THE NUMERICAL STRUCTURE OF AL-KHALĪLĪ'S AUXILIARY TABLES

by

Glen R. Van Brummelen

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Name: Glen R. Van Brummelen

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Examining Committee:

Chairman: Dr. G. Bojadziev

Dr. J. L. Berggren
Senior Supervisor

Dr. R. Harrop

Dr. R. Russell

Dr. T. Swartz

Dr. Hannah Gay
External Examiner
Department of History

Date Approved: April 11, 1988

Title of Thesis/Project/Extended Essay

The Numerical Structure of al-Khalīlī's Auxiliary Tables

Author:

(signature)

Glen Van Brummelen

(name)

Apr. 11, 1988

(date)

ABSTRACT

One of the major achievements of medieval Islamic science is the construction of tables of values for certain functions relating to astronomy. These tables range in size from a few to 250 000 entries and are generally based on trigonometric formulae. Al-Khalīlī's auxiliary tables, for example, contain over 13 000 entries and give values for the functions $f(\phi,\theta) = \frac{R \text{ Sin } \theta}{\text{Cos } \phi}$, $g(\phi,\theta) = \frac{\text{Sin } \theta \text{ Tan } \phi}{R}$, and $G(x,y) = \text{arc Cos}\left[\frac{Rx}{\text{Cos } y}\right]$ that are accurate to the equivalent of three or four significant decimal digits. The applications of these functions to problems of spherical astronomy are known; however, the texts are silent concerning how the entries were actually calculated.

The purpose of this study is to develop computer-based methods implementing statistical tests to discover the numerical structure of al-Khalīlī's auxiliary tables. We have discovered an interpolation grid on the $g(\phi,\theta)$ tables, as well as a likely interpolation scheme. Al-Khalīlī then used an equation, based on the sine addition formula, to generate the values of $f(\phi,\theta)$ from corresponding entries in the $g(\phi,\theta)$ table. Both of the above tables were constructed using trigonometric values rounded to two sexagesimal digits. Finally, the extent of the work done on the $G(x,y)$ table reveals a curious lack of concern for accuracy early in the calculation combined with a higher level of accuracy at a later stage.

It is hoped that the techniques used in this study as well as other methods can be used to determine the structure of many other astronomical tables and so reveal a clear picture of the evolution of numerical techniques in medieval Islam.

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CHAPTER 1

INTRODUCTION

1.1 Types of Medieval Islamic Astronomical Tables

No account of the history of Islamic science would be complete without an extensive examination of the various types of astronomical tables that appear in the manuscripts. These tables, which often appear not only in the astronomical handbooks (called **zījes**) but also in other contexts as well as independently, constitute a pinnacle of astronomical research of the medieval period and attest to the prodigious numerical ability and sheer patience of their constructors. While the methods of construction of most of these tables are unknown, their usefulness can be exemplified by the fact that the fourteenth century astronomer Shams al-Dīn al-Khalīlī's hour-angle and prayer tables were used until the late nineteenth century.[1] The tables have widely varying purposes, but can be roughly grouped into five categories: tables dealing directly with planetary and spherical astronomy, tables aiding religious ritual, tables to help in the construction of astronomical instruments, mathematically based astrological tables[2] and

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[1] D. A. King, "al-Khalīlī", Dictionary of Scientific Biography (New York: Charles Scribner's Sons, 1978), p. 259. We will study al-Khalīlī's auxiliary tables in detail in Chapter 3.

[2] We shall not deal with these tables here. A description may be found in E. S. Kennedy, "Mathematics Applied to Astrology", in Proceedings of the Sixteenth International Congress of the History of Science: C. Meetings on Specialized Topics, Aug. 26 - Sept. 3, 1981, Bucharest, Romania, pp. 246-250.

finally purely mathematical auxiliary tables used to assist in the computation of more complex functions.

1.1.1 Tables of Spherical and Planetary Astronomy

Every Islamic zīj contains a collection of tables of functions of spherical astronomy, the science which describes rules for changes of coordinates in the celestial sphere. An example of one of the most common functions to appear in tabular form is the **oblique ascension,** the angular distance along the celestial equator (taken in the direction opposite to that of the daily rotation) from the vernal equinox (♈) to the horizon, as a function of the observer's terrestrial latitude and the angular distance along the ecliptic between ♈ and the horizon. (See Fig. 1.1.) Tables of oblique ascensions with varying degrees of accuracy appear as early as the ninth century in Ḥabash al-Ḥāsib's Zīj,[3] and reach their peak in the fourteenth century with the work of al-Kāshī and particularly of Ulugh Beg, whose values were calculated to the equivalent of seven decimal digits. Other tables include functions such as the **right ascension**[4] of various points of the ecliptic, the **solar**

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[3]M.-T. Debarnot, "The *Zīj* of Ḥabash al-Ḥāsib: A Survey of MS Istanbul Yeni Cami 784/2", in Eds. D. A. King and G. Saliba, From Deferent to Equant: A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E. S. Kennedy (Annals of the New York Academy of Sciences v. 500, 1987), p. 47.

[4]The angular distance along the celestial equator from the vernal equinox to the perpendicular projection of the ecliptic point onto the equator. The right ascension and the **declination**, the length of this projection, were used as the **equatorial** system of coordinates on the celestial sphere. See Fig. 1.2.

azimuth[5] as a function of solar altitude, celestial longitude, and terrestrial latitude, and the longitude of the **ascendant**[6] as a function of solar altitude.[7]

Another group of tables distinct from the above functions are those relating to planetary astronomy. Other than the sun, the five visible planets and the moon are seen to make a path through the sphere of fixed stars, staying within a few degrees on either side of the ecliptic. Many Islamic astronomers constructed tables based on Ptolemaic models describing the paths of these planets. Their longitudinal motion was generally decomposed into two parts, as follows:

$$\lambda(t) = \bar{\lambda}(t) + e(t) \tag{1.1}$$

where $\lambda(t)$ represents the true longitude at time t, $\bar{\lambda}(t)$ is a linear function of t describing the mean longitudinal motion, and $e(t)$ is a correction factor used to account for variations and retrogradations in the object's path, which in the medieval period was termed the **equation**. Planetary equation tables formed an integral part of the zījes.[8]

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[5]The angular distance along the horizon from the north point to the perpendicular projection of the sun onto the horizon. See Fig. 1.3.

[6]The location of the intersection of the ecliptic with the horizon.

[7]D. A. King, "On the Astronomical Tables of the Islamic Middle Ages", Studia Copernicana 13 (1975), p. 45.

[8]E. S. Kennedy, A Survey of Islamic Astronomical Tables, Transactions of the American Philosophical Society, New Series (Philadelphia, 1956), vol. 56, pt. 2, p. 142.

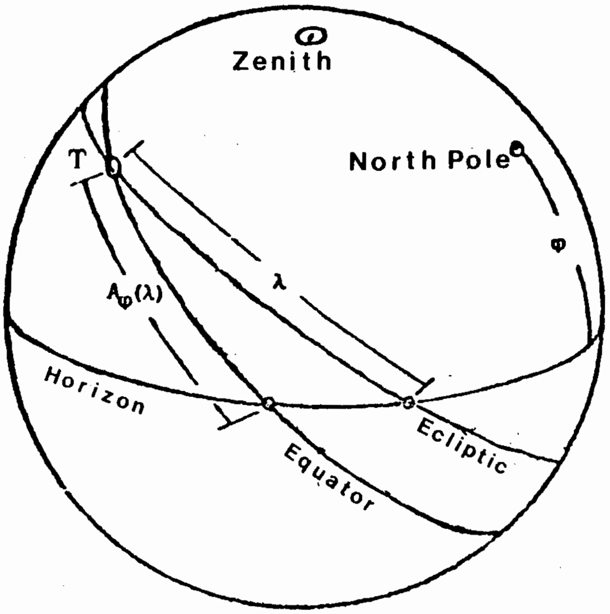

Fig. 1.1: The oblique ascension $A_\phi(\lambda)$ of celestial longitude λ, where ϕ is the the observer's terrestrial latitude.

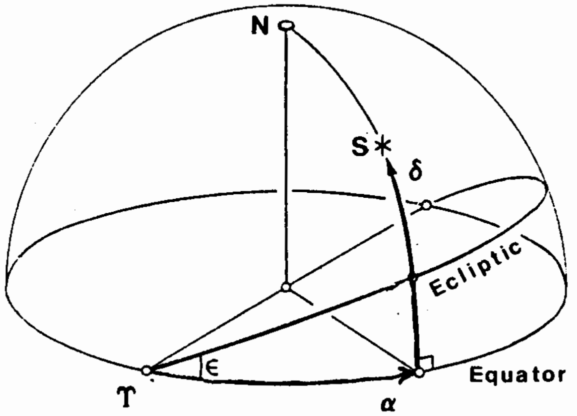

Fig. 1.2: The right ascension α and declination δ of star S, where ϵ is the obliquity of the ecliptic, and T is the vernal equinoctial point.

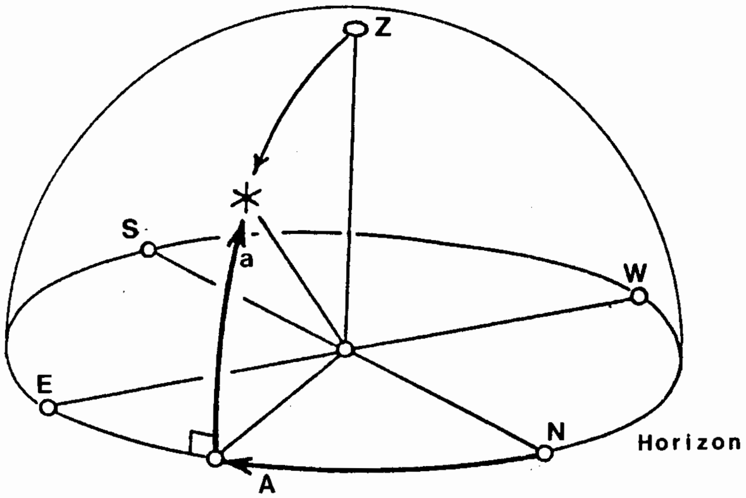

Fig. 1.3: The solar azimuth A and solar altitude a. Here N is the north point on the horizon.

1.1.2 Tables with Religious Significance

One of the principal features of the religion of Islam is the strict observance of certain rituals by the faithful around the world. These involve in particular the five daily prayers at specified times, the injunction to face the holy city of Mecca, the site of the Kaaba, and the observance of fasting during daylight hours in the sacred month of Ramadan. All three of these requirements give rise to non-trivial problems in spherical astronomy. The five daily prayer times are defined astronomically and must be strictly observed; the local direction of Mecca, known as the **qibla**, is required in order to orient the prayer walls of local mosques as well as the individual worshipper; and the beginning of each month in the Muslim calendar depends on the visibility of the lunar crescent when it is near the sun.

Finding the qibla given the worshipper's terrestrial coordinates is one of the more complicated problems in spherical astronomy. Approximate methods of solution were described as early as AD 900 and were used as late as the fourteenth century.[9] A modern exact solution to the qibla problem is given by

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[9]D. A. King, "al-Khalīlī's Qibla Table", Journal of Near Eastern Studies 34 (1975), note 3 p. 81, pp. 120-122. See also D. A. King, "The Earliest Islamic Mathematical Methods and Tables for Finding the Direction of Mecca", Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften 3 (1986), pp. 82-149.

$$q = \text{arc cot} \left\{ \frac{\sin \phi \cos \Delta L - \cos \phi \tan \phi_M}{\sin \Delta L} \right\} \quad (1.2)$$

where ϕ and ϕ_M are the latitudes of the observer and of Mecca respectively, ΔL is the difference in longitude between the observer and Mecca, and q (the qibla) is the angular direction east or west of due south, depending on the longitude of the observer.[10] Exact solutions in medieval times were achieved through either geometric constructions[11] or trigonometric formulae. Typical of medieval trigonometric solutions is that of al-Marrākushī, a thirteenth century Moroccan astronomer:

$$q = \text{arc Cos} \left\{ \frac{R \left[\frac{\text{Sin } h \text{ Tan } \phi}{R} \right] - \frac{R \text{ Sin } \phi_M}{\text{Cos } \phi}}{\text{Cos } h} \right\}, \quad (1.3)$$

where h, the height of the zenith of Mecca in the observer's sky, is determined by

$$\text{Sin } h = \text{Sin}(\bar{\phi} + \phi_M) - \text{Vers } \Delta L \frac{\text{Cos } \phi_M \text{ Cos } \phi}{R^2}. \quad (1.4)$$ [12]

D. A. King[13] has suggested that a method based on this formula may have been used by al-Khalīlī to create his qibla table of 2880 entries, which we will discuss later.

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[10]D. A. King, "al-Khalīlī's Qibla Table", p. 82.

[11]J. L. Berggren, "A Comparison of Four Analemmas for Determining the Azimuth of the Qibla", Journal for the History of Arabic Science 4 (1980), pp. 69-80.

[12]D. A. King, "al-Khalīlī's Qibla Table", pp. 101, 104. See Sec. 2.2 of this thesis for an explanation of the trigonometrical functions listed.

[13]D. A. King, "al-Khalīlī's Qibla Table", p. 99.

Prayer times depend on the location of the sun throughout the day and thus admit to an astronomical determination; hence, tables regulating prayer times fall under the category of astronomical timekeeping. Problems in this area involve the determination of time since the rising of the sun or a particular star given the object's current celestial coordinates and the observer's location. One Egyptian table, constructed by the thirteenth century astronomer Najm al-Dīn al-Miṣrī, serves for all latitudes and contains over 250 000 entries.[14] Prior to the thirteenth century these tasks were performed by the **muezzin**, using primarily the basics of folk astronomy. After this time, however, the new occupation of **muwaqqit** (=timekeeper) originated in Egypt. These scholars were hired by mosques expressly to solve problems of timekeeping and the direction of the qibla.

The prediction of the first visibility of the lunar crescent signifying the beginning of an Islamic month is perhaps the most difficult problem tackled by medieval astronomers. The moon makes a complete revolution around the celestial sphere inside a narrow band about ten degrees wide centred on the ecliptic every 29 or 30 days, and hence passes the sun about once a month. The light of the sun blots out the moon from view when the angular distance between them is less than about $9\frac{1}{2}$ degrees. The first sighting of the moon as it emerges from the sun's light defines the beginning of a new month. Further complications add to the

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[14] D. A. King, "On the Astronomical Tables of the Islamic Middle Ages", pp. 44-45.

difficulty of the problem: the exact position of the moon in the band around the ecliptic will of course be crucial and will involve the use of lunar latitude tables; the apparent size of the moon will affect the visibility; and even seasonal conditions can alter the time of first sighting.[15] Tables were constructed using methods of varying complexity and taking into account different factors, but the exact time could never be determined until the actual sighting. Even today the prediction of the first glimpse of the lunar crescent cannot always be made with complete accuracy.

1.1.3 Instrument Making Tables

Islamic astronomy did not, of course, consist entirely of table construction and use. Instruments such as astrolabes, quadrants and sundials were regularly used to obtain measurements and calculations both for immediate purposes and for use as arguments of certain astronomical functions. The construction of the best of these tools involves precise workmanship and, especially, accurate markings and curves. In particular, tables giving the locations of the standard curves found on sundials are relatively common. The marking of certain curves on the astrolabe also require precision, and tables giving the location of these curves for some terrestrial latitudes may also be found in the literature.

