FACTORIAL AND FRACTIONAL FACTORIAL DESIGNS WITH RANDOMIZATION RESTRICTIONS - A PROJECTIVE GEOMETRIC APPROACH

by

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Abstract

Two-level factorial and fractional factorial designs have played a prominent role in the theory and practice of experimental design. Though commonly used in industrial experiments to identify the significant effects, it is often undesirable to perform the trials of a factorial design (or, fractional factorial design) in a completely random order. Instead, restrictions are imposed on the randomization of experimental runs.

In recent years, considerable attention has been devoted to factorial and fractional factorial plans with different randomization restrictions (e.g., nested designs, split-plot designs, split-split-plot designs, strip-plot designs, split-lot designs, and combinations thereof). Bingham et al. (2006) proposed an approach to represent the randomization structure of factorial designs with randomization restrictions. This thesis introduces a related, but more general, representation referred to as *randomization defining contrast subspaces* (RDCSS). The RDCSS is a projective geometric formulation of *randomization defining contrast subgroups* (RDCSG) defined in Bingham et al. (2006) and allows for theoretical study.

For factorial designs with different randomization structures, the mere existence of a design is not straightforward. Here, the theoretical results are developed for the existence of factorial designs with randomization restrictions within this unified framework. Our theory brings together results from finite projective geometry to establish the existence and construction of such designs. Specifically, for the existence of a set of disjoint RDCSSs, several results are proposed using $(t-1)$-spreads and

partial $(t-1)$-spreads of $PG(p-1,2)$. Furthermore, the theory developed here offers a systematic approach for the construction of two-level full factorial designs and regular fractional factorial designs with randomization restrictions.

Finally, when the conditions for the existence of a set of disjoint RDCSSs are violated, the data analysis is highly influenced from the overlapping pattern among the RDCSSs. Under these circumstances, a geometric structure called *star* is proposed for a set of $(t-1)$-dimensional subspaces of $PG(p-1,q)$, where $1 < t < p$. This experimental plan permits the assessment of a relatively larger number of factorial effects. The necessary and sufficient conditions for the existence of stars and a collection of stars are also developed here. In particular, stars constitute useful designs for practitioners because of their flexible structure and easy construction.

Dedication

To my teachers, parents and sisters.

Acknowledgments

There are many people who deserve thanks for helping me in many different ways to pursue my career in academics. The last four years in this department has been an enjoyable and unforgettable experience for me.

First and foremost, I cannot thank enough to my senior supervisor, Dr. Derek Bingham, for his support, guidance and encouragement in every possible way. In particular, I will always be grateful to him for his friendship. Many thanks to my committee members, Dr. Kenny Ye, Dr. Tom Loughin, Dr. Boxin Tang and Dr. Randy Sitter for their useful comments and suggestions that led to significant improvement in the thesis. I would thank Dr. Petr Lisonek for his help on getting me started in the area of Projective Geometry.

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Chapter 1

Introduction

In the initial stages of experimentation, factorial and fractional factorial designs are commonly used to help assess the impact of several factors on a process. Ideally one would prefer to perform the experimental trials in a completely random order. However, in many applications, experimenters impose restrictions on the randomization of the trials. These restrictions are often due to limited resources or the nature of the experiment. Thus, it is often infeasible or impractical to completely randomize the trials. In recent years, experimenters have devoted considerable attention to factorial and regular fractional factorial layouts with restricted randomization such as blocked designs, split-plot designs, strip-plot designs and split-lot designs. The treatment structure of these factorial designs is the same as that of their completely randomized counterpart, but they differ in their randomization structure. Furthermore, because of the different randomization restrictions the factorial designs have to be analyzed differently.

A review of the literature reveals that separate approaches have been taken to construct the common designs with randomization restrictions following factorial structure. For example, strip-plot designs have been constructed using Latin square fractions (e.g., Miller, 1997), while graphical techniques were used to construct split-lot designs (e.g., Taguchi, 1987; Mee and Bates, 1998; Butler 2004). Blocking in different factorial designs have been extensively studied using different methods (e.g., Sitter, Chen and Feder, 1997; Mukerjee and Wu, 1999), and different split-plot designs have been provided by Huang, Chen and Voelkel (1998); Bingham and Sitter (1999); Bisgaard (2000); and Butler (2004).

Occasionally, attempts have been made to study factorial designs with several different randomization restrictions in an unified framework. For instance, Patterson and Bailey (1978) used "design keys" to construct factorial designs with randomization restrictions defined by blocked, nested, crossed structure and combinations thereof. The notion of design keys was first introduced by Patterson (1965). Recently, Bingham et al. (2006) proposed an approach to represent the randomization structure of factorial designs with different randomization restrictions. This approach unifies the representation of such designs, and can be viewed as a generalization of the block defining contrast subgroup (Sun, Wu and Chen, 1997), except that there is a randomization defining contrast subgroup (RDCSG) for each stage of randomization. The formulation proposed in Bingham et al. (2006) uses *randomization restriction factors* instead of blocking factors.

This thesis proposes a related but, more general structure referred to as *randomization defining contrast subspace* (RDCSS). The RDCSS methodology is a projective geometric formulation of RDCSG defined in Bingham et al. (2006), and allows for theoretical development of such designs. The RDCSS formulation allows us to study these designs under this unified framework. For instance, it turns out that in some cases the existence of good factorial designs with randomization restrictions is non-trivial. In this thesis, we establish the necessary and sufficient conditions for the existence of such designs. Of course, these designs are useful from a practitioner's viewpoint only if they can be constructed. Assuming the existence, we develop construction algorithms for full factorial and regular fractional factorial designs with different randomization restrictions. On the other hand, when a desired factorial design does not exist, alternative designs are proposed.

To find designs for a particular randomization structure, and establish whether or not a design even exists, Bingham et al. (2006) had used an exhaustive computer search. The formulation presented in this thesis does not require an exhaustive search to conclude the existence of a desired design. In some cases, both the existence and construction can be established directly, whereas in other cases, one can search for the desired design in a reduced search space.

The designs obtained by Bingham et al. (2006) frequently did not allow the assessment of all the factorial effects. This is because of the desire to use half-normal plots to assess the effects, but many of the effects have a different variance. When there are too few effects with identical null distribution one must sacrifice the assessment of some of the effects. We propose new designs called *stars* and *galaxies* that are aimed at assessing as many effects as possible. The results proposed here cover a wide range of settings with both small and large run-size.

It is worth noting that designs with randomization restrictions often have larger run-size than completely randomized designs. This is because at each stage of randomization multiple experimental units are processed simultaneously, thus typically reducing cost and time. For example, Jones and Goos (2007) used a 128-run D-optimal split-split plot design to analyze the cheese-making experiment described in Schoen (1999), and in the polypropylene experiment, Jones and Goos (2006) used a 100-run design. Mee and Bates (1998) have considered 64-wafer designs and 81-wafer designs for the integrated circuit experiment. To identify the significant factors in the battery cell experiment, Vivacqua and Bisgaard (2004) performed a 64-run design. Bingham and Sitter (2001) have used a 64-run design for the wood product experiment, and Bingham et al. (2006) have used a 32-run design to analyze the plutonium alloy experiment.

This thesis is organized in the following manner. The next chapter starts with an overview of common factorial designs with different randomization restrictions and then a review of the finite projective geometric representation of factorial designs. Later in Chapter 2, we elaborate on the notion of RDCSS. A framework is proposed in Chapter 3 that can be used to express the response models for factorial designs with different randomization restrictions under the unified notion first introduced in Bingham et al. (2006). Furthermore, the impact of RDCSS structure on the linear regression model for factorial designs is discussed. The main results of this chapter demonstrate that the distribution of an effect estimate depends upon its presence in different RDCSSs. This in turn motivates one to find disjoint subspaces of the effect space $\mathcal{P}$ that can be used to construct RDCSSs (where $\mathcal{P}$ is the set of all factorial effects in a 2^p full factorial design, or a 2^{n-k} regular fractional factorial design with $p = n - k$). In Chapter 4, conditions for the existence of a set of disjoint subspaces of $\mathcal{P}$ are derived. The construction algorithms are also developed here for factorial designs claimed to exist. When these necessary and sufficient conditions are violated, overlapping among the RDCSSs cannot be avoided. Since the assessment of factorial effects on a process is the objective of the experimentation, it may appear that the overlapping among the RDCSSs is a problem. This is often the case, but it turns out that one can propose design strategies that use the overlap among different RDCSSs as an advantage. Both the existence and construction of such designs are developed in Chapter 5.

Finally, the work done for this thesis focuses on full factorial layouts, however the main results are easily extendable to regular fractional factorial designs. This is briefly outlined at the end of Chapter 4. Moreover, the results developed in Chapter 4 and 5 are presented for two-level factorial designs only. These results can be easily generalized to q-level factorial designs.

Chapter 2

Preliminaries and Notations

Two-level full factorial and fractional factorial designs are widely used in industrial (Box, Hunter and Hunter, 1978) and agricultural (Kempthorne, 1952; Cochran and Cox, 1957) experiments to assess the impact of factorial effects on a process. Though an ideal choice, when designing a factorial experiment, it is often impossible or impractical to completely randomize the experimental units. The resulting experimental plans have randomization restrictions on the trials, which impacts the data analysis.

We first provide an overview of the two-level factorial and fractional factorial designs in Section 2.1. Then, a review of factorial designs with common randomization restrictions (e.g., blocked designs, split-plot designs, strip-plot designs, split-lot designs and combinations thereof) is presented in Section 2.2. In Section 2.3, a finite projective geometric representation of factorial designs is outlined. This representation is specifically useful for unifying the factorial and fractional factorial designs with different randomization restrictions, which is outlined in Section 2.4.

2.1 Factorial and fractional factorial designs

Factorial designs are widely used in experiments involving several factors where it is necessary to study the impact of the factors or factor combinations on a process. Special cases of the general factorial designs are widely used in scientific endeavors and they form the basis for other designs of considerable practical value. The most important among these special cases is the factorial design with p factors, each having two levels. These levels may be quantitative or qualitative with levels corresponding to the "high" and "low" levels of a factor, or perhaps the presence and absence of a chemical. A full replicate of such a design requires 2^p observations and is called a 2^p *full factorial design.* The set of all level combinations can be represented by a $2^p \times p$ matrix of -1's and $+1$'s, where ± 1's represent the two levels of each factor, respectively.

Example 2.1. Consider a factorial design with 3 two-level factors. The set of all level combinations for the 3 independent factors can be written as:

$$\mathcal{D} = \begin{pmatrix} A & B & C \\ -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

In general, for p independent factors, the matrix $\mathcal{D}$ obtained in a similar fashion is called the 2^p *full factorial design matrix.* The set of columns corresponding to all the main effects and interactions is called the 2^p *full factorial model matrix*, denoted by

X. The corresponding model matrix X for the 2^3 design is given by

$$\mathbf{X} = \begin{pmatrix} A & B & C & AB & AC & BC & ABC \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This representation of the factor level combinations is convenient since the columns of X denote the linear contrasts that estimate the main effects and interactions in a normal linear regression model by $X'Y/2^p$, where Y is the vector of observations corresponding to the factor level settings of each row of $\mathcal{D}$ (for details on the response model of interest, see Chapter 3).

2.1.1 Fractional factorial designs

As the number of factors in a 2^p factorial design increases, the number of trials required for a full replicate of the design rapidly outgrows the resources available for many experiments. In such cases, one cannot perform a full replicate of the design and a fractional factorial design has to be run. If the experimenter can reasonably assume that certain interactions involving a large number of factors are negligible, information on the lower order effects can be obtained by running a suitable fraction of the 2^p full factorial design.

Two-level fractional factorial designs are broadly divided into regular and non-regular fractional factorial designs (e.g., Tang and Deng, 1999). A regular fractional factorial design can be specified in terms of a set of defining contrasts. For example, if there are only enough resources for 2^{p-k} experimental trials, then the choice of trials to be performed is determined by assigning k of the factors to the interaction columns of the 2^{p-k} full factorial model matrix. These $p-k$ factors are frequently called *basic factors* and the additional k factors are referred to as *added factors* (e.g., Franklin and Bailey, 1977; Cheng and Li, 1993; Bingham and Sitter, 1999). That is, a 2^{p-k} regular fractional factorial design is constructed from the full factorial design generated from the $p-k$ basic factors, which we call the *base factorial design.* For the results presented in this thesis, we only consider regular fractional factorial designs.

Example 2.2. Suppose a two-level factorial design with 5 factors has to be performed in 8 runs. That is, the design of interest is a 2^{5-2} regular fractional factorial design. The 3 basic factors in a 2^{5-2} fractional factorial design are the three independent factors (A, B, C) of the base factorial design (a 2^3 full factorial design). The two added factors (D, E) are assigned to columns chosen from the remaining columns of the model matrix for the base factorial design. One possible assignment is $D = AC$ and $E = BC$. That is, the level settings of D and E are determined by the columns corresponding to AC and BC, respectively. Let I be the identity element (or, the column of 1's for the mean). Then,

$$I = ACD \quad \text{and} \quad I = BCE$$

are called the *fractional generators.* From every k independently chosen fractional generators, $2^k - k - 1$ more relations are derived. For example, $I = ABDE$ is derived from $I = ACD$ and $I = BCE$. The entire set of $2^k - 1$ relations,

$$I = ACD = BCE = ABDE,$$

forms the *defining contrast subgroup,* and the terms ACD, BCE and $ABDE$ are called *words.* The number of factors in a word is called the *length* of a word (or *word-length*).

Thus, a 2^{p-k} regular fractional factorial design is constructed by choosing k independent fractional generators from the set of all factorial effects in a 2^p full factorial layout. Two distinct sets of fractional generators (or equivalently, defining contrast subgroups) generate distinct 2^{-k} fractions of a 2^p full factorial design. That further introduces the notion of ranking among different 2^{-k} fractions of a 2^p full factorial design. The ranking criteria are generally based on a few operating assumptions that are common to many experiments:

- *The effect sparsity principle*: only a few effects in a factorial experiment are likely to be significant.
- *The hierarchical ordering principle*: lower order effects are more likely to be significant than higher order effects.
- *The effect heredity principle*: interactions involving significant main effects are more likely to be active than other interactions.

Many of the ranking criteria are functions of the sequence of word-lengths (known as *word-length pattern*) in the defining contrast subgroup. The conventional criteria for ranking two-level regular fractional factorial designs are (i) maximum resolution (Box and Hunter, 1961), (ii) minimum aberration (Fries and Hunter, 1980), and (iii) maximum number of clear effects (Chen, Sun and Wu, 1993; Wu and Chen, 1992).

The procedure for assessing the significance of the main effects and interactions does not depend on the "goodness" of the fraction. If the design used is a replicated factorial or fractional factorial design, the assessment of the factorial effects can be done by using the usual hypothesis tests based on the analysis of variance. For unreplicated factorial and fractional factorial designs, the significant factorial effects can be identified using approaches such as half-normal plots (Daniel, 1959) or, for example, permutation tests (Loughin and Noble, 1997; Loeppky and Sitter, 2002).

Half-normal plots were introduced by Daniel (1959) for assessing the significance of factorial effects in unreplicated 2^p factorial and fractional factorial experiments. This is a plot of the ordered absolute value of effect estimates against the percentiles of the half-normal distribution, where all the factorial effects are negligible under the null hypothesis and the data is assumed to be i.i.d. normal. In this thesis, we assume that the data comes from a normal distribution and the half-normal plot will be used as the main analysis tool. The following example (Montgomery, 2001) illustrates the use of a half-normal plot for identifying significant effects in a factorial experiment.

Example 2.3. An unreplicated full factorial experiment is carried out in a pilot plant to study the factors expected to influence the filtration rate of a chemical product produced in a pressure vessel. The 4 two-level factors are temperature (A), pressure (B), concentration of formaldehyde (C) and stirring rate (D). Table 2.1 displays the effect estimates for the 15 factorial effects obtained from the unreplicated 2^4 completely randomized full factorial design.

Table 2.1: Factorial effect estimates for the chemical experiment.

Effects	Estimates	Effects	Estimates
A	21.625	B	3.125
C	9.875	D	14.625
AB	0.125	AC	-18.125
AD	16.625	BC	2.375
BD	-0.375	CD	-1.125
ABC	1.875	ABD	4.125
ACD	-1.625	BCD	-2.625
ABCD	1.375		

The corresponding half-normal plot is shown in Figure 2.1. If none of the effects are important, the effect estimates should all fall on a straight line. The effects detected to be far away from the straight line suggested by the bulk of the estimates can be considered significant. In Figure 2.1, all the effects except A, C, D, AC and AD appear to fall on a straight line. These five effects would be considered active.

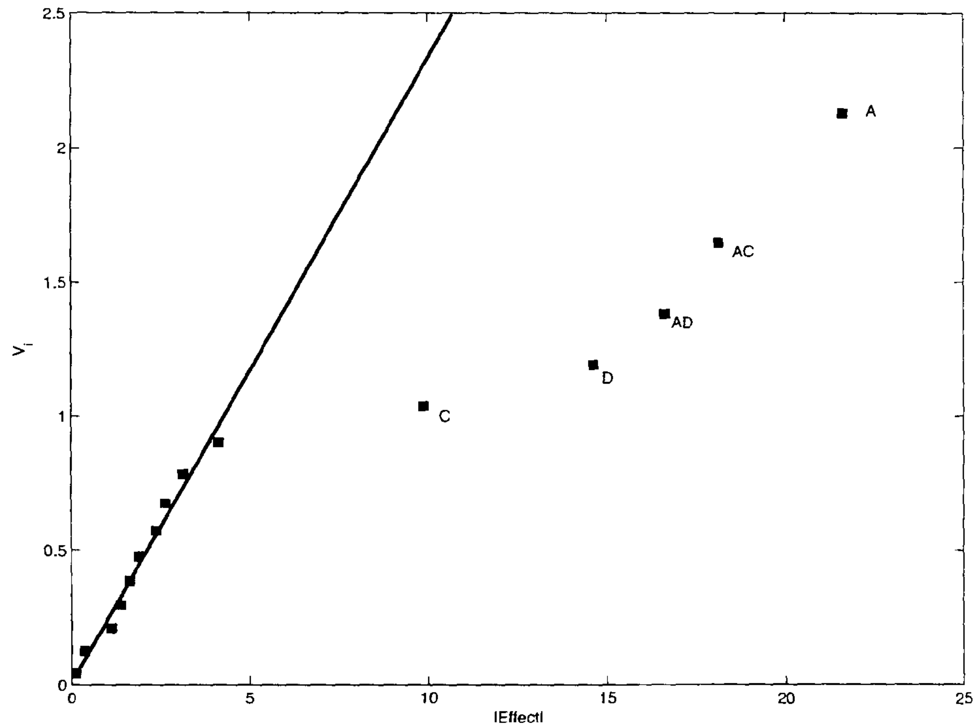

Figure 2.1: The half-normal plot for the 15 factorial effects.

An important assumption of the half-normal plot approach is that all the effects used in a half-normal plot have the same variance with mean zero (i.e., under the null hypothesis of no active effects, all the effect estimates are i.i.d. normal).

For the above example, it was assumed that the trials were performed in a completely random order, which ensures that the effect estimates are independent and identically distributed under the null hypothesis. Thus, only one half-normal plot is required to assess the significance of all the factorial effects. If there are restrictions on the randomization of the experimental runs, the i.i.d. assumption is likely to be violated. To assess the significance of effects in the restricted randomization case, one would use separate half-normal plots for sets of effects having identical distributions under the null hypothesis. Indeed, this a very important issue that motivates much of the work in this thesis. As a matter of choice, one would elect to run a design where half-normal plots are constructed with a reasonable number of effects per plot.

2.2 Factorial and fractional factorial designs with randomization restrictions

The inability to perform the trials of a factorial experiment in a completely random order is often due to imposed randomization restrictions on the experiment trials. In recent years, considerable attention has been devoted to factorial and fractional factorial layouts with restricted randomization, such as blocked designs (Bisgaard, 1994; Sitter, Chen and Feder, 1997; Sun, Wu and Chen, 1997; Cheng, Li and Ye, 2004), split-plot designs (Addelman, 1964; Box and Jones, 1992; Huang, Chen and Voelkel, 1998; Bingham and Sitter, 1999; Bisgaard, 2000; Trinca and Gilmour, 2001; Kowalski, Cornell and Vining, 2002; Ju and Lucas, 2002; Jones and Goos, 2006), strip-plot designs (Miller, 1997), and split-lot designs (Mee and Bates, 1998; Butler, 2004). Although the treatment structure of these designs are identical, they differ in the randomization structures. These designs are often larger than the completely randomized designs. The following is a brief review of some common designs.

2.2.1 Block designs

In many situations it is impossible to perform all of the trials of an experiment under homogeneous conditions. In other cases, it might be desirable to deliberately vary the experimental conditions to ensure that the treatments are equally effective (or, robust) across different situations that are likely to be encountered in practice. The design technique frequently used in such situations is *blocking*. Because the only randomization of treatments is within the blocks, the blocks are said to represent the *restrictions on randomization*.

Common block designs are randomized complete block designs (RCBD), Latin square designs (LSD) and Graeco-Latin square designs (GLSD). In particular, for an unreplicated 2^p factorial experiment, blocking induces incomplete block designs (ICBD) called *blocked factorial designs*. The technique used for arranging the trials of a 2^p factorial design in blocks is known as *confounding*. This technique causes information about certain factorial effects (preferably higher order interactions) to be confounded with blocks. To be precise, in a 2^p factorial design with blocks of size 2^{p-k} each, $2^k - 1$ factorial effects become confounded with blocks. The technique used for partitioning the 2^p experimental units into 2^k blocks is similar to the construction of a 2^{p-k} regular fractional factorial design. Indeed, Lorenzen and Wincek (1992) refer to blocking as a special case of fractionation. The following example illustrates the construction of a blocked factorial design.

Example 2.4. Consider a 2^6 factorial experiment, where the available resources consist of batches of only 16 homogeneous experimental units. Thus, one has to run a blocked factorial design in 4 blocks of size 16 units each. Let $b_1 = ABCD$ and $b_2 = CDEF$ be the two independent *blocking factors* (Bisgaard, 1994; Sitter, Chen and Feder, 1997). Then, the third blocking factor is derived from the two independent ones: $b_1 b_2 = ABEF$. The resulting treatment structure is shown in Table 2.2.

Table 2.2: The arrangement of 64 experimental units in 4 blocks.

(1)	ab	a	b	c	abc	e	abe
cd	ef	acd	aef	d	cef	cde	f
ace	acf	ce	cf	ae	af	ac	acef
ade	adf	de	df	acde	acdf	ad	adef
bce	bcf	abce	abcf	be	bf	bc	bcef
bde	bdf	abde	abdf	bcde	bcdf	bd	bdef
cdef	abcd	acdef	bcd	def	abd	cdf	abcde
abef	abcdef	bef	bcdef	abcef	abdef	abf	abcdf

The presence of factorial structure in a blocked factorial design makes the analysis relatively easy compared to other incomplete block designs. The blocking arrangement for this example causes the 3 four-factor interactions ($b_1 = ABCD, b_2 = CDEF$ and $b_1b_2 = ABEF$) to be confounded with the block effects. The orthogonality among the columns of the model matrix ensures that the error variances of the remaining factorial effects are not impacted by the blocking effects. So, the significance of the remaining 60 effects can be assessed using a half-normal plot.

2.2.2 Split-plot designs

In many applications, an ideal choice is to run all possible treatment combinations in a completely randomized order. However, it may be difficult to change the levels for some of the factors. In such situations, the experimenter restricts the randomization by fixing the levels of the hard-to-change factors and then run all combinations or a fraction of all combinations of the remaining factors. Such a strategy may lead to a split-plot design. As a convention, the hard-to-change factors are called the whole-plot factors and the easy-to-change factors are called the sub-plot (or split-plot) factors. This design was first developed and used for mainly agronomic experiments (Yates, 1937), but is applicable in many fields of experimental research.

In a 2^p split-plot design there are two types of factors: p_1 whole-plot (WP) factors and p_2 sub-plot (SP) factors, where $p = p_1 + p_2$. The experimental units where WP factors are applied are called whole-plots, and the experimental units where SP factors are applied are called sub-plots. We describe a 2^p full factorial split-plot design using a simple example.

Example 2.5. Consider the cheese-making example in Bingham, Schoen and Sitter (2004). Here, the authors studied the quality characteristics for the production of cheeses, where the cheese making process consists of two stages. In the first stage, milk is processed into batches of curds. These curds are then processed to produce cheese. The 4 two-level factors A, B, q and r were suspected to be responsible for the poor quality of cheese, where $p_1 = 2$ of these factors (A, B) affected the processing of the milk in the first stage, and the remaining $p_2 = 2$ factors (q, r) were related to the processing conditions to generate the curds used to make cheese. A designed experiment was used to investigate the impact of these factors on the resulting cheese quality. Since milk in a single tank gives rise to several batches of curds, they treated the milk in a tank under a randomly selected setting of A, B as whole-plots, and the randomly selected settings of the processing conditions q, r as sub-plot factors (see Figure 2.2).

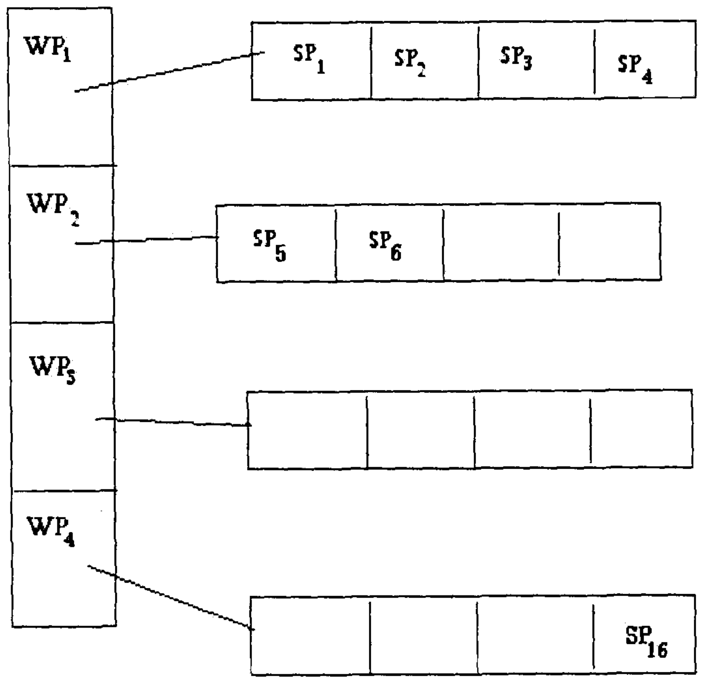

Figure 2.2: The split-plot design configuration.

