THEORETICAL STUDY OF THE NONLINEAR CUBIC-QUINTIC SCHRÖDINGER EQUATION

by

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THEORETICAL STUDY OF THE NONLINEAR

CUBIC-QUINTIC SCHRÖDINGER EQUATION

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ABSTRACT

The one dimensional propagation of intense electromagnetic waves in a nonlinear medium with an intensity dependent refractive index $n = n_0 + n_2|\underline{E}|^2 + n_4|\underline{E}|^4$ is examined theoretically. The nonlinear cubic-quintic Schrödinger equation (NLCQSE) governing the dynamics of the electromagnetic field in the medium is derived. Three conservation laws and the Galilean invariance of the equation are obtained. The Lagrangian formulation for the field equation is developed. The solitary wave solutions for the NLCQSE are obtained for all possible cases corresponding to different signs of n_2 and n_4. Two analytical techniques i.e. the Bäcklund transformation and the inverse scattering transform method are used to test the stability of the solitary wave solutions i.e. to find multi-soliton solutions. These two approaches seem to imply that the solitary waves are not true solitons. However, numerical simulation shows that quasi-soliton behaviour is found to persist over wide regions of parameter space.

DEDICATION

To

The Unsolved Mysteries of Nature

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CHAPTER 1

Introduction

Although a great deal of nature can be accurately described by linear fields, nevertheless, nature in its most general and complete sense, is nonlinear. We are familiar with the small amplitude approximation that we make in order to obtain linear field equations for vibratory motion [1]. Similarly in electrodynamics, for a weak enough field propagating through a dielectric medium, the response of the medium is linear and the electric displacement vector depends linearly on the applied electric field E, viz,

$$\underline{D} = \varepsilon^{(0)}\underline{E} \tag{1.1}$$

where $\varepsilon^{(0)}$ is the linear dielectric constant of the medium. However, with the invention of lasers, it is now possible to generate very intense light pulses with peak electric fields in excess of 10^9v/m [2]. Materials that show a linear response to weak fields, eventually show nonlinear behaviour at high enough field strength as the electronic or ionic oscillators are driven to large amplitudes. In the nonlinear regime the linear relation (1.1) is modified, eg., to

$$\underline{D} = [\varepsilon^{(0)} + \varepsilon^{(2)}|\underline{E}|^2 + \text{higher order nonlinear terms}]\underline{E} \tag{1.2}$$

where $\varepsilon^{(2)}$ is the second order dielectric constant and so on. The nonlinearity can be equivalently expressed in terms of the refractive index of the medium. As early as the mid nineteen sixties, it was well known [3] that when an electromagnetic wave propagates through a nonlinear dispersive medium, a solitary wave can be obtained as a result of the interaction of nonlinear and dispersive effects.

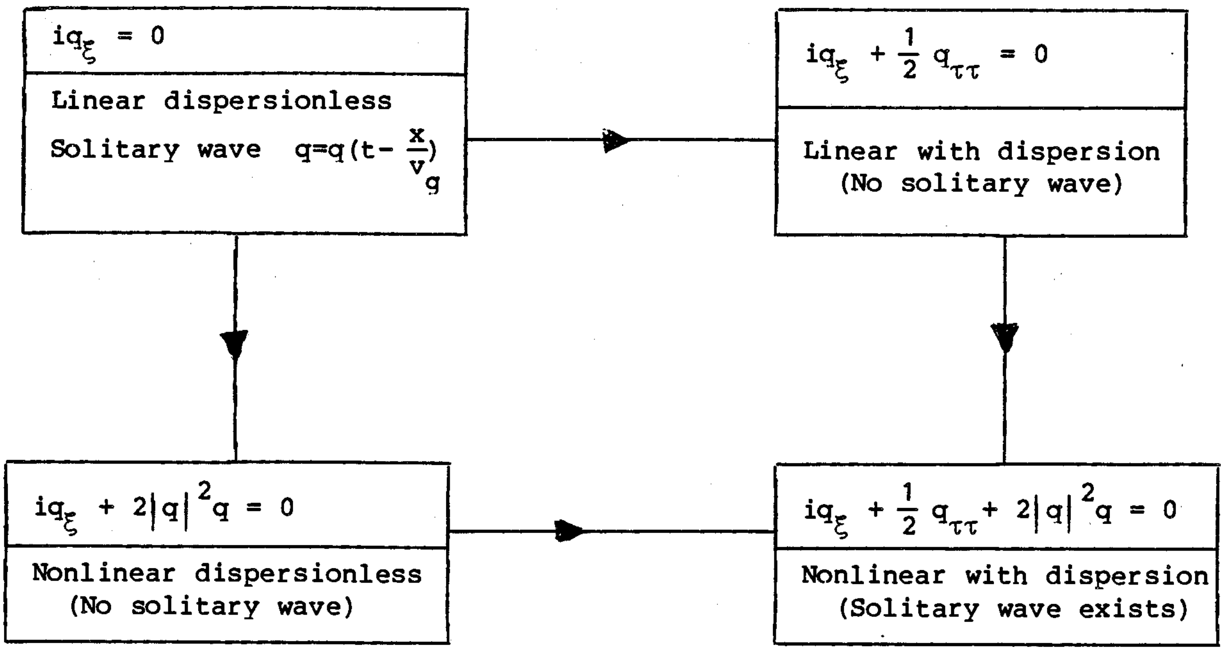
The nonlinearity of the index of refraction [4]

$$n = n_0 + n_2|E|^2 \tag{1.3}$$

could be used to compensate the pulse broadening effect. When the pulse spreading due to dispersion and the pulse squeezing due to the nonlinearity of the refractive index are balanced, the light pulse tends to form a localized pulse which holds its shape and travels at constant velocity. Such a pulse is referred to as "a solitary wave". Starting from (1.3), one can derive the following dynamical equation for the field, viz,

$$iq_\xi + \frac{1}{2} q_{\tau\tau} + 2|q|^2 q = 0 \tag{1.4}$$

which is called the cubic nonlinear Schrödinger equation (NLSE). Here subscripts indicate partial differentiation w.r.t. the indicated variable. $q(\xi,\tau)$ corresponds to the electric field strength, ξ is proportional to the distance x along the propagation direction and $\tau \propto t - \frac{x}{v_g}$, where t is time and v_g is group velocity. The conditions under which a solitary wave can be obtained are shown in the following diagram:



In equation (1.4), the second term describes the effect of dispersion and the third term the effect of nonlinearity. When the second and third terms are absent, any localized solution of the equation travels without changing shape and therefore is a (trivial example of a) solitary wave. The effect of introducing dispersion without nonlinearity is to eliminate the possibility of solitary waves because different Fourier components will propagate at different velocities causing a spreading effect. Introducing nonlinearity without dispersion again rules out the possibility for solitary waves because the pulse energy is continually injected into higher frequency modes which by the uncertainty principle causes a squeezing effect. But with both dispersion and nonlinearity present, new (non trivial) solitary waves can again be obtained that can be qualitatively understood as the balance between nonlinearity and dispersion. If on collision, the solitary waves pass through each other (interacting nonlinearly as they do so) and come out with the same shapes and velocities as before the collision, they are said to be solitons ie. solitons are stable solitary waves. In 1970, Zakharov and Shabat [5] were the first to solve equation (1.4) for multi-soliton solutions using the inverse scattering transform method (ISTM) and hence demonstrating that the solitary wave solutions of (1.4) are solitons. The ISTM was first discovered by Krushkal et al [6] in 1967 and was used to find the soliton solutions of the historically famous KdV equation describing shallow water waves in a rectangular canal.

In 1973, Hasegawa and Tappert [7] pointed out that the nonlinearity in the refractive index (1.3) could make it possible to transmit picosecond duration light pulses without distortion in an optical fiber having appropriate dispersion. The experimental observations of such solitons were reported [8,9,10] from the Bell Laboratories in the early eighties. These optical solitons are now considered to have a potential application in the development of a high-bit-rate transmission system [11]. For these solitons, the (small) fiber loss is the only factor that contributes to the distortion of the stable pulse by broadening the pulse width and decreasing the amplitude. In this case, equation (1.4) is modified to include damping and one must study the perturbed NLS equation [12]. In a series of papers [13,14,15,16] Hasegawa and Kodama showed that an optical soliton deformed by the fiber loss can be reshaped by appropriate pumping to a narrower and higher pulse during the course of transmission through the fiber.

One may wonder what would happen if one could find a material with refractive index whose fourth order nonlinearity also becomes important for sufficiently intense electromagnetic fields (below the dielectric breakdown limit). For a higher order positive nonlinearity the narrowing effect is expected to be even stronger. But whether the balance between the total nonlinearity and dispersion actually occurs and is stable or not, is not obvious. If it does, we shall get narrower and higher solitons and hence it will make the communication system more effective by increasing the bit-rate of transmission. Very little has been done in this direction and the detailed calculations have not been yet carried out. In this thesis we shall assume the nonlinear refractive index to be of the form

$$n = n_0 + n_2|\underline{E}|^2 + n_4|\underline{E}|^4 \tag{1.5}$$

Pushkarov et al. [26] have written down solitary wave solutions corresponding to (1.5) (without giving any derivation) for the situation when n_2 is positive and n_4 is positive or negative. They call their solitary wave solutions solitons without ever checking their stability.

In chapter 2, we define our model explicitly and by Taylor expanding the wave number k around the carrier frequency ω_0 (= $2\pi c/\lambda$, where λ is the vacuum wave length) and in powers of the electric field we derive a nonlinear evolution equation (the nonlinear cubic-quintic Schrödinger equation) that governs the dynamics of the electromagnetic pulse in the medium characterized by (1.5). We give an alternative derivation of the same equation starting with Maxwell's equations in Appendix A. In chapter 3, we obtain three conservation laws by inspection for the nonlinear cubic-quintic Schrödinger equation (NLCQSE) as well as showing that it is Galilean invariant. These are important properties of the equation and two of the conservation laws are used to check the accuracy of the numerical scheme employed in chapter 7 for checking the stability of the solitary wave solutions. It has been noticed [17] that if one knows the Lagrangian density corresponding to the field equation, one can possibly derive the infinite number of conservation laws making use of the Bäcklund transformation for the equation. Thus in chapter 3, we also develop the Lagrangian formalism and show that the three conservation laws can also be obtained from the Lagrangian density.

In chapter 4, we solve the NLCQSE analytically and obtain the solitary wave solutions for all possible signs of n_2 and n_4. We show that the solutions reported by Pushkarov et al. form a subset of our solutions. A special case is relegated to Appendix B. In subsequent chapters we attempt to determine whether the solitary wave solutions are solitons or not. Presently there are two different but interconnected analytic methods to obtain multi-soliton solutions if they exist. These methods are the Bäcklund transformation and the ISTM [18,19]. If we derive the Bäcklund transformation we can obtain an eigenvalue problem to solve the equation by the ISTM. If, on the other hand, we are able to establish the ISTM, we can obtain the Bäcklund transformation from the eigenvalue problem of the ISTM. Thus, in chapter 5, we make an attempt to derive the Bäcklund transformation for the evolution equation by a method due to Clairin. In chapter 6, we explore the possibility that the NLCQSE belongs to the class of equations that can be solved by the ISTM using the Ablowitz-Kaup-Newell-Segur (AKNS) eigenvalue problem. The results of these two approaches lead us to believe that the solitary wave solutions of the NLCQSE are probably not solitons. Since we cannot absolutely rule out the possibility of solitons, we finally examine the stability of the solitary waves numerically in chapter 7. The general conclusion is that the solitary wave solutions are not solitons, but "quasi-soliton" and other interesting behaviour is observed. Experimental difficulties aside, some aspects of our theoretical results may possibly be experimentally testable. We discuss these possibilities in chapter 8 and present conclusions of this thesis in chapter 9.

CHAPTER 2

Formulation of the problem

In this chapter, we derive the nonlinear (cubic-quintic) Schrödinger equation (NLCQSE) that describes the propagation of an optical pulse in a nonlinear dispersive medium. In section 2.1, we introduce the refractive index which characterizes our model. Beginning with this model in section 2.2, we derive the nonlinear differential equation governing the dynamics of the electric field in the medium. Finally we obtain the NLCQSE by moving into the group velocity co-ordinate system.

2.1 The Model

The physical process under investigation is the (one dimensional) propagation of intense electromagnetic waves in a nonlinear dispersive isotropic medium characterized by a refractive index given by

$$n = \frac{ck}{\omega} = n_o(\omega) + i\chi(\omega) + n_2|\underline{E}|^2 + n_4|\underline{E}|^4 \qquad (2.1)$$

where E is the electric field intensity, n_o is the linear index of refraction, n_2 and n_4 are higher order coefficients of the refractive index and χ is the imaginary part of the linear refractive index that accounts for any damping. In (2.1) we have neglected any variation of n_2 and n_4 with ω assuming that we are far away from any resonance as far as these coefficients are concerned. At this level our model is a phenomenological one and we will not go into possible microscopic contributions to n_2 and n_4.

As mentioned in the introduction, the effect of the second order nonlinearity has already been studied in detail, the nonlinear Schrödinger equation resulting when the $|\underline{E}|^4$ contribution is neglected. When the electric field is sufficiently intense (assuming that it's below the dielectric breakdown limit) such that $E \sim (\frac{n_2}{n_4})^{\frac{1}{2}}$, the fourth order nonlinearity in equation (2.1) becomes comparable to the second order nonlinearity and plays an important role. The nonlinear dependence of refractive index on the electric field intensity gives rise to a pulse compression or pulse broadening effect depending upon the signs of n_2 and n_4. The dependence of the refractive index on the frequency (ω) causes dispersion. The nonlinear combination of both effects determines the shape of the optical pulse as it propagates.

Under the assumption that the diameter of the medium guide (e.g. an optical fiber) is much larger than the wave length of the radiation, the electric field can be written in terms of a (slowly varying) complex amplitude ϕ times a plane wave, viz;

$$E(x,t) = R_e\{\phi(x,t)e^{-i\{k_o x - \omega_o t\}}\} \qquad (2.2)$$

where R_e means real part and k_o and ω_o are the central wave number and angular frequency respectively.

2.2 Derivation of the Basic Nonlinear Dynamical Equation

The dynamical equation that describes the development of the amplitude function $\phi(x,t)$ in the non linear medium may be derived as follows. (An alternate derivation starting directly from Maxwell's equations is given in Appendix A).

Expanding $k = k(\omega, |E|^2, |E|^4)$ around the carrier frequency $\omega_o \left(= \frac{2\pi c}{\lambda}\right)$ and zero electric field, we obtain

$$k - k_o = i\chi \frac{k_o}{n_o} + \frac{\partial k}{\partial \omega}\bigg|_o (\omega-\omega_o) + \frac{\partial k}{\partial |\underline{E}|^2}\bigg|_o |E|^2 + \frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}\bigg|_o (\omega-\omega_o)^2 + \frac{1}{6}\frac{\partial^3 k}{\partial \omega^3}\bigg|_o (\omega-\omega_o)^3$$

$$+ \frac{\partial^2 k}{\partial \omega \partial |\underline{E}|^2}\bigg|_o (\omega-\omega_o)|\underline{E}|^2 + \frac{\partial k}{\partial |\underline{E}|^4}\bigg|_o |\underline{E}|^4 + \frac{1}{2}\frac{\partial^3 k}{\partial \omega^2 \partial |\underline{E}|^2}\bigg|_o |\underline{E}|^2 (\omega-\omega_o)^2$$

$$+ \frac{1}{24}\frac{\partial^4 k}{\partial \omega^4}\bigg|_o (\omega-\omega_o)^4 \qquad (2.3)$$

We have consistently kept all terms up to fourth order in the expansion.

From (2.1) we find that

$$\frac{\partial^3 k}{\partial \omega^2\, \partial |\underline{E}|^2}\bigg|_o = 0$$

$$\frac{\partial^2 k}{\partial \omega \partial |\underline{E}|^2}\bigg|_o = \frac{n_2}{c}$$

$$\frac{\partial k}{\partial |\underline{E}|^2}\bigg|_o = \frac{2\pi n_2}{\lambda} \qquad (2.4)$$

$$\frac{\partial k}{\partial |\underline{E}|^4}\bigg|_o = \frac{2\pi n_4}{\lambda}$$

Writing $\left.\frac{\partial k}{\partial \omega}\right|_o = k'$, $\left.\frac{\partial^2 k}{\partial \omega^2}\right|_o = k''$ etc. and using (2.4), (2.3) may be re-expressed as

$$k-k_o = i\chi \frac{k_o}{n_o} + k'(\omega-\omega_o) + \frac{2\pi n_2}{\lambda}|\underline{E}|^2 + \frac{1}{2}k''(\omega-\omega_o)^2 + \frac{1}{6}k'''(\omega-\omega_o)^3$$

$$+ \frac{n_2}{c}|\underline{E}|^2(\omega-\omega_o) + \frac{2\pi n_4}{\lambda}|\underline{E}|^4 + \frac{1}{24}k''''(\omega-\omega_o)^4 \qquad (2.5)$$

We now write the electric field $E(x,t)$ in its Fourier integral form as

$$E(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon(k,\omega)e^{i(kx-\omega t)}dk\, d\omega \qquad (2.6)$$

so that equation (2.2) can be written for $\phi(x,t)$ as

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\, d\omega \qquad (2.7)$$

From (2.7) we obtain

$$i\frac{\partial \phi}{\partial x} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{-(k-k_o)\}\varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\, d\omega$$

$$i\chi\frac{k_o}{n_o}\phi = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} i\chi\frac{k_o}{n_o}\varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\, d\omega$$

$$ik'\frac{\partial\phi}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k'(\omega-\omega_o)\varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\ d\omega$$

$$\frac{2\pi n_2}{\lambda}|\underline{E}|^2\phi = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{2\pi n_2}{\lambda}|\underline{E}|^2\varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\ d\omega$$

$$-\frac{1}{2}k''\frac{\partial^2\phi}{\partial t^2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2}k''(\omega-\omega_o)^2\varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\ d\omega$$

(2.8)

$$-\frac{i}{6}k'''\frac{\partial^3\phi}{\partial t^3} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{k'''}{6}(\omega-\omega_o)^3\varepsilon(k,\omega)e^{i\{(k-k_o)x-(w-\omega_o)t\}}dk\ d\omega$$

$$i\frac{n_2}{c}|\underline{E}|^2\frac{\partial\phi}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{n_2}{c}(\omega-\omega_o)|\underline{E}|^2\varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\ d\omega$$

$$\frac{2\pi n_4}{\lambda}|\underline{E}|^4\phi = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{2\pi n_4}{\lambda}|\underline{E}|^4\varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\ d\omega$$

$$\frac{1}{24}k''''\frac{\partial^4\phi}{\partial t^4} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{24}k''''(\omega-\omega_o)^4\varepsilon(k,\omega)e^{i\{(k-k_o)x-(\omega-\omega_o)t\}}dk\ d\omega$$

Adding all the equations of (2.8) and using (2.5) we obtain

$$i\left\{\frac{\partial\phi}{\partial x} + r\phi + k'\frac{\partial\phi}{\partial t}\right\} - \frac{1}{2}k''\frac{\partial^2\phi}{\partial t^2} - \frac{i}{6}k'''\frac{\partial^3\phi}{\partial t^3} + i\frac{n_2}{c}|\phi|^2\frac{\partial\phi}{\partial t}$$

$$+\frac{1}{24}k''''\frac{\partial^4\phi}{\partial t^4} + \frac{2\pi n_2}{\lambda}|\phi|^2\phi + \frac{2\pi n_4}{\lambda}|\phi|^4\phi = 0 \qquad (2.9)$$

where $r = \chi \frac{2\pi}{\lambda}$.

Equation (2.9) governs the dynamics of the electric field envelope $\phi(x,t)$ in the medium and it contains the terms accounting for effects of group dispersion, medium losses and pulse compression etc. In equation (2.9), if we switch off the loss term, non linearity and group dispersion i.e. $r = k'' = k''' = k'''' = n_2 = n_4 = 0$, the equation reduces to

$$\frac{\partial\phi}{\partial x} + \frac{1}{v_g}\frac{\partial\phi}{\partial t} = 0 \qquad (2.10)$$

where $v_g = \frac{\partial\omega}{\partial k} \equiv \frac{1}{k'}$ is the group velocity. From equation (2.10) it's clear that

$$\phi(x,t) = \phi\left(t - \frac{x}{v_g}\right)$$

This suggests that the dynamical evolution of ϕ may be best seen by moving into the group velocity co-ordinates (ξ,τ) where $\xi \propto x$ and $\tau \propto \left(t - \frac{x}{v_g}\right)$. We also, for convenience, normalize the distance x, time t and the electric field amplitude ϕ as follows:

$$\xi = 10^{-9} \frac{x}{\lambda}$$

$$\tau = \frac{10^{-4.5}}{(-\lambda k'')^{\frac{1}{2}}} \left(t - \frac{x}{v_g}\right) \qquad (2.11)$$

$$q = 10^{4.5} (\pi n_2)^{1/2} \phi$$

where in defining τ it's assumed that k'' is negative [20]. To give the reader a feeling for these normalized quantities we take the nominal example of a glass fiber discussed by Hasegawa and Kodama [21].

For $\lambda = 1.5\mu m$

$$n_2 = 1.2 \times 10^{-22} m^2/v^2$$

and the group dispersion

$$\lambda^2 \frac{\partial^2 n}{\partial \lambda^2} = -8.13 \times 10^{-3}$$

Using these values we calculate

$$-\lambda k'' = -\lambda \frac{\partial^2 k}{\partial \omega^2} = -\frac{\lambda^2}{2\pi c^2} \left(\lambda^2 \frac{\partial^2 n}{\partial \lambda^2}\right) = 3.23 \times 10^{-32} s^2$$

For these values, we find that in (2.11)

$$\xi = 1 \quad \text{corresponds to} \quad x = 1.5\text{km}$$

$$q = 1 \quad \text{corresponds to} \quad \phi = 1.62 \times 10^{6}\text{v/m}$$

$$\tau = 1 \quad \text{corresponds to} \quad t - \frac{x}{v_g} = 5.68\text{psec.}$$

From (2.11), we can write the transformation of the spatial and temporal operators as follows

$$\frac{\partial}{\partial x} = \frac{10^{-9}}{\lambda}\frac{\partial}{\partial \xi} - \frac{10^{-4.5}}{(-\lambda k'')^{1/2}} k' \frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial t} = \frac{10^{-4.5}}{(-\lambda k'')^{1/2}} \frac{\partial}{\partial \tau} \qquad (2.12)$$

$$\frac{\partial^2}{\partial t^2} = \frac{10^{-9}}{-\lambda k''}\frac{\partial^2}{\partial \tau^2}$$

Using (2.11) and (2.12), (2.9) transforms to

$$i\frac{\partial q}{\partial \xi} + \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + 2|q|^2 q + \delta|q|^4 q = -i\Gamma q - i\beta_1\frac{\partial^3 q}{\partial \tau^3} - i\beta_2|q|^2\frac{\partial q}{\partial \tau} - \beta_3\frac{\partial^4 q}{\partial \tau^4} \qquad (2.13)$$

where

$$\delta = \frac{2n_4}{\pi n_2^2} 10^{-9}$$

$$\Gamma = 10^9 \lambda r$$

$$\beta_1 = \frac{1}{6} \frac{k'''}{k''} \frac{10^{-4.5}}{\sqrt{-\lambda k''}} \tag{2.14}$$

$$\beta_2 = \frac{10^{-4.5}}{\pi c \sqrt{-\lambda k''}} \lambda$$

$$\beta_3 = \frac{1}{24} \frac{k''''}{\lambda (k'')^2} 10^{-9}$$

We now wish to show that the terms on the right hand side of (2.13) are very small and may be neglected. Let us estimate the coefficients β_1, β_2, β_3 and Γ in (2.13).

We have already calculated

$$-\lambda k'' = 3.23 \times 10^{-32} s^2 \quad \text{for} \quad \lambda = 1.5\mu m \quad \text{so that}$$

$$k'' = -2.15 \times 10^{-26} s^2/m$$

Now
$$k''' = \frac{\partial k''}{\partial \omega} = \frac{\partial k''}{\partial \lambda} \frac{\partial \lambda}{\partial \omega} = - \frac{\lambda^2}{2\pi c} \frac{\partial k''}{\partial \lambda} \simeq - \frac{\lambda}{2\pi c} k''$$

$$= 1.71 \times 10^{-41} s^3/m.$$

Similarly we estimate

$$k'''' \simeq -1.36 \times 10^{-56} s^4/m$$

Using these values we obtain from (2.14)

$$\beta_1 = -2.34 \times 10^{-5}, \quad \beta_2 = 2.8 \times 10^{-4}, \quad \beta_3 = -8.17 \times 10^{-10}$$

i.e. these coefficients are very small.

For, say, a quartz fiber with loss rate 0.2dB/km [22]

$$\Gamma = 10^9 \lambda r = 10^9 (1.5) 10^{-6} \times \frac{0.2}{20} (\ell n 10) \times 10^{-3} = 3.45 \times 10^{-2}$$

so that the damping coefficient is also small. Now making the crude approximation that $\frac{\partial q}{\partial \tau} \sim \frac{q}{\tau}$, etc., and assuming that q, τ and $\xi \sim 1$, the L.H.S. of (2.13) is of order unity and the R.H.S. negligible. Thus our dynamical equation finally becomes

$$i \frac{\partial q}{\partial \xi} + \frac{1}{2} \frac{\partial^2 q}{\partial \tau^2} + 2|q|^2 q + \delta |q|^4 q = 0 \tag{2.15}$$

which we call the NLCQSE.

For $\delta = 0$, (2.15) reduces to the well-known NLSE which has soliton solutions for $n_2 > 0$. It should be noted that if $n_2 < 0$ the cubic non linear term in (2.15) would have a negative coefficient and the sign of δ depends upon the sign of n_4.

