

**A CASE STUDY OF A COMPUTER-BASED LABORATORY  
COURSE IN CALCULUS AT DALAT UNIVERSITY**

by

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**THESIS SUBMITTED IN PARTIAL FULFILLMENT OF  
THE REQUIREMENTS FOR THE DEGREE OF  
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in the Faculty

of

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## **Abstract**

In recent years, computers have begun to appear quite frequently in Vietnamese universities. But the development of computing technology is especially concentrated on serving teachers and students of computing science. Computer technology has not been applied extensively for study and research in other areas such as mathematics, physics, chemistry or biology. There are not many studies and evaluations involving computers in the mathematics and science education communities in Vietnam.

This study investigates how students learn mathematics in a computer aided instructional environment. The study involved designing a computer-based calculus course for students in the First-Year Basic Science. A sample of six first-year students of Dalat University used computers during six sessions for five weeks as part of this laboratory course. Participant observation, students' lab papers and interviews were used to create descriptions and explanations of how students learned mathematics using computers. Two questionnaires were distributed to students to investigate their attitudes toward mathematics and the computer.

The study reveals that computers help students to reinforce mathematical concepts, and enrich the lectures they experience in class. Communicating with the computer seems to motivate students to learn new mathematical concepts. The computing environment enables students to develop skills of problem solving and computation and to improve abilities of mathematical intuition and communication. Graphical features of the computer help students to visualize mathematical relations, and approach some concepts. Students believed that the computer should be used in schools, that it would help students to learn mathematics better and develop positive attitudes toward mathematics and computers.

The study suggests that, in the design of mathematics curricula with a computer-based approach, concept presentation should be emphasized. In order to help students develop reasoning and problem solving skills, teachers need to require students to explain

and evaluate the mathematical reasonableness of the solutions given by the computer. In addition, the curriculum also should help students to develop awareness of their learning techniques in a computer-assisted environment.

## **Dedication**

**For my parents.**

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# **Chapter 1**

## **Introduction**

Following the reunification of Vietnam in 1975, massive changes have occurred in the country's educational system. The first five years after reunification were also marked by a severe shortage of well qualified people in the general work force and poor economic conditions, and there were very few resources for education (World Education Services, 1994, p. 6). The government undertook a massive national effort to alleviate illiteracy with considerable success, yet there remain some persistent problems in the remote mountain areas and in the Mekong Delta area, where there are inadequate schools and a shortage of teachers.

At this time there were three separate ministries controlling education in Vietnam. The Ministry of Education controlled basic and secondary education, the General Department for Vocational Training administered vocational and trade training, and the Ministry of Higher and Secondary Technical Education was in charge of universities, colleges, and secondary technical education. In 1987, the Secondary Technical and Vocational Training Ministries came together to form the Ministry of Higher Technical and Vocational Training. Finally, in 1990, this and the Ministry of Education combined to form the Ministry of Education and Training (Bo Giao Duc va Dao Tao).

In an effort to improve the national education system, the Ministry of Education and Training has undertaken some educational reforms. One of the notable reforms is the development of a plan in 1990 to restructure post-secondary education in Vietnam. The broad intention of this plan is to establish a "University Credit System," which includes two stages of work for undergraduate students. After a time of discussions, amendments and revisions that lasted until 1995 this plan was implemented in universities in Vietnam.

At present, the undergraduate degree programs of the Vietnamese universities follow the credit system and consist of two stages. In the first stage, students take a core

curriculum made up courses in humanities, languages, arts, sciences and mathematics. At the end of the second year, students must take a selection examination before proceeding on to the second stage, which consists of two or three years of study in a specialization. At the end of the fourth or fifth year, all students must take a national examination, with the exception of those very bright students chosen to write a thesis instead. For undergraduate students of science, the first stage is seen as a “Basic Science” component, offered at both community colleges and universities according to the policy of increased access to higher education through the community colleges, while the University Credit System enables further study of science and specialization in applied contexts.

While the broad purpose for the restructuring is to increase access to basic science education, and, therefore, to improve the scientific and technological literacy of the Vietnamese citizenry, there are many problems and conditions that help to shape the specific nature of the impending reforms. Over the past twenty years, the country has been somewhat isolated from the professional science and science-mathematics education communities; textbooks and teaching methods reflecting current understandings in science and mathematics are lacking. The condition of teaching laboratories and the technical tools for teaching are very poor in most areas of the country. These, in part, have led to a rather “rhetorical” science and mathematics education, that is, one which relies heavily on rote memorization and routinization, at the expense of “broad and deep understanding” of the subject matter - the fundamental principles of science. Frequently, the need to learn by memorization is exacerbated by the lack of practical, concrete laboratory and applied activities that are relevant and motivating for students. This is seen to be one of the main reasons why many students do not show much interest in learning science and mathematics.

University science and mathematics education reform requires close analysis in terms of the purposes, methods and varying approaches in Vietnamese universities. There is a need to develop a program of studies in science and mathematics that reflects the true

nature of the scientific enterprise, the role of mathematics in scientific study, as well as the interrelations among science, mathematics, technology, and society.

Recent revisions in the science and mathematics curricula for the basic science program in Vietnam include some resequencing and regrouping of topics, but, the general approach to science curriculum development and pedagogy remains unchanged. This state of affairs seems to run parallel to the situation in North America in the 1980's. When analyzing the sources of reform issues, for example, Paul DeHart Hurd (1986) states that,

The central problem is the gap between the existing school curriculum in science and demands of living in a scientifically and technologically driven economy. (p. 354)

Basic science and mathematics education in Vietnamese universities has been based on textbooks which are out-of-date, and which have ignored relationships between science, mathematics and society, particularly in terms of technological applications.

Certainly, there is a need to develop students' awareness of the interaction of science, mathematics and technology in bringing about social change in Vietnam. This is due in part to a need to present an authentic view of science. As Hurd writes,

Science and technology have become an integrated system for research and development, indistinguishable in their methodologies. (p. 354)

Derek de Solla Price, a philosopher of science (quoted in Hurd, 1986, p. 356), agrees with this appraisal of science in his description of modern science as "applied technology." He says that the reform of science and mathematics education must allow the curriculum to evolve in a manner that allows students to understand the nature and culture of modern science.

In the Western models of education, there is a great deal of research on the use of technology as a teaching tool in science and mathematics classrooms. This research has involved the role of technological tools in science and mathematics education. However, such studies are still relatively new for science and mathematics education communities in

Vietnam. There is a need for thoughtful evaluations of the applications and effects of technology in science and mathematics education in Vietnamese universities.

Bates (1992) suggests that there is a need “to prepare students for a technological society” (p. 10), and therefore that technology should be used both as content (the object of study), and for the development of skills in using technology. In addition, he states “technology can enable teachers and students to do things that would be difficult or impossible without the technology” (p. 10). He also claims that

Well-designed technology-based learning materials can enable students to learn more quickly, or teach knowledge or skills that would be difficult or extremely time-consuming for an individual teacher. Also, well-designed learning materials can be fun and hence motivate learners (p. 10).

Kozma and Johnston (1991), of the National Center for Research to Improve Postsecondary Teaching and Learning (NCRIPAL) in the U.S., believe that “the computer has launched a revolution in learning and teaching in higher education” (p. 19).

Computing technology has already had marked influences in mathematics and science education. The Conference Board of the Mathematical Sciences (1983) discusses some of these influences. The report states that “computing technology may require the redefinition of what is fundamental mathematics and there are new views of what it means to think and learn mathematics and science” (quoted in Shumway, 1986, p. 121).

Shumway (1986) concludes from his own studies of computer applications in science and mathematics education that,

We have powerful new tools that will make dramatic changes in mathematics, science and how we learn mathematics and science... There are still many questions, but it is clear that our responsibility is to help the youth of today learn to use these powerful tools to do mathematics and science. (p. 134)

### ***Background to the study***

In recent years, personal computers became more common in Vietnam. Government offices, businesses and the universities have begun to use them, especially in two



applications: word processing and file management in clerical work. In some big cities, such as Ho Chi Minh City, there are a few high schools that have also been equipped with computers.

In education the computer is strongly desired, particularly by young Vietnamese people - students and teachers in all levels of schooling. Besides learning in schools, some students attend Computing Clubs for basic computer classes. Teachers and students in universities have also tried to go to Computing Clubs or Computing Science Centers to learn more about computing on their own. There is great hope that this powerful and exciting tool can help people to learn and study more effectively. The rapidly increasing number of Computing Clubs and Computing Science Centers attests to the interest in computers.

There have been some curriculum projects in university education that have involved computer technology. All students at Dalat University, for example, have been required in the past few years to take some basic courses in computer applications. Before this innovation, these courses were required only by students majoring in Computer Science. Further, Computer Science, originally established as a specialization in the Mathematics Department, is now a separate department. Although there are many difficulties in the design and development of the curricula as well as in the preparation of human resources for this new department, its establishment is generally seen as a good measure that aims at satisfying the study demands of students and the research demands of teachers.

The development of computing technology is especially aimed at serving teachers and students of computing science. Computer technology has yet to be integrated in study and research in other departments such as mathematics, physics, chemistry and biology. This does not mean the teachers who are working in these departments do not know about the computer's usefulness, but rather, that such knowledge is limited to those applications that are currently used such as word processing or file management. There are many

instructors and professors who do not use computers in preparing lectures, in writing course notes, or in evaluating students' marks. It is difficult for students and professors to envision how the computer might be used and how its effects may be understood in scientific activities. As a result, relatively few studies and evaluations have been conducted in the use of computers in mathematics and science education in Vietnam.

As Hurd (1986) states, "school science courses continue to be bounded by a concept of discrete disciplines" (p. 354). Indeed, most professors in Vietnam have not yet developed a profound vision of the interrelations between computing technology and other disciplines. My desire to explore the applications of computers in the teaching of mathematics at Dalat University is driven by my motivation to learn about such applications.

There were many other reasons why I became interested in this problem. Two years ago, there was a chemistry student at Dalat University who came to see me about a problem he had encountered during a chemistry project. It was a mathematical problem that related to linear programming, but he was at a loss for its solution. His chemistry teacher advised him to audit the mathematics course in Optimization Theory, which could take considerable time, especially for one without an advanced background in mathematics. I suggested that he try using a computer spreadsheet program to solve his problem. I spent about ten hours teaching him how run and use the spreadsheet software Lotus 1-2-3 on a personal computer, and fortunately this helped him to finish the project completely and effectively. After that, he told me he was interested in using the computer for modeling chemical problems. He said the process of working with the computer had helped him to understand more deeply and clearly the chemical concepts involved in his project. I wondered why we did not combine computer technology with mathematics and science education to help our students learn.

We had our own problems at Dalat University. One and a half years ago my colleagues in the mathematics department and I had a meeting to discuss problems of lack

of relevance in the curriculum and poor achievement among students in the mathematics department. We analyzed many different factors contributing to this situation and submitted our ideas about possible solutions. I suggested that in addition to teaching mathematics by lecture and textbook, we should try to use computers to provide opportunities for our students to explore different ways for discovering and understanding mathematics. I suggested this measure would make the study of mathematics more interesting as well. Following this meeting, I cooperated with a teacher of Numerical Analysis in guiding tutorial groups in computer programming in selected topics. We observed that students were more lively and much more interested in the study of mathematics. At the same time, working with computers helped students to understand the mathematical contents of the lecture more deeply. On this basis we expect that, with the assistance of computers, students will be more motivated and more effective in their studies.

### ***Purpose of the study***

This thesis examines the use of computers in the teaching of mathematics to First-Year Basic Science students at Dalat University.

The first part of the study deals with designing a computer-based calculus course for students in First-Year Basic Science. The laboratory course was designed to enable students to explore and reinforce mathematical concepts of calculus, develop skills of mathematical reasoning, problem solving and computation, and develop abilities in mathematical intuition and communication in a computer-augmented learning environment.

The second part of the study investigates how students learn calculus with the computer-based laboratory course. A qualitative case study is used to develop an understanding of the nature of students' experiences learning mathematics in computer aided instruction. The way in which the computer-based approach influences students' understanding of mathematics is examined in terms of learning mathematical concepts, developing skills of mathematical reasoning, problem solving and computation; improving

mathematical intuition and communication abilities; and developing positive attitudes toward mathematics and computers.

Implications are derived for the design of a computer laboratory for the Mathematics Department at Dalat University. The advantages and limitations of this design will also be discussed in order to draw reasonable conclusions and implications for the use of computers in mathematics education, particularly in present conditions in Vietnam.

### ***Significance of the study***

It is hoped that this study brings a useful application for mathematics teachers in Vietnam. The study provides a picture of how students can learn mathematics in a computer-based learning environment, and, at the same time, makes visible some effects of using this tool in mathematics education. There is a need for Vietnamese teachers to explore diversifying the teaching-learning process, to consider the effects and impacts of computers and other technologies on mathematics education. There is also a need to assist Vietnamese students in the learning of mathematics. This study is seen as an important part of the educational reform to improve the quality of education in Vietnam.

It is also hoped that this study will pave the way for further studies. The study will encourage the author as well as other colleagues to explore and research different applications of computers in teaching other subjects such as physics, chemistry and biology. I hope in the near future such research and applications will be realized at all levels of education in Vietnam.

### ***Limitations of the study***

This study is devoted to understanding a few students' experiences of learning mathematics with a computer-based approach at Dalat University, Vietnam. It is based on the performance of six lab sessions of the computer-based calculus course with participation of only six students. Participants were not randomly selected; all are first-year

basic science students at Dalat University. The number of participants and their non-random selection preclude generalizing results of the study to the whole population of undergraduate students. The students' learning of mathematics is examined in the framework of the calculus course with the specific software. Interpretations and conclusions drawn from the study, hence, cannot be generalized to other mathematics courses.

### *Organization of the thesis*

Chapter One describes the general topic of the thesis. Some characteristic aspects of education in Vietnam are presented to show the national context in which the study takes place. Some literature is cited to frame the study. Simultaneously, Chapter One also shows that, although some research on the topic has been done in Western countries, the important and basic questions still have not been answered for Vietnam education. My reasons for being interested in this particular topic are given, and lead into an outline of the purpose of the study. The significance of the thesis is also outlined, as are its limitations.

The literature review presented in Chapter Two examines studies that have been done on some related topics including the role of computers in mathematics education, the influences of using computers in mathematics education, and studies of using computers in mathematics teaching in university. The chapter presents the conclusions from those studies in addition to outlining the questions that are given in the thesis.

Chapter Three reports on the methodology of the study. It includes information about the concrete context of the study, the process of designing the Computer-Based Laboratory Course, and the qualitative method used in the study. The theoretical base and the procedure that are applied to design the Lab Course are presented in detail. The selection of subjects and materials, and the features of software for the study are considered. The experiment and data collection are described, and the methods used for analyzing qualitative data are discussed.

Chapter Four consists of analysis and evaluation of data for the study, which are drawn from different sources: my observations, students' papers, interviews and videotaped images of the learning activities of students. The chapter gives interpretations, comments and evaluations of students using computers when learning mathematics as well as their attitudes toward the use of computers in learning mathematics.

The final chapter presents the conclusions, limitations and implications of the study. Specifically, the chapter considers how computers affected students' learning of mathematics, how the students reacted to the use of computers, and how computers should be used to increase effectiveness in mathematics teaching at Dalat University and in mathematics instruction generally. Limitations of interpretation and generalization of the study results are discussed and a listing of suggestions for future investigations is offered.

## **Chapter 2**

### **Review of Related Literature**

A portion of educational literature deals with the interaction between technology and education. Specifically, the effects of computer technology in science education in general, and in mathematics education in particular, are considered and evaluated carefully. The viewpoints and objectives of the use of computers in education are given and discussed. Methods of combining computers with traditional teaching are suggested and evaluated. The quality of educational software has been improved, and new software is being designed to more effectively meet the requirements of educators as well as of learners. It has been suggested that computer technology significantly impacts on our lives and that it will continue to do so at an ever-increasing rate (Reif, 1985; Day, 1987). Mathematics education cannot escape this impact.

The literature presented in this chapter is selected to concentrate on three aspects as follows:

- The roles of computers in mathematics education.
- The effects of using computers in mathematics education.
- Studies of the use of computers in mathematics teaching in university.

#### ***The roles of computers in mathematics education***

At present, computers are used to perform diverse tasks and services in the business world and in homes. Many mathematicians and mathematics educators agree that mathematics education and the societal arena should be integrated, not regarded as separate entities. Thus, there is a need to emphasize roles and objectives of computers in mathematics education.

A noted research study done by Ediger (1989) examined the roles of computers through four schools of thought in the teaching of mathematics.

The first school of thought is essentialism. Essentialism emphasizes a core of subject matter for mastery by all students. It requires that all students master a common set of mathematical goals.

Software stresses the concepts of tutorial (new sequential content for students), drill and practice (review to emphasise recall of subject matter), and games (enjoyable ways of learning the new, as well as reviewing previously presented ideas in mathematics). (p. 3)

The second school of thought about computers in the teaching of mathematics emphasizes problem solving. It advocates students solving life-like problems in mathematics. Flexible steps in problem solving include clarity in problem selection, gathering data to arrive at a tentative solution, developing tentative solutions (hypotheses), testing the hypotheses, and revising these when needed.

Software emphasizing simulation can also be life-like and real. Adequate opportunities are provided students to participate in simulated experiences. Problem solving becomes a major objective when using simulated software content. (p. 4)

The third school of thought about the use of computers in teaching mathematics emphasizes an "idea centered" mathematics curriculum. An idea centered curriculum supposes content in mathematics is learned for its own sake. The broad goal is for students to attain cognitive abilities in mathematics. Mental development of the learner is paramount in guiding students to attain relevant mathematical knowledge and understanding.

Quality software will assist students to achieve vital subject matter in an idea centered mathematics curriculum... Experimenting with computer and appropriate software to explore ideas in mathematics is the heart of an idea centered curriculum... Tutorial software sequentially presents new subject matter in an idea centered curriculum. Drill and practice may be utilized, as the need arises, in an idea centered mathematics curriculum. Simulations and games should be emphasized only if vital mathematics content is emphasized in an idea centered curriculum. (p. 4)

The fourth school of thought stresses a decision making model. It is advocated that students learn to make choices and selections from various alternatives.



The student sequences his/her own learning opportunities. Thus, a psychological, not logical, mathematics curriculum is in evidence. Sequence resides within the student in learning, and not in textbooks, workbooks, or predetermined measurably stated ends... With students making choices pertaining to sequential software packages, diverse kinds of program may be selected, from among alternatives. These include: drill and practice, tutorial, simulation, and games. (p. 4)

In utilization of mathematics software, Ediger (1986) supposes that mathematics teachers are decision-makers. They need to select software so that students may attain specific objectives, and the software chosen must be on the understanding level of students. These objectives are also seen as purposes based upon which mathematics teachers perform selections of software and build corresponding teaching strategies. Ediger analyzed four purposes of using computers in mathematics education including drill and practice, tutorial mode of instruction, games and simulation.

One purpose in utilizing computer instruction is to emphasize drill and practice. There are vital facts that students need to rehearse. A quality mathematics program can be highly effective here... Students tend to forget specific facts unless opportunities for drill and practice are given. Drill and practice programs should increase students' interest in learning. Ample opportunities need to be present in using computers to have students make responses. Feedback given by the computer must be ample so that learners know if they responded correctly.

A second purpose in using computers emphasizes a tutorial mode of instruction. New mathematics content is then presented to students... The order of presentation should harmonize with the student's present level of achievement. Thus, ideas presented on the monitor are sequential from the student's own unique perception...

A third purpose in computer usage is games. Two or more students may interact with a program in mathematics - each person playing to win the game... As each student plays to win for scoring the highest number of points when responding correctly to mathematics content on the screen, motivation for learning is in evidence... Wholesome attitudes toward mathematics and toward each other should be stressed in a game program.

A fourth purpose in computer use stresses simulation. In simulated experiences, students play roles. Two or more students may participate in a simulation program. Choices and decisions are made by learners. Feedback is obtained pertaining to the quality of decisions made in the role playing activity. (pp. 5-6)

Ediger (1986) also discussed the potential of computer managed instruction (CMI) in mathematics education.

With CMI, selected tests may be scored... The mathematics teacher here may analyse the kinds of errors made by students in class. The common errors revealed may become the basic for remedial work for the entire class or a committee. Specific errors made by a few students can be handled in a small group, whereas individual problems in mathematics should be handled on a one to one basis.

With CMI printouts, the mathematics teacher receives feedback on item analysis. Weak test items can then be revised. Easy test items that all or nearly all students get correct may be augmented in difficulty in measuring a more complex goal attainment. (p. 6)

The National Council of Teachers of Mathematics (1989) and the National Research Council (1989) identified problem solving as a primary focus of the mathematics curriculum (Mayes, 1992). On that basis, Mayes writes

The integration of problem solving into the mathematics curriculum requires an effective pedagogy for the instruction of problem solving... The computer environment has received broad attention as another aspect of a pedagogy of the instruction of problem solving. (p. 243)

Hamm and Adams (1988) purport that thinking mathematically requires more than large amounts of exposure to math content. In learning mathematics, students need direct decision making experiences, and the application of mathematics to these decision making experiences is seen to broaden young minds. Application is regarded as the key to understanding mathematics. Mathematical problem solving requires flexibility and resourcefulness, the ability to use knowledge efficiently and understand the rules which underlie domains of knowledge. Hamm and Adams reason that these attributes comprise "mathematical behaviour," that the ways we teach mathematics and the lessons students abstract from their experiences in doing mathematics are equally important in shaping mathematical behaviour. If we want students to learn problem solving skills then we must focus on the development of these skills. According to Hamm and Adams, then, the development of students' abstractions from concrete experiences with math manipulatives is necessary in shaping mathematical problem solving behaviour. Learning is seen to progress along a continuum from concrete to abstract. They suggested that,

The computer is a powerful tool used for the manipulation and animation of abstract representation - from animated concrete, semi-concrete, to animated

abstract explorations. In fact, electronic technology has the potential for making the abstract more concrete. (p. 14)

It is proposed that mathematics teachers now have sufficient computer hardware and software to change their approaches to instruction designed to promote problem solving. New computer software, video technology, and various combinations of the two can be harnessed for problem solving and mathematical learning. Hamm and Adams (1988) write that,

When students are able to see relationships more graphically and witness theory in electronic representation of practice, a deeper understanding of underlying principles emerges.

Empowering students with the ability to visualize abstract concepts helps them relate to the real physical meaning and make applications. Visualization can help students understand and see things they can not in a lecture (helps combine visualization with instruction). Whether used in class, after class, or in a class demonstration the computer makes abstractions more concrete. The animated graphing of functions, working simulated problem sets, demonstrating constructions, adding colour, movement, and manipulation helps bring concepts to an interactive visual level. Mathematics can come alive in ways that textbooks and still frames can not demonstrate. (p. 15)

It could be said that computer technology has the potential to breathe life into mathematical abstractions in ways that sharpen and, perhaps, begin to redefine the teaching and learning of mathematics. Computer control can bring mathematical abstractions alive in a more dynamic way than static charts or formulas. Hamm and Adams (1988) believe that,

If mathematics teachers can learn to use these extraordinary tools then students are more likely to find joy in mathematics instruction. (p. 15)

The literature represented above strongly supports the use of computer as the new tool of technology in mathematics education. Yet, there is a clear need to investigate the effectiveness of computer technologies and to compare their effects with traditional teaching.

### ***The effects of using computers in mathematics education***

In considering the possibilities for computer technologies in mathematics education, Blum and Niss (1991) observe that,

New possibilities have become available for making mathematical contents accessible to learners, for advancing the acquisition of mathematics concepts, for promoting the intended aims, for relieving mathematics teaching of some tedious activities and thus making it more efficient. (pp. 57-58)

They suggest aspects that could emerge in process of using computers in mathematics instruction:

- More complex applied problems with more realistic data become accessible to mathematics instruction at earlier stages and more easily than before.
- The relief from tedious routine makes it possible to concentrate better on the applicational and problem solving processes, and thus to advance important process-oriented qualifications with learners.
- Problems can be analysed and understood better by varying parameters and studying the resulting effects numerically, algebraically or graphically (corresponding to the so-called operative principle: "What happens, if ... ?")
- Problems which are inaccessible from a given theoretical basis, for instance by being too complex or mathematically too demanding for a given age group, may be simulated numerically or graphically. (p. 58)

Blum and Niss also argue that the computer has implications not only for methods of instruction, but for goals and content of curriculum as well.

- Routine computational skills are becoming increasingly devalued, whereas problem solving abilities such as building, applying and interpreting models, experimenting, simulating, thinking algorithmically or performing computational modelling are becoming re-emphasized.
- New types of content which are particularly close to applications can be treated more easily now, e.g., difference and differential equations or data analysis at the upper secondary level, and statistics, optimization, dynamic systems or chaos theory at the tertiary level.
- By virtue of various relieving effects produced by computers, there is simply more room for modelling and applications in the curriculum.
- It is possible and because of their increasing relevance in the application practice also necessary to deal with such applied problems that necessitate the use of computers to a considerable extent. (p. 58)

A study of the influence of problem solving computer software on the attitudes of high school mathematics students toward mathematics was conducted by Funkhouser

(1993). The 40 participants in the study were enrolled in a geometry or second-year algebra course in a public high school. A rigorous schedule of computer-based and noncomputer-based student activities was developed. The students were given the National Assessment of Educational Progress (NAEP) and a skills-based test of problem solving ability developed by Mayer et al. (1986), before and after instruction that was augmented by computer software. The computer applications Building Perspective™ and Blockers and Finders™ were chosen because they were designed to develop problem solving abilities. Building Perspective challenged students to construct a view of a group of buildings from above. Blockers and Finders challenged the students to find a variety of obstacles in a maze. Previous piloting and research with these programs suggested that they were especially well suited to develop problem solving ability (Funkhouser, 1990). These researchers concluded that

Students who use problem solving software tend to develop a more positive view of their own mathematical abilities and a more positive disposition toward mathematics as a subject. (p. 345)

This project further suggested that,

Augmenting regular mathematics courseware with computer software to enhance attitudes toward mathematics does not have an “opportunity cost” for content mastery or other mathematics skills; on the contrary, such software appears to enhance both content mastery and problem solving ability to a significant degree. (p. 345)

Research reports represented thus far in this review seem very positive and promising. However, it should be recognized that computers may also present problems and risks. Blum and Niss (1991) give some indication of these:

- Arithmetic and geometric skills and abilities of learners may atrophy, though they are still indispensable, also for applied problem solving and real world applications.
- The devaluation of routine skills which hitherto have helped students to pass tests and examinations will make mathematics instruction more demanding for all students and too demanding for some of them, for proper problem solving and modelling are ambitious activities, with or without computers.

- Paradoxically perhaps, teaching and learning may become even more remote from real life before, because real life may now only enter the classroom through computers, simulations may replace real experiments, computer graphics may serve as substitutes for real objects.
- The use of ready-made software in applied problem solving may put the emphasis on routine and recipe-like modelling, thus neglecting essential activities such as critically analysing and comparing different models, choosing adequate ones, or reflecting upon the meaning and suitability of concepts and results within a mathematical model. To put it more succinctly and more generally: Intellectual efforts and activities of students may be replaced by mere button pressing.
- More and more mathematics teachers are becoming interested in computers instead of in modelling and applications, and more and more students are being prevented (or rather: like to be prevented) from reflecting on challenging mathematical problems (pure or applied) by being engaged in outward technological problems (which would not exist without computers) or by spending their time on constructing technically elegant programmes. So, the growing interest in computers and their increasingly easy availability in the classroom may, in some cases, even act to detriment of modelling and applications in mathematics instruction. (pp. 58-59)

Blum and Niss suggest that there are no easy recipes for solving these problems.

Perhaps the most important remedy is a very elementary one: teachers and students should become fully aware of these problems. Such an awareness would contribute toward one of the vital goals of mathematics instruction, namely the acquisition of critical competence in and “meta-knowledge” of mathematics, its relations to applications, and the advantages and risks of its tools.

The “gender factor” should also be considered carefully when applying computers in mathematics education. There are some studies which investigate this aspect, and their results have been mixed. Hativa and Shorer (1989), in a study of the effects of different variables on computer-based instruction arithmetic achievement of junior high school students ( $n=211$ ), reported that males had higher performance and larger learning gains than females. However, Webb (1985) reported no gender differences for achievement or attitude for junior high school students learning computer programming. Munger and Loyd (1989) reported that the Computer Based Instruction (CBI) mathematics achievement of

high school students of either gender with positive attitudes towards computers performed best.

Clariana and Schultz (1993) conducted a study of the effects of individualized mathematics CBI for males versus females of different ability levels. Mathematics CBI was expected to produce a differential effect for males and females. The study involved eighth-grade students from two inner-city junior high schools in a summer remedial program. There were 24 females and 26 males. The Wide Range Achievement Test served as the mathematics posttest measure. The factors analyzed by mixed analysis of variance included gender, ability and mathematics content. The four-way interaction of gender, ability, test, and content was significant. The results showed that the low females made small pretest to posttest gains compared to the low males, high males, and high females. The authors concluded that,

This finding supports the premise of this study that mathematics CBI may produce differential achievement effects for males and females. (p. 286)

### *Studies of using computer in mathematics teaching in university*

In the past few years much attention has been directed to reform undergraduate calculus curricula in universities. A number of mathematicians and mathematics educators have described the problems encountered (Douglas, 1986; Steen, 1988) and suggested remedies for the problems (Small & Horsack, 1986; Stoutemyer, 1985; Wilf, 1983; Zorn, 1986, 1987). Central to these recommendations has been the suggestion to use computer algebra systems software in order to de-emphasize computations and to emphasize the concepts of calculus.

In order to evaluate the effects of computer algebra system on concept and skill acquisition in calculus, Palmiter (1991) performed a study involving 78 engineering students enrolled in an integral calculus course at a large university. The subjects were randomly assigned to an experimental or a control group. The control group was taught the

traditional integral calculus course using paper-and-pencil methods to compute antiderivatives and definite integrals. The experimental group was taught to compute integrals by using MACSYMA, a computer algebra system. Both groups were taught about the definition of integral, the fundamental theorem of calculus, inverse functions, techniques of integration, and applications of the integral.

This study investigated whether there was a significant difference between the two groups of students in

- (a) knowledge of calculus concepts
- (b) knowledge of calculus procedures, and
- (c) grades in subsequent calculus courses.

The study compared the performance of the “traditional lecture” method in calculus to the performance of the “computer-augmented lecture method.”

Palmiter reported that students who were taught calculus using the MACSYMA had higher scores on a test of conceptual knowledge of calculus than the students taught by a traditional method. Students in the computers class also had higher scores on a calculus computational exam using the computer algebra system than students in the traditional class using paper and pencil. At the same time, 73% of the students in the computer group responded that they had learned more than usual in the course when compared to previous mathematics courses. In the traditional group 48% of the students gave the same response. Ninety-five percent of the students in computer group indicated they would sign up for a course with computer algebra systems again. Forty-three percent indicated that their attitude toward mathematics and computers had improved.

Another remarkable study of using computers in calculus teaching was done by Judson (1991). Judson studied the effectiveness of the computer algebra laboratory at Trinity University, which was established with the fundamental purpose of improving the teaching of undergraduate calculus. Judson designed a laboratory course for teaching Calculus I. The course was designed to reinforce the basic concepts of calculus as well as



to demonstrate the value of curve plotting, example generation, and exploration to achieve a deeper understanding. The software used in the experiment was the computer algebra system MAPLE , which is a command driven system that forces students to communicate correctly and to know exactly which operation they want to perform.

Fourteen Calculus I students registered for the lab and completed all laboratory sessions at a fixed time with the instructor present, or on an independent study basis. After finishing the laboratory course, students were asked to complete a questionnaire. An analysis of items on the questionnaire followed. The author judged the first formal laboratory experience to be an unqualified success, and enthusiastically recommended that others try the optional laboratory approach. Judson concluded that,

Students earn one credit hour while reinforcing and exploring the concepts of whatever mathematics they are studying. They discuss mathematics with one another and they enjoy learning. They develop a sort of comradeship, and they have fun. Teaching the laboratory was truly a remarkable experience and one that the author looks forward to repeating, not only for Calculus I, but for Calculus II and Calculus III, and perhaps Linear Algebra. (p. 40)

Many educators believe that the relationship between attitudes toward mathematics and their effect on learning mathematics is critical (Kulm, 1980). Clement (1981) noted that the affective reactions of students toward the presence of microcomputers in the teaching-learning process might be an essential factor to explore in successful implementation of this technology. Ganguli (1992) investigated the effect on student attitudes toward mathematics and using the microcomputer as a teaching aid in mathematics instruction.

Ganguli's study was designed to combine the strengths of both the teacher and the computer in the normal classroom setting so that the computer does not replace the teacher but, rather, supplements instruction. Ganguli used a sample consisting of 110 college students enrolled in four sections of an intermediate algebra class offered by the open-admissions undergraduate unit of a large midwestern state university. The focus of instruction in the study was to develop the concept of relationship between the shape of a

graph and its function. In order to measure students' attitudes the Mathematics Attitude Inventory (MAI) with six subscales (Sandman, 1980) was selected.

The results indicated that the attitudes of the experimental group, which was taught with computer aid, were significantly changed in a positive direction, whereas the control group that was taught without computer aid failed to show a similar result. The significant treatment effects on MAI subscales showed that the students in the microcomputer treatment group experienced a more positive self-concept in mathematics, more enjoyment of mathematics, and more motivation to do mathematics than their counterparts in the control group. Simultaneously, from the accounts of the two instructors who participated in the study, it was clear that the computer-generated graphics led to more active classroom discussions in experimental sections and consequently created more rapport between the teacher and the students than in the control sections. The instructional strategies in the study showed that the computer could serve as an electronic chalkboard for delivering instruction in a normal classroom setting. The study suggests that the use of the computer in teaching favourably influenced students' attitudes.