Here, WP_1 represents the first tank of milk and SP_1,... ,SP_4 denote the four batches of curds obtained from the first tank of milk. In a completely randomized design, the variation is only due to the variability between plots. However, in split-plot designs, there are two sources of variation: between plot variability (WP variability) and within plot variability (SP variability). In matrix notation, a regression model for the split-plot experimental design is

$$Y = X\beta + \varepsilon^{wp} + \varepsilon^{sp},$$

where the mean term $X\beta$ consists of the regression parameters for all the factorial effects, and ε^{wp}, ε^{sp} are the whole-plot and sub-plot error vectors respectively. It is assumed that the error terms are mutually independent and normally distributed random variables. Furthermore, the analysis of a split-plot design is different than that of the completely randomized design. The complete analysis of variance table for a $2^{p_1+p_2}$ factorial split-plot design with r replicates is shown in Table 2.3.

Table 2.3: The analysis of variance table for a split-plot design.

Sources of Variation	df
Replicates	$r-1$
Whole-plot analysis:	
WP effects	$2^{p_1}-1$
WP error	$(r-1)(2^{p_1}-1)$
Subplot analysis:	
WP*SP interaction effects	$2^p - 2^{p_1}$
SP error	$(r-1)(2^p - 2^{p_1})$
Total	$r2^p - 1$

Similar to block designs, the set of n experimental units are divided into subsets (sub-plots). However, no prior information is available regarding the significance of the factors used for partitioning the experimental units. Thus all the factorial effects have to be assessed. If the design is replicated $r > 1$ times, then the usual ANOVA based hypothesis tests can be performed. If the design is unreplicated, two separate half-normal plots are required to assess the significance of the $2^p - 1$ factorial effects (one for the WP effects and one for the SP effects).

2.2.3 Strip-plot designs

Strip-plot configurations can be an economically attractive option in situations where the process being investigated can be separated into two distinct stages, and it is possible to apply the second stage simultaneously to groups of first-stage outcomes. It is common to represent a strip-plot structure as a rectangular array of experimental units where one set of factors is applied to the rows and another set of factors is applied to the columns (e.g., see Mead 1988). These designs are also known as row-column designs. Strip-plot designs are also called strip-block designs (e.g., Vivacqua and Bisgaard, 2004).

Example 2.6. Consider the washer-dryer example in Miller (1997). Here, a manufacturer of household appliances wanted to investigate different methods of reducing the wrinkling of clothes being laundered. In the first stage of the experiment, sets of cloth samples were run through one of four washing machines. Once the cloth samples were washed, the samples were divided into four groups such that each group contained exactly one sample from each washer. In the second stage, each group of samples were assigned to one of four dryers. Once dry, the extent of wrinkling on each sample was evaluated. Let the washer configuration be represented by a 2^2 design in factors (A, B), and the dryer configurations by a 2^2 design in factors (a, b). The design structure for this experiment is illustrated in Figure 2.3.

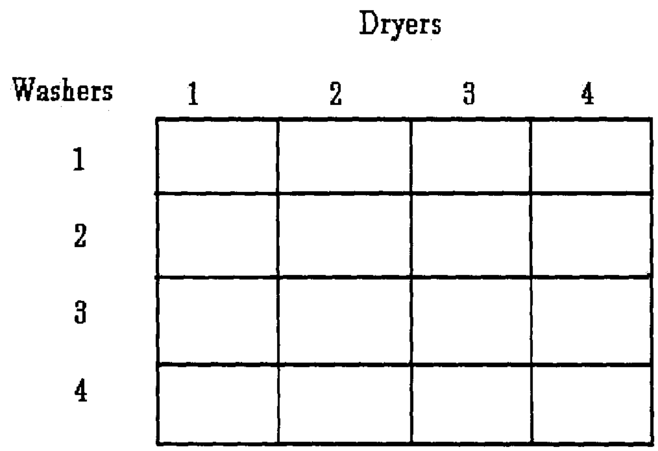

Figure 2.3: The row-column design arrangement.

The entire experiment requires only four washer loads and four dryer loads, producing 16 observations, whereas a completely randomized design using four washer loads and four dryer loads would only produce four observations. That is, the strip-plot design allows a larger number of treatment combinations to be investigated for the same amount of experimental resources. A model to describe this setting is

$$Y = X\beta + \varepsilon^r + \varepsilon^c + \varepsilon,$$

where $X\beta$ is the mean term consisting of the regression parameters for the factorial effects, ε^r and ε^c are the error vectors associated with the rows and columns respectively, and ε is the replication error associated with the experimental units. All three error terms are assumed to be mutually independent and normally distributed. While convenient in resource usage, the analysis of the data obtained from the experiment can be relatively complex compared to the case of a completely randomized design. The analysis of variance table of a replicated $2^{p_1+p_2}$ factorial strip-plot design with p_1 row factors, p_2 column factors and r replicates is shown in Table 2.4.

Table 2.4: The analysis of variance table for a strip-plot design.

Sources of Variation	df
Replicates	$r-1$
Row analysis:	
Row effects	$2^{p_1}-1$
Error (row)	$(r-1)(2^{p_1}-1)$
Column analysis:	
Column effects	$2^{p_2}-1$
Error (column)	$(r-1)(2^{p_2}-1)$
Unit analysis:	
Row*Column interaction effects	$(2^{p_1}-1)(2^{p_2}-1)$
Error (unit)	$(r-1)(2^{p_1}-1)(2^{p_2}-1)$
Total	$r2^p-1$

Similar to split-plot designs, the factorial effects involved in the grouping of the experimental units into rows and columns are analyzed separately.

2.2.4 Split-lot designs

Split-lot factorial designs are useful for experiments where the product is formed in a number of distinct processing stages with each stage containing a certain number of factors. This can be viewed as a generalization of strip-plot designs with 2 or more stages. The design is set up so that the settings of the factors at each processing stage are used on multiple experimental units. Consequently, at each processing stage the design has a split-plot (or split-unit) structure. As with split-plot designs, the split-lot structure allows for economical use of resources with some additional analysis complexity.

A review of the literature indicates that split-lot factorial designs were first considered by Taguchi (1987) under the name of multiway split-unit designs. The construction of split-lot factorial designs was pioneered by Mee and Bates (1998). They used split-lot designs for the fabrication of integrated circuits in the semiconductor industry. In their article, Mee and Bates found designs in cases where there are many processing stages, with potentially many factors. Another important application of split-lot designs is in product assembly. In (Bisgaard, 1997), the processing stages were the various parts of a product and the factors were certain specifications of each part. At each processing stage, the design is equivalent to a split-plot design where the factors for that stage are the whole-plot factors. The experimental units are thus separated at each stage into whole-plots (or sub-lots), which are processed together for that stage. More recently, Butler (2004) proposed a construction method for split-lot designs using a *grid representation* technique. The designs found by Butler (2004) have minimum aberration under the split-lot structure and in some sense minimize the confounding of main effects and two-factor interactions with the sub-lots.

Example 2.7. Consider a 2^4 full factorial experiment where the experimental units are processed together in 3 stages. At Stage 1, the 16 experimental units are split into two sub-lots (B_{11}, B_{12}) consisting of eight units each. These two sub-lots are processed separately and in random order: one at the low level of A, and the other at the high level of A. Once the processing is done for Stage 1, all of the experimental units move to Stage 2. The 16 units are again split into two sub-lots (B_{21}, B_{22}) of size eight each such that B_{21} consists of four units chosen from B_{11} and four units randomly chosen from B_{12}. Then, the two sub-lots (B_{21}, B_{22}) are processed separately at the low level and high level of B. Similarly, at Stage 3, the sub-lots (B_{31}, B_{32}) of size eight each are formed such that B_{31} consists of four units from B_{11} that are a combination of two units each from B_{21} and B_{22}. Finally, the other four units of B_{31} are from B_{12} such that there are two units each from B_{21} and B_{22}. A realization of the allotment of all the experimental units in different sub-lots is shown in Figure 2.4.

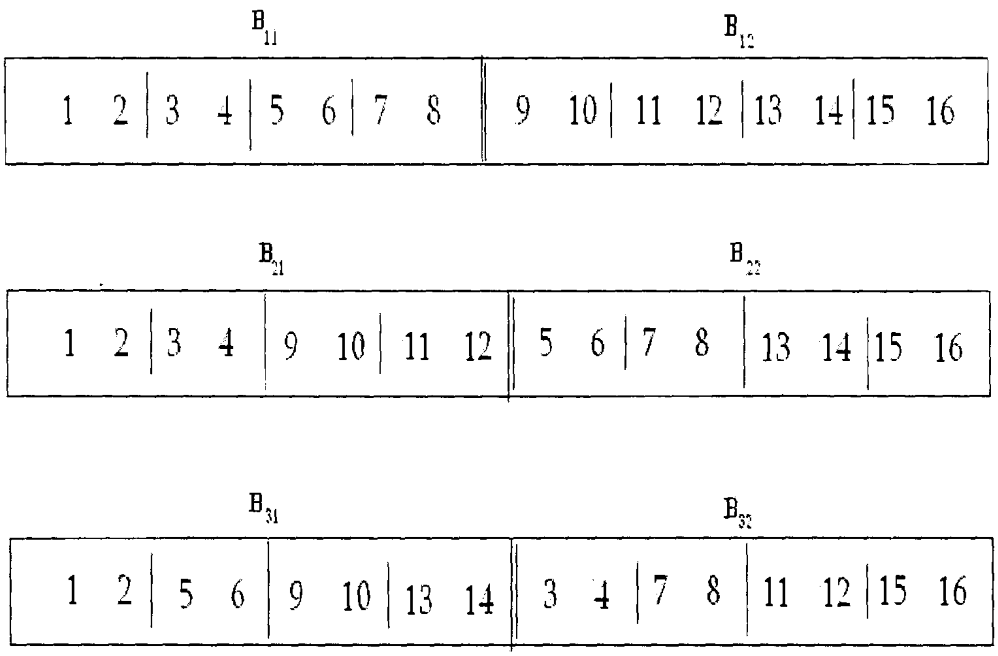

Figure 2.4: A split-lot design structure for a three-stage process.

The numbers $\{1, ..., 16\}$ in Figure 2.4 denote the 16 experimental units. Although the design configuration shown in Figure 2.4 seems simple, splitting the experimental units into sub-lots using the method described can sometimes be challenging. The following methodology is a general approach for splitting the experimental units into different sub-lots.

In Example 2.7, let all of the 16 factor combinations be randomly assigned to the 16 experimental units as indicated in Table 2.5.

Table 2.5: A design matrix for a 2^4 full factorial experiment.

	A	B	C	D
y_1	-1	-1	-1	-1
y_2	-1	-1	-1	1
y_3	-1	-1	1	-1
y_4	-1	-1	1	1
y_5	-1	1	-1	-1
y_6	-1	1	-1	1
y_7	-1	1	1	-1
y_8	-1	1	1	1
y_9	1	-1	-1	-1
y_{10}	1	-1	-1	1
y_{11}	1	-1	1	-1
y_{12}	1	-1	1	1
y_{13}	1	1	-1	-1
y_{14}	1	1	-1	1
y_{15}	1	1	1	-1
y_{16}	1	1	1	1

Then, the experimental units can be assigned to the sub-lots using the following rule:

$$B_{11} = \{y_i : \theta_A(i) = -1\}, \qquad B_{12} = \{y_i : \theta_A(i) = 1\},$$
$$B_{21} = \{y_i : \theta_B(i) = -1\}, \qquad B_{22} = \{y_i : \theta_B(i) = 1\},$$
$$B_{31} = \{y_i : \theta_C(i) = -1\}, \qquad B_{32} = \{y_i : \theta_C(i) = 1\},$$

where $\theta_\delta(i)$ is the entry in the i-th row of the column corresponding to the factorial effect δ in the model matrix X. This technique can be used to construct sub-lots for complex situations. One can view this as blocking or split-plotting at each stage. More complex examples on factorial and fractional factorial split-lot designs are given in Chapter 4 and 5. The following model can be used to describe a split-lot design with m levels of randomization

$$Y = X\beta + \varepsilon^1 + \varepsilon^2 + \cdots + \varepsilon^m + \varepsilon,$$

where the $n \times 1$ vector ε^k represents the error associated with the k-th stage of randomization and ε is the replication error vector. It is assumed that the $m + 1$

error terms are mutually independent and normally distributed. While useful for large experiments, the analysis of the data obtained becomes somewhat more complex than for a completely randomized design. The analysis of variance table for the 2^3 split-lot example is shown in Table 2.6.

Table 2.6: The analysis of variance table for the 2^3 split-lot example.

Sources of Variation	df
Stage 1 analysis:	
Effects (A, BC)	2
Stage 2 analysis:	
Effects (B, CA)	2
Stage 3 analysis:	
Effects (C, AB)	2
Other effects:	
Effects (ABC)	1
Total	7

In a 2^p factorial split-lot designs, the factorial effects used in the partitioning of n experimental units into sub-lots are analyzed together. The total number of separate analyses depends on the structure of the sets of effects used at each of the m stages of randomization. Nonetheless, at least m separate analyses have to be done.

2.3 Finite projective geometric representation

We use the finite projective geometric representation of factorial designs (Bose, 1947) to develop results for the existence and construction of factorial designs with randomization restrictions. Consider a factorial experiment involving p factors $F_1, ..., F_p$, each having q levels, where $q \geq 2$ is a prime or prime power. Let $GF(q)$ be a finite field with q elements. Here q is called the order of the field. Let V_q^p be the p-dimensional vector space over $GF(q)$, i.e., $V_q^p = \{(v_1, ..., v_p) : v_i \in GF(q) \text{ for } i = 1, ..., p\}$. The canonical basis elements of the vector space V_q^p can be identified with the p factors of a q^p factorial experiment. A factorial effect δ can be expressed in the form

$$\delta = F_1^{v_1} \cdots F_p^{v_p}, \quad \text{where } v_i \in GF(q) \text{ for } i = 1, ..., p.$$

The effect δ is an r-factor interaction if exactly r entries of the vector $v = (v_1, ..., v_p)$ are nonzero (δ is a main effect or a factor if $r = 1$). For instance, if $q = 2$ and $p = 3$, then (100), (001) and (101) represent A, C and AC respectively. Any t independent effects $\delta_1, ..., \delta_t$ (or equivalently, t linearly independent vectors in V_q^p) generate a subspace of size q^t contained in the vector space V_q^p.

The *projective space* $PG(p-1, q)$ is the geometry whose {points, lines, planes, ... , hyperplanes} are the subspaces of V_q^p of rank $\{1, 2, 3, ..., p-1\}$. A $(t-1)$-dimensional subspace of $PG(p-1, q)$ is a t-dimensional subspace of V_q^p. The 1-dimensional subspaces of V_q^p are the points, and the 2-dimensional subspaces are the lines of $PG(p-1, q)$. Each point of $PG(p-1, q)$ can be represented by a non-zero vector u of V_q^p, provided any non-zero scalar multiple of u represents the same point. That is, for two vectors u and v in V_q^p, if there exists $\alpha \in GF(q)$ and $\alpha \neq 0$ such that $u = \alpha v$, then u and v are said to be equivalent. In general, the $(t-1)$-dimensional objects described by t-dimensional subspaces of V_q^p are also known as $(t-1)$-flats or $(t-1)$-dimensional subspaces in $PG(p-1, q)$. The number of points in $PG(p-1, q)$ is equal to $(q^p - 1)/(q-1) = q^{p-1} + \cdots + 1$, and the number of distinct $(t-1)$-flats in $PG(p-1, q)$, called the *Gaussian number* $\left[\begin{smallmatrix} p \\ t \end{smallmatrix}\right]_q$, is given by:

$$\begin{bmatrix} p \\ t \end{bmatrix}_q = \frac{(q^p - 1)(q^{p-1} - 1) \cdots (q^{p-t+1} - 1)}{(q^t - 1)(q^{t-1} - 1) \cdots (q - 1)}.$$

Thus, for $t = 1$, the number of points (or, 0-flats) in $PG(p-1, q)$ is $\left[\begin{smallmatrix} p \\ 1 \end{smallmatrix}\right]_q$. For a detailed discussion on finite projective spaces, see Hirschfeld (1998). The following example explains the geometric structure of $PG(p-1, q)$.

Example 2.8. As a simple illustration consider the classical projective space $PG(2,2)$ with the smallest finite field $GF(2)$. The number of $(t-1)$-dimensional subspaces in $PG(2,2)$ can be computed using the Gaussian number formula. The number of points $(t=1)$ in $PG(2,2)$ is 7, and the number of lines $(t=2)$ is 7. Each line has 3 points, and each point is on three lines. In addition, each pair of distinct points is on a unique line, and any pair of two distinct lines meets at a unique point. The resulting geometric structure, frequently called the Fano plane, is shown in Figure 2.5. Here, each line represents a 1-dimensional subspace of $PG(2,2)$.

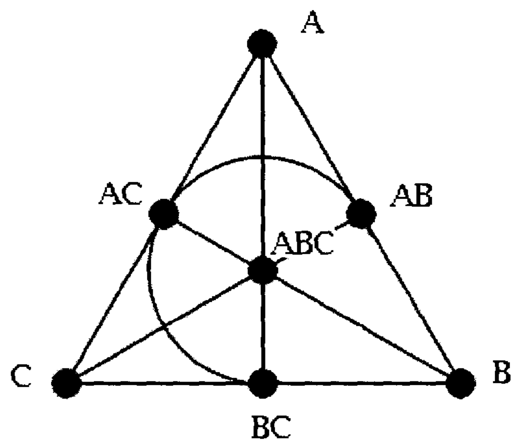

Figure 2.5: The Fano plane.

The points are denoted by 3-dimensional vectors in V_2^3: $\{(100),(010),(001),(110),\ldots,(111)\}$, and the lines (1-flats) are the 2-dimensional subspaces of V_2^3: $\{(100,110,010),(100,101,001),(010,011,001),\ldots,(110,011,101)\}$. In other words, for a factorial experiment with 3 two-level factors A, B, and C, the points correspond to $\{A,B,C,AB,\ldots,ABC\}$, and the 1-dimensional projective subspaces are $\{(A,AB,B),(A,AC,C),(B,BC,C),(A,BC,ABC),(B,AC,ABC),(C,AB,ABC),(AB,BC,AC)\}$. It is obvious to see from Figure 2.5 that there does not exist two disjoint subspaces of size 3 each in a 2^3 full factorial layout, as that would require 2 lines that do not intersect.

For applications of projective geometry in factorial designs see Bose (1947); Dey and Mukerjee (1999); and Mukerjee and Wu (2001). In factorial designs, these projective points are also referred to as pencils. A typical pencil belonging to a factorial effect is a non-null p-dimensional vector b over $GF(q)$. For $\alpha \neq 0 \in GF(q)$, b and αb represent the same pencil carrying $q-1$ degrees of freedom. A pencil b represents an r-factor interaction if b has exactly r nonzero elements (e.g., Bose, 1947; Dey and Mukerjee, 1999, Ch.8). Therefore, the set of all p-dimensional pencils over $GF(q)$ forms a $(p-1)$-dimensional finite projective geometry, denoted by $PG(p-1, q)$.

Since the two-level factorial designs are the most common designs in practice, this thesis will focus on $q = 2$, though most of the results presented in Chapters 4 and 5 hold for general q. For $q = 2$, a pencil b with r nonzero elements corresponds to an unique r-factor interaction in a 2^p factorial design. Thus, the set of all effects (excluding the grand mean) of a 2^p factorial design is equivalent to $PG(p-1, 2)$, which we call the effect space $\mathcal{P}$.

2.4 Randomization restrictions and subspaces

Suppose an experiment with p factors each at two levels is to be performed. An ideal choice is a 2^p factorial experiment with the trials performed in completely random order. However, it is not always possible to perform the experimental trials in a completely random order, and often randomization restrictions are imposed. So far the bulk of the literature focuses on different approaches for constructing regular factorial designs with different randomization restrictions. For example, Taguchi (1987) used linear graphs for the construction of split-lot designs while, Mee and Bates (1998) developed separate tools for different run-size factorial experiments under the split-lot design setting. Butler (2004) uses a grid-representation technique to construct some specific split-lot designs. Miller (1997) discusses the construction of strip-plot designs via Latin square fractions, and split-plot designs have also been found in a variety of ways (Huang, Chen and Voelkel, 1998; Bingham and Sitter, 1999; Bisgaard, 2000). Blocking in factorial and fractional factorial designs have also been studied in many different ways (e.g., Sitter, Chen and Feder, 1997; Mukerjee and Wu, 1999; Chen and Cheng, 1999).

Imposing restrictions on the randomization of experimental runs amounts to grouping the experimental units into sets of trials. We consider the usual approach of forming these sets for factorial experiments by using independent effects from $\mathcal{P}$. For example, blocked factorial designs use the 2^t $(t < p)$ combinations of t blocking factors (independent effects from $\mathcal{P}$) to divide 2^p treatment combinations into 2^t blocks (e.g., Lorenzen and Wincek, 1992).

Example 2.9. Consider a 2^6 full factorial design with four blocks, where the six factors are given by $(A, B, ..., F)$. Let the two independent blocking factors be $b_1 = ABCD$ and $b_2 = CDEF$. Then, the 64 experimental units are partitioned into 4 blocks $\mathcal{B}_i, i = 1, ..., 4$ of size 16 each. The block $\mathcal{B}_1$ consists of experimental units given by

$$\{y_i : (\theta_{b_1}(i), \theta_{b_2}(i)) = (0,0)\}.$$

Recall that, $\theta_\delta(i)$ is the i-th row entry of the column corresponding to the effect δ in the model matrix X. The remaining experimental units are assigned to the three blocks $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ such that $(\theta_{b_1}(i), \theta_{b_2}(i)) = (1,0), (0,1)$ and $(1,1)$, respectively.

Similarly, we consider the setting where 2^p experimental runs are partitioned into sets of trials (e.g., blocks, batches, lots, or sub-plots) by using a set of t independent effects of $\mathcal{P}$ that represent the imposed randomization restrictions, or equivalently the t randomization restriction factors (Bingham et al., 2006).

The set of all non-null linear combinations of these t randomization restriction factors in $\mathcal{P}$ over $GF(2)$ forms a $(t-1)$-dimensional subspace of $\mathcal{P} = PG(p-1, 2)$.

We define such subspaces as *randomization defining contrast subspace* (RDCSS). The RDCSS structure can be used to study factorial and fractional factorial designs with different randomization restrictions under one framework. As an alternative, but related, one could use "design keys" proposed by Patterson and Bailey (1978) to study factorial and regular fractional factorial designs with several different randomization restrictions.

Frequently, there is more than one stage of randomization in a factorial experiment, where the randomization structure can be characterized by its RDCSSs. For a 2^p factorial design with m levels of randomization, the m RDCSSs can be denoted by the projective subspaces $S_1, \ldots, S_m$ contained in the corresponding effect space $\mathcal{P} = PG(p-1, 2)$. Let the size of S_i be $2^{t_i} - 1$ for $0 < t_i < p$. Then, the experimental units are partitioned into 2^{t_i} sets (e.g., batches or blocks) due to S_i, where the size of each set is $(|\mathcal{P}| + 1)/(|S_i| + 1)$.

Example 2.10. Consider a 2^5 factorial experiment with randomization structure defined by a split-plot design, where A,B are whole-plot factors, and C, D,E are subplot factors. This is sometimes referred to as a 2^{2+3} factorial experiment (Bisgaard, 2000). Under this setting, the effect space is $\mathcal{P} = \langle A, B, C, D, E\rangle$ and the only RDCSS, S_1, is given by $S_1 = \langle A, B\rangle$. Here $\langle a_1, \ldots, a_k\rangle$ denotes the projective space spanned by $a_1, \ldots, a_k$. That is, $S_1 = \{A, B, AB\}$. Since, $|S_1| = 2^2 - 1$, the set of all experimental units are partitioned into 4 subsets (batches) and each subset consists of $2^5/2^2$ experimental units. These four subsets $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ consist of experimental units corresponding to $(\theta_A(i), \theta_B(i)) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$, respectively.

Example 2.11. Consider a 2^5 factorial experiment with randomization structure characterized by a strip-plot design (Miller 1997), where the row configurations are represented by a 2^2 design in factors (A, B), and the column configurations are represented by a 2^3 design in factors (C, D, E). Under this setting, the RDCSSs are $S_1 = \langle A, B \rangle$, $S_2 = \langle C, D, E \rangle$, and the effect space is $\mathcal{P} = \langle A, B, C, D, E \rangle$.

Although the treatment structure for both examples are same, the randomization restriction induces different error structures (Milliken and Johnson, 1984, Ch.4). Therefore, the distribution of the factorial effect estimates are different. Consequently, the half-normal plot procedure for assessing the significance of the factorial effects in the effect space $\mathcal{P}$ will be different in these two examples. That is, the number of half-normal plots and the sets of factorial effects for these plots are likely to be different. We elaborate on this in the next chapter.

Chapter 3

Linear Regression Model and RDCSSs

The normal linear regression model is typically used for the analysis of factorial designs. These statistical models are a way of characterizing relationships between the response variable, y, and a set of p independent factors, $x = (x_1, ..., x_p)$. A regression model for the data is a combination of the systematic part of the relationship between x and y, along with the variation, or noise in the measurement of the response.

When the experimental trials are performed in a completely random order, the regression model usually contains one source of variability, the replication error. If restrictions are imposed on the randomization of the experiment, variation in the observations is a combination of several components. This impacts the distribution of the parameter estimates of the regression model. It turns out that the distribution of parameter estimates can be characterized by the underlying RDCSS structure of the factorial design.

In this chapter, we first propose a framework in Section 3.1 that can be used to express the response models for the factorial designs with different randomization restrictions under the unified notion (Section 2.5) first introduced in Bingham et al. (2006). Next, the impact of the RDCSS structure on linear regression models for factorial designs is discussed. The main result of this chapter indicates that the distribution of an effect estimate depends upon its presence in different RDCSSs. The corresponding analysis using half-normal plots motivates a design strategy. In particular, we desire non-overlapping subspaces of the effect space $\mathcal{P}$ that can be used for constructing RDCSSs. This is illustrated through an example in Section 3.2.

3.1 Unified Model

Consider an unreplicated two-level regular full factorial design with p independent factors. The response model of interest is the linear regression model,

$$Y = X\beta + \varepsilon, \tag{3.1}$$

where X denotes the $n \times 2^p$ model matrix and $\beta = (\beta_0, \beta_1, ..., \beta_{2^p-1})'$ is the $2^p \times 1$ vector of parameters corresponding to the factorial effects of the 2^p factorial design. Since the trials are performed using an unreplicated full factorial design, the number of experimental units n is 2^p. Without loss of generality, the columns of X can be written as $X = \{c_0, c_1, \ldots, c_p, c_{p+1}, \ldots, c_{n-1}\}$, where c_0 is a column vector of all 0's corresponding to the grand mean, columns labelled $c_1, \ldots, c_p$, refer to the p independent factors and the remaining columns of X represent the interactions obtained via addition of subsets of $\{c_1, \ldots, c_p\}$ modulo 2. For the results in this section, we recode the factor levels 0 and 1 as $+1$ and -1, respectively.