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[15] O. Neugebauer, The Exact Sciences in Antiquity, 2nd ed. (New York: Dover, 1969), pp. 106-110.

1.1.4 Mathematical Tables

We come now to the class of tables that is the source of the principal object of study in this thesis – the auxiliary tables. Most astronomical functions determined by medieval astronomers were found using exact or approximate trigonometric formulae. In order to relieve some of the tedium of repeated calculation as well as to provide tools for further research, various types of purely mathematical tables were constructed. The simplest of these are the sexagesimal (base 60) multiplication tables, which usually give the products $a \cdot b$, where a and $b = 1, 2, \ldots, 60$. These tables are convenient for use in sexagesimal multiplication. Other examples are trigonometric tables, generally giving the sine and tangent functions. These appear as early as the ninth century, resulting from contact with Indian mathematics. These tables reach their pinnacle in the work of Ulugh Beg, who in 1440 compiled sine and tangent tables for every minute of argument between 0° and 90°, to the equivalent of nine decimal places.[16]

The most interesting use of purely mathematical tables in medieval Islam, however, is found in the class of tables that

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[16] These tables are reproduced in C. Schoy, Die Trigonometrischen Lehren Des Persischen Astronomen Abu ʿl-Raiḥān Muḥammed Aḥmad al-Bīrūnī (Hannover: Orient-Buchhandlung Heinz Lafaire K.-G., 1927), pp. 92-108. The magnitude of Ulugh Beg's feat can be seen by the fact that Isaac Newton attempted the same task (except to 15 decimal places) over 200 years later and gave up in frustration because "the sheer drudgery of the project exhausted his patience". He completed nine entries. See Richard E. Westfall, Never at Rest: A Biography of Isaac Newton (Cambridge: Cambridge, 1980), p. 112.

give values for **auxiliary functions.** Early in the development of functions for use in spherical and astronomical timekeeping one finds many recurrences of mathematical expressions that appear as parts of different functions; for instance, multiples of Sin ϵ (where ϵ is the obliquity of the ecliptic) are useful in calculating solar declinations.[17] Repeated calculation of these quantities for different functions would quickly become an exercise in monotony. As early as the mid-ninth century Islamic scientists began to construct tables of these mathematical building blocks in order to simplify their own and their readers' calculations, often reducing large and cumbersome equations to straightforward combinations of values taken from these tables. Some of the applications of al-Khalīlī's auxiliary tables, for example, are described in Sec. 3.2.1. The methods of calculation of some of these auxiliary tables will form the central object of this study.

1.2 Uses of the Digital Computer in the Analysis of Tables

The advent of the digital computer has revolutionized almost every scientific field; hence, it is not surprising that its tremendous computational power has propelled forward the study of ancient and medieval astronomy. E. S. Kennedy[18] and

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[17] D. A. King, The Astronomical Works of Ibn Yūnus (Yale: unpublished doctoral dissertation, 1972), p. 96. Ibn Yūnus constructed tables of (n/R) · Sin ϵ and (n/R) · Cos ϵ for n = 1, 2, ..., 60.

[18] E. S. Kennedy, "The Digital Computer and the History of the Exact Sciences", Centaurus 12 (1967), pp. 107-113.

O. Gingerich[19] in 1967 introduced the computer to the field and described its use in recomputation of astronomical and mathematical tables. In this way the accuracy of these tables can be checked easily over a large number of values. Kennedy has also, with the aid of a computer, compiled a list of geographical coordinates for certain locations given in Islamic astronomical works and has organized this large amount of data into alphabetical order as well as according to increasing longitude and latitude.[20]

Since these early advances, however, while the digital computer has been transformed into a tool of incredible speed and potential, its use in the history of Islamic science has remained restricted to recomputation of tabular values. There remain many unexplored prospects in the reconstruction of astronomical and mathematical tables and their underlying parameters. For example, J. Hogendijk has recently begun to investigate this area by describing a method to determine the parameters behind lunar crescent visibility tables.[21] Many opportunities remain, however, for the use of the statistical and numerical tools provided by the computer.

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[19]O. Gingerich, "Applications of High-Speed Computers to the History of Astronomy", Vistas in Astronomy 9 (1967), pp. 229-236.

[20]E. S. Kennedy and M. H. Kennedy, Geographical Coordinates of Localities from Islamic Sources (Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften, 1987).

[21]J. Hogendijk, "Three Islamic Lunar Crescent Visibility Tables" (unpublished, 1987).

1.3 The Central Problem

The numerical methods used by medieval Islamic astronomers for table computations, for example, have been largely ignored in current research. Theoretical presentations showing the applications of geometry and trigonometry in spherical astronomy abound, both in the medieval manuscripts and in current analysis. The methods used to generate the vast number of tables that appear in the zījes and other treatises, however, remain a mystery. This gap may owe something to a bias, both past and present, in favour of mathematical theory over computational methods: medieval scientists generally carefully justified the formulas used to solve problems based on their tables without explicitly describing and verifying the accuracy of the methods used to create the tables themselves, and modern analysis consists essentially of recomputation of tabular values to determine their accuracy. Certainly the determination of a mode of calculation solely from the tabular values would be a daunting task without the aid of a digital computer, but current technology allows for the application of mathematical and statistical analysis without having to perform thousands of computations by hand. This study will use these tools in an attempt to determine the methods of calculation used in the auxiliary tables of Shams al-Dīn al-Khalīlī.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

2.1 Arabic Arithmetic

Sexagesimal arithmetic was the astronomers' mode of calculation from long before the medieval period. Its origins in numeration date back as far as the Old Babylonian period, c. 2000 BC, and its use by the Alexandrian astronomer Ptolemy in the mid-second century AD was responsible for its application to medieval astronomy and trigonometry. The Hellenistic version of sexagesimal representation used by Ptolemy (also used widely by Islamic astronomers) used the sexagesimal base only for fractional parts, while retaining decimal notation for the integral part. The characters used to represent individual sexagesimal digits were simply the letters of the Arabic alphabet in order corresponding to the values 1,2,...,9,10,20,...,50. This system, known as **abjad** numeration,[1] lends itself to some confusion due to the similarity of certain characters. Handwriting variations can render the symbols for 13, 18, 53, and 58, for example, virtually indistinguishable. This inevitably leads to a greater likelihood of scribal error in transcription than what would be encountered with other systems. The notation we will use for sexagesimal numbers is now conventional and accurately reflects

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[1]A detailed description of the Arabic numeral system may be found in R. A. K. Irani, "Arabic Numeral Forms", Centaurus 4 (1955), pp. 1-12.

how the numbers appear in the texts. The value

$$106 \; ; \; 13 \; , \; 48$$

represents $106 + \frac{13}{60} + \frac{48}{60^2}$, with the semicolon denoting the sexagesimal point and the comma separating consecutive sexagesimal digits.

The usefulness of the sexagesimal system in the computations required by astronomy and trigonometry becomes clear when one considers the long list of divisors of 60, but arithmetical procedures are not as easy in sexagesimal as in decimal arithmetic (or any other system with a reasonably small base). Addition and subtraction may be carried out with no difficulty analogously to decimal procedures, but multiplication and division are different matters. Multiplication of two numbers with, say, three sexagesimal digits each requires nine separate multiplications of two integers between 0 and 59. The decimal multiplication table can be memorized by any elementary school student, but the average reckoner would not be instantly able to determine the product of 47 and 54. The sexagesimal multiplication tables described in Sec. 1.1.4 were often used to speed calculation, but even then nine separate table searches would be required in order to compute the product discussed above. As a condition for ease of use of a hypothesized numerical method, then, we shall in further chapters prefer those methods that minimize the number of multiplications required to solve the problem.

2.2 Trigonometry

The earliest surviving example of a full-fledged trigonometric function and table occurs in Ptolemy's Almagest. In this book Ptolemy defines a function which gives the value of the length of a chord subtended by an arc θ on a circle of radius R = 60 (see Fig. 2.1). Using an arc sum and half-arc formula and a clever method of estimation of the chord of 1°, Ptolemy calculates to three sexagesimal digits the value of the length of the chord (which we call **Crd** θ) for arcs $\theta = \frac{1}{2}°$, 1°, $1\frac{1}{2}°$, ..., 180°.[2] This table becomes the basis of all the trigonometrical procedures carried out in the Almagest.

After some work with the chord function in plane and spherical trigonometry, it soon becomes clear that Ptolemy's chord function is not ideal. Often the value required is not the chord of the angle, but rather some multiple of the chord of double the angle. In fact, it is easy to see from Fig. 2.1 that

$$R \sin \theta = \frac{1}{2} \text{Crd } 2\theta, \qquad (2.1)$$

and so Ptolemy's chord function can be easily transformed into the much more useful sine function. This observation, however, was apparently never made by Hellenistic mathematicians and it was left to their Indian counterparts to invent the sine. The first extant sine table is found in Surya Siddhanta and Aryabhatiya, c. 400 AD, with only 24 entries corresponding to

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[2] For a full description of Ptolemy's method see G.J. Toomer (tr.), Ptolemy's Almagest (New York: Springer-Verlag, 1984), pp. 48-60.

increments of 3;45 = $3\frac{3}{4}°$. The radius of the base circle is R = 3438 parts,[3] and the sine values given are integral multiples of 1 part.

The introduction of Indian science to the Islamic world signified, among many other advances, the beginning of the most productive era in the history of trigonometry. The use of a base circle with radius R = 60 became standard, and in the ninth century Ḥabash al-Ḥāsib composed the first known table of tangents.[4] The cosine and cotangent, and the less popular secant and cosecant, all gained acceptance, and most of the common trigonometric identities were discovered. Each trigonometric function was defined not as a ratio of sides, but as lengths of the appropriate lines in the base circle for the sine and cosine, and as shadow lengths for the tangent and cotangent. Each Islamic trigonometric function based on a circle with R = 60, consequently, is sixty times the modern version.[5] We shall use the conventional capitalized notation to represent the medieval functions; i.e.,

$$\text{Sin }\theta = R \sin \theta; \quad \text{Tan }\theta = R \tan \theta; \quad \text{etc.} \qquad (2.2)$$

Islamic trigonometric tables generally give values for only the

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[3]This radius value, according to E. S. Kennedy, was likely chosen so that the length of one minute of arc on the base circle would have a length of one part (using the Indian value of π).

[4]S. Tekeli, "Ḥabash al-Ḥāsib", Dictionary of Scientific Biography (New York: Charles Scribner's Sons, 1972), p. 612.

[5]A base of 60 was not universal, however. Both Abu l-Wafa' and Abū Naṣr Manṣūr, for example, used R = 1, and R = 10 and 20 were occasionally used by al-Khalīlī. But we shall assume R = 60 unless otherwise stated.

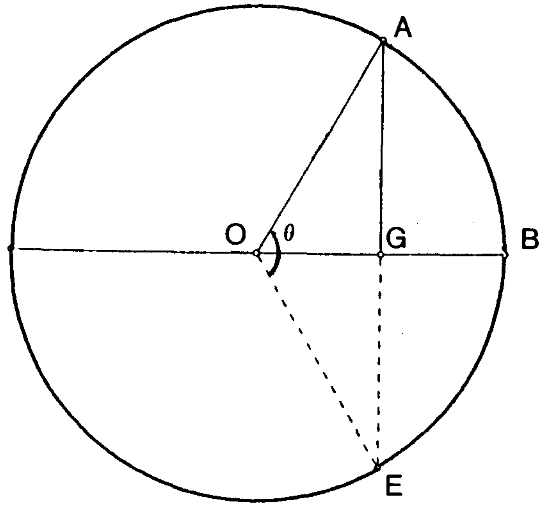

Fig. 2.1: The relationship between the sine and chord of an arc. If the radius OB of the circle is R units in length, the chord of θ is defined to be the length AE. The modern sine of $\theta/2$ is, of course, AG/R, or AE/2R.

sine and the tangent, since the two other common functions can be easily derived by the relations

$$\text{Cos } \theta = \text{Sin } (90° - \theta) \quad \text{and} \quad \text{Cot } \theta = \frac{R^2}{\text{Tan } \theta}. \tag{2.3}$$

1.3 Interpolation Methods

The use of interpolation methods to determine values of functions whose arguments lie between successive tabular entries as well as to create tabular entries within a grid of directly tabulated values dates back as far back as the astronomers of ancient Babylon.[6] There is little evidence, however, to suggest that Hellenistic scientists went much beyond linear interpolation. The theoretical development of higher order schemes came about through the efforts of others. Thus, among

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[6] O. Neugebauer, The Exact Sciences in Antiquity, 2nd ed. (New York: Dover, 1969), pp. 28, 135-136.

Liu Cho (AD 544-610) generated a second order scheme to apply to equally spaced intervals between nodes, and I-Hsing (AD 683-727) discovered a more general method to apply for unequally spaced nodes.[7] Islamic methods described in such treatises as Ibn Yūnus' Hākimī Zīj and the anonymous Dastūr al-Munajjimīn[8] included linear, second and even third order schemes as well as inverse linear and quadratic schemes to determine, say, the arc Sine from a Sine table. Linear methods, however, were often considered too trivial to note, and third order schemes were exotic and are rarely found in the literature. Interpolation schemes based on functions other than polynomials such as the sine function exist, but are rare in medieval mathematics. Hence, almost every interpolation scheme described by Islamic authors is of second order.

Not every interpolation scheme found in Islamic texts, however, is equivalent to passing a parabola through three given points. The most famous instance is al-Bīrūnī's failed attempt to use second differences to generate a better interpolation formula. Given two tabular entries $(x_0, f(x_0))$ and $(x_1, f(x_1))$ the standard formula to approximate $f(x)$ (where $x_0 < x < x_1$) with linear interpolation is

$$f(x) = f(x_0) + \frac{x - x_0}{x_1 - x_0} \Delta f_0, \qquad (2.4)$$

where $\Delta f_0 = f(x_1) - f(x_0)$, the forward difference. Al-Bīrūnī attempted to extend this method to account for second order

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[7]J. Hamadanizadeh, Medieval Interpolation Theory (Columbia: unpublished doctoral dissertation, 1976), pp. 20-21, 22, 24-26.

[8]J. Hamadanizadeh, Medieval Interpolation Theory, p. 31.

differences by using the additional point $(x_{-1}, f(x_{-1}))$ and the formula

$$f(x) = f(x_0) + \frac{x - x_0}{x_1 - x_0}\left[\Delta f_{-1} + \frac{x - x_0}{x_1 - x_0}\Delta^2 f_{-1}\right], \qquad (2.5)$$

where $\Delta^2 f_{-1} = \Delta f_0 - \Delta f_{-1}$. This intuitively appealing extension from linear to quadratic interpolation is easily seen to pass through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ but misses $(x_{-1}, f(x_{-1}))$ dramatically. In fact, J. Hamadanizadeh has remarked that the difference between al-Bīrūnī's parabola and the true parabola in the domain $[x_0, x_1]$ is equal to the difference between the true parabola and the line joining $(x_0, f(x_0))$ to $(x_1, f(x_1))$; i.e., al-Bīrūnī's formula is precisely as distant from true second order interpolation as is linear interpolation.[9]

Another example of an interpolation scheme differing from direct polynomial interpolation can be found in the Dastūr al-Munajjimīn.[10] The unknown author attributes this scheme to the tenth century mathematician Abū Ja·far al-Khāzin, and applies it to determining planetary longitudes on days between the directly computed values which are spaced ten days apart. Given endpoints (x_0, λ_0) and (x_{10}, λ_{10}), the author generates a second order formula based on the three points (x_{-1}, λ_{-1}), (x_0, λ_0) and $(x_{10}, \lambda_{10} + 5e')$ (where e' is a certain second difference) to calculate the values $\lambda_1,\ldots,\lambda_5$. Another parabola is used to join (x_5, λ_5) and (x_{10}, λ_{10}). The resulting 'bent'

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[9]J. Hamadanizadeh, Medieval Interpolation Theory, p. 121.

