Classification of Walks in Wedges

by David Fernand Laferrière

B.Sc., University of Victoria, 2005

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department
of
Mathematics

SIMON FRASER UNIVERSITY
2007

APPROVAL

Name: David Fernand Laferrière

Degree: Master of Science

Title of Thesis: Classification of Walks in Wedges

Examining Committee: Dr. Jason Bell
Chair

____

Dr. Marni Mishna
Senior Supervisor

____

Dr. Petr Lisonek
Supervisory Committee

____

Dr. Andrew Rechnitzer
External Examiner

Date of Defense: July 25, 2007

Abstract

Planar lattice walks are combinatorial objects which arise in statistical mechanics in both the modeling of polymers and percolation theory. It has been shown previously that lattice walks restricted to a half-plane have algebraic generating functions. Much work has been done to classify the generating functions of walks restricted to the first quadrant quarter-plane as algebraic, D-finite, or non-D-finite. We consider walks restricted to two regions: an eighth-plane wedge and a three-quarter-plane region. We find combinatorial criteria to define families of walks with algebraic generating functions in those regions, as well as an isomorphism that maps nearly one fourth of the walks in the eighth-plane to walks in the quarter-plane. Further, we find evidence of a family of walks whose generating functions are non-D-finite in any wedge smaller than a half-plane.

Keywords: Dyck paths; formal power series; generating functions; planar lattice paths; statistical mechanics

Subject Terms: Combinatorial Enumeration Problems; Generating Functions; Statistical Mechanics

Dedication

For my grandfather, William.

Acknowledgments

Dr. Marni Mishna encouraged me and found me a challenging problem, and she helped me keep my eyes on the prize. SFU and NSERC generously supported me, which allowed me to finish on time. Dr. Petr Lisonek's very thorough consideration of my thesis allowed me to make it polished and consistent. Dr. Andrew Rechnitzer helped me understand the statistical mechanics involved in this research.

Erin has been unwavering in her patience and understanding, and she was correct: it was worth all the trouble.

My parents each supported me in their own way, and for that I am grateful.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Planar lattice walks

Many problems in statistical mechanics can be modeled using combinatorial objects. An example of such an object is the random walk. In this thesis, we examine one form of random walk: the *planar lattice path* with next nearest neighbour steps.

A planar lattice path (also known as a *planar lattice walk*) of length n is a succession of n steps from one point in $\mathbb{Z} \times \mathbb{Z}$ to another, where each step is from a fixed set (called a *step set*) $\mathcal{Y} \subseteq \mathbb{Z} \times \mathbb{Z}$. Unless otherwise stated, a lattice walk starts at the origin.

We denote the eight steps $(0,1)$, $(1,1)$, $(1,0)$, $(1,-1)$, $(0,-1)$, $(-1,-1)$, $(-1,0)$, and $(-1,1)$ by N, NE, E, SE, S, SW, W, and NW, respectively. We are interested in walk sets given by subsets $\mathcal{Y}$ of $\{\mathsf{N}, \mathsf{NE}, \ldots, \mathsf{W}, \mathsf{NW}\}$. Since the above steps never change the x- and y- coordinate by more than one, we call them *unit steps* or *next nearest neigbour steps.* Figure 1.1 shows an example of a path generated by the step set $\mathcal{Y} = \{\mathsf{N}, \mathsf{E}, \mathsf{S}, \mathsf{W}\}$.

The length generating functions of walks in the half-plane $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} | i \geq 0\}$,

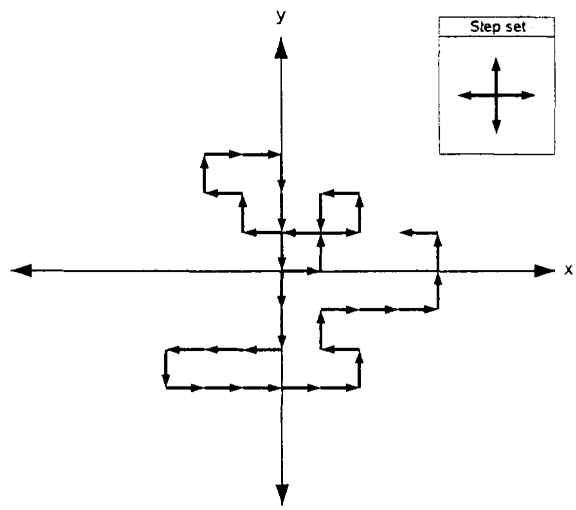


Figure 1.1: Planar lattice path of length 36 generated by $\mathcal{Y} = \{\mathsf{N}, \mathsf{E}, \mathsf{S}, \mathsf{W}\}$.

which we denote by $\mathcal{R}_{\pi}$, have been characterized by Banderier and Flajolet [1]. The generating functions of walks in the first quadrant $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} | i \geq 0, j \geq 0\}$, which we denote by $\mathcal{R}_{\pi/2}$, have been the subject of much study (see [4],[5],[6],[20],[21]). The additional constraint on walks in $\mathcal{R}_{\pi/2}$ causes the behaviour of generating functions of walks in $\mathcal{R}_{\pi/2}$ to be different from that of generating functions of walks in $\mathcal{R}_{\pi}$.

This thesis is motivated primarily by the question, "For a given step set $\mathcal{Y}$, what effect does the choice of a region have on the generating function of the walks in that region?" We therefore consider walks in three different regions of the planar lattice: the quarter-plane $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} | i \geq 0, j \geq 0\}$, the one-eighth-plane $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} | i \geq 0, i \geq j\}$, and the three-quarter-plane bounded by the negative x-axis and the negative y-axis. These three regions we denote by $\mathcal{R}_{\pi/2}$, $\mathcal{R}_{\pi/4}$, and $\mathcal{R}_{3\pi/2}$, respectively. We consider these regions specifically because $\mathcal{R}_{\pi/4}$ is smaller than $\mathcal{R}_{\pi/2}$, and $\mathcal{R}_{3\pi/2}$ is bigger than $\mathcal{R}_{\pi}$.

The initial part of this research is an endeavour to classify the generating functions of each of the walks described above as algebraic, D-finite, or neither for the regions $\mathcal{R}_{\pi/4}$ and $\mathcal{R}_{3\pi/2}$. Algebraic generating functions are further classified as rational, trivial (meaning the steps generate no walks), or neither. We then examine how results on $\mathcal{R}_{\pi/2}, \mathcal{R}_{\pi/4}$, and $\mathcal{R}_{3\pi/2}$ can be extrapolated to general regions.

We begin by first describing the most ubiquitous lattice path, the *Dyck path.*

1.2 Dyck paths

Dyck paths are planar lattice paths of length $2n$ made up of the steps NE and SE that start at the origin, remain in the region $y \geq 0$, and end at the point $(2n, 0)$. Figure 1.2 shows an example of a Dyck path of length 22. These paths are counted by the Catalan numbers, which are a very well studied integer sequence in combinatorics. The nth Catalan number, denoted by C_n is defined as

$$C_n = \binom{2n}{n}\frac{1}{n+1}.$$

Its entry in the On-Line Encyclopedia of Integer Sequences is A000108 [22]. Stanley's book *Enumerative Combinatorics Volume 2* [23] describes sixty-six different interpretations of the Catalan numbers.

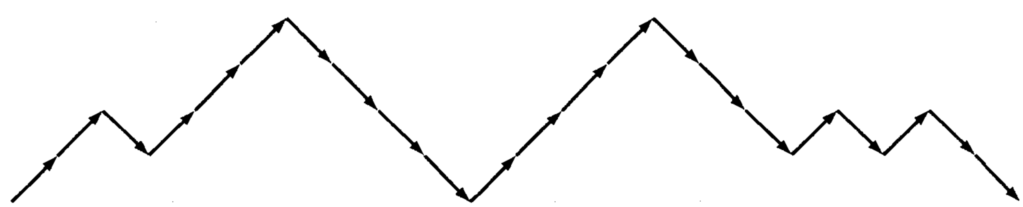


Figure 1.2: Dyck path of length 22

Dyck paths are a very important set of paths because they model physical phenomena well, and techniques used in their enumeration have many applications in the enumeration of other planar lattice paths. They are somewhat simple in that they are only restricted by one boundary. We will examine in Chapter 3 the complications that arise when we consider walks that are restricted by two boundaries. However we must first develop some of the tools we use to enumerate walks.

1.3 Enumerative methods in combinatorics

Stanley [23] writes in his *Enumerative Combinatorics Volume I*,

> The basic problem of enumerative combinatorics is that of counting the number of elements of a finite set. Usually we are given an infinite class of finite sets S_i where i ranges over some index set $I \ldots$, and we wish to count the number $f(i)$ of elements of each S_i "simultaneously."

The combinatorial objects examined in this thesis can be quite complicated and are rarely counted directly. We must use more powerful tools of enumeration in order to gain the desired information.

1.3.1 Ordinary generating functions

Let $I = \mathbb{N}$. Then $f : \mathbb{N} \to \mathbb{N}$, where $f(n) \in \mathbb{N}$ is the number of objects of size n. The *ordinary generating function* of f is the formal power series

$$\sum_{n \geq 0} f(n) x^n.$$

For example, let S_n be the set of Dyck paths of length $2n$. Then

$f(n) = C_n = \binom{2n}{n}/(n+1)$, and the ordinary generating function $F(x)$ of f is

$$F(x) = \sum_{n \geq 0} \binom{2n}{n} \frac{1}{n+1} x^n.$$

For many applications it is sufficient to express a generating function $F(x)$ in terms of a functional equation as opposed to a series in terms of $f(n)$. For example, $F(x) = \sum 2^n x^n$ can be expressed as $F(x) = 1 + 2xF(x)$.

Hierarchy of ordinary generating functions

For the sake of simplicity, we will let η denote an arbitrary ordinary generating function. We classify η as *rational*, *algebraic*, *D-finite*, or *non-D-finite*. We will define these terms below using the notation defined in Appendix A. Throughout this thesis K will denote a field of characteristic zero. In practice, however, we generally assume K to be the complex numbers.

The smallest class of generating functions we consider is the class of rational generating functions.

Definition 1.1. A formal power series $\eta \in K[[x]]$ is said to be *rational* if there exist polynomials $Q(x)$ and $P(x)$, such that $Q(x) \neq 0$ and $\eta = \frac{P(x)}{Q(x)}$.

The rational generating functions have a natural extension to the class of algebraic generating functions, which we define below.

Definition 1.2. A formal power series $\eta \in K[[x]]$ is said to be *algebraic* if there exist polynomials $P_0(x), \ldots, P_d(x) \in K[x]$, not all 0, such that

$$P_0(x) + P_1(x)\eta + \ldots + P_d(x)\eta^d = 0. \tag{1.3.1}$$

The smallest positive integer d for which Equation (1.3.1) holds is called the *degree* of η.

Example 1.3. Consider the Dyck paths: the generating function $D(x)$ that counts Dyck paths satisfies the equation

$$1 - D(x) + x^2 D(x)^2 = 0, \tag{1.3.2}$$

so $D(x)$ is algebraic.

If a generating function $\eta = \sum a(n)x^n$ is algebraic, then the asymptotic growth of $a(n)$ must be of the form $a(n) \sim \alpha\mu^n n^\gamma$, where α and μ are algebraic over $\mathbb{Q}$, and $\gamma \in \mathbb{Q} \setminus \{-1, -2, \ldots\}$ [28].

The rational generating functions are easily shown to be algebraic generating functions.

The class of algebraic generating functions extends further to the class of *differentiably finite*, which we will write as *D-finite* from now on. Stanley defines D-finite as follows:

Definition 1.4. Let $u \in K[[x]]$. If there exist polynomials $p_0(x), \ldots, p_d(x) \in K[x]$ with $p_d(x) \neq 0$, such that

$$p_d(x)u^{(d)} + p_{d-1}(x)u^{(d-1)} + \cdots + p_1(x)u' + p_0(x)u = 0,$$

where $u^{(j)} = d^j u/dx^j$, then we say that u is a D-finite power series.

Another word for D-finite is *holonomic.* An example of a D-finite function is $u(x) = x^2e^x$, since $u'(x) = (1 + \frac{2}{x})u$. However, x^2e^x is not an algebraic series, so the D-finite functions form a class which is not equal to the class of algebraic functions. If $u \in K[[x]]$ is not D-finite, we say it is non-D-finite.

There is another way of determining if a generating function η is D-finite that depends on a property of its coefficients. We first define a function $f : \mathbb{N} \to K$ to be *polynomially recursive*, or *P-recursive*, if there exist polynomials $P_0, \ldots, P_e \in K[n]$

with $P_e \neq 0$, such that

$$P_e(n)f(n+e) + P_{e-1}f(n+e-1) + \cdots + P_0(n)f(n) = 0,$$

for all $n \in \mathbb{N}$. The following proposition from Stanley [23] relates D-finite series and P-recursive functions.

Proposition 1.1. *Let $\eta = \sum_{n\geq 0} f(n)x^n \in K[[x]]$. Then η is D-finite if and only if f is P-recursive.*

Closure properties of generating functions

Let $u, v \in K[[x]]$ be rational power series, and let $\alpha, \beta \in K$. Then uv and $\alpha u + \beta v$ are also rational. Similarly, if $u, v \in K_{\text{alg}}[[x]]$, then $\alpha u + \beta v, uv \in K_{\text{alg}}[[x]]$. This is because $K_{\text{alg}}[[x]]$ forms a subalgebra of $K[[x]]$ [23].

Stanley [23] also states the closure properties of D-finite power series in the form of two theorems, which we summarize:

1. If u, v are D-finite, and $\alpha, \beta \in K$, then $\alpha u + \beta v$ and uv are both D-finite.

2. If u is D-finite, and $v \in K_{\text{alg}}[[x]]$ with $v(0) = 0$, then $u(v(x))$ is D-finite.

1.3.2 Multivariate generating functions

Oftentimes we are interested in counting combinatorial objects in terms of more than one property. For example, we might like to know how many Dyck paths of length n there are that intersect the x-axis k times. A generating function in a single variable could not contain the information we require, so we instead use a generalized form of the ordinary generating function called the *multivariate* generating function.

A multivariate generating function is a formal power series over a set of variables $x_1, \ldots, x_m$ where each x_i encodes a property of the combinatorial object. For this thesis we will always use the variable t as our 'counting' variable, *i.e.*, the variable t will correspond to the size of the objects we enumerate.

Continuing our previous example, let $D(t;z)$ be the generating function that counts the Dyck paths of length n that have k intersections with the x-axis. This example has statistical mechanical properties that are examined in Example 2.2. We let the power of t denote the length of the walk, and we let the power of z denote the number of time it intersects the x-axis. Therefore, we define $D(t;z)$ by

$$D(t;z) = \sum_{n\geq 0} t^n \sum_{k\geq 0} a_{n,k} z^k,$$

where $a_{n,k}$ denotes the number of walks of length n that intersect the axis k times. We find by examininination that

$$D(t;z) = z + z^2t^2 + (z^2+z^3)t^4 + (z^2+2z^3+z^4)t^6 + O(t^8).$$

We classify the multivariate generating functions in a similar way to the univariate generating functions. A series $\eta \in K[[x_1,\ldots,x_m]]$ is classified as rational, algebraic, D-finite, or non-D-finite. A series η is rational if there exist polynomials $P, Q \in K[x_1,\ldots,x_m]$, $Q \neq 0$, such that $Q\eta = P$.

The series η is algebraic over the field $K(x_1,\ldots,x_m)$ if there exist polynomials $P_0,\ldots,P_k \in K[x_1,\ldots,x_m]$ not all equal to 0 such that $P_0+P_1\eta+P_2\eta^2+\cdots+P_k\eta^k = 0$.

The series η is D-finite over $K(x_1,\ldots,x_m)$ if, for $1 \leq i \leq m$, η satisfies a system of non-trivial partial differential equations of the form

$$\sum_{j=0}^{d_i} P_{j,i} \frac{\partial^j \eta}{\partial x_i^j} = 0,$$

where $P_{j,i} \in K[x_1,\ldots,x_m]$, and d_i is the order of the partial differential equation in the variable x_i.

1.3.3 Noncommutative generating functions

One tool we often use to show that a power series $\eta \in K[[x_1, \ldots, x_m]]$ is algebraic is the theory of *noncommutative* formal series in several variables. Stanley [23] describes this theory in detail, although the theory of algebraic formal power series itself is due to Chomsky and Schützenberger [12].

Again, let K denote a fixed field. Let X be a set, called an *alphabet*, and let X^* be the set of all finite strings of zero or more elements from X. We denote the empty word (word string of zero elements) by ϵ. Finally, define $X^+ = X^* \setminus \{\epsilon\}$.

Example 1.5. Let $X = \{x, y\}$. The *Dyck language*, denoted by $\mathcal{D}$, is the set of all words $w \in X^*$ satisfying the following conditions:

(a) The number of x's in w is equal to the number of y's in w.

(b) For any factorization $w = uv$, the number of x's in u is at least as large as the number of y's.

The Dyck words of length six or less are given by

$$\epsilon \quad xy \quad x^2y^2 \quad xyxy \quad x^3y^3 \quad x^2yxy^2 \quad x^2y^2xy \quad xyx^2y^2 \quad xyxyxy.$$

If we map x to the step NE and map y to SE, it is clear that the Dyck language is isomorphic to the set of Dyck paths described in Example 1.3.

Definition 1.6. A *formal noncommutative (power) series* in X over K is a function $S : X^* \to K$. We write $\langle S, w\rangle$ for $S(w)$ and then write

$$S = \sum_{w \in X^*} \langle S, w\rangle w.$$

The set of all non-commutative formal series in X is denoted $K\langle\langle X\rangle\rangle$.

Example 1.7. Let $X = \{x, y\}$, and for $w \in X^*$ define $\langle S, w\rangle$ as

$$\langle S, w\rangle = \begin{cases} 1 \text{ if } w \in \mathcal{D} \\ 0 \text{ otherwise.} \end{cases}$$

Then

$$S = 1 + xy + x^2y^2 + xyxy + x^3y^3 + x^2yxy^2 + x^2y^2xy + xyx^2y^2 + xyxyxy + \cdots .$$

Definition 1.8. Let $Z = \{z_1, \ldots, z_n\}$ be an alphabet disjoint from X. A *proper algebraic system* is a set of equations $z_i = p_i$, $1 \leq i \leq n$, where:

(a) $p_i \in K\langle X, Z\rangle$ (i.e., p_i is a non-commutative polynomial in the alphabet $X \bigcup Z$);

(b) $\langle p_i, 1\rangle = 0$ and $\langle p_i, z_j\rangle = 0$ (i.e., p_i has no constant term and no terms $c_j z_j, 0 \neq c_j \in K$).

A *solution* to a proper algebraic system $(p_1, \ldots p_n)$ is an n-tuple $(R_1, \ldots, R_n) \in K\langle\langle X\rangle\rangle^n$ of formal series in X with zero constant term satisfying

$$R_i = p_i(X, Z)_{z_i = R_i}. \tag{1.3.3}$$

Each R_i is called a *component* of the system $(p_1, \ldots p_n)$.

Definition 1.9.

(a) A series $S \in K\langle\langle X\rangle\rangle$ is *algebraic* if $S - \langle S, 1\rangle$ is a component of a proper algebraic system. The set of all algebraic series $S \in K\langle\langle X\rangle\rangle$ is denoted $K_{alg}\langle\langle X\rangle\rangle$.

(b) The *support* of a series $S = \sum\langle S, w\rangle w \in K\langle\langle X\rangle\rangle$ is defined by

$$supp(S) = \{w \in X^* : \langle S, w\rangle \neq 0\}.$$

A *language* is a subset of X^*. A language $\mathcal{L}$ is said to be *algebraic* if it is the support of an algebraic series. An algebraic language is also called *context-free.*

Example 1.10. Consider the Dyck language, $\mathcal{D}$. The non-empty words in $\mathcal{D}$ are described by the following recursive property. If $w \in \mathcal{D}^+$ then it begins with an x, followed by a Dyck word, then a y, and finally another Dyck word. Thus $\mathcal{D}$ is a solution to the proper algebraic system

$$z = 1 + xzyz, \tag{1.3.4}$$

and is therefore algebraic.

We would like to relate algebraic formal series to *commutative* algebraic generating functions. First, we let $\phi : K\langle\langle X\rangle\rangle \to K[[X]]$ be the continuous algebra homomorphism defined by $\phi(x) = x$ for all $x \in X$. Thus $\phi(S)$ can be thought of as the "abelianization" of S.

Example 1.11. Let $X = \{x, y\}$, and for $w \in X*$ and $S \in K\langle\langle X\rangle\rangle$ define $\langle S, w\rangle$ as

$$\langle S, w\rangle = \begin{cases} 1 \text{ if } w \in \mathcal{D} \\ 0 \text{ otherwise.} \end{cases}$$

Then $\phi(S) = 1 + xy + 2x^2y^2 + 5x^3y^3 + \cdots$.

We now state without proof a theorem from Stanley [23] which allows us to relate algebraic formal series to algebraic generating functions.

Theorem 1.2. *Let $S \in K_{alg}\langle\langle X\rangle\rangle$, where X is a finite alphabet. Then $\phi(S)$ is algebraic over the field $K(x)$ of rational functions in the commuting variables X.*

This theorem is particularly useful because many combinatorial objects are isomorphic to languages.

Example 1.12. We now apply Theorem 1.2 to Equation (1.3.4) to obtain the generating function which counts the Dyck words. We first abelianize the noncommutative generating function of the Dyck words. This new ordinary generating function $D(x, y)$ must satisfy the functional equation

$$D(x,y) = 1 + xyD(x,y)^2.$$

Essentially what we have done is replace each x and y in a word $w \in \mathcal{D}$ with commutative variables to obtain the ordinary generating function $D(x, y)$. We now let $x = y = t$ to obtain the length generating function $D(t)$:

$$\begin{aligned} D(t) &= 1 + t^2 D(t)^2 \\ &= \frac{1 - \sqrt{1 - 4t^2}}{2t^2}. \end{aligned}$$

1.3.4 The importance of D-finiteness

One might ask why it is relevant that the generating function of a combinatorial object is D-finite or algebraic. According to Flajolet, Gerhold, and Salvy [13],

> ...a rough heuristic in this range of problem is the following: *Almost anything is [non-D-finite] unless it is [D-finite] by design.*

In general, we expect a combinatorial object with a D-finite generating function to have a "nice" structure. Furthermore, the asymptotic growth rate of the coefficients of a D-finite series is of the form

$$a(n) \sim \zeta^{-n} e^{P(n^{(1/r)})} n^{\theta} \log^{\ell} n,$$

where $\ell, r \in \mathbb{N}$, θ is algebraic over $\mathbb{Q}$, and P is a polynomial [28]. This form is important because typical applications of lattice path enumeration are interested in asymptotic behaviour. Chapter 2 gives some examples of such applications in statistical mechanics.

On the other hand, given a combinatorial object, we would like to determine its generating function exactly. Guttmann [16] describes a method to distinguish between lattice-based problems that are likely to be "solvable", and those which are not. Typically explicit solutions of such problems are given as solutions to low order differential equations. If a problem is unsolvable, its solution (generating function) can be neither algebraic nor D-finite.

Guttman's method takes advantage of the following fact: if $f(z) = \sum_{n\geq 0} a_n(x)z^n$ with coefficients $a_n(x)$ in the field $K = \mathbb{C}(x)$ is algebraic or D-finite, then the poles of $a_n(x)$ that lie in a bounded region cannot become dense on the boundary of that as n increases. For example, the function $f(z) = \sum_{n\geq 0} z^n/(1-x^n)$ has poles that become dense on the unit circle. However, not every non-D-finite function displays this behaviour. For example, Guttmann [16] gives the following as an example of a non-D-finite function whose singularities do not become dense on a boundary:

$$\begin{aligned} f(x,z) &= e^{x(e^{z/(1-x)}-1)} \\ &= 1 + \frac{xz}{1-x} + \frac{x(1+x)z^2}{2(1-x)^2} + \frac{x(1+3x+x^2)z^3}{6(1-x)^3} + O(z^4). \end{aligned}$$

Bousquet-Mélou and Petkovšek [6] used this fact about the singularities of non-D-finite generating functions to find the first example of a set of walks restricted to $\mathcal{R}_{\pi/4}$ whose length generating function is non-D-finite. Up to that point, it had been conjectured that all walks restricted to $\mathcal{R}_{\pi/4}$ had D-finite generating functions. Bousquet-Mélou and Petkovšek proved the following results about walks generated by $\{(2,1),(-1,2)\}$ (these walks are known as *knight walks*):

1. The length generating functions of knight walks in $\mathcal{R}_{\pi/2}$ that begin at $(1,1)$ is non-D-finite.

2. The generating function that counts the knight walks in $\mathcal{R}_{\pi/2}$ that begin at $(1,1)$ and end on the line $y = x$ is non-D-finite.