For a factorial design with m levels of randomization, where the RDCSSs are denoted by S_i, $i = 1, \ldots, m$, the error ε in model (3.1) can be divided into $m + 1$ independent error terms, $\varepsilon = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_m$. The $n \times 1$ vector ε_0 denotes the replication error, and the vector ε_i ($1 \leq i \leq m$) is the error vector associated with the randomization restriction characterized by S_i, where $|S_i| = 2^{t_i} - 1$. The restriction defined by S_i creates a partition of the set of n experimental units into $|S_i| + 1$

batches (or blocks, for example). Thus, the error vector ε_i $(1 \leq i \leq m)$ can be further simplified to $N_i\epsilon_i$, where ϵ_i is a $2^{t_i} \times 1$ vector corresponding to the error associated with each of the 2^{t_i} batches,

$$\begin{aligned} \varepsilon &= \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_m, &(3.2)\\ &= \epsilon_0 + N_1\epsilon_1 + \cdots + N_m\epsilon_m, &(3.3) \end{aligned}$$

and $\epsilon_0 = \varepsilon_0$ is the vector of replication errors. The coefficient N_i is an $n \times 2^{t_i}$ matrix referred to as the i-th *incidence matrix*, with elements defined as:

$$\begin{aligned} (N_i)_{rl} &= 1, \quad \text{if } r\text{-th experimental unit belongs to the } l\text{-th batch at } i\text{-th} \\ &\qquad\quad \text{stage of randomization,} &(3.4)\\ &= 0, \quad \text{otherwise,} \end{aligned}$$

for $i = 1, \ldots, m$; $l = 1, \ldots, 2^{t_i}$ and $r = 1, \ldots, n$. The following example illustrates the different parts of the model.

Example 3.1. Consider a 2^4 full factorial design with the effect space $\mathcal{P} = \langle A, B, C, D\rangle$, where the randomization structure is characterized by the subspaces $S_1 = \langle A, B, C\rangle$ and $S_2 = \langle B, C, D\rangle$. Under these settings, the design matrix $\mathcal{D}$ is given by:

$$\mathcal{D} = \{c_1, \ldots, c_4\} = \begin{pmatrix} A & B & C & D \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix},$$

and the incidence matrix, N_1, for the first stage of randomization can be written as

$$N_1 = \begin{array}{c} \begin{array}{cccccccc} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} & \mathcal{B}_{14} & \mathcal{B}_{15} & \mathcal{B}_{16} & \mathcal{B}_{17} & \mathcal{B}_{18} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Here, $\mathcal{B}_{1j}$ denotes the j-th batch formed due to the randomization restriction defined by subspace S_1. Since the size of S_1 is $2^3-1=7$, the experimental units are partitioned into 8 batches of 2 experimental units, and therefore the restriction error associated with the batches formed due to S_1 is $\epsilon_1 = (\epsilon_{11}, \ldots, \epsilon_{18})'$. Note that N_1 indicates which experimental unit appears in which batch. Similarly, $\epsilon_2 = (\epsilon_{21}, \ldots, \epsilon_{28})'$ is the restriction error associated with the batches formed due to S_2. The error ε_1 associated with the experimental units due to the randomization restriction defined by the subspace S_1 is given by $\varepsilon_1 = \{\epsilon_{11}, \epsilon_{11}, \epsilon_{12}, \epsilon_{12}, \epsilon_{13}, \epsilon_{13}, \epsilon_{14}, \epsilon_{14}, \epsilon_{15}, \epsilon_{15}, \epsilon_{16}, \epsilon_{16}, \epsilon_{17}, \epsilon_{17}, \epsilon_{18}, \epsilon_{18}\}$. Similarly, $\varepsilon_2 = \{\epsilon_{21}, \epsilon_{22}, \epsilon_{23}, \epsilon_{24}, \epsilon_{25}, \epsilon_{26}, \epsilon_{27}, \epsilon_{28}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}, \epsilon_{24}, \epsilon_{25}, \epsilon_{26}, \epsilon_{27}, \epsilon_{28}\}$ is the error associated with the experimental units due to S_2.

We now use the incidence matrices to help derive the distribution of parameter estimates corresponding to the factorial effects in the model. The most natural way to estimate the regression parameters is using the *generalized least square* (GLS) estimator $\hat{\beta} = (X'\Sigma_y^{-1}X)^{-1}X'\Sigma_y^{-1}Y$, where,

$$\Sigma_y = \mathrm{Var}(\varepsilon) = \sigma^2 I + \sum_{i=1}^{m} \sigma_i^2 N_i N_i'. \tag{3.5}$$

The independence and normality assumptions among the restriction errors implies that the distribution of the parameter estimate vector $\hat{\beta}$ is normal with mean β and variance $(X'\Sigma_y^{-1}X)^{-1}$.

Note that finding the distribution of individual effect estimates involves computation of the inverse of $X'\Sigma_y^{-1}X$. It turns out that one can avoid the inversion by using the *ordinary least square* (OLS) estimator of β, $\tilde{\beta} = (X'X)^{-1}X'Y$. The equality of the two estimators of β can be established by verifying necessary and sufficient conditions (Anderson, 1948; Watson, 1955; Zyskind, 1967; Rao, 1967; Alalouf and Styan, 1984; Puntanen and Styan, 1989;). In the next result, we propose to use one such condition to establish the equality of the estimators.

Theorem 3.1. *For an unreplicated 2^p full factorial design, $\hat{\beta} = \tilde{\beta}$ under model (3.1).*

Proof: Let X be the model matrix for the factorial design and Y be the column vector of all the observations arising from model (3.1). Then, the GLS estimator of $X\beta$ can be written in terms of the OLS estimator of $X\beta$, as,

$$X\hat{\beta} = X\tilde{\beta} - H\Sigma_y M(M\Sigma_y M)^{+}Y,$$

where Σ_y is the variance covariance matrix (3.5), $H = X(X'X)^{-1}X'$, $M = I - H$, and $(M\Sigma_y M)^{+}$ is the Moore-Penrose inverse of $M\Sigma_y M$ (e.g., see Albert, 1973; Rao, 1973; Pukelsheim, 1977; Baksalary and Kala, 1978 for details). For a 2^p full factorial design, the model matrix X can be viewed as a Hadamard matrix of order n. Therefore, $X'X = nI$ and $XX' = nI$ implies that $H = I$ (or equivalently, $M = 0$), i.e., $H\Sigma_y M = 0$. Since the Moore-Penrose inverse of a null matrix is its transpose (Harville, 1997, Ch. 20), $M = 0$ implies that $(M\Sigma_y M)^{+} = 0$. Hence, the equality of GLS and OLS estimators of $X\beta$ is verified. Since the model matrix X has full column rank and the covariance matrix Σ_y is positive definite, then $\hat{\beta} = \tilde{\beta}$. □

Theorem 3.1 shows that the regression coefficients of model (3.1) can be estimated by OLS. Consequently, the variance of the effect estimates is $\mathrm{Var}(\hat{\beta}) = \mathrm{Var}(\tilde{\beta}) = (X'\Sigma_y X)/n^2$, and thus $\hat{\beta} \sim N(\beta, X'\Sigma_y X/n^2)$. This theorem is useful for finding the distribution of individual effect estimates in so far as we now only need to consider the OLS estimator.

For a 2^p factorial design with $r > 1$ replicates, the hat matrix, H, in Theorem 3.1 simplifies to $\frac{1}{r}\left(J_{r\times r} \otimes I_{2^p\times 2^p}\right)$, where $\otimes$ is the Kronecker product. Although $M \neq 0$ for this case, simple calculation using a Kronecker representation of the corresponding incidence matrices (equation (3.2)-(3.4)) in the covariance matrix Σ_y (equation (3.5)) shows that $H\Sigma_y M = 0$. This further implies that $H\Sigma_y M(M\Sigma_y M)^+ = 0$. Haberman (1975) showed that the condition $H\Sigma_y^{-1}M = 0$ is a necessary and sufficient condition for the equality of OLS and GLS estimators of $X\beta$. This involves inversion of the covariance matrix, which we wanted to avoid. Thus, the equality of OLS and GLS estimators is ensured from the condition used in the proof of Theorem 3.1 even if the design is replicated.

The presence of $N_i N_i'$ in the expression of Σ_y (equation (3.5)) suggests that the distribution of the effect estimates, or equivalently the simplification of $\mathrm{Var}(\tilde{\beta})$, depends on the overlapping structure among the S_i's. Since the S_i's are subspaces contained in $\mathcal{P}$, it may be possible to have $S_{ij} = S_i \cap S_j \neq \phi$. While not obvious at the moment, these cases are of specific interest in our setting. It turns out that when this condition does not hold, the variances of the effects in S_{ij} will be impacted by both σ_i^2 and σ_j^2. On the other hand, we show that when $S_{ij} = \phi$, the variances of all the effects in S_i are not functions of σ_j^2. We now propose results to formally explain the impact of overlapping patterns among the RDCSSs on the distribution of individual effect estimates.

Theorem 3.2. *Consider a 2^p full factorial design, where the randomization restrictions are defined by subspaces $S_1, \ldots, S_m$ in $\mathcal{P}$. Then, for any two effects E_1 and*

E_2 in the effect space $\mathcal{P}$, the corresponding parameter estimators $\hat{\beta}_{E_1}$ and $\hat{\beta}_{E_2}$ have independent normal distributions.

Proof: Since $\hat{\beta}$ has a multivariate normal distribution, it is enough to show that $\text{cov}(\hat{\beta}_{E_1}, \hat{\beta}_{E_2}) = 0$. From equation (3.5) and the fact that $X'X = nI$, the variance of $\hat{\beta}$ can be written as a product of $1/n^2$ and

$$X'\Sigma_y X = n\sigma^2 I + \sum_{i=1}^{m} \sigma_i^2 X' N_i N_i' X.$$

Let δ_s denote the factorial effect corresponding to s-th column ($s > 1$) of X. Then, by applying the definition of N_i,

$$\begin{aligned}(X'N_i)_{st} &= \pm n_i \ , \quad \text{if } \delta_s \in S_i, \\ &= 0 \ , \quad \text{otherwise,}\end{aligned}$$

where $n_i = 2^{p-t_i}$ is the number of 1's in each column of N_i. The positive and negative sign of n_i varies with the columns of N_i. Thus, entries of the s-th row of $X'N_i$ are $\pm n_i$ if δ_s is contained in S_i, and zero otherwise. This further implies that the s-th diagonal entry of $(X'N_i)(N_i'X)$ is $n_i^2 2^{t_i} = n \cdot n_i$, if $\delta_s \in S_i$. For $s \neq t$, $s, t > 1$, and $1 \leq i \leq m$, orthogonality of the two columns X_s and X_t implies that the (s,t)-th entry of $(X'N_i)(N_i'X)$ is zero. That is, $X'\Sigma_y X$ is a sum of diagonal matrices and thus, $\text{cov}(\hat{\beta}_{E_1}, \hat{\beta}_{E_2}) = 0$. □

The effect estimates, therefore, follow independent normal distributions. However, the distributions of all the factorial effects are not necessarily identical. Next, we propose the main result of this section which establishes the relationship between the variance of the effect estimates and the presence of effects in different RDCSSs.

Theorem 3.3. *Consider a 2^p full factorial design, where the randomization restrictions are defined by $S_1, \ldots, S_m$ in $\mathcal{P}$. Define a sequence of index sets $\{T_E, E \in \mathcal{P}\}$*

such that $T_E = \{i \,:\, 1 \leq i \leq m,\ E \in S_i\}$. *Then, for any given effect* $E \in \mathcal{P}$,

$$Var(\hat{\beta}_E) = \begin{cases} \frac{\sigma^2}{n} + \sum_{\{i\,:\,i\in T_E\}} \frac{n_i}{n}\sigma_i^2, & \text{if } E \in \{S_1 \cup \cdots \cup S_m\}, \\ \frac{\sigma^2}{n}, & \text{if } E \in \mathcal{P}\backslash\{S_1 \cup \cdots \cup S_m\}, \end{cases}$$

where σ^2 *is the replication error variance and* σ_i^2 *is the* i*-th restriction error variance.*

Proof: Define an $2^p \times 1$ column vector η_E such that $(\eta_E)_s = 1$ if the s-th column of X corresponds to effect E and zero otherwise. Then, for a given effect $E \in \mathcal{P}$, $\eta_E' X' N_i N_i' X \eta_E = nn_i$ whenever $E \in S_i$, for $i \in \{1, ..., m\}$. From equation (3.5) and the multivariate normal distribution of $\hat{\beta}$, we get $\text{Var}(\hat{\beta}_E) = \text{Var}(\delta_E'\hat{\beta}) = \frac{\sigma^2}{n} + \sum_{\{i\,:\,i\in T_E\}} \frac{n_i}{n}\, \sigma_i^2$. If instead $E \in \mathcal{P}\backslash\{S_1 \cup \cdots \cup S_m\}$, $\eta_E' X' N_i N_i' X \eta_E = 0$ for all i in $\{1, ..., m\}$. As a result, $\text{Var}(\hat{\beta}_E) = \frac{\sigma^2}{n}$. □

Corollary 3.1. *Consider a* 2^p *full factorial design, where the randomization restrictions are defined by subspaces* $S_1, \ldots, S_m$ *in* $\mathcal{P}$ *and* $S_i \cap S_j = \phi$ *for all* $i \neq j$. *Then, for any given effect* $E \in \mathcal{P}$,

$$Var(\hat{\beta}_E) = \begin{cases} \frac{\sigma^2}{n} + \frac{n_i}{n}\sigma_i^2, & \text{if } E \in S_i, \\ \frac{\sigma^2}{n}, & \text{if } E \in \mathcal{P}\backslash\{S_1 \cup \cdots \cup S_m\} \end{cases}.$$

Proof: Note that, for any effect $E \in \mathcal{P}$, there exists a unique i such that $E \in S_i$. That is, $\eta_E' X' N_i N_i' X \eta_E$ is nonzero for a unique $i \in \{1, ..., m\}$, hence the result. □

These results show that the distribution of effect estimates depends on an effect's presence in different RDCSSs. For instance, consider the plutonium example setup in Bingham et al. (2006). Here, the authors performed a split-lot design with 3 levels of randomization in a 2^5 full factorial experiment, where $S_1 = \{A, B, AB, CDE, ACDE, BCDE, ABCDE\}$, $S_2 = \{C, AD, BE, ACD, BCE, ABDE, ABCDE\}$, $S_3 = \{D, E, DE, ABC, ABCD, ABCE, ABCDE\}$, and the effect space is $P = \langle A, B, C, D, E\rangle$. Then, from Theorem 3.3, the variance of parameter estimates corresponding to the factorial effects in S_i's are given by

$$\mathrm{Var}(\hat{\delta}) = \begin{cases} \frac{4}{32}\sigma_1^2 + \frac{1}{32}\sigma^2 & \text{for } \delta \in S_1 \backslash (S_1 \cap S_2 \cap S_3) \\ \frac{4}{32}\sigma_2^2 + \frac{1}{32}\sigma^2 & \text{for } \delta \in S_2 \backslash (S_1 \cap S_2 \cap S_3) \\ \frac{4}{32}\sigma_3^2 + \frac{1}{32}\sigma^2 & \text{for } \delta \in S_3 \backslash (S_1 \cap S_2 \cap S_3) \\ \frac{4}{32}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{1}{32}\sigma^2 & \text{for } \delta \in S_1 \cap S_2 \cap S_3 \end{cases},$$

whereas for the rest of the effects in $\mathcal{P}$, $\mathrm{Var}(\hat{\delta}) = \sigma^2/32$. For effects in unreplicated factorial experiments with randomization restrictions, separate analyses (for instance half-normal plots) are required. That is, Theorem 3.3 categorizes the factorial effects for separate analyses based on the overlapping pattern among the RDCSSs. Next, we discuss the impact of the size of the overlap among the RDCSSs on the analysis.

3.2 Motivation for disjoint RDCSSs

A common strategy for the analysis of factorial designs is the use of half-normal plots (Daniel, 1959). To do this, the effects appearing on the same plot must have the same error variance. From Theorem 3.3 and Corollary 3.1, m separate half-normal plots are required if S_i's are pairwise disjoint and $\mathcal{P} = \{\cup_{i=1}^m S_i\}$. If instead $\mathcal{P}\backslash\{\cup_{i=1}^m S_i\} \neq \phi$, $m + 1$ such plots have to be constructed to assess the significance of the effects. On the other hand, if $S_{ij} = S_i \cap S_j \neq \phi$ for some $i, j \in \{1, ..., m\}$, then the effects in S_{ij} will have a variance that is a linear combination of σ_i^2 and σ_j^2.

In the plutonium example (Bingham et al., 2006) setup described above, since the three RDCSSs S_1, S_2 and S_3 overlap, the effect space can be categorized into five groups $G_1, ..., G_5$, with effects having identical distributions within groups, which should therefore be analyzed together (see Table 3.1).

Table 3.1: The ANOVA table for the 2^5 split-lot design in a two-stage process.

Effects	Variance	Degrees of Freedom
$G_1 = S_1 \cap S_2 \cap S_3$	$\frac{4}{32}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{1}{32}\sigma^2$	1
$G_2 = S_1 \backslash (S_1 \cap S_2 \cap S_3)$	$\frac{4}{32}\sigma_1^2 + \frac{1}{32}\sigma^2$	6
$G_3 = S_2 \backslash (S_1 \cap S_2 \cap S_3)$	$\frac{4}{32}\sigma_2^2 + \frac{1}{32}\sigma^2$	6
$G_4 = S_3 \backslash (S_1 \cap S_2 \cap S_3)$	$\frac{4}{32}\sigma_3^2 + \frac{1}{32}\sigma^2$	6
$G_5 = \mathcal{P} \backslash (S_1 \cup S_2 \cup S_3)$	$\frac{1}{32}\sigma^2$	12

The factorial effects in G_2, G_3, G_4 and G_5 were assessed using four separate half-normal plots. Clearly, G_1 has too few effects for constructing a useful half-normal plot. Therefore, one sacrifices the ability to assess the significance of effects in G_1.

Indeed, for unreplicated experiments, assessing the significance of effects in S_{ij} may have to be sacrificed due to a lack of degrees of freedom. To get the most out of the experiment it is preferable to have $S_{ij} = \phi$. That is, one prefers disjoint RDCSSs. Finding such a design is equivalent to finding a set of disjoint subspaces that satisfies the experimenter's requirement. It turns out that this is not always easy. In the next chapter, we develop results that specify the conditions for the existence of designs with non-overlapping RDCSSs.

Chapter 4

Factorial designs and Disjoint Subspaces

Several half-normal plots are required to assess the significance of the factorial effects in an unreplicated factorial experiment with randomization restrictions. Only effects with the same variance may appear together on the same plot. From the discussion in Chapter 3, it is preferable to find disjoint subspaces for constructing RDCSSs. In most applications, the desired number and size of the RDCSSs, or equivalently the subspaces of the effect space $\mathcal{P} = PG(p-1, 2)$, are pre-determined. Surprisingly, determining the existence of disjoint subspaces of $\mathcal{P}$ for constructing RDCSSs (or equivalently, finding the design with the desired randomization and analysis properties) is a fairly complex task.

In this chapter, the conditions for the existence of disjoint subspaces of $\mathcal{P}$ are first derived. The results presented focus on the existence of a set of disjoint subspaces of both equal and unequal sizes that span the entire effect space $\mathcal{P}$ of a two-level factorial design. Next, the main theoretical result is developed in Section 4.1.2. Construction methods are proposed in Section 4.2. Finally, in Section 4.3, we develop existence results for regular fractional factorial designs with randomization restrictions. The results are constructive and thus allow experimenters to find designs in practice. It should be noted that the results presented here for two-level factorial designs can easily be extended to q-levels.

4.1 Existence of RDCSSs

We first start by exploring the available geometric structures that can be used to establish the existence of a set of disjoint subspaces. These results focus on equal sized subspaces or experiments with blocks (or batches) of the same size at the m stages of randomization. In Section 4.1.2, new results are developed for disjoint subspaces of different sizes.

4.1.1 RDCSSs and $(t-1)$-spreads

In many applications, the number of stages of randomization (m) is pre-specified by the experimenter. Thus, if one can obtain a set of pairwise disjoint equal sized subspaces (say $\mathcal{S}$) with $|\mathcal{S}| \geq m$, an appropriate subset of $\mathcal{S}$ can be selected that satisfies the criteria of the RDCSSs required by the experimenter. It turns out that one can establish conditions for the existence of a set of disjoint $(t-1)$-dimensional subspaces where a set $\mathcal{S}$ partitions the effect space $\mathcal{P} = PG(p-1,2)$. The next definition is due to André (1954).

Definition 4.1. *For $1 \leq t \leq p$, a $(t-1)$-spread of the effect space $\mathcal{P}$ is a set $\mathcal{S}$ of $(t-1)$-dimensional subspaces of $\mathcal{P}$ which partitions $\mathcal{P}$.*

That is, every element of $\mathcal{P}$ is contained in exactly one of the $(t-1)$-dimensional subspaces. A $(t-1)$-spread $\mathcal{S}$ is said to be nontrivial if $t > 1$. In other words, $\mathcal{S}$ is nontrivial if the size of every element of $\mathcal{S}$ is at least 3. When a $(t-1)$-spread of $\mathcal{P}$

exists, the size of $\mathcal{S}$ is $|\mathcal{S}| = (2^p - 1)/(2^t - 1)$, which is the maximum size of a set of disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$. For example, in a 2^6 full factorial design (or equivalently in $PG(5,2)$), there exists up to 9 disjoint subspaces of size 7 each. If the required number of RDCSSs, m, is less than $|\mathcal{S}|$, then one can select a subset of $\mathcal{S}$ to construct the RDCSSs. However, the existence of a $(t-1)$-spread in general is not guaranteed and depends on a necessary and sufficient condition established by André (1954).

Lemma 4.1. *A $(t-1)$-spread $\mathcal{S}$ of $PG(p-1,2)$ exists if and only if t divides p.*

That is, if p is a prime number (e.g., in 2^5, 2^7 factorial experiments), there does not exist any nontrivial $(t-1)$-spread $\mathcal{S}$ of $\mathcal{P}$. Nevertheless, the required number of disjoint subspaces is determined by the experimental setting. If there does not exist a $(t-1)$-spread of $\mathcal{P}$, one would be interested in knowing the maximum number of disjoint $(t-1)$-dimensional subspaces that can be obtained in the effect space $\mathcal{P}$. This is called a partial $(t-1)$-spread in finite projective geometry.

Definition 4.2. *A partial $(t-1)$-spread $\mathcal{S}$ of the effect space $\mathcal{P}$ is a set of $(t-1)$-dimensional subspaces of $\mathcal{P}$ that are pairwise disjoint.*

Similar to $(t-1)$-spreads, effort has been devoted in establishing the existence of a maximal partial $(t-1)$-spread of $\mathcal{P}$ (e.g., Beutelspacher, 1975; Drake and Freeman, 1979; Eisfeld and Storme, 2000; Govaerts, 2005). The following result summarizes the upper bounds available on the maximum number of pairwise disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$ (see Govaerts, 2005).

Lemma 4.2. *Let $\mathcal{P}$ be the projective space $PG(p-1,2)$, with $p = kt + r$, for positive integers k, t, r such that $r < t < p$, and $\mathcal{S}$ be a partial $(t-1)$-spread of $\mathcal{P}$ with $|\mathcal{S}| = 2^r \frac{2^{kt}-1}{2^t-1} - s$, where s is known as the deficiency. Then,*

(a) $s \geq 2^r - 1$, if $r = 1$.

(b) $s \geq 2^{r-1} - 1$, if $r > 1$ and $t \geq 2r$.

(c) $s \geq 2^{r-1} - 2^{2r-t-1} + 1$, if $r > 1$ and $t < 2r$.

Lemma 4.2 provides upper bounds on the maximum number of disjoint $(t-1)$-dimensional subspaces of $PG(p-1,2)$ for different combinations of t and r. This is of particular interest when no $(t-1)$-spread exists (i.e., t does not divide p). It is worth noting that these bounds may not be tight.

Example 4.1. Consider a 2^5 full factorial experiment with randomization restrictions defined by S_1, S_2 and S_3, such that $S_1 \supset \{A, B\}$, $S_2 \supset \{C\}$ and $S_3 \supset \{D, E\}$. From the discussion in Chapter 3, one needs at least three half-normal plots. The exact number depends on the overlapping pattern among the S_i's. To use a half-normal plot for assessing significant effects one requires at least six or seven effects for each plot (Schoen, 1999). In this setting, only 1 or 2 effects are assumed to be more active than others. Therefore, since the S_i's are subspaces, one useful randomization structure would be where $|S_i| = 2^3 - 1$ for all i, and the S_i's are all pairwise disjoint. Here, $p = 5$ and $t = 3$, so Lemma 4.1 implies that there does not exist a 2-spread of $\mathcal{P} = PG(4,2)$. Moreover, from Lemma 4.2, $k = 1$ and $r = 2$ implies that the maximum number of disjoint 2-dimensional subspaces of $\mathcal{P}$ is bounded above by 2. However, there is no certainty from the theorem regarding the existence of even two disjoint 2-dimensional subspaces, indeed, there is not.

This example motivates the need for further exploration of the subspace structure in

$\mathcal{P}$. In the next section, we develop results for the existence of sets of pairwise disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$ when a spread does not exist. In practice this means that effects appearing in multiple RDCSSs will inherent the variance component from each of the overlapping subspaces. Though the set of disjoint subspaces may not be maximal, the designs obtained using the results in the next section can be easily constructed and are thus useful to experimenters.

4.1.2 RDCSSs and disjoint subspaces

First, necessary and sufficient conditions for the existence of a set of disjoint $(t-1)$-dimensional subspaces are established. Then, these conditions are generalized for the existence of sets of m disjoint subspaces of unequal sizes (i.e., different size RDCSSs). This latter case is important in multistage experiments, where the number of units in a batch or block are not the same at each stage.

Theorem 4.1. *Let $\mathcal{P}$ be the projective space $PG(p-1, 2)$ and S_1, S_2 be two distinct $(t-1)$-dimensional subspaces of $\mathcal{P}$, for $0 < t < p$.*

(a) If $t \leq p/2$, there exists S_1 and S_2 such that $S_1 \cap S_2 = \phi$.