Another study by Alkalay (1993) supports Ganguli's conclusions. In this study, computers were used as a laboratory extension of precalculus course. The purpose of the study was to determine the effects of using the computer for independent exploration on student attitudes toward mathematics and on student attitudes toward computers. Two classes, a total of 27 students, participated in the study. The Mathematics Attitude and Computer Attitude Scales were administered at the beginning (pretest) and conclusion (posttest) of the study. The software Pre-Calculus by Kemeny and Kurtz was chosen. The students completed three units in the laboratory manual that were divided into four laboratory sessions. Unit I covered Shifting, Stretching, Flattening, and Reflecting Functions; Unit II covered Inverse Functions; and Unit III covered Polynomial Functions. An analysis of the attitude scales revealed that,

The findings of the student evaluations of the individual units indicate highly positive reactions toward using the computer in a laboratory setting for precalculus. Students agreed that the computer and instructional units were appropriate learning tools. (p.117)

A study of the exploration of personal computer animation for the mathematics classroom was conducted by Kaljumagi (1992). This study focused on whether or not the current software is of practical use to a teacher who wants to create his or her own animations, taking into account the typical school teacher's limited time and resources. The author worked with two animation programs: an animation art program, Ani ST (1987), and an animation broad-based program, Mathematica (1988). These programs were used to construct animations demonstrating mathematical concepts for such topics in calculus as linear equations, quadratic equations, trigonometric functions, polynomial functions, analytic geometry and polar coordinates.

From this investigation, it was found that computer animation has the potential to become a practical and powerful tool in the teaching of mathematics. Animations help teachers demonstrate many calculus concepts that are difficult to explain verbally or to show with static pictures. The realization of this potential appears to be dependent on the educational benefit to the student compared with other modes of instruction.

Kaljumagi concluded,

After working with Ani ST and Mathematica, my conviction that the personal computer can be used for animations and that these animations may have educational value has been strengthened... I believe that animation is a potentially feasible tool for the classroom, at least for certain topics. (p. 375)

McGivney (1990) investigated the use of computer software in teaching a mathematics course for a general university audience. The study investigated software titled Discrete Mathematics and several programs written by the researcher to perform lab exercises for a course on Contemporary Mathematics. This course was designed specially for a class of students from a variety of academic disciplines and backgrounds. Students worked in teams of two or three in the lab sessions designed to complement the course

lectures. McGivney taught three classes of Contemporary Math (over 60 students). In most cases, the software helped appreciably in solving the problems for topics of Contemporary Math, including set theory, voting methods, graph theory, recursion and linear programming.

With respect to the specific factors that seem promising in this approach, McGivney (1990) stated that,

First, the software allows students to grasp the climate in which mathematicians work... The software invites experimentation, and experimentation can lead to insight, and insight can develop appreciation and confidence.

Second, the laboratory environment substantially changes the student-teacher-relationship. Lecturing is minimized. In a lab the teacher is no longer in control as the omniscient dispenser of information, but functions more as a colleague working to help and encourage the student. Alternate solutions are not only tolerated, but encouraged. A teacher does not always have a ready explanation for questions that arise. (p. 53)

At the same time, McGivney emphasized,

Students observe that software, as helpful as it has become, is not a panacea. There are still many problems that currently available software cannot help them solve, and there are problems that probably won't be solved by any means in our lifetime. Students see that the software we use has limits and wonder if it could be extended to solve other problems they encounter in music, art, business, or education. And they learn that some problems are easier to do with pencil and paper than with a computer. (p. 54)

Through his study of teaching a software aided mathematics course, McGivney concluded that,

The experience of teaching this new version of contemporary math has convinced me that software, together with well-designed lab experiments, can make the learning of mathematics more effective and more enjoyable. When students work together using material and equipment that allow them to use their imagination, mathematics can be discovered and remembered (and even enjoyed). (p. 57)

A comparison of computer-assisted instruction (CAI) and traditional instruction in a college algebra course was analyzed in a study by Tilidetzke (1992). This study investigated the use of a tutorial package in a course of instruction in College Algebra, taught at the University of North Carolina. This course serves as a precalculus course for

those students planning to take calculus. A tutorial software package published by Addison-Wesley was used. The primary research question was to determine whether the students using CAI in the selected topical areas would do as well as or better than the students receiving classroom instruction on a posttest and on a delayed posttest. There were two instructors participating in the study. Each instructor taught one control class (using the usual classroom instruction) and one experimental class (using computer assisted instruction by means of tutorials in three selected topics). There were 21 students in each class involved in the study. The selected topics included (1) the multiplication and division of complex numbers; (2) completing the square; and (3) solving linear equations. The conclusions from this study were,

The CAI section performed as well as or better than the control section on, (1) a posttest and (2) a delayed posttest embedded in the final exam, covering the three topics of this study. (p. 59)

Moreover, the study also showed that,

For each instructor no significant difference was found between the control section and the CAI section on both the posttest and delayed posttest. This indicates that computer-assisted instruction is a viable alternative to classroom instruction on certain topics. The CAI treatment was found to be effective, independent of instructors. (p. 60)

A portion of literature deals with different applications of computers in teaching various mathematics topics.

Walton and Walton (1992) studied a computer program designed to reinforce the concept and definition of definite integral. They gave three approaches to calculate the value of the definite integral for functions of one or more variables: numerical approximation, the "Monte Carlo method," and purely mathematical means. All three methods gave similar results. These researchers suggested that,

The computer is a powerful tool that can be used by students in elementary calculus classes to enrich and reinforce concepts in pure mathematics. Definitions and ideas that initially seem elusive can be made more concrete when the awesome number-crunching capability of the computer is used to perform calculations which approximate the results determined by purely theoretical means. (p. 401)

The random number generator of the computer was used to find Monte Carlo method approximations of both the single and double integral. The study illustrated the simulation role of the computer for randomized processes, and the concretization of abstract definitions and ideas in pure mathematics.

Wieder (1991) described how the software MathCAD can be used as a lecture-demonstration tool to teach general calculus. Wieder gave some applications of MathCAD in teaching a maximum-minimum problem, a problem involving a p-series and a geometric series, and a problem involving the computation of an area between curves in polar coordinates. The proper design of MathCAD worksheets was discussed with emphasis on answering the “what if” question. Discovering the relationship between input parameters and the outcomes helped students to understand the mathematical meaning of problems.

Thomas (1990) reported on an approach to computational matrix algebra that takes advantage of a high quality, low cost software package called MATLAB. Matrix algebra is a powerful mathematical tool, providing convenient ways to represent many complex problems and the means for solving them. Thomas supposed that the difficulty with introducing students to matrix algebra has always been the burden of computation associated with the subject. Working with MATLAB students may investigate matrix algebra inductively, letting the computer do the laborious computations. The MATLAB programs free students from a burden of dreary computation that can get in the way of concept development. Thomas concluded that,

The benefits include (1) a reduction in instructional time otherwise consumed by computation, (2) an increase in time available for concept development, and (3) the opportunity to attack real world problems in a computing environment that is not intimidated by large numbers of variables, fractions, or decimals. (p. 74)

Two applications discussed in Thomas's article focus on matrix multiplication as a means of representing geometric transformations. One uses MATLAB to represent reflections. The other one uses MATLAB to represent a series of rotations in matrix form. Thomas emphasized the use of concrete models to help visualize and internalize a problem.

The computer software handles computation, while students focus on concept development and strategic planning.

Solution of second order differential equations represents a large portion of the problems which are studied in undergraduate mathematics, science and engineering. Mathews (1991) studied the use of Computer Algebra System (CAS) such as MAPLE and Mathematica in teaching students how to solve differential equations. Numerical methods have been used with computers for about half a century to produce approximate numerical solutions, but with CAS it is possible to obtain the exact analytic solution to problems. Mathews designed subroutines to create symbolic solutions to differential equations, using high-resolution graphical tools to plot the solutions. Mathews' study illustrates how CAS helps to solve differential equations, and how, by taking advantage of their capabilities, students can easily solve many equations and investigate the underlying theory. Mathews remarked that,

In comparison to numerical solutions, the CAS solution involves the exact formula, which can be used to obtain a graph. CAS tools such as MAPLE and Mathematica will soon be accepted for everyday use in mathematics, engineering and science. (p. 84)

The literature reviewed in this section of the analysis represents different ways of applying computers in teaching university courses in mathematics. Such mathematical investigations are not common; most have occurred in the context of mathematics teaching in high school. In the past, the development of mathematical software and hardware was not sufficient to meet the complicated level of mathematical content taught in university. At the present time, however, there are some software applications such as MAPLE and Mathematica that are powerful enough to satisfy the requirements of university courses in mathematics. Preliminary research in the use of computers in university mathematics teaching show promising results.

## **Chapter 3**

### **Method of Research**

This chapter describes the methods employed in the study through a discussion of the context of the study, the design of the computer-based laboratory course, and the qualitative approach to the research design.

#### ***Context of the study***

The study was conducted at Dalat University, located in the “highlands” of Vietnam with a population of approximately 4200 undergraduate students. There are four faculties in Dalat University; the Faculty of General Education, the Faculty of Science and Engineering, the Faculty of Sociology and Humanities and the Faculty of Education. The Faculty of Sociology and Humanities consists of four departments: Arts, History, Business Administration and Foreign Languages. The Faculty of Science and Engineering is composed by five departments: Mathematics, Physics, Chemistry, Biology and Computer Science. Particularly, the Department of Computer Science and the Faculty of Education are new divisions, having been established during the past year.

The undergraduate degree program at Dalat University is divided into two stages. Students study a core curriculum called the General Education Program during the first three semesters, followed by a specialization in the second stage of five semesters. Each semester lasts for fifteen weeks of study and three weeks of examinations. There are seven core curricula in the General Education Program that are based on seven subject groups determined by the Ministry of Education and Training. Students can select one among these seven curricula to study in the first stage. Students who want to enroll in the sciences take courses of the first, second or third curriculum. Students who study Mathematics, Physics or Computer Science in the second stage choose the first core curriculum. The second core



curriculum is reserved for students who intend to learn chemistry in the second stage, while the third core curriculum is for students who specialize in biology.

The year of the present study marked the introduction of the credit system and General Education Program at Dalat University. The number of students entering the first year of study at Dalat University increased unexpectedly as well, from 1300 to 1900 students. These factors have caused many difficulties for the university, particularly for first year students.

In Dalat University generally and in the Faculty of Science and Engineering specifically, the majority of professors and instructors use the strategy of direct teaching with lectures. Courses involving practical laboratory work were combined with their theoretical components in the past. Beginning this year, however, they are organized as separate courses in order to be compatible with the credit system. All departments in the Faculty of Science and Engineering have their own labs except the Mathematics Department. In fact, not only in Dalat University but generally speaking in Vietnam, few labs have been used in mathematics education at all levels. Parallel to establishing the Computer Science Department, Dalat University developed the Center of Computing Services. The center organizes a large computer lab with many personal computers connected in a local area network. The lab serves all students and teachers in the university, and, at the same time, it is used as the lab of the Computer Science Department.

The main instructional media at Dalat University are blackboards. Technology tools for instruction such as overhead projectors, computers, analog devices, video disc or video tapes have rarely been used in teaching sciences and mathematics. Hence, most teachers in Dalat University do not have experience, knowledge, and skills of using instructional technologies, particularly computers, in their teaching. Similarly, there has been scant educational research on using and evaluating technological means in teaching science and mathematics.

In the past year there has been an emphasis on using video in teaching at Dalat University. This caused much debate among teachers, students and parents. In order to cope with the situation of increasing enrollments Dalat University has been using videos in teaching first- and second-year courses. The university built some large classrooms, each with seating for over one hundred students. The students are distributed into several classrooms with video monitors, while there is only one teacher, whose lecture is transmitted to other classrooms. Technical employees are responsible to video all the teaching activities of that teacher, and all of these pictures are transmitted live to monitors of other classrooms involved concurrently. Thus, in classrooms without teachers, students learn by watching the screens of monitors. This model of teaching has been applied for the past year with many different opinions about this innovation. There has yet to be conducted any research or formal evaluation of the effects and impacts of this teaching strategy.

In the usual way of learning mathematics at Dalat University, students go to classrooms to listen to teachers present lectures, and write notes. At the end of a topic of the lectures, they will have some homework given by the teacher. Homework is not often marked for student achievement but it helps students to understand the lectures more deeply, and do some problem solving. There are neither teacher assistants nor tutorial classes. During lectures students are allowed to ask teachers questions concerning the lectures, or difficult problems in homework assignments. Student achievement for the entire semester is based on one or two in-term exams and a final exam, with the greatest weighting on the final.

Students tend to study mathematics individually at home and relate with teachers directly about their problems in class. However, this arrangement is unsatisfactory because teachers need the majority of time for presenting lectures, and the student-teacher ratio is large for each mathematical subject. These factors have affected students' attitudes toward mathematics as well as their mathematics achievement. In such a learning environment, students do not have many chances to work and learn with each other. Many students do

not have habits of representing, sharing their mathematical ideas or learning with other students; they prefer to rely on teachers for help with mathematical problems.

Consequently, students feel uncomfortable when asked to listen, exchange or appreciate ideas different from theirs, and in most problems they expect advice or answers from teachers rather than from their classmates. All of these factors have contributed to limit the mathematics achievement of students, and fosters competition rather than cooperative learning patterns. Learning mathematics is a great challenge for many students.

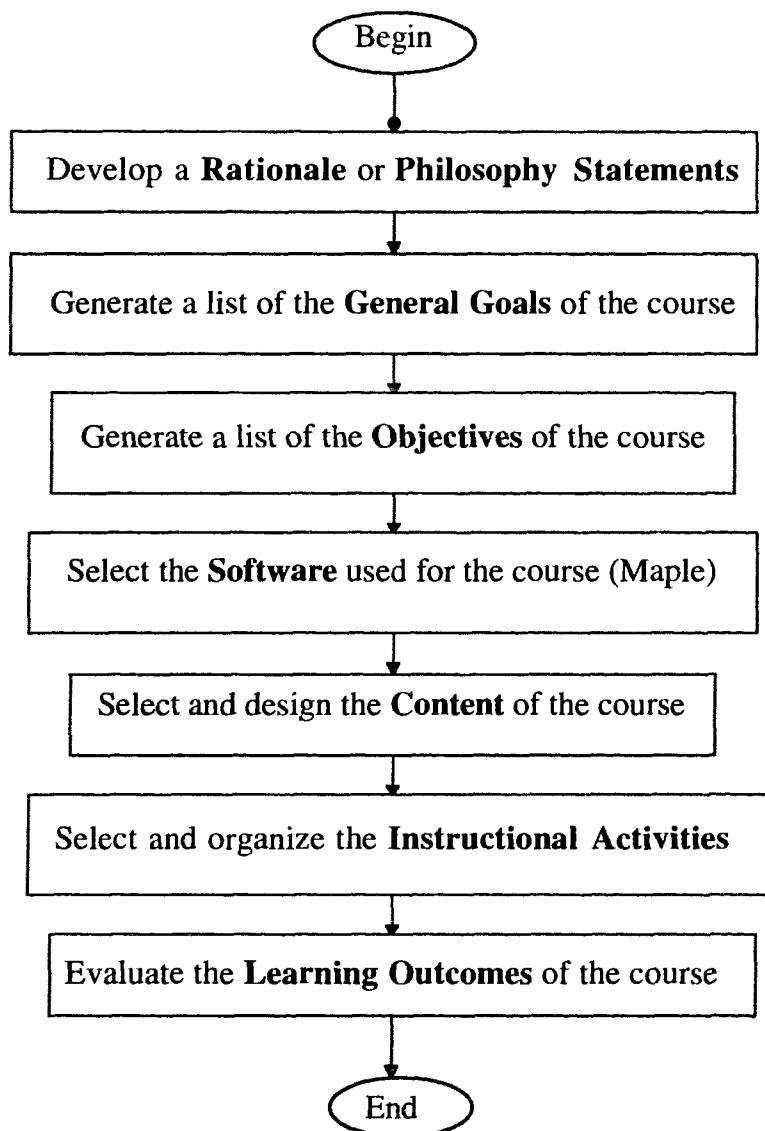
For science teachers in general, and mathematics teachers in particular at Dalat University, it is possible to say that the model of teaching used is traditional. Most teachers combine theoretical lectures with correcting, solving problems or answering questions of students, and the lectures are presented chapter by chapter from the textbook. Teachers play a center role in class; they talk continuously from their own framework, they do not appreciate students' background, needs, or feelings. Students do listen carefully and passively, and some of them try to remember all of the teachers' talk. For them, the knowledge contained in the lectures is seen as an information system. The teaching process is performed linearly in one direction, from mathematical knowledge of teachers to students. In addition, mathematical lectures tend to build concepts, with little emphasis on problem solving, and no concern for social and technology issues related to mathematical understanding.

In a learning context such as that described above students do not have different modes available that enable them to explore and practice mathematics. They do not have many chances to develop problem solving skills and communication abilities. They cannot see how technology is used in solving mathematical problems. And finally, they do not develop positive attitudes toward learning mathematics. In order to partly overcome these conditions, the computer-based lab course was designed and investigated in this study.

### *Designing the computer-based laboratory course*

The approach of Logical-Rational Development was used to design the Computer-Based Lab Course in calculus. In this approach the developer follows a step-by-step procedure to derive an instructional program. The most widely acknowledged outline of the rational process for development is that proposed by Tyler (1950). Since then a number of eminent curriculum theorists have built refinements on that basic plan. The lab course was designed on the basis of the outline presented in Figure 1.

Figure 1: Diagram of design of the Lab Course.



### *Philosophy Statements*

The philosophical ideas that were used as a basis for the goals of the Computer-Based Lab Course were drawn from research reports of Emmer (1993), Bergsten (1993), Pinchback (1994), and Judson (1991).

In order to answer the question of what must be changed in the content of traditional courses in mathematics to meet the needs of the technical revolution of recent years, Emmer (1993) writes

We are assuming here that something must be changed since, in our opinion, courses based on the use of new technologies can offer to the students more than traditional 'chalk & blackboard' courses do. Probably this assumption is not generally accepted but it is clear that the spread of informatic tools throughout our society cannot leave the school apart, so in some sense we are forced to introduce computers in schools either as a new subject of study or as new teaching instrument. (p. 222)

Emmer thought both Calculus and Advanced Calculus were fundamental courses devoted to developing logical-deductive skills while presenting mathematical techniques and results. Traditionally, the organization of these courses consists of general lectures and practical exercise sessions. The introduction of computers brought some modifications in both, since computers have been used to present and visualize mathematical phenomena as well as to solve problems from a numerical point of view.

Emmer emphasized the role of graphic facilities offered by personal computers. The visualization of some aspects of mathematics using graphics helps students to understand difficult concepts. Emmer writes

Visualization of mathematical phenomena significantly increases the rate of assimilation; the students acquire basic knowledge more quickly and develop mathematical intuition. (pp. 227-228)

However, Emmer also notes that,

Of course it is important to recall that the emphasis on applications and on the relevance of visualization does not mean at all that the abstract and rigorous approach to mathematical problems should be abandoned in favor of simulations and heuristic reasoning. (p. 228)

Bergsten (1993) gave some valuable opinions about the use of computers in mathematics education. He writes

There are at least two obvious ways of using a computer algebra system (CAS) like Maple or Mathematica in a mathematics course, i.e., as a computational tool or as an educational tool (learning tool). This means, in other words, to generate results, and to generate ideas. As a parallel, mathematics has two "faces", logic and intuition, as emphasized by many mathematicians (e.g., Hilbert and Poincare). Deduction is characterized by a sequential order of steps in a logical chain, while intuition is working more with a simultaneous view of relations, with patterns. (p. 133)

Bergsten thought these distinctions led into the practice of mathematics and the development of conceptual understanding. That this parallels the use of a CAS as an

educational tool or as a computational tool, might be one of the reasons that such systems often are advocated in mathematics education. In analyzing the use of CAS in teaching analysis courses, Bergsten writes that,

A CAS can provide numerical and graphical support to the result (as well as the result itself), but does not give the user the “finger-tip” (i.e., a physical) experience of controlling the quantities that the mathematical symbols represent. To work on such problems with a CAS you must know the essence of the definition in question, a knowledge that might, on the other hand, need this kind of “by hand” estimation to develop. In analysis one must learn to control the mathematical symbols by respecting their properties, a respect that grows out of increased familiarity and understanding. (p. 134)

For the educational use of a CAS in analysis, he emphasized there was a need for a sense of estimating the symbolic and graphic computer outputs. He said

To develop rich concept images in analysis computer graphics is a superb tool, and a CAS provides a mathematical environment that supports the linking of numerical, algebraic and graphical information. Nevertheless, without a developed sense of estimating the symbolic and graphic computer outputs, the black box effect, as well as the screen itself as an external physical unity, will put the user outside the real events, without control of what is going on, and why. And getting control is what learning is about. Therefore, one of the crucial questions, when discussing the use of a CAS as an educational tool in mathematics, and in analysis in particular, concerns the development of making symbolic and graphic computational estimations. (p. 135)

Bergsten’s conclusions from his study of the role of a computer algebra system in an analysis course were that:

CAS might enlighten the complexity inherited in symbolic (algebraic) computations and visualize the results, support imagery and versatile learning, help to build concept images and create links to, and display, the power of algebraic computations, and be used as a powerful problem solving tool. (p. 138)

In considering the teaching of mathematics with the computer algebra system Maple, Pinchback (1994) analyzes different points of view around this issue,

Some view the development of CAS as a great opportunity, claiming it will remove the drudgery of educationally contentless algebraic manipulation, and thus provide the chance to explore mathematical issues that were unfeasible to address in the classroom before. Others see a great danger that students will become mathematically illiterate, will treat these software tools as though they are perfectly implemented black boxes, and will end up with

little or no understanding of the mathematics that those black boxes purport to implement. (p. 1)

Pinchback supposed that, in teaching mathematics, the more formal and rigorous concepts were treated as black boxes, if not explicitly, then by unmentioned assumption; black boxes were used to hide confusing complexity from students. They were not inherently bad things. Much of the process of designing any kind of course could be viewed as a process of deciding which black boxes to present and which to ignore. Pinchback was also critical of the “black box” theme: In particular, when students were denied access to the inside of a black box, or when they did not understand how the black box worked, or they had no reason to care about the internal workings of a box containing concepts critical to a subject. In order to avoid these situations Pinchback suggests that,

With Maple we have such options: in this case we can use the *student* package to perform smaller steps in problem solving, for example by using the *intparts* routine for doing integration by parts. For other situations instructors are exploring the Maple worksheet interface as a mechanism for presenting the steps in an algorithm, instead of just providing pre-written subroutines as a fait accompli. Students modify portions of the worksheet for a given problem and step their way through the algorithm, and are thus more involved with the CAS-based presentation of the material ... Just because you have a piece of software, even a really good piece of software with lots of online help available, students still need a teacher to help them understand a subject. The teacher knows from experience which black boxes are important to open, how to explain what the internal contents are, and explain why those contents were put there in the first place. A teacher provides context, relevance, and details as-need explanations ... To put it in simple terms, if you expect your students to use Maple and learn something, then you need to be sure that you are prepared to help and guide them through that process. (pp. 2-3)

Judson (1991) conducted a one credit hour mathematics laboratory course for Calculus I students at Trinity University. He evaluated students’ enjoyment, feelings, motivation and cooperative learning while they took this lab:

Students enjoyed working as partners. They talked about math, they asked each other questions, and they helped one another. They particularly liked working as partners if they each had their own machine. In this way they could both experiment and compare their results ... Students earn one credit hour while reinforcing and exploring the concepts of whatever mathematics they are studying. They discuss mathematics with one another and they enjoy learning. They develop a sort of comradeship, and they have fun. (pp. 39-40)



In summary, the philosophical underpinnings of the computer-based lab course designed for this study were:

- Computers have the ability to present a visual portrayal of mathematical phenomena. This helps students to acquire basic knowledge more quickly and develop mathematical intuition.
- A computer algebra system (Maple or Mathematica) can be used as a computational tool and a learning tool in mathematics courses. It helps students in algebraic computations and it is a powerful problem solving tool.
- For the use of the computer algebra system as an educational tool, there is a need to develop symbolic and graphic computational estimations.
- To teach mathematics effectively with the computer algebra system, teachers need to help and guide students to understand the mathematical contents as well as the mechanism of the algorithm hidden in subroutines and commands of the software that are seen as implemented black boxes.
- The computer lab course offers to students a computer-based approach to learning mathematics that is complementary to traditional mathematics courses.
- The computer lab course helps students to develop abilities of communication and cooperation in learning mathematics.
- The computer lab course helps students to develop positive attitudes toward mathematics and computers.

### *General Goals*

The computer lab course is designed to provide students with new ways of approaching mathematics. A computer algebra system is suggested for use as an environment in which students can practice mathematics, reinforce and extend conceptual understanding, develop mathematical intuition, solve problems, and develop computational skills. The course, therefore, is seen as a complement and extension to the lecture

mathematics course. Presentation and visualization of mathematical concepts by using the graphical facilities of the computer allow students to view and explore the concepts intuitively. In addition, the course is designed to create a working environment that enables students to develop collaborative problem solving and mathematical communication skills, as well as high motivation for learning mathematics.

### *Objectives*

The Computer-Based Lab Course is intended for First-Year Basic Science students who study Calculus (A1). The lab course provides students with knowledge and skills of using the computer algebra system, “Maple”. Maple offers students an interactive environment in the form of commands and worksheets that enable them to:

- Explore and reinforce mathematical concepts to achieve a deeper understanding of the course. (The concepts include functions, limits and continuity, derivation and integration of functions in one real variable.)
- Visualize mathematical concepts more easily, and develop mathematical intuition, high motivation for learning mathematics. (The concepts that can be approached visually include functions and graphs, relations between graphs and roots of equations, implicit functions, inverse functions, continuity of functions, derivatives and geometric meanings, derivatives and local extrema, Taylor’s formula and remainders, area problems and definition of definite integrals, Riemann’s sum and Darboux’s sum, and numerical approximation methods for calculating definite integrals.)
- Learn how to do calculus computations using computers, including explaining the theoretical basis and computational algorithms, and estimating symbolic and graphic outputs. (The computational problems include solving equations, computing limits of sequences and functions, investigating continuity of functions, computing derivatives and higher-order derivatives, doing Taylor

expansions and analyzing remainders, computing indefinite and definite integrals, computing improper integrals, computing numerical approximations of definite integrals by using the rectangular rule, the trapezoidal rule, and Simpson's rule. Computations involving limits, derivatives and integrals are performed in two ways: using the definition to understand the concepts, and using Maple's direct commands to get results quickly.)

- Develop calculus problem solving strategies. (The problems investigated consist of applying derivatives to solve maximum and minimum problems, applying definite integrals to calculate areas, arc lengths, volumes and surface areas of solid of revolutions.)
- Learn and practice skills of cooperative learning, collaborative problem solving, and mathematical communication and reasoning.

*The software used for the course: Maple*

Maple is relatively easy to learn and use. It is intended for the scientist, mathematician, engineer, and student who need to use mathematics in solving problems. The strength of Maple is its ability to solve problems in symbolic form. Maple can symbolically operate on fractions, factors and expanded polynomials, give exact solutions to equations (polynomial, exponential, trigonometric, logarithmic, etc.), graph various functions, compute limits, symbolically calculate derivatives, give exact solutions to indefinite and definite integrals, solve certain differential equations, and calculate power series. Maple manipulates both real and complex numbers, and has an impressive linear algebra package that allows the user many matrix-manipulating commands.

Maple normally performs exact, rational arithmetic with no round-off or truncation errors. It also simplifies those mathematical expressions that lie in the domain of rational functions. Maple can be used as a sophisticated calculator for algebraic or numerical computations, and it has a wide range of built-in operators and functions. However, the

real power of Maple lies in the fact it is a programming language in which all the operators and functions can be used to write user-developed procedures to solve problems.

### *Content of the course*

The content of the course is based on the syllabus of Calculus A1 (Appendix A), for first-year basic science students. The course consists of four lab assignments. The first assignment introduces Maple to students, and the remaining assignments represent the topics included in Calculus A1:

- Lab #0: Introduction to using Maple.
  - Introduction to Maple.
  - Starting a Maple session.
  - Some Notes in Communication between User-Maple.
  - Numerical Evaluations and Approximations.
  - Expressions and Equations.
  - Graphical Representations.
  - Polynomials.
  - Understanding Inequalities through Plotting.
- Lab #1: Functions, limits and continuity.
  - Functions.
  - Implicit Functions.
  - Inverse Functions.
  - Limits of Sequences.
  - Limits of Functions.
  - Properties of Limits.
  - Continuity and One-Sided Limits.
  - The Intermediate Value Theorem.

- Lab #2: Derivation of functions in one variable.
  - Derivative and Geometrical Meaning.
  - Differentiation Formulas.
  - The Fundamental Theorems (Fermat, Rolle and Lagrange Theorems).
  - The Taylor Formula.
  - Monotone Functions, Concavity and the Points of Inflection.
  - Applied Maximum and Minimum Problems.
- Lab #3: Integration of functions in one variable.
  - The Antiderivative and Indefinite Integrals.
  - Indefinite Integration Techniques.
  - The Definite Integral.
  - Definite Integration Techniques.
  - Numerical Approximations of Definite Integral.
  - Some Applications of Integration.
  - Improper Integrals.

*Activities and instructions*

In order to complete the lab course, students need to perform the four assignments on computers for fifteen weeks and take a midterm and final exam. Each lab assignment is divided into several parts, each designed for one lab session. Students attend a lab session once a week for three hours. Lab assignments are performed after students have theoretical concepts from lectures. Thus students can take the lab course either concurrently or consecutively.

Laboratory exercises and support materials (e.g., worksheets) are designed so that students can perform them by themselves on computers. I was present to assist students on

their assignments. Beginning presentations provide an overview of the lab and help students to focus on their work.

During the entire lab, the instructor is available for all individual consultations. However, these consultations are carried out so that students do not become too passive in their learning. The instructor's answers should motivate students to think more deeply about their mathematical problems, or help them to discover mistakes in their mathematical reasoning rather than providing them correct solutions quickly. The instructor can also return the previous week's student reports to discuss common misconceptions or mistakes either on an individual basis, or with groups of students. In this process, the instructor encourages students to discuss mathematics with each other.

The worksheets given out in a lab session often include instructions and exercises. These two parts are represented alternately in every problem of the lab worksheets. After the lab session students save their worksheets on their floppy disks, and print them for the lab instructor.

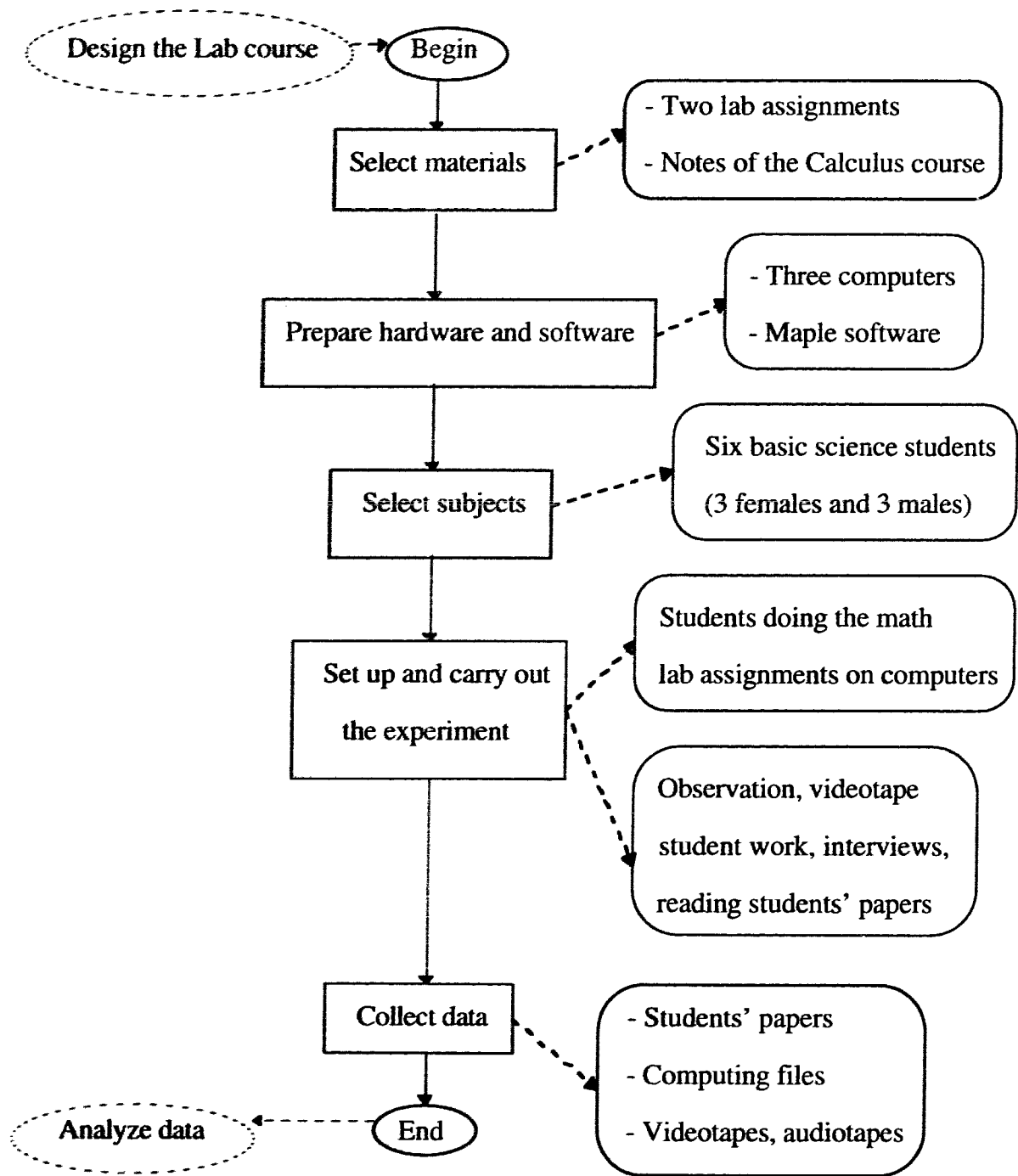
#### *Evaluation of the course*

Students' achievement in the lab course is evaluated on the basis of the lab reports, and their results on the midterm and final exams. The two exams include questions requiring computations and problem solving on the computer, and multiple choice and open ended questions used for testing understanding of concepts. The marking scales on the lab assignments and two exams can be different; they depend on the structure as well as the number of questions in each. The lab instructor transforms these marks to a scale of 10. The proportional grading is 50% for assignments, 20% for the midterm and 30% for the final exam.

