SUM-FREE SETS AND SEQUENCES

by

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ABSTRACT

In this thesis, we survey some results on sum-free sets of integers, sum-free sets in finite groups, and sum-free sequences, especially the reciprocal sum of the elements of a sum-free sequence. By studying locally maximal sum-free sets, we derive(known) bounds on Ramsey numbers $R_n(3, 2)$. Also some generalizations of Schur's theorem are discussed. We use group-theoretic and number-theoretic results in the thesis. As a final chapter, we present some open problems.

DEDICATION

This thesis is dedicated to my best friend and my wife, Emine Nur Özgener

QUOTATION

"*Eppur si muove.*"

Galileo Galilei

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INTRODUCTION

The purpose of this thesis is to survey the results on sum-free sets of integers, sum-free sets in finite groups, and sum-free sequences. Sum-free sets were introduced first by I. Schur while he was giving a simple proof of Fermat's Last Theorem over a finite field. A sum-free set is a set in which no element can be expressed as the sum of two elements of the set. Schur showed then it is impossible to partition the positive integers into finitely many classes so that each class is sum-free. This is known as Schur's theorem. It is the first example, other than Dirichlet's pigeon-hole principle, of what is called as Ramsey type theorems.

In Chapter 1, we give the necessary definitions and notations. We state, without proofs, two versions of Ramsey's theorem. We illustrate, by an example, how one can use sum-free sets to obtain bounds on the Ramsey number $R(3, 3, \ldots, 3, 2)$ which will be denoted by $R_n(3, 2)$ for short. The Ramsey number $R_n(3, 2)$ is the smallest positive integer such that any coloring of the edges of the complete graph with $R_n(3, 2)$ vertices with n colors forces the existence of a monochromatic $\mathbf{K}_3$.

Chapter 2 is the chapter in which we deal with sum-free sets of integers. We start with some historical background, and give some motivation as to why we study sum-free sets. We state the problem of Schur which can be phrased as "What is the largest integer $f(n)$ for which there exists some way of partitioning the set $\{1, 2, \ldots, f(n)\}$ into n sets, each of which is sum-free?" The function $f(n)$ is known as the Schur's function. We give the known upper and lower bounds on the Schur's function $f(n)$. We give some generalizations of Schur's theorem, and define the corresponding functions for these generalizations. Using these functions, we get better bounds on the Ramsey numbers $R_n(3, 2)$.

In Chapter 3, we state and give detailed proofs of some fundamental addition theorems of groups of finite order such as the Cauchy-Davenport theorem, Vosper's theorem, and Kneser's theorem. These theorems are of vital importance in the study of sum-free sets in groups of finite order.We also give some more theorems about sum-free sets in finite abelian groups of special orders, such as groups of prime power orders, or groups of order divisible by a certain prime. In certain cases, depending upon $| G |$, the order of the group G, the stuructures and sizes of sum-free sets are fully determined. At the end of the chapter, we mention about sum-free sets in non-abelian groups. By $\lambda(G)$, we denote the cardinality of a largest maximal sum-free set in G. We give upper and lower bounds on $\lambda(G)$.

The fourth chapter deals with sum-free sequences, especially with the reciprocal sum of the elements of a sum-free sequence. We define a special class called χ- sequences whose counting function satisfies a certain inequality. We give upper and lower bounds on the reciprocal sum of the elements of a sum-free sequence. A conjecture of Erdös and its positive solution are presented in details.

In the last chapter, we give a list of what are, to the best of my knowledge,unsolved problems and conjectures.

After the last chapter, we have an appendix in which we present four tables. In these tables, we list the groups of "small order" and a representative of a maximal sum-free set from each isomorphism class

We try to keep the notation standard. The only non standard one, I think, is the ues of $\{ x_1, x_2, \ldots, x_{m_1} \}_<$, to mean the set $\{ x_1, x_2, \ldots, x_{m_1} \}$ in which $x_1 < x_2 < \ldots < x_{m_1}$. We used the O and o notations as well. Other notations are self-explanatory.

Chapter 1

PRELIMINARIES

Definition 1.1. Let G be an additive semigroup and let S and T be subsets of G. We define

$$S + T = \{ s + t \mid s \in S, t \in T \}$$

to be the sum of S and T. In particular,

$$S + S = \{ s_1 + s_2 \mid s_1, s_2 \in S \}.$$

Note that $s + s \in S + S$ for all $s \in S$.

A subset S of an additive semigroup G will be called a sum-free set if and only if $S \cap (S + S) = \emptyset$, the empty set, or equivalently if and only if the equation $x_1 + x_2 - x_3 = 0$ has no solution with $x_1, x_2, x_3 \in S$.

Sum-free sets have been studied in several contexts but mainly because of their connection with the Ramsey numbers, $R(k_1, k_2, \ldots, k_n, r)$ which will be defined later.

Definition 1.2. We call a set S an s-set if S contains s elements. Naturally, if T is a subset of S and T contains r elements, T is said to be an r-subset of S. We will denote the cardinality of the set S by $|S|$, and we will denote that T is a subset of the set S by $T \subseteq S$.

Definition.1.3. Let S be an s-set and let $\prod_r(S)$ denote the collection of all r-subsets of S.

$$\prod_r(S) = \{ T \mid T \subseteq S \text{ and } |T| = r \}.$$

Further, let

$$\prod_r(S) = S_1 \cup S_2 \cup \ldots \cup S_n$$

be a partition of $\prod_r(S)$ into n mutually disjoint subsets.

Suppose that for some $k \geq r$, there exists a k-subset K of S such that all the r-subsets of K belong to the same S_i for some i. Then we call K a (k,S_i)-subset of S with respect to the given partition.

Now we can state two versions of Ramsey's theorem without proofs. The proofs can be found in Graham et. al [12].

Theorem (Ramsey [19]). Let n, $k_1, k_2, \ldots, k_n$, r be positive integers with $k_i \geq r$, $1 \leq i \leq n$. Then there exists a least integer $R(k_1, \ldots, k_n, r)$ such that the following statement is true for any $s \geq R(k_1, \ldots, k_n, r)$.

> For any s-set S and for any partition of $\prod_r(S)$ into n subsets $S_1, \ldots, S_n$ there exists a subset K_i which is a (k_i, S_i)-subset of S for some i, $1 \leq i \leq n$.

We will denote the complete graph on k vertices by $\mathbf{K}_k$ and we will denote the set of vertices of a graph H by V(H).

For r = 2, we can restate Ramsey's theorem in the language of graph theory.

Theorem (Ramsey). Given positive integers n, $k_1, \ldots, k_n$ with each $k_i \geq 2$, there exists a least positive integer $R(k_1, \ldots, k_n, 2)$ such that the following statement is true for every $s \geq R(k_1, \ldots, k_n, 2)$.

> For any edge-coloring of K_s with n colors there exists an i, $1 \leq i \leq n$, and a subset L of $V(K_s)$ of size k_i such that the complete graph on L is (edge) monochromatic of color i.

If $k_1 = k_2 = \ldots = k_n = k \geq 2$, then we write $R_n(k, 2)$ for $R(k_1, \ldots, k_n, 2)$. Thus $R_n(k, 2)$ is the smallest positive integer such that any coloring of the edges of the complete graph on $R_n(k, 2)$ vertices with n colors forces the existence of a monochromatic $\mathbf{K}_k$.

Definition 1.4. If S is a sum-free set in a group G and $|T| \leq |S|$ for every subset T of G which is sum-free, then we say S is a maximum sum-free set in G and we write $|S| = \lambda(G)$. Thus $\lambda(G)$ denotes the cardinality of a maximum sum-free set in G. A maximal sum-free set is one to which no new elements can be added so that the new set is still sum-free.

Definition 1.5. The Schur function f(n) is defined as follows. For each n, f(n) is the largest integer such that it is possible to partition the integers $\{1, 2, \ldots, f(n)\}$ into n sets, none of which contains a solution to the equation $x_1 + x_2 - x_3 = 0$; i.e., into n sum-free sets.

We can generalize the idea of the Schur function. This generalization is due to P.Turán.

Definition 1.6. If m, n are positive integers, f(m, n) is defined to be the largest integer such that the set $\{ m, m+1, \dots, m+f(m,n) \}$ can be partitioned into n sum-free sets.

We can also consider the function g(n), the largest positive integer such that it is possible to partition the integers $\{ 1, 2, \dots, g(n) \}$ into n sets, none of which contains a solution to the equation $\sum_{i=1}^{m} a_i x_i = 0$ where the a_i are given integers, or we could define a Schur function on a given system of simultaneous linear equations.

Definition 1.7. (Rado). The equation $\sum_i a_i x_i = 0$ is n-fold regular if there exists a least positive integer h(n) such that whenever $\{1, 2, \dots, h(n)+1\}$ is partitioned into n classes in any manner, at least one of the classes contains a solution to the given equation. The equation is said to be regular if it is n-fold regular for every n.

We will obtain some bounds on the Schur function f(n). The lower bound on f(n) has been improved by considering various generalization of the problem. If we have a system (S) of simultaneous linear equations, we proceed by partitioning sets of integers into (S)-free sets, that is, into sets which contain no solution to the system (S).

In the following example, we illustrate the use of sum-free sets in order to find a bound on the Ramsey number $R_2(3, 2)$.

Let $G=\mathbb{Z}_5$, the integers modulo 5. Suppose that we partition $\mathbb{Z}_5^*$, the non-zero elements of G, into two disjoint sum-free sets, $S_1 = \{ 1, 4 \}$ and $S_2 = \{ 2, 3 \}$, and assign to the set S_k the color C_k for $k = 1, 2$. Let $\mathbf{K}_5$ be the complete graph on $v_0, v_1, \dots, v_4$, and color the edge from v_i to v_j in color C_k if $i - j \in S_k$. Since $S_k = -S_k$, this induces a well-defined edge-coloring of the graph.

Let v_r, v_m, v_n be any three vertices of $\mathbf{K}_5$ and consider the triangle on these vertices. Suppose that two of its edges $\{ v_r, v_m \}$ and $\{ v_m, v_n \}$ are colored C_k. This means that $r - m, m - n \in S_k$. But since S_k is sum-free, we have then $r - n = (r - m) + (m - n) \notin S_k$ so the edge $\{ v_r, v_n \}$ is colored in the other color and no monochromatic triangle can occur. This shows that $R_2(3, 2) > 5$. It is easy to show that $R_2(3, 2) = 6$. Suppose we color the edges of $\mathbf{K}_6$ with two colors, say purple and pink. Denote the vertex set by $V(\mathbf{K}_6) = \{ v_1, v_2, v_3, v_4, v_5, v_6 \}$. If we consider a vertex, say v_1, at least three edges incident with it are of the same color, say purple. Suppose these edges are $\{ v_1, v_2 \}$, $\{ v_1, v_3 \}$, and $\{ v_1, v_4 \}$. If we have one edge among the vertices v_2, v_3, and v_4 with the same color, then we have a purple triangle. If there is no such an edge, then they all have to be colored pink, and hence we have a pink triangle. If we color the edges of $\mathbf{K}_6$ with two colors, there will be a monochromatic triangle.

All the applications of sum-free sets to estimating Ramsey numbers are similar to this example, in that they all depend on partitioning a group or a set of positive integers into a pairwise disjoint union of sum-free sets.

Chapter 2

Sum-Free Sets of Integers

1. Introduction

Motivation: Pierre de Fermat conjectured, circa 1637, that " it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or in any power higher than the second into powers of like degree."

That is, he conjectured that the equation

$$x^n + y^n = z^n$$

where n is a positive integer greater than 2, has no solution in integers all different from zero.

Dickson [7] considered the following conjecture which is stronger than Fermat's conjecture.

If p,q are odd primes, then the congruence

$$x^p + y^p + z^p \equiv 0 \pmod{q}$$

does not have a non-trivial integer solution. The congruence is Fermat's equation over the field $\mathbf{F}_q$ with q elements.

Note that if, given p, there exist infinitely many primes q such that the above congruence does not have a solution, then Fermat's theorem would hold for n = p. For a good account of Fermat's Last Thereom, the reader is advised to take a look at the books by Edwards [8] and Ribenboim [22].

In 1909, Leonard Eugene Dickson disproved his conjecture by proving that the given congruence has a solution modulo q provided that

$$(p - 1)^2(p - 2)^2 + 6p - 2 \le q.$$

I. Schur [23] has simplified Dickson's proof in 1916 and in his proof he used the idea of sum-free sets.

2. SCHUR'S PROBLEM

We need a lemma to start with.

Lemma 2.1. For any integer $n \geq 2$,

$$\lfloor n!e \rfloor = n! \sum_{j=0}^{n} \frac{1}{j!} \tag{2.1}$$

$$\lfloor (n+1)!\,e \rfloor = (n+1)\lfloor n!\,e \rfloor + 1. \tag{2.2}$$

Proof: The proof follows from Taylor's theorem.■

Let us restate the problem of Schur. What is the largest integer f(n) for which there exists a partitioning of the set { 1, 2, ... , f(n) } into n sets, each of which is sum-free?

Only the first four values of f(n) are known and for f(5) we have f(5) ≥ 157. The known values are f(1) = 1, f(2) = 4, f(3) = 13, f(4) = 44. To see f(2) ≥ 4 observe that { 1, 4 }, { 2, 3 } is a sum-free partition and f(3) ≥ 13 follows from the sets { 1, 4, 10, 13 }, {2, 3, 11, 12 } and {5, 6, 8, 9 }, where 7 can be placed in any one of these three sets.

L. Baumert [4] has found f(4) = 44 and the first two sum-free partitions in 1965 using a back-track programming technique. His first partition is given below.

A = { 1, 3, 5, 15, 17, 19, 26, 28, 40, 42, 44 },

B = { 2, 7, 8, 18, 21, 24, 27, 33, 37, 38, 43 },

C = { 4, 6, 13, 20, 22, 23, 25, 30, 32, 39, 41 },

D = { 9, 10, 11, 12, 14, 16, 29, 31, 34, 35, 36 }.

Baumert's method showed that there is no possibility of placing the first 45 positive integers into four sets with the required condition. Also, A. S. Fraenkel (unpublished) independently verified that f(4) = 44. He has a list of 273 partitions for this case and he believes that it is an exhaustive collection.

We will discuss the construction of such sets later on. In 1978 Harold Fredricksen [11] verified that f(5) ≥ 157 by using a back-track search technique. We give a partition below.

A = { 1, 4, 10, 16, 21, 23, 28, 34, 40, 43, 45, 48, 54, 60, 98, 104, 110,
113, 115, 118, 124, 130, 135, 137, 142, 148, 154, 157 },

B = { 2, 3, 8, 9, 14, 19, 20, 24, 25, 30, 31, 37, 42, 47, 52, 65, 70, 88,
93, 106, 111, 116, 121, 127, 128, 133, 134, 138, 139, 144, 149,150, 155, 156 },

C = { 5, 11, 12, 13, 15, 29, 32, 33, 35, 36, 39, 53, 55, 56, 57, 59, 77, 79, 81, 99, 101,
102, 103, 105, 119, 122, 123, 125, 126, 129, 143, 145, 146, 147, 153 },

D = { 6, 7, 17, 18, 22, 26, 27, 38, 41, 46, 50, 51, 75, 83, 107, 108,
112, 117, 120, 131, 132, 136, 140, 141, 151, 152 },

E = { 44, 49, 58, 61, 62, 63, 64, 66, 67, 68, 69, 71, 72, 73, 74, 76, 78,
80, 82, 84, 85, 86, 87, 89, 90, 91, 92, 94, 95, 96, 97, 100, 109, 114 }.

Before investigating the problem of partitioning the integers into n sum-free sets we attemp to get some idea of the magnitude of the problem. We now state the first theorem.

Theorem 2.2. (Schur [23]). $\frac{3^n - 1}{2} \leq f(n) \leq \lfloor n!\, e \rfloor - 1.$

Proof: (a) We will show first $f(n) \leq \lfloor n!\, e \rfloor - 1$. Suppose that the set $\{ 1, 2, \dots, N \}$ can be partitioned into n sum-free sets $S_1, S_2, \dots, S_n$. Without loss of generality, we will assume that

$$m_1 = |S_1| \geq |S_i| \qquad \text{for } 2 \leq i \leq n$$

and note that

$$N \leq m_1 n. \tag{2.3}$$

Let $S_1 = \{ x_1, x_2, \dots, x_{m_1} \}_<$. Look at the $m_1 - 1$ differences

$$x_2 - x_1, x_3 - x_1, \dots, x_{m_1} - x_1.$$

They belong to the set $\{ 1, 2, \dots, N \}$ and since S_1 is sum-free, they must be distributed among the (n - 1) sets $S_2, \dots, S_n$. (If $x_j - x_1 \in S_1$ for some j, $2 \leq j \leq m_1$, then we would have, since S_1 is sum-free,

$$(x_j - x_1) + x_1 = x_j \notin S_1$$

which is not true since $x_j \in S_1$.) Let S_2 be the set containing the largest number, call it m_2, of these $m_1 - 1$ differences $x_j - x_1$ where $j \in \{ i_1, i_2, \dots, i_{m_2} \}_<$.

As before,

$$m_1 - 1 \leq m_2 (n - 1) \tag{2.4}$$

The differences $x_{i_j} - x_{i_1}$, $2 \le j \le m_2$ must be distributed among the (n - 2) sets $S_3, \dots, S_n$ and let S_3 be the set containing the largest number, call it m_3, of these $m_2 - 1$ differences. We have then

$$m_2 - 1 \le m_3(n - 2). \tag{2.5}$$

Continuing in this fashion we get, for each integer υ, m_υ such that

$$m_\upsilon - 1 \le m_{\upsilon+1} (n - \upsilon). \tag{2.6}$$

and dividing both sides by $(n - \upsilon)!$ and rearranging yields

$$\frac{m_\upsilon}{(n - \upsilon)!} \le \frac{m_{\upsilon+1}}{(n - \upsilon - 1)!} + \frac{1}{(n - \upsilon)!}$$

We will eventually arrive at a case where $\upsilon = k$ and $m_k = 1$ for some $k \le n$, so using (2. 3) and (2. 6) we get:

$$\frac{m_1}{(n - 1)!} \le \frac{m_2}{(n - 2)!} + \frac{1}{(n - 1)!}$$
$$\frac{m_2}{(n - 2)!} \le \frac{m_3}{(n - 3)!} + \frac{1}{(n - 2)!}$$
$$\vdots$$
$$\frac{m_{k-1}}{(n - k + 1)!} \le \frac{m_k}{(n - k)!} + \frac{1}{(n - k + 1)!},$$

where $m_k = 1$. Therefore, and summing over all υ yields

$$\begin{aligned} N &\le n! \left(\frac{1}{(n - 1)!} + \frac{1}{(n - 2)!} + \dots + \frac{1}{(n - k)!} \right) \\ &\le n! \left(\frac{1}{(n - 1)!} + \frac{1}{(n - 2)!} + \dots + \frac{1}{(n - k)!} + \frac{1}{(n - k + 1)!} + \dots + 1 \right) \\ &= \lfloor n!\, e \rfloor - 1, \end{aligned}$$

by Lemma 2. 1.

(b) Given a partition of the set { 1, 2, ..., f(n) } into n sum-free sets $S_1, S_2, \dots, S_n$ we can get a partition of { 1, 2, ..., 3f(n) + 1 } into n + 1 sets as follows. (This construction is due to I.Schur [23].)

Let $S_1 = \{ x_{11}, x_{12}, \dots, x_{1,t_1} \}$, $S_2 = \{ x_{21}, x_{22}, \dots, x_{2,t_2} \}$, ..., $S_n = \{ x_{n1}, x_{n2}, \dots, x_{n,t_n} \}$.