CHAPTER 3

Galilean Invariance and Conservation Laws for the NLCQSE

In this chapter, we shall investigate some of the important properties of the NLCQSE. In section 3.1, we derive by inspection the first three conservation laws for this equation. To obtain additional conservation laws or to establish that there are an infinite number of them, we clearly cannot proceed by inspection but must follow a more general approach e.g. a Lagrangian formulation. It is generally believed that the existence of an infinite number of conservation laws, a Bäcklund transformation and an ISTM are intimately connected. The existence of one implies the existence of others. If we could develop the Lagrangian formulation for our problem, it would be possible to find an infinite number of conserved densities and hence an infinite number of conservation laws provided the Bäcklund transformation for the equation is known. So in section 3.2, we develop the Lagrangian formalism for the problem. In section 3.3, we obtain the same three conservation laws from the Lagrangian density. Finally, the invariance of the NLCQSE under a Galilean transformation is demonstrated in section 3.4.

3.1 Derivation of the Conservation Laws

Scott et al. in their well known paper [23] point out the importance of distinguishing between those nonlinear wave equations that dissipate energy and those that do not. The latter ones are often referred to in the engineering literature as "conservative". Now as we have derived the NLCQSE as an approximate description of the system, it is not a priori obvious that the energy is conserved. So, it is important to find the conservation laws, if any, including the energy conservation. Zakharov and

Shabat [24] have found, by the inverse scattering technique, an infinite set of conservation laws for the nonlinear cubic Schrödinger equation. However, this doesn't imply that the NLCQSE also has an infinite number (or any) of conservation laws. By inspection, we have found three conservation laws for the equation

$$iq_{\xi} + \frac{1}{2} q_{\tau\tau} + 2|q|^2 q + \delta|q|^4 q = 0 \tag{3.1}$$

The First Conservation Law

From (3.1) we write

$$q_{\xi} = \frac{1}{2} iq_{\tau\tau} + 2i|q|^2 q + i\delta|q|^4 q \tag{3.2a}$$

and

$$q^*_{\xi} = -\frac{1}{2} iq^*_{\tau\tau} - 2i|q|^2 q^* - i\delta|q|^4 q^* \tag{3.2b}$$

Let's define the integral

$$I_1 = \int_{-\infty}^{+\infty} |q|^2 d\tau \tag{3.3}$$

so that

$$\frac{dI_1}{d\xi} = \int_{-\infty}^{+\infty} \{q_{\xi} q^* + qq^*_{\xi}\} d\tau \tag{3.4}$$

Using (3.2), (3.4) becomes

$$\frac{dI_1}{d\xi} = \frac{i}{2} \int_{-\infty}^{+\infty} \{q_{\tau\tau} q^* - qq^*_{\tau\tau}\} d\tau \tag{3.4a}$$

Now by the method of integration by parts

$$\int_{-\infty}^{+\infty} q_{\tau\tau}q^*d\tau = q_\tau q^* \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} q_\tau q_\tau^* d\tau$$

$$= - \int_{-\infty}^{+\infty} q_\tau q_\tau^* d\tau = -q_\tau^* q \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} q \ q_{\tau\tau}^* d\tau$$

Thus

$$\int_{-\infty}^{+\infty} q_{\tau\tau}q^*d_\tau = \int_{-\infty}^{+\infty} q \ q_{\tau\tau}^* d\tau \qquad (3.4b)$$

where we have used the condition

$$q \to 0 \quad \text{as} \quad \tau \to \pm \infty \qquad (3.4c)$$

Making use of (3.4b), (3.4a) becomes

$$\frac{dI_1}{d\xi} = 0 \qquad (3.5)$$

This is what we call the first conservation law, I_1 being the conserved quantity.

The Second Conservation Law

From (3.2), we obtain

$$q_{\xi\tau} = \frac{1}{2} iq_{\tau\tau\tau} + 2iq^2q_\tau^* + 4iq^*q_\tau + 3i\delta q^{*2}q^2q_\tau + 2i\delta q^3q^*q_\tau^* \qquad (3.6a)$$

$$q_{\xi\tau}^* = -\frac{1}{2} iq_{\tau\tau\tau}^* - 2iq^{*2}q_\tau - 4iqq_\tau^* - 3i\delta q^2q^{*2}q_\tau^* - 2i\delta q^{*3}qq_\tau \qquad (3.6b)$$

Now define the quantity

$$I_2 = \int_{-\infty}^{+\infty} \{q^* q_\tau - q q_\tau^*\} d\tau \tag{3.7}$$

Therefore

$$\frac{dI_2}{d\xi} = \int_{-\infty}^{+\infty} \{q_\xi^* q_\tau + q^* q_{\xi\tau} - q_\xi q_\tau^* - q q_{\xi\tau}^*\} d\tau \tag{3.8}$$

Making use of (3.2) and (3.6), following the same procedure that we used to derive the first conservation law, we obtain

$$\frac{dI_2}{d\xi} = 0 \tag{3.9}$$

This is the second conservation law for the equation (3.1).

The Third Conservation Law

Let's now define the integral

$$I_3 = \int_{-\infty}^{+\infty} \{|q_\tau|^2 - 2|q|^4 - \frac{2}{3}\delta|q|^6\} d\tau \tag{3.10}$$

Following the same procedure we obtain

$$\frac{dI_3}{d\xi} = 0 \tag{3.11}$$

which is the third conservation law.

(3.5), (3.9) and (3.11) can be written in their explicit form as

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty}|q|^2 d\tau = 0 \tag{3.12a}$$

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty}\{q^*q_\tau - qq^*_\tau\}d\tau = 0 \tag{3.12b}$$

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty}\{|q_\tau|^2 - 2|q|^4 - \frac{2}{3}\delta|q|^6\}d\tau = 0 \tag{3.12c}$$

These conservation laws have significant physical meaning i.e. they represent the conservation of number, conservation of momentum and the conservation of energy respectively according to the terminology used by Zakharov et al. in the case of the NLSE [5]. We have obtained the three basic conservation laws for the NLCQSE, but the question whether the equation has infinite number of conservation laws is still open. We will address this issue in the coming chapters.

3.2 The Lagrangian Formulation of the NLCQSE

The Lagrangian formalism is another way to explore the conservative nature of a system. The Lagrangian density is a useful concept. If the Lagrangian density, no matter how we find it, has the form of that for a conservative system, then the corresponding wave system may be considered as conservative in the conventional sense of the term. In the NLCQSE, q is a complex field. So the field is described by two independent field variables q and q^*. The Lagrangian density of a one dimensional complex field can be taken, in general, of the form:

$$L = L(q, q^*, q_\tau, q^*_\tau, q_\xi, q^*_\xi, \xi, \tau) \tag{3.13}$$

We take the total derivative of L with respect to x_μ $\{$where $x_1 = \xi$, $x_2 = \tau\}$.

$$\frac{dL}{dx_\mu} = \frac{\partial L}{\partial q_i} q_{i,\mu} + \frac{\partial L}{\partial q_{i,\nu}} q_{i,\mu\nu} + \frac{\partial L}{\partial x_\mu} \tag{3.13a}$$

where $q_1 = q$, $q_2 = q^*$, $q_{i,\mu} \equiv \frac{dq_i}{dx_\mu}$ etc. and the sum over repeated indices is implied. The Euler-Lagrange equations corresponding to a Lagrangian of the form (3.13) are

$$\frac{d}{dx_\nu}\left(\frac{\partial L}{\partial q_{i,\nu}}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1,2 \tag{3.14}$$

Making use of (3.14), (3.13a) becomes

$$\frac{dL}{dx_\mu} = \frac{d}{dx_\nu}\left(\frac{\partial L}{\partial q_{i,\nu}}\right) q_{i,\mu} + \frac{\partial L}{\partial q_{i,\nu}} \frac{dq_{i,\mu}}{dx_\nu} + \frac{\partial L}{\partial x_\mu}$$

$$= \frac{d}{dx_\nu}\left(\frac{\partial L}{\partial q_{i,\nu}} q_{i,\mu}\right) + \frac{\partial L}{\partial x_\mu}$$

Combining total derivatives on both sides, this can be written as

$$\frac{d}{dx_\nu}\left\{\frac{\partial L}{\partial q_{i,\nu}} q_{i,\mu} - L\delta_{\mu\nu}\right\} = -\frac{\partial L}{\partial x_\mu} \tag{3.15}$$

If L does not depend explicity on x_μ, $\frac{\partial L}{\partial x_\mu} = 0$ and (3.15) becomes

$$\frac{dT_{\mu\nu}}{dx_\nu} = 0 \qquad (3.16)$$

where

$$T_{\mu\nu} = \frac{\partial L}{\partial q_{i,\nu}} \; q_{i,\mu} - L\delta_{\mu\nu} \qquad (3.17)$$

$T_{\mu\nu}$ is in general a four-tensor of the second rank for a three dimensional field. But we are dealing, however, with only one dimensional field. So the indices μ, ν would run over ξ and τ only. Consider the Lagrangian density

$$L = \frac{i}{2} \{qq^*_\xi - q^* q_\xi\} + \frac{1}{2} q_\tau q^*_\tau - q^2 q^{*2} - \frac{\delta}{3} q^3 q^{*3}$$

$$\equiv \frac{i}{2} \{qq^*_\xi - q^* q_\xi\} + \frac{1}{2}|q_\tau|^2 - |q|^4 - \frac{\delta}{3}|q|^6 \qquad (3.18)$$

Using the Lagrangian density (3.18), the Lagrange equations (3.14) yield the field equations

$$- iq^*_\xi + \frac{1}{2} q^*_{\tau\tau} + 2|q|^2 q^* + \delta|q|^4 q^* = 0 \qquad (3.19)$$

$$iq_\xi + \frac{1}{2} q_{\tau\tau} + 2|q|^2 q + \delta|q|^4 q = 0 \qquad (3.20)$$

which are indeed the NLCQS equations.

Hence the Lagrangian density given by (3.18) qualifies to be the Lagrangian density for our problem.

3.3 Derivation of the Conservation Laws from the Lagrangian Formalism

We can now obtain the three conservation laws systematically from the Langrangian formalism developed in section 3.2.

We can rewrite (3.16) as

$$\frac{dT_{\mu o}}{d\tau} + \frac{dT_{\mu j}}{dx_j} = 0 \tag{3.21}$$

and in our problem, x_j has just one component and that's ξ. (3.21) has the structure of an equation of continuity, which says that the time rate of change of some density plus the divergence of some corresponding flux or current density vanishes. The equation of continuity

$$\frac{\partial D}{\partial \xi} + \frac{\partial P}{\partial \tau} = 0 \tag{3.22}$$

implies the conservation of integral quantities [25]

$$I = \int_{-\infty}^{+\infty} D \, d\tau \tag{3.22a}$$

provided the integral exists and the integrand satisfies the appropriate boundary conditions.

Now for our problem, (3.21) is a set of two equations i.e.

$$\frac{dT_{\xi o}}{d\tau} + \frac{dT_{\xi\xi}}{d\xi} = 0 \tag{3.23a}$$

$$\frac{dT_{oo}}{d\tau} + \frac{dT_{o\xi}}{d\xi} = 0 \tag{3.23b}$$

where according to Goldstein [25] $T_{\xi\xi}$ is the energy density and $T_{o\xi}$ is the momentum density. Now from (3.17)

$$T_{\xi o} = \frac{\partial L}{\partial q_{i,\tau}} q_{i,\xi} = \frac{\partial L}{\partial q_\tau} q_\xi + \frac{\partial L}{\partial q_\tau^*} q_\xi^*$$

$$= \frac{1}{2}\{q_\xi q_\tau^* + q_\tau q_\xi^*\}$$

and

$$T_{\xi\xi} = \frac{\partial L}{\partial q_{i,\xi}} q_{i,\xi} - L = \frac{\partial L}{\partial q_\xi} q_\xi + \frac{\partial L}{\partial q_\xi^*} q_\xi^* - L$$

$$= -\frac{1}{2}\{|q_\tau|^2 - 2|q|^4 - \frac{2}{3}\delta|q|^6\} \tag{3.24}$$

Similarly

$$T_{oo} = -\frac{i}{2}\{qq_\xi^* - q^*q_\xi\} + \frac{1}{2}|q_\tau|^2 + |q|^4 + \frac{\delta}{3}|q|^6$$

$$T_{o\xi} = -\frac{i}{2}\{q^*q_\tau - q^*q_\tau\}$$

From (3.23) and (3.24) we obtain the second and third conservation laws

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty}\{q^*q_\tau - qq_\tau^*\}\, d\tau = 0 \tag{3.12b}$$

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty}\{|q_\tau|^2 - 2|q|^4 - \frac{2}{3}\delta|q|^6\}\, d\tau = 0 \tag{3.12c}$$

It's worth noticing that the conserved quantity in (3.12c) is the Hamiltonian corresponding to the Lagrangian of our problem with the Hamiltonian density defined as viz;

$$H = \pi - L \tag{3.24a}$$

where the canonical momentum π is given as;

$$\pi = \frac{\partial L}{\partial q_{i,\xi}} = \frac{\partial L}{\partial q_\xi} q_\xi + \frac{\partial L}{\partial q^*_\xi} q^*_\xi \tag{3.24b}$$

Now from (3.14), we write

$$\frac{d}{d\xi}\left(\frac{\partial L}{\partial q^*_\xi}\right) + \frac{d}{d\tau}\left(\frac{\partial L}{\partial q^*_\tau}\right) - \frac{\partial L}{\partial q^*} = 0$$

or

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty} \frac{\partial L}{\partial q^*_\xi} q^* \, d\tau - \int_{-\infty}^{+\infty} \frac{\partial L}{\partial q^*_\xi} \frac{dq^*}{d\xi} d\tau = -\int_{-\infty}^{+\infty} \frac{d}{d\tau}\left(\frac{\partial L}{\partial q^*_\tau}\right) q^* d\tau + \int_{-\infty}^{+\infty} \frac{\partial L}{\partial q^*} q^* d\tau \tag{3.25a}$$

And from the second Lagrange's equation

$$\frac{d}{d\xi}\left(\frac{\partial L}{\partial q_\xi}\right) + \frac{d}{d\tau}\left(\frac{\partial L}{\partial q_\tau}\right) - \frac{\partial L}{\partial q} = 0$$

or

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty} \frac{\partial L}{\partial q_\xi} q \, d\tau - \int_{-\infty}^{+\infty} \frac{\partial L}{\partial q_\xi} \frac{dq}{d\xi} d\tau = -\int_{-\infty}^{+\infty} \frac{d}{d\tau}\left(\frac{\partial L}{\partial q_\tau}\right) q \, d\tau + \int_{-\infty}^{+\infty} \frac{\partial L}{\partial q} q \, d\tau \tag{3.25b}$$

Subtracting (3.25b) from (3.25a) and making use of (3.18), we obtain

$$\frac{i}{2}\frac{d}{d\xi}\int_{-\infty}^{+\infty} |q|^2 d\tau = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{q^*_{\tau\tau} q - q_{\tau\tau} q^*\} d\tau = 0 \quad \text{(using 3.4c)}$$

So

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty} |q|^2 d\tau = 0 \tag{3.12a}$$

which is the first conservation law.

The equation of continuity (3.22) exists for all solutions q. If the Bäcklund transformation for the NLCQSE exists (i.e. if we can find it) then it would be possible to find an infinite number of conserved densities and hence an infinite number of conservation laws following basically the same approach as applied by Scott for the Sine Gordon equation [23]. This is the basic motivation behind developing the Lagrangian formulation of the problem. We will return to this issue again in chapter 5.

3.4 Invariance Under Galilean Transformation

We demonstrate here that the NLCQSE (3.1) is invariant under the Galilean transformation

$$\begin{aligned} \xi' &= \xi \\ \tau' &= \tau - v\xi \\ q(\xi,\tau) &= q'(\xi',\tau')e^{i\left\{v\tau' + \frac{v^2}{2}\xi'\right\}} \end{aligned} \tag{3.26}$$

The corresponding operators transform accordingly as

$$\begin{aligned} \frac{\partial}{\partial\xi} &= \frac{\partial}{\partial\xi'} - v\,\frac{\partial}{\partial\tau'} \\ \frac{\partial}{\partial\tau} &= \frac{\partial}{\partial\tau'} \end{aligned} \tag{3.27}$$

From (3.26) and (3.27), we obtain

$$i\,\frac{\partial q}{\partial \xi} = \{i\,\frac{\partial q'}{\partial \xi'} - iv\,\frac{\partial q'}{\partial \tau'} + \frac{v^2}{2}\,q'\}e^{i\{v\tau' + \frac{v^2}{2}\xi'\}}$$

$$\frac{1}{2}\,\frac{\partial^2 q}{\partial \tau^2} = \{\frac{1}{2}\,\frac{\partial^2 q'}{\partial \tau'^2} + iv\,\frac{\partial q'}{\partial \tau'} - \frac{v^2}{2}\,q'\}e^{i\{v\tau' + \frac{v^2}{2}\xi'\}} \qquad (3.28)$$

Thus equation (3.1) transforms to

$$i\,\frac{\partial q'}{\partial \xi'} + \frac{1}{2}\,\frac{\partial^2 q'}{\partial \tau'^2} + 2|q'|^2 q' + \delta|q'|^4 q' = 0$$

Hence the NLCQSE is invariant under the Galilean transformation. This invariance reveals that if $q(\xi,\tau)$ is a solution of (3.1) then $q(\xi,\tau - v\xi)e^{i\{v\tau - \frac{v^2}{2}\xi\}}$ is also a solution. If $q(\xi,\tau)$ is a solitary wave solution then the other solution represents the solitary wave moving with relative velocity v. It's interesting to note that the Lagrangian density (3.18) is also invariant under the Galilean transformation.

CHAPTER 4

Solitary Wave Solutions for the NLCQSE

In this chapter, we solve the NLCQSE to obtain the solitary wave solutions. Some of these solutions have been quoted in a different form without derivation by Pushkarov et al [26]. In section 4.1, we make the distinction between the terms solitary wave and soliton. In section 4.2 we derive a general solitary wave solution for the NLCQSE. The solitary wave solutions for all possible special cases (ie. all signs of n_2 and n_4) are deduced in section 4.3 and 4.4. Finally in section 4.5, we qualitatively discuss the role of nonlinearities and the dispersion in producing the necessary balance for solitary waves to exist.

4.1 Solitary Waves and Solitons

A solitary wave is a localized shape that propagates at constant velocity without change of form. If two or more solitary waves after suffering a collision, come out with exactly the same shape and velocity, they are called solitons and the collision is called a perfectly elastic collision. The ending "on" is Greek for particle and the word soliton means the particle-like behaviour of the solitary wave. Not all solitary waves exhibit soliton behaviour. Some equations may have solitary wave solutions that have approximate soliton behavior in the sense that when two such solitary waves collide, they re-emerge with a slight change in shape and/or velocity, leaving a small amount of energy behind in the form of oscillations ("radiation"). Such solitary waves are said to exhibit soliton-like or quasi-soliton behavior and such collisions are referred to as being only partially elastic. However, conventions differ from one area of physics to another. For instance in particle physics and solid state physics, the transparency of the waves to one another is not so important relative to other particle-like properties such as localisability and finite energy. Consequently some models are used in which the waves are not strictly solitons in the sense defined above but nevertheless they are still called solitons by workers in those fields [27]. However we shall stick to our strict definition stated above. Accordingly, what we obtain in this chapter are referred to as solitary wave solutions and their soliton nature will be investigated in the upcoming chapters.

4.2 The General Solitary Wave Solution for the NLCQSE

The solitary wave solutions for the cubic nonlinear Schrödinger equation which is a special case of the NLCQSE, are well known. We present here a method to solve the NLCQSE for solitary wave solutions.

The NLCQS equation is

$$i \frac{\partial q}{\partial \xi} + \frac{1}{2} \frac{\partial^2 q}{\partial \tau^2} + 2|q|^2 q + \delta |q|^4 q = 0 \qquad (4.1)$$

We assume the solitary wave solution to be of the form

$$q = e^{i\{P\xi + w\tau\}} F(\tau - w\xi) \qquad (4.2)$$

where P and w are real constants which can be determined from initial conditions; and F is a real function. w can be interpreted as the velocity relative to the group velocity.

From (4.2), we obtain

$$i\,\frac{\partial q}{\partial \xi} = \left\{-PF + i\frac{\partial F}{\partial \xi}\right\} e^{i\{P\xi + w\tau\}}$$

$$\frac{1}{2}\,\frac{\partial^2 q}{\partial \tau^2} = \left\{\frac{1}{2}\,\frac{\partial^2 F}{\partial \tau^2} + iw\,\frac{\partial F}{\partial \tau} - \frac{1}{2}\,w^2 F\right\} e^{i\{P\xi + w\tau\}} \tag{4.3}$$

Using (4.2) and (4.3), (4.1) leads to

$$-PF + \frac{1}{2}\,\frac{\partial^2 F}{\partial \tau^2} + i\,\frac{\partial F}{\partial \xi} + iw\,\frac{\partial F}{\partial \tau} - \frac{1}{2}\,w^2 F + 2F^3 + \delta F^5 = 0 \tag{4.4}$$

We can convert this partial differential equation into an ordinary differential equation by substituting $\tau - w\xi = t$ (not to be confused with the lab time) so that

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial \xi} = -\,w\,\frac{\partial}{\partial t}$$

With this substitution, equation (4.4) becomes;

$$\frac{1}{2}\,\frac{d^2 F}{dt^2} - \left(P + \frac{w^2}{2}\right)F + 2F^3 + \delta F^5 = 0 \tag{4.5}$$

It's now straightforward to integrate this second order ordinary differential equation. Multiplying equation (4.5) by $\frac{dF}{dt} \equiv F'$ and integrating once, we obtain

$$\left(\frac{dF}{dt}\right)^2 = (2P + w^2)F^2 - 2F^4 - \frac{2}{3}\delta F^6 + C_1 \tag{4.6}$$

where C_1 is the integration constant.

Now for a solitary wave of localized shape

$$F \text{ and } F' \to 0, \quad \text{as} \quad t \to \pm \infty$$

so from (4.6), $C_1 = 0$. However more generally, non-localized travelling wave solutions to (4.6) can be obtained for arbitrary C_1.

Equation (4.6) now becomes

$$\frac{dF}{dt} = F\{(2P + w^2) - 2F^2 - \gamma F^4\}^{1/2} \tag{4.7}$$

where $\gamma = \frac{2\delta}{3}$ which on integration gives

$$\int \frac{dF}{F\{(2P + w^2) - 2F^2 - \gamma F^4\}^{1/2}} = t + C_2 \tag{4.8}$$

where C_2 is the integration constant.

We rewrite (4.8) as

$$\int \frac{dX}{X\{(2P + w^2) - 2X - \gamma X^2\}^{1/2}} = 2(t + C_2) \tag{4.9}$$

where $X = F^2$.

However, we have assumed that F is real. For this to be true for all values of δ we recognize that

$$2P + w^2 > 0$$

Assuming this, we solve the integral in (4.9) and obtain [28]

$$\frac{(2P + w^2) - X}{(2P + w^2)^{1/2}\{(2P + w^2) - 2x - \gamma x^2\}^{1/2}} = - \tanh\{2(2P + w^2)^{1/2}(t + C_2)\} \tag{4.10}$$

which can be rewritten as

$$\{1 + \gamma(2P + w^2)\tanh[\theta(t + C_2)]\}X^2 - 2(2P + w^2)\{1 - \tanh[\theta(t + C_2)]\}X$$

$$+ (2P + w^2)^2\{1 - \tanh^2[\theta(t + C_2)]\} = 0 \tag{4.11}$$

where $\theta \equiv 2(2P + w^2)^{1/2}$.

Equation (4.11) is a quadratic equation in X. We obtain the solution

$$X = \frac{(2P + w^2)\{1 \pm i[1 + \gamma(2P + w^2)]^{1/2}\sinh[\theta(t + C_2)]\}}{[1 + \gamma(2P + w^2)]\cosh^2[\theta(t + C_2)] - \gamma(2P + w^2)} \tag{4.12}$$

If we choose (this does not restrict the generality of our results.)

$$C_2 = \frac{i\pi}{2\theta}$$

then

$$\sinh[\theta(t + C_2)] = \sinh[\theta t + i\,\tfrac{\pi}{2}] = \sinh(\theta t)\cosh(i\,\tfrac{\pi}{2}) + \cosh(\theta t)\sinh(i\,\tfrac{\pi}{2})$$

$$= \sinh(\theta t)\cos(\tfrac{\pi}{2}) + \cosh(\theta t)\; i\, \sin(\tfrac{\pi}{2})$$

$$= i\, \cosh(\theta t) \tag{4.13}$$

Similarly we obtain

$$\mathrm{Cosh}^2[\theta(t + C_2)] = -\sinh^2(\theta t) \tag{4.14}$$

Making use of (4.13), (4.14) and of appropriate trigonometric identities, (4.12) can be rewritten as

$$X = \frac{(2P + w^2)\left[1 \pm \{1 + \gamma(2P + w^2)\}^{1/2}\cosh(\theta t)\right]}{1 - \left[1 + \gamma(2P + w^2)\right]\cosh^2(\theta t)} \tag{4.15}$$

There are two values of X in (4.15). We shall keep the one that makes F real. From (4.15) we write

$$F = \sqrt{X} = \frac{\left[2P + w^2\right]^{1/2}\left[1 \pm \{1 + \gamma(2P + w^2)\}^{1/2}\cosh\theta t\right]^{1/2}}{\left[1 - \{1 + \gamma(2P + w^2)\}\cosh^2(\theta t)\right]^{1/2}} \tag{4.16}$$

Thus we can write the solitary wave solution for the equation (4.1) as

$$q = Fe^{i[P\xi + w\tau]} \tag{4.2}$$

where F is given by (4.16).

(4.16) is the general solution in the sense that we have not assigned any positive or negative signs to δ. From this general solution we shall obtain particular solutions for different possible special cases, in the following sections.