### ***Research design with a qualitative approach***

The focus of the study was to investigate how students could learn calculus with the computer-based laboratory course. The study involved qualitative methods focusing on the nature of students' experiences learning calculus in a computer-augmented learning environment. The following diagram shows the main activities of the study.

**Figure 2:** Diagram of design of the study.





### *Selection of hardware and software*

The study took place in Dalat University. The computer lab used is in the Mathematics Department. The selection of Maple software for the study was based on consultation with instructors in the Department of Mathematics and Statistics, and the Faculty of Education of Simon Fraser University. However, it should be noted that any computer algebra system such as MACSYMA, muMATH, or Mathematica could be used as well.

### *Selection of students*

From the population of first-year basic science students of Dalat University, six students were selected and invited to participate in the study.

How were these students selected? First, I sought from the office of the Faculty of General Education of Dalat University a list of 306 first-year basic science students (230 male; 76 female). 21 male and 16 female students representing “high” to “average” groups were given the questionnaire. Of the 37 students selected, 22 returned the questionnaire. The content of the questionnaire is presented in Appendix B. It included questions to determine students’ mathematics achievement in high school, English background and computing skills. From data collected through the questionnaire, I selected six students, three females and three males, and invited them to participate in the study.

These students were selected with different levels of mathematical background and in consideration of a balance of gender. In addition, the software Maple used in the experiment was an English version so it required the students know English to be able to understand basic commands and messages appearing on the computer’ screen. Three of the students selected were novice computer users because of limited exposure in their high school programs.

The students were informed of the nature and purpose of the study. They were given Consent Forms before participating in the experiment. They understood that they could withdraw their participation in the study at any time, and that their participation was completely voluntarily.

The following is a description of the students who participated in the experiment, including their marks in three subjects (Mathematics, Physics and Chemistry) of the Dalat University entrance examination, their mathematics ability in high school, English background, and their ability of using computers. I have used pseudonyms in this document.

- **The first student:** Le Thanh (Male)
  - Mathematics: 10      Physics: 10      Chemistry: 9.5
  - High school mathematics achievement: Very good.
  - English ability: Finished English grade twelve with good performance.
  - Computing skill: Learned basic skills in using computers.
- **The second student:** Nguyen Hai (Male)
  - Mathematics: 8      Physics: 7.5      Chemistry: 1.5
  - High school mathematics achievement: Good.
  - English ability: Finished English grade twelve with fairly good performance.
  - Computing skill: Never learned how to use computers.
- **The third student:** Hoang Phan (Male)
  - Mathematics: 4      Physics: 5      Chemistry: 3
  - High school mathematics achievement: Average.
  - English ability: Finished English grade twelve with average performance.
  - Computing skill: Only knew how to use keyboard.
- **The fourth student:** Le Thi Mai (Female)
  - Mathematics: 9      Physics: 9      Chemistry: 6.5
  - High school mathematics achievement: Very good.

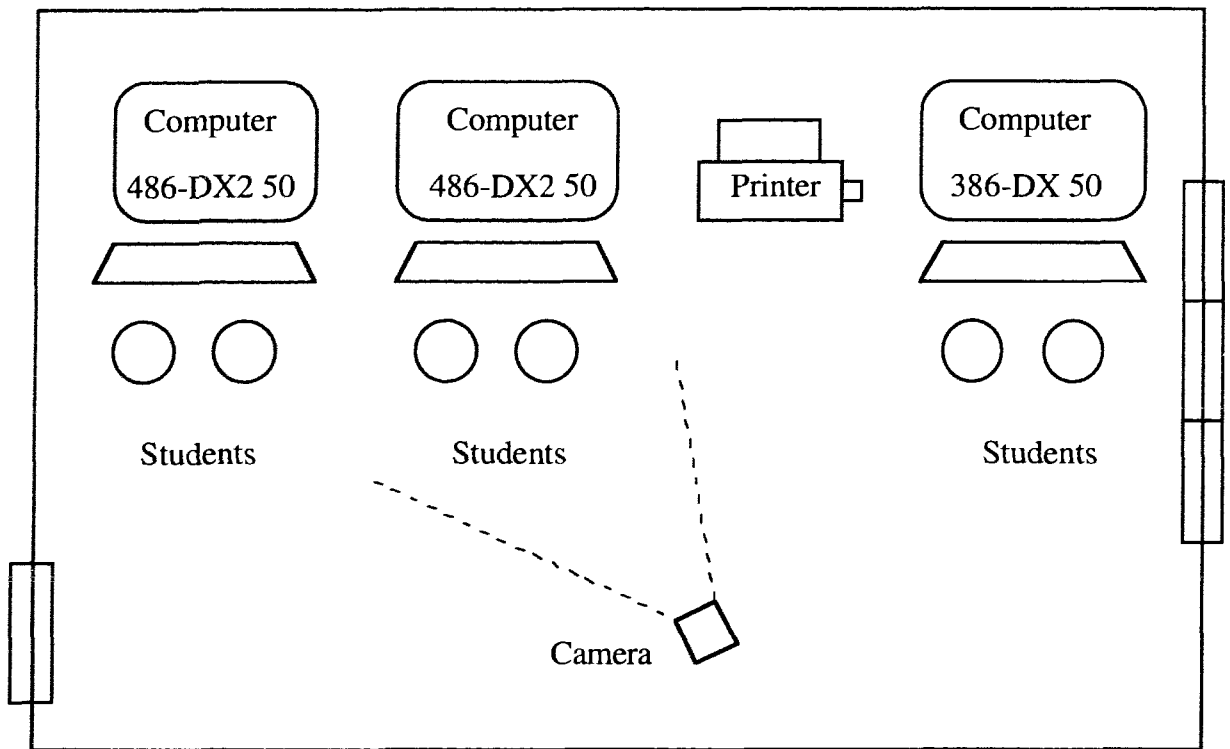
- English ability: Finished English grade twelve with average performance.
- Computing skill: Never learned how to use computers.
- The fifth student: Nguyen Thi Thu (Female)
  - Mathematics: 4      Physics: 7      Chemistry: 7
  - High school mathematics achievement: Average.
  - English ability: Finished English grade twelve with fairly good performance.
  - Computing skill: Learned basic skills in using computers.
- The sixth student: Ngo Thi Trang (Female)
  - Mathematics: 5      Physics: 6.5      Chemistry: 4
  - High school mathematics achievement: Good.
  - English ability: Finished English grade twelve with good performance.
  - Computing skill: Learned basic skills in using computers.

All of the students had just finished high school, and were in their first year of university. Their ages were from seventeen to nineteen years old. Three of them studied in urban high schools, and the other three students in suburban high schools. Two students lived in Dalat city, the city at which the university located, and the others were from neighboring provinces.

#### *Setting up and carrying out the experiment*

In order to be able to carry out the experiment, the computer lab of the Mathematics Department of Dalat University was reorganized as shown in Figure 3.

**Figure 3:** Diagram of the Computer Lab



Three computers and one printer were put along the length of the lab room. A video camera was placed in front of the computers. The camera could be focused on students' learning activities as well as output data displayed on screens of the computers.

The data were collected for a period of five weeks, beginning in the middle of September, and ending in the middle of October, 1995. The students performed six lab sessions in that period of time, two sessions in the first week and one every subsequent week. Six students were divided into three groups of two, each pair working at a computer. The choice of partners for the groups was arbitrary; the students could choose partners whom they felt comfortable to work with. In the first two lab sessions the groups included group one with Thanh and Trang, group two with Phan and Hai and group three with Mai and Thu. After a time of working together the students knew each other and they changed their partners. In every lab session I made detailed observation of how students learned mathematics through working with computers and their partners. Data for the study include

my field notes taken during the observation period, videotapes of session 3, 4 and 5, and audio-taped interviews with the students. The data collection schedule is presented in Table 1, below:

**Table 1: Schedule for observation, videotapes and interviews.**

Lab Session	Group 1 Thanh - Trang	Group 2 Phan - Hai	Group 3 Mai - Thu
1 September 14, 1995	Observation	Observation	Observation
2 September 16, 1995	Observation	Observation	Observation
	Group 1 Thanh - Phan	Group 2 Hai - Thu	Group 3 Mai - Trang
3 September 21, 1995	Videotaped Interviewed	Observation Interviewed	Observation Interviewed
4 September 30, 1995	Observation	Videotaped	Observation
5 October 7, 1995	Observation Interviewed	Observation Interviewed	Videotaped Interviewed
6 October 14, 1995	Observation	Observation	Observation

In order to consider students' attitudes toward mathematics and computers after they had completed all of the sessions, the "Mathematics Attitude Scale" and the "Computer Attitude Scale" were given to the students. These were given in the form of a questionnaire and consisted of questions to be answered from Strongly Disagree, Disagree, Undecided, Agree to Strongly Agree. The contents of the tests were built based on studies of Aiken (1979), Popovich, Hyde, Zakrajsek (1987), and Sandman (1980). They are presented in Table 2 and Table 3.

Table 2: The Mathematics Attitude Scale

SD : Strongly Disagree - D : Disagree

U : Undecided - A : Agree - SA : Strongly Agree

SD	D	U	A	SA
<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>

Enjoyment of Mathematics

1. Mathematics is not a very interesting subject.
2. I have usually enjoyed studying mathematics in school.
3. I have seldom liked studying mathematics.
4. Mathematics is enjoyable and stimulating to me.
5. Mathematics is dull and boring.
6. I like trying to solve new problems in mathematics.

Freedom from Fear of Mathematics

1. Mathematics makes me feel nervous and uncomfortable.
2. I am very calm when studying mathematics.
3. Mathematics makes me feel uneasy and confused.
4. Trying to understand mathematics doesn't make me anxious.
5. Mathematics is one of my most dreaded subjects.
6. I don't get upset when trying to do mathematics lessons.

Perceptions of Mathematics

1. Mathematics is a very worthwhile and necessary subject.
2. Other subjects are more important to people than mathematics.
3. Mathematics helps to develop the mind and teaches a person to think.
4. Mathematics is not especially important in everyday life.
5. Mathematics has contributed greatly to the advancement of civilization.
6. Mathematics is not one of the most important subjects for people to study.



Table 3: The Computer Attitude Scale

SD : Strongly Disagree - D : Disagree

U : Undecided - A : Agree - SA : Strongly Agree

SD	D	U	A	SA
<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>

Enjoyment of Computers

1. Computer is not a very interesting subject.
2. I enjoy computer work.
3. I have seldom liked working on computers.
4. Computer is enjoyable and stimulating to me.
5. Learning computers is boring to me.
6. I like learning on a computer.

Freedom from Fear of Computers

1. Computers make me feel nervous and uncomfortable.
2. I am very calm when studying computer.
3. Computers make me feel uneasy and confused.
4. Trying to understand computers doesn't make me anxious.
5. Computer is one of my most dreaded subjects.
6. I don't get upset when trying to do computer lessons.

Perceptions of Computers in Mathematics Education

1. I feel that the use of computers in schools will help students to learn mathematics.
2. I feel that the use of computers in schools will negatively affect students' mathematical learning abilities.
3. I think that computers and other technological advances have helped to improve mathematics education.
4. I think that computers are not worthwhile and necessary in mathematics education.

Each test was considered in three dimensions: enjoyment of mathematics / computers, freedom from fear of mathematics / computers, and perceptions of mathematics / computers in mathematics education. Each dimension had items stated in positive statements, and negative statements. The items were mixed randomly to avoid biases. Only five students completed these scales, as one withdrew from this portion of the study.

During the study, I was the lab instructor. My work as lab instructor was guided by instructional activities described in lab course. At the beginning of every lab session, I gave lab assignments to students and presented objectives as well as requirements of lab work for that session. Students then performed the lab work on computers in groups. During the sessions, students could consult with me anytime for help, either with computing problems or mathematical questions. For mathematical questions, I encouraged students to explore, discuss or share ideas with each other in the process of searching out solutions. In the case of difficult concepts or problems I tried to use inquiry teaching instead of providing solutions directly. I also tried to help students construct new concepts based on their mathematical background rather than simply memorizing the formulae. At the end of each lab session, I guided students to save their lab work on files on their disks, and at the same time printed these worksheets out to hand in to me for assessment and evaluation.

### *Collection of data*

Three pairs of students performed six lab sessions and in each session every group had to complete a lab paper, thus there were eighteen lab papers that were collected. Learning activities of the three groups were recorded on video tapes, each lasting approximately one hundred and twenty minutes. Two interviews were conducted with each group and these were recorded on audio tapes of thirty minutes. All of these data were collected for the analysis stage.

### *Methods used for analyzing qualitative data*

Protocols of the students' activities, dialogues and statements were sorted, selected and transcribed by the author. These transcriptions were coded in terms of significant events, actions and reactions. The students' lab papers were considered and evaluated to find consistent explanations of students' ways of doing mathematics with computers. The information that was drawn from the lab papers was coded using the same scheme that was used for the transcription data.

The coding scheme was developed by Bogdan and Biklen (1992). The codes chosen were context, situation, activity, strategy, and relationship to social structure. Activity and strategy coding was focused on activities involving the learning of mathematical concepts, reasoning, problem solving and computational skills, developing of mathematical visualization, intuition and communication abilities, and positive attitudes toward mathematics and computers.

Through analyzing and conceptualizing the data I tried to build rational explanations about the students' learning and thinking mathematics, as well as their experiences with the computer-based approach.

## Chapter 4

### Analysis of Data

This chapter presents an analysis of data collected and describes the major findings of the study. The study attempts to answer the question: How do students learn mathematics with a computer-based laboratory course? Students' learning is examined in terms of learning mathematical concepts; developing skills of mathematical reasoning, problem solving and computation; improving abilities in mathematical intuition and communication; and developing positive attitudes toward mathematics and computers. Each of these is dealt with primarily on the basis of discussion and analysis of the student lab papers, questionnaires and interview data, and where pertinent, on data from the observations and videotapes.

#### *Learning mathematical concepts*

I investigated how a computing environment enabled students to reinforce mathematical concepts. I observed students doing the second exercise of the sixth topic of lab #0. The exercise defined a set of function  $\{e^x, e^{-x}, \sin x, x^2\}$  and required students to plot this set on the interval  $[-\pi .. \pi]$  and on the same coordinate system, and then guess which graph corresponded with which function. All groups plotted the given set of functions. For example

>  $y := \sin(x)$ ;

$y := \sin(x)$

>  $f := x^2$ ;

$f := x^2$

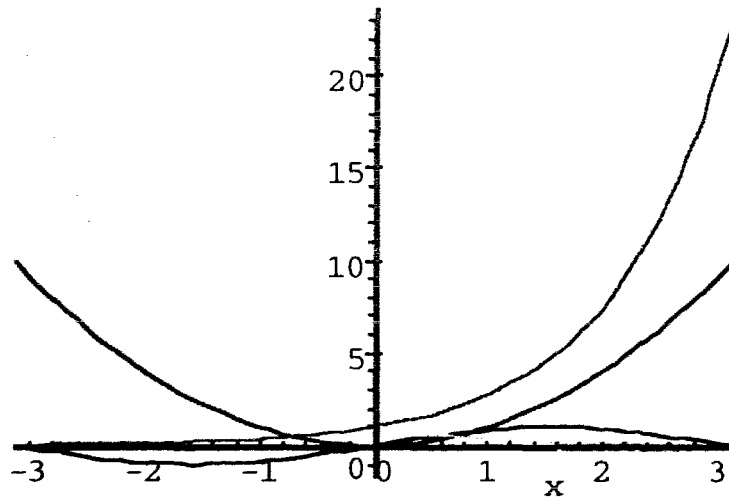
>  $\exp(x)$ ;

$e^x$

>  $\exp(-x)$ ;

$$e^{(-x)}$$

```
> plot({exp(x), exp(-x), y, f}, x=-Pi..Pi);
```



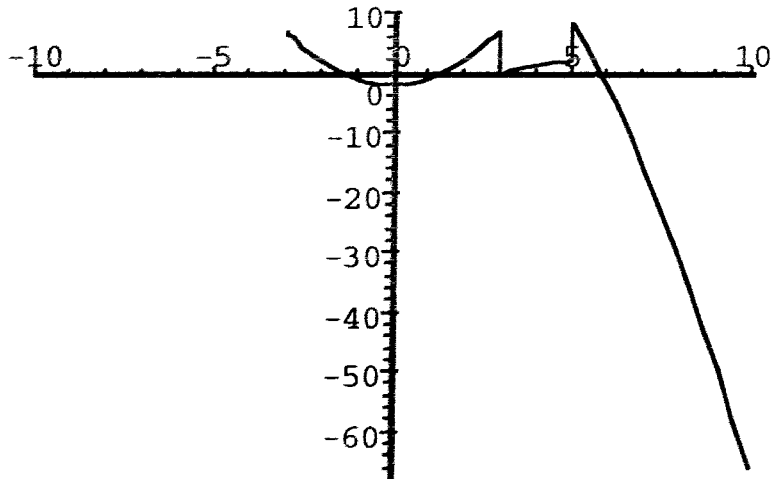
(Lab paper of Hai & Phan, lab session 2, September 19, 1995)

Some students did not recognize the graphs of functions of  $e^x$  and  $e^{-x}$ . I saw them ask classmates for help, and the students who understood explained the graphs to them. Students reinforced their knowledge of basic functions. Such circumstances occurred very often in the study. The following is a similar example of students reinforced mathematical concepts.

After plotting the piecewise function  $h(x)$  defined as

$$h(x) = \begin{cases} x^2 - 2, & -3 \leq x \leq 3 \\ x - 3, & 3 < x \leq 5 \\ -x^2 + 34, & x > 5 \end{cases}$$

```
( > plot(h, -10 .. 10); )
```



Thanh and Phan asked me:

Phan: Is there any case where the domains overlap?

Author: What do you mean by overlap?

Thanh: He means, for example, can we define a function like  $h(x)$  with some function in the interval from -3 to 3, and with another function in the interval from 2 to 5?

Author: Oh yes, I understand. Do you think you can?

The two students thought for a while and they could not answer the question.

Author: What do you think the graph would look like if you could plot the function you describe?

Phan pointed to the graph displayed on the computer's screen and said: We will have two curves on the overlapping interval.

Author: So if you choose a point in the overlapping interval, how many values of  $y$  can you evaluate?

Phan thought for a little while and then pointed to the graph and answered: I will have two values of  $y$  because it will have two curves here.

Thanh: Because it has two formulas to evaluate  $y$ -values.

Author: So far, one value of  $x$  we will have two different values of  $y$ . Do you think that is a function?

Thanh and Phan were silent and thought.

Author: Could you tell me the definition of a function?

Thanh answered haltingly: Hum ... a function is a map.

Author: So could you define what a map is?

Thanh and Phan did not state the definition completely. I helped them to remember. Students discovered that there were mathematical concepts they learned in high school, but which they did not understand clearly. The opportunities, such as those above, helped them to reinforce and understand the concepts more deeply. At the same time, their reasoning skills were developed.

Phan: There are many things we learned in high school but we don't remember and understand them clearly. I think when we do it on a computer, it helps us to remember and understand more deeply.

Trang: There are many things we learn in school that we don't remember or understand clearly, and I think computers, will help us to remember and understand more deeply.

(Interview 2, October 7, 1995)

Interaction between computer and students motivates students to learn new mathematical concepts. Learning becomes a need rather than a requirement to satisfy the teacher. The following anecdote demonstrates this.

While solving polynomial equations in the seventh topic of lab #0, students met solutions represented in form of complex numbers.

> dt2 := x^5/5 - 5\*x^3/3 + 4\*x;

$$dt2 := \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x$$

> solve(dt2 = 0, x);

$$0, \frac{1}{6}\sqrt{150 + 61\sqrt{95}}, -\frac{1}{6}\sqrt{150 + 61\sqrt{95}}, \frac{1}{6}\sqrt{150 - 61\sqrt{95}}, -\frac{1}{6}\sqrt{150 - 61\sqrt{95}}$$

(Lab paper of Thanh & Trang, lab session 2, September 19th, 1995)

They did not understand the solution because they had not yet studied the concept of complex numbers. They asked me for an explanation. I set aside ten minutes to present the

concept and gave them several exercises. I saw the students knew how to apply the definition of complex number for solving the exercises.

This phenomenon does not happen much in the lecture class where the teacher is usually active in providing mathematical knowledge to students and where students listen passively. Students learn new mathematical concepts from direct motivation of teachers. In computing environment the position of students is changed. They become users while computers offer tools and also challenges for them. Observing students doing mathematics on computers, I saw that they always wanted to maintain an ongoing process of interaction between themselves and computers. This motivated them to study responses from the computer. Through this, they explored and learned new mathematical concepts.

One important thing I found out through observation of the lab participants was that although they could do mathematics on computers, they were not sure they understood the mathematical concepts involved. In fact, this phenomenon also occurred when students learned mathematics in class. I saw many students in high school who could calculate definite integrals fluently, but did not understand the concept of definite integral.

I observed participants calculated the limit of a sequence as follows:

> f := n -> n/(n+1);

$$f := n \rightarrow \frac{n}{n+1}$$

> Limit(f(n), n=infinity);

$$\lim_{n \rightarrow \infty} \frac{n}{n+1}$$

> value("");

1

(Lab paper of Thanh & Phan, lab session 4, September 30, 1995)

I asked students to explain how they understood the definition of the limit of a sequence, but not all of them could do that. So how can a computer help students to learn mathematical concepts? I think this depends very much on the design of the lab course.



For example, in order to help students understand the concept of the limit of a sequence in terms of epsilon and a large number N, I gave a limit problem in the lab assignment #1 and solved it using the definition of limit of sequence.

Problem: Find limit of the sequence  $x_n = \frac{n}{n+1}$ .

> f := n -> n/(n+1);

$$f := n \rightarrow \frac{n}{n+1}$$

We give a number epsilon, denoted by e

> e := 0.001;

$$e := .001$$

We try finding whether there is a number N, such that from the (N+1)th number on we have  $|x_n - 1| < 0.001$ . We consider the difference

> d := abs(f(n) - 1) < e;

$$d := \left| \frac{n}{n+1} - 1 \right| < .001$$

We see it as an equality for n. Solve the inequality

> normal(d);

$$\frac{1}{|n+1|} < .001$$

> isolate(", abs(n+1));

$$|n+1| < -1000.$$

> solve(", n);

$$\{n < -1001.\}, \{999. < n\}$$

Because n is a natural number so we get the solution  $n > 999$ . Thus, if we choose

> N := 999;

$$N := 999$$

then from the 1000th number on we have  $|x_n - 1| < 0.001$ .

(The fourth topic, Lab Assignment #1)

I interpreted this problem to students. Afterward, I saw them study the example command by command and discuss it with each other. Sometimes they asked me for help. I talked to them and I thought they understood the concept of the limit of a sequence. One interesting point here was that they experimented using  $n$  bigger than 999 to see how the value of  $|x_n - 1|$  was smaller than the epsilon 0.001.

```
> evalf(abs(f(1000) - 1));  
                                .0009990009990
```

```
> evalf(abs(f(1005) - 1));  
                                .0009940357853
```

This helped them to see concretely how the definition worked. I suggested they make the value of the epsilon smaller, and repeating the commands, they got another value of  $N$ . I asked them to explain this.

Thanh: When we give the epsilon smaller the value of  $N$  is bigger.

Phan: The value of  $N$  changes and it depends on the epsilon.

Thanh: It means from this  $N$  on the values of  $|x_n - 1|$  will be smaller than the epsilon.

They discovered dependence of  $N$  on the epsilon. Through experimentation, they understood the concept more deeply. I think these learning activities would have been difficult to perform in class without a computer because the calculations would take too long. Besides providing students with the Maple commands that enable them to do quick computations, I think the lab course can help students study mathematical concepts.

The following citation demonstrates how computer graphics help students to approach mathematical concepts. In calculus, when learning the limit of a function one of the concepts that students find very difficult to understand is that of the nonexistence of a

limit. Most of them do not think that there are functions that have no limits even though their ranges can lie in some interval, or they cannot imagine what a function has no limit is like.

Thanh and Phan considered a special function given in the fifth topic of Lab #1:

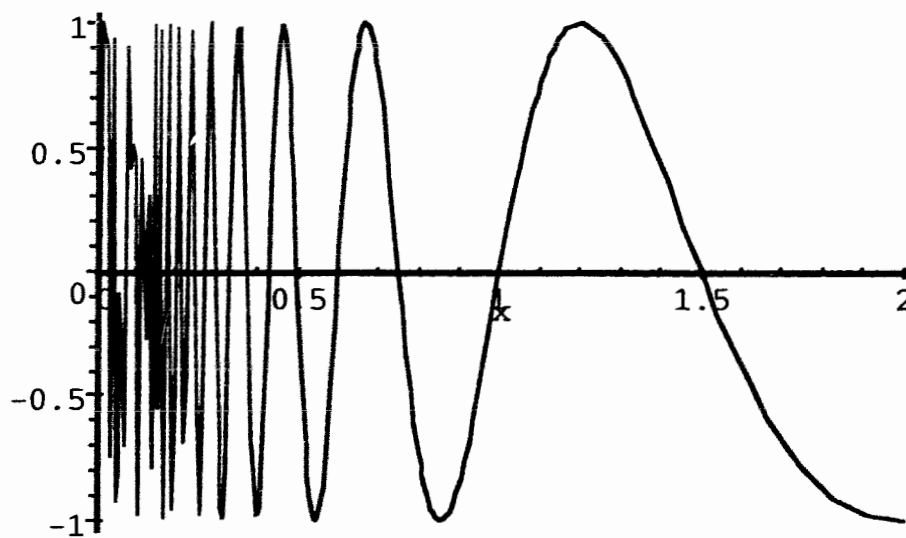
```
> f := x -> sin(3*Pi/x);
```

$$f := x \rightarrow \sin\left(3 \frac{\pi}{x}\right)$$

```
> f(x);
```

$$\sin\left(3 \frac{\pi}{x}\right)$$

```
> plot(f(x), x=0 .. 2);
```



```
> limit(f(x), x=0);
```

-1 .. 1

After they finished, I asked them some questions.

Author: What do you think about the graph?

Thanh: It looks like the graph of the function  $\sin x$ , but it's not well proportioned.

Phan: The values of  $y$  run from minus one to plus one.

Author: What do you think about  $y$  when  $x$  tends to zero?

They looked at the graph and thought.

Thanh: When  $x$  tends to zero then  $y$  will run between minus one and plus one.

Author: You mean  $y$  vacillates in the interval of minus one and plus one.

Thanh: Yes.

Author: So do you think  $y$  will be close to a specific value when  $x$  tends to zero?

They looked at the graph again and thought.

Thanh: No, I don't think so.

Author: What do you think about the limit of  $y$  when  $x$  tends to zero?

Phan: It does not have a limit, does it?

I think that the above process of exploration helped the students to realize a function can have no limit when  $x$  tends to some value. Clearly, it is difficult to perform such learning activities in traditional teaching environment, and without the assistance of computers.

### ***Developing mathematical reasoning skills***

Learning mathematics in class, students develop deductive reasoning skills. With computers, students develop inductive reasoning skills. I saw that computers enabled students to do many different problems effectively and then observe commonalities which they could conceptualize as general mathematical properties. The inductive process helped them approach mathematical concepts more easily. The following example illustrates this.

To discover the property of the combination function of the function and its inverse function, I guided students to perform some problems.

```
> f1 := x -> sin(x);
```

```
f1 := sin
```

```
> f2 := x -> arcsin(x);
```

```
f2 := arcsin
```

> g := f1 @ f2;

$g := \sin @ \arcsin$

> g(x);

$x$

> f1 := x -> x+1;

$f1 := x \rightarrow x + 1$

> f2 := x -> x-1;

$f2 := x \rightarrow x - 1$

> g := f1 @ f2;

$g := f1 @ f2$

> g(x);

$x$

Reasoning inductively from the results, I saw the students conclude that the combination function  $(f @ f^{-1})(x) = x$ .

Mai: Thus, if we combine a function and its inverse function then we will have the function  $x$ .

I believe that when students make such discoveries, they see mathematical relationships in concrete terms and believe the reasonableness of the result.

I used this method to help students to understand the definition of the limit of a function in terms of the limit of a sequence through performing the following example.

> f := x -> x\*sin(1/x);

$f := x \rightarrow x \sin\left(\frac{1}{x}\right)$

> a := (n) -> 1/n;

$a := n \rightarrow \frac{1}{n}$

> limit(a(n), n=infinity);

0

> f(a(n));

$$\frac{\sin(n)}{n}$$

> limit(f(a(n)), n=infinity);

0

> b := (n) -> 1/3^n;

$$b := n \rightarrow \frac{1}{3^n}$$

> limit(b(n), n=infinity);

0

> f(b(n));

$$\frac{\sin(3^n)}{3^n}$$

> limit(f(b(n)), n=infinity);

0

(Lab paper of Thanh & Phan, lab session 6, October 14th, 1995)

In next example, with another function, students built two different sequences,  $a_n$  and  $b_n$ , that both tended to 2 (when  $n$  tended to infinity) and investigated the limits of the corresponding sequences  $f(a_n)$  and  $f(b_n)$  (when  $n$  tended to infinity).

> f := x -> x+1;

$$f := x \rightarrow x + 1$$

> limit(f(x), x=2);

3

> a := n -> 2 + 1/n;

$$a := n \rightarrow 2 + \frac{1}{n}$$

> limit(a(n), n=infinity);

2

> limit(f(a(n)), n=infinity);

3

> b := n -> 2 + 1/3^n;

$$b := n \rightarrow 2 + \frac{1}{3^n}$$

> limit(b(n), n=infinity);

2

> limit(f(b(n)), n=infinity);

3

(Lab paper of Thanh & Phan, lab session 6, October 14th, 1995)

The inductive process helped students to approach the definition:

“The function  $f(x)$  has the limit  $L$  when  $x$  tends to  $a$ , if for any sequence  $\{x_n\}$  such that  $x_n \neq a$  as  $x_n \rightarrow a$  (when  $n \rightarrow \infty$ ) we have the corresponding sequence  $f(x_n) \rightarrow L$  (when  $n \rightarrow \infty$ )”. When I talked to students I thought they understood the definition.

### ***Developing problem solving skills***

A computing environment enables students to explore mathematics and, in the process, improve problem solving skills. I saw that students did not perform exactly what was presented in the lab assignments, but liked to add, or revise data, commands or assumptions as they aimed to experiment and explore.

For example, in the sixth topic of Lab #0, there was a paragraph involving computing coordinates of intersection points of two graphs written as follows:

> f := x^2 + 3;

$$f := x^2 + 3$$

> g := 7 + 3\*x - 5\*x^2;

$$g := 7 + 3x - 5x^2$$

> q := solve(f=g, x);

$$q := \frac{1}{4} + \frac{1}{12}\sqrt{105}, \frac{1}{4} - \frac{1}{12}\sqrt{105}$$

> q[1];

$$\frac{1}{4} + \frac{1}{12}\sqrt{105}$$

> q[2];

$$\frac{1}{4} - \frac{1}{12}\sqrt{105}$$

> y1 := subs(x=q[1], f);

$$y1 := \left(\frac{1}{4} + \frac{1}{12}\sqrt{105}\right)^2 + 3$$

> y2 := subs(x=q[2], f);

$$y2 := \left(\frac{1}{4} - \frac{1}{12}\sqrt{105}\right)^2 + 3$$

After doing this paragraph, the group of Thanh and Trang inserted three more commands to compute values of y-coordinates of the intersection points, to check if they were equal. This helped them to see a clearer relationship between the roots of the equation and the graph.

> y3 := subs(x=q[2], g);

$$y3 := \frac{31}{4} - \frac{1}{4}\sqrt{105} - 5\left(\frac{1}{4} - \frac{1}{12}\sqrt{105}\right)^2$$

> evalf(y2);

$$3.364710385$$

> evalf(y3);

$$3.364710381$$

(Lab paper of Thanh & Trang, Lab Session 1, September 14, 1995)

The desire to explore mathematics was also expressed in the statements of the students in the interview.

**Author:** When I read your papers, I saw other commands besides commands I wrote in the lab assignments. Could you give me an explanation of this?



Thanh: Oh, yes. They are commands that I wanted to try to see how Maple worked. I think we should try to solve other exercises besides the lab assignments. And these make learning more interesting.

Phan: When we try this we feel very interested. We can see whether or not computers do what we guessed they would. I also bring some of my exercises to try but I don't always have enough time.

(Interview 2, October 10, 1995)

Such exploration does not often occur in traditional way of mathematics learning in Vietnam. In class, students have to follow the teacher's presentation of lectures and their roles are often passive rather than active. It is difficult for them to stop to check their understanding or feel free to explore hunches during lectures. On the other hand, if students have the chance to ask the teacher questions, it is difficult to give them solutions quickly and precisely without taking too much time, particularly for questions involving computations.

