SOME RESULTS ON THE STRUCTURE OF THE Σ_2 ENUMERATION DEGREES

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SOME RESULTS ON THE STRUCTURE OF THE

Σ_2 ENUMERATION DEGREES

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ABSTRACT

Cooper and McEvoy have defined a *jump* operator on the enumeration degrees (e-degrees) and have shown that the set of e-degrees of Σ_2 sets is the same as the set of e-degrees below $\mathbf{0}'_e$. They have also defined the concept of a *low* e-degree (in the natural way). Cooper has shown that the Σ_2 e-degrees are dense. Gutteridge has proved the existence of a minimal pair of Σ_2 e-degrees.

We have proved the following results about e-degrees:

Theorem 1. *For every finite partial order* $(\mathcal{P},\leq^*)$, *if* $p_0 <^* p_1 <^* \dots <^* p_n \in \mathcal{P}$, $\mathbf{a}_0 <_e \mathbf{a}_1 <_e \dots <_e \mathbf{a}_n \leq_e \mathbf{0}'_e$, $p_0 \neq 0$ *implies* $\mathbf{a}_0 \neq \mathbf{0}_e$ *and* $p_n \neq 1$ *implies* $\mathbf{a}_n \neq \mathbf{0}'_e$, *then there exists an embedding* f *of* $\mathcal{P}$ *in the* Σ_2 *degrees such that* $f(p_i) = \mathbf{a}_i$ *for every* $i \leq n$.

Definition. A degree $\mathbf{a}$ is said to be *splitting* if there exists a pair of degrees $\mathbf{b}$ and $\mathbf{c}$ strictly below $\mathbf{a}$ with $\mathbf{a} = \mathbf{b} \vee \mathbf{c}$.

Theorem 2. *There exists a non-zero low non-splitting degree.*

Theorem 3. *For every non-zero low degree* $\mathbf{a}$ *there exists a* Σ_2 *degree* $\mathbf{b}$ *such that* $\mathbf{a} \perp_e \mathbf{b}$ *and for every* $\mathbf{z} \leq_e \mathbf{a}$, *either* $\mathbf{z} \leq_e \mathbf{b}$ *or there exists* $\mathbf{y} <_e \mathbf{a}$ *such that* $\mathbf{y} \vee \mathbf{z} = \mathbf{a}$ *and* $\mathbf{y} \leq_e \mathbf{b}$.

Corollary 4. *There exists a pair of incomparable* Σ_2 *degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that for every* $\mathbf{z} <_e \mathbf{a}$, $\mathbf{z} \leq_e \mathbf{b}$.

Theorem 5. *For every pair of distinct* Σ_2 *degrees* $\mathbf{a}$ *and* $\mathbf{b}$, $\{\mathbf{z}: \mathbf{z} <_e \mathbf{a}\} \neq \{\mathbf{z}: \mathbf{z} <_e \mathbf{b}\}$.

Theorem 6 (Diamond). *There exists a pair of low degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}_e$ *and* $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

To my parents,

Farhat and Iqbal Ahmad

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CHAPTER I

INTRODUCTION AND TERMINOLOGY

§1.1 INTRODUCTION

Enumeration reducibility is a very natural reducibility between subsets of ω. It was first defined by Friedberg and Rogers [1959]. Intuitively, A is enumeration reducible to B (written $A \leq_e B$) if there is an effective procedure for producing an enumeration of A from *any* enumeration of B. For a more detailed discussion of the intuition, see Rogers [1967] (pp. 145-147). Turing reducibility is often viewed as a reducibility between *total* (everywhere defined) functions. The beauty of enumeration reducibility is that if we identify functions with their *graphs* (defined in §1.2) then e-reducibility extends Turing reducibility to the set of *partial* functions.

Enumeration degrees are defined in a manner analogous to Turing degrees. $\mathbf{0_e}$ denotes the least e-degree, consisting of all the r.e. sets. Gutteridge [1971] has shown that there are no minimal e-degrees, hence the enumeration degrees are not elementarily equivalent to the Turing degrees. Of particular interest are those e-degrees containing a

Σ_2 set. It is easy to work with these degrees as Σ_2 sets allow effective approximations (see §1.3); Cooper [1984] has shown that the Σ_2 e-degrees are dense, and are precisely the degrees below $\mathbf{0}'_{\mathbf{e}}$ (defined in §1.3), hence they possess the nice property that they are closed downwards under $\leq_e$. Some of these features suggest an analogy between the Σ_2 e-degrees and the r.e. Turing degrees. In fact Cooper has asked if these two classes are elementarily equivalent. This is answered in the negative by the Diamond Theorem (see Chapter V) which contrasts with Lachlan's [1966] Non-Diamond Theorem for the r.e. Turing degrees. Cooper and Copestake [ta] have shown that the Σ_2 e-degrees properly contain the simpler class of Δ_2 e-degrees, hence the former is a *proper* class.

The results in this thesis may be viewed in the following context. Let $\mathfrak{D}^+_{\mathbf{e}}(\Sigma_2)$ denote the Σ_2 e-degrees with least element $\mathbf{0}_{\mathbf{e}}$ and greatest element $\mathbf{0}'_{\mathbf{e}}$; let $\mathrm{Th}(\mathfrak{D}^+_{\mathbf{e}}(\Sigma_2))$ denote the theory of $\mathfrak{D}^+_{\mathbf{e}}(\Sigma_2)$ in $L = \{\leq, 0, 1\}$, the language of partial order with least and greatest elements. By Theorem 2.1.1 the sentences in $\mathrm{Th}(\mathfrak{D}^+_{\mathbf{e}}(\Sigma_2))$ of the form $(\forall x)(\exists y_1)\ldots(\exists y_n)\alpha(x,y_1,\ldots,y_n)$, where α is quantifier-free, are decidable; in fact, any such sentence which is consistent is true in $\mathfrak{D}^+_{\mathbf{e}}(\Sigma_2)$. An answer to Question 1 (see Appendix) combined with the results in this thesis would suffice to decide all sentences of the form $(\forall x_1)(\forall x_2)(\exists y)\alpha(x_1,x_2,y)$, ($\alpha$ quantifier-free).

§1.2 BASIC NOTATION

ω is the set of non-negative integers. ψ and φ denote partial functions while f denotes total functions. Other lower case italic letters range over elements of ω or $\omega \cup \{-1\}$. $x \dot{-} y =_{dfn} x - y$ if $x \geq y$, and 0 otherwise. 2^{ω} denotes the power set of ω. Upper case italic letters range over subsets of ω with D, E and F being reserved for finite sets. χ_A denotes the characteristic function of A: we write $A(x)$ for $\chi_A(x)$. $A\restriction x$ denotes $\{y \in A: y < x\}$, while $A\restriction[>x]$ denotes $\{y \in A: y > x\}$. $|A|$ denotes the cardinality of A. $\max F$ denotes the greatest element of F if $F \neq \emptyset$, and -1 otherwise. Analogously, $\min A$ denotes the least element of A if $A \neq \emptyset$, and ∞ otherwise.

$A \subset B$ means $A \subseteq B$ but $A \neq B$. $\overline{A}$ denotes the complement of A and $A - B =_{dfn} A \cap \overline{B}$. $A =^{*} B$ means that the symmetric difference of A and B is finite. $A \oplus B =_{dfn} \{2x: x \in A\} \cup \{2x+1: x \in B\}$. $C = A \sqcup B$ means $C = A \cup B$ and A and B are disjoint.

$\langle x,y\rangle =_{dfn} \frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$. $\langle\cdot,\cdot\rangle$ is a recursive bijection from $\omega\times\omega$ to ω (see Rogers [1967] (p. 64)). Note that $\max\{x, y\} \leq \langle x,y\rangle$. $\langle x,y,z\rangle$ denotes $\langle\langle x,y\rangle,z\rangle$; $\langle\cdot,\cdot,\cdot\rangle$ is a recursive bijection from $\omega\times\omega\times\omega$ to ω. $(x)_0$ and $(x)_1$ are defined by $x = \langle (x)_0, (x)_1\rangle$.

$A^{[y]}$ denotes $\{x \in A: (x)_1 = y\}$ and $A^{\{y\}}$ denotes $\{(x)_0: x \in A^{[y]}\}$. $A^{[<y]}$ denotes $\bigcup\{A^{[z]}: z < y\}$; $A^{[\leq y]}$, $A^{[>y]}$ and

$A^{[\geq y]}$ are defined analogously. For a function φ, graph $\varphi =_{\mathrm{dfn}}$ $\{\langle x,y\rangle: \varphi(x) = y\}$.

If $D = \{x_0 < x_1 < \ldots < x_n\}$ then the canonical index of D is $2^{x_0} + 2^{x_1} + \ldots + 2^{x_n}$; the canonical index of $\emptyset$ is 0. D_z denotes the set with canonical index z. We often identify a finite set with its canonical index, and write $\langle D,x\rangle$ for $\langle z,x\rangle$ where $D = D_z$.

${}^{<\omega}2$ is the set of finite sequences of 0's and 1's. ${}^{<\omega}\omega$ and ${}^{<\omega}(\omega \cup \{-1\})$ are the sets of finite sequences of elements of ω and $\omega \cup \{-1\}$ respectively. σ, τ and γ range over elements of ${}^{<\omega}2$, ${}^{<\omega}\omega$ or ${}^{<\omega}(\omega \cup \{-1\})$. The *length* of σ (written $\mathrm{lh}(\sigma)$) is $|\mathrm{dom}\,\sigma|$. ${}^{n}2$ is the set of finite sequences of 0's and 1's of length n. $\sigma\restriction x$ denotes the restriction of σ to $\{y: y < x\}$. $e(\sigma) =_{\mathrm{dfn}} \mathrm{lh}(\sigma) \dot{-} 1$ and $\sigma^- =_{\mathrm{dfn}} \sigma\restriction e(\sigma)$. $\sigma \subseteq \tau$ means that σ is an initial segment of τ while $\sigma \subset \tau$ means $\sigma \subseteq \tau$ but $\sigma \neq \tau$. $\sigma\hat{\ }\tau$ denotes the concatenation of σ followed by τ.

Lower case boldface letters range over e-degrees. In Chapter II they also range over elements of arbitrary partial orders. $\mathbf{a} \vee \mathbf{b}$ denotes the least upper bound of $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ the greatest lower bound.

Read "is defined" for $\downarrow$, "is undefined" for $\uparrow$, "the least x such that" for μx and "there exist infinitely many x such that" for $\exists^{\infty} x$. $\bigvee$ denotes an infinite disjunction. $\square$ and $\blacksquare$ mark the end of a proof.

§1.3 ENUMERATION REDUCIBILITY

We assume that the reader is familiar with the basic concepts of recursion theory as found in Part A of Soare [1987].

Definition 1.3.1.

.1. A sequence of finite sets $\{F^s\}_{s\in\omega}$ is called a *recursive sequence* or *strong array* if there exists a recursive function $f(s)$ such that $F^s = D_{f(s)}$ for every s.

.2. A recursive sequence of finite sets $\{A^s\}_{s\in\omega}$ is called a *recursive enumeration* of an r.e. set A if $A^s \subseteq A^{s+1}$ for every s, and $A = \bigcup_{s\in\omega} A^s$.

$\{W_e\}_{e\in\omega}$ denotes a fixed acceptable numbering of the r.e. sets and $\{W_e^s\}_{e,s\in\omega}$ denotes a fixed standard enumeration of the r.e. sets. The symbol K is reserved for $\{e: e \in W_e\}$ which has Turing degree $\mathbf{0}'$.

Intuitively, A is enumeration reducible to B if there is an effective procedure for producing an enumeration of A from *any* enumeration of B. There is a natural one-one correspondence between all such procedures and the r.e. sets (see Rogers [1967] (pp. 145-147)). Hence the i-th *enumeration operator* (*e-operator*) is defined by

$$\Psi_i(B) = \{x: \langle z,x\rangle \in W_i \text{ and } D_z \subseteq B\}.$$

Θ, Ω, Ψ and Φ range over e-operators. Note that for every i, Ψ_i is a mapping from 2^ω to 2^ω which is *monotone*, that is, if $A \subseteq B$ then $\Psi_i(A) \subseteq \Psi_i(B)$. We identify an e-operator with its associated r.e. set and write $W_i(B)$ for $\Psi_i(B)$. Formally:

Definition 1.3.2.

.1. A is *enumeration reducible* (*e-reducible*) to B ($A \leq_e B$) if $A = W_i(B)$ for some i.

.2. $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.

.3. A and B are *incomparable* $(A \perp_e B)$ if $A \nleq_e B$ and $B \nleq_e A$.

It is clear that $\equiv_e$ is an equivalence relation. An *enumeration degree* (*e-degree*) is an equivalence class under $\equiv_e$. Lower case boldface letters range over e-degrees, and occasionally over Turing degrees. $\deg_e A$ denotes the e-degree containing A. If P is a property of sets then an e-degree has property P if it contains a set with property P. The e-degrees form an upper semi-lattice: $\mathbf{a} \vee \mathbf{b} = \deg_e(A \oplus B)$, where $A \in \mathbf{a}$ and $B \in \mathbf{b}$. We also define a join operation on e-operators by: $(\Psi_i \oplus \Psi_j)(A) = \Psi_i(A) \oplus \Psi_j(A)$.

If we identify a function with its graph then e-reducibility may also be viewed as a reducibility between functions, both total and partial. The equivalence classes of partial functions are called *partial degrees*. The e-degrees and partial degrees are isomorphic as every e-degree contains the graph of a function: if $A \in \mathbf{a}$ then consider $\{\langle x,1\rangle: x \in A\}$. A is *total* if it is the graph of a total function. It is easily proved that $A \leq_T B$ if and only if $\chi_A \leq_e \chi_B$ (see Rogers [1967] (pp. 151-153)). Hence the e-degrees restricted to the total degrees are isomorphic to the Turing degrees. We denote the upper semi-lattice isomorphism $\deg_T A \to \deg_e \chi_A$ by f^*. When we speak of a Turing degree as an e-degree we are referring to its image under f^*.

0_e denotes the least e-degree which consists of all the r.e. sets. If A is r.e. then $\bar{A} \equiv_e \chi_A$. Hence the r.e. Turing degrees are isomorphic to the Π_1 e-degrees.

Definition 1.3.3. An e-degree **a** is *quasi-minimal* if $\mathbf{a} >_e \mathbf{0_e}$ and for every non-zero $\mathbf{b} \leq_e \mathbf{a}$, **b** is non-total.

Medvedev [1955] proved that there are quasi-minimal e-degrees, thereby showing that the e-degrees are indeed a proper extension of the Turing degrees. Case [1971] showed that the e-degrees do not form a lattice, that there is a minimal pair of e-degrees and that no total e-degree is minimal. Gutteridge [1971] showed that there is a minimal pair of r.e. Turing degrees which is a minimal pair of Π_1 e-degrees. He also proved that there are no minimal e-degrees and relativised to show that no total e-degree has a minimal cover; furthermore he showed that any e-degree has at most countably many minimal covers.

Cooper [1984] has shown that the Σ_2 e-degrees are dense. McEvoy and Cooper [1985] have proved that every low minimal pair of r.e. Turing degrees is a minimal pair of Π_1 e-degrees but that there is a minimal pair of high r.e. Turing degrees which is not a minimal pair of e-degrees.

Cooper [1984] and McEvoy [1985] have defined a jump operator on the e-degrees:

Definition 1.3.4. $(\deg_e A)' = \deg_e J(A)$, where $J(A) = \chi_{K_A}$ and $K_A = \{e: e \in W_e(A)\}$.

$\mathbf{a}'$ denotes the jump of $\mathbf{a}$. McEvoy has shown that the jump is preserved under the isomorphism f^*. Hence $\mathbf{0}_e' = \deg_e \chi_K$. Cooper has shown that the set of Σ_2 e-degrees is exactly the set of e-degrees below $\mathbf{0}_e'$.

Definition 1.3.5. A sequence of recursive sets $\{A^s\}_{s\in\omega}$ is *uniformly recursive* if there is a recursive function $f(s,x)$ such that $A^s(x) = f(s,x)$ for every x, s.