Chapter 2

Statistical Mechanics

2.1 Statistical mechanics and combinatorics

There has recently been an increased interest in the enumeration of planar lattice paths under different restrictions (see [3],[4],[5],[6],[7],[20],[21]). An example of a restricted lattice path is a self-avoiding walk. This is in large part due to the suitability of these combinatorial objects for the modeling of physical systems that arise in statistical mechanics. Most significantly, self-avoiding walks have been used to model polymers in solution under different physical conditions (see [9],[10],[11],[25],[26],[27]). These conditions usually take the form of a region in which the polymer must remain and how the polymer interacts with the boundaries of that region.

Much of the information presented in Section 2.1.1 is taken from *Introductory Statistical Mechanics*[8] by Roger Bowley and Mariana Sánchez.

2.1.1 Introduction to statistical mechanics

Statistical mechanics is an area of physics which examines the bulk properties of matter under the assumption that matter is composed of a very large number of particles (typically on the order of 10^{23}), such as atoms or molecules. It is also assumed that each particle obeys the laws of mechanics. These laws may be those of classical, quantum, relativistic, or some other physics. The behaviour of an individual particle is not considered, as statistical methods are used. Since such large numbers of particles are considered, typical applications of combinatorics to statistical mechanics involve the asymptotic behaviour of combinatorial objects, *i.e.*, the behaviour of their generating functions as $n \to \infty$.

The bulk properties of a system of particles is called the *macrostate* of the system. Some examples of macrostate properties are pressure and temperature of a gas. When we consider the macrostate of a system, we are not concerned with an individual particle of that system. If we are interested in an individual particle, we consider its *microstate*. The energy and momentum of a particle are examples of microstates.

In statistical physics, an important property of a system is its entropy. Entropy can be thought of as a measure of the disorder of a system. The more disorganized a system is, the higher its entropy. For example, think of a container of water molecules as our system. When the water is frozen, its molecules are packed together in a regular lattice in a very orderly manner. When the water is in the form of steam, each molecule moves about nearly independent of the other molecules. Therefore, the system has lower entropy when the water is frozen than when the water is in gas form.

When combinatorial objects are used in statistical mechanics, the information most often obtained relates to the entropy of a system. Therefore the bulk of this chapter will be devoted to the methods used to consider the entropy of a physical system.

In statistical mechanics, the entropy of a system, denoted by S, is

$$S = k \log W \tag{2.1.1}$$

where $k = 1.38066 \times 10^{-23}$ J K^{-1} is Boltzmann's constant and W is the number of possible microstates of the system. The units in k are joules (J) and degrees Kelvin (K). Boltzmann's constant is named for thermodynamicist Ludwig Boltzmann. He proved that Equation (2.1.1) gives a measure of the entropy of a system of atoms and molecules in the gas phase.

There is another way to consider the entropy of a system: we may think of the entropy of a system as the amount of "mixupedness" (a word coined by the physicist Gibbs) which remains about a system after its macrostate has been determined. Given a system's macrostate, its entropy measures the degree to which the probability of the system is spread out over different possible microstates. The higher the number of possible microstates, the higher a system's entropy.

2.1.2 The canonical ensemble

In statistical mechanics, an *ensemble* is a collection of systems all prepared in the same way, *i.e.*, they have the same macroscopic properties: they each have the same number of particles, energy, volume, shape, magnetic field, et cetera. However, two systems prepared in the same manner may not be in the same quantum state, *i.e.*, their microscopic properties may differ. The probability that a system is in a particular quantum state is the fraction of the ensemble in this state. The *microcanonical ensemble* is the physical model that assumes that each quantum state is equally probable. This is perhaps not the most realistic model, but it is a reasonable starting point.

The *canonical ensemble* is similar to the microcanonical in that each system is identically prepared, but the energy of each system is not constant. In this ensemble, energy may be passed from one system to its neighbours, so the energy of a given system fluctuates. Each system is in contact with the other systems, which act as a *heat bath* for the system. A heat bath is a system so large that when it loses or gains heat from some other system its energy remains constant. The combined system is thermally isolated so its total energy, U_T is constant.

The partition function

Definition 2.1. Let E_j denote the total energy of a system in microstate j. The *partition function* is given by the equation

$$Z = \sum_j e^{-\beta E_j},$$

where the sum is taken over all the different quantum microstates, and the inverse temperature β is conventionally defined as

$$\beta \equiv \frac{1}{kT},$$

where T is the temperature in degrees Kelvin.

As a simple example, suppose we have a system A of four particles where no two particles have the same energy. Next, suppose the energies of the particles are $E_1 = 0, E_2 = 1.4 \times 10^{-23} J, E_3 = 2.8 \times 10^{-23} J, E_4 = 5.6 \times 10^{-23} J$. Finally, suppose the heat bath of the system has a temperature of 4 K. Then the partition function is

$$Z = e^0 + e^{-1/4} + e^{-1/2} + e^{-1} = 2.75.$$

Alternatively, one may write Z as a sum over energy levels, E_n. If there are g_n quantum states with energy E_n, then Z may be written as

$$Z = \sum_n g_n e^{-\beta E_n},$$

where the sum is over all the different energy levels of the system. The quantity g_n is called the *degeneracy* of the energy level E_n. If no two microstates have the same energy, then we say the system is *non-degenerate.*

Entropy in the canonical ensemble

In the canonical ensemble, the entropy of a system A is calculated by considering $M-1$ replica systems in contact with each other and with A; when M is taken to be very large, these systems act as a heat bath for A. The collection of systems is thermally isolated.

Each of the replica systems is identical to the rest, but we assume that each of the systems is easily distinguished from one another based on their position. A typical example of a system would be a large classical object containing many particles such as a bar of lead. We say that a system, *i.e.*, bar of lead, is in a particular quantum mechanical state ψ_i, and let n_i be the number of systems in the quantum state ψ_i. The M systems can be arranged in $M!$ ways. Since swapping the positions of two replica systems in the state i would not change the macrostate whatsoever, and there are $n_i!$ ways of arranging the systems in state i, the number of arrangements of systems, denoted by W, is

$$W = \frac{M!}{n_1!n_2!n_3!\cdots}.$$

If we take M to be so large that each n_i is huge, we may use Stirling's approximation and the identity $M\log(M) = \sum_i n_i \log(M)$ to express the entropy for the M systems, S_m, as

$$S_m = k\log(W) = -kM\left(\sum_i \left(\frac{n_i}{M}\right)\log\left(\frac{n_i}{M}\right)\right).$$

As $M \to \infty$, n_i/M approaches the probability p_i of finding the system in state ψ_i.

Thus the average entropy per system, S, may be written as

$$S = \frac{S}{S_m} = -k \sum_i p_i \log(p_i). \tag{2.1.2}$$

If all the probabilities are equal, then $p_i = 1/W$, and

$$S = -k \sum_{i=1}^{W} \frac{1}{W} \log\left(\frac{1}{W}\right) = k \log(W),$$

which is Boltzmann's entropy for a system.

In the canonical ensemble the probability of a system being in the state ψ_i is

$$p_i = \frac{e^{-\beta E_i}}{Z},$$

so

$$\log(p_i) = -\beta E_i - \log(Z). \tag{2.1.3}$$

Suppose these probabilities are not equal. We substitute Equation (2.1.3) into Equation (2.1.2) to see that

$$S = k \sum_i p_i(\beta E_i + \log(Z)).$$

The average energy is denoted $\bar{U}$ and is defined by

$$\bar{U} = \sum_i p_i E_i.$$

Therefore,

$$S = \frac{\bar{U}}{T} + k \log(Z),$$

which we rewrite as

$$\bar{U} - TS = -kT \log(Z). \tag{2.1.4}$$

The quantity $(\bar{U} - TS)$ is the mean value of the *Hemlholtz free energy*, an important quantity in statistical mechanics. We denote this quantity by F.

2.1.3 Combinatorics and entropy

Since the focus of enumerative combinatorics is counting, it is ideally suited to determine the entropy of systems in some models. The main tool used is the ordinary generating function, which is used to count arrangements of microstates and is written

$$G_w(x) = \sum_{n=0}^{\infty} w_n x^n, \tag{2.1.5}$$

where w_n is the number of arrangements of n microstates. When combinatorics is used in statistical mechanics, we eliminate all multiplicative constants and units of measurement such as k and J, respectively. This is done for the sake of simplicity, and for the reason that it would be entirely arbitrary to assign physical units to a planar lattice path, which is a purely mathematical object. Therefore we say that the entropy of a system of n particles is simply $\log(w_n)$.

A combinatorial object that has been used in statistical mechanics is the random planar walk. An *interacting* model is one in which the vertices of the walk interact with each other, *i.e.*, they are not independent. For example, a *self-avoiding* walk (a walk which never visits the same point more than once) is an interacting model. In an interacting model, each walk has an energy associated with it which depends on some property of the walk. An example of such a property is the interactions of a walk with some boundary. As a result, the generating functions will depend on a new parameter which will be conjugate to the energy. We let $p_n(m)$ be the number of a family of walks of length n counted with respect to some property of size m. For example, m may be the number of times a walk hits the x-axis. If a walk is counted by $p_n(m)$, we say it has *energy* m.

The *canonical partition function* of the model is

$$p_n(z) = \sum_{m \geq 0} p_n(m) z^m,$$

where z is an *activity* conjugate to the energy.

In combinatorial models, a property of interest is the *free energy density* (free energy per vertex or edge), which is defined as followed:

$$F_n(z) = \frac{1}{n}\log p_n(z).$$

The *limiting free energy density* (per vertex or edge) is defined by

$$\mathcal{F}(z) = \lim_{n\to\infty} F_n(z) = \lim_{n\to\infty}\frac{1}{n}\log p_n(z) \tag{2.1.6}$$

for values of z where this limit exists.

The generating function of a model with partition function $p_n(z)$ is

$$G(x,z) = \sum_{n\geq 0} p_n(z)x^n = \sum_{n\geq 0}\sum_{m\geq 0} p_n(m)z^m x^n.$$

If the limiting free energy exists, it can be computed from the radius of convergence of $G(x,z)$, denoted $x_c(z)$, using the Cauchy-Hadamard theorem, which we state below.

Theorem 2.1 (Cauchy-Hadamard). *The radius of convergence r of the Taylor series*

$$\sum_{n=0}^{\infty} a_n z^n$$

is

$$r = \left(\overline{\lim_{n\to\infty}}(|a_n|)^{\frac{1}{n}}\right)^{-1}.$$

Thus, from Equation (2.1.6) we find that

$$\frac{1}{x_c(z)} = \lim_{n\to\infty}[p_n(z)]^{1/n} = e^{\mathcal{F}(z)} \tag{2.1.7}$$

where $\mathcal{F}(z)$ exists.

Example 2.2 ***Staircase walks adsorbing on the main diagonal.*** An example of a combinatorial object with statistical mechanics properties is given by van Rensburg[25]: a staircase walk above the main diagonal is a random walk in the square lattice made up of N and E steps that starts at the origin and stays on or above the line $y = x$. We say that such a walk *adsorbs* where it intersects the main diagonal. If we rotate these walks by 45°, we see that the ones that end on the diagonal are isomorphic to Dyck paths (see Figure 2.2). Therefore the generating function of such walks, denoted by G_D, is

$$G_D(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x^2}.$$

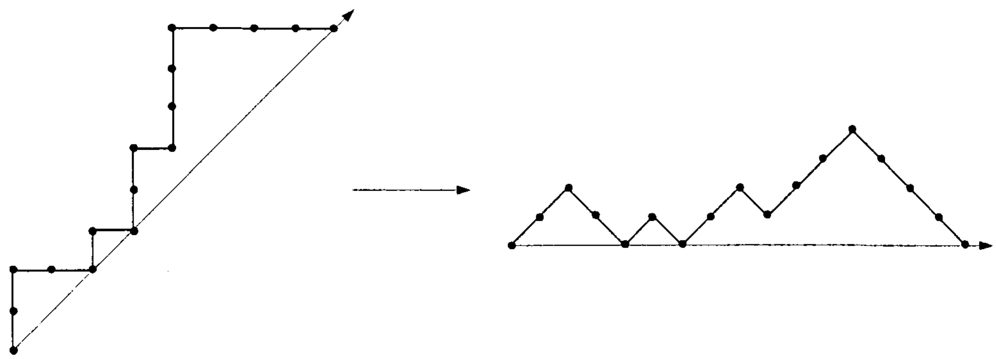


Figure 2.1: Staircase walk rotated to form a Dyck path.

An *excursion* is a staircase walk above the main diagonal with only its end points on the diagonal. Such walks can be thought of as a step N followed by a Dyck path and ending with a step E. Therefore the generating function of excursions, denoted by G_E is

$$G_E = x^2 G_D(x) = \frac{1 - \sqrt{1 - 4x^2}}{2}.$$

We now use the generating functions for $G_D(x)$, and $G_E(x)$ to construct a model for Dyck paths adsorbing on (attaching to) the main diagonal. To do this, we introduce a second variable z, which shall be the activity conjugate to the number of times a walk hits the main diagonal. A Dyck path is either a single visit of weight z, or an excursion followed by a Dyck path. Thus we obtain the generating function

$$\begin{aligned} G_D(x,z) &= z + zG_E(x)G_D(x,z) \\ &= \frac{z}{1 - z(1-\sqrt{1-4x^2})/2}. \end{aligned}$$

The radius of convergence $x_c(z)$ of $G_D(x,z)$ is determined from the singularities of $G_D(x,z)$, *i.e.*, when its denominator is zero, or when the square root term in the denominator is zero. Therefore $x_c(z)$ depends on $z(1-\sqrt{1-4x^2})/2$. If $1 > z(1-\sqrt{1-4x^2})/2$, then $x_c(z) = 1/2$. If $1 = z(1-\sqrt{1-4x^2})/2$, then we solve for x to find $x_c(z)$. From this we determine that

$$x_c(z) = \begin{cases} \frac{1}{2}, & \text{if } z \leq 2; \\ \frac{\sqrt{z-1}}{z}, & \text{if } z > 2. \end{cases}$$

This defines the free energy $\mathcal{F}_D(z) = -\log x_c(z)$ for this model.

Physically this can be thought of as follows: when $z \leq 2$, the free energy of the system is independent of z, and therefore the systems interactions with the surface are ignored. For $z > 2$, an increase in z corresponds to a more attractive diagonal boundary, and thus a higher free energy.

2.1.4 Planar lattice walks and statistical mechanics

Linear polymers

A polymer is a molecule made up of many copies of a single smaller molecule called a monomer. A linear polymer is simply a chain of monomers, *i.e.*, two monomers have one neighbour, and the rest each have two. Self-avoiding walks are a natural choice of combinatorial object to utilise as possible models for polymers. In particular, the thermodynamical properties of self-avoiding walks in confined geometries have been investigated.

At first glance, it may seem that such a model would be far too idealized and simple to have any practical application in physics. However, it may be that model systems that idealize a physical systems may exhibit the same *critical behaviour* (behaviour as $n \to \infty$) as the physical system they simulate. This would provide insight into the properties that determine phase behaviour of physical systems.

Forces in square lattice directed paths in a wedge

Consider the walks generated by $\{\mathsf{N}, \mathsf{E}\}$ confined to an arbitrary wedge between the y-axis and the line $y = rx$, where r is a non-negative real number. We call such a wedge an *r-wedge*. These walks can be used as a simple model of a linear polymer in a confined space. The thermodynamical properties of these walks have been investigated by van Rensburg and Ye [27]. Figure 2.1.4 shows a general representation of the walks in which we are interested.

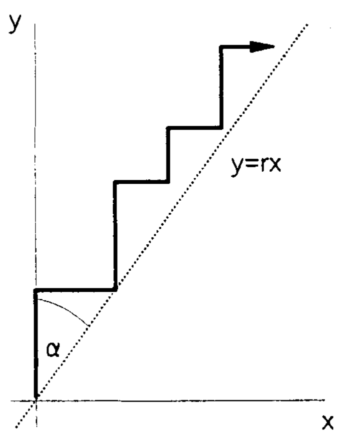


Figure 2.2: Arbitrary confined walk

Let $c_n^{(r)}$ be the number of directed paths from the origin of length n confined to the r-wedge. Then their generating function, denoted by g_r, is defined as

$$g_r = \sum_{n=0}^{\infty} c_n^{(r)} t^n.$$

Although they did not find g_r for an arbitrary r, van Rensburg and Ye were able to prove that

$$\lim_{n\to\infty} [c_n^{(r)}]^{1/n} = \begin{cases} 2, & \text{if } r \leq 1, \\ \frac{1+r}{r^{r/(1+r)}}, & \text{if } r > 1. \end{cases}$$

Therefore, by Theorem 2.1 the radius of convergence of the generating function g_r is given by

$$t_r = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq r \leq 1, \\ \frac{r^{r/(1+r)}}{1+r}, & \text{if } r > 1. \end{cases}$$

From this they are able to determine the free energy per vertex of the walk of infinite length:

$$\mathcal{F}_r = -\log t_r = \log(1+r) - \frac{r \log r}{1+r}.$$

The derivative of $\mathcal{F}_r$, denoted by F_r, represents the entropic force exerted as the wedge is closed by increasing r. This force can be thought of as a spring force exerted by the walk on the walls. This corresponds to the entropic force a polymer molecule would exert on its enclosing boundaries. The expression of F_r is given by

$$F_r = \begin{cases} 0, & \text{if } 0 \leq r < 1, \\ \frac{\log r}{(1+r)^2}, & \text{if } r \geq 1. \end{cases}$$

The entropic force can also be expressed in terms of the angle α between the y-axis and the line $y = rx$. In these terms the force is expressed

$$F_\alpha = \begin{cases} \left[\frac{1+\cot^2\alpha}{(1+\cot\alpha)^2}\right] \log(\cot\alpha), & \text{if } 0 \leq \alpha < \pi/4, \\ 0, & \text{if } \alpha \geq \pi/4. \end{cases}$$

With the use of MAPLE, van Rensburg and Ye found F_r to be maximum when $r = 2.09349\ldots$ [27]. This corresponds to a wedge angle $\alpha = 0.445624612\ldots$, which is near $\pi/7$. It should be noted that F_r is a measure of vertical force against the line $y = rx$.

What does all this mean to a non-physicist? According to this model, if we have a polymer in a wedge, we can 'squeeze' it by decreasing α. As α decreases from $\pi/2$ to $\pi/4$, we encounter no resistance, *i.e.*, no force against the boundary of the wedge. Further closing the wedge, however, causes a force which increases the smaller the wedge becomes. The vertical force reaches a maximum near $\pi/7$ and then decreases with α.

The above results are in part what motivates our study of walks in different restricted regions of the planar lattice. The combinatorial model in [27] is much easier to examine than the walks we consider, and it illustrates the importance of the choice of boundaries. The fact that $F_\alpha = 0$ for $\alpha \in (\pi/4, \pi/2)$ leads us to hypothesize that the behaviour of the generating functions of the walks in $\mathcal{R}_{\pi/4}$ will not differ significantly from that of the walks in $\mathcal{R}_{\pi/2}$.

Other models

There are many other restricted self-avoiding walks whose thermodynamical properties have been studied. Van Rensburg [25] has written a survey that covers a wide variety of self-avoiding walk models and their thermodynamical properties. Other combinatorial objects whose statistical mechanical properties have been investigated are interacting polygons, animals and vesicles; van Rensburg's [24] book on the subject is a thorough resource with many examples explained in detail.

Chapter 3

Lattice Walk Classification

We now examine general lattice walks in four different regions. Section 3.1 recalls the half-plane, Section 3.2 the quarter-plane, Section 3.3 the one-eight-plane, and Section 3.4 the three-quarter plane.

3.1 Reduction to half-plane

There has been much work done recently to classify the generating functions of lattice paths in various restricted regions. For a given region, let $\mathcal{L}(\mathcal{Y})$ denote the set of all walks generated by a step set $\mathcal{Y}$ that remain in that region. It is natural to think of lattice walks in terms of languages. Let $\mathcal{Y} = \{A_1, \ldots, A_n\}$ be a step set of cardinality n. Then $\mathcal{L}(\mathcal{Y})$ is isomorphic to some language $\mathcal{Z} \subseteq Z^*$, where $Z = \{a_1, \ldots, a_n\}$.

Let S be the formal noncommutative power series that sums all the words in $\mathbb{Z}$. In order to determine the complete generating function $Q_{\mathcal{Y}}(x, y; t)$, we apply the function $\phi : K\langle\langle Z \rangle\rangle \rightarrow K[[x, y, t]]$ defined as follows: if a_k is the letter corresponding to the step $A_k = (x^i, y^j)$, then $\phi(a_k) = x^i y^j t$. For example, if $\mathcal{Y} = \{\mathsf{N}\}$ and $Z = \{a\}$, then $\phi(a) = yt$.

For each $\mathcal{L}(\mathcal{Y})$, we actually consider two generating functions. The *complete generating function*, denoted by $Q_\mathcal{Y}(x, y; t)$ is defined by

$$Q_\mathcal{Y}(x, y; t) = \sum_{i,j \in \mathbb{Z}, n \geq 0} a_{i,j}(n) x^i y^j t^n,$$

where $a_{i,j}(n)$ counts the number of walks of length n that end at the point (i, j). The *length generating function*, denoted by $Q_\mathcal{Y}(t)$ counts the walks of length n irrespective of their end point, *i.e.*, $Q_\mathcal{Y}(t) = Q_\mathcal{Y}(1, 1; t)$. If $Q_\mathcal{Y}(x, y; t)$ is algebraic (resp. D-finite), then $Q_\mathcal{Y}(t)$ is also algebraic (resp. D-finite).

3.1.1 Walks in the half-plane

We say that a step set $\mathcal{Y}$ is *simple* if each step in $\mathcal{Y}$ is of the form $(1, a)$, where $a \in \mathbb{Z}$. A walk generated by a simple step set is called *semi-directed* because it always moves in the positive x direction. Banderier and Flajolet [1] proved that, in $\mathcal{R}_\pi$, for any simple step set $\mathcal{Y}$ the complete generating function for walks that stay on or above the x-axis is algebraic. By assigning a weight to each step in a simple step set, we adapt their result to show that for *any* $\mathcal{Y}$, the length generating function for the walks generated by $\mathcal{Y}$ that remain in the half-plane $\mathcal{R}_\pi$ are algebraic. We state this result formally in Lemma 3.2.

For the purpose of this section, we will think of a step set $\mathcal{Y}$ of size m as an ordered m-tuple, although we will still call it a step set. With a step set $\mathcal{Y} = ((a_1, b_1), \ldots, (a_m, b_m))$ we can associate an m-tuple of *weights* $\Pi = (w_1, \ldots, w_m)$, where the weight $w_i > 0$ is associated with the step (a_i, b_i). The weight of a given path is defined as the product of the weights of its individual steps. Let $a(n)$ be the sum of the weights of all paths of length n.

There are several ways weights can be used with lattice paths. If each $w_i = 1$ then $a(n)$ is the total number of paths of length n. Another situation of interest is when $\sum w_i = 1$, as this corresponds to a probabilistic model of walks where, at each step in a walk, each step (a_i, b_i) is taken with probability w_i. Finally, we may use weights to enumerate coloured paths: for example, $w_i = 2$ would mean that the step (a_i, b_i) can be coloured in one of two ways. In this case $a(n)$ is again the total number of paths of length n.

Example 3.1. Suppose we wanted to count the number of Dyck paths where we colour each of the steps either red or blue. Then $\mathcal{Y} = ((1,1),(1,-1))$, $\Pi = (2,2)$. In this case the weights correspond to the number of ways of taking a step.

Let $\mathcal{S} = ((1,b_1),\dots,(1,b_m))$ be a simple step set, with $\Pi = (w_1,\dots,w_m)$ a corresponding m-tuple of weights. Banderier and Flajolet give $Q_{\mathcal{S}}(t)$ as a function of the *characteristic polynomial* of $\mathcal{S}$, which is a Laurent polynomial in u denoted by $P(u)$ and defined as

$$P(u) := \sum_{j=1}^{m} w_j u^{b_j}.$$

The characteristic polynomial associated with the coloured Dyck paths in Example 3.1 is $P(u) = 2u + 2u^{-1}$.