Form the following sets

$$S_1' = \{ 3x_{11}, 3x_{11}-1, 3x_{12}, 3x_{12}-1, \ldots, 3x_{1,t_1}, 3x_{1,t_1}-1 \}$$
$$S_2' = \{ 3x_{21}, 3x_{21}-1, 3x_{22}, 3x_{22}-1, \ldots, 3x_{2,t_2}, 3x_{2,t_2}-1 \}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$S_n' = \{ 3x_{n1}, 3x_{n1}-1, 3x_{n2}, 3x_{n2}-1, \ldots, 3x_{n,t_n}, 3x_{n,t_n}-1 \}$$
$$S_{n+1}' = \{ 1, 4, 7, \ldots, 3f(n) + 1 \}.$$

It is easy to see that if any of the first n of the n + 1 sets are not sum-free, then the corresponding set of the original n sets would not be sum-free. The set S_{n+1}' is sum-free as all its elements are congruent to 1 modulo 3. Therefore we have

$$3 f(n) + 1 \leq f(n + 1).$$

Since $f(1) = 1$,

$$f(n) \geq 1 + 3 + 3^2 + 3^3 + \ldots + 3^{n-1} = \frac{3^n - 1}{2}.$$

We can improve this lower bound a little bit by using Fredricksen's result [11] which is that $f(5) \geq 157$. Thus

$$f(6) \geq 3 f(5) + 1 \geq 3 (157) + 1$$

and hence

$$f(n) \geq 3^{n-5} (157) + 3^{n-6} + \ldots + 1 \qquad \text{for } n \geq 5$$
$$= 3^{n-5} (157) + (3^{n-5} - 1)/2$$
$$= \frac{3^{n-5} (315) - 1}{2}.$$

Therefore

$$f(n) \geq \frac{3^{n-5} (315) - 1}{2} = \frac{3^n \left(\frac{315}{243} \right) - 1}{2}. \blacksquare \qquad (2.7)$$

Remark: Whitehead improved the upper bound slightly to $f(n) \leq \lfloor n! (e - 1/24) \rfloor - 1$.

3. AN IMPROVED LOWER BOUND

We can improve the bound given in (2. 7) above, but we need a definiton first.

Definition 2.3. Let g(r) be the smallest number of sum-free sets into which the set of integers$\{1, 2, \ldots, r\}$ can be partitioned. Equivalently, we say that if $f(n - 1) < r \le f(n)$, then $g(r) = n$.

Lemma 2. 4. For all $r \ge 9{,}300{,}217$,

$$g(r) < \log r. \tag{2.8}$$

Proof: Given r, choose n so that

$$\frac{3^{n-6}(315)-1}{2} \le r < \frac{3^{n-5}(315)-1}{2}. \tag{2.9}$$

Now

$$r < \frac{3^{n-5}(315)-1}{2} \le f(n)$$

implies

$$g(r) \le n,$$

and

$$e^n < \frac{3^{n-6}(315)-1}{2} \le r, \qquad \text{for } n \ge 16.$$

Therefore

$$n < \log r,$$

so

$$g(r) < \log r \qquad \text{for } r \ge 472.\blacksquare$$

For the following theorem, we need a definition.

Definition 2. 5. Let m, k be positive integers and let $X = 2f(m) + 1$. Write the numbers 1, 2, ... , $X^k - 1$ in base X so that we have the following representation for each integer a:

$$a = a_0 + a_1X + a_2X^2 + \ldots + a_{k-1}X^{k-1},$$

where $0 \le a_i \le 2f(m)$ for $0 \le i \le k - 1$.

We call the integer a good if $a_i \le f(m)$ for each i, and bad if $a_i \ge f(m) + 1$ for at least one value of i.

Theorem 2. 6.(Abbott and Hanson [2]) For all positive integers m and k

$$f(km + g(kf(m))) \geq (2f(m) + 1)^k - 1. \qquad (2.10.)$$

Proof: We will show that the good numbers in $\{1, 2, \dots, X^k - 1\}$ can be partitioned into $g(kf(m))$ sum-free sets and the bad numbers into km sum-free sets. The theorem will then follow.

Let $g(kf(m)) = N$. We know that the set of integers $\{1, 2, \dots, kf(m)\}$ can be partitioned into disjoint sum-free sets $A_1, A_2, \dots, A_N$.

This partition induces a partition of the good integers in $\{1, 2, \dots, X^k - 1\}$ into N sum-free sets $B_1, B_2, \dots, B_N$ in the following manner.

For every a, $1 \leq a \leq X^k - 1$, define

$$\sigma(a) = \sum_{i=0}^{k-1} a_i, \text{ where } a = a_0 + a_1X + a_2X^2 + \dots + a_{k-1}X^{k-1}$$

If a is a good integer, then $a_i \leq f(m)$ for each i; therefore $\sigma(a) \leq kf(m)$ and hence $\sigma(a) \in A_j$ for some j, $1 \leq j \leq N$. For each j, $1 \leq j \leq N$, let

$$B_j = \{ a \mid 1 \leq a \leq X^k - 1, \text{a is good and } \sigma(a) \in A_j \}$$

It is not difficult to see that each B_j is sum-free. For if a, b $\in B_j$, then either a + b is a bad integer so belongs to none of the B_j, or $a_i + b_i \leq f(m)$ for every i. In this case suppose that $a + b \in B_j$. Then $\sigma(a)$, $\sigma(b)$ and $\sigma(a + b) = \sigma(a) + \sigma(b)$ are integers from the set $\{1, 2, \dots, k f(m)\}$ and all belonging to the set A_j. Since A_j is sum-free this is a contradiction, and hence B_j is sum-free.

We now consider the bad integers in $\{1, 2, \dots, X^k - 1\}$. Divide the bad integers into k classes $C_{-1}, C_0, \dots, C_{k-2}$ by placing $a = \sum_{i=0}^{k-1} a_iX^i$ in class C_j, $-1 \leq j \leq k-2$ if $a_i \leq f(m)$ assume $a_{-1} = 0$ for $-1 \leq i \leq j$ and $a_{j+1} \geq f(m) + 1$. Next divide each of $C_{-1}, C_0, \dots, C_{k-2}$ into m sets as follows.

Let $D_1, D_2, \dots, D_m$ be a sum-free partition of the set $\{1, 2, \dots, f(m)\}$ and split the numbers in C_j into m sets $D_{j1}, D_{j2}, \dots, D_{jm}$ in the following way.

If $a \in C_j$, then $f(m) + 1 \leq a_{j+1} \leq 2f(m)$, and we assign a to the set D_{js} if and only if

$a_{j+1} \equiv - u \pmod X$ for some $u \in D_s$. Since a_{j+1} is one of the numbers $f(m) + 1, f(m + 2, \ldots, 2 f(m)$ exactly one such u can be found, and the partition is well-defined. It remains to show that D_{js} is sum-free. Suppose that we can find $a, b, c \in D_{js}$ such that $a + b = c$. We have

$$a = \sum_{i=0}^{k-1} a_i X^i, \quad b = \sum_{i=0}^{k-1} b_i X^i, \; c = \sum_{i=0}^{k-1} c_i X^i,$$

where $a_i, b_i, c_i \leq f(m)$ for $i= 0, 1, \ldots, j$, $a_{j+1}, b_{j+1}, c_{j+1} \geq f(m) + 1$, and

$$a_{j+1} \equiv - u \pmod X, \qquad b_{j+1} \equiv - v \pmod X, \qquad c_{j+1} \equiv - w \pmod X$$

where $u, v, w \in D_s$.

Since

$$a_{j+1} + b_{j+1} = c_{j+1} + X,$$

it follows that

$$u + v \equiv w \pmod X$$

and since $u, v, w \leq f(m)$ we must have $u + v = w$. However, this contradicts the fact that D_s is sum-free. Hence we have shown that D_{js} is sum-free. So we have partitioned the bad integers into km sum-free sets. We previously partitioned the good integers into N sum-free sets and the theorem follows.■

Corollary 2.7. For all sufficiently large n, we have

$$f(n) > 315^{n/5-c\log n}$$

where c is some positive absolute constant.

Proof: For large k, we have

$$f(5k + g(kf(5))) \geq (2(157) + 1)^k - 1 = 315^k - 1.$$

Let n be large. Choose k so that

$$5k + g(kf(5)) \leq n < 5(k + 1) + g((k + 1)f(5)).$$

Then $f(n \geq 315^k - 1$, and

$$n < 5(k + 1) + g((k + 1)f(5))$$

or

$$\frac{n}{5} < k + 1 + \frac{g((k + 1)f(5))}{5}.$$

Hence by solving for k we get

$$k > \frac{n}{5} - 1 - \frac{g((k+1)f(5))}{5} > \frac{n}{5} - 1 - \frac{\log((k+1)f(5))}{5} > \frac{n}{5} - 1 - d\log n,$$

where d is a constant and k is sufficiently large. The corollary then follows.■

While the best upper and lower bounds for f(n) are quite far apart, we can still gain a little more insight into the behaviour of f(n). Using Theorem 2. 6 we show that $\lim_{n \to \infty} f(n)^{1/n}$ exists and equals L, although it is not known whether L is finite or infinite. We can state the following as a corollary to Theorem 2. 6.

Corollary 2. 8. $\lim_{n \to \infty} f(n)^{1/n}$ exists.

Proof: Let $\alpha = \liminf_{n \to \infty} f(n)^{1/n} \le \limsup_{n \to \infty} f(n)^{1/n} = \beta$.

Suppose first that β is finite. Let $\varepsilon > 0$ be given, and let m be the smallest integer for which

$$f(m)^{1/m} > \beta - \varepsilon. \qquad (2.\ 11.)$$

By (2. 8), for sufficiently large k,

$$g(kf(m)) < \log(kf(m)) = \log k + \log f(m).$$

Hence, for fixed m

$$\frac{g(kf(m))}{k} \to 0 \qquad \text{as} \qquad k \to \infty,$$

i.e. , $g(kf(m)) = o(k)$. Since $g(kf(m)) = o(k)$, there exists an integer $k_0 = k_0(\varepsilon)$, such that for $k \ge k_0$, we have

$$km + g(kf(m)) < \lfloor km(1+\varepsilon) \rfloor. \qquad (2.\ 12\ )$$

For any $n \ge \lfloor k_0 m(1+\varepsilon) \rfloor$, define k by

$$\lfloor km(1+\varepsilon) \rfloor \le n < \lfloor (k+1)m(1+\varepsilon) \rfloor. \qquad (2.\ 13\ )$$

Hence, by using (2. 10), (2. 12), and (2. 13),

$$f(n) \ge f(\lfloor km(1+\varepsilon) \rfloor) > f(km + g(kf(m))) \ge (2f(m)+1)^k - 1 > f(m)^k.$$

In order to write this down, we used the facts that f(x) is an increasing function of x, f(x) is an increasing function of x, the theorem, and a rough estimate, in turn. This implies that

$$f(n)^{1/n} > f(m)^{k/n} > (\beta - \varepsilon)^{km/n}$$

by (2. 11). Hence, by (2. 13),

$$\liminf_{n \to \infty} f(n)^{1/n} \ge (\beta - \varepsilon)^{1/(1+\varepsilon) - m/n}$$

It follows that $\alpha = \beta$. A similar argument deals with the case where β is infinite.■

4. APPLICATIONS OF SUM-FREE SETS TO ESTIMATES OF THE RAMSEY NUMBERS

We will be considering the second statement of Ramsey's theorem from Chapter 1.

Specifically, we are going to deal with $R_n(3, 2)$, where $R_n(3, 2)$ is the smallest positive integer such that coloring the edges of the complete graph on $R_n(3, 2)$ vertices in n colors forces the existence of a monochromatic triangle. We begin with a well known results.

Theorem 2. 9. For all sufficiently large n,

$$R_n(3, 2) > 315^{n/5 - c\log n} + 2.$$

Proof: We prove

$$R_n(3, 2) - 1 \geq f(n) + 1. \qquad (2.\ 14.)$$

from which by Corollary 2. 7, the theorem follows.

To prove (2. 14), let $A_1, A_2, \dots, A_n$ be a sum-free partition of the set $\{ 1, 2, \dots, f(n) \}$. Let $\mathbf{K} = \mathbf{K}_{f(n)+1}$ be the complete graph on $f(n) + 1$ vertices $x_0, x_1, \dots, x_{f(n)}$.

We color the the edges of **K** with n colors $C_1, C_2, \dots, C_n$ by coloring the edge e_{ij} joining the vertex x_i to the vertex x_j by the color C_m if $|i - j| \in A_m$. Suppose that this coloring gives us a triangle with vertices x_i, x_j, x_k all of whose edges are monochromatic under C_m. Assume without loss of generality that $i > j > k$. Then $i - j, i - k, j - k \in A_m$ but $(i - j) + (j - k) = i - k$ which contradicts the fact that A_m is sum-free.

Therefore

$$R_n(3, 2) - 1 \geq f(n) + 1 > 315^{n/5 - c\log n}. ■$$

Theorem 2. 10. $R_{n+1}(3, 2) \leq (n + 1)(R_n(3, 2) - 1) + 2$, where $n \geq 1$.

Proof: Let **K** be the complete graph on $(n + 1)(R_n(3, 2) - 1) + 2$ vertices and consider a coloring of **K** with $(n + 1)$ colors. Choose a vertex v of **K.** Of the $(n + 1)(R_n(3, 2) - 1) + 1$ edges ending at v, at least $R_n(3, 2)$ must have the same color. Suppose these join v to the vertices $x_1, x_2, \dots, x_s$, where $s \geq R_n(3, 2)$. Consider the edges e_{ij} where $1 \leq i < j \leq s$. If any one of them has the original color, then the triangle $\{ x_i, x_j, v \}$ is monochromatic. If none of them has the original color then the complete graph $\mathbf{K}_s$ on $x_1, x_2, \ldots, x_s$ must be colored in the other n colors. But by the choice of s, this forces the existence of a monochromatic triangle in $\mathbf{K}_s$, and hence in K.■

Corollary 2. 11. Since $R_1(3, 2) = 3$, the theorem implies that

$$R_n(3, 2) \leq 3(n!), \text{ for all } n \geq 1.$$

This theorem can be improved:

Corollary 2. 12. $R_n(3, 2) \leq [n!\, e] + 1$, for all $n \geq 1$.

Proof: We have

$$R_1(3, 2) \leq [1!\, e] + 1 = 3,$$

which is the starting point of our induction. If $n \geq 1$ and

$$R_n(3, 2) \leq [n!\, e] + 1,$$

then by Theorem 2. 10 and (2. 2)

$$R_{n+1}(3, 2) \leq (n + 1)[n!\, e] + 2 = [(n + 1)!\, e] + 1.$$■

5.GENERALIZATION OF SCHUR'S THEOREM

In this section, we will study some generalizations of Schur's problem and give some better bounds on f(n). The following results are due to Abbott and Hanson [2].

We consider the system (S) of $\binom{k-1}{2}$ linear equations in $\binom{k}{2}$ unknowns

$$x_{i,j} + x_{j,j+1} = x_{i,j+1} \qquad \text{for } 1 \leq i < j \leq k - 1.$$

We define the generalizations $f_k(n)$ and $g_k(m)$ of f(n) and g(m), given in Definition 1. 5 and Definition 2. 3, respectively.

Definition 2. 13. Let $\mathcal{A}$ be a set of positive integers. $\mathcal{A}$ is called (S)-free if and only if it contains no solution to the system (S).

By Rado's theorem, see Graham et.al.[12], the system is regular.

Define $f_k(n)$ as the largest positive integer so that the set $\{1, 2, \ldots, f_k(n)\}$ can be partitioned into n (S)-free sets.

Define $g_k(m)$ as the smallest number of (S)-free sets into which the set $\{1, 2, \ldots, m\}$ can be partitioned; or as before, if $f_k(n - 1) < m \leq f_k(n)$, then $g_k(m) = n$.

Remark: When $k = 3$ we obtain the functions $f(n)$ and $g(m)$. Similar theorems for $f_k(n)$ and $g_k(m)$ can be proven. Also similar estimates for other Ramsey numbers can be given by using $f_k(n)$.

We now give without proof, a theorem concerning $f_k(n)$.

Theorem 2. 14. For all positive n and m,

$$f_k(n+m) \geq (2f_k(m)+1)f_k(n) + f_k(m).$$

Corollary 2. 15. For all positive n and m,

$$f(n+m) \geq (2f(m)+1)f(n) + f(m).$$

Corollary 2. 16. For $n \geq 5$, and for some absolute constant c, we have

$$f(n) \geq c\, 315^{n/5}.$$

The proof of Corollary 2. 16 follows by induction on n by using Corollary 2. 15 with $m = 5$.

Note that the above lower bound is an improvement over the one we have in Corollary 2. 7.

Corollary 2. 17. For $n \geq 1$, and for some constant c_k, $c_k = c(k)$, we have

$$f_k(n) \geq c_k\,(2k-3)^n.$$

Theorem 2. 18. Let the system (S) be given and let the function $f_k(n)$ be defined. Hence

$$R_n(k, 2) \geq f_k(n) + 2,$$

and for $n \geq 1$, $k \geq 2$ and for some constant c_k, $c_k = c(k)$,

$$R_n(k, 2) \geq c_k\,(2k-3)^n.$$

Corollary 2. 19. For $n \geq 5$, and for some absolute constant c, we have

$$R_n(3, 2) \geq c\, 315^{n/5}.$$

We now consider the function $f(m, n)$ which is defined to be the largest positive integer such that the set $\{ m, m+1, \dots, m+f(m, n) \}$ can be partitioned into n sum-free sets.

For $m = 1$, we get $f(1, n) = f(n) - 1$. We also have

$$f(m, n) \leq m\, f(n) - 1,$$

since the set $\{ m, 2m, \dots, m\,(f(n)+1) \}$ cannot be partitioned into n sum-free sets.

So we have

$$f(m, n) \leq m\,[n!\, e] - m - 1.$$

We will get some lower bound on $f(m, n)$ as well.

Definition 2. 20. A set S of positive integers is called strongly sum-free if and only if it contains no solution to either of the equations

$$a + b = c, \tag{2. 15}$$

$$a + b + 1 = c. \tag{2. 16}$$

Definition 2. 21. For any positive integer n, we define $\phi(n)$ to be the largest positive integer for which the set $\{ 1, 2, \ldots, \phi(n) \}$ can be partitioned into n strongly sum-free sets.

This function is well-defined by Rado's theorem. A lower bound on $\phi(n + m)$ is given without a proof in the following theorem.

Theorem 2. 22. For m and n positive,

$$\phi(n + m) \geq 2f(m)\, \phi(n) + f(m) + \phi(n)$$

where $f(n)$ is the Schur function for equation (2. 15).

We use this theorem to get a lower bound on $f(m, n)$.

Theorem 2. 23. For m and n positive,

$$f(m, n) \geq m\, \phi(n) - 1.$$

Corollary 2. 24. For m and n positive,

$$f(m, n) \geq m\,(3\, f(n - 1) + 1) - 1.$$

To prove Corollary 2. 24 we take $n = 1$ and $m = n - 1$ in Theorem 2. 22 obtaining

$$\phi(n) \geq 3\, f(n - 1) + 1.$$

Then use Theorem 2. 23.■

Corollary 2. 25. For m and n positive and an absolute constant c,

$$f(m, n) > c\, m\, 315^{n/5}.$$

In this chapter, we studied the problem of Schur. We saw the connection between sum-free sets and the Ramsey numbers $R_n(3, 2)$. We gave upper and lower bound on the Schur's function $f(n)$. Also some generalization of Schur's theorem have been introduced.

Chapter 3

Addition Theorems for Groups and Sum-Free Sets in Groups

1. Introduction

In this chapter we will study sum-free sets in groups in particular we look at maximal sum-free sets in abelian groups of specific order; maximal sum-free sets in groups; and a little bit of Group Ramsey Theory. We will be interested in finding the sizes and stuructures of sum-free sets in abelian groups of specific order

We have to have some results about the addition of subsets of group elements. To start with, we will fix the notation and some definitions.

In this chapter we only consider additive groups of finite order.