It is easily seen that A is Σ_2 if and only if there is a uniformly recursive sequence $\{A^s\}_{s\in\omega}$ such that

$$A = \{x: \exists t\ (\forall s > t)\ [x \in A^s]\}.$$

Such a sequence is called a Σ_2*-approximation* to A. If, in addition, $\lim_s A^s(x)$ exists for every x, it is called a Δ_2*-approximation*.

Clearly A is Δ_2 if and only if A has a Δ_2-approximation. Cooper and Copestake [ta] have constructed a Σ_2 e-degree which is not Δ_2.

McEvoy and Cooper [1985] have extended the concept of lowness to the e-degrees:

Definition 1.3.6.

.1. A is *low* if $J(A) \in \mathbf{0}_e'$.

.2. A *low approximation* to A is a Δ_2-approximation $\{A^s\}_{s\in\omega}$ such that for every e, $\{W_e^s(A^s)\}_{s\in\omega}$ is a Δ_2-approximation to $W_e(A)$.

McEvoy has shown that a set is low if and only if it has a low approximation. A low approximation to A is equivalent to a Δ_2-approximation to $K_A^0 =_{dfn} \{\langle x,e\rangle : x \in W_e(A)\}$, hence A is low if and only if K_A^0 is Δ_2. Since the enumeration jump is an extension of the Turing jump every low Turing degree is a low e-degree, however McEvoy [1985] has shown that there is a low quasi-minimal degree, hence the low e-degrees are a proper extension of the low Turing degrees.

§1.4 SOME TECHNICAL TOOLS

The main results in this thesis involve the construction of Σ_2 sets and e-operators, or equivalently, r.e. sets. We do this using the finite injury priority method, or methods similar to it in those cases where the requirements are infinitary. Traditionally these methods are used to construct a recursive enumeration of an r.e. set, however the similarity between a strong array and a uniformly recursive sequence allows us to use the same methods to construct a Σ_2-approximation to a Σ_2 set.

Typically we begin with a recursive list of conditions involving the set(s) to be constructed. Each such condition is called a *requirement*. We order the requirements in descending order of *priority*. Hence if R_m and R_n denote the m-th and n-th requirements with $m < n$ then R_m has *higher priority* than R_n, or equivalently, R_n has *lower priority* than R_m. We then outline a recursive procedure for constructing the recursive sequence which we call a *construction*. We think of the s-th member of the sequence as being constructed at *stage* s. We say that a requirement is *satisfied* or *met* if it holds at the end of the construction. Since requirement R_0 has highest priority, our goal is to ensure that it is satisfied, then R_1 and so on. Hence we may take an action at stage s to satisfy requirement R_m $(m < n)$ even if it means undoing an action taken at a previous stage in order to satisfy R_n, thereby *injuring* requirement R_n at stage s. If each requirement is injured only finitely often this is called the finite injury priority method. All the constructions in this thesis fit into this general framework though not all are finite injury.

At each stage we would like our actions to be based on *true* information about the various sets involved. Hence the following concept and related results are useful:

Definition 1.4.1. Let $\{A^s\}_{s\in\omega}$ be a Σ_2-approximation to $A \in \Sigma_2$. We say that s is a *true stage* in the approximation if $A^s \subseteq A$.

Cooper has proved the following result, though the proof in [1984] is less direct than the one given here.

Proposition 1.4.2. *Every* Σ_2 *set has a* Σ_2*-approximation with infinitely many true stages.*

Proof. Let $A \in \Sigma_2$ and $\{A^s\}_{s\in\omega}$ be a Σ_2-approximation to A. We can assume that A^s is finite for every s by replacing it with $A^s \restriction s$ if necessary. For every n, set $\upsilon(n) = \langle n,0\rangle = \mu\upsilon\ [(\upsilon)_0 = n]$ by definition of $\langle\cdot,\cdot\rangle$. For every s, set $B^s = \bigcap \{A^t : \upsilon((s)_0) \leq t \leq s\}$. The desired Σ_2-approximation $\{\tilde{A}^s\}_{s\in\omega}$ is defined by:

$$\tilde{A}^s = \begin{cases} A^s, & \text{if } (\exists t < s)\ [(t)_0 = (s)_0 \text{ and } B^t = B^s], \\ B^s, & \text{otherwise.} \end{cases}$$

Clearly $B^s \subseteq \tilde{A}^s \subseteq A^s$ and if $t < s$ and $(t)_0 = (s)_0$ then $B^t \supseteq B^s$. Hence for every n, $B_n = \lim_{(s)_0=n} B^s$ exists, therefore $t(n) = \mu t\ [(t)_0 = n \text{ and } (\forall s \geq t)\ [(s)_0 = n \Rightarrow B^s = B_n]]$ is defined.

Let $\tilde{A}$ denote the Σ_2 set to which $\{\tilde{A}^s\}_{s\in\omega}$ is an approximation. Then $\tilde{A} \subseteq A$, since $\tilde{A}^s \subseteq A^s$ for every s. Suppose $x \in A$. Then we can choose s' such that $x \in A^s$ for every $s > s'$. Let $F = \{(s)_0 : s \leq s'\}$ and $s^* = \max \{s'\} \cup \{t(n)+1 : n \in F\}$. Suppose $s > s^*$. If $(s)_0 \in F$ then $x \in \tilde{A}^s = A^s$ by choice of s^*. Otherwise $\upsilon((s)_0) > s^* \geq s'$, hence $x \in B^s \subseteq \tilde{A}^s$ by definition of B^s and choice of s'. Therefore $x \in \tilde{A}$, so $\tilde{A} = A$.

For every n, $\tilde{A}^{t(n)} = B_n \subseteq \cap \{A^s : s \geq \upsilon(n)\} \subseteq A$, hence $\{\tilde{A}^s\}_{s\in\omega}$ contains infinitely many true stages. ∎

Sometimes it is more convenient to take action on behalf of a requirement R_n at pre-designated stages, say stages $s+1$ where $(s)_0 = n$.

Proposition 1.4.3. *For every $A \in \Sigma_2$ there exists a Σ_2-approximation $\{A^s\}_{s\in\omega}$ to A such that for every n, $\{A^t\}_{(t)_0=n}$ is a Σ_2-approximation to A with infinitely many true stages.*

Proof. Let $A \in \Sigma_2$ and $\{A^s\}_{s\in\omega}$ be a Σ_2-approximation to A with infinitely many true stages (Proposition 1.4.2). Set $\tilde{A}^s = A^{(s)_0+(s)_1}$ for every s. Then $\{\tilde{A}^t\}_{(t)_0=n} = \{A^{n+(t)_1}\}_{(t)_0=n} = \{A^{n+k}\}_{k\in\omega}$ since $\langle n,\cdot\rangle$ is an increasing function; hence $\{\tilde{A}^t\}_{(t)_0=n}$ is a Σ_2-approximation to A with infinitely many true stages.

Suppose $x \in A$. Then we can choose s' such that $x \in A^s$ for every $s > s'$. Choose s'' such that $(s)_0 + (s)_1 > s'$ for every $s > s''$. Then $x \in A^{(s)_0+(s)_1} = \tilde{A}^s$ for every $s > s''$. Hence $\{\tilde{A}^s\}_{s\in\omega}$

is a Σ_2-approximation to A. ∎

If $x \in W_i(A)$ then there must be $\langle D,x\rangle \in W_i$ such that $D \subseteq A$. We think of D as the *reason* that $x \in W_i(A)$. If $A \in \Sigma_2$ and $\{A^s\}_{s\in\omega}$ is a Σ_2-approximation to A, then *use functions* help us keep track of the reason that an element $x \in W_i^s(A^s)$ or a finite set $F \subseteq W_i^s(A^s)$.

Use functions.

Let θ be an e-operator and $X \in \Sigma_2$. Let $\{\theta^s\}_{s\in\omega}$ be a fixed recursive enumeration of θ and $\{X^s\}_{s\in\omega}$ a fixed Σ_2-approximation to X.

$$h(\theta,X,y,s) = \begin{cases} \uparrow & \text{if } y \notin \theta^s(X^s) \\ (\mu t \leq s)\,\exists D\,[y \in \theta^t(D) \text{ and } \forall u\,[t \leq u \leq s \Rightarrow D \subseteq X^u]], & \text{otherwise} \end{cases}$$

$$u(\theta,X,y,s) = \begin{cases} \uparrow & \text{if } h(\theta,X,y,s)\uparrow \\ D_z & \text{where } z = \mu x\,[y \in \theta^{h(\theta,X,y,s)}(D_x) \text{ and} \\ & \forall u\,[h(\theta,X,y,s) \leq u \leq s \Rightarrow D_x \subseteq X^u]], \text{ otherwise} \end{cases}$$

$$H(\theta,X,F,s) = \begin{cases} \uparrow & \text{if } F \not\subseteq W_e^s(X^s), \\ (\mu t \leq s)\,\exists D\,[F \subseteq W_e^t(D) \text{ and } \forall u\,[t \leq u \leq s \Rightarrow D \subseteq X^u]], & \text{otherwise.} \end{cases}$$

$$U(\theta,X,F,s) = \begin{cases} \uparrow & \text{if } H(\theta,X,F,s)\uparrow, \\ D_z & \text{where } z = \mu i\ [F \subseteq W_e^{H(\theta,X,F,s)}(D_i) \text{ and } \\ & \forall u\ [H(\theta,X,F,s) \leq u \leq s \Rightarrow D_i \subseteq X^u]], \text{ otherwise.} \end{cases}$$

$h(\theta,X,y,s)$ and $H(\theta,X,y,s)$ are called **history functions.**

Remark. If $u(\theta,X,y,s)\downarrow \subseteq X^{s+1}$ $(U(\theta,X,F,s)\downarrow \subseteq X^{s+1})$ then $h(\theta,X,y,s+1)\downarrow = h(\theta,X,y,s)$ and $u(\theta,X,y,s+1)\downarrow = u(\theta,X,y,s)$ $(H(\theta,X,F,s+1)\downarrow = H(\theta,X,F,s)$ and $U(\theta,X,F,s+1)\downarrow = U(\theta,X,F,s))$. Hence if $y \in \theta(X)$ $(F \subseteq \theta(X))$, then $h(\theta,X,y,s)$ and $u(\theta,X,y,s)$ $(H(\theta,X,F,s)$ and $U(\theta,X,F,s))$ reach limits denoted by $h(\theta,X,y)$ and $u(\theta,X,y)$ $(H(\theta,X,F)$ and $U(\theta,X,F))$ respectively.

Note that the definition of $h(\theta,X,y,s)$ and $u(\theta,X,y,s)$ only depends on $\{\theta^t\}_{t\leq s}$ and $\{X^t\}_{t\leq s}$. Hence given recursive sequences $\{\theta^t\}_{t\leq s}$ and $\{X^t\}_{t\leq s}$, $h(\theta,X,y,s)$ and $u(\theta,X,y,s)$ are defined as above.

CHAPTER II

EMBEDDING PARTIAL ORDERS IN THE Σ_2 E-DEGREES

§2.1 INTRODUCTION

We denote the least and greatest elements of a partial order by **0** and **1**, respectively. Cooper has shown that the Σ_2 e-degrees are dense. Using essentially the same method as that used in [1984], we generalize this to:

Theorem 2.1.1. *For every finite partial order* $(\mathcal{P},\leq^*)$, *if* $\mathbf{p}_0 <^* \mathbf{p}_1 <^* \dots <^* \mathbf{p}_n \in \mathcal{P}$, $\mathbf{a}_0 <_e \mathbf{a}_1 <_e \dots <_e \mathbf{a}_n \leq_e \mathbf{0}_e'$, $\mathbf{p}_0 \neq \mathbf{0}$ *implies* $\mathbf{a}_0 \neq \mathbf{0}_e$ *and* $\mathbf{p}_n \neq \mathbf{1}$ *implies* $\mathbf{a}_n \neq \mathbf{0}_e'$, *then there exists an embedding* f *of* $\mathcal{P}$ *in* $\mathcal{D}_e(\Sigma_2)$ *such that* $f(\mathbf{p}_i) = \mathbf{a}_i$ *for every* $i \leq n$.

§2.2 PROOF OF THEOREM

Suppose $(\mathcal{P}, \leq^*)$, $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$, and $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ satisfy the hypothesis of the theorem. W.l.o.g. we can assume that $\mathcal{P}$ contains least and greatest elements. For simplicity we first assume that $\mathbf{p}_n = \mathbf{1}$ and $\mathbf{a}_0 >_e \mathbf{0}_e$. Let $\mathcal{Q} = \mathcal{P} - \{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n\}$. We partition $\mathcal{Q}$ into sets $\mathcal{Q}_i$, where $i \leq n$ and $\mathcal{Q}_i = \{\mathbf{q} \in \mathcal{Q}\colon i = \mu j\ [\mathbf{q} \leq^* \mathbf{p}_j]\}$. Let $A_i \in \mathbf{a}_i$ for $i \leq n$. For every $\mathbf{q} \in \mathcal{Q}$ we construct a Σ_2 set $B_\mathbf{q}$ such that for every $i \leq n$, $\mathbf{q} \in \mathcal{Q}_i$,

$$B_\mathbf{q} \leq_e A_i, \tag{2.2.1}$$

and the following *maximal independence* properties hold:

$$A_i \not\leq_e \Big(\bigoplus_{j<i} A_j\Big) \oplus \Big(\bigoplus_{\mathbf{r}\in\mathcal{Q}} B_\mathbf{r}\Big), \tag{2.2.2}$$

and

$$B_\mathbf{q} \not\leq_e \Big(\bigoplus_{j<i} A_j\Big) \oplus \Big(\bigoplus_{\substack{\mathbf{r}\in\mathcal{Q}\\ \mathbf{r}\neq\mathbf{q}}} B_\mathbf{r}\Big). \tag{2.2.3}$$

For every $\mathbf{s} \in \mathcal{S}$, set

$$f(\mathbf{s}) = \deg_e ((\bigoplus_{\substack{j \leq n \\ \mathbf{p}_j \leq^* \mathbf{s}}} A_j) \oplus (\bigoplus_{\substack{\mathbf{r} \in \mathcal{Q} \\ \mathbf{r} \leq^* \mathbf{s}}} B_\mathbf{r})).$$

Then

$$f(\mathbf{p}_i) = \deg_e ((\bigoplus_{j \leq i} A_j) \oplus (\bigoplus_{\substack{\mathbf{r} \in \mathcal{Q}_j \\ j \leq i}} B_\mathbf{r})) = \deg_e A_i = \mathbf{a}_i.$$

Now it is clear that $\mathbf{s} \leq^* \mathbf{t}$ implies $f(\mathbf{s}) \leq_e f(\mathbf{t})$. Suppose $\mathbf{t} \not\leq^* \mathbf{s}$.

Assume $\mathbf{t} = \mathbf{p}_i$ $(i \leq n)$. Then $f(\mathbf{t}) = \mathbf{a}_i$ and

$$f(\mathbf{s}) \leq_e \deg_e ((\bigoplus_{j<i} A_j) \oplus (\bigoplus_{\mathbf{r} \in \mathcal{Q}} B_\mathbf{r})).$$

By 2.2.2 $f(\mathbf{t}) = \mathbf{a}_i \not\leq_e f(\mathbf{s})$.

Assume $\mathbf{t} = \mathbf{q}$, where $\mathbf{q} \in \mathcal{Q}_i$ $(i \leq n)$. Then $\deg_e B_\mathbf{q} \leq_e f(\mathbf{t})$ and

$$f(\mathbf{s}) \leq_e \deg_e ((\bigoplus_{j<i} A_j) \oplus (\bigoplus_{\substack{\mathbf{r} \in \mathcal{Q} \\ \mathbf{r} \neq \mathbf{q}}} B_\mathbf{r})).$$

By 2.2.3 $\deg_e B_\mathbf{q} \not\leq_e f(\mathbf{s})$, therefore $f(\mathbf{t}) \not\leq_e f(\mathbf{s})$. Hence f is the desired embedding.