Sometimes, student explorations got me very interested. For example, when considering the graphical relationship between graphs of the function and its inverse function, Thanh and Phan did the following:

> f := 1 + sqrt(1+x^2);

$$f := 1 + \sqrt{1 + x^2}$$

> q := solve(f=y, x);

$$q := \sqrt{-1 + (-1 + y)^2}, -\sqrt{-1 + (-1 + y)^2}$$

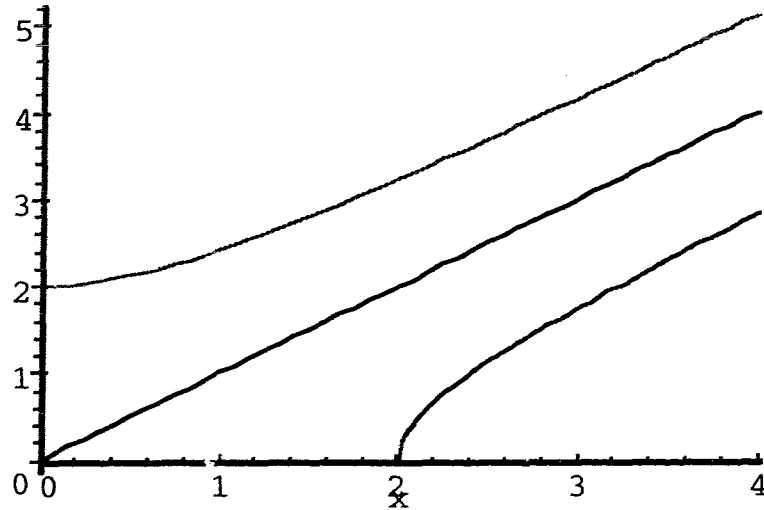
> q[1];

$$\sqrt{-1 + (-1 + y)^2}$$

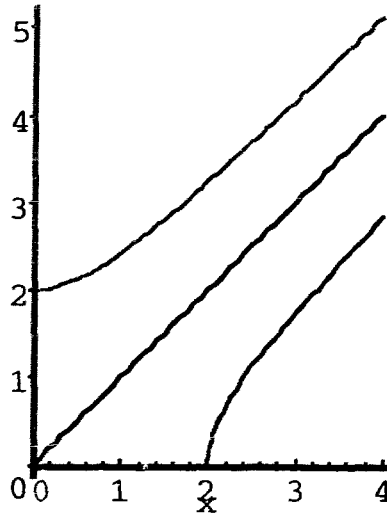
> g := subs(y=x, q[1]);

$$g := \sqrt{-1 + (-1 + x)^2}$$

> plot({x, f, g}, x=0..4);



But the result did not look good, it did not clearly express the graphical relationship. They suggested I choose the option 1:1 of the graphical window to make the scales of y- and x-axes equal, so that the graphs would look better. I had not thought of this before.



This example shows the changed role of the teacher in a computing environment. The teacher becomes a person who provides students with working conditions, and generally guides them. Students do not expect the teacher to know everything. I saw this made students feel interested and motivated. Such explorations helped students to improve skills of using Maple in problem solving.

I saw students used pattern recognition in solving problems with computers very well. I think they were affected by a style of learning mathematics from high school which had developed this skill. For instance, the lab assignment gave a problem and presented steps and commands to solve it, then if there was an exercise similar to the given problem, students knew how to apply the pattern they learned to solve the exercise.

In the seventh topic of lab #0, there was the following example of finding roots of the polynomial:

> p2 := x^5/5 - 5\*x^3/3 + 4\*x;

$$p2 := \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x$$

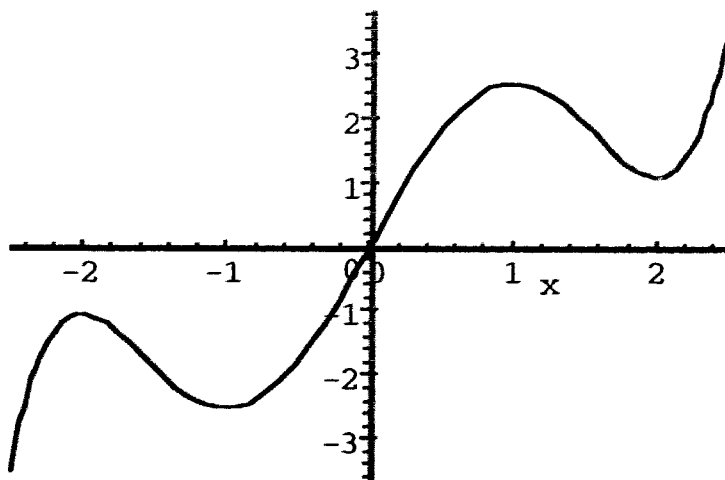
> roots(p2);

[[0, 1]]

> factor(p2);

$$\frac{1}{15}x(3x^4 - 25x^2 + 60)$$

> plot(p2, x=-2.5 .. 2.5);



> solve(p2=0, x);

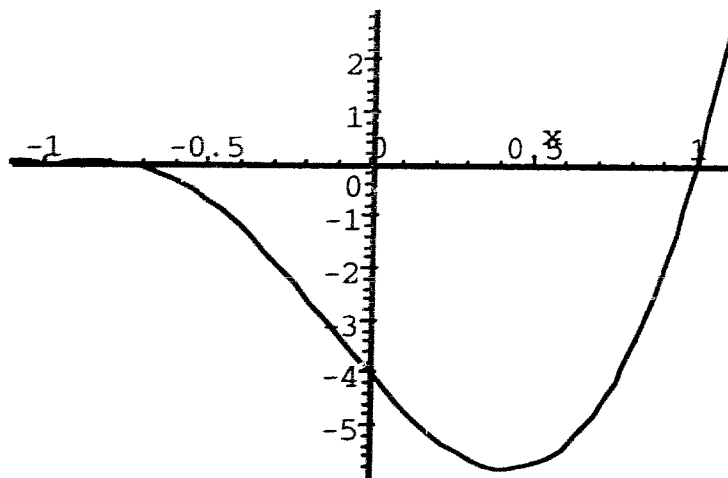
$$0, \frac{1}{6}\sqrt{150 + 61\sqrt{95}}, -\frac{1}{6}\sqrt{150 + 61\sqrt{95}}, \frac{1}{6}\sqrt{150 - 61\sqrt{95}}, -\frac{1}{6}\sqrt{150 - 61\sqrt{95}}$$

Thanh and Trang applied this pattern to find roots of the polynomial given in another exercise:

> dathuc := x^6 - x^5 + 8\*x^3 + 3\*x^2 - 7\*x - 4;

$$dathuc := x^6 - x^5 + 8x^3 + 3x^2 - 7x - 4$$

> plot(dathuc, x=-1.1 .. 1.1);



> nghiem := solve(dathuc=0, x);

$$\begin{aligned} \text{nghiem} := & 1, -\%1^{1/3} + \frac{5}{9} \frac{1}{\%1^{1/3}} + \frac{2}{3}, \\ & \frac{1}{2} \%1^{1/3} - \frac{5}{18} \frac{1}{\%1^{1/3}} + \frac{2}{3} + \frac{1}{2} I \sqrt{3} \left\{ -\%1^{1/3} - \frac{5}{9} \frac{1}{\%1^{1/3}} \right\}, \\ & \frac{1}{2} \%1^{1/3} - \frac{5}{18} \frac{1}{\%1^{1/3}} + \frac{2}{3} - \frac{1}{2} I \sqrt{3} \left\{ -\%1^{1/3} - \frac{5}{9} \frac{1}{\%1^{1/3}} \right\}, -1, -1 \\ \%1 := & \frac{73}{27} + \frac{1}{9} \sqrt{606} \end{aligned}$$

> evalf(nghiem);

1., -.7760454353, 1.388022718 - 1.796588525 I, 1.388022718 + 1.796588525 I, -1., -1.

```
> evalf(subs(x=nghiem[5], dathuc));
```

O

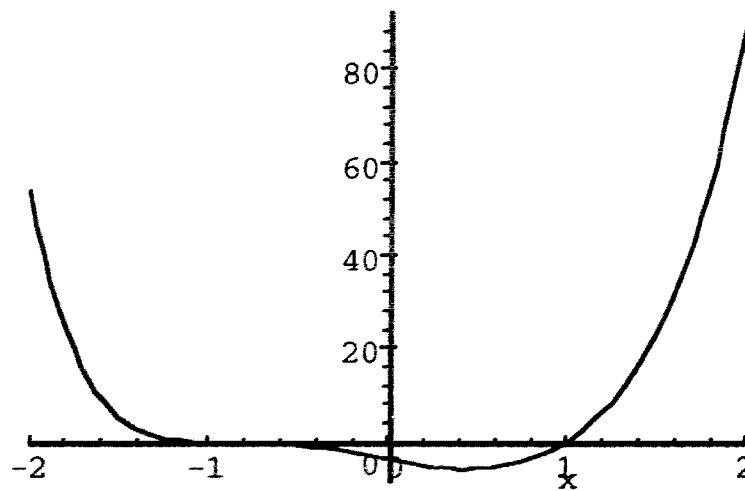
(Lab paper of Thanh & Trang, lab session 2, September 16, 1995)

Mai and Thu also applied patterns in a similar way.

```
> dathuc := x^6 - x^5 + 8*x^3 + 3*x^2 - 7*x - 4;
```

$$dathuc := x^6 - x^5 + 8x^3 + 3x^2 - 7x - 4$$

```
> plot(dathuc, x=-2 .. 2);
```



```
> solve(dathuc=0, x);
```

$$\begin{aligned} &1, -\%1^{1/3} + \frac{5}{9} \frac{1}{\%1^{1/3}} + \frac{2}{3} \frac{1}{2} \%1^{1/3} - \frac{5}{18} \frac{1}{\%1^{1/3}} + \frac{2}{3} + \frac{1}{2} I \sqrt{3} \left( -\%1^{1/3} - \frac{5}{9} \frac{1}{\%1^{1/3}} \right) \\ &\frac{1}{2} \%1^{1/3} - \frac{5}{18} \frac{1}{\%1^{1/3}} + \frac{2}{3} - \frac{1}{2} I \sqrt{3} \left( -\%1^{1/3} - \frac{5}{9} \frac{1}{\%1^{1/3}} \right), -1, -1 \\ &\%1 := \frac{73}{27} + \frac{1}{9} \sqrt{606} \end{aligned}$$

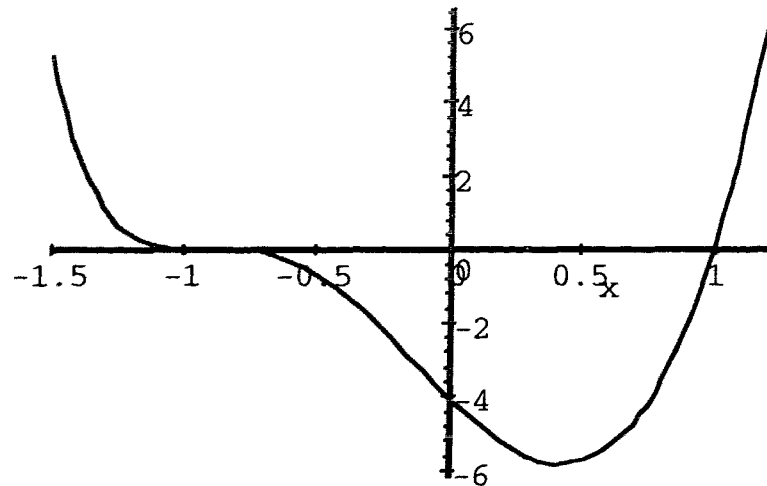
```
> roots(dathuc);
```

$$[[-1, 2], [1, 1]]$$

```
> factor(dathuc);
```

$$(x-1)(x^3-2x^2+3x+4)(x+1)^2$$

```
> plot(dathuc, x=-1.5 .. 1.2);
```



(Lab paper of Mai & Thu, lab session 2, September 16, 1995)

These students tried to use the `plot()` command with different domains of  $x$  based on the solutions of the `solve()` command to finally get a good graph that showed the roots of the polynomial more clearly.

Solving the above problem also illustrates that problem solving approaches using computers have characteristics different from the traditional approach. In class students solve the problem of finding the roots of a polynomial by applying algebraic transformations till they find simpler equations, or they use ready-made formulas to calculate solutions. By using computers, students first plot the polynomial on different domains to find out which domains contain roots, and then solve the polynomial on appropriate domains. Or students can solve the polynomial first, and then plot it to check the roots on graph. Each method has its own advantages. The first helps students to see the process of solving a problem in detail, and the second one helps them to see the relationship between the graph and roots of an equation. Of course, the second one could also work in class but it takes a great deal of time and effort for teacher and students, and it is not animated as the computer. With computers, students can change several parameters and see graphs in many different aspects.

I think students need to know different ways of problem solving, and they need to get practical experience in all these ways. For example, through doing the lab the students accumulated some experiences of plotting:

Trang: First, I plot the function on an arbitrary domain of  $x$ , and then I see somewhere it is not clear or it can cut the  $x$ -axis. I will note those intervals, and then I change values of  $x$  and  $y$  here (she shows on the screen where she changes the domains of  $x$  and  $y$  in the `plot()` command) and I plot it again. But if the graph is not good again I will choose a smaller interval for  $x$ . When it is good I move the cursor to its intersection point with the  $x$ -axis and I can see its approximate value, and then I estimate the interval around it and I use the `fsolve()` command to calculate on that interval.  
(Interview 1, September 21, 1995)

There has been some concern that when using computers for problem solving, students do not see the reasoning process; or if calculating between the problem given and solution obtained can cause students to not develop reasoning skills. I saw this happen in the study. I observed that participants found a combination of two functions of  $f(x) = x^3$  and  $g(x) = 3x^2 - 2$  by typing commands, and Maple immediately delivered a solution

> f := x -> x^3;

$$f := x \rightarrow x^3$$

> g := x -> 3\*x^2 - 2;

$$g := x \rightarrow 3x^2 - 2$$

> (f@g)(u);

$$(3u^2 - 2)^3$$

(Lab paper of Thu & Hai, lab session 3, September 21, 1995)

I asked them to explain how to calculate the result to check whether they understood the definition of combination function. Some students could not do that. If doing the problem with pencil and paper students would have to perform

$$(f@g)(u) = f[g(u)]$$

$$= [g(u)]^3$$

$$= (3u^2 - 2)^3$$

This way they can see now to get the result and understand the definition of combination function better.

Thus, the computer acts as a black box. It takes input data from students and provides them output data. The calculation process is hidden. In order to help students develop problem solving skills completely in computer-based learning environment the curriculum needs to be designed to help students make a study of the problem solving processes.

I presented to students the definition of combination function and showed them how to apply this definition to find a combination of two functions. Then, students used computers to calculate a combination of three functions as follows:

> f := x -> x / (x+1);

$$f := x \rightarrow \frac{x}{x+1}$$

> g := x -> x^10;

$$g := x \rightarrow x^{10}$$

> h := x -> x + 3;

$$h := x \rightarrow x + 3$$

> (f @ g @ h)(x);

$$\frac{(x+3)^{10}}{(x+3)^{10} + 1}$$

(Lab paper of Trang & Mai, lab session 3, September 21, 1995)

The lab assignments should be designed so that, besides computational tasks, students are required to interpret computed results reasonably. I think the interpretation helps students to understand the problem solving process and develop reasoning skills. Problems that require student interpretation should be selected so that they are not too complicated in terms of computations, but highlight necessary factors for interpretation. For instance, to let students interpret ways of finding a combination function, it is not necessary



to give complicated component functions. Once students can conceptualize the problem solving process, they can use the computer to solve more complicated problems. In the above situation, if the teacher asks students to explain the results students will have chance to make a study of the solving problem process and they will see how the definition of a combination function works. Once they understand, they can use a computer to calculate a combination of, not two, but many functions as in the above example.

In addition, students' learning strategies also play an important role in using computers to learn mathematics in general, and learning problem solving skills in particular. I argue that students need to be aware of characteristics, abilities and limitations of computers when learning mathematics with computers.

I saw that in the lab sessions at the beginning of the study, students paid attention to typing commands correctly and executing them to get results. If the computer did not show any error, then students felt assured about the results and continued. They did not think much about how to get the results. In other words, their learning strategies were oriented to the results rather than the process. In later sessions I asked them to interpret solutions and present ways of getting to the results. I saw that they found this difficult, and in interpreting the problems, they discovered that there were things they thought they understood well but as a matter of fact understood only vaguely. They began to realize that the problems were not as simple as they thought. They changed their study strategies. They concentrated on thinking more about the problem solving process, discussed results with each other and sometimes checked results using paper and pen. Their speed of learning was slower but the learning was of higher quality. Students realized that this was how they should learn mathematics in a computer-augmented environment.

**Thu:** I think that we should combine studying in class with using computers. It is better because computers don't show us how we calculate solutions.

**Author:** Could you give me an example of that?

Thu: Yes, hum ... It's like when we solve a quadratic equation, if we solve it by computers then we don't know how to solve, but if we solve it by hand, step by step, then we understand how it works.

Phan: I think we need to study in class to understand the theory, and then solve problems on computer, because if we don't know the theory we can't understand the results from the computer.

Trang: Yes, if after learning theory in class we practice what we learn on the computer then our knowledge will be consolidated, because we will understand and remember longer.

(Interview 2, October 7, 1995)

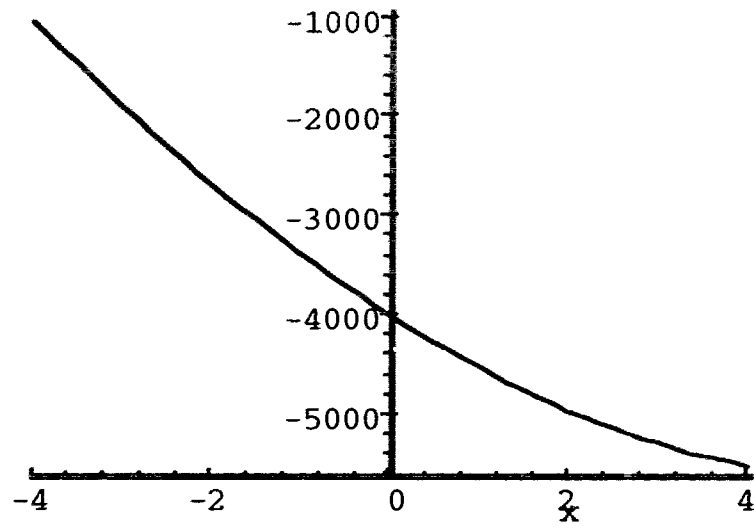
Thus, to help students to develop problem solving skills when learning mathematics with computers, the teacher needs to create opportunities for students to discuss not only computers or mathematics, but also learning strategies and experiences in a computer-augmented environment.

In doing problem solving with computers, students need to evaluate numerical and graphical solutions given by the computer. For example, I observed students solve the third exercise in the sixth topic of lab #0. This exercise plotted the function  $y = x^3 + 45x^2 - 574x - 4007$  alternately on three different intervals of  $-4$  to  $4$ ,  $-50$  to  $10$ , and  $-75$  to  $25$ .

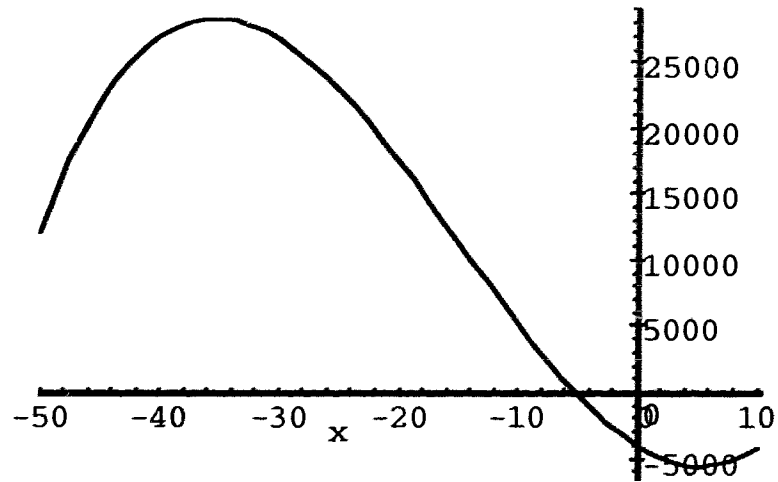
```
> y := x^3 + 45*x^2 - 574*x - 4007;
```

$$y := x^3 + 45x^2 - 574x - 4007$$

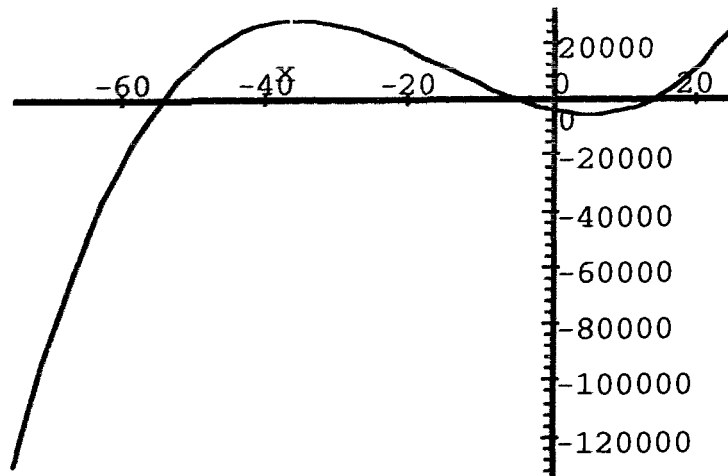
```
> plot(y, x=-4..4);
```



```
> plot(y, x=-50..10);
```



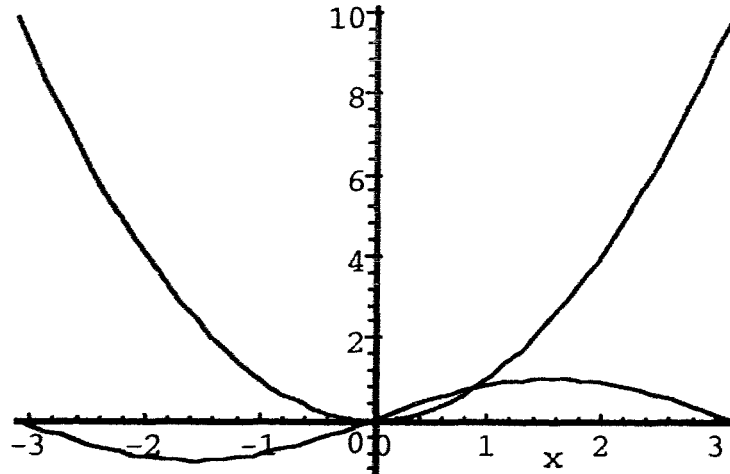
```
> plot(y, x=-75..25);
```



Students had to adjust the domain of  $x$  a number of times in order to get right graph, or more precisely right shape of the graph for the given function. If students accepted and did not evaluate the first graph then they could get a wrong picture of the graph of a degree three polynomial. To estimate solutions students have to have mathematical knowledge and practical experience using Maple. I think these two processes affect each other, because through estimating, students' mathematical understanding and skills in using Maple are both upgraded.

In another circumstance, I observed the students plot the graphs of two functions  $y = x^2$  and  $y = \sin x$  on the same coordinate system.

```
> plot({x^2, sin(x)}, x=-Pi..Pi);
```



Students saw that two graphs cut each other at two distinct points so the equation of  $x^2 = \sin x$  would have two distinct roots in the interval of  $[-\pi .. \pi]$ . But solving this equation

```
> fsolve(x^2 = sin(x), x=-Pi..Pi);
```

.8767262154

only gave the value of one root, and they had to solve the equation one more time on a narrower domain to get the other root.

```
> fsolve(x^2 = sin(x), x=-0.5..0.5);
```

0

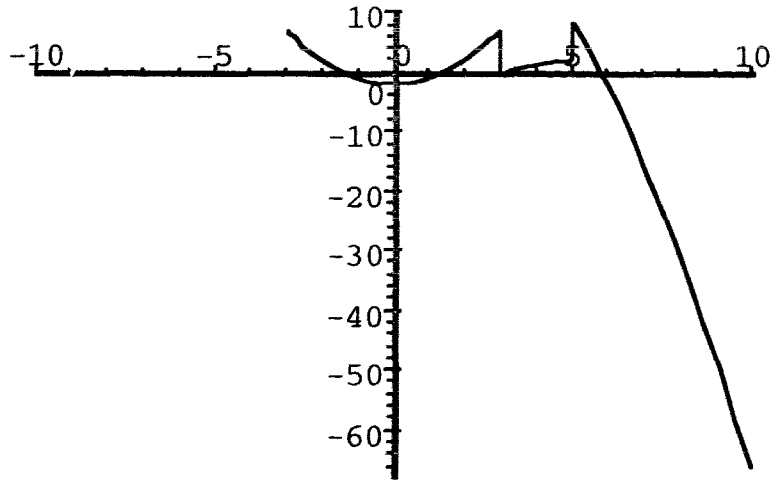
That was a property of the `fsolve()` command, which did not necessarily find all roots of the equation.

I saw the students apply their understanding of relationship between graph and roots of equation to evaluate the solution given by the `fsolve()` command. At the same time, after I explained the properties of this command, students had experience using it in later exercises. Through such cases, I saw the students got in the habit of evaluating solutions and accumulating experiences in using Maple. After doing an exercise or problem, they stopped for a little while to consider the results, and sometimes they added, deleted or changed data or parameters to have a more complete evaluation of solutions.

I said that there was a need for students to make a study of the problem solving process, but in fact it is not a process that is obvious and easy for students to understand. For instance, in the above example it is difficult to solve the equation of  $x^2 = \sin x$  by elementary transformations, and I think students cannot interpret how Maple solves the equation given their mathematical background. Thus the teacher needs to decide which problem is appropriate to students studying the mechanism of problem solving.

We consider another case in which the solution given by Maple is not really mathematically correct. In the first topic of lab #1, students plotted a piecewise defined function as following

```
> h := proc(x)
>   if -3<=x and x<=3 then x^2 - 2
>   elif 3<x and x<=5 then x - 3
>   elif x>5 then -x^2 + 34
>   else undefined
>   fi
> end;
h := proc(x)
    if -3 <= x and x <= 3 then x^2-2
    elif 3 < x and x <= 5 then x-3
    elif 5 < x then -x^2+34
    else undefined
    fi
end
> plot(h, -10..10);
```



(Lab paper of Trang & Mai, lab session 3, September 21, 1995)

The graphical solution showed unwanted lines connecting portions of the graph that should not be connected. When I asked students to remark on the solution, nobody indicated these lines. But when I guided them to solve the problem by themselves, they understood and saw the weakness of computing solution. When they solved it using paper and pencil methods, the lines did not appear because the logic of the reasoning process guaranteed the reasonableness of the solution. After performing this problem, we had an interesting dialogue:

Hai: So how do we know when the solution is right, and when it is wrong?

Author: Do you think in this case the computer gave us the wrong solution?

Phan: No, I think it's not wrong but it's not precise.

Hai: So when do we know it's not precise?

Thanh: We have to think about it.

Hai: So the computer is also limited.

Thus, we can see that in reality there are students who might think computers knew all the answers and believe that the computer's answers cannot be wrong. This kind of thinking is dangerous, because it makes students believe computers are superhuman. Such

distortions of the relationship between humans and computers should be avoided. Consideration of solutions helps students to realize the abilities and limitations of computers, and develop better problem solving and reasoning skills.

I found that students did not always know how to transfer the mathematical concepts in their mind to the commands of Maple. I observed Phan and Thanh solving the third exercise of the first topic of lab #1. The exercise was given as following:

"Find a second-degree polynomial such that its graph passes through the three points [0, 1], [2, 4], and [3, 1]. Plot the graph of the polynomial."

This is a common exercise that many students met in high schools so Thanh and Phan knew how to solve it by pen and paper methods. However they did not know how to begin to solve it using Maple so they decided to do it by hand (paper and pen) first and then to transfer their work to Maple commands. But in fact, after they finished solving it by hand they still found it difficult to do with Maple, and they had to ask me for help. I think it takes time to overcome this difficulty. If students have the chance to practice with Maple they will master it. The exercise was solved with Maple as follows:

```
> y := a*x^2 + b*x + c;
```

$$y := a x^2 + b x + c$$

```
> eq1 := subs(x=0, y) = 1;
```

$$eq1 := c = 1$$

```
> eq2 := subs(x=2, y) = 4;
```

$$eq2 := 4 a + 2 b + c = 4$$

```
> eq3 := subs(x=3, y) = 1;
```

$$eq3 := 9 a + 3 b + c = 1$$

```
> solve({eq1, eq2, eq3});
```

$$\left\{ c = 1, a = \frac{-3}{2}, b = \frac{9}{2} \right\}$$

```
> subs(", y);
```



$$-\frac{3}{2}x^2 + \frac{9}{2}x + 1$$

> f := ";

$$f := -\frac{3}{2}x^2 + \frac{9}{2}x + 1$$

There were different characteristics evident in students' problem solving strategies when solving by hand and with computers, but the students themselves did not realize this yet. For instance, when solving the above problem using pen and paper they had to take care of everything including putting a second-degree function into general form, replacing coordinate values to set up a system of three equations, calculating to solve this system by replacing roots from the first function to get a concrete function satisfying the conditions of the problem and then plotting this function. In each such step they had to do a series of detailed computations. This work devoured students' time and energy, and students found it difficult to recognize key ideas or factors in the process of problem solving.

With computers, students did not have to pay attention to computational operations, but instead had to think about strategies or algorithms and to decide which factors or arguments were key to solving the problem. They needed to think about which parts could be solved easily by computers. At the same time they had to evaluate the reasonableness of results given by computers.

However, students were in the habit of working with paper and pen. I think their ways of thinking as well as their learning behaviors depended partly on the tools which they used. In some cases I saw that they found a way of applying Maple to match their old ways of thinking rather than using Maple as a powerful computational tool. For instance, in order to substitute the values of  $x=0$  and  $y=1$  into the expression  $y = a*x^2 + b*x + c$  they commanded  $1 = a*(0)^2 + b*0 + c$  instead of  $\text{subs}(x=0, y) = 1$ . In order to solve the following system of three equations:

$$c = 1$$

$$4*a + 2*b + c = 4$$

$$9*a + 3*b + c = 1$$

they tried to substitute the value of  $c=1$  into the other two equations

$$4*a + 2*b + c = 4$$

$$9*a + 3*b + c = 1$$

rather than use the command

```
solve({eq1, eq2, eq3});
```

Generally speaking, students could direct Maple to calculate an expression, plot a function, or solve a simple equation, but they could not combine commands to solve a complicated problem. I think in order to use Maple effectively for solving complicated problems students need time to drill and practise with Maple.

Another characteristic typical of the use of computers in problem solving was that students did not see the mathematical assumptions needed in the process of problem solving. While working with paper and pen these assumptions are clear. For example in the second topic of lab #1 that dealt with the implicit function, students using Maple found two branches of the function defined implicitly in form of the quadratic expression  $q$  by solving the equation  $q=0$  as following

```
> q := 17*y^2 + 12*x*y + 8*x^2 - 46*y - 28*x + 17;
```

$$q := 17 y^2 + 12 x y + 8 x^2 - 46 y - 28 x + 17$$

```
> r := solve(q=0, y);
```

$$r := -\frac{6}{17}x + \frac{23}{17} + \frac{2}{17}\sqrt{-25x^2 + 50x + 60}, -\frac{6}{17}x + \frac{23}{17} - \frac{2}{17}\sqrt{-25x^2 + 50x + 60}$$

Two branches of the implicit function  $q(x, y)$  were

```
> y1 := r[1];
```

$$y1 := -\frac{6}{17}x + \frac{23}{17} + \frac{2}{17}\sqrt{-25x^2 + 50x + 60}$$

> y2 := r[2];

$$y2 := -\frac{6}{17}x + \frac{23}{17} - \frac{2}{17}\sqrt{-25x^2 + 50x + 60}$$

Students got the solutions for y1 and y2 quickly, but the condition of existence of two solutions y1 and y2 was not seen in the process of finding the solutions. Meanwhile, if they solved the equation of q=0 by hand, then with normal logic they would always check the condition of existence of roots, that is  $-25x^2 + 50x + 60 \geq 0$ , before proceeding to calculate the roots concretely.

A similar circumstance arose when students solved the second exercise in the third topic of lab #1. The exercise was given as follows: "Consider the function  $y = x^2 + 3x$ . Under which conditions does the function has an inverse function? Find the inverse function". Most students knew how to find the inverse function.

> f := x^2 + 3\*x;

$$f := x^2 + 3x$$

> h := solve(y=f, x);

$$h := -\frac{3}{2} - \frac{1}{2}\sqrt{9+4y}, -\frac{3}{2} + \frac{1}{2}\sqrt{9+4y}$$

> h[1];

$$-\frac{3}{2} - \frac{1}{2}\sqrt{9+4y}$$

> h[2];

$$-\frac{3}{2} + \frac{1}{2}\sqrt{9+4y}$$

> subs(y=x, h[1]);

$$-\frac{3}{2} - \frac{1}{2}\sqrt{9+4x}$$

> subs(y=x, h[2]);

$$-\frac{3}{2} + \frac{1}{2}\sqrt{9+4x}$$

(Lab paper of Trang & Mai, lab session 4, September 30, 1995)

But they did not answer the question about the mathematical conditions under which the inverse function exists. Maple did not show students the mathematical assumptions used in problem solving; they had to check these for themselves. Hence, there is a need for students to be aware of necessary mathematical assumptions when learning problem solving using computers.

The computer does not give students formal proofs. For instance, it cannot show students how a mathematical theorem is proved. One student (Mai) asked me an interesting question - "Can we use Maple to prove an inequality?". I did not think Maple could do it, but from that question I considered using Maple as a "word processor" for mathematics, to present normal proofs. I tried to use this in helping students to solve an exercise involving sequences.

To prove the sequence of  $f_n$  was decreasing I guided students to perform the following:

> f := n -> (n+2)^2/2^(n+2);

$$f := n \rightarrow \frac{(n+2)^2}{2^{(n+2)}}$$

> g := f(n+1) / f(n);

$$g := \frac{(n+3)^2 2^{(n+2)}}{2^{(n+3)} (n+2)^2}$$

> h := simplify(y);

$$h := \frac{1 (n+3)^2}{2 (n+2)^2}$$

> h;

$$\frac{1 (n+3)^2}{2 (n+2)^2}$$

> h < 1;

$$\frac{1}{2} \frac{(n+3)^2}{(n+2)^2} < 1$$

> h\*2\*(n+2)^2 < 2\*(n+2)^2;

$$(n+3)^2 < 2(n+2)^2$$

> expand("");

$$n^2 + 6n + 9 < 2n^2 + 8n + 8$$

> rhs("") - lhs("") > 0;

$$0 < n^2 + 2n - 1$$

> solve("", n);

$$\{n < -1 - \sqrt{2}\}, \{-1 + \sqrt{2} < n\}$$

> evalf("");

$$\{n < -2.414213562\}, \{.414213562 < n\}$$

I saw students were very interested in doing this exercise. Although we still had to think how to prove the inequality  $f(n+1) / f(n)$  less than 1 we used Maple to present the proof process to students, and it helped us to do all computations which appeared during proving. From that experience, I think teachers can use Maple to present the process of problem solving to students before providing them the direct commands of Maple that enable them to get solutions quickly.