(b) If $t > p/2$, for every $S_1, S_2 \in \mathcal{P}$, $|S_1 \cap S_2| \geq 2^{2t-p} - 1$ and there exists S_1, S_2 such that the equality holds.

The proof of Theorem 4.1 will be shown in a more general setup (Theorem 4.3). Along with the conditions for the existence of disjoint subspaces, the result proposed in Theorem 4.1 also provides the size of minimum overlap when there does not exist even two $(t-1)$-dimensional subspaces. It turns out that when $t \leq p/2$, one can obtain more than two disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$. From Section 3.2, it is obvious that the subspaces required for constructing RDCSSs should be large enough to construct useful half-normal plots. This indicates that in two-level factorial designs, $t \geq 3$ is desirable, which further implies that if $t \leq p/2$, the value of p is bounded below by 6. Since designs with randomization restrictions are often larger than completely randomized designs, these results are useful to a practitioner.

When t does not divide p, one can assume that $p = kt + r$ for positive integers k, t, r satisfying $0 < r < t < p$ and $k \geq 1$. It can be tempting to work with a $(t-1)$-spread $\mathcal{S}_0$ (say) of $PG(kt-1, 2)$, which is embedded in $\mathcal{P}$. The following new result demonstrates the existence of a set of disjoint subspaces based on $\mathcal{S}_0$.

Lemma 4.3. *Let $\mathcal{P}$ be the projective space $PG(p-1, 2)$ for $p = kt + r$. Then, there exists m subspaces $S_1, \ldots, S_m$ in $\mathcal{P}$ such that $|S_i| = 2^t - 1$, $i = 1, ..., m$, where $m = \frac{2^{kt}-1}{2^t-1}$, and the S_i's are pairwise disjoint. Furthermore, there exists S_{m+1} such that $|S_{m+1}| = 2^r - 1$ and $S_{m+1} \cap S_i = \phi$ for all $i = 1, \ldots, m$.*

Proof of Lemma 4.3 follows from the existence of a $(t-1)$-spread of $PG(kt-1, 2)$. Since $\mathcal{S}_0$ is constructed from a $(t-1)$-spread of a *subspace* which is a proper subset of $\mathcal{P}$, the set of disjoint $(t-1)$-subspaces in $\mathcal{P}$ can be expanded. The following result due to Eisfeld and Storme (2000) ensures the existence of a relatively larger set of disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$.

Lemma 4.4. *Let $\mathcal{P}$ be the projective space $PG(p-1, 2)$, for $p = kt + r$. Then, there exists a partial $(t-1)$-spread $\mathcal{S}$ of $\mathcal{P}$ with $|\mathcal{S}| = 2^r \frac{2^{kt}-1}{2^t-1} - 2^r + 1$.*

That is, there always exists a set of disjoint $(t-1)$-dimensional subspaces of cardinality $|\mathcal{S}|$. The proof developed below is more concise than the one provided in Eisfeld and Storme (2000). Most importantly, the proof is useful insofar as it outlines the construction of the partial $(t-1)$-spread of $\mathcal{P}$ claimed to exist in the lemma.

Proof: Define a sequence $\{s_i\}_{i=1}^k$ such that $s_i = it + r - 1$ for $i = 1, \ldots, k$, and $\mathcal{P}_i = PG(2s_i + 1, 2)$ for $i = 1, \ldots, k-1$. Note that $s_i + t = s_{i+1}$ for $i = 1, \ldots, k-1$ implies that the effect space $\mathcal{P}$ is $PG(s_{k-1} + t, 2) = \mathcal{P}'_k$. Let U_{k-1} be an (s_{k-1})-dimensional subspace of $\mathcal{P}'_k$. Then, from Lemma 4.1, there exists an (s_{k-1})-spread $\mathcal{S}'_{k-1}$ of $\mathcal{P}_{k-1}$. Moreover, it is possible to construct an $\mathcal{S}'_{k-1}$ that contains U_{k-1}, which will be shown in Section 4.2.2. So, $\mathcal{S}_{k-1} = \left\{S \cap \mathcal{P} : S \in \mathcal{S}'_{k-1} \backslash \{U_{k-1}\}\right\}$ is a set of disjoint $(t-1)$-dimensional subspaces in $\mathcal{P}$. Next, define $\mathcal{P}'_{k-1} = U_{k-1}$ and let U_{k-2} be an (s_{k-2})-dimensional subspace of $\mathcal{P}'_{k-1}$. Assuming the existence of a (s_{k-2})-spread $\mathcal{S}'_{k-2}$ of $\mathcal{P}_{k-2}$ that contains U_{k-2}, define $\mathcal{S}_{k-2} = \left\{S \cap \mathcal{P} : S \in \mathcal{S}'_{k-2} \backslash \{U_{k-2}\}\right\}$. Following the recursion steps in a similarly fashion, we obtain the sets $\mathcal{S}_1, \ldots \mathcal{S}_{k-1}$, such that the set of disjoint $(t-1)$-spaces of $\mathcal{P}$ is

$$\mathcal{S} = \left(\bigcup_{i=1}^{k-1} \mathcal{S}_i \right) \cup \mathcal{S}_0,$$

where $\mathcal{S}_0$ is a $(t-1)$-dimensional space of U_1. Therefore, the number of disjoint $(t-1)$-dimensional subspaces in $\mathcal{P}$ is given by $|\mathcal{S}| = 1 + \sum_{i=1}^{k-1} \left(\frac{2^{2(s_i+1)}-1}{2^{(s_i+1)}-1} - 1 \right) = 1 + \sum_{i=1}^{k-1} 2^{(s_i+1)} = 2^r \frac{2^{kt}-1}{2^t-1} - 2^r + 1$. □

Though the proof provided above is constructive for the most part, the steps where one has to obtain spreads that satisfy certain requirements, imposed by the experimenter, are nontrivial and thus elaborated on in Section 4.2.2. The following example summarizes the results on partial $(t-1)$-spreads.

Example 4.2. Consider a 2^8 full factorial design with m stages of randomization characterized by RDCSSs given by $S_1, ..., S_m$. For analyzing the data, the size of each RDCSS should be more than six or seven, i.e., $|S_i| \geq 7$ for $i = 1, .., m$. Since $t = 3, r = 2$, and we are interested in 2-dimensional subspaces of $\mathcal{P}$, the result presented in Lemma 4.2(c) indicates the size of the maximal partial 2-spread of $\mathcal{P}$ is bounded above by 34. Lemma 4.3 guarantees the existence of only 9 disjoint subspaces of size 7 each, whereas from Lemma 4.4, the existence of a partial 2-spread with 33 disjoint subspaces is ensured. Since the disjoint subspaces obtained in Lemma 4.3 are constructed from a 2-spread of $PG(5,2)$ which is a proper subset of $PG(7,2)$, and Lemma 4.4 finds a set of disjoint 2-dimensional subspaces in $PG(7,2)$, there is such a difference. This example illustrates that either the bound in Lemma 4.2(c) is not tight or there exist more disjoint 2-spaces of $\mathcal{P}$.

For $t = 2$ and p odd (e.g., $p = 2k+1$ for some positive integer k), Addleman (1962) proved that the bound $|\mathcal{S}| \leq (2^p - 5)/3$ is tight (same as $|\mathcal{S}|$ in Lemma 4.4). Thus, the bound provided in Lemma 4.2(c) is not tight at least for general t, k and r. A construction of $(2^p - 5)/3$ disjoint 1-dimensional subspaces of $\mathcal{P} = PG(2k, 2)$, proposed in Wu (1989), is based on the existence of two permutations of the effect space satisfying certain properties. These results were established in the context of constructing $2^m 4^n$ factorial designs (for non-negative integers m and n) using two-level factorial designs. The construction provided in Wu (1989) is only for $t = 2$ and $q = 2$, whereas, Lemma 4.4 holds for general t and is easily extendable for arbitrary prime, or prime power q in $PG(p-1, q)$.

The results discussed so far in this chapter focus on the existence of disjoint subspaces of the same size, however, it is likely to have requirements for disjoint subspaces of different sizes (e.g., the battery cell experiment in Vivacqua and Bisgaard, 2004; the plutonium example in Bingham et al., 2006). Before developing conditions for the existence of a set of disjoint subspaces of unequal sizes, we propose a useful intermediate result.

Theorem 4.2. *Let $\mathcal{P}$ be the projective space $PG(p-1,2)$ and S_i be a $(t_i - 1)$-dimensional subspace of $\mathcal{P}$, where $0 < t_i < p$ for $i = 1, 2$. Then, $|\langle S_1, S_2 \rangle| = 2^p - 1$,* *whenever* $|S_1 \cap S_2| = 2^{t_1+t_2-p} - 1$ *for* $t_1 + t_2 > p$.

Proof: Let $\bar{S}$ denote a set of factorial effects (or points) contained in the subspace S that generates S, i.e., $\langle \bar{S} \rangle = S$. Also, for any two non-disjoint subspaces S_i and S_j, let $\overline{(S_i \cap S_j)} \subset \bar{S}_i$ for $i \neq j$. Then,

$$A_1 = \langle \bar{S}_1 \setminus \overline{(S_1 \cap S_2)} \rangle, \; A_2 = \langle \overline{(S_1 \cap S_2)} \rangle \quad \text{and} \quad A_3 = \langle \bar{S}_2 \setminus \overline{(S_1 \cap S_2)} \rangle$$

are pairwise disjoint subspaces. If $|S_1 \cap S_2| = 2^{t_1+t_2-p} - 1$, then A_2 is equivalent to a $PG(t_1 + t_2 - p - 1, 2)$ contained in the effect space $\mathcal{P}$. Similarly, A_1 and A_2 are equivalent to $PG(p - t_2 - 1, 2)$ and $PG(p - t_1 - 1, 2)$ respectively. Since A_i's are pairwise disjoint subspaces, the span of S_1 and S_2 is $\langle S_1, S_2 \rangle = \langle A_1, A_2, A_3 \rangle = PG(p-1, 2)$. □

This theorem implies that if $t_1 + t_2 > p$ and $|S_1 \cap S_2| = 2^{t_1+t_2-p} - 1$, then $\langle S_1 \cup S_2 \rangle$ covers the entire effect space $\mathcal{P}$. Furthermore, it is clear from the proof that if $|S_1 \cap S_2| > 2^{t_1+t_2-p} - 1$, the size of $\langle S_1 \cup S_2 \rangle$ is less than $2^p - 1$ and thus $\langle S_1, S_2 \rangle$ is a proper subset of $\mathcal{P}$. Next, we develop conditions for the existence of a pair of unequal sized disjoint subspaces of the effect space $\mathcal{P}$.

Theorem 4.3. *Let* $\mathcal{P}$ *be the projective space* $PG(p - 1, 2)$ *and* S_i *be a* $(t_i - 1)$-*dimensional subspace of* $\mathcal{P}$, *where* $0 < t_i < p$ *for* $i = 1, 2$.

(a) If $t_1 + t_2 \leq p$, *there exists* S_1 *and* S_2 *such that* $S_1 \cap S_2 = \phi$.

(b) If $t_1 + t_2 > p$, *for every* S_1, S_2 *in* $\mathcal{P}$, $|S_1 \cap S_2| \geq 2^{t_1+t_2-p} - 1$ *and there exists* S_1, S_2 *such that the equality holds.*

Proof: Let the effect space be $\mathcal{P} = \langle F_1, \ldots, F_p \rangle$, where the F_i's are the independent factors of a 2^p full factorial design. Since $t_1 + t_2 \leq p$, part (a) holds by defining $S_1 = \langle F_1, \ldots, F_{t_1} \rangle$ and $S_2 = \langle F_{t_1+1}, \ldots, F_{t_1+t_2} \rangle$. For part (b), $S_1 = \langle F_1, \ldots, F_{t_1} \rangle$

and $S_2 = \langle F_{p-t_2+1}, \ldots, F_{t_1}, F_{t_1+1}, \ldots, F_p \rangle$ provides the minimum possible overlap of $S_1 \cap S_2 = \langle F_{p-t_2+1}, \ldots, F_{t_1} \rangle$ with $|S_1 \cap S_2| = |PG(t_1 + t_2 - p - 1, 2)| = 2^{t_1+t_2-p} - 1$. In addition, if S_1, S_2 are such that $t_1 + t_2 > p$ and $|S_1 \cap S_2| < 2^{t_1+t_2-p} - 1$, then according to Theorem 4.2, $|\langle S_1, S_2 \rangle| > 2^p - 1$. This contradicts the fact that if $S_1 \subset \mathcal{P}$ and $S_2 \subset \mathcal{P}$, then $\langle S_1, S_2 \rangle$ should also be contained in $\mathcal{P}$. □

This theorem is directly applicable for designs with two stages of randomization, for example, row-column designs, strip-plot designs, two-stage split-lot designs. For $t_1 = t_2 = t$, this theorem simplifies to Theorem 4.1. It is easy to verify that one can have at most one $(t-1)$-space with $t > p/2$. For instance, in a 2^5 factorial experiment (Example 4.1), there does not exist even two disjoint subspaces of size 7 each. Bingham et al. (2006) discovered this through an exhaustive computer search, whereas Theorem 4.1 identifies this directly. It turns out that when $t_1 + t_2 \leq p$, one can expect more disjoint subspaces of size $2^t - 1$ if $t < p - \max(t_1, t_2)$. The next theorem is the main new result of this section.

Theorem 4.4. *Let $\mathcal{P}$ be the projective space $PG(p-1, 2)$ and S_1 be a $(t_1 - 1)$-dimensional subspace of $\mathcal{P}$ with $p > t_1 > p/2$. Then, there exists $m - 1$ subspaces $S_2, \ldots, S_m$ such that $|S_i| = 2^{t_i} - 1$ for $t_i \leq p - t_1$, $2 \leq i \leq m$, and S_i, $i = 1, ..., m$ are all pairwise disjoint, where $m = 2^{t_1} + 1$.*

Proof: Define $s = t_1 - 1$ and $t = (p - t_1) - 1$. Then, the effect space $\mathcal{P}$ is a $PG(s + t + 1, 2)$ and S_1 is an s-dimensional subspace of $\mathcal{P}$. Since $s > t$, define $\mathcal{P}' = PG(2s + 1, 2)$ so that $\mathcal{P}' \supseteq \mathcal{P}$, and let $\mathcal{S}'$ be an s-spread of $\mathcal{P}'$ that contains S_1. The construction of such a spread is non-trivial, and is shown in Section 4.2.3. Then the set of disjoint t-dimensional subspaces of $\mathcal{P}$ is given by $\mathcal{S} = \{S \cap \mathcal{P} : S \in \mathcal{S}' \backslash \{S_1\}\}$, which further implies that the elements of $\mathcal{S}$ can be denoted by $S_2, S_3, \ldots, S_m$ for

$m = \frac{|PG(2s+1,2)|}{|PG(s,2)|}$. As required, the experimenter can obtain a $(t_i - 1)$-dimensional subspace of S_i if $t_i - 1 \leq t$ (or equivalently, $t_i \leq p - t_1$) for $i = 2, \ldots, m$. □

Theorem 4.4 proposes the existence of $2^{t_1}+1$ disjoint subspaces of $\mathcal{P}$ with one (t_1-1)-dimensional subspace ($t_1 > p/2$) and 2^{t_1} disjoint subspaces S_i's with $|S_i| = 2^{t_i} - 1$, where $t_i \leq p - t_1$. Thus, according to the requirements of the experiment, one can construct designs with the randomization restriction defined by up to $2^{t_1}+1$ RDCSSs of different sizes. Furthermore, as we shall see, the proof points to a construction strategy for $2^{t_1}+1$ disjoint subspaces of unequal sizes (see Section 4.2.3 for an elaborate construction). Though Lemma 4.4 is not a special case of Theorem 4.4, the two construction techniques are similar (see Sections 4.2.2 and 4.2.3).

Thus far, we have established necessary and sufficient conditions for the existence of a set of disjoint subspaces of the same and also different sizes. If the desired number of stages of randomization (m) is less than or equal to the number of subspaces guaranteed to exist from one of the results, one can obtain an appropriate subset of $\mathcal{S}$ that satisfies the restrictions imposed by the experimenter. Next, we propose a construction approach for factorial designs with m levels of randomization.

4.2 Construction of Disjoint Subspaces

First, the construction for equal sized subspaces is presented, followed by the construction of disjoint subspaces of different sizes. The subspaces themselves have no statistical meaning until the factors have been assigned to columns of the design matrix, or equivalently to points in $PG(p-1, 2)$. The set of disjoint subspaces obtained from an arbitrary assignment may not directly satisfy the experimenter's restrictions on RDCSSs. Consequently, we propose an algorithm that transforms a set of disjoint subspaces obtained from the construction to another set of disjoint subspaces that satisfies the properties of the desired experimental design.

4.2.1 RDCSSs and $(t-1)$-spreads

When t divides p, the existence of a $(t-1)$-spread of $\mathcal{P} = PG(p-1, 2)$ is guaranteed from Lemma 4.1. The construction of a spread starts with writing the $2^p - 1$ nonzero elements of $GF(2^p)$ in cycles of length N (Hirschfeld, 1998). For any prime or prime power q, an element w is called primitive if $\{w^i : i = 0, 1, ..., q-2\} = GF(q)\backslash\{0\}$. A primitive element of $GF(2^p)$ is a root of a primitive polynomial of degree p for over $GF(2)$ (for details see Artin, 1991). The $2^p - 1$ elements of the effect space $\mathcal{P}$, or equivalently, the nonzero elements of $GF(2^p)$, are w^i, $i = 0, ..., 2^p - 2$, where w^i can be written as a linear combination of the basis polynomials $w^0, ..., w^{p-1}$. The element $w^i = \alpha_0 w^{p-1} + \alpha_1 w^{p-2} + \cdots + \alpha_{p-1}$ represents an r-factor interaction $\delta = (\alpha_0, \alpha_1, ..., \alpha_{p-1})$, for $\alpha_i \in GF(2)$, if exactly r entries of δ are nonzero. For example, let $p = 4$ and the primitive polynomial be $w^4 + w + 1$. Then,

$$\begin{aligned}
w^0 = 1 &= (0001) = D, \\
w^1 = w &= (0010) = C, \\
w^2 = w^2 &= (0100) = B, \\
w^3 = w^3 &= (1000) = A, \\
w^4 = w + 1 &= (0011) = CD, \\
w^5 = w^2 + w &= (0110) = BC, \\
&\vdots \\
w^{14} = w^3 + 1 &= (1001) = AD.
\end{aligned}$$

Following this representation for the factorial effects in $\mathcal{P}$ and using shorthand notation k for w^k, the cycles of length N can be written as shown in Table 4.1. Here, θ is the number of distinct cycles and the entry $(iN + j)$ denotes w^{iN+j} for $0 \leq i \leq \theta - 1$, $0 \leq j \leq N - 1$.

Table 4.1: The elements of $\mathcal{P}$ using cyclic construction.

S_1	S_2	......	S_N
0	1	......	$N-1$
N	$N+1$	......	$2N-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\theta-1)N$	$(\theta-1)N+1$	......	$\theta N-1$

The following result due to Hirschfeld (1998, Ch.4) presents a necessary and sufficient condition for the existence of a set of $(t-1)$-dimensional subspaces of size N which depends on the greatest common divisor (gcd) of t and p.

Lemma 4.5. *There exists a $(t-1)$-space of cycle N less than $|PG(p-1,q)|$ if and only if $gcd(t,p) > 1$, where, $N = \frac{|PG(p-1,q)|}{|PG(l-1,q)|}$ and $l = gcd(t,p)$.*

Since t divides p, there exists $2^t - 1$ cycles of length N each. The S_i's are therefore pairwise disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$. That is, the subspaces S_1,...,S_N, constitute a $(t-1)$-spread $\mathcal{S}$ of the effect space $\mathcal{P} = PG(p-1,2)$. Given the spread, an experimenter must now assign factors to the points in $PG(p-1,2)$ to achieve the desired design.

A $(t-1)$-spread of $PG(p-1,2)$, obtained above, distributes all the main effects (or factors) evenly among all the $|\mathcal{S}|$ disjoint subspaces. However, restrictions on the m stages of randomization are usually pre-specified by the experimenter. Indeed, for a block design, an RDCSS will contain no main effects, whereas for a split-lot design, one or more factors may be assigned to the subspace representing an RDCSS. For example, consider a 2^6 full factorial experiment with the randomization structure determined by a *blocked split-lot design*, where the trials have to be run in blocks of size 8 each. Further suppose that the experimenter wishes to specify the factorial effects ABC, BDE and CEF to be confounded with the blocks. In addition, let the experimental units have to be processed into two steps, where the restrictions imposed by the experimenter on the two steps of randomization are such that $S_1^* \supset \{A, B\}$ and $S_2^* \supset \{D\}$. As a result, there are three restrictions on the randomization of the experiment, one due to blocking the experimental units and the other two due to splitting the experimental units into sub-lots. To use half-normal plots for the assessment of the factorial effects on the process, it is desirable to have three disjoint subspaces of size more than six or seven each, where the subspaces should satisfy the restrictions on the three RDCSSs given by S_1^*, S_2^* and $S_3^* = \langle ABC, BDE, CEF \rangle$. It turns out that one can relabel the points of $\mathcal{P}$ such that the spread $\mathcal{S}^*$ obtained from the transformed space contains three disjoint subspaces satisfying the experimenter's requirement on the RDCSSs.

Although it is tempting to use an exhaustive search to find an appropriate relabelling of $\mathcal{P}$ that meets the experimenter's requirement, if the number of independent factors is large, computation time and resources can be expensive. A simpler approach which works in many cases uses the structure of a $(t-1)$-spread to our advantage and reduces the search space. Instead of randomly relabelling the points of $\mathcal{P}$ (or equivalently the columns of the model matrix X), if we find a relabelling that preserves the geometric structure among the points, the search space is significantly reduced. For this purpose, a *collineation* (e.g., Coxeter, 1974; Batten, 1997) of the projective space $\mathcal{P}$ is used to relabel its points. A collineation of $PG(p-1, q)$ is a permutation f of its points such that $(t-1)$-dimensional subspaces are mapped to $(t-1)$-dimensional subspaces, for $1 \leq t \leq p$,

$$f : PG(p-1, q) \longrightarrow PG(p-1, q).$$

For example, in a 2^3 full factorial design, the set of seven factorial effects forms a $PG(2, 2)$, where the points $\{C_1, ..., C_7\}$ can be denoted by $\{A, B, AB, C, ..., ABC\}$. A feasible configuration for the set of lines of $PG(2, 2)$ is $\{(A, B, AB), (B, C, BC), (A, C, AC), (A, BC, ABC), (B, AC, ABC), (C, AB, ABC), (AB, BC, AC)\}$. Figure 4.1 displays a collineation of the projective space $PG(2, 2)$.

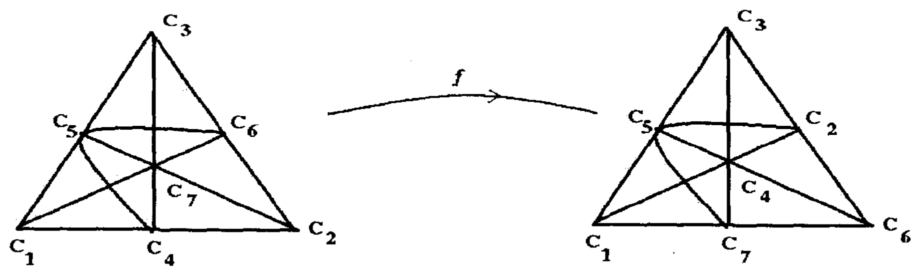

Figure 4.1: A collineation of $PG(2,2)$.

The existence of a collineation f that transforms a spread $\mathcal{S}$ to $\mathcal{S}^*$ (or equivalently, the effect space $\mathcal{P}$ to $\mathcal{P}^*$), can be established by the existence of a $p \times p$ matrix $\mathcal{M}$, such that for every given $z \in \mathcal{S}$, there is an unique $z' \in \mathcal{S}^*$ that satisfies $z\mathcal{M} = z'$. The collineation matrix for the transformation in this example is

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the transformation of a spread amounts to relabelling the columns of the model matrix, there may not exist an appropriate collineation under several circumstances. For instance, one cannot find a collineation matrix $\mathcal{M}$ if the experimenter's requirement is not achievable. For example, in a 2^5 full factorial split-lot design with 3 levels of randomization, if the restrictions imposed on the three RDCSSs are $S_1 \supset \{A, B\}$, $S_2 \supset \{C, D\}$ and $S_3 \supset \{E, AD\}$, then there does not exist a collineation that meets the requirements. Moreover, if the desired set of subspaces is non-isomorphic to the spread we started with, then also there does not exist any relabelling of $\mathcal{P}$ to obtain the desired design. However, finding an appropriate collineation matrix whenever it exists is also nontrivial. Next, we propose an algorithm that finds a collineation matrix $\mathcal{M}$, if it exists, and concludes the nonexistence if one does not exist. The algorithm is illustrated through an example.

Consider the earlier setup of a 2^6 full factorial experiment with the blocked split-lot design, where the RDCSSs are characterized by $S_1^* \supset \{A, B\}$, $S_2^* \supset \{D\}$ and $S_3^* \supset \{ABC, BDE, CEF\}$. For constructing useful half-normal plots, RDCSSs should satisfy $|S_i^*| \geq 7$, for $i = 1, ..., 3$ and hence $t = 3$. Since t divides p, there exist 7 cycles of length 9 each, or equivalently, 9 disjoint subspaces of size 7 each (i.e., a 2-spread of $\mathcal{P}$). The 2-spread $\mathcal{S} = \{S_1, ..., S_9\}$ obtained using the primitive polynomial, $w^6 + w + 1$ is shown in Table 4.2.

Table 4.2: The 2-spread obtained using the cyclic construction.

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
F	E	D	C	B	A	EF	DE	CD
BC	AB	AEF	DF	CE	BD	AC	BEF	ADE
CDEF	BCDE	ABCD	ABCEF	ABDF	ACF	BF	AE	DEF
CDE	BCD	ABC	ABEF	ADF	CF	BE	AD	CEF
BDE	ACD	BCEF	ABDE	ACDEF	BCDF	ABCE	ABDEF	ACDF
BCF	ABE	ADEF	CDF	BCE	ABD	ACEF	BDF	ACE
BDEF	ACDE	BCDEF	ABCDE	ABCDEF	ABCDF	ABCF	ABF	AF

Note that each element of $\mathcal{S}$ contains at most one main effect. To obtain a set of disjoint subspaces satisfying the restrictions imposed on the 3 stages of randomization, one has to find an appropriate 6×6 collineation matrix $\mathcal{M}$. An algorithm for finding the matrix $\mathcal{M}$ is outlined as follows:

1. Select one of the $\binom{9}{3}$ possible choices for a set of three disjoint subspaces from the spread $\mathcal{S}$. For example, S_1, S_3 and S_7 are chosen such that, $S_1 \longrightarrow S_1^*$, $S_3 \longrightarrow S_2^*$ and $S_7 \longrightarrow S_3^*$.