Lemma 2.3.1 below is the key to the proof of the theorem. In order to apply the lemma set $k_i = |\mathcal{Q}_i|$ for $i \leq n$, and $\mathcal{Q}_i = \{\mathbf{q}_{i,0}, \mathbf{q}_{i,1}, \ldots, \mathbf{q}_{i,k_i-1}\}$. Then $B_{\mathbf{q}_{i,j}}$ is $B_{i,j}$ of the lemma. Let $\mathcal{Q}_i^* = \bigcup_{j \leq i} \mathcal{Q}_j$ for $i \leq n$. The lemma is a slightly stronger result than needed because it states that given $\{B_\mathbf{q}: \mathbf{q} \in \mathcal{Q}_k^*\}$, for some $k < n$, such that for every $i \leq k$, $\mathbf{q} \in \mathcal{Q}_i$, 2.2.1, 2.2.2 and 2.2.3 hold with $\mathcal{Q}$ replaced by $\mathcal{Q}_k^*$, this set can be extended to $\{B_\mathbf{q}: \mathbf{q} \in \mathcal{Q}_{k+1}^*\}$ such that for every $i \leq k+1$, $\mathbf{q} \in \mathcal{Q}_i$, 2.2.1, 2.2.2 and 2.2.3 hold with $\mathcal{Q}$ replaced by $\mathcal{Q}_{k+1}^*$.

If $\mathbf{p}_n \neq 1$, then $\mathbf{a}_n \neq \mathbf{0}'_\mathbf{e}$, and we consider the extended sequences $\mathbf{p}_0 <^* \mathbf{p}_1 <^* \ldots <^* \mathbf{p}_n <^* 1$ and $\mathbf{a}_0 <_e \mathbf{a}_1 <_e \cdots <_e \mathbf{a}_n <_e \mathbf{0}'_\mathbf{e}$. If $\mathbf{a}_0 = \mathbf{0}_\mathbf{e}$, then $\mathbf{p}_0 = 0$, and we set $f(\mathbf{p}_0) = \mathbf{0}_\mathbf{e}$, $\mathcal{Q} = \mathcal{P} - \{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n\}$, and consider the truncated sequences $\mathbf{p}_1 <^* \mathbf{p}_2 <^* \ldots <^* \mathbf{p}_n$ and $\mathbf{a}_1 <_e \mathbf{a}_2 <_e \cdots <_e \mathbf{a}_n$.

§2.3 THE KEY LEMMA

Lemma 2.3.1. *Given* n, $k_0, k_1, \ldots, k_n$, Σ_2 *sets* $A_0 <_e A_1 < \ldots <_e A_n$ *and* $B_{i,j}$, *for* $i < n$, $j < k_i$, *such that for every* $i < n$,

.1. $B_{i,j} \leq_e A_i$ *for every* $j < k_i$,

.2. $A_i \not\leq_e (\bigoplus_{l<i} A_l) \oplus (\bigoplus_{\substack{l<n \\ m<k_l}} B_{l,m})$,

.3. $B_{i,j} \not\leq_e (\bigoplus_{l<i} A_l) \oplus (\bigoplus_{\substack{l<n, m<k_l \\ l \neq i \text{ or } m \neq j}} B_{l,m})$ *for every* $j < k_i$,

there exist $B_{n,j}$, *for* $j < k_n$, *such that* $B_{n,j} \leq_e A_n$ *for every* $j < k_n$, *and for every* $i \leq n$,

.4. $A_i \not\leq_e (\bigoplus_{l<i} A_l) \oplus (\bigoplus_{\substack{l\leq n \\ m<k_l}} B_{l,m})$,

.5. $B_{i,j} \not\leq_e (\bigoplus_{l<i} A_l) \oplus (\bigoplus_{\substack{l\leq n, m<k_l \\ l \neq i \text{ or } m \neq j}} B_{l,m})$ *for every* $j < k_i$.

Proof. Fix n, $k_0, \ldots, k_n$. Assume that A_i $(i \leq n)$ and $B_{i,j}$ $(i < n,\ j < k_i)$ satisfy the hypothesis.

Let C be defined by:

$$C^{\{2i\}} = \begin{cases} A_i, & \text{if } i \le n, \\ \emptyset, & \text{otherwise,} \end{cases} \quad \text{and}$$

$$C^{\{2\langle i,j\rangle+1\}} = \begin{cases} B_{i,j}, & \text{if } i < n,\ j < k_i, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Note that $C \equiv_e A_n$.

We will construct e-operators Θ_j, for $j < k_n$, satisfying certain requirements and will set $B_{n,j} = \Theta_j(C)$. In terms of the Θ_j $(j < k_n)$ we define other e-operators $\tilde{\Theta}$ and $\hat{\Theta}_j$ by:

$$\tilde{\Theta} = \{\langle F,\langle x,j\rangle\rangle \colon j < k_n \text{ and } \langle F,x\rangle \in \Theta_j\}$$

and

$$\hat{\Theta}_j = \{\langle F,\langle x,k\rangle\rangle \colon k < k_n,\ k \ne j \text{ and } \langle F,x\rangle \in \Theta_k\},$$

for $j < k_n$. Note that $\tilde{\Theta}(C) \equiv_e \bigoplus_{j<k_n} \Theta_j(C)$ and $\hat{\Theta}_j(C) \equiv_e \bigoplus_{\substack{k<k_n \\ k\ne j}} \Theta_k(C)$.

Elements $i(r)$, $j(r)$ and $e(r)$, Σ_2 sets C_r and $C_{r,z}$, and requirement r are defined as follows.

Case 1. $r \equiv 0 \bmod 2$.

Subcase 1.1. $r = 4\langle i,e\rangle$.

If $i \leq n$, then $i(r) = i$, $e(r) = e$ and $j(r)$ is undefined.

Also:

$$C_r^{\{2l\}} = \begin{cases} \emptyset, & \text{if } i \leq l \leq n, \\ C^{\{2l\}}, & \text{otherwise,} \end{cases} \quad \text{and} \quad C_r^{\{2l+1\}} = C^{\{2l+1\}};$$

$$C_{r,z}^{\{l\}} = \begin{cases} C^{\{l\}}\restriction z, & \text{if } l = 2i, \\ C_r^{\{l\}}, & \text{otherwise.} \end{cases}$$

Requirement r is:

$$\text{"}A_i \neq W_e(C_r \oplus \tilde{\theta}(C))\text{".}$$

Note $C_r \equiv_e (\bigoplus_{l<i} A_l) \oplus (\bigoplus_{\substack{l<n \\ m<k_l}} B_{l,m})$. Therefore $A_i \not\leq_e C_r$, and

$C_{r,z} \equiv_e C_r \oplus A_i\restriction z \equiv_e C_r$.

If $i > n$ then $i(r)$, $j(r)$, $e(r)$, C_r and $C_{r,z}$ are undefined and requirement r is the empty requirement.

Subcase 1.2. $r = 4\langle i,j,e\rangle+2$.

If $i < n$ and $j < k_i$ then $i(r) = i$, $j(r) = j$ and $e(r) = e$.

Also:

$$C_r^{\{2l\}} = \begin{cases} \emptyset, & \text{if } i \le l \le n, \\ C^{\{2l\}}, & \text{otherwise}, \end{cases} \quad \text{and} \quad C_r^{\{2l+1\}} = \begin{cases} \emptyset, & \text{if } l = \langle i,j\rangle, \\ C^{\{2l+1\}}, & \text{otherwise}; \end{cases}$$

$$C_{r,z}^{\{l\}} = \begin{cases} C^{\{l\}}\upharpoonright z, & \text{if } l = 2\langle i,j\rangle+1, \\ C_r^{\{l\}}, & \text{otherwise}. \end{cases}$$

Requirement r is:

$$\text{"}B_{i,j} \ne W_e(C_r \oplus \tilde{\theta}(C))\text{"}.$$

Note $C_r \equiv_e (\bigoplus_{l<i} A_l) \oplus (\bigoplus_{\substack{l<n, m<k_l \\ l\ne i \text{ or } m\ne j}} B_{l,m})$. Therefore $B_{i,j} \not\le_e C_r$, and $C_{r,z} \equiv_e C_r \oplus B_{i,j}\upharpoonright z \equiv_e C_r$.

If $i \ge n$ or $j \ge k_i$ then $i(r)$, $j(r)$, $e(r)$, C_r and $C_{r,z}$ are undefined and requirement r is the empty requirement.

Case 2. $r = 2\langle j,e\rangle+1$.

If $j < k_n$ then $j(r) = j$, $e(r) = e$ and $i(r)$ is undefined.

Also:

$$C_r^{\{l\}} = \begin{cases} \emptyset, & \text{if } l = 2n, \\ C^{\{l\}}, & \text{otherwise}; \end{cases}$$

$$C_{r,z}{}^{\{l\}} = \begin{cases} C^{\{l\}}\restriction z, & \text{if } l = 2n, \\ C_r{}^{\{l\}}, & \text{otherwise.} \end{cases}$$

Requirement r is:

$$\text{"}\theta_j(C)^{\{r\}} \neq W_e(C_r \oplus \hat{\theta}_j(C))^{\{r\}}\text{"}.$$

Note $C_r \equiv_e (\bigoplus_{l<n} A_l) \oplus (\bigoplus_{\substack{l<n \\ m<k_l}} B_{l,m}) \equiv_e A_{n-1}$. Therefore $A_n \not\leq_e C_r$, and $C_{r,z} \equiv_e C_r \oplus A_n\restriction z \equiv_e C_r$.

If $j \geq k_n$ then $i(r)$, $j(r)$, $e(r)$, C_r and $C_{r,z}$ are undefined and requirement r is the empty requirement.

The natural order of the requirements is the order of priority. It is easily checked that if all the requirements are met then the lemma is proved. In order to satisfy requirement r, we construct e-operators $\theta_{j,r}$, for $j < k_n$, and set

$$\theta_j = \bigcup_r \theta_{j,r}.$$

$\theta^s_{j,r}$ is the set of instructions $\langle F,x\rangle$ which have been enumerated into $\theta_{j,r}$ by the end of stage s; $\{\theta^s_{j,r}\}_{s\in\omega}$ is a recursive enumeration of

$\theta_{j,r}$. Set

$$\theta_{j,<r} = \bigcup_{q<r} \theta_{j,q}$$

and $\theta_{j,\geq r} = \theta_j - \theta_{j,<r}$, and let $\tilde{\theta}_{<r}$ and $\hat{\theta}_{j,<r}$ be defined analogously. $\{\tilde{\theta}^s\}_{s\in\omega}$, $\{\hat{\theta}^s_j\}_{s\in\omega}$, $\{\tilde{\theta}^s_{<r}\}_{s\in\omega}$, $\{\hat{\theta}^s_{j,<r}\}_{s\in\omega}$, $\{\theta^s_{j,<r}\}_{s\in\omega}$ and $\{\theta^s_{j,\geq r}\}_{s\in\omega}$ denote the natural recursive enumerations of $\tilde{\theta}$, $\hat{\theta}_j$, $\tilde{\theta}_{<r}$, $\hat{\theta}_{j,<r}$, $\theta_{j,<r}$ and $\theta_{j,\geq r}$ respectively, generated by $\{\theta^s_{k,q}\}_{s\in\omega}$ $(k < k_n)$.

Choose e-operators Ω_i, for $i < n$, and $\Psi_{i,j}$, for $i < n$, $j < k_i$, such that $A_i = \Omega_i(A_n)$ and $B_{i,j} = \Psi_{i,j}(A_n)$. Let $\{\Omega^s_i\}_{s\in\omega}$ and $\{\Psi^s_{i,j}\}_{s\in\omega}$ be recursive enumerations of Ω_i and $\Psi_{i,j}$ respectively. Let $\{A^s_n\}_{s\in\omega}$ be a Σ_2-approximation to A_n such that for every r, $\{A^s_n\}_{(s)_0=r}$ is a Σ_2-approximation to A_n with infinitely many true stages. $\{A^s_i\}_{s\in\omega}$ and $\{B^s_{i,j}\}_{s\in\omega}$ are Σ_2-approximations to A_i and $B_{i,j}$ respectively, where $A^s_i = \Omega^s_i(A^s_n)$ and $B^s_{i,j} = \Psi^s_{i,j}(A^s_n)$. $\{C^s\}_{s\in\omega}$, $\{C^s_r\}_{s\in\omega}$ and $\{C^s_{r,z}\}_{s\in\omega}$ are the natural Σ_2-approximations to C, C_r and $C_{r,z}$ respectively, generated by $\{A^s_i\}_{s\in\omega}$ and $\{B^s_{i,j}\}_{s\in\omega}$.

Length of agreement functions.

If requirement r is empty then $L(r,s)\uparrow$.

Assume requirement r is not empty.

Case 1. $r \equiv 0 \bmod 2$.

Subcase 1.1. $r = 4\langle i,e\rangle$.

$$L(r,s) = \mu z\,[z = s \text{ or } A_i^s(z) \neq W_e^s(C_r^s \oplus \tilde{\theta}^s(C^s))(z)].$$

Subcase 1.2. $r = 4\langle i,j,e\rangle+2$.

$$L(r,s) = \mu z\,[z = s \text{ or } B_{i,j}^s(z) \neq W_e^s(C_r^s \oplus \tilde{\theta}^s(C^s))(z)].$$

Case 2. $r = 2\langle j,e\rangle+1$.

$$L(r,s) = \mu z\,[z = s \text{ or } \theta_j^s(C^s)^{\{r\}}(z) \neq W_e^s(C_r^s \oplus \hat{\theta}_j^s(C^s))^{\{r\}}(z)].$$

We attend to requirement r at stages $s+1$, where $(s)_0 = r$. Let T denote the set of true stages in $\{A_n^s\}_{s\in\omega}$. Note that T is also a set of true stages in $\{C^s\}_{s\in\omega}$, $\{C_r^s\}_{s\in\omega}$ and $\{C_{r,z}^s\}_{s\in\omega}$.

Suppose $r = 4\langle i,e\rangle$ $(i \leq n)$. We arrange that for $z \in W_e^s(C_r^s \oplus \tilde{\theta}^s(C^s))\restriction L(r,s)$, $z \in W_e^s(C_r^s \oplus \tilde{\theta}^s(F^s \cup C_{r,z}^s))$, where $F^s \subseteq C^s$ is finite, and $\lim_{s\in T} F^s = F \subseteq C$ (F finite). This is done in such a way that if $L(r,s) \to \infty$ as s increases in T, then $z \in W_e(C_r \oplus \tilde{\theta}(F \cup C_{r,z}))$ for every $z \in W_e(C_r \oplus \tilde{\theta}(C))$. If requirement r fails, $A_i \leq_e C_r$, by a kind of *back and forth* construction as follows, which is a contradiction. Begin enumerating $W_e(C_r \oplus \tilde{\theta}(F \cup C_{r,z})) \subseteq A_i$, for $z = 0$. As elements enter this set add them to $C_r^{\{2i\}}$ to build $C_{r,z}$ for increasing values of z, and continue enumerating $W_e(C_r \oplus \tilde{\theta}(F \cup C_{r,z})) \subseteq A_i$. Since $z \in W_e(C_r \oplus \tilde{\theta}(F \cup C_{r,z}))$ for every $z \in W_e(C_r \oplus \tilde{\theta}(C)) = A_i$, every element of A_i is enumerated.

If $r = 4\langle i,j,e\rangle+2$ $(i < n,\ j < k_i)$, we use the above strategy, with A_i replaced by $B_{i,j}$ and $C_r^{\{2i\}}$ replaced by $C_r^{\{2\langle i,j\rangle+1\}}$.

Suppose $r = 2\langle j,e\rangle+1$ $(j < k_n)$. We code $A_n^s\restriction(L(r,s)+1)$ into $\theta_j^s(C^s)^{\{r\}}$; in addition, as in the previous strategy, we arrange that for $z \in W_e^s(C_r^s \oplus \hat{\theta}_j^s(C^s))^{\{r\}}\restriction L(r,s)$, $z \in W_e^s(C_r^s \oplus \hat{\theta}_j^s(F^s \cup C_{r,z}^s))^{\{r\}}$ where $F^s \subseteq C^s$ is finite, and $\lim_{s\in T} F^s = F \subseteq C$ (F finite). Again, this is done in such a way that if $L(r,s) \to \infty$ as s increases in T, then $z \in W_e(C_r \oplus \hat{\theta}_j(F \cup C_{r,z}))^{\{r\}}$ for every $z \in W_e(C_r \oplus \hat{\theta}_j(C))^{\{r\}}$. If requirement r fails, $A_n \leq_e C_r$ as follows, which is a contradiction. There exists m such that $\theta_j(C)^{\{r\}}\restriction[>m] = A_n\restriction[>m]$. Enumerate elements of $A_n\restriction(m+1)$ and begin enumerating $W_e(C_r \oplus \hat{\theta}_j(F \cup C_{r,z}))^{\{r\}}\restriction[>m] \subseteq \theta_j(C)^{\{r\}}\restriction[>m] \subseteq A_n$, for $z = m+1$. As elements enter this set add them to $C_r^{\{2n\}}$ to build $C_{r,z}$ for increasing values of z. Continue enumerating $W_e(C_r \oplus \hat{\theta}_j(F \cup C_{r,z}))^{\{r\}}\restriction[>m] \subseteq A_n$. Since $z \in W_e(C_r \oplus \hat{\theta}_j(F \cup C_{r,z}))^{\{r\}}$ for every $z \in W_e(C_r \oplus \hat{\theta}_j(C))^{\{r\}}\restriction[>m] = \theta_j(C)^{\{r\}}\restriction[>m] = A_n\restriction[>m]$, every element of A_n is enumerated.