[10]For a detailed discussion see J. Hamadanizadeh, "Interpolation Schemes in *Dastūr al-Munajjimīn*", Centaurus 22 (1978), pp. 44-52.

union of two parabolae generally produces values between linear and second order interpolation; perhaps this change was due to a comparison with true longitudinal values of sample periods.

For ease of reference, we shall refer to the entries that are computed directly as **nodes**, and to the set of nodes in a given table as the **interpolation grid.** A typical span of entries between two successive nodes will be called an **internodal block.**

2.4 Definitions of Terms Used in the Text

In the succeeding chapters certain notations will be used which may be unfamiliar to the reader. In order to clarify the meaning of these symbols, they are defined below.

It will be convenient to use functional notation to indicate rounding procedures. To this end, we define

$$r_n(x) = \frac{\mathrm{Int}(x \cdot \beta^n + \frac{1}{2})}{\beta^n}, \tag{2.6}$$

where β is the base of the number system ($\beta = 60$ unless otherwise noted), Int(x) is the greatest integer less than or equal to x, and n is a positive integer giving the number of digits after the sexagesimal point to which x is to be rounded; for instance,

$$r_2(36;13,48,30) = 36;13,49.$$

In order to describe and compare the levels of sophistication of the various tables, it will be useful to introduce a term which measures the size of the error compared to the number of digits displayed in the table. Let $\hat{x}$ be the approximation to x, and let n be the number of digits displayed after the decimal point. We say $\hat{x}$ approximates x with **k digits of error**, where

$$k = 0 \qquad \text{if } \hat{x} = r_n(x);$$
$$k = 1 + \log_\beta\left\{\frac{|\hat{x} - r_n(x)|}{\beta^{-n}}\right\} \qquad \text{otherwise.}$$

The meaning of this term is made clear by example. If $r_3(x)$ = 47; 8,34,14 and $\hat{x}$ = 47; 8,34,15, then $\hat{x}$ differs from $r_3(x)$ by 1 in the last place and hence approximates x with 1 digit of error. If, say, $\hat{x}$ = 47; 8,33,14, then $\hat{x}$ approximates x with precisely 2 digits of error, and so on. (Note, however, that the number of digits in error is generally not an integer.) This figure, then, describes how many meaningless digits appear in the approximation.

The error representation scheme used in the text may be described as follows. The (rounded) exact value of a given function is written as usual:

$$f(14°,69°) = 57;44.$$

The value of the function given in the table itself is indicated by the use of a "T" preceding the symbol denoting the function, and if appropriate, the error in the final digit is shown in square brackets immediately after the function value. This error is calculated as follows:

$$\text{error} = \text{text} - r_n(\text{exact value}), \qquad (2.7)$$

where n is the number of digits given in the table after the sexagesimal point. Thus

$$Tf(14°,69°) = 57;43\ [-1].$$

CHAPTER 3

AL-KHALĪLĪ'S AUXILIARY TABLES

3.1 al-Khalīlī's Life and Work

Little is known about the life of Shams al-Dīn al-Khalīlī, other than that he was a contemporary of Ibn al-Shaṭīr in the late fourteenth century. All his known works deal with the science of astronomical timekeeping,[1] presumably written in connection with his occupation as muwaqqit at the Umayyad mosque in Damascus. Other than a treatise on the use of a trigonometric quadrant, all of his known works are tables related to various functions of astronomical timekeeping. These include auxiliary tables to aid in keeping time by the sun for all latitudes as well as complete timekeeping tables for the latitude of Damascus, tables giving times of prayer for Damascus, an extensive qibla table, and tables converting ecliptic coordinates to equatorial coordinates for use in computations relating to lunar crescent visibility. But perhaps his most interesting works are his tables of major auxiliary functions to solve various problems of spherical astronomy. These relatively simple combinations of trigonometric functions solve nothing when taken individually, but when combined in certain ways they lead to the solution of a host of problems in

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[1]D. A. King, "Astronomical Timekeeping in Fourteenth Century Syria", in Proceedings of the First International Symposium for the History of Arabic Science (Aleppo: Institute for the History of Science, 1976), Vol. 2, p. 80.

spherical astronomy.

3.2 The Auxiliary Functions

The first two of the three auxiliary functions calculated by al-Khalīlī are quite similar in nature, and are called the "first and second functions" in the manuscripts. Translated into modern notation, the first function is defined by

$$f(\phi,\theta) = \frac{R \text{ Sin } \theta}{\text{Cos } \phi}, \qquad (3.1)$$

and the second function by

$$g(\phi,\theta) = \frac{\text{Sin } \theta \text{ Tan } \phi}{R}, \qquad (3.2)$$

where ϕ is the local latitude and θ is some other value depending on the application. Most of the texts do not define all three of the functions explicitly, but one of the manuscripts describes their mathematical form in a marginal note. The first and second functions are calculated for the following arguments:

$\theta = 1°, 2°, \ldots, 90°$

$\phi = 1°, 2°, \ldots, 55°$, and 21;30° (the latitude of Mecca) and 33;30° (the latitude of Damascus),

producing a total of 5130 entries in each of the two tables. On each page of the document, the two functions are tabulated side by side for a fixed value of ϕ and all values of θ, arranged in columns of thirty entries. Fig. 3.1 below gives a schematic layout of one of these pages. In the following discussions, we shall refer to the argument that varies as one moves horizontally through the table as the **horizontal argument**, and to the other as the **vertical argument.** In the $f(\phi,\theta)$ and $g(\phi,\theta)$ tables, then, ϕ is the horizontal argument and θ is the vertical argument.

$\phi = 36°$								
θ Value			$f(\phi,\theta)$			$g(\phi,\theta)$		
1	31	61	1;18	38;13	64;52	0;46	22;27	38; 7
2	32	62	2;36	39;19	65;29	1;32	23; 6	38;28
3	33	63	3;54	40;24	66; 6	2;18	23;45	38;49
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
29	59	89	35;58	63;34	74;10	21; 9	37;22	43;35
30	60	90	37; 6	64;14	74;11	21;48	37;46	43;36

Fig. 3.1: A schematic layout of one of the pages of al-Khalīlī's tables showing values of $f(\phi,\theta)$ and $g(\phi,\theta)$ for a given value of ϕ

The third auxiliary table represents perhaps the greatest feat of calculation of the three, due to the nature of the function it describes. It is defined as follows:

$$G(x,y) = \text{arc Cos} \left\{\frac{Rx}{\text{Cos } y}\right\}, \qquad (3.3)$$

where the horizontal argument x is the "jayb al-tartīb", or the "auxiliary Sine". This function is calculated for the arguments

$x = 1,2,\ldots,59$

$y = 0°,1°,\ldots, \text{Int}(\text{arc Cos } Rx)$,

which results in 3420 entries. For larger values of y the argument $\frac{Rx}{\text{Cos } y}$ is greater than 60 and the function value does not exist. In the manuscript these entries are filled in as 0; 0, referring to an empty place. Thus, while the first two tables if written on a single very large page would be rectangular, the third table would be missing a curved area in the bottom right corner.

All three functions are tabulated to two sexagesimal digits, one following the sexagesimal point. The values are for the most part reasonably accurate: about 50% of the entries agree with the correct (rounded) value, and almost all are in error by less than 5 in the second place. The manuscript used throughout, MS. Paris Bibliothèque Nationale, ar. 2558, fols. 61v-104r, is the oldest of the known manuscripts (dated 1408), and is carefully and elegantly copied. Other than the columns for the latitude of Mecca in the $f(\phi,\theta)$ and $g(\phi,\theta)$ tables, it is also complete. Appendix A contains some sample columns of all three tables.

3.2.1 Some Uses of the Auxiliary Functions

Since many of al-Khalīlī's formulae for use in astronomy can be derived from the cosine law of spherical trigonometry (although there is no direct evidence that al-Khalīlī was familiar with it), it is not surprising that they have similar mathematical structure. These similarities lend themselves to the implementation of auxiliary functions in order to facilitate their computation. Several uses of the auxiliary functions outlined by al-Khalīlī are described below.[2]

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[2]The discussion in this section is taken primarily from D. A. King, "al-Khalīlī's Auxiliary Tables for Solving Problems of Spherical Astronomy", Journal for the History of Astronomy 4 (1973), pp.99-110.

The formula to find the altitude of a celestial object in the prime vertical[3] is easily seen to be

$$h_0 = \text{arc Sin}\left[\frac{R \text{ Sin } \delta}{\text{Sin } \phi}\right], \tag{3.4}$$

where δ is the declination of the object and ϕ is the local latitude. al-Khalīlī gives two different solutions to this problem using his auxiliary tables. Firstly,

$$h_0 = \text{arc Sin}\left[\frac{R \text{ Sin } \delta}{\text{Sin } \phi}\right]$$

$$= 90° - \text{arc Cos}\left[\frac{\text{Sin } \delta \text{ Tan } 45°}{\text{Cos } \bar{\phi}}\right]$$

$$= 90° - G[g(45°,\delta),\ \bar{\phi}], \tag{3.5}$$

where $\bar{\phi} = 90° - \phi$. Alternatively,

$$\frac{\text{Sin } h_0 \text{ Tan } \phi}{R} = \frac{R \text{ Sin } \delta}{\text{Cos } \phi},$$

so h_0 can be found by solving the equation

$$g(\phi,h_0) = f(\phi,\delta). \tag{3.6}$$

In order to determine the solar azimuth (measured from the meridian), a precise formula is

$$a(h,\delta,\phi) = \text{arc Cos}\left[\frac{\text{Sin } h \text{ Tan } \phi - \dfrac{R^2 \text{ Sin } \delta}{\text{Cos } \phi}}{\text{Cos } h}\right], \tag{3.7}$$

where h is the solar altitude, δ is the declination, and ϕ is the local latitude. This is clearly equivalent to

$$a(h,\delta,\phi) = G[g(\phi,h) - f(\phi,\delta),\ h]. \tag{3.8}$$

An important corollary of this result is the qibla formula

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[3]The **prime vertical** is the great circle passing through the celestial north pole and the east and west points on the horizon.

derived by al-Marrākushī and quoted by al-Khalīlī. Inserting the zenith of Mecca in place of the sun in the observer's sky (so that h is the height of the zenith of Mecca and $\delta = \phi_M$), we derive (1.3), which is equivalent to

$$q = G[g(\phi,h) - f(\phi,\phi_M), h]. \qquad (3.9)$$

Thus al-Khalīlī could have used his auxiliary tables to compute his qibla table, and in fact he mentions in his introduction to the qibla table that al-Marrākushī's method is the best solution to the qibla problem that he knows. King has already voiced his doubts regarding this possibility;[4] we shall discuss it further in Sec. 3.7.

3.3 The Derivation of the $f(\phi,\theta)$ Table from the $g(\phi,\theta)$ Table

Approximately 50 to 55% of the entries in the $f(\phi,\theta)$ table are correct to both sexagesimal digits; the remaining entries are for the most part in error by one or two in the second sexagesimal place. The errors are, however, not uniformly distributed throughout the table: the entries in certain columns contain generally larger errors than those in other columns. Often within a given column in all three of al-Khalīlī's tables the errors appear to change continuously as the vertical argument varies. This may be a sign of interpolation, but it may also be caused either by a flawed value which in some way affects every entry in the column or by more indirect factors.

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[4]D. A. King, "al-Khalīlī's Qibla Table", Journal of Near Eastern Studies 34 (1975), p. 106.

Two regions of the $f(\phi,\theta)$ table reveal distinct patterns. The first column, corresponding to $\phi = 1°$, agrees in all but two entries with a two sexagesimal digit Sine table. That the entries are close to the Sine values is not surprising, since

$$f(1°,\theta) = \frac{R \text{ Sin } \theta}{\text{Cos } 1°} \simeq \frac{60;\ 0}{59;59} \text{ Sin } \theta,$$

which is only very slightly larger than Sin θ. However, 23 of the 90 entries in the $\phi = 1°$ column have an error of -1 (using the error representation system described in Sec. 2.4), while none of the entries err on the positive side. Since 88 of the 90 entries (including the 23 in error) agree with $r_1(\text{Sin } \theta)$, it seems clear that al-Khalīlī simply approximated $f(1°,\theta)$ with Sin θ.

The two entries that fail to fit this pattern are those corresponding to $\theta = 17°$ and $\theta = 89°$. In the case of $\theta = 17°$, we have $Tf(1°,17°) = 17;32$ [-1], whereas $r_1(\text{Sin } 17°) = 17;33$. This discrepancy may be a copying error, for, in those sections of the $g(\phi,\theta)$ table that were calculated using

$$g(\phi,\theta) = \frac{1}{R} \cdot r_1(\text{Sin } \theta) \cdot r_1(\text{Tan } \phi), \qquad (3.10)$$

the value used for $r_1(\text{Sin } 17°)$ seems to be 17;32, the value found in $Tf(1°,17°)$. We will discuss this further in Sec. 3.4.

The discrepancy for $\theta = 89°$ is due to the second pattern found in the table. Where $\theta = \bar{\phi} = 90° - \phi$, the value of $f(\phi,\bar{\phi})$ is

$$f(\phi,\bar{\phi}) = \frac{R \text{ Sin } \bar{\phi}}{\text{Cos } \phi} = \frac{R \text{ Cos } \phi}{\text{Cos } \phi} = R = 60;\ 0. \qquad (3.11)$$

If interpolation had been used over a fixed grid throughout the table, only a small number of the entries on the diagonal through the table determined by $\theta = \bar{\phi}$ would be calculated directly as nodal values. But in fact every tabular entry on this diagonal (other than Tf(8°,82°) = 60; 5 [+5], an obvious scribal error) is precisely 60; 0.[5] This does not provide strong evidence against interpolation, for it is possible that al-Khalīlī superimposed this diagonal after completing the table. It is clear, however, that he recognized that $f(\phi,\bar{\phi}) = R = 60;0$ and used this result in the construction of this portion of the table.

Al-Khalīlī may have completed the bulk of the table in a variety of ways. Due to the excessive work involved in multiplication and division, however, it would have been in his interest to choose a method which minimizes the number of directly calculated entries. Thus the method of interpolation recommends itself, since only a small fraction of entries would need to be directly calculated. Attempts to find an interpolation grid (in the same fashion as the corresponding efforts for the $g(\phi,\theta)$ table, described in Sec. 3.4.2), however, fail to produce any recognizable patterns. Other possible methods include treating each individual column as a multiple of a Sine table; i.e.,

$$f(\phi,\theta) = \frac{R}{\text{Cos } \phi} \cdot \text{Sin } \theta. \qquad (3.12)$$

The constant $\frac{R}{\text{Cos } \phi}$ may be evaluated once for an entire column, or taken from a secant table. This method does not reduce the

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[5]Of course, the columns corresponding to $\phi = 21;30°$ and $\phi = 33;30°$ do not have entries for $\theta = \bar{\phi}$. In the following discussions, we shall largely ignore these two columns.

number of multiplications required, but it does mechanize the procedure and it uses the same value as a multiplicand for 90 consecutive entries. The analogous method working over rows as opposed to columns is also possible, but given the representation of the table in the manuscripts this is unlikely. Attempts to reproduce the table via these and other methods fail to give a higher percentage agreement with the $f(\phi,\theta)$ table than the percentage of correct entries in the table itself.