Another thing I found out in the study was that, when using computers to do mathematics, students could reuse a sequence of commands from a certain algorithm to solve problem. For instance, after solving the exercise: "Find a second-degree polynomial such that its graph passes through three points [0, 1], [2, 4], and [3, 1]. Plot the graph of the polynomial", I saw some students reuse the sequence of commands

> y := a\*x^2 + b\*x + c;

> eq1 := subs(x=0, y) = 1;

> eq2 := subs(x=2, y) = 4;

```
> eq3 := subs(x=3, y) = 1;  
> solve({eq1, eq2, eq3});  
> subs("", y);  
> f := “;
```

to solve a similar problem. They only substituted the data in the commands. In other words, they reused the process of solving to solve a class of similar problems. I think this is very effective. It helps students to not spend a great deal of time repeating the same problems many times.

In addition, there was a problem in the study involving evaluation of learning outcomes. While using computers to solve equations one student (Hai) asked me, " Could I use this way to do exercises given in class?" He wanted to know if he could use the commands of Maple like solve() to solve equations. I answered "Why not? You can do it any way you like, and I think it's good if you know as many different ways as possible." I wanted to encourage him to explore different ways of problem solving, but I did not really understand his question. I only understood it after he asked the next question "So would the teacher mark it if I solve by this way?" I could not answer this immediately. I thought for a little while and said, "I don't know. But, anyway I think that if later this lab is taught on a regular basis, your teacher will let you know about this."

This made me think, and I tried to find out some colleagues' opinions. We discussed this problem and they had different opinions, although all of us agreed that the lab course should be taught to students. I think I have to use the verbs "solve", "prove" or "calculate" more carefully in the assignment labs. At the same time I needed to explain the meanings of these words to students in different contexts. I think a good way to help students to clarify these terms is to indicate the means students can use when doing mathematics, for instance "using Maple to solve the equation ...", or "solve the equation ... by using algebraic transformations."

### *Developing computational skills*

In the process of performing the study I saw students develop computational skills. In a computing environment, I think we should not understand the term “computational skills” as the ability to do addition, subtraction, multiplication, division or algebraic transformations but instead the ability to use computers to do complicated computational tasks. At the end of the study, I saw students had experience in using Maple to do mathematics. They realized good points as well as weak points of using computers to do mathematics. They knew how to use the commands and symbols of Maple to calculate algebraic expressions, solve equations, describe a function, plot a function, calculate values of a function.

```
> pt := x^3 + 4*x^2 + x - 6;
```

$$pt := x^3 + 4x^2 + x - 6$$

```
> factor("");
```

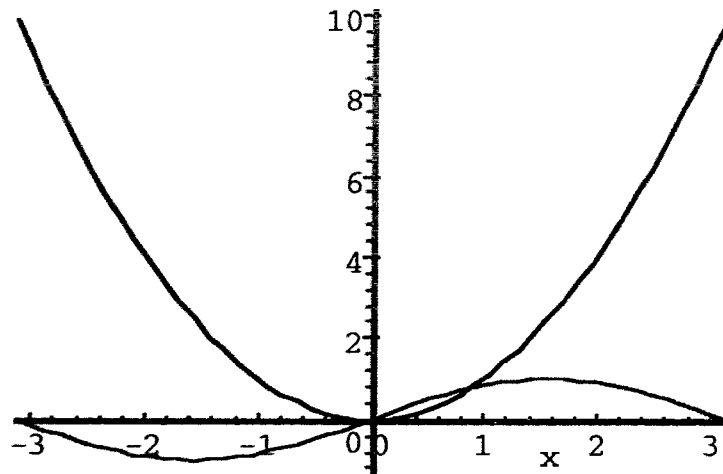
$$(x - 1)(x + 3)(x + 2)$$

```
> solve(pt, x);
```

$$1, -3, -2$$

(Lab paper of Hai & Phan, lab session 1, September 14, 1995)

```
> plot({x^2, sin(x)}, x=-Pi..Pi);
```



> fsolve(x^2 = sin(x), x=-Pi .. Pi);

.8767262154

> fsolve(x^2 = sin(x), x=-0.5 .. 0.5);

0

(Lab paper of Thanh & Trang, lab session 2, September 16, 1995)

> dathuc := x^6 - x^5 + 8\*x^3 + 3\*x^2 - 7\*x - 4;

$$dathuc := x^6 - x^5 + 8x^3 + 3x^2 - 7x - 4$$

> solve(dathuc=0, x);

$$1, -\%1^{1/3} + \frac{5}{9} \frac{1}{\%1^{1/3}} + \frac{2}{3} \frac{1}{2} \%1^{1/3} - \frac{5}{18} \frac{1}{\%1^{1/3}} + \frac{2}{3} + \frac{1}{2} I \sqrt{3} \left( -\%1^{1/3} - \frac{5}{9} \frac{1}{\%1^{1/3}} \right),$$

$$\frac{1}{2} \%1^{1/3} - \frac{5}{18} \frac{1}{\%1^{1/3}} + \frac{2}{3} - \frac{1}{2} I \sqrt{3} \left( -\%1^{1/3} - \frac{5}{9} \frac{1}{\%1^{1/3}} \right), -1, -1$$

$$\%1 := \frac{73}{27} + \frac{1}{9} \sqrt{606}$$

> roots(dathuc);

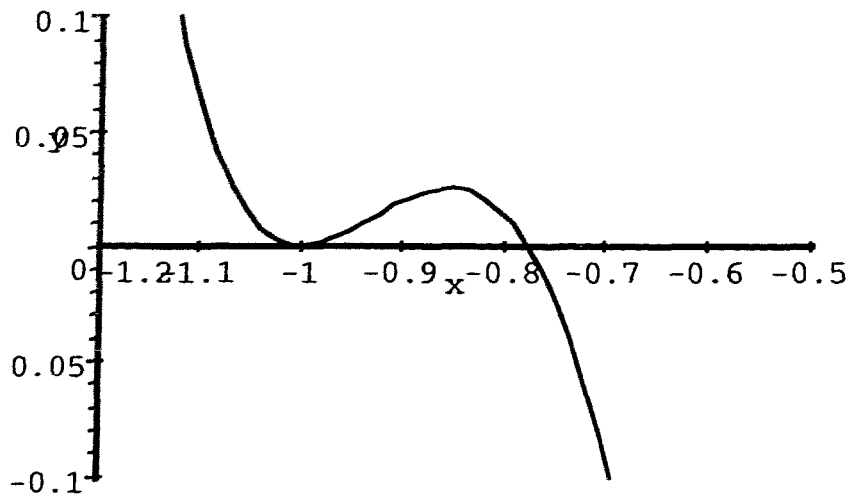
[[ -1, 2], [ 1, 1]]



> factor(dathuc);

$$(x-1)(x^3-2x^2+3x+4)(x+1)^2$$

> plot(dathuc, x=-1.5 .. 1.2);



(Lab paper of Mai & Thu, lab session 2, September 16, 1995)

Students knew how solve equations, calculate limit of a sequence or function with Maple. They could combine many commands of Maple to solve bigger problems.

> f := n -> 2^(sqrt(n));

$$f := n \rightarrow 2^{\sqrt{n}}$$

> f(n);

$$2^{\sqrt{n}}$$

> Limit(f(n), n=infinity);

$$\lim_{n \rightarrow \infty} 2^{\sqrt{n}}$$

> value("");

$\infty$

(Lab paper of Mai & Trang, lab session 6, October 14, 1995)

> f := n -> Sum('1/2^k', 'k'=1..n);

$$f := n \rightarrow \sum_{k=1}^n \frac{1}{2^k}$$

> limit(f(n), n=infinity);

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{2^k}$$

> value("");

1

(Lab paper of Thanh & Phan, lab session 6, October 14, 1995)

In these ways, students' computational skills were upgraded.

### ***Improving mathematical intuition***

In Vietnam, models of teaching mathematics are based mainly on presentation of lectures, and teachers believe that the logical proof is the heart of mathematics. With this approach, students pay attention to developing skills of mathematical reasoning and deductive proof. They do not do much mathematical conjecturing, and their mathematical intuition ability is not well developed. The following examples illustrate this comment.

In the first topic of lab #1, there was a paragraph involved plotting functions written as

> f := x-> sqrt(x+1);

$$f := x \rightarrow \sqrt{x+1}$$

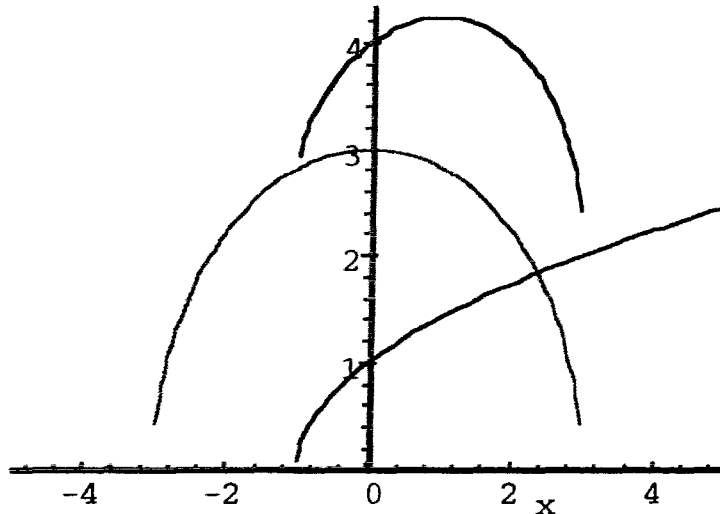
> g := x-> sqrt(9-x^2);

$$g := x \rightarrow \sqrt{9-x^2}$$

> (f+g)(x), (f-g)(x), (f/g)(x);

$$\sqrt{x+1} + \sqrt{9-x^2}, \sqrt{x+1} - \sqrt{9-x^2}, \frac{\sqrt{x+1}}{\sqrt{9-x^2}}$$

> plot({f(x), g(x), (f+g)(x)}, 'x'=-5..5);



After plotting, the students were asked to answer the question: "Can you indicate which of the preceding functions corresponds to which line on the graph?" Nobody could answer that question although they learned how to plot functions in high school.

Afterward, I guided them to consider the domains of the given functions and compare them with lines on the graph. They understood immediately, and then conjectured which line belonged to which function.

Trang: Because the domain of the function  $f(x) = \sqrt{x+1}$  is  $x \geq -1$  so its graph is this line (She points out the graph on the screen). And the domain of the function  $g(x) = \sqrt{9-x^2}$  is  $x \geq -3$  and  $x \leq 3$  so its graph is this line (She points out the graph on the screen). We imply the rest line is the graph of the function  $f(x) + g(x)$ .

The question was not really difficult for them, but they did not have experience in matching algebraic expressions with graphic representation of them. They were not in the habit of finding the main factors in a problem, and seeing a problem at a conceptual level.

I think that opportunities, as in the above example, are very necessary to help students improve their ability to make mathematical conjectures, and through it their

mathematical intuitions may be developed. I saw that participants were very interested while doing the above example.

After finishing the first topic of lab #1, the mathematical intuition of some participants became sharper. They saw important points while working with mathematical phenomena. I observed Hai and Thu solving the following exercise:

"Define a function as an absolute value

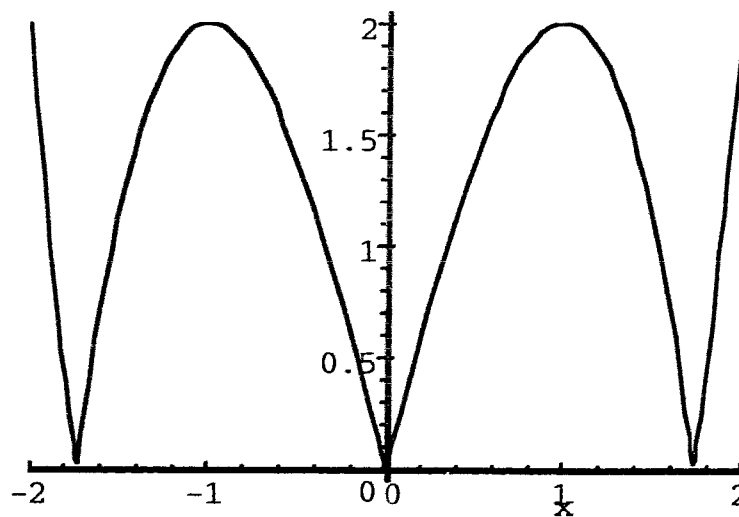
```
> f := x->abs(x^3 - 3*x);
```

Plot this function on the interval -2 .. 2. Then explain how the graph is shaped from the corresponding function without the absolute value."

Hai typed the command to define the function and then plotted it.

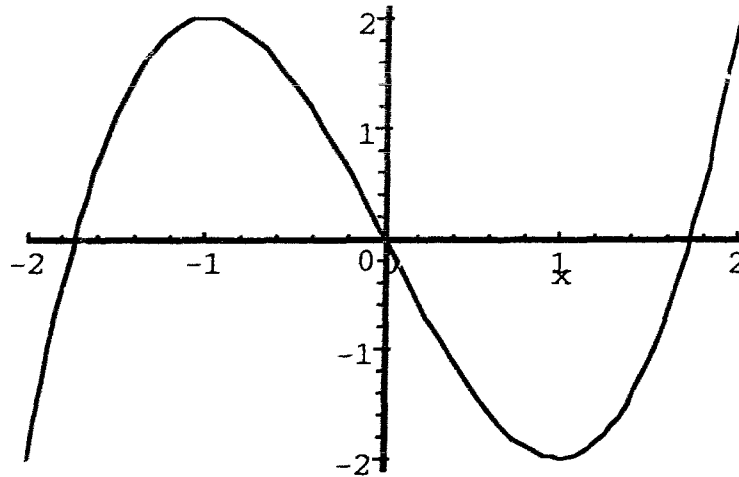
```
> f := x->abs(x^3 - 3*x);
```

```
> plot(f(x), x=-2 .. 2);
```



Hai and Thu said that they had met this problem in grade 12, but the rule for plotting was very long and difficult to remember. They observed the graph and read the exercise again and then they decided to plot the corresponding function without the absolute value. Hai executed the command

```
> plot(x^3 - 3*x, x=-2 .. 2);
```



They observed and compared the results, and talked about how the graphs were related.

(I reopen the window that has the first graph)

Author: What do you think about this graph?

Hai: It always lies above the x-axis.

Author: Why?

Hai: Because the absolute value function is always positive or equal to zero.

(I reopen the window that has the second graph)

Author: And how about this graph?

Hai: This is the graph of a third-degree function.

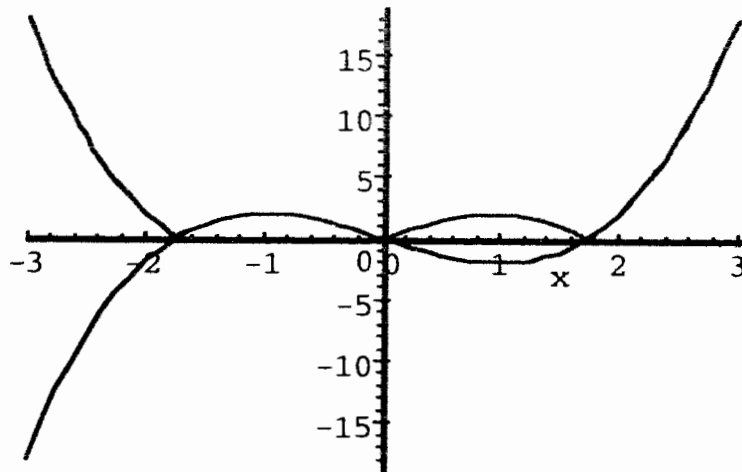
Thu: It can have both positive and negative values.

Author: Can you tell me how they relate with each other?

Thu: If we take these branches symmetrically through the x-axis we will have the graph of the absolute value function. (She shows the branches that lay under the x-axis)

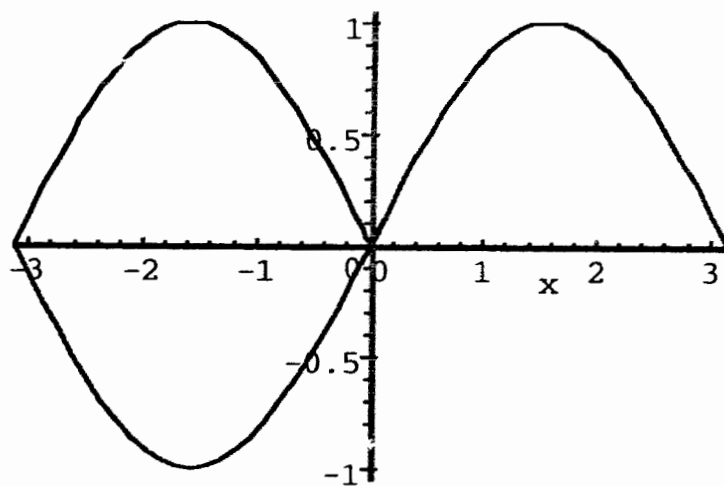
Trang and Mai saw this graphical relationship more clearly by plotting the two functions on the same coordinate system.

```
> plot({x^3 - 3*x, abs(x^3 - 3*x)}, x=-3 .. 3);
```

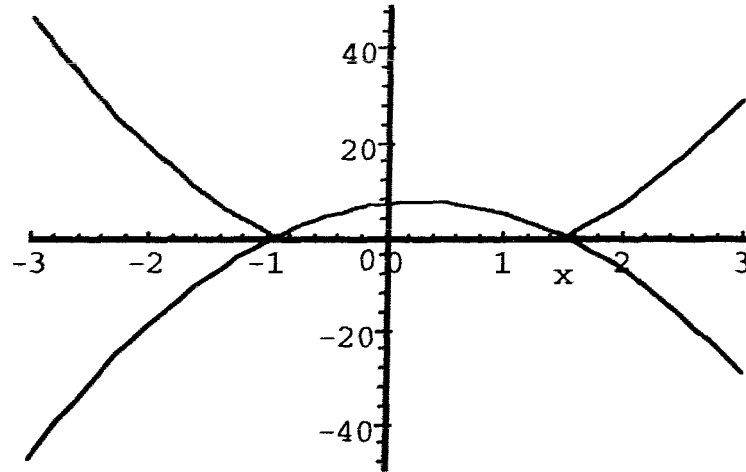


The above example shows that computers help students see graphical relationship of the functions  $y = f(x)$  and  $y = |f(x)|$ . In order to help participants explain the mathematical nature of this relationship, I suggested they plot various functions such as

```
> plot({sin(x), abs(sin(x))}, x=-Pi .. Pi);
```



```
> plot({7 + 3*x - 5*x^2, abs(7 + 3*x - 5*x^2)}, x=-3..3);
```



They observed and compared graphs to find commonalities. Then, they generalized about the graphical relationship of general functions of  $y = f(x)$  and  $y = |f(x)|$ .

Thanh: We can imply the graph of the function  $y = |f(x)|$  from the graph of the function  $y = f(x)$  by retaining the positive parts of this graph and then taking symmetrically through the x-axis the negative parts of the graph. We will get whole of the graph of the absolute value function.

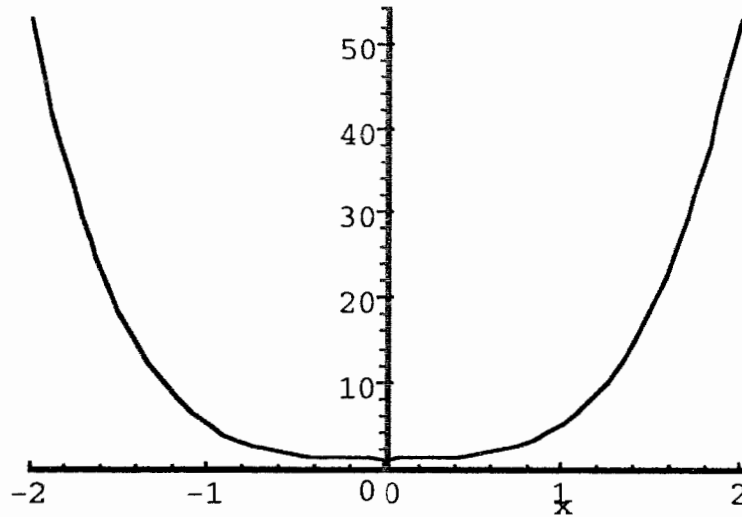
Afterward, I guided them to prove the property they had conjectured by using the definition of the absolute value.

$$y = |f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$

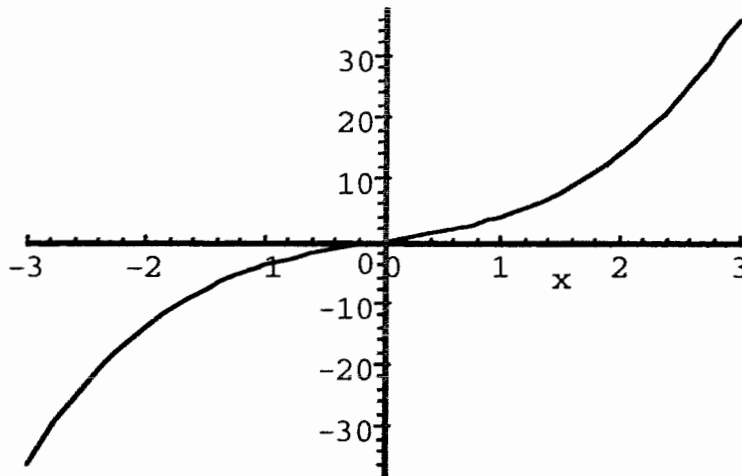
One more time, we see here the inductive process was used to help students capture graphical relationships.

A similar situation occurred with Trang and Mai when they were asked to explain graphical properties of odd and even functions. After checking the odd or even property of two functions of  $f(x) = 3x^4 + x^2 + 1$  and  $g(x) = x^3 + 3x$ , they drew the graphs as follows

> plot(x^2 + 3\*x^4 + 1, x=-2 .. 2);



> plot(x^3 + 3\*x, x=-3 .. 3);



But they did not explain their results. I suggested they plot several odd and even functions which they knew. Based on induction from the results, they explained the given question.

**Mai:** The graph of an even function is symmetric through the y-axis because we have  $f(x) = f(-x)$ . If we take a value of  $x$  then the value of  $-x$  is symmetric through the y-axis but they have same y-value. (She shows on the screen how the graph is created)



## ***Improving mathematical communication***

Besides mathematical intuition, mathematical communication is also an important part of mathematics. There is a need for students to develop their abilities for mathematical communication. The activity of transferring mathematical knowledge from teachers to students is one of communicating. Students receive and understand communications, and they communicate back results. Communication can appear in many different forms. It may be oral, written, or it may take other forms such as drawing a picture or building a model.

In a lecture-based approach, students almost always communicate the mathematical problem to be solved or the theorem to be proved in writing, sometimes with words and often only with mathematical symbols. They do not have the chance to communicate mathematically with each other, using language or other means, to express their discoveries and beliefs about mathematics. They do not have the habit of discussing mathematics, and they are fearful of presenting their mathematical ideas to others.

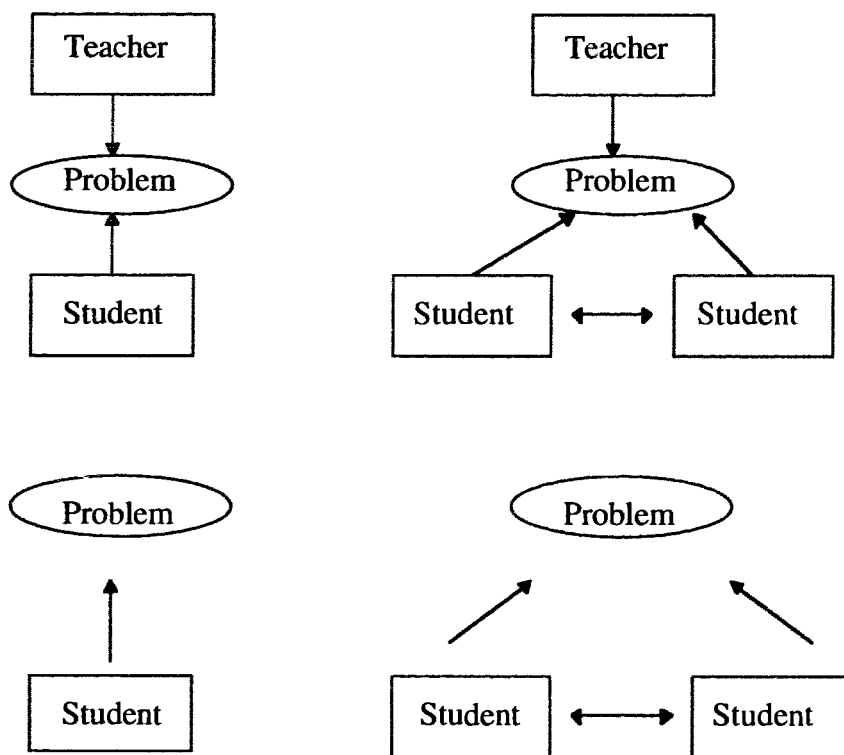
When I asked participants to state and explain the definition of limit of a sequence, one student (Thanh) said "I understand it, but I don't know how to say it." In another case, after I explained an exercise examining the monotonicity of a sequence to one group, there was another group having trouble solving this exercise. I encouraged them to ask the group that I had helped. One student (Trang) in this group told me that "I know how to do it, but I can't explain it. I'm afraid when I say it they will not understand."

In the process of doing the lab, I saw that once the students met a question requiring them to explain results they found it difficult to express their understanding, and they did not use mathematical terms precisely. I had to encourage them to talk to me as well as to the others about mathematics.

After working for some time, the students' mathematical communication ability improved, and their communicating behaviors also changed. For instance, in the first part of the experiment the students were not in the habit of talking together about mathematics.

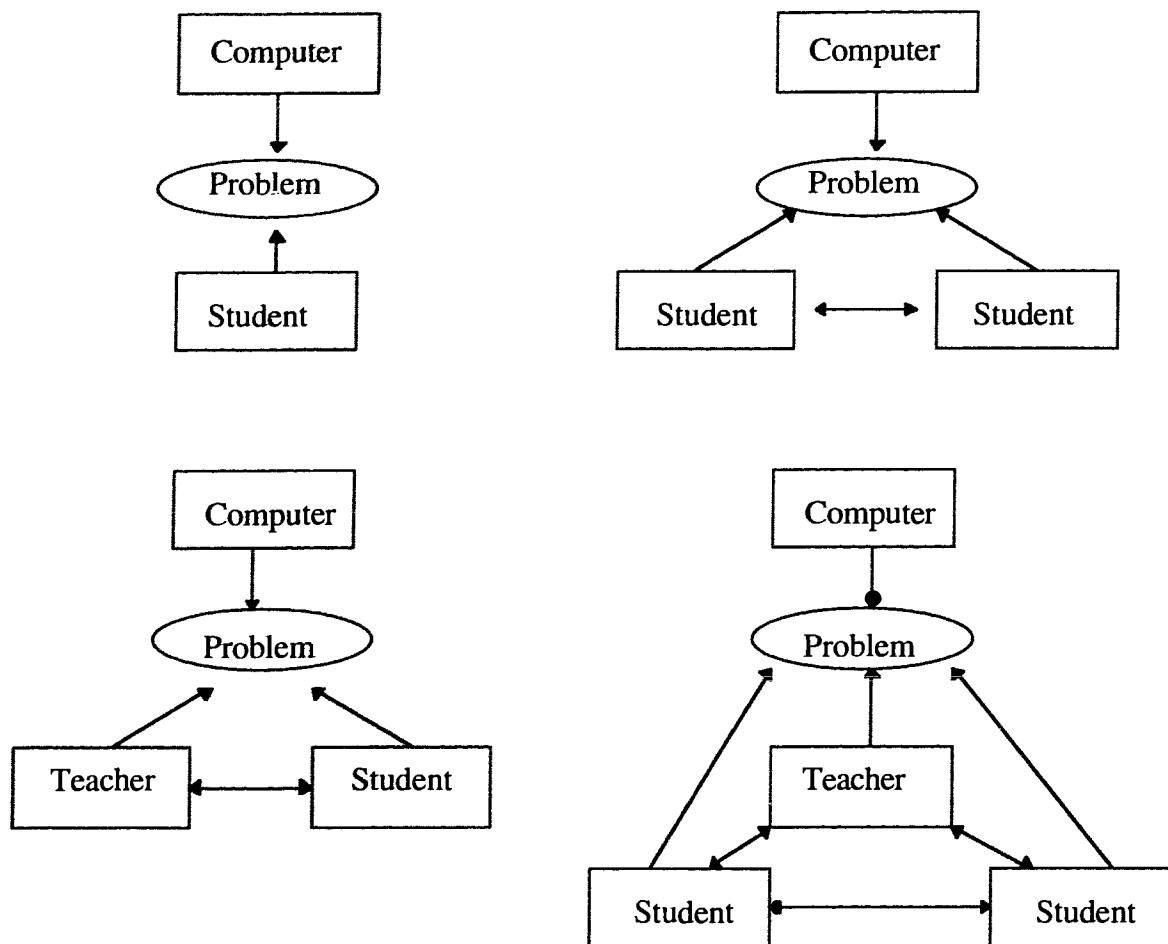
Each member in the group liked to read his or her own lab assignment, and then perform independently on the computer. One student executed some commands, and then gave the computer up to the other who executed some commands, and conversely. Later, each member in group still read his or her own lab assignment, but both performed together on the computer. It took some time before they read one lab assignment together, and discussed what they should do on the computer. Sometimes, students in one group went to another group to ask some questions, exchange information, or share results.

The computing environment elicited its own kinds of mathematical communication formalities. In a class environment, the models of communication I often see are



Teacher and student(s) communicate with each other on a mathematical problem. Students work on their own, or many students work together on a mathematical problem.

When learning mathematics with a computer-based approach, I discovered some other communication models as follows:



Student(s) and computer communicate with each other on a mathematical problem. Teacher and student(s) work together to communicate with the computer.

Each communication model has its own characteristics. In class when a student asks the teacher some problem, the teacher can assess the students' mathematical background and give appropriate guidance to help the student solve the problem. The computer is not intelligent enough to do so. On the other hand, the computer can provide students with a concrete solution to a complicated computational problem which would take the teacher or student a great deal of time and effort to complete. The communication between teacher and student is a human interaction while the communication between computer and student is a technical interaction. In the communication process with teachers, students can get feedback on the problem solving process. Teachers can indicate which methods or approaches are

better to solve the problem for students, or they can show students what is wrong in their mathematical reasoning. The interaction between students and computers currently does not give students such feedback. Students can receive computational results in the form of either numbers, symbols or pictures from computers. All communicating models are valuable, and they complement each other.

Communication in class is helpful in learning mathematics at a conceptual level. For instance, in the lab I explained to students the concept of the limit of a function by giving an example on the computer I saw that they were having difficulty in following and did not understand clearly. I changed and presented the idea using paper and pen, and I saw they understood more easily. I felt more comfortable because I could investigate how the students were thinking and why they did not understand, I could draw some pictures on paper to demonstrate the concept, or I could use some appropriate metaphor to help the students capture the concept.

***Developing positive attitudes toward mathematics and computers***

The result of the Mathematical Attitude Scale Test is presented in Table 4, below:

---

**Table 4: The result of the test of the Mathematics Attitude Scale**

SD : Strongly Disagree - D : Disagree

U : Undecided - A : Agree - SA : Strongly Agree

	SD	D	U	A	SA
<b><u>Enjoyment of Mathematics</u></b>					
1. Mathematics is not a very interesting subject.	4/5	1/5	0/5	0/5	0/5
2. I have usually enjoyed studying mathematics in school.	0/5	0/5	0/5	2/5	3/5
3. I have seldom liked studying mathematics.	1/5	4/5	0/5	0/5	0/5

4. Mathematics is enjoyable and stimulating to me.	0/5	0/5	0/5	5/5	0/5
5. Mathematics is dull and boring.	2/5	3/5	0/5	0/5	0/5
6. I like trying to solve new problems in mathematics.	0/5	0/5	1/5	3/5	1/5

Freedom from Fear of Mathematics

1. Mathematics makes me feel nervous and uncomfortable.	0/5	5/5	0/5	0/5	0/5
2. I am very calm when studying mathematics.	0/5	4/5	0/5	1/5	0/5
3. Mathematics makes me feel uneasy and confused.	1/5	1/5	3/5	0/5	0/5
4. Trying to understand mathematics doesn't make me anxious.	0/5	1/5	0/5	2/5	2/5
5. Mathematics is one of my most dreaded subjects.	0/5	4/5	1/5	0/5	0/5
6. I don't get upset when trying to do mathematics lessons.	0/5	1/5	3/5	1/5	0/5

Perceptions of Mathematics

1. Mathematics is a very worthwhile and necessary subject.	0/5	0/5	0/5	3/5	2/5
2. Other subjects are more important to people than mathematics.	2/5	1/5	2/5	0/5	0/5
3. Mathematics helps to develop the mind and teaches a person to think.	0/5	0/5	0/5	1/5	4/5
4. Mathematics is not especially important in everyday life.	2/5	1/5	2/5	0/5	0/5
5. Mathematics has contributed greatly to the	1/5	0/5	0/5	2/5	2/5

advancement of civilization.

6. Mathematics is not one of the most important subjects for people to study.      1/5      1/5      3/5      0/5      0/5

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In terms of enjoyment of mathematics, all of the participants liked mathematics. All respondents agreed with the statement "Mathematics is enjoyable and stimulating to me", and with regard to the statement, "Mathematics is not a very interesting subject" all disagreed or strongly disagreed. Their positive attitudes toward mathematics went back at least as far as high school. The following quotes show this:

Author: How did you feel when you studied math in high school?

Mai: I enjoyed it very much.

Trang: It was the subject I liked most.

Phan: It was the subject that I spent the most time on because I liked it.

(Interview 1, September 21, 1995)

In Vietnamese high schools mathematics is taught very much better than other subjects. It takes an important position in the high school curriculum, and it is part of all major examinations. Everybody realizes this, and school boards select good teachers to teach math, particularly for important classes in the end level of educational phases such as grade nine and twelve. Besides learning math in schools, students usually go to Learning Centers to learn more math with famous teachers, with the hope that they will get good mathematics teaching to enable them to pass important examinations. Working with such teachers is why the students liked mathematics and enjoyed learning it in high school. They maintained this attitude throughout the study.