Construction.

Stage 0.

Do nothing.

Stage $s+1$.

Let $r = (s)_0$. If requirement r is empty, do nothing.

Otherwise do the following. For every z, set

$$E^s_z = U(\tilde{\theta}_{<r}, C, \tilde{\theta}^s_{<r}(C^s), s) \cup C^s_{r,z}.$$

Case 1. $r \equiv 0 \bmod 2$.

Let $e = e(r)$.

For every z, x such that

1.1. $z \in W^s_e(C^s_r \oplus \tilde{\theta}^s(C^s)) \restriction L(r,s) - W^s_e(C^s_r \oplus \tilde{\theta}^s(E^s_z))$,

1.2. $x \in u(W_e, C_r \oplus \tilde{\theta}(C), z, s) - C^s_r \oplus \tilde{\theta}^s(E^s_z)$,

enumerate $\langle E^s_z, w\rangle$ into $\theta_{k,r}$, where $x = 2\langle w,k\rangle+1$ for some k; (note $k < k_n$). If z satisfies 1.1 and $E^s_z \subseteq C$, we say $s+1$ is (r,z)-*active*.

Case 2. $r \equiv 1 \bmod 2$.

Let $j = j(r)$ and $e = e(r)$.

For every $z \in A^s_n \restriction (L(r,s)+1)$, enumerate $\langle C^s, \langle z,r\rangle\rangle$ into $\theta_{j,r}$.

For every z, x such that

2.1. $z \in W^s_e(C^s_r \oplus \hat{\theta}^s_j(C^s))^{\{r\}} \restriction L(r,s) - W^s_e(C^s_r \oplus \hat{\theta}^s_j(E^s_z))^{\{r\}}$,

2.2. $x \in u(W_e, C_r \oplus \hat{\theta}_j(C), \langle z,r\rangle, s) - C^s_r \oplus \hat{\theta}^s_j(E^s_z)$,

enumerate $\langle E^s_z, w\rangle$ into $\theta_{k,r}$, where $x = 2\langle w,k\rangle+1$ for some k; (note

$k < k_n$ and $k \neq j$). If z satisfies 2.1 and $E_z^s \subseteq C$, we say $s+1$ is (r,z)-*active*.

End of construction.

Proposition 1. *For every* $k < k_n$, r, $\theta_{k,r}(\omega) \subseteq \omega^{[\geq r]}$.

Proof. It suffices to show that for every $k < k_n$, r, s, $\theta_{k,r}^s(\omega) \subseteq \omega^{[\geq r]}$. The proof is by induction on s. $\theta_{k,r}^0 = \emptyset$ for every $k < k_n$, r. Assume $\theta_{k,r}^s(\omega) \subseteq \omega^{[\geq r]}$ for every $k < k_n$, r. If $r \neq (s)_0$ then $\theta_{k,r}^{s+1} = \theta_{k,r}^s$ for every $k < k_n$. Suppose $r = (s)_0$ and $\langle F,w\rangle$ is enumerated into $\theta_{k,r}$ at stage $s+1$, where $k < k_n$. From the construction, requirement r cannot be empty.

Case 1. $r \equiv 0 \bmod 2$.

Let $e = e(r)$. Then $2\langle w,k\rangle+1 \in u(W_e, C_r \oplus \tilde{\theta}(C), z, s) - C_r^s \oplus \tilde{\theta}^s(E_z^s)$ for some $z \in W_e^s(C_r^s \oplus \tilde{\theta}^s(C^s)) \upharpoonright L(r,s)$. Hence $\langle w,k\rangle \in \tilde{\theta}^s(C^s) - \tilde{\theta}^s(E_z^s)$. Now $\tilde{\theta}_{<r}^s(C^s) \subseteq \tilde{\theta}_{<r}^s(E_z^s)$ (Proposition 2). Therefore $w \in \theta_k^s(C^s) - \theta_{k,<r}^s(C^s) = \theta_{k,\geq r}^s(C^s) \subseteq \theta_{k,\geq r}^s(\omega) \subseteq \omega^{[\geq r]}$ by the induction hypothesis.

Case 2. $r \equiv 1 \bmod 2$.

Let $j = j(r)$ and $e = e(r)$. If $k = j$, then $w = \langle z,r\rangle \in \omega^{[\geq r]}$ for some z. Otherwise $2\langle w,k\rangle+1 \in$ $u(W_e, C_r \oplus \hat{\theta}_j(C), \langle z,r\rangle, s) - C_r^s \oplus \hat{\theta}_j^s(E_z^s)$ for some $z \in$

$W^s_e(C^s_r \oplus \hat{\theta}^s_j(C^s))^{\{r\}} \upharpoonright L(r,s)$. Hence $\langle w,k\rangle \in \hat{\theta}^s_j(C^s) - \hat{\theta}^s_j(E^s_z)$. The rest goes as in Case 1. □

Proposition 2. *For every* r, z, *if requirement* r *is not empty and* $r = (s)_0$ *then,*

.1. $\tilde{\theta}^s_{\langle r}(C^s) \subseteq \tilde{\theta}^s_{\langle r}(E^s_z)$,

.2. *if* $r \equiv 0 \bmod 2$ *and* $z \in W^s_{e(r)}(C^s_r \oplus \tilde{\theta}^s(C^s)) \upharpoonright L(r,s)$ *then*

.1. $z \in W^s_{e(r)}(C^s_r \oplus \tilde{\theta}^{s+1}(E^s_z))$,

.2. *if* $s+1$ *is* (r,z)*-active then* $C^s_r \subseteq C_r$, $C^s_{r,z} \subseteq C_{r,z}$, $\tilde{\theta}^s_{\langle r}(C^s) \subseteq \tilde{\theta}_{\langle r}(C)$ *and* $z \in W_{e(r)}(C_r \oplus \tilde{\theta}(C))$,

.3. *if* $r \equiv 1 \bmod 2$ *and* $z \in W^s_{e(r)}(C^s_r \oplus \hat{\theta}^s_{j(r)}(C^s))^{\{r\}} \upharpoonright L(r,s)$ *then*

.1. $z \in W^s_{e(r)}(C^s_r \oplus \hat{\theta}^{s+1}_{j(r)}(E^s_z))^{\{r\}}$,

.2. *if* $s+1$ *is* (r,z)*-active then* $C^s_r \subseteq C_r$, $C^s_{r,z} \subseteq C_{r,z}$, $\tilde{\theta}^s_{\langle r}(C^s) \subseteq \tilde{\theta}_{\langle r}(C)$ *and* $z \in W_{e(r)}(C_r \oplus \hat{\theta}_{j(r)}(C))^{\{r\}}$.

Proof. Assume requirement r is not empty and $r = (s)_0$. From the construction, $U(\tilde{\theta}_{\langle r}, C, \tilde{\theta}^s_{\langle r}(C^s), s) \subseteq E^s_z$, therefore 1 holds.

Suppose $r \equiv 0 \bmod 2$ and $z \in W^s_{e(r)}(C^s_r \oplus \tilde{\theta}^s(C^s)) \upharpoonright L(r,s)$. Let $e = e(r)$. Then either $z \in W^s_e(C^s_r \oplus \tilde{\theta}^s(E^s_z))$ or by the action taken at stage $s+1$, $u(W_e, C_r \oplus \tilde{\theta}(C), z, s) \subseteq C^s_r \oplus \tilde{\theta}^s(E^s_z) \cup C^s_r \oplus \tilde{\theta}^{s+1}(E^s_z) \subseteq C^s_r \oplus \tilde{\theta}^{s+1}(E^s_z)$. Therefore 2.1 holds.

Assume in addition that $s+1$ is (r,z)-active. Then $C^s_r \subseteq C^s_{r,z} \subseteq E^s_z \subseteq C$. Therefore $C^s_r \subseteq C_r$ and $C^s_{r,z} \subseteq C_{r,z}$. By 1, $\tilde{\theta}^s_{\langle r}(C^s) \subseteq \tilde{\theta}^s_{\langle r}(E^s_z) \subseteq \tilde{\theta}_{\langle r}(C)$ and by 2.1, $z \in W^s_e(C^s_r \oplus \tilde{\theta}^{s+1}(E^s_z)) \subseteq W_e(C_r \oplus \tilde{\theta}(C))$.

The proof of 3 is similar. □

Proposition 3. *For every* r,

.1. *requirement* r *is satisfied,*

.2. $\theta_{k,r}(C)$ *is finite for every* $k < k_n$.

Proof. The proof is by induction on r. Assume 1-2 hold for every $r < q$. We show that 1-2 hold for $r = q$. If requirement q is empty then from the construction $\theta_{k,q} = \emptyset$ for every $k < k_n$, so we are done.

Assume requirement q is not empty. It follows from the induction hypothesis that $U(\tilde{\theta}_{<q}, C, \tilde{\theta}_{<q}(C))$ is finite.

Case 1. $q \equiv 0 \bmod 2$.

Let $i = i(q)$ and $e = e(q)$. If $q \equiv 0 \bmod 4$ set $X = A_i$ and $l' = 2i$. If $q \equiv 2 \bmod 4$ let $j = j(q)$; set $X = B_{i,j}$ and $l' = 2\langle i,j\rangle+1$. Then requirement q is "$X \neq W_e(C_q \oplus \tilde{\theta}(C))$". Suppose $X = W_e(C_q \oplus \tilde{\theta}(C))$. We define enumerations, $\{X'^{s}\}_{s\in\omega}$ and $\{C_q'^{s}\}_{s\in\omega}$, of sets X' and C_q' respectively as follows.

$$C_q'^{0} = U(\tilde{\theta}_{<q}, C, \tilde{\theta}_{<q}(C)) \cup C_q; \quad X'^{0} = \emptyset.$$

$$(C_q'^{s+1})^{\{l\}} = \begin{cases} (C_q'^{s})^{\{l\}} \cup X'^{s}, & \text{if } l = l', \\ (C_q'^{s})^{\{l\}}, & \text{otherwise.} \end{cases}$$

$$X'^{s+1} = X'^{s} \cup W_e(C_q \oplus \tilde{\theta}(C_q'^{s+1})).$$

It is clear that $X' \leq_e C_q$.

Claim 1. $X' = X$.

Proof. We show that $X'^s \subseteq X$ and $C_q'^s \subseteq C$ for every s, by induction. Hence $X' \subseteq X$. $C_q'^0 \subseteq C$, since $C_q \subseteq C$, and $X'^0 = \emptyset$. Assume $C_q'^s \subseteq C$ and $X'^s \subseteq X$. Then $C_q'^{s+1} \subseteq C$ by definition of C. Therefore $X'^{s+1} \subseteq X \cup W_e(C_q \oplus \tilde{\theta}(C)) = X$, and we are done.

Since $W_e(C_q \oplus \tilde{\theta}(C_q'^{s+1})) \subseteq X'^{s+1}$ for every s, $W_e(C_q \oplus \tilde{\theta}(C_q')) \subseteq X'$. We show that $X \upharpoonright z \subseteq X'$ for every z, by induction. Hence $X \subseteq X'$.

Assume $X \upharpoonright z \subseteq X'$. If $z \notin X$ then we are done.

Suppose $z \in X = W_e(C_q \oplus \tilde{\theta}(C))$. Choose $s \in T$ such that $(s)_0 = q$, $z \in W_e^s(C_q^s \oplus \tilde{\theta}^s(C^s)) \upharpoonright L(q,s)$, $\tilde{\theta}^s_{<q}(C^s) = \tilde{\theta}_{<q}(C)$ and $U(\tilde{\theta}_{<q}, C, \tilde{\theta}^s_{<q}(C^s), s) = U(\tilde{\theta}_{<q}, C, \tilde{\theta}_{<q}(C))$. Then $C_q^s \subseteq C_q$ and $C^s_{q,z} \subseteq C_{q,z}$, since $s \in T$. Therefore $E_z^s \subseteq U(\tilde{\theta}_{<q}, C, \tilde{\theta}_{<q}(C)) \cup C_{q,z} \subseteq C_q'$ by the induction hypothesis. Now $z \in W_e^s(C_q^s \oplus \tilde{\theta}^{s+1}(E_z^s))$ (Proposition 2), so $z \in W_e(C_q \oplus \tilde{\theta}(C_q')) \subseteq X'$. Therefore $X \upharpoonright (z+1) \subseteq X'$, and we are done.

□

$X \leq_e C_q$ by Claim 1, which is a contradiction. Therefore $X \neq W_e(C_q \oplus \tilde{\theta}(C))$, so 1 holds.

Claim 2. *For every z, there are only finitely many (q,z)-active stages.*

Proof. Suppose not. Choose z giving a contradiction and a (q,z)-active stage $t+1$ such that $U(\tilde{\theta}_{<q}, C, \tilde{\theta}_{<q}(C), s)$ has reached a limit by stage t. Then $\tilde{\theta}^t_{<q}(C^t) = \tilde{\theta}_{<q}(C)$ (Proposition 2), therefore $E^t_z = U(\tilde{\theta}_{<q}, C, \tilde{\theta}_{<q}(C)) \cup C^t_{q,z}$, and $z \in W^t_e(C^t_q \oplus \tilde{\theta}^{t+1}(E^t_z))$ (Proposition 2). Since $C^t_q \subseteq C_q$ and $C^t_{q,z} \subseteq C_{q,z}$ (Proposition 2), we can choose a stage $t' > t$ such that $C^t_q \subseteq C^s_q$ and $C^t_{q,z} \subseteq C^s_{q,z}$ for every $s > t'$. Let $s+1 > t'$ be a (q,z)-active stage. It follows from the definition of E^s_z and the choice of t and t' that $E^t_z \subseteq E^s_z$. Therefore $z \in W^s_e(C^s_q \oplus \tilde{\theta}^s(E^s_z))$, which is a contradiction. □

By 1 we can choose a least y such that $X(y) \neq W_e(C_q \oplus \tilde{\theta}(C))(y)$. Choose a stage t such that

$$(\forall s \geq t)\ [X \upharpoonright (y+1) \subseteq X^s \text{ and } W_e(C_q \oplus \tilde{\theta}(C)) \upharpoonright (y+1) \subseteq W^s_e(C^s_q \oplus \tilde{\theta}^s(C^s))].$$

Claim 3. *For every $z > y$, there are no (q,z)-active stages after stage t.*

Proof. Suppose not. Choose $z > y$ and a (q,z)-active stage $s+1 > t$. Note $L(q,s) > z > y$ and from the construction, $E^s_y \subseteq E^s_z \subseteq C$.

Case 1. $y \in X$.

Then $y \in W^s_e(C^s_q \oplus \tilde{\theta}^s(C^s)) \upharpoonright L(q,s)$. Therefore $y \in W_e(C_q \oplus \tilde{\theta}(C))$ (Proposition 2), which contradicts the choice of y.

Case 2. $y \in W_e(C_q \oplus \tilde{\theta}(C))$.

Then $y \in X^s$. Therefore $y \in X^s \restriction z = (C^s_{q,z})^{\{l'\}} \subseteq C^{\{l'\}}_{q,z} \subseteq X$ (Proposition 2), which contradicts the choice of y. □

Let $k < k_n$. From the construction $w \in \theta_{k,q}(C)$ if and only if there exist z, t such that $\langle E^t_z, w\rangle$ is enumerated into $\theta_{k,q}$ at (q,z)-active stage $t+1$. The set of stages t for which there exists x such that t is (q,x)-active is finite (Claims 2 and 3). Since only finitely many instructions are enumerated into $\theta_{k,q}$ at each stage, 2 holds.