Based on the diagonal corresponding to $f(\phi,\bar{\phi})$, however, it is possible to construct a method which is by far the most efficient and most easily applied of those methods thus far considered and which also produces a remarkably high percentage agreement over most of the table. Applying the angle addition formula for Sines to $f(\phi,\bar{\phi} + n)$,[6] we get

$$\begin{aligned} f(\phi,\bar{\phi} + n) &= \frac{R \operatorname{Sin} (\bar{\phi} + n)}{\operatorname{Cos} \phi} \\ &= \frac{R \cdot \frac{1}{R} [\operatorname{Sin} \bar{\phi} \operatorname{Cos} n + \operatorname{Sin} n \operatorname{Cos} \bar{\phi}]}{\operatorname{Cos} \phi} \\ &= \frac{\operatorname{Sin} \bar{\phi}}{\operatorname{Cos} \phi} \operatorname{Cos} n + \frac{\operatorname{Tan} \phi \operatorname{Sin} n}{R} \\ &= \operatorname{Cos} n + g(\phi,n). \qquad (3.13) \end{aligned}$$

Using the angle subtraction formula for Sines we get a similar formula for $f(\phi,\bar{\phi} - n)$. So, another possible way to generate the $f(\phi,\theta)$ table is from the $g(\phi,\theta)$ table using the equation

$$f(\phi,\bar{\phi} \pm n) = \operatorname{Cos} n \pm g(\phi,n). \qquad (3.14)$$

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[6]This formula was well-known to Islamic scientists at this time. See J. L. Berggren, *Episodes in the Mathematics of Medieval Islam* (New York: Springer-Verlag, 1986), pp. 135-138 for a description of the proof given by Abu l-Wafa'.

This method has several advantages. The obvious rationale in favour of its use is the fact that it completely avoids the task of sexagesimal multiplication; a comparatively easy addition or subtraction is all that is required to generate an entry. It is also a stable algorithm: the errors involved in rounding Cos n and in using the tabular value for $g(\phi,n)$ are not magnified by simply adding or subtracting the two quantities. Finally, given the layout of the table in the manuscripts, this method allows the reckoner to compute $f(\phi,\theta)$ directly from the appropriate entry in the adjacent $g(\phi,--)$ column on the same page.

It is of course easy to test this hypothesis by taking the two digit rounded value for $r_1(\text{Cos } n)$, applying (3.14) using al-Khalīlī's $g(\phi,n)$ value, and comparing the result obtained for $f(\phi,\bar{\phi} \pm n)$ to the entry in the table. The results were discouraging for the first nine columns and two areas within the columns for $\phi = 46°$ through $\phi = 49°$.[7] In the remaining 80% of the table, the percentage agreement is high enough to be consistent with the hypothesis that al-Khalīlī used this method, given the possibility of scribal and computational errors. Of 781 tabular entries checked in this area (see Fig. 3.2), 749, or 95.9%, matched with the calculation above.[8]

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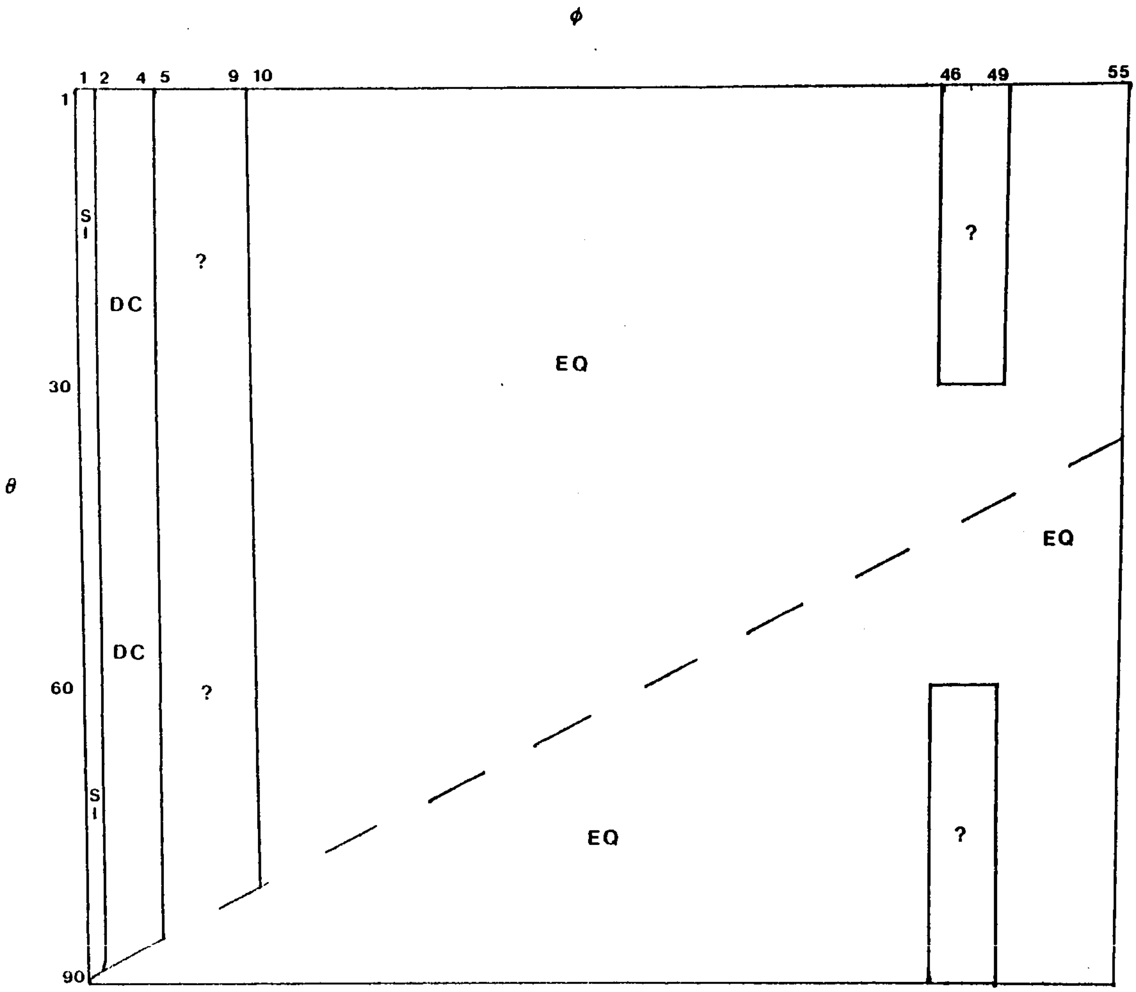
[7] These areas are for $\theta = 1°,\ldots,30°$, and $\theta = 61°,\ldots,90°$, the first and third columns in the manuscript. The entries here present a considerably higher error level than elsewhere in the table, up to 5 in the second place. Perhaps these entries were copied from an earlier set of tables al-Khalīlī may have computed.

[8]This comparison is successful both for those entries that are



SI = Sine Values

DC = Direct Computation (using two sexagesimal digit trigonometric values)

EQ = Equation (3.14)

Fig. 3.2: Schematic diagram of the methods used to generate al-Khalīlī's $f(\phi,\theta)$ table

One may suppose that since the number of possible methods of calculation is bounded only by the imagination, it is impossible to prove rigorously that al-Khalīlī actually used (3.14) or a mathematically equivalent formula. This is, of course, true; however, there are compelling reasons to believe that no reasonable method other than (3.14) could possibly generate such a high match with the table. For, calculation of $f(\phi,\theta)$ according to (3.14) introduces two errors: the rounding of Cos n, and the use of the tabular value $g(\phi,n)$. Each of these two factors causes a particular error pattern over the 4000 entries in question, and it seems unlikely that either pattern could be generated by any function other than one mathematically equivalent to $r_1(\text{Cos } n)$ or $Tg(\phi,n)$. Yet the addition of the two error factors produces a match of 96% with the tabular values. A method that is truly distinct would not contain one or both of these factors and would introduce its own, caused by rounding and other means. The final error pattern produced by this method, while it may agree with that in the table for a certain percentage of entries,[9] has a probability of fitting the error pattern for a large number of entries comparable to the chance

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[8](cont'd) accurate to two sexagesimal digits, and for those that are not. Of the accurate entries, 399 of 416 agree with (3.14), and of the inaccurate entries, 350 of 365 agree. Both of these figures correspond to a match of 95.9%.

[9]Suppose (for simplicity) that 50% of the tabular entries are correct, and that the remaining entries err by 1 in the last place, 25% in each direction. An independent method with the same error distribution has a

$$(.5)^2 + (.25)^2 + (.25)^2 = .375,$$

or 37.5% probability of agreeing with a given entry. Clearly over a large number of entries the percentage agreement will converge to this figure.

that a randomly chosen house key has of opening a given lock.

While (3.14) can be used to generate entries of $g(\phi,\theta)$ from the $f(\phi,\theta)$ table as well as vice versa, there are several reasons for believing that $g(\phi,\theta)$, the "second function", is actually the table originally calculated. Firstly, whenever $\bar{\phi} + n > 90°$ and $\bar{\phi} - n < 0$; i.e., $\phi < n$ and $\bar{\phi} < n$, (3.14) is useless for computing g from f, so a large area corresponding to approximately one quarter of the $g(\phi,\theta)$ table is inaccessible from $f(\phi,\theta)$ using this formula. Secondly, (3.14) produces from a single entry of the $g(\phi,\theta)$ table two distinct entries in the $f(\phi,\theta)$ table (provided $\theta \neq \bar{\phi}$). Finally, the $g(\phi,\theta)$ table contains an interpolation grid not found in the $f(\phi,\theta)$ table, as we shall see in Sec. 3.4.2.

3.4 The Construction of the $g(\phi,\theta)$ Table

The error levels in the $g(\phi,\theta)$ table are, of course, roughly the same as those in the $f(\phi,\theta)$ table. But whereas there are small zones in the $f(\phi,\theta)$ table (for $\phi = 46°$ through $49°$) where the errors reach 4 in the last place consistently, the $g(\phi,\theta)$ table has none of these zones. Also, although for some values of the arguments the $g(\phi,\theta)$ table simplifies to a straightforward function this fact does not in general appear to have been utilized by al-Khalīlī. For instance, where $\theta = 90°$,

$$g(\phi,90°) = \frac{\text{Sin } 90° \text{ Tan } \phi}{R} = \text{Tan } \phi, \qquad (3.15)$$

and where $\theta = \bar{\phi} = 90° - \phi$,

$$g(\phi,\bar{\phi}) = \frac{\text{Sin } \bar{\phi} \text{ Tan } \phi}{R} = \text{Sin } \phi, \qquad (3.16)$$

but neither of the corresponding areas show any more accurate entries than anywhere else in the table. The column for $\phi = 45°$ is, however, a Sine table accurate to two sexagesimal digits. This, incidentally, shows that al-Khalīlī had access to sine values with this accuracy, even though his extant sine table is slightly less accurate. Considering the accuracy of the trigonometric tables of the fourteenth century, however, this is not surprising.

3.4.1 A Correlation Method to Determine the Rounding Procedure

In order to determine the method of computation of some or all of the entries, it is important to be able to ascertain the accuracy to which some of the intermediate parameters were rounded. Explicit testing of all the various rounding techniques; for instance, $g(\phi,\theta) = \frac{1}{R} \cdot r_m(\text{Sin } \theta) \cdot r_n(\text{Tan } \phi)$ for pairs of values m and n; has several drawbacks. Firstly, in general this brute force method is very time-consuming; secondly, if an interpolation grid or similar method were used it could be difficult to spot the slightly increased percentage agreement as significant when the correct rounding procedure for the nodal entries is used; and finally, the usual advantage of a brute force method – a guarantee of success – does not apply here. It is possible, for instance, that al-Khalīlī used, say, three sexagesimal digit values for trigonometric arguments that are flawed in the last digit. In this case brute force will not only fail, but may mislead one into more closely examining those hypotheses that by sheer chance exhibit a slightly higher percentage agreement. Clearly a more systematic method is required.

Consider a given constant ϕ and a hypothesized rounding procedure r_n for Tan ϕ. The value al-Khalīlī would use for Tan ϕ is then altered by the amount

$$\Delta t = r_n(\mathrm{Tan}\ \phi) - \mathrm{Tan}\ \phi. \qquad (3.17)$$

This results in a function value shift given by

$$Tg(\phi,\theta) = \frac{\mathrm{Sin}\ \theta\ (\mathrm{Tan}\ \phi + \Delta t)}{R}$$

$$= \frac{\mathrm{Sin}\ \theta\ \mathrm{Tan}\ \phi}{R} + \Delta t\ \frac{\mathrm{Sin}\ \theta}{R}. \qquad (3.18)$$

Of course the rounding of Sin θ will also cause an error but for each ϕ we assume the application of the same set of values of Sin θ and hence each column should be affected equally by this rounding. What (3.18) demonstrates is that if al-Khalīlī had used the hypothesized rounding procedure, the value of Δt should be linearly related to the average signed level of error found in the column.

This linear relation will, of course, be complicated by several factors. Firstly, the final rounding of $g(\phi,\theta)$ to two sexagesimal digits will alter the final error, perhaps significantly. Secondly, if al-Khalīlī used an interpolation scheme, only the nodal entries would be affected directly: the effect on the internodal entries would only be indirectly felt, through the values at the nodes. Finally, if the rounding procedure is sufficiently precise (Δ is very small), its effect on the final value will be minimal or unnoticeable. In the latter case it may be possible to find a method of direct calculation or an interpolation grid simply by assuming that al-Khalīlī's value of Tan ϕ is accurate.

Thus the linear relation may only be seen in a statistical sense, if at all. The statistical tool to determine whether the connection exists is, of course, the correlation coefficient. The measure of the total error in a given column that we will use is simply the sum of the signed errors in the last digit over the 90 entries in the column. Table 3.1 gives the values of Δt for the hypothesized rounding procedure r_1 and the total column error for 17 scattered values of ϕ.

ϕ	Δt	Total Column Error
1°	+0; 0, 9,42	+6
2°	+0; 0,17, 7	+6
10°	+0; 0,13,22	+4
16°	-0; 0,17, 1	-67
20°	-0; 0,17,35	-47
21°	+0; 0, 5,22	+10
22°	-0; 0,29,40	-41
23°	-0; 0, 6,34	-33
24°	+0; 0,10,36	-40
25°	+0; 0,17,32	-10
30°	-0; 0,27,40	-115
36°	+0; 0,26,48	0
42°	-0; 0,27,18	+1
46°	+0; 0, 5,25	+13
47°	+0; 0,28,19	-2
48°	-0; 0,12,21	-60
55°	-0; 0,20, 2	-23

Table 3.1: Values of Δt and total column error for selected values of ϕ in the $g(\phi,\theta)$ table

The correlation coefficient between columns 2 and 3 in the above table is 0.6286. Assuming the two columns of data have a zero correlation, the probability of observing data with a correlation coefficient as high as this or higher is less than 0.5%.[10] This result, then, provides good statistical evidence that al-Khalīlī used a two sexagesimal digit value of Tan ϕ.