4.3 Solitary Wave Solutions for Positive n_2.

This case has three special sub cases

(i) $n_2 > 0$, $n_4 > 0$ i.e. $\gamma > 0$

From (4.16) we obtain, on taking minus sign,

$$F = \frac{\left[2P + w^2\right]^{1/2}}{\left[1 + \left\{1 + \gamma(2P + w^2)\right\}^{1/2}\cosh(\theta t)\right]^{1/2}} \tag{4.17}$$

The other solution corresponding to taking the plus sign makes F imaginary, so we reject it as being inconsistent with our assumption that F is real. Thus we write the solution of the equation (4.1) as

$$q(\xi,\tau) = [2P+w^2]^{1/2}\left[1+\left\{1+\gamma(2P+w^2)\right\}^{1/2}\cosh[2(2P+w^2)^{1/2}(\tau-w\xi)]\right]^{-1/2}e^{i\left\{P\xi+w\tau\right\}} \tag{4.18}$$

Let $\quad [2P + w^2]^{1/2} = C$

Then $\quad [1 + \gamma(2P + w^2)]^{1/2} = [1 + \gamma C^2]^{1/2}$

and $\quad P = \dfrac{C^2 - w^2}{2}$.

And we can write (4.18) as

$$q(\xi,\tau) = C[1 + (1 + \gamma C^2)^{1/2}\cosh[2C(\tau - w\xi)]]^{-1/2} e^{i[(C^2 - w^2)\frac{\xi}{2} + w\tau]} \tag{4.19}$$

The constant C, can be determined by initial conditions. Assume that initially i.e. $\xi = \tau = 0$

$$|q| = q_0$$

Then from (4.19) we find

$$C = q_o(2 + \gamma q_o^2)^{1/2} \tag{4.20}$$

where q_o is given by (2.11) as

$$q_o = 10^{4.5}\sqrt{\pi n_2}\,\phi_o \tag{4.21}$$

We can convert our solution (4.19) to the form quoted by Pushkarov et al [26]. Consider

$$\int_{-\infty}^{+\infty} |q|^2 d\tau = \varepsilon_o \tag{4.22}$$

We know from (3.12a) that ε_o is a conserved quantity. Substituting for q as given by (4.19), (4.22) becomes

$$\varepsilon_o = C^2 \int_{-\infty}^{+\infty} \frac{d\tau}{\{1 + (1 + \gamma C^2)^{1/2} \cosh[2C(\tau - w\xi)]\}}$$

Solving the integral we obtain

$$\varepsilon_o = \frac{2}{\sqrt{\gamma}} \tan^{-1}\left\{\frac{\sqrt{\gamma}\, C}{1 + (1 + \gamma C^2)^{1/2}}\right\}$$

or $$\frac{\sqrt{\gamma}\, C}{1 + (1 + \gamma C^2)^{1/2}} = \tan\left[\frac{\sqrt{\gamma}}{2}\, \varepsilon_o\right] \qquad (4.23)$$

We solve (4.23) for C and obtain

$$C = \frac{1}{\sqrt{\gamma}} \tan\left[\sqrt{\gamma}\, \varepsilon_o\right]$$

define $$\eta = \sqrt{\gamma}\, \varepsilon_o$$

then $$C = \varepsilon_o \frac{\tan\eta}{\eta}$$

Thus (4.19) becomes

$$q(\xi,\tau) = \varepsilon_o \frac{\tan\eta}{\eta}\left\{1 + \sec\eta \cosh\left[2\varepsilon_o \frac{\tan\eta}{\eta}(\tau - w\xi)\right]\right\}^{-1/2} e^{i\left\{\left(\varepsilon_o^2 \frac{\tan^2\eta}{\eta^2} - w^2\right)\frac{\xi}{2} + w\tau\right\}} \qquad (4.24)$$

This is the form of solution reported by Pushkarov. We shall prefer to write the solution in the form (4.19). The width of the solitary wave would be

$$L = \frac{1}{2C} = \frac{1}{2q_o\{2 + \gamma q_o^2\}^{1/2}} \tag{4.25}$$

Thus for $n_4 > 0$ the solitary waves will be narrower than those which occur when $n_4 = 0$.

(ii) $n_2 > 0$, $n_4 < 0$ i.e. $\gamma < 0$

For this case the solution (4.16) becomes

$$F = \frac{C\{1 \pm [1 - |\gamma|C^2]^{1/2}\cosh(2Ct)\}^{1/2}}{\{1 - [(1 - |\gamma|C^2)\cosh^2(2Ct)]\}^{1/2}} \tag{4.26}$$

The only solution consistent with F real is

$$F = C\{1 + (1 - |\gamma|C^2)^{1/2}\cosh[2C(\tau - w\xi)]\}^{-1/2}$$

Thus

$$q(\xi,\tau) = C\{1 + (1 - |\gamma|C^2)^{1/2}\cosh[2C(\tau - w\xi)]\}^{-1/2} e^{i\{(C^2 - w^2)\frac{\xi}{2} + w\tau\}} \tag{4.27}$$

We calculate C by initial conditions and obtain

$$C = q_o(2 - |\gamma|q_o^2)^{1/2} \tag{4.28}$$

Thus the width of the solitary wave is

$$L = \frac{1}{2C} = \frac{1}{2q_o(2 - |\gamma|q_0^2)^{1/2}} \tag{4.29}$$

Thus for $n_4 < 0$ the solitary wave will be broader than that with $n_4 = 0$.

(iii) $n_4 = 0$, $n_2 > 0$ i.e. $\gamma = 0$

For this case the NLCQSE reduces to the NLS equation and the solution (4.16) becomes

$$F = \frac{C[1 \pm \cosh(2Ct)]^{+1/2}}{[1 - \cosh^2(2Ct)]^{1/2}}$$

The only solution consistent with F real is

$$F = \frac{C[1 - \cosh(2Ct)]^{1/2}}{[1 - \cosh^2(2Ct)]^{1/2}} = \frac{C}{[1 + \cosh(2Ct)]^{1/2}}$$

$$= \frac{C}{\sqrt{2}\cosh(Ct)} = \frac{C}{\sqrt{2}}\operatorname{sech}[C(\tau - w\xi)]$$

Thus

$$q(\xi,\tau) = \frac{C}{\sqrt{2}}\operatorname{sech}[C(\tau - w\xi)]e^{i\{(C^2 - w^2)\frac{\xi}{2} + w\tau\}} \tag{4.30}$$

This is the well known soliton solution [29] for the NLS equation.

4.4 Solitary wave solutions for $n_2 \leq 0$

Here we again discuss three subcases:

(i) $n_2 < 0,\ n_4 < 0.$

For this, the NLCQSE becomes

$$iq_\xi + \frac{1}{2} q_{\tau\tau} - 2|q|^2 q - |\delta||q|^4 q = 0 \tag{4.31}$$

We solve this equation following the same procedure and obtain

$$\begin{aligned} F &= C\{(1 - |\gamma|C^2)^{1/2}\cosh(2Ct) - 1\}^{-1/2} \\ &= C\{(1 - |\gamma|C^2)^{1/2}\cosh[2C(\tau - w\xi)] - 1\}^{-1/2} \end{aligned} \tag{4.32}$$

This solution is unacceptable because at $\tau = \xi = 0$, F becomes imaginary in contradiction to our assumption that F is real. Therefore a solitary wave solution does not exist for $n_2 < 0$, $n_4 < 0$.

(ii) $n_2 < 0.\ n_4 = 0$

For this case the NLCQSE reduces to

$$iq_\xi + \frac{1}{2} q_{\tau\tau} - 2|q|^2 q = 0 \tag{4.33}$$

and the solution (4.32) reduces to

$$F = C\{\cosh[2C(\tau - w\xi)] - 1\}^{-1/2}$$

$$= C\{2\sinh^2[2C(\tau - w\xi)]\}^{-1/2} \tag{4.34}$$

$$= \frac{C}{\sqrt{2}}\operatorname{csch}[2C(\tau - w\xi)]$$

This is certainly not a solitary wave solution because it doesn't have a finite amplitude at $\tau = \xi = 0$.

(iii) $n_2 \leq 0,\ n_4 > 0$

For $n_2 < 0,\ n_4 > 0$ the NLCQSE becomes

$$iq_\xi + \frac{1}{2}q_{\tau\tau} - 2|q|^2 q + \delta|q|^4 q = 0 \tag{4.35}$$

and the solution is

$$F = \frac{C\{1 \pm (1 + \gamma C^2)^{1/2}\cosh[2C(\tau - w\xi)]\}^{+1/2}}{\{(1 + \gamma C^2)\cosh^2[2C(\tau - w\xi)] - 1\}^{1/2}}$$

The solution consistent with F real is

$$F = \frac{C}{\{(1 + \gamma C^2)^{1/2}\cosh[2C(\tau - w\xi)] - 1\}^{1/2}}$$

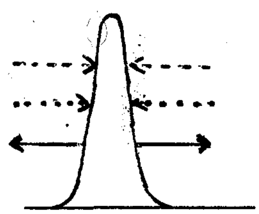
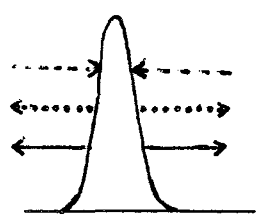
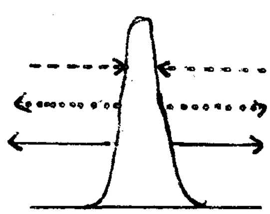
Thus

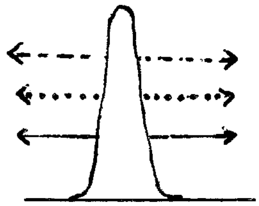
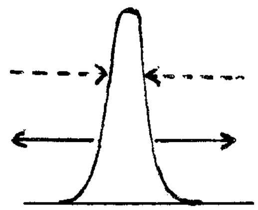
$$q = \frac{C}{\{(1 + \gamma C^2)^{1/2}\cosh[2C(\tau - w\xi)] - 1\}^{1/2}}\, e^{i\{(C^2 - w^2)\frac{\xi}{2} + w\tau\}} \tag{4.36}$$

For $n_2 = 0$, $n_4 > 0$, our equation NLCQSE reduces to the "higher" NLS equation whose solution has been quoted by Kodama et al. [30]. We derive this solution by our method in Appendix B.

4.5 Discussion

We can understand from a qualitative, i.e. hand waving, argument how the dispersion and nonlinear terms can balance to yield solitary waves. We can summarize these effects in diagrams as follows where the solid arrow corresponds to the effect of dispersion, the dashed arrow to the n_2 contribution and the dotted one to the n_4 contribution.

	n_2	n_4	Diagram	Expectation
a)	+	+		Solitary wave may exist.
b)	+	-		Solitary wave may exist.
c)	-	+		Solitary wave may exist.

	n_2	n_4	Diagram	Expectation
d)	-	-		Solitary wave can't exist.
e)	+	0		Solitary wave may exist.

The n_2 and n_4 contributions cause pulse compression or pulse broadening depending upon whether they take on the positive or negative sign respectively and the dispersion term (for $k'' < 0$) always causes spreading. The possible balance between squeezing and spreading determines the possibility of the existence of solitary waves. The solitary wave in case e) is known to be a soliton, but the soliton behavior of the solitary waves for other cases is still to be explored. Various solitary wave solutions obtained in this chapter are plotted in the following diagrams. In fig. 4.4, the energy content of the pulse, i.e. I_1, is kept fixed for all the three plots.

Fig 4.1) Plot for solitary wave solution (4.19). Input parameters are $\delta = 30$, $C = 0.5$, $n_2 > 0$.

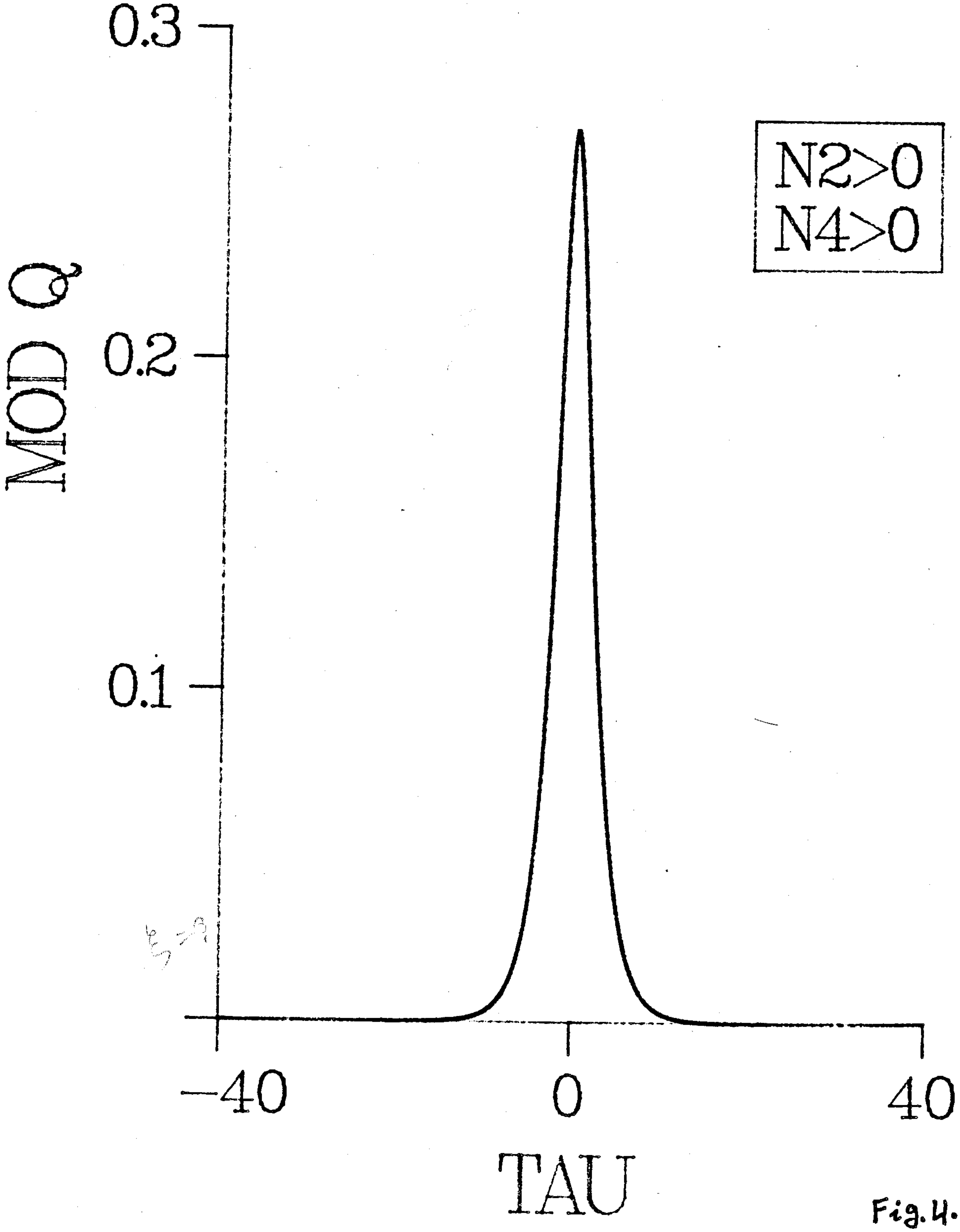


Fig. 4.1

Fig 4.2) Plot for solitary wave solution (4.27). Input parameters are $\delta = -5.0$, $C = 2.0$, $n_2 > 0$.

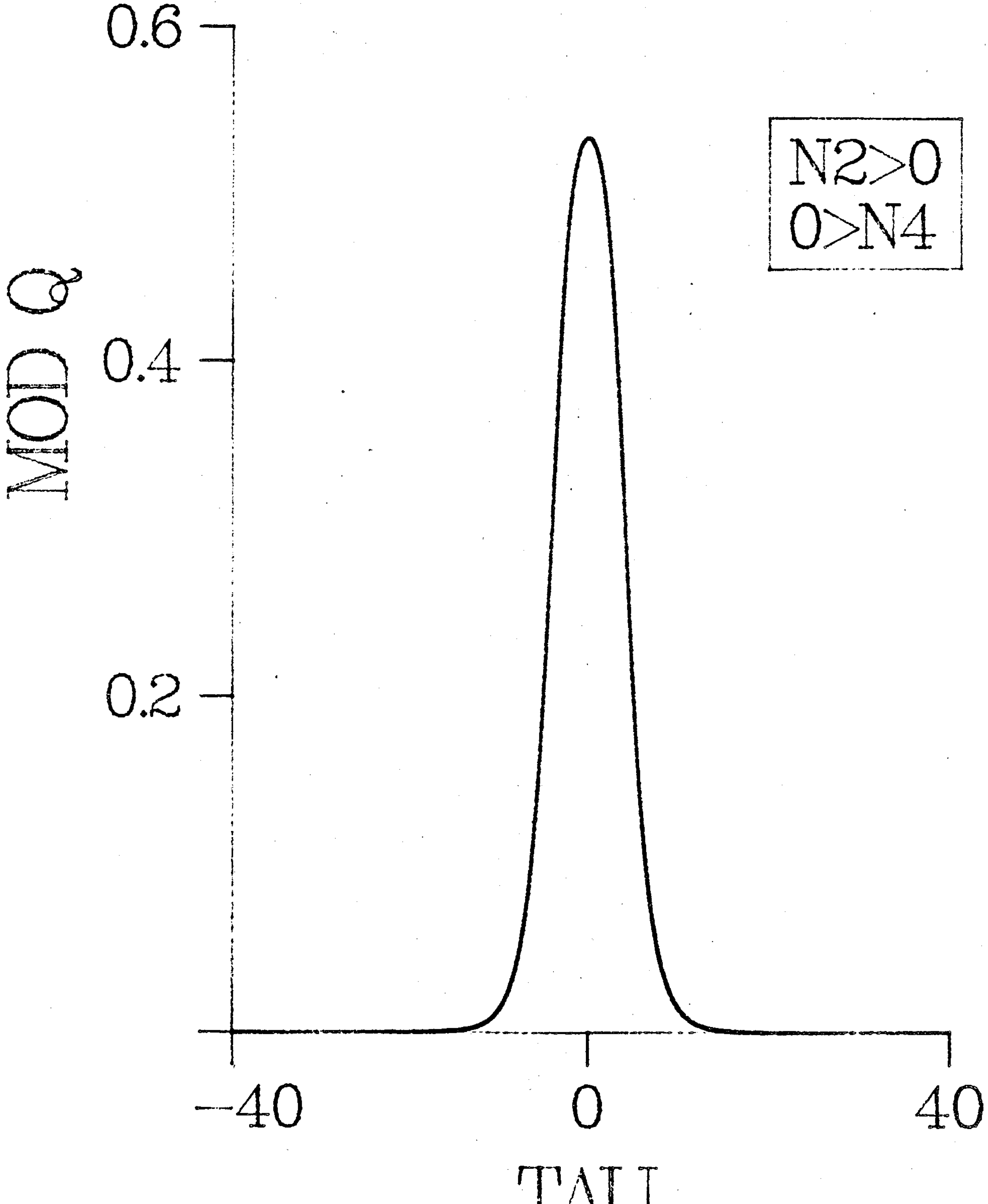


Fig. 4.2

Fig 4.3) Plot for solitary wave solution (4.36). Input parameters are $\delta = 8$, $C = 0.7$, $n_2 < 0$.

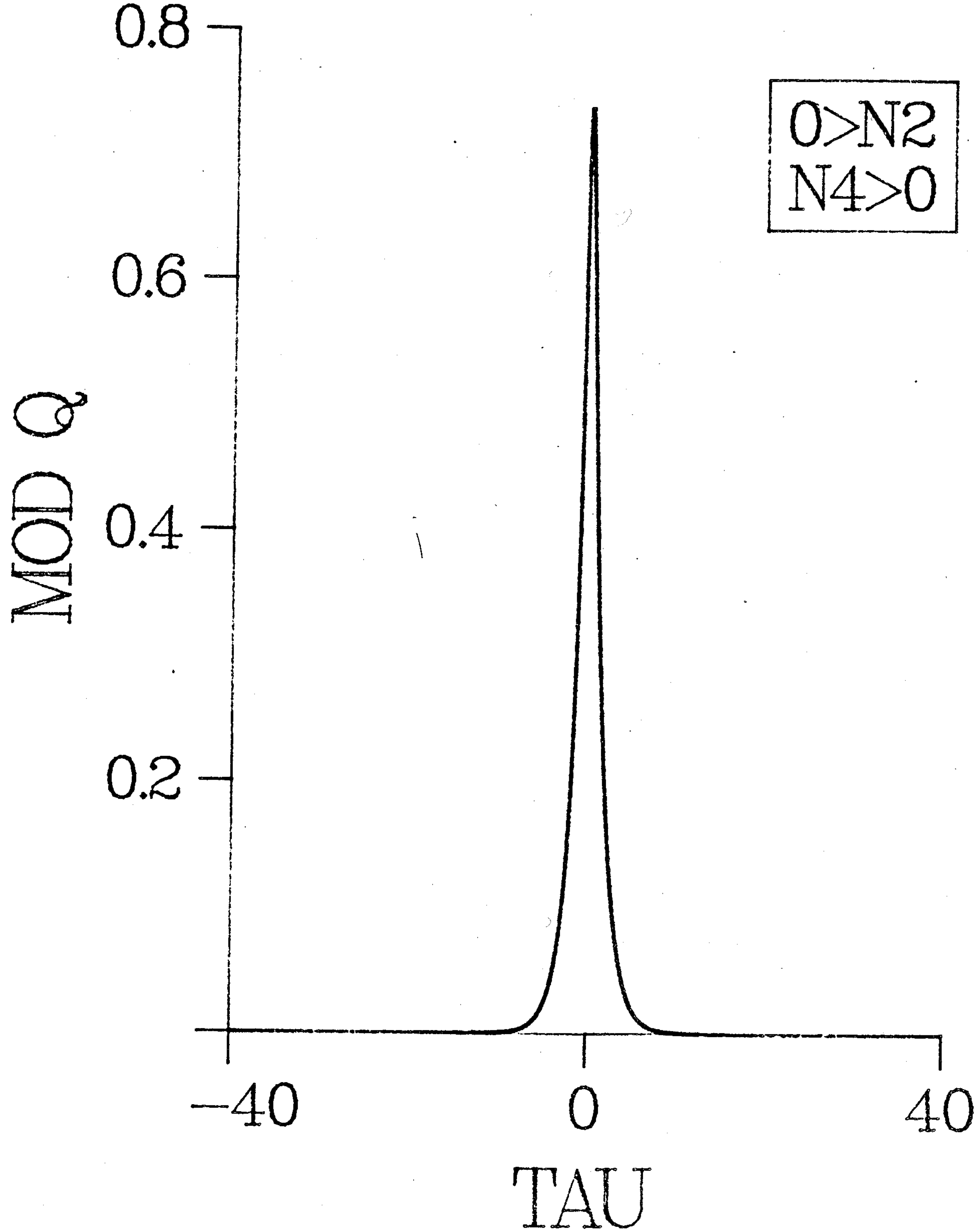


Fig. 4.3

Fig 4.4) Comparative plots of solitary wave solutions for $n_4 > 0$, $n_4 = 0$ and $n_4 < 0$. Input parameters are $I_1 = 1.0$, $|\delta| = 1.5$, $n_2 > 0$.

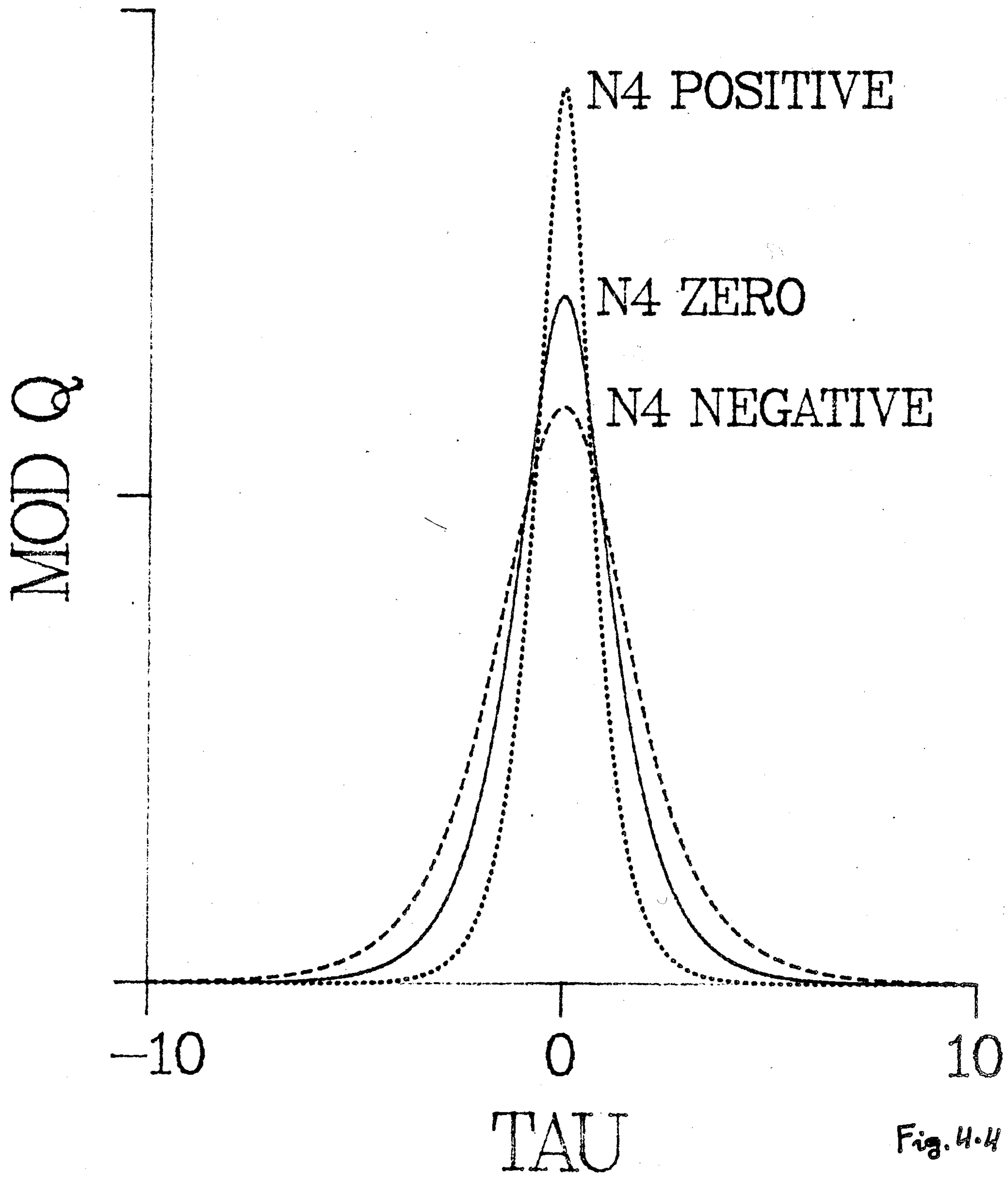


Fig. 4.4

CHAPTER 5

Search for Multi-Soliton Solutions for the NLCQSE- The Bäcklund Transformation

In the previous chapter, we obtained the solitary wave solutions for the NLCQSE. Now, we address the issue of whether the solitary waves are stable, ie. whether our equation, the NLCQSE, has multi-soliton solutions. There are two possible analytical techniques which may be used to obtain the soliton solutions i.e. the Bäcklund transformation and the inverse scattering transform method (ISTM). However, the two techniques are interrelated [31]. The eigenvalue problem in the ISTM is transformable to the Bäcklund transformation [32] and conversely if the Bäcklund transformation for the given evolution equation is known, one can deduce the eigenvalue problem. A number of nonlinear evolution equations such as the Korteweg-deVries (KdV) equation, the NLS equation and the Sine-Gordon equation belong to a class of equations that can be solved by the ISTM [33] and the related eigenvalue problem can be deduced from the Bäcklund transformation [31] for these equations. We wonder if the NLCQSE has a Bäcklund transformation and hence multi-soliton solutions.