Case 2. $q \equiv 1 \bmod 2$.

Let $j = j(q)$ and $e = e(q)$. Then requirement q is "$\theta_j(C)^{\{q\}} \neq W_e(C_q \oplus \hat{\theta}_j(C))^{\{q\}}$". Suppose $\theta_j(C)^{\{q\}} = W_e(C_q \oplus \hat{\theta}_j(C))^{\{q\}}$.

Since $\theta_{j,<q}(C)$ is finite, (by the induction hypothesis), $m = \max\ (\theta_{j,<q}(C)^{\{q\}})$ is defined.

Claim 4. *For every* z, *if* $z \in \theta^s_{j,q}(C^s)^{\{q\}}$ *then* $z \in A^s_n$.

Proof. Assume $z \in \theta^s_{j,q}(C^s)^{\{q\}}$. Then there exists an instruction $\langle F, \langle z,q\rangle\rangle \in \theta^s_{j,q}$ such that $F \subseteq C^s$. Suppose such an instruction is enumerated into $\theta_{j,q}$ at stage $t+1 \leq s$. An inspection of the construction shows that $q = (t)_0$, $z \in A^t_n \restriction L(q,t)$ and $F = C^t \subseteq C^s$. Since $(C^t)^{\{2n\}} = A^t_n$, $z \in (C^t)^{\{2n\}} \subseteq (C^s)^{\{2n\}} = A^s_n$. □

Claim 5. $\theta_j(C)^{\{q\}}\restriction[>m] = A_n\restriction[>m]$.

Proof. Let $z > m$.

Suppose $z \in \theta_j(C)^{\{q\}}$. Then $z \notin \theta_{j,<q}(C)^{\{q\}}$, so $z \in \theta_{j,q}(C)^{\{q\}}$ (Proposition 1). Choose $s \in T$ such that $z \in \theta^s_{j,q}(C^s)^{\{q\}}$. Then $z \in A^s_n \subseteq A$ (Claim 4).

Suppose $z \in A_n$. Choose $s \in T$ such that $(s)_0 = q$ and $z \in A^s_n\restriction(L(q,s)+1)$. Then $C^s \subseteq C$ and $\langle C^s, \langle z,q\rangle\rangle$ is enumerated into $\theta_{j,q}$ at stage $s+1$. Therefore $z \in \theta_j(C)^{\{q\}}$. □

We define enumerations, $\{A'^s_n\}_{s\in\omega}$ and $\{C'^s_q\}_{s\in\omega}$, of sets A'_n and C'_q respectively as follows.

$$C'^0_q = U(\tilde{\theta}_{<q}, C, \tilde{\theta}_{<q}(C)) \cup C_q; \quad A'^0_n = A_n\restriction(m+1).$$

$$(C'^{s+1}_q)^{\{l\}} = \begin{cases} (C'^s_q)^{\{l\}} \cup (A'^s_n), & \text{if } l = 2n, \\ (C'^s_q)^{\{l\}}, & \text{otherwise.} \end{cases}$$

$$A'^{s+1}_n = A'^s_n \cup W_e(C_q \oplus \hat{\theta}_j(C'^{s+1}_q))^{\{q\}}\restriction[>m].$$

It is clear that $A'_n \leq_e C_q$.

Claim 6. $A'_n = A_n$.

Proof. We show that $A'^s_n \subseteq A_n$ and $C'^s_q \subseteq C$ for every s, by induction. Hence $A'_n \subseteq A_n$. Clearly $C'^0_q \subseteq C$ and $A'^0_n \subseteq A_n$. Assume

$C_q'^{s} \subseteq C$ and $A_n'^{s} \subseteq A_n$. Then $C_q'^{s+1} \subseteq C$ by definition of C. Therefore $A_n'^{s+1} \subseteq A_n \cup W_e(C_q \oplus \hat{\theta}_j(C))^{\{q\}}\restriction[>m] = A_n \cup \theta_j(C)^{\{q\}}\restriction[>m] = A_n$ (Claim 5), and we are done.

Since $W_e^s(C_q \oplus \hat{\theta}_j(C_q'^{s+1}))\restriction[>m] \subseteq A_n'^{s+1}$ for every s, $W_e(C_q \oplus \hat{\theta}_j(C_q'))^{\{q\}}\restriction[>m] \subseteq A_n'$. We show that $A\restriction z \subseteq A_n'$ for every z, by induction. Hence $A_n \subseteq A_n'$.

Assume $A_n\restriction z \subseteq A_n'$. By definition of $A_n'^{0}$, we can assume $z > m$. If $z \notin A_n$, we are done.

Suppose $z \in A_n\restriction[>m] = \theta_j(C)^{\{q\}}\restriction[>m] = W_e(C_q \oplus \hat{\theta}_j(C))^{\{q\}}\restriction[>m]$. Choose $s \in T$ such that $(s)_0 = q$, $z \in W_e^s(C_q^s \oplus \hat{\theta}_j^s(C^s))^{\{q\}}\restriction L(q,s)$, $\tilde{\theta}_{<q}^s(C^s) = \tilde{\theta}_{<q}(C)$ and $U(\tilde{\theta}_{<q},C,\tilde{\theta}_{<q}^s(C^s),s) = U(\tilde{\theta}_{<q},C,\tilde{\theta}_{<q}(C))$. Then $C_q^s \subseteq C_q$ and $C_{q,z}^s \subseteq C_{q,z}$. Therefore $E_z^s \subseteq U(\tilde{\theta}_{<q},C,\tilde{\theta}_{<q}(C)) \cup C_{q,z} \subseteq C_q'$ by the induction hypothesis. $z \in W_e^s(C_q^s \oplus \hat{\theta}_j^{s+1}(E_z^s))^{\{q\}}$ (Proposition 2), therefore $z \in W_e(C_q \oplus \hat{\theta}_j(C_q'))^{\{q\}}\restriction[>m] \subseteq A_n'$. So $A_n\restriction(z+1) \subseteq A_n'$, and we are done. □

Hence $A_n \leq_e C_q$, which is a contradiction. Therefore $\theta_j(C)^{\{q\}} \neq W_e(C_q \oplus \hat{\theta}_j(C))^{\{q\}}$, so 1 holds.

By 1 we can choose a least y such that $\theta_j(C)^{\{q\}}(y) \neq W_e(C_q \oplus \hat{\theta}_j(C))^{\{q\}}(y)$.

From the construction $w \in \theta_{j,q}(C)$ if and only if there exist $s \in T$, z such that $(s)_0 = q$, $w = \langle z,q \rangle$ and $z \in A_n^s\restriction(L(q,s)+1)$. However $\lim_{s \in T} L(q,s) = y$, therefore $\theta_{j,q}(C)$ is finite.

As in Case 1, Claim 2, we can show that for every z, there are only finitely many (q,z)-active stages. Choose a stage t such that

$$(\forall s \geq t)\ [\Theta_j(C)^{\{q\}}\restriction(y+1) \subseteq \Theta_j^s(C^s)^{\{q\}} \text{ and } W_e(C_q \oplus \hat{\Theta}_j(C))^{\{q\}}\restriction(y+1) \subseteq W_e^s(C_q^s \oplus \hat{\Theta}_j^s(C^s))^{\{q\}}].$$

Claim 7. *For every* $z > y$, *there are no* (q,z)-*active stages after stage* t.

Proof. Suppose not. Choose $z > y$ and a (q,z)-active stage $s+1 > t$. Note $L(q,s) > z > y$ and $E_y^s \subseteq E_z^s \subseteq C$.

Case 1. $y \in \Theta_j(C)^{\{q\}}$.

Then $y \in W_e^s(C_q^s \oplus \hat{\Theta}_j^s(C^s))^{\{q\}}\restriction L(q,s)$. Therefore $y \in W_e(C_q \oplus \hat{\Theta}_j(C))^{\{q\}}$ (Proposition 2), which contradicts the choice of y.

Case 2. $y \in W_e(C_q \oplus \hat{\Theta}_j(C))^{\{q\}}$.

Then $y \in \Theta_j^s(C^s)^{\{q\}}$. If $y \in \Theta_{j,<q}^s(C^s)^{\{q\}}$ then $\langle\langle y,q\rangle,j\rangle \in \tilde{\Theta}_{<q}^s(C^s) \subseteq \tilde{\Theta}_{<q}(C)$ (Proposition 2), so $y \in \Theta_{j,<q}(C)^{\{q\}} \subseteq \Theta_j(C)^{\{q\}}$, which contradicts the choice of y. Therefore $y \in \Theta_{j,q}^s(C^s)^{\{q\}}$ (Proposition 1). Then $y \in A_n^s$ (Claim 4), so $y \in (C_{q,z}^s)^{\{2n\}} \subseteq C_{q,z}^{\{2n\}} \subseteq A_n$ (Proposition 2). Choose $u \in T$ such that $q = (u)_0$, $L(q,u) = y$ and $y \in A_n^u$. Then $\langle C^u, \langle y,q\rangle\rangle$ is enumerated into $\Theta_{j,q}$ at stage $u+1$ with $C^u \subseteq C$. Therefore $y \in \Theta_j(C)^{\{q\}}$ which is a contradiction. □

Therefore $\theta_{k,q}(C)$ is finite for every $k < k_n$, $k \neq j$, so 2 holds. □

Hence all requirements are satisfied. ∎

Corollary 2.3.2. *For every pair of incomparable* Σ_2 *e-degrees* **a** *and* **b** *there exists a* Σ_2 *e-degree* **c** *such that* $\mathbf{c} \perp_e \mathbf{a}$ *and* $\mathbf{c} \perp_e \mathbf{b}$.

Proof. This is easily proved using the same techniques as those used to prove Lemma 2.3.1. ∎

CHAPTER III

A NON-SPLITTING E-DEGREE

§3.1 INTRODUCTION

Definition 3.1.1. A degree **a** is said to be *splitting* if there exists a pair of degrees **b** and **c** strictly below **a** with $\mathbf{a} = \mathbf{b} \vee \mathbf{c}$.

Every r.e. Turing degree is a splitting degree by the Sacks Splitting Theorem (see Soare [1987] (pp. 124-126)). In contrast to this, for the Σ_2 e-degrees, we have:

Theorem 3.1.2. *There exists a non-zero low non-splitting e-degree.*

The *lowness* of the non-splitting degree is needed to prove Theorem 4.1.1.

A feature of the proof worth noting is that while constructing a Σ_2 set A, we simultaneously attempt to construct e-reductions of A to $W(A)$ for various e-operators W. In general, given $W(A)$, the task of constructing an e-reduction of A to $W(A)$, (given that one exists), is a difficult one.

§3.2 PROOF OF THEOREM

Definition. Let $n = \langle n_0, n_1, n_2\rangle$. $V_n = W_{n_0}$, $\Omega_n = W_{n_1}$ and $\Theta_n = W_{n_2}$.

We show that there exists a non-r.e. low set A such that

$$\neg\left(\bigvee_{i=0}^{\infty} [A = V_i(\Omega_i(A) \oplus \Theta_i(A)) \text{ and } \Omega_i(A) \perp_e \Theta_i(A)]\right).$$

Specifically, we construct a Σ_2-approximation $\{A^s\}_{s\in\omega}$ to A and attempt to satisfy the following requirements, listed in order of priority.

N_0: $A \neq W_0$.

P_0: $\exists^{\infty} s\, [k \in W_j^s(A^s)] \Rightarrow k \in W_j(A)$, where $0 = \langle k, j\rangle$.

N_1: $A \neq W_1$.

P_1: $\exists^{\infty} s\, [k \in W_j^s(A^s)] \Rightarrow k \in W_j(A)$, where $1 = \langle k, j\rangle$.

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In addition, for every i, we construct e-operators Ψ_i and Φ_i and attempt to meet requirement R_i.

$$R_i: \ A \neq V_i(\Omega_i(A) \oplus \Theta_i(A)) \ \text{ or } \ A =^* \Psi_i(\Omega_i(A)) \ \text{ or } \ A =^* \Phi_i(\Theta_i(A)).$$

If all requirements N_i, P_i and R_i are met then it is clear that $\deg_e A$ satisfies the conditions of the theorem.

Definitions. $\{\Psi_i^s\}_{s\in\omega}$, $\{\Phi_i^s\}_{s\in\omega}$ and $\{\tilde{A}^s\}_{s\in\omega}$ are recursive enumerations of Ψ_i, Φ_i and $\tilde{A}$ respectively. At certain stages of the construction it is necessary to *dump* (permanently put) elements into A for the sake of requirements P_i and R_i. $\tilde{A}$ consists of all such elements. $\tilde{A}^s$ is the set of elements which have been enumerated into $\tilde{A}$ by the end of stage s.

F_Ω and F_Θ are binary partial recursive functions with range the set of finite sets. $\{F_\Omega^s\}_{s\in\omega}$ and $\{F_\Theta^s\}_{s\in\omega}$ are recursive approximations to F_Ω and F_Θ respectively, defined as follows. $F_\Omega^s(i,x)\downarrow = D$ ($F_\Theta^s(i,x)\downarrow = D$) if there is a stage $t \leq s$ at which $F_\Omega(i,x)$ ($F_\Theta(i,x)$) is explicitly defined to be D.

$$\Psi_i^s = \{\langle x,\emptyset\rangle : x \in \tilde{A}^s\} \cup \{\langle x,F_\Omega^s(i,x)\rangle : F_\Omega^s(i,x)\downarrow\}$$

and

$$\Phi_i^s = \{\langle x,\emptyset\rangle : x \in \tilde{A}^s\} \cup \{\langle x,F_\Theta^s(i,x)\rangle : F_\Theta^s(i,x)\downarrow\}.$$

Each requirement N_i has a *witness*. If x is a witness for N_i then we attempt to arrange that $A(x) \neq W_i(x)$. B denotes the set of all witnesses (at the end of the construction). $\{B^s\}_{s\in\omega}$ is a Δ_2-approximation to B, where

$$B^s = \left[\bigcup_{t\leq s} A^t\right] - \tilde{A}^s;$$

hence once an element is enumerated into $\tilde{A}$, it cannot be a witness for any requirement N_i. x_e^s denotes the e-th element of B^s (in the natural order), and is a witness for requirement N_e at stage s. $B^s_{\leq e} = \{x_n^s : n \leq e\}$. $B^s_{<e}$, $B^s_{\geq e}$ and $B^s_{>e}$ are defined analogously.

At every stage each witness has an associated i-state for every i. An i-state is an element of ${}^{i+1}2$ and is a technical tool for meeting requirements R_j, where $j \leq i$. It is not the ususal i-state of the maximal set construction. For every i, σ_i is a binary partial recursive function with range $\{0, 1\}$. $\sigma_i(e,s)\downarrow$ for every i if and only if $x_e^s\downarrow$; if $x_e^{s+1}\downarrow$ then $\sigma_i(e,s+1) = \sigma_i(e,s)$, unless otherwise specified. If $x = x_e^s$ for some e then the *i-state of x at stage* s is $(\sigma_0(e,s),\sigma_1(e,s),\ldots,\sigma_i(e,s))$. For $i \leq e$, $\sigma_i(e,s) = 0$ indicates that at stage s, x_e^s is part of our strategy to achieve $A =^* \Psi_i(\Omega_i(A))$. That is, we hope to arrange that $A(x_e^s) = \Psi_i(\Omega_i(A))(x_e^s)$. If $\sigma_i(e,s) = 1$ then x_e^s is part of our strategy to achieve $A =^* \Phi_i(\Theta_i(A))$. For a fixed i, i-states are ordered lexicographically.

At stage s+1 we say that x is *unused* if $x \notin \bigcup_{t\le s} A^t$.

Requirements R_i present the greatest difficulty. In order to gain insight into the full construction, suppose that we were only interested in satisfying requirements N_i, P_i for every i, and the single requirement R_0. Let us see how we could meet R_0 in a manner which would also allow us to meet the other requirements. For notational ease we omit the subscript 0 from the various e-operators.