Under the heading of freedom from fear of mathematics, participants all disagreed that mathematics made them feel nervous and uncomfortable, although three out of five were undecided about whether or not math made them feel uneasy and confused.

When teaching problem solving in high school, teachers have a tendency to teach students to do mathematics following patterns. At the same time, in order to help them cope with examinations, teachers often teach students tricks in doing mathematics. Students attempt to memorize mechanisms and tricks to solve problem, rather than studying the mathematical nature of the problem. For instance, in the study the students explained how to find an inverse function, but they could not define what the inverse function was. They did not state how it related to its original function.

Students felt comfortable with this learning style. They could solve problems without deeply understanding the concepts. I think the participants brought this belief into the study. In lab assignments, I often asked them to explain concepts, results or mathematical phenomena. I think this partly changed their beliefs, and sometimes made them feel uneasy and confused about mathematics; nonetheless, they still worked very hard despite being confused at times. On the survey, almost all of them agreed that trying to understand mathematics did not made them anxious, and mathematics was not one of their most dreaded subjects.

What did the students think about mathematics as a discipline? The result of the test items on perceptions of mathematics show that all participants agreed mathematics was a very worthwhile and necessary subject, that it helped to develop the mind and taught people to think.

However, when asked if math was not one of the most important subjects for people to study, three of five students were undecided. In Vietnam, mathematics is taught in isolation separate from social and technological problems. The model of Society-Technology-Science has not yet been developed and applied in science and mathematics education. The societal role of mathematics and its applications in everyday life are seldom mentioned in mathematics teaching. This led to the situation where students did not appreciate how mathematics was important for people to study.

The Table 5 below presents the results of the Computer Attitude Scale test.

**Table 5:** The result of the test of the Computer Attitude Scale

SD : Strongly Disagree - D : Disagree

U : Undecided - A : Agree - SA : Strongly Agree

	SD	D	U	A	SA
<b><u>Enjoyment of Computers</u></b>					
1. Computer is not a very interesting subject.	3/5	2/5	0/5	0/5	0/5
2. I enjoy computer work.	0/5	0/5	0/5	1/5	4/5
3. I have seldom liked working on computers.	4/5	1/5	0/5	0/5	0/5
4. Computer is enjoyable and stimulating to me.	0/5	1/5	0/5	1/5	3/5
5. Learning computers is boring to me.	3/5	2/5	0/5	0/5	0/5
6. I like learning on a computer.	0/5	0/5	0/5	3/5	2/5
<b><u>Freedom from Fear of Computers</u></b>					
1. Computers make me feel nervous and uncomfortable.	1/5	2/5	2/5	0/5	0/5
2. I am very calm when studying computer.	0/5	3/5	2/5	0/5	0/5
3. Computers make me feel uneasy and confused.	0/5	2/5	2/5	1/5	0/5
4. Trying to understand computer doesn't make me anxious.	0/5	2/5	1/5	1/5	1/5
5. Computer is one of my most dreaded subjects.	4/5	1/5	0/5	0/5	0/5
6. I don't get upset when trying to do computer lessons.	0/5	0/5	3/5	1/5	1/5
<b><u>Perceptions of Computers in Mathematics Education</u></b>					
1. I feel that the use of computers in schools will	0/5	1/5	1/5	3/5	0/5



help students to learn mathematics.

- |  |     |     |     |     |     |
|--|-----|-----|-----|-----|-----|
| 2. I feel that the use of computers in schools will negatively affect students' mathematical learning abilities. | 1/5 | 2/5 | 1/5 | 1/5 | 0/5 |
| 3. I think that computers and other technological advances have helped to improve mathematics education.         | 0/5 | 0/5 | 3/5 | 1/5 | 1/5 |
| 4. I think that computers are not worthwhile and necessary in mathematics education.                             | 4/5 | 1/5 | 0/5 | 0/5 | 0/5 |

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The survey showed that the students liked working on computers. All of them all answered with agreement or strongly agreement to the statements, "I enjoy computer work" and "I like learning on a computer." These were consistent with my observations. The participants completed the lab sessions enthusiastically. They always came on time and worked very hard although I knew that they had a heavy course load and did not have much extra time. At the end of the experiment they asked me to install Maple software on the computers of the Center of Computing Services, and to give them copies of the rest of the lab assignments so they could work on them after I left.

In terms of freedom from fear of computers, the students' responses indicated that they felt computers were not really easy and could make them anxious. At present in Vietnam, computers are still something new and draw a great deal of attention. Students have seen many television commercials and heard many stories about them. In the meantime, high school education has lagged behind societal technological innovations.

In such a context, students know little about the real abilities of computers, and they cannot visualize how they could learn with computers. Almost of them all believe that computers will make everything much easier and that computers can solve any problem. However, through doing the lab assignments the students realized that there were exercises they could not solve easily with computers without a good understanding of mathematics, and they had to learn new computing skills to be able to master and use the computer

effectively. I think these discoveries changed their previous beliefs. Moreover, among the participants there were students who had never worked with computers before, so that during this study they had to learn computer skills, Maple, and mathematics simultaneously. They were introduced to many confusing new terms in a short time. I think this contributed to their anxiety and left them undecided about whether they became upset when working on computer lessons. For instance, two students told me the following:

Mai: Earlier on, I intended to withdraw and not to participate in the experiment because I did not know anything about computers and I was afraid that I could not work in the lab. But because I like math I also wanted to study computers so I decide to participate in the experiment.

Phan: I think if I had learned how to use computers before I would have done the lab assignments much better. Before I had never heard of files, directories, windows or menus. Sometimes they confuse me.  
(Interview 1, September 21, 1995)

Nonetheless, all the students agreed that computer science was not one of their most dreaded subjects.

In the section on the perception of computers in mathematics education, the responses showed that the students realized the need for computers in mathematics education but they did not evaluate how computers affected the learning of mathematics.

Most participants were in strong disagreement (4/5 strongly disagreed and 1/5 disagreed) with the statement, "I think that computers are not worthwhile and necessary in mathematics education." But for the statement, "I think that computers and other technological advances have helped to improve mathematics education" there were three out of five students undecided, and two who agreed and strongly agreed.

I think it is reasonable that some students were undecided about this statement in the Vietnamese context. Students had heard general information about computers but the majority of them had not had many chances to use computers previously. They realized that computer science is an independent subject like chemistry or biology which required study. They had never seen mathematics teachers use computers. They had never seen people learning mathematics or other subjects with computers. They had no clear evidence to

evaluate how computers affected the learning of mathematics. For these reasons they were undecided about whether computers can help improve mathematics education.

Nonetheless, the common belief of most participants was that computers would help students to learn math in schools. For the statement, "I feel that the use of computers in schools will help students to learn mathematics" there were three out of five who agreed. There was one student who disagreed with this statement, and another undecided. There were similar results with the statement, "I feel that the use of computers in schools will negatively affect students' mathematical learning ability." I think the fact that some were undecided or thought that computers did not help students to learn math or could cause negative effects in students' math learning ability might be the result of thinking that computers would do all the work for students.

Thu: I think if we use computers in high schools students will like learning mathematics more. But I am afraid that students may be lazier because the computer does almost everything for us.  
(Interview 2, October 7, 1995)

Generally, only a limited amount of time was available for the students to work in the computer-based laboratory, and I think it was not long enough for them to investigate in detail how computers affected their learning of mathematics and their attitudes toward mathematics. I only can say that observation showed participants were interested in doing the lab course, and their attitudes toward mathematics and computers were positive throughout the study. At the same time, the participants believed that computers could help students learn mathematics and develop positive attitudes toward mathematics. The following quotes exemplify this belief:

Thu: I think if we use computers in high school students will like learning mathematics more.

Mai: I also think that computers should be used in school, but it would have to be combined with learning mathematics in class.

Trang: Yes, I agree.

Phan: I think if computers are used in school they will make students who don't like math study math.

Hai: I think if computers are used in high school we will learn mathematics easier.

Thanh: I think if we use computers to learn mathematics in school, we will not waste time on computational work, and we can concentrate on learning other more important things.

Trang: There are many things we learn in school that we don't remember or understand clearly, and I think computers, will help us to remember and understand more deeply.

(Interview 2, October 7, 1995)

*Some other factors (Learning pace, reading skills and English background)*

A computing environment enables students to learn mathematics at their own pace. Observation showed that various groups' rates in performing the lab assignments were different. There were examples in the lab assignments where one group repeated a study two or three times while another did it only once. Or there were paragraphs in the lab assignments that one group performed and other passed over. During the study the groups recognized these differences, but it did not worry them enough to make slower group rush or omit sections to catch up with the faster one. The group work was based on the abilities of the members of the group.

In contrast, in mathematical lectures in class at Dalat University, teachers usually set standard pace during lectures, and require students to attain this standard regardless of how their backgrounds differ. This causes some students to memorize a great deal or to jump over parts of the material they misunderstand, so as to catch up with the pace of the lectures. This causes gaps in their knowledge. I think a computing environment can help us overcome this situation. With the computer-based approach, students have the chance to find out what they do not understand in the lectures and they can work on these areas. At the same time, practicing mathematics in a computing environment helps them to enrich their studying of their lecture classes.

Mai: There are things we learn in class that we don't understand much because the teacher teaches so fast, but when we practice on computer we can follow up on our ideas so we can understand.

Trang: After learning theory in class, if we practice what we learn on computer then our knowledge will be consolidated, and we will understand and remember longer.

(Interview 2, October 7, 1995)

The factors of reading skill and English background are also considered in the study because I think they can affect students' learning mathematics in a computing environment. Learning mathematics with computers does not mean students working purely with computers. I observed that students also had to read a lot. They had to read lab assignments to know how to do the lab, to read messages and results which appeared on the computers, and sometimes the help pages of the on-line help software. Reading occupied a good deal of lab time.

I discovered that participants did not have strong skills in reading written documents. They often glanced at parts of the text dealing with concepts, definitions or explanations, but liked to concentrate on learning commands so they could operate the computer. I think this was partly because they liked to practice mathematics on computers but also because they had difficulty reading the written lab. There were many cases in which they typed and executed commands, but the computer failed to deliver the desired solutions and then they had to reread instructions. There were also cases where they did not understand the results given by the computer, and they had to reread the explanations in the lab assignments. They preferred to refer to the lab instructor rather than reading for themselves. Some students comments:

Hai: I feel the written lab assignments were difficult to understand.

Author: Could you give me some examples?

Hai was embarrassed for a little while and could not give any example, but then said: I think we should have time to read the lab assignment at home and make a study of what it means so we can do the lab better.

Thanh: They are written too precisely so they are a little difficult to understand, and they have some words I've never met before so I don't understand what they mean.

Author: Could you tell me what words you don't understand?

Thanh: For example, I don't know what the word "reference" means.

Mai: In high school we didn't have to read anything like this.

Trang: Yes, the teachers usually told us what we had to do.

(Interview 1, September 21, 1995)

But after doing the lab for some time, I saw that their reading skills became better. They did not complain any more. They concentrated on reading the assignments more carefully and performed the lab better.

In Vietnam, almost all computing software is imported from America and it is all in English. The exception in recent years has been some word processing software in Vietnamese. Many students believe that if their English is not good, they cannot learn anything related to computers. This had led to a situation where some students whose English is barely satisfactory or marginal feel a fear of learning or working with computers, although their attitudes toward computers are not negative. Conversely, some students think that if their English is good, they can master everything related to computers. We will see that this thinking is not at all reasonable.

When I asked students about their experiences learning mathematics with computers after doing two lab sessions, they stated that:

Trang: The lab is not difficult, but I learn slowly, because there are English words I do not know.

Thu: There are words whose everyday meanings I understand, but I don't understand their mathematical meanings.

Hai: When I search for them in a dictionary, they don't have those meanings listed.

Phan: I think the lab is not so difficult, but we don't know many English words.

(Interview 1, September 21, 1995)

Particularly, there was one student (Hai) who very much liked to study English during the lab. When I explained English words he noted them very carefully, and asked many questions involving English rather than mathematics. Although he tried to remember as many words as possible, he did not really concentrate on mathematics as much as the other students. After a few lab sessions, when the mathematical difficulty increased, he found it difficult to understand the lab and his motivation for learning decreased.

Although the students were very worried about their English in the beginning of the study, after many lab sessions where they solved successfully problems by understanding mathematical concepts and thinking mathematically, their perspective on English changed. My observation showed that in later lab sessions, students concentrated on mathematics more than English. While working, if they got a error message or unusual results, they learned to read the message and review their use of commands based on their mathematical background, rather than stopping and making a study of the English meanings of each word in the message. After completing the experiment, students told me that:

Thanh: I think English is not so important, because if we understand the mathematics we can interpret results.

Phan: I think if we know English well we can read messages more easily but it is not enough, because if we don't study the lecture we can't do mathematics on computers.

Mai: Generally speaking, there are not many difficult words (she means mathematics terms and computing terms) and they are repeated many times so we can understand them. I think it does not affect learning on computers very much.

Thu: I think if students learn English well but not mathematics they can't do well this computer lab.

(Interview 2, October 7, 1995)

All students agreed that the lab was not as easy as they thought at the beginning. I think the main reason is the limitations of the context of the study. I had supposed that students would take the lab course after they had listened to lectures in class. But in fact, because of difficulties at Dalat University, students only heard the relevant lectures when

my study was about to finish. Hence, there were parts in the assignments they had to explore by themselves, and they found this way difficult.

### ***Summary***

This chapter provided an analysis of students' experiences of learning mathematics with a computer-based approach. The chapter described and interpreted how students learned mathematical concepts, developed skills of mathematical reasoning, problem solving and computation in a computing environment. It was also mentioned that the use of computers helped students to improve mathematical intuition and communication. The chapter investigated students' attitudes toward mathematics and computers. The attitudes were analysed in terms of enjoyment of mathematics and computers, freedom from fear of mathematics and computers, and perceptions of mathematics and computers in mathematics education. Finally, some other factors such as learning pace, reading skills and English background are also considered in the study.



## **Chapter 5**

### **Conclusions, Limitations and Implications**

This chapter discusses the major results arising from the study. Following this discussion, the limitations of the study are mentioned. Some limitations of Maple drawn from the study are discussed. Finally, the implications of the research including questions for further research are presented.

#### ***Conclusions***

The study shows that a computer-assisted learning environment enables students to reinforce mathematical concepts. The process of doing mathematics with Maple in the study helped students to understand more deeply the concepts of equation, function, limit of a sequence, and limit of a function. Students reinforced their mathematical knowledge through performing examples written in the lab course, solving exercises with Maple, explaining the results given by Maple, or communicating mathematically with each other or with the teacher.

The process of interaction between students and computer may stimulate students to learn new mathematical concepts. Students asked the teacher to present any new mathematical concepts which arose during their work on computers.

The study also showed that students could apply the commands of Maple mechanically to do mathematics without clearly understanding the mathematical concepts involved. In order to help students learn mathematical concepts, the teacher can use Maple as a mathematical word processor to present a mathematical problem as in the lecture. I applied this method in the study to illustrate to students how the definition of limit of a sequence worked. I saw the students approach this concept enthusiastically. Thus, the curriculum should not only show students how to use the computer to perform fast calculations but also to support the presentation of concepts.

In particular, the computer's graphic capabilities can help students in their approach to some mathematical concepts and relationships. For instance, students realized that it is possible that function  $y = f(x)$  may have no limit when  $x$  tends to some value through consideration of the graph of the function  $y = \sin(3\pi/x)$  when  $x$  tends to 0. Students saw the mathematical relationship between solutions of an equation and intersection points of graphs when they plotted the functions  $y = \sin(x)$  and  $y = x^2$ , and chose appropriate intervals on which they solved the equation of  $\sin(x) = x^2$ . Students also discovered the graphical relationship of two functions  $y = f(x)$  and  $y = |f(x)|$  through observing the graphs of some absolute value functions.

Students can develop reasoning skills when learning with a computer-based laboratory course. In the study, the students improved their reasoning skills through activities such as problem solving, evaluating solutions, interpreting problem solving processes and communicating mathematically. The computer is particularly appropriate in enabling students to do many different problems and observe commonalities which can be conceptualize as general mathematical properties. Through this process, students develop inductive reasoning skills. For instance, participants learned the definition of the limit of a function in terms of the limit of a sequence by calculating the limits of some different functions and then generalizing these results.

The computer can help students develop problem solving and computational skills. Throughout the experiment, students knew how to direct Maple to calculate an expression, solve an equation, plot a function, and find the limit of a sequence or a function. However, because of the limited time available for the study, the students did not develop enough ability to combine many commands to solve a complicated problem. Computational skills are understood here in terms of using Maple to do mathematics. Students learn how to use commands appropriately to solve mathematical problems effectively.

The computing environment enables students to explore mathematics. In the process of doing mathematics with Maple, besides working on problems and exercises presented in

the lab course, students performed other calculations to see what results Maple would produce, or to check their thinking or mathematical conjectures with solutions given by Maple. Such explorations helped students to understand mathematics more deeply, and at the same time upgrade skills in using Maple to solve problems.

The study indicates that when students do problem solving with computers, they may not see clearly how the problem is solved because the computer's calculation process is hidden. In the study there were circumstances where students could not interpret how the solutions were obtained. In order to overcome this, the curriculum needs to emphasize the appropriateness of the process leading to the results, as well as the reasonableness of the results given by the computer. The study showed that when students were required to interpret computed results they noticed more about problem solving processes. Through interpretation, the students understood problem solving processes, reinforced concepts, and developed their problem solving and reasoning skills.

In addition, the study also showed that when students used computers to solve problems they had chance to investigate key factors of a problem and to build appropriate solutions, since time-consuming computational work is undertaken by the computer. Problem solving skills are developed in terms of accumulating experience in using the computer as a computational tool, and building problem solving strategies that take full advantage of the abilities of the computer.

The study indicated that the computer did not identify the mathematical assumptions used in the process of calculating solutions. Teachers need to help students relate their lectures to the mathematical assumptions underlying the problem solving technique in Maple. Understanding the conditions under which solutions exist helps students to understand more clearly the mathematical nature of the solutions. At the same time their reasoning ability is also upgraded.

Maple cannot perform formal proofs. It cannot help students to prove a mathematical statement or theorem. But students can use Maple as a mathematical word

processor to express their mathematical ideas. The curriculum can use Maple in this way to present to students demonstrations of the process of problem solving. In the study I used Maple to show students how I proved an inequality.

Teachers should create opportunities to help students think about their strategies for learning mathematics with computers. It is also very useful to enable them to share their experiences of doing computer-assisted mathematics. Developing awareness of learning techniques in a computing environment helps students to improve their skills in problem solving and reasoning. In the study, when I encouraged students to present their thinking as well as experiences about how they could learn mathematics effectively with computers I observed their learning strategies changed and they learned better.

The computer has the ability to give solutions quickly without presenting computational details. Students can explore mathematics, and check solutions given by computers against their understanding. Such activities help them to improve their mathematical intuition. Solving problem with computers, students are released from heavy computational tasks and they can concentrate on algorithms and mathematical reasoning. It helps them capture important features of a problem. In this way, students develop mathematical intuition.

Observations from the study show that graphics are one of the facilities of the computer which students find very interesting. Computer graphics can help students to see some mathematical relations. Students can see a mathematical concept or property in two dimensions, the symbolic and the imagistic. For instance, the students in the study found the roots of a third-degree equation by factoring the equation, and at the same time they saw the roots by plotting the corresponding third-degree function. Seeing a mathematical problem in its different aspects helps students improve their mathematical intuition.

It should be noted that the numeric and graphical solutions given by Maple depend partly on properties and parameters in the commands that are used. They also depend on the algorithms used by the software designer. In addition, because of limitations of the

hardware, in some cases the graphical solutions are not always perfect. Hence, students need to evaluate the results and make a study of their mathematical meanings. The evaluation process helps students to understand the use of the commands more clearly.

Learning mathematics in a lecture class, students develop communication abilities in terms of writing and listening. Through working in the laboratory, students have the chance to communicate with the computer as well as with people. In the study students learned how to get the computer to solve a mathematical problem, and how to explain and evaluate solutions given by the computer. At the same time, in the laboratory environment they had the chance to represent mathematics to the teacher and the other students, and they discussed mathematics with each other. Their abilities in mathematics communication were developed.

In a computing environment, the position and function of the teacher are changed. The teacher decides which problem solving processes should be opened to students, which results students need to evaluate or interpret in detail, and which mathematical assumptions students need to be considered. The teacher becomes the person who provides students with a working environment and conditions for learning. He/she does not give students mathematical knowledge directly, but discusses mathematics with them. The teacher and students can share skills, experiences and explorations, involving doing mathematics with the computer.

In this study, all of the students enjoyed mathematics. The results of the Mathematics Attitude Scale test showed that the students did not feel that mathematics made them feel nervous, but it might make them feel uneasy and confused. They realized that mathematics was a very worthwhile and necessary subject. They agreed that the role of mathematics is to teach people to think, and to help people to develop their minds. But they did not see how mathematics could be applied in everyday life. They did not realize how mathematics relates to society and technology.

Observation from the study showed that all of the students liked working on the computer. The results of the Computer Attitude Scale test showed students felt computers were not easy and might make them anxious. Some of them were undecided about whether computers could improve mathematics education, because they had never seen people use computers to teach mathematics in schools. But interview results show that all of the students believe the use of computers in schools will help students to learn mathematics and develop positive attitudes toward mathematics.

Working with computers in the study, students learned mathematics at their own pace. They sequenced their own learning based on their background and experiences. Their understandings were built on practical activities. It was possible for them to avoid learning mathematics through rote memorization and routinization.

In learning mathematics in a lecture class, students need listening and note taking skills. To learn mathematics with a computer students need to have reading skills. The study showed that students who did not have good skills in reading written documents had difficulty when they did mathematics independently with a computer in a laboratory course. But after working for some time, as their reading skills improved, their learning became easier. The computing environment helped students develop skills of reading mathematical documents.

With grade 12 English reading ability, students can communicate with Maple. However, students in the study realized that a good background in English could help them to understand operations on the computer, but it is not a crucial factor for learning mathematics in a computer environment.

### ***Limitations of the study***

The purpose of the study was to investigate students' experiences learning mathematics in computer aided instruction. The results of the study were based primarily on the performance of six students in a laboratory course. Participants were drawn from the

population of first-year basic science students of Dalat University, and they were not randomly selected. The learning experiences of students was analyzed in the framework of calculus knowledge and the software Maple. These created limitations in the generalizability of the results of the study. Some conclusions drawn from the study might be not consistent for other students in the population, or for other mathematics knowledge, or for other mathematical software.

In addition, the factor of gender was not investigated in this study. The effect of students' differing levels of mathematical background on their learning of mathematics in a computing environment were not examined either. The study did not compare the effectiveness in term of student mathematical achievement of learning mathematics with a computer-based approach with learning mathematics in a class without computers.

### ***Limitations of Maple***

Some limitations of Maple were illustrated by the study. I discovered that Maple did not use sequences to prove the function  $f(x) = \sin(1/x)$  had no limit when  $x$  tended to 0. We considered the function

> f := x -> sin(1/x);

$$f := x \rightarrow \sin\left(\frac{1}{x}\right)$$

> limit(f(x), x=0);

-1 .. 1

We knew that the given function had no limit when  $x$  tends to 0. I built two different sequences that tended to 0 when  $n$  tends to infinity.

> an := n -> 1/(n\*Pi);

$$an := n \rightarrow \frac{1}{n \pi}$$

> bn := n -> 1/(2\*n\*Pi + Pi/2);

$$bn := n \rightarrow \frac{1}{2n\pi + \frac{1}{2}\pi}$$

> limit(an(n), n=infinity);

0

> limit(bn(n), n=infinity);

0

We considered two corresponding sequences  $f(a_n)$  and  $f(b_n)$ .

> f(an(n));

$\sin(n\pi)$

> f(bn(n));

$\cos(2n\pi)$

We used Maple to find the limits of these two sequences when  $n$  tended to infinity.

> limit(f(an(n)), n=infinity);

-1 .. 1

> limit(f(bn(n)), n=infinity);

-1 .. 1

The results showed that these had no limit when  $n$  tended to infinity. But in fact we had  $\sin(n\pi) = 0$  and  $\cos(2n\pi) = 1$  so two limits had to be 0 and 1, and because these two limit values were different, the given function had no limit when  $x$  tended to 0. If our work were based on the results given by Maple, we would not be able to prove the problem. I think the reason for this is that Maple does not understand the meaning of  $n$  as we do. When we used  $n$ , we meant  $n$  is a natural number, but Maple used it as a real number. We need to clarify this when guiding students in using Maple.

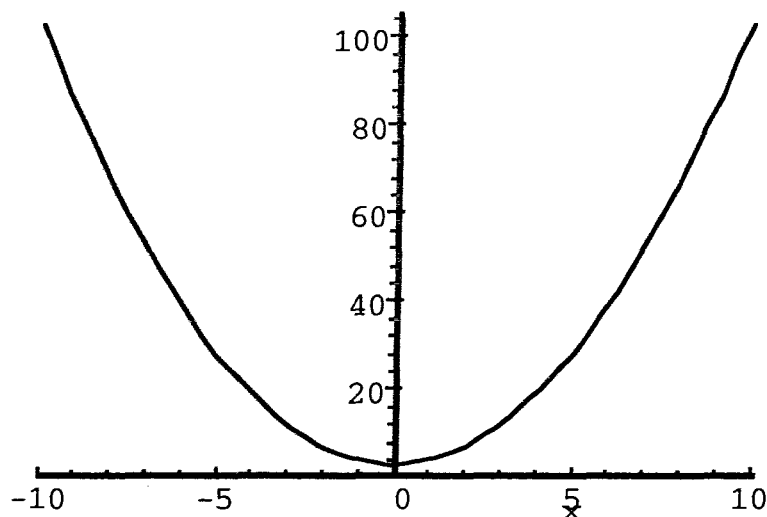
It should be noted that there are mathematical concepts used in Maple which do not exist in the high school mathematical curriculum. For instance, in Maple we have the concept of the graph of an expression and students can execute commands



```
> y := x^2 + 3;
```

$$y := x^2 + 3$$

```
> plot(y, x);
```



in which  $y$  is defined as the name of the expression  $(x^2 + 3)$ , and  $\text{plot}(y, x)$  creates "the graph of the expression  $y$ ". In high school students only learn the concept of graph of a function, and they do not see an expression as a function. Hence, there is a need to clarify this when designing the lab course.

In addition, the results given by Maple do not always convey mathematical concepts completely. For example, using Maple to find the limit of function  $f(x) = \sin\left(\frac{3\pi}{x}\right)$  when  $x$  tends to 0 students performed

```
> f := x -> sin(3*Pi/x);
```

$$f := x \rightarrow \sin\left(3 \frac{\pi}{x}\right)$$

```
> limit(f(x), x=0);
```

-1 .. 1

This function has no limit value when  $x$  tends to 0, and the concept of nonexistence of limit was expressed by the result -1 .. 1. One student (Thanh) asked me that why Maple did not give the solution as "no limit" instead of -1 .. 1. I explained that it was because it

wanted to emphasize the given function oscillated in the interval from -1 to 1. I think that in this case students needed to understand more clearly the concept of the nonexistence of a limit.

### ***Implications***

Throughout the experiment, the dean of the Mathematics Department at Dalat University followed my work and gave me some ideas involving the design of the lab course. After I finished the experiment he asked me to report briefly on the results of the study. He suggested I continue on to write a complete lab course, and study how the course could be developed in the context of Dalat University for both calculus and linear algebra courses.

I think some research questions need to be clarified in further studies before we can develop a computer-based learning environment that will help students to learn mathematics more effectively at Dalat University. The research questions include:

- Is there a significant difference in student achievement between students who learn mathematics in a computing environment and students who learn mathematics in class without computers?
- How does the factor of gender affect learning mathematics in a computing environment? The effects should be examined in terms of student achievement and attitudes toward mathematics and computers.
- Study other models for teaching mathematics that use computers, for instance teaching with lectures that combine with demonstrations on a computer in class. Consider the feasibility of other models in the context of Dalat University.

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## Appendix A

### The syllabus of the course Calculus A1 - 5 Credits

This course introduces the knowledge of calculus for functions in one real variable and linear algebra. The content of the calculus part of the course includes limits of sequences, limits of functions, derivation and integration. The content of the linear algebra part includes the concepts of vector spaces, matrices, linear mappings, determinants and systems of linear equations.

#### ***Part 1: CALCULUS***

##### *Chapter 1: Limits. 15(10, 5)*

##### 1.1 Introduction.

1.1.1 Sentences. Operations.

1.1.2 Sets. Operations.

1.1.3 Mappings.

##### 1.2 Real numbers.

1.2.1 Axioms of the set of real numbers.

1.2.2 Some properties of the set of real numbers.

##### 1.3 Limits of sequences of real numbers.

1.3.1 Definitions: sequences, subsequences, limits, proper limits.

1.3.2 Theorems of the limits of real number sequences.

##### 1.4 Limits of real functions.

1.4.1 Definition.

1.4.2 Theorems of the limits of real functions.

1.4.3 Infinite largeness. Infinite smallness.

##### 1.5 Continuous functions.



1.5.1 Definition.

1.5.2 Main theorems.

*Chapter 2: Derivation and differentiation of one variable function. 15(10, 5)*

2.1 Derivative and differential of first order.

2.1.1 Definitions.

2.1.2 Geometric meaning and mechanical meaning of derivation.

2.1.3 Derivation and differentiation formulas.

2.1.4 Derivatives of basic elementary functions.

2.2 Basic theorems.

2.2.1 Fermat's theorem.

2.2.2 Rolle's theorem.

2.2.3 Cauchy's theorem.

2.3 Derivative and differential of higher order.

2.3.1 Definition.

2.3.2 Examples.

2.4 Taylor's formula.

2.4.1 Taylor's formula with remainder forms.

2.4.2 Approximating function by polynomial.

2.5 Some applications of derivation.

2.5.1 L'Hospital's rule.

2.5.2 Investigating functions.

*Chapter 3: Integration. 15(10, 5)*

3.1 Indefinite integrals.

3.1.1 The concepts of antiderivative and indefinite integrals.

3.1.2 Method of changing variables, method of integration by parts.

3.1.3 Indefinite integrals of some basic elementary functions.

3.1.4 Indefinite integrals of rational, irrational, and trigonometric functions.

3.2 Definite integrals.

3.2.1 Definition of definite integral: Riemann sum, Darboux sum.

3.2.2 Integrable conditions.

3.2.3 Some classes of integrable functions.

3.2.4 Properties of definite integrals.

3.2.5 Newton-Leibnitz's formula.

3.2.6 Method of changing variable, method of integration by parts.

3.2.7 Numerical approximations of definite integrals.

3.3 Applications of definite integrals.

3.3.1 Evaluating arc lengths.

3.3.2 Evaluating areas.

3.3.3 Evaluating volumes and surface areas of a solid of revolutions.

3.4 Improper integrals.

3.4.1 Integrals with infinite boundaries: definition, properties, convergence condition.

3.4.2 Integrals of unbounded functions: definition, properties, convergence condition.

## ***Part 2: LINEAR ALGEBRA***

*Chapter 4: Vector Space. 8(6, 2)*

4.1 Complex numbers.

4.2 Vector space.

4.2.1 Definition. Examples.

4.2.2 Basic properties.

### 4.3 Subspace.

4.3.1 Definition. Examples.

4.3.2 Vector subspace generated by set. Its characteristics.

### 4.4 Linear dependence. Linear independence.

4.4.1 Definitions. Properties.

4.4.2 Systems of equivalent linear independent vector.

### 4.5 Base. Finite dimension vector space.

4.5.1 Generated set. Finite dimension vector space.

4.5.2 Base. Theorem of existence of the vector space base.

4.5.3 Number of dimension of the vector space.

4.5.4 Coordinate.

## *Chapter 5: Matrices. 8(6, 2)*

### 5.1 Concept of matrices

### 5.2 Operations on matrices.

5.2.1 Matrix addition.

5.2.2 Matrix multiplication with number.

5.2.3 Matrix multiplication.

### 5.3 Forms of matrices.

### 5.4 Matrix rank.

### 5.5 Elementary transformation on matrices.

## *Chapter 6: Determinants. 8(5, 3)*

### 6.1 Permutation.

### 6.2 Definition of determinants.

6.2.1 Definition.

6.2.2 Examples.

6.3 Basic properties of determinants.

6.4 Evaluating determinants by expanded formulas.

6.4.1 Expanded formulas along row or column.

6.4.2 Laplace's rule.

6.5 Applying determinants to find inverse matrices.

6.5.1 Adjoint matrices.

6.5.2 Conditions of existence of inverse matrices. Find the inverse matrices.

6.6 Criteria of determining matrix rank.

*Chapter 7: Systems of linear equations. 6(4, 2)*

7.1 Concepts of the systems of linear equations.

7.2 The necessary and sufficient condition for existence of roots.

7.3 Solve systems of linear equations by Cramer's method.

7.4 Solve systems of linear equations by Gauss's method.

Note:

The symbol  $n(x, y)$  means there are  $n$  hours for teaching the chapter, including  $x$  hours for lectures and  $y$  hours for exercises.

## **Appendix B**

### **The Questionnaire**

**Dear Friends,**

I am a graduate student of the Faculty of Education of Simon Fraser University, Canada. At present, I am doing my thesis in science education. My thesis involves the study of applications of computers in learning and teaching mathematics. I am hoping to have your help. Some of you will be invited to participate in an experiment as part of my study from September 3, 1995 to October 30, 1995. The purpose of this questionnaire is to help me to have some information about you, and based on which I will invite you to participate in the study.

In order to help me, you please answer the questions given in the next part. Your answers will be used only for this study. Please note that it is not only students with high marks who will be invited to participate. I will make selection based on my own needs in terms of educational research.

I hope that you will answer this questionnaire fully. Thank you very much for your cooperations, and good luck in your academic studies.