In section 5.1, we introduce the concept of the Bäcklund transformation and discuss it in the context of an illustrative example. In section 5.2, we make an attempt to derive the Bäcklund transformation for the NLCQSE. The conclusion from the result of our attempt is drawn in section 5.3.

5.1 The Bäcklund transformation

Let z satisfy a differential equation. The Bäcklund transformation will yield another solution say z' satisfying the same form of the equation. Define the transformation between the two solutions as

$$\frac{\partial z}{\partial x} \equiv p = f(x',y',z',p',g')$$
$$\frac{\partial z}{\partial y} \equiv g = \phi(x',y',z',p',g') \tag{5.1}$$

where z is a function of two independent coordinates x and y eg. x may be temporal and y may be a spatial coordinate, $x = x'$, $y = y'$ and $p' = \frac{\partial z'}{\partial x'}$ etc.

The integrability condition for z requires

$$\frac{\partial p}{\partial y} = \frac{\partial g}{\partial x}$$

which on defining $\Omega = \frac{\partial p}{\partial y} - \frac{\partial g}{\partial x}$ becomes

$$\Omega = f_{y'} - \phi_{x'} + f_{z'}g' - \phi_{z'}p' + (f_{p'} - \phi_{g'})s' + f_{g'}t' + \phi_{p'}r' = 0 \tag{5.2}$$

where $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$ etc. (5.3)

Now, the integrability condition (5.2) can be satisfied in either of two ways

(i) We can satisfy (5.2) identically i.e.

$$f_{p'} - \phi_{g'} = f_{g'} = \phi_{p'} = f_{y'} - \phi_{x'} + f_{z'}g' - \phi_{z'}p' = 0$$

In this case the transformation is called a contact transformation.

(ii) The equation (5.2) can be satisfied if z' is its solution. In this case the transformation is called the Bäcklund transformation.

For example, consider the sine-Gordon equation

$$\phi_{xt} = \sin\phi \tag{5.4a}$$

This equation has the Bäcklund transformation

$$\phi'_x = \phi_x - 2a\sin\left(\frac{\phi + \phi'}{2}\right)$$

$$\phi'_t = -\phi_t + \frac{2}{a}\sin\left(\frac{\phi - \phi'}{2}\right) \tag{5.4b}$$

$$x = x', \quad t = t'$$

where a is an arbitrary constant.

The application of the integrability condition

$$\frac{\partial}{\partial t}\left(\phi'_x\right) - \frac{\partial}{\partial x}\left(\phi'_t\right) = 0$$

to (5.4b) gives equation (5.4a) and the integrability condition

$$\frac{\partial}{\partial t}\left(\phi_x\right) - \frac{\partial}{\partial x}\left(\phi_t\right) = 0$$

yields

$$\phi'_{xt} = \sin\phi' \tag{5.4c}$$

Thus the sine-Gordon equation is invariant under the Bäcklund transformation (5.4b) i.e. if ϕ is a solution of the sine-Gordon equation, so is ϕ'. The Bäcklund transformation may be used to generate additional solutions of (5.4a) by inserting the known solution into (5.4b). For example, $\phi_0' = 0$ is a trivial "vacuum" solution of (5.4c), Substituting this into (5.4b), we obtain a pair of equations viz;

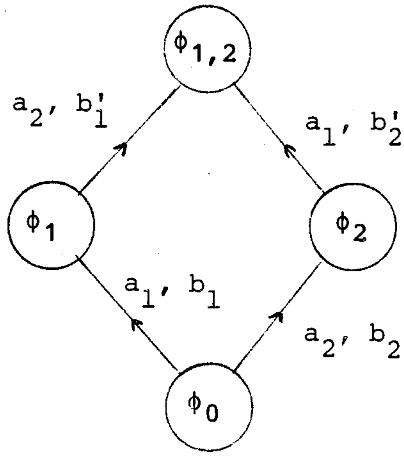
$$\phi_x = 2a \sin\left(\frac{\phi}{2}\right)$$
$$\phi_t = \frac{2}{a} \sin\left(\frac{\phi}{2}\right) \tag{5.4d}$$

which may be solved to obtain a second (nontrivial) solution, viz;

$$\phi = 4\tan^{-1}\left\{e^{(ax + t/a + b)}\right\} \tag{5.4e}$$

where b is a constant of integration.

(5.4e) can be shown to be a "one soliton" solution of the sine-Gordon equation. Let ϕ_1, ϕ_2 be two such one soliton solutions derived from the vacuum solution ϕ_0 by applying the Bäcklund transformation with parameters a_1 and a_2 and integration constants b_1 and b_2 respectively. It is possible to choose the appropriate integration constants say b_1' and b_2' such that we obtain the same solution $\phi_{1,2}$ by further applying the Bäcklund transformation with parameter a_1 to ϕ_2 and a_2 to ϕ_1 as shown schematically by a commutative Bianchi diagram, viz;

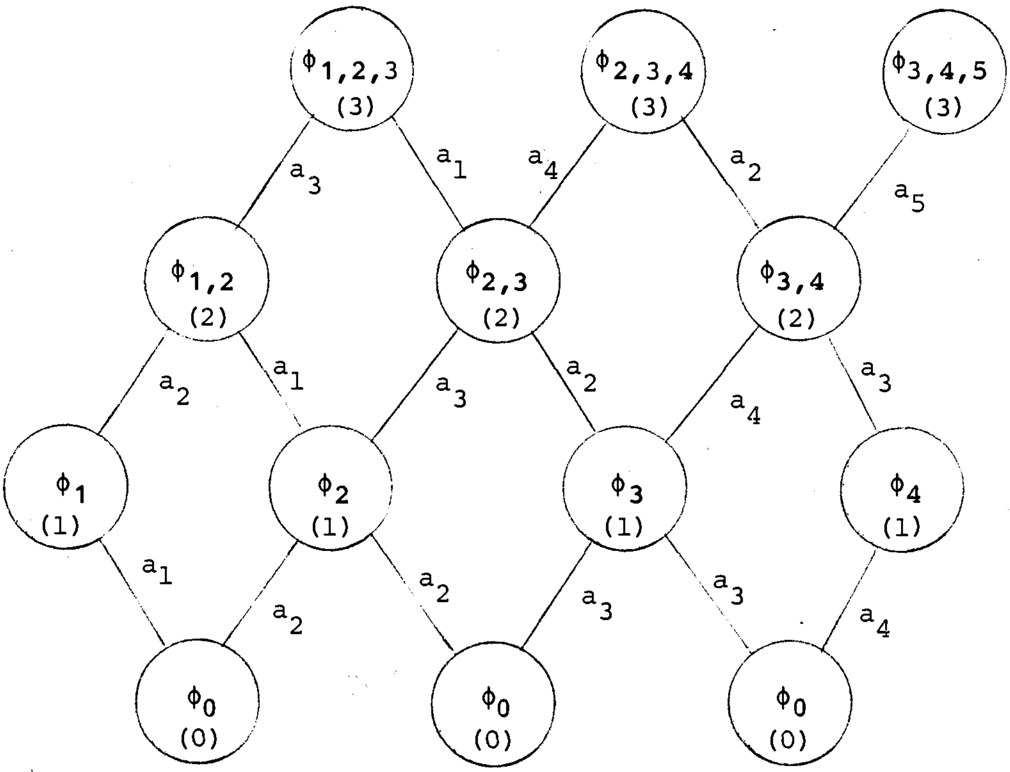


$\phi_{1,2}$ would be a two soliton solution.

Starting from the Bäcklund transformation (5.4b), one can prove the following theorem

$$\phi_{n_1+1} = \phi_{n_2+1} = 4\tan^{-1}\left[\left(\frac{a_1 + a_2}{a_1 - a_2}\right) \tan\left\{\frac{\phi_{n_1} - \phi_{n_2}}{4}\right\}\right] + \phi_{n-1} \qquad (5.4f)$$

where ϕ_{n_1} and ϕ_{n_2} are solutions of (5.4a) generated by application of the Bäcklund transformation (5.4b) to a known solution ϕ_{n-1} with parameters a_1 and a_2 respectively. In (5.4f) ϕ_{n-1} represents an (n-1) soliton solution, ϕ_{n_1} and ϕ_{n_2} represent n soliton solutions and ϕ_{n_1+1} and ϕ_{n_2+1} represent (n+1) soliton solutions. Thus knowing the one soliton solution, an infinite sequence of additional solutions may be generated without further recourse to integration, by making use of (5.4f). The generation of N-soliton solutions is shown in the extended Lamb diagram given below where the integration constants b have been suppressed for brevity. The number in parentheses indicates the number of solitons.



5.2 The Bäcklund Transformation for the NLCQSE.

G.L. Lamb [31] has derived the Bäcklund transformation for the NLS equation by a method due to Clairin [34]. Here we attempt to derive the Bäcklund transformation for the NLCQSE by using the same method. The NLCQSE and its complex conjugate are

$$iq_{\xi} + \frac{1}{2} q_{\tau\tau} + 2|q|^4 q + \delta|q|^4 q = 0 \tag{3.19}$$

$$-i\bar{q}_{\xi} + \frac{1}{2} \bar{q}_{\tau\tau} + 2|q|^2 \bar{q} + \delta|q|^4 \bar{q} = 0 \tag{3.20}$$

where the bar indicates complex conjugate.

Let's make the substitutions

$$2\xi = y, \quad 2\tau = x, \quad q = z, \quad \frac{\delta}{2} = A, \quad \frac{\partial z}{\partial y} = g$$

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial y^2} = t, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s \tag{5.5}$$

With these substitutions, the evolution equations (3.19) and (3.20) become

$$ig + r + z^2 \bar{z} + Az^3 \bar{z}^2 = 0 \tag{5.6a}$$

$$-i\bar{g} + \bar{r} + \bar{z}^2 z + A\bar{z}^3 z^2 = 0 \tag{5.6b}$$

The general form adopted for the Bäcklund transformation is

$$p = f(z,\bar{z},z',\bar{z}',p',\bar{p}')$$

$$g = \phi(z,\bar{z},z',\bar{z}',g',\bar{g}',p',\bar{p}') \tag{5.7}$$

with $x = x'$, $y = y'$ and also the complex conjugate transformation

$$\bar{p} = \bar{f}$$

$$\bar{g} = \bar{\phi} \tag{5.8}$$

The integrability condition for z requires

$$\frac{\partial p}{\partial y} - \frac{\partial g}{\partial x} \equiv \Omega = 0 \tag{5.9}$$

From (5.7) we obtain

$$\frac{\partial p}{\partial y} = f_z g + f_{\bar z}\bar g + f_{z'} g' + f_{\bar z'}\bar g' + f_{p'} s' + f_{\bar p'}\bar s' \tag{5.10a}$$

$$\frac{\partial g}{\partial x} = \phi_z p + \phi_{\bar z}\bar p + \phi_{z'} p' + \phi_{\bar z'}\bar p' + \phi_{g'} s' + \phi_{\bar g'}\bar s' + \phi_{p'} r' + \phi_{\bar p'}\bar r' \tag{5.10b}$$

Using (5.10), (5.9) becomes

$$\Omega = f_z\phi + f_{\bar z}\bar\phi + f_{z'} g' + f_{\bar z'}\bar g' + (f_{p'} - \phi_{g'})s' + (f_{\bar p'} - \phi_{\bar g'})\bar s' - \phi_z f - \phi_{\bar z}\bar f$$

$$- \phi_{z'} p' - \phi_{\bar z'}\bar p' - \phi_{p'} r' - \phi_{\bar p'}\bar r' = 0 \tag{5.11}$$

Let the transformed solution z' also satisfy the equation of the same form as (5.6) i.e.

$$ig' + r' + \bar z' z'^2 + Az'^3\bar z^2 = 0 \tag{5.12a}$$

$$-i\bar g' + \bar r' + \bar z'^2 z' + A\bar z'^3 z^2 = 0 \tag{5.12b}$$

Making use of (5.12), (5.11) becomes

$$\Omega = f_z\phi + f_{\bar z}\bar\phi + f_{z'}g' + f_{\bar z'}\bar g' + (f_{p'} - \phi_{g'})s' + (f_{\bar p'} - \phi_{\bar g'})\bar s' - \phi_z f - \phi_{\bar z}\bar f - \phi_{z'}p'$$

$$- \phi_{\bar z'}\bar p' + \phi_{p'}(ig' + z'^2\bar z' + Az'^3\bar z'^2) + \phi_{\bar p'}(-i\bar g' + \bar z'^2 z' + A\bar z'^3 z'^2) = 0 \tag{5.13}$$

From (5.13), we obtain

$$\Omega_{s'} = f_{p'} - \phi_{q'} = 0$$

(5.14)

$$\Omega_{\bar{s}'} = f_{\bar{p}'} - \phi_{\bar{g}'} = 0$$

Using (5.14), (5.13) becomes

$$\Omega = f_z\phi + f_{\bar{z}}\bar{\phi} + f_{z'}g' + f_{\bar{z}'}\bar{g}' - \phi_z f - \phi_{\bar{z}}\bar{f} - \phi_{z'}p' - \phi_{\bar{z}'}\bar{p}'$$

$$+ \phi_{p'}(ig' + z'^2\bar{z}' + Az'^3\bar{z}'^2) + \phi_{\bar{p}'}(-i\bar{g}' + \bar{z}'^2z' + A\bar{z}'^3z'^2)$$

(5.15)

From (5.14), we have

$$\Omega_{s'g'} = f_{p'g'} - \phi_{g'g'} = 0$$

But f is not an explicit function of g' or $\bar{g}'$ so

$$f_{p'g'} = 0, \quad f_{\bar{p}'\bar{q}'} = 0 \qquad (5.15a)$$

it follows that

$$\phi_{g'g'} = 0 \qquad (5.16a)$$

Similarly from (5.14) we obtain

$$\phi_{\bar{g}'\bar{g}'} = 0 \qquad (5.16b)$$

Now using the fact that f is not an explicit function of g' or $\overline{g}'$ and making use of (5.16a) and (5.16b), we obtain from (5.15)

$$\Omega_{g'g'} = 2i\phi_{p'g'} = 2i\phi_{p'p'} = 0 \tag{5.17}$$

and

$$\Omega_{\overline{g}'\overline{g}'} = -2i\phi_{\overline{g}'\overline{p}'} = -2if_{\overline{p}'\overline{p}'} = 0 \tag{5.18}$$

Making use of (5.14), (5.17) and (5.18), we obtain from (5.15)

$$\Omega_{g'p'} = i\phi_{p'p'} = 0$$

$$\Omega_{\overline{g}'\overline{p}'} = -i\phi_{\overline{p}'\overline{p}'} = 0$$

or

$$\phi_{p'p'} = 0 \tag{5.19a}$$

and

$$\phi_{\overline{p}'\overline{p}'} = 0 \tag{5.19b}$$

Let Z stand for the set of four independent variables $z, z', \overline{z}, \overline{z}'$. We note from (5.7) that ϕ, in general, is a function of Z, $g', \overline{g}', p'$ and $\overline{p}'$. From (5.17) we notice that $\phi_{g'}$ is independent of p'. From (5.14) and (5.7) we conclude that $\phi_{g'}$ is independent of g' and $\overline{g}'$. So combining these facts we obtain

$$\phi_{g'} = f_{p'} = F(Z, \overline{p}') \tag{5.20}$$

Equation (5.7) tells us that f is, in general, a function of Z, p' and $\overline{p}'$. (5.17) implies that f is only a linear function of p' and (5.18) tells us that it can also be only a linear function of $\overline{p}'$. Combining these facts, we can write

$$f = kp'\overline{p}' + \ell p' + m\overline{p}' + n \tag{5.21}$$

where k, ℓ, m and n are arbitrary functions of Z.

Now from (5.21) and (5.14), we obtain

$$\begin{aligned} \phi_{g'} &= f_{\overline{p}'} = k\overline{p}' + \ell \\ \phi_{\overline{g}'} &= f_{\overline{p}'} = kp' + m \end{aligned} \tag{5.22}$$

which imply that ϕ must be of the form

$$\phi = k(\overline{p}'g' + p'\overline{g}') + \ell g' + m\overline{g}' + X \tag{5.22a}$$

where X is independent of g' and $\overline{g}'$. Thus X is possibly a function of Z, p' and $\overline{p}'$. But from (5.19) it implies that ϕ can depend only linearly on p' or $\overline{p}'$. Therefore we can write X as

$$X = \sigma p'\overline{p}' + \tau p' + \theta\overline{p}' + \chi$$

where σ, τ, θ and χ are functions of Z.

Thus (5.22a) becomes

$$\phi = k(\overline{p}'g' + p'\overline{g}') + \ell g' + m\overline{g}' + \sigma p'\overline{p}' + \tau p' + \theta\overline{p}' + \chi \tag{5.23}$$

From (5.1) and (5.5) we obtain, making use of (5.21)

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x} = \frac{\partial f}{\partial x} = kp'\bar{r}' + k\bar{p}'r' + \ell r' + m\bar{r}' + p'\bar{p}'k_x + p'\ell_x + \bar{p}'m_x + n_x$$

and

$$ig + r = i\phi + \frac{\partial f}{\partial x} = k\bar{p}'(ig'+r) + kp'(i\bar{g}'+\bar{r}') + \ell(ig'+r') + m(i\bar{g}'+\bar{r}')$$
$$+ i\sigma p'\bar{p}' + i\tau p' + i\theta\bar{p}' + i\chi + p'\bar{p}'k_x + p'\ell_x + \bar{p}'m_x + n_x \qquad (5.24)$$

However from (5.6a) and (5.12a) we have

$$ig + r = -z^2\bar{z} - Az^3\bar{z}^2$$
$$(5.25)$$
$$ig' + r' = -z'^2\bar{z}' - Az'^3\bar{z}'^2$$

Therefore comparing (5.25) and (5.24) we obtain

$$-z^2\bar{z} - Az^3\bar{z}^2 = k\bar{p}'(-z'^2\bar{z}' - Az'^3\bar{z}'^2) + kp'(i\bar{g}' + \bar{r}') + \ell(-z'^2\bar{z}' - Az'^3\bar{z}'^2)$$
$$+ m(i\bar{g}' + \bar{r}') + i\sigma p'\bar{p}' + i\tau p' + i\theta\bar{p}' + i\chi + p'\bar{p}'k_x + p'\ell_x + \bar{p}'m_x + n_x \qquad (5.25a)$$

Now the variables z, p', $\bar{p}'$, $\bar{q}'$, $\bar{r}'$ are independent variables. Thus equation (5.25a) must be satisfied identically. First, we compare the coefficients of $\bar{q}'$ and $\bar{r}'$ on both sides of (5.25a) and obtain

$$k = m = 0 \qquad (5.25b)$$

Hence (5.25a) is reduced to

$$-z^2\bar{z} - Az^3\bar{z}^2 = -\ell(z'^2\bar{z}' + Az'^3\bar{z}'^2) + i\sigma p'\bar{p}' + i\tau p' + i\theta\bar{p}' + i\chi + p'\ell_x + n_x \tag{5.26}$$

Now

$$\ell_x = \ell_z z_x + \ell_{\bar{z}}\bar{z}_x + \ell_{z'}z'_x + \ell_{\bar{z}'}\bar{z}'_x = \ell_z p + \ell_{\bar{z}}\bar{p} + \ell_{z'}p' + \ell_{\bar{z}'}\bar{p}' \tag{5.26a}$$

From (5.21) and (5.25b) we have

$$p = f = \ell p' + n$$
$$\bar{p} = \bar{f} = \bar{\ell}\,\bar{p}' + \bar{n} \tag{5.26b}$$

Thus (5.26a) becomes

$$\ell_x = \ell_z\ell p' + \ell_z n + \ell_{\bar{z}}\bar{\ell}\,\bar{p}' + \ell_{\bar{z}}\bar{n} + \ell_{z'}p' + \ell_{\bar{z}'}\bar{p}' \tag{5.26c}$$

Similarly we obtain

$$n_x = n_z\ell p' + nn_z + \bar{\ell}n_{\bar{z}}\bar{p}' + n_{\bar{z}}\bar{n} + n_{z'}p' + n_{\bar{z}'}\bar{p}' \tag{5.26d}$$

Substituting (5.26c) and (5.26d) into (5.26) we obtain

$$-z^2\bar{z} - Az^3\bar{z}^2 = p'\bar{p}'\{i\sigma + \ell_{\bar{z}}\bar{\ell} + \ell_{\bar{z}'}\} + p'\{i\tau + \ell_z n + \ell_{\bar{z}}\bar{n} + \ell n_z + n_{z'}\}$$

$$+ \bar{p}'\{i\theta + \bar{\ell}n_{\bar{z}} + n_{\bar{z}'}\} + p'^2\bar{p}'\{o\} + p'^2\{\ell_{z'} + \ell\ell_z\} + \bar{p}'^2\{o\}$$

$$+ nn_z + \bar{n}n_{\bar{z}} - \ell z'^2\bar{z}' - \ell Az'^3\bar{z}'^2 + i\chi \tag{5.27}$$

Now the requirement that (5.27) is satisfied identically, leads us to write

$$i\sigma + \ell_{\bar{z}}\bar{\ell} + \ell_{\bar{z}'} = 0 \qquad \text{(a)}$$

$$i\tau + \ell_z n + \ell_{\bar{z}}\bar{n} + \ell n_z + n_{z'} = 0 \qquad \text{(b)}$$

$$i\theta + \bar{\ell}n_{\bar{z}} + n_{\bar{z}'} = 0 \qquad \text{(c)} \qquad (5.28)$$

$$\ell_{z'} + \ell\ell_z = 0 \qquad \text{(d)}$$

$$nn_z + \bar{n}n_{\bar{z}} - \ell z'^2\bar{z}' - \ell Az'^3\bar{z}'^2 + i\chi = -z^2\bar{z} - Az^3\bar{z}^2 \qquad \text{(e)}$$

Notice that we have already developed the Bäcklund transformation from (5.7) into the form (5.21) and (5.23). Now our job is to calculate k, ℓ, m, n, τ, θ, σ and χ that we know can depend upon z only. If we look at (5.28) carefully, it seems promising to start with equation (5.28d) that involves only ℓ. Now (5.28d) is satisfied if ℓ is a constant.