All elements of ω enter A, in order, as witnesses. So $\bigcup_t A^t = \omega$; it follows from the definition of B that $\overline{\tilde{A}} = B$. $\tilde{A} \subseteq \Psi(\emptyset) \cap \Phi(\emptyset)$ from the definition of Ψ and Φ. So if we can arrange that in the case $A = V(\Omega(A) \oplus \Theta(A))$, $A \cap B =^* \Psi(\Omega(A)) \cap B$ or $A \cap B =^* \Phi(\Theta(A)) \cap B$, then R_0 is satisfied. Rather than monitoring lengths of agreement between A^s and $V^s(\Omega^s(A^s) \oplus \Theta^s(A^s))$, we simply base our actions on the assumption that $A = V(\Omega(A) \oplus \Theta(A))$. Assume $x \in B$. When x first enters A we try to arrange that $A(x) = \Psi(\Omega(A))(x)$, so the initial 0-state of x is (0). If there follows a stage s such that $A^s(x) = V^s(\Omega^s(A^s) \oplus \Theta^s(A^s))(x) = 1$, then at stage s+1 we set $F_\Omega(0,x) = \Omega^s(A^s)$ and $F_\Theta(0,x) = \Theta^s(A^s)$. Note that $x \in V(F_\Omega(0,x) \oplus F_\Theta(0,x))$. In addition we enumerate the members of A^s which are strictly greater than x into $\tilde{A}$. Now if there is a stage $t > s$ such that $A^{t+1}\upharpoonright x \ne A^t\upharpoonright x$, then x is enumerated into $\tilde{A}$ at stage $t+1$. But $x \notin \tilde{A}$ since $x \in B$, so $A^t\upharpoonright x = A^{s+1}\upharpoonright x = A\upharpoonright x$ for every $t > s$. Then $F_\Omega(0,x) \subseteq \Omega(A \cup \{x\})$; if $x \notin A$ we would like

$F_\Omega(0,x) \not\subseteq \Omega(A)$. If this holds then $A(x) = \Psi(\Omega(A))(x)$, from the definition of Ψ, and we can either put x into A, or remove it, for the sake of a requirement N_j. However, if at a later stage t we find that $x \notin A^t$ and $F_\Omega(0,x) \subseteq \Omega^t(A^t)$ then we next try to arrange that $A(x) = \Phi(\Theta(A))(x)$. So at stage $t+1$ we change the 0-state of x to (1), and we enumerate the members of A^t greater than x into $\tilde{A}$. Then $F_\Omega(0,x) \subseteq \Omega(A)$ and from the action taken at stage $s+1$ we have $F_\Theta(0,x) \subseteq \Theta(A \cup \{x\})$. Now assume $A = V(\Omega(A) \oplus \Theta(A))$. Since $x \in V(F_\Omega(0,x) \oplus F_\Theta(0,x))$, $x \notin A$ implies $F_\Theta(0,x) \not\subseteq \Theta(A)$. So $A(x) = \Phi(\Theta(A))(x)$ from the definition of Φ.

Once a witness y achieves a 0-state of (1), if $A = V(\Omega(A) \oplus \Theta(A))$ then $A(y) = \Phi(\Theta(A))(y)$. Due to this *foolproof* quality, witnesses of 0-state (0) are replaced by witnesses of 0-state (1) when these are available and associated with lower priority requirements. Hence at the end of the construction, either all but finitely many members of B have 0-state (0), or every member of B has 0-state (1). So requirement R_0 is satisfied. Once the witnesses associated with higher priority requirements have settled down, requirement P_j can be satisfied by dumping (if necessary) finitely many elements into A.

To satisfy requirement R_1 we simply repeat the same strategy on $B - \{\min B\}$, and so on. The use of i-states captures this idea.

Construction.

Stage 0. $A^0 = \emptyset$.

Stage s+1 (s ≡ 0 mod 6).

Choose the least $e \leq s$ such that $x_e^s\uparrow$. Set $A^{s+1} = A^s \cup \{x\}$ where x is the least unused element. Set $\sigma_i(e,s+1) = 0$ for every i.

Stage s+1 (s ≡ 1 mod 6).

Choose the least $e \leq s$ such that there exists $m > e$ such that x_m^s has a strictly greater e-state than x_e^s at stage s. Choose the least such m. Enumerate the members of $B_{\geq e}^s - \{x_m^s\}$ into $\tilde{A}$. Set $A^{s+1} = A^s \cup \tilde{A}^{s+1} \cup \{x_m^s\}$ and $\sigma_i(e,s+1) = \sigma_i(m,s)$ for every i.

Stage s+1 (s ≡ 2 mod 6).

Choose the least $e \leq s$ such that $A^s(x_e^s) = W_e^s(x_e^s) = 1$. Enumerate the members of $B_{>e}^s$ into $\tilde{A}$. Set $A^{s+1} = (A^s - \{x_e^s\}) \cup \tilde{A}^{s+1}$.

Stage s+1 (s ≡ 3 mod 6).

Let $i = (s)_0$. Choose the least e, $i \leq e \leq s$ such that $x_e^s \in A^s$, $x_e^s \in V_i^s(\Omega_i^s(A^s) \oplus \Theta_i^s(A^s))$ and $F_\Omega^s(i,x_e^s)\uparrow$. Set $F_\Omega(i,x_e^s) = \Omega_i^s(A^s)$ and $F_\Theta(i,x_e^s) = \Theta_i^s(A^s)$. Enumerate the members of $B_{>e}^s$ into $\tilde{A}$. Set $A^{s+1} = A^s \cup \tilde{A}^{s+1}$. We call s+1 an (x_e^s,Ψ_i)-*stage*.

Stage s+1 (s ≡ 4 mod 6).

Let $i = (s)_0$. Choose the least e, $i \leq e \leq s$ such that $x_e^s \notin A^s$, $x_e^s \in \Psi_i^s(\Omega_i^s(A^s))$ and $\sigma_i(e,s) = 0$. Enumerate the members of $B_{>e}^s$

into $\tilde{A}$. Set $A^{s+1} = A^s \cup \tilde{A}^{s+1}$ and $\sigma_i(e,s+1) = 1$. We call s+1 an (x_e^s,Φ_i)-*stage*.

Stage s+1 (s ≡ 5 mod 6).

Choose the least $e \leq s$ such that $k \in W_j^s(A^s \cup B_{>e}^s)$ and $k \notin W_j^s([B_{\leq e}^s \cap A^s] \cup \tilde{A}^s)$, where $e = \langle k,j\rangle$. Enumerate the members of $B_{>e}^s$ into $\tilde{A}$. Set $A^{s+1} = A^s \cup \tilde{A}^{s+1}$. We say P_e *receives attention* at stage s+1.

Note. At each stage s+1 we are asked to choose an element e satisfying a given set of conditions. Henceforth we refer to this element as $e(s)$. If $e(s)\uparrow$, we do nothing at stage s+1.

End of construction.

Proposition 1.

.1. $\bar{\tilde{A}} = B$.

.2. *For every* $x \in B$ *there exist* e *and* t *such that* $x = x_e^s$ *for every* $s \geq t$.

Proof. By the action taken at stages s+1 (s ≡ 0 mod 6) of the construction, $\bigcup_t A^t = \omega$. 1 is immediate from the definition of $\{B^s\}_{s\in\omega}$. 2 follows from the fact that this is a Δ_2-approximation to B. □

Proposition 2.

.1. $\tilde{A} \subseteq A$.

.2. $\{A^s\}_{s\in\omega}$ *is a* Δ_2*-approximation to* A.

Proof. From the construction $\tilde{A}^t \subseteq A^s$ for every $s \geq t$. So 1 holds.

Fix $x \in B$. Choose e and t such that $x = x_e^s$ for every $s \geq t$ (Proposition 1). It suffices to show that for every $s \geq t$, if $x \notin A^s$ then $x \notin A^{s+1}$. Suppose not. Choose $s \geq t$ such that $x \notin A^s$ and $x \in A^{s+1}$. An inspection of the construction shows that $s \equiv 1 \bmod 6$, $e(s)\downarrow < e$ and $x_{e(s)}^{s+1} = x$. This contradicts the choice of e and t.
□

Proposition 3. *For every* m, e, t, *if* $m < e$, $x_e^t\downarrow \in B$ *and* $x_m^t \in A^t$ *then for every* $s \geq t$, $x_m^t \in A^s$.

Proof. Suppose not. Choose m, e and t satisfying the hypothesis and a least s such that $s \geq t$ and $x_m^t \notin A^{s+1}$. Then $x_m^t \notin \tilde{A}^s$ and since $x_e^t \in B$, $x_e^t \notin \tilde{A}^s$. Therefore $x_m^t = x_{m'}^s$ and $x_e^t = x_{e'}^s$ for some $m' < e'$. By the choice of s, $x_{m'}^s \in A^s$ and $x_{m'}^s \notin A^{s+1}$. An inspection of the construction shows that $s \equiv 2 \bmod 6$, $e(s)\downarrow = m'$ and $B_{>m'}^s \subseteq \tilde{A}^{s+1}$. But then $x_{e'}^s \in \tilde{A}^{s+1}$, which is a contradiction. □

Proposition 4.

.1. F_Ω *and* F_Θ *are well-defined.*

.2. *For every* x, i, *there is at most one* (x,Ψ_i)*-stage.*

Proof. 1–2 follow from the observation that $s+1$ is an (x,Ψ_i)-stage if and only if $F_\Omega^s(i,x)\uparrow$, $F_\Theta^s(i,x)\uparrow$, $F_\Omega^{s+1}(i,x)\downarrow$ and $F_\Theta^{s+1}(i,x)\downarrow$. $\square$

Proposition 5. *For every* x, t, *if* $x \in B$ *and* $t+1$ *is an* (x,Ψ_i)*-stage for some* i *then for every* $s > t$, $A^t - \{x\} \subseteq A^s$.

Proof. Assume $x \in B$, $t+1$ is an (x,Ψ_i)-stage and $s > t$. From stage $t+1$ we have $x = x^t_{e(t)}$ and $B^t_{>e(t)} \subseteq \tilde{A}^{t+1} \subseteq A^s$. Suppose $m < e(t)$ and $x^t_m \in A^t$. Then $x^t_m \in A^s$ (Proposition 3). Therefore $A^t - \{x\} \subseteq A^s$. $\square$

Proposition 6. *For every* x, s, i, *if* $x \in B$, $x = x^s_e$ *for some* e *and* $\sigma_i(e,s) = 1$ *then there exist* u *and* v, $u < v < s$ *such that*

.1. $v+1$ *is an* (x,Φ_i)*-stage,*

.2. $u+1$ *is an* (x,Ψ_i)*-stage,*

.3. $F_\Omega^{u+1}(i,x)\downarrow$ *and* $F_\Theta^{u+1}(i,x)\downarrow$,

.4. $F_\Omega(i,x) \subseteq \Omega_i^v(A^v)$,

.5. $A^v \subseteq A^t$ *for every* $t \geq v$.

Proof. Assume x, s and i satisfy the hypothesis. Choose $\upsilon < s$ least such that $x = x_m^{\upsilon+1}$ for some m and $\sigma_i(m,\upsilon+1) = 1$. An inspection of the construction shows that $\upsilon+1$ must be an (x,Φ_i)-stage, $x = x_m^{\upsilon}$, $x \notin A^{\upsilon}$, $x \in \Psi_i^{\upsilon}(\Omega_i^{\upsilon}(A^{\upsilon}))$ and $B_{>m}^{\upsilon} \subseteq \tilde{A}^{\upsilon+1}$. So Ψ_i^{υ} must contain an instruction $\langle x,F\rangle$ where $F \subseteq \Omega_i^{\upsilon}(A^{\upsilon})$. Since $x \notin \tilde{A}$, it follows from the definition of Ψ_i^{υ} that $F_{\Omega}^{\upsilon}(i,x)\downarrow$ and $F = F_{\Omega}(i,x)$. Choose u least such that $F_{\Omega}^{u+1}(i,x)\downarrow$. From the proof of Proposition 4, $u+1$ must be an (x,Ψ_i)-stage and $F_{\theta}^{u+1}(i,x)\downarrow$.

Fix $t \geq \upsilon$. Since $x \notin A^{\upsilon}$ and $B_{>m}^{\upsilon} \subseteq \tilde{A}^{\upsilon+1}$, in order to show that $A^{\upsilon} \subseteq A^t$ it suffices to show that $B_{<m}^{\upsilon} \cap A^{\upsilon} \subseteq A^t$. This follows from Proposition 3. □

Proposition 7. *For every* e,

.1. $x_e =_{dfn} \lim_s x_e^s$ *exists,*

.2. $\sigma_i^e =_{dfn} \lim_s \sigma_i(e,s)$ *exists for every* $i \leq e$,

.3. $A(x_e) \neq W_e(x_e)$,

.4. *requirement* P_e *receives attention only finitely often,*

.5. $\exists^{\infty} s\ [k \in W_j^s(A^s)] \Rightarrow k \in W_j(A)$, *where* $e = \langle k,j\rangle$.

Proof. Assume that 1-5 hold for every $e < e'$. We show that 1-5 hold for $e = e'$. Choose a stage υ such that for every $e < e'$:

1'. x_e^s has reached a limit by stage υ,

2'. for every $i \leq e$, $\sigma_i(e,s)$ has reached a limit by stage υ, 3'. $A^s(x_e)$ and $W_e^s(x_e)$ have reached a limit by stage υ (Proposition 2) and $A(x_e) \neq W_e(x_e)$,

4'. requirement P_e does not receive attention after stage υ,

5'. for every $i \leq e$, there are no (x_e, Ψ_i)-stages after stage υ (Proposition 4).

An inspection of stages s+1 ($s \equiv 0 \bmod 6$), where $s > \upsilon$ shows that $x_{e'}^{s+1}\downarrow$ for each such stage. Now if $s > \upsilon$ and $x_{e'}^{s+1} \neq x_{e'}^{s}\downarrow$ then $s \equiv 1 \bmod 6$, $x_{e'}^{s+1}\downarrow$ and the e'-state of $x_{e'}^{s+1}$ is strictly greater than the e'-state of $x_{e'}^{s}$. Also if $s > \upsilon$, $x_{e'}^{s+1} = x_{e'}^{s}\downarrow$ and the e'-state of $x_{e'}^{s}$ at stage s+1 is different from its e'-state at stage s, then $s \equiv 4 \bmod 6$ and the e'-state of $x_{e'}^{s}$ at stage s+1 is greater than its e'-state at stage s. The last two observations follow from the choice of υ and an inspection of the construction. Since there are only finitely many e'-states, 1-2 hold.

By 1 we can choose a least t such that $x_{e'}^{s} = x_{e'}^{t+1}$ for every $s > t$. From an inspection of the construction: $e(t)\downarrow = e'$, and $t \equiv 0 \bmod 6$ or $t \equiv 1 \bmod 6$. In either case $x_{e'}^{t+1} \in A^{t+1}$. If $x_{e'} \notin W_{e'}$, then $x_{e'}$ is never later removed from A. Otherwise, for every $s > \max\{\upsilon, t\}$ ($s \equiv 2 \bmod 3$), if $x_{e'} \in W_{e'}^{s}$, then $x_{e'} \notin A^{s+1}$, so $x_{e'} \notin A$. Therefore 3 holds.

Hence we may assume that υ is chosen in such a way that in addition 1'-3' hold for $e = e'$. Let $k = (e')_0$ and $j = (e')_1$.

Claim.

.1. *If* $P_{e'}$ *receives attention at stage* s *then* $k \in W_j^s([B^s_{\le e'} \cap A^s] \cup \tilde{A}^s)$.

.2. *If* $k \in W_j^t([B^t_{\le e'} \cap A^t] \cup \tilde{A}^t)$ *for some* $t > v$ *then*

.1. $k \in W_j^s([B^s_{\le e'} \cap A^s] \cup \tilde{A}^s)$ *for every* $s \ge t$,

.2. $k \in W_j(A)$.

Proof. 1 follows from an inspection of the construction. By the choice of v, $B^s_{\le e'} \cap A^s = \{x_m: m \le e'\} \cap A$ for every $s > v$. Therefore 2 holds. □

4 follows from the claim and the observation that if $P_{e'}$ receives attention at stage s+1 then $k \notin W_j^s([B^s_{\le e'} \cap A^s] \cup \tilde{A}^s)$.