2. Choose two effects from S_1, one effect from S_3 and three effects from S_7 to relabel these to the desired effects $(A, B), D$ and (ABC, BDE, CEF) in S_1^*, S_2^* and S_3^* respectively. For example, one choice among $\binom{7}{2}\binom{7}{1}\binom{7}{3}$ different options is $\{CDE, BCF, D, EF, AC, BF\}$. The collineation matrix is defined by the mapping induced from $CDE \rightarrow A, BCF \rightarrow B$, $D \rightarrow D$, ..., $BF \rightarrow CEF$.

3. Construct a $p^2 \times p^2$ matrix $\mathcal{A}$ and a $p^2 \times 1$ vector δ as follows. Denote the (i, j)-th entry of the $p \times p$ matrix $\mathcal{M}$ as x_k, where $k = j + (i-1)p$. Then, define the rows of matrix $\mathcal{A}$ and vector δ in the order of restrictions on the transformation. For the example under consideration, the first (in general, s-th) restriction $(CDE)\mathcal{M} = A$ can be written as:

$$(CDE)\mathcal{M} = (100000)'. \tag{4.1}$$

Then, the first (s-th) set of six (in general p) rows of δ are given by the right side of equation (4.1). The corresponding rows of $\mathcal{A}$ can be written by first denoting $CDE = (001110)'$ and defining

$$\begin{aligned} \mathcal{A}_{il} &= 1, \quad \text{if } l = i + (\tau - 1)p \text{ and the } \tau\text{-th entry of } (001110) \text{ is nonzero,} \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

for $p(s-1)+1 \leq i \leq ps$, $1 \leq s \leq p$. Similarly, all the rows of the matrix $\mathcal{A}$ and vector δ can be expressed using the p restrictions on the transformation.

4. If there exists a solution of $\mathcal{A}x = \delta$, reconstruct the matrix $\mathcal{M}$ from the solution $x = \mathcal{A}^{-L}\delta$ and exit the algorithm, where $\mathcal{A}^{-L}$ is a left inverse of $\mathcal{A}$.

5. If there does not exist a solution of $\mathcal{A}x = \delta$, go to Step 2 and if possible, choose a different set of effects from the subspaces selected in Step 1.

6. If all possible choices for the set of effects from these three subspaces have been exhausted, then go to step 1 and choose a different set of three subspaces.

7. If all the $\binom{9}{3}$ different choices for a set of subspaces have been used and still a solution does not exist, then either the two spreads $\mathcal{S}$ and $\mathcal{S}^*$ are non-isomorphic, or the experimenter's requirement is not achievable. Thus, the desired spread cannot be obtained from $\mathcal{S}$.

In the illustration used here, the factorial effects chosen for relabelling the columns to achieve the desired design provide a feasible solution to $\mathcal{A}x = \delta$. The collineation matrix $\mathcal{M}$, reconstructed from the solution $x = \mathcal{A}^{-L}\delta$, is given by

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

For the example under consideration, an exhaustive search found that 45.7% of all possible choices give a feasible solution to the equation $\mathcal{A}x = \delta$. That is, an arbitrary choice of p independent effects from $\mathcal{S}$ (according to Steps 1 and 2) results in a feasible design only 45.7% of the time. The rest of the time, an arbitrarily chosen set of effects lead to an infeasible solution by turning a full factorial design into a replicated fractional factorial design. Note that the search space can be further reduced by improving Step 2 to choose independent effects compared to an arbitrary set of effects from the subspace S_i.

Though necessary to search for a feasible choice of collineation matrix, the spread acts as a template for the search to make it faster than the exhaustive relabelling of all the factorial effects to find the design satisfying the experimenter's requirement. For this example, our algorithm may require at most $\binom{9}{3}\binom{7}{2}\binom{7}{1}\binom{7}{3}$ different relabellings, whereas an exhaustive relabelling approach can require up to $(2^6 - 1)!$ different relabellings. To find the proportion of feasible relabellings out of $\binom{9}{3}\binom{7}{2}\binom{7}{1}\binom{7}{3}$ different choices, our Matlab 7.0.4 implementation of the algorithm took almost 67 hours on a Pentium(R) 4 processor machine running Windows XP. The algorithm finds the first feasible collineation matrix in 5.34 seconds on the same machine. It is worth noting that the computation involved in the algorithm uses modular arithmetic.

In many cases, whenever t does not divide p, there does not exist a $(t-1)$-spread of $\mathcal{P} = PG(p-1,2)$. However, a partial $(t-1)$-spread $\mathcal{S}$ of $\mathcal{P}$ may be available. Recall that if the number of stages of randomization (m) is less than $|\mathcal{S}|$, then a set of m disjoint subspaces can be constructed that satisfies the randomization restrictions. Next we propose a construction for RDCSSs if $m < |\mathcal{S}|$, and there does not exist a $(t-1)$-spread of $\mathcal{P}$.

4.2.2 Partial $(t-1)$-spreads

When t does not divide p, Lemma 4.4 guarantees the existence of $|\mathcal{S}| = 2^r\frac{2^{kt}-1}{2^t-1} - 2^r + 1$ disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$, where $p = kt + r$. For constructing these subspaces, one can use the steps outlined in the proof of Lemma 4.4 for the most part. However, the proof assumes the existence of an (s_i)-spread $\mathcal{S}_i'$ of $\mathcal{P}_i$ that contains U_i, where U_i is an (s_i)-dimensional subspace of $\mathcal{P}_{i+1}'$, for $s_i = it + r - 1$, $\mathcal{P}_i = PG(2s_i + 1, 2)$, and $\mathcal{P}_{i+1}' = PG(s_i + t, 2)$, $i = 1, \ldots, k-1$. The construction of the spread $\mathcal{S}_i'$ is nontrivial, and we develop a two step construction method: (a) construct a (s_i)-spread $\mathcal{S}_i''$ of $\mathcal{P}_i$ as described in Section 4.2.1, and then (b) transform the spread $\mathcal{S}_i''$ to $\mathcal{S}_i'$ by finding an appropriate collineation such that $U_i \in \mathcal{S}_i'$. Thus, we can construct a set of $|\mathcal{S}|$ disjoint $(t-1)$-dimensional subspaces, or, a partial $(t-1)$-spread $\mathcal{S}$ of $\mathcal{P}$, using the recursive construction method described in the proof of Lemma 4.4. Finally, this partial spread $\mathcal{S}$ has to be transformed using an appropriate collineation to obtain the m RDCSSs satisfying the experimenter's requirement.

4.2.3 Disjoint subspaces of different sizes

A more general setting is when the RDCSSs are allowed to have different sizes. For a 2^p full factorial design, Theorem 4.4 guarantees the existence of only one subspace S_1 of size $2^{t_1} - 1$ with t_1 greater than $p/2$, and 2^{t_1} subspaces of size bounded above by $2^t - 1$ where $t \le p - t_1$. For constructing these $2^{t_1} + 1$ pairwise disjoint subspaces of $\mathcal{P}$, the proof of Theorem 4.4 requires constructing a $(t_1 - 1)$-spread $\mathcal{S}'$ of $PG(2t_1 - 1, 2)$

that contains S_1. The spread $\mathcal{S}'$ can be obtained by first constructing a (t_1-1)-spread of $PG(2t_1-1,2)$ and then by applying the appropriate collineation $\mathcal{M}_0$ found by the algorithm described in Section 4.2.1. After $\mathcal{S} = \{S \cap \mathcal{P} : S \in \mathcal{S}' \backslash \{S_1\}\}$ is obtained, one has to find a suitable collineation $\mathcal{M}_1$ so that the final set of subspaces satisfy the experimenter's restrictions on RDCSSs. The steps of the construction are illustrated through an example.

Consider a 2^7 full factorial design with 3 stages of randomization. Let the restrictions imposed on the three RDCSSs be $S_1 \supset \{A,B,C,D\}$, $S_2 \supset \{E,F\}$ and $S_3 \supset \{G\}$. Following the notation of Theorem 4.4, since $p = 7$ and $t_1 = 4$ there exists 17 pairwise disjoint subspaces with $|S_i| = 2^{t_i} - 1$ for $i = 1, ..., 17$, where $t_1 = 4$ and $t_i \le 3$ for $i = 2, ..., 17$. Then, a 3-spread $\mathcal{S}''$ of $PG(7,2)$ is constructed using the method described in Section 4.2.1, and an appropriate collineation matrix $\mathcal{M}_0$ is found which transforms $\mathcal{S}''$ to $\mathcal{S}'$ such that $\mathcal{S}'$ contains $S_1 = \langle A,B,C,D \rangle$. Table 4.3 contains some of the elements of $\mathcal{S}'$.

Table 4.3: The 3-spread $\mathcal{S}'$ obtained after applying $\mathcal{M}_0$ on $\mathcal{S}''$.

S_1	S_2	$\cdots$	S_{16}	S_{17}
A	BFGH	$\cdots$	AH	BCFGH
B	DH	$\cdots$	ACDEF	H
C	CDEF	$\cdots$	ABDFH	ABCDEF
D	ADFH	$\cdots$	BCDFG	ABCDFH
AB	BDFG	$\cdots$	CDEFH	BCFG
BC	CEFH	$\cdots$	BCEH	ABCDEFH
CD	ACEH	$\cdots$	ACGH	EH
ABD	ABGH	$\cdots$	BEGH	ADGH
AC	BCDEGH	$\cdots$	BDF	ADEGH
BD	AF	$\cdots$	ABEG	ABCDF
ABC	BCEG	$\cdots$	ABCE	ADEG
BCD	ACDE	$\cdots$	DEFGH	E
ABCD	ABCDEFGH	$\cdots$	ADEFG	BCEFGH
ACD	ABCEFG	$\cdots$	CG	BCEFG
AD	ABDG	$\cdots$	ABCDFGH	ADG

Given the spread $\mathcal{S}'$, we first obtain $\mathcal{S} = \{S \cap \mathcal{P} : S \in \mathcal{S}' \backslash \{S_1\}\}$, and then the collineation matrix $\mathcal{M}_1$ is obtained to accommodate other restrictions on $S_2, ..., S_m$.

The two collineation matrices used for the transformations are as follows:

$$\mathcal{M}_0 = \begin{pmatrix} 1&1&1&1&1&1&1&0\\ 0&1&0&0&1&1&0&0\\ 1&0&1&1&1&0&1&0\\ 1&1&1&0&1&0&0&1\\ 0&1&0&1&0&1&1&1\\ 0&1&1&1&1&1&0&0\\ 0&1&0&0&0&1&1&1\\ 1&0&0&0&0&0&0&0 \end{pmatrix}, \quad \mathcal{M}_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&1&0&0&1\\ 0&0&1&0&1&1&1&1\\ 1&1&1&1&1&0&1&0\\ 1&0&0&1&1&1&0&1 \end{pmatrix}.$$

As a result, the three disjoint subspaces that satisfy the experimenter's requirements are $S_1 = \langle A, B, C, D\rangle$, $S_2 = \langle E, F, CG\rangle$ and $S_3 = \langle G, BCF, ABCDEF\rangle$. Since the construction algorithm does not involve any recursion, it can be made more efficient by combining the problem of finding the two collineation matrices into one problem. When transforming the 3-spread $\mathcal{S}''$ to $\mathcal{S}'$ containing S_1, we can impose other restrictions ($S_2 \supset \{E, F\}$ and $S_3 \supset \{G\}$) in this step itself. Thus, $\{S \cap \mathcal{P} : S \in \mathcal{S}'\backslash\{S_1\}\} \cup S_1$ contains the required set of subspaces $S_1, ..., S_3$, for the 3 stages of randomization. The grouping of effects based on its null distribution is shown in Table 4.4.

Table 4.4: The ANOVA table for the 2^7 full factorial design.

Effects	Variance	Degrees of Freedom
S_1	$\frac{2^3}{2^7}\sigma_1^2 + \frac{1}{2^7}\sigma^2$	15
S_2	$\frac{2^4}{2^7}\sigma_2^2 + \frac{1}{2^7}\sigma^2$	7
S_3	$\frac{2^4}{2^7}\sigma_3^2 + \frac{1}{2^7}\sigma^2$	7
$\mathcal{P}\backslash(S_1 \cup S_2 \cup S_3)$	$\frac{1}{2^7}\sigma^2$	98

The assessment of all the 127 effects can be done by using 4 half-normal plots.

The designs discussed so far in this chapter focus on full factorial experiments. Nevertheless, fractional factorial designs are often desirable for experiments involving a large number of factors, and are therefore of interest. It turns out that the results developed here for the existence and construction can easily be adapted for regular fractional factorial designs with different randomization restrictions. In addition, the RDCSS structure can be used to unify the fractionation of two-level regular factorial designs with different randomization restrictions. We present a brief discussion on such designs in the following section.

4.3 Fractional factorial designs

In this section, we first establish the existence of two-level regular fractional factorial designs by constructing these designs using the existence results and construction techniques developed so far in this chapter. Then, we focus on different ways of fractionating a 2^p full factorial design.

If the number of factors in a two-level factorial experiment is p and the resources are enough for only a 2^{-k} fraction of the complete set of 2^p treatment combinations, a 2^{p-k} regular fractional factorial design can be constructed. A 2^{p-k} regular fractional factorial design is constructed by assigning the k additional factors (*added factors*) to the columns of the model matrix corresponding to (preferably) the higher order interactions of the two-level full factorial design generated with $p - k$ *basic factors.*

Recall from Chapter 2 that a full factorial design with randomization restrictions can be characterized by its RDCSS structure. It turns out that one can use the set of disjoint subspaces in the effect space of the *base factorial design* to construct a regular fractional factorial design. In some cases, the fractional generators have to be chosen from the RDCSSs of the base factorial design, whereas there are cases when a distinct disjoint subspace is preferred to choose fractional generators from. Thus, the results developed so far for a maximal set of disjoint subspaces of both equal and unequal sizes can be used to construct regular fractional factorial designs with randomization restrictions. The following examples illustrate the construction in both situations.

Example 4.3. Consider a 2^{8-2} fractional factorial experiment with randomization structure characterized by a split-lot design. Further suppose that the experimental units have to be processed in 4 stages with randomization restrictions defined by $S_1 \supset \{A, B\}$, $S_2 \supset \{C, D\}$, $S_3 \supset \{E, F\}$ and $S_4 \supset \{G, H\}$. Then, the 6 (or, in general, $p - k$) independent basic factors and their interactions, $\mathcal{P} = \langle A, B, ..., F \rangle$, form a 2^6 full factorial split-lot design. Lemma 4.1 guarantees the existence of a 2-spread of $\mathcal{P}$, and the construction method outlined in Section 4.2.1 can be used to construct 3 RDCSSs that satisfies the restrictions defined by S_1, S_2 and S_3. Table 4.5 shows the transformed spread $\mathcal{S} = \{S_1^*, ..., S_9^*\}$, where $S_1 = S_1^*$, $S_2 = S_3^*$ and $S_3 = S_7^*$.

Table 4.5: The 2-spread of $PG(5, 2)$ after transformation.

S_1^*	S_2^*	S_3^*	S_4^*	S_5^*	S_6^*	S_7^*	S_8^*	S_9^*
DF	BCD	ABCEF	BE	DEF	BF	BCF	ADEF	ACF
BDF	BDE	C	ABCDE	CDE	ABCD	EF	BCDE	ABDE
AB	ABE	ABCDEF	BCDF	AC	DE	E	CDF	AE
ABDF	ACDE	D	CDEF	ACDEF	BDEF	BCEF	ACE	CEF
A	ABC	CD	ABF	AF	ACEF	BC	ABD	ABCDF
B	CE	ABEF	ACD	CF	ACDF	BCE	ABCF	BCDEF
ADF	AD	ABDEF	AEF	ADE	ABCE	F	BEF	BD

The collineation matrix used to transform the 2-spread (shown in Table 4.2) obtained from the cyclic construction to $\mathcal{S} = \{S_1^*, ..., S_9^*\}$ is given by

$$\mathcal{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Since a 2-spread of $\mathcal{P}$ consists of nine disjoint subspaces of size 7 each, S_4 can be constructed using a subspace from the remaining six disjoint subspaces, $\mathcal{S} \backslash \{S_1, S_2, S_3\}$, and then by assigning two interactions to the two added factors G and H. For example, if we choose $S_4 = S_8^*$ and $G = CDF$, $H = BEF$, then the fraction defining contrast subgroup (FDCS) is

$$I = CDFG = BEFH = BCDEGH,$$

where the resulting design is of resolution IV. Of course, there are several options for the two generators which further leads to different designs. These designs can be ranked using different criteria, such as minimum aberration (Fries and Hunter, 1980), maximum number of clear effects (Chen, Sun and Wu, 1993; Wu and Chen, 1992) and V-criterion (Bingham et al., 2006). The technique used here for constructing a fractional factorial design is simply an approach to label the higher order effects to the added factors. To get all designs, or designs that are optimal according to some criterion, one can avoid all possible relabellings by using the spread structure which serves as a template to reduce the search space.

The above example presents a scenario where the availability of more than 3 disjoint subspaces in $\mathcal{P}$ has been used to construct a regular fractional factorial design. In this setup with 6 basic factors, one can have up to nine stages of randomization and disjoint RDCSSs with S_i's large enough to perform useful half-normal plots. However, if more than nine stages of randomization are required, overlapping among the RDCSSs cannot be avoided. The next example presents a scenario where the added factors have to be assigned to higher order factorial effects in the RDCSSs of the base factorial design.

Example 4.4. Consider a 2^{8-2} regular fractional factorial design with the requirement of 3 stages of randomization, where the imposed restrictions on the RDCSSs are defined by $S_1 \supset \{A, B\}$, $S_2 \supset \{C, D, E\}$ and $S_3 \supset \{F, G, H\}$. In this case also, one can start with the algorithm in Section 4.2.1 to construct a 2-spread of the effect space for the base factorial design such that the spread consists of three disjoint subspaces satisfying $S_1 \supset \{A, B\}$, $S_2 \supset \{C, D, E\}$ and $S_3 \supset \{F\}$. After transforming the 2-spread (shown in Table 4.2) obtained from the cyclic construction, the resulting spread

$\mathcal{S} = \{S_1, ..., S_9\}$ that satisfies the experimenter's requirement for the base factorial design is shown in Table 4.6.

Table 4.6: The 2-spread of $PG(5, 2)$ after applying the collineation matrix $\mathcal{M}$.

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
A	C	BE	DF	BDF	BF	AC	BCE	BDEF
B	D	ABCF	ABE	CDF	DEF	BD	ABCDF	CEF
ABCDEF	CE	BEF	ACF	ABDE	ABD	ABDF	BCF	ABCE
BCDEF	E	F	ACD	AEF	ADF	BCDF	EF	ACDF
CDEF	DE	ABC	BCDE	ACDE	AE	CF	ABCDE	ADE
AB	CD	ACEF	ABDEF	BC	BDE	ABCD	ADEF	BCD
ACDEF	CDE	ACE	BCEF	ABCEF	ABEF	AF	AD	ABF

The collineation matrix $\mathcal{M}$ used for the transformation is given by

$$\mathcal{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Next, one can fractionate the subspace S_3 by choosing two generators (or points) from this subspace. For example, the two added factors G and H can be assigned to the columns corresponding to interactions BEF and ACE respectively. As a result, the fraction defining contrast subgroup is

$$I = BEFG = ACEH = ABCFGH.$$

The fractional factorial design obtained is of resolution IV, and the word length pattern for this design is $(0, 2, 0, 1)$. Similar to Example 4.3, designs obtained as a result of different choices of feasible collineation matrices and (G, H) from the corresponding S_3's can be ranked using a criteria that suits the experimenter.

The two cases, (i) when a new RDCSS has to be constructed to assign the added factors (Example 4.3), and (ii) when the added factors are chosen from the RDCSSs of the base factorial design (Example 4.4), do not cover all possible types of fractional factorial design. In fact, one of the most common design, a fractional factorial split-plot (FFSP) design is different than the previous two types of fractionation. In this case, the added factors are assigned to the interactions of basic factors contained in S_i's and $\mathcal{P}\backslash(\cup_{i=1}^{m} S_i)$, where S_i's are the RDCSSs of the base factorial design. For example, in a $2^{(4+4)-(1+1)}$ FFSP design, the base factorial design is a 2^{3+3} full factorial split-plot design. To construct the $2^{(4+4)-(1+1)}$ FFSP design, one needs to choose one generator each from $S_1 = \langle A, B, C\rangle$ and $\mathcal{P}\backslash S_1$, where $\mathcal{P} = \langle A, B, C, D, E, F\rangle$. If the two added factors G and H are assigned to the columns of the model matrix corresponding to ABC and $CDEF$ respectively, then the fraction defining contrast subgroup is

$$I = ABCG = CDEFH = ABDEFGH.$$

The resulting $2^{(4+4)-(1+1)}$ FFSP design is of resolution IV, and the corresponding word length pattern is $(0, 1, 1, 0, 1)$.

The ranking of fractional factorial designs using different criteria is often computationally expensive. Several efficient algorithms have been proposed in the past to obtain fractional factorial designs with randomization restrictions that are optimal in some sense (e.g., Bingham and Sitter, 1999; Butler, 2004). The RDCSS structure can be used to shorten the computer search for finding such optimal designs. The complexity of the algorithm can be further reduced by using the collineation matrices for relabelling the effect space.

4.4 Further applications

In this section, we provide a few illustrative industrial examples. The examples presented in this section bring out some of the main features of the theory developed here that can be used in practical settings.

Example 4.5. Consider the battery cell experiment in Vivacqua and Bisgaard (2004). A company manufacturing electric batteries had problems in keeping the open circuit voltage (OCV) within specification limit. In this experiment, the authors sorted 6 two-level factors that potentially could have impact on OCV. It turns out that the batteries are manufactured in a two-stage process: (a) assembly process, and (b) curing process. Vivacqua and Bisgaard (2004) performed a 2^6 full factorial experiment with 4 factors (A, B, C, D) at the assembly process stage and 2 factors (E, F) at the curing process stage. After investigating some options, they chose a strip-block arrangement to optimize the resources.

Note that the effect space for this factorial layout is $\mathcal{P} = \langle A, ..., F\rangle$, and the two stages of randomization are characterized by subspaces $S_1 = \langle A, ..., D\rangle$ and $S_2 = \langle E, F\rangle$. Vivacqua and Bisgaard (2004) chose a design where they could not assess the significance of the effects in S_2, because S_2 was not large enough to construct useful half-normal plot (see Table 4.7).

Table 4.7: The ANOVA table for the battery cell experiment.

Effects	Variance	Degrees of Freedom
S_1	$\frac{2^2}{2^6}\sigma_1^2 + \frac{1}{2^6}\sigma^2$	15
S_2	$\frac{2^4}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	3
$\mathcal{P}\backslash(S_1 \cup S_2)$	$\frac{1}{2^6}\sigma^2$	45

In cases like this, one can use the strategies developed here to construct designs that will allow assessment of more factorial effects. As discussed earlier in this thesis, to construct useful half-normal plots, the set of effects with equal variance should contain more than six or seven effects. This can be done by introducing an extra blocking factor δ at the second stage of the process, i.e., $S_2 = \langle E, F, \delta\rangle$. However, from Theorem 4.3(a), there does not exist two disjoint subspaces S_1 and S_2 of size $2^4 - 1$ and $2^3 - 1$ respectively. In addition, Theorem 4.3(b) indicates that the overlap between S_1 and S_2 is at least $2^{4+3-6} - 1$. Keeping this is mind, one chooses δ to be a higher order interaction in S_1, for example $\delta = ABCD$. The corresponding analysis of variance table would be as shown in Table 4.8.

Table 4.8: The grouping of factorial effects for the battery cell experiment.

Effects	Variance	Degrees of Freedom
$S_1 \cap S_2$	$\frac{2^2}{2^6}\sigma_1^2 + \frac{2^3}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	1
$S_1 \backslash (S_1 \cap S_2)$	$\frac{2^2}{2^6}\sigma_1^2 + \frac{1}{2^6}\sigma^2$	14
$S_2 \backslash (S_1 \cap S_2)$	$\frac{2^3}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	6
$\mathcal{P} \backslash (S_1 \cup S_2)$	$\frac{1}{2^6}\sigma^2$	42

One can use 3 separate half-normal plots to assess the significance of all the factorial effects, but information about the 4-factor interaction $ABCD$ is sacrificed.

Example 4.6. Consider the setup of the chemical experiment in Schoen (1999). The goal of this experiment was to identify significant factors from a list of potential candidates that were suspected to impact the yield of a catalyst synthesized on gauze. This experimental procedure involved 5 stages: (i) Gauze preparation (H, J), (ii) Mixing components $(D, E, G, P, K, L, M, N, Q)$, (iii) Treatment of mixture (A, B), (iv) Synthesis (C) and (v) End of synthesis (O, F), where the letters in the bracket represent the factors associated with each stage of the experiment. There were a total of 16 two-level factors to be screened, and it was decided to run 32 trials. They performed a fractional factorial block design using 8 blocks of size 4 each, the data collected was analyzed using two half-normal plots. The distribution of effects according to their variance is shown in Table 4.9.

Table 4.9: The ANOVA table for the chemical experiment.

Effects	Variance	Degrees of Freedom
Between block effects	$\frac{2^3}{2^6}\sigma_1^2 + \frac{1}{2^6}\sigma^2$	7
Other effects	$\frac{1}{2^6}\sigma^2$	24

This experimental setting and its nature is an ideal scenario for a fractional factorial split-lot design with 5 stages of randomization. The 5 stages of randomization can be represented by subspaces $S_1', ..., S_5'$ contained in the effect space $\mathcal{P}$ of the corresponding base factorial design. The 5 stages of the process imposes restrictions on the randomization of the trials: $S_1' \supset \{H, J\}$, $S_2' \supset \{D, E, G, P, K, L, M, N, Q\}$, $S_3' \supset \{A, B\}$, $S_4' \supset \{C\}$ and $S_5' \supset \{O, F\}$. In order to construct useful half-normal plots, the subspaces should contain more than six or seven effects, i.e., $|S_i'| \geq 2^3 - 1$. Since S_2' should consists of at least 9 effects, one must construct S_2' with $|S_2'| \geq 2^4 - 1$. However, we know from Theorem 4.3(a) that there does not exist two disjoint subspaces of size 7 each in $\mathcal{P} = PG(4, 2)$. Thus, there does not exist an appropriate design that can be used to analyze this experiment in 32 runs.