Now, let $\mathcal{Y} = ((a_1,b_1),\dots(a_m,b_m))$ be an arbitrary step set. For a walk w generated by $\mathcal{Y}$ to remain in $\mathcal{R}_\pi$, the following must be true of any prefix u of w: the sum of all the steps in u must have a non-negative x-coordinate. We show these walks have an algebraic length generating function by mapping $\mathcal{Y}$ to an ordered pair $(\mathcal{S},\Pi)$, where $\mathcal{S}$ is a simple step set and Π is a system of weights. We map $\mathcal{Y}$ to $(\mathcal{S},\Pi)$ as follows: first, map each step (a_i,b_i) to the step $(1,a_i)$, and let k be the number of distinct a_i's. Next, let $\mathcal{S} = ((1,c_1),\dots(1,c_k))$ be a k-tuple that contains each of the distinct $(1,a_i)$'s. Finally, let $\Pi = (w_1,\dots,w_k)$, where each w_j is the number of steps in $\mathcal{Y}$ with x-coordinate c_j.

Essentially we are mapping w to a semi-directed walk with possibly different coloured steps that stays above the x-axis. For example, the step set $\mathcal{Y} = ((2,2),(2,1),(-1,3),(-3,2))$ has two steps with x-coordinate 2, one step with

x-coordinate -1, and one step with x-coordinate -3. Therefore $\mathcal{Y}$ is mapped to the ordered pair $(\mathcal{S}, \Pi)$, where $\mathcal{S} = ((1,2),(1,-1),(1,-3))$, and $\Pi = (2,1,1)$.

This shows that each walk generated by $\mathcal{Y}$ is isomorphic to a semi-directed walk with coloured steps, and therefore $Q_{\mathcal{Y}}(t) = Q_{\mathcal{S}}(t)$, which is algebraic. However, in order to show the algebraicity of the complete generating function for step sets that are not simple, we use other techniques, which we describe below.

3.1.2 Single restriction lemma

We begin by proving a lemma which states that walks isomorphic to those generated in $\mathcal{R}_\pi$ by $\mathcal{Y}$ whose steps are all unit steps have an algebraic complete generating function. In order to do so we utilize both the Dyck language and its extension: the *Motzkin language.*

Motzkin paths are similar to Dyck paths: they are lattice paths generated by $\{(1,1),(1,0),(1,-1)\}$ that end on the x-axis and cannot step below it. Before we define the language corresponding to the Motzkin paths, we require the following definition: let u be a word in X^* and let $\alpha \in X$. We define $|u|_\alpha$ to be the number of occurences of the letter α in the word u.

The words that correspond to Motzkin paths are defined as follows: a word $u \in \{a,b,c\}^*$ is *Motzkin word* if $|u|_a = |u|_b$ and for any factorization $u = wv$ we have $|w|_a \geq |w|_b$. For example, $u = aaacbccbbcacbaaabbcccb$ is a Motzkin word. We call the set of all Motzkin words the Motzkin language and denote it by $\mathcal{M}$.

Like Dyck words, Motzkin words have a simple unique decomposition. If the first letter of a non-empty Motzkin word is c, the word can be thought of as a c followed by a Motzkin word. If its first letter is an a, the decomposition is similar to that of a Dyck word: again we consider the factorization $u = awbv$ where w is the shortest word possible such that $|awb|_a = |awb|_b$. For example, the words $u_1 = aaacbccbbcacbaaabbcccb$

and $u_2 = caaacbccbbcacbaaabbcccb$ are factored as shown:

$$u_1 = a(aacbccb)b(cacbaaabbcccb)$$
$$u_2 = c(aaacbccbbcacbaaabbcccb).$$

Thus $\mathcal{M}$ is a solution to the proper algebraic system

$$z = 1 + cz + azbz,$$

and $\mathcal{M}$ is algebraic. To obtain the length generating function of Motzkin words, $m(t)$, we let $a = b = c = t$:

$$m(t) = 1 + tm(t) + t^2 m(t)^2.$$

To obtain the complete generating function of Motzkin paths, which we denote by $m(x, y; t)$, we let $a = xyt$, $b = xt/y$, and $c = xt$:

$$m(x, y; t) = 1 + txm(x, y; t) + t^2 x^2 m(x, y; t)^2.$$

A Motzkin-like language applied to lattice walks

Each walk set can be considered in terms of the restrictions on its walks. For example, the set of walks generated by the step set $\{\mathsf{S}, \mathsf{NE}, \mathsf{E}\}$ in $\mathcal{R}_{\pi/2}$ is unrestricted. Walks generated by $\{\mathsf{W}, \mathsf{N}, \mathsf{NE}\}$, on the other hand, have the following restriction: for any i, the first i steps of a walk must have at least as many NE steps as S steps and at least as many NE steps as W. This representation of planar lattice walks has similar structure to Motzkin words and Dyck words.

For example, we may represent walks generated by $\{\mathsf{W}, \mathsf{S}, \mathsf{NE}\}$ with words made up of the letters $\{x_1, y_1, y_2\}$, where x_1 represents NE, y_1 represents S, and y_2 represents W. The set of words representing these walks is the set

$$\{u \in \{x_1, y_1, y_2\}^* : |w|_{y_1} \leq |w|_{x_1}, |w|_{y_2} \leq |w|_{x_1} \forall wv = u\}.$$

When we consider sets of walks in terms of a language we are able to classify the generating functions for many different step sets as algebraic. We now state and prove a lemma that gives a necessary condition for algebraicity of $Q_{\mathcal{Y}}(x, y; t)$.

Lemma 3.1. *Let $\mathcal{Y} = \{A_1, \ldots, A_n\}$ be a step set of cardinality n, and let the language $\mathcal{Z} \subseteq \{a_1, \ldots, a_n\}^*$ be isomorphic to $\mathcal{L}(\mathcal{Y})$. Suppose there exists some permutation ρ and integers $1 \leq i < j \leq m \leq n$ such that we can define $\mathcal{Z}$ entirely by the condition that for any factorization $z = uv$ of a word $z \in \mathcal{Z}$,*

$$|u|_{a_{\rho(1)}} + \cdots + |u|_{a_{\rho(i)}} + 2(|u|_{a_{\rho(i+1)}} + \cdots + |u|_{a_{\rho(j)}}) \geq |u|_{a_{\rho(j+1)}} + \cdots + |u|_{a_{\rho(k)}} + 2(|u|_{a_{\rho(k+1)}} + \cdots + |u|_{a_{\rho(m)}}).$$

Then $\mathcal{Z}$ is algebraic, and the complete generating function $Q_{\mathcal{Y}}(x, y; t)$ is also algebraic. If $\mathcal{Z}$ can be defined in such a manner, it is called singular.

Proof. Let $\mathcal{W} \subseteq \mathcal{Z}$ be the set of all words $w \in \{a, b, c, d, e\}^*$ such that $|w|_a + 2|w|_b = |w|_c + 2|w|_d$. These words are isomorphic to the walks generated by $\{(1, 1), (1, 2), (1, -1), (1, -2), (1, 0)\}$ in $\mathcal{R}_{\pi/2}$ that end on the x-axis. In order to prove that $\mathcal{W}$ is algebraic, we must first define three other sets of words.

Let $\mathcal{W}_2$ be the set of all words $w \in \{a, b, c, d, e\}^*$ such that $|w|_a + 2|w|_b = |w|_c + 2|w|_d - 1$ and such that for any prefix u of w, $|u|_a + 2|u|_b \geq |u|_c + 2|u|_d - 1$. These words correspond to walks that stay above and end on the line $y = -1$.

Let $\mathcal{W}_3 \subseteq \mathcal{Z}$ be the set of all words $w \in \{a, b, c, d, e\}^*$ such that $|w|_a + 2|w|_b = |w|_c + 2|w|_d + 1$. These words correspond to walks that stay above $y = 0$ end on the line $y = 1$.

Finally, let $\mathcal{W}_4$ be the set of all words $w \in \{a, b, c, d, e\}^*$ that satisfy the following three conditions:

- $|w|_a + 2|w|_b = |w|_c + 2|w|_d$;

- there exists a factorization $w = uv$ such that $|u|_a + 2|u|_b = |u|_c + 2|u|_d - 1$;
- for any prefix u of w, $|u|_a + 2|u|_b \geq |u|_c + 2|u|_d - 1$.

The words in $\mathcal{W}_4$ correspond to walks that end on the x-axis but touch the line $y = -1$ at some point. For the purposes of this lemma, we will speak of walks and their corresponding words as if they are the same object.

In manner similar to that of Motzkin paths, we decompose a non-trivial w based on its first return to the x-axis. The decomposition depends on both w's first step and its first step back to the axis. If w begins with an e, we count that single step as both its first step and its first return to the axis. Therefore we have five cases:

Case 1. If w begins with e, it is factored as $w = ev$, where $v \in \mathcal{W}$.

Case 2. If w begins with a and first returns to the x-axis with c, it is factored as $w = aucv$, where $u, v \in \mathcal{W}$.

Case 3. If w begins with a and first returns with d, it is factored as $w = aw_3du$, where $u \in \mathcal{W}$, and $w_3 \in \mathcal{W}_3$.

Case 4. If w begins with b and first returns with c, it is factored as $w = bw_2cu$, where $u \in \mathcal{W}$, and $w_2 \in \mathcal{W}_2$.

Case 5. If w begins with b and first returns with d, it is factored as $w = bw_4du + bvdv'$, where $u, v, v' \in \mathcal{W}$, and $w_4 \in \mathcal{W}_4$. The plus sign in such a decomposition represents an exclusive or, *i.e.*, w satisfies precisely one of the two possible decompositions. This case is different from the others because w might never touch the line $y = 1$ between its first step and its first return.

Figure 3.1 shows the decomposition of Case 4 in terms of a walk.

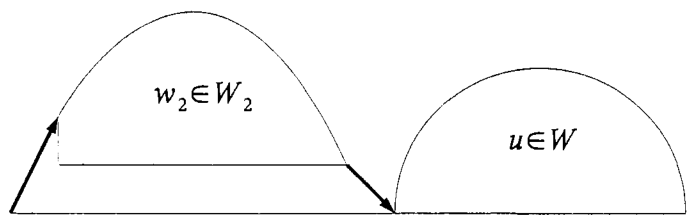


Figure 3.1: Decomposition of w in Case 4.

We next decompose a word $w_2 \in \mathcal{W}_2$ uniquely based on its first step onto the line $y = -1$. If its first step to $y = -1$ is c, we factor w_2 as $w_2 = ucv$, where $u, v \in \mathcal{W}$. Otherwise we factor w_2 as $w_2 = w_3 dw$, where $w \in \mathcal{W}$, and $w_3 \in \mathcal{W}_3$.

In contrast to $\mathcal{W}_2$, we uniquely decompose $w_3 \in \mathcal{W}_3$ based on its last return to the $x - axis$. If that step is a, we factor w_3 as $w_3 = uav$, where $u, v \in \mathcal{W}$. Otherwise we factor w_3 as $w_3 = wbw_2$, where $w \in \mathcal{W}$, and $w_2 \in \mathcal{W}_2$.

Finally, we consider $w_4 \in \mathcal{W}_4$. We know that w_4 must step down to the line $y = -1$, so we uniquely decompose it based on its final step from that line. Again we have two possibilities: if the last step from $y = -1$ is a, we factor w_4 as $w_4 = w_2 aw$, where $w \in \mathcal{W}$, and $w_2 \in \mathcal{W}_2$. Otherwise we factor w_4 as $w_4 = u_2 bv_2$, where $u_2, v_2 \in \mathcal{W}_2$.

From the above unique decompositions, we determine that $(\mathcal{W}, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4)$ is a solution to the proper algebraic system

$$\begin{aligned} w &= 1 + ew + awcw + aw_3dw + bw_2cw + bw_4dw + bwdw \\ w_2 &= wcw + w_3dw \\ w_3 &= waw + wbw_2 \\ w_4 &= w_2(aw + bw_2). \end{aligned}$$

Therefore W is an algebraic language.

We now prove that the language $\mathcal{X} \subset \{a,b,c,d,e\}^*$ defined by the restriction $|u|_a + 2|u|_b \geq |u|_c + 2|u|_d$ is algebraic. In order to do so, we must define a second language $\bar{\mathcal{X}} = \{u \in \{a,b,c,d,e\}^* : |u|_a + 2|u|_b \geq |u|_c + 2|u|_d - 1\}$.

For $z \in \mathcal{X}$ there is a unique decomposition based on the *last* time z leaves the $x - axis$. This decomposition depends on the step taken by z from the axis for the last time. If z ends on the axis, it is in $\mathcal{W}$. Otherwise it leaves the axis for the last time with either c or d. If its last step from the axis is c, it factors as $z = waz'$, where $w \in \mathcal{W}$, and $z' \in \mathcal{X}$. Otherwise it factors as $z = wbz'$ or $z = wb\bar{z}$, where $w \in \mathcal{W}$, $z' \in \mathcal{X}$, and $\bar{z} \in \bar{\mathcal{X}}$.

For $\bar{z} \in \bar{\mathcal{X}}$ there is a unique decomposition based on the last time $\bar{z}$ steps to the region above the x-axis from below. If this last step is from the axis, $\bar{z}$ factors as $\bar{z} = w_4az$, $\bar{z} = w_4bz$, or $\bar{z} = w_4b\bar{z}'$, where $w_4 \in \mathcal{W}_4$, $z \in \mathcal{X}$, and $\bar{z}' \in \bar{\mathcal{X}}$. Otherwise $\bar{z}$ factors as $\bar{z} = w_2bz$, where $w_2 \in \mathcal{W}_2$, and $z \in \mathcal{X}$. Therefore $(\mathcal{X}, \bar{\mathcal{X}})$ is a solution to the proper algebraic system

$$z = w + waz + wb\bar{z} + wbz$$

$$\bar{z} = w_4(az + b\bar{z} + bz) + w_2bz,$$

so $\mathcal{X}$ is an algebraic language. Figure 3.1 shows the decomposition of z in terms of a walk.

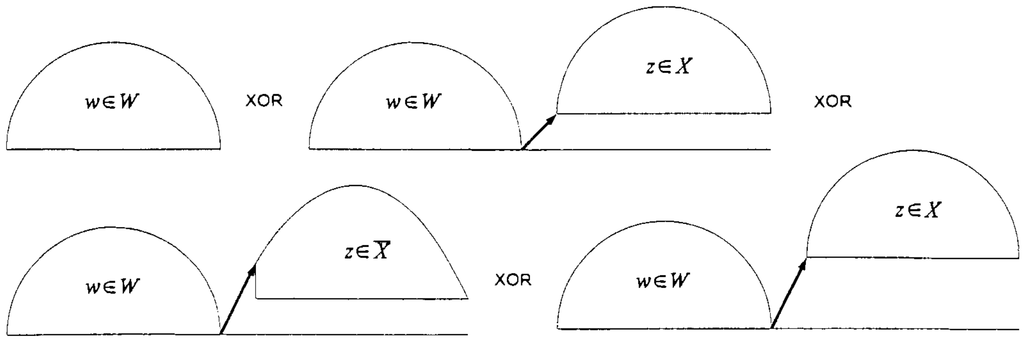


Figure 3.2: Decomposition of z.

Finally, to obtain the algebraic language $\mathcal{Z}$ isomorphic to $\mathcal{L}(\mathcal{Y})$ we simply let $a = a_{\rho(1)} + \cdots + a_{\rho(i)}$, let $b = a_{\rho(i+1)} + \cdots + a_{\rho(j)}$, let $c = a_{\rho(j+1)} + \cdots + a_{\rho(k)}$, let $d = a_{\rho(k+1)} + \cdots + a_{\rho(m)}$, and let $e = a_{\rho(m+1)} + \cdots + a_{\rho(n)}$.

Given a step set $\mathcal{Y}$, we let $Q_{\mathcal{Y}}(x, y; t) = \phi(\mathcal{Z})$, which is algebraic by Theorem 1.2.

□

Example 3.2. Let $\mathcal{Y} = \{\mathsf{NE}, \mathsf{SE}, \mathsf{E}\}$ and consider the walks $\mathcal{L}(\mathcal{Y})$ generated by $\mathcal{Y}$ in the region $\mathcal{R}_{\pi/2}$. Then $\mathcal{L}(\mathcal{Y})$ is characterized by the single restriction $|u|_{\mathsf{NE}} \geq |u|_{\mathsf{SE}}$. By Lemma 3.1, $\mathcal{L}(\mathcal{Y})$ is isomorphic to a language $\mathcal{Z} \subseteq \{a, b, c, d, e\}^*$ defined by the following proper algebraic system:

$$w = 1 + bw + awcw$$

$$z = w + waz.$$

We apply ψ to obtain the following generating functions:

$$\begin{aligned} W(x;t) &= 1 + xtW(x;t) + t^2 W(x;t)^2 \\ Q(x,y;t) &= W(x;t) + xytW(x;t)Q(x,y;t) \\ &= 1 + (xy + x)t + (2x^2 + x^2y^2 + 2x^2y)t^2 + O(t^3). \end{aligned}$$

Notice that $W(x;t) = Q(x,0;t)$ is the complete generating function for Motzkin paths.

Lemma 3.1 applies to walks whose step sets are of any cardinality and whose steps are not necessarily unit steps. For example, the walks in the first quadrant generated by the step set $\{(2,2),(-2,1),(0,1)\}$ have an algebraic complete generating function and are counted by an algebraic length generating function.

Banderier and Flajolet's result for simple walks in $\mathcal{R}_{\pi/2}$ can also be extended to walks in $\mathcal{R}_{\pi}$ that are not simple, and it can be stated in terms of a language.

Lemma 3.2. *Let $\mathcal{Y} = \{A_1, \ldots, A_n\}$ be a step set of cardinality n, and let the language $\mathcal{Z} \subseteq \{a_1, \ldots, a_n\}^*$ be isomorphic to $\mathcal{L}(\mathcal{Y})$. Suppose there exists a permutation ρ, positive integers $1 \leq j \leq m \leq n$, and a set of positive integers $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ such that we can define $\mathcal{Z}$ entirely by the condition that for any factorization $z = uv$ of a word $z \in \mathcal{Z}$,*

$$\gamma_1|u|_{a_{\rho(1)}} + \cdots + \gamma_j|u|_{a_{\rho(j)}} \geq \gamma_{j+1}|u|_{a_{\rho(j+1)}} + \cdots + \gamma_m|u|_{a_{\rho(m)}}. \tag{3.1.1}$$

Then the length generating function $Q_{\mathcal{Y}}(t)$ is algebraic.

Proof. Because of Banderier and Flajolet [1], it is sufficient to show that $\mathcal{Z}$ is isomorphic to a set of walks generated by a step set $\mathcal{X}$ that must remain in $\mathcal{R}_{\pi}$. We first define a function $\theta : \mathcal{Y} \to \mathbb{Z} \times \mathbb{Z}$ as follows:

$$\theta(A_{\rho(i)}) = \begin{cases} (\gamma_i, i) & \text{if } 1 \leq i \leq m, \\ (0, i) & \text{if } m+1 \leq i \leq n. \end{cases}$$

Essentially θ maps a step $A_{\rho(i)}$ to a step with a unique y coordinate that steps away from or towards the y-axis by a distance of γ_i. Denote $\theta(A_{\rho(i)})$ by B_i, and let $\mathcal{X} = \theta(\mathcal{Y})$. Then a walk w generated by $\mathcal{X}$ is in $\mathcal{R}_{\pi}$ if and only if, for every factorization $w = uv$, the following inequality is satisfied:

$$\gamma_1|u|_{B_1} + \cdots + \gamma_j|u|_{B_j} \geq \gamma_{j+1}|u|_{B_{j+1}} + \cdots + \gamma_m|u|_{B_m}. \tag{3.1.2}$$

Inequality (3.1.2) is the same as Inequality (3.1.1), so the walks generated by $\mathcal{X}$ in $\mathcal{R}_\pi$ are isomorphic to $\mathcal{Z}$. □

Lemma 3.2 is a more general result than Lemma 3.1, but our proof of Lemma 3.1 is constructive and relies solely on the combinatorial properties of algebraic languages. In contrast, the proof of Lemma 3.2 relies on Flajolet and Banderier's [1] result whose proof uses combinatorial arguments combined with complex analysis and Laurent polynomials.

The language in Lemma 3.1 is defined by a specific case of the Inequality (3.1.1) where $\Gamma = \{1, 2\}$. The method used to prove Lemma 3.1 can be used for any specific set of positive integers Γ, but the larger Γ is, the more complicated the construction.

3.2 Walks in the quarter-plane

Recall that the quarter-plane is defined as $\mathcal{R}_{\pi/2} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} | i \geq 0, j \geq 0\}$. This region is a well studied one, and this section outlines some of the results proven about walks in $\mathcal{R}_{\pi/2}$. As mentioned earlier, Dyck paths are easy to investigate because they are really only bounded by the x-axis.

The next step is to consider a step set that generates walks which may interact with both boundaries in $\mathcal{R}_{\pi/2}$. A walk that may interact with two boundaries is defined by two restrictions as opposed to the single restriction of Dyck paths. One such step set is $\mathcal{Y} = \{\mathsf{NE}, \mathsf{W}, \mathsf{S}\}$. The walks generated by $\mathcal{Y}$ are known as *Kreweras' walks* and are named for Germain Kreweras.

3.2.1 Kreweras' walks

Kreweras' walks are planar lattice walks that start at the origin, remain in $\mathcal{R}_{\pi/2}$, and are composed of three types of steps: NE, W, and S. Figure 3.3 shows an example of a Kreweras' walk of length 24.

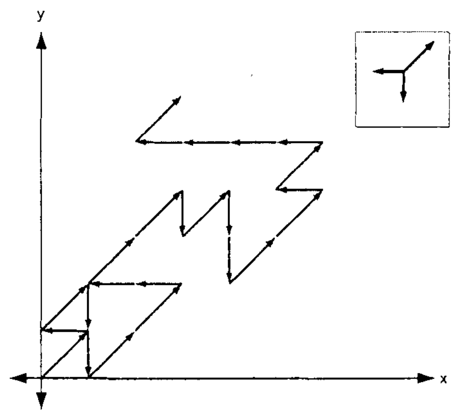


Figure 3.3: Example of a Kreweras' walk of length 24

In 1965 [18] Kreweras proved that the number of planar walks that begin at $(0,0)$, consist of $3n$ unit steps, any of which can be NE, S, or W and always remain in $\mathcal{R}_{\pi/2}$ is

$$a(3n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}. \tag{3.2.1}$$

Gessel [15] proved in 1986 that the generating function $Q(t)$ of the numbers $a(3n)$ is algebraic. However, the methods used by both Kreweras and Gessel involved first guessing the correct solution then proving it. The fact that this generating function is algebraic is interesting because $Q(t)$ cannot be constructed with a proper algebraic system, a result which can be proven with the "pumping lemma" for algebraic series [17]. The pumping lemma states a property that every context-free algebra must have, and the generating function for Kreweras' walks cannot have that property.

In 2005 Bousquet-Mélou [5] further investigated the enumeration of planar lattice walks using NE, S, or W steps. She proved that the complete generating function of those walks is algebraic. Unlike Kreweras and Gessel, she derived the solution constructively rather than conjecture and verify a solution. Her methods and results are summarized below.

Bernardi [2] recently constructed a purely combinatorial proof for Equation (3.2.1). He did so by finding a bijection between Kreweras' walks that end at the origin and loopless triangulations.

Enumeration of Kreweras' walks of length n

Consider planar lattice walks that start at $(0, 0)$, use NE, S, or W steps, and always remain $\mathcal{R}_{\pi/2}$ Let $a_{i,j}(n)$ be the number of n-step walks of this type ending at (i, j). Denote by $Q(x, y; t)$ the *complete generating function* of walks:

$$Q(x, y; t) := \sum_{i,j,n \geq 0} a_{i,j}(n) x^i y^j t^n.$$

We construct these walks recursively by starting considering a walk as a shorter walk with a single step appended to it. This recursive definition gives the functional equation

$$Q(x, y; t) = 1 + t \left(\frac{1}{x} + \frac{1}{y} + xy \right) Q(x, y; t) - \frac{t}{y} Q(x, 0; t) - \frac{t.}{x} Q(0, y; t). \qquad (3.2.2)$$

The first term in the right-hand side counts the empty walk. The next term shows the three ways one can add a step at the end of a walk. For example, $\frac{t}{x} Q(x, y; t)$ represents adding a step west at the end of a walk. Since the walk must remain in $\mathcal{R}_{\pi/2}$, we cannot add a S step to a walk ending on the x-axis, and we cannot add a W step to a walk ending on the y-axis. The last two terms in the right-hand side subtract the contributions of these forbidden moves.