So $\qquad \ell = a \qquad$ (constant) $\qquad$ (5.29)

Considering ℓ as constant, (5.28a) gives

$$\sigma = 0 \qquad (5.30)$$

Now substituting the values of k, m, ℓ and σ from (5.25b), (5.29) and (5.30) into (5.21) and (5.23), we obtain

$$p = f = ap' + n \qquad \text{(a)}$$

$$g = \phi = ag' + \tau p' + \theta \bar{p}' + \chi \qquad \text{(b)} \tag{5.31}$$

(5.31) is the Bäcklund transformation derived to this point. η, τ, θ and χ are still to be determined. Keeping in mind that p', $\bar{p}'$, q' and z are independent variables; we obtain from (5.31)

$$\phi_z = \tau_z p' + \theta_z \bar{p}' + \chi_z$$

$$\phi_{\bar{z}} = \tau_{\bar{z}} p' + \theta_{\bar{z}} \bar{p}' + \chi_{\bar{z}}$$

$$\phi_{z'} = \tau_{z'} p' + \theta_{z'} \bar{p}' + \chi_{z'} \tag{5.32a}$$

$$\phi_{\bar{z}'} = \tau_{\bar{z}'} p' + \theta_{\bar{z}'} \bar{p}' + \chi_{\bar{z}'}$$

and

$$f_z = n_z$$

$$f_{\bar{z}} = n_{\bar{z}}$$

$$f_{z'} = n_{z'} \tag{5.32b}$$

$$f_{\bar{z}'} = n_{\bar{z}'}$$

Substituting (5.32) and (5.31) into (5.11) we obtain,

$$\Omega = g'\{an_z + n_{z'} + i\tau\} + p'\{\tau n_z + n_{\bar z}\bar\theta - a\chi_z - \tau_{\bar z}\bar n - \chi_{z'} - n\tau_z\}$$

$$+ \bar p'\{n_z\theta + n_{\bar z}\bar\tau - \theta_z n - \bar n\theta_{\bar z} - \bar a\chi_{\bar z} - \chi_{\bar z'}\} + \bar g'\{\bar a n_{\bar z} + n_{\bar z'} - i\theta\}$$

$$+ p'^2\{-a\tau_z - \tau_{z'}\} + p'\bar p'\{-\bar a\tau_{\bar z} - \tau_{\bar z'}\} + \bar p'^2\{-\bar a\theta_{\bar z}\}$$

$$+ \chi n_z + \bar\chi n_{\bar z} - n\chi_z - \bar n\chi_{\bar z} + \tau z'^2\bar z' + \tau A z'^3\bar z'^2 + \theta(\bar z'^2 z' + A\bar z'^3 z'^2)$$

$$\equiv Jg' + Kp' + L\bar p' + M + N\bar g' + Pp'^2 + Qp'\bar p' + R\bar p'^2 = 0 \qquad (5.33)$$

Now comparing the coefficients of g', p', $\bar p'$, $\bar g'$, p'^2, $p'\bar p'$ and $\bar p'^2$ on both sides we obtain,

$$J = K = L = M = N = P = Q = R = 0 \qquad (5.33a)$$

$$N = 0 \text{ implies } \bar a n_{\bar z} + n_{\bar z'} - i\theta = 0 \qquad (5.33b)$$

We can rewrite (5.28c) as

$$\bar a n_{\bar z} + n_{\bar z'} + i\theta = 0 \qquad (5.33c)$$

Comparing (5.33b) and (5.33c) we obtain,

$$\theta = 0 \qquad (5.34)$$

Substituting (5.34), into (5.33a) yields

$$J \equiv an_z + n_{z'} + i\tau = 0 \qquad \text{(a)}$$

$$K \equiv \tau n_z - a\chi_z - n\tau_z - \bar{n}\tau_{\bar{z}} - \chi_{z'} = 0 \qquad \text{(b)}$$

$$L \equiv n_{\bar{z}}\bar{\tau} - \bar{a}\chi_{\bar{z}} - \chi_{\bar{z}'} = 0 \qquad \text{(c)}$$

$$M \equiv \chi n_z + \bar{\chi} n_{\bar{z}} - n\chi_z - \bar{n}\chi_{\bar{z}} + \tau z'^2 \bar{z}' + \tau A z'^3 \bar{z}'^2 = 0 \qquad \text{(d)}$$

$$(5.35)$$

$$N \equiv \bar{a} n_{\bar{z}} + n_{\bar{z}'} = 0 \qquad \text{(e)}$$

$$P \equiv -(a\tau_z + \tau_{z'}) = 0 \qquad \text{(f)}$$

$$Q \equiv -(\bar{a}\tau_{\bar{z}} + \tau_{\bar{z}'}) = 0 \qquad \text{(g)}$$

$$R = 0$$

Using (5.34), we rewrite (5.31) as

$$\begin{aligned} p &= ap' + n \\ q &= ag' + \tau p' + \chi \end{aligned} \qquad (5.36)$$

Thus the Bäcklund transformation has now been developed to the form (5.36). Our task has been reduced to determining n, τ and χ only. For this purpose we try to solve the equations (5.28e) and (5.35). G.L. Lamb [31] has solved equations of this form for the NLS equation for the case $a = 1$. Following him we solve (5.28e) and (5.35) for the NLCQSE choosing $a = 1$. For this purpose we define the new variables.

$$\omega = z + z', \quad \bar{\omega} = \bar{z} + \bar{z}', \quad \nu = z - z', \quad \bar{\nu} = \bar{z} - \bar{z}' \qquad (5.37)$$

It follows that

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial \omega} + \frac{\partial}{\partial \nu} \qquad \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \bar{\omega}} + \frac{\partial}{\partial \bar{\nu}}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial \omega} - \frac{\partial}{\partial \nu} \qquad \frac{\partial}{\partial \bar{z}'} = \frac{\partial}{\partial \bar{\omega}} - \frac{\partial}{\partial \bar{\nu}} \tag{5.38}$$

Thus (5.35f) and (5.35g) become

$$\tau_\omega = \tau_{\bar{\omega}} = 0 \tag{5.39}$$

In general τ can be a function of ν, $\bar{\nu}$, ω, $\bar{\omega}$. But (5.39) tells us that τ is independent of ω and $\bar{\omega}$. So we can write

$$\tau = \tau(\nu, \bar{\nu}) \tag{5.40a}$$

and (5.35e) becomes

$$\frac{\partial n}{\partial \bar{\omega}} = 0$$

which implies that

$$n = n(\nu, \bar{\nu}, \omega) \tag{5.40b}$$

Next, (5.35a) gives

$$\frac{\partial n}{\partial \omega} = -\frac{i\tau}{2} \tag{5.41}$$

which on integration yields

$$n = -\frac{i\tau}{2}\omega + \gamma(\nu,\bar{\nu}) \tag{5.40c}$$

where γ is the constant of integration which is to be determined.

Now

$$n_{\bar{z}} = n_{\bar{\nu}} = -\frac{i}{2}\omega\,\tau_{\bar{\nu}} + \gamma_{\bar{\nu}}$$

$$n_z = -\frac{i}{2}\tau - \frac{i}{2}\omega\,\tau_\nu + \gamma_\nu \tag{5.40d}$$

Similarly using (5.40) and (5.39), (5.35c) yields

$$\chi_{\bar{\omega}} = \frac{1}{2}n_{\bar{z}}\bar{\tau} = \frac{\bar{\tau}}{2}(-\frac{i}{2}\omega\,\tau_{\bar{\nu}} + \gamma_{\bar{\nu}})$$

or

$$\chi = \frac{\bar{\omega}\,\bar{\tau}}{2}\,(-\frac{i}{2}\omega\,\tau_{\bar{\nu}} + \gamma_{\bar{\nu}}) + \frac{\zeta}{2}(\omega,\nu,\bar{\nu}) \tag{5.40e}$$

where $\frac{\zeta}{2}$ is the constant of integration, also to be determined.

From (5.40e) we obtain

$$\chi_z = \frac{\partial x}{\partial\omega}\frac{\partial\omega}{\partial z} + \frac{\partial x}{\partial\nu}\frac{\partial\nu}{\partial z}$$

$$= \frac{1}{2}\left\{-\frac{i}{2}\bar{\omega}\,\bar{\tau}\,\tau_{\bar{\nu}} + \zeta_\omega\right\} + \chi_\nu \tag{5.42}$$

$$\chi_{z'} = \frac{1}{2}\left\{-\frac{i}{2}\bar{\omega}\,\bar{\tau}\,\tau_{\bar{\nu}} + \zeta_\omega\right\} - \chi_\nu$$

From (5.40a)

$$\tau_{\bar z} = \tau_{\bar \nu}$$
$$\tau_z = \tau_\nu \qquad (5.43)$$

Using (5.40d), (5.42) and (5.43), we obtain from the equation (5.35b)

$$\zeta_\omega = -\frac{i}{2}\tau^2 - \bar\gamma\tau_{\bar\nu} - \gamma\tau_\nu + \tau\gamma_\nu$$

Thus

$$\zeta = \omega\{-\frac{i}{2}\tau^2 + \tau\gamma_\nu - \gamma\tau_\nu - \bar\gamma\tau_{\bar\nu}\} + \eta(\nu,\bar\nu) \qquad (5.44)$$

where η is the integration constant.

Substituting the value of ζ from (5.44) into (5.40e), we obtain

$$\chi = \frac{\bar\omega\,\bar\tau}{2}\{-\frac{i}{2}\omega\tau_{\bar\nu} + \gamma_{\bar\nu}\} + \frac{\omega}{2}\{-\frac{i}{2}\tau^2 + \tau\gamma_\nu - \gamma\tau_\nu - \bar\gamma\tau_{\bar\nu}\} + \frac{1}{2}\eta \qquad (5.45)$$

Making use of (5.45), (5.40d) and (5.37), the equation (5.28e) becomes

$$\omega^0\{-\frac{i}{2}\tau\gamma + \gamma\gamma_\nu + \bar\gamma\gamma_{\bar\nu} + \frac{1}{2}\eta + \frac{1}{4}\nu^2\bar\nu + \frac{A}{16}\bar\nu^2\nu^3\} + \omega\{-i\gamma\tau_\nu - i\bar\gamma\tau_{\bar\nu}\}$$
$$+ \omega^2\{-\frac{1}{4}\tau\tau_\nu + \frac{1}{4}\bar\nu + \frac{3}{16}A\nu\bar\nu^2\} + \omega\bar\omega\{\frac{1}{2}\bar\tau\tau_{\bar\nu} + \frac{\nu}{2} + \frac{3}{8}A\nu^2\bar\nu\}$$
$$+ \bar\omega\{\frac{i}{2}\bar\tau\gamma_{\bar\nu}\} + \bar\omega\omega^2\{\frac{3}{8}A\nu\bar\nu\} + \bar\omega^2\omega^2\{\frac{3}{16}A\nu\} + \bar\omega^2\{\frac{A}{16}\nu^3\} = 0 \qquad (5.46)$$

Comparing the coefficients of equal powers and combinations of ω and $\bar{\omega}$ on both sides of (5.46), yields:

$$
\begin{aligned}
& -\frac{i}{2}\tau\gamma + \gamma\gamma_{\nu} + \bar{\gamma}\gamma_{\bar{\nu}} + \frac{i}{2}\eta = -\frac{1}{4}\nu^{2}\bar{\nu} - \frac{A}{16}\nu^{3}\bar{\nu}^{2} && \text{(a)} \\
& \gamma\tau_{\nu} + \bar{\gamma}\tau_{\bar{\nu}} = 0 && \text{(b)} \\
& \tau\tau_{\nu} = \bar{\nu} + \frac{3}{4}A\nu\bar{\nu}^{2} && \text{(c)} \\
& \bar{\tau}\tau_{\bar{\nu}} = -\nu - \frac{3}{4}A\nu^{2}\bar{\nu} && \text{(d)} \\
& \bar{\tau}\gamma_{\bar{\nu}} = 0 && \text{(e)} \\
& A\nu\bar{\nu} = 0 && \text{(f)} \\
& A\nu = 0 && \text{(g)} \\
& A\nu^{3} = 0 && \text{(h)}
\end{aligned}
\qquad (5.47)
$$

The motivation here is to solve equations (5.47) and determine γ, η, τ in terms of ν and $\bar{\nu}$. Then we can find χ from (5.45) and the Bäcklund transformation (5.36) would be developed to its final form. But look at (5.47f), (5.47g), and (5.47h). These equations imply that

(i) for non zero ν and $\bar{\nu}$

$$A = 0$$

This is equivalent to switching off fifth order nonlinear term in the NLCQS equation and hence reducing it to the NLS equation. The Bäcklund transformation for the NLS equation can then be obtained as outlined below.

For $A = 0$, (5.47) becomes

$$-\frac{i}{2}\tau\gamma + \gamma\gamma_{\nu} + \bar{\gamma}\gamma_{\bar{\nu}} + \frac{i}{2}\eta = -\frac{1}{4}\nu^{2}\bar{\nu} \qquad \text{(a)}$$

$$\gamma\tau_{\nu} + \bar{\gamma}\tau_{\bar{\nu}} = 0 \qquad \text{(b)}$$

$$\tau\tau_{\nu} = \bar{\nu} \qquad \text{(c)} \qquad (5.48)$$

$$\bar{\tau}\tau_{\bar{\nu}} = -\nu \qquad \text{(d)}$$

$$\bar{\tau}\gamma_{\bar{\nu}} = 0 \qquad \text{(e)}$$

(5.48c) and (5.48d) are satisfied if

$$\tau = i(b - 2\nu\bar{\nu})^{1/2} \qquad (5.49)$$

where b is a real constant.

Equation (5.48b) is satisfied by

$$\gamma = ik\nu \qquad (5.50)$$

where k is real constant. Thus (5.48a) yields

$$\eta = i\nu\left\{\frac{1}{2}\,|\nu|^{2} + k\tau - 2k^{2}\right\} \qquad (5.51)$$

Making use of (5.51) and (5.44), we obtain from (5.40e)

$$\chi = -k\eta + \frac{1}{2}\tau\eta + \frac{1}{4} i\nu(|\omega|^2 + |\nu|^2) \quad (5.52)$$

Thus the Bäcklund transformation (5.36) now takes the form

$$p = p' - \frac{i}{2}\omega\tau + ik\nu$$

$$g = g' + \frac{1}{2}\tau(p+p') - k\eta + \frac{1}{4} i\nu(|\omega|^2 + |\nu|^2) \quad (5.53)$$

This is the Bäcklund transformation for the NLSE.

(ii) for non-zero A, $\nu = 0$ so that

$$z = z'$$

or/and $\bar{\nu} = 0$

so that $\bar{z} = \bar{z}'$

This last case means that the solution of the NLCQSE (ie. $A \neq 0$) is transformed into itself. We can't generate another solution. The purpose in deriving the Bäcklund transformation was to obtain the multisoliton solutions and that purpose seems to be defeated here. We say that the Bäcklund transformation breaks down or there is no Bäcklund transformation for the NLCQSE at least in the existing framework that we are using. In the calculation presented here we have taken $a = 1$, but we have carried out calculations for other values of a. In all such attempts, we were unable to find the other solution.

5.3 Conclusion

If there is no Bäcklund transformation for the NLCQSE we can expect that the equation has no soliton (or multisoliton) solutions. But, one can argue that there may be some other method (i.e. a different value of constant a could be chosen or a method entirely different than the one due to Clairin) to derive the Bäcklund transformation for our equation. Thus we can speculate, but we can't deduce here that the NLCQSE has no soliton solution. In the next chapter we shall try the other technique i.e. the ISTM to look for soliton solutions.

CHAPTER 6

Search for Multi-Soliton Solutions

The Inverse Scattering Transform Method

In chapter 5, we made an attempt to derive the multi-soliton solutions to the NLCQSE using the Bäcklund transformation. An alternative approach is to use the inverse scattering transform method (ISTM). In section 6.1, we give a brief introduction to the ISTM technique. In section 6.2, we make an attempt to construct an inverse scattering framework for the NLCQSE. The conclusion is given in section 6.3.

6.1 Introduction

The inverse scattering method is very important in the sense that it allows us to use linear techniques to solve certain nonlinear evolution equations and to discover multi-soliton solutions. The method was developed by Gardner et al. [6] in 1967 and was used to solve the KdV equation. A general formulation of the method by P.D. Lax [35] soon followed (1968) and this is what we briefly outline here.

Consider a general nonlinear equation

$$\phi_t = K(\phi) \tag{6.1}$$

where K is a nonlinear operator on some suitable space of functions. Suppose that we can find linear operators L and B' which depend upon ϕ, a solution of (6.1), and satisfy the following operator equation

$$iL_t = [B',L] \equiv B'L - LB' \tag{6.2}$$

If B' is self adjoint, equation (6.2) implies that the eigenvalues ζ of L which appear in

$$L\phi = \zeta\phi \tag{6.3}$$

are independent of time. Also it follows from (6.2) that the eigenfunction ϕ evolves in time according to the equation

$$i\phi_t = B'\phi \tag{6.4}$$

Assume, further, that we can associate a scattering problem with the operator L. Then given the initial shape $\phi(x,o)$, we can find $\phi(x,t)$ by carrying out the following linear steps.

(1) **The Direct Problem**

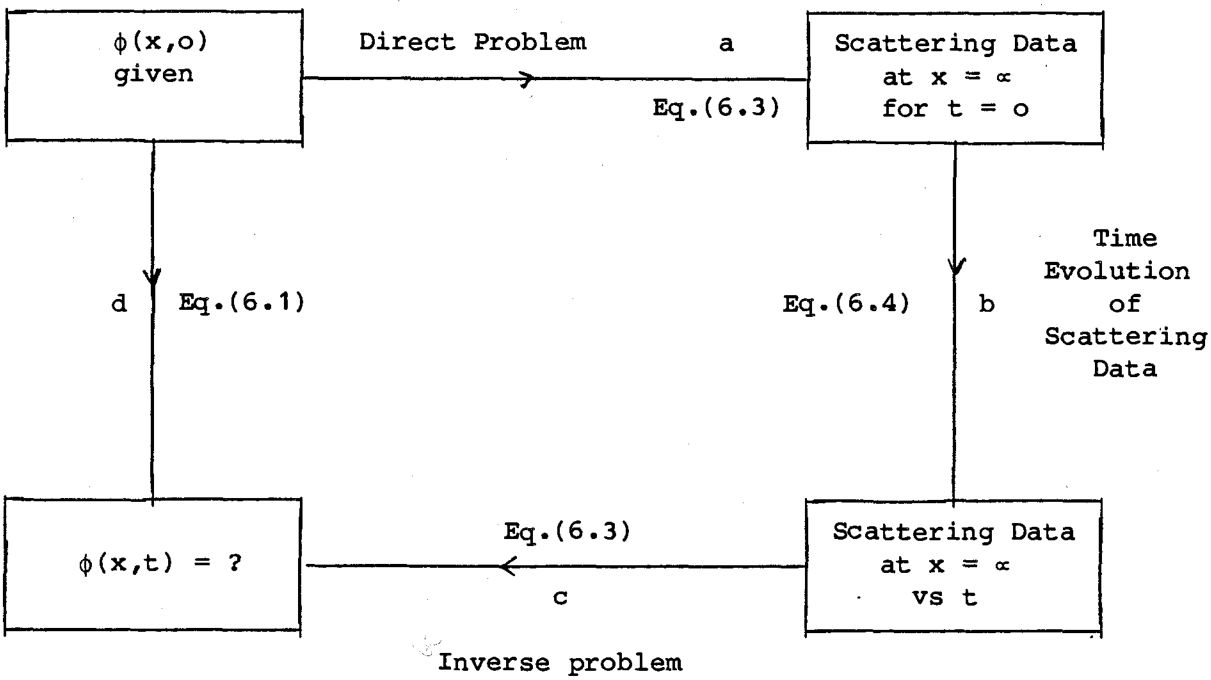
Using equation (6.3) we calculate scattering parameters (e.g. reflection and transmission coefficients of L) for ϕ at $x = \infty$ and $t = o$ from a knowledge of $\phi(x,o)$.

(2) **Time Evolution of the Scattering Data**

We use equation (6.4) together with the asymptotic form of B' at $x = \infty$ to calculate the time evolution of the scattering data.

(3) **The Inverse Problem**

From the knowledge of the scattering data of L as a function of time, we can construct $\phi(x,t)$. The following figure summarizes the inverse scattering method. The idea is to avoid path d i.e. to avoid solving equation (6.1) directly. Instead we solve equation (6.1) by going through linear computations of steps a,b and c.



There are many potential technical difficulties with the procedure outlined above, eg. we may not be able to find operators L and B' which satisfy equation (6.2), we may not be able to solve the inverse problem for the operator L etc.

6.2 Application to the NLCQSE.

In order to carry out the first step, we should be able to write the appropriate eigenvalue problem i.e. equation (6.3). So far, no general method has been developed to derive the eigenvalue problem corresponding to a given evolution equation. It's well known that certain evolution equations, eg. KdV, Sine Gordon and NLS equations belong to a class of equations that correspond to the Zakharov-Shabat [24] and other eigenvalue problems which are special cases of a more general eigenvalue problem due to Ablowitz, Kaup, Newell and Segur (AKNS) [33] viz:

$$v_{1x} + i\zeta v_1 = q(x,t)v_2$$
$$v_{2x} - i\zeta v_2 = r(x,t)v_1 \tag{6.5}$$

This can be rewritten in the form (6.3) as

$$\begin{bmatrix} i\frac{d}{dx} & -iq \\ ir & -i\frac{d}{dx} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \zeta \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{6.5a}$$

where v_1 and v_2 are the components of the eigenfunction $\psi = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $q(x,t)$, $r(x,t)$ are the solutions of a coupled pair of nonlinear evolution equations. Now, firstly, the NLCQSE is an extension of the NLS equation in the sense that if we switch off the fifth order nonlinear term the equation reduces to the NLS equation and secondly, the AKNS eigenvalue problem is very general. Thus it's reasonable to expect that if the NLCQSE does have soliton solutions, it possibly can be associated with the AKNS eigenvalue problem. Under this assumption our first step becomes to check the possibility that the NLCQSE corresponds to the AKNS eigenvalue problem (6.5). We assume that $q(x,0)$ and $r(x,0)$ are given. In general, the eigenvalues and eigenfunctions of (6.5) will evolve in time as the potentials $q(x,t)$ and $r(x,t)$ evolve according to some evolution equation. We choose the time dependence of $v_1(x,t)$ and $v_2(x,t)$ as

$$v_{1t} = A(x,t,\zeta)v_1 + B(x,t,\zeta)v_2$$
$$v_{2t} = C(x,t,\zeta)v_1 + D(x,t,\zeta)v_2 \tag{6.6}$$

This is equivalent to (6.4), with $\phi = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $B' = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

But as pointed out in the previous section, we insist that the eigenvalue be independent of time. From cross differentiation of (6.5) and (6.6) we obtain

$$D = -A + d(t)$$

where $d(t)$ is an integration constant and we can set it equal to zero without any loss of generality. We thus obtain

$$
\begin{aligned}
&D = -A && \text{(a)} \\
&A_x = qC - rB && \text{(b)} \\
&B_x + 2i\zeta B = q_t - 2Aq && \text{(c)} \\
&C_x - 2i\zeta C = r_t + 2Ar && \text{(d)}
\end{aligned}
\qquad (6.7)
$$

Equations (6.7) can give us possible evolution equations for q and r. First we find A,B,C and D. These coefficients are functions of x,t and ζ. We expand them in powers of ζ and we will systematically calculate the expansion coefficients.

$$
\begin{aligned}
&A = \sum_{n=0}^{N} A^{(n)}\zeta^{(n)} && \text{(a)} \\
&B = \sum_{n=0}^{N} B^{(n)}\zeta^{n} && \text{(b)} \\
&C = \sum_{n=0}^{N} C^{(n)}\zeta^{n} && \text{(c)}
\end{aligned}
\qquad (6.8)
$$

We set $N = 4$

From (6.8) and (6.7) we obtain

$$A^{(4)} = a_4$$

$$A^{(3)} = a_3$$

$$A^{(2)} = \frac{a_4}{2} qr + a_2 \tag{6.9}$$

$$A^{(1)} = \frac{ia_4}{4}\left[rq_x - qr_x\right] + \frac{a_3}{2} qr + a_1$$

$$A^{(0)} = \frac{a_4}{8}\left[q_x r_x - rq_{xx} - qr_{xx}\right] + \frac{3}{8} a_4\left[q^2 r^2\right] + i\,\frac{a_3}{4}\left[rq_x - qr_x\right] + \frac{a_2}{2} qr + a_o$$

$$B^{(4)} = 0$$

$$B^{(3)} = iqa_4$$

$$B^{(2)} = iqa_3 - \frac{a_4}{2} q_x \tag{6.10}$$

$$B^{(1)} = \frac{a_4}{2} iq^2 r + ia_2 q - \frac{a_3}{2} q_x - i\,\frac{a_4}{4} q_{xx}$$

$$B^{(0)} = -\frac{3}{4} a_4 qrq_x + i\,\frac{a_3}{2} q^2 r - i\,\frac{a_3}{4} q_{xx} - \frac{a_2}{2} q_x + ia_1 q + \frac{a_4}{8} q_{xxx}$$

$$C^{(4)} = 0$$

$$C^{(3)} = ira_4$$

$$C^{(2)} = \frac{a_4}{2} r_x + ira_3 \tag{6.11}$$

$$C^{(1)} = -\frac{ia_4}{4} r_{xx} + \frac{a_3}{2} r_x + \frac{ia_4}{2} qr^2 + ia_2 r$$

$$C^{(0)} = -\frac{a_4}{8} r_{xxx} + \frac{3}{4} a_4 qrr_x - i\frac{a_3}{4} r_{xx} + \frac{ia_3}{2} qr^2 + \frac{a_2}{2} r_x + ira_1$$

where a_0, a_1, a_2, a_3 and a_4 are independent of x but may depend upon t. Along with these equations also we obtain

$$q_t = 2A^{(0)} q + B_x^{(0)} \qquad \text{(a)}$$

$$\tag{6.12}$$

$$r_t = -2A^{(0)} r + C_x^{(0)} \qquad \text{(b)}$$

Making use of (6.9), (6.10) and (6.11), we rewrite (6.12a) as

$$q_t = a_4\{-\frac{1}{2} qq_x r_x - qrq_{xx} - \frac{1}{4} q^2 r_{xx} + \frac{3}{4} q^3 r^2 - \frac{3}{4} rq_x q_x + \frac{1}{8} q_{xxxx}\}$$

$$\tag{6.13}$$

$$+ ia_3\{\frac{3}{2} qrq_x - \frac{1}{4} q_{xxx}\} + a_2\{q^2 r - \frac{1}{2} q_{xx}\} + ia_1 q_x + 2a_0 q$$

Now, if we choose

$$a_0 = a_1 = a_3 = a_4 = 0, \quad a_2 = -i, \quad r = -q^*$$

the eigenvalue problem (6.5) reduces to for this special case,

$$v_{1x} + i\zeta v_1 = qv_2$$

$$v_{2x} - i\zeta v_2 = -q^* v_1 \qquad (6.13a)$$

which is the Zakharov-Shabat eigenvalue problem and the evolution equation (6.13) becomes

$$iq_t + \frac{1}{2} q_{xx} + |q|^2 q = 0 \qquad (6.14)$$

which is the NLS equation.

We are looking for the possibility of deriving the NLCQSE out of (6.13). In order to obtain the higher order nonlinear term we can't set $a_4 = 0$. Let's make the suitable choice $a_0 = a_1 = a_3 = 0$, $a_2 = -i$, $a_4 = \frac{i}{3}\delta$, $r = -2q^*$.