If a 64-run design is performed instead, one can construct a design that satisfies the requirements. Let, $a, ..., f$ be the 6 independent basic factors, and $\mathcal{P} = \langle a, ..., f\rangle$ be the effect space for the corresponding base factorial design. For two subspaces S_1^*, S_2^* in $\mathcal{P}$ with $|S_1^*| = 2^5 - 1$ and $|S_2^*| = 2^4 - 1$, Theorem 4.3(b) implies that $|S_1^* \cap S_2^*| \geq 2^3 - 1$. The most obvious choice for S_1^* and S_2^* are $S_1^* = \langle a, b, c, d, e\rangle$ and $S_2^* = \langle a, b, c, f\rangle$. Now, define $S_2 = S_2^*$ and construct $S_i, i = 1, 3, 4, 5$ from S_1^* such that the overlaps $S_i \cap S_j$ for $i \neq j$ are avoided. For instance, $S_1 = \langle b, d, ce\rangle$, $S_3 = \langle b, e, acd\rangle$, $S_4 = \langle b, cd, ade\rangle$ and $S_5 = \langle b, de, ae\rangle$ provide the minimum pairwise overlap of only one effect. These subspaces can now be mapped into subspaces containing the original factors by relabelling: $a \rightarrow D, b \rightarrow CDEGH, c \rightarrow HC, d \rightarrow H, e \rightarrow A$ and $f \rightarrow E$. Of course, one could use collineation matrix approach to find an appropriate relabelling such that the RDCSSs meet the experimenter's requirements. By defining $S_i' = S_i$ for all i, the subspaces $S_1' = \langle CDEGH, H, ACH\rangle$, $S_2' = \langle CDEGH, D, E, HC\rangle$, $S_3' = \langle CDEGH, A, CD\rangle$, $S_4' = \langle CDEGH, C, ADH\rangle$ and $S_5' = \langle CDEGH, AH, AD\rangle$ satisfy the size requirements, which allow the assessment of significance for all the factorial effects except $CDEGH$. The analysis of variance table is shown in Table 4.10.

Table 4.10: The grouping of effects for the chemical experiment.

Effects	Variance	Degrees of Freedom
$\{CDEGH\}$	$\frac{2^2}{2^6}\sigma_2^2 + \frac{2^3}{2^6}(\sigma_1^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2) + \frac{1}{2^6}\sigma^2$	1
$S_1'\backslash\{CDEGH\}$	$\frac{2^3}{2^6}\sigma_1^2 + \frac{1}{2^6}\sigma^2$	6
$S_2'\backslash\{CDEGH\}$	$\frac{2^2}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	14
$S_3'\backslash\{CDEGH\}$	$\frac{2^3}{2^6}\sigma_3^2 + \frac{1}{2^6}\sigma^2$	6
$S_4'\backslash\{CDEGH\}$	$\frac{2^3}{2^6}\sigma_4^2 + \frac{1}{2^6}\sigma^2$	6
$S_5'\backslash\{CDEGH\}$	$\frac{2^3}{2^6}\sigma_5^2 + \frac{1}{2^6}\sigma^2$	6
$\mathcal{P}\backslash(\cup_{i=1}^5 S_i')$	$\frac{1}{2^6}\sigma^2$	24

To assign the 10 additional factors to higher order interactions in $S_i', i = 1, ..., 5$, one should choose one fractional generator from S_1', six from S_2', one from S_3' and two from S_5'. Note that the choice of generators should not include $CDEGH$ because this effect cannot be assessed for significance. One set of generators is given by

$$B = ACD,\ F = AH,\ J = CDEG,\ K = CGH,\ L = DEG,$$
$$M = CDH,\ N = CDEH,\ O = ACEGH,\ P = CEH,\ Q = EG.$$

These generators may not be the best possible set of generators. One can choose a different sets of generators using the rule (one generator from S_1', six from S_2', one from S_3' and two from S_5') to construct an optimal 2^{16-6} fractional factorial split-lot design in this setting.

The results discussed here help an experimenter in determining when a design exists, and how to construct one if it exists. The focus of this chapter was on results and algorithms related to the existence and construction of disjoint subspaces. When the conditions for the existence of a set of disjoint subspaces are not met, overlap among many pairs of subspaces cannot be avoided. Under these circumstance, one must be careful in deciding on the size of the overlap as well as the factorial effects that belong to the intersecting set. In the next chapter, we develop factorial and fractional factorial designs with randomization restrictions where the required number of RDCSSs, m, is greater than the size of a maximal set of disjoint subspaces, $|\mathcal{S}|$.

Chapter 5

Factorial Designs and Stars

An ideal choice for the randomization structure of a 2^p full factorial design is to have disjoint RDCSSs such that the S_i's corresponding to the RDCSSs are large enough to construct useful half-normal plots. Often, there are limitations on the number and size of the disjoint subspaces contained in the effect space $\mathcal{P} = PG(p-1,2)$. As described in the previous chapter, under these circumstances one would like to find a set of disjoint subspaces for constructing RDCSSs with different sizes. For the existence of a set of m disjoint $(t-1)$-dimensional subspaces, the conditions developed in Chapter 4 are based on the decomposition of p as $p = kt + s$, where k, t and s are nonnegative integers. In this chapter, we assume that s is strictly positive, i.e., there does not exist a $(t-1)$-spread of $\mathcal{P}$.

The results developed in Chapter 4 focus on the factorial designs where overlaps among the RDCSSs are avoided. However, the desired number of disjoint subspaces in the effect space $\mathcal{P}$ often can exceed the size of a maximal partial $(t-1)$-spread, which further causes RDCSSs to overlap. So, a few of the RDCSSs for different stages of randomization must share some of the randomization restriction factors. This is the main focus of this chapter.

When a $(t-1)$-spread of $PG(p-1,2)$ does not exist and $m > |\mathcal{S}|$, the overlap among at least a few of the RDCSSs cannot be avoided. Given this situation, one possibility is to maximize the number of disjoint RDCSSs, and then obtain a set of subspaces that minimize the size of the overlap among the non-disjoint RDCSSs. This combination of disjoint and overlapping subspaces of $PG(p-1,2)$ resembles the geometric structure called a $(t-1)$-cover of $\mathcal{P}$ (Beutelspacher, 1975).

Recall that assessing the factorial effects for an unreplicated factorial experiment requires constructing half-normal plots of size more than six or seven each. Since a $(t-1)$-cover approach minimizes the overlap, one may have to sacrifice the assessment of factorial effects present in multiple RDCSSs. For full factorial designs, if the effects present in multiple RDCSSs are higher order interactions, one may not be too concerned. However, if the number of effects in the intersection is large, then the loss of information relating to lower order effects cannot be avoided. In this case, sacrificing the assessment of all the effects in the overlap is not desirable.

It may appear that overlap among RDCSSs is a problem for the analysis of unreplicated factorial designs with randomization restrictions. It turns out that one can use an alternative strategy that uses overlapping among distinct subspaces as an advantage, and allows one to assess the significance of all the factorial effects in the effect space. For this purpose, we propose a geometric structure called a *star*, which consists of a set of distinct $(t-1)$-dimensional subspaces of $PG(p-1,2)$ with a common overlap on a $(r-1)$-dimensional subspace in $\mathcal{P}$.

This chapter is organized as follows. In Section 5.1, the focus is on the use of $(t-1)$-covers of the effect space $\mathcal{P}$ to construct designs when $m > |\mathcal{S}|$. The existence and construction of stars are developed in Section 5.2.1. The relationship between stars and $(t-1)$-covers is established in Section 5.2.2. A closer look at the class of 2^p factorial designs with $p = kt + s$ shows that the designs can be classified into two different groups: (a) $k = 1$ and (b) $k > 1$. In the first case, Theorem 4.1 shows that there does not exist even two disjoint $(t-1)$-dimensional subspaces. Stars are specifically beneficial for such cases. For the case $k > 1$, the maximum number of disjoint $(t-1)$-dimensional subspaces available in $PG(p-1,2)$ is often large (for details, see Lemma 4.4). Therefore, for smaller experiments, the desired number of RDCSSs (m) is usually less than the size of a maximal partial $(t-1)$-spread $\mathcal{S}$. In contrast, for full factorial experiments with large run-size and fractional factorial experiments with many factors, m can exceed $|\mathcal{S}|$. A generalization of stars which entertains large designs, called a *finite galaxy*, is proposed in Section 5.2.3. Again, the results developed here focus on only two level factorial designs, but are easily extended for q level factorial and regular fractional factorial designs.

5.1 Minimum overlap

In this section, geometric structures available in $PG(p-1,2)$ are used to construct designs that maximize the number of disjoint subspaces for constructing RDCSSs, and minimize the size of overlaps among the intersecting subspaces. A closely related geometric structure is called a $(t-1)$-cover (Eisfeld and Storme, 2000) of $\mathcal{P}$. A *cover* of the effect space $\mathcal{P}$ is a set of distinct subspaces in $\mathcal{P}$ that contains all the factorial effects.

Definition 5.1. *A $(t-1)$-cover $\mathcal{C}$ of $PG(p-1,2)$ is a set of $(t-1)$-dimensional subspaces of $PG(p-1,2)$ which covers all the points of $PG(p-1,2)$.*

Finding a set of subspaces that covers the entire effect space can be a stronger requirement compared to finding a pre-specified number of distinct subspaces. Nonetheless, if it is easy to construct a larger set of subspaces, one can always obtain an appropriate subset to construct RDCSSs as per the requirement. For example, Lemma 4.4 guarantees the existence of 17 disjoint subspaces of size 7 each in the base factorial design of a 2^{20-13} regular fractional factorial layout. A 2-cover $\mathcal{C}$ of the base factorial design with maximum number of disjoint subspaces consists of 16 disjoint subspaces and a set of 3 intersecting subspaces. Thus, if the experimenter needs less than 19 RDCSSs, one can take an appropriate subset of $\mathcal{C}$. Recall that, for the discussion in this chapter, m is supposed to be larger than the size of a maximal partial $(t-1)$-spread of $\mathcal{P}$. Similar to Chapter 4, the subspaces obtained from a standard $(t-1)$-cover construction technique may not satisfy the requirements for RDCSSs. Thus, the columns of the model matrix require relabelling to get the desired design.

From the definition of a $(t-1)$-cover, it is apparent that there exists more than one set of $(t-1)$-dimensional subspaces that covers the effect space $\mathcal{P}$. However, we are interested in $(t-1)$-covers that maximize the number of disjoint subspaces. These $(t-1)$-covers are called minimal $(t-1)$-covers of $\mathcal{P}$ (Eisfeld and Storme, 2000).

Definition 5.2. *A set of $(t-1)$-dimensional subspaces of $\mathcal{P} = PG(p-1, 2)$ is said to be a minimal $(t-1)$-cover $\mathcal{C}$ of $\mathcal{P}$ if there does not exist a $(t-1)$-cover $\mathcal{C}'$ of $\mathcal{P}$ such that $\mathcal{C}'$ is a proper subset of $\mathcal{C}$.*

In other words, the set of subspaces in a minimal $(t-1)$-cover cannot be further shortened and still form a cover. Consequently, a minimal $(t-1)$-cover $\mathcal{C}$ consists of a maximum number of disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$ that forms a cover of $\mathcal{P}$. The following result due to Eisfeld and Storme (2000) provides a lower bound on the size of a $(t-1)$-cover.

Lemma 5.1. *A $(t-1)$-cover of $\mathcal{P} = PG(p-1, 2)$ contains at least $2^s \frac{2^{kt}-1}{2^t-1} + 1$ elements, where $p = kt + s$ for $0 < s < t < p$.*

A minimal $(t-1)$-cover that attains this lower bound can be constructed using construction techniques similar to that of a partial $(t-1)$-spread developed in Section

4.2.2. The next example illustrates the use of a minimal $(t-1)$-cover in constructing factorial designs when the desired number of subspaces for RDCSSs (m) is more than the maximum number of disjoint subspaces $(|\mathcal{S}|)$ and less than the size of a minimal $(t-1)$-cover $\mathcal{C}$.

Note that, for a regular fractional factorial design with at most 5 basic factors, there does not exist even a pair of disjoint subspaces large enough to construct useful half-normal plots. The regular fractional factorial designs with 6 basic factors is not considered here because there exists a 2-spread of $\mathcal{P}$, which is not the focus of this chapter. Therefore, a two-level regular fractional factorial design, which allows construction of at least two disjoint RDCSSs large enough to perform useful half-normal plots where a $(t-1)$-spread does not exist, consists of at least 7 basic factors. Since multiple experimental units are processed together at each stage of randomization, designs with randomization restrictions have usually much larger run-size than completely randomized designs. Therefore, these designs are useful in practice.

Example 5.1. Consider a 2^{20-13} fractional factorial split-lot design with 18 stages of randomization. Suppose that the restrictions imposed by the experimenter on different stages of randomization are characterized by $S_1 \supset \{F_1, F_2, F_3\}$ and $S_i \supset \{F_{i+2}\}$ for $i = 2, ..., 18$. To get useful half-normal plots, each RDCSS should contain the necessary number of effects. Recall that the corresponding *base factorial design* is the full factorial design constructed from the basic factors. By using Lemma 4.4 for the base factorial design, $p = 7$ and $t = 3$ implies that there exist only $2(63/7) - 2 + 1 = 17$ disjoint 2-dimensional subspaces. Therefore, for constructing 18 RDCSSs of size 7 each, one can have at most 16 disjoint subspaces. The other two 2-dimensional subspaces must overlap.

It turns out that there exists a minimal 2-cover of $\mathcal{P}$ which consists of 16 disjoint subspaces and a set of 3 non-disjoint subspaces overlapping on a common subspace of size 3. Thus, 2 out of the 3 intersecting subspaces have to chosen to construct the desired RDCSS. However, the significance of the factorial effects contained in the 2 intersecting RDCSSs cannot be assessed if the factorial experiment is unreplicated. Let S_i, $i = 1, ..., 16$ represent the disjoint RDCSSs, and S_{17}, S_{18} be the two overlapping RDCSSs. Then, the analysis of variance is shown in Table 5.1.

Table 5.1: The ANOVA table for the 2^{20-13} split-lot design in a 18-stage process.

Effects	Variance	Degrees of Freedom
S_1	$\frac{2^4}{2^7}\sigma_1^2 + \frac{1}{2^7}\sigma^2$	7
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
S_{16}	$\frac{2^4}{2^7}\sigma_{16}^2 + \frac{1}{2^7}\sigma^2$	7
$S_{17}\backslash(S_{17} \cap S_{18})$	$\frac{2^4}{2^7}\sigma_{17}^2 + \frac{1}{2^7}\sigma^2$	4
$S_{18}\backslash(S_{17} \cap S_{18})$	$\frac{2^4}{2^7}\sigma_{18}^2 + \frac{1}{2^7}\sigma^2$	4
$S_{17} \cap S_{18}$	$\frac{2^4}{2^7}(\sigma_{17}^2 + \sigma_{18}^2) + \frac{1}{2^7}\sigma^2$	3
$\mathcal{P}\backslash(\cup_{i=1}^{18} S_i)$	$\frac{1}{2^7}\sigma^2$	4

Since the total number of distinct $(t-1)$-dimensional subspaces in a *minimal* $(t-1)$-cover $\mathcal{C}$ is less than any other $(t-1)$-cover, the non-disjoint subspaces overlap on a smallest possible intersecting set. If the size of the common overlap in Example 5.1 was smaller (e.g., $|S_{17} \cap S_{18}| = 1$), then by assigning a higher order interaction to the effect in the intersecting set one could sacrifice the assessment of this one effect and assess the significance for the rest of the effects. Here, it is unlikely that all 15 effects in $S_{17} \cup S_{18}$ and $\mathcal{P}\backslash(\cup_{i=1}^{18} S_i)$ are negligible. Thus, one would not want to sacrifice the assessment of all these effects. In particular, for constructing regular fractional factorial designs, it is often preferable to assign *added factors* to higher order interactions of the corresponding *base factorial design.* Therefore, it is desirable to develop a new strategy to assess the significance of more factorial effects.

Overlap among the RDCSSs may appear to cause problems in assessing the significance of factorial effects if the factorial design is unreplicated. Next, we develop a new overlapping strategy resulting in a geometric structure called a star. When $k = 1$ (i.e., there does not exist even a pair of disjoint $(t-1)$-dimensional subspaces), a star is geometrically similar to a minimal $(t-1)$-cover but flexible enough to allow different sizes of the common overlap.

5.2 Overlapping strategy

In this section, we first highlight the features of the RDCSS structure of a factorial design that are required to efficiently assess the significance of factorial effects. This further motivates the geometric structure of the new design called a star. Necessary and sufficient conditions will be developed to establish the existence of stars. Next, an algorithm is proposed for constructing stars. Since the geometry of stars is similar to that of a minimal $(t-1)$-cover, we establish a relationship between the two geometric structures. Finally, the notion of stars is generalized to accommodate larger designs.

In order to use the overlap among the RDCSSs to our advantage, the size of the overlaps themselves should be large enough. The idea here is that when an overlap must occur, we shall require the number of effects in the overlap to be large enough to construct a separate half-normal plot. Furthermore, one must remember that the variance of an effect estimate depends on its presence in different RDCSSs (Theorem 3.3). The following properties summarize the requirements of a good factorial design when overlap among RDCSSs cannot be avoided.

- *The size of each overlap should be more than six or seven.* Recall from Chapter 2 that the factorial effects with equal variance are plotted on separate half-normal plots. In addition, more than six or seven effects are required to construct an informative half-normal plot (Schoen, 1999). Therefore, from Theorem 3.3, the effects contained in an overlap have to be plotted together on a separate half-normal plot. If $S_{ij} = S_i \cap S_j$ is non-null, then the size of S_{ij} should be at least

$2^3 - 1$. As a result, the size of S_i and S_j should be more than $2^4 - 1$.

- *All non-disjoint subspaces are preferred to have a common overlap.* Let S_i, S_j and S_k be three RDCSSs such that S_{ij}, S_{ik} and S_{jk} are non-empty, where $S_{i_1 i_2} = S_{i_1} \cap S_{i_2}$, for $i_1, i_2 \in \{i, j, k\}$. Then, the factorial effects in $S_i \backslash (S_{ij} \cup S_{ik})$ have distribution that differs from those of the factorial effects in S_{ij} or S_{ik} (Theorem 3.3). Thus, if all the pairwise intersections among the m RDCSSs are different, $\binom{m}{2} + m$ separate half-normal plots are required. The geometric structure formed as a result is known as the *conclave of planes* (Shaw and Maks, 2003). If all the overlaps are identical, only $m + 1$ distinct half-normal plots are needed to assess the significance of factorial effects contained in the RDCSSs.

In addition to the inefficiency in assessing the factorial effects on a process, a minimal $(t-1)$-cover approach addresses subspaces of equal size only. The RDCSSs are often characterized by the experimenters and are likely to be of different sizes. The next example (Vivacqua and Bisgaard, 2004) presents a scenario where subspaces of different sizes are desirable.

Example 5.2. Consider the battery cell experiment described in Example 4.5. Here, the experimenter had to sacrifice the assessment of the effect in overlap between S_1 and S_2. There exists a better strategy that uses the overlapping between subspaces as an advantage, and leads one to construct a design that allows the assessment of all the factorial effects in the effect space. Of course, this is not a big issue because it is likely that the 4-factor interaction ($ABCD$) is negligible. However, if this was an 8-factor design with 64 runs with two additional factors G and H in the curing stage, one would have to choose two fractional generators from S_2. Under these circumstances, assigning two interactions from $S_2 = \langle E, F, ABCD \rangle$, considered in Example 4.5, may cause $ABCD$ to be aliased with a 2-factor interaction. Since the size of overlap between S_1 and S_2 is too small to construct half-normal plots, one would have to sacrifice information on a 2-factor interaction. Instead, one can allow a larger overlap between S_1 and S_2 to construct useful half-normal plots. For example, by defining $S_1 = \langle A, BC, CD, AB \rangle$ and $S_2 = \langle E, F, BC, CD, AB \rangle$ with the additional factor being $G = ABEF$ and $H = CDF$, the resulting design allows more enlightening analysis. The grouping of effects based on their distribution under the null hypothesis is shown in Table 5.2. Specifically, notice that all of the factorial effects can be assessed using 4 half-normal plots.

Table 5.2: The distribution of factorial effects for the battery cell experiment.

Effects	Variance	Degrees of Freedom
$S_1 \cap S_2$	$\frac{2^2}{2^6}\sigma_1^2 + \frac{2^1}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	7
$S_1 \backslash (S_1 \cap S_2)$	$\frac{2^2}{2^6}\sigma_1^2 + \frac{1}{2^6}\sigma^2$	8
$S_2 \backslash (S_1 \cap S_2)$	$\frac{2^1}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	24
$\mathcal{P} \backslash (S_1 \cup S_2)$	$\frac{1}{2^6}\sigma^2$	24

Other than the two properties described above, it is preferable to have a factorial design that entertains unequal sized RDCSSs. Considering the three features (two properties on the overlapping pattern among the RDCSSs, and the flexibility among the sizes of the different RDCSSs), we propose stars for full factorial and regular fractional factorial designs with p basic factors.

5.2.1 Stars

The notion of stars was first introduced by Shaw and Maks (2003) in a specific context for a set of 1-dimensional subspaces with a common overlap on a point in $\mathcal{P}$. In this section, we formalize the notion of stars and further generalize this concept for $(t-1)$-dimensional subspaces of $\mathcal{P} = PG(p-1, 2)$. First, we discuss the different components of a star for both equal and unequal sized subspaces, then the existence and construction of stars are established.

A star consists of two components: (a) a set of $(t-1)$-dimensional subspaces (π_t's) in $\mathcal{P}$, that are referred to as *rays* of the star, and (b) the common overlap on a $(r-1)$-dimensional subspace (π_r) is called the *nucleus* of the star, where $r < t$. The star formed from these subspaces (or rays) constitutes a $(t-1)$-cover of $\mathcal{P}$ if these subspaces span the effect space $\mathcal{P}$. Next, we define the geometric structure called a star in a general setup.

Definition 5.3. *A star $St(\mu, \pi_t, \pi_r)$ is a set of μ rays consisting of $(t-1)$-dimensional subspaces (π_t's) in $\mathcal{P}$, and the nucleus π_r, a $(r-1)$-dimensional subspace, where $r < t$.*

If a star $St(\mu, \pi_t, \pi_r)$ exists, the maximum number of rays in $St(\mu, \pi_t, \pi_r)$ is given by $\mu = (2^p - 2^r)/(2^t - 2^r)$. Consequently, the smaller the nucleus is, the fewer the number of rays (μ). The following example illustrates the details of stars.

Example 5.3. Consider the setup of the plutonium example in Bingham et al. (2006). The authors performed a designed experiment to identify the factors which have significant impact on the plutonium alloy. They used a 2^5 full factorial design with 3 stages of randomization characterized by $S_1 \supset \{A, B\}$, $S_2 \supset \{C\}$ and $S_3 \supset \{D, E\}$. The factors (A, B) represent the casting mechanism for creating a type of plutonium alloy, and (C, D, E) are the heat treatments applied to the three stages of the manufacturing process. The data analysis using a half-normal plot approach requires each RDCSS to have more than six or seven effects. From Theorem 4.1 , it is obvious that there does not exist even two disjoint subspaces of size 7 each in this effect space. Bingham et al. (2006) used an exhaustive computer search to reach this conclusion. They chose to sacrifice the assessment of one effect $ABCDE$. The design proposed by Bingham et al. (2006) is equivalent to a $St(5, \pi_3, \pi_1)$. By defining the nucleus of a star to be the 0-dimensional subspace, $\pi_1 = \{ABCDE\}$, and assuming that the rays of the star are 2-dimensional subspaces of $\mathcal{P}$, the maximum number of rays is $\mu = \frac{2^5-2^1}{2^3-2^1} = 5$. The five rays $S_1 = \langle A, B, \pi_r \rangle$, $S_2 = \langle C, AD, \pi_r \rangle$, $S_3 = \langle D, E, \pi_r \rangle$, $S_4 = \langle AC, AE, \pi_r \rangle$ and $S_5 = \langle BC, BD, \pi_r \rangle$ constitute the star. The data analysis was done using four separate half-normal plots for the four sets of effects given by $S_i \backslash \pi_r$, for $i = 1, ..., 3$ and $\mathcal{P} \backslash (\cup_{i=1}^3 S_i)$ (see Table 5.3).

Table 5.3: The ANOVA table for the plutonium alloy experiment.

Effects	Variance	Degrees of Freedom
$S_1 \backslash \{ABCDE\}$	$\frac{2^2}{2^5}\sigma_1^2 + \frac{1}{2^5}\sigma^2$	6
$S_2 \backslash \{ABCDE\}$	$\frac{2^2}{2^5}\sigma_2^2 + \frac{1}{2^5}\sigma^2$	6
$S_3 \backslash \{ABCDE\}$	$\frac{2^2}{2^5}\sigma_3^2 + \frac{1}{2^5}\sigma^2$	6
$\{ABCDE\}$	$\frac{2^2}{2^5}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{1}{2^5}\sigma^2$	1
$\mathcal{P} \backslash (S_1 \cup S_2 \cup S_3)$	$\frac{1}{2^5}\sigma^2$	12

Instead of sacrificing the assessment of one effect, if all the factorial effects are to be assessed, the size of the common overlap among the RDCSSs has to be large enough, e.g., $|\pi_r| \geq 7$ and that further implies that $|S_i| \geq 15$. It turns out that one can construct a star with the desired features. For $r = 3$ and $t = 4$, the number of rays is bounded above by $\mu = \frac{2^5-2^3}{2^4-2^3} = 3$. Let the nucleus be $\pi_r = \langle AB, DE, ACD \rangle$. Then, one feasible choice for the set of three rays is $S_1 = \langle A, \pi_r \rangle$, $S_2 = \langle C, \pi_r \rangle$ and $S_3 = \langle D, \pi_r \rangle$. Since the resulting star $St(3, \pi_4, \pi_3)$ covers $\mathcal{P}$, only 4 half-normal plots are required to analyze the data. The analysis of variance is shown in Table 5.4.

Table 5.4: The sets of effects having equal variance in the 2^5 split-lot design.

