (b1_up - 1) * dt;
tau1_down = (b1_down - 1) * dt;

anti_tau1_up = (anti_b1_up - 1) * dt;
anti_tau1_down = (anti_b1_down - 1) * dt;

tau2 = (b2 - 1) * dt;
anti_tau2 = (anti_b2 - 1) * dt;

if tau1_up < T
    bond1_up(i) = 0;
else
    bond1_up(i) = exp(-r*T);
end

if tau1_down < T
    bond1_down(i) = 0;
else
    bond1_down(i) = exp(-r*T);
end

if anti_tau1_up < T
    anti_bond1_up(i) = 0;
else
    anti_bond1_up(i) = exp(-r*T);
end

if anti_tau1_down < T
    anti_bond1_down(i) = 0;
else
    anti_bond1_down(i) = exp(-r*T);
end

if (tau1_up == T) & (tau2 == T)
    senior_up(i) = exp(-r*T);
    equity_up(i) = exp(-r*T);
elseif (tau1_up < T) & (tau2 < T)
    senior_up(i) = 0;
    equity_up(i) = 0;
else
    senior_up(i) = exp(-r*T);
    equity_up(i) = 0;
end
```

```matlab
        if (tau1_down == T) & (tau2 == T)
            senior_down(i) = exp(-r*T);
            equity_down(i) = exp(-r*T);
        elseif (tau1_down < T) & (tau2 < T)
            senior_down(i) = 0;
            equity_down(i) = 0;
        else
            senior_down(i) = exp(-r*T);
            equity_down(i) = 0;
        end

        if (anti_tau1_up == T) & (anti_tau2 == T)
            anti_senior_up(i) = exp(-r*T);
            anti_equity_up(i) = exp(-r*T);
        elseif (anti_tau1_up < T) & (anti_tau2 < T)
            anti_senior_up(i) = 0;
            anti_equity_up(i) = 0;
        else
            anti_senior_up(i) = exp(-r*T);
            anti_equity_up(i) = 0;
        end

        if (anti_tau1_down == T) & (anti_tau2 == T)
            anti_senior_down(i) = exp(-r*T);
            anti_equity_down(i) = exp(-r*T);
        elseif (anti_tau1_down < T) & (anti_tau2 < T)
            anti_senior_down(i) = 0;
            anti_equity_down(i) = 0;
        else
            anti_senior_down(i) = exp(-r*T);
            anti_equity_down(i) = 0;
        end
end

f_bond1_up = (bond1_up + anti_bond1_up)/2;
f_bond1_down = (bond1_down + anti_bond1_down)/2;

f_senior_up = (senior_up + anti_senior_up)/2;
f_senior_down = (senior_down + anti_senior_down)/2;

f_equity_up = (equity_up + anti_equity_up)/2;
f_equity_down = (equity_down + anti_equity_down)/2;

gamma=sqrt(kappa^2 + 2*sigma^2);
C = @(t) (kappa+gamma) * (exp(gamma*t)-1) + 2*gamma;
A = @(t) 2 * gamma * exp((kappa+gamma)*t/2) / C(t);
B = @(t) 2*(exp(gamma*t)-1) / C(t);
psurvival = @(t,y) A(t)^(2*kappa*theta/(sigma^2)) * exp(-B(t)*y);

cont_var = (f_bond1_up - f_bond1_down) - exp(-
r*T)*(psurvival(T,lambda1_0 + h/2) - psurvival(T,lambda1_0 - h/2));

cov_senior = cov(cont_var, f_senior_up - f_senior_down);
b_senior = cov_senior(1,2) / cov_senior(1,1);
```

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```
cov_equity = cov(cont_var, f_equity_up - f_equity_down);
b_equity = cov_equity(1,2) / cov_equity(1,1);


senior_deriv1_cont = f_senior_up - f_senior_down - b_senior * cont_var;
equity_deriv1_cont = f_equity_up - f_equity_down - b_equity * cont_var;


senior_derivative1 = (1/h) * mean(senior_deriv1_cont)
stdev_senior_derivative1 = std(senior_deriv1_cont)/(sqrt(N)*h)


equity_derivative1 = (1/h) * mean(equity_deriv1_cont)
stdev_equity_derivative1 = std(equity_deriv1_cont)/(sqrt(N)*h)


l2=etime(clock,t0)/60



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Calculates the CDS premium and derivative with numerical
% integration and differentiation.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


T = 5;
M = T*100;
dt = 1 / 100;


lambda_0 = 0.03;


theta = 0.03;
kappa = 0.25;
r = 0.05;
sigma = 0.17;


h = 0.00001;


f1 = zeros(M+1,1);
area1 = zeros(M+1,1);


f2 = zeros(M+1,1);
area2 = zeros(M+1,1);


f1_up = zeros(M+1,1);
area1_up = zeros(M+1,1);


f2_up = zeros(M+1,1);
area2_up = zeros(M+1,1);


f1_down = zeros(M+1,1);
area1_down = zeros(M+1,1);


f2_down = zeros(M+1,1);
area2_down = zeros(M+1,1);


gamma=sqrt(kappa^2 + 2*sigma^2);
```