Let G denote an additive group. We reserve the notation $\subseteq$ for subsets. A^c will denote the set-theoretic complement of A in G. A + B is defined as before, noting that $A + \emptyset = \emptyset$. For $A \subseteq G$, we define the following sets in the natural way $-A = \{ -a \mid a \in A \}$, $kA = \{ ka \mid a \in A \}$ where k is an integer, $A \setminus B = \{ g \in G \mid g \in A, g \notin B \}$.

Definition 3. 1. A sum-free set S is maximal if for every sum-free set T where $S \subseteq T \subseteq G$, we have S = T. Let $\Lambda(G)$ be the set of cardinalities of all maximal sum-free sets in G and let $\lambda(G) = \max \Lambda(G)$. Clearly, S is a maximal sum-free set if and only if $S \cup \{ g \}$ is not sum-free for any $g \in S^c$.

A symmetric sum-free partition of $G^* = G \setminus \{ 0 \}$ is a partition

$$G^* = \bigcup_{i=1}^{n} S_i$$

where $S_i = -S_i$ and S_i is sum-free, $1 \le i \le n$. From Greenwood and Gleason's paper [15], we know that

$$R_n(3,2) \geq |G| + 1$$

if such a partition for G* exists. From Street and Whitehead's paper [32], it suffices to study the maximal sum-free sets in order to estimate the Ramsey numbers. Exploiting the nature of Ramsey numbers, we can find an upper bound on the size of symmetric sum-free sets.

A sum-free covering of G* is a collection of sum-free sets $\mathcal{S} = \{ S_1, S_2, \ldots, S_n \}$ such that

$$G^* = \bigcup_{i=1}^{n} S_i.$$

If $\mathcal{S}$ and $\mathcal{T} = \{ T_1, T_2, \ldots, T_n \}$ are two sum-free coverings of G such that $S_i \subseteq T_i$ for all $1 \leq i \leq n$, we say that $\mathcal{S}$ is embedded in $\mathcal{T}$.

We will also discuss the cardinality $\mu(G) = \min \Lambda(G)$ of the smallest possible maximal sum-free set. We will also discuss specifically lower bounds on $\mu(G)$ when G is an elementary abelian 2-group. This is a good opportunity to give the definition of an elementary abelian p-group, where p is any prime.

An abelian p-group G is an abelian group in which the order of each element is a power of p. A known fact is that an abelian p-group G is the direct product of cyclic subgroups H_i, $1 \leq i \leq n$. Moreover, the integer n and the orders of the H_i are uniquely determined, up to ordering, by G. If the order of H_i is p^{e_i}, we say that G is of type $(p^{e_1}, p^{e_2}, \ldots, p^{e_n})$. In particular, if G is of type $(p, \ldots, p)$, G is called an elementary abelian p-group.

We now prove some preliminary results.

Theorem 3. 2. (Mann [21]) Let G be a finite abelian group and let A and B be subsets of G. Then either $G = A + B$ or $|G| \geq |A| + |B|$.

Proof: Suppose $G \neq A + B$. Since $(A + B)^c \neq \emptyset$, we can find an element g in $(A + B)^c$. Define the set B' as follows.

$$B' = \{ g - b \mid b \in B \} = \{ g \} - B = g - B.$$

The last equality is given just for simplicity of notation. From this definition of B' we have immediately $|B'| = |B|$, and $B' \subseteq G$.

Suppose that $A \cap B' \neq \emptyset$. Then there exists $a \in A \cap B'$. Hence $a = g - b$ or $g = a + b$ which is a contradiction. Therefore A must be disjoint from B'. This implies the following.

$$|G| \geq |A| + |B'| = |A| + |B|. \blacksquare$$

As a consequence, we have the following corollary.

Corollary 3. 3. Let S be a largest maximal sum-free set in a finite group G. Then

$$\lambda(G) = |S| \leq \frac{|G|}{2}.$$

This upper bound is due to Erdös.

Note that this upper bound is best possible since it is achieved if we take $G = \mathbb{Z}_2$.

Definition 3. 4. Let G be an abelian group and let A and B be subsets of G, and $g \in G$. Then the transform of the pair (A, B) by g is the pair (A^g, B_g) where

$$A^g = A \cup (B + g), \qquad B_g = B \cap (A - g).$$

The transform we use here is similar to one that was introduced by Cauchy [5]. The next lemma will give some idea about the connection between the pair (A, B) and the pair (A^g, B_g).

Lemma 3. 5. Let G be an abelian group, let A and B be subsets of G, and let $g \in G$. Let (A^g, B_g) be the transformed pair. Then we have the following.

(i) $|A^g| + |B_g| = |A| + |B|$;

(ii) $A^g + B_g \subseteq A + B$ where, in particular, $A^g + B_g = \emptyset$ if $B_g = \emptyset$.

Proof: (i) We will use the definition of A^g and B_g.

$$\begin{aligned} |A^g| + |B_g| &= |A \cup (B + g)| + |B \cap (A - g)| \\ &= |A \cup (B + g)| + |(B + g) \cap A| \\ &= |A| + |B + g| - |(B + g) \cap A| + |(B + g) \cap A| \\ &= |A| + |B + g| \\ &= |A| + |B|. \end{aligned}$$

(ii) Let $a \in A^g, b \in B_g$. So $b \in B \cap (A - g)$ i.e., $b \in B$. If $a \in A$, we have $a + b \in A + B$ and therefore $A^g + B_g \subseteq A + B$. If $a \notin A$, then $a \in B + g$, so $a = b_1 + g$ for some $b_1 \in B$. By definition $B_g \subseteq A - g$, so $b = a_1 - g$ for some $a_1 \in A$. Hence $a + b = b_1 + g + a_1 - g = b_1 + a_1 = a_1 + b_1 \in A + B$, since we are in an abelian group.

In each case, we have $A^g + B_g \subseteq A + B. \blacksquare$

2. CRITICAL PAIRS AND VOSPER'S THEOREM

Theorem 3. 6 is a fundamental inequality which was first proved by Cauchy [5] and was later rediscovered by Davenport.

After proving this theorem we will turn to the main business of this section which is to characterize those pairs A, B (called <u>critical pairs</u>) for which the inequality in the Cauchy-Davenport theorem is an equality. This characterization is the content of Theorem 3. 10, Vosper's theorem [33].

Theorem 3. 6. (Cauchy-Davenport) Let G be the group of residues modulo p, where p is a prime, and let A and B be subsets of G. Then

$$|A + B| \geq \min(p, |A| + |B| - 1).$$

Proof: If $\min(|A|, |B|) = 1$, then the theorem is obviously true. If $|A| + |B| > p$, then Theorem 3.2. tells us we must have $G = A + B$, so $|A + B| = p$; the theorem is still valid in this case.

Hence from now on we can assume $|A| + |B| \leq p$, $\min(|A|, |B|) \geq 2$. Assume furthermore, without loss of generality, that $0 \in B$. We will prove the theorem by induction on $|B|$.

(i) Claim: $A \neq A + B$.

To prove this claim, we choose $b \in B \setminus \{0\}$. Fix one element a of A, then consider $a, a + b, \ldots, a + kb$ for every k. If we had $A = A + B$, then the elements $a, a + b, \ldots, a + kb$ would be in A as well. Since this would be true for every k, then it would be true even for $k = p$ which tells us that A is the whole group, which is impossible by the assumptions $|A| + |B| \leq p$ and $\min(|A|, |B|) \geq 2$. Then the claim follows.

(ii) Claim: For some element a of A, $|B_a| < |B|$. If $|B_a| = |B|$ for every a in A, then $B \subseteq A - a$ or $B + a \subseteq A$ holds for every a in A. Hence $A + B \subseteq A$. We have also $0 \in B$, therefore $A \subseteq A + B$ which gives the equality $A = A + B$ which contradicts (i). So the claim follows.

We are now ready to begin the induction proof.

(iii) If $|B| = 2$, we want to show $|A+B| \geq \min(p, |A|+1)$, or more precisely $|A+B| \geq |A|+1$. (Since $|A|+2 \leq p$, we have $|A|+1 = \min(p, |A|+1)$.)

Suppose $|A+B| \leq |A|$. We know that $A \subseteq A+B$ and we then have $A = A+B$ which contradicts (i).

(iv) Assume the theorem is valid for $|B| < n$. Choose B with $|B| = n$ and then choose an element a of A in such a way that $|B_a| < |B|$.

We have by Lemma 3. 5 (ii),

$$\begin{aligned} |A+B| \geq |A^a + B_a| &\geq \min(p, |A^a|+|B_a|-1) \\ &\geq |A^a|+|B_a|-1 \\ &= |A|+|B|-1. \end{aligned}$$

Observe that by our earlier comment, if $\min(p, |A^a|+|B_a|-1) = p$, then $A^a + B_a = G$ and by Lemma 3. 5 (ii), $A+B = G$ and we are done. So we assume

$\min(p, |A^a|+|B_a|-1) = |A^a|+|B_a|-1$

we have used the induction hypothesis and Lemma 3. 5 (i) in the second and the third step of the above computation, respectively.■

In Vosper's theorem, Theorem 3. 10, we will give necessary and sufficient conditions for which

$$|A+B| = \min(p, |A|+|B|-1).$$

We need some terminology at this step.

Definition 3. 7. Let G be the group of residues modulo p, where p is a prime, and let A and B be subsets of G. The pair (A, B) is called a critical pair if and only if

$$|A+B| = \min(p, |A|+|B|-1).$$

Let G be an abelian group and let A and B be subsets of G. A is called an arithmetic progression with difference d or a standard set with difference d if and only if

$$A = \{ a+id \mid i = 0, 1, \ldots, |A|-1 \}, \qquad \text{for some } a, d \in G, d \neq 0.$$

The pair (A, B) is called a standard pair with difference d if and only if both A and B are standard sets with difference d.

If A is a subset of the cyclic group with elements $\{ 0, 1, 2, \ldots, t-1 \}$ and with the addition modulo t, we can define the gaps in A as follows. If $a, a+n+1 \in A$, but $a+1, a+2, \ldots, a+n \in A^c$, then we say that there is a gap of length n in A, occuring between a and $a+n+1$.

It should be noted that, when we say A/H is a standard set with difference d we mean that

$$A = \{ a + id \mid i = 0, 1, \ldots, |A| - 1 \} + H, \text{ for some } a, d \in G, d \neq 0.$$

We will need the following lemma in order to prove Vosper's theorem.

Lemma 3. 8. Let G be an abelian group and let A, B, C and D be subsets of G. Suppose $A - B = C - D$. Then

$$A \cap B = \emptyset \qquad \text{if and only if} \qquad C \cap D = \emptyset.$$

Corollary 3. 9. Let G be an abelian group and let K, L, M, and N be subsets of G. Then

(i) $(K+L) \cap M = \emptyset$ if and only if $K \cap (M-L) = \emptyset$;

(ii) $(K-L) \cap (M+N) = \emptyset$ if and only if $(K-M) \cap (L+N) = \emptyset$.

Theorem 3. 10. (Vosper [33]) Let G be the additive group of residues modulo p, where p is a prime, and let A and B be subsets of G. Then the pair (A, B) is critical if and only if one of the following is satisfied.

(i) $|A| + |B| > p$,

(ii) $\min(|A|, |B|) = 1$,

(iii) $A = (g - B)^c$, for some g in G, or

(iv) (A, B) is a standard pair.

Proof: First we will prove that each of the conditions (i)-(iv) will suffice for the pair (A, B) to be critical.

If we have $|A| + |B| > p$, then by Theorem 3.2 we have $G = A + B$. In other words, $|A + B| = p$, and hence we have

$$|A + B| = \min(p, |A| + |B| - 1) = p,$$

so the pair (A, B) is critical.

If we have $\min(|A|, |B|) = 1$, then $|A| + |B| - 1 \leq p$ and hence $|A + B| = \max(|A|, |B|) = |A| + |B| - 1 = \min(p, |A| + |B| - 1)$. Hence the pair (A, B) is critical.

If $A = (g - B)^c$, then we have $A \cap (g - B) = \emptyset$. So by Corollary 3. 9 (i), $(A + B) \cap \{g\} = \emptyset$ which tells us $|A + B| \leq p - 1$. Since $A = (g - B)^c$, we have $|A| = |B^c| = p - |B|$. Therefore $|A| + |B| = p$. From Theorem 3. 6,

$$|A + B| \geq \min(p, p - 1) = p - 1.$$

So we have

$$|A + B| = p - 1 = |A| + |B| - 1$$

which means that the pair (A, B) is critical.

If the pair (A, B) is a standard pair, then we have

$$A = \{ a + id \mid i = 0, 1, \ldots, |A| - 1 \} \qquad \text{and} \qquad B = \{ b + id \mid i = 0, 1, \ldots, |B| - 1 \}$$

for some $a, b, d \in G, d \neq 0$, hence

$$A + B = \{ a + b + id \mid i = 0, 1, \ldots, |A| + |B| - 2 \}.$$

Therefore, we have

$$|A + B| = \min(p, |A| + |B| - 1).$$

So the pair (A, B) is a critical pair.

Next, we assume that the pair (A, B) is a critical pair. Then we will show that one of the conditions (i)-(iv) is satified.

If $|A| + |B| > p$, then we have (i). If $\min(|A|, |B|) = 1$, then condition (ii) is obtained.

If $|A| + |B| = p$, then we have $|A + B| = p - 1$ because $|A + B| = \min(p, p - 1)$. So $A + B = \{g\}^c$ for some g in G. Hence $(A + B) \cap \{g\} = \emptyset$, and by Corollary 3. 9 (i) $A \cap (g - B) = \emptyset$ yielding $A \subseteq (g - B)^c$. On the other hand, we have $|A| = p - |B| = |B^c| = |(g - B)^c|$. This tells us that $A = (g - B)^c$. So we have (iii).

Now we assume that $|A| + |B| < p$ and $\min(|A|, |B|) > 1$. We will prove that condition (iv) holds. The proof will be given in several steps.

First we list our claims and then we will prove them one by one.

Claim 1: The pair (A, B) is standard if A is a standard set.

Claim 2: Let $D = (A + B)^c$. Then the pair $(-A, D)$ is a critical pair.

Claim 3: The pair (A, B) is standard if $A + B$ is a standard set.

Claim 4: If $| B | \geq 3$ and $0 \in B$, then there exists $a \in A$ such that

$$| B | > | B_a | \geq 2.$$

Proof of Claim 1: For simplicity, we can assume that $A = \{ 0, 1, \ldots, | A | - 1 \}$. Let us consider the gaps in B.

If $b \in B$, then $\{ b, b + 1, \ldots, b + | A | - 1 \} \subseteq A + B$. We have $| A + B | < p - 1$ which means that there are some elements of G not in $A + B$ and in B there must be at least one gap of length at least $| A |$.

Suppose now that B has at least one other gap. Then $A + B$ contains all the elements of B together with at least $| A | - 1$ elements from the first gap together with at least one element from the second gap. Whence we have

$$| A + B | \geq | B | + (| A | - 1) + 1 = | A | + | B |$$

which is impossible since the pair (A, B) is critical.

A fortiori B has only one gap, i. e., B is in arithmetic progression with difference 1, and the pair (A, B) is a standard pair.

Corollary to Claim 1: If $\min (| A |, | B |) = 2$, then the pair (A, B) is a standard pair.

Proof of Claim 2: We are given $D = (A + B)^c$. Therefore $(A + B) \cap D = \emptyset$, so from Corollary 3. 9 (i), we have $B \cap (D - A) = \emptyset$ which implies that $B \subseteq (D - A)^c$.

Let $(D - A)^c = E$. So $E \cap (D - A) = \emptyset$ implies $(E + A) \cap D = \emptyset$. Hence we have $E + A \subseteq D^c = A + B$. On the other hand, we have $B \subseteq E$ which implies $A + B \subseteq A + E$, so we have equality, i. e., $A + B = A + E$. Since the pair (A, B) is critical and $| A | + | B | < p$, we have

$$p - 1 \geq | A | + | B | - 1 = | A + B | = | A + E | \geq \min (p, | A | + | E | - 1) = | A | + | E | - 1.$$

The latter inequality yields $| B | \geq | E |$. Therefore $B = E$, and so $B^c = D - A$. Now we can find

$| D - A |$ by using the last equality.

$$| D - A | = | B^c | = p - | B | \text{ and}$$

$$|D| = p - |A+B| = p - |A| - |B| + 1.$$

So

$$|D - A| = \min(p, |D| + |-A| - 1).$$

By definition the pair (-A, D) is critical.

Proof of Claim 3: If A + B is a standard set, so is $D = (A+B)^c$. By the second claim, the pair (-A, D) is critical. We also have , since $|B| > 1$,

$$|-A| + |D| = |A| + p - |A| - |B| + 1 = p - |B| + 1 < p,$$

and

$$\min(|-A|, |D|) > 1.$$

By the first claim, (-A, D) is a standard pair, since D is a standard set. Hence -A is a standard set too. Then A is standard and by the first claim, the pair (A, B) is a standard pair.

Proof of Claim 4: Let us define the following set.

$$Y = \{ a \in A \mid |B| > |B_a| \}$$

We will show that $|Y| \geq 2$. Two cases may arise.

(a) $Y = A$. Then obviously $|Y| \geq 2$.

(b) $Y \neq A$. Then, let $Z = A \setminus Y$, and $Z \neq \emptyset$. For all $z \in Z$, $B_z \subseteq B$ and $|B_z| \geq |B|$, so $B_z = B$. Therefore

$$B_z = B \cap (A - z) = B \qquad \text{for all } z \in Z.$$

From the last equality, we have $B \subseteq A - z$, so $B + z \subseteq A$. Since this last inclusion is true for all $z \in Z$, we have $B + Z \subseteq A$.

Therefore using Theorem 3. 6 and the hypothesis $|B| \geq 3$

$$p > |A| \geq |B + Z| \geq |B| + |Z| - 1 \geq |Z| + 2.$$

So we have,

$$|Y| = |A| - |Z| \geq 2.$$

Now we want to show that for some $a \in Y$, $|B_a| \geq 2$. Assume the contrary, i. e., for every $a \in Y$, $|B_a| < 2$.

This assumption and the assumption that $0 \in B$ implies that

$$B \cap (A - a) = \{0\}.$$

Let us denote $B \setminus \{0\}$ by E so that $E \cap (A - a) = \emptyset$. From Corollary 3. 9 (i), we have $(E + a) \cap A = \emptyset$. Since this is true for every $a \in Y$, we then have $(E + Y) \cap A = \emptyset$.

We know that $E + Y \subseteq A + B$, and $A \subseteq A + B$ so we get

$$|E + Y| \leq |A + B| - |A| = |A| + |B| - 1 - |A| = |B| - 1 = |E|;$$

here we used the fact that the pair (A, B) is critical.

On the other hand, from Theorem 3. 6, and the facts that $Y \subseteq A$ and $E \subseteq B$, and $|Y| \geq 2$,

$$p > |E + Y| \geq |E| + |Y| - 1 \geq |E| + 1.$$

since $|Y| \geq 2$. This is a contradiction, so $|B_a| \geq 2$ for some $a \in Y$.

This completes the proof of the claims.

We now prove the statement of the theorem by using induction on the size of B.

By the corollary to Claim 1, when $|B| = 2$, (A, B) is a standard pair.

The induction hypothesis is that the pair (A, B) is a standard pair for $2 \leq |B| \leq k$; note that our initial hypotheses are still in effect, that is, $|A| + |B| < p$ and $\min(|A|, |B|) > 1$.

Let $3 \leq |B| \leq k + 1$. Since $3 \leq |B|$, by the last claim, we have $|B| > |B_a| \geq 2$ for some $a \in A$. The idea is to show that the pair (A^a, B_a) is a critical pair. Since the pair (A, B) is critical and we assume that $|A| + |B| < p$

$$\begin{aligned} p - 1 > |A| + |B| - 1 &= |A + B| \\ &\geq |A^a + B_a| && \text{by Lemma 3. 5 (ii)} \\ &\geq |A^a| + |B_a| - 1 && \text{by Theorem 3. 6} \\ &= |A| + |B| - 1. \end{aligned}$$

Hence

$$|A^a + B_a| = |A^a| + |B_a| - 1$$

which says that the pair (A^a, B_a) is a critical pair.