There is a correspondence between walks ending on the x-axis and those ending on the y-axis. If we take a walk that ends on the x-axis and substitute a S step for each W step and vice versa, we effectively reflect the walk across the line $y = x$ to get a walk of the same length that ends on the y-axis. Therefore $Q(x, 0; t) = Q(0, x; t)$

and $Q(0, y; t) = Q(y, 0; t)$. Notice that $xy - t(x + y + x^2y^2)$ is symmetric in x and y. Thus we can swap x and y in Equation (3.2.2) to show that $Q(x, y; t) = Q(y, x; t)$, so $Q(x, y; t)$ is symmetric in x and y.

Equation (3.2.2) is equivalent to

$$\left(xy - t(x + y + x^2y^2)\right) Q(x, y) = xy - R(x) - R(y), \tag{3.2.3}$$

where

$$R(x) = xtQ(x, 0; t).$$

Because of the symmetry properties of $Q(x, y; t)$, $R(x)$ is well-defined.

In order to state Theorem 3.3 we first recall from Table A.1 that the coeffecient of x^i in $f(x)$ is denoted by by $[x^i]f(x)$.

Theorem 3.3 (Bousquet-Mélou [5]). *Let $W \equiv W(t)$ be the power series in t defined by*

$$W = t(2 + W^3).$$

Then the generating function of Kreweras' walks ending on the x-axis is

$$Q(x, 0; t) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{W} - \frac{1}{x}\right)\sqrt{1 - xW^2}\right).$$

Consequently, the length generating function of walks ending at $(i, 0)$ is

$$[x^i]Q(x, 0; t) = \frac{W^{2i+1}}{2 \cdot 4^i t}\left(C_i - \frac{C_{i+1}W^3}{4}\right),$$

where $C_i = \binom{2i}{i}/(i + 1)$ is the ith Catalan number. The Lagrange inversion formula gives the number of such walks of length $3n + 2i$ as

$$a_{i,0}(3n + 2i) = \frac{4^n(2i + 1)}{(n + i + 1)(2n + 2i + 1)}\binom{2i}{i}\binom{3n + 2i}{n}.$$

The complete generating function $Q(x, y; t)$ can be recovered using Equation (3.2.3):

$$Q(x, y; t) = \frac{(1/W - \bar{x})\sqrt{1 - xW^2} + (1/W - \bar{y})\sqrt{1 - yW^2}}{xy - t(x + y + x^2y^2)} - \frac{1}{xyt},$$

where $\bar{x} = 1/x$ and $\bar{y} = 1/y$.

The technique used by Bousquet-Mélou to prove Theorem 3.3 is called the *obstinate kernel method.* This method couples the variables x and y so as to cancel the *kernel* $K(x, y) = (xy - t(x + y + x^2y^2))$ to find information about the series $R(x)$. She then uses a tool called the *algebraic kernel method*, which we describe in detail below. With the algebraic kernel method, she found $Q_d(x) = \sum a_{i,i}(n)x^n$, the *diagonal* of $Q(x, y; t)$ that counts the Kreweras' walks of length n that end on the diagonal $y = x$. By finding $Q_d(x)$, she was then able to determine $Q(x, y; t)$. We will first use Dyck paths to give an example of the obstinate kernel method.

Obstinate kernel method applied to Dyck paths

Let $\mathcal{L}(\mathcal{Y})$ be the set of walks generated by $\mathcal{Y} = \{\mathsf{NE}, \mathsf{SE}\}$ in $\mathcal{R}_{\pi/2}$, and consider the complete generating function $Q_{\mathcal{Y}}(x, y; t)$ of $\mathcal{L}(\mathcal{Y})$. The Dyck paths are a subset of $\mathcal{L}(\mathcal{Y})$, and the complete generating function of the Dyck paths is $Q_{\mathcal{Y}}(x, 0; t)$. We construct the walks recursively in a manner similar to that of Kreweras' walks to obtain the functional equation

$$Q_{\mathcal{Y}}(x, y; t) = 1 + t(xy + x/y)Q_{\mathcal{Y}}(x, y; t) - \frac{tx}{y}Q_{\mathcal{Y}}(x, 0; t). \tag{3.2.4}$$

Equation (3.2.4) can equivalently be written as

$$(xy - tx^2y^2 - tx^2)Q_{\mathcal{Y}}(x, y; t) = xy - x^2tQ_{\mathcal{Y}}(x, 0; t). \tag{3.2.5}$$

We call $(xy - tx^2y^2 - tx^2)$ the *kernel* of $Q_{\mathcal{Y}}(x, y; t)$ and denote it by $K(x, y)$. We treat $K(x, y)$ as a function of y and find its roots:

$$\begin{aligned} y &= \frac{-x \pm \sqrt{x^2 - 4t^2x^4}}{-2tx^2} \\ &= \frac{1 \pm \sqrt{1 - 4t^2x^2}}{2tx}. \end{aligned}$$

Let $Y_0 = \frac{1-\sqrt{1-4t^2x^2}}{2tx}$, and let $Y_1 = \frac{1+\sqrt{1-4t^2x^2}}{2tx}$. Since Y_0 is a formal power series in t, we may substitute it into Equation (3.2.5) to obtain

$$\begin{aligned} 0 &= xY_0 - x^2tQ_{\mathcal{Y}}(x, 0; t) \\ Q_{\mathcal{Y}}(x, 0; t) &= \frac{xY_0}{x^2t} \\ &= \frac{1 - \sqrt{1 - 4t^2x^2}}{2t^2x^2}. \end{aligned} \tag{3.2.6}$$

Finally, we substitute Equation (3.2.6) into Equation (3.2.5) and divide by $K(x, y)$ to obtain

$$Q_{\mathcal{Y}}(x, y; t) = \frac{1 - 2x^3yt^2 - \sqrt{1 - 4t^2x^2}}{2t^2x^2(tx^2y^2 + tx^2 - xy)}. \tag{3.2.7}$$

Algeraic kernel method applied to Kreweras' walks

Equation (3.2.3) can be rewritten as

$$K(x, y)Q(x, y) = xy - R(x) - R(y).$$

The algebraic kernel method couples the x and y variables and uses the algebraic properties of $K(x, y)$ to determine Q_d. Equivalently,

$$xyK_r(x, y)Q(x, y) = xy - R(x) - R(y), \tag{3.2.8}$$

where $K_r(x, y) = 1 - t(\bar{x} + \bar{y} + xy)$ is the rational version of the kernel K. K_r has an invariance property:

$$K_r(x, y) = K_r(\bar{x}\bar{y}, y) = K_r(x, \bar{x}\bar{y}) \equiv K_r.$$

Iterative application of the involutive transformations $\Phi : (x, y) \mapsto (\bar{x}\bar{y}, y)$ and $\Psi : (x, y) \mapsto (x, \bar{x}\bar{y})$ gives us the pairs (x, y), $(\bar{x}\bar{y}, y)$, $(\bar{x}\bar{y}, x)$, $(x, \bar{x}\bar{y})$, and $(y, \bar{x}\bar{y})$, all of which can be substituted for (x, y) in Equation (3.2.8).

These substitutions give three equations:

$$\begin{aligned} xyK_rQ(x, y) &= xy - R(x) - R(y), \\ \bar{x}K_rQ(\bar{x}\bar{y}, y) &= \bar{x} - R(\bar{x}\bar{y}) - R(y), \\ \bar{y}K_rQ(x, \bar{x}\bar{y}) &= \bar{y} - R(x) - R(\bar{x}\bar{y}). \end{aligned}$$

Summing the first and third equations and subtracting the second one gives:

$$\begin{aligned} K_r\left(xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(x, \bar{x}\bar{y})\right) &= xy - \bar{x} + \bar{y} - 2R(x) \\ &= \frac{1 - K_r}{t} - 2\bar{x} - 2R(x). \end{aligned}$$

Equivalently,

$$xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(x, \bar{x}\bar{y}) + \frac{1}{t} = \frac{1}{K_r}\left(\frac{1}{t} - 2\bar{x} - 2R(x)\right). \qquad (3.2.9)$$

We now consider Equation (3.2.9) in terms of formal power series in t and examine the constant term in y, *i.e.*, the sum of all the terms that contain no powers of y. On the left hand side of Equation (3.2.9), the only terms without a power of y as a coefficient are in $\bar{x}Q(\bar{x}\bar{y}, y; t)$ and $1/t$, so we may ignore the other terms. Recall that

$$xQ(xy, \bar{y}; t) = x \sum_{i,j,n\geq 0} (xy)^i (\bar{y})^j t^n a_{i,j}(n).$$

The constant terms with respect to y are those in which $i = j$, *i.e.*,

$$[y^0]xQ(xy, \bar{y}; t) = x \sum_{i=j,n\geq 0} x^i t^n a_{i,i}(n).$$

Walks for which $i = j$ are those that end on the diagonal $y = x$. These walks are encoded by the series

$$\sum_{i=j,n\geq 0} x^i t^n a_{i,i}(n),$$

which we denote by $Q_d(x)$. Thus, the constant term with respect to y on the left hand side of Equation (3.2.9) is $\bar{x}Q_d(\bar{x}) + 1/t$. To determine Q_d we need only to extract the terms constant in y on the right hand side of Equation (3.2.9).

As a polynomial in y, $K(x,y)$ has two roots:

$$Y_0(x) = \frac{1 - t\bar{x} - \sqrt{(1-t\bar{x})^2 - 4t^2x}}{2tx}$$

$$Y_1(x) = \frac{1 - t\bar{x} + \sqrt{(1-t\bar{x})^2 - 4t^2x}}{2tx}.$$

Using Y_0 and Y_1, $K(x,y)$ factors as $-tx^2(y-Y_0)(y-Y_1)$, so $K_r = -tx\bar{y}(y-Y_0)(y-Y_1)$. Below we use partial fractions of y to express K_r as a Laurent series in y:

$$K_r = txY_1(1 - Y_0\bar{y})(1 - y/Y_1)$$
$$\frac{1}{K_r} = \frac{A}{txY_1} + \frac{B}{1 - Y_0\bar{y}} + \frac{C}{1 - y/Y_1}.$$

Gathering like powers of y yields the system of equations

$$1 = A + A\frac{Y_0}{Y_1} + BtxY_1 + CtxY_1$$
$$0 = -Y_0A - Ct$$
$$0 = \frac{-1}{Y_1} - Btx,$$

which we solve to determine

$$A = \frac{Y_1}{Y_0 - Y_1}\text{, and } B = C = \frac{-1}{tx(Y_0 - Y_1)}.$$

Let $\Delta(x) = (1 - t\bar{x})^2 - 4t^2x$, which is the descriminant of Y_0 and Y_1. Since $Y_0 - Y_1 = -\sqrt{\Delta(x)}/tx$, there is the following expression for $1/K_r$:

$$\frac{1}{K_r} = \frac{1}{\sqrt{\Delta(x)}}\left(\frac{1}{1 - \bar{y}Y_0} + \frac{1}{1 - y/Y_1} - 1\right)$$
$$= \frac{1}{\sqrt{\Delta(x)}}\left(\sum_{n\geq 0} y^{-n}Y_0^n + \sum_{n\geq 1} y^nY_1^{-n}\right). \qquad (3.2.10)$$

Each of $\Delta(x)$, Y_0, and Y_1 can be expressed as a formal power series in t with coefficients in $\mathbb{Q}[x,\bar{x}]$, so the above expansion of $1/K_r$ is valid in the set of formal power series in t with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$. Therefore from Equation (3.2.9), we determine

$$-\bar{x}Q_d(\bar{x}) + \frac{1}{t} = \frac{1/t - 2\bar{x} - 2R(x)}{\sqrt{\Delta(x)}}.$$

As a Laurent polynomial in x, $\Delta(x)$ has three roots. Two of which (X_0 and X_1) are formal power series in $\sqrt{t}$; the other (X_2) is a Laurent series in t. The coefficients of these series can be computed inductively:

$$\begin{aligned} (1-t\bar{x})^2 - 4t^2x &= 0 \\ (x-t)^2 - 4t^2x^3 &= 0 \\ x &= \pm 2tx^{3/2} + t. \end{aligned}$$

Thus X_0 is defined by the relation $x = t + 2tx^{3/2}$, and X_1 is defined by $x = t - 2tx^{3/2}$.

To calculate X_2 we solve for the "other" x:

$$\begin{aligned} (1-t\bar{x})^2 - 4t^2x &= 0 \\ x &= \frac{x^2 - 4t^2x^3 - t^2}{2t}. \end{aligned} \tag{3.2.11}$$

Therefore X_2 is defined by the relation shown in Equation (3.2.11).

We now develop the initial series for X_0, X_1, and X_2 below by iteritave expansion:

$$\begin{aligned} X_0 &= t + 2t^2\sqrt{t} + 6t^4 + 21t^5\sqrt{t} + 80t^7 + \frac{1287}{4}t^8\sqrt{t} + \cdots, \\ X_1 &= t - 2t^2\sqrt{t} + 6t^4 - 21t^5\sqrt{t} + 80t^7 - \frac{1287}{4}t^8\sqrt{t} + \cdots, \\ X_2 &= \frac{1}{4t^2} - 2t - 12t^4 - 160t^7 - 2688t^{10} - 50688t^{13} + \cdots. \end{aligned}$$

In order to proceed further, we must first state an important lemma proven by Bousquet-Mélou and Schaeffer.

Lemma 3.4 (Factorization Lemma [3, 7]). *Let $\Delta(x;t)$ be a series in t with coefficients in $\mathbb{R}[x,\bar{x}]$, and assume $\Delta(x;0) = 1$. There exists a unique triple $(\Delta_0(t), \Delta_+(x;t), \Delta_-(\bar{x};t)) \equiv (\Delta_0, \Delta_+(x), \Delta_-(\bar{x}))$ of formal power series in t that satisfies the following conditions:*

1. $\Delta(x) = \Delta_0\Delta_+(x)\Delta_-(\bar{x})$;
2. *the coefficients of* Δ_0 *are in* $\mathbb{R}$;
3. *the coefficients of* $\Delta_+(x)$ *are in* $\mathbb{R}[x]$;
4. *the coefficients of* $\Delta_-(\bar{x})$ *are in* $\mathbb{R}[\bar{x}]$;
5. $\Delta(0) = \Delta_+(0;t) = \Delta_-(0;t) = \Delta_+(x;0) = \Delta_-(\bar{x};0) = 1$.

The proof of Lemma 3.4 gives an explicit construction of how to factor $\Delta(x)$ based on the series expansions of its roots as series over t. We use this proof to factor $\Delta(x)$:

$$\Delta(x) = \Delta_0\Delta_+(x)\Delta_-(\bar{x})$$

with

$$\Delta_0 = 4t^2X_2\ ,\ \Delta_+(x) = 1 - x/X_2\ ,\ \Delta_-(\bar{x}) = (1-\bar{x}X_0)(1-\bar{x}X_1).$$

Recall that W is defined by the formula $W = t(2+W^3)$. Though it is not obvious, $X_2 = 1/W^2$:

$$\begin{aligned} W &= t(2+W^3) \\ W - W^3t &= 2t \\ W &= \left(\frac{2t}{1-W^2t}\right) \\ \frac{1}{W^2} &= \left(\frac{1-W^2t}{2t}\right)^2. \end{aligned}$$

Note that $1/W^2$ has the same iterative definition as X_2. We use Maple to expand $1/W^2$:

$$\frac{1}{W^2} = \frac{1}{4t^2} - 2t - 12t^4 - 160t^7 + O(t^{10}).$$

Since the first terms of $1/W^2$ are the same as those of X_2, and $1/W^2$ satisfies the same recurrence as X_2, we may conclude that the two series are indeed equal. Therefore

$$\Delta_0 = 4t^2/W^2 \text{ and } \Delta_+(x) = 1 - xW^2.$$

We proceed with the canonical factorization of $\Delta(x)$ and the fact that $X_2 = 1/W^2$ to determine

$$\sqrt{\Delta_-(\bar{x})}\left(-\bar{x}Q_d(\bar{x}) + \frac{1}{t}\right) = \frac{1/t - 2\bar{x} - 2R(x)}{\sqrt{\Delta_0\Delta_+(x)}},$$

Extracting the nonnegative powers of x gives $R(x)$, from which $Q(x, 0; t)$ is determined:

$$Q(x, 0; t) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x}\left(\frac{1}{W} - \frac{1}{x}\right)\sqrt{1 - xW^2}\right).$$

From $Q(x, 0; t)$, $Q(x, y; t)$ can now be completely determined, and Theorem 3.3 follows.

3.2.2 Reverse Kreweras' walks

We now develop in more detail an example from Mishna [20] to further illustrate the importance of the algebraic kernel method of lattice path enumeration. Consider planar lattice walks that start at $(0, 0)$, use SW, N, or E steps and remain in $\mathcal{R}_{\pi/2}$. Walks created using these steps are simply Kreweras' walks done in reverse, so we call them *reverse Kreweras' walks*.

Enumeration of reverse Kreweras' walks of length n

Let $P(x, y; t)$ be the complete generating function for reverse Kreweras' walks. We obtain the functional equation

$$P(x, y; t) = 1 + t\left(\frac{1}{xy} + x + y\right) P(x, y; t) - \frac{t}{xy}P(0, y; t) - \frac{t}{xy}P(x, 0; t) + \frac{t}{xy}P(0, 0; t),$$

or equivalently

$$\left(xy - t(1 + x^2y + y^2x)\right) P(x, y; t) = xy - tP(0, y; t) - tP(x, 0; t) + tP(0, 0; t).$$

We let $L(x, y) = xy - t(1 + x^2y + y^2x)$ and let

$$L_r(x, y) = \frac{1}{xy}L(x, y) = 1 - t(\bar{x}\bar{y} + x + y). \tag{3.2.12}$$

This rational kernel has the same invariance property as the rational kernel for Kreweras' Walks, *i.e.*,

$$L_r(x, y) = L_r(\bar{x}\bar{y}, y) = L_r(x, \bar{x}\bar{y}) \equiv L_r.$$

Again, iterative application of the involutive transformations $\Phi : (x, y) \mapsto (\bar{x}\bar{y}, y)$ and $\Psi : (x, y) \mapsto (x, \bar{x}\bar{y})$ gives us the pairs (x, y), $(\bar{x}\bar{y}, y)$, $(\bar{x}\bar{y}, x)$, $(x, \bar{x}\bar{y})$, and $(y, \bar{x}\bar{y})$.

We now define a counterpart to L_r:

$$\bar{L}_r := L_r(\bar{x}, \bar{y}) = L_r(\bar{x}, xy) = L_r(xy, \bar{y}).$$

Substitution of the pairs $(\bar{x}, \bar{y})$, $(\bar{x}, xy)$, and $(xy, \bar{y})$ into Equation (3.2.12) yields the following system of equations:

$$\begin{aligned}
\bar{x}\bar{y}\bar{L}_r P(\bar{x}, \bar{y}; t) &= \bar{x}\bar{y} - tP(\bar{x}, 0; t) - tP(0, \bar{y}; t) + tP(0, 0; t)\\
y\bar{L}_r P(\bar{x}, xy; t) &= y - tP(\bar{x}, 0; t) - tP(0, xy; t) + tP(0, 0; t)\\
x\bar{L}_r P(xy, \bar{y}; t) &= x - tP(xy, 0; t) - tP(0, \bar{y}; t) + tP(0, 0; t).
\end{aligned}$$

Summing the first and second equation and subtracting the third one gives

$$\bar{x}\bar{y}P(\bar{x},\bar{y};t) + yP(\bar{x},xy;t) - xP(xy,\bar{y};t) = \frac{1}{\bar{L}_r}\left(\bar{x}\bar{y} + y - 2tP(\bar{x},0;t) + tP(0,0;t)\right). \tag{3.2.13}$$

We now apply the fact that the rational kernel $K_r(x,y)$ from the proof of Theorem 3.3 is equal to $\bar{L}_r(x,y)$ to express the right hand side of Equation (3.2.13):

$$\begin{aligned}&\frac{1}{K_r}(\bar{x}\bar{y} + y - x2tP(\bar{x},0;t) + tP(0,0;t))\\ &\qquad = \frac{1}{\sqrt{\Delta(x)}}\left(\sum_{n\geq 0} y^{-n}Y_0^n + \sum_{n\geq 1} y^n Y_1^{-n}\right)\left(\bar{x}\bar{y} + y - x2tP(\bar{x},0;t) + tP(0,0;t)\right).\end{aligned}$$

As in the case of Equation (3.2.9), extraction of the constant term of Equation (3.2.13) with respect to y on the left hand side yields $xP_d(x)$, where $P_d(x)$ is the generating function that counts the number of reverse Kreweras' walks ending on the diagonal. The constant term in y on the right hand side is more difficult to extract:

$$\begin{aligned}&\frac{1}{\sqrt{\Delta(x)}}\left(\sum_{n\geq 0} y^{-n}Y_0^n + \sum_{n\geq 1} y^n Y_1^{-n}\right)\left(\bar{x}\bar{y} + y - x2tP(\bar{x},0;t) + tP(0,0;t)\right)\\ &= \frac{1}{\sqrt{\Delta(x)}}\left(1 + \bar{y}Y_0 + \frac{y}{Y_1} + \frac{y^2}{Y_1^2} + \dots\right)\left(\bar{x}\bar{y} + y - x2tP(\bar{x},0;t) + tP(0,0;t)\right)\\ &= \frac{1}{\sqrt{\Delta(x)}}\left(\frac{\bar{x}}{Y_1} + Y_0 - x2tP(\bar{x},0;t) + tP(0,0;t)\right).\end{aligned}$$

$Y_0Y_1 = \bar{x}$, so extraction of the constant term in y from Equation (3.2.13) yields

$$\begin{aligned}xP_d(x) &= \frac{1}{\sqrt{\Delta(x)}}\left(\frac{1}{xY_1} + Y_0 - x - 2tP(\bar{x},0;t) + tP(0,0;t)\right)\\ &= \frac{1}{\sqrt{\Delta(x)}}\left(2Y_0 - x - 2tP(\bar{x},0;t) + tP(0,0;t)\right).\end{aligned} \tag{3.2.14}$$

For a series $f(x,\bar{x},t)$ in $\mathbb{C}[x,\bar{x}][[t]]$, let $f^{\leq}$ denote the series obtained from f by keeping only those terms whose coefficients of t^n are polynomials in $\mathbb{C}[\bar{x}]$. We now use the canonical factorization of $\Delta(x)$, use the fact that $\Delta_0 = 4t^2/W^2$ and $\Delta_+(x) = 1 - xW^2$, and extract the non-positive powers of x to determine $P(\bar{x}, 0; t)$:

$$0 = \frac{-2tP(\bar{x},0,t) + tP(0,0,t)}{\sqrt{\Delta_0\Delta_-(\bar{x})}} - \left(\frac{x - 2Y_0}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leq}. \tag{3.2.15}$$

First we calculate

$$\frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}} = \frac{x}{\sqrt{\Delta_0}}\frac{1}{\sqrt{\Delta_-(\bar{x})}}.$$

Bousquet-Mélou[5] showed that $\Delta_-(\bar{x})$ can be expressed as $1 - \bar{x}W(1 + W^3/4) + \bar{x}^2W^2/4$. If we let $\mathcal{X} = \bar{x}W^2/4 - W(1 + W^3/4)$, then $\sqrt{\Delta_-(\bar{x})} = \sqrt{1 - \bar{x}\mathcal{X}}$ Note that $\mathcal{X} \in k[[t]]$. Therefore

$$\begin{aligned}
\frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}} &= \frac{x}{\sqrt{\Delta_0}}\frac{1}{\sqrt{1-\bar{x}\mathcal{X}}} \\
&= \frac{x}{\sqrt{\Delta_0}}\left(1 - \frac{\bar{x}\mathcal{X}}{2} + O(\bar{x}^2)\right) \\
&= \frac{x}{\sqrt{\Delta_0}} - \frac{\mathcal{X}}{2\sqrt{\Delta_0}} + O(\bar{x}) \\
\Rightarrow \left(\frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leq} &= \frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}} - \frac{x}{\sqrt{\Delta_0}}.
\end{aligned}$$