Effects	Variance	Degrees of Freedom
$S_1 \backslash \pi_r$	$\frac{2^1}{2^5}\sigma_1^2 + \frac{1}{2^5}\sigma^2$	8
$S_2 \backslash \pi_r$	$\frac{2^1}{2^5}\sigma_2^2 + \frac{1}{2^5}\sigma^2$	8
$S_3 \backslash \pi_r$	$\frac{2^1}{2^5}\sigma_3^2 + \frac{1}{2^5}\sigma^2$	8
π_r	$\frac{2^1}{2^5}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{1}{2^5}\sigma^2$	7

The overlapping among the RDCSSs turned out to be an advantage for the assessment of factorial effects. However, the effects in the common overlap (π_r) have relatively large variance. That is, there is a tradeoff between the ability to assess the significance of factorial effects and the variance of the effect estimates. Thus, if the design under consideration is an unreplicated full factorial, one may prefer to sacrifice a few effects by minimizing the overlap. In some cases, availability of stars with different sized nuclei can be useful. For instance, when a regular fractional factorial design has to be constructed from the base factorial design (e.g., in a three-stage 2^{6-1} split-lot design), the added factors are assigned to the columns corresponding to preferably higher order interactions of the basic factors.

The notion of stars can be further generalized for a set of subspaces of unequal sizes with a common overlap. Without loss of generality, let μ_i be the number of $(t_i - 1)$-dimensional rays in $\mathcal{P} = PG(p-1, 2)$, for $i = 1, ..., k$, and the common overlap be a $(r-1)$-dimensional subspace in $\mathcal{P}$. Such a star can be denoted by $St(\mu_1, ..., \mu_k, \pi_{t_1}, ..., \pi_{t_k}, \pi_r)$. Recall that if $t_i + t_j \leq p$ for any pair i, j, then there exists a set of disjoint subspaces (Theorem 4.3), which is not the focus in this chapter, and thus we assume that $0 < r < t_i < p$ and $t_i > p/2$ for all $i \in \{1, ..., k\}$.

A star is said to be *balanced* if all of its rays are of same size, while a star with different sized rays is called an *unbalanced star*. The geometric structure of two stars can be compared by ordering their rays according to its size. Without loss of generality, let Ω be a star $St(\mu_1, ..., \mu_k, \pi_{t_1}, ..., \pi_{t_k}, \pi_r)$ in $\mathcal{P} = PG(p-1, 2)$ such that $r < t_1 < t_2 < \cdots < t_k < p$. Next, we develop the geometric equivalence between two stars Ω_1 and Ω_2.

Definition 5.4. *Two stars Ω_1 and Ω_2 in $PG(p-1, 2)$, with nuclei of same size, are said to be geometrically equivalent if*

$$(t_1^{(1)}, ..., t_k^{(1)}) = (t_1^{(2)}, ..., t_k^{(2)}) \quad \textit{and} \quad (\mu_1^{(1)}, ..., \mu_k^{(1)}) = (\mu_1^{(2)}, ..., \mu_k^{(2)}).$$

Here, the superscripts (1) and (2) correspond to the parameters of star Ω_1 and Ω_2 respectively. Although the stars have a flexible geometric structure that uses overlapping among the RDCSSs to our advantage, and are generalizable for subspaces of different dimensions, the existence of stars is non-trivial. Even for a balanced star, the existence of a star $St(\mu, \pi_t, \pi_r)$ is not guaranteed for any t and r. For example, there does not exist a balanced star with 5-dimensional rays and a 2-dimensional nucleus that covers the effect space $\mathcal{P} = PG(6, 2)$.

Next, we propose conditions for the existence of stars. As illustrated in Example 5.3, if there exists a star that covers the entire effect space, one can select an appropriate subset of rays to construct the desired set of RDCSSs. Thus, the result presented here focus on the existence of stars that cover $\mathcal{P}$.

Theorem 5.1. *If there exists a star $St(\mu_1, ..., \mu_k, \pi_{t_1}, ..., \pi_{t_k}, \pi_r)$ in $\mathcal{P} = PG(p-1, 2)$, the positive integers μ_i, t_i, $i = 1, ..., k$ and r satisfy the following relation:*

$$(2^{p-r} - 1) = \mu_1(2^{t_1-r} - 1) + \mu_2(2^{t_2-r} - 1) + \cdots + \mu_k(2^{t_k-r} - 1).$$

Proof: Suppose there exists a star $St(\mu_1, ..., \mu_k, \pi_{t_1}, ..., \pi_{t_k}, \pi_r)$ that is also a cover of the effect space $\mathcal{P}$. Then,

$$\frac{2^p - 1}{2 - 1} = \mu_1 \frac{2^{t_1} - 2^r}{2 - 1} + \cdots + \mu_k \frac{2^{t_k} - 2^r}{2 - 1} + \frac{2^r - 1}{2 - 1}, \tag{5.1}$$

which simplifies to $(2^{p-r} - 1) = \sum_{i=1}^{k} \mu_i(2^{t_i-r} - 1)$. □

The total number of rays in a star $St(\mu_1, ..., \mu_k, \pi_{t_1}, ..., \pi_{t_k}, \pi_r)$ is $\mu = \mu_1 + \cdots + \mu_k$. That is, at most μ distinct RDCSSs can be constructed using the rays of a star $St(\mu_1, ..., \mu_k, \pi_{t_1}, ..., \pi_{t_k}, \pi_r)$. Note that the condition in Theorem 5.1 is a necessary condition and may not be sufficient. That is, the existence of positive integers μ_i, t_i for $i = 1, ..., k$ and r which satisfy $(2^{p-r} - 1) = \sum_{i=1}^{k} \mu_i(2^{t_i-r} - 1)$ does not guarantee the existence of a star $St(\mu_1, ..., \mu_k, \pi_{t_1}, ..., \pi_{t_k}, \pi_r)$. The following example illustrates the underlying reason.

Example 5.4. Consider a 2^6 full factorial design with 3 stages of randomization. Let the RDCSSs be such that $|S_1| = 7$ and $|S_2| = |S_3| = 15$. From Theorem 4.1, it is obvious that overlapping among the RDCSSs cannot be avoided. Although the quantities $\mu_1 = 1, \mu_2 = 4, t_1 = 3, t_2 = 4$ and $r = 1$ satisfy the relation: $2^p - 1 = \mu_1(2^{t_1} - 2^r) + \mu_2(2^{t_2} - 2^r) + 2^r - 1$, there does not exist a $St(1, 4, \pi_3, \pi_4, \pi_1)$. This is obvious from Theorem 4.1, which says that the minimum overlap between the two subspaces S_2 and S_3 is at least 3. However, as we shall see, all is not lost.

By imposing a stronger condition to the special case ($t_1 = \cdots = t_k = t$), the result can be further refined to become both necessary and sufficient. This modified result has similar spirit as the necessary and sufficient condition (André 1954) for the existence of a $(t-1)$-spread of $PG(p-1, 2)$.

Theorem 5.2. *There exists a star $St(\mu, \pi_t, \pi_r)$ in $\mathcal{P} = PG(p-1, 2)$, if and only if $(t-r)$ divides $(p-r)$, for $0 < r < t \leq p$. Furthermore, if $(t-r)$ divides $(p-r)$, the number of rays is $\mu = (2^{p-r} - 1)/(2^{t-r} - 1)$.*

Proof: If there exists a star $St(\mu, \pi_t, \pi_r)$ in $\mathcal{P}$, then the maximum number of rays is

$$\mu = \frac{|PG(p-1,2)| - |PG(r-1,2)|}{|PG(t-1,2)| - |PG(r-1,2)|} = \frac{2^p - 2^r}{2^t - 2^r} = \frac{2^{p-r} - 1}{2^{t-r} - 1}.$$

Note that μ is an integer if and only if $(t-r)$ divides $(p-r)$. Since $\mu(|PG(t-1,2)| - |PG(r-1,2)|) + |PG(r-1,2)| = |PG(p-1,2)|$, the star $St(\mu, \pi_t, \pi_r)$ is a $(t-1)$-cover of $\mathcal{P} = PG(p-1, 2)$.

From Theorem 4.3, there exists an $(r-1)$-dimensional subspace $\mathcal{U}_1$ in $\mathcal{P} = PG(p-1,2)$ that is disjoint from an $(p-r-1)$-dimensional subspace $\mathcal{U}_2$ in $\mathcal{P}$. When $(t-r)$ divides $(p-r)$, Lemma 4.1 determines the existence of a $(t-r-1)$-spread $\mathcal{S}$ of a $\mathcal{U}_2$ with $|\mathcal{S}| = (2^{p-r}-1)/(2^{t-r}-1) = \mu$. Thus, the μ distinct $(t-1)$-dimensional rays of $St(\mu, \pi_t, \pi_r)$ can be constructed by combining the individual elements of the spread $\mathcal{S}$ with the nucleus $\pi_r = \mathcal{U}_1$. □

Corollary 5.1. *For positive integers $t < p$ and $r = t-1$, there always exists a star $St(\mu, \pi_t, \pi_r)$ contained in $\mathcal{P}$, where $\mu = |PG(p-t, 2)|$.*

For instance, both sets of parameters in Example 5.2 ($t = 3, r = 1, p = 5$ and $t = 4, r = 3, p = 5$) satisfy the condition $(t-r)$ divides $(p-r)$. Of course, these new designs called stars are useful to a practitioner only if they can be constructed. Assuming the existence of a star, we propose an algorithm to construct a star Ω, where all the μ rays are $(t-1)$-dimensional subspaces of $PG(p-1,2)$.

Construction 5.1. Let Ω be a star in $\mathcal{P} = PG(p-1,2)$, which consists of μ rays denoted by $\{S_i\}_{i=1}^{\mu}$, and a nucleus π_r, where $|S_i| = 2^t - 1$, for all i and $r < t$. The following is the outline of an algorithm for constructing the star Ω.

1. Choose r independent factorial effects from the effect space $\mathcal{P}$ to construct the nucleus π_r of size $2^r - 1$.

2. Construct a star $\Omega_0 = St(\mu_0, \pi_{r+1}, \pi_r)$ by defining a nucleus $R_0 = \pi_r$ and $\mu_0 = 2^{p-r} - 1$ distinct rays $R_j = \langle \delta_j, \pi_r \rangle$, where $\delta_j \in \mathcal{P} \backslash \left(\cup_{l=0}^{j-1} R_l \right)$, $j = 1, ..., \mu_0$. There exists a set of δ_i's such that $\mathcal{U}_2 = \{\delta_1, ..., \delta_{\mu_0}\}$ is a $(p-r-1)$-dimensional subspace of $\mathcal{P}$ that is disjoint from π_r. This can instead be obtained by arbitrarily constructing a $(p-r-1)$-dimensional subspace $\mathcal{U}_2$ that is disjoint from $\mathcal{U}_1 = \pi_r$, and then by relabelling the points of $\mathcal{P}$ to get the desired rays.

3. Since $(t-r)$ divides $(p-r)$, there exists a $(t-r-1)$-spread $\mathcal{S}$ of $\mathcal{U}_2$ with $|\mathcal{S}| = (2^{p-r}-1)/(2^{t-r}-1) = \mu$. Let $J_1, ..., J_\mu$ be the elements of $\mathcal{S}$. This spread $\mathcal{S}$ can be constructed using the technique shown in Section 4.2.1.

4. The required set of μ rays are $S_i = \langle J_i, \pi_r \rangle$, $i = 1, ..., \mu$.

The resulting structure is the desired star $\Omega = St(\mu, \pi_t, \pi_r)$. One might be tempted to take a similar approach for constructing an unbalanced star. Instead of using a spread of $\mathcal{U}_2$, if a sequential approach is taken for constructing a set of disjoint J_i's from the elements of $\mathcal{U}_2$, it may lead to overlap among the J_i's. The following example illustrates the construction of a balanced star $St(\mu, \pi_t, \pi_r)$.

Example 5.5. Consider the setup in Example 5.3. Here, the existence of a star $St(3, \pi_4, \pi_3)$ in $\mathcal{P} = PG(4, 2)$ is guaranteed since it satisfies the sufficiency condition $(t-r)$ divides $(p-r)$ of Theorem 5.2. The experimenter's requirement for the three RDCSSs were $S_1 \supset \{A, B\}$, $S_2 \supset \{C\}$ and $S_3 \supset \{D, E\}$. Thus, having the freedom to construct the nucleus first, one can choose r independent higher order effects to construct a $(r-1)$-dimensional subspace. For example, consider $R_0 = \pi_r = \langle AB, DE, ACD \rangle$. The effects $\delta_1, ..., \delta_3$ can be chosen sequentially as described in Step 2. Considering the experimenter's requirement the obvious choice for $\delta_1 \in \mathcal{P} \backslash R_0$ would be $\delta_1 = A$. Then, $\delta_2 \in \mathcal{P} \backslash (R_0 \cup R_1)$ can be chosen to be $\delta_2 = C$, which matches the requirement imposed on the RDCSS defined by S_2. Lastly, the effects in $\mathcal{P} \backslash (R_0 \cup R_1 \cup R_2)$ forms a subspaces that satisfies the desired criterion on the third RDCSS. As a result, the subspaces $S_1 = \langle \delta_1, \pi_r \rangle$, $S_1 = \langle \delta_2, \pi_r \rangle$ and $S_1 = \langle \delta_3, \pi_r \rangle$ constitute a star $St(3, \pi_4, \pi_3)$.

This star can also be constructed by selecting the two disjoint subspaces $\mathcal{U}_1 = \langle AB, DE, ACD \rangle$ and $\mathcal{U}_2 = \langle A, C \rangle$ as mentioned in the proof of Theorem 5.2. Since $p - r = 2$ and $t - r = 1$, the only 0-spread of $\mathcal{U}_2$ is the trivial spread, the set of all points of $\mathcal{U}_2$. Hence, the rays of the star would be $S_1 = \langle A, \mathcal{U}_1 \rangle$, $S_1 = \langle C, \mathcal{U}_1 \rangle$ and

$S_1 = \langle AC, \mathcal{U}_1 \rangle$, which is the same as above.

In Example 5.5, the choice of $\mathcal{U}_1$ and $\mathcal{U}_2$ do not have to be so specific. One can start with an obvious choice and then use an appropriate relabelling to get the desired design. For the rays constructed here, all of the factorial effects (δ_i's) were chosen to be main effects. However, based on the imposed restrictions one can choose main effects or interactions. Different choices of factorial effects in the construction of RDCSSs lead to different randomization restrictions. For example, in block designs RDCSSs do not contain main effects, whereas for a split-lot designs, one or more factors are assigned to the subspaces representing RDCSSs. The construction provided above is very useful, because one can use the restrictions imposed on the RDCSSs to choose the factorial effects for constructing rays of a star.

Although the experimenter has some control over the choice of effects in constructing a nucleus π_r and the star $\Omega_0 = St(\mu_0, \pi_{r+1}, \pi_r)$, the construction of spread required in Step 3 limits the choices to some extent. Thus, if necessary, one can find an appropriate relabelling in a similar manner as described in Section 4.2.1 to transform the star (Ω) such that the resulting star (Ω') satisfies the desired features. The next example demonstrates the usefulness of stars in a real application.

Example 5.6. In the chemical experiment presented in Example 4.6, the original experimental setting required $|S_i'| \geq 2^3 - 1$ for $i = 1, 3, 4, 5$ and $|S_2'| \geq 2^4 - 1$. Assuming that the allowed run-size is 64, Theorem 5.2 guarantees the existence of a star $St(5, \pi_4, \pi_2)$. The rays of this star can be used to construct S_i's for the base factorial design. Any two distinct S_i overlaps on the 1-dimensional nucleus of the star. One can use the fractionation technique described in Section 4.3 to choose a good set of fractional generators. The ANOVA table is shown in Table 5.5. This design is specifically better if suppose more additional factors are introduced in other stages of the process. In the design proposed in Example 4.6, only S_2 contains enough interactions to choose fractional generators from. While, in the design proposed here, one can choose fractional generators from any of the five RDCSSs.

Table 5.5: The ANOVA table for the battery cell experiment.

Effects	Variance	Degrees of Freedom
$\cap_{i=1}^{5} S_i'$	$\frac{2^2}{2^6}(\sigma_1^2 + \cdots \sigma_5^2) + \frac{1}{2^6}\sigma^2$	3
$S_1' \backslash (\cap_{i=1}^{5} S_i')$	$\frac{2^2}{2^6}\sigma_1^2 + \frac{1}{2^6}\sigma^2$	12
$S_2' \backslash (\cap_{i=1}^{5} S_i')$	$\frac{2^2}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	12
$S_3' \backslash (\cap_{i=1}^{5} S_i')$	$\frac{2^2}{2^6}\sigma_3^2 + \frac{1}{2^6}\sigma^2$	12
$S_4' \backslash (\cap_{i=1}^{5} S_i')$	$\frac{2^2}{2^6}\sigma_4^2 + \frac{1}{2^6}\sigma^2$	12
$S_5' \backslash (\cap_{i=1}^{5} S_i')$	$\frac{2^2}{2^6}\sigma_5^2 + \frac{1}{2^6}\sigma^2$	12

Since the common overlap is not large enough to construct useful half-normal plots, one has to sacrifice the assessment of the three effects contained in $\cap_{i=1}^{5} S_i'$. The significance for the rest of the effects can easily be assessed using half-normal plots. The construction of S_i's for the five stages of randomization follows from Construction 5.1. The algorithm starts by first choosing a 1-dimensional nucleus π_2. Without loss of generality, let $\pi_2 = \langle e, f \rangle$. Then, $S_0' = \langle a, b, c, d \rangle$ is disjoint from π_2. Lemma 4.1 implies that there exists a 1-spread $\mathcal{S}$ of S_0'. The elements of the 1-spread $\mathcal{S}$ are shown in Table 5.6.

Table 5.6: The elements of $\mathcal{S}$ using cyclic construction.

S_1''	S_2''	S_3''	S_4''	S_5''
d	c	b	a	cd
bc	ab	acd	bd	ac
bcd	abc	$abcd$	abd	ad

The subspaces $S_i = \langle S_i'', \pi_2 \rangle$ for $i = 1, \ldots, 5$, are 3-dimensional subspaces of $\mathcal{P}$, and the pairwise overlap among S_i's is π_2. To bring this construction into our setting, we relabel the factors as: $a \to C, b \to A, c \to D, d \to H, e \to HDO$ and $f \to ACDE$. This relabelling results in $\pi_2^* = \langle HDO, ACDE \rangle$. Note that the relabelling is not arbitrary, and it depends on the requirement on the restrictions on different stages of randomization in the experiment. The relabelled spread $\mathcal{S}^*$ is presented in Table 5.7.

Table 5.7: The elements of the relabelled spread.

S_1^*	S_2^*	S_3^*	S_4^*	S_5^*
H	D	A	C	HD
AD	AC	CDH	AH	CD
ADH	ACD	$ACDH$	ACH	CH

The required RDCSSs $S_i', i = 1, ..., 5$ are now given by $S_i' = \langle S_i^*, \pi_2^* \rangle$, for all i. Lastly, these S_i's have to be fractionated by choosing 1 generator from S_1', 7 from S_2', 1 from S_3' and 1 from S_4'. The resulting structure is the required design. Of course, one has to be careful in selecting these fractional generators, because they will impact the word-length pattern and hence the optimality criteria. As mentioned earlier, assessment on only three effects $(HDO, ACDE, ACEHO)$ have to be sacrificed, and the rest of the factorial effects in $\mathcal{P}$ can be assessed using 5 half-normal plots.

So far in this section, we assumed that there does not exist even two disjoint RDCSSs in the effect space. For equal sized RDCSSs, this is equivalent to the assumption $k = 1$, where the effect space is the set of all factorial effects in a 2^p full factorial layout for $p = kt + s$ and $0 < s < t$. As a result, all the $(t-1)$-dimensional subspaces in any $(t-1)$-cover are also non-disjoint. Under these circumstances, the geometric structure of a minimal $(t-1)$-cover is similar to that of a balanced star that covers the effect space $\mathcal{P}$. Next, we establish the relationship between minimal $(t-1)$-covers and balanced stars.

5.2.2 Balanced stars and minimal $(t-1)$-covers

This section focuses on the relationship between balanced stars and minimal $(t-1)$-covers of $PG(p-1, 2)$. In a 2^p factorial layout with $p = kt + s$, if $k = 1$ then we show that a minimal $(t-1)$-cover $\mathcal{C}$ of $PG(p-1, 2)$ is a special case of a balanced star $St(\mu, \pi_t, \pi_r)$. That is, there exists a positive integer r such that $(t-r)$ divides $(p-r)$, and any two elements of $\mathcal{C}$ intersect on a common subspace of size $2^r - 1$. First, we establish the relationship between the two geometric structures. Then, for the $k > 1$ case, we propose the use of balanced stars to modify a minimal $(t-1)$-cover to construct designs that are more efficient than a standard minimal $(t-1)$-cover for assessing the significance of the factorial effects.

Theorem 5.3. *For a projective space $\mathcal{P} = PG(p-1, 2)$, if $p = kt + s$ and $t > p/2$ then a minimal $(t-1)$-cover of $\mathcal{P}$ is equivalent to a star $St(2^s + 1, \pi_t, \pi_{t-s})$ in $\mathcal{P}$.*

The proof is shown in a more general setup (Theorem 5.4). According to this theorem, a minimal $(t-1)$-cover of $\mathcal{P}$, for $t > p/2$, is geometrically equivalent to a star. Subsequently, the requirement for the geometric structure we call a star may seem questionable. Recall that a minimal $(t-1)$-cover assumes that the smaller the size of the overlap is, the smaller the requirement is for the number of distinct $(t-1)$-dimensional subspaces to cover the entire effect space. Therefore, a minimal $(t-1)$-cover consists of minimum size overlap ($|\pi_{t-s}|$). This overlap may not be large enough to obtain a useful half-normal plot for the assessment of factorial effects if the experiment is unreplicated. In contrast, the stars with different sized nuclei provide a variety of good designs. The following example illustrates the benefits of a star over a minimal $(t-1)$-cover of $\mathcal{P}$.

Example 5.7. Consider a 2^7 full factorial experiment where the desired RDCSSs are characterized by $S_1, ..., S_m$, where $|S_i| = 2^4 - 1$ for all i. From Theorem 4.1(b), $|S_i \cap S_j| \geq 2^{8-7} - 1$, for all $i \neq j$. According to Lemma 5.1, the number of distinct 3-dimensional subspaces in a minimal 3-cover of $\mathcal{P}$ is $2^3 + 1$, and Theorem 5.3 implies that the common overlap (say S_0) among all these distinct subspaces is of size 1. To assess the impact of factorial effects on the process, one has to plot m half-normal plots of size 14 each for the effects in $S_i \backslash S_0$, $i = 1, ..., m$, and one half-normal plot of size $(14(9 - m))$ for the effects not contained in any of the RDCSSs. On the down-side, the assessment for the effect in the common overlap has to be sacrificed, and the maximum number of levels of randomization is bounded above by 9. This can be important for constructing fractional factorial designs with 7 *basic factors* and S_i's with $|S_i| = 2^4 - 1$, $i = 1, ..., m$. Instead of using a minimal $(t-1)$-cover, a star $St(\mu, \pi_4, \pi_3)$ can be used to construct up to 15 RDCSSs in a fractional factorial setup. In addition, the size of the common overlap (S_0) is 7, which allows assessment of all the factorial effects in $\mathcal{P}$. The assessment of factorial effects is done by using m half-normal plots of size 8 each for the effects in $(S_i \backslash S_0)$'s, one plot of size 7 for the effects in the overlap, and one half-normal plot of size $(8(15 - m))$ for rest of the effects in $\mathcal{P}$.

In summary, the RDCSSs constructed using minimal $(t - 1)$-covers of $\mathcal{P}$ are forced to have a fixed sized overlap (π_{t-s}), whereas stars provide different sized overlaps for RDCSSs. Furthermore, the number of $(t - 1)$-dimensional rays in a star with nucleus larger than $|\pi_{t-s}|$, is greater than the number of $(t - 1)$-dimensional subspaces in a $(t - 1)$-cover of $PG(t + s - 1, 2)$. More importantly, different size RDCSSs can be constructed using stars, whereas the minimal cover approach focuses on equal size subspaces. Thus, stars support a bigger class of factorial and fractional factorial designs with randomization restrictions.

It turns out that the geometric structure of a minimal $(t - 1)$-cover of $PG(kt +$

$s-1,2)$, for $k>1$, is also related to a balanced star in a particular way. Before going in to the details of the role of a balanced star in a minimal $(t-1)$-cover of $\mathcal{P}$ with $k>1$, it should be noted that we are interested in RDCSSs of size greater than or equal to 2^3-1, i.e., $t\geq 3$. This is required for constructing useful half-normal plots to assess the significance of factorial effects. Under the assumption that there does not exist a $(t-1)$-spread of $\mathcal{P}$, p must be at least 7 (i.e., $k=2, s=1$). This implies that factorial experiments of at least 128 runs are of interest. So far in this chapter, most of the results focused on designs with small run-sizes. Here onwards, the results and discussion are targeted to designs that allow at least 2^7 experimental trials. These designs can be useful for applications where the number of units can be quite large (e.g., microchip industries and microarray experiments).

The next result establishes the relationship between a balanced star $St(\mu,\pi_t,\pi_r)$ and a minimal $(t-1)$-cover of the effect space $\mathcal{P}=PG(kt+s-1,2)$. Although the result holds for any set of positive integers k,t and s, the theorem has useful applications for large factorial designs.

Theorem 5.4. *A minimal $(t-1)$-cover $\mathcal{C}$ of $\mathcal{P}=PG(kt+s-1,2)$, for $k>1$ and $0<s<t$, is a union of $2^s\left(\frac{2^{kt}-1}{2^t-1}-1\right)$ disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$ and a star $St(2^s+1,\pi_t,\pi_{t-s})$ contained in $\mathcal{P}$.*

Proof: From the construction shown in Section 4.2.2, the effect space $PG(p-1,2)$, for $p=kt+s$, can be written as a disjoint union of $2^s\,\frac{2^{kt}-1}{2^t-1}-2^s$ disjoint $(t-1)$-dimensional subspaces and a $(t+s-1)$-dimensional subspace $\mathcal{U}$ contained in $\mathcal{P}$. From Theorem 5.2, there exists a star $St(\mu,\pi_t,\pi_{t-s})$ contained in $\mathcal{U}$, that is also a cover of $\mathcal{U}$. Since the maximum number of rays in this star is $\mu=(2^{t+s}-2^{t-s})/(2^t-2^{t-s})=2^s+1$, all the disjoint $(t-1)$-dimensional subspaces and the star $St(2^s+1,\pi_t,\pi_{t-s})$ constitutes a minimal $(t-1)$-cover of $\mathcal{P}$ (Lemma 5.1). $\square$

Theorem 5.3 is a special case of this theorem. Since the common overlap among the non-disjoint elements of $\mathcal{C}$ is a $(t-s-1)$-dimensional subspace, if $t-s=1$ for a full factorial design, one can assign a higher order interaction to the effects in the overlap and assume it to be negligible. In a regular fractional factorial design, or a full factorial design with $t-s=2$, one would not want to sacrifice the assessment of all the factorial effects in the overlaps. In fact, the assessment of other factorial effects can also be affected (see Example 5.1). To avoid this problem, we propose a similar structure to a $(t-1)$-cover but not minimal.