So the pair (A^a, B_a) is a standard pair by the induction hypothesis, since $|B| > |B_a| \geq 2$, and $|A^a| + |B_a| < p$. This implies that $A^a + B_a$ is a standard set.

Above we obtained

$$| A^a + B_a | = | A + B |.$$

From Lemma 3. 5 (ii), we have $A^a + B_a \subseteq A + B$. Hence, we have the equality $A^a + B_a = A + B$. Since $A^a + B_a$ is a standard set, so is $A + B$. Now we can refer to the third claim so showing the pair (A, B) is a standard pair, and this proves the theorem.■

We need a theorem of Kneser. Though we will not provide a proof here, it can be found in Mann's book [21].

Theorem 3. 11. (Kneser) Let G be an abelian group, and let A and B be finite subsets of G. Then there exists a subgroup H of G such that

$$A + B + H = A + B,$$

and

$$| A + B | \geq | A + H | + | B + H | - | H |.$$

3. GENERALIZATIONS OF THE CAUCHY-DAVENPORT THEOREM AND VOSPER'S THEOREM

In this section we will present some generalizations of the Cauchy-Davenport theorem and Vosper's theorem. We need them to study sum-free sets in groups. This work was initiated by J. H. B. Kemperman [17] and M. Kneser [18].

Before stating our first lemma of this section, we will give some definitions.

Definition 3. 12. Let G be an abelian group. Let C be a subset of G. If H is a non-trivial subgroup of G such that $C + H = C$, then C is a union of cosets of H in G. In this case C is called periodic with period H. Note that H is not uniquely determined from this definition. Since

$$H(C) = \{ g \in G \mid C + g = C \}$$

is a subgroup of G, it is clear that H(C) is the largest period (stabilizer) of C.

In the case where $C + H = C$ implies that $H = \{ 0 \}$, C is called aperiodic.

A subset C of G is called quasi-periodic if there exists a subgroup H of G of order $| H | \geq 2$ such that $C = C' \cup C''$ where C' is the disjoint union of cosets of H in G and C'' is contained in another coset of H in G, i. e., $C'' \subset c + H \quad c \in C''$. We call C'' residual. The subgroup H is called the quasi-period of C.

Note that if C is quasi-periodic, $2 \leq |H| \leq |C|$ for each quasi-period H of C. If each element g of $G \setminus \{0\} = G^*$ is of order greater than $|C|$, then C cannot be quasi-periodic. As the terminology suggests, each periodic set is quasi-periodic as well.

Now we can state the first lemma of this section.

Lemma 3. 13. (Kemperman [17]) Let G be an abelian group. Suppose that a finite subset C of G is the union of the proper non-empty subsets $C_0, C_1, \ldots, C_n$, $n \geq 1$, such that for $i = 0, 1, \ldots, n$

$$|C| < |C_i| + |H(C_i)|. \tag{3.1}$$

Then

(i) $$|C| + |H(C)| \geq |C_i| + |H(C_i)|. \tag{3.2}$$

for at least one $i = 0, 1, \ldots, n$, and

(ii) either C is quasi-periodic or there exists $c \in C$ for which $C - c = H_1 \cup H_2$ where H_1, H_2 are finite subgroups of G of the same order with $H_1 \cap H_2 = \{0\}$.

We omit the proof.

Definition 3. 14. Let G be an abelian group, and let A and B be non-empty subsets of G. Then we say the sum A + B is small when

$$|A + B| \leq |A| + |B| - 1.$$

Now we will give a theorem due to Kneser [18] which is a generalization of the Cauchy-Davenport theorem.

Theorem 3. 15. (Kneser) Let G be an abelian group, and let A and B be finite subsets of G. Assume that

$$|A + B| \leq |A| + |B| - 1. \tag{3.3}$$

Let $H = H(A + B)$ denote the largest period of A + B. Then H satisfies

$$|A + B| + |H| = |A + H| + |B + H|. \tag{3.4}$$

We omit the proof.

4. A FURTHER VARIATION OF VOSPER'S THEOREM

Next we will give a variation of Vosper's theorem. They will provide some information concerning the order of a sum-set which is small compared with the orders of the summand sets.

Theorem 3. 16. (Kemperman [17]) Let G be an abelian group, and let A and B be finite subsets of G with

$$|A|, |B| \geq 2, \qquad \text{and} \qquad |A+B| \leq |A|+|B|-1.$$

Then either A + B is a standard set or A + B is quasi-periodic.

The proof is too complicated and hence will be omitted.

5. CHARACTERIZATION OF PAIRS (A, B) WHERE A + B IS SMALL

Given the pair (A, B) where the sum A + B is small is it possible to characterize the pair (A, B)? Is it also possible to construct such pairs? The forthcoming theorem will answer the above questions. It shows that it is sufficient to consider the case when the sum A + B is aperiodic.

Theorem 3. 17. Let G be an abelian group. The following construction produces exactly all the pairs (A, B) of finite non-empty subsets of G where the sum A + B is small, i. e., $|A+B| \leq |A|+|B|-1$.

Construction: Pick a proper finite subgroup H of G and let ν denote the natural mapping $\nu : G \longrightarrow G/H$. Next choose finite, non-empty subsets A^*, B^* of G/H so that $A^* + B^*$ is aperiodic and

$$|A^*+B^*| = |A^*|+|B^*|-1 \qquad (3.5)$$

Finally, we let A and B be any subsets of $\nu^{-1}A^*$ and $\nu^{-1}B^*$, respectively, with

$$|\nu^{-1}A^* \cap A^c| + |\nu^{-1}B^* \cap B^c| < |H| \qquad (3.6)$$

Then this construction generates a pair (A, B) satisfying (3. 3), and any pair satisfying (3. 3) may be constructed in this way.

We omit the proof.

6. PAIRS (A, B) WHERE A + B IS SMALL AND IS A STANDARD SET

In this section we will investigate the following problem.

Let G be an abelian group, and let A and B be finite non-empty subsets of G so that the sum A + B small, and min ($|A|$, $|B|$) $\geq$ 2. We know from Theorem 3. 15 and Theorem 3. 16 that either A + B is a standard set or A + B is quasi-periodic. Given such information on A + B, what can we say about the pair (A, B)? We are only interested in the case where A + B is a standard set . The following results are due to Kemperman [17].

Lemma 3. 18. Let G be an abelian group of order n, and let A and B be finite non-empty subsets of G where the sum A + B is a standard set with difference d, and $|A + B| < n$. Then

$$|A+B| \geq |A|+|B| - 1.$$

Lemma 3. 19. Let G be an abelian group of order n, and let A and B be non-empty subsets of G with the following properties. The sum A + B is small, A + B is a standard set with difference d and $|A + B| \leq n - 2$. Here n is the order of the element d. Then A and B are standard sets of difference d, and we have

$$|A+B|=|A|+|B| - 1.$$

We omit the proofs.

An immediate corollary is the following.

Corollary 3. 20. Let G be an abelian group, and let A and B be non-empty subsets of G. Suppose that min ($|A|$, $|B|$) $\geq$ 2, the sum A + B is small and every $g \in G \setminus \{ 0 \}$ has order at least $|A + B| + 2$. Then each of A, B and A + B are standard sets with difference d.

Note that the above corollary for the special case that G is a cyclic group of prime order is due to Vosper [34], and was later rediscovered by S. Chowla and E. G. Straus.

7 MAIN RESULTS: SUM-FREE SETS IN GROUPS

In the previous sections, we have prepared ourselves for the real meat of this chapter which is sum-free sets in groups.

We gave earlier in Corollary 3. 3 an upper bound on $\lambda(G)$, the cardinality of a largest maximal sum-free set in G. Now we give a lower bound on $\lambda(G)$, also due to Erdös [11]. If G is any finite abelian group, then

$$\frac{2|G|}{7} \leq \lambda(G).$$

In the introduction to this chapter, we noted that the upper bound is best possible and is attainable if we take $G = \mathbb{Z}_2$. Now we will show that this lower bound is also best possible and attainable if we take $G = \mathbb{Z}_7$.

Claim: In $\mathbb{Z}_7$ a maximal sum-free set cannot have more than 2 elements.

Proof of the claim: In $\mathbb{Z}_7$, we can show, with some computations, that the 2-element sum-free sets in $\mathbb{Z}_7$ are $\{1, 3\}$, $\{1, 5\}$, $\{1, 6\}$, $\{2, 3\}$, $\{2, 5\}$, $\{2, 6\}$, $\{3, 4\}$, $\{4, 5\}$, and $\{4, 6\}$. We can divide these nine sets into classes as follows. We take the set $S = \{1, 3\}$ and consider kS, where $2 \leq k \leq 6$ and we are doing the arithmetic modulo 7. We find that

$2S = \{2, 6\}$, $3S = \{3, 2\}$, $4S = \{4, 5\}$,

$5S = \{5, 1\}$, $6S = \{6, 4\}$.

So we have a class

$$C_1 = \{kS \mid 1 \leq k \leq 6\} = \langle \{1, 3\} \rangle.$$

Now we take $T = \{1, 6\}$ and consider kT where $2 \leq k \leq 3$. We get in this case

$2T = \{2, 5\}$, $3T = \{3, 4\}$.

Hence we have another class

$$C_2 = \{kT \mid 1 \leq k \leq 3\} = \langle \{1, 6\} \rangle.$$

It suffices to show that a representative of each class C_i is maximal in order to show that these nine sets are maximal. We will take $S = \{1, 3\}$ and $T = \{1, 6\}$ as representatives of classes C_1 and C_2, respectively.

Let us look at the entire list of sum-free sets in $\mathbb{Z}_7$.

We say that an element a is compatible with an element b if $\{a, b\}$ is a sum-free set.

According to the above definition and by inspection, we see that except 3, 5, and 6 no element is compatible with 1. Again by inspection, we see that 1, 2, and 4 are compatible with 3. Since the set of compatible elements with 1 and the set of compatible elements with 3 have no element in common, we cannot add a third element to the set S to obtain a larger sum-free set.

Similarly, we can show that we cannot add a third element to the set T to obtain a larger sum-free set.

Therefore the classes C_1 and C_2 contain only maximal sum-free sets in $\mathbb{Z}_7$. Hence $\lambda(\mathbb{Z}_7) = 2$ which attains the lower bound.■

In the theorem and corollary below, we will answer the following question.

Can one find a necessary and sufficient condition on G so that the upper bound on $\lambda(G)$ is attained, i. e., $\lambda(G) = \frac{|G|}{2}$?

Although we are mainly interested in abelian, finite groups the next theorem is proven for all groups.

Theorem 3. 21. Let S be a finite subset of a group G. Then $|S + S| = |S|$ if and only if there exists a finite subgroup H of G so that

$$S + H = S = H + S \quad \text{and} \quad S - S = H = -S + S.$$

Proof: Assume for a finite subset S of G we have $|S + S| = |S|$. In order to show the existence of a finite subgroup H of G, we choose $s_1, s_2 \in S$ and define

$$H_1 = -s_1 + S, \qquad H_2 = S - s_2.$$

Then

$$|H_1 + H_2| = |-s_1 + S + S - s_2| = |S + S| = |S| = |H_1| = |H_2|$$

and so is finite.

Now consider $(-s_1 + s_1) + (s_2 - s_2) = 0$. This implies that $0 \in H_1$ and $0 \in H_2$, so $0 \in H_1 + H_2$.

Then, since $H_1 \subseteq H_1 + H_2$ and $H_2 \subseteq H_1 + H_2$ we have $H_1 \cup H_2 \subseteq H_1 + H_2$. So

$$|H_1 \cup H_2| \leq |H_1 + H_2| = |H_1| = |H_2|.$$

Then we have $H_1 + H_2 = H_1 = H_2$. Let us call this set H. We would like to show that H is finite subgroup of G.

Suppose $h \in H$. Then $h = -s_1 + s_3$, and $h = s_4 - s_2$ for some $s_3, s_4 \in S$. So we have

$-h = -s_3 + s_1$, and $-h = s_2 - s_4$ which imply $-h \in H$. We already have $0 \in H$ and H is closed under addition, so H is a finite subgroup of G.

Since $H_1 = H_2 = H$, we have $H = -s + S = S - s$ for all $s \in S$. Adding s to both sides of the last equality.

$$s + H = S = H + s$$

for all $s \in S$. Eventually, we get

$$S + H = S = H + S.$$

Now assume the converse. Since $S + H = H + S = S$, S is a union of left or right cosets of H in G. That is,

$$S = \bigcup_{s \in S} (s + H) = \bigcup_{s \in S} (H + s).$$

Hence $| S |$ is some multiple of $| H |$, say $| S | = k| H |$, where k is the number of cosets of H in G, $k \geq 1$.

Since $H = S - S$, we have $| H | = | S - S | \geq | S |$. Therefore we have $| H | = | S |$ and S is both a left and a right coset of H.

Hence

$$S = s + H = H + s$$

for all $s \in S$. Then we get

$$S + S = s + H + H + s = s + H + s,$$

and so

$$| S + S | = | H |.$$

In other words

$$| S + S | = | S |.\blacksquare$$

Corollary 3. 22. Let G be an abelian group and let $| G | = 2m$. Then $\lambda(G) = m$ if and only if G has a subgroup H of order m, and in this case the maximal sum-free set is the coset aH, $a \notin H$.

Proof: This follows directly from Theorem 3. 21 and the fact that every abelian group of order 2m has a normal subgroup of order m.■

A lower bound on $\lambda(G)$ when G is abelian is hard to achieve. We will try to determine the size and the structure of maximal sum-free sets S in arbitrary abelian groups G. We will study certain cases depending on the prime divisors of $|G|$, the order of G, to find a lower bound on $\lambda(G)$.

Except for Theorem 3. 21 G will denote an abelian group.

In Definition 3. 12, we have defined the largest period $H(C)$ for any subset C of G. Some facts about $H(C)$ are listed below.

(I) $C + H(C) = C$,

(II) if $C + K = C$ for some subgroup K of G, then $K \leq H(C)$,

(III) the subgroup generated by $H(C)$ and $H(D)$, $\langle H(C), H(D) \rangle$, is contained in $H(C + D)$,

(IV) $H(S) = H(-S)$.

We will show that there exists a subgroup H of G such that $H = H(S + S) = H(S) = H(S - S)$ for a maximal sum-free set S in G.

The subgroup $H = H(S + S)$ so that $S + S + H = S + S$ exists by Theorem 3. 11. Hence we have by Theorem 3. 15 either

$$|S + S| \geq 2|S|$$

or

$$|S + S| = 2|S + H| - |H|. \tag{3.7}$$

Lemma 3. 23. Let S be a maximal sum-free set in G, and let $H = H(S + S)$. Then $S + H$ is a sum-free set in G and therefore $S + H = S$.

The proofs of the lemma and the following corollaries will be omitted.

Corollary 3. 24. Let S be a maximal sum-free set in G. Let $H = H(S + S)$. Then

$$H = H(S + S) = H(S) = H(S - S).$$

Corollary 3. 25. Let S be a maximal sum-free set in G. Let $H = H(S + S)$.

Then either

$$|S + S| \geq 2|S|$$

or

$$| S + S | = 2| S | - | H |,$$

and either

$$| S - S | \geq 2| S |$$

or

$$| S - S | = 2| S | - | H |.$$

In the next lemma, we will give some upper bounds on $\lambda(G)$ depending on the order of G.

Lemma 3. 26.(Diananda and Yap [7]) Let G be a finite group. We will consider the following cases.

(i) $| G |$ has at least one prime factor p of the form $3n + 2$; without loss of generality we may assume p is the smallest such prime,

(ii) no prime p of the form $3n + 2$ divides $| G |$, but $3 \mid | G |$,

(iii) $| G |$ is a product of primes each of which is of the form $3n + 1$.

Then

$$\lambda(G) \leq \begin{cases} \dfrac{| G |(p+1)}{3p} & \text{in case (i)} \\ \dfrac{| G |}{3} & \text{in case (ii)} \\ \dfrac{| G | - 1}{3} & \text{in case (iii).} \end{cases}$$

The proof of Lemma 3. 26 is omitted.

Note that the cases considered in the above lemma exhaust all possibilities for $| G |$ and they are mutually exclusive.

Remark: In cases (i) and (ii), the structures and sizes of maximal sum-free sets are fully determined. That is, the upper bounds given above in the first two cases are exact values. Yet we know very little about the third case. In the last case, even the size of the maximal sum-free sets is known for special cases only.

We will demonstrate the first two cases by giving examples later on.

The following theorem deals with the case when $| G |$ is divisible by a prime p of the form $3n + 2$.

Theorem 3. 27. In the first case of Lemma 3. 26 the upper bound is attainable. Also, if S is a maximal sum-free set in G, then S is a union of cosets of some subgroup H of index p in G, S/H is a standard set in G/H and $S \cup (S+S) = G$.

Proof: (i) Let us denote p by 3n - 1. Consider $G = \mathbb{Z}_p$. By Lemma 3. 26, $\lambda(G) \leq n$. If we look at the set $S = \{ n, n+1, \ldots, 2n-1 \}$, we see that S is sum-free and $|S| = n$. This gives $\lambda(G) = n$.

The Cauchy-Davenport theorem (Theorem 3. 6) gives $|S+S| \geq 2n-1$, and since S is sum-free we have $|S+S| \leq 2n-1$. So $|S+S| = 2n-1$. Hence the pair (S, S) is critical and by Vosper's theorem (Theorem 3. 11) the pair (S, S) is a standard pair, so S is a standard set.

Without loss of generality since p is prime, we can take d = 1, the common difference of the progression. This gives us that, up to automorphism, $S = \{ n, n+1, \ldots, 2n-1 \}$ is the only possible set.

(ii) We can generalize the idea in (i). Let K be a subgroup G of index p, and let g be an element of order p so that

$$G = K \cup (K+g) \cup \ldots \cup (K+(p-1)g).$$

Consider the set

$$T = \bigcup_{j=n}^{2n-1} (K+jg).$$

To show that T is sum-free, consider k + jg and k' +ig, where $k, k' \in K$ and $n \leq i \leq j \leq 2n-1$. If we take the smallest possible value, n, for i and j, we end up with $k + k' + 2ng$ which is not in T. For larger values of i and j, we have the same conclusion. The size of T is $\frac{|G|(p+1)}{3p}$, since there are n cosets and the size of K is $\frac{|G|}{p}$. Therefore T is a maximal sum-free set in G and $\lambda(G) = \frac{|G|(p+1)}{3p}$.

Assume now S is a maximal sum-free set with $\frac{|G|}{3}\frac{(p+1)}{p}$ elements. Assume H is a subgroup of G for which the second option in (3. 7) holds. Then, we have

$$\frac{|G|(p+1)}{3p} \leq \frac{|H|}{3} \left[\frac{|G|}{|H|} + 1 \right],$$

so that $|H| = \frac{|G|}{p}$.

We know, from Lemma 3. 23, that S is a union of cosets of H. By the assumption, $|S| = \frac{|G|(p+1)}{3p}$, and (3. 7) we get $|S+S| + |S| \geq |G|$. Since S is sum-free, we get

$|S+S| + |S| \leq |G|$, hence we have the equality. Therefore $S \cup (S+S) = G$,

$|S+S| = 2|S| - |H|$. In view of the last equality we get $|(S/H) + (S/H)| = 2|S/H| - 1$, where

S/H is a subset of the factor group G/H. By an extension of Vosper's theorem (Diananda [6]),

we have S/H as a standard set, and this subset as in (i) is isomorphic to the set

$\{ n, n+1, \dots, 2n-1 \}$.■

Example: Take p = 17. Then in $\mathbb{Z}_{17}$, $\lambda(\mathbb{Z}_{17}) = 6$ and $S = \{ 6, 7, 8, 9, 10, 11 \}$ is a

maximal sum-free set .