With this choice (6.13) becomes

$$iq_t + \frac{1}{2} q_{xx} + 2|q|^2 q + \delta|q|^4 q = -\frac{\delta}{3}\{q|q_x|^2 + 2|q|^2 q + \frac{1}{2} q^2 q^*_{xx} + \frac{3}{2} q^* q_x^2 + \frac{1}{8} q_{xxxx}\} \qquad (6.15)$$

We get all the terms we need on the L.H.S., but we also get terms on R.H.S. that our equation does not have. There is no apparent way that a proper choice for a_4 could lead to the NLCQSE. This shows that the NLCQSE may not belong to the AKNS eigenvalue problem.

6.3 Conclusion

We have shown that the NLCQE does not belong to the very general AKNS eigenvalue problem. It was plausible to expect that if the NLCQSE does have soliton solutions, it should correspond to the AKNS framework. The possibility that it belongs to a different eigenvalue problem appears to be remote. Thus we can conclude that, maybe, the NLCQSE does not possibly have soliton solutions. This is consistent with our failure to find a Bäcklund transformation. This speculation is supported by numerical simulation given in the next chapter. By the way, if equation (6.15) should correspond to some physical system, it would have soliton solutions.

CHAPTER 7

The Numerical Simulations

In previous chapters we attempted to analytically investigate the stability of the solitary wave solutions. The indication is that we don't have true solitons but the possibility of having quasi-soliton behaviour can't be ruled out. In this chapter, we study numerically the collisions of two solitary waves whose input shapes are calculated from the analytical expressions in Chapter 4. In section 7.1, we discuss the numerical scheme that was used. In section 7.2 we present our numerical results for the various combinations of signs of n_2 and n_4. The conclusions are given in section 7.3.

7.1 The Numerical Scheme

To solve the NLCQSE

$$iq_\xi + \frac{1}{2} q_{\tau\tau} + 2|q|^2 q + \delta |q|^4 q = 0 \tag{7.1}$$

we replace the derivatives q_ξ and $q_{\tau\tau}$ in (7.1) by finite difference approximations [1,36]. Consider the function $q(\xi,\tau)$. We write the Taylor expansion of q at $(\xi + \Delta\xi)$ and $(\xi - \Delta\xi)$ around ξ.

$$q(\xi + \Delta\xi,\tau) = q(\xi,\tau) + (\Delta\xi)q_\xi(\xi,\tau) + \frac{1}{2}(\Delta\xi)^2 q_{\xi\xi}(\xi,\tau) + O[(\Delta\xi)^3] \tag{7.2a}$$

$$q(\xi - \Delta\xi,\tau) = q(\xi,\tau) - (\Delta\xi)q_\xi(\xi,\tau) + \frac{1}{2}(\Delta\xi)^2 q_{\xi\xi}(\xi,\tau) + O[(\Delta\xi)^3] \tag{7.2b}$$

Subtract (7.2b) from (7.2a)

$$q(\xi + \Delta\xi,\tau) - q(\xi - \Delta\xi,\tau) = 2(\Delta\xi)q_\xi(\xi,\tau) + O[(\Delta\xi)^3)$$

or

$$\frac{q(\xi + \Delta\xi,\tau) - q(\xi - \Delta\xi,\tau)}{2\Delta\xi} = q_\xi + O(\Delta\xi)^2 \tag{7.2c}$$

which is the central difference approximation (CDA) to q_ξ. Note that the forward difference approximation (FDA), viz;

$$q\ (\xi,\tau) = \frac{q(\xi + \Delta\xi,\tau) - q(\xi,\tau)}{\Delta\xi} + O(\Delta\xi)$$

is not as accurate for a given $\Delta\xi$. Also it's found that numerical instability occurs if the FDA is used for the NLCQSE. For sufficiently small $\Delta\xi$, the higher order terms in (7.2c) may be neglected to obtain

$$q_\xi(\xi,\tau) = \frac{q(\xi + \Delta\xi,\tau) - q(\xi - \Delta\xi,\tau)}{2\Delta\xi} \tag{7.3}$$

In our numerical runs the value of $\Delta\xi$ used is ~ 0.003. Now to obtain $q_{\tau\tau}$, we write the Taylor expansion of q at $\tau + \Delta\tau$ and $\tau - \Delta\tau$ around τ viz;

$$q(\xi,\tau + \Delta\tau) = q(\xi,\tau) + (\Delta\tau)q_\tau(\xi,\tau) + \frac{(\Delta\tau)^2}{2} q_{\tau\tau}(\xi,\tau) + O\left((\Delta\tau)^3\right) \tag{7.4a}$$

$$q(\xi,\tau - \Delta\tau) = q(\xi,\tau) - (\Delta\tau)q_\tau(\xi,\tau) + \frac{(\Delta\tau)^2}{2} q_{\tau\tau}(\xi,\tau) + O\left((\Delta\tau)^3\right) \tag{7.4b}$$

Adding (7.4b) to (7.4a) yields

$$\frac{q(\xi,\tau + \Delta\tau) + q(\xi,\tau - \Delta\tau) - 2q(\xi,\tau)}{(\Delta\tau)^2} = q_{\tau\tau} + O\left((\Delta\tau)^2\right) \tag{7.4c}$$

Neglecting higher order terms for sufficiently small $(\Delta\tau)$ we have

$$q_{\tau\tau} = \frac{q(\xi,\tau + \Delta\tau) + q(\xi,\tau - \Delta\tau) - 2q(\xi,\tau)}{(\Delta\tau)^2} \tag{7.5}$$

We choose $\Delta\tau \sim 0.1$ in our simulations.

Substituting (7.5) and (7.3) into (7.1) we obtain

$$i\left\{\frac{q(\xi + \Delta\xi,\tau) - q(\xi,\tau)}{2\Delta\xi}\right\} + \frac{q(\xi,\tau + \Delta\tau) + q(\xi,\tau - \Delta\tau) - 2q(\xi,\tau)}{2(\Delta\tau)^2}$$

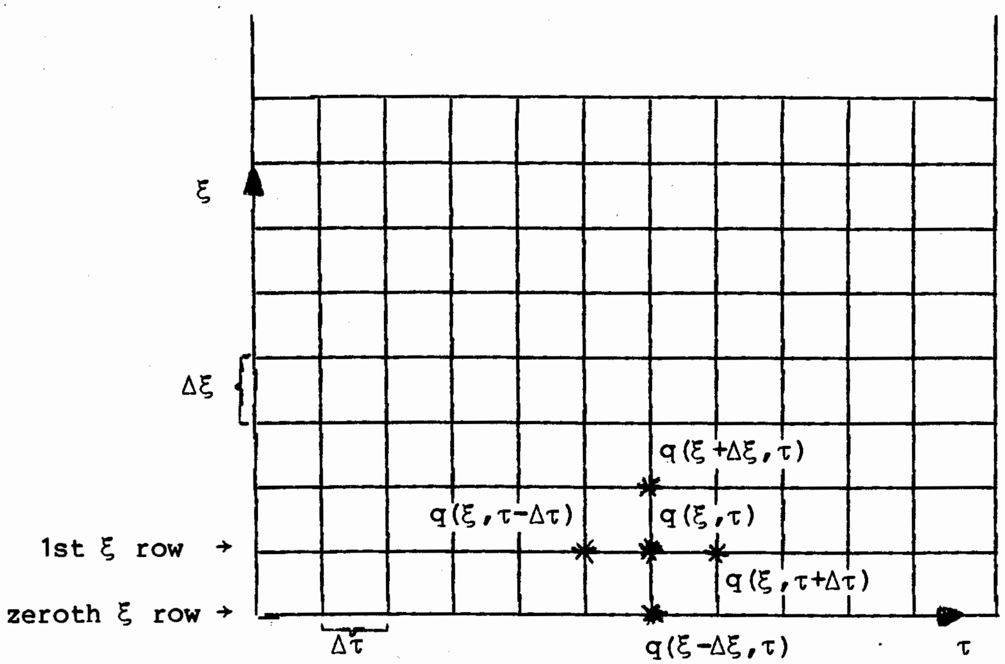
$$+ 2|q(\xi,\tau)|^2 q(\xi,\tau) + \delta|q(\xi,\tau)|^4 q(\xi,\tau) = 0$$

or

$$q(\xi + \Delta\xi,\tau) = q(\xi - \Delta\xi,\tau) + \frac{i\Delta\xi}{(\Delta\tau)^2}\{q(\xi,\tau + \Delta\tau) + q(\xi,\tau-\Delta\tau) - 2q(\xi,\tau)\}$$

$$+ 4i(\Delta\xi)|q(\xi,\tau)|^2 q(\xi,\tau) + 2i\delta(\Delta\xi)|q(\xi,\tau)|^4 q(\xi,\tau) = 0 \tag{7.6}$$

Knowing the terms on R.H.S. we can obtain the L.H.S. and advance in ξ. The terms in this finite-difference formulas are indicated schematically in the following figure. Note that we cannot use the CDA on the first ξ step. For the first step i.e. to go from zero'th ξ row to 1st ξ row $\Delta\xi$ was further subdivided into 600 steps and a forward



difference approximation was used. This approach yielded extremely accurate values of q on the first ξ row to use in the CDA. The FDA could not be used for larger ξ because it involved too much computing time and was found to be unstable. The program for this finite difference scheme was written by Stuart Cowan. Periodic boundary conditions were imposed by taking the extreme right and left mesh pts to be adjacent. Each plot shown involved of the order of 100 million floating point multiplications. The accuracy of the numerical runs was checked by continually monitoring the conserved quantities

$$I_1 = \int_{-\infty}^{+\infty} |q|^2 d\tau$$

$$I_3 = \int_{-\infty}^{+\infty} [|q_\tau|^2 - 2|q|^4 - \frac{2}{3}\delta|q|^6] d\tau$$

which were derived in Chapter 3.

7.2 Numerical Plots and Discussion

We now present our results for different possible cases depending upon the signs of n_2 and n_4. In discussing the results, we have labeled the pulse initially on the Left (Right) as L(R). Although a wide variety of relative pulse heights and velocities were considered, here the pulses are taken to be identical and (except for Fig.7.1a) the velocities w symmetric. Also, except for Fig.7.4c, each ξ step shown corresponds to $\Delta\xi = 1$. Fig.7.0 shows the accuracy of the numerical scheme including the periodic boundary conditions. The solitary wave disappears on the right edge and reappears on the left due to the periodic boundary conditions. Below, we discuss the results for the various combinations of n_2 and n_4.

(i) $n_2 > 0,\ n_4 < 0$

For this case, quasi-soliton behaviour was observed over the entire range of parameter space consistent with our derivation i.e. the solitary wave solutions are relatively stable. Some typical results are shown in Figs. 7.1 and 7.2. In Fig.7.1 we have taken $\delta = -1.3$ and $q_o = 1$ while in Fig.7.2, $\delta = -5$, $q_o = 0.5$ where q_o is the maximum input q. In terms of the conserved quantity I_1 of the pulses, $I_1 = 3$ in the first case and 1.4 in the second. Fig.7.1, therefore, involves the collision of more energetic pulses. A sizable radiative peak emerges between the two quasi-solitons, the peak shedding successive oscillations. As a result of the periodic boundary conditions, the quasi-solitons in this case are unphysically running back into the radiative oscillations and producing the noisy ripple. For the less energetic pulses of Fig.7.2, the radiation is less pronounced, appearing as an oscillatory plateau between the quasi-solitons which eventually dies away.

(ii) $n_2 > 0, \; n_4 > 0$.

In this case both the nonlinear terms contribute to pulse compression and a much richer spectrum of possible behaviour is found. Quasi-soliton behaviour is, of course expected for $R \ll 1$ where R is the ratio of the fifth order to third order contributions i.e. $R = |\delta||q|^2/2$. Quasi-soliton behaviour is also found to persist for larger R values, eg. Fig.7.3 with $R \simeq 0.26$ for $|q| = q_o$ where only a small radiative oscillation is quickly shed during the interaction period. But for $R \simeq 1$ small changes in parameters q_o, δ or the initial velocities can lead to wildly different scenarios as, eg. illustrated in Fig. 7.4a,b,c. Only q_o has been varied, taking on the values 0.27, 0.31 and 0.35 with the corresponding R values 1.13, 1.48 and 1.89. In Fig.7.4a, the two solitary waves simply flatten out after the interaction. In Fig.7.4b they also flatten out but a sharp spike forms in the middle. In Fig.7.4c, the pulses barely meet before a rapid transition to explosive behaviour is observed. Our numerical results show that for $R \sim 1$, the behaviour is not quasi-soliton, the stability of the solitary waves being very weak.

(iii) $n_2 < 0, \; n_4 > 0$

As $n_4 \to 0$ (i.e. $\delta \to 0$) we find that larger and larger electric field amplitudes are required to sustain the solitary wave as δ decreases. For $\delta \ll 1$, the solitary wave solution is very sharply peaked and physically either entirely unrealizable (requiring maximum electric field amplitudes beyond dielectric breakdown) or at least highly unstable. For larger values of δ, the solitary wave solutions are less sharply peaked, but are subject to an unstable fifth order compression effect. No quasi-soliton behaviour was observed for this case. Typically, the two colliding solitary waves produced only radiative or explosive behaviour.The stability in this case is extremely weak. As in the previous case, for $R \simeq 1$, the results were extremely sensitive to the input velocities w etc. In Fig.7.5 and 7.6, $R \simeq 2.5$ for $|q| = q_o$ but the initial size of the nonlinear terms is about 7 times larger in Fig.7.5 than in Fig.7.6. Explosive behaviour is observed in Fig.7.6, while in Fig.7.5 only chaotic radiation is produced. This is because the velocities w were larger in Fig.7.5 so that the pulses spent less time interacting with each other and hence prevented an explosion. If the velocities w are decreased, an explosion occurs.

7.3 Conclusion

The numerical simulations show that the solitary wave solutions that we derived in Chapter 4, are not solitons and hence our tentative conclusion reached in the previous chapters is supported. However quasi-soliton behaviour for $n_2 > 0$, $n_4 > 0$ and $n_2 > 0$, $n_4 < 0$ cases persist over wide regions of parameter space. But for the $n_2 > 0$, $n_4 > 0$ case numerical results show that for $R \sim 1$, the behaviour is not even quasi-soliton, the stability of the solitary waves being very weak. Thus, on theoretical grounds, we can rule out the possibility of obtaining substantially narrower solitons by finding a material with a large δ value and/or a large dielectric strength.

Fig 7.0) Propagation of a solitary wave. The input parameters are: $C = 0.5$, $\delta = 10.00$, $n_2 > 0$. The solitary wave disappears on the right edge and reappears on the left due to the periodic boundary conditions. Max. $|\Delta I_3/I_3| = 0.21\%$.

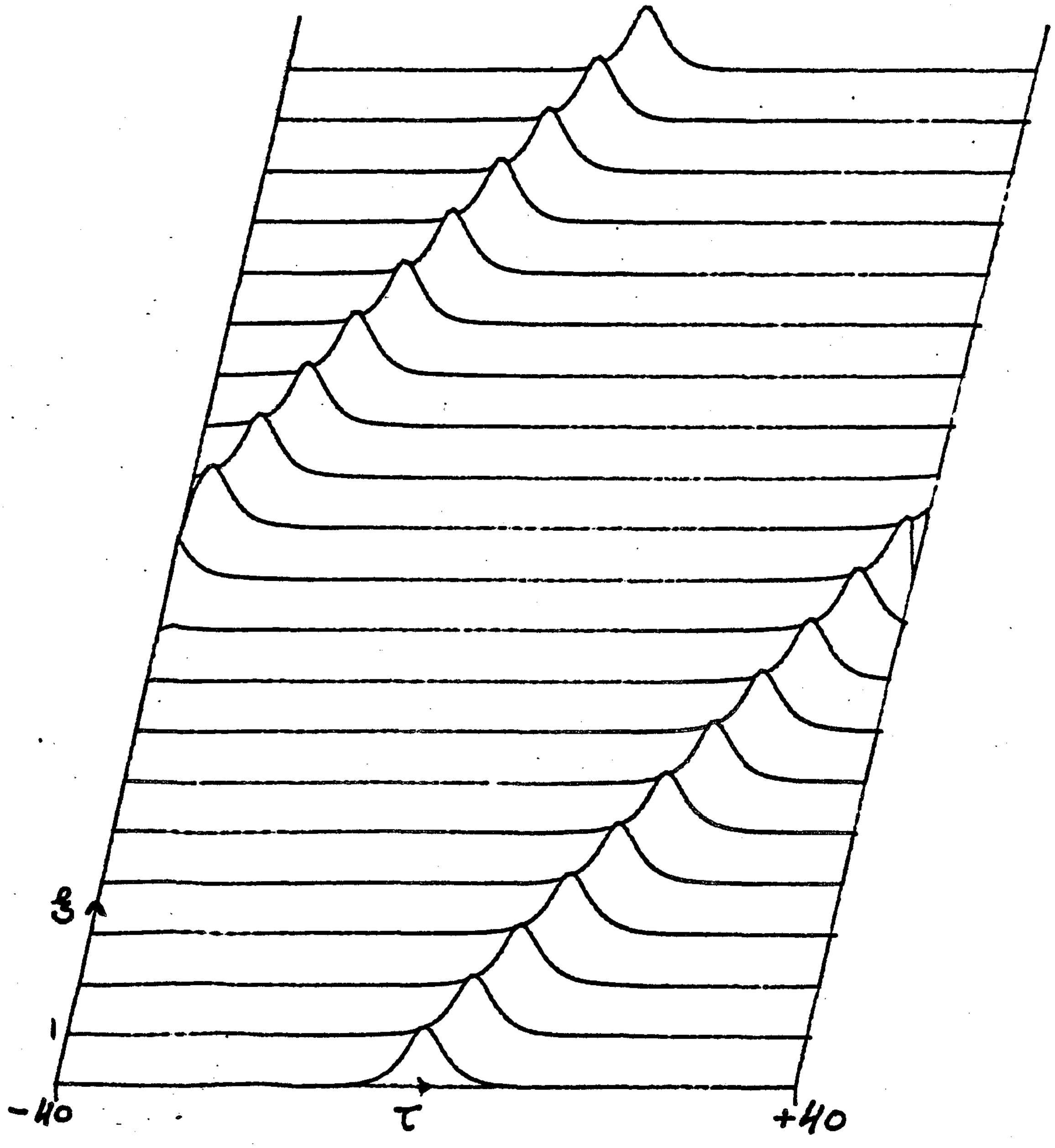


Fig 7

Fig. 7.1.a) Quasi-soliton behaviour for $n_2 > 0$, $n_4 < 0$. Input parameters are $C = 1.07$, $\delta = -1.3$ for each pulse and $w_L = +3.2$, $w_R = 0$. Max. $|\Delta I_3/I_3| = 0.25\%$. The arrow indicates that the quasi-soliton disappears on the right edge and reappears on the left due to the periodic boundary conditions.

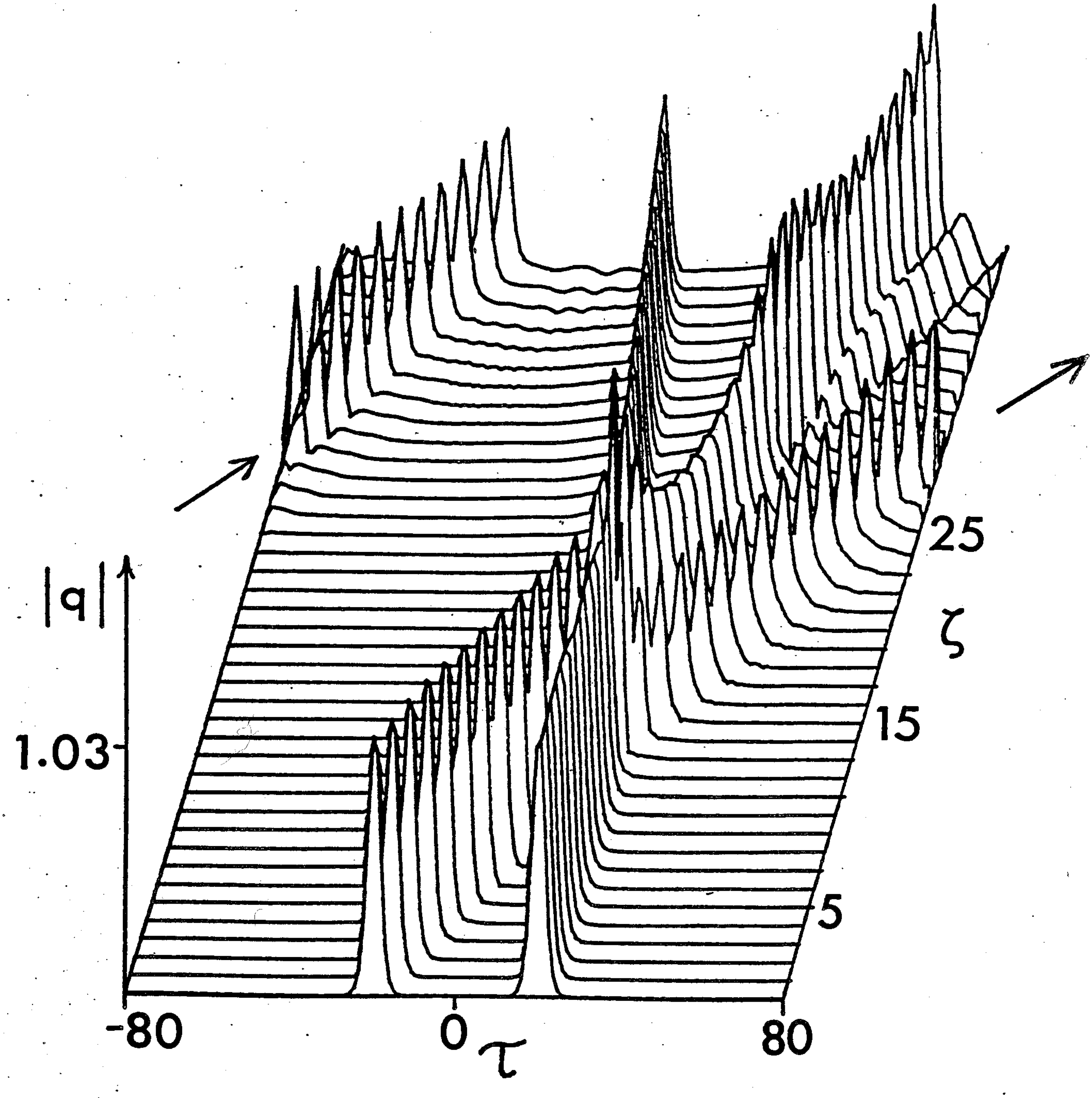


Fig 7.1a

Fig. 7.1.b) Quasi-soliton behaviour for $n_2 > 0$, $n_4 < 0$. Input parameters: $C = 1.07$, $\delta = -1.3$, $w_L = +1.6$, $w_R = -1.6$ Max. $|\Delta I_3/I_3| = 0.60\%$. The arrows indicate that the quasi-solitons reappear on the opposite edges due to the periodic boundary conditions.

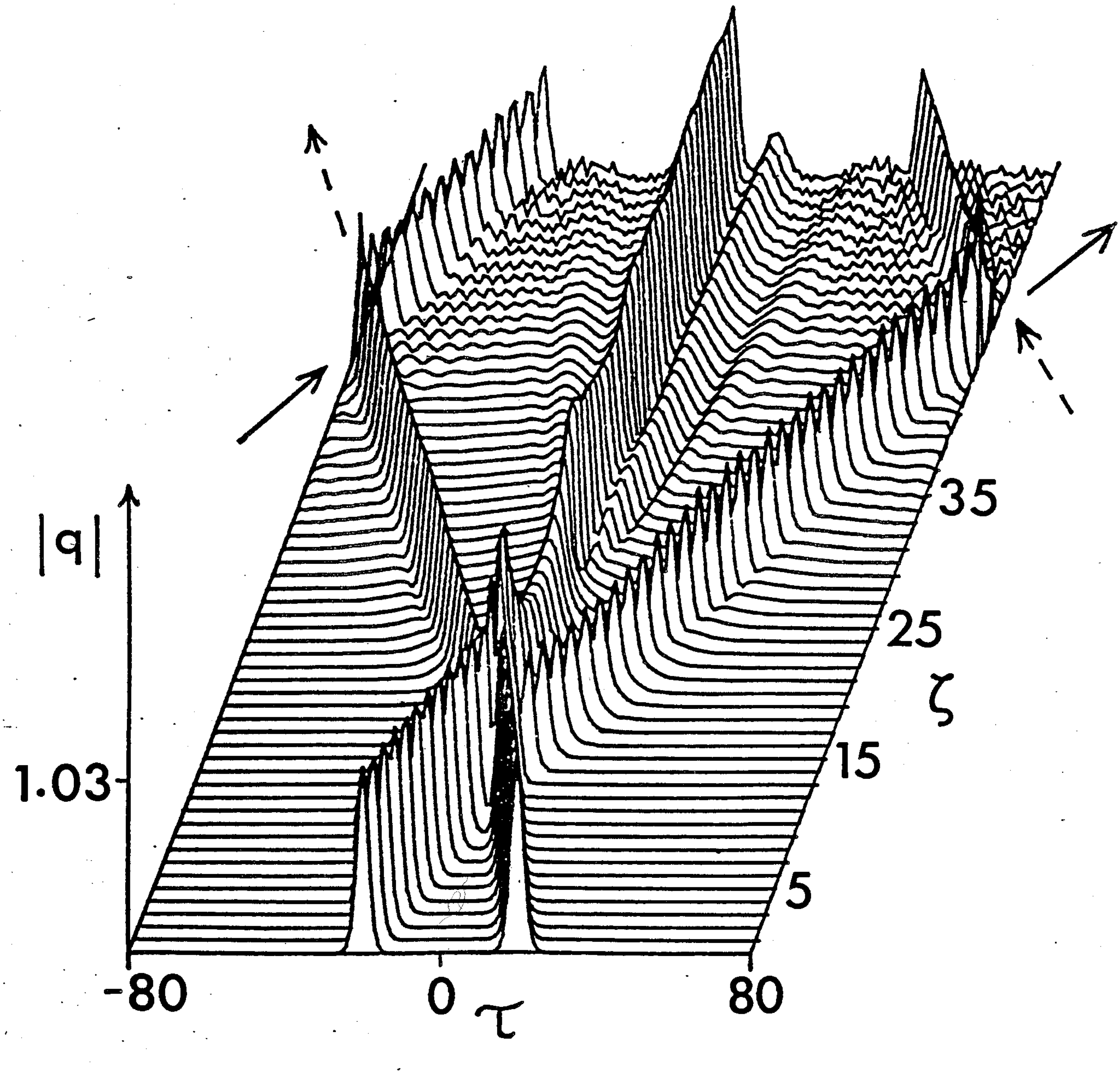


Fig. 7.1 b

Fig. 7.2) Quasi-soliton behaviour for $n_2 > 0$, $n_4 < 0$. Input parameters: $C = 0.54$, $\delta = -5.0$, $w_L = +1.6$, $w_R = -1.6$. Max. $|\Delta I_3/I_3| = 0.20\%$.