If the star $St(2^s+1, \pi_t, \pi_{t-s})$ in a minimal $(t-1)$-cover $\mathcal{C}$ is replaced by a star with larger nucleus, the number of disjoint subspaces may decrease. However, the size of the overlap among the non-disjoint subspaces will become large enough for the assessment of all the factorial effects in $\mathcal{P}$. We call this a *modified minimal* $(t-1)$-*cover* of the effect space $\mathcal{P}$. In addition to the ability of assessing the significance of more factorial effects, replacement of the star in a minimal $(t-1)$-cover by a star with bigger nucleus increases the total number of $(t-1)$-dimensional subspaces. This can be used to construct more RDCSSs if required.

Consider a 2^7 factorial setup with minimal 2-cover. For instance, in Example 5.1, $\mathcal{U}$ is a 3-dimensional subspace of $\mathcal{P}$, and thus the overlap between any pair of 2-dimensional subspaces contained in $\mathcal{U}$ is at least $2^{3+3-4}-1$ (Theorem 4.1). The size of the overlap for this minimal 2-cover cannot be increased, because the dimension of any ray is one more than the dimension of the nucleus. Thus, we have to consider $t=4$ instead of $t=3$ to gain the advantage of a modified minimal $(t-1)$-cover. Lemma 4.1 guarantees the existence of a 3-spread of $PG(7,2)$. Since this chapter focuses only on the case when $(t-1)$ does not divide $(p-1)$, we are not discussing the $t=4$ case. Moving up the ladder, if we consider a factorial setup with $p=9$ and $t=4$, a minimal 3-cover consists of 33 disjoint 3-dimensional subspaces and a star $St(3, \pi_4, \pi_3)$. The effects in the common overlap (or nucleus) for this case can easily be assessed using one half-normal plot because the overlap contains 7 factorial effects. Thus, there is no need for improvement. The importance of the modified minimal $(t-1)$-cover over a minimal $(t-1)$-cover becomes apparent for the first time in a 2^{10} factorial setup. A minimal 3-cover of the corresponding effect space $\mathcal{P}$ consists of 65 disjoint 3-dimensional subspaces and a star $St(5, \pi_4, \pi_2)$. If we use a star $St(7, \pi_4, \pi_3)$ instead of a star $St(5, \pi_4, \pi_2)$, the resulting geometric structure is not a minimal 3-cover but allows the assessment of all the factorial effects in $\mathcal{P}$.

Note that the new proposed design may not be very useful for experiments in say the auto industry or chemical industries. These designs have potential applications in microchip industries or perhaps microarray experiments where the number of units can be quite large. The availability of large numbers of trials (or points in $\mathcal{P}$) allows construction of different designs. In the next section, we propose one such structure called a finite galaxy. A finite galaxy is a collection of disjoint stars with some useful statistical properties. As an alternative to a modified minimal $(t-1)$-cover, we propose finite galaxies for constructing full factorial and regular fractional factorial designs where $|\mathcal{S}|$ is large. Although the results proposed in the next section focus on balanced stars, they are easily extended to unbalanced stars.

5.2.3 Finite galaxies

In this section, we first establish the necessary and sufficient conditions for the existence of a maximal set of disjoint stars. This provides a set of $(t-1)$-dimensional subspaces that can be relatively larger than the one obtained from a modified minimal $(t-1)$-cover of $\mathcal{P}$. Then, an algorithm is developed for constructing these sets of disjoint stars. We define a finite galaxy to be a collection of stars with specific properties.

Definition 5.5. *A finite galaxy $\mathcal{G}$ is a set of disjoint stars contained in the effect space $\mathcal{P} = PG(p-1, 2)$ that covers $\mathcal{P}$.*

A finite galaxy $\mathcal{G}$ is said to be *homogeneous* if all the stars in $\mathcal{G}$ are geometrically equivalent (Definition 5.4). All the stars in a finite galaxy are assumed to be balanced. Denote a homogeneous finite galaxy $\mathcal{G}$ by $\mathcal{G}(\nu, t^*-1, t-1)$, where $\nu = (2^p-1)/(2^{t^*}-1)$ is the number of disjoint stars with $(t-1)$-dimensional rays and $(r-1)$-dimensional nuclei for suitable positive integers $r < t$ and $t^* \leq p$. Each star $St(\mu, \pi_t, \pi_r)$ in $\mathcal{G}(\nu, t^*-1, t-1)$ is assumed to be a $(t-1)$-cover of $PG(t^*-1, 2) \subset \mathcal{P}$. As expected, the existence of such a geometry is not so trivial, and requires verification of a necessary and sufficient condition. The following result establishes the existence of a homogeneous finite galaxy that is also a $(t-1)$-cover of the effect space $\mathcal{P}$.

Theorem 5.5. *There exists a homogeneous finite galaxy $\mathcal{G}(\nu, t^*-1, t-1)$ in $\mathcal{P} = PG(p-1, 2)$ with $\nu = (2^p-1)/(2^{t^*}-1)$ disjoint stars if and only if there exists positive integers t and t^* such that $t < t^* \leq \frac{p}{2}$ and t^* divides p.*

Proof: Suppose there exists a homogeneous finite galaxy $\mathcal{G}$ that spans the effect space $\mathcal{P}$, then the number of disjoint stars in $\mathcal{G}$,

$$\nu = \frac{|PG(p-1,2)|}{|St(\mu, \pi_t, \pi_r)|}$$

is an integer. Since every star $St(\mu, \pi_t, \pi_r)$ is a $(t-1)$-cover of $PG(t^*-1, 2) \subset \mathcal{P}$ for some $t < t^* \leq p$, $|St(\mu, \pi_t, \pi_r)| = |PG(t^*-1, 2)|$, and thus ν is equal to $(2^p-1)/(2^{t^*}-1)$. Furthermore, $(2^p-1)/(2^{t^*}-1)$ is an integer if and only if t^* divides p. Consequently, $t^* \leq p/2$ and hence the existence of desired positive integers t and t^*.

On the other hand, if there exists positive integers t and t^* such that $t < t^* \leq p/2$ and t^* divides p, then there exists a (t^*-1)-spread of $\mathcal{P}$ (Lemma 4.1). From Theorem 5.2 and Corollary 5.1, there exists a star $St(\mu, \pi_t, \pi_r)$ in $PG(t^*-1, 2)$ for at least one choice of r. Hence, the existence of a finite galaxy $\mathcal{G}(\nu, t^*-1, t-1)$ is established. □

For constructing large factorial and fractional factorial designs, use of a homogeneous finite galaxy instead of a modified minimal $(t-1)$-cover can sometimes be more advantageous. Recall that for constructing a minimal $(t-1)$-cover, one has to search for collineation matrices in a recursive manner. Instead, the construction of stars is relatively straightforward and does not require any search for finding collineation matrices. For constructing RDCSSs, the number of subspaces obtained from a homogeneous finite galaxy can be much larger than from a minimal $(t-1)$-cover of $\mathcal{P}$. The following example illustrates the difference between the two geometries.

Example 5.8. Consider a 2^{15-5} regular fractional factorial design with blocked split-lot structure. Let the RDCSSs be defined by S_i, $i = 1, ..., m$, where $|S_i| = 2^4 - 1$ for all i. Here, the number of base factors p is 10, and the size of each RDCSS is $2^4 - 1$. Since $t = 4$ and $t^* = 5$ satisfy the conditions in Theorem 5.5, there exists a homogeneous finite galaxy $\mathcal{G}(\nu, 4, 3)$. There exists $\nu = \frac{2^{10}-1}{2^5-1} = 33$ disjoint stars, where every star $St(\mu, \pi_4, \pi_r)$ is contained in a $PG(4, 2)$ of $\mathcal{P}$. These stars constitute a $(t^* - 1)$-spread of $\mathcal{P}$. From Theorem 5.2, there exists a star $St(\mu, \pi_4, \pi_r)$ in $PG(4, 2)$ if and only if $(4 - r)$ divides $(5 - r)$. That is, there exists only one geometrically distinct balanced star, given by $r = 3$. The number of rays in each star is $\mu = (2^{5-3} - 1)/(2 - 1) = 3$. As a result, up to $\mu \cdot \nu = 99$ distinct RDCSSs of size 15 each can be constructed using this galaxy. The size of overlap for any pair of intersecting RDCSSs is 7, which is the same as the size of the nucleus of a star $St(3, \pi_4, \pi_3)$.

The size of a minimal 4-cover in a 2^{10} factorial layout is 69, and if modified by a star $St(7, \pi_4, \pi_3)$ instead of a star $St(5, \pi_4, \pi_2)$, the size of the modified minimal 4-cover obtained would be 71. A total of 99 subspaces are obtained using a homogeneous finite galaxy in Example 5.8. Therefore, if the number of RDCSSs required by the experimenter is large, a finite galaxy can be more useful.

Even though the construction of stars is straightforward and does not require searching for collineation matrices, the construction of a finite galaxy satisfying the experimenter's requirement involves constructing a $(t^* - 1)$-spread of $\mathcal{P}$. Since the spread construction technique shown in Section 4.2.1 often requires transformation of $\mathcal{P}$ to get the desired design, the construction of a finite galaxy may involve relabelling of columns of the model matrix (or equivalently, the points of $\mathcal{P}$).

Construction 5.2. Recall that the existence of a finite homogeneous galaxy $\mathcal{G}(\nu, t^* - 1, t - 1)$ assume that t and t^* satisfy (a) $t < t^* \leq p/2$, and (b) t^* divides p. The following steps can be used to construct a $\mathcal{G}(\nu, t^* - 1, t - 1)$.

1. Construct a $(t^* - 1)$-spread $\mathcal{S}$ of $\mathcal{P}$ using the methodology shown in Section 4.2.1. Define $\mathcal{S} = \{S_1, ..., S_\nu\}$.
2. Set $i = 1$.
3. Construct a star $\Omega_i = St(\mu, \pi_t, \pi_r)$ such that $\Omega_i \subset S_i$, and Ω_i is a cover of S_i.
4. Stop if $i = \nu$, otherwise assign $i = i + 1$ and go to Step 3.

Certainly, the experimenter has some control over the assignment of factorial effects in the RDCSSs that come from the construction of ν disjoint stars. However, the construction technique shown in Section 4.2.1 for a $(t^* - 1)$-spread distributes all the main effects evenly among the elements of the spread. This feature is not desirable in many cases. As a result, one may need to use a collineation matrix to relabel the columns of the model matrix, or equivalently the points of $PG(p - 1, q)$, to get the desired design. The following example illustrates the algorithm for constructing a homogeneous finite galaxy.

Example 5.9. Consider a 2^{10-4} fractional factorial design with m stages of randomization. The corresponding base factorial design has 6 basic factors. Since $t^* = 3$ and

$t = 2$ satisfies the necessary conditions in Theorem 5.5, the existence of a homogeneous finite galaxy $\mathcal{G}(9, 2, 1)$ is guaranteed. The 2-spread obtained from the construction method described in Section 4.2.1 provides $S_1, ..., S_9$ (shown in Table 4.2). A star $St(3, \pi_2, \pi_1)$ is then constructed in each S_i. Although the 2-spread is pre-specified, when constructing these stars, one can select the effects that are common in multiple RDCSSs. A realization of the homogeneous finite galaxy obtained from this is shown in Figure 5.1.

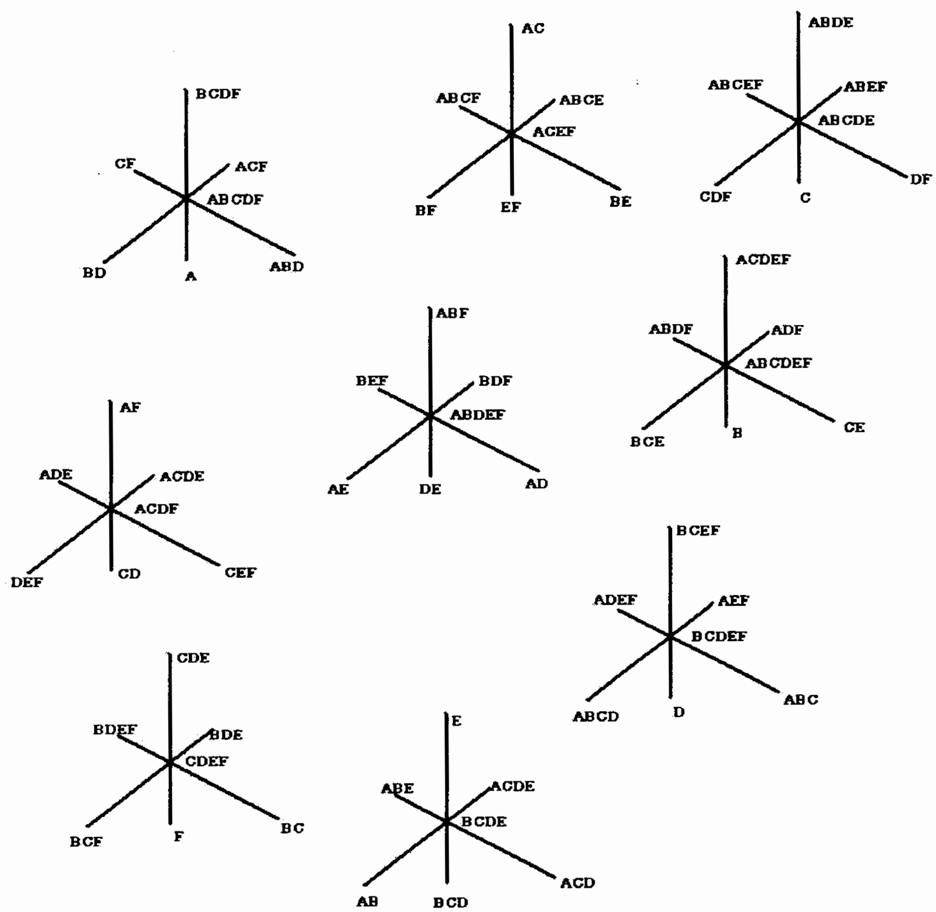

Figure 5.1: A homogeneous finite galaxy $\mathcal{G}(9, 2, 1)$ in $PG(5, 2)$.

The algorithm proposed in Section 4.2.1 can be used for finding an appropriate collineation matrix for transforming the 2-spread $\mathcal{S} = \{S_1, ..., S_9\}$, or equivalently, the finite galaxy constructed using $\mathcal{S}$. Nonetheless, one must remember that at most p independent relabellings can be done for the transformation of the projective space $PG(p-1, 2)$. Thus, one should use the flexibility in the construction of stars to get a good design. For instance, in Example 5.9 the nuclei of all the stars is the largest possible interaction in each star.

5.3 Discussion

Though the existence results discussed in this chapter focus on two-level factorial designs, all the results and their proofs can be generalized to q levels simply by replacing $PG(p-1, 2)$ with $PG(p-1, q)$. For example, in Theorem 5.2, there exists a star $St(\mu, \pi_t, \pi_r)$ with $\mu = (q^{p-r}-1)/(q^{t-r}-1)$ rays in $PG(p-1, q)$ if and only if $(t-r)$ divides $(p-r)$. In addition, the construction of a star $St(\mu, \pi_t, \pi_r)$ in $PG(p-1, q)$ is also similar to the one shown for the $q = 2$ case in Construction 5.1.

In short, for assessing the significance of effects in factorial designs with small run-size or fewer RDCSSs, stars are more efficient than minimal $(t-1)$-covers. In experiments with large two-level full factorial or regular fractional factorial designs, one should either use a modified minimal cover, or a finite galaxy depending on the requirements of the experiment. The results proposed for the existence and construction of finite galaxies focus on the homogenous balanced stars. However, the existence results can easily be extended to the *heterogeneous* case where stars are not necessarily geometrically equivalent. These results are also adaptable to the homogeneous case with unbalanced stars. The algorithms described in Constructions 5.1 and 5.2 can also be extended for both of these cases.

For example, consider a 2^{15-5} fractional factorial experiment with m stages of randomization $S_1, ..., S_m$, where $|S_i| \geq 7$. Let $\mathcal{P}$ be the effect space for the corresponding base factorial design. For $t^* = 5$, there exists a $(t^* - 1)$-spread $\mathcal{S}$ of $\mathcal{P}$ with

$|\mathcal{S}| = 33$. Distinct stars can be used to cover each element of $\mathcal{S}$. Since the desired RDCSSs must contain at least 7 factorial effects, we will focus on stars with at least 2-dimensional rays. Following the notation in Theorem 5.2, the options for balanced stars are $St(3, \pi_4, \pi_3)$, $St(5, \pi_3, \pi_1)$ and $St(7, \pi_3, \pi_2)$. The geometric structure of these stars is shown Figure 5.2.

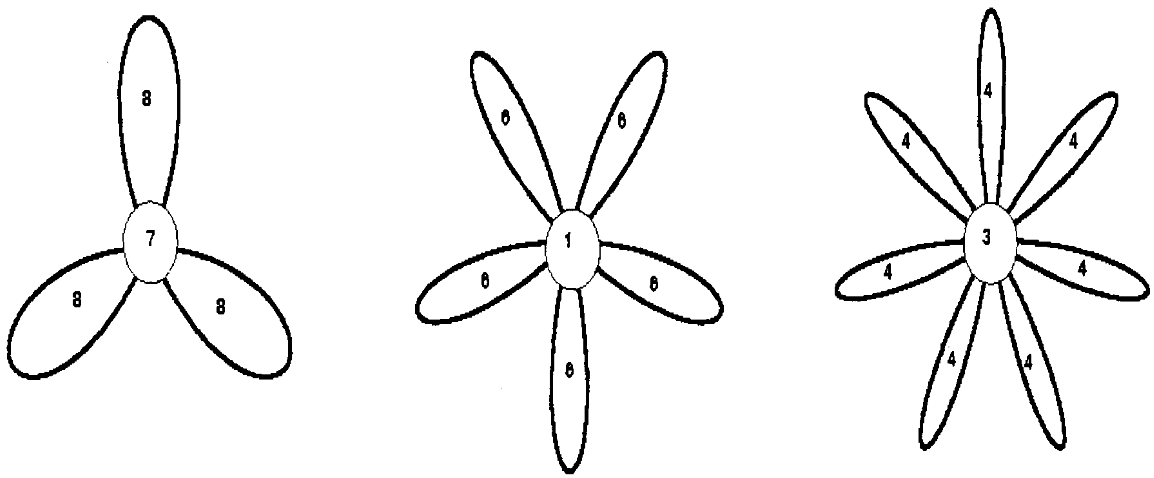

Figure 5.2: Balanced stars; The numbers $\{1, 3, 4, 6, 7, 8\}$ represent the number of effects in the rays and the common overlap.

Due to limitation of the space, the factorial effects are not explicitly written in the figures displayed here, and therefore have different representations than the one used for Figure 5.1. The star on the left is a $St(3, \pi_4, \pi_3)$ with a common overlap of size 7, the one in the middle is a $St(5, \pi_3, \pi_1)$ with the overlap of size 1, and the star on the right represents a $St(7, \pi_3, \pi_2)$. Recall that a useful half-normal plot requires more than six or seven factorial effects. If a star in the finite galaxy is a balanced star $St(5, \pi_3, \pi_1)$, one would have to sacrifice the assessment of only one factorial effect per such star. If the star $St(7, \pi_3, \pi_2)$ is used for constructing a finite galaxy, none of the effects can be assessed. This turns out to be the worst case among all three options. In conclusion, for this particular example, the two stars $St(3, \pi_4, \pi_3)$ and $St(5, \pi_3, \pi_1)$ seem to be the better choices for constructing a finite galaxy.

Chapter 6

Summary and Future Work

Two-level full factorial and regular fractional factorial designs have played a prominent role in the theory and practice of experimental design. In the initial stages of experimentation, these designs are commonly used to help assess the impact of several factors on a process. Ideally one would prefer to perform the experimental trials in a completely random order. In many applications, restrictions are imposed on the randomization of experimental runs. This thesis has developed general results for the existence and construction of designs with randomization restrictions under the unified framework first introduced by Bingham et al. (2006).

Results for the linear regression model are developed in Chapter 3 that express the response model for factorial designs with different randomization restrictions under the unified framework. Under the assumptions of model (3.1), the main result of this chapter (Theorem 3.3) demonstrates how the distribution of an effect estimate depends upon its presence in different RDCSSs. This in turn motivates one to find disjoint subspaces of the effect space $\mathcal{P}$ that can be used to construct RDCSSs.

Though preferred, the existence of a set of m disjoint subspaces of the effect space $\mathcal{P}$ may not be possible. In Chapter 4, conditions for the existence of a set of disjoint subspaces of $\mathcal{P}$ are derived. In the general case, Theorem 4.4 presents a sufficient condition for the existence of a set of disjoint subspace of different sizes. These subspaces are then used to construct RDCSSs of both equal and unequal sizes that are often needed by the experimenter. The designs obtained here are specifically useful to practitioners as the construction algorithms are also developed.

When the existence conditions for a set of disjoint subspaces are violated, overlap among the RDCSSs cannot be avoided. Since the assessment of factorial effects on a process is the objective of the experimentation, in Chapter 5, we propose designs that allow for the assessment of significance of as many effects as possible. The design strategies (stars and galaxies) proposed in this chapter use the overlap among different RDCSSs as an advantage, which seemed like a problem using the minimal $(t-1)$-cover approach. The existence conditions are proposed for balanced stars, unbalanced stars and finite galaxies. Significantly, construction algorithms are developed for the designs obtained from stars and galaxies. The experimenter has more control on the construction of these designs compared to the construction developed in Section 4.2. Since the designs obtained using finite galaxies are typically big, one might question the usefulness of such designs in practice. Note that the large designs may be uncommon in full factorial and fractional factorial designs if the trials are performed in a completely random order. If randomization restrictions are imposed on the trials, large designs are useful in many applications (e.g., Vivacqua and Bisgaard, 2004; Jones and Goos, 2006; Jones and Goos, 2007).

There are a few additional issues that require further mention. Firstly, the designs used in this dissertation for illustrating both the existence results and construction algorithms are all two-level full factorial and regular fractional factorial designs. The existence results and their proofs in Chapters 4 and 5 can be easily generalized to q levels by replacing $PG(p-1,2)$ with $PG(p-1,q)$ and some minor modifications. In addition, the construction of a $(t-1)$-spread of $PG(p-1,q)$ is similar to the $q=2$ case shown in Section 4.2.1. The construction of stars and galaxies are also generalizable to q-level factorial designs, where $q>2$. However, there are some results that may be non-trivial to establish. For example, the results developed in Chapter 3 use the properties of Hadamard matrix representation of the model matrix X. To establish similar results for the distribution of the effect estimates in q-level full factorial and regular fractional factorial designs, one may have to use some of the results on more general orthogonal arrays.

Secondly, the results developed for the distribution of effect estimates assume that the underlying designs are full factorial and regular fractional factorial designs. If one considers some non-regular designs, we cannot use the geometric structure of a full factorial design to categorize the factorial effects into sets of effects having equal variance for performing half-normal plots. To understand the complexity of the problem it is worth noting that there does not even exist a corresponding *base factorial design.* Moreover, the results on the distribution of effect estimates developed in Chapter 3 may not hold either. For instance, it is unlikely that the two estimators OLS and GLS of regression coefficients β are equal. Under these circumstances, one has to work with the GLS estimator which requires the inversion of the covariance matrix Σ_y. It turns out that the inverse of Σ_y can be written in a closed form, conditional on some assumptions on the overlapping pattern among RDCSSs.

The result developed in Theorem 5.1 only provides a necessary condition for the existence of an unbalanced star. The sufficiency condition for the existence needs further exploration. However, considering the nature of the necessary and sufficient condition for a balanced star (Theorem 5.2), one suspects that the sufficiency of an unbalanced star $St(\mu_1, ..., \mu_k, t_1, ..., t_k, \pi_r)$ should depend on "$g(t_1 - r, ..., t_k - r)$ divides $(p - r)$", for some function g. It is expected that once the existence of an unbalanced star is established its construction should be fairly straightforward.

Furthermore, the results developed for finite galaxies (Section 5.2.3) focus on homogeneous stars. The necessary and sufficient conditions for the existence of a heterogeneous galaxy requires further investigation. Stars are specifically useful to the practitioner because of their easier construction.

Finally, construction algorithms for both overlapping and disjoint subspaces of equal and different sizes are proposed. One of the important steps of these algorithms is to transform a set of disjoint subspaces (often a $(t-1)$-spread of the effect space $\mathcal{P} = PG(p-1,q)$) to another set of disjoint subspaces such that the transformed set has the features of the desired design. Starting with the $(t-1)$-spread obtained from the cyclic construction method (Section 4.2.1), it is possible that none of the collineation matrices lead to the desired set of subspaces. This does not imply that the experimenter's requirement is impossible to meet. This occurs when the two spreads (the one we started with and the one we are searching for) are non-isomorphic, and thus the desired spread cannot be obtained by a linear transformation. Consequently, a lurking mathematical problem is to find all non-isomorphic spreads, or if easier, one can first find all possible spreads and then use collineation matrices to filter out the isomorphic ones. In the special case of $t = p/2$, some results are known for the complete classification of spreads (e.g., Dempwolff 1994).

The set of all non-isomorphic $(t-1)$-spreads of $PG(p-1,q)$ is also required for finding regular fractional factorial designs that are optimal under different criteria, such as minimum aberration (Fries and Hunter, 1980), maximum number of clear effects (Chen, Sun and Wu, 1993; Wu and Chen, 1992) and the V-criterion (Bingham et al., 2006). Traditionally, some of the commonly used good designs have been catalogued for the convenience of practitioners. To provide such a catalogue for fractional factorial designs with different randomization restrictions, one needs to find all possible designs and then rank them using the desired criterion.

As an alternative, one might consider the search table approach developed in Franklin and Bailey (1977) which can be generalized to generate candidate designs in our setting. The sequential updating approach developed in Chen, Sun and Wu (1993) can be used to avoid an exhaustive search. The use of these two approaches to more efficiently construct a catalogue of fractional factorial split-plot designs is shown in Bingham and Sitter (1999). These algorithms require isomorphism checks for a candidate design. It turns out that the isomorphism check is computationally expensive, and efficient algorithms have been developed to improve the efficiency of the isomorphism check algorithm (e.g., Clark and Dean, 2001; Lin and Sitter, 2006). Furthermore, the RDCSS structure can be used to shorten the candidate designs and generalize the isomorphism check algorithm for fractional factorial designs with different randomization restrictions. Future work will focus on developing an efficient isomorphism check algorithm for generating the set of all non-isomorphic fractional factorial designs for specific randomization structures.

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