Assume $\exists^\infty s\ [k \in W_j^s(A^s)]$. Then we can choose $t > v$ $(t \equiv 5 \bmod 6)$ such that $k \in W_j^t(A^t \cup B^t_{>e'})$. From stage t+1, either $k \in W_j^t([B^t_{\le e'} \cap A^t] \cup \tilde{A}^t)$ or $P_{e'}$ receives attention. So 5 follows from the claim. □

Definition. If $x = x_e$ for some e and $i \le e$ then the *eventual* i-state of x is $(\sigma_0^e,\dots,\sigma_i^e) = \lim_s (\sigma_0(e,s),\dots,\sigma_i(e,s))$.

Proposition 8. *For every* i, e, e', *if* $i \le e < e'$ *then the eventual* i-*state of* x_e *is greater than or equal to the eventual* i-*state of* $x_{e'}$.

Proof. Suppose not. Choose $i \leq e < e'$ giving a contradiction. Choose a stage $t \equiv 1 \bmod 6$ such that x_e^s, $x_{e'}^s$, and the i-states of x_e and $x_{e'}$ have reached a limit by stage t (Proposition 7). Then the e-state of $x_{e'}^t$ is greater than the e-state of x_e^t at stage t. From stage $t+1$ we see that $e(t)\downarrow \leq e$. Then $x_e^{t+1} \neq x_e^t$ by the action taken at stage $t+1$, which contradicts the choice of t. □

Corollary 9. *For every i there exists $e' \geq i$ such that for every $e > e'$, $\sigma_i^e = \sigma_i^{e'}$.*

Proof. It suffices to show that there exists $e' \geq i$ such that for every $e > e'$ the eventual i-states of x_e and $x_{e'}$ are the same. This follows from Proposition 8 and the fact that there are only finitely many i-states. □

Proposition 10. *For every i, $A \neq V_i(\Omega_i(A) \oplus \Theta_i(A))$ or $A =^* \Psi_i(\Omega_i(A))$ or $A =^* \Phi_i(\Theta_i(A))$.*

Proof. Fix i. If $A \neq V_i(\Omega_i(A) \oplus \Theta_i(A))$ we are done, so assume equality. Choose $e' \geq i$ such that for every $e > e'$, $\sigma_i^e = \sigma_i^{e'}$ (Corollary 9).

Case 1. $\sigma_i^{e'} = 0$.

Then $A =^* \Psi_i(\Omega_i(A))$. For every x, $x \in \tilde{A}$ or $x = x_e$ for some e (Proposition 1). Since $\tilde{A} \subseteq \Psi_i(\emptyset)$ it suffices to show that for every $e \geq e'$,

$$A(x_e) = \Psi_i(\Omega_i(A))(x_e).$$

Let $e \geq e'$. Choose a stage v such that for every $k \leq e$, x_k^s and the k-states of x_k have reached a limit by stage v (Proposition 7), and there are no (x_k, Ψ_i)-stages after stage v (Proposition 4). Since $\{A^s\}_{s \in \omega}$ is a low approximation (Proposition 7), we can assume in addition that for every $k \leq e$, $A^s(x_k)$, $V_i^s(\Omega_i^s(A^s) \oplus \Theta_i^s(A^s))(x_k)$ and $\Psi_i^s(\Omega_i^s(A^s))(x_k)$ have reached a limit by stage v.

Assume $x_e \in A$. We show that $F_\Omega(i, x_e)\downarrow$. Choose a stage $t > v$ ($t \equiv 3 \bmod 6$) such that $(t)_0 = i$. By the choice of v, $x_e \in A^t$, $x_e \in V_i^t(\Omega_i^t(A^t) \oplus \Theta_i^t(A^t))$ and $t+1$ is not an (x_k, Ψ_i)-stage for any $k \leq e$. From stage $t+1$ we see that $F_\Omega^t(i, x_e^t)\downarrow$. Let $u+1$ be the unique (x_e, Ψ_i)-stage (Proposition 4). Then $F_\Omega(i, x_e) = \Omega_i^u(A^u)$. $x_e \in A$ implies $A^u \subseteq A$ (Proposition 5). Therefore $x_e \in \Psi_i(\Omega_i(A))$.

Assume $x_e \notin A$ and $x_e \in \Psi_i(\Omega_i(A))$. Choose a stage $t > v$ ($t \equiv 4 \bmod 6$) such that $(t)_0 = i$. By the choice of v, $x_e \notin A^t$, $x_e \in \Psi_i^t(\Omega_i^t(A^t))$ and $\sigma_i(e, t) = 0$. Inspecting stage $t+1$ of the construction, we see that $i \leq e(t)\downarrow \leq e$, $\sigma_i(e(t), t) = 0$ and $\sigma_i(e(t), t+1) = 1$. But this contradicts the choice of v. Therefore $x_e \notin \Psi_i(\Omega_i(A))$.

Case 2. $\sigma_i^{e'} = 1$.

We show that $A =^* \Phi_i(\theta_i(A))$. As in case 1 it suffices to show that for every $e \geq e'$, $A(x_e) = \Phi_i(\theta_i(A))(x_e)$. Let $e \geq e'$. Choose a stage v such that x_e^s and $\sigma_i(e,s)$ have reached a limit by stage v. Then there exists an (x_e, Ψ_i)-stage $u+1$ (Proposition 6), and $F_\theta(i, x_e) = \theta_i^u(A^u)$.

Assume $x_e \in A$. Then $A^u \subseteq A$ (Proposition 5). Therefore $x_e \in \Phi_i(\theta_i(A))$.

Assume $x_e \notin A$ and $x_e \in \Phi_i(\theta_i(A))$. Inspecting stage $u+1$ of the construction, we see that $x_e \in V_i^u(F_\Omega(i, x_e) \oplus F_\theta(i, x_e))$. $F_\Omega(i, x_e) \subseteq \Omega_i(A)$ (Proposition 6). Since $x \notin \tilde{A}$, $x_e \in \Phi_i(\theta_i(A))$ implies $F_\theta(i, x_e) \subseteq \theta_i(A)$. But then $x_e \in V_i(\Omega_i(A) \oplus \theta_i(A))$. This contradicts the assumption that $A = V_i(\Omega_i(A) \oplus \theta_i(A))$. Therefore $x_e \notin \Phi_i(\theta_i(A))$.

□

Hence all requirements N_i, P_i and R_i are satisfied (Propositions 7 and 10).

CHAPTER IV

A SPECIAL PAIR OF Σ_2 E-DEGREES

§4.1 INTRODUCTION

Theorem 3.1.2 suggests that it may be possible to prove:

Theorem 4.1.1. *There exists a pair of incomparable* Σ_2 *e-degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that for every* $\mathbf{z} <_e \mathbf{a}$, $\mathbf{z} \leq_e \mathbf{b}$.

This naturally leads to the question of whether such a situation can be symmetric. This is answered by:

Theorem 4.1.2. *For every pair of distinct* Σ_2 *e-degrees* $\mathbf{a}$ *and* $\mathbf{b}$, $\{\mathbf{z}: \mathbf{z} <_e \mathbf{a}\} \neq \{\mathbf{z}: \mathbf{z} <_e \mathbf{b}\}$.

§4.2 PROOF OF THEOREM 4.1.1

Definition 4.2.1. A sequence of Σ_2 sets $\{B_n\}_{n\in\omega}$ is *uniformly* Σ_2 if there is a recursive function $f(n,s,x)$ such that for every n, $\{B_n^s\}_{s\in\omega}$ is a Σ_2-approximation to B_n, where $B_n^s(x) = f(n,s,x)$.

We let **a** be a low non-splitting e-degree (Theorem 3.1.2). Due to the density of the Σ_2 e-degrees it suffices to show that there exists a Σ_2 e-degree **b** such that $\mathbf{a} \not\leq_e \mathbf{b}$ and for every $\mathbf{z} <_e \mathbf{a}$, $\mathbf{z} \leq_e \mathbf{b}$.

Let $A \in \mathbf{a}$. We first construct a uniformly Σ_2 sequence of Σ_2 sets, $B_0 \leq_e B_1 \leq_e \ldots$, strictly below A, such that for every $Z <_e A$, $Z \leq_e B_i$ for some i. We then construct a Σ_2 set B such that $A \not\leq_e B$ and $B_i \leq_e B$ for all i. $\mathbf{b} = \deg_e B$ is the desired degree.

In Lemmas 4.2.2 and 4.2.3 the constructions are carried out in a more general setting, yielding Corollary 4.2.4, of which the theorem is an immediate consequence.

Lemma 4.2.2. *For every non-r.e. low set* A *there exists a uniformly* Σ_2 *sequence of* Σ_2 *sets*, $B_0 \leq_e B_1 \leq_e \ldots$, *strictly below* A, *such that for every* $Z \leq_e A$, $Z \leq_e B_i$ *or* $B_i \oplus Z \equiv_e A$ *for some* i.

Proof. Let A be non-r.e. and low. For notational convenience, rather than constructing each set B_i separately, we construct a single Σ_2 set B, and show that there exists a uniformly Σ_2 sequence of Σ_2 sets, $B_0 \leq_e B_1 \leq_e \dots$, below B such that for every i,

$$B_i <_e A \text{ and } \exists j\, [W_i(A) \leq_e B_j \text{ or } A \leq_e B_j \oplus W_i(A)],$$

which proves the lemma.

Ideally, we would like to set $B_i = Y_i$ where $\{Y_i\}_{i\in\omega}$ is defined as follows. Set $Y_0 = \emptyset$. Set $Y_1 = Y_0 \oplus W_0(A)$ if $A \nleq_e Y_0 \oplus W_0(A)$, and Y_0 otherwise. Set $Y_2 = Y_1 \oplus W_1(A)$ if $A \nleq_e Y_1 \oplus W_1(A)$, and Y_1 otherwise, and so on. It is clear that $\{B_i\}_{i\in\omega}$ would satisfy all the requirements of the lemma except, possibly, that the sequence be uniformly Σ_2. However, this obstacle can be overcome by noting that it suffices to have for every i, $B_i \leq_e Y_j$ for some j, and $Y_i \leq_e B_k$ for some k.

Definitions.

$$p_i = \begin{cases} -1, & \text{if } A \nleq_e Y_i \oplus W_i(A), \\ \mu k\, [A = W_k(Y_i \oplus W_i(A))], & \text{otherwise.} \end{cases}$$

$\tau_{-1} = \emptyset$ and $\tau_i = (p_0, \dots, p_i)$. $\mathcal{S} = {}^{<\omega}(\omega \cup \{-1\})$ consists of all possible values of τ_i and $\mathcal{S}' = \mathcal{S} - \{\emptyset\}$. $\sigma \to c(\sigma)$ denotes an arbitrary fixed recursive bijection from $\mathcal{S}'$ to ω, and $i \to \sigma_i$ its inverse.

We construct B so that $B^{\{j\}} = W_k(A)$ if $\sigma_j = \tau_k$ and $p_k = -1$, and is finite otherwise. $B_i = \bigoplus_{lh(\sigma_j) \leq i+1} B^{\{j\}}$. It is easy to check that for every i, $B_i \leq_e Y_j$ for some j, and $Y_i \leq_e B_k$ for some k.

For each $\sigma \in \mathcal{S}$ we define an e-operator Ψ_σ by induction on $lh(\sigma)$.

$\Psi_\emptyset = \emptyset$.

$lh(\sigma) > 0$:

$$\Psi_\sigma = \begin{cases} \Psi_{\sigma^-}, & \text{if } \sigma(e(\sigma)) > -1, \\ \Psi_{\sigma^-} \oplus W_{e(\sigma)}, & \text{otherwise.} \end{cases}$$

Proposition 1. $Y_i = \Psi_{\tau_{i-1}}(A)$.

Proof. The proof is by induction on i. $\tau_{-1} = \emptyset$, so $\Psi_{\tau_{-1}}(A) = \emptyset = Y_0$. Assume $Y_i = \Psi_{\tau_{i-1}}(A)$. If $A \not\leq_e Y_i \oplus W_i(A)$, then $Y_{i+1} = Y_i \oplus W_i(A)$, $\tau_i(i) = p_i = -1$, and $\Psi_{\tau_i}(A) = \Psi_{\tau_{i-1}}(A) \oplus W_i(A) = Y_i \oplus W_i(A)$ by the induction hypothesis. Otherwise $Y_{i+1} = Y_i$, $\tau_i(i) = p_i > -1$, and $\Psi_{\tau_i}(A) = \Psi_{\tau_{i-1}}(A) = Y_i$ by the induction hypothesis. □

Requirements.

We construct a Σ_2-approximation $\{B^s\}_{s\in\omega}$ to B and attempt to satisfy the following requirements.

$$P_i:\quad B^{\{c(\tau_i)\}} = \begin{cases} W_i(A), & \text{if } p_i = -1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

N_i: $(\forall \gamma \in \mathcal{S}')\,[\mathrm{lh}(\gamma) \geq i+1,\ \tau_{i-1} \subseteq \gamma$ and $\gamma(i) \neq p_i \Rightarrow B^{\{c(\gamma)\}}$ is finite].

In order of priority the requirements are $P_0, N_0, P_1, N_1, \ldots$ If these requirements are met then it is clear from the preceding discussion that $\{B_i\}_{i\in\omega}$ satisfies the conclusion of the lemma.

Let $\{A^s\}_{s\in\omega}$ be a low approximation to A, and for every $\sigma \in \mathcal{S}$, let $\{\Psi^s_\sigma\}_{s\in\omega}$ denote the natural recursive enumeration of Ψ_σ.

Length of agreement functions.

For every $\sigma \in \mathcal{S}'$,

$$l(\sigma,s) = \begin{cases} s, & \text{if } \sigma(e(\sigma)) = -1, \\ \mu z\,[z = s \text{ or } A^s(z) \neq W^s_{\sigma(e(\sigma))}(\Psi^s_{\sigma^-}(A^s) \oplus W^s_{e(\sigma)}(A^s))(z)], & \text{otherwise.} \end{cases}$$

$$m(\sigma,s) = \max\,\{l(\sigma,t): t \leq s\}.$$

$$L(\sigma,s) = \min\,\{m(\gamma,s): \gamma \in \mathcal{S}' \text{ and } \gamma \subseteq \sigma\}.$$

Definition. For $\sigma \in \mathcal{S}'$, we say that stage s is *σ-expansionary* if $(s)_0 = c(\sigma)$ and for every $t < s$, $(t)_0 = c(\sigma)$ implies $L(\sigma,t) < L(\sigma,s)$.

At a σ-expansionary stage there is *evidence* that $\tau_i = \sigma$ where $i = e(\sigma)$. So at stages succeeding σ-expansionary ones, we take action based on the assumption that $\tau_i = \sigma$.

Construction.

Stage 0.

$B^0 = \emptyset$.

Stage s+1.

Let $\sigma = \sigma_{(s)_0}$ and $i = e(\sigma)$.

Case 1. s is σ-expansionary.

Subcase 1.1. $\sigma(i) = -1$.

$(B^{s+1})^{\{c(\sigma)\}} = W_i^s(A^s)$ and $(B^{s+1})^{\{j\}} = (B^s)^{\{j\}}$ for $j \neq c(\sigma)$.

Subcase 1.2. $\sigma(i) > -1$.

$(B^{s+1})^{\{j\}} = (B^s)^{\{j\}}\restriction\sigma(i)$ for every j such that $\mathrm{lh}(\sigma_j) \geq i+1$, $\sigma^- \subsetneq \sigma_j$ and $\sigma_j(i) = -1$.

$(B^{s+1})^{\{j\}} = \emptyset$ for every j such that $\mathrm{lh}(\sigma_j) \geq i+1$, $\sigma^- \subsetneq \sigma_j$ and $\sigma_j(i) > \sigma(i)$.

$(B^{s+1})^{\{j\}} = (B^s)^{\{j\}}$ for all remaining j.

Case 2. Otherwise,

$B^{s+1} = B^s$.

End of construction.

Remark. For every $\sigma \in \mathcal{S}'$, if $\sigma(e(\sigma)) \neq -1$ then $B^{\{c(\sigma)\}} = \emptyset$.

Proposition 2. *For every* $\sigma \in \mathcal{S}'$, $\{m(\sigma,s): s \in \omega\}$ *is infinite if and only if* $\sigma(e(\sigma)) = -1$ *or* $A = W_{\sigma(e(\sigma))}(\Psi_{\sigma^-}(A) \oplus W_{e(\sigma)}(A))$.