3.4.2 The Location of the Interpolation Grid

The correlation argument above produces a good statistical reason for the use of $r_1(\text{Tan } \phi)$ in further research, but it leaves open several possibilities regarding its application. Considering the relative rarity of the tangent function, it is certainly possible that al-Khalīlī used Sine values more accurate than the r_1 values. Given the size of the table it is unlikely that al-Khalīlī computed every entry directly, but it is not impossible and one must never underestimate what a dedicated individual can do. If an interpolation scheme were used, there are still several possibilities regarding the location and spacing of the nodes, and as we have seen with the $f(\phi,\theta)$ table, al-Khalīlī may have used different methods in different areas of the table.

The obvious first hypothesis to attempt is, of course, the rounding procedure defined by

$$g(\phi,\theta) = r_1\left[\frac{r_1(\text{Sin } \theta)\ r_1(\text{Tan } \phi)}{R}\right]. \tag{3.19}$$

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[10] D. V. Lindley & W. F. Scott, New Cambridge Elementary Statistical Tables, Cambridge, 1984, p. 56.

Table 3.2 below shows, for a span of six columns corresponding to $\phi = 20°,\ldots,25°$ and for each value of θ, the number N of tabular entries agreeing with calculation according to (3.19).

θ	N	θ	N	θ	N	θ	N	θ	N	θ	N
1	3	16	5	31	6	46	5	61	2	76	3
2	2	17	4	32	3	47	4	62	2	77	0
3	4	18	5	33	4	48	4	63	3	78	2
4	3	19	4	34	4	49	6	64	2	79	2
5	5	20	5	35	5	50	6	65	6	80	6
6	3	21	2	36	5	51	4	66	4	81	5
7	2	22	2	37	2	52	4	67	2	82	2
8	3	23	3	38	3	53	4	68	3	83	2
9	3	24	2	39	2	54	5	69	3	84	2
10	5	25	6	40	6	55	6	70	6	85	6
11	3	26	4	41	1	56	2	71	2	86	4
12	4	27	3	42	2	57	3	72	2	87	3
13	2	28	4	43	6	58	2	73	1	88	3
14	1	29	6	44	6	59	3	74	2	89	5
15	4	30	6	45	6	60	6	75	6	90	5

Table 3.2: The number of tabular values of $g(\phi,\theta)$ that agree with (3.19) for $\phi = 20°,\ldots,25°$

Of the entries whose θ values are divisible by 5, 101 of 108, or 93.5%, agree with calculation according to (3.19). Of the entries whose θ values are not divisible by 5, only 228 of 432, or 52.8%, agree with (3.19). So at least over $\phi = 20°,\ldots,25°$, we have conclusive evidence that al-Khalīlī used an interpolation grid with nodes separated by 5° of θ.

Extending this study over the entire table, the same results are generated for most values of ϕ. A comparison for all values of ϕ is given in Table 3.3. For $\phi < 45°$, the distribution of the numbers in Table 3.3 is clearly not random. 32 of the columns have a match of 15 out of 18 or better with (3.19).[11] Of the other thirteen columns, nine have matches of eight or less of 18 nodes. (See Fig. 3.3 for a histogram.) For $\phi > 45°$ the pattern changes: almost all of these columns exhibit a failure of a sufficiently high level to reject the possibility of the use of $r_1(\mathrm{Tan}\ \phi)$.

The extremely small percentage of failure over the nodal values compared with the percentage of failure over the internodal entries is firm evidence in favour of the hypothesized interpolation grid. But in order for the hypothesis to explain satisfactorily the nodal entries, the cause of the failure over the remaining columns needs to be shown. The fact that the columns that fail to match are scattered randomly seems to indicate that these columns should have been calculated as the surrounding columns were; possibly the error was caused by a different value of Tan ϕ. The third column of Table 3.3 reveals a strong pattern that supports this theory: of those columns that fail comparison with (3.19) on the nodal entries, the failures are almost entirely to one side, positive or negative, of the expected value.

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[11] Assuming a 50% probability of a match with an independent method, the probability of a given column matching this well or better are less than 0.4%. Comparison of (3.19) with the internodal values in these same columns again reveals only approximately a 50% match.

ϕ	Nodes Correct	+/-		ϕ	Nodes Correct	+/-
1	12	1/5		29	5	0/13
2	15			30	5	0/13
3	18			31	17	
4	17			32	7	0/11
5	17			33	18	
6	17			33.5	4	0/14
7	18			34	18	
8	15			35	17	
9	14			36	16	
10	14			37	17/17	
11	17			38	17	
12	17			39	18	
13	17/17			40	7	0/11
14	16			41	18	
15	7/16	0/9		42	8	10/0
16	7/15	1/7		43	16	
17	6/15	0/9		44	16	
18	18			45	18	
19	18			46	12/17	2/3
20	18			47	11/16	1/4
21	18			48	10	1/7
22	17			49	14	
23	17			50	11	7/0
24	14			51	8	0/10
25	17			52	4	0/14
26	17			53	7	0/11
27	18			54	5/17	0/12
28	5	1/12		55	14	

Table 3.3: Comparison of the number of nodal entries agreeing with (3.19) in the $g(\phi,\theta)$ table, and the direction of the error in those columns with 5 or more failures

```
 1- 4 *
 5- 8 *********
 9-12 *
13-16 *********
17-18 **************************
```

Fig. 3.3: Histogram of column 2 of Table 3.3 for $\phi < 45°$

Table 3.4 below shows the results of recomputation of the nodal entries with a tangent value shifted up or down one minute, as suggested by the errors in Table 3.3.

ϕ	r_1(Tan ϕ)	Match with r_1(Tan ϕ)	New Tan ϕ Value	Match with New Value
15	16; 5	7/16	16; 4	15/17
16	17;12	7/15	17;11	15/17
17	18;21	6/15	18;20	15/15
28	31;54	5/18	31;53	17/18
29	33;16	5/18	33;15	16/18
30	34;38	5/18	34;37	17/18
32	37;30	7/18	37;29	16/18
40	50;21	7/18	50;20	16/17
42	54; 1	8/18	54; 2	14/18
48	66;38	10/18	66;37	10/18
50	71;30	11/18	71;31	16/18
51	74; 6	8/18	74; 5	16/18
52	76;48	4/18	76;47	16/18
53	79;37	7/18	79;36	18/18
54	82;35	5/17	82;34	16/18

Table 3.4: Comparison of nodal entries in the $g(\phi,\theta)$ table with (3.19), using a new Tan ϕ value suggested by column 3 of Table 3.3

The resulting match over the columns above is 233 of 264, or 88.3%. On its own this result may not seem surprising: if some entries are too small, the use of a slightly higher value of Tan ϕ should shift some entries up and hence produce a higher level of agreement. However, it is not hard to see that the match is too high to be attributed to this fact. In any case, the high match guarantees that the value of Tan ϕ found by multiplying each nodal entry in the column by Sin θ and taking the mean will be almost precisely the altered value used in Table 3.4. The reconstructed Tan ϕ values, with errors illustrated, are shown in Table 3.5.[12]

ϕ	Tan ϕ		ϕ	Tan ϕ
1	1; 3		29	33;15 [-1]
2	2; 6		30	34;37 [-1]
3	3; 9		31	36; 3
4	4;12		32	37;29 [-1]
5	5;15		33	38;58
6	6;18		34	40;28
7	7;22		35	42; 1
8	8;26		36	43;36
9	9;30		37	45;13
10	10;35		38	46;53
11	11;40		39	48;35
12	12;45		40	50;20 [-1]
13	13;51		41	52; 9
14	14;58		42	54; 2 [+1]
15	16; 4 [-1]		43	55;57
16	17;11 [-1]		44	57;56
17	18;20 [-1]		45	60; 0
18	19;30		46	62; 8
19	20;40		47	64;21
20	21;50		48	-----
21	23; 2		49	69; 1
22	24;14		50	71;31 [+1]
23	25;28		51	74; 5 [-1]
24	26;43		52	76;47 [-1]
25	27;59		53	79;36 [-1]
26	29;16		54	82;34 [-1]
27	30;34		55	85;41
28	31;53 [-1]			

Table 3.5: The reconstructed tangent values used by al-Khalīlī in the construction of the $g(\phi,\theta)$ table

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[12]For small values of ϕ it is hard to verify any hypothesis, since the function values are very small. But for $\phi > 10°$, a comparison over the nodal values in 24 selected columns reveals a match of 261 out of 285, or 91.6%, between (3.19) and al-Khalīlī's accurate tabular entries. The same comparison over the nodal values for al-Khalīlī's inaccurate entries produces a match of 130 out of 139, or 93.5%.

An objection may be raised that the discovery of incorrect tangent values invalidates the test in Sec. 3.4.1, which used correct tangent values. Only a small percentage of the tangent values are incorrect, however, affecting the correlation only minimally. If the test were performed with the new tangent values, undoubtedly the correlation would be even higher.

3.4.3 The Interpolation Scheme

A comparison of the tabular values with direct computation according to (3.19), using the reconstructed tangent values, reveals two small regions with a high enough percentage agreement to support the direct calculation hypothesis over these entries. A small block near the centre and at the bottom of the table has a percentage agreement of 97.8%, and an area comprising most of the columns for high values of ϕ has a 95.2% match.[13] See Fig. 3.4 for an outline of these regions. The vast majority of the internodal values, however, remain unexplained.

An examination of the first differences throughout the entire table reveals no consistent pattern. It is immediately clear, however, that linear and second order interpolation (the most likely possibilities from an historical viewpoint) fail to account for even small sections. Statistical tests searching for the effect of $r_n(\text{Sin } \theta)$ on internodal values for different values of n also failed to find a relation.

Over a large area covering almost 50% of the table, however, a uniform pattern emerges. This area has rather clearly defined boundaries, but the edges described in Fig. 3.4 are not to be taken as completely rigid. The method of interpolation that

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[13] In the latter area, it appears that al-Khalīlī is using Sin 17° = 17;32 [-1], the same value that occurs in the first column of the $f(\phi,\theta)$ table. This value agrees with that in a sine table known to al-Khalīlī. See D. A. King, Shams al-Dīn al-Khalīlī and the Culmination of the Islamic Science of Astronomical Timekeeping (unpublished: Frankfurt University, 1987), Table 3.1A, p. 177.

matches the values in the table is a variant of linear interpolation, and may be described as follows. Let $x = Tg(\phi,\theta + 5°) - Tg(\phi,\theta)$ in sixtieths, where θ is divisible by 5. Then x is the difference in tabular values at the ends of a typical span between two consecutive nodes. Let $n + \frac{m}{5} = \frac{x}{5}$, where m and n are integers and $0 \le m \le 4$. For the first m entries, add n + 1 minutes to the previous entry; for the remaining entries, add n minutes to the previous entry. The agreement over the area in question is 327 of 357 blocks of entries between successive nodes, or 91.6%. This corresponds to a per entry agreement of 97.8%.[14]

This variant of linear interpolation has several advantages. First, the values it produces are not likely to wander from the true $g(\phi,\theta)$ values; that is, the interpolation function is stable. Second, these values are best for use with a function that is close to linear but has a small negative second derivative, precisely the nature of $g(\phi,\theta)$ for fixed ϕ. Finally, the method is very easy to execute: only a simple addition is required for each tabular value after the average first difference has been found. These considerations show that this scheme is ideal for application to the calculation of the $g(\phi,\theta)$ table.

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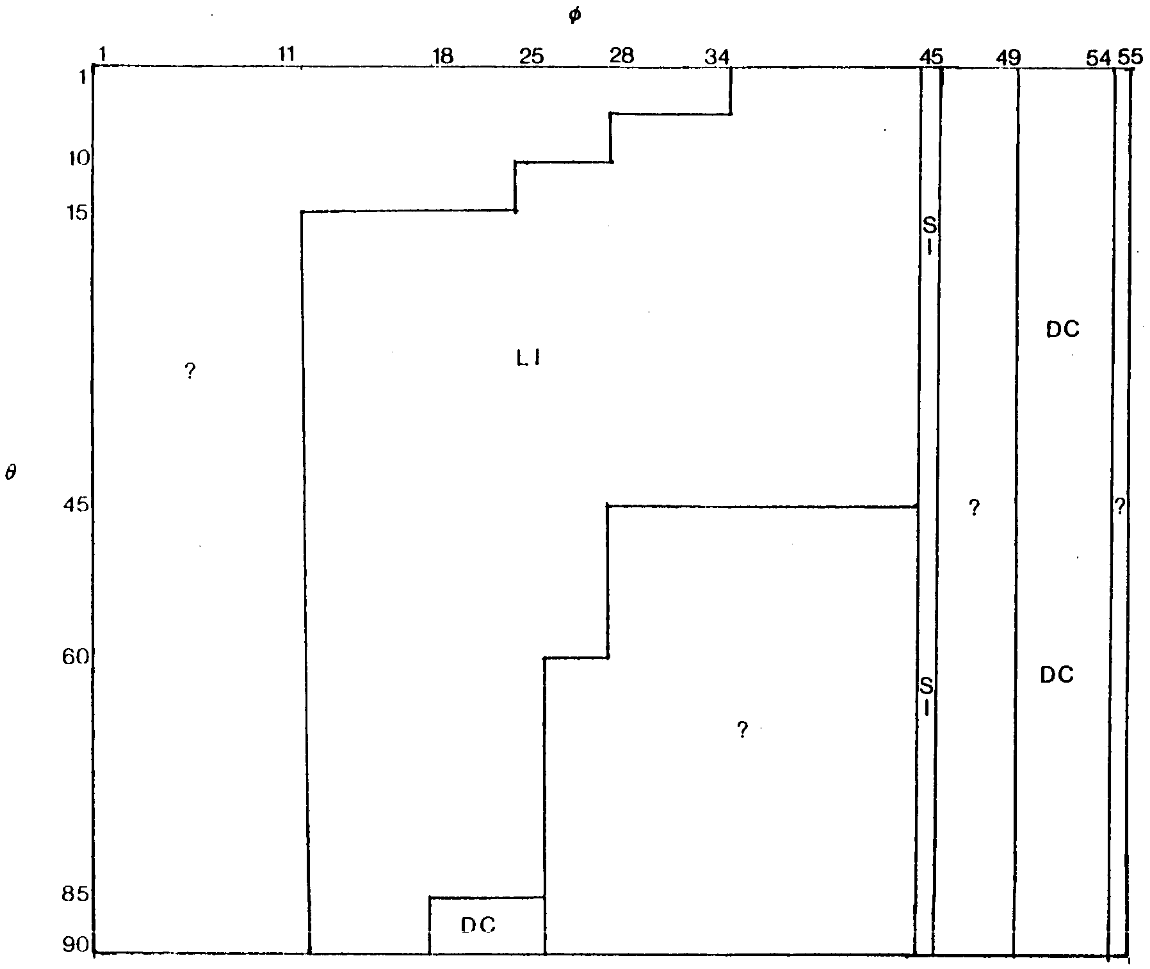
[14]The 30 blocks that fail to match the hypothesized interpolation scheme imply at least 30 errors were made over the 120 entries in these blocks (assuming the hypothesis to be true). Since there is no reliable way of knowing what number of the remaining 90 entries contain errors, we remove them from consideration. This gives 30 errors in $327 \cdot 4 + 30 = 1352$ entries, or a 97.8% agreement.