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```
C = @(t) (kappa+gamma) * (exp(gamma*t)-1) + 2*gamma;
A = @(t) 2 * gamma * exp((kappa+gamma)*t/2) / C(t);
B = @(t) 2*(exp(gamma*t)-1) / C(t);
psurvival = @(t,y) A(t)^(2*kappa*theta/(sigma^2)) * exp(-B(t)*y);
partial_A = @(t) (gamma*(kappa+gamma)*exp((kappa+gamma)*t/2) * C(t) -
gamma*(kappa+gamma)*exp(gamma*t)*2*gamma*exp((kappa+gamma)*t/2))/
C(t)^2;
partial_B = @(t) (2*gamma*exp(gamma*t)*C(t) -
2*gamma*(gamma+kappa)*exp(gamma*t)*(exp(gamma*t)-1)) / C(t)^2;
pdensity = @(t,y) -(2*kappa*theta/(sigma^2)) *
A(t)^(2*kappa*theta/(sigma^2) - 1) * partial_A(t) * exp(-B(t)*y) +
A(t)^(2*kappa*theta/(sigma^2)) * y * partial_B(t) * exp(-B(t)*y);


func1 = @(t,y) exp(-r*t)*psurvival(t,y);
func2 = @(t,y) exp(-r*t)*pdensity(t,y);


f1(1) = func1(0,lambda_0);
f2(1) = func2(0,lambda_0);


f1_up(1) = func1(0,lambda_0 + h/2);
f2_up(1) = func2(0,lambda_0 + h/2);


f1_down(1) = func1(0,lambda_0 - h/2);
f2_down(1) = func2(0,lambda_0 - h/2);


for j=1:M
    f1(j+1) = func1(j*dt,lambda_0);
    area1(j+1) = area1(j) + dt * (f1(j+1) + f1(j))/2;

    f2(j+1) = func2(j*dt,lambda_0);
    area2(j+1) = area2(j) + dt * (f2(j+1) + f2(j))/2;

    f1_up(j+1) = func1(j*dt,lambda_0 + h/2);
    area1_up(j+1) = area1_up(j) + dt * (f1_up(j+1) + f1_up(j))/2;

    f2_up(j+1) = func2(j*dt,lambda_0 + h/2);
    area2_up(j+1) = area2_up(j) + dt * (f2_up(j+1) + f2_up(j))/2;

    f1_down(j+1) = func1(j*dt,lambda_0 - h/2);
    area1_down(j+1) = area1_down(j) + dt * (f1_down(j+1) +
f1_down(j))/2;

    f2_down(j+1) = func2(j*dt,lambda_0 - h/2);
    area2_down(j+1) = area2_down(j) + dt * (f2_down(j+1) +
f2_down(j))/2;
end

c = area2(M+1) / area1(M+1)

((area2_up(M+1) - c * area1_up(M+1))-(area2_down(M+1) - c *
area1_down(M+1)))/h
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Calculates the hedge ratios and prices numerically for the
% zero correlation case.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


T = 10;


lambda1_0 = 0.03;
lambda2_0 = 0.03;


theta = 0.03;
kappa = 0.25;
r = 0.05;
sigma = 0.17;


h = 0.00001;


CDS_1s = 1.4439;
CDS_1l = 2.3227;