Investigator

Nguyen Huu Tan

**Please answer the following questions:**

- Student Full Name: \_\_\_\_\_
- Sex:      Male  Female
- Your marks in the following subjects on the entrance examination of Dalat University:  
    Mathematics: \_\_\_\_\_ Physics: \_\_\_\_\_ Chemistry: \_\_\_\_\_
- Your mathematics achievement in Grade 12 of high school:  
    Poor  Average  Good  Very good
- Your English background:
  - You have never learned English
  - You have studied English Grade 12 
    - Poor
    - Average
    - Good
    - Very good
  - You have got the English Certification in A level
  - You have got the English Certification in B level
- Your basic computer skills:
  - You have never learned computers
  - You know how to use keyboard and mouse
  - You have learned the basics of using computers already
  - You have got the Computing Certification A level
  - You have got the Computing Certification B level
- Please let me know your address or phone number so that I can contact you.
  - Phone number: \_\_\_\_\_
  - Address: \_\_\_\_\_

## Appendix C

### Calculus A1 - Lab #0

#### INTRODUCTION TO USING MAPLE

##### Objectives

Lab #0 introduces students some basic skills of using the mathematical software Maple. Through this lab, students will know how to start Maple and do mathematics with Maple's interface system. Using computer-based approach, students will perform some mathematical problems that they are used to meeting in high schools such as computing numerical evaluations and approximations, manipulating algebraic expressions, solving equations and inequalities and plotting graphs. In addition, through investigating polynomials students will make a study of relationships between graphs and existence of roots as well as properties of roots of equations.

Lab #0 helps students to review some high school mathematical concepts with the new approach based on computers beside the traditional way of studying mathematics in classrooms. At the same time, working in a computing environment enables students to practice problem solving and computational skills and develop mathematics intuition.

---

##### 1. Introduction to Maple

Maple is a computer algebra system software which is relatively easy to learn and use. Maple's use is intended for the scientist, mathematician, engineer or student who could use the assistance of a mathematical expert in solving certain mathematical problems.

Maple has superior ability to solve problems in symbolic form. Maple can operate symbolically on fractions, factor and expand polynomials, give exact solutions to equations, graph various functions, compute limits, symbolically calculate derivatives, give exact solutions to indefinite and definite integrals, solve certain differential equations and

calculate power series. Maple manipulates both real and complex numbers. It also has an impressive linear algebra package that allows the user many matrix manipulating commands.

Another power of Maple lies in the fact it is a programming language. The operators and functions of Maple can be used to write user-developed procedures to solve problems.

Starting a Maple session is also very simple. To initiate Maple, simply open a worksheet. Maple has an interface system in form of menu that is similar to the Windows operating system and other commonly-used software, so users can adapt quickly and work comfortably in it. Maple is really a mathematical expert system. It is a powerful computer-based mathematical tool.

---

## **2. Starting a Maple Session**

Suppose that you have a prompt from DOS operating system: Now type the command **C:\WIN** to initiate Windows, and the Program Manager window is opened. From this window, you double click the mouse on the icon with the title **Maple V Release 3** to open the window of this application. From the Maple V Release 3 window, you double click the mouse on the icon with the title **Maple V R3 for Windows**. Maple opens a worksheet and it is ready to use. You will see in this worksheet a symbol "**>**", which is a Maple prompt. After this prompt you can type commands and functions of Maple to do mathematics. Results will appear in next line. Now you know to open Maple. I hope that you will enjoy it!

---

## **3. Some notes in communication between user and Maple**

Maple is designed to help us evaluate mathematical expressions. However the main means of entering data into computer is a keyboard with some limitations of characters and numeric symbols. Hence, there is a need of some conventions when entering mathematical expressions from the keyboard:



- Ordinary names are used instead of Greek letters to represent variables, unknown quantities, or special actions. For example, summations are denoted by **sum(i, i=1 .. n)** or **Sum(i, i=1.. n)** rather than using symbol  $\Sigma$ . The distinction between capitals and lowercase letters is important, as explained later.
- Multiplication is indicated by the symbol “\*”.
- $a \div b$  is entered as **a/b**.
- Subscripts, such as in  $x_n$ , are written **x[n]**.
- Exponents or powers such as  $x^n$  are written **x^n** or **x\*\*n**.
- Command lines appear in the text as lines beginning with the symbol **>**, as in  
**> a+b;**  
 The results generated by Maple usually appear immediately following the command.
- Complicated expressions may result in the need to enter a single command across several lines of input. To accommodate this, Maple does not complete and evaluate an expression until it sees a “;” somewhere on the line.

**> a + b;**

**and**

**> a**

**> +**

**> b**

**> ;**

both cause the system to evaluate the expression  $a + b$ .

The use of the semicolon also allows us to enter two or more expression on the same line, as in

**> a + b; 1/c;**

#### 4. Numerical evaluations and approximations

Most simply Maple can be used like a calculator. You can use Maple to perform basic arithmetic by entering any required expression, ending it by the symbol ";" and pressing the Enter key. The actual value of the expression appears immediately following the command line.

```
> (3+4)*12;
```

You often need to refer to the value of the previous expression. You can do this by using the double quote symbol "" as a name for the last computed value. Thus, for example, the reciprocal of the preceding expression is

```
> 1/";
```

Observe that Maple supports true rational arithmetic. Things get even more interesting when you discover that there is no practical limit on the size of the integers used.

You can compute

```
> 100!
```

```
> %/(99!);
```

By default, Maple includes a full ten digits in its floating-point approximations but makes provision for you to request more. The more digits you request, the more accuracy you can expect in your answer. All Maple's exact numerical quantities can be approximated by using the command `evalf()`. This command means "evaluate using floating-point numbers".

```
> evalf(5/17);
```

```
> evalf(5/17, 30);
```

```
> evalf(5/17, 100);
```

Of course, not all real numbers are rational numbers; for example,  $\sqrt{2}$ ,  $\pi$  and  $e$ . Maple represent these numbers exactly as

```
> sqrt(2), Pi, E;
```

In Maple,  $\pi$  is denoted as **Pi**, and  $e$  is denoted as **E**. Floating-point approximations for these can be obtained to an arbitrarily high degree of precision.

```
> evalf(sqrt(2), 40);
```

```
> evalf(Pi, 100);
```

```
> evalf(E, 30);
```

A second way to generate more digits is to make a global in the number of digits.

Use the system variable **Digits**.

```
> Digits := 30;
```

```
> evalf(sqrt(2));
```

```
> evalf(Pi);
```

The default value of digits is 10. To restore the default value, set digits back to 10.

```
> Digits := 10;
```

---

## Exercises

- Use Maple to evaluate the following expressions:

(a)  $1/5 + 2/3$

(b)  $2^{40}$

(c)  $1/e + 1/\pi$

(d) Compare two numbers  $e^\pi$  and  $\pi^e$

- Evaluate the following roots with 12 digits:

(a)  $4^{1/5}$

(b)  $(-4)^{1/5}$

(c)  $5^{2/3}$

(d)  $(-5)^{2/3}$

- Try to evaluate  $1/0$  to see how Maple manipulates. You can also try with  $0/0$ .

---

## 5. Expressions and equations

The real power of Maple is to allow you deal with expressions involved with unknown quantities. Consider, for example, the expression

```
> x + z;
```

You can raise this to a power, expand the result fully, and factor it again. All using facilities provided by Maple through two commands **expand()** and **factor()**.

```
> ""^6;  
> expand("");  
> factor("");
```

It is often convenient to name an intermediate expression. This is done as follows

```
> y := 4*x - 3;
```

The symbol “:=“ is used to label the expression  $4x - 3$  with the name  $y$ . The value of  $y$  becomes  $4x - 3$ , and whatever you do with  $y$  you actually do to its value.

```
> y;  
> y^2 + 5*x + 3;  
> expand(y^3);
```

You frequently need to replace a particular unknown (or even an entire subexpression) in an existing expression. You do this via use of the **subs()** command.

```
> y;  
> subs(x=t, y);
```

The first argument to **subs()** is always an equation in which the left-hand side is the object to be replaced and the right-hand side is its new value. Similarly, you can replace  $z$  by  $s+t$  in the expression  $(x + z)^6$ .

```
> (x+z)^6;  
> subs(z=(s+t), "");
```

You create equations simply by indicating that one expression be equal to another.

```
> 4*x + 3 = x + 7;
```

Like expressions, you can name equations by using **:=** as in

```
> eq := "";
```

Give an equation, you can extract the left and right sides of the equation by using the command **lhs()** and **rhs()**.

> leftside := lhs(eq);

> rightside := rhs(eq);

You can solve an equation in a way you used to do with pencil and paper. Maple is designed so that you can do mathematics as you have done it before.

> leftside - rightside = 0;

> " + (4=4);

> " / 3;

> subs(" , eq);

Once you understand how to solve a given type of equation, you can reduce the process of isolating  $x$  to a single command **isolate()**.

> with(student);

> isolate(leftside = rightside, x);

( The **isolate()** command can be read as “isolate  $x$  in the equation  $4x + 3 = x + 7$ ”. This command is in the package named “student”, so in order to use it you need to load the package by the **with()** command).

Maple provides you the **solve()** command to solve equations. This command can be used for finding exact solutions to systems of one or more equations in one or several unknowns.

> solve(eq, x);            The command gives result of the value of  $x$ .

> solve(eq, {x});        The command gives result of the set of values of  $x$ .

---

## Exercises

- Expand some equalities that you are familiar with:

(a)  $(x + y)^2$                       (b)  $(x - y)^2$

(c)  $(x + y)^3$                       (d)  $(x - y)^3$

- Expand the following expressions:

(a)  $\cos(a+b)$                       (b)  $\cos(a-b)$

(c)  $\sin(a+b)$

(d)  $\sin(a-b)$

- Factor the expression  $eq = x^3 + 4x^2 + x - 6$ . Can you guess the roots of the equation  $eq = 0$ ? Solve the given equation.
- Try to perform the following commands:  
> `expand((x^3+y^3)/(x+y));`  
> `normal("");`

Can you guess the effect of the `normal()` command? Try to command ? `expand`, and ? `normal`.

---

## 6. Graphical representations

> `restart;`

In order to begin a session you can command **restart**. This command erases all of names and releases all memory used. This helps you to avoid errors caused by using same names while practicing.

Maple provides you the **plot()** command that allows you create graphs. The process of drawing graphs Maple refers to as plotting.

> `y := x^2 + 3;`

> `plot(y, x);`

In this plot you do not specify a range for the values of either  $x$  or the expression  $y$ . By default,  $x$  is given values in the interval  $-10 .. 10$ , and the range of  $y$  values is chosen to accommodate all the computed-points. This is done automatically, but you can also control the scaling of both  $x$ -axis and the  $y$ -axis by giving the desired range of values explicitly.

> `plot(y, 'x'=-3..3, 'y'=0..20);`

Here you use single forward-slanting quotes to the names  $x$  and  $y$  to guarantee that their names, rather than their values are considered. To understand this you can compare the difference in results for the following two statements:

> `y;`

```
> 'y';
```

Try the following example.

```
> restart;
```

```
> f := x^2;
```

```
> plot(f);
```

You will have no graph, because the domain is not given. The correct syntax for the `plot()` command

```
> plot(f, x = -3..3);
```

```
> plot(f, x = -3..3, y = -1..15);
```

You can give the domain of  $x$  values that you want to plot in that domain, and at the same time define the graph view window by using both a horizontal and a vertical range.

```
> plot([x, f, x = -3..3], x = -8..8, y = -1..15);
```

You can plot several graphs on same axes.

```
> g := 7 + 3*x - 5*x^2;
```

```
> plot({f, g}, x = -3..3);
```

You can also command

```
> setfunc := {f, g};
```

```
> plot(setfunc, x = -3..3);
```

In order to recognize which graph corresponds to which function, you have to use your mathematical knowledge. However, an another way easier to recognize them is to assign colors to graphs, and then use `display()` command to plot them. This command belongs to the package `plots`, hence you need to load it before using `display()`.

```
> with(plots):
```

```
> gr1 := plot(f, x = -3..3, color = blue):
```

```
> gr2 := plot(g, x = -3..3, color = red):
```

```
> display({gr1, gr2});
```

There are two intersection points on the graphs. To compute the points exactly, you solve the equation  $f$  equals to  $g$ .

```
> solve(f = g, x);
```

The `solve()` command returns an expression sequence containing two roots.

Generally, Maple gives three data structures.

Expression sequence:  $a, b, c$

List:  $[a, b, c]$

Set:  $\{a, b, c\}$

The expression sequence is the content of both the list and the set.

The list retains order and multiplicity. The lists  $[a, b, c]$  and  $[a, a, b, c]$  are distinct, as are the lists  $[a, b, c]$  and  $[a, c, b]$ .

The set is mathematically faithful. The sets  $\{a, b, c\}$  and  $\{a, a, b, c\}$  are the same set, as are the sets  $\{a, b, c\}$  and  $\{a, c, b\}$ .

To reference an individual member of any of these three data structures, it is convenient to tag the data structure with a referencing name and to use selector brackets.

```
q1 := a, b, c      q1[2] is b
```

```
q2 := [a, b, c]   q2[2] is b
```

```
q3 := {a, b, c}   q3[2] is b
```

You can tag the output of the previous `solve()` command as follows:

```
> q := solve(f=g, x);
```

Then `q` is the name of the expression sequence (the list of roots in form of an expression sequence), and you can refer the individual members of the expression sequence `q`.

```
> q[1];
```

```
> q[2];
```

You can evaluate approximately these roots by using function `evalf()`.

```
> evalf(q[1]);
```



```
> evalf(q[2]);  
> evalf(q);  
> evalf({q});  
> evalf({q});
```

You can obtain the y-coordinates corresponding to the two x-coordinates just calculated.

```
> y1 := subs(x=q[1], f);  
> y2 := subs(x=q[2], f);
```

You can try to expand the expressions  $y_1$  and  $y_2$ , or you can try to simplify these expressions.

```
> simplify(y1);  
> expand(y1);
```

Sometimes it is necessary to work completely numerically. Begin by obtaining more digits in the floating-point versions of the x-coordinates at the intersections, and then evaluating y-coordinates.

```
> x1 := evalf(q[1], 30);  
> subs(x=x1, f);
```

You can do the direct numeric calculation of roots of equations. Maple has the **fsolve()** command to do this.

```
> fsolve(f=g, x);      Compare results with evalf().
```

In this instance, **fsolve()** delivered both solutions. In the event that **fsolve()** should fail to deliver all solutions or fail to deliver the desired solution, there is an option for telling **fsolve()** where to look for the desired solution.

```
> fsolve(f=g, x, 0..2);
```

---

## Exercises

- Plot the graph of function  $y = \sin x$ , on the interval  $[-\pi, \pi]$ . Then plot graphs of functions  $y = \sin x$  and  $y = x^2$  on the same coordinate system on the interval  $[-\pi, \pi]$ , and compare with the previous graph.
- Define a set of functions  $\{ e^x, e^{-x}, \sin x, x^2 \}$ . Plot functions of this set on the same coordinate system, on the interval  $[-\pi, \pi]$ . Guess which graph is corresponding with which function.
- Consider the function  $y = x^3 + 45x^2 - 574x - 4007$ . Plot this function on the interval of -4 to 4. Based on your knowledge of the shape of third degree function, what remarks do you have about the graph just plotted? Try to plot the previous function on the interval of -50 to 10, and on the interval of -75 to 25. Try to draw experience of the plotting function from the previous cases.
- Plot the graph of function  $y = x^3 - 3x - 4$  in turn on the interval of -4 to 4, and -1000 to 1000. What are your remarks of these two graphs? Do you think the point  $(0, 0)$  lies on the graph?
- Plot the graphs of two functions  $y = x^2$  and  $y = \sin x$  on the same coordinate system. Guess how many real roots the equation  $x^2 = \sin x$  has. You can check this by solving the given equation.

---

## 7. Polynomials

Through investigating polynomials, this part will help you understand the commands of finding roots of equations. Consider the polynomial

```
> p1 := 40*x^8 - 700*x^7 + 5250*x^6 - 21985*x^5 + 56065*x^4 - 88890*x^3 +  
85320*x^2 - 45225*x + 10125;
```

```
> f := factor(p1);
```

From the result, can you guess how many real roots the polynomial  $p1$  has? and which one is single root, which one is double root, and which one is multiple root? Do you remember the graphical properties of single root, double root and multiple root?

We plot the previous polynomial.

```
> plot(p1, x=-5..5);
```

In this plot, we can not see clearly how the plot cuts the  $x$ -axis, so it is difficult to guess roots of the polynomial. You can try to narrow the domain of  $x$  values.

```
> plot(p1, x=-1..4);
```

The result is not better. Try

```
> plot(p1, x=0.5..4);
```

It seems the graph is a little better, but perhaps you need to restrict the range of  $y$  values.

```
> plot(p1, x=0.5..4, y=-5..10);
```

You see the graph better much in this time. Compare the graph with your conjecture about the polynomial's roots. To find roots of a polynomial, Maple has the `roots()` command.

```
> roots(p1);
```

The result is a list of roots and numbers that indicate the multiplicities of the roots. This means if this number is one then the corresponding root is a single root, if this number is two then the corresponding root is a double root, and if this number is three then the corresponding root is a multiple root.

The `roots()` command works only with polynomials. The `solve()` command is also used to find roots of equations but more general it is applied for many kinds of equations, not only for polynomials.

```
> solve(p1=0, x);
```

You consider an interested example. Finding roots of the polynomial

```
> p2 := x^5/5 - 5*x^3/3 + 4*x;
```

```
> roots(p2);
```

The result shows that the polynomial has a single root 0. Try to factor the polynomial and plot it.

```
> factor(p2);
```

```
> plot(p2, x=-2.5..2.5);
```

The result matches with the result that roots() provided. Now, use the solve() to find roots.

```
> solve(p2=0, x);
```

This time, we have five roots in which one is real, and the others are complex. Thus, the roots() command has no ability to find complex roots if they exist.

One more example shows a different property of roots() command. Find the roots of the following polynomial:

```
> p3 := x^3 + 2*x^2 + 1;
```

```
> roots(p3);
```

The result shows that the polynomial has no root. Do you feel there is something wrong here? Remember that in high school you learned that the graph of third degree function always cuts the x-axis at least at one point. This means a third degree polynomial always has at least one real root. Try to find roots by using the solve().

```
> r := solve(p3=0, x);
```

What is the symbol 1%? It is a label that Maple uses to define a sub-expression in a larger expression, in order to shorten the amount of space required to print the large expression. In this case you see there are three roots involving 1%. If you look closely, you will see there are two complex roots, and one real one. You also see that the real root is not rational root, and this thing explains why roots() cannot find this root. A property of the roots() command is it can only find rational roots.

You can check this real root.

```
> r1 := r[1];
```

```
> subs(x=r1, p3);
```

```
> simplify("");
```

You can evaluate the approximate root.

```
> appr := evalf(r1);
```

```
> subs(x = appr, p3);
```

```
> evalf(subs(x=appr, p3), 20);
```

You see the result is approximately 0 (not exactly 0).

Another example of finding roots of polynomial.

```
> p4 := x^4 + x^3 - 2*x^2 + 3;
```

```
> factor(p4);
```

```
> roots(p4);
```

It has no factors, and no roots either. From the last experience, we see that the polynomial can have either rational roots or complex roots. We try to find roots with `solve()`.

```
> r := solve(p4=0, x);
```

What is the **RootOf()**? It is a placeholder Maple uses to represent all the roots of an equation, including the complex ones. **RootOf** is a shorthand which saves a lot of space.

We try to substitute the **RootOf** back into the polynomial.

```
> subs(x=r, p4);
```

```
> simplify("");
```

Indeed, the **RootOf** contains the roots of the given polynomial. If we really want to know the values contained in **RootOf**, we use **allvalues()**.

```
> allvalues(r);
```

In this case we obtain four roots which are complex. Hence, the polynomial's graph does not cut the x-axis. Plot to see this:

```
> plot (p4, x=-2..1.5);
```

Consider the following expression. Although it is not polynomial, it will show you another property of the solve().

```
> e := x^2 - 1 - exp(x/10)/10;
```

Clearly you cannot use the roots() to find roots of the equation  $e=0$  in this case. Let's try with solve().

```
> solve(e=0, x);
```

Do you think this equation has no root? Plot the polynomial.

```
> plot(e, 'x'=-2..2);
```

The graph seems to show that the polynomial has two roots,  $x=-1$  and  $x=1$ . The problem is these two roots not exact  $x=-1$  and  $x=1$ , because of the component  $\exp(x/10)/10$ . The solve() command works only for exact solutions, and in this case it cannot find any exact solutions. There is another command in Maple helps to find approximate solutions, that is **fsolve()**.

```
> fsolve(e=0, x);
```

The result shows only one root. Why? That is a property of the fsolve() command; it does not necessarily find all roots. The user can provide the command a domain that is expected to contain a certain root, when using it to find solutions.

For example, to find the root near  $x=-1$ .

```
> fsolve(e=0, x, -1.2..-0.8);
```

To find the root near  $x=1$ .

```
> fsolve(e=0, x, 0.8..1.2);
```

Does the equation have more solutions? We see that when  $x$  tends to positive infinity,  $\exp(x/10)$  tends to positive infinity too and tends faster than  $x^2$ ; thus the expression will tend to negative infinity. Consider the graph

```
> plot(e, x=-infinity..infinity);
```

We see the right branch of the graph has to cross the  $x$ -axis to go down negative infinity when  $x$  is large enough. Hence, the equation has to have a third zero. To find this

third zero we have to provide `fsolve()` an appropriate domain so it can find. We can explore this domain by using `plot`.

```
> plot(e, x=0..10);
```

```
> plot(e, x=0..100);
```

```
> plot(e, x=0..200);
```

```
> plot(e, x=0..140);
```

Finally, we find the zero that lies between 110 and 130. Now, we can try this domain with `fsolve()`.

```
> fsolve(e=0, x, 110..130);
```

We check the solution.

```
> evalf(subs(x=", e));
```

The result is very near 0.

---

## Exercises

- Consider the polynomial defined by

```
> p := x^6 - x^5 + 8*x^3 + 3*x^2 - 7*x - 4;
```

Find all the roots of the polynomial, find exact rational roots and approximate irrational roots. If there are approximate roots, substitute them back into the polynomial to check.

Plot the polynomial on an appropriate domain to be able to show all the solutions.

- Doing the same as in the previous exercise, but with the polynomial

```
> p := x^5 - 2*x^4 + x^3 - x^2 + x - 1;
```

- Create a seven degree polynomial so it has five real roots. Plot the polynomial on an appropriate domain to show all the roots.

---

## 8. Understanding inequalities through plotting

This gives you some examples of using plotting as an aid to understanding solutions of inequalities.

Solve the inequality  $3x - 5 < 4$ .

```
> restart;  
> y := 3*x - 5;  
> plot(y-4, x);
```

In this case the default plotting domain of  $-10..10$  captures the interesting part of the graph, i.e., the part that allows determination of where the graph crosses the x-axis, so there is no need to specify the domain. From the graph you can read off the algebraic solution of the inequality:  $\{x: x < 3\}$ .

You can obtain this solution algebraically by using `solve()`.

```
> solve(y-4<0, {x});
```

You see that the use of `solve()` here gives you the solution quickly, but it does not show you the algebraic techniques used to derive this solution. However, the emphasis here is on constructing a graphical interpretation of inequalities.

Consider the problem of comparing two or more expressions through solving the inequality  $3x - 11 > -3x + 11$ .

```
> restart;  
> y := 3*x - 11;  
> z := -3*x + 11;  
> set1 := {y, z};  
> plot(set1, x);
```

In this case, the actual solution is “y is greater than z for all x such that  $x > 11/3$ ”. The x value ( $x=11/3$ ) corresponds to the intersection of the two lines, and is the solution of the equation  $y=z$ .

```
> y=z;  
> solve("", {x});
```

Again, the `solve()` command confirms the previous solution.

```
> solve(y-z > 0, x);
```



An alternative graphical approach is to plot a graph of the difference of the two expressions as in:

```
> plot(y-z, x);
```

Here the solution to the inequality is the value of  $x$  for which this line lies above the  $x$ -axis.

Maple also supports set notation and set operations. This can help in setting up the initial graphs and manipulating the resulting solutions. The expression `set1` is a set in the usual mathematical sense. To test for membership we can use the command **member()**.

```
> member(y, set1);
```

```
> member(x, set1);
```

Maple has commands **union()**, **intersection()**, and **difference()** (**minus()**) to help you to do common set operations.

```
> {y} union {z};
```

(Try the Maple help commands `? sets` and `? solve`)

Solve the inequality  $\frac{x-1}{x+1} > 0$ .

```
> restart;
```

```
> (x-1)/(x+1) > 0;
```

```
> plot((x-1)/(x+1), x);
```

The graph produced is not very informative near  $x = -1$  because of the extremely small values found in the denominator. The `plot()` command works by sampling the curve at various points and then using straight lines or other smooth curves in between the true sample points. Sometimes, as in this example, this leads to a very poor approximation for parts of the curve. When this happens you just describe the graph more precisely, as in:

```
> plot((x-1)/(x+1), 'x'=-3..3, 'y'=-5..5);
```

Based on the graph, can you give the solution to this equality? Its algebraic solution is

```
> solve((x-1)/(x+1) > 0, x);
```

A closer examination of the graph shows that the points where the subexpressions  $x-1$  and  $x+1$  are zero play a crucial role. The whole problem can be reformulated as one of determining the signs of the two subexpressions  $x-1$  and  $x+1$ . Thus, you have another way to solve that is to plot both two lines  $x-1$  and  $x+1$ .

```
> plot({x+1, x-1}, x);
```

For example, both these straight lines are negative at  $x = -5$ , so the value of the original expression must be positive at  $x = -5$ .

Consider the inequality

```
> restart;
```

```
> (x^2 - 5*x + 6) / (x^2 + 3*x + 2) < 0;
```

```
> f := lhs("");
```

You can factor both the numerator and the denominator of  $f$ . The numerator is given by the command **numer()** and can be factored.

```
> numer(f);
```

```
> p := factor("");
```

Similarly, the command **denom()** is used to obtain the denominator, and then it is factored.

```
> denom(f);
```

```
> q := factor("");
```

The combined set of factors of both the numerator and the denominator are given by

```
> e1 := convert(p, set) union convert(q, set);
```

The result is the set of expressions whose signs for a given  $x$  value determine the sign of the entire expression. The following is a graph of this set of expressions.

```
> plot(e1, x=-5..5);
```

From the graph you see that the solution to the equality does not include  $x=0$ . To find locations where the sign of a particular expression can change, you must find the zeros of the corresponding graph. For example, the first expression in e1 and its zero are

```
> e1[1];  
> solve("=0, x);
```

Also, the changes in sign of the expression exactly match the changes in sign of the graph of the product of all the individual factors.

```
> e1;  
> e2 := convert(e1, `*`);  
> plot(e2, 'x'=-5..5, 'y'=-5..15);
```

From the graph, you can obtain the solution of the inequality  $f < 0$ .

---

### Exercises

- Use the graphical method to consider the following inequality. Explain the commands and indicate the solution of the inequality.

```
> r := (x+5) / (x+1) > (x+2) / (x-2);  
> lhs(r) - rhs(r);  
> plot(" ", x=-10..10, y=-20..20);
```

- You can also use the command solve() to solve an equation system consists of several equations. Try to do the following commands:

```
> e1 := 3*x + 4*y = 12;  
> e2 := 4*x - 3*y = 10;  
> solve({e1, e2});
```

- Solve the previous equation system by using the graphical method.
-

## **Appendix D**

### **Calculus A1 - Lab #1**

#### **FUNCTIONS, LIMITS AND CONTINUITY**

##### **Objectives**

The Lab #1 represents students knowledge involved functions, limit of sequence and function, and the continuity of function.

For function part, through the Lab #1 students know how to define a function in Maple. A function can be defined by an expression or different expressions on different domains. The Lab helps students reinforce basic knowledge of function as odd function, even function, functional operators, composition of functions, implicit function and inverse function.

For limit and continuity part, the Lab #1 represents students how to find limits of sequences, limits of functions, and one-sided limits in Maple. Based on results of finding limits, students investigate the continuity of functions. However, before learning how to compute limits the studying of formal definitions of concepts of sequence's limit and function's limit is specially emphasized. These are represented detail aimed at helping students can approach the concepts easily. Besides that, two important theorems involved this part, the Squeeze Theorem and the Intermediate Value Theorem, are introduced to students under the application point of view. Through that, students study problem solving skills.

In process of doing the Lab #1, the graphical methods are used constantly as a mean of visual mathematics that helps students in decreasing difficulties while approaching abstract mathematical concepts involved limits and continuity.

---

## 1. Functions

> restart; with(student);

First, you need to distinguish between defining a function and an expression in Maple. You can assign a name to an expression by using := so you knew as in

> e := x^2;

But then, you can not compute value of the expression e at point  $x = 2$  by commanding e(2). This proves that e is not denoted as a function of variable x. However you can compute value of the expression for x by using the command subs() you knew.

> e(2);

> subs(x=2, e);

To define a function of variable x that is represented as in mathematics, you use the syntax as follows:

> f := (x) -> x^2; (or shortly f := x->x^2);

The previous command defines the function  $f(x) = x^2$ , and then you can compute

> f(x), f(2), f(x+1), f(a+b);

You can try ? -> to know more about defining a function in Maple. However, the distinction between function and expression as above is only formal distinction, because when you have understood the concept of function, you can use either an expression or a definition of function in Maple to represent a function. For example, if you see the expression e as a function of a variable x you can draw its graph as in

> plot(e, 'x'=-10..10, 'y'=-5..20);

Similarly, you can draw the graph of the function f so you defined.

> plot(f(x), 'x'=-10..10, 'y'=-5..20);

Two results are the same, if there is a difference then it is syntax using in the plot().

You can also build a function based on the expression defined by using the command **unapply()**. In other word, this command helps to convert an expression to a function.

```
> e := x^3 + x^5;
> f := unapply(e, x);
> f(x);
> f(2);
```

Sometimes an equation such as  $f(x) = x^2$  may occur as the result of solving an equation for  $f(x)$ , as in

```
> eq := x^2 + 3 = 3*g(x);
> isolate("", g(x));
```

There are many common functions built by Maple called built-in functions, as  $\sin(x)$ ,  $\cos(x)$ ,  $\log(x)$ ,  $\sqrt{x}$ ,  $\exp(x)$ ,  $\text{abs}(x)$ , ... You can use these but need not to define them.

```
> f := x -> x*log(x);
```

Whenever you need to check your function  $f$ , you only type  $f(x)$ . Then, you can compute values of function at particular point in its domain, as in

```
> f(x);
> f(3);
> evalf("");
> f(1+a);
```

Domain of a function may be restricted to interval such as  $0 \leq x \leq 3$ , i.e values of the function are computed only in this interval. Such function can be defined in Maple by using a slightly longer form through the command `proc()`, as illustrated here.

```
> g := proc(x)
> if 0 <= x and x <= 3 then x^2
> else ERROR(x, `not in domain`)
> fi
> end;
```

Then you can use this function inside the domain but when you use it outside the domain you obtain an error.

```
> g(0), g(2), g(1.5);  
> g(4);
```

This approach also allows you to define functions that use different rules on different subintervals. These are often referred to as piecewise defined functions.

```
> h := proc(x)  
> if -3<=x and x<=3 then x^2 -2  
> elif 3<x and x<=5 then x - 3  
> elif x>5 then -x^2 + 34  
> else undefined  
> fi  
> end;
```

You try plotting the function defined.

```
> plot(h, -10..10);
```

The actual domain of this function is from  $x=-3$  to  $\infty$ . The `plot()` command is clever enough to ignore the points outside its domain and still continue with the plot. However, another weakness that is inherent in graphing by sampling shows up here as unwanted lines connecting portions of the graph that should not be connected. Can you find them on the graph, and give an explanation?

Do you remember how a function is called odd function, even function? You try following examples.

```
> f := x -> x^2 + 3*x^4 + 1;  
> f(2), f(-2);  
> f(x) = f(-x);  
> g := x -> x^3 + 3*x;  
> g(3), g(-3);
```

```
> g(x) = -g(-x);
```

Generally, you can check whether a function is odd or even by computing

```
> evalf(f(x) - f(-x));
```

```
> evalf(g(x) + g(-x));
```

You draw the graphs of two previous functions. Observing the graphs you can remember and explain graphical properties of odd function and even function.

The following examples shows that Maple also supports the standard mathematical notion for some of the common operators on given functions to create new functions.

However, you have to be careful in determining the new domains and ranges of these new functions.

```
> f := 'f', g := 'g';
```

```
> f(x), g(x);
```

```
> f+g;
```

```
> (f+g)(x);
```

```
> (f-g)(x);
```

```
> (f*g)(x);
```

```
> (f/g)(x);
```

Consider the following two functions:

```
> restart;
```

```
> f := x -> sqrt(x+1);
```

```
> g := x -> sqrt(9-x^2);
```

```
> (f+g)(x), (f-g)(x), (f/g)(x);
```

You see that the domain of the function  $f$  is  $x \geq -1$ , of the function  $g$  is  $-3 \leq x \leq 3$ .

The new domain of the function  $f+g$  is  $-1 \leq x \leq 3$ .

```
> plot( {f(x), g(x), (f+g)(x)}, 'x' = -5..5 );
```

Can you indicate which of the preceding functions correspond to which line on the graph?



The usual mathematical notation for the composition of  $f$  and  $g$  is  $f \circ g$ , but because of the limitations of the typewriter keyboard we use  $f @ g$  to represent composite function.

```
> restart;  
> f(x), g(x);  
> (f @ g)(x);
```

We define  $y$  as a function of a variable  $u$ .

```
> y := u -> u^3;
```

We have  $u$  is a function of a variable  $x$ .

```
> u := x -> sin(x) - cos(x);
```

By doing the composition of two functions  $y$  and  $u$  in that order, we have a new function  $g$  that is a function of a variable  $x$ .

```
> g := y @ u;  
> g(x);
```

Compute all the possible composite functions of the two functions shown below.

```
> f := x -> x^3;  
> g := x -> 3*x^2 - 2;  
> (f @ g)(u);  
> (g @ f)(u);  
> (f @ f)(u);  
> (g @ g)(u);
```

You explain the above results. Do you think the composition of two functions has commutativity? It means if we have  $f @ g = g @ f$ ?