SI = Sine Values

DC = Direct Computation (using two sexagesimal digit trigonometric values)

LI = Variant of linear interpolation described on p. 46

Fig. 3.4: Schematic diagram of the methods used to generate al-Khalīlī's $g(\phi,\theta)$ table

The objection may be raised that the area in question has an unusual shape, and in fact over a region similar to this one the probability that an independent method will match the suggested pattern is highest. This latter fact can be seen as follows: given any span of four entries between consecutive nodes $g(\phi,\theta)$ and $g(\phi,\theta + 5°)$, if $x = g(\phi,\theta + 5°) - g(\phi,\theta)$ is divisible by 5 the hypothesized scheme produces three second differences of zero; if x is not divisible by 5 it will produce one second difference of -1 and two second differences of zero. This produces an average second difference of $-\frac{1}{3} \cdot \frac{4}{5} + 0 \cdot \frac{1}{5} = \frac{-4}{15}$ minutes. This corresponds to the following average second derivative with respect to θ:

$$D_\theta{}^2 g(\phi,\theta) \simeq \frac{-4/15'}{1°} = \frac{-4}{15} \cdot \frac{1}{60} = -\frac{1}{225}. \qquad (3.20)$$

But

$$D_\theta{}^2 g(\phi,\theta) = \left\{\frac{\text{Sin } \theta \text{ Tan } \phi}{R}\right\}'' = -\left[\frac{\pi}{180}\right]^2 \cdot g(\phi,\theta). \qquad (3.21)$$

So the area of the table where the average second differences best correspond to the nature of the function is the area for which

$$-\left[\frac{\pi}{180}\right]^2 \cdot g(\phi,\theta) = -\frac{1}{225},$$

$$\text{or } g(\phi,\theta) \simeq 14.59 = 14;35.$$

The band of the table where the entries are of this magnitude starts at the bottom, where $\phi \simeq 13°$ and $\theta = 90°$, and extends diagonally up to the right edge where $\phi = 55°$ and $\theta \simeq 10°$. Near this zone a reasonably accurate independent method is most likely to agree with the hypothesized scheme, with the probability of agreement continuously decreasing as the entries diverge from this curve.

While this argument does demonstrate that the probability of a match should be higher over this area, it is not sufficient to explain the match in the table. Firstly, the area that does match does not correspond very well to the curve where the probability of a match is greatest, particularly for $\phi > 45°$. Secondly, even where the probability of an independent method matching our scheme is highest, it is still very small. An example is the column for $\phi = 33\frac{1}{2}°$, which was independently computed, and the column for $\phi = 45°$, which is just a copy of $r_1(\text{Sin }\theta)$ for all θ. For both of these independently calculated columns, not one of the 18 internodal blocks matches with the hypothesized scheme. So clearly the fact that 91.6% of the blocks match in the area in question is far from coincidental. I cannot, however, satisfactorily explain why al-Khalīlī would have used this method in the central area and not in those areas immediately adjacent.

3.5 Possible Reconstructions of G(x,y)

The G(x,y) table is probably the greatest computational feat of the three auxiliary functions. While the G(x,y) table contains less than 70% of the number of entries in either of the two other tables, each entry is considerably harder to compute. After the division $\frac{Rx}{\text{Cos } y}$ is performed, an arc Cosine is required, usually of an argument which cannot be read directly from a Sine or arc Sine table. The location of the final entry of any given column also cannot be determined trivially. Finally, the G(x,y) function is not so well-behaved as either $f(\phi,\theta)$ or $g(\phi,\theta)$: while each column of the first two functions is a constant multiple of the Sine table, the curve produced by G(x,y) varies in slope and concavity in different regions of the table. This forces any fixed interpolation method simultaneously to match many different types of curves. Thus one might expect either a variety of interpolation methods, or possibly a finer grid.

Certain values of x and y result in simplifications of the function G(x,y). For x = 0,

$$G(x,y) = \text{arc Cos}\left[\frac{Rx}{\text{Cos } y}\right] = \text{arc Cos } 0 = 90^\circ, \qquad (3.22)$$

but al-Khalīlī does not include x = 0 as part of the domain of the table. For y = 0,

$$G(x,y) = \text{arc Cos}\left[\frac{Rx}{\text{Cos } 0^\circ}\right] = \text{arc Cos } x, \qquad (3.23)$$

and for y = 60,

$$G(x,y) = \text{arc Cos}\left[\frac{Rx}{\text{Cos } 60^\circ}\right] = \text{arc Cos } 2x. \qquad (3.24)$$

But study of the two rows corresponding to these values of y fail to reveal any structure to the errors in these rows; in fact, even the pairs of values G(2x,0) and G(x,60) fail to match.

3.5.1 The Cause of Anomalous Errors in the G(x,y) Table

The errors in the G(x,y) table show a similar distribution to those found in the first two tables. Approximately one half of the entries are accurate to the two sexagesimal digits displayed in the table, and most of the remaining entries err by one minute to either side. Some entries, however, show considerably larger errors of up to 30 minutes. Most of these so-called "anomalous errors"[15] are found in regions corresponding to small values of x and large values of y, a fact consistent with the observation that G(x,y) changes most rapidly there with respect to y.

The cause of these errors is easily found. Where the errors are large, the function G is very unstable with respect to y. The only computation performed using y in the evaluation of G, however, is the use of Cos y. Considering al-Khalīlī's heavy use of two sexagesimal digit trigonometric values in the first two tables, the obvious first attempt is to compare the tabular values with computation according to

$$G(x,y) = \text{arc Cos}\left[\frac{Rx}{r_1(\text{Cos } y)}\right], \qquad (3.25)$$

using accurate division and arc Cosine functions.

The results of this comparison over entries in the table with an error of five or more minutes are shown in Table 3.6 below. As the rightmost column giving the difference between

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[15]D. A. King, "al-Khalīlī's Auxiliary Tables for Solving Problems of Spherical Astronomy", pp. 101, 105.

the tabular values and reconstruction via (3.25) demonstrates, the error pattern caused by rounding Cos y to two sexagesimal digits is almost precisely that which occurs in the table. (It is easily checked that those entries with errors of less than five minutes show approximately the same agreement as those that appear in Table 3.6.) This result verifies not only that al-Khalīlī used two sexagesimal digit rounded values of Cos y, but also that after this point any other errors introduced have relatively minor effects.

x	y	True Value	al-Khalīlī	Recomputed	Difference
1	88	61;28	61;34 [+6]	61;34	0
1	89	17;15	17;45 [+30]	17;45	0
2	87	50;26	50;20 [-6]	50;20	0
2	88	17;14	17;45 [+31]	17;45	0
3	87	17;11	16;46 [-25]	16;46	0
4	86	17; 7	17; 1 [-6]	17; 2	1
5	85	17; 2	17;10 [+8]	17;10	0
6	84	16;56	16;46 [-10]	16;46	0
7	83	16;48	16;55 [+7]	16;55	0
12	78	15;51	15;43 [-8]	15;44	1
17	73	14;17	14;23 [+6]	14;23	0
18	72	13;53	13;46 [-7]	13;47	1
22	68	11;49	11;54 [+5]	11;54	0
25	65	9;38	9;32 [-6]	9;32	0
27	63	7;36	7;30 [-6]	7;30	0
29	61	4;28	4;20 [-8]	4;20	0
36	53	4;27	4;36 [+9]	4;36	0
43	44	4;56	5; 3 [+7]	5; 2	-1
55	21	10;55	11; 1 [+6]	10;56	-5

The second and seventh entries above have been reconstructed as if they were scribal errors. The entries that actually appear in the table here are 17;15 and 16;10 respectively.

Table 3.6: Comparison of computation via (3.25) with tabular values for those entries with anomalous errors in the G(x,y) table

3.5.2 The Rounding Procedure in the Argument of the Arc Cosine

After the computation of Cos y, two steps remain to complete the evaluation of G(x,y): first, the value $\frac{Rx}{\text{Cos } y}$, hereafter called the **argument** of the arc Cosine, must be found; then the arc Cosine of the argument must be taken. Both of these procedures may be accomplished in a variety of ways. The argument may be easily calculated by simply dividing Cos y into Rx, but D. King[16] has suggested that al-Khalīlī may have consulted a table for

$$\frac{\text{Sec } y}{R} = \frac{R}{\text{Cos } y}, \qquad y = 1°,\ldots,89°, \qquad (3.26)$$

presumably using two-digit values of Cos y as we have found, and then multiplied the appropriate value in this table by x for each entry of the G(x,y) table. King notes that "no independent table of the Secant is contained in any known Islamic source",[17] but we cannot ignore the possibility that al-Khalīlī was innovative. The arc Cosine operation is more difficult to deal with, since the argument is usually not an integer. Linear interpolation from an arc Cosine table or inverse linear interpolation from a Cosine table are possible, and these tables may well have had varying levels of accuracy and different spacings between entries. Also, we have no information concerning the rounding of the argument prior to the application of the arc Cosine algorithm and hence we cannot even be sure of

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[16] D. A. King, "al-Khalīlī's Auxiliary Tables for Solving Problems of Spherical Astronomy", p. 109.

[17] "al-Khalīlī's Auxiliary Tables for Solving Problems of Spherical Astronomy", p. 109.

the exact value that al-Khalīlī started with when taking the arc Cosine.

Before we begin to attempt to determine the method that al-Khalīlī used to calculate entries of G(x,y) directly, however, we must choose an appropriate subset of the original table for use as a data set. Given that the probability that al-Khalīlī used some form of interpolation is high, those areas where interpolation may have been used to generate the entries should be avoided. Also, those areas where the function is most sensitive to changes in the arguments are to be preferred, since these entries leave clearer traces of the errors caused in calculation. Fortunately these two considerations lead to the use of the same data set, those entries near the bottom curved edge of the table. Our data set, then, is defined as follows: for each column, take those entries whose y arguments are greater than or equal to the highest value of y divisible by 5. This gives 132 entries, a suitably large number.[18]

Calculation of the argument via either King's hypothesis or direct division leads to several possible results, given different levels of rounding. The results of direct division may be rounded to different levels:

$$\text{Argument} = r_n\left[\frac{Rx}{r_1(\text{Cos } y)}\right], \qquad (3.27)$$

for some value of n. The method of calculation with the use of

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[18]Over the entire table this method produces 181 entries. At the time the tests were performed, however, the columns for x > 43 had not yet been translated.

a secant table also entails this set of possibilities, but it has an intermediate step where roundoff also occurs. This method may be represented by

$$\text{Argument} = r_n\left\{x - r_m\left[\frac{R}{r_1(\text{Cos } y)}\right]\right\}, \qquad (3.28)$$

for values of m and n. If $m \geq 2$ in the latter formula (i.e., al-Khalīlī's secant table is accurate to at least three sexagesimal digits), the values produced by the two equations (3.27) and (3.28) are virtually identical, and the effect of the differences on the function values on the data set is insignificant. So we must consider two sets of possibilities: either $m = 1$ in (3.28), or $n = 1,2,...$ in (3.27).[19]

The hypothesis of the use of (3.28) with $m = 1$ is easily refuted. Since the arc Cosine is a decreasing function, arrangement of the tabular entries in ascending order by function value should correspond to a descending order in their respective arguments. But Table 3.7 illustrates that this is not the case with (3.28) and $m = 1$. Table 3.7 shows only a subset of the entire data set, but the somewhat random pattern in the rightmost column holds true throughout the data set.

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[19]Note that rejection of the first case would not eliminate the possibility that al-Khalīlī used a secant table: it only implies that the secant values would not have been calculated with $m = 1$.

x	y	Tabular Value	(3.28) With m = 1	(3.28) With m = 2
11	79	16; 7 [+2]	57;34	57;39
10	80	16;15 [-3]	57;40	57;36
40	46	16;21 [+2]	57;20	57;35
37	50	16;23	57;21	57;34
9	81	16;26 [-3]	57;36	57;33
33	55	16;30 [+1]	57;45	57;32
27	62	16;32 [-2]	57;36	57;31
8	82	16;39	57;28	57;29
3	87	16;46 [-25]	57;27	57;27
6	84	16;46 [-10]	57;24	57;27
7	83	16;55 [+7]	57;24	57;24

Table 3.7: Comparison of the tabular values of G(x,y) with the argument calculated via (3.28) with m = 1 and m = 2 for selected entries

Of the remaining possible ways described to compute the argument, all but one produce values so close to $\frac{Rx}{r_1(\text{Cos } y)}$ that they cause no distinguishable effect on the data entries. This final possibility to consider is the use of direct division and rounding to only two sexagesimal digits – (3.27) with n = 1. The error traces caused by this coarse rounding procedure should be noticeable on the data set, since these entries are sensitive to error. It is also the most reasonable hypothesis to consider, since al-Khalīlī has so far shown a preference for two values with two sexagesimal digits.

In order to determine whether the two digit hypothesis is valid, we assume the hypothesis and calculate the effect it should have on the tabular values. For notational simplicity, define

$$f(x) = \text{arc Cos}(x),$$

$$z = \frac{Rx}{r_1(\text{Cos } y)},$$

$$\hat{z} = r_1(z),$$

$$\text{and } \Delta z = \hat{z} - z.$$

Then z is the exact value of the argument obtained by using al-Khalīlī's two digit cosine values, $\hat{z}$ is the argument rounded by our hypothesis, and Δz is the change in the argument caused by rounding.

We wish to determine whether Δz correlates with the error in the tabular entries (measuring the latter error with the assumption that $r_1(\text{Cos } y)$ is the correct Cosine), much as we did with the columns in the $g(\phi,\theta)$ table. However, in the previous case, the hypothesized rounding error had a linear relationship with the errors in the tabular entries (see (3.18)), an advantage not available in the present situation. But

$$f(\hat{z}) - f(z) \simeq \Delta z \cdot f'(z), \tag{3.29}$$

since $(\hat{z} - z)$ is small, or

$$\Delta z \simeq \frac{\Delta f(z)}{f'(z)}, \tag{3.30}$$

where $\Delta f(z) = f(\hat{z}) - f(z)$. So Δz should be linearly correlated with $\frac{\Delta f(z)}{f'(z)}$, not with $\Delta f(z)$.

The situation, however, presents additional complications. Two additional errors are generated when proceeding from z to the tabular entry. First, al-Khalīlī's unknown method of determining arc Cosines certainly introduces its own error, thus disturbing the correlation. Call $\hat{f}(z)$ the function that gives al-Khalīlī's arc Cosine for argument z. Then the tabular entry is $r_1(\hat{f}(\hat{z}))$. This final rounding to two sexagesimal digits is the second error, and will also disturb the original correlation.

The final rounding error is easily simulated, but the effect of using $\hat{f}$ instead of f is more difficult to copy, since $\hat{f}$ is unknown. Using the implicit assumption of the two sexagesimal digit hypothesis, however, the overall effect of the error caused by $\hat{f}$ may be found, albeit rather discretely, by calculating

$$e(f) = r_1(\hat{f}(\hat{z})) - r_1(\text{arc Cos}(\hat{z})) \qquad (3.31)$$

for each value in the data set. (Note that $r_1(\hat{f}(\hat{z}))$ is al-Khalīlī's tabular value.) Table 3.8 below gives the results of these calculations.