The following theorem deals with the case when $|G|$ is divisible by 3 but not by any prime p

of the form 3n + 2

Theorem 3. 28. In the second case of Lemma 3. 26 the upper bound is attainable. Also, if

S is a maximal sum-free set in G, then S is a union of cosets of some subgroup H of G, such that

G/H is the cyclic subgroup $\mathbb{Z}_{3m}$ for some m, S/H is a standard set in G/H and

$$|S+S| = 2|S| - |H|.$$

Proof: (i) Obviously G has a subgroup K of order $\frac{|G|}{3}$ and an element g of order 3 such that

$$G = K \cup (K+g) \cup (K+2g).$$

Then it is easy to see that the set $T = K+g$ is sum-free and has $\frac{|G|}{3}$ elements. So T is maximal by

Lemma 3. 26, hence $\lambda(G) = \frac{|G|}{3}$.

(ii) We now let S be a maximal sum-free set in G with $|S| = \frac{|G|}{3}$. Moreover, let H be a

subgroup of G which is the largest period of S + S. Hence Corollary 3. 24 it is the largest period

of S by , so S is a union of cosets of H, and $|H| = \frac{|G|}{3m}$, for some m.

Since $S \cap (S+S) = \emptyset$, we have

$$|S+S| \leq |G| - |S| = |G| - \frac{|G|}{3} = \frac{2|G|}{3}.$$

But Corollary 3. 25 tells us that we have either

$$|S+S| \geq 2|S|$$

or

$$|S+S| = 2|S| - |H|.$$

Therefore we have to consider the following two cases.

(a) $|S+S| = 2|S| - |H|$

and

(b) $|S+S| = 2|S|$.

Our claim is that (b) cannot happen. Actually, this was conjectured by Yap[41] and proved by Street[28].

Observe that if A is any subset of G with the property that $A = -A$, then $|A|$ is odd if and only if $0 \in A$ since $|G|$ is odd.

Since S is sum-free and no sum-free set can contain 0 and $|S|$ is odd, then by the above observation $S \neq -S$. But $0 \in (S-S) = -(S-S)$ and hence $|S-S|$ is odd. Now $S \cap (S+S) = \emptyset$ and so by Corollary 3. 9, we have $(S-S) \cap S = \emptyset$. Hence

$$\begin{aligned} S \cap (S+S) &= ((S-S) \cap S) \cup (S \cap (S-S)) \\ &= ((S-S) \cap S) \cup ((S-S) \cap (-S)) \\ &= (S \cup (-S)) \cap (S-S) \\ &= \emptyset. \end{aligned}$$

By Corollary 3. 25, we have same possibilities for S - S that is, either

(a') $|S-S| = 2|S| - |H|$

or

(b') $|S-S| = 2|S|$.

Since $|S-S|$ is odd (b') is not possible. Hence we have (a').

Let us call the factor group G/H, G~, and its subset S/H, S~. Obviously S~ is a maximal sum-free set in G~.

From Corollary 3. 24, we have

$$H = H(S) = H(S+S) = H(S-S).$$

Therefore both S~ and S~ - S~ are aperiodic. Note that

$$|S\sim - S\sim| = 2|S\sim| - 1 = 2m - 1 \qquad (3.8)$$

and

$$| S\sim \cup (S\sim - S\sim) | = | S\sim | + | S\sim - S\sim | - | S\sim \cap (S\sim - S\sim) |$$

$$= m + 2m - 1 = 3m - 1 = | G\sim | - 1.$$

Hence Theorem 3. 16, and (3. 8) say that $S\sim - S\sim$ is either quasi-periodic or a standard set.

Suppose now that $S\sim - S\sim$ is quasi-periodic. That is,

$$S\sim - S\sim = T' \cup T''$$

where

$$T' = T' + U\sim, \qquad T'' \subseteq t + U\sim,$$

and $U\sim$ is a subgroup of $G\sim$ and $t \in T''$. Since $S\sim - S\sim = -(S\sim - S\sim)$, we have $T'' \subseteq U\sim$. If $S\sim \cap U\sim \neq \emptyset$, then from $S\sim$ being sum-free we deduce that no complete coset of $U\sim$ is contained in $S\sim$. But we assumed that $S\sim - S\sim$ is quasi-periodic, so we must have $S\sim \cap U\sim = \emptyset$. But this implies that $S\sim$ is periodic with period $U\sim$, which is another contradiction. Therefore $S\sim - S\sim$ must be a standard set with difference d. Now (3. 8) tells us that the order of d is 3m.

From Lemma 3. 19, we know that $S\sim$ is also a standard set with difference d, so

$$| S\sim + S\sim | = 2 | S\sim | - 1$$

and

$$| S + S | = 2 | S | - | H |.$$

Since $| G\sim | = 3m$, and $G\sim$ contains an element d of order 3m, hence $G\sim$ turns out to be the cyclic group $\mathbb{Z}_{3m}$. If $\eta \in \mathcal{A}ut(G\sim)$ such that $\eta(d) = 1$, then $\eta(S\sim) = \{ m, m + 1, \dots , 2m - 1 \}$.■

We give an example below.

Example: To illustrate the theorem, we take $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Then $\lambda(G) = \frac{|G|}{3} = 3$. Let $H = \{ (0, 0), (0, 1), (0, 2) \}$ and define $S = \{ (1, 0), (1, 1), (1, 2) \}$. Then $S = (1, 0) + H$, so $S/H = \{ (1, 0) \}$. If we compute $S + S$, we see that S is sum-free and $| S + S | = 2| S | - | H |$. Also $S + S = \{ (2, 0), (2, 1), (2, 2) \} = (2, 0) + H$. So $G = H \cup ((1, 0) + H) \cup ((2, 0) + H)$.

Corollary 3. 29. Let $G = \mathbb{Z}_m$ where $m = 3^n$ for some n. Then there are precisely n non-isomorphic maximal sum-free sets in G.

Corollary 3. 30. Let G be an elementary abelian 3-group. If S is a maximal sum-free set in G, then S is a coset of a maximal subgroup of G.

Proof: Since $|G| = 3^n$ for some k, then G has a subgroup K of index 3, and an element g of order 3 such that

$$G = K \cup (K + g) \cup (K + 2g).$$

Since the index of K in G is 3, we have $|K| = 3^{n-1}$ so K is maximal subgroup of G, and the set $S = K + g$ is obviously sum-free.■

We note that if we drop the adjective "elementary" in Corollary 3. 30, a maximal sum-free set S in a 3-group G is not necessarily a union of cosets of a maximal subgroup H of G.

For this we take $G = \mathbb{Z}_9$, then $\lambda(G) = 3$. The set $S = \{2, 3, 7\}$ is a maximal sum-free set in $\mathbb{Z}_9$, but it is not a coset of $H = \{0, 3, 6\}$ which is a maximal subgroup of $\mathbb{Z}_9$.

Corollary 3. 31. Let p be a prime of the form 3n + 1.

(a) Consider $G = \mathbb{Z}_{3p}$. If S is a maximal sum-free set in G, then

(i) S is a coset of the subgroup H of order p, and

(ii) S may be mapped under some h to the set $\{p, p + 1, \dots, 2p - 1\}$ where $\eta \in \mathcal{A}ut(G)$.

(b) Consider $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_p$. If S is a maximal sum-free set in G, then

(i) S is a coset of the subgroup H of order 3p,

(ii) S is the union of p cosets of a subgroup K of G of order 3 where G/K is cyclic and S/K is a maximal sum-free set in G/K.

The proof is omitted.

The following theorem deals with the case when $|G|$ is divisible by primes p of the form 3n + 1.

Theorem 3. 32. In the third case of Lemma 3. 26, let m be the exponent of G; i. e., m is the smallest positive integer such that mg = 0 for all $g \in G$.

Then

$$\frac{(m-1)|G|}{3m} \leq \lambda(G) \leq \frac{|G| - 1}{3}.$$

Proof: It suffices to establish the lower bound since we have the upper bound by Lemma 3. 26.

In G, there is a subgroup K of index m and an element g with order m where $g \in G \setminus K$. Then we have a partition of G as follows

$$G = K \cup (K + g) \cup \ldots \cup (K + (m-1)g).$$

Then the following set is manifestly sum-free

$$T = (K + 2g) \cup (K + 5g) \cup \ldots \cup (K + (m-2)g).$$

There are $\frac{m-1}{3}$ terms in the union; the order of K is $\frac{|G|}{3}$, hence $|T| = \frac{(m-1)|G|}{3m}$, which is the required lower bound.■

It was conjectured in Diananda and Yap's paper [7] that in this case $\lambda(G)$ equals its lower bound. Rhemtulla and Street [23] proved this conjecture for elementary abelian p-groups. This will be our Theorem 3. 35.

In particular, if G is a cyclic group, then $|G| = m$ and m is the exponent of G. Hence we have the following corollary.

Corollary 3. 33. If G is a cyclic group $G = \mathbb{Z}_m$, where m is the product of primes of the form $3n + 1$, then

$$\lambda(G) = \frac{m-1}{3}.$$

Remark: If $|G| = m$, where m is a product of primes p each of which is of the form $3n + 1$, the minimum value of the quotient $\frac{m-1}{3m}$ is $\frac{2}{7}$, which is Erdös' lower bound.

We need some more terminology at this step.

Definition 3. 34. Let G be a group, H a subgroup of G and S a maximal sum-free set in G. Then we say S avoids H if and only if $S \cap H = \emptyset$, and S covers H if and only if $S \cap H$ is a maximal sum-free set in H.

We know the size and structure of maximal sum-free sets in a group G where all divisors of $|G|$ are congruent to 1 modulo 3 for the cyclic groups only. The next theorem will tell us about the size of a maximal sum-free set in an elementary abelian p-group where p is a prime of the form $3n + 1$.

Theorem 3. 35. Let G be an elementary abelian p-group where p is a prime of the form $3n + 1$. Let $|G| = p^k$ for some k. Then $\lambda(G) = np^{k-1}$.

The next task is to determine the structure of the maximal sum-free sets in the group of order p, $p = 3n + 1$. But first we will give an example to demonstrate the preceding theorem.

Example: To illustrate the theorem, take $n = 2, k = 5$; so $G = \mathbb{Z}_7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_7$. Let $S = \{2, 3\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_7$. That is, $S = \{(x, y_1, y_2, y_3, y_4) \mid x \in \{2, 3\}, y_i \in \mathbb{Z}_7\}$. Then, obviously, S is a maximal sum-free set in G and $\lambda(G) = |S| = 2.7^4$.

Lemma 3. 36. Let $G = \mathbb{Z}_n$, where $n = 3k + 1$ is not necessarily prime. Let S be a sum-free set in G satisfying

$$|S| = k, \qquad S^c = S + S, \quad \text{and} \qquad S = -S. \tag{3.9}$$

Then

(i) $(S + g) \cap S = \emptyset$ if and only if $g \in S$,

(ii) if $|(S + g) \cap S| = 1$ for some $g \in G$, then $|(S + g') \cap S| \geq k - 3$, where $g' = \frac{3g}{2}$ and $\pm\frac{g}{2} \in S$, and

(iii) if $|(S + g) \cap S| = \lambda > 1$ for some $g \in G$, then there exists $g' \in G$ such that $|(S + g') \cap S| \geq k - (\lambda + 1)$.

The next theorem will tell us about the structure of a maximal sum-free set in the group $G = \mathbb{Z}_p$, where $p = 3n + 1, n > 2$. The proof will not be given here. The reader is advised to refer to the paper by Rhemtulla and Street [24].

Theorem 3. 37. Let $G = \mathbb{Z}_p$ and $p = 3n + 1, n > 2$. Then any maximal sum-free set S in G may be mapped, under some automorphism of G, to one of the following sets:

$$A = \{n, n + 2, \ldots, 2n - 1, 2n + 1\},$$

$$B = \{n, n + 1, \ldots, 2n - 1\},$$

$$C = \{n + 1, n + 2, \ldots, 2n\}.$$

If $p = 7$, i. e. ,when $n = 2$, sets of type A cannot occur.

In order to characterise the maximal sum-free sets in elementary abelian p-groups, $p = 3n + 1$;

we need one additional lemma.

Lemma 3. 38. Let $G = \mathbb{Z}_p$ and $p = 3n + 1$, p prime (and consequently n even). Let S be a maximal sum-free set in G with

(i) S is isomorphic to $C = \{ n+1, n+2, \ldots, 2n \}$, and
(ii) $S \subseteq \{ \frac{n}{2}+1, \ldots, \frac{5n}{2} \}$.

Then either

$$S = C \qquad \text{or} \qquad S = C' = \{ \tfrac{n}{2}+1, \ldots, n, 2n+1, \ldots, \tfrac{5n}{2} \}.$$

We now are in a position that we can characterise the maximal sum-free sets S in an elementary abelian p-group G. This characterization will be given in the next theorem without a proof. However, the proof can be found in Rhemtulla and Street [24].

Theorem 3. 39. Let G be an elementary abelian p-group and $|G| = p^k$, $p = 3n + 1$, p prime and $n > 2$. Let S be a maximal sum-free set in G. Let, moreover, G have the following representation

$$\{ (i_1, i_2, \ldots, i_k) \mid i_j \in \mathbb{Z}_p, \ 1 \leq j \leq k \}.$$

Then, under some automorphism of G, S can be mapped one of the following $2k + 1$ sets:

${}^kA_k = \{ (i_1, i_2, \ldots, i_k) \mid i_k \in A \}$;

${}^kA_{k-r} = \{ (i_1, i_2, \ldots, i_k) \mid \text{not all } i_1, \ldots, i_r = 0, i_k \in C \} \cup \{ (0, \ldots, 0, i_{r+1}, \ldots, i_k) \mid i_k \in A \}$

for $1 \leq r \leq k-1$;

${}^kB_k = \{ (i_1, i_2, \ldots, i_k) \mid i_k \in B \}$;

${}^kB_{k-r} = \{ (i_1, i_2, \ldots, i_k) \mid \text{not all } i_1, \ldots, i_r = 0, i_k \in C \} \cup \{ (0, \ldots, 0, i_{r+1}, \ldots, i_k) \mid i_k \in B \}$

for $1 \leq r \leq k-1$;

${}^kC = \{ (i_1, i_2, \ldots, i_k) \mid i_k \in C \} = {}^kA_0 = {}^kB_0$,

where the sets A, B, and C were defined in Theorem 3. 37.■

Remark: We noted earlier that if $n = 2$, then the sets of type A do not occur. If $|G| = 7^k$, then there are $k + 1$ non-isomorphic maximal sum-free sets, namely ${}^kB_{k-r}$ and ${}^kC$ where $0 \leq r \leq k-1$.

We will state a theorem without a proof which characterises the maximal sum-free sets in cyclic groups of prime power order, for primes congruent to 1 modulo 3. A prime number p is said to be bad if it is congruent to 1 modulo 3.

Lemma 3. 40. Let $G = \mathbb{Z}_n$, $n = 3k + 1$. Let S be a maximal sum-free set in G and let H be a subgroup of G of order m. Let S_i be the subset of H for which

$$S_i + i = S \cap (H + i)$$

where H + 1 generates G/H. Then the cosets of H, more than half of whose elements belong to S, form a sum-free set in G/H.

Theorem 3. 41. (Yap) Let $G = \mathbb{Z}_n$, $n = p^e = 3k + 1$ and p is a bad prime. Then any maximal sum-free set S may be mapped, under some automorphism of G, to one of the following sets

$$A = \{ k, k + 2, \dots , 2k - 1, 2k + 1 \};$$

$$B = \{ k, k + 1, \dots , 2k - 1 \};$$

$$C = \{ k + 1, k + 2, \dots , 2k \}.$$

The proof of this theorem is quite long, it uses the Theorems 3. 15, 3. 16, 3. 17, the Lemmas 3. 23, 3. 36, 3. 40, and the Corollaries 3. 24.,and 3. 33.

A complete characterisation of maximal sum-free sets in abelian groups of order $3^m n$ where $m \geq 1$ and every prime divisor p of n (if $n > 1$) is bad is given by H. P. Yap [45].

Theorem 3. 42. (Yap) Let G be an abelian group of order $3^m n$ ($m \geq 1$) where every prime divisor p of n is bad.

Then either there exists a non-trivial subgroup H, of order $\frac{|G|}{3q}$, where $3q \mid |G|$, of G such that S is a union of cosets of H and S/H is maximal sum-free set in G/H or $|S + S^*| = |S| + |S^*| - 1$ where $S^* = -S \cup S$, and thus $S = (S + S^*)^c$ is a standard set.

Corollary 3. 43. Under the same hypotheses of Theorem 3. 42, if G has exponent less than or equal to $\frac{|G|}{3}$, then S is a union of cosets of a non-trivial subgroup H, of order $\frac{|G|}{3q}$, of G and S/H is a maximal sum-free set in G/H.

Proof: Suppose that

$$S = \{ s + id \mid i = 0, 1, \dots, 3^{m-1}n - 1 \}, \text{ for some } s, d \in G, d \neq 0.$$

If the order of $d \in G$ is $\frac{|G|}{3}$, then S is a coset of a subgroup H, of order $\frac{|G|}{3}$, of G. If the order of d is strictly less than $\frac{|G|}{3}$, then $|S| < \frac{|G|}{3}$ which is not possible. So we have the corollary.■

We will give the statements of two theorems only.

Theorem 3. 44. Let $G = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$, where $p = 3k + 1$. Let $H_0 = \langle (p, 0) \rangle \oplus \langle (0, 1) \rangle$, and $H_i = \langle (1, i) \rangle$ for $1 \le i \le p$, $K = \langle (p, 0) \rangle$. Let S be a maximal sum-free set in G. If $|S| > kp(p + 1)$, then there exists a λ such that

$$kp + \lambda = \max_{0 \le i \le p} \{ |S \cap H_i| \}$$

and

$$|S \cap K| = m < \lambda < k.$$

Theorem 3. 45. Assume the hypotheses of Theorem 3. 44. Let $|S \cap H_1| = kp + \lambda$, $H_1 = \bigcup_{i=0}^{p-1} K_i$ where $K_0 = K$, $K_i = x_i + K$, $x_1 + x_1 = x_2$, $x_1 + x_2 = x_3$, ..., and let moreover $x_i + S_i = S \cap K_i$ for $0 \le i \le p - 1$. Then

$$\max \{ |S_1|, |S_2|, \ldots, |S_{p-1}| \} \ge k + 2,$$

and at least one of the S_i is empty.

For the proofs, see Yap [45].

One can show by using Theorem 3. 44, and Theorem 3. 45 that $\lambda(\mathbb{Z}_{7^2} \oplus \mathbb{Z}_7) = 112$.

8. GROUP RAMSEY THEORY

In this section, we will give a brief review of Group Ramsey Theory which deals with finding the smallest number of sum-free sets needed to partition $G^* = G \setminus \{ 0 \}$.

The reader is advised to refer to Definition 3. 1. We will denote $R_n(3, 2)$ by R_n.

Theorem 3. 46. Let G be an additive group. Every sum-free partition of G^* can be embedded in at least one covering of G^* by maximal sum-free sets.

Proof: Let $\mathcal{S} = \{ S_1, S_2, \ldots, S_n \}$ be a sum-free partition of G^*. Then for each i we adjoin elements of G^* to S_i, provided that S_i is still sum-free, until a maximal sum-free set, T_i, is obtained. Note that T_i may not be unique. Now $\mathcal{T} = \{ T_1, T_2, \ldots, T_n \}$ is a covering of G^* by maximal sum-free sets, each $S_i \subseteq T_i$ and so $\mathcal{S}$ is embedded in $\mathcal{T}$.■

Corollary 3. 47. Each of the maximal sum-free sets of the previous theorem has cardinality less than R_{n-1}.