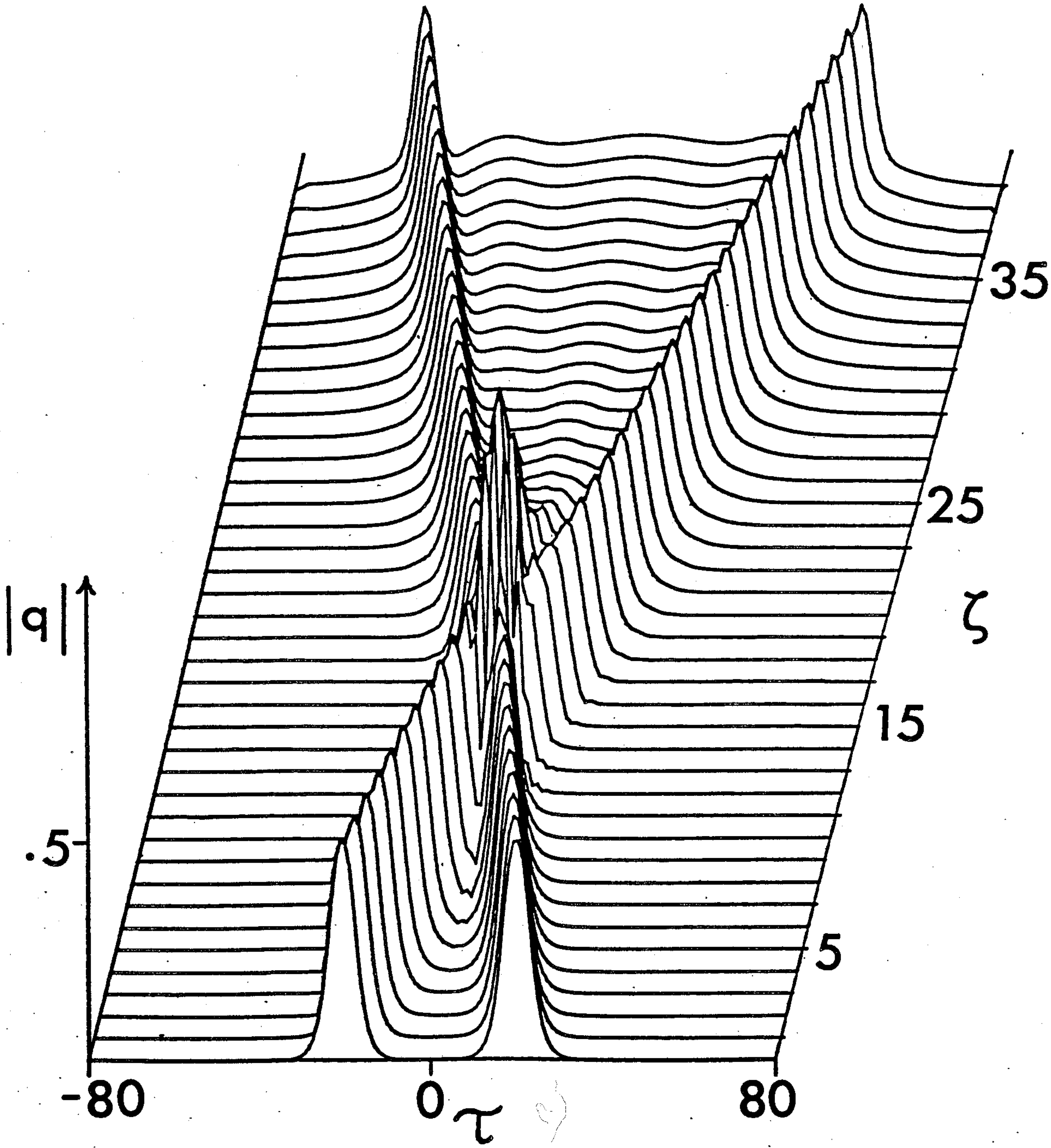


Fig. 7.2

Fig 7.3) Quasi-soliton behaviour for $n_2 > 0$, $n_4 > 0$. Input parameters: $C = 0.65$, $\delta = +3.0$, $w_L = +1.6$, $w_R = -1.6$. Max. $|\Delta I_3/I_3| = 1.22\%$.

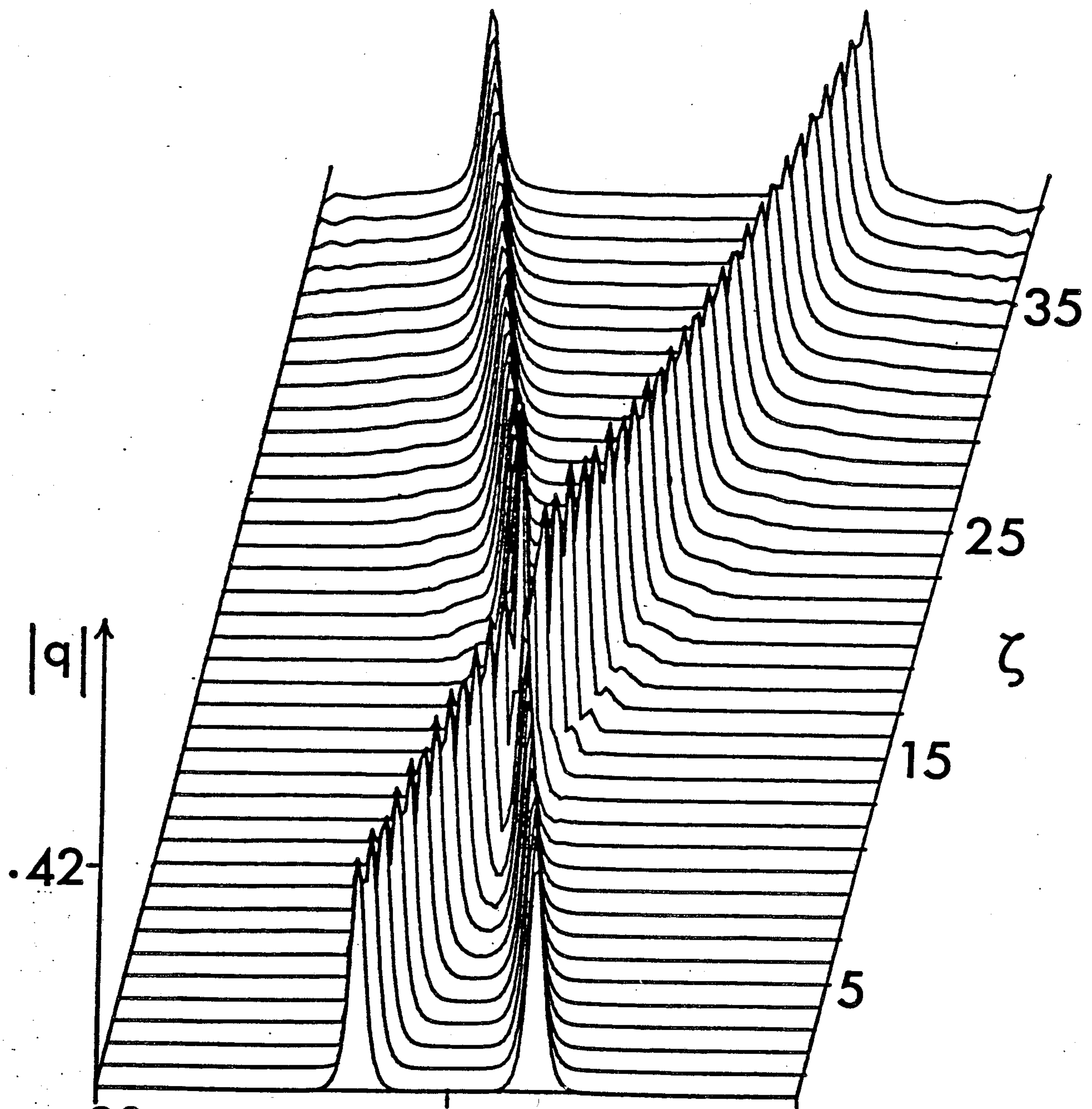


Fig.7.3

Fig 7.4.a) Radiative (dispersive) behaviour for $n_2 > 0$, $n_4 > 0$.

Input parameters: $C = 0.50$, $\delta = 30.9$, $w_L = +1.6$, $w_R = -1.6$.

Max. $|\Delta I_3/I_3| = 1.40\%$.

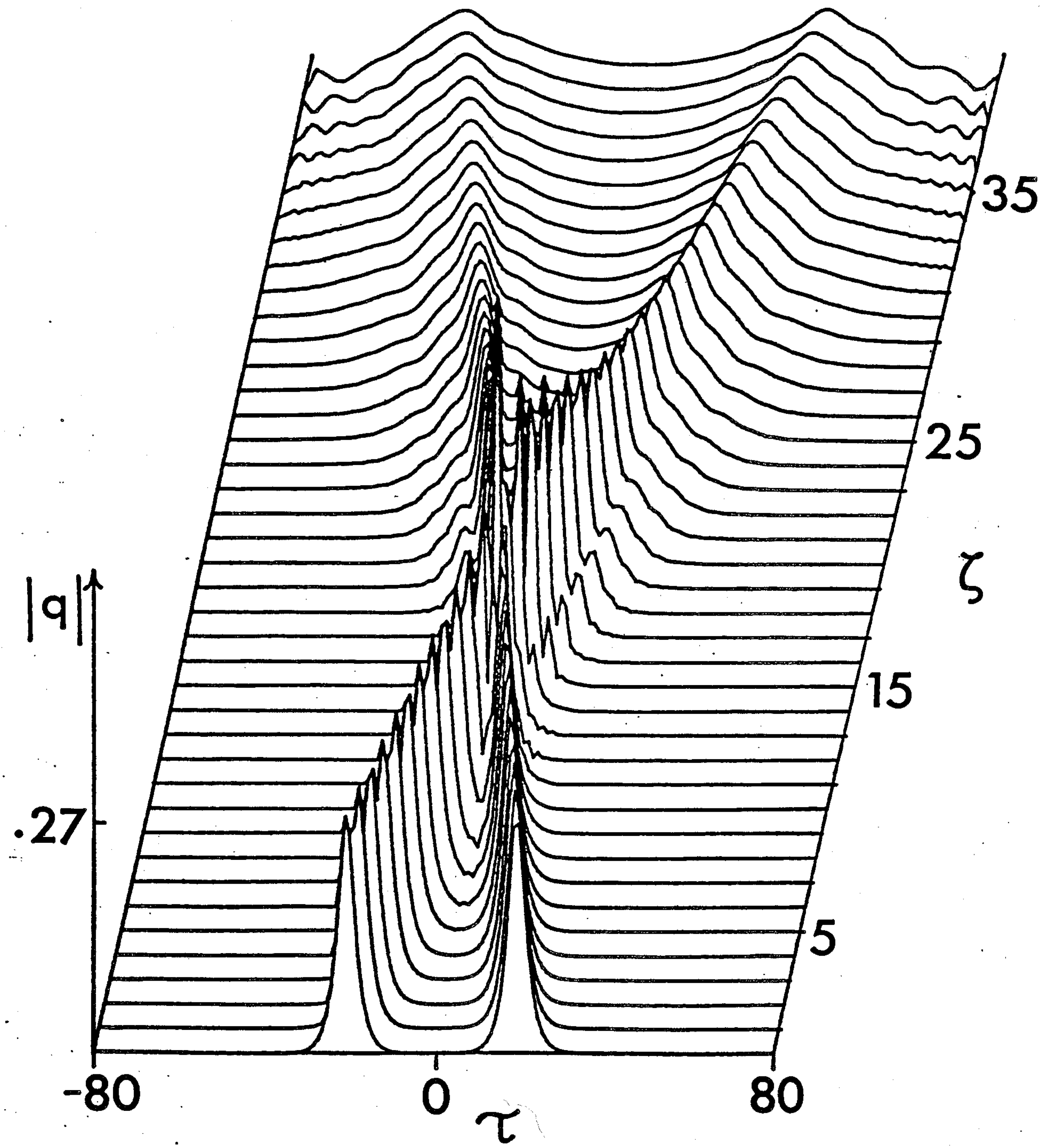


Fig. 7.4 a

Fig 7.4.b) Radiative and spiking behaviour for $n_2 > 0$, $n_4 > 0$. Input parameters: $C = 0.61$, $\delta = 30.9$, $w_L = +1.6$, $w_R = -1.6$. Max. $|\Delta I_3/I_3| = 5.24\%$.

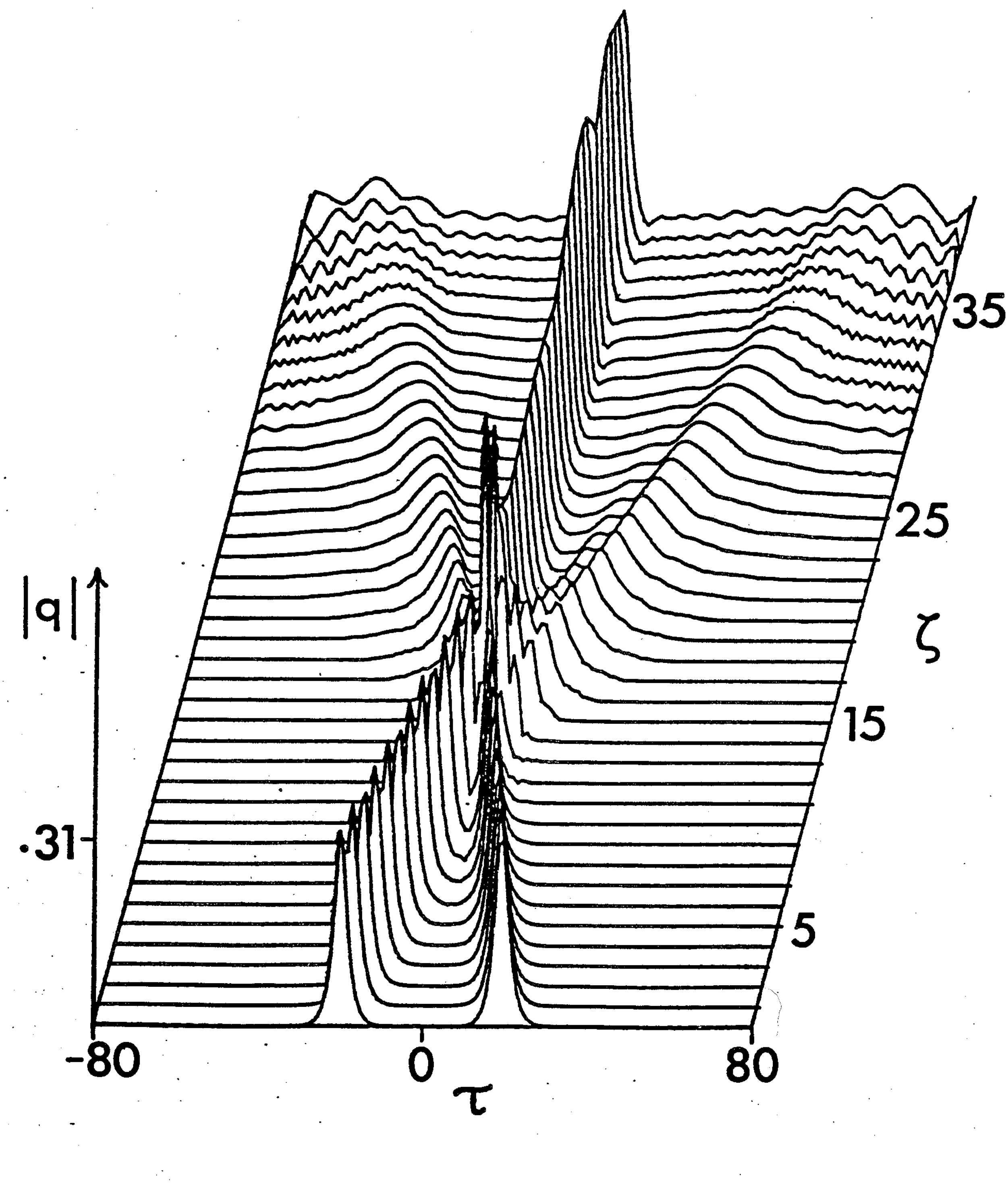


Fig.4 b

Fig 7.4.c) Explosive behaviour for $n_2 > 0$, $n_4 > 0$, Input parameters: $C = 0.75$, $\delta = 30.9$, $w_L = +1.6$, $w_R = -1.6$. Max. $\left|\Delta I_3 / I_3\right| = 0.79\%$.

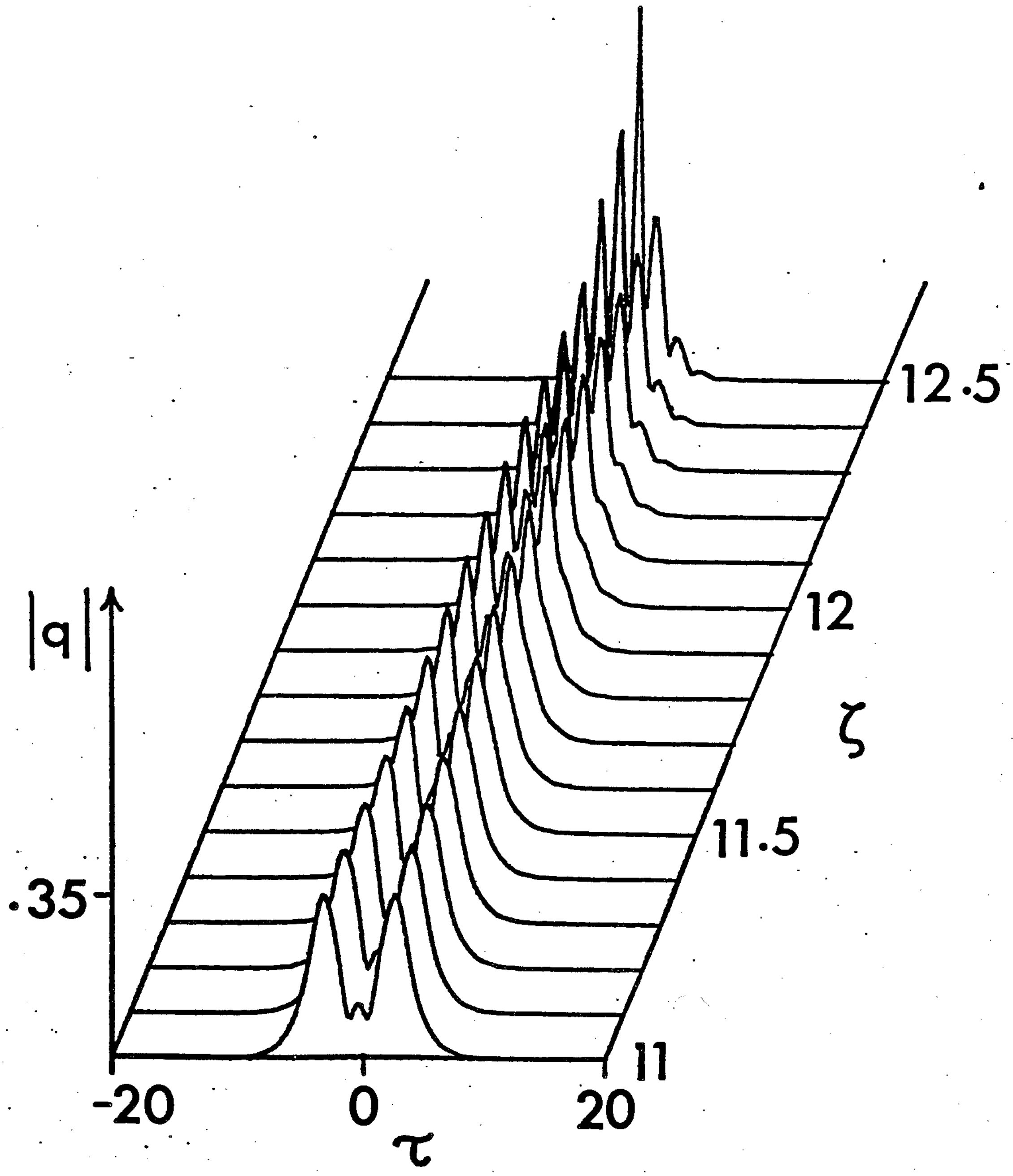


Fig.7.4 c

Fig 7.5) Radiative behaviour for $n_2 < 0$, $n_4 > 0$. Input parameters:
$C = 0.50$, $\delta = 30.9$, $w_L = +4.0$, $w_R = -4.0$
Max. $|\Delta I_3/I_3| = 0.42\%$

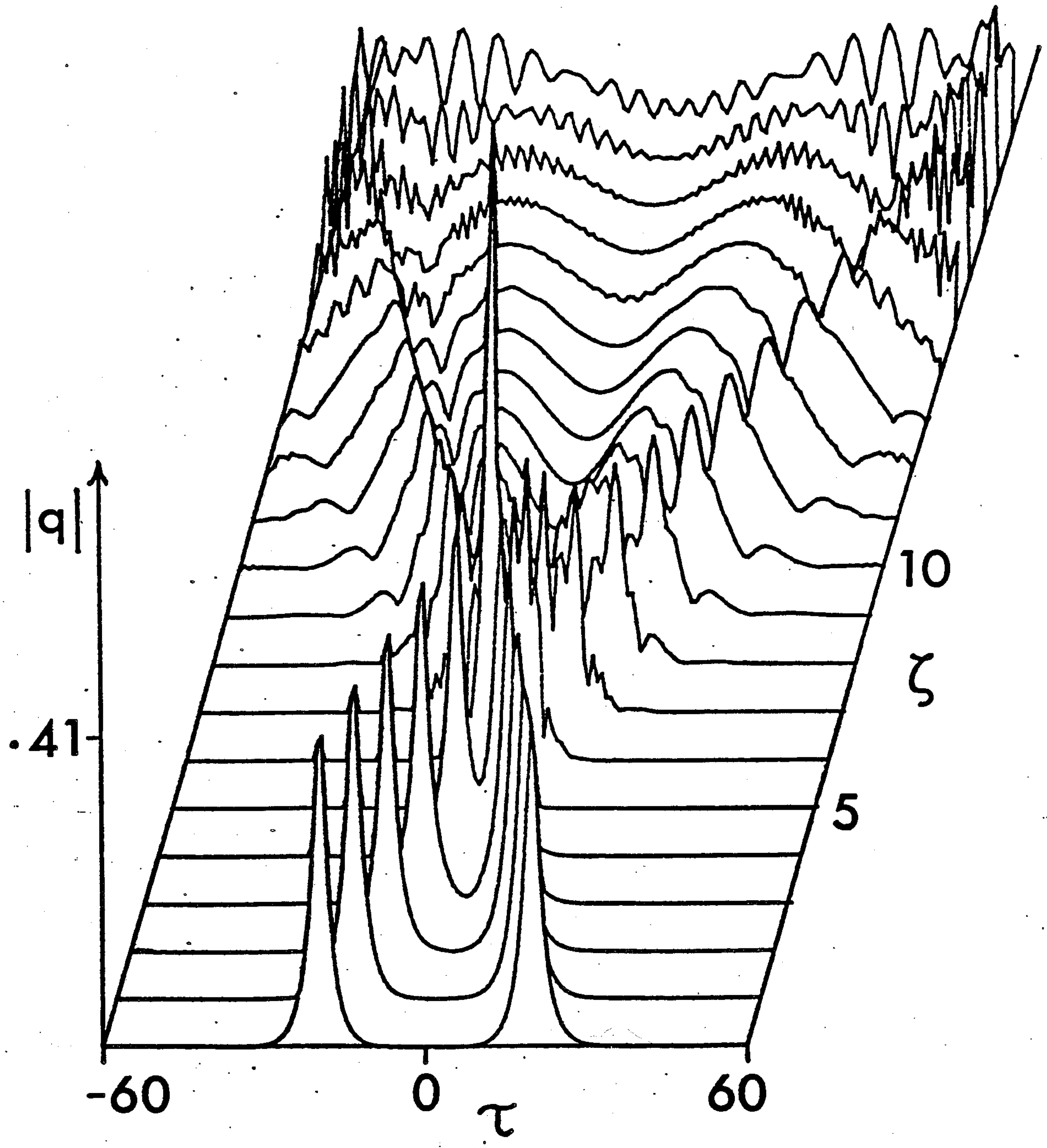


Fig. 7.5

Fig 7.6) Explosive behaviour for $n_2 < 0$, $n_4 > 0$. Input parameters $C = 0.25$, $\delta = 100.0$, $w_L = +1.6$, $w_R = -1.6$. Max. $|\Delta I_3 / I_3| = 0.51\%$.