---

## Exercises

- Define a function has an absolute value as follows:

```
> f := x -> abs(x^3 - 3*x);
```

Plot this function on the interval  $-2 .. 2$ . Explain how the graph is shaped from the corresponding function without the absolute value.

- Compute composite functions  $f \circ g \circ h$  and  $f \circ h \circ g$  of the functions  $f$ ,  $g$ , and  $h$  defined as follows:

```
> f := x -> x / (x+1);
```

```
> g := x -> x^10;
```

```
> h := x -> x + 3;
```

Explain the way creates the results, and check if the composition of functions has associativity.

- Find a two-degree polynomial such that its graph passes through three points  $[0,1]$ ,  $[2, 4]$ , and  $[3, 1]$ . Plot the graph of the polynomial.
- You have an another way of command to solve the above problem. Perform the following commands to enhance skills of using Maple.

```
> y := a*x^2 + b*x + c;
```

```
> X := [0, 2, 3];
```

```
> Y := [1, 4, 1];
```

```
> for k from 1 to 3 do e.k := subs(x=X[k], y) = Y[k] od;
```

```
> q := solve({e1, e2, e3}, {a, b, c});
```

```
> assign(q);
```

```
> a, b, c;
```

```
> y;
```

```
> plot(y, 'x' = -5..5);
```

( The **assign()** command is very useful when you want to convert an equation in form  $a = b$  to an assignment in form  $a := b$ )

- Define a function  $h$  by using **proc()** where  $h$  is defined as follows:

$h(x) = 2*x - 5$  if  $x < 0$ , but  $h(x) = x^2 + 7*x - 2$  if  $x \geq 0$ . Compute  $h(3)$ ,  $h(-3)$ .

## 2. Implicit Function

Consider the following quadratic expression

```
> q := 17*y^2 + 12*x*y + 8*x^2 - 46*y - 28*x + 17;
```

Setting the equation  $q = 0$  will define the function  $y = y(x)$  implicitly. We attempt to plot this implicit function  $y(x)$ . We solve the equation  $q = 0$  for  $y$  so that  $y$  appears on the left and only  $x$ 's appear on the right, i.e finding the zero  $y = y(x)$ .

```
> r := solve(q=0, y);
```

There are two branches to this implicitly defined function. You can assign names

```
> y1 := r[1];
```

```
> y2 := r[2];
```

What is the domain of for each branch of the implicit function  $y(x)$ ? This question can be answered if you deduce which values for  $x$  keep the square root in either branch real.

```
> solve(-25*x^2 + 50*x + 60 > 0, x);
```

```
> evalf("");
```

The above result helps to define the domain you can plot. Now you can plot each branch of the implicitly defined function  $y(x)$ .

```
> plot({y1, y2}, x = -1..3);
```

On the graph the branches do not quite touch because of the way Maple's `plot()` command generates points at which the functions  $y1$  and  $y2$  are evaluated.

Maple also supports the command `implicitplot()`, in the package `plots`, helps you draw a graph of a implicit function.

```
> with(plots):
```

```
> implicitplot(q, x=-3..3, y=-1..3);
```

---

## Exercises

- Given a line  $y = x + 1$  and a quadratic equation  $17y^2 + 12xy + 8x^2 - 46y - 28x + 17 = 0$ . Draw the graphs of both the line and the quadratic equation on the same coordinate system. Calculate the points of intersection of the two curves.
- 

### 3. Inverse Function

Given a function  $y = f(x)$ . To obtain the inverse function, first solve the equation  $f(x) = y$  for the variable  $x$ . The result produced is the function  $x = g(y)$ . Then you switch the letters to obtain  $y = g(x)$ . Do you remember when a function has an inverse function, and how is the graphical relationship between the function and its inverse function?

Consider the function

```
> f := 1 + sqrt(1 + x^2);
```

```
> plot(f, x = -1..1);
```

From the graph you see that  $f$  is a continuous function but it is not a monotone function (Why?) on its domain. However, if we restrict the function  $f$  on the domain  $(0, +\infty)$  or  $(-\infty, 0)$  so  $f$  is a monotone function. If a function is continuous and monotone then it has an inverse function. So to find an inverse function we choose the branch for which  $x$  is nonnegative (similarly with the branch for which  $x$  is nonpositive). Solve the equation

```
> q := solve(f=y, x);
```

There are two solutions  $x = g(y)$ . Because  $x$  is nonnegative we want the first solution.

```
> q[1];
```

To write in form of convention of function, we replace the  $y$  with  $x$  in the expression.

```
> g := subs(y=x, q[1]);
```

Plot of  $f(x)$  and  $g(x)$ .

```
> plot([x, f, x=0..3], [x, g, x=2..4]);
```

To see more clearly the graphical relationship of the function and its inverse function, you can add the line  $y = x$ .

```
> plot({ [x, f, x=0..3], [x, g, x=2..4], [x, x, x=0..4] });
```

---

### Exercises

- Consider the function  $y = \sin(x)$ . Find the inverse function of this function, and then plot graphs of these two functions on the same coordinate system. Compute all the possible composite functions of the function  $y = \sin(x)$  and its inverse function. Do you have any comment?
  - Given the function  $y = x^2 + 3x$ . With which condition does the given function have an inverse function, and then find the inverse function? Plot graphs of the function and its inverse function on an appropriate domain.
- 

## 4. Limit of Sequence

The sequence of real numbers is a map of the set of natural numbers  $\mathbf{N}$  into the set of real numbers  $\mathbf{R}$ . We can see a sequence as a function of natural number  $n \in \mathbf{N}$ . In Maple we have two ways to describe a sequence, describe a sequence as a function of  $n$ , or describe a sequence as a list of individual members of the sequence by using the command `seq()`.

For example with the sequence  $x_n = \frac{n}{n+1}$  we can describe a list of ten first members of the sequence.

```
> s := [seq(n/(n+1), n=1..10)];
```

Describe the sequence as a function of  $n$ .

```
> n := 'n';
```

```
> f := n -> n / (n+1);
```

Use this function to re-create the list of the sequence's members.

```
> s := [seq(f(n), n=1..10)];
```

To see intuitively how the previous sequence tends to a particular limit when  $n$  tends to infinity, we will plot the graph to show the points of the sequence as a list of pairs  $[i, f(i)]$  as follows

```
> p := [seq( [i, f(i)], i=1..30 )];
```

```
> plot({p}, x=1 ..30, y=0..1, style=point);
```

The graph shows that  $y$ -coordinates of the points, that are  $x_n$  values, has tendency of tending to 1. To see more clearly this we draw more the line  $y=1$ .

```
> plot({1, p}, x=1 ..30, y=0..1, style=point);
```

Limit of sequence is calculated in Maple by using the command **Limit()**.

```
> n := 'n';
```

```
> Limit(f(n), n=infinity);
```

This command only helps us write a limit expression in form of mathematics, but it does not calculate the limit. The actual value of this limit is calculated by applying the command **value()** to the above resulting expression.

```
> value("");
```

However, rather than entering two commands as above we can use the single command **limit()** (with a lowercase "l") to ask Maple to compute the value of the limit immediately.

```
> limit(f(n), n=infinity);
```

In addition, from the graph it is also easy to see that this an increase sequence (do you remember how a sequence is called an increase sequence, or a decrease sequence?). To check this we can compute the difference

```
> f(n+1) - f(n);
```

```
> normal("");
```

The result is a positive expression, thus  $f(n+1) > f(n)$  proves that the given sequence is increase.

You can find limit of a given sequence easily by using the command `limit()`, however to understand the mathematical concept of limit we return to make a study the definition of limit of a sequence.

The number  $a \in \mathbf{R}$  is called the limit of the sequence  $\{x_n\}$  if for any  $\varepsilon > 0$  there exists a number  $N$  such that for any  $n > N$  implies  $|x_n - a| < \varepsilon$ .

For example, we reconsider the previous sequence  $x_n = \frac{n}{n+1}$ . We give a number  $\varepsilon$ ,

denoted by  $e$ , is

```
> e := 0.001;
```

We try finding whether there is a number  $N$ , such that from the  $(N+1)$ th number on we have  $|x_n - 1| < 0.001$ . If we can do so we will conclude that 1 is the limit of the sequence  $\{x_n\}$ . We consider the difference

```
> d := abs(f(n)-1) < e;
```

We see it as an inequality for  $n$ . Solve the inequality

```
> normal(d);
```

```
> isolate(" ", abs(n+1));
```

```
> solve(" ", n);
```

Because  $n$  is a natural number so we get the solution  $n > 999$ . Thus, if we choose

```
> N := 999;
```

then from the 1000th number on we have  $|x_n - 1| < 0.001$ . To be easy to see this we try computing several members.

```
> d := abs(f(n)-1);
```

```
> for k from 1 to 10 do e.k := evalf( subs(n=999+k, d) ); od;
```

In practice, we often meet a sequence defined in term of a sum. Maple supports the command `Sum()` allows us to define a sequence in this style. For example, find limit of a sequence defined as follows:

$$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$$

> f := n -> Sum('1/k^2', 'k=1..n');

> f(n);

> limit(f(n), n=infinity);

> value("");

### Exercises

- Examine the monotony of the following sequences:

(a)  $x_n = \frac{5-n}{5n-1}$

(b)  $x_n = \frac{4^n}{n!}$

(c)  $x_n = \frac{(n+2)^2}{2^{n+2}}$

- Find limits of the following real sequences:

(a)  $x_n = \frac{n}{2^n}$

(b)  $x_n = \sqrt[n]{n}$

(c)  $x_n = 2^{\sqrt{n}}$

- Prove the following equalities:

(a)  $\lim_{n \rightarrow \infty} \frac{1}{n^2 + 1} = 0$

(b)  $\lim_{n \rightarrow \infty} \frac{5n-3}{6n+2} = \frac{5}{6}$

- Using the definition:

The number  $+\infty$  is called the limit of the sequence  $\{x_n\}$  if for any number  $M > 0$ ,

there exists the number  $N$  such that for any number  $n > N$  implies  $x_n > M$ .

Prove that  $\lim_{n \rightarrow \infty} 2^{\sqrt{n}} = +\infty$ .

- Prove that the following sequence are convergent.

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \dots + \frac{\sin n}{n^2}$$

- Prove that the following sequence are nonconvergent (divergent).

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$



## 5. Limit of Function

Given the function

```
> restart; with(student):
```

```
> f := x -> (x^3 - 3*x^2 + 2*x - 6) / (x-3);
```

This function is not defined at the point  $x = 3$ .

```
> f(3);
```

Finding the limit of the function is used to explore how the values of the function are when  $x$  gets values near this point  $x = 3$ , or in other word when  $x$  tends to 3. We try computing a table of values of  $f(x)$  when  $x$  tends to 3.

```
> [f(2.9), f(2.95), f(2.97), f(2.99), f(2.995), f(3.1), f(3.05), f(3.03), f(3.01), f(3.005)];
```

Observing the result we see that the values of  $f(x)$  seems to tend a limit value of 11. In other word, for  $x$  values that are very closed to 3 so the  $y$  values will be very closed to 11.

An another approach to discover the limit of the function is to build a sequence  $\{x_n\}$  such that  $x_n \rightarrow 3$  (when  $n \rightarrow \infty$ ) then the according sequence  $\{f(x_n)\}$  will tend to what limit (when  $n \rightarrow \infty$ ).

Consider the sequence  $\{x_n\}$

```
> xn := n -> 3 + 1/n^2;
```

```
> limit(xn(n), n=infinity);
```

We see that this sequence tends to 3 when  $n$  tends to infinity. Consider the sequence  $\{y_n\}$  where  $y_n = f(x_n)$

```
> yn := n -> f(xn(n));
```

```
> yn(n);
```

```
> limit(yn(n), n=infinity);
```

The result is the according sequence  $\{f(x_n)\}$  tends to a limit of 11.

Now we make a study the formal definition of limit of function.

We say that  $\lim_{x \rightarrow a} f(x) = L$ , if for any  $\epsilon > 0$  there exists a  $\delta$  such that for any  $x$  so  $|x - a| < \delta$  implies  $|f(x) - L| < \epsilon$ .

We see  $x$  in the domain  $|x - a| < \delta$  that means  $x$  very closed to  $a$ , then the value  $f(x)$  is in the domain  $|f(x) - L| < \epsilon$  that means  $f(x)$  closed to  $L$ .

We will use this definition primarily to prove that a chosen limit value is correct.

The way we apply this definition is like a game between two people. The first person chooses a value for  $\epsilon$ . Their goal is to choose an  $\epsilon$  that is as nasty as possible (usually very small). The second person - you - has to choose a suitable value of  $\delta$  (as requested by the definition) for use in conjunction with the specified  $\epsilon$ . The value of  $\delta$  can be seen a bound of  $|x - a|$ .

Prove that  $\lim_{x \rightarrow 1} x^2 + 1 = 2$ .

The function is defined by

```
> f := x -> x^2 + 1;
```

and the proposed limit value is

```
> L := 2;
```

We must find a suitable restriction on  $|x - 1|$  so that the following inequality is satisfied when those values of  $x$  are used:

```
> abs( f(x) - L ) < epsilon;
```

We need an inequality involving  $|x - 1|$ , hence it is a good idea to try rearranging the previous expression into a different form that contains  $|x - 1|$ .

```
> factor(lhs("")) < epsilon;
```

```
> expand(lhs("")) < epsilon;
```

```
> isolate("", abs(x-1));
```

The right-hand side of the result inequality is

```
> bound := rhs("");
```

The above expression is a bound of  $|x-1|$ . However the problem with it is that its value depends on the value of  $x$ . The bound  $\delta$  we are looking for must work independent of our choice of  $x$ . That is, it must depend only on  $\epsilon$ . The following crucial observations allows us to eliminate any reference to  $x$  in the above bound expression.

- We can always choose the smaller of two bounds. The rule we are seeking need only specify what happens near  $x = a$ , for example as the region, say,  $|x-1| < 1/2$ . Thus, we can ignore any values of  $x$  outside this region even if such  $x$  values place  $f(x)$  near  $L = 2$ .
- In the region  $|x-1| < 1/2$ , we can systematically underestimate the value of bound by using instead a single value of bound that is the worst we can find (i.e the smallest) over this entire interval.

In the numerator of the expression bound,  $\epsilon$  is a constant, having already been chosen by our opponent earlier in the game. Thus we can estimate the bound by examining the behavior of

`> e := subs(epsilon = 1, bound);`

on the interval  $|x-1| \leq 1/2$  and scaling the result by  $\epsilon$ . It is clear from the graph of  $e$  on this interval

`> plot(e, x=1-1/2 .. 1+1/2);`

that its minimum value occurs at  $x = 3/2$ . We can choose the value  $\delta$  is:

`> delta := min(1/2, subs(x=3/2, bound);`

For this bound we have  $|f(x) - L| < \epsilon$  whenever  $|x-1| < \delta = \min(1/2, 2/5\epsilon)$  since

$$|x-1| < \delta = \min(1/2, 2/5\epsilon) \Rightarrow |x-1| < \epsilon / |x+1|$$

$$\Rightarrow |x-1| |x+1| = |x^2-1| = |x^2+1 - L| < \epsilon$$

We conclude that the limit of the function when  $x$  tends to 1 is 2.

In addition, there is an another definition of limit of function based on the concept of limit of sequence. From this definition, we can prove back the definition given above.

The definition is:

We say that  $\lim_{x \rightarrow a} f(x) = L$ , if for any sequence  $\{x_n\}$  such that  $x_n \neq a$  so  $x_n \rightarrow a$  (when  $n \rightarrow \infty$ ) we have the according sequence  $f(x_n) \rightarrow L$  (when  $n \rightarrow \infty$ ).

We make a study this definition through the following example. Consider the function

```
> f := x -> x*sin(1/x);
```

```
> limit(f(x), x=0);
```

The limit of the function tends to 0 when x tends to 0. We consider a sequence

```
> a := (n) -> 1/n;
```

```
> limit(a(n), n=infinity);
```

This sequence tends to 0, and the limit of according sequence  $f(a_n)$  is

```
> limit(f(a(n)), n=infinity);
```

An another sequence

```
> b := (n) -> 1/3^n;
```

```
> limit(b(n), n=infinity);
```

This sequence also tends to 0, and the limit of according sequence  $f(b_n)$  is

```
> limit(f(b(n)), n=infinity);
```

With the two sequences  $\{a_n\}$  and  $\{b_n\}$  given above only aims at illustrating how the definition works, it is not used to prove the limit of the given function when x tends to 0 is 0. Can you explain why?

In practice, when using Maple to find limits of functions you can use the command **Limit()** or **limit()**. For instance, return to the two previous examples.

```
> g := x -> x*sin(1/x);
```

```
> Limit(g(x), x=0);
```

```
> value(?);
```

Or shortly,

```
> limit(g(x), x=0);
```

Consider the function

```
> f := x -> (x^3 - 3*x^2 + 2*x - 6) / (x-3);  
> limit(f(x), x=3);
```

We try seeing how the shape of the graph of the function in a region near the point  $x = 3$ .

```
> plot (f, 1..4);
```

If you look closely you will see that there is a point missing from the graph at  $x = 3$ . Except for that single troublesome point, the resulting graph is identical to the graph of the expression

```
> normal(f(x));
```

Can you explain this? Besides that, you can know how the command **normal()** works if you take the numerator of the function  $f$ , factor it and then compare it with  $f$ .

```
> e := numer(f(x));  
> factor("");  
> f(x);
```

We often meet a function exist no limit as a number even though the function value always lies in a narrow range such as  $[-1..1]$ . Consider the function

```
> f := x -> sin(3*Pi/x);  
> f(x);  
> plot(f(x), x = 0..1);
```

From the graph you see the magnitude of  $f(x)$  oscillates in the interval from  $-1$  to  $1$  at the points near  $x = 0$ . You try finding whether the function has a limit when  $x$  tends to  $0$ .

```
> limit(f(x), x = 0);
```

The result shows that the function has no limit value when  $x$  tends to  $0$ , even though the function's range lies in the interval  $[-1..1]$ .

There are functions whose values are unbounded and not exist a finite limit either.

```
> f := x -> 1/x^2;
```

> plot(f(x), x = -1..1, y = 0..20);

> limit(f(x), x=0);

---

### Exercises

- Given a function  $g(x) = (x^2 + 1) / (1 - 3x^2)$ . Compute the values of  $g(x)$  at the points of 100, 1000, 10000, 100000. Guess the limit of the function  $g$  when  $x$  tends to infinity.

Plot the graph and compute the limit to check your judgement.

> plot( g(x), 'x'=-5..5, 'y'=-6..6, numpoints=1000);

> Limit( g(x), x = infinity );

By using - infinity compute limit when  $x \rightarrow -\infty$ .

> Limit( g(x), x = - infinity );

- Given a function  $f(x) = (x^2 - 1) / (x + 1)$ . Find limit of the function when  $x$  tends to -1.

Plot graph of the function in the interval contained the point  $x = -1$ . Do you know this graph, except the point  $x = -1$ , is graph of what function?

- Prove that  $\lim_{x \rightarrow 2} 23x^5 + 105x^4 - 10x^2 + 17x = 2410$ .

- Given a function defined by  $f(x) = \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$ .

Find limits of  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$ .

- Find the following limits

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \text{ and } \lim_{x \rightarrow \infty} \frac{\sin x}{x}.$$

---

## 6. Properties of Limits

When computing limits we often make use of their general algebraic properties to simplify computations. These properties are used to break large problems down into smaller, more manageable ones. Maple helps you remember back these properties.

> restart : with(student) :

The limit of a constant is that constant.

> Limit(3, x=a);

> expand("");

Constants can be factored right out of a limit computation.

> Limit( c \* f(x), x=a);

> expand("");

Limits can also “map” onto each of sums, products, and exponents, as in

> Limit( f(x) + g(x), x=a );

> expand("");

> Limit( f(x) \* g(x), x=a );

> expand("");

> Limit( f(x) ^ g(x), x=a );

> expand("");

The next rule is really a proper case of a power form.

> Limit( sqrt(f(x), x=a );

> expand("");

The problem is in what conditions so we have the above rules. You observe the following example.

Given two functions

> f1 := x -> x;

> f2 := x -> sin(1/x);

Find limit of f1 \* f2 when x tends to 0.

> Limit(f1(x) \* f2(x), x=0);

If applying the rules we can write

> expand("");

> value("");

The result is not to compute the limit, even though the expression  $x \sin(1/x)$  really has the limit when  $x \rightarrow 0$ .

```
> limit(f1(x) * f2(x), x=0);
```

So why we can not apply the rule in this case? To understand this we try finding the limits.

```
> limit(f1(x), x=0);
```

```
> limit(f2(x), x=0);
```

We see that  $\sin(1/x)$  has no limit when  $x$  tends to 0. Thus the more precise state is if there exists  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  then we have the rule

$\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$ . We have similar results for other cases.

Applying the preceding rules, you can rewrite the following limits in terms of limits of simpler expressions. Of course, you have to make sure by yourself that limits of individual members are exist.

```
> e1 := (x^3 + 2*x^2 - 1) / (5 - 3*x) :
```

```
> Limit(e1, x=-2);
```

```
> expand("");
```

You explain the results that Maple gave. Compute limit of the preceding expression we have

```
> value("");
```

You can also change variable in limit computation. You try doing the following example:

```
> e2 := ((1+x)^(1/3) - 1) / x;
```

```
> Limit(e2, x=0);
```

The numerator and denominator of  $e2$  are

```
> p := numer(e2);
```

```
> q := denom(e2);
```

If we let  $1+x = a^3$ , then  $p$  and  $q$  can be rewritten as:



```
> p = a-1;
```

```
> q = a^3-1;
```

In this form, q can be factored as

```
> factor("");
```

and this can be used to simplify the limit computation.

```
> subs(p=a-1, q=a^3-1, p/q);
```

```
> factor("");
```

and since  $a \rightarrow 1$  as  $x \rightarrow 0$ , the value of the limit can be rewritten as

```
> Limit("", a=1);
```

```
> value("");
```

However, you know that it can be evaluated directly by Maple as in

```
> l := Limit(e2, x=0);
```

```
> value(l);
```

In addition, Maple supports the command **changevar()** helps to perform the change of variable. All can be reformulated at once by the command

```
> changevar(1+x = a^3, l, a);
```

```
> value("");
```

We consider an important limit theorem, that is the Squeeze theorem.

The Squeeze theorem says that

If  $g(x) \leq f(x) \leq h(x)$  for all  $x$  near  $a$ , and

$$\lim_{x \rightarrow a} g(x) = L = \lim_{x \rightarrow a} h(x)$$

then

$$\lim_{x \rightarrow a} f(x) = L$$

You consider the problem of proving the  $\lim_{x \rightarrow 0} x \sin(1/x) = 0$ .

First, we examine the plot of this function near 0.

```
> f := x -> x*sin(1/x);
```

```
> plot(f(x), x=-0.5 .. 0.5);
```

The resulting graph is only approximate, but it would appear that the limit is 0 at  $x=0$ . We can compute this limit exactly at  $x=0$  because  $-|x| \leq x \sin(x^{-1}) \leq |x|$  (you try proving this inequality) for all  $x$  near 0.

```
> plot( {abs(x), f(x) }, x=-1.0 .. 1.0 );
```

The preceding graph shows that  $|x|$  is an upper bound. Similarly,  $-|x|$  is a lower bound. We plot all three functions together.

```
> plot( { abs(x), f(x), -abs(x) }, x=-1.0 .. 1.0 );
```

You can compute the limits

```
> Limit(abs(x), x=0);
```

```
> value("");
```

```
> Limit(-abs(x), x=0);
```

```
> value("");
```

Applying the Squeeze theorem we imply that  $\lim_{x \rightarrow 0} x \sin(1/x) = 0$ .

## Exercises

- Using the properties of limit rewrite limit of the following expression when  $x$  tends to 1 in terms of limits of more simple expressions and explain the results.

```
> e := (x^2 - x)^(1/5) + (x^3 + x)^9;
```

- Given the function

```
> f := (t) -> (sqrt(t) - 3) / (t-9);
```

Find limit of the function when  $t \rightarrow 9$  by changing variable  $t = x^2$ .

- Find limit  $\lim_{x \rightarrow 0} \sin(1/x)$ . Plot the function  $\sin(1/x)$  on the closed interval  $[-1..1]$  to see a sense of the Maple's answer.
- Find whether the following limits really exist? (Try drawing the graphs!)

$$\lim_{x \rightarrow \infty} \left(1 + \frac{\sin x}{x}\right)^x \quad \text{vs} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{\sin x}{x}\right)^{x/\sin x}$$

- Investigate shape of the function  $\frac{\sin(x^2)}{x^3}$  by using the graph and limits when  $x$  tends to infinity. Using the upper and lower bound, and applying the Squeeze theorem to prove the resulting limit.

## 7. Continuity and One-Sided Limit

The right-hand limit of  $f(x)$  when  $x$  tends to  $a$  from the right-hand, i.e  $x$  tends to  $a$  with values that are greater than  $a$ , if exists then it will be denoted by  $\lim_{x \rightarrow a^+} f(x)$ .

The left-hand limit of  $f(x)$  when  $x$  tends to  $a$  from the left-hand, i.e  $x$  tends to  $a$  with values that are smaller than  $a$ , if exists then it will be denoted by  $\lim_{x \rightarrow a^-} f(x)$ .

In Maple these notations become

> restart: with(student):

> Limit(f(x), x=a, right);

> Limit(f(x), x=a, left);

If a function has the left-hand limit, and the right-hand limit when  $x$  tends to  $a$ , and these two limits are equal then the function has a limit when  $x$  tends to  $a$ . Then, we have

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$$

There are some cases  $\lim_{x \rightarrow a} f(x)$  does not exist. For example there are two cases, that we knew, are function tends to infinity, and function oscillates in certain range. Now we make a study one more case.

Consider a function defined on two different domains, and each domain accords to a particular function. At the point that joints the two domains, the values of these functions are not equal. For example, we define a function as in:

$$f(x) = \begin{cases} \sqrt{x} + 2, & x > 3 \\ -\sqrt{x} + 2, & x \leq 3 \end{cases}$$

We can use two functions  $f_1$  and  $f_2$  to do this. For  $x > 3$  we will see  $f$  as the function  $f_1$ , otherwise for  $x \leq 3$  we will see  $f$  as the function  $f_2$ .

```
> f1 := x -> if is(x>3) then sqrt(x)+2 else undefined fi;
```

```
> f2 := x -> if is(x<=3) then -sqrt(x) + 2 else undefined fi;
```

By using the `is()` command to define  $f_1$  and  $f_2$  we allow  $f_1$  and  $f_2$  to return different formulas depending on the assumptions placed on symbolic argument. For example, if no assumptions are placed on  $x$ , then  $f_1(x)$  returns unevaluated, as in

```
> f1(x);
```

However, for the case

```
> assume(x>3);
```

We find that  $f_1(x)$  evaluates as

```
> f1(x);
```

We plot both  $f_1$  and  $f_2$  ( we see them as the plot of the function  $f$ ).

```
> plot({ f1, f2 }, 0..5);
```

From the graph you see that the value of limit you expect at the point  $x=3$  depends on the side (left or right) of  $x=3$  from which you want  $x$  to tend to 3. At the same time, the graph also shows that the function is discontinuous at  $x=3$ .

We find the one-sided limits of the function  $f$ .

```
> assume(x>3):
```

```
> limit(f1(x), x=3, right);
```

```
> assume(x<=3):
```

```
> limit(f2(x), x=3, left);
```

The one-sided limits are not equal, so the function  $f$  has no limit when  $x \rightarrow 3$  and  $f$  is discontinuous at  $x=3$ .

However, there also is a case of a function that has a limit when  $x$  tends to a particular point, but it is discontinuous at that point. We consider a function defined particularly at  $x=a$  such that the value of that function

```
> h := x -> x^2 + 3;
```

```
> h(2) := 1;
```

The value of the function  $h$  is defined as 1 at  $x=2$ , while

```
> limit(h(x), x=2);
```

shows the limit of  $h$  when  $x$  tends to 2 is 7. We plot

```
> plot(h, 0..5);
```

You see that the `plot()` command will often miss discontinuities because it generates a curve primarily by sampling function values and assuming continuity between sample points. Hence, if you only see the graph in this case then it is difficult to know the function is discontinuous at  $x=2$ .

Through above observations we see that concept of limit allows us to understand concept of continuity more precisely.

A function  $f$  is said to be continuous at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

Function  $f$  will be discontinuous at the point  $a$  if  $f$  is not defined at  $x=a$ , or if its limit at  $x=a$  does not exist, or if  $f(a)$  is different from the limit value of  $f$  when  $x$  tends to  $a$ .

---

## Exercises

- Investigate the continuity of function defined by

```
> f := x -> abs(x-1)/(x-1);
```

Plot graph of the function on the interval  $[-2..2]$ . From the graph guess whether the function is continuous at  $x=1$ . Find the one-sided limits of  $f$  when  $x$  tends to 1, then let know whether the function is continuous at  $x=1$ .

- Investigate the limit

```
> f := x -> 1 / (x-1)^2 / (x+1);
```

```
> Limit(f(x), x = -1);
```

using directional limits, and draw an appropriate graph illustrating this limit. Does the limit exist? Do the directional limits exist? If so, what are their values?

- Given the function

$$f(x) := x \rightarrow \frac{\text{abs}(x^2-1) + \text{abs}(x-1)}{x-1};$$

Draw the graph of the function on the interval  $[-2, 2]$ . Guess if the function is continuous at  $x=1$ . You find the limit of the function when  $x$  tends to 1. Then find the left-hand limit and the right-hand limit when  $x$  tends to 1. Give an explanation.

- Find  $a$  such that the following function is continuous.

$$f(x) = \begin{cases} e^x, & x < 0 \\ a + x, & x \geq 0 \end{cases}$$

- Consider the following functions:

$$f(x) := \begin{cases} \sqrt{x}, & x \geq 1 \\ 1, & x < 1 \end{cases}$$

$$g(x) := \begin{cases} x^2 - 2, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

Define function  $h$  as  $f$  if  $x \geq 1$  and  $h$  as  $g$  in the reverse case. Draw the graph of  $h$  in the domain  $-2 \dots 2$ . Find the directional limits of  $h$  when  $x$  tends to 1. Let know whether  $h$  is continuous at  $x=1$ ?

- Consider the function  $f(x) = x^{-1/x}$ .

Draw the graph of the function on different intervals of the positive part of  $x$ -axis to see if the limits  $\lim_{x \rightarrow 0^+} f(x)$  and  $\lim_{x \rightarrow \infty} f(x)$  exist?

Evaluate the values  $f(0.5)$ ,  $f(0.2)$ ,  $f(0.1)$ ,  $f(0.01)$ . Guess the limit of  $f$  when  $x$  tends to  $0^+$ .

Evaluate the values  $f(10)$ ,  $f(20)$ ,  $f(50)$ ,  $f(100)$ . Guess the limit of  $f$  when  $x$  tends to infinity.

Evaluate the above limits and compare to your judgment.

- Given function  $g(x) = \frac{1-2^{1/x}}{1+2^{1/x}}$  Evaluate the limits of the function  $g$  when  $x$  tends to  $+\infty$  and  $-\infty$ . Does the function have limit when  $x$  tends to 0? Explain why.

## 8. The Intermediate Value Theorem

**Theorem:** If  $f$  is continuous on the closed interval  $[a, b]$ , and  $f(a) \leq N \leq f(b)$  then there exists a number  $c$  in  $[a, b]$  such that  $f(c) = N$ .

This theorem helps you answer some following problems. You consider the problem:

Find out whether  $f(x) = 0$  has a solution between  $x=1$  and  $x=2$  if  $f$  is defined by

$$> f := x \rightarrow 4*x^3 - 6*x^2 + 3*x - 2;$$

The function  $f$  is continuous since it is defined in terms of a polynomial, so we can use the theorem.

$$> f(1);$$

$$> f(2);$$

Since  $f(1) < 0 < f(2)$ , the theorem guarantees that there must be a  $c$  in  $[1, 2]$  that satisfies the equation  $f(c) = 0$ . The theorem did not produce the exact solution, but at least we know roughly where to look to find one. In this particular example we can compute an exact solution algebraically.

$$> \text{solve}(f(x)=0, x);$$

The values of which are approximately

$$> \text{evalf}(", 5);$$

Not all polynomial equations can be easily solved. Then the information provided by the mean value theorem becomes even more important.

Solve the polynomial equation  $f(x) = 0$  where  $f$  is defined by

$$> f := x \rightarrow 79*x^{14} + 56*x^{12} + 49*x^9 + 63*x^8 + 57*x^7 - 59*x^3;$$

First we attempt to solve this using Maple's built-in `solve()` command.

$$> \text{solve}(f(x)=0, x);$$

You see that Maple was able to find only three roots explicitly, all of them at  $x=0$ . The `RootOf()` expression indicates that the remaining roots are all roots of the indicated

polynomial. You can still find approximations to more of these roots if you know the locations of both positive and negative function values. In this case we have

> f(1/2);

> f(1);

Implying that there is a root between  $x=1/2$  and  $x=1$ . By continually breaking the interval size in half, you can get as close as you like to a solution to  $f(x) = 0$ . For example, at the midpoint of the interval  $[1/2, 1]$  we have

> f(3/4);

so you know that there is a root somewhere in  $[3/4, 1]$  (Can you explain this clearer?).

Similarly, since

> f(7/8);

is positive, you know that there is a solution in the interval  $[3/4, 7/8]$ . In Maple, an accurate estimate of this root can be obtained by the command

> fsolve(f(x)=0, x, 3/4..7/8);

---

## Exercises

- Use the intermediate value theorem to prove that there is a  $c$  such that  $f(c) = 10$  when  $f$  is defined by  
> f := x -> x^3 - 5\*x^2 + x;
  - Explain why the intermediate value theorem cannot be used to find a solution to  $f(c) = 3$  for  $f(x) = (x-2)^4 + 5$ .
  - Given function  $f(x) = x^3 + 4x^2 - 6$ . Draw the graph of this function. From the graph let know how many roots the equation  $x^3 + 4x^2 - 6 = 0$  has and define intervals contained the roots. Find the estimates of these roots in the intervals just defined.
-