From this we determine

$$\begin{aligned}
\left(\frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leq}\left(\sqrt{\Delta_0\Delta_-(\bar{x})}\right) &= \left(\frac{x}{\sqrt{\Delta_0\Delta_-(\bar{x})}} - \frac{x}{\sqrt{\Delta_0}}\right)\left(\sqrt{\Delta_0\Delta_-(\bar{x})}\right) \\
&= x\left(1 - \sqrt{\Delta_-(\bar{x})}\right).
\end{aligned}$$

The other term in the right hand side of Equation (3.2.15) is more difficult to determine:

$$\begin{aligned}\left(\frac{2Y_0}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leq} &= \left(\frac{1-t\bar{x}-\sqrt{\Delta(x)}}{tx\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leq} \\ &= \left(\frac{1-t\bar{x}}{tx\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leq} - \left(\frac{\sqrt{\Delta_+(x)}}{xt}\right)^{\leq} \\ &= \frac{\bar{x}\left(\frac{1-t\bar{x}}{t}\right)}{\sqrt{\Delta_0\Delta_-(\bar{x})}} - \left(\frac{\sqrt{\Delta_+(x)}}{xt}\right)^{\leq}. \end{aligned} \tag{3.2.16}$$

We next develop $\frac{\bar{x}}{t}\sqrt{\Delta_+(x)}$ as a series in x:

$$\begin{aligned}\left(\frac{\sqrt{\Delta_+(x)}}{xt}\right)^{\leq} &= \left(\frac{\sqrt{1-xW^2}}{xt}\right)^{\leq} \\ &= \left(\frac{\bar{x}}{t}\left(1-\frac{xW^2}{2}-O(x^2)\right)\right)^{\leq} \\ &= \frac{\bar{x}}{t}-\frac{W^2}{2t}. \end{aligned} \tag{3.2.17}$$

Combine Equations (3.2.16) and (3.2.17):

$$\left(\frac{2Y_0}{\sqrt{\Delta_0\Delta_-(\bar{x})}}\right)^{\leq}\sqrt{\Delta_0\Delta_-(\bar{x})} = \frac{1}{t}\left(\bar{x}(1-t\bar{x}) - \left(\bar{x}-\frac{W^2}{2}\right)\sqrt{\Delta_0\Delta_-(\bar{x})}\right).$$

Therefore,

$$2P(\bar{x},0;t) = P(0,0;t) - \frac{1}{t}\left(x\left(1-\sqrt{\Delta_-(\bar{x})}\right) - \frac{1}{t}\left(\bar{x}(1-t\bar{x}) - \left(\bar{x}-\frac{W^2}{2}\right)\sqrt{\Delta_0\Delta_-(\bar{x})}\right)\right),$$

so

$$\begin{aligned}2P(x,0;t) =& P(0,0;t) - \frac{1}{t}\left(\bar{x}\left(1-\sqrt{\Delta_-(x)}\right) - \frac{1}{t}\left(x(1-tx) - \left(x-\frac{W^2}{2}\right)\sqrt{\Delta_0\Delta_-(\bar{x})}\right)\right)\\ =& P(0,0;t) - \frac{1}{t}\left(\bar{x}\left(1-\sqrt{\Delta_-(x)}\right) - \frac{1}{t}\left(x(1-tx) - \left(x-\frac{W^2}{2}\right)\frac{2t}{W}\sqrt{\Delta_-(x)}\right)\right)\\ =& P(0,0;t) - \left(\frac{1}{xt}+\frac{x^2}{t}-\frac{x}{t^2}\right) - \sqrt{\Delta_-(x)}\left(-\frac{1}{xt}+\frac{2x}{tW}-\frac{W}{t}\right).\end{aligned}$$

The only remaining unknown term is $P(0,0;t)$, which counts the number of reverse Kreweras' walks ending at $(0,0)$. Fortunately, each reverse Kreweras' walk that ends in the origin is simply a Kreweras' walk done in reverse. Thus we can extract $P(0,0;t)$ from Theorem 3.3:

$$P(0,0;t) = \frac{W}{2t}\left(1-\frac{W^3}{4}\right).$$

We now have a complete expression for $P(x,0;t)$, and thus $P(x,y;t)$.

Theorem 3.5. *Let $W \equiv W(t)$ be the power series in t defined by*

$$W = t(2+W^3),$$

and let $V \equiv V(t)$ be the power series in t defined by

$$V = 1 - xW(1+W^3/4) + x^2W^2/4.$$

Then the generating function of reverse Kreweras' walks ending on the x-axis is

$$P(x,0;t) = \frac{1}{2}\left(\frac{W}{2t}\left(1-\frac{W^3}{4}\right) - \left(\frac{1}{xt}+\frac{x^2}{t}-\frac{x}{t^2}\right) - \sqrt{V}\left(-\frac{1}{xt}+\frac{2x}{tW}-\frac{W}{t}\right)\right).$$

Corollary 3.6. *The complete generating function for reverse Kreweras' walks is*

$$P(x,y;t) = \frac{xy - U(x) - U(y) + U(0)}{xy - t(1 + x^2y + y^2x)},$$

where

$$U(x) = tP(x,0;t).$$

Corollary 3.7. *Let $W(t)$ and $V(t)$ be defined as in Theorem 3.5. The length generating function for the reverse Kreweras' walks is*

$$P(1,1;t) = \frac{1 - 2U(1) + U(0)}{(1-3t)},$$

where

$$U(x) = tP(x,0;t).$$

3.2.3 Double Kreweras' walks

The complete and length generating functions of both Kreweras' and reverse Kreweras' walks are algebraic, which leads us to wonder if there exist other generalizations of the Kreweras' walks with algebraic complete and length generating functions.

We consider the walks generated by the union of the steps that generate the Kreweras' and reverse Kreweras' walks. That is, let $\mathcal{Y} = \{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{S}, \mathsf{SW}, \mathsf{W}\}$. We call the walks generated by $\mathcal{Y}$ in $\mathcal{R}_{\pi/2}$ *double Kreweras'* walks, and denote the complete generating function of these walks by $Q_d(x,y;t)$. We write $Q_d(x,y;t)$ in terms of its iterative definition below:

$$\begin{aligned} Q_d(x,y;t) =& 1 + t(y + xy + x + \bar{y} + \bar{x}\bar{y} + \bar{x})Q_de(x,y;t) \\ &- t(\bar{x} + \bar{x}\bar{y})Q_d(0,y;t) - t(\bar{y} + \bar{x}\bar{y})Q_d(x,0;t) + t\bar{x}\bar{y}Q_d(0,0;t). \end{aligned} \quad (3.2.18)$$

Unfortunately, we have been unable to successfully apply the algebraic kernel method to Equation (3.2.18). One problem is that the step set for double Kreweras' walks is "too symmetric." The kernel of $Q_d(x, y; t)$ is invariant over the same x, y pairs as the Kreweras' and reverse Kreweras' walks, but substituting the different pairs into Equation (3.2.18) does not yield enough different equations to determine anything about $Q_d(x, y; t)$. However, with the aid of the *GFUN* package in MAPLE (see Appendix B) we hypothesize that the length generating function for the double Kreweras' walks, which we denote by $Q_d(t)$, satisfies the following two equations:

$$0 = 2t^3(6t-1)Q_d(t)^4 + 4t^2(6t-1)Q_d(t)^3 + 3t(6t-1)Q_d(t)^2 + (6t-1)Q_d(t) + 1 \tag{3.2.19}$$

$$0 = (6t-1)(2t+1)tQ_d'(t) + (12t^2 + 6t - 1)Q_d(t) + 1. \tag{3.2.20}$$

Therefore, it is our conjecture that the length generating function $Q_d(t)$ is algebraic.

We rearrange Equation (3.2.19):

$$Q_d(t) = \frac{1}{1-6t} - 3tQ_d(t)^2 - 4t^2Q_d(t)^3 - 2t^3Q_d(t)^4. \tag{3.2.21}$$

Since the generating function for unrestricted walks generated by $\mathcal{Y}$ in the planar lattice is $\frac{1}{1-6t}$, it appears that $Q_d(t)$ may simply be the subtraction of the generating function of walks that leave $\mathcal{R}_{\pi/2}$ from the unrestricted generating function. However, we have yet to find a proof that this is the case. The simplicity of $Q_d(t)$ is encouraging in that, if it were correct, there may be a nice combinatorial argument for a proof. From Equation (3.2.20) we also determine the following recursion for the coefficients $a(n)$ of t^n in $Q_d(t)$:

$$a_0 = 1,$$

$$a_1 = 3,$$

$$a_n = \frac{1}{n+1}(12(n-1)a_{n-2} + (4n+2)a_{n-1}) \text{ for } n \geq 2.$$

3.2.4 Classes of walks

There are many instances of different step sets that lead to isomorphic walks. For example, the walks generated by a step set $\mathcal{Y}$ are isomorphic to those generated by the set made up of the steps in $\mathcal{Y}$ reflected across the line $y = x$. Instead of examining each step set individually, it is useful to collect the step sets into classes according to the length generating functions of their walks in a given region. Table 3.1 gives the isomorphic classes of step sets of cardinality three whose walk sets are non-trivial.

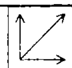
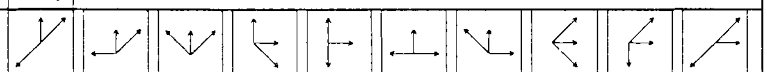
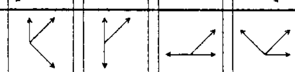
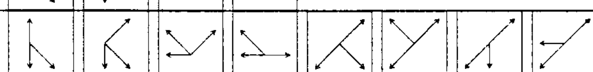
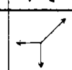
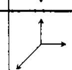
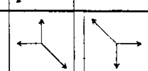
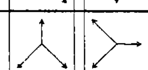
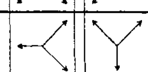
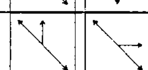
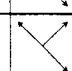
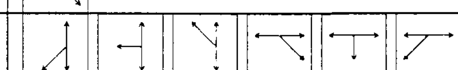
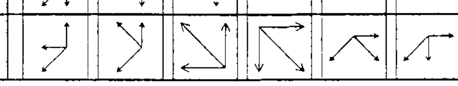
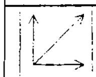
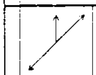
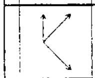
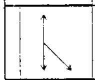
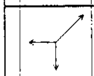
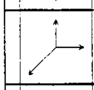
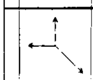
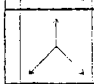
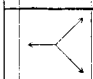
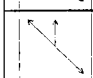
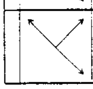
Class	$\mathcal{Y}$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	

Table 3.1: $\mathcal{R}_{\pi/2}$ step set classes

The generating functions for all walks generated by $\mathcal{Y}$ in the first quadrant where

$|\mathcal{Y}| = 3$ have been classified by Mishna [20]. Table 3.2 summarizes her results.

$\mathcal{Y}$	Counting GF	$Q_{\mathcal{Y}}(x, y; t) = \sum a_{ij}(n) x^i y^j t^n$
	$(1-3t)^{-1}$	$(1 - t(x+y+xy))^{-1}$
	$\frac{1-4t-\sqrt{1-8t^2}}{4t(3t-1)}$	$\frac{1-yt-\sqrt{1-2yt+t^2y^2-4t^2}}{t(2t-yx+y^2xt+yx\sqrt{1-2yt+t^2y^2-4t^2})}$
	$\frac{1-3t-\sqrt{1-2t-3t^2}}{2t(3t-1)}$	$\frac{1-\sqrt{1-4xt^2-4x^2t^2}}{t(2xt-y+y\sqrt{1-4xt^2-4x^2t^2})}$
	$\frac{1-2t-\sqrt{1-8t^2}}{2t(3t-1)}$	$\frac{1-\sqrt{1-4t^2-4xt^2}}{t(2t+2xt-y+y\sqrt{1-4t^2-4xt^2})}$
	$\frac{W(1-t)+2t(W-1)\sqrt{1-W^2}}{tW(3t-1)}$	$\frac{xy-R(x,t)-R(y,t)}{xy-t(x+y+x^2y^2)}$
	$\frac{(W^2t+W-2t)\sqrt{(1-W)(1+W^2/r+W^3/4)}+W+Wt}{tW(3t-1)}$	$\frac{xy-U(x,t)-U(y,t)}{xy-t(x+y+x^2y^2)}$
	$\frac{1-t-\sqrt{(1+t)(1-3t)}}{2t^2}$	$a_{i,j}(n) = \frac{(i+1)(j+1)(i+j+2)n!}{(\frac{n-i-2j}{3})!(\frac{n-i+j}{3}+1)!(\frac{n+2i+j}{3}+2)!}$
	D-Finite	
	D-Finite	
	non-D-Finite	
	non-D-Finite	

$W \equiv W(t)$ is the power series in t defined by $W = t(2 + W^3)$

$$V = 1 - xW(1 + W^3/4) + x^2W^2/4$$

$$R(x,t) = \frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{W} - \frac{1}{x}\right)\sqrt{1 - xW^2}$$

$$U(x,t) = \left(\frac{-2x}{W}\left(1 - \frac{W^2}{2x}\right) + \frac{1}{tx}\right)\frac{\sqrt{V}}{2} + \left(1 - tx - \frac{t}{x^2}\right)\frac{x}{2t}$$

Table 3.2: Classification of walks in $\mathcal{R}_{\pi/2}$ [20]

3.2.5 Step sets of cardinality greater than three

Mishna [20] states a conjecture which applies to step sets of all cardinalities that are made up unit steps. In order to state her conjecture, we must first define two operators on steps: rev and $\mathsf{reflect}$. The rev operator reverses the direction of a step, *i.e.*, $\mathsf{rev}(x, y) = (-x, -y)$. The $\mathsf{reflect}$ operator reflects the step across the line $y = x$, *i.e.*, $\mathsf{reflect}(x, y) = (y, x)$. We now state Mishna's conjecture:

The generating function $Q_{\mathcal{Y}}$ is holonomic if and only if at least one of the following holds:

- $\mathcal{L}(\mathcal{Y})$ are the Kreweras' or reverse Kreweras' walks;
- $\mathcal{Y}$ is singular;
- $\mathcal{Y}$ is symmetric across the x- or y-axis;
- $\mathcal{Y} = \mathsf{rev}(\mathcal{Y})$;
- $\mathcal{Y} = \mathsf{reflect}(\mathsf{rev}(\mathcal{Y}))$.

3.2.6 Gessel's walks

Gessel conjectured that the length generating function $Q_{\mathcal{Y}}(t)$ for $\mathcal{Y} = \{\mathsf{N}, \mathsf{SE}, \mathsf{S}, \mathsf{NW}\}$ is D-finite, which is also guessed by MAPLE using the GFUN package. The GFUN package guesses that $Q_{\mathcal{Y}}(t)$ satisfies the following differential equation:

$$(-28x^3 - x^2 + 6x - 1)xQ'_{\mathcal{Y}}(t) + (-28x^3 - 12x^2 + 9x - 1)Q_{\mathcal{Y}}(t) - 5x + 1 = 0.$$

However, Zeilberger has conjectured that the complete generating function $Q_{\mathcal{Y}}(x, y; t)$ is not D-finite. Gessel's walks are interesting because if both Gessel and Zeilberger are correct, the walks would represent a first case in $\mathcal{R}_{\pi/2}$ where the complete generating function was not in the same class as the length generating function.

3.3 The eighth-plane

Recall that the $\frac{1}{8}$-plane is the region $\mathcal{R}_{\pi/4} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} | i \geq 0, i \geq j\}$. We consider this region because we would like to see if shrinking the lattice region in question to one smaller than $\mathcal{R}_{\pi/2}$ results in a significant change in the behaviour of the generating functions of walks. We specifically choose $\mathcal{R}_{\pi/4}$ for two reasons: the first is that it lacks any reflective symmetry in the planar lattice, whereas $\mathcal{R}_{\pi/2}$ is symmetric across the line $y = x$. The second reason is that the boundaries of $\mathcal{R}_{\pi/4}$ are each parallel to a unit step.

3.3.1 $\mathcal{R}_{\pi/2}$ to $\mathcal{R}_{\pi/4}$ isomorphism

When we consider walks in terms of their restrictions, it is clear that some walks in $\mathcal{R}_{\pi/4}$ are isomorphic to walks in $\mathcal{R}_{\pi/2}$. For example, the walks generated in $\mathcal{R}_{\pi/4}$ by $\{\mathsf{NE}, \mathsf{S}, \mathsf{W}\}$ are isomorphic to the walks generated by $\{\mathsf{NE}, \mathsf{SE}, \mathsf{W}\}$ in $\mathcal{R}_{\pi/2}$ because both are defined by the restriction $|u|_a \geq |u|_b \geq |u|_c$. In this section we define a natural isomorphism between walks in $\mathcal{R}_{\pi/4}$ and walks in $\mathcal{R}_{\pi/2}$.

Define $\psi : \mathcal{R}_{\pi/2} \rightarrow \mathcal{R}_{\pi/4}$ by $\psi((x,y)) = (x+y, y)$. Let $w = w_0, w_1, \ldots, w_n$ be a planar lattice walk. Define the mapping Ψ on the set of planar lattice walks by $\Psi(w) = \psi(w_0), \psi(w_1), \ldots, \psi(w_n)$. Since ψ is a function, it is obvious that Ψ is well defined.

Lemma 3.8. *The mapping Ψ is an isomorphism that maps walks from $\mathcal{R}_{\pi/2}$ to $\mathcal{R}_{\pi/4}$.*

Proof. We first show that ψ is a bijection from $\mathbb{Z} \times \mathbb{Z}$ to itself. Suppose $\psi((x,y)) = \psi((a,b))$. Then $(x+y, y) = (a+b, b)$, which implies that $y = b$ and $x = a$. Therefore ψ is one-to-one. Let $(x,y) \in \mathbb{Z} \times \mathbb{Z}$. Then $(x-y, y) \in \mathbb{Z} \times \mathbb{Z}$, and therefore ψ is onto. Thus ψ is a bijection from the set of all lattice steps to itself.

Since both w and $\Psi(w)$ are a sequence of points in $\mathbb{Z} \times \mathbb{Z}$, both represent a planar lattice walk of length n. It remains to show that $\Psi(w)$ is one-to-one and onto.

Let $w = w_0, w_1, \ldots, w_n$ and $v = v_0, v_1, \ldots, v_n$ be walks in $\mathcal{R}_{\pi/4}$, and suppose that $\Psi(w) = \Psi(v)$. Then for $0 \leq i \leq n$, $\psi(w_i) = \psi(v_i)$, which implies that $w = v$. Therefore, Ψ is one-to-one. Next, let $w = w_0, w_1, \ldots, w_n$ be a walk in $\mathcal{R}_{\pi/4}$, where $w_i = (a_i, b_i)$. Then for $0 \leq i \leq n$, $0 \leq b_i \leq a_i$, $\psi^{-1}((a_i, b_i)) = (a_i - b_i, b_i)$ is a point in $\mathcal{R}_{\pi/2}$. Thus Ψ is onto and is therefore an isomorphism.

Finally, note that $\psi((0,0)) = (0,0)$. Therefore if w is a walk that begins at the origin, so is $\Psi(w)$.

□

Table 3.3.1 shows the mapping of the steps $\{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{SE}, \mathsf{S}, \mathsf{SW}, \mathsf{W}, \mathsf{NW}\}$ in $\mathcal{R}_{\pi/4}$ to $\mathcal{R}_{\pi/2}$ by Ψ.

Step in $\mathcal{R}_{\pi/2}$-plane	Ψ(step)
$\mathsf{N} = (0,1)$	$\mathsf{NE} = (1,1)$
$\mathsf{NE} = (1,1)$	$(2,1)$
$\mathsf{E} = (1,0)$	$\mathsf{E} = (1,0)$
$\mathsf{SE} = (1,-1)$	$\mathsf{S} = (0,-1)$
$\mathsf{S} = (0,-1)$	$\mathsf{SW} = (-1,-1)$
$\mathsf{SW} = (-1,-1)$	$(-2,-1)$
$\mathsf{W} = (-1,0)$	$\mathsf{W} = (-1,0)$
$\mathsf{NW} = (-1,1)$	$\mathsf{N} = (0,1)$

Table 3.3: $\Psi\{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{SE}, \mathsf{S}, \mathsf{SW}, \mathsf{W}, \mathsf{NW}\}$

From these mappings we can see that a set of walks in $\mathcal{R}_{\pi/4}$ whose step set is a subset of $\{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{S}, \mathsf{SW}, \mathsf{W}\}$ is isomorphic to a set of walks in $\mathcal{R}_{\pi/2}$ whose step set is a subset of $\{\mathsf{N}, \mathsf{E}, \mathsf{SE}, \mathsf{S}, \mathsf{W}, \mathsf{NW}\}$.

For example, the set of walks in $\mathcal{R}_{\pi/4}$ generated by the step set $\{\mathsf{NE}, \mathsf{S}, \mathsf{W}\}$ is isomorphic to the set of walks in $\mathcal{R}_{\pi/2}$ generated by the step set $\{\mathsf{N}, \mathsf{SE}, \mathsf{W}\}$.

Another way to think of the isomorphism Ψ^{-1} is as a transformation on $\mathcal{R}_{\pi/2}$ itself. This transformation is illustrated in Figure 3.4.

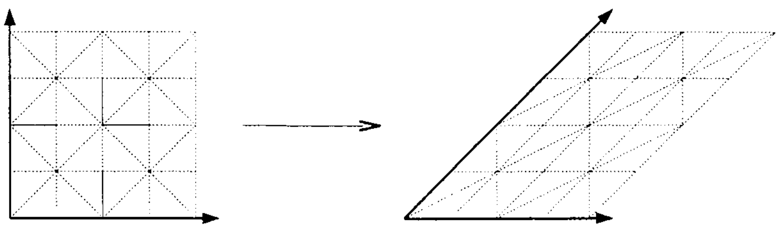


Figure 3.4: $\mathcal{R}_{\pi/2}$ mapped by Ψ^{-1} to $\mathcal{R}_{\pi/4}$

The isomorphism Ψ allows us to classify (as algebraic, D-finite, or neither) the generating function of every set of walks in $\mathcal{R}_{\pi/2}$ whose step set is of cardinality three and whose step set does not contain either NW or SE based on the classifications of generating functions of walks in the first quadrant. Table 3.3.1 summarizes the classification of walks in $\mathcal{R}_{\pi/2}$ we are able to classify using Ψ.

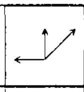
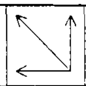
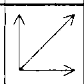
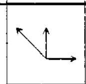
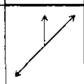
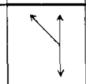
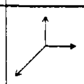
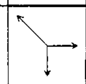
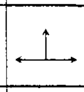
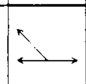
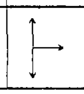
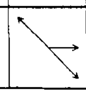
$\mathcal{Y}$ in $\frac{1}{8}$-plane	$\Psi(\mathcal{Y})$	Classification of $Q_{\mathcal{Y}}(x, y; t)$
		Rational
		Algebraic
		Algebraic
		Algebraic
		Algebraic
		Non-D-Finite

Table 3.4: Walks in $\mathcal{R}_{\pi/4}$ mapped by Ψ

We are further able to prove that two other classes of walks in $\mathcal{R}_{\pi/4}$ have algebraic complete generating functions. The length generating functions of these walks are algebraic by Banderier and Flajolet [1], but we find their complete generating functions and prove that they are algebraic.

Walks generated by $\mathcal{Y} = \{\mathsf{NW}, \mathsf{NE}, \mathsf{E}\}$

The walks generated by the step set $\{\mathsf{NW}, \mathsf{NE}, \mathsf{E}\}$ are isomorphic to the set $\mathcal{W}$ of all words $w \in \{a, b, c\}^*$ such that for all factorizations $w = uv$, $|u|_a \geq 2|u|_b$. That is to say, for every b in w, there must be at least two previous a's. The complete generating function of these words is algebraic, and thus so is its length generating function.