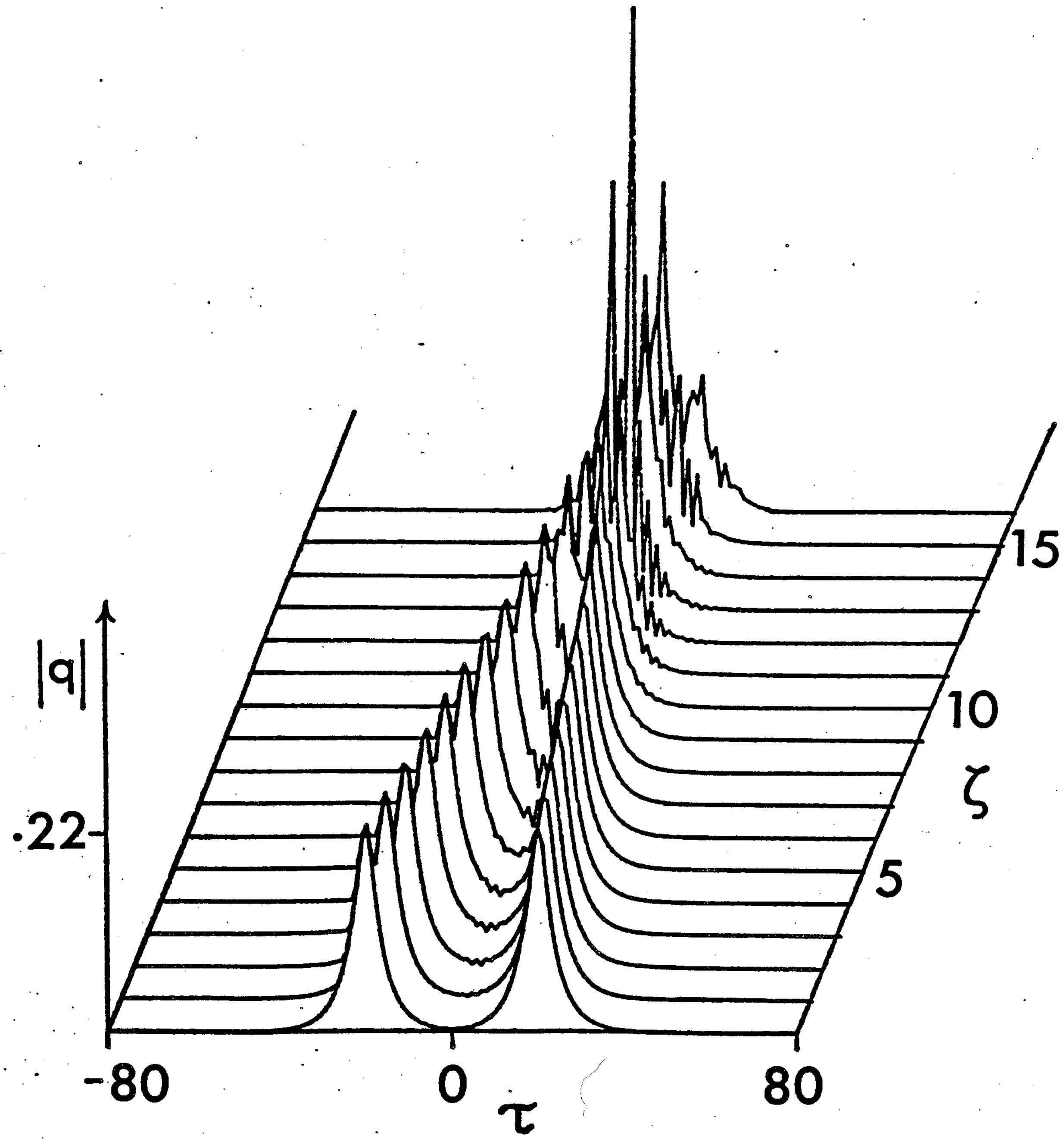


Fig.7.6

CHAPTER 8

Possible Comparison with Experiments

From an experimental viewpoint two central interrelated questions remain to be answered:

1) How big is n_4 and therefore δ?

2) Can one ever test the effect of including the n_4 contribution?

There are laser sources capable of generating sufficiently intense pulses, the only limitation being the dielectric break down limit of the medium i.e. the electric field in the medium should be below the dielectric strength. The dielectric strengths of the various media are given in Table 8.1. In section 8.1 we attempt to estimate the order of magnitude of n_4 for liquids and solids from the known value of n_4 in Rb vapour. In section 8.2, we show that the n_4 contribution is probably negligible in glass fibers. In section 8.3, we point out that it may be possible to test our results experimentally in Rb and possibly other vapours.

8.1 An Estimate of n_4 for Dielectric Solids and Liquids

The microscopic origin of n_2 and n_4 has been discussed by Grishkowsky et al. [37] and Lehmberg et al. [38]. However, to the best of our knowledge, no first-principle-estimate of n_4 has been given in the literature. The third order nonlinear effect is well known and well understood in the literature and the values of n_2 for solids, liquids and vapors are available and are given in table 8.2. Note that n_2 is of the same order of magnitude for all solids and liquids. However, because the study of the fifth order nonlinear effect has not been carried out in detail yet, the values of n_4 for various dielectric materials are not available. In the absence of any better method, we estimate here the value of n_4 for solids and liquids by connecting it to its microscopic origin i.e. the polarizability of the molecule. The molecular polarizability γ is related to the refractive index n by the Clausius-Mossotti equation, viz,

$$\gamma = \frac{3}{4\pi N} \left(\frac{n^2-1}{n^2+2}\right) \tag{8.1}$$

where N is the number of molecules per unit volume. (8.1) may be rewritten as

$$n^2(1-X\gamma) = 1 + 2X\gamma \tag{8.2}$$

where $X = \frac{4\pi N}{3}$.

The average dipole moment $\underline{P}$ of the molecules is approximately proportional to the electric field acting on the molecule, viz,

$$\underline{P} = \gamma\underline{E} \tag{8.3}$$

Thus the vector $\underline{P}$ is sensitive to the direction of the electric field. This forces us to write the molecular polarizability in its nonlinear form as, viz;

$$\gamma = \gamma_0 + \gamma_2|\underline{E}|^2 + \gamma_4|\underline{E}|^4 \tag{8.4}$$

The nonlinear refractive index in our model is assumed to be of the form

$$n = n_o + n_2|\underline{E}|^2 + n_4|\underline{E}|^4 \tag{1.5}$$

Substituting (8.4) and (1.5) into (8.2) and comparing equal powers of $|\underline{E}|$ on both sides, we obtain

$$\frac{\gamma_2}{\gamma_0} = \frac{6n_0 n_2}{(n_0^2+2)(n_0^2-1)} \tag{8.5}$$

and

$$\frac{\gamma_4}{\gamma_2} = \frac{n_4}{n_2}\left\{1+A\,\frac{n_2^2}{n_4}\right\} \tag{8.6}$$

with $A = \dfrac{2-3n_0^2}{2n_0(n_0^2+2)}$

Now for the case of Rb vapour we have information from the experiment done by Puell et al. [39], viz. $n_2 = 3.8 \times 10^{-31}N$ esu and $n_4 = -3.1 \times 10^{-40}N$ esu, where N is the number density of Rb atoms. The experiment was performed at $N \sim 10^8$ atoms/cm^3. Thus from (8.6) we obtain $|\gamma_4/\gamma_2| \sim 10^{-9}$esu. Since n_2 and n_0 are about the same for all liquids and solids, the relation (8.5) tells us that $|\gamma_2/\gamma_0|$ is of the same order of magnitude for all molecules. Knowing that γ is the microscopic property of the molecule, we extend our argument to assume that $|\gamma_4/\gamma_2|$ is of the same order of magnitude for all molecules. Hence $|\gamma_4/\gamma_2| \sim 10^{-9}$esu. Now, it is known that $n_2 \sim 10^{-13}$esu for all solids and liquids. Substituting the values of $|\gamma_4/\gamma_2|$ and n_2 in (8.6), we obtain $|n_4| \sim 10^{-22}$esu in solids and liquids.

8.2 Experiments on Glass Fibers

Experiments have been performed [8] at Bell Labs on glass fibers and solitons have been observed. The experiments were done with the power P of the order of 10W in a glass fiber with radius r of the order of a micrometer. Specifically, let us consider a typical set of values used, viz;

$$P = 11.4\text{W} \quad \text{and} \quad r = 4.66\mu\text{m}$$

Thus the corresponding intensity $I = \frac{P}{\pi r^2} = 1.67\times10^{11}\text{W/m}^2 = 1.67\times10^{14}$ esu. and the magnitude of the electric field

$$|\phi| = |\underline{E}| = \left[\frac{8\pi I}{c}\right]^{1/2} = 3.74\times10^2 \text{ Stat Volt/cm} \quad \text{or} \quad 1.12\times10^7\text{V/m}.$$

Thus from the definition

$$|q| = 10^{4.5}(\pi n_2)^{1/2}|\phi|$$

we obtain $|q| = 6.88$ using $n_2 = 1.2\times10^{-22}(\text{m/V})^2$ or 1.08×10^{-13}esu for the glass fiber. As argued in the previous section, $n_4 \sim 10^{-22}$esu so that $|\delta| \sim 10^{-5}$. Thus the ratio R of the fifth order nonlinear term to the third order term in the NLCQSE is

$$R = \frac{\delta|q|^4 q}{2|q|^2 q} = \frac{\delta}{2}|q|^2 \sim 10^{-4}$$

For such a small R value, the quasi-soliton behaviour predicted by our theory would be indistinguishable from the "true" soliton behaviour (ie.

when the fifth order term is completely neglected). To obtain a larger R value, the electric field $\underline{E}$ must be increased. From Table 8.1, the dielectric strength of glass is about 1.4×10^7V/m which corresponds to $|q| = 8.6$ and $R \sim 10^{-4}$. Thus even at the largest electric fields below the dielectric breakdown, the fifth order contribution is probably negligible in glass fibers. Of course, our estimate of n_4 was crude and also the dielectric strength of some materials may be higher than in a glass fiber, so we cannot absolutely rule out the possibility of observing the fifth order contribution in solids and liquids.

8.3 Experiments on Rb Vapour

For gases, n_2 and n_4 depend upon the number density N (atoms/cm^3). From section 8.1, for Rb vapour (at $\lambda = 1.06\mu$m)

$$\delta = \frac{2n_4}{\pi n_2^2} \times 10^{-9} = \frac{-1.37\times10^{12}}{N} \tag{8.7}$$

Puell et al. have actually carried out a self-focusing (as opposed to pulse compression considered here) experiment with $N = 10^8$atoms/cm^3 which corresponds to $\delta = -1.37\times10^4$. Since, δ is negative, the fifth order term is negative and causes defocusing while the third order term produces a focusing effect. The two competing effects roughly cancel when $\frac{|\delta||q|^2}{2} \sim 1$ i.e. the fifth order and third order nonlinear terms are of comparable magnitudes. This implies that $|q| = 1.21\times10^{-1}$ which corresponds to $|\underline{E}| = 3.5 \times 10^4$ Stat Volts/cm. and the intensity $I = 1.5\times10^{11}$W/cm^2. A cancellation at this intensity was indeed observed by Puell et al. [39].

It seems possible, in principle, to experimentally verify the quasi-soliton behaviour predicted by our theoretical results for $n_2 > 0$, $n_4 < 0$ even for large δ values, in Rb vapour. From (8.7) $\delta = -1$, eg., corresponds to $N \sim 10^{12}$ atoms/cm^3. Since it is possible to reach the sufficiently high value of δ in vapours, it seems possible to test some other theoretical results as well, in some suitable vapour.

Table 8.1

Dielectric constant and strength of various materials

Material	Dielectric Constant K	Dielectric Strength V/m
Air	1.00059	3×10^6
Bakelite	4.9	2.4×10^7
Glass (pyrex)	5.6	1.4×10^7
Mica	5.4	$(1-10)10^7$
Neoprene	6.9	1.2×10^7
Paper	3.7	1.6×10^7
Paraffin	2.1-2.5	1×10^7
Plexiglas	3.4	4×10^7
Polystyrene	2.55	2.4×10^7
Porcelain	7	5.7×10^7
Transformer oil	2.24	1.2×10^7
Water (20°C)	80	---
Fused quartz	3.8	8×10^6
Teflon	2.1	6×10^7
Amber	2.7	9×10^7

Ref: [41], [42]

Table 8.2

Values of n_2 for Various Materials

Material	$n_2 \times 10^{-13}$ esu ($\times \frac{10^{-8}}{9}$ MKS)
Fused quartz	1.2-1.4
Ruby	1.5
Lucite	2.7
NaCl	6.5
CCl_4	2.5
Toluene	45
Benzene	20-25
CS_2	110-200
Water	1.4
Air (1atm)	0.041
Air (100atm)	4.1
Glass (heavy silicate flint)	0.9
Calcite	0.8
Sapphire	0.2

Ref: [43] and [4]

CHAPTER 9

Conclusions

We have derived the nonlinear (cubic-quintic) Schrödinger equation that describes the dynamics of the propagation of intense electromagnetic pulses in a nonlinear dispersive medium characterized by a refractive index $n = n_0 + n_2|\underline{E}|^2 + n_4|\underline{E}|^4$ and have obtained solitary wave solutions for this equation for all possible signs of n_2 and n_4. To determine whether the solitary waves are solitons or not, two analytic approaches to obtaining multi-soliton solutions were investigated, viz. the Bäcklund transformation and the inverse scattering transform method. These approaches seemed to indicate that the solitary waves were not solitons. This speculation was found to be well supported by numerical simulations. However, quasi-soliton behaviour was found to persist over a wide region of parameter space. Other interesting behaviour was also observed in the numerical simulations. Some aspects of our theoretical results may be experimentally testable. We have ruled out on theoretical grounds the possibility of obtaining substantially narrower solitons (of relevance to the development of high bit rate transmission system) in eg. a glass fiber or in any other material which might have a large δ value and/or a large dielectric strength.

APPENDIX A

Alternate derivation of the NLCQSE

Consider a dielectric medium with a nonlinear dielectric constant given by

$$\varepsilon = \varepsilon_0 + \varepsilon_2 |\underline{E}|^2 + \varepsilon_4 |\underline{E}|^4 \tag{A.1}$$

where ε_o is the linear dielectric constant and ε_2 and ε_4 are higher order coefficients. The electric field in the medium can be taken to be of the same form as in chapter 2 i.e.

$$E(x,t) = \mathrm{Re}\left\{\phi(x,t) e^{i[kx-\omega t]}\right\} \tag{A.2}$$

or equivalently

$$\underline{E}(x,t) = \frac{\hat{e}}{2} \left\{\phi e^{i[kx-\omega t]} + \phi^* e^{-i[kx-\omega t]}\right\} \tag{A.3}$$

Maxwell's equations in the dielectric medium can be written as

$$\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial}{\partial t}\underline{B} \tag{A.4}$$

$$\nabla \cdot \underline{E} = 0 \tag{A.5}$$

$$\nabla \cdot \underline{B} = 0 \tag{A.6}$$

$$\nabla \times \underline{B} = \frac{1}{c}\frac{\partial \underline{D}}{\partial t} \tag{A.7}$$

In writing equation (A.5), the reasonable assumption $\varepsilon_2 \ll \varepsilon_o$ and $\varepsilon_4 \ll \varepsilon_0$ has been made.

From (A.4) and (A.7), we obtain

$$\nabla x(\nabla x \underline{E}) = -\frac{1}{c^2}\frac{\partial^2 \underline{D}}{\partial t^2} \tag{A.8}$$

In a medium characterized by (A.1) the electric displacement vector will be written as

$$\underline{D} = \{\varepsilon_0 + \varepsilon_2 |\underline{E}|^2 + \varepsilon_4 |\underline{E}|^4\}\underline{E} \tag{A.9}$$

Making use of (A.5) and (A.9), we obtain from (A.8)

$$\frac{\partial^2 \underline{E}}{\partial x^2} - \frac{\varepsilon_0}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} - \frac{\varepsilon_2}{c^2}\frac{\partial^2(|\underline{E}|^2\underline{E})}{\partial t^2} - \frac{\varepsilon_4}{c^2}\frac{\partial^2}{\partial t^2}(|\underline{E}|^4\underline{E}) = 0 \tag{A.10}$$

From (A.3) we obtain

$$|\underline{E}|^2\underline{E} = \hat{e}\,\frac{3}{4}|\phi|^2\left\{\frac{1}{2}\,\phi e^{i(kx-\omega t)} + \frac{1}{2}\phi^* e^{-i(kx-\omega t)}\right\}$$

$$+ \frac{\hat{e}}{8}\left\{\phi^{*3} e^{-3i(kx-\omega t)} + \phi^3 e^{3i(kx-\omega t)}\right\}$$

Now, neglecting the third harmonic terms we obtain

$$|\underline{E}|^2\underline{E} = \frac{3}{4}\,|\phi|^2\underline{E} \tag{A.11}$$

Similarly $$|\underline{E}|^4\underline{E} = \frac{5}{8}\,|\phi|^4\underline{E} \tag{A.12}$$

From (A.11) and (A.3), we obtain

$$\frac{\partial^2}{\partial t^2}(|\underline{E}|^2\underline{E}) = \hat{e}\,\frac{3}{4}\,e^{i\{kx-\omega t\}}\left[-\frac{\omega^2}{2}\,|\phi|^2\phi - 2i\omega|\phi|^2\phi_t + \phi^*\phi_t^2 - i\omega\phi^2\phi_t^* + |\phi|^2\phi_{tt}\right.$$

$$\left. + \frac{\phi^2}{2}\,\phi_{tt}^* + 2\phi|\phi_t|^2\right] + \text{c.c.} \qquad \text{(A.13)}$$

Now let us estimate the terms on R.H.S. in (A.13) for picosecond pulses eg. $\Delta t = 5.68 \times 10^{-12}$s for $\lambda = 1.5 \times 10^{-6}$m as considered in chapter 2. Also we make the crude approximation $\phi_t \sim \frac{\phi}{\Delta t}$. Noting that $|\phi|^2\phi$ is common in all terms, we estimate its coefficients, viz;

$$\omega^2 = (2\pi)^2\left(\frac{c}{\lambda}\right)^2 = 1.57 \times 10^{30}\,s^{-2}$$

$$\frac{\omega}{\Delta t} = \frac{2\pi c}{\lambda\Delta t} = 2.2 \times 10^{26}\,s^{-2}$$

$$\frac{1}{\Delta t^2} = 3.09 \times 10^{22}\,s^{-2}$$

Thus all terms on R.H.S. of (A.13) are negligible compared to the first one. Therefore (A.13) is reduced to

$$\frac{\partial^2}{\partial t^2}(|\underline{E}|^2\underline{E}) = -\,\hat{e}\,\frac{3}{8}\,\omega^2|\phi|^2\phi e^{i(kx-\omega t)} - \hat{e}\,\frac{3}{8}\,\omega^2|\phi|^2\phi^* e^{-i(kx-\omega t)} \qquad \text{(A.14)}$$

Similarly

$$\frac{\partial^2}{\partial t^2}(|\underline{E}|^4\underline{E}) = -\,\hat{e}\,\frac{5}{16}\,\omega^2|\phi|^4\phi e^{i(kx-\omega t)} - \hat{e}\,\frac{5}{16}\,\omega^2|\phi|^4\phi^* e^{-i(kx-\omega t)} \qquad \text{(A.15)}$$

From (A.3) we obtain

$$\frac{\partial^2 \underline{E}}{\partial x^2} = e^{i(kx-\omega t)}\hat{e}\left\{ik \frac{\partial \phi}{\partial x} - \frac{k^2}{2} \phi\right\} + \hat{e}e^{-i(kx-\omega t)}\left\{-ik \frac{\partial \phi^*}{\partial x} - \frac{k^2}{2} \phi^*\right\} \tag{A.16}$$

where we have neglected the $\frac{\partial^2 \phi}{\partial x^2}$ term by making the slowly varying envelope approximation. Note that for picosecond pulses we cannot neglect the second time derivative. Thus

$$\frac{\partial^2 \underline{E}}{\partial t^2} = e^{i(kx-\omega t)}\hat{e}\left\{\frac{1}{2} \frac{\partial^2 \phi}{\partial t^2} - i\omega \frac{\partial \phi}{\partial t} - \frac{\omega^2}{2} \phi\right\}$$

$$+ \hat{e}e^{-i(kx-\omega t)}\left\{\frac{1}{2} \frac{\partial^2 \phi}{\partial t^2} + i\omega \frac{\partial \phi^*}{\partial t} - \frac{\omega^2}{2} \phi^*\right\} \tag{A.17}$$

Now, substituting (A.17), (A.15), (A.14) and (A.16) into (A.10) and using

$\frac{\varepsilon_0}{c^2} = \frac{1}{v^2}$ and $\omega = kv$, we obtain the following

$$e^{i(kx-\omega t)}\left\{i\phi_x + \frac{i}{v} \phi_t - \frac{1}{2\omega v} \phi_{tt} + \frac{3}{8} \frac{\varepsilon_2 \omega^2}{kc^2} |\phi|^2 \phi + \frac{5}{16} \frac{\varepsilon_4 \omega^2}{kc^2} |\phi|^4 \phi\right\}$$

$$+ e^{-i(kx-\omega t)}\left\{-i\phi_x^* - \frac{i}{v} \phi_t^* - \frac{1}{2\omega v} \phi_{tt}^* + \frac{3}{8} \frac{\varepsilon_2 \omega^2}{kc^2} |\phi|^2 \phi^* + \frac{5}{16} \frac{\varepsilon_4 \omega^2}{kc^2} |\phi|^4 \phi^*\right\} = 0 \tag{A.18}$$

Comparing the coefficients of $e^{i(kx-\omega t)}$ and $e^{-i(kx-\omega t)}$ on both sides of (A.18) we obtain

$$i\phi_x + \frac{i}{v}\phi_t - \frac{1}{2\omega v}\phi_{tt} + \frac{3}{8}\frac{\varepsilon_2\omega^2}{kc^2}|\phi|^2\phi + \frac{5}{16}\frac{\varepsilon_4\omega^2}{kc^2}|\phi|^4\phi = 0 \qquad (A.19)$$

and

$$-i\phi^*_x - \frac{i}{v}\phi^*_t - \frac{1}{2\omega v}\phi^*_{tt} + \frac{3}{8}\frac{\varepsilon_2\omega^2}{kc^2}|\phi|^2\phi^* + \frac{5}{16}\frac{\varepsilon_4\omega^2}{kc^2}|\phi|^4\phi^* = 0 \qquad (A.20)$$

Now we move to the group velocity coordinate system defined by (unlike chapter 2 we will not bother normalizing the new coordinates)

$$\xi = x$$

$$\tau = t - \frac{x}{v} \qquad (A.21)$$

$$\phi(x,t) = q(\xi,\tau)$$

Thus (A.19) and (A.20) become

$$iq_\xi - \frac{1}{2\omega v}q_{\tau\tau} + \frac{3}{8}\frac{\varepsilon_2\omega^2}{kc^2}|q|^2q + \frac{5}{16}\frac{\varepsilon_4\omega^2}{kc^2}|\phi|^4\phi = 0 \qquad (A.22)$$

$$-iq^*_\xi - \frac{1}{2\omega v}q^*_{\tau\tau} + \frac{3}{8}\frac{\varepsilon_2\omega^2}{kc^2}|q|^2q^* + \frac{5}{16}\frac{\varepsilon_4\omega^2}{kc^2}|q|^4q^* = 0 \qquad (A.23)$$

These equations are equivalent to the NLCQSE. We do not obtain any higher order dispersion term in this derivation because we did not consider the frequency variation. We have also not included damping.

APPENDIX B

Solitary Wave Solutions of the "Higher" NLSE.

For $n_2 = 0$ and neglecting the higher dispersion terms i.e. $k''' = k'''' = 0$ as well as damping i.e. $\gamma = 0$, the dynamical equation (2.9) becomes

$$i\left[\frac{\partial\phi}{\partial x} + k'\,\frac{\partial\phi}{\partial t}\right] - \frac{1}{2}\,k''\frac{\partial^2\phi}{\partial t^2} + \frac{2\pi n_4}{\lambda}\,|\phi|^4\phi = 0 \qquad \text{(B.1)}$$

Now we move to the group velocity coordinate system defined by

$$\xi = 10^{-9}\,\frac{x}{\lambda}$$

$$\tau = \frac{10^{-4.5}}{(-\lambda k'')^{1/2}}\left(t - \frac{x}{v}\right)$$

$$q = 10^{2.25}(\pi n_4)^{1/4}\phi$$

Note that the normalization of q is different than in chapter 2. Thus the equation (B.1) becomes

$$i\,\frac{\partial q}{\partial \xi} + \frac{1}{2}\,\frac{\partial^2 q}{\partial \tau^2} + 2|q|^4 q = 0 \qquad \text{(B.2)}$$

This is what Kodama et al. called the "higher" NLS equation [30] and appears as a special case out of our model. Let us apply the same method as in chapter 4 to obtain the solitary wave solution for equation (B.2)

Assuming

$$q = F(\tau - w\xi)e^{i\{P\xi + w\tau\}} \tag{B.3}$$

and paralleling chapter 4, (B.2) yields

$$\frac{1}{2}\int \frac{dx}{x\sqrt{(2P+w^2) - \frac{4}{3}x^2}} = \int dt \tag{B.4}$$

where $t = \tau - w\xi$ and $x = F^2$

Setting $\sqrt{2P+w^2} = C$ and performing the integration in (B.4) we obtain

$$\frac{C}{\left[C^2 - \frac{4}{3}x^2\right]^{1/2}} = -\tanh\left[2C(t+C_1)\right] \tag{B.5}$$

where C_1 is integration constant. Choosing $2CC_1 = i\frac{\pi}{2}$ we obtain from (B.5)

$$x = \sqrt{\frac{3}{4}}\, C \operatorname{sech}(2ct) \tag{B.6}$$

Thus

$$q(\xi,\tau) = \left(\frac{3}{4}\right)^{1/4}\left[c \operatorname{sech}\{2c(\tau - w\xi)\}\right]^{1/2} e^{i\{(c^2-w^2)\frac{\xi}{2} + w\tau\}} \tag{B.7}$$

This solution is of the same form as that quoted by Kodama et al. [30].

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