Proof: Consider the following collection of maximal sum-free sets.

$$\mathcal{L} = \{ T \mid S_i \subseteq T \text{ for some } i \}.$$

$\mathcal{L}$ has a maximal element, say T_0. Hence for every $T \in \mathcal{L}$, $|T| \leq |T_0|$.

We may assume, without loss of generality, $S_1 \subseteq T_0$. Now, we form the collection

$$\mathcal{R} = \{ T_0, S_2 \setminus T_0, \ldots, S_n \setminus T_0 \}.$$

For all $i = 2, 3, \ldots, n$, we have $T_0 \cap (S_i \setminus T_0) = \emptyset$ and $S_1 \subseteq T_0$. This collection is a sum-free partition of G^*. We know, from Section 1, that $|G| \leq R_{n-1}$. This proves the corollary..■

Since every sum-free partition of G^* can be embedded into at least one covering of G^* by maximal sum-free sets, we need only consider coverings of G by maximal sum-free sets.

Greenwood and Gleason gave sum-free partitions of $\mathbb{Z}_5$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_{41}$; and Whitehead gave the sum-free partitions of $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_7 \oplus \mathbb{Z}_7$. Also, we have

$$\mathbb{Z}_{13} = \{4, 6, 7, 9\} \cup \{1, 5, 8, 12\} \cup \{2, 3, 10, 11\}.$$

All the sets which appear in these partitions of G are maximal sum-free sets S_i. These sets have one additional property and that is that $S_i \cup (S_i + S_i) = G$. We need the following definition.

Definition 3. 48. Let G be a group, and let S be a maximal sum-free set in G. S is said to fill G if $G^* \subseteq (S + S) \cup S$. If every maximal sum-free set S in G fills G, then G is called a filled group.

Note that if S fills G, we can have

$(S + S) \cup S = G^*$	if and only if	$S \cap (-S) = \emptyset$,
$(S + S) \cup S = G$	otherwise.	

If G is a finite abelian group, then the necessary and sufficient conditions for G to be a filled group are known. If G is a finite non-abelian group, then only necessary conditions are known.

Theorem 3. 49. A finite abelian group G is filled if and only if it is

(i) an elementary abelian 2-group, or

(ii) $\mathbb{Z}_3$, or

(iii) $\mathbb{Z}_5$.

Theorem 3. 50. Let G be a filled finite non-abelian group. Then

(i) for any normal subgroup H of G, G/H is filled, and

(ii) if G' denotes the commutator subgroup of G, then G = G' or G/G' is an elementary abelian 2-group or $G/G' \cong \mathbb{Z}_5$ and $|G|$ is even.

We are not going to provide the proofs here, they can be found in Street and Whitehead [32].

We will give some examples to show that the conditions in Theorem 3. 51 are not sufficient for G to be filled.

(1) Take $G = \mathbf{D}_n$, the dihedral group of order 2n. Let $n = 6k + 1 \geq 2$ and

$$\mathbf{D}_n = \langle s, t \mid s^n = t^2 = 1, sts = s^{-1} \rangle.$$

If we choose

$$S = \{ s^{2k-1}, \ldots, s^{4k}, s^{2k+1}t, \ldots, s^{4k}t \},$$

then S is a maximal sum-free set , which does not fill G.

(2) Take $G = \mathbf{Q}$, the quaternion group of order 8; where

$$\mathbf{Q} = \langle s, t \mid s^4 = 1, s^2 = t^2, sts = t \rangle.$$

Let S be the set of consisting of the only element of order 2. Then S is a maximalsum-free set which does not fill G.

(3) When $G = G' = \mathcal{A}_5$, the alternating group of order five, G is not filled by the maximal sum-free set

$$S = \{ (14)(23), (12)(35), (13)(45), (15)(24), (25)(34), (12)(34),$$
$$(15)(23), (14)(25), (24)(35), (14)(35), (123), (245) \}.$$

By Theorem 3. 51 (i), **SL**(2, 5) is not a filled group, either.

We will now introduce method called isomorph rejection which is effective for generating the family of maximal sum-free sets. This will be done by computing the family for $(\mathbb{Z}_2)^4$ in detail.

Consider $\mathcal{A}ut((\mathbb{Z}_2)^4)$. This group can be viewed as a vector space of dimension 4 over $\mathbf{GF}(2)$ and hence $\mathcal{A}ut((\mathbb{Z}_2)^4) \cong \mathbf{GL}(4, 2)$.

Since any sum-free 1-set consists of a non-identity element, so it is isomorphic to $\{ 0001 \}$. Any sum-free 2-set must generate a subgroup of order 4 and so is isomorphic to $\{ 0001, 0010 \}$. Similarly, any sum-free 3-set generates a subgroup of order 8 andso is isomorphic to $\{ 0001, 0010, 0100 \}$. If we have a sum-free 4-set, then either this set generates the whole group, in which case it is isomorphic to

$$A = \{ 0001, 0010, 0100, 1000 \}$$

or it is contained in a subgroup of order 8 and so by Corollary 3.30 is isomorphic to

$$B = \{ 0001, 0010, 0100, 0111 \}.$$

Any sum-free set which contains more than 4 elements must contain a subset isomorphic to either A or B.

We see that A + A contains all the elements of $(\mathbb{Z}_2)^4$ which have exactly two ones. By adjoining any one of 0111, 1011, 1101, 1110, and 1111 to A we can preserve its sum-freeness. By a simple observation, we see that if we adjoin 1111 and any one of the other four, we cannot have a sum-free set. But, on the other hand, we get sum-free sets as follows.

$$A_5 = A \cup \{ 1111 \}$$

and

$$A_8 = A \cup \{ 0111, 1011, 1101, 1110 \}$$

which are both maximal. Hence $5, 8 \in \Lambda((\mathbb{Z}_2)^4)$.

Now, consider the subgroup $H = \{ 0000, 0011, 0101, 0110 \}$. Clearly $H + 0001 = B$ and $B + B = H$. Take an element of another coset of H, say $B' = B \cup \{ 1000 \}$, and adjoin it to B, then $B' + B' = H \cup (H + 1001)$, then we can adjoin the remaining elements of $H + 1000$ to B'; this construction gives $B_8 = (H + 0001) \cup (H + 1000)$, we see that $B_8 = A_8$.

Take now an element of the other coset of H, say $B'' = B \cup \{ 1001 \}$, then $B'' + B'' = H \cup (H + 1000)$, and by adjoining the remaining elements of $H + 1001$ to B'' we obtain $C_8 = (H + 0001) \cup (H + 1001)$. We see that $C_8 = T(B_8)$, where $T \in \mathbf{GL}(4, 2)$ and

$$
T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},
$$

when we consider the elements of $(\mathbb{Z}_2)^4$ as column vectors.

Hence we have only A_5 and A_8.

In Table 4 of the Appendix we have only two non-abelian groups of order 16 which can be partitioned into three sum-free sets. They are G_4 and G_5. Partitions of $(G_4)^*$ and $(G_5)^*$is given below.

$$(G_4)^* = \{ r, t, st, ts, rsts \} \cup \{ s, rs, rt, (st)^2, sts \} \cup \{ rst, rts, tst, r(st)^2, rtst \},$$
$$(G_5)^* = \{ r, r^3, s, s^3, r^2s^2 \} \cup \{ r^2, rs^3, r^2s, r^2s^3, r^3s \} \cup \{ s^2, rs, rs^2, r^3s^2, r^3s^3 \}.$$

This partition of G_4 gives a monochromatic triangle-free coloring of $\mathbf{K}_{16}$ isomorphic to the coloring that one can obtain from a partition of $(\mathbb{Z}_4)^2$. This partition of G_5 gives a monochromatic triangle-free coloring of $\mathbf{K}_{16}$ isomorphic to the coloring that one can obtain from a partition of $(\mathbb{Z}_2)^4$. These two non-abelian groups are the only ones which can be partitioned into three sum-free sets.(See Whitehead [39].)

For a discussion concerning sum-free sets and difference sets, see Street and Whitehead [31]. Also, in the same paper they determined some sum-free cyclotomic classes in finite fields and via those classes they constructed difference sets, association schemes and block designs. Also, they give a characterisation of sum-free sets in $\mathbf{GF}(q)$ for $q = p^{2m}$ where $p^m \equiv 1 \pmod 3$ and m is a positive integer. As a corollary, they obtained

$$p^{2m} + 1 \le R_e(3, 2) = R_e$$

where $e = p^m - 1$.

The following lemma gives restriction the range of the set $\Lambda(G)$ if G is an abelian group of order 4n for some n.

Lemma 3. 51. Let G be an abelian group of order 4n. Then

(i) if $n \geq 3$, then $2n - 1 \notin \Lambda(G)$, and

(i) if $n \geq 6$, then $2n - 2 \notin \Lambda(G)$.

Now consider $\mathbb{Z}_{13}$. There are at least three ways of partitioning $\mathbb{Z}_{13}$ into maximal sum-free sets. We can consider the cubic residues and their multiplicative cosets in **GF**(13). This gives us the partition we had before. We can consider the quartic residues and their multiplicative cosets in **GF**(13). This gives us the following partition for $\mathbb{Z}_{13}$

$$\mathbb{Z}_{13} = \{ 1, 3, 9 \} \cup \{ 2, 5, 6 \} \cup \{ 4, 10, 12 \} \cup \{ 7, 8, 11 \}.$$

Thirdly, we consider the difference set $\{ 0, 1, 3, 9 \}$ in $\mathbb{Z}_{13}$ and its shifts which contain 0, we get

$$\mathbb{Z}_{13} = \{ 1, 3, 9 \} \cup \{ 2, 8, 12 \} \cup \{ 4, 5, 7 \} \cup \{ 6, 10, 11 \}.$$

We now ask the following question. In how many ways can a group G be partitioned into maximal sum-free sets?

From Table 1 of the Appendix, we see that, for $\mathbf{Q}$, the quaternion group of order 8, $\mu(\mathbf{Q}) = 1$ and $\lambda(\mathbf{Q}) = 4$. It is also an open problem to have bounds for $\mu(G)$. We know the following about $\mu(G)$.

Let $G = \mathbb{Z}_n$ and let $g \in G^*$. Consider the sum-free set $S = \{ g, \ldots, 2g - 1 \}$. Hence

$$S + S = \{ 2g, \ldots, 4g - 2 \} \quad \text{or} \quad \{ 2g, \ldots, 0, \ldots, 4g - n - 2 \},$$

and S is maximal sum-free if

(1) $g + 2g - 1 \leq n$, or $3g \leq n + 1$ and

(2) $3(g + 1) > n + 1$. Thus $\mu(\mathbb{Z}_n) \leq \left\lceil \frac{n+1}{3} \right\rceil$.

We can also think about a generalisation of Lemma 3. 51 to non-abelian groups. This question has no solution at present. It would be desirable to generalise Lemma 3. 52 even for abelian groups, in the following fashion.

Let $| G | = 4n$. Does there exist a function $\Theta(k)$, such that if $n \geq k$, we have

$$2n - \Theta(k), \ldots, 2n - 1 \notin \Lambda(G), \text{ but } 2n - \Theta(k) - 1 \in \Lambda(G)?$$

Lemma 3. 51 would imply that $\Theta(k) = 0$ for $k = 0, 1, 2$ and $\Theta(k) = 1$ for $k = 3, 4, 5$, $\Theta(k) \geq 2$ for $k \geq 6$.

We will discuss $\mu(G)$ when G is an elementary abelian 2-group. We will denote G by $(\mathbb{Z}_2)^n$, we know that the group G is filled.

Theorem 3. 52. If m is the smallest positive integer for which

$$|((\mathbb{Z}_2)^n)^*| \leq m + \binom{m}{2},$$

then

$$\mu((\mathbb{Z}_2)^n) \geq m. \blacksquare$$

We can now obtain lower bounds for $\mu((\mathbb{Z}_2)^n)$ when n = 3, 4, 5, 6. These are

$$4 \leq \mu((\mathbb{Z}_2)^3), \qquad 5 \leq \mu((\mathbb{Z}_2)^4),$$

$$8 \leq \mu((\mathbb{Z}_2)^5), \qquad 11 \leq \mu((\mathbb{Z}_2)^6).$$

The following computer results allow us to determine $\mu((\mathbb{Z}_2)^n)$ for $n \leq 4$.

$$\{4\} = \Lambda((\mathbb{Z}_2)^3),$$

$$\{5, 8\} = \Lambda((\mathbb{Z}_2)^4),$$

$$\{9, 10, 16\} \subseteq \Lambda((\mathbb{Z}_2)^5),$$

$$\{13, 17, 18, 20, 32\} \subseteq \Lambda((\mathbb{Z}_2)^6).$$

Before closing this chapter, we will give a theorem and a conjecture concerning the sum-free sets in non-abelian groups.

Theorem 3. 53. Let G a non-abelian group of order 3p, where p is a bad prime. If S is a maximal sum-free set in G, then S is a coset of the subgroup H of order p.

From the section on the main results, we know the following lower bound for a non-trivial abelian group G.

$$\frac{2|G|}{7} \leq \lambda(G).$$

For non-abelain groups no such lower bound is known. If the commutator subgroup G' is smaller than G, then

$$\lambda(G) \geq \lambda(G/G')\,|G| \geq \frac{2|G|}{7}.$$

But if we have G = G', then there is no known non-trivial lower bound on $\lambda(G)$. There is a conjecture related to this instance.

Conjecture: For $\mathcal{A}_n$, the alternating group of degree n, we have

$$\lambda(\mathcal{A}_n) = \frac{(n-1)!}{2}.$$

Since any coset of a proper subgroup is sum-free, we have

$$\lambda(\mathcal{A}_n) \geq \frac{(n-1)!}{2}..$$

But in reality, it is not hopeful to restrict the problem to cosets only. Because, for n = 5, we have other sum-free sets besides the cosets. The following two sets can be given as examples $\mathcal{A}_5$.

$$S_1 = \{(12345), (15432), (12543), (13452), (13425), (15243),$$
$$(13245), (15423), (12453), (13542), (12435), (15342))\},$$
$$S_2 = \{ (14)(23), (15)(24), (15)(23), (14)(35), (12)(35), (25)(34),$$
$$(14)(25), (13)(45), (12)(34), (24)(35), (123), (245))\}.$$

Chapter 4

Sum-Free Sequences

1. Introduction

In this chapter, we will study sum-free sequences of positive integers. We will , in particular, be interested in finding bounds on the reciprocal sum of the elements of a sum-free sequence.

We will start with a definition, and fix the notation. Unless otherwise stated, in this section, when we say "a sequence," we mean a strictly increasing sequence of positive integers.

Definition 4. 1. A sequence $\mathcal{A}$ with terms $a_1 < a_2 < \ldots$ is called sum-free if

$$a_n \neq \sum_{k=1}^{n-1} \varepsilon_k a_k \qquad \text{with } \varepsilon_k = 0,\ 1,$$

in other words, none of the terms $a_1 < a_2 < \ldots$ is the sum of other terms in $\mathcal{A}$.

Paul Erdös [10] proved the following inequality.

Let $\mathcal{A}$ be a sum-free sequence, then

$$\sum_{a_k \in \mathcal{A}} \frac{1}{a_k} < 103.$$

We will define

$$\rho(\mathcal{A}) = \sum_{a \in \mathcal{A}} \frac{1}{a}, \qquad \lambda = \sup_{\mathcal{A}} \rho(\mathcal{A})$$

where the supremum is taken over all sum-free sequences $\mathcal{A}$.

If we take $\mathcal{A} = \{ 2^n \}$ for $n \geq 0$, we see that $\mathcal{A}$ is sum-free; hence $\rho(\mathcal{A}) = 2$. Therefore we have $2 \leq \lambda < 103$.

Since we have $\rho(\mathcal{A}) = 2$ for $\mathcal{A}$ as above, one might think that the reciprocal sum of any other sum-free sequence is dominated by $\rho(\mathcal{A})$. But we can give the following example to show that this is not the case.

We now define a sequence $\mathcal{U} = \{ u_k \}$ as follows: For $1 \leq k \leq 14$, let u_k be given

1, 2, 4, 8, 19, 37, 55, 73, 91, 109, 127, 145, 163, 181.

Let

$$u_{15} = 1 + \sum_{k=1}^{14} u_k = 1016, \qquad \text{then for } k > 15 \qquad u_k = 2^{k-15} u_{15}.$$

One easily sees that $\mathcal{U}$ is a sum-free sequence. On the other hand, we have

$$\rho(\mathcal{U}) = \sum_{k=1}^{14} \frac{1}{u_k} + \frac{2}{u_{15}} = 2.03510128\ldots .$$

Hence we have $2.035 < \lambda < 103$. E. Levine and J. O'Sullivan [18] showed that $\lambda < 4$. Later H. L. Abbott [1] established the lower bound $2.0648 < \lambda$. We will present his construction later. Levine and O'Sullivan conjectured that λ is much closer to 2 than to 4. Abbott's construction is an evidence for such a conjecture.

Notation. Let $\mathcal{A}$ be a sum-free sequence with terms $a_1 < a_2 < \ldots$. The counting function $A(x)$ of such a sequence is

$$A(x) = \sum_{a_k \le x} 1 .$$

Erdös [10] proved the following inequality for $A(x)$.

$$A(x) \le \frac{x}{k+1} + \sum_{i=1}^{k} a_i + k \qquad (k \ge 1, x \ge 0).$$

Levine and O'Sullivan improved this inequality. This improvement is given in the next theorem without a proof.

Theorem 4. 2. If $\mathcal{A}$ is sum-free, then

$$A(x) \le \frac{x}{k+1} + \frac{1}{k+1} \sum_{i=1}^{k} a_i + \frac{k}{2} \qquad (k \ge 1, x \ge 0).$$

Instead of this inequality, we will use a weaker version of it which is given below.

Theorem 4. 3. If $\mathcal{A}$ is sum-free, then

$$A(x) \le \frac{x}{k+1} + a_k \tag{4. 1}$$

for $k \ge 1, x \ge 0$.

Proof: Although we did not give a proof of Theorem 4. 2, we will use Theorem 4. 2 to prove Theorem 4. 3.

From Definition 4. 1, we know that the sequence is strictly increasing. Hence, for

$i \le k$, $a_i \le a_k - (k - i)$.

Hence, by Theorem 4. 2,

$$
\begin{aligned}
A(x) &\le \frac{x}{k+1} + \frac{1}{k+1}\left[\sum_{i=1}^{k} (a_k - k + i)\right] + \frac{k}{2} \\
&= \frac{x}{k+1} + \frac{k}{k+1} a_k + \frac{k}{k+1} \\
&\le \frac{x}{k+1} + a_k,
\end{aligned}
$$

in the last line, we used $k \le a_k$.■

Definition 4. 4. We call a sequence a χ-sequence if it satisfies inequality (4. 1).

Note that every sum-free sequence is a χ-sequence. We will denote the supremum of $\rho(\mathcal{A})$ by μ, where supremum is taken over all χ-sequences. In this case, it is obvious that $\lambda \le \mu$. Our aim is to establish $\mu < 4$.

We give an example of a sequence which is a χ-sequence, but not sum-free; Levine and O'Sullivan made a conjecture based on this example.

Define the sequence $\wp$ with terms $p_1, p_2, \ldots$ as follows. Let $p_1 = 1$. Assume now $p_1, p_2, \ldots, p_{n-1}$ have been defined. We define p_n as follows. Let p_n be the least integer so that inequality (4. 1) is not violated for $k = 1, 2, \ldots, n - 1$, *viz.*

$$p_n = \max_{1 \le k \le n} (k+1)(n - p_k).$$

Hence we have

$$\wp = \{ 1, 2, 4, 6, 9, 12, 15, 18, 21, 24, 28, 32, \ldots \}.$$

For $\wp$, we have $\rho(\wp) \cong 3.01$. Levine and O'Sullivan believe that $\rho(\wp)$ dominates the reciprocal sum of any other χ-sequence.

Conjecture (Levine & O'Sullivan [18]). $\mu = \rho(\wp)$.