Minutes	Number of entries
5	2
4	0
3	3
2	12
1	27
0	58
-1	24
-2	4
-3	0
-4	2

Table 3.8: The number of entries in the data set with e(f) equal to the number of minutes displayed

So, our problem is now to determine the level of correlation between Δz and $\frac{\Delta G}{f'(z)}$ (where ΔG is the actual error in the tabular entry) after the two additional error factors described above are taken into account. The procedure we shall use is as follows: take 132 data points (x,y) randomly scattered in the domain in which the data set is located.
Calculate $z = \frac{Rx}{r_1(\text{Cos } y)}$, and find $\hat{z} = r_1(z)$ and $\Delta z = \hat{z} - z$. To obtain the effect of al-Khalīlī's arc Cosine operation, take the arc Cosine and add a number of minutes randomly chosen according to the probability distribution represented in Table 3.8. Finally, round the result to two sexagesimal digits. This gives 132 data points $(\Delta z, \frac{\Delta G}{f'(z)})$, from which the correlation coefficient may be easily found.

It is true that al-Khalīlī's arc Cosine operation is not likely to produce a randomly distributed error: certain values of $\hat{z}$ may be more likely to result in particular errors in the evaluation of $\hat{f}(\hat{z})$ than others, but this will not affect the test. The correlation we are attempting to find uses the values of Δz, not $\hat{z}$, as the abscissas. Given any $\hat{z}$, the set of possible values of z is the interval $(\hat{z} - (0;0,30), \hat{z} + (0;0,30)]$; thus, any value of Δz is equally likely to produce the given value of $\hat{z}$. Hence Δz is independent of $\hat{z}$, and the fact that the value of $\hat{z}$ may influence the error caused by the use of $\hat{f}$ does not change the expected correlation coefficient from the one produced by our test.

The test described above was run 100 times and gave a set of correlation coefficients loosely fitting a normal distribution. A histogram of these coefficients appears in Fig. 3.5 below.

```
0.46-0.50 **
0.50-0.54 ******
0.54-0.58 ****
0.58-0.62 *******************
0.62-0.66 *******************************
0.66-0.70 *******************
0.70-0.74 ************
0.74-0.78 *******
```

Fig. 3.5: Histogram of correlation coefficients produced by the above test

The coefficients range between 0.4929 and 0.7798, with a mean of 0.6447. The actual correlation coefficient between Δz and $\frac{\Delta G}{f'(z)}$, where the value of G(x,y) is taken from al-Khalīlī's table, is 0.1078.

Clearly the correlation coefficient found from al-Khalīlī's tabular values is sufficiently small immediately to reject the two sexagesimal digit hypothesis over the entire data set. If the 132 data points are in fact independent, the probability of obtaining a correlation coefficient as high or higher than 0.1078 is approximately 11%.[20] This probability is certainly much too high to allow us to conclude a relation between the data, but it leaves open the possibility that a small subset of the original data set is in fact correlated. Examination of the various subsets of the original data set most likely to have been calculated directly, however, consistently give small or

[20]New Cambridge Elementary Statistical Tables, pp. 56, 34.

negative correlations.

The two-digit hypothesis is now eliminated, and the logical next step is to test the three-digit hypothesis; i.e., (3.27) with n = 2. A histogram of the 100 correlation coefficients produced by the program applied to this new hypothesis is given in Fig. 3.6 below.

```
-0.3 — -0.2 ***
-0.2 — -0.1 ************
-0.1 — +0.0 *************************************
+0.0 — +0.1 **************************************
+0.1 — +0.2 ************
```

Fig. 3.6: Histogram of the statistical test described above

This set of coefficients has mean -0.00634, virtually zero. Hence, the effects of the use of a three digit argument as opposed to the use of the true argument $\frac{Rx}{r_1(\mathrm{Cos}\ y)}$ are so small that our statistical test could not detect them.

While our statistical test firmly rejects the two digit hypothesis, in fact a more direct means may have been used to cause some doubt regarding its feasibility. For tabular entries that have very small values, the corresponding arguments are close to 60; 0 and the arc Cosine function is extremely sensitive to changes in the argument for these values. In fact, given the two-digit value for the argument, up to ten different possible two-digit tabular values may have this number as a Cosine. Ten of the tabular entries with the smallest values are given below in Table 3.9. Column 4 gives the different possible tabular values whose cosines correspond to the two-digit argument of column 3. Column 5 gives the 'true' value of G(x,y) according to (3.25), and the value found in al-Khalīlī's table appears in column 6.

x	y	2-Digit Argument	Possible arc Cosines	True G(x,y)	al-Khalīlī
31	58	58;29	12;53-12;56	12;53	12;53
32	57	58;45	11;41-11;45	11;44	11;44
33	56	59; 1	10;21-10;25	10;23	10;23
34	55	59;16	8;55- 9; 1	8;56	8;56
35	54	59;33	6;58- 7; 5	7; 3	7; 3
36	53	59;48	4;35- 4;46	4;36	4;36
37	51	58;47	11;32-11;35	11;34	11;34
38	50	59; 7	9;48- 9;53	9;50	9;50
39	49	59;26	7;50- 7;56	7;50	7;50
40	48	59;47	4;47- 4;57	4;57	4;56

Table 3.9: A closer comparison of (3.25) with al-Khalīlī's tabular values for entries of G(x,y) with small values

Note that the values for G(x,y) given by the use of a precise argument also fall within the ranges of column 4, but in fact they are considerably closer to the values in al-Khalīlī's table than would be expected by chance.

Had the ranges found in column 4 of Table 3.9 above been of a similar size for a larger and more equally distributed set of tabular entries, the statistical test performed earlier would have been redundant; however, for most of the tabular entries the range defined as in column 4 is only one or two possible values. The results of the statistical test combined with Table 3.9 are nevertheless enough to conclude that al-Khalīlī's values for the argument were taken to at least three sexagesimal digits, a curious fact since the denominator of the argument (Cos y) was taken to only two digits. Likewise, had al-Khalīlī used a Cosine table to generate arc Cosines the values in this table must have been accurate to at least three sexagesimal digits. Finally, the possibility that al-Khalīlī used an arc Cosine (or arc Sine) table is remote. Had he used such a table its values for integral arguments would surely have occurred along the first row of the table, for entries corresponding to G(x,0). But the errors contained in the first row are not reflected in the other tabular entries, whose values would rely on those in the first row, as we would expect if this arc Cosine table had been used.

To proceed any further than this point is virtually impossible. Calculation via

$$G(x,y) = \text{arc Cos}\left\{r_2\left[\frac{Rx}{r_1(\text{Cos } y)}\right]\right\} \quad (3.32)$$

already produces a 70% agreement. In order to work any further according to the likely hypothesis that al-Khalīlī used a reasonably accurate Cosine table, we require the Cosine values that al-Khalīlī would have used. Many such tables existed in the fourteenth century with the required accuracy, but the Cosine values they contain are of course not in perfect agreement.

3.6 Acceptable Error Levels in al-Khalīlī's Table

Given a numerical table of the size of al-Khalīlī's tables entirely computed by hand, we may expect two types of error to alter the entries from the values that the algorithm used by the constructor should produce. Computational errors should occur at a level roughly proportional to the difficulty of the numerical operation, and scribal errors should cause randomly distributed errors. Scribal errors in the first digits are in general easy to detect, but those that occur in the final digit are usually impossible to distinguish from computational errors. It will be useful to check whether the error levels derived from the methods we have discovered agree with these considerations. Table 3.10 below gives the number of scribal errors found in the first three digits of the entries in those areas of al-Khalīlī's tables that have been explained, using the Paris manuscript as the sole source. (In this section only, **digits** shall refer to the individual characters; so 57;34, for instance, has four digits.)

	G	f	g	Total
1st Digit	1	1	0	2
2nd Digit	2	1	1	4
3rd Digit	6	6	12	24

Table 3.10: The location of scribal errors in the explained regions of al-Khalīlī's auxiliary tables

The large proportion of scribal errors in the third digit is to be expected, since the magnitude of an entry with an incorrect third digit is not markedly different from the correct value, and a scribe is less likely to catch it. Assuming as a very rough guess that the number of scribal errors in the fourth digit is between twice and three times the number in the third digit, we arrive at a figure of 48 to 72 scribal errors in the fourth digit of the 6600 entries, or 0.727% - 1.09%. So we shall use the figure of 1% as an estimate of the number of hidden scribal errors.

Fortunately, the operations that we know al-Khalīlī used to calculate entries in his table correspond to either a single addition or a single multiplication. The calculation of $f(\phi,\theta)$ from $g(\phi,\theta)$ via

$$f(\phi,\bar{\phi} \pm n) = \text{Cos } n \pm g(\phi,n) \tag{3.33}$$

generated a 95.9% success rate. Thus, assuming our 1% hidden scribal errors, al-Khalīlī's error rate with respect to addition is approximately 3%. The combined success rate in the $g(\phi,\theta)$ table over the interpolation nodes and areas where direct calculation was used is 94.0%, corresponding to a 5% error rate with respect to multiplication. Considering the relative difficulty of multiplication as opposed to addition and the damping effect of serious errors being caught by observation, these figures are appropriate.

3.7 The Feasibility of the Construction of al-Khalīlī's Qibla Table from his Auxiliary Tables

Al-Khalīlī's qibla table, probably his most impressive accomplishment, represents a vast amount of calculation. For the arguments

$$\phi = 10°, 11°, \ldots, 56°, \text{ and } 33;30°,$$

$$\text{and } \Delta L = 1°, 2°, \ldots, 60°,$$

where ϕ is the worshipper's latitude and ΔL is his longitudinal difference from Mecca, the table gives the direction of Mecca relative to the meridian at the worshipper's location in degrees and minutes, for a total of 2880 entries. Roughly $\frac{1}{4}$ of the entries are correct to the two sexagesimal digits displayed, while most of the others are in error by less than five minutes. As the locations of the entries move nearer to Mecca, however, the errors increase to a maximum of 41 minutes as the function becomes more sensitive to small changes in the argument.

While he does not explicitly state which formula he used, al-Khalīlī declares that he knows of no better method of qibla calculation than that of al-Marrākushī, who used the equation

$$q = \text{arc Cos}\left\{\frac{R\left[\frac{\text{Sin } h \text{ Tan } \phi}{R}\right] - \frac{R \text{ Sin } \phi_M}{\text{Cos } \phi}}{\text{Cos } h}\right\}, \qquad (3.34)$$

where h, the height of the zenith of Mecca in the observer's sky, is given by

$$\text{Sin } h = \text{Sin}(\bar{\phi} + \phi_M) - \text{Vers } \Delta L \frac{\text{Cos } \phi_M \text{ Cos } \phi}{R^2}. \qquad (3.35)$$

The nature of the above qibla formula immediately suggests the possibility of the use of al-Khalīlī's auxiliary tables to generate his qibla table using

$$q = G\{[g(\phi,h) - f(\phi,\phi_M)],\ h\}. \qquad (3.36)$$

King notes that "al-Khalīlī's qibla values are generally more accurate than those which can be derived from his auxiliary tables in this way. Thus the possibility that he computed his qibla values independently of the auxiliary tables cannot be ruled out."[21] I have not extensively examined the possibility of the use of the auxiliary tables, but several simple observations are enough to show that what King suggests is in fact a rather strong possibility. The obvious first reason is that the domain of the qibla table extends to $\phi = 56°$, while the auxiliary tables end at 55°. A more serious cause for doubt that the auxiliary tables were used is the fact that the errors in the $f(\phi,\theta)$ table are not reflected in the qibla table. Since ϕ_M is constant, the same value $f(\phi,\phi_M) = f(\phi,21;30)$ should be used for an entire column of the qibla table. Several of the columns of the $f(\phi,\theta)$ table contain entries for $\theta = 21°$ and 22° that are both in error by several minutes. It is easily seen that an error of this size in the value of $f(\phi,\phi_M)$ should result in an error in the qibla value of approximately the same magnitude, but these errors do not exist in the qibla table. The possibility of the use of the other two tables would require more examination, but I find it unlikely that the two-digit values in the auxiliary tables are enough to produce an accuracy

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[21]D. A. King, "al-Khalīlī's Qibla Table", p. 108.

level close to the level found in the qibla table.

CHAPTER 4

CONCLUSION

The study of the history of computational methods has in the past suffered from a lack of scholarly analysis, reflecting perhaps the outlook of the ancient and medieval mathematicians themselves, which placed it out of the realm of mathematics and into the category of common sense, self-taught practice. Ptolemy, for example, after the preliminary chapters of the Almagest, states that he has completed the discussion of all the mathematics required for the book without describing how any of his numerous tables were computed, even though Glowatzki and Göttsche[1] have demonstrated that Ptolemy did more extensive calculation than he states on the chord table. This outlook continued through the Islamic medieval period, with the result that mathematical and astronomical tables, often fraught with numerical inaccuracies, were presented in the zījes as a *fait accompli*. The study presented here has described some methods to determine the numerical procedures used by Shams al-Dīn al-Khalīlī. Future work in the same vein should help to uncover the currently unknown numerical practices of scientists from a variety of historical periods.

Even a cursory examination of the results presented here, connected with some initial observations of other auxiliary tables, shows a level of improvement from the tenth to the

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[1]E. Glowatzki and H. Göttsche, Die Sehnentafel des Klaudios Ptolemaios (Munich: Oldenbourg, 1976), pp. 60-71.

fourteenth centuries. The sine tables of Ḥabash al-Ḥāsib from the ninth century[2] and the mathematician Abū Naṣr Manṣūr from the tenth century,[3] for example, are taken by simply dividing each entry of Ptolemy's three sexagesimal digit chord table by two and displaying four digits, with the result that the last digit is always either 0 or 30. We have found that another table in Abū Naṣr's Table of Minutes, defined by $f_1(\phi) = \sin \epsilon \cos \phi$ (where ϵ is the obliquity of the ecliptic) was calculated by rounding Ptolemy's value of 2ϵ to two sexagesimal digits and using linear interpolation to obtain a value for $\sin \epsilon$ from the chord table. The values in the table, however, are displayed to four sexagesimal digits. This produces an average of 2.6 digits in error. Also, these tables (and most of the tables from this period) are quite small, consisting of a few hundred entries at most. This is not surprising, since early indications show that they were computed directly, entry-for-entry, without using timesaving techniques like interpolation.

Al-Khalīlī's tables, on the other hand, may not be as accurate as the tables of Abū Naṣr or Ḥabash al-Ḥāsib, since they are given to only two sexagesimal digits, but the

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[2]M.-T. Debarnot, "The *Zīj* of Ḥabash al-Ḥāsib: A Survey of MS Istanbul Yeni Cami 784/2", in Eds. D. A. King and G. Saliba, From Deferent to Equant: A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E. S. Kennedy (Annals of the New York Academy of Sciences v. 500, 1987), p. 46.

[3]See further C. Jensen, "Abū Naṣr's Approach to Spherical Trigonometry as Developed in His Treatise *The Table of Minutes*", Centaurus 16 (1971), pp. 1-19.

techniques and parameters used show considerably more sophistication. The tables contain an average of only 0.5 digits in error; in fact, about half of the entries are exact to the two digits displayed. The trigonometric tables on which the tables are based are almost sufficient to produce accurate tabular values: the sine table consists of the correct sine values rounded to two sexagesimal digits, and about 75% of the tangent values are also correct to two digits. The methods that we have discovered al-Khalīlī used to compute the tables themselves exhibit a great deal of thought and foresight, since they are both stable and easy to implement. The interpolation scheme suggested for the $g(\phi,\theta)$ table, for instance, is essentially the simplest possible method, but it nicely suits the nature of the function as well. The formula used to generate entries of the $f(\phi,\theta)$ table from corresponding values of $g(\phi,\theta)$ is far from obvious, but it results in computations even quicker than those required for the $g(\phi,\theta)$ table and produces entries that are generally no less accurate than the entries from which they were generated. Finally, the great care taken in the application of the arc Cosine in the calculation of $G(x,y)$ shows that al-Khalīlī had some idea of the instability involved with this function. Clearly al-Khalīlī spent a significant amount of effort in considering how to calculate all three of his tables; yet, none of this work is recorded or even mentioned.