CDS_2s = 1.4439;
CDS_2l = 2.3227;



gamma=sqrt(kappa^2 + 2*sigma^2);
C = @(t) (kappa+gamma) * (exp(gamma*t)-1) + 2*gamma;
A = @(t) 2 * gamma * exp((kappa+gamma)*t/2) / C(t);
B = @(t) 2*(exp(gamma*t)-1) / C(t);
psurvival = @(t,y) A(t)^(2*kappa*theta/(sigma^2)) * exp(-B(t)*y);



EQ_p =@(t,y1,y2) exp(-r*t)*psurvival(t,y1)*psurvival(t,y2);
SN_p =@(t,y1,y2) exp(-r*t)*(1-(1-psurvival(t,y1))*(1-psurvival(t,y2)));


SN = SN_p(T,lambda1_0,lambda2_0)
EQ = EQ_p(T,lambda1_0,lambda2_0)

SN1 = (SN_p(T,lambda1_0 + h/2,lambda2_0) - SN_p(T,lambda1_0 -
h/2,lambda2_0))/h
EQ1 = (EQ_p(T,lambda1_0 + h/2,lambda2_0) - EQ_p(T,lambda1_0 -
h/2,lambda2_0))/h

SN2 = (SN_p(T,lambda1_0,lambda2_0 + h/2) - SN_p(T,lambda1_0,lambda2_0 -
h/2))/h
EQ2 = (EQ_p(T,lambda1_0,lambda2_0 + h/2) - EQ_p(T,lambda1_0,lambda2_0 -
h/2))/h



B1 = exp(-r*T)*psurvival(T,lambda1_0);
B2 = exp(-r*T)*psurvival(T,lambda2_0);



A = [1 1 0 0;CDS_1s CDS_1l 0 0;0 0 1 1;0 0 CDS_2s CDS_2l];
```

```
inv_A = inv(A);

B_EQ = [EQ;-EQ1;EQ;-EQ2];
Hedge_EQ = inv_A * B_EQ

B_SN = [SN-B2;-SN1;SN-B1;-SN2];
Hedge_SN = inv_A * B_SN


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Calculates the hedge ratios and their standard errors.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

T = 10;

lambda1_0 = 0.03;
lambda2_0 = 0.03;

theta = 0.03;
kappa = 0.25;
r = 0.05;
rho_cop = 0;
rho_diff = 0.8;
sigma = 0.17;


CDS_1s = 1.4439;
CDS_1l = 2.3227;

CDS_2s = 1.4439;
CDS_2l = 2.3227;


SN = 0.5603;
ERR_SN = 5.0873e-004;

EQ = 0.3593;
ERR_EQ = 9.4252e-004;


SN1 = -0.4082;
ERR_SN1 = 0.0066;

EQ1 = -1.0835;
ERR_EQ1 = 0.0066;


SN2 = -0.4082;
ERR_SN2 = 0.0066;

EQ2 = -1.0835;
ERR_EQ2 = 0.0066;
```

```
gamma=sqrt(kappa^2 + 2*sigma^2);
C = @(t) (kappa+gamma) * (exp(gamma*t)-1) + 2*gamma;
A = @(t) 2 * gamma * exp((kappa+gamma)*t/2) / C(t);
B = @(t) 2*(exp(gamma*t)-1) / C(t);
psurvival = @(t,y) A(t)^(2*kappa*theta/(sigma^2)) * exp(-B(t)*y);


B1 = exp(-r*T)*psurvival(T,lambda1_0);
B2 = exp(-r*T)*psurvival(T,lambda2_0);


A = [1 1 0 0;CDS_1s CDS_1l 0 0;0 0 1 1;0 0 CDS_2s CDS_2l];
inv_A = inv(A);


B_EQ = [EQ;-EQ1;EQ;-EQ2];
ERR_B_EQ = [ERR_EQ;ERR_EQ1;ERR_EQ;ERR_EQ2];


Hedge_EQ = inv_A * B_EQ
VAR_Hedge_EQ = (inv_A.^2)*(ERR_B_EQ.^2);
ERR_Hedge_EQ = VAR_Hedge_EQ.^(1/2)


B_SN = [SN-B2;-SN1;SN-B1;-SN2];
ERR_B_SN = [ERR_SN;ERR_SN1;ERR_SN;ERR_SN2];


Hedge_SN = inv_A * B_SN
VAR_Hedge_SN = (inv_A.^2)*(ERR_B_SN.^2);
ERR_Hedge_SN = VAR_Hedge_SN.^(1/2)
```

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