2. AN ESTIMATE FOR μ

In this section we will estimate μ. We will deal with χ-sequences. We will start with a theorem.

Theorem 4. 5.(Levine & O'Sullivan [18]) Let $\mathcal{A}$ be a χ-sequence, let N, M, and H be positive integers so that M = 2H. Then

$$\rho(\mathcal{A}) \le \sum_{r=0}^{N}\rho(r) + 2\sqrt{\frac{3R^2 + 11R + 16}{(N+1)\,2^R}} + \frac{1}{2^{H-1}} + \sum_{i=1}^{M}\left(\rho(r_i) - \frac{\gamma}{r_i(r_i+1)}\right)$$

where

$$R = N + H, \qquad \gamma = \sqrt{\frac{(N+1)(3R^2 + 11R + 16)}{2^R}}.$$

Since we will later give a theorem which is proven in a similar way to this, we omit the proof.

Corollary 4. 6.(Levine & O'Sullivan [18]) Keeping the notation the same as in the previous theorem, we get

$$\rho(\mathcal{A}) \le \sum_{r=0}^{N}\rho(r) + 2\sqrt{\frac{3R^2 + 11R + 16}{(N+1)\,2^R}} + \frac{1}{2^{H-1}} + \sum_{i=1}^{M}\rho(r_i).$$

Since $A(n) \le n$, we have

$$\rho(r) = \sum_{n\in J(r)}\frac{A(n)}{n(n+1)} \le \sum_{n=2^{r-1}+1}^{2^r}\frac{1}{n+1} < \log 2.$$

Let N = 4, and M = 2 in the above corollary to get

$$\rho(\mathcal{A}) \le \sum_{r=0}^{4}\rho(r) + 1.92 + 1 + 2\log 2 < \sum_{n=1}^{16}\frac{1}{n+1} + 4.32 < 6.76$$

for any χ-sequence. Hence

$$\mu < 6.76. \tag{4. 2}$$

3. A DETAILED STUDY OF $\wp$

For the χ-sequence $\wp$, all the terms less than 2^{18} were determined by a computer. We have the following results due to Levine & O'Sullivan [18].

$$A(2^{18}) = 3360,$$

$$\sum_{i=1}^{3360}\frac{1}{p_i} = 3.008466\ldots,$$

$$\sum_{r=0}^{18} \rho(r) = 2.995648\ldots \ .$$

We find a bound on $\rho(r)$ which is given in the following lemma without a proof.

Lemma 4. 7.(Levine & O'Sullivan [18]) For the χ-sequence $\mathcal{A}$,

$$\rho(r) < \frac{\log 2}{k+1} + \frac{a_k}{2^r} \tag{4. 3}$$

for $r \geq 0, k \geq 1$.

Using inequality (4. 3) for the χ-sequence $\wp$, we have

$$\sum_{i=1}^{M} \rho(r_i) \leq \frac{M \log 2}{k+1} + p_k \sum_{i=1}^{M} \frac{1}{2^{r_i}} < \frac{M \log 2}{k+1} + \frac{p_k}{2^N} \tag{4. 4}$$

where we made the use of the fact that each $r_i > N$ for $i = 1, 2, \ldots, M$.

Now we use inequality in Corollary 4. 6 to estimate $\rho(\wp)$ with $N = 18$, and $M = 20$. The choice of the latter is quite arbitrary. So we have

$$\rho(\wp) < 2.99565 + 0.00145 + \frac{1}{2^9} + \sum_{i=1}^{M} \rho(r_i) < 3 + \sum_{i=1}^{M} \rho(r_i).$$

But (4. 3) gives

$$\sum_{i=1}^{20} \rho(r_i) < \frac{20 \log 2}{k+1} + \frac{p_k}{2^{18}}.$$

To make the right-hand-side small, we may choose $k = 410$. This gives $p_{410} = 8964$ so that

$$\rho(\wp) < 3 + \frac{20 \log 2}{411} + \frac{8964}{2^{18}}$$

$$\rho(\wp) < 3.0679\ldots \ .$$

Hence we have

$$3 < \rho(\wp) < 3.0679.$$

4. MORE ON χ-SEQUENCES

If we consider (4.2), we see a large discrepancy between it and the conjectured value $\mu = \rho(\wp)$. In this section, we will try to narrow this gap. We give some lemmas and theorems without proofs, and we show by means of them that if a χ-sequence has a large reciprocal sum, then its first three terms must be the same as those of $\wp$. The work in this section is entirely taken from Levine & O'Sullivan [18].

The first lemma of this section is a special case of Theorem 4. 3. We restrict x to the terms of the sequence $\mathcal{A}$.

Lemma 4. 8. $\mathcal{A}$ is a χ-sequence if and only if

$$i \leq \frac{a_i}{k+1} + a_k \qquad \text{for } k \geq 1, i \geq 1.$$

Lemma 4. 9. Let h, w, r, m be integers with $r > 0$, and $0 \leq h < w < m + r$ so that

(i) $\frac{r-1}{r} \leq \frac{h+1}{w+1}$, and

(ii) $m \leq rp_{h+1} - (r-1)(p_h + 1)$.

Let $\mathcal{B}$ be a χ-sequence with terms $b_1 < b_2 < \ldots$ such that $b_i = p_i$, $i = 1, 2, \ldots, h$ and $b_w = t \geq p_w$. Let $\mathcal{A}$ be a sequence obtained from $\mathcal{B}$ as follows.

Replace the terms $b_{h+1}, \ldots, b_w$ by $p_{h+1}, \ldots, p_w$; and delete $d = t - p_{h+1}$ terms b_{rq+m} where $q = 1, 2, \ldots, d$.

Then $\mathcal{A}$ is a χ-sequence and

$$\rho(\mathcal{A}) - \rho(\mathcal{B}) = \sum_{i=h+1}^{w} \left(\frac{1}{p_i} - \frac{1}{b_i}\right) - \sum_{q=1}^{d} \frac{1}{b_{rq+m}}.$$

Theorem 4. 10. Let $\mathcal{B}$ be a χ-sequence. Then there exists a χ-sequence $\mathcal{A}$ with $a_1 = 1$ such that $\rho(\mathcal{A}) \geq \rho(\mathcal{B})$.

Theorem 4. 11. Let $\mathcal{B}$ be a χ-sequence. Then there exists a χ-sequence $\mathcal{A}$ with $a_1 = 1$, $a_2 = 2$ such that $\rho(\mathcal{A}) \geq \rho(\mathcal{B})$.

Theorem 4. 12. Let $\mathcal{B}$ be a χ-sequence. Then there exists a χ-sequence $\mathcal{A}$ with $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ such that $\rho(\mathcal{A}) \geq \rho(\mathcal{B})$.

Unfortunately, we do not get any further theorems like Theorems 4. 10 - 4. 12. There is no general procedure for doing so. Even it is not possible to show for a χ-sequence $\mathcal{A}$ with large reciprocal sum should have $a_4 = 6$.

By modifying the proofs of Theorems 4. 10 - 4. 12 one would show for a χ-sequence $\mathcal{A}$ so that $\rho(\mathcal{A}) > \mu - \varepsilon$ with ε sufficiently small must have $a_1 = 1$, $a_2 = 2$, and $a_3 = 4$. If ε is small enough, it is possible to show that either $a_4 = 6$ or $28 \le a_4 \le 64$.

Theorem 4. 13. Let $\mathcal{B}$ be a χ-sequence. Then there exists a χ-sequence $\mathcal{A}$ with $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and either $a_4 = 6$ or $28 \le a_4 \le 64$ such that $\rho(\mathcal{A}) \ge \rho(\mathcal{B})$.

5. A BETTER ESTIMATE FOR μ

In this section, we will improve the bound (4. 2). Keeping Theorem 4. 12 in mind, we take a χ-sequence $\mathcal{A}$ with $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and either $a_4 = 6$ or $a_4 \ge 28$. So we have $A(1) = 1$, $A(2) = 2$, $A(3) = 2$, and $A(4) = 3$. The work in this section is entirely taken from Levine & O'Sullivan [18].

Then

$$\sum_{r=0}^{2} \rho(r) = \sum_{n=1}^{4} \frac{A(n)}{n(n+1)} = \frac{23}{20}.$$

From Theorem 4. 5, by taking N = 6, and M = 6, we get

$$\rho(\mathcal{A}) \le \sum_{r=0}^{6} \rho(r) + \frac{1}{8}\sqrt{\frac{179}{7}} + \frac{1}{4} + \sum_{i=1}^{6}\left(\rho(r_i) - \frac{\gamma}{r_i(r_i+1)}\right)$$

where $\gamma = \frac{1}{16}\sqrt{1253} \cong 2.212$ and $r_i \ge 7$ for $i = 1, 2, \ldots, 6$.

Since

$$\sum_{r=0}^{6} \rho(r) = \sum_{r=0}^{2} \rho(r) + \sum_{r=3}^{6} \rho(r) = \frac{23}{20} + \sum_{r=3}^{6} \rho(r)$$
$$= \frac{23}{20} + \sum_{n=5}^{64} \frac{A(n)}{n(n+1)},$$

we have

$$\rho(\mathcal{A}) \le 2.0231 + \sum_{n=5}^{64} \frac{A(n)}{n(n+1)} + \sum_{i=1}^{6}\left(\rho(r_i) - \frac{\gamma}{r_i(r_i+1)}\right).$$

Now we have two cases to consider.

Case 1: Assume $a_4 \ge 28$. For $5 \le n \le 27$, $A(n) = 3$. If $n > 27$, then

$A(n) \leq A(n-1) + 1$, so that $A(n) \leq n - 24$. From (4. 1), we have for $k = 3$, $A(n) \leq \frac{n}{4} + 4$.

Hence

$$A(n) \leq \min\left(n - 24, \left[\frac{n}{4} + 4\right]\right) = s_n, \text{ for } n > 27.$$

Therefore

$$\sum_{n=5}^{64} \frac{A(n)}{n(n+1)} \leq \sum_{n=5}^{27} \frac{3}{n(n+1)} + \sum_{n=5}^{64} \frac{s_n}{n(n+1)} < .75.$$

From Lemma 4. 7, with $k = 3$

$$\rho(r) - \frac{\gamma}{r(r+1)} < \frac{\log 2}{4} + \left(\frac{4}{2^r} - \frac{\gamma}{r(r+1)}\right).$$

Since $\gamma \cong 2.212$, we get, for $r \geq 7$,

$$\frac{4}{2^r} - \frac{\gamma}{r(r+1)} < 0.$$

So

$$\rho(r) - \frac{\gamma}{r(r+1)} < \frac{\log 2}{4}, \quad \text{for } r \geq 7. \qquad (4.5)$$

Hence

$$\rho(\mathcal{A}) \leq 2.0322 + .75 + \frac{3 \log 2}{2} < 3.84.$$

Case 2: Assume now $a_4 = 6$. Again, from (4. 1), we get for $k = 2, 3, 4$ $A(n) \leq \frac{n}{3} + 2$, $A(n) \leq \frac{n}{4} + 4$, and $A(n) \leq \frac{n}{5} + 6$, respectively. So we have

$$\sum_{n=5}^{64} \frac{A(n)}{n(n+1)} \leq \sum_{n=5}^{23} \frac{\left[\frac{n}{3} + 2\right]}{n(n+1)} + \sum_{n=24}^{39} \frac{\left[\frac{n}{4} + 4\right]}{n(n+1)} + \sum_{n=40}^{64} \frac{\left[\frac{n}{5} + 6\right]}{n(n+1)} < 1.0926.$$

As in Case 1, we have

$$\rho(r) - \frac{\gamma}{r(r+1)} < \frac{\log 2}{5}, \quad \text{for } r \geq 8.$$

We also have (4. 13) as well. Since $r_1 \geq 7$, and $r_i \geq 8$ for $i \geq 2$, we get

$$\sum_{i=1}^{6} \left(\rho(r_i) - \frac{\gamma}{r_i(r_i+1)}\right) < \frac{\log 2}{5} + \frac{5 \log 2}{5} = \frac{5 \log 2}{4} < .875.$$

Hence, we have

$$\rho(\mathcal{A}) \leq 2.0322 + 1.0926 + .875 = 3.9998.$$

Therefore we conclude the section with

$$\mu < 4.$$

6.A "CONJECTURE" OF ERDÖS

Erdös [12] conjectured that if $a_1 < a_2 < \ldots$ is a sum-free sequence $\mathcal{A}$ with $a_1 \geq n$, then $\rho(\mathcal{A}) < \log 2 + \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$.

E. Levine [19] established Erdös' conjecture positively. He showed if $\mathcal{A}$ is any sequence which satisfies (4. 1), then $\rho(\mathcal{A}) < \log 2 + O(a^{-1/3})$, where $a = a_1$.

By taking the first n + 1 terms as n, n + 1, ..., 2n, and the remaining terms as s, 2s, 4s, ... where $s = 1 + \sum_{i=0}^{n} (n + i)$, we see that the constant log2 is best possible.

Theorem 4. 14.(Levine [19]) Let $\mathcal{A}$ be a sequence of integers with $a_1 < a_2 < \ldots$ whose counting function A(x) satisfies (4. 1). Then

$$\rho(\mathcal{A}) < \log 2 + O(a^{-1/3}), \qquad \text{where } a = a_1.$$

Proof: We partition the positive integers into intervals

$$J(r) = \{ n \mid 2^r < n \leq 2^{r+1} \}, \qquad r = -1, 0, 1, 2, \ldots .$$

We introduce N(r) as $A(2^{r+1}) - A(2^r)$ and $\rho(r)$ as $\sum_{a_k \in J(r)} \frac{1}{a_k}$, let t be such that $a \in J(t)$.

Hence we have

$$2^t < a \leq 2^{t+1}$$

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \sum_{r=t}^{\infty} \rho(r)$$

$$N(r) \leq 2^{r+1},$$

and

$$\rho(r) \leq \frac{N(r)}{2^r}$$

We define the sets Δ and Γ as follows.

$$\Delta = \{ r \geq t \mid N(r) \leq 2^{(2r/3)+2} \},$$

and

$$\Gamma = \{ r \geq t \mid N(r) > 2^{(2r/3)+2} \}.$$

Hence

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \sum_{r \in \Delta} \rho(r) + \sum_{r \in \Gamma} \rho(r).$$

We now estimate the first term on the right-hand-side of the above.

$$\sum_{r \in \Delta} \rho(r) \le \sum_{r \in \Delta} \frac{N(r)}{2^r} \le 4 \sum_{r=t}^{\infty} 2^{-r/3} = O(a^{-1/3})$$

Let $\Gamma = \{ s = r_1, r_2, r_3, \dots \}_<$. Hence we have

$$s \ge \max(t, 4),$$
$$r_i - r_j \ge i - j \qquad (i \ge j),$$
$$r_i \ge i + s - 1.$$

Now consider r_j, and let $q = [2^{2r_j/3}]$. Since $r_j \in \Gamma$, we have

$$q < N(r_j) \le A(2^{r_j+1}).$$

So we have $a_q < 2^{r_j+1}$. Hence for any r_i, we get

$$\rho(r_i) \le \frac{N(r_i)}{2^{r_i}} \le \frac{A(2^{r_i+1})}{2^{r_i}} \le \frac{2}{q+1} + \frac{a_q}{2^{r_i}},$$

and

$$\rho(r_i) < 2^{1-(2r_j/3)} + 2^{1+r_j-r_i}. \tag{4. 6.}$$

We now consider the following partition for Γ,

$$\Gamma_1 = \{ r \in \Gamma \mid r \le \frac{4s}{3} \},$$
$$\Gamma_2 = \{ r_i \in \Gamma \mid r_i > \frac{4s}{3}, \text{ and } i \le t \},$$
$$\Gamma_3 = \Gamma \setminus (\Gamma_1 \cup \Gamma_2).$$

So we get

$$\sum_{k=1}^{\infty} \frac{1}{a_k} \le \sum_{d=1}^{3} \left(\sum_{r \in \Gamma_d} \rho(r) \right) + O(a^{-1/3})$$

We want to estimate the first term right-hand-side of the above for d = 3, 2, and 1, in turn. If $r_i \in \Gamma_3$, by letting $j = \left[\frac{i}{2}\right]$, we get

$$r_j \ge \left[\frac{i}{2}\right] + 3 > \frac{i+3}{2},$$
$$r_i - r_j \ge i - \left[\frac{i}{2}\right] \ge \frac{i}{2}.$$

From (4. 6),

$$\rho(r_i) < 2^{-i/3} + 2^{1-i/2} < 2^{2-(i/3)}.$$

If we sum the above over r_i and keeping in mind that $r_i \in \Gamma_3$ implies $t < i$, we get

$$\sum_{r_i \in \Gamma_3} \rho(r_i) < \sum_{i > t} 2^{2-(i/3)} = O(2^{-t/3}).$$

Therefore

$$\sum_{r_i \in \Gamma_3} \rho(r_i) < O(a^{-1/3}).$$

From (4. 6) by letting j = 1, one gets

$$\rho(r_i) < 2^{1-(2s/3)} + 2^{1+s-r_i}.$$

So we have

$$\sum_{r_i \in \Gamma_2} \rho(r_i) < \sum_{i \leq t} 2^{1-(2s/3)} + \sum_{r_i > \frac{4s}{3}} 2^{1+s-r_i}$$

$$= O(|t|\, 2^{-2s/3} + 2^{-t/3})$$

$$= O(a^{-2/3}\log(a+1) + a^{-1/3}),$$

since $2^{-2s/3} = O(a^{-2/3})$. Hence

$$\sum_{r_i \in \Gamma_2} \rho(r_i) = O(a^{-1/3}).$$

Finally, we estimate the last case where d = 1. We let

$$p = A(2^s) + \left[2^{(2s/3)+2}\right],$$

$$b = a_p,$$

$$m = \text{largest integer in } \Gamma_1,$$

so that $p < A(2^{s+1})$. Hence

$$\sum_{r \in \Gamma_1} \rho(r) \leq \sum_{2^s < a_p \leq 2^{m+1}} \frac{1}{a_k} = \sum_{2^s < a_p \leq b} \frac{1}{a_k} + \sum_{b < a_p \leq 2^{m+1}} \frac{1}{a_k}.$$

Then

$$\sum_{r \in \Gamma_1} \rho(r) < \frac{A(b) - A(2^s)}{2^s} + \sum_{b < a_p \leq 2^{m+1}} \frac{1}{a_k}. \qquad (4.7.)$$

Since $A(b) = A(2^s) + \left[2^{(2s/3)+2}\right]$, we get

$$\frac{A(b) - A(2^s)}{2^s} = \frac{[2^{(2s/3)+2}]}{2^s} = O(2^{-s/3}) = O(a^{-1/3}).$$

Our aim is to show that the second term on the right-hand-side of (4. 7) is bounded by log2. Since m is the largest integer in Γ_1, we have $m \le \frac{4s}{3}$, also $p \ge [2^{(2s/3)+2}]$, and $s \ge 4$, so we get

$$\frac{2^{m+1}}{p+1} - p < 0.$$

Then, by (4. 1), we get

$$A(2^{m+1}) - A(b) \le \frac{2^{m+1}}{p+1} + a_p - p < a_p = b,$$

from which we conclude that, in the last term of (4. 7), we are adding at most b distinct integers so that each a_k is larger than a_p. On the other hand, that sum cannot exceed

$$\sum_{b < n \le 2b} \frac{1}{n}.$$

Whence

$$\sum_{b < a_p \le 2^{m+1}} \frac{1}{a_k} \le \sum_{b < n \le 2b} \frac{1}{n} < \log 2.$$

Therefore, we get

$$\sum_{r \in \Gamma_1} \rho(r) < \log 2 + O(a^{-1/3}).$$

By putting necessary parts together, we arrive at

$$\sum_{k=1}^{\infty} \frac{1}{a_k} < \log 2 + O(a^{-1/3}). \blacksquare$$

7. AN IMPROVED LOWER BOUND FOR λ

In this section,we present a construction due to H. L. Abbott [1]. He improved the lower bound given by Levine and O'Sullivan, which is $\lambda > 2.0351$, to $\lambda > 2.0648$. His construction is given in the following theorem without a proof.