To prove that $Q_{\mathcal{Y}}(x, y; t)$ is algebraic, we simply apply Lemma 3.1. Let $\mathcal{Z} \subset \{a, b, c\}^*$ be the language isomorphic to $\mathcal{L}(\mathcal{Y})$. Then $(\mathcal{W}, \mathcal{W}_3, \mathcal{Z})$ is a solution to the proper algebraic system

$$w = 1 + aw + aw_3bw$$
$$w_3 = waw$$
$$z = w + waz.$$

Therefore both $\mathcal{Z}$ and $Q_{\mathcal{Y}}(x, y; t)$ are algebraic. Recall that $\phi(a) = xt$, $\phi(b) = \bar{x}\bar{y}t$, and $\phi(c) = xyt$ to obtain the following system:

$$W(x, y; t) = 1 + xytW(x, y; t) + xyt^3W(x, y; t)^3,$$
$$Q_{\mathcal{Y}}(x, y; t) = W(x, y; t) + xtW(x, y; t)Q_{\mathcal{Y}}(x, y; t).$$

Note that $W(x, y; t)$ counts the walks that end on the line $y = x$. We find the first few terms of $Q_{\mathcal{Y}}$:

$$Q_{\mathcal{Y}}(x, y; t) = 1 + (xy + x)t + (x^2 + 2x^2y + x^2y^2)t^2 + O(t^3).$$

Walks generated by $\mathcal{Y} = \{\mathsf{W}, \mathsf{NW}, \mathsf{E}\}$

The walks generated by the step set $\{\mathsf{W}, \mathsf{NW}, \mathsf{E}\}$ are isomorphic to the set $\mathcal{W}$ of all words $w \in \{a, b, c\}^*$ such that for all factorizations $w = uv$, $|u|_b + 2|u|_c \leq |u|_a$. Again, the complete generating function of these words is algebraic, and thus so is its length generating function.

To prove that $Q_{\mathcal{Y}}(x, y; t)$ is algebraic, we apply Lemma 3.1. Let $\mathcal{Z} \subset \{a, b, c\}^*$ be the language isomorphic to $\mathcal{L}(\mathcal{Y})$. Then $(\mathcal{W}, \mathcal{W}_3, \mathcal{Z})$ is a solution to the proper algebraic system

$$w = 1 + awcw + aw_3cw$$
$$w_3 = waw$$
$$z = w + waz.$$

Therefore both $\mathcal{Z}$ and $Q_{\mathcal{Y}}(x, y; t)$ are algebraic. We again recall that $\phi(a) = xt$, $\phi(b) = \bar{x}t$, and $\phi(c) = \bar{x}\bar{y}t$ to obtain the following system:

$$W(x, y; t) = 1 + t^2 W(x, y; t)^2 + xyt^3 W(x, y; t)^3,$$
$$Q_{\mathcal{Y}}(x, y; t) = W(x, y; t) + xtW(x, y; t)Q_{\mathcal{Y}}(x, y; t).$$

Again, $W(x, y; t)$ counts the walks that end on the line $y = x$. We find the first few terms of $Q_{\mathcal{Y}}$:

$$Q_{\mathcal{Y}}(x, y; t) = 1 + xt + (1 + x^2)t^2 + (2x + xy + x^3)t^3 + O(t^4)$$

3.3.2 Unclassified cases

Two interesting unclassified cases in $\mathcal{R}_{\pi/4}$ are $\mathcal{Y}_1 = \{\mathsf{NW}, \mathsf{SW}, \mathsf{E}\}$, which is defined by the restriction $2|u|_c \leq 2|u|_b \leq |u|_a$, and $\mathcal{Y}_2 = \{\mathsf{NE}, \mathsf{SE}, \mathsf{W}\}$, which is defined by the restriction $|u|_c \leq 2|u|_b \leq 2|u|_a$. These walks are interesting because their structure is very similar to that of the walks generated by $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in $\mathcal{R}_{\pi/2}$.

In $\mathcal{R}_{\pi/2}$, $Q_{\mathcal{Y}_1}(t)$ and $Q_{\mathcal{Y}_2}(t)$ are both D-finite via Theorem 3.9, which was proved by Bousquet-Mélou and Petkovšek [6]. Note that Theorem 3.9 does not apply to Gessel's walks.

Theorem 3.9 (Bousquet-Mélou and Petkovšek). *Let $\mathcal{Y}$ be a step set, and suppose that $(x, y) \in \mathcal{Y} \Rightarrow (-x, y) \in \mathcal{Y}$, and suppose that $|y| \leq 1$ for all $(x, y) \in \mathcal{Y}$. Then the complete generating function $Q_{\mathcal{Y}}(x, y; t)$ for the walks generated by $\mathcal{Y}$ in $\mathcal{R}_{\pi/2}$ is D-finite.*

Unfortunately we are unable to extend Theorem 3.9 to $\mathcal{R}_{\pi/4}$. Bousquet-Mélou and Petkovšek proved their theorem by describing a "correspondence" between certain walks in $\mathcal{R}_{\pi}$ and walks in $\mathcal{R}_{\pi/2}$. However, the proof of D-finiteness depends on the fact that the complete generating functions of walks in the half-plane are algebraic. To extend their method to $\mathcal{R}_{\pi/2}$ would require that all walks in the region $\mathcal{R}^*_{\pi/2}$ between $y = x$ and $y = -x$ have algebraic generating functions, and that region does not have that property (see Chapter 4).

However, the walks generated in $\mathcal{R}^*_{\pi/2}$ by $\mathcal{Y}_1$ are isomorphic to walks similar to Kreweras' walks, and the walks generated in $\mathcal{R}^*_{\pi/2}$ by Y_2 are isomorphic to walks similar to reverse Kreweras' walks. Therefore, we believe that in $\mathcal{R}_{\pi/4}$, $Q_{\mathcal{Y}_1}(t)$ and $Q_{\mathcal{Y}_2}(t)$ are both D-finite.

The remaining unclassified cases in $\mathcal{R}_{\pi/4}$ for $\mathcal{Y}$ of cardinality three are all very similar. Each of them is defined by a set of restrictions of the form $\alpha|u|_a \leq \beta|u|_b \leq \gamma|u|_a + \delta|u|_c$, where $\alpha, \beta, \gamma, \delta \in \mathbb{N}$. What that means is that any walk subject to these restrictions can never return to the origin. Mishna and Rechnitzer [21] have proven that walks defined by $|u|_a \leq |u|_b \leq |u|_a + |u|_c$ have a non-D-finite length generating function, and we believe that their result is evidence that our remaining unclassified walks in $\mathcal{R}_{\pi/2}$ also have non-D-finite length generating functions.

Table 3.3.2 outlines the walks classified in this section, as well as the conjectures for the unclassified walks. The column labeled "Classification" contains the class into which the complete generating function $Q_{\mathcal{Y}}(x, y; t)$ falls. Conjectures for the generating functions are denoted by an asterisk.

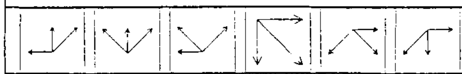
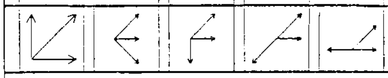
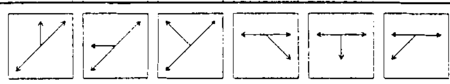
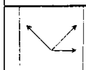
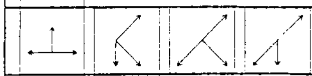
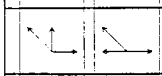
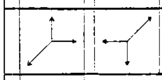
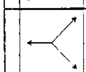
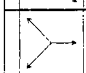
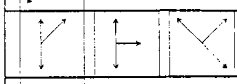
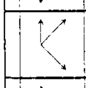
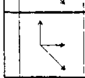
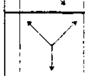
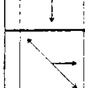
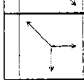
Class	Restrictions	Classification	cf.
	$\lvert u\rvert_a = \lvert u\rvert_b = 0$	Rational	$\frac{1}{1-t}$
	$\lvert u\rvert_b \leq \lvert u\rvert_a$	Algebraic	§ 3.2.4
	$\lvert u\rvert_b \leq \lvert u\rvert_a$, $\lvert u\rvert_c = 0$	Algebraic	§ 3.2.4
	$2\lvert u\rvert_b \leq \lvert u\rvert_a$	Algebraic	§ 3.2.4
	$\lvert u\rvert_b + \lvert u\rvert_c \leq \lvert u\rvert_a$	Algebraic	§ 3.2.4
	$\lvert u\rvert_b + 2\lvert u\rvert_c \leq \lvert u\rvert_a$	Algebraic	§ 3.2.4
	$\lvert u\rvert_c \leq \lvert u\rvert_b \leq \lvert u\rvert_a$	Algebraic	§ 3.2.4
	$\lvert u\rvert_c \leq 2\lvert u\rvert_b \leq 2\lvert u\rvert_a$	D-finite*	
	$2\lvert u\rvert_c \leq 2\lvert u\rvert_b \leq \lvert u\rvert_a$	D-finite*	
	$\lvert u\rvert_a \leq \lvert u\rvert_c \leq \lvert u\rvert_a + \lvert u\rvert_b$	non-D-finite	§ 3.2.4
	$\frac{1}{2}\lvert u\rvert_a \leq \lvert u\rvert_c \leq \lvert u\rvert_a + \lvert u\rvert_b$	non-D-finite*	
	$\lvert u\rvert_c \leq \lvert u\rvert_a \leq 2\lvert u\rvert_c + \lvert u\rvert_b$	non-D-finite*	
	$2\lvert u\rvert_c \leq \lvert u\rvert_b \leq \lvert u\rvert_a + \lvert u\rvert_c$	non-D-finite*	
	$2\lvert u\rvert_b \leq 2\lvert u\rvert_c \leq 2\lvert u\rvert_b + \lvert u\rvert_a$	non-D-finite*	
	$2\lvert u\rvert_b \leq 2\lvert u\rvert_c \leq \lvert u\rvert_b + \lvert u\rvert_a$	non-D-finite*	

Table 3.5: $\mathcal{R}_{\pi/4}$ step set classes (conjectures denoted by $*$)

In both $\mathcal{R}_{\pi/4}$ and $\mathcal{R}_{\pi/2}$ there are step sets that yield algebraic generating functions, and there are step sets that yield non-D-finite generating functions. We also have an explicit isomorphism which maps walks between $\mathcal{R}_{\pi/4}$ and $\mathcal{R}_{\pi/2}$. Therefore, we hypothesize that the generating functions for walks in different regions no bigger than

$\mathcal{R}_{\pi/2}$ do not differ significantly in their behaviour as long as their boundaries have rational slopes.

3.4 The three-quarter-plane

Since Banderier and Flajolet [1] proved that all walks in $\mathcal{R}_{\pi}$ have algebraic generating function, we are interested in whether or not this is true for regions "larger" than $\mathcal{R}_{\pi}$. A natural place to begin is the three-quarter-plane region $\mathcal{R}_{3\pi/2}$ bounded by the negative x- and y-axes. This region is a good choice for two regions: the boundaries are parallel to some of the unit steps, and the region is symmetric across the line $y = x$. These two facts allow for simplification in many cases.

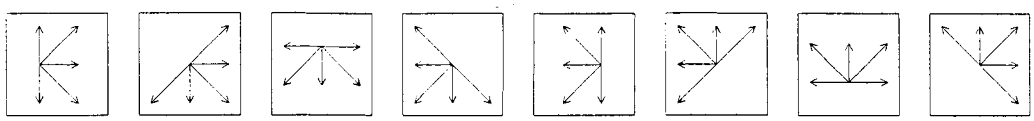
We first investigate walks in $\mathcal{R}_{3\pi/2}$ by using MAPLE to count the number of walks of length one to fifty generated by different $\mathcal{Y}$'s of cardinality three. From this enumeration we use the GFUN package to try to guess three things: $Q_{\mathcal{Y}}(t)$, an algebraic equation satisfied by $Q_{\mathcal{Y}}(t)$, and a linear differential equation satisfied by $Q_{\mathcal{Y}}(t)$. The code we use is adapted from code developed by Mishna; a sample of the code is given in Appendix B.

The only $\mathcal{Y}$'s for which MAPLE is able to return any guesses all fall into a specific class of step sets, which we describe in Section 3.4.1.

3.4.1 Half-planar step sets

Definition 3.3. A step set is *half-planar* if the walks it generates in the unrestricted planar lattice remain in a half-plane.

An example of a half-planar step set is $\{\mathsf{N}, \mathsf{NE}, \mathsf{W}, \mathsf{NW}\}$. Figure 3.5 shows all the maximal half-planar step sets.



(a) (b) (c) (d) (e) (f) (g) (h)

Figure 3.5: Maximal half-planar step sets

When we examined the generating functions of walks generated by half-planar step sets in $\mathcal{R}_{3\pi/2}$, we noticed that they were algebraic and their walks fell into one of three categories:

1. their walks were unrestricted;
2. their walks only interacted with one of the boundaries of $\mathcal{R}_{3\pi/2}$;
3. their walks could potentially interact with either boundary, but once they stepped off a line through the origin, they could only afterwards interact with a single boundary.

We give an example of each of the three possibilities below.

Example 3.4. Consider the walks generated by $\mathcal{Y} = \{\mathsf{NW}, \mathsf{NE}, \mathsf{SE}\}$ in $\mathcal{R}_{3\pi/2}$. These walks are unrestricted, so their length generating function is $Q_{\mathcal{Y}}(t) = \frac{1}{1-3t}$, which is of course algebraic.

Example 3.5. Let $\mathcal{Y} = \{\mathsf{NE}, \mathsf{SW}, \mathsf{NW}\}$. The walks generated by $\mathcal{Y}$ in $\mathcal{R}_{3\pi/2}$ are defined by the restriction

$$|u|_{\mathsf{NE}} + |u|_{\mathsf{NW}} \geq |u|_{\mathsf{SW}}.$$

Therefore, by Lemma 3.1, $Q_{\mathcal{Y}}(t)$ is defined by the system

$$W(t) = 1 + 2t^2 W(t)^2$$

$$Q_{\mathcal{Y}}(t) = W(t) + 2tW(t)Q_{\mathcal{Y}}(t).$$

Therefore $Q_{\mathcal{Y}}(t)$ is algebraic.

Example 3.6. Let $\mathcal{Y} = \{\mathsf{W}, \mathsf{S}, \mathsf{E}\}$. Then walks in $\mathcal{L}(\mathcal{Y})$ may end either in the fourth quadrant or the second quadrant. If $w \in \mathcal{L}(\mathcal{Y})$ ends in the fourth quadrant, it is factored as $w = vu_1$, where v is the longest prefix of w that ends at the origin with a step W (v may also be empty), and u_1 is a walk in the half-plane $y \geq 0$. The generating function $V(t)$ for walks in $\mathcal{L}(\mathcal{Y})$ that end at the origin with a step W is $V(t) = \frac{1}{2} + \frac{1}{2\sqrt{1-4t^2}}$. Since u_1 cannot step into the region $x > 0$, the generating function $U_1(t)$ for walks like u_1 is defined by the system

$$D(t) = 1 + t^2 D(t)^2$$
$$U_1(t) = D(t) + tD(t)U_1(t).$$

Therefore $U_1(t)$ is algebraic, as is $Q_1(t) = V(t)U_1(t)$.

If $w \in \mathcal{L}(\mathcal{Y})$ ends in the second quadrant, it is factored as $w = vu_2$, where v is the longest prefix of w that ends at the origin with a step E (again, v may be empty), and u_q is a walk in the half-plane $x \geq 0$. The generating function for v is again $V(t)$ as defined above. The walks in the half-plane $x \geq 0$ are defined by the restriction $|u|_{\mathsf{W}} \geq |u|_{\mathsf{E}}$, so by Lemma 3.1 their generating function $U_2(t)$ is defined by the system

$$M(t) = 1 + tM(t) + t^2 M(t)^2$$
$$U_2(t) = M(t) + tM(t)U_2(t).$$

Therefore $U_2(t)$ and $Q_2(t) = V(t)U_2(t)$ are also algebraic.

We are thus able to determine the algebraic length generating function for walks in $\mathcal{L}(\mathcal{Y})$:

$$Q_{\mathcal{Y}}(t) = Q_1(t) + Q_2(t) - P(t).$$

We use MAPLE to determine the $Q_{\mathcal{Y}}(t)$ explicitly, and the result is given in Table 3.6, as well as the length generating functions for every class of half-planar step sets of cardinality three in $\mathcal{R}_{3\pi/2}$. These generating functions were determined with Theorem 3.10.

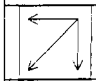
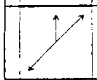
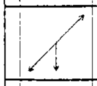
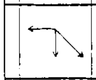
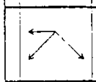
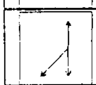
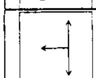
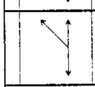
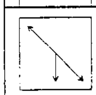
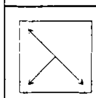
$\mathcal{Y}$	$Q_{\mathcal{Y}}(x,y;t)$
	$\frac{2}{1-t}$
	$\frac{1-\sqrt{1-8t^2}}{2t(2t-1+\sqrt{1-8t^2})}$
	$\frac{1-t-\sqrt{1-2t-3t^2}}{t\left(3t-1+\sqrt{1-2t-3t^2}\right)}$
	$\frac{1-2t+3t^3+(t^2+t-1)\sqrt{1-2t-3t^2}}{(t-1)t(1-3t-\sqrt{1-2t-3t^2})}$
	$\frac{1-t-t^2+4t^3+(t^2+t-1)\sqrt{1-8t^2}}{t(1-t)(4t-1+\sqrt{1-8t^2})}$
	$\frac{2\left(t+(1-2t)\sqrt{1-4t^2}-\sqrt{(1-4t^2)(1-8t^2)}+t\sqrt{1-8t^2}\right)}{\sqrt{1-4t^2}\left(1-2t-\sqrt{1-4t^2}\right)\left(1-4t-\sqrt{1-8t^2}\right)}$
	$\frac{2\left((1-2t)\sqrt{1-4t^2}+t^2-\sqrt{(1-4t^2)(1-2t-3t^2)}+t\sqrt{1-2t-3t^2}\right)}{\sqrt{1-4t^2}\left(1-3t-\sqrt{1-2t-3t^2}\right)\left(1-2t-\sqrt{1-4t^2}\right)}$
	$\frac{1-2t+(3-4t)\sqrt{1-4t^2}-3\sqrt{(1-4t^2)(1-8t^2)}+(1+2t)\sqrt{1-8t^2}}{2\sqrt{1-4t^2}\left(1-2t-\sqrt{1-4t^2}\right)\left(1-2t-\sqrt{1-8t^2}\right)}$
	$\left(\frac{1}{2}+\frac{1}{2\sqrt{1-4t^2}}\right)\left(\frac{1-t-\sqrt{1-2t-3t^2}}{2t^2\left(1-\frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right)}+\frac{1-\sqrt{1-8t^2}}{4t^2\left(1-\frac{1-\sqrt{1-8t^2}}{4t}\right)}\right)-\frac{1}{\sqrt{1-4t^2}}$
	$\frac{\left(1+\frac{1}{\sqrt{1-4t^2}}\right)(1-\sqrt{1-8t^2})}{4t^2\left(1-\frac{1-\sqrt{1-8t^2}}{t}\right)}-\frac{1}{\sqrt{1-4t^2}}$

Table 3.6: Half-planar length generating functions in $\mathcal{R}_{3\pi/2}$ for $|\mathcal{Y}|=3$

Rather than considering just $\mathcal{R}_{3\pi/2}$, we consider a more general region larger than the half-plane. Define the region as follows: let one of its boundaries be either the positive y-axis or the line $y=m_1x$, where m_1 is a non-positive rational number, and let the second boundary be either the negative y-axis or the line $y=m_2x$, where m_2 is a rational number. We call such a region a *rational region*, and denote it by $\mathcal{R}_{\mathsf{frac}}$. We also assume that $m_2>m_1$. Figure 3.4.1 shows two possible $\mathcal{R}_{\mathsf{frac}}$'s, where the shaded area is excluded from $\mathcal{R}_{\mathsf{frac}}$. We call the region excluded from $\mathcal{R}_{\mathsf{frac}}$ the *forbidden region.*

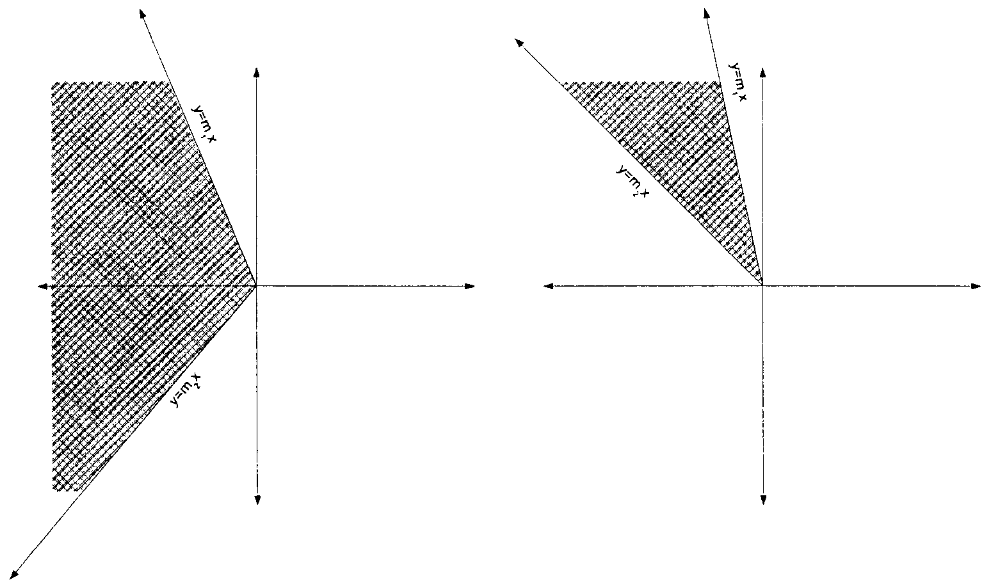


Figure 3.6: Two possible rational regions

First write m_1 and m_2 as fractions in lowest terms, *i.e.*, $m_1 = p_1/q_1$ and $m_2 = p_2/q_2$, where q_1 and q_2 are positive. Then a walk w in $\mathcal{R}_{\mathsf{frac}}$ generated by a step set $\mathcal{Y}$ must satisfy at least one of two conditions based on m_1 and m_2: for any prefix u of w, either

$$\begin{aligned} & q_1(\delta_{\mathsf{NW}}|u|_{\mathsf{NW}} + \delta_{\mathsf{N}}|u|_{\mathsf{N}} + \delta_{\mathsf{NE}}|u|_{\mathsf{NE}} - \delta_{\mathsf{SW}}|u|_{\mathsf{SW}} - \delta_{\mathsf{S}}|u|_{\mathsf{S}} - \delta_{\mathsf{SE}}|u|_{\mathsf{SE}}) \\ & \geq p_1(\delta_{\mathsf{NE}}|u|_{\mathsf{NE}} + \delta_{\mathsf{E}}|u|_{\mathsf{E}} + \delta_{\mathsf{SE}}|u|_{\mathsf{SE}} - \delta_{\mathsf{SW}}|u|_{\mathsf{SW}} - \delta_{\mathsf{W}}|u|_{\mathsf{W}} - \delta_{\mathsf{NW}}|u|_{\mathsf{NW}}), \end{aligned} \tag{3.4.1}$$

or

$$\begin{aligned} & q_2(\delta_{\mathsf{NW}}|u|_{\mathsf{NW}} + \delta_{\mathsf{N}}|u|_{\mathsf{N}} + \delta_{\mathsf{NE}}|u|_{\mathsf{NE}} - \delta_{\mathsf{SW}}|u|_{\mathsf{SW}} - \delta_{\mathsf{S}}|u|_{\mathsf{S}} - \delta_{\mathsf{SE}}|u|_{\mathsf{SE}}) \\ & \leq p_2(\delta_{\mathsf{NE}}|u|_{\mathsf{NE}} + \delta_{\mathsf{E}}|u|_{\mathsf{E}} + \delta_{\mathsf{SE}}|u|_{\mathsf{SE}} - \delta_{\mathsf{SW}}|u|_{\mathsf{SW}} - \delta_{\mathsf{W}}|u|_{\mathsf{W}} - \delta_{\mathsf{NW}}|u|_{\mathsf{NW}}), \end{aligned} \tag{3.4.2}$$

where

$$\delta_x = \begin{cases} 1 & \text{if } x \in \mathcal{Y} \\ 0 & \text{otherwise.} \end{cases}$$

The first condition is simply that the walk is above the line $y = m_1 x$, *i.e.*, $q_1 y \geq p_1 x$. The second condition is that the walk must be below the line $y = m_2 x$. Note that both conditions may be satisfied simultaneously. For the case when $\mathcal{R}_{\mathsf{frac}}$ is bounded by the negative y-axis, simply let $q_2 = 0$ in Equation (3.4.2).