Al-Khalīlī's tables and others of his period display the application of more sophisticated procedures than earlier works, but when observed from a modern viewpoint they still contain some rudimentary errors. The use of tangent values less accurate than the number of digits desired in the tables, especially considering that better tangent values must have been available, is an oversight noticeable even to the untrained eye. Even more curious is the fact that while al-Khalīlī calculated the arc Cosine very carefully, and to at least three sexagesimal digits, he used only two digit Cosine values to compute the arguments for the arc Cosines, resulting in rather large errors in certain portions of the G(x,y) table. So it appears that while the art of numerical calculation was more highly developed in al-Khalīlī's tables than in those of Abū Naṣr, it had not moved beyond practical, behind-the-scenes operation to a more systematic approach.

The above speculations should give some indication of the usefulness of more advanced computational techniques in the study of the history of mathematics. Until now, the only published effort made to determine the numerical structure of mathematical tables of historical interest is the work of Glowatzki and Göttsche,[4] which is somewhat limited in scope.[5]

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[4]E. Glowatzki and H. Göttsche, Die Sehnentafel des Klaudios Ptolemaios, pp. 60-71. Glowatzki and Göttsche's main argument is that recalculation of chord values according to a method decribed in the Almagest requires at least five sexagesimal digits to achieve the accuracy found in Ptolemy's chord table.

[5]G. J. Toomer, review of E. Glowatzki and H. Göttsche, in Centaurus 21 (1977), pp. 321-323.

This thesis uncovers some information of interest on al-Khalīlī's auxiliary tables but falls short of presenting a full account of methods that could be used to find the numerical structure of other tables. The use of more advanced statistical tools would lead to more comprehensive methods, presenting the opportunity to explore the evolution of numerical techniques in the scientific works of medieval Islam and other cultures.

APPENDIX A: SELECTED COLUMNS OF AL-KHALĪLĪ'S AUXILIARY TABLES

The error representation system used in the tables below, described fully in Sec. 2.4, is as follows:

$$\text{error} = \text{text} - r_1(\text{exact value})$$

$$f(\phi,\theta) = \frac{R \text{ Sin } \theta}{\text{Cos } \phi} \text{ for } \phi = 1°$$

θ	$f(\phi,\theta)$	θ	$f(\phi,\theta)$	θ	$f(\phi,\theta)$
1	1; 3	31	30;54	61	52;29
2	2; 6	32	31;48	62	52;59
3	3; 8	33	32;41	63	53;28
4	4;11	34	33;33	64	53;56
5	5;14	35	34;25	65	54;23
6	6;16	36	35;16	66	54;49
7	7;19	37	36; 7	67	55;14
8	8;21	38	36;56 [-1]	68	55;38
9	9;23	39	37;46	69	56; 1
10	10;25	40	38;34	70	56;23
11	11;27	41	39;22	71	56;44
12	12;28 [-1]	42	40; 9	72	57; 4
13	13;30	43	40;55 [-1]	73	57;23
14	14;31	44	41;41	74	57;41
15	15;32	45	42;26	75	57;57 [-1]
16	16;32	46	43;10	76	58;13 [-1]
17	17;32 [-1]	47	43;53	77	58;28
18	18;32 [-1]	48	44;35 [-1]	78	58;41 [-1]
19	19;32	49	45;17	79	58;54
20	20;31	50	45;58	80	59; 5 [-1]
21	21;30	51	46;38	81	59;16
22	22;29	52	47;17	82	59;25 [-1]
23	23;27	53	47;55 [-1]	83	59;33 [-1]
24	24;24	54	48;32 [-1]	84	59;40 [-1]
25	25;21 [-1]	55	49; 9	85	59;46 [-1]
26	26;18	56	49;45	86	59;51 [-1]
27	27;14 [-1]	57	50;19 [-1]	87	59;55 [-1]
28	28;10	58	50;53	88	59;58
29	29; 5 [-1]	59	51;26	89	60; 0
30	30; 0	60	51;58	90	60; 0 [-1]

$$g(\phi,\theta) = \frac{\text{Sin } \theta \text{ Tan } \phi}{R} \text{ for } \phi = 35°$$

θ	$g(\phi,\theta)$	θ	$g(\phi,\theta)$	θ	$g(\phi,\theta)$
1	0;44	31	21;40 [+2]	61	36;45
2	1;28	32	22;17 [+1]	62	37; 6
3	2;12	33	22;54 [+1]	63	37;26
4	2;56	34	23;30	64	37;46
5	3;40	35	24; 6	65	38; 5
6	4;24 [+1]	36	24;41 [-1]	66	38;23
7	5; 8 [+1]	37	25;16 [-1]	67	38;40
8	5;52 [+1]	38	25;51 [-1]	68	38;57
9	6;35 [+1]	39	26;26	69	39;13
10	7;18	40	27; 0	70	39;29
11	8; 1	41	27;33 [-1]	71	39;44 [+1]
12	8;44	42	28; 6 [-1]	72	39;58 [+1]
13	9;27	43	28;39	73	40;11
14	10;10	44	29;11	74	40;23
15	10;53 [+1]	45	29;43 [+1]	75	40;35
16	11;35	46	30;14 [+1]	76	40;46
17	12;17	47	30;44	77	40;56
18	12;59	48	31;14 [+1]	78	41; 6
19	13;41	49	31;43 [+1]	79	41;15 [+1]
20	14;22	50	32;11	80	40;23 [+1]
21	15; 3	51	32;38 [-1]	81	41;30
22	15;44	52	33; 5 [-1]	82	41;36
23	16;25	53	33;32 [-1]	83	41;42
24	17; 5	54	33;59	84	41;47
25	17;45	55	34;25	85	41;51
26	18;25	56	34;50	86	41;55
27	19; 4	57	35;14	87	41;58 [+1]
28	19;43	58	35;37 [-1]	88	41;59
29	20;22	59	36; 0 [-1]	89	42; 0
30	21; 1 [+1]	60	36;23	90	42; 1

$$G(x,y) = \text{arc Cos}\left[\frac{Rx}{\text{Cos } y}\right] \text{ for } x = 40$$

y	G(x,y)	y	G(x,y)
0	48;10 [-1]	30	39;40
1	48;10 [-1]	31	38;57
2	48; 9 [-1]	32	38;10 [-1]
3	48; 7	33	37;21
4	48; 4	34	36;29 [+1]
5	48; 0	35	35;32
6	47;54	36	34;30
7	47;48	37	33;24 [-1]
8	47;41	38	32;13
9	47;33	39	30;56
10	47;25 [+1]	40	29;31
11	47;14 [+1]	41	27;57
12	47; 2	42	26;12 [-1]
13	46;49 [-1]	43	24;17
14	46;35 [-1]	44	22; 5 [+1]
15	46;21	45	19;30 [+2]
16	46; 6 [+1]	46	16;21 [+2]
17	45;49 [+1]	47	12; 9 [-1]
18	45;30	48	4;56 [+1]
19	45;10		
20	44;49		
21	44;26		
22	44; 2		
23	43;35 [-1]		
24	43; 8		
25	42;39		
26	42; 8 [+1]		
27	41;34		
28	40;58		
29	40;20		

APPENDIX B: A SEXAGESIMAL DESK CALCULATOR PROGRAM

During the course of my research on al-Khalīlī's auxiliary tables, I found it extremely useful to have access to a program that emulates a desk calculator, but works in a sexagesimal rather than a decimal base. Included below is the core of the program, written in pseudo-code. The user will find it easy to modify the code below to any base and to suit virtually any application.

The program stores numbers in fixed-point format, and relies on machine arithmetic only for operations with integers. A given number is stored as a vector with 13 integer elements. The first digit is a sign indicator (0 for positive, 1 for negative), the next six elements represent the integer portion of the number, and the remaining six store the fractional part. Thus x = -36,22;14,8 is stored as follows:

$$\bar{x} = (1,0,0,0,0,36,22,14,8,0,0,0,0)$$

The use of six digits before and after the sexagesimal point should ensure sufficient accuracy for most purposes.

For the sake of brevity I have included only the routines for addition, subtraction, and multiplication, as well as one machine-dependent sine function for illustration. All unary operations (functions of one variable, such as the sine or square root) should be coded in the program according to the generic example immediately following the sine evaluation function. All binary operations may be included in the "Case" statement near the end of the code labelled "Operation Entry Mode".

The square bracket notation used in the code signifies certain characters or sets of characters that the user must specify to represent the given operation. In an interactive context, the command

Input (char)

instructs the computer to wait until a character is received. Finally, the three rightmost columns on the screen are reserved to output the names of the operations performed, and are called the operation column.

```
Program SixtyCalc

(*      Variables: vector variables representing numbers
(*      are represented by a vertical bar above the name.
(*      x̄: current number being evaluated.
(*      ȳ: in binary operations, the first number entered.
(*      m̄: the memory variable.
(*      char: current character being evaluated.
```

```
(*      op: code number or string signifying the binary
(*      to be used.


(*      Initialization of Variables

   x̄ := 0, ȳ := 0, m̄ := 0, op := (nothing)
   Input(char)


(*      Operation Entry Mode

   If char = (digit or [+/-] or [.]) then
      ȳ := x̄
      goto number entry mode

   If char = [Clear] then
      x̄ := 0, ȳ := 0, op := (nothing)
      write("CLR": operation column)

   If char = [Clear Entry] then
      x̄ := ȳ, ȳ := 0
      write("CE": operation column)

   If char = [Store in Memory] then
      m̄ := x̄
      write("STO": operation column)

   If char = [Recall Memory] then
      If op ≠ (nothing) then ȳ := x̄
      x̄ := m̄
      write("RCL": operation column)
      display(x̄)

   If char = [Sine] then
      z := decvalue(x̄)
      f := sin(z)
      x̄ := sixtyval(z)
      write("SIN": operation column)
      display(x̄)

(*      The statement below is a generic example of a monic
(*      operation.  The user may specify to requirements.

   If char = [Monic operation] then
      x̄ := monic operation(x̄)
      write("[Monic op]": operation column)
      display(x̄)

   If char = [a binary operation] then
      op := (operation)
      write(op: operation column)
```

```
(*     User-supplied binary operations may be included
(*     in the Case statement below.

   If char = [=] then
      Case
         op = add: z̄ := add(x̄,ȳ)
         op = mult: z̄ := mult(x̄,ȳ)
             ⋮
         op = (nothing): z̄ := x̄

(*     The following line is currently written for ease of
(*     use in sums of series; e.g., "5 + 6 = 3 =" will
(*     produce the two values 11 and 14.  The user should
(*     alter this line according to preference.

      ȳ := x̄, x̄ := z̄

   Input(char)
   Goto beginning of Operation Entry Mode


Procedure display(x̄)

   Move cursor one line down
   write (x̄: main screen)   (*  Include punctuation.
   End


(*     Number Entry Mode

(*     This code allows entry of sexagesimal digits using
(*     the digits 0 through 9.  In batch operation,
(*     remove all write statements in this section.
(*     currdig: The location (in x̄) of the sexagesimal
(*     digit currently being entered.

   If char ≠ digit then goto Operation Entry Mode

   If currdig > 13 then
      Input(char)
      Goto beginning of Number Entry Mode

   If digit > 5 then
      val := digit
   Else
      val := digit * 10
      Input(digit2)
      val := val + digit2

(*     Now that we have the sexagesimal digit, print the
(*     value and update variables.

   If currdig > 7 then   (*  Fractional part.
```

```
      write(val: main screen)
      x(currdig) := val
      currdig := currdig + 1
   Else   (*  Integer part.
      Do t = 3 to 7
         x(t - 1) := x(t)
      x(7) := val
      write(integer digits: main screen)

   Input(char)
   Goto beginning of Number Entry Mode


Real function decvalue(x̄)

(*     For use, along with sixtyval, in machine-dependent
(*     routines required by the user.  Use according to
(*     the example given for the sine in Operation Entry Mode.

   val := 0
   Do t = 13 to 2 by -1
      val := val + x(t) * 60^(7-t)
   If x(1) = 1 then val := -val
   decvalue := val
   End


Vector function sixtyval(z)

   r̄ := 0
   If z < 0 then r(1) := 1
   z := abs(z)
   If z > 60^6 then
      write("Overflow Error")
      sixtyval := 0
   Else
      Do t = 5 to 6 by -1
         r(7 - t) := Int(z/(60^t))
         z := z - r(7 - t) * 60^t
      sixtyval := r̄
   End


Vector function add(x̄,ȳ)

(*     This routine performs additions using machine arithmetic
(*     only for addition of integers less than 60.

   If x(1) = y(1) then    (*  Same sign: add absolute values.
      z(1) = x(1)
      carry := 0

      Do t = 13 to 2 by -1
```

```
        z(t) := x(t) + y(t) + carry
        carry := 0
        If z(t) > 59 then
           carry := 1
           z(t) := z(t) - 60

     If carry = 1 then write("Warning: Overflow")
     add := z̄
  Else   (*  Different signs: subtract the two numbers.

(*     The following If statement allows us to assume that x̄
(*     has at least as large absolute value as ȳ.

     If abs(ȳ) > abs(x̄) then
        add := add(x̄,ȳ)
     Else
        z(1) := x(1)
        carry := 0

        Do t = 13 to 2 by -1
           z(t) := x(t) - y(t) - carry
           carry := 0
           If z(t) < 0 then
              carry := 1
              z(t) := z(t) + 60

        If carry = 1 then write("Warning: Overflow")
        add := z̄
  End


Vector function subtract(ȳ,x̄)
   x(1) := 1 - x(1)
   subtract := add(ȳ,x̄)


Vector function mult(x̄,ȳ)

(*     This routine uses the Hindu and Arabic "gelosia" method
(*     to multiply two numbers.
(*     mat(1-12,1-24): used to store the two sexagesimal digit
(*     products of pairs of single digits of the operands.

(*     Set up the product matrix.

   Do t = 1 to 12
      Do u = 1 to 12
         pdt := y(t + 1) * x(t + 1)
         mat(t,2u-1) := Int(pdt/60)
         mat(t,2u) := pdt - 60 * mat(t,2u-1)

(*     Begin sum of products: first, diagonals that miss the
```

```
(*      upper row.
   carry := 0
   Do t = 13 to 8 by -1
      z(t) := carry
      Do u = 24 to (2t - 14) by -1
         z(t) := z(t) + mat(t + 5 - Int(u/2),u)
      carry := 0
      If z(t) > 59 then
         carry := Int(z(t)/60)
         z(t) := z(t) - 60 * carry

(*      Now continue to sum the products using diagonals that
(*      reach the top row.

   Do t = 7 to 2 by -1
      z(t) := carry
      Do u = 1 to (2t + 9)
         z(t) := z(t) + mat(t + 5 - Int(u/2),u)
      carry := 0
      If z(t) > 59 then
         carry := Int(z(t)/60)
         z(t) := z(t) - 60 * carry

   If carry > 0 then write("Warning: Overflow")
   z(1) := abs{x(1) - y(1)}
   mult := z̄
   End
```

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