Theorem 4. 15. Let $\mathcal{A}$ be a (finite) sum-free set. Let $\sigma = \sum_{a \in \mathcal{A}} a$, and τ be an integer exceeding σ. Define integers k, m, n, r, and p as below.

$k = \binom{\tau - \sigma + 2}{2}$, $\qquad m = \binom{\tau - \sigma + 1}{2}$

$$n = \left[\frac{k - 1 + \sigma}{\tau}\right], \qquad r = k - n\tau - 1, \qquad p = \binom{k+1}{2}\binom{r+1}{2} + n.$$

We choose and $\mathcal{A}$ in such a way that $r > 0$. Define the sets $\mathcal{B}$ and $\mathcal{C}$ as below.

$$\mathcal{B} = \{\ \mu\tau + 1 \mid \mu = 1, 2, \ldots, k\ \},$$

$$\mathcal{C} = \{\ (p + \nu)\tau + 1 \mid \nu = 1, 2, \ldots, m + 1\ \}.$$

Then $\mathcal{S} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is a sum-free set.■

Abbott computed $\sum_{s \in \mathcal{S}} \frac{1}{s}$ for various sets $\mathcal{A}$ and various choices of τ. He noticed that if $\mathcal{A} = \{\ 1, 2, 4, 8\ \}$ and $\tau = 24.$, then

$$\sum_{s \in \mathcal{S}} \frac{1}{s} > 2.0648.$$

In this case, we obtain the following values

$\sigma = 15, \qquad \tau = 24, \qquad k = 55, \qquad m = 45,$

$n = 2, \qquad r = 6, \qquad p = 1521.$

$$\mathcal{B} = \{\ 24\mu + 1 \mid \mu = 1, 2, \ldots, 55\ \},$$

$$\mathcal{C} = \{\ 24(\ 1521 + \nu\ ) + 1 \mid \nu = 1, 2, \ldots, 46\ \}.$$

Then

$$\sum_{s \in \mathcal{S}} \frac{1}{s} > 2.0648.■$$

In this chapter, our main concern was to study a special class of sequences, namely sum-free sequences of positive integers. We also studied a related class of sequences, namely χ-sequences. We looked at the reciprocal sum of the elements of such sequences. In this way we can control the lememnts of a χ-sequence.

Chapter 5

UNSOLVED PROBLEMS

In this last chapter, we present some unsolved problems related to sum-free sets.

(1) Finding the values of the Schur function f(n) is presently an unsolved problem. The last "value" was found about ten years ago. It may be possible to find the exact value of f(5) with the aid of a high-speed computer.

(2) We know from Chapter 2, Corollary 2. 8, that the limit $\lim_{n \to \infty} f(n)^{1/n} = L$ exists where f(n) is the Schur function. It is not known if L is finite or infinite. P. Erdös is offering $100 for the answer.

(3) Another problem about sum-free set is the following. Denote by h(n) the largest m for which there exists some way of partitioning the set { 1, 2, ..., m } into n sets which are sum-free modulo m + 1; that is, they contain no solution of $x + y \equiv z \pmod{m+1}$. The partitions which give the values f(1) =1, f(2) = 4, f(3) =13, f(4) = 44 are sum-free moduli 2, 5, 14, and 45, respectively. The conjecture is that h(n) = f(n) for all n.

(4) It is shown that the limit $\lim_{n \to \infty} h(n)^{1/n} = L^*$ exists. It would be interesting to show that $L = L^*$.

(5) In Theorem 3. 32, we gave the bounds

$$\frac{(m-1)|G|}{3m} \leq \lambda(G) \leq \frac{|G|-1}{3}.$$

Diananda and Yap [7] conjectured that $\lambda(G) = \frac{(m-1)|G|}{3m}$ if G is an abelian group of order divisible by bad primes and of exponent m.

(6) The lower bound $\frac{2|G|}{7} \leq \lambda(G)$ is known for finite abelian groups. For any finite group, the conjecture is that $\frac{\lambda(G)}{|G|}$ can be arbitrarily small.

(7) For what values of n does $\Lambda(\mathbf{D}_n) = \Lambda(\mathbb{Z}_2 \oplus \mathbb{Z}_n)$, where $\mathbf{D}_n$ is the dihedral group of order 2n? We know that this is true for n = 2, 3, 6, 7.

(8) It is desirable to find bounds on $\mu(G)$, the cardinality of the smallest sum-free set.

(9) From Theorem 3. 46, we know that every group can be by locally maximal sum-free sets. Which groups can be partitioned into maximal sum-free sets?

(10) A result similar to that of Theorem 4. 10 must be true for sum-free sequences. But no proof is known yet.

(11) Determine the exact value of λ and those sum-free sequences $\mathcal{A}$ for which $\rho(\mathcal{A}) = \lambda$.

(12) Prove that if a sum-free sequence $\mathcal{A}$ is such that $\rho(\mathcal{A})$ is very close to λ, then $a_1 = 1$.

(13) Intead of integers, consider the class of real-valued sequences $\mathcal{A}$ with the counting function A(x) and terms $0 \le a_1 \le a_2 \le \ldots$ satisfying

$$A(x) \le \frac{x}{k+1} + a_k \qquad (k \ge 1, x \ge 0).$$

What is the best bound for $\rho(\mathcal{A})$ over this class of sequences?

The following problem, which I call the "reciprocal version of Schur's problem," is of special interest. The following is not known about it.

Assume that we partition the positive integers into finitely many classes, that is $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_n$. Then tthe equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ has a solution in C_i.

Appendix

In the following table, we give non-isomorphic maximal sum-free sets in groups of orders 2, 3, ... , 11, 13, 14, and 16. We will write the non-abelian groups multiplicatively. We will extend Table 1 to Table 2 in which we can have orders 12, 15, 16 (all abelian cases), 32 and 64 (for elementary abelian cases). A group of isomorphism acting on a family of sets partitions the family of sets into isomorphism classes, i. e. , into equivalence classes. A transversal is a set containing exactly one member of each equivalence class.

We will denote the direct product of m copies of $\mathbb{Z}_n$ by $(\mathbb{Z}_n)^m$ for simplicity; the dihedral group by $\mathbf{D}_6$; the non-abelian group of order 12 by Γ, where

$$\Gamma = < s, t \mid s^4 = t^3 = 1, ts = st^2 >.$$

For the elementary abelian groups of orders 16, 32, and 64, we will use B_n to denote the generating set { 00...01, 00...010, ... , 10...00 } of the group $(\mathbb{Z}_2)^n$. The direct product notation indicates cosets.

These are taken from Street and Whitehaed[30] and Whitehaed[37].

TABLE 1

$\lvert G\rvert$	G	$\Lambda(G)$	Transversals
2	$\mathbb{Z}_2$	{ 1 }	{ 1 }
3	$\mathbb{Z}_3$	{ 1 }	{ 1 }
4	$\mathbb{Z}_4$	{ 1, 2 }	{ 2 }, { 1, 3 }
	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	{ 2 }	{ 01, 11 }
5	$\mathbb{Z}_5$	{ 2 }	{ 1, 4 }
6	$\mathbb{Z}_6$	{ 2, 3 }	{ 2, 3}, { 2, 5 }, { 1, 3, 5 }
	$\mathbf{S}_3$	{ 2, 3 }	{ (123), (12) }, { (12), (13), (23) }
7	$\mathbb{Z}_7$	{ 2 }	{ 2, 3 }, { 3, 4 }
8	$\mathbb{Z}_8$	{ 2, 3, 4 }	{ 1, 6}, { 2, 6 }, { 3, 4, 5 }, { 1, 3, 5, 7 }
	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	{ 2, 4 }	{ 20, 01 }, { 01, 11, 21, 31 }, { 10, 11, 30, 31 }
	$(\mathbb{Z}_2)^3$	{ 4 }	{ 001, 011, 101, 111 }
	$\mathbf{D}_4$	{ 3, 4 }	{ s^2, t, st }, { t, st, s^2t, s^3t }, {s, s^3, t, s^2t }
	$\mathbf{Q}$	{1, 4 }	{ s^2 }, { t, st, s^2t, s^3t }
9	$\mathbb{Z}_9$	{ 3 }	{ 3, 4, 5 }, { 1, 4, 7 }
	$(\mathbb{Z}_3)^2$	{ 3 }	{ 01, 11, 21 }
10	$\mathbb{Z}_{10}$	{ 3, 4, 5 }	{ 3, 4, 5 }, { 4, 5, 6}, { 1, 4, 6, 9 }, { 1, 3, 5, 7, 9 }
	$\mathbf{D}_5$	{ 4, 5 }	{ s^2, s^3, s^2t, s^3t }, { t, st, s^2t, s^3t , s^4t }
11	$\mathbb{Z}_{11}$	{ 3, 4 }	{ 3, 4, 5 }, { 4, 5, 6, 7 }
13	$\mathbb{Z}_{13}$	{ 3, 4 }	{ 3, 4, 5 }, { 1, 3, 9 }, { 4, 6, 7, 9 }, { 4, 5, 6, 7 }, { 5, 6, 7, 8 }
14	$\mathbb{Z}_{14}$	{4, 5, 7 }	{ 2, 5, 9, 13 }, { 4, 5, 6, 7 }, { 5, 6, 7, 8, 9 }, {1, 3, 5, 7, 9, 11, 13 }
	$\mathbf{D}_7$	{ 4, 5, 7 }	{ s, st, s^3t , s^6t }, { s^2, s^3, t, st }, { s^3, s^4, t, st, s^2t }, { t, st, s^2t, s^3t , s^4t, s^5t , s^6t }

TABLE 2

$\|G\|$	G	$\Lambda(G)$	Transversals
12	$\mathbb{Z}_{12}$	{ 3, 4, 6 }	{ 2, 6, 10 }, { 3, 4, 5 }, { 5, 6, 7, 8 }, { 1, 4, 7, 10 }, {1, 3, 5, 7, 9, 11 }
	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$	{ 4, 6 }	{ 01, 04, 11, 14 }, { 10, 11, 12, 13, 14, 15 }
	Γ	{ 2, 3, 4, 6 }	{ s^2, t }, { s^2, s^2t, s^2t^2 }, { s, s^3, t, s^2t }, { s, s^3, s^2t, s^2t^2 }, { s, s^3, st, s^3t, st^2, s^3t^2 }
	D_6	{ 4, 6 }	{ s, s^4, st, s^4t }, { s, s^5, st, s^5t }, { s^2, s^3, s^2t, s^3t }, { s^3, st, s^2t, s^3t }, { s, s^3, s^5, st, s^3t, s^5t }, { t, st, s^2t, s^3t, s^4t, s^5t }
	$\mathcal{A}_4$	{ 3, 4 }	{ (123), (12)(34), (13)(24) }, { (12)(34), (243), (143) }, { (123), (243), (142), (134) }
15	$\mathbb{Z}_{15}$	{ 3, 4, 5, 6 }	{ 1, 3, 11 }, { 1, 3, 5, 7 }, { 1, 3, 7, 12 }, { 1, 3, 5, 12, 14 }, { 1, 4, 7, 10, 13 }, { 1, 4, 6, 9, 11, 14 }
16	$\mathbb{Z}_{16}$	{ 4, 5, 6, 8 }	{ 1, 3, 10, 12 }, { 1, 4, 6, 9 }, { 1, 4, 6, 15 } { 1, 4, 9, 14 }, { 1, 6, 9, 14 }, { 1, 6, 10, 14 }, { 2, 6, 10, 14 }, { 1, 3, 8, 10, 15 }, { 1, 6, 8, 13, 15 }, { 1, 6, 8, 10, 15 }, { 1, 4, 7, 9, 12, 15 } { 1, 3, 5, 7, 9, 11, 13, 15 }
	$\mathbb{Z}_2 \oplus \mathbb{Z}_8$	{ 4, 6, 8 }	{ 01, 06, 11, 16 }, {01, 06, 11, 14 } { 01, 11, 14, 16 }, {01, 06, 10, 14 } {02, 06, 10, 14 }, { 01, 06, 12, 16 } {02, 06, 12, 16 }, { 01, 07, 11, 17, 14, 04 } { 01, 03, 05, 07, 11, 13, 15, 17 } { 01, 03, 05, 07, 10, 12, 14, 16 }
	$(\mathbb{Z}_4)^2$	{ 3, 4, 5, 8 }	{ 01, 10, 33 }, { 01, 03, 22, 20 } { 01, 03, 10, 30, 22 }, { 01, 03, 10, 30, 12, 32, 21, 23 }
	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$	{ 4, 5, 6 }	{ 112, 102, 012, 002 } { 001, 003, 112, 102, 012 } { 001, 003, 011, 013, 101, 103, 111, 113 } { 001, 003, 011, 013, 112, 110, 102, 100 }
	$(\mathbb{Z}_2)^4$	{ 5, 8 }	$B_4 \cup$ { 1111 }, $(\mathbb{Z}_2)^3 \oplus 1000$

TABLE 3

$\lvert G \rvert$	G	$\Lambda(G)$	Transversals
32	$(\mathbb{Z}_2)^5$	$\{ 9, 10, 16 \}$	$B_5 \cup \{ 01111, 10011, 10101, 10110 \}$
			$\mathbb{Z}_2 \oplus \{ B_4 \cup \{ 1111 \} \}$
			$(\mathbb{Z}_2)^4 \oplus 10000$
64	$(\mathbb{Z}_2)^6$	$\{ 13, 17, 18, 20, 32 \}$	
			$B_6 \cup \{ 001101, 010111, 011011, 011110, 100111, 110010, 110101 \}$
			$B_6 \cup \{ 001011, 001101, 001110, 010101, 010110, 011001, 011010, 011100, 011111, 100111 \}$
			$\mathbb{Z}_2 \oplus \{ B_5 \cup \{ 01111, 10011, 10101, 10110 \} \}$
			$(\mathbb{Z}_2)^2 \oplus \{ B_4 \cup \{ 1111 \} \}$
			$(\mathbb{Z}_2)^5 \oplus 100000$

Whitehead [1975] gave a list of locally maximal sum-free sets in non-abelian groups of order 16. We have nine such groups. We will introduce the groups as below. We will represent the groups by G_i where $1 \le i \le 9$.

$$G_1 = < r, s \mid r^2 = s^2, (rs)^2 = e >,$$

$$G_2 = < r, s, t \mid r^2 = s^2 = t^2, rst = str = trs >,$$

$$G_3 = < s, t \mid s^4 = t^4 = e, t^{-1}st^2 = s^{-1} >,$$

$$G_4 = < r, s, t \mid r^2 = s^2 = t^2 = (st)^4 = (tr)^2 = (rs)^2 = e >,$$

$$G_5 = < r, s \mid r^4 = s^4 = (rs)^2 = (r^{-1}s)^2 = e >,$$

$$G_6 = < r, s, t \mid r^2 = s^2 = t^2 = (st)^2 = (tr)^2 = (rs)^2 >,$$

$$G_7 = < s, t \mid s^8 = t^2 = (st)^2 = e >,$$

$$G_8 = < r, s \mid r^2 s^4 = (rs)^2 = e >,$$

$$G_9 = < s, t \mid s^4 = t^2 = (st)^2 >.$$

In the next table we will give the non-abelian groups of order 16 and their transversals.

TABLE 4

G	$\Lambda(G)$	Transversals
G_1	$\{ 3, 4, 5, 8 \}$	$\{ s, s^6, s^3rs \}, \{ s, s^4, s^7rs \}, \{ s, s^6, rs, s^4rs \},$ $\{ s, s^6, s^2rs, s^6rs \}, \{ s^2, s^6, rs, s^4rs \},$ $\{ s^2, s^6, s^2rs, s^6rs \}, \{ s, s^4, s^7, srs, s^3rs \},$ $\{ s, s^3, s^5, s^7, rs, s^2rs, s^4rs, s^6rs \},$ $\{ s, s^3, s^5, s^7, srs, s^3rs, s^5rs, s^7rs \}$
G_2	$\{ 4, 8 \}$	$\{ r, s, t, (rs)^2 \}, \{ r, s, t, rsr, rst, rtr, rts, srs \},$ $\{ r, s, rt, tr,, st, ts, rsr, srs \},$ $\{ r, rs, sr, rt, tr, rst, rts, srs \}$
G_3	$\{ 2, 4, 5, 6, 8 \}$	$\{ s^2, t^2 \}, \{ s, s^3, t^2, s^2t^2 \}, \{ s, s^3, t, t^3, s^2t^2 \},$ $\{ s^2, t, t^3, st, st^3, s^2t^2 \}, \{ s, s^3, t, t^3, st^2, s^2t, s^2t^3, s^3t^2 \}$ $\{ t, t^3, st, st^3, s^2t, s^2t^3, s^3t, s^3t^3 \}$
G_4	$\{ 5, 6, 8 \}$	$\{ r, t, st, ts, rsts \}, \{ s, t, rt, (st)^2, rtst \},$ $\{ r, s, t, sts, rtst \}, \{ r, s, t, (st)^2, rsts, rtst \},$ $\{ r, t, rs, st, ts, sts, r(st)^2, rtst \},$ $\{ t, rt, st, ts, rst, rts, sts, rsts \},$ $\{ s, t, rs, rt, sts, tst, rsts, rtst \}$
G_5	$\{ 4, 5, 6, 8 \}$	$\{ r^2, s^2, rs, r^3s^3 \}, \{ r, r^3, s, s^3, r^2s^2 \},$ $\{ r, r^3, s^2, rs, r^3s^3 \}, \{ r, r^3, s^2, rs, r^2s^2, r^3s \},$ $\{ r, r^3, rs, rs^3, r^2s^2 \}, \{ r, r^3, s, s^3, rs^2, r^2s, r^2s^3, r^3s^2 \},$ $\{ r, r^3, rs, rs^2, rs^3, r^3s, r^3s^2, r^3s^3 \}$
G_6	$\{ 2, 8 \}$	$\{ r^2, t \}, \{ r, r^3, s, s^3, t, r^2t, rst, (rs)^3t \},$ $\{ r, r^3, s, s^3, rt, r^3t, st, s^3t \}$
G_7	$\{ 4, 5, 6, 8 \}$	$\{ s, s^6, t, s^3t \}, \{ s, s^6, t, s^4t \}, \{ s^4, t, st, s^2t, s^3t \},$ $\{ s, s^4, s^7, t, s^2t, s^5t \}, \{ s^2, s^6, t, st, s^4t, s^5t \},$ $\{ s, s^3, s^5, s^7, t, s^2t, s^4t, s^6t \},$ $\{ t, st, s^2t, s^3t, s^4t, s^5t, s^6t, s^7t \}$
G_8	$\{ 4, 5, 6, 8 \}$	$\{ s, s^6, r, rs^4 \}, \{ s, s^6, rs^3, rs^7 \}, \{ s, s^4, s^7, rs, rs^3 \},$ $\{ s, s^6, r, rs^3, rs^4, rs^7 \}, \{ s, s^3, s^5, s^7, r, rs^2, rs^4, rs^6 \},$ $\{ s, s^3, s^5, s^7, rs, rs^3, rs^5, rs^7 \},$ $\{ r, rs, rs^2, rs^3, rs^4, rs^5, rs^6, rs^7 \}$
G_9	$\{ 3, 4, 6, 8 \}$	$\{ s, s^4, s^7 \}, \{ s, s^6, t, t^3 \}, \{ s^2, s^6, t, t^3, st, s^5t \},$ $\{ s, s^3, s^5, s^7, t, t^3, s^2t, s^6t \},$ $\{ t, t^3, st, s^2t, s^3t, s^4t, s^5t, s^6t, s^7t \}$

For the groups $(\mathbb{Z}_2)^5$ and $(\mathbb{Z}_2)^6$, it is not known if we have the complete solutions.

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