For a half-planar step set $\mathcal{Y}$, we define its *bounding edge* as a line that unrestricted walks generated by $\mathcal{Y}$ cannot cross. For example, walks generated by $\mathcal{Y} = \{\mathsf{N}, \mathsf{W}, \mathsf{S}\}$ cannot cross the y-axis. Note that the bounding edge may not be unique. In fact, it is only unique in the case where $\mathcal{Y}$ contains two parallel steps, *i.e.*, two steps that point in opposite directions. In $\mathcal{R}_{\mathsf{frac}}$, walks generated by a half-planar step set have a special property: once a walk steps off its bounding edge, it cannot return.

As an example, let $\mathcal{R}_{\mathsf{frac}} = \mathcal{R}_{3\pi/2}$, and let $\mathcal{Y} = \{\mathsf{NW}, \mathsf{W}, \mathsf{S}, \mathsf{SE}\}$. Then the bounding edge of $\mathcal{Y}$ is the line $y = -x$. Once a walk w leaves the bounding line, it must remain in either the the second quadrant or the fourth quadrant permanently. This is because a walk may never step towards the bounding line by definition of a half-planar step set. We use that fact to prove the following theorem.

Theorem 3.10. *Let $\mathcal{Y}$ be half-planar, and let $\mathcal{L}(\mathcal{Y})$ be the walks generated by $\mathcal{Y}$ in a rational region $\mathcal{R}_{\mathsf{frac}}$. Then the length generating function $Q_{\mathcal{Y}}(t)$ for $\mathcal{L}(\mathcal{Y})$ is algebraic.*

Proof. We find $Q_{\mathcal{Y}}(t)$ be decomposing a walk $w \in \mathcal{L}(\mathcal{Y})$ based on the region in which it terminates. Given a half-planar $\mathcal{Y}$ and a region $\mathcal{R}_{\mathsf{frac}}$, there are three possibilities for how walks may interact with the boundaries of $\mathcal{R}_{\mathsf{frac}}$:

1. walks in $\mathcal{L}(\mathcal{Y})$ cannot interact with either boundary except at the origin;
2. walks in $\mathcal{L}(\mathcal{Y})$ are restricted by one boundary but can only interact with the second boundary at the origin;

3. walks in $\mathcal{L}(\mathcal{Y})$ may interact with either boundary depending on whether they step off the boundary line at $x > 0$ or at $x < 0$.

In the first case the walks in $\mathcal{L}(\mathcal{Y})$ are unrestricted, and therefore $Q_{\mathcal{Y}}(t) = \frac{1}{1-|\mathcal{Y}|t}$.

Next, assume that walks in $\mathcal{L}(\mathcal{Y})$ are only restricted by one boundary. Without loss of generality assume the boundary to be the line $y = m_1 x$ (if $\mathcal{L}(\mathcal{Y})$ actually interacts with the other boundary, we simply reflect the boundary lines and the step set across the x-axis). Then a walk $w \in \mathcal{L}(\mathcal{Y})$ is defined by the restriction that for any prefix u of w,

$$\begin{aligned} &q_1(\delta_{\mathsf{NW}}|u|_{\mathsf{NW}} + \delta_{\mathsf{N}}|u|_{\mathsf{N}} + \delta_{\mathsf{NE}}|u|_{\mathsf{NE}} - \delta_{\mathsf{SW}}|u|_{\mathsf{SW}} - \delta_{\mathsf{S}}|u|_{\mathsf{S}} - \delta_{\mathsf{SE}}|u|_{\mathsf{SE}}) \\ &\geq p_1(\delta_{\mathsf{NE}}|u|_{\mathsf{NE}} + \delta_{\mathsf{E}}|u|_{\mathsf{E}} + \delta_{\mathsf{SE}}|u|_{\mathsf{SE}} - \delta_{\mathsf{SW}}|u|_{\mathsf{SW}} - \delta_{\mathsf{W}}|u|_{\mathsf{W}} - \delta_{\mathsf{NW}}|u|_{\mathsf{NW}}). \end{aligned} \tag{3.4.3}$$

Therefore by Lemma 3.2, $Q_{\mathcal{Y}}(t)$ is algebraic.

Finally, consider the case where $w \in \mathcal{L}(\mathcal{Y})$ may interact with either boundary of $\mathcal{R}_{\mathsf{frac}}$. If w steps off the bounding line, we decompose w in a *nearly* unique manner based on where it leaves its bounding line. If it does not leave the bounding line, we decompose w based on where it ends. For the sake of simplicity, if w leaves the bounding line or terminates in the region above $y = m_1 x$, we say that w ends above the forbidden region. If it leaves the bounding line or terminates in the region below $y = m_2 x$, we say that w ends below the forbidden region.

Suppose w ends above the forbidden region. Then we may uniquely decompose w as $w = vu_1$ based on its last return to the origin from *below* the forbidden region. In this decomposition v is the longest walk possible that ends at the origin with a step from below the forbidden region, and u_1 is defined by the inequality given in Equation (3.4.1). If $\mathcal{Y}$ contains two steps (a, b) and $(-a, -b)$, then the boundary line is unique and v is a walk made up solely of (a, b) and $(-a, -b)$ steps that ends at the origin with a step up out from below the forbidden region. It is plain that v is isomorphic to a semi-directed walk that ends on the x-axis with a step up. If the boundary line for $\mathcal{Y}$ is not unique, then v is the empty walk. The generating function $P(t)$ for semi-directed walks that end on the x-axis is

$$P(t) = \frac{1}{\sqrt{1-4t^2}},$$

which is algebraic. Since precisely half of the non-trivial walks counted by $P(t)$ end with a step up, the generating function for walks like v is denoted by $V(t)$ and given by

$$V(t) = \begin{cases} \frac{1}{2} + \frac{1}{2\sqrt{1-4t^2}} & \text{if the boundary line is unique,} \\ 1 & \text{otherwise.} \end{cases}$$

Since u_1 is defined by a single restriction, its corresponding generating function $U_1(t)$ is also algebraic by Lemma 3.2. Therefore the generating function $Q_1(t)$ of walks that end above the forbidden region is algebraic and defined as

$$Q_1(t) = V(t)U_1(t).$$

If w ends below the forbidden region, its decomposition is nearly identical to the above case. We decompose w as $w = vu_2$ based on its last return to the origin from above the forbidden region. The generating function that counts v is again $V(t)$. The suffix u_2 is defined by the inequality in Equation (3.4.2) and is counted by $U_2(t)$, which is algebraic by Lemma 3.2. Therefore the generating function $Q_2(t)$ of walks that end below the forbidden region is algebraic and defined as

$$Q_2(t) = V(t)U_2(t).$$

As we mentioned before, the above decomposition is *nearly* unique. The only walks that are overcounted are those that end at the origin, which are counted exactly twice. Therefore the length generating function for $\mathcal{L}(\mathcal{Y})$ in $\mathcal{R}_{\mathsf{frac}}$ is

$$Q_{\mathcal{Y}}(t) = Q_1(t) + Q_2(t) - P(t),$$

and $Q_{\mathcal{Y}}(t)$ is algebraic. □

We realize that the proof of Theorem 3.10 is technical, but the essential idea behind it is that half-planar walks fall into one of the three categories illustrated by Examples 3.4, 3.5, and 3.6.

If *both* the boundaries of $\mathcal{R}_{\mathsf{frac}}$ are parallel to unit steps, we say that $\mathcal{R}_{\mathsf{frac}}$ is *normal*. If $\mathcal{R}_{\mathsf{frac}}$ is normal, then we may actually use Lemma 3.1 instead of Lemma 3.2 in the method used to Theorem 3.10. The reason for this is that when $\mathcal{R}_{\mathsf{frac}}$ has both boundaries parallel to unit steps, the restrictions on the walks all have either 1 or 2 as coefficients, and therefore Lemma 3.1 is applicable. The advantage of this method is that we are able to determine not only the length generating functions, but the complete generating functions of the walks in $\mathcal{R}_{\mathsf{frac}}$. Tables 3.7 and 3.8 show the restrictions on walks generated by each maximal half-planar step set in all normal regions larger than $\mathcal{R}_{\pi}$ (up to translation).

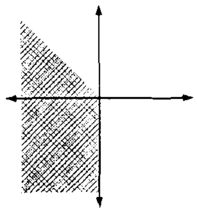
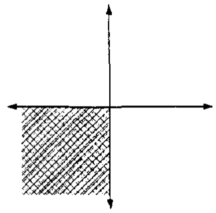
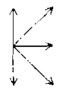
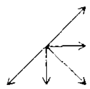
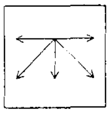
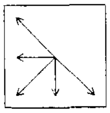
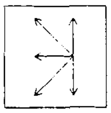
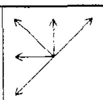
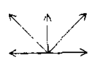
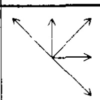
		
	Restrictions	
	None	None
	$\lvert u\rvert_a + \lvert u\rvert_b + \lvert u\rvert_c \geq \lvert u\rvert_e$	$\lvert u\rvert_a + \lvert u\rvert_b + \lvert u\rvert_c \geq \lvert u\rvert_e$
	$\lvert u\rvert_a + \lvert u\rvert_b \geq \lvert u\rvert_d \geq \lvert u\rvert_e$	$\lvert u\rvert_a + \lvert u\rvert_b \geq \lvert u\rvert_d + \lvert u\rvert_e$ OR $\lvert u\rvert_b = \lvert u\rvert_c = \lvert u\rvert_d = 0$
	$\lvert u\rvert_a \geq \lvert u\rvert_c + \lvert u\rvert_d + \lvert u\rvert_e$ OR $\lvert u\rvert_b = \lvert u\rvert_c = \lvert u\rvert_d = 0$	$\lvert u\rvert_a \geq \lvert u\rvert_c + \lvert u\rvert_d + \lvert u\rvert_e$ OR $\lvert u\rvert_e \geq \lvert u\rvert_a + \lvert u\rvert_b + \lvert u\rvert_c$
	$\lvert u\rvert_a \geq \lvert u\rvert_b + 2\lvert u\rvert_c + \lvert u\rvert_d$ OR $\lvert u\rvert_b = \lvert u\rvert_c = \lvert u\rvert_d = 0$	$\lvert u\rvert_a + \lvert u\rvert_e \geq \lvert u\rvert_b + \lvert u\rvert_c$ OR $\lvert u\rvert_b = \lvert u\rvert_c = \lvert u\rvert_d = 0$
	$2\lvert u\rvert_a + \lvert u\rvert_e \geq 2\lvert u\rvert_b + \lvert u\rvert_c$	$\lvert u\rvert_a + \lvert u\rvert_e \geq \lvert u\rvert_b + \lvert u\rvert_c$
	$\lvert u\rvert_a + 2\lvert u\rvert_b + \lvert u\rvert_c \geq \lvert u\rvert_d$	None
	None	None

Table 3.7: Restrictions on half-planar step sets in $\mathcal{R}_{5\pi/8}$, $\mathcal{R}_{3\pi/2}$

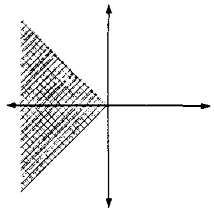
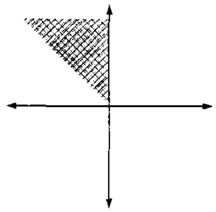
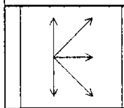
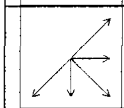
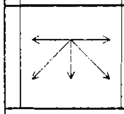
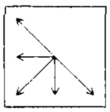
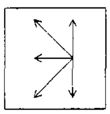
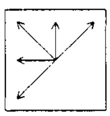
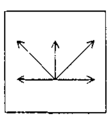
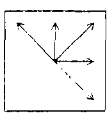
		
	Restrictions	
	None	None
	None	None
	$\|u\|_a + 2\|u\|_b + \|u\|_c \geq \|u\|_e$	None
	$2\|u\|_a + \|u\|_b \geq \|u\|_d + 2\|u\|_e$ OR $\|u\|_b = \|u\|_c = \|u\|_d = 0$	None
	$\|u\|_a \geq \|u\|_b + 2\|u\|_c + \|u\|_d$ OR $\|u\|_b \geq \|u\|_a + \|u\|_c + 2\|u\|_e$	$\|u\|_b + 2\|u\|_c + \|u\|_d \geq \|u\|_a$ OR $\|u\|_b = \|u\|_c = \|u\|_d = 0$
	$2\|u\|_a + \|u\|_e \geq 2\|u\|_b + \|u\|_c$	$\|u\|_a \geq \|u\|_c + \|u\|_d + \|u\|_e$ OR $2\|u\|_b + \|u\|_c \geq \|u\|_a \geq 2\|u\|_b$
	$\|u\|_a + 2\|u\|_b + \|u\|_c \geq \|u\|_d$	$\|u\|_b + \|u\|_c \geq \|u\|_d + \|u\|_e$ OR $\|u\|_d \geq \|u\|_a + 2\|u\|_b + \|u\|_c$
	None	$\|u\|_b + \|u\|_c + \|u\|_d \geq \|u\|_e$ OR $\|u\|_b = \|u\|_c = \|u\|_d = 0$

Table 3.8: Restrictions on half-planar step sets in $\mathcal{R}^*_{3\pi/2}$, $\mathcal{R}_{7\pi/4}$

3.4.2 Non-half-planar step sets in $\mathcal{R}_{3\pi/2}$

In general, for a given $\mathcal{Y}$ the walks generated by $\mathcal{Y}$ must satisfy one of two conditions: either $\delta_{\mathsf{N}}|u|_{\mathsf{N}} + \delta_{\mathsf{NE}}|u|_{\mathsf{NE}} + \delta_{\mathsf{NW}}|u|_{\mathsf{NW}} \geq \delta_{\mathsf{SE}}|u|_{\mathsf{SE}} + \delta_{\mathsf{S}}|u|_{\mathsf{S}} + \delta_{\mathsf{SW}}|u|_{\mathsf{SW}}$ OR $\delta_{\mathsf{NE}}|u|_{\mathsf{NE}} + \delta_{\mathsf{E}}|u|_{\mathsf{E}} + \delta_{\mathsf{SE}}|u|_{\mathsf{SE}} \geq \delta_{\mathsf{SW}}|u|_{\mathsf{SW}} + \delta_{\mathsf{W}}|u|_{\mathsf{W}} + \delta_{\mathsf{NW}}|u|_{\mathsf{NW}}$. What this essentially means is that the walk must either be in the half-plane $y \geq 0$ or the half-plane $x \geq 0$. Note that this is not an exclusive or, *i.e.*, both conditions may simultaneously be satisfied.

The most significant difference between step sets that are half-planar and those that are not is that a walk generated by a half-planar $\mathcal{Y}$ may *only* satisfy both conditions when it is at the origin, but a walk generated by a non-half-planar $\mathcal{Y}$ may intersect any point in $\mathcal{R}_{3\pi/2}$. The freedom enjoyed by a walk generated by non-half-planar $\mathcal{Y}$ greatly complicates things.

The fact that walks are unable to enter the region $(\mathbb{Z} \times \mathbb{Z})\backslash\mathcal{R}_{3\pi/2}$ keeps us from utilizing the method that Bousquet-Mélou [3] used to prove that all walks made up of unit steps in the *slit plane*[1], denoted by $\mathcal{R}_{2\pi}$, have algebraic functions because her method uses the fact that walks that begin and end on the x-axis in $\mathcal{R}_{3\pi/2}$ may step into the half-plane $x < 0$.

Another stumbling block is the fact that MAPLE has been unable to guess for any of the non-planar step sets $\mathcal{Y}$ of cardinality three, let alone higher cardinalities. We have used MAPLE to search for algebraic equations of order seven or lower and differential equations of order seven or lower with coefficients of degree seven to no avail.

However, because non-half-planar walks in $\mathcal{R}_{3\pi/2}$ are able to return to the origin, we believe that they have a "cycling" behaviour similar to that of walks in $\mathcal{R}_{2\pi}$ which would give them a nice decomposition and thus D-finite generating functions.

[1]Walks in the slit plane begin at the origin and may step anywhere in $\mathbb{Z} \times \mathbb{Z}$ except for the non-positive x-axis.

Chapter 4

Observations on More General Wedges

4.1 A class of non-D-finite walks

There exist walks in $\mathcal{R}_{\pi/4}$ whose length generating functions are non-D-finite, but there are none in $\mathcal{R}_{\pi}$. This leads us to pursue a general class of non-D-finite walks in an arbitrary region "smaller" than $\mathcal{R}_{\pi}$. First, consider the step set $\mathcal{Y} = \{\mathsf{NE}, \mathsf{SE}, \mathsf{NW}\}$ in $\mathcal{R}_{\pi/4}$. Mishna and Rechnitzer[21] proved the following theorem:

Theorem 4.1 ([21]). *The complete generating function $Q_{\mathcal{Y}}(x, y; t)$ and the length generating function $Q_{\mathcal{Y}}(1, 1; t)$ are both non-D-finite.*

These walks have two things in common with the knight's walks of Bousquet-Mélou and Petkovšek:

- they may interact with both boundaries of $\mathcal{R}_{\pi/4}$;
- they are unable to return to the origin.

Therefore a good place to start our search for a class of non-D-finite walks is a generalization of the walks in Theorem 4.1.

4.1.1 Walks in a quarter-plane wedge

Let $\mathcal{X} = \{\mathsf{N}, \mathsf{S}, \mathsf{E}\}$ and consider the walks generated by $\mathcal{X}$ that remain in the wedge bounded by $y = \pm x$. Let

$$Q_{\mathcal{X}}(x, y; t) = \sum_{n,i,j\geq 0} b_{i,j}(n) x^i y^j t^n,$$

where $b_{i,j}(n)$ is the number of walks generated by $\mathcal{X}$ of length n that end at (i, j). If we map $\mathcal{Y}$ onto $\mathcal{X}$ by mapping NE to E, SE to S, and NW to N, we see that

$$\begin{aligned} &Q_{\mathcal{X}}(x, y; t) = Q_{\mathcal{Y}}(x^{1/2}y^{-1/2}, x^{1/2}y^{1/2}), \text{and} \\ &Q_{\mathcal{X}}(1, 1; t) = Q_{\mathcal{Y}}(1, 1; t). \end{aligned}$$

Therefore each walk generated by $\mathcal{Y}$ in the quarter-plane is isomorphic to a walk generated by $\mathcal{X}$ in the wedge $0 \leq |y| \leq x$, and Theorem 4.1 implies that neither $Q_{\mathcal{X}}(x, y; t)$ nor $Q_{\mathcal{X}}(1, 1; t)$ are D-finite.

Let $B_{k,j}(t) = \sum_{n\geq 0} b_{k,j}(n) t^n$ be the power series in t that counts the number of walks ending at (k, j). Then

$$B_k(y, t) = \sum_{j=-k}^{k} B_{k,j}(t) y^j \tag{4.1.1}$$

encodes the walks that end on $x = k$ where y marks the final height of the walk. From this, we rewrite $Q_{\mathcal{X}}(x, y; t)$ as

$$Q_{\mathcal{X}}(x, y; t) = \sum_{k\geq 0} B_k(y, t) x^k. \tag{4.1.2}$$

In order to determine $B_k(t)$, we uniquely decompose a walk w generated by $\mathcal{X}$ in terms of its steps E. A walk w that ends on the line $x = k$ can be decomposed as

$w = uv$ where u is a walk that ends on the line $x = k - 1$ followed by a step E, and v is a walk made up solely of N and S steps that starts at (k, i), ends at (k, j), and remains between $y = x$ and $y = -x$. Therefore v is isomorphic to a generalized Dyck path (called a *directed path*) that begins at height i, ends at height j, and remains in a strip of height $2k$. Figure 4.1.1 (adapted from [21]) illustrates this decomposition.

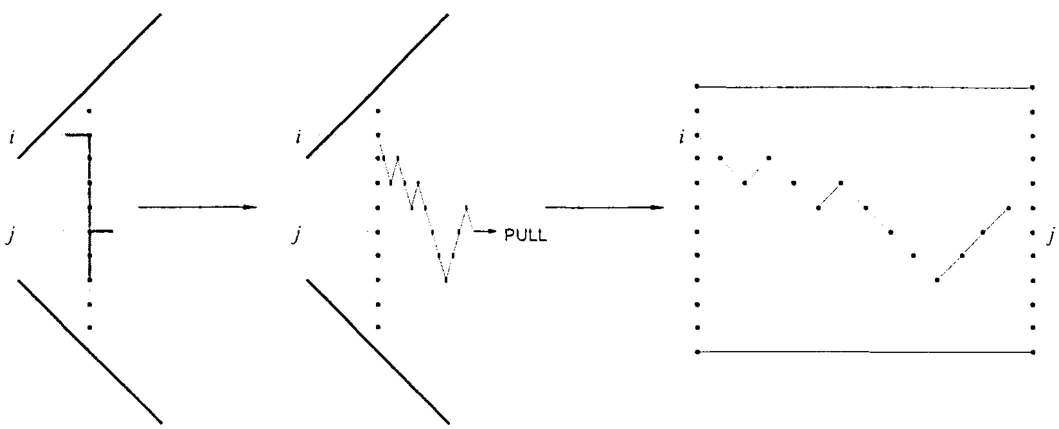


Figure 4.1: Directed path obtained from decomposition of a walk in the wedge

We denote the length generating function of such generalized Dyck paths by $H^{2k}_{i,j}(t)$. This generating function is given as Example 11 in [14] and is written in terms of a generalized Fibonacci polynomial

$$f_n(t) = f_{n-1}(t) - t^2 f_{n-2}(t).$$

We have

$$H^{k}_{i,j}(t) = t^{j-i}\frac{f_{i+1}(t) f_{k-j+1}(t)}{f_{k+2}(t)}, \tag{4.1.3}$$

for $j \leq i$.

Therefore,

$$\begin{aligned} B_k(y,t) &= \sum_{j=-k}^{k} B_{k,j}(t)y^j \\ &= \sum_{j=-k}^{k} y^j \sum_{i=-(k-1)}^{k-1} tB_{k-1,i}(t)H^{2k}_{i+k,j+k}(t) \\ &= \sum_{i=-(k-1)}^{k-1} B_{k-1,i}(t) \sum_{j=-k}^{k} y^j t H^{2k}_{i+k,j+k}(t). \end{aligned} \tag{4.1.4}$$

Next consider $B_k(y;t)$ under the transformation $t \mapsto \frac{q}{1+q^2}$. We compute

$$H^{2k}_{i,j}(t) = H^{2k}_{i,j}\left(\frac{q}{1+q^2}\right) = q^{j-i+1}\frac{(1-q^{2i+2})(1-q^{4k-2j+2})}{(1-q^2)(1-q^{4k+4})},$$

and substitute into Equation (4.1.4), which results in the following expression for $B_k(y;t) = B_k(y, \frac{q}{1+q^2})$:

$$\begin{aligned} B_k(y;t) &= \sum_{i=1-k}^{k-1} B_{k-1,i}(t)\left(\sum_{j=-k}^{i} y^j q^{j-i+1}\frac{(1-q^{2i+2})(1-q^{4k-2j+2})}{(1-q^2)(1-q^{4k+4})}\right. \\ &\quad \left. + \sum_{j=i+1}^{k} y^j q^{i-j+1}\frac{(1-q^{2j+2})(1-q^{4k-2i+2})}{(1-q^2)(1-q^{4k+4})}\right) \\ &= \sum_{i=1-k}^{k-1} B_{k-1,i}(t)\left(\sum_{j=-k}^{i} y^j q^{j-i+1}\frac{(1-q^{2i+2})(1-q^{4k-2j+2})}{(1-q^2)(1-q^{4k+4})}\right. \\ &\quad \left. + \sum_{j=-k}^{i+1} y^j q^{i+j+1}\frac{(1-q^{-2j+2})(1-q^{4k-2i+2})}{(1-q^2)(1-q^{4k+4})}\right). \end{aligned} \tag{4.1.5}$$

Let $w \in \mathcal{L}(\mathcal{X})$ be a walk that ends at the point (i,j). The reflection of w across the x-axis is also in $\mathcal{L}(\mathcal{X})$, so $B_k(y;t) = B_k(\frac{1}{y};t)$. We take advantage of this fact and use maple to simplify Equation (4.1.5):

$$\begin{aligned} B_k(y,\frac{q}{1+q^2}) =& \frac{y(1+q^2)B_{k-1}(y,\frac{q}{1+q^2})}{(q-y)(yq-1)} \\ &+ \frac{yq^k(q^2-yq+qy^{2k+1}+1)(1+q^2)B_{k-1}(q,\frac{q}{1+q^2})}{y^k(q-y)(yq-1)(1+q^{2k+2})}. \end{aligned} \tag{4.1.6}$$

Since we are primarily interested in the length generating function of the walks in $\mathcal{L}(\mathcal{X})$, let $y = 1$:

$$\begin{aligned} B_k(1, \frac{q}{1+q^2}) =& \frac{(1+q^2)B_{k-1}(1, \frac{q}{1+q^2})}{(q-1)(q-1)} \\ &+ \frac{q^k(q^2-q+q+1)(1+q^2)B_{k-1}(q, \frac{q}{1+q^2})}{(q-1)(q-1)(1+q^{2k+2})} \\ =& \frac{(1+q^2)B_{k-1}(1, \frac{q}{1+q^2})}{(1-q)^2} + \frac{q^k(1+q^2)^2 B_{k-1}(q, \frac{q}{1+q^2})}{(1-q)^2(1+q^{2k+2})}. \end{aligned} \tag{4.1.7}$$

Since $B_k(1)$ is rational, it appears that each of the $(2k+2)$-th roots of -1 is a pole of $B_k(1)$, and the set of poles of $B_k(1)$ taken over all k is dense in the unit circle. If this were the case, the following theorem applied to the generating function $Q_{\mathcal{X}}(y, y; \frac{q}{1+q^2}) = \sum B_k(1; \frac{q}{1+q^2})y^k$ would allow us to conclude that $Q_{\mathcal{X}}(x, y; t)$ is non-D-finite.

Theorem 4.2. *[19] Let $f(x;t) = \sum_n c_n(x)t^n$ be a D-finite power series in $\mathbb{C}(x)[[t]]$ with rational coefficients in x. For $n \geq 0$, let S_n be the set of poles of $c_n(y)$, and let $S = \bigcup S_n$. Then S has only a finite number of accumulation points.*

Unfortunately, Mishna and Rechnitzer were unable to prove directly that the singularities in question did not cancel. However, using an iterated kernel method they were able to prove that $Q_{\mathcal{X}}(x, y; t)$ is indeed non-D-finite. Therefore the singularities in Equation (4.1.7) do *not* cancel, for otherwise $Q_{\mathcal{X}}(x, y; \frac{q}{1+q^2})$ would be rational with a finite number of poles, and therefore algebraic.

4.1.2 An extension to more general wedges

A natural question is whether Theorem 4.1 can be extended to the generating functions of walks generated by $\mathcal{X}$ in an arbitrary region between $y = ax$ and $y = -bx$, where $a, b \in \mathbb{R}^+$ and $x \geq 0$. We first explore the region bounded by $y \pm mx$, where $m \in \mathbb{Z}$.

Let $Q_{\mathcal{X}_m}(x, y; t) = \sum_{n,i,j\geq 0} b_{i,j}(n)x^i y^j t^n$ be the complete generating function of the walks generated by $\mathcal{X}$ in the region $0 \leq |y| \leq mx$, and consider each walk in terms of its steps E as in the case of $m = 1$. Therefore,

$$Q_{\mathcal{X}_m}(x, y; t) = \sum_{k\geq 0} C_k(y, t)x^k, \tag{4.1.8}$$

where

$$C_k(y, t) = \sum_{j=-mk}^{mk} C_{k,j}(t)y^j \tag{4.1.9}$$

$$= \sum_{j=-mk}^{mk} y^j \sum_{i=-(k-1)}^{k-1} tC_{k-1,i}(t)H^{2mk}_{i+mk,j+mk} \tag{4.1.10}$$

$$= \sum_{i=-(k-1)}^{k-1} C_{k-1,i}(t) \sum_{j=-mk}^{mk} y^j t H^{2mk}_{i+mk,j+mk}. \tag{4.1.11}$$

Next, consider $C_k(y; t)$ under the transformation $t \mapsto \frac{q}{1+q^2}$. We compute

$$H^{2mk}_{i+mk,j+mk}(t) = H^{2mk}_{i+mk,j+mk}\left(\frac{q}{1+q^2}\right) = q^{j-i+1}\frac{(1-q^{2i+2mk+2})(1-q^{2mk-2j+2})}{(1-q^2)(1-q^{4mk+4})},$$

and substitute into Equation (4.1.4). Let $y = 1$ and simplify with MAPLE, which results in the following expression for $C_k(1; t) = C_k(1, \frac{q}{1+q^2})$:

$$C_k(1; t) = \frac{(1+q^2)C_{k-1}(1, \frac{q}{1+q^2})}{(1-q)^2} + \frac{(1+q^2)(1-q^{2mk-2})C_{k-1}(q, \frac{q}{1+q^2})}{(1-q)^2(1+q^{2mk+2})q^{k-2}}. \tag{4.1.12}$$

Similarly to the case where $m = 1$, the poles of $C_k(1, t)$ appear to be the $(2mk + 2)$-th roots of -1. Barring some miraculous cancellations, it would appear that the set of poles of $C_k(1, t)$ taken over all k is dense on the unit circle. We have unfortunately been unable to prove that this is the case. However, we have used MAPLE to plot these singularities for various values of m and k, and they do appear to be growing dense on the unit circle. Of course, these plots prove nothing in of themselves as there may be cancellation for some much larger value of k. They are encouraging, though. Table 4.1.2 shows some of the plots done with MAPLE.

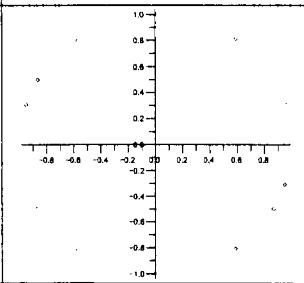
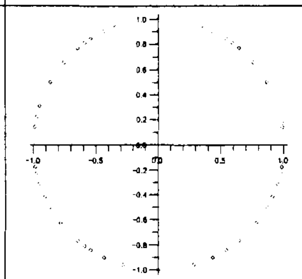
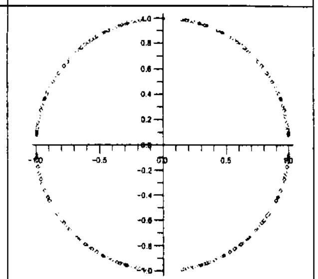
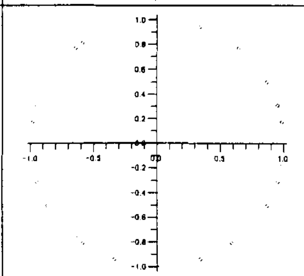
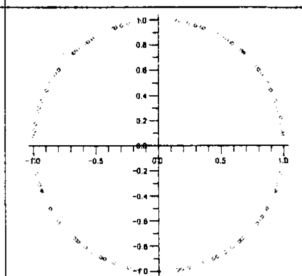
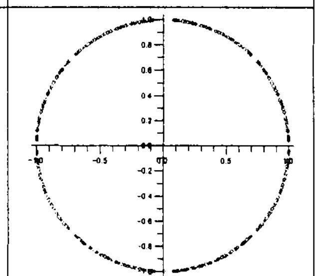
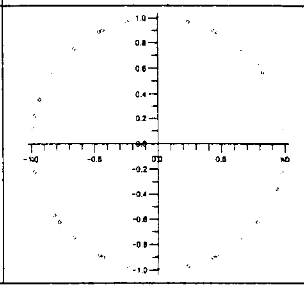
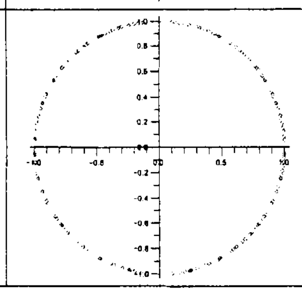
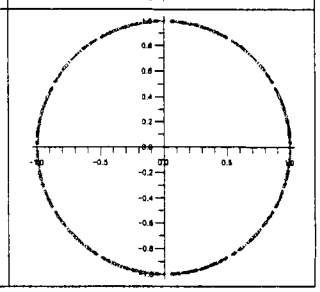
	$k = 2$	$k = 5$	$k = 10$
$m = 2$			
$m = 4$			
$m = 6$			

Table 4.1: Plots of roots of the denominator of $B_k(1, \frac{q}{1+q^2})$

Chapter 5

Conclusions and Future Directions

5.1 Conclusions

In Chapter 1 we said that the motivation of this thesis is, "For a given step set $\mathcal{Y}$, what effect does our choice of the boundaries of a region have on the generating function of the walks in that region?" While we have not answered the question entirely, we have made significant progress towards that goal.

Lemma 3.8 leads us to conclude that there is not a significant difference between $\mathcal{R}_{\pi/4}$ and $\mathcal{R}_{\pi/2}$ in terms of the behaviour of generating functions of walks.

Theorem 3.10 gives a large class of step sets that have algebraic complete generating functions in *any region* $\mathcal{R}_{\mathsf{frac}}$ at least as large as $\mathcal{R}_{\pi}$ with rational lines for boundaries. Furthermore, Therem 3.10 in conjunction with Lemma 3.1 gives a construction for these generating functions in regions bounded by two lines parallel to unit steps. The fact that we can construct these generating functions explicitly will allow us analyze their asymptotic behaviour.

5.2 Future work

In the course of this research many questions and conjectures arose that are possible directions for further study. We outline some of these below.

5.2.1 A necessary condition for D-finiteness?

When we examine the complete generating functions classified by Mishna[20] (see Table 3.2), we find two conditions that separate the two step sets whose complete generating functions are non-D-finite from the others. First, $\mathcal{Y}_1 = \{\mathsf{NW}, \mathsf{NE}, \mathsf{SE}\}$ and $\mathcal{Y}_2 = \{\mathsf{NW}, \mathsf{N}, \mathsf{SE}\}$ generate no non-trivial walks that end at the origin. Second, the walks generated by $\mathcal{Y}_1$ and $\mathcal{Y}_2$ may intersect both the x- and y-axes. That is, the walks may interact with either of the regional boundaries. From these conditions we formulate Conjecture 1.

Conjecture 1. *Let R be the convex region bounded by the line $y = ax$ and the line $y = bx$, and let $\mathcal{Y}$ be a step set. Suppose that the walks $\mathcal{L}(\mathcal{Y})$ generated by $\mathcal{Y}$ that remain in R may interact with both boundaries, and suppose that the only walk in $\mathcal{L}(\mathcal{Y})$ that returns to the origin is the trivial walk. Then the length generating function $Q_{\mathcal{Y}}(t) = \sum a(n)t^n$ of the walks in $\mathcal{L}(\mathcal{Y})$ is non-D-finite.*

If Conjecture 1 is true, then it further implies that the complete generating function $Q_{\mathcal{Y}}(x, y; t)$ of $\mathcal{L}(\mathcal{Y})$ is also non-D-finite.

Recall from Section 3.2.6 that Zeilberger conjectured that the complete generating function of $\mathcal{L}(\mathcal{Y})$ for $\mathcal{Y} = \{\mathsf{N}, \mathsf{S}, \mathsf{SE}, \mathsf{NW}\}$ is non-D-finite. If he is correct, then Conjecture 1 clearly cannot be a *sufficient* condition for D-finiteness. Therefore we choose to conjecture in a manner that does not conflict with Zeilberger.

5.2.2 Double Kreweras' walks

Conjecture 2. *The length generating function $Q_d(t)$ of the walks generated by $\{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{S}, \mathsf{SW}, \mathsf{W}\}$ in $\mathcal{R}_{\pi/2}$is algebraic and is defined by the system*

$$0 = 2t^3(6t-1)Q_d(t)^4 + 4t^2(6t-1)Q_d(t)^3 + 3t(6t-1)Q_d(t)^2 + (6t-1)Q_d(t) + 1.$$

5.2.3 Walks in a region larger than $\mathcal{R}_\pi$

Though we were unable to find evidence that all step sets in $\mathcal{R}_{3\pi/2}$ give D-finite length generating functions, we belive that that is the case. The intuition behind the D-finiteness for non-half-planar step sets is the fact that walks generated by a non-half-planar step may return to the origin as many times as they like. This indicates that there may be some sort of cyclic construction of these walks that yield a D-finite generating function similar to that of walks in the slit plane. This construction would differ from that of D-finite walks in $\mathcal{R}_{\pi/2}$.

Conjecture 3. *For any step set $\mathcal{Y} \subseteq \{0, \pm 1\} \times \{0, \pm 1\}$, the length generating function $Q_{\mathcal{Y}}(t)$ for walks generated by $\mathcal{Y}$ in $\mathcal{R}_{3\pi/2}$ is D-finite.*

5.2.4 A class of non-D-finite walks

In Chapter 4 we describe a potential class of non-D-finite walks.

Conjecture 4. *Let $\mathcal{Y} = \{\mathsf{N}, \mathsf{S}, \mathsf{E}\}$ and let $m \in \mathbb{N}$. Then the complete generating function $Q_{\mathcal{Y}}(x, y; t)$ for walks in the wedge between the lines $y \pm mx$ is non-D-finite.*

5.2.5 Extensions of proven results

There is also more work that can be done with the results we have already proven. For example, any of the walks we have directly enumerated can be examined in terms of their asymptotic behaviour. This asymptotic data can then be used to determine the statistical mechanical properties of many walks using the methods discussed in Chapter 2.

Asymptotic techniques such as those used by van Rensburg in [26] may also shed light on the behaviour of the generating functions of walks generated by a half-planar $\mathcal{Y}$ in *any* region larger than $\mathcal{R}_\pi$. This would include regions whose boundaries do not have rational slopes.

Another worthwhile task is to generalize the construction used in Lemma 3.1 to all the walks whose generating functions are algebraic by Lemma 3.2.

Finally, it is of interest to consider regions of the planar lattice that are "bigger" than the slit plane. For example, suppose walks are allowed to walk anywhere in $\mathbb{Z} \times \mathbb{Z}$ with the following restriction: a walk which crosses the negative x-axis from south to north (resp. north to south) cannot do so again without first crossing the negative x-axis from north to south (resp. south to north).

We close with a quote from Kurt Vonnegut, Jr.:

> New knowledge is the most valuable commodity on earth. The more truth we have to work with, the richer we become.

Appendices

A Table of notation

$K[x]$	ring of polynomials in x
$K(x)$	field of rational functions in x (the quotient field of $K[x]$)
$K[[x]]$	ring of formal power series in x
$K((x))$	field of Laurent series in x (the quotient field of $K[[x]]$)
$K_{\text{alg}}[[x]]$	ring of algebraic power series in x over $K(x)$
$K_{\text{alg}}((x))$	field of algebraic Laurent series in x over $K(x)$
$K[x_1, \ldots, x_i]$	ring of polynomials in $x_1, \ldots, x_i$
$K\langle X\rangle$	ring of noncommutative polynomials in the alphabet X
$K_{\text{rat}}\langle\langle X\rangle\rangle$	ring of rational noncommutative series in the alphabet X
$K\langle\langle X\rangle\rangle$	ring of formal (noncommutative) series in the alphabet X
$K_{\text{alg}}\langle\langle X\rangle\rangle$	ring of (noncommutative) algebraic series in the alphabet X
$[x^n]F(x)$	the coefficient of x^n in the power series $F(x)$

Table A.1: Power Series Notation

B Sample MAPLE code for counting walks

The MAPLE code below was used to generate sequences that counted the number of walks of length 1 to n generated by a given step set that remain in the three-quarter plane. We then used the GFUN package to guess the length generating function of the walks generated then guess if the generating function was D-finite or algebraic. Similar code was used for the same purpose in the quarter and one-eighth plane.

The first MAPLE procedure we define is `Count270`. It counts the walks of length n which are generated by a given step set $\mathcal{Y}$ and end at the point (i, j). It does so by starting at the point (i, j) and recursively working back to the origin by subtracting steps in $\mathcal{Y}$. Each successful return to the origin adds 1 to the value returned by `Count270`. The inputs a and b in `Count270` are used to keep track of the last point in the walk before the current point. This ensures that walks which step across the forbidden region are not mistakenly counted.

```
#--Count270 counts the walks of length <<n>> generated
#--by the step set <<Steps>> ending at point (<<i>>,<<j>>)
Count270 := proc (i, j, a, b, n, Steps)
  option remember;
  #--if walk steps into forbidden region
  if i < 0 and j < 0 or n < 0 then 0
  #--if walk crosses forbidden region
  elif (i = -1 and j = 0 and a = 0 and b = -1)
    or (i = 0 and j = -1 and a = -1 and b = 0) then 0
  #--if walk has returned to origin and is of length <<n>>
  elif n = 0 and j = 0 and i = 0 then 1
  #--if walk is in the allowed region, but not at the origin
  else add(Count270(i-s[1], j-s[2], i, j, n-1, Steps), s = Steps)
  end if:
end proc:
```

Next we define the procedure `GenSeries270`, which has two uses. It can be used to generate the first n coefficients of the length generating function $Q_{\mathcal{Y}}(t)$ for walks generated by a given step set $\mathcal{Y}$, or it can be used to generate the first n coefficients of the length generating function for walks that end at a given point (i, j). Which task it undertakes depends on the input it receives.

In order to calculate the coefficients of the generating function for walks that end at (i, j), it simply creates a sequence where each term in the sequence is determined by `Count270`. In order to calculate the first n coefficients of the length generating function, it calculates a double sum for each $0 \leq i \leq n$ which exhaustively counts all the walks of length i generated by $\mathcal{Y}$.

```
#--Generate the sequence of number of walks generated by <<Steps>> of length
#--<<n>> ending at point <<final>>, where <<Steps>> is a list of 2-vectors
GenSeries270 := proc (Steps, N, final)
  local k;
  if nargs = 3 then [seq(Count270(op(final), k, Steps), k = 0 .. N)]
  else [seq(add(add(Count270(i, j, i, j, k, Steps), j = -k .. k),
    i = -k .. k), k = 0 .. N)]
  end if:
end proc:
```

We now give an example of an application of `GenSeries270` and the GFUN package for the step set $\mathcal{Y} = \{\mathsf{N}, \mathsf{SE}, \mathsf{W}\}$.

```
#--Generate the sequence that counts the walks generated by W,N,SW
L:=GenSeries270([[0,1],[-1,-1],[-1,0]],50);

L:=[1, 2, 5, 13, 35, 96, 267, 750, 2123, 6046, 17303, 49721, 143365, 414584,
1201917, 3492117, 10165779, 29643870, 86574831, 253188111, 741365049,
2173243128, 6377181825, 18730782252, 55062586341, 161995031226, 476941691177,
1405155255055, 4142457992363, 12219350698880, 36064309311811, 106495542464222,
314626865716275, 929947027802118, 2749838618630271, 8134527149366543,
24072650378629801, 71264483181775040, 211043432825804129, 625190642719667122,
1852627179112970417, 5491513337424989754, 16282402094173127445,
48290501472790543731, 143257222282210719471, 425087187921124738344,
1261657886652854734743, 3745441438578943092225, 11121367816301815115037,
33029606975710986335682, 98114921186644851532353]
```

Finally, we use the GFUN package to guess the if $Q_{\mathcal{Y}}(t)$ satisfies an algebraic equation and a differential equation.

```
#--Use gfun package to guess generating function
with(gfun):
guessgf(L,x);

                                  1/2
                 1    1/2 I (x + 1)
              [- --- + ---------------, ogf]
                 2 x               1/2
                      (-1 + 3 x)     x
```

```
#--Use gfun package to guess differential equation
listtodiffeq(L,y(x));

                      2      /d        \                          2
 [{(-1 + 2 x + 3 x ) x |-- y(x)| + 1 + (4 x - 1 + 3 x ) y(x), y(0) = 1}, ogf]
                             \dx       /

listtoalgeq(L,y(x));

                                                        2       2
                  [1 + (-1 + 3 x) y(x) + (-x + 3 x ) y(x) , ogf]
```

MAPLE guesses that the length generating function for the walks generated by $\mathcal{Y} = \{\mathsf{N}, \mathsf{W}, \mathsf{SW}\}$ satisfies the equation

$$1 + (3t - 1)Q_{\mathcal{Y}}(t) + (3t^2 - t)Q_{\mathcal{Y}}(t)^2 = 0,$$

which is indeed the correct solution.

Since `GenSeries270` generates the initial terms of an integer sequences, it is natural to check if these sequences are included in the On-Line Encyclopedia of Integer Sequences [22]. Unfortunately, we were unable to classify any previously unclassified walks in this manner.

Bibliography

[1] Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.*, 281(1-2):37–80, 2002.

[2] Olivier Bernardi. Bijective counting of kreweras walks and loopless triangulations. FPSAC, 2006. available at arXiv.org/math/0605320.

[3] Mireille Bousquet-Mélou. Walks on the slit plane: Other approaches. *Adv. in App. Math.*, 27(1):243–288, 2001.

[4] Mireille Bousquet-Mélou. Counting walks in the quarter plane. In *Mathematics and computer science, II (Versailles, 2002)*, Trends Math., pages 49–67. Birkhäuser, Basel, 2002.

[5] Mireille Bousquet-Mélou. Walks in the quarter plane: Kreweras' algebraic model. *Ann. Appl. Probab.*, 15(2):1451–1491, 2005.

[6] Mireille Bousquet-Mélou and Marko Petkovšek. Walks confined in a quadrant are not always D-finite. *Theoret. Comput. Sci.*, 307(2):257–276, 2003. Random generation of combinatorial objects and bijective combinatorics.

[7] Mireille Bousquet-Mélou and Gilles Schaeffer. Walks on the slit plane. *Probab. Theory Related Fields*, 124(3):305–344, 2002.

[8] Roger Bowley and Maria Sanchez. *Introductory statistical mechanics.* Oxford University Press, Oxford, 2000.

[9] Richard Brak, Anthony J. Guttmann, and Stuart G. Whittington. A collapse transition in a directed walk model. *J. Phys. A*, 25(9):2437–2446, 1992.

[10] Richard Brak, Gary K. Iliev, Andrew Rechnitzer, and Stuart G. Whittington. Motzkin path models of long chain polymers in slits. *J. Phys. A: Math. Gen.*, 40:4415–4437, 2007.

[11] Richard Brak, Aleksander L. Owczarek, Andrew Rechnitzer, and Stuart G. Whittington. A directed walk model of a long chain polymer in a slit with attractive walls. *J. Phys. A*, 38(20):4309–4325, 2005.

[12] Noam Chomsky and Marcel-Paul Schützenberger. The algebraic theory of context-free languages. In *Computer programming and formal systems*, pages 118–161. North-Holland, Amsterdam, 1963.

[13] Philippe Flajolet, Stefan Gerhold, and Bruno Salvy. On the non-holonomic character of logarithms, powers, and the nth prime function. *Elec. J. Comb.*, 11(2):1–16, 2005.

[14] Phillipe Flajolet and Robert Sedgewick. Analytic combinatorics: functional equations, rational and algebraic functions. *INRIA Rapport do Recherche no. 4103*, 2001.

[15] Ira M. Gessel. A probabilistic method for lattice path enumeration. *J. Statist. Plann. Inference*, 14(1):49–58, 1986.

[16] Anthony J. Guttmann. Indicators of solvability for lattice models. *Disc. Math.*, 217(1–3):167–189, 2000.

[17] John E. Hopcroft and Jeffrey D. Ullman. *Formal languages and their relation to automata*. Addison-Wesley, 1969.

[18] Germain Kreweras. Sur une classe de problèms liés au treillis des partitions d'entiers. *Cahiers du B.U.R.O.*, 6:5–105, 1965.

[19] Leonard Lipshitz. *D*-finite power series. *J. Algebra*, 122(2):353–373, 1989.

[20] Marni Mishna. Classifying lattice walks restricted to the quarter plane. FPSAC, 2007. available at arXiv.org/math/0611651.

[21] Marni Mishna and Andrew Rechnitzer. Two non-holonomic lattice walks in the quarter plane, 2007. preprint, available at arXiv.org/math/0701800.

[22] Neil J. A. Sloane. The on-line encyclopedia of integer sequences, 2007. www.research.att.com/~njas/sequences/.

[23] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999.

[24] E. J. Janse van Rensburg. *The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles*, volume 18 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2000.

[25] E. J. Janse van Rensburg. Statistial mechanics of directed models of polymers in the square lattice. *J. Phys. A: Math. Gen.*, 36(15):R11–R61, 2003.

[26] E. J. Janse van Rensburg. Forces in motzkin paths in a wedge. *J. Phys. A: Math. Gen.*, 36:1581–1608, 2006.

[27] E. J. Janse van Rensburg and L Ye. Forces in square lattice directed paths in a wedge. *J. Phys. A: Math. Gen.*, 38(40):8493–8503, 2005.

[28] Jet Wimp and Doron Zeilberger. Resurrecting the asymptotics of linear recurrences. *J. Math. Anal. Appl.*, 111(1):162–176, 1985.