Proof. This is immediate from the definition of $l(\sigma,s)$ and the fact that $\{A^s\}_{s\in\omega}$ is a low approximation. □

Corollary 3. *For every* $\sigma \in \mathcal{S}'$, *if*

.1. $lh(\sigma) = i+1$,

.2. $\tau_{i-1} \subseteq \sigma$,

.3. $\sigma(i) \neq p_i$,

.4. $p_i = -1$ *or* $-1 < \sigma(i) < p_i$,

then $\{m(\sigma,s): s \in \omega\}$ *is finite.*

Proof. Suppose not. Choose $\sigma \in \mathcal{S}'$ and i satisfying 1-4 with

$\{m(\sigma,s)\colon s \in \omega\}$ infinite. Then $\sigma(i) \neq -1$ by 3 and 4, therefore $A = W_{\sigma(i)}(\Psi_{\sigma^-}(A) \oplus W_i(A))$ (Proposition 2). But $\Psi_{\sigma^-}(A) = \Psi_{\tau_{i-1}}(A) = Y_i$ by 1, 2 and Proposition 1. So $A = W_{\sigma(i)}(Y_i \oplus W_i(A))$.

Case 1. $p_i = -1$.

Then $A \not\leq_e Y_i \oplus W_i(A)$ by definition of p_i, which is a contradiction.

Case 2. $p_i > -1$.

Then $\sigma(i) < p_i$, by 4, which contradicts $p_i = \mu k\ [A = W_k(Y_i \oplus W_i(A))]$. □

Proposition 4. *Requirements* P_i *and* N_i *are satisfied.*

Proof. Fix i. It follows from the definition of τ_i and Propositions 1 and 2 that there are infinitely many τ_i-expansionary stages. Suppose $\gamma \in \mathcal{S}'$, $lh(\gamma) \geq i+1$, $\tau_{i-1} \subseteq \gamma$, and $\gamma(i) \neq p_i$. Let $\gamma' = \gamma \restriction (i+1)$.

Case 1. $p_i = -1$.

Then $\{m(\gamma',s)\colon s \in \omega\}$ is finite (Corollary 3). Hence $\{L(\gamma,s)\colon s \in \omega\}$ is bounded, by definition of $L(\gamma,s)$, so there are only finitely many γ-expansionary stages. Since $B^{\{c(\gamma)\}}$ can only grow at stages succeeding γ-expansionary ones, it must be finite. Therefore N_i is satisfied.

An inspection of the construction shows that $(B^{s+1})^{\{c(\tau_i)\}} = W_i^s(A^s)$ for infinitely many s. Fix j. Since $\{A^s\}_{s\in\omega}$ is a low approximation, $(B^{s+1})^{\{c(\tau_i)\}}\restriction j = W_i(A)\restriction j$ for infinitely many s. If we can show that $W_i(A)\restriction j \subseteq B^{\{c(\tau_i)\}}$, then $B^{\{c(\tau_i)\}}\restriction j = W_i(A)\restriction j$. Inspecting the construction, we see that it suffices to show that

$$\{L(\sigma,s)\colon \sigma \in \mathcal{S}',\ \ i+1 \geq \mathrm{lh}(\sigma),\ \ \sigma^- \subseteq \tau_i,\ \ p_{e(\sigma)} = -1 \text{ and } j > \sigma(e(\sigma)) > -1\}$$

and

$$\{L(\sigma,s)\colon \sigma \in \mathcal{S}',\ \ i+1 \geq \mathrm{lh}(\sigma),\ \ \sigma^- \subseteq \tau_i \text{ and } p_{e(\sigma)} > \sigma(e(\sigma)) > -1\}$$

are finite. But this follows from Corollary 3. Since the choice of j was arbitrary, $B^{\{c(\tau_i)\}} = W_i(A)$, so P_i is satisfied.

Case 2. $p_i > -1$.

Subcase 2.1. $\gamma'(i) = -1$.

By the action taken at stages succeeding τ_i-expansionary ones, $B^{\{c(\gamma)\}} = B^{\{c(\gamma)\}}\restriction p_i$.

Subcase 2.2. $-1 < \gamma'(i) < p_i$.

As in the proof that N_i is satisfied in Case 1, $B^{\{c(\gamma)\}}$ is finite.

Subcase 2.3. $p_i < \tau'(i)$.

By the action taken at stages succeeding τ_i-expansionary ones, $B^{\{c(\tau)\}} = \emptyset$.

Therefore N_i is satisfied.

By the Remark following the construction, $B^{\{c(\sigma)\}} = \emptyset$ for every $\sigma \in \mathcal{S}'$ such that $\sigma(e(\sigma)) \neq -1$. Therefore P_i is satisfied. □

Hence the lemma is proved. ∎

Lemma 4.2.3. *If* $B_0 \leq_e B_1 \leq_e \dots$ *is a uniformly* Σ_2 *sequence of* Σ_2 *sets,* $A \in \Sigma_2$ *and* $A \not\leq_e B_i$ *for all* i, *then there exists a* Σ_2 *set* B *such that* $A \not\leq_e B$ *and* $B_i \leq_e B$ *for all* i.

Proof. Assume A and $\{B_i\}_{i\in\omega}$ satisfy the hypothesis. We first define an auxiliary set $\tilde{B} \subseteq B$, by setting $\tilde{B}^{\{i\}} = B_i$ for every i. B is obtained by adding finitely many elements to each column of $\tilde{B}$, in order to satisfy $A \not\leq_e B$, as follows.

Since $A \not\leq_e B_i$, A is non-r.e., so $A \neq W_0(\omega)$. Set $z_0 = \mu z\,[A(z) \neq W_0(\omega)(z)]$. Choose a finite set F_1 such that $W_0(\omega)(z_0) = W_0(F_1)(z_0)$, and put the elements of F_1 into B. No other elements are added to the 0-th column of B. Now $B^{\{0\}} =^* \tilde{B}^{\{0\}} = B_0$. Since

$A \not\leq_e B_0$ and $B^{\{0\}} \equiv_e B^{[0]} \cup \omega^{[\geq 1]}$, $A \neq W_1(B^{[0]} \cup \omega^{[\geq 1]})$. Set $z_1 = \mu z\,[A(z) \neq W_1(B^{[0]} \cup \omega^{[\geq 1]})(z)]$. Choose a finite set $F_2 \subseteq \omega^{[\geq 1]}$ such that $W_1(B^{[0]} \cup \omega^{[\geq 1]})(z_1) = W_1(B^{[0]} \cup F_2)(z_1)$, and put the elements of F_2 into B, and so on. For every i, it is clear that $(B - \tilde{B})^{[i]}$ is finite, and $W_i(B)(z_i) = W_i(B^{[<i]} \cup \omega^{[\geq i]})(z_i) \neq A(z_i)$. So $A \not\leq_e B$ and $B_i \leq_e B$ for every i, since $B_i = \tilde{B}^{\{i\}} =^* B^{\{i\}}$.

We construct a Σ_2-approximation $\{B^s\}_{s\in\omega}$ to B and attempt to satisfy the following requirements, listed in order of priority.

S: $\tilde{B} \subseteq B$.

P_0: $A \neq W_0(B)$.

N_0: $(B - \tilde{B})^{[0]}$ is finite.

P_1: $A \neq W_1(B)$.

N_1: $(B - \tilde{B})^{[1]}$ is finite.

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From the definition of $\tilde{B}$ it is clear that if these requirements are met then the lemma is proved.

Let $\{A^s \oplus \tilde{B}^s\}_{s\in\omega}$ be a Σ_2-approximation to $A \oplus \tilde{B}$ with infinitely many true stages, T.

Construction.

Stage 0.

$B^0 = \emptyset$.

Stage s+1.

For each $i \leq s+1$, we define a finite set F_i^{s+1} as follows.

$$F_0^{s+1} = \tilde{B}^s \cup (\cup \{B^{t+1}: t < s \text{ and } A^t \oplus \tilde{B}^t \subseteq A^s \oplus \tilde{B}^s\}).$$

In order to define F_{i+1}^{s+1}, set $E_{i,s+1}^t = B^t$ for every $t \leq s$, and $E_{i,s+1}^{s+1} = F_0^{s+1} \cup \omega^{[\geq i]}$. Then

$$F_{i+1}^{s+1} = \cup \{u(W_i, E_{i,s+1}, x, s+1) - F_0^{s+1}: x \leq z_i^s \text{ and } u(W_i, E_{i,s+1}, x, s+1)\downarrow\},$$

where

$$z_i^s = \mu z\, [z = s \text{ or } A^s(z) \neq W_i^{s+1}(E_{i,s+1}^{s+1})(z)].$$

Now $B^{s+1} = \bigcup_{i \leq s+1} F_i^{s+1}$.

End of construction.

Definition. Let $B' = \bigcup_{s \in T} B^{s+1}$.

Proposition 1.

.1. $\tilde{B} \subseteq B$.

.2. $B' = B$.

Proof. 1 is immediate from the construction.

$B \subseteq B'$ since $B = \{x: \exists t\ (\forall s > t)\ [x \in B^s]\}$ and T is infinite. Assume $u \in T$. Choose t such that $A^u \oplus \tilde{B}^u \subseteq A^s \oplus \tilde{B}^s$ for every $s > t$. Then $B^u \subseteq F_0^{s+1} \subseteq B^{s+1}$ for every $s > t$, by definition of F_0^{s+1} in the construction. So $B' \subseteq B$. □

Proposition 2. *For every* i

.1. $A \not\equiv W_i(B)$,

.2. $\exists t\ (\forall s > t)\ [s \in T \Rightarrow F_{i+1}^{s+1} = \emptyset]$,

.3. $(B - \tilde{B})^{[i]}$ *is finite.*

Proof. The proof is by induction on i. Assume 1-3 hold for every $i < m$. We show that 1-3 hold for $i = m$. Since $\tilde{B} \subseteq B$ (Proposition 1) and $(B - \tilde{B})^{[<m]}$ is finite by the induction hypothesis, $\tilde{B}^{[<m]} \equiv_e B^{[<m]} \equiv_e B \cup \omega^{[\geq m]}$. Now $\tilde{B}^{[<m]} \equiv_e B_0 \oplus B_1 \oplus \dots \oplus B_{m-1} \leq_e B_{m-1}$, so $A \not\leq_e \tilde{B}^{[<m]}$. Therefore $A \not\leq_e B \cup \omega^{[\geq m]}$. Choose z least such that $A(z) \neq W_m(B \cup \omega^{[\geq m]})(z)$.

Claim.

.1. *For every* $s \in T$, $W_m^{s+1}(E_{m,s+1}^{s+1})\restriction(z_m^s+1) \subseteq W_m(B)$.

.2. $W_m(B \cup \omega^{[\geq m]})\restriction(z+1) = \lim_{s \in T} W_m^{s+1}(E_{m,s+1}^{s+1})\restriction(z+1) = W_m(B)\restriction(z+1)$.

.3. $\lim_{s \in T} z_m^s = z$.

Proof. Suppose $s \in T$ and $x \in W_m^{s+1}(E_{m,s+1}^{s+1})\restriction(z_m^s+1)$. Then $u(W_m, E_{m,s+1}, x, s+1) \subseteq F_0^{s+1} \cup F_{m+1}^{s+1} \subseteq B^{s+1}$ by definition of F_{m+1}^{s+1} and B^{s+1}. But $B^{s+1} \subseteq B' = B$. Therefore $x \in W_m(B)$, so 1 holds.

Choose a stage $t' \in T$ such that $W_m(B \cup \omega^{[\geq m]})\restriction(z+1) \subseteq W_m^{t'+1}(B^{t'+1} \cup \omega^{[\geq m]})$. Choose $t > \max\{t', z\}$ such that for every $s > t$, $A\restriction(z+1) \subseteq A^s$ and $A^{t'} \oplus \tilde{B}^{t'} \subseteq A^s \oplus \tilde{B}^s$.

Suppose $s > t$ and $s \in T$. Then $B^{t'+1} \subseteq F_0^{s+1} \subseteq E_{m,s+1}^{s+1}$ from the construction. Therefore $W_m(B \cup \omega^{[\geq m]})\restriction(z+1) \subseteq W_m^{s+1}(E_{m,s+1}^{s+1})$. Suppose $W_m^{s+1}(E_{m,s+1}^{s+1})\restriction(z+1) - W_m(B \cup \omega^{[\geq m]}) \neq \emptyset$. Choose x least such that $x \in W_m^{s+1}(E_{m,s+1}^{s+1})\restriction(z+1) - W_m(B \cup \omega^{[\geq m]})$. If $x < z$ then $z_m^s = x$ since $A^s\restriction z = A\restriction z = W_m(B \cup \omega^{[\geq m]})\restriction z$. If $x = z$ then $z_m^s > x$ since $A^s(z) = A(z) \neq W_m(B \cup \omega^{[\geq m]})(z)$. In either case $x \in W_m(B)$ by 1, which contradicts $x \notin W_m(B \cup \omega^{(\geq m)})$. Therefore $W_m(B \cup \omega^{[\geq m]})\restriction(z+1) = W_m^{s+1}(E_m^{s+1})\restriction(z+1)$, $z_m^s = z$ and $W_m^{s+1}(E_m^{s+1})\restriction(z+1) \subseteq W_m(B)$ by 1. Therefore 2–3 hold. □

Since $A(z) \neq W_m(B \cup \omega^{[\geq m]})(z)$, 1 follows from the Claim.

Choose $t \in T$ such that z_m^s and $W_m^{s+1}(E_{m,s+1}^{s+1})\restriction(z+1)$ have reached a limit (on stages $s \in T$) by stage t, and for every $x \in W_m(B)\restriction(z+1)$, $t > h(W_m, B, x)$. Suppose $s > t$ and $s \in T$. Then $z_m^s = z$ and since $E_{m,s+1}^u = B^u$ for every $u \leq s$, for every $x \in W_m^{s+1}(E_{m,s+1}^{s+1})\restriction(z+1) = W_m(B)\restriction(z+1)$, we have $u(W_m, E_{m,s+1}, x, s) = u(W_m, B, x, s)\downarrow \subseteq B^{t+1} \subseteq F_0^{s+1} \subseteq E_{m,s+1}^{s+1}$; therefore $u(W_m, E_{m,s+1}, x, s+1) = u(W_m, E_{m,s+1}, x, s) \subseteq F_0^{s+1}$. Hence $F_m^{s+1} = \emptyset$. So 2 holds.

Note that if $s \in T$, $t < s$, and $A^t \oplus \tilde{B}^t \subseteq A^s \oplus \tilde{B}^s$, then $t \in T$. Also $F_{i+1}^{s+1} \subseteq \omega^{[\geq i]}$ for every i. Therefore

$$
\begin{aligned}
(B - \tilde{B})^{[m]} &= \left[\bigcup_{s\in T} (B^{s+1})^{[m]}\right] - \tilde{B} \quad \text{(Proposition 1)} \\
&= \left[\bigcup_{s\in T} \left[(B^{s+1})^{[m]} - \bigcup_{t\in T\restriction s} B^{t+1}\right]\right] - \tilde{B} \\
&= \bigcup_{s\in T} \left[(B^{s+1})^{[m]} - \left[\bigcup_{t\in T\restriction s} B^{t+1}\right] - \tilde{B}\right] \\
&\subseteq \bigcup_{s\in T} \left[\left[\bigcup_{k\leq m+1} F_k^{s+1}\right] - F_0^{s+1}\right] \quad \left(\text{since } F_0^{s+1} \subseteq \left[\bigcup_{t\in T\restriction s} B^{t+1}\right] \cup \tilde{B}\right) \\
&\subseteq \bigcup_{s\in T} \left[\bigcup_{k\leq m} F_{k+1}^{s+1}\right],
\end{aligned}
$$

which is finite from 2. Therefore 3 holds. □

Hence requirements S, P_i and N_i are satisfied (Propositions 1 and 2). ∎

Corollary 4.2.4. *For every non-zero low e-degree* $\mathbf{a}$ *there exists a* Σ_2 *e-degree* $\mathbf{b}$ *such that* $\mathbf{a} \not\leq_e \mathbf{b}$ *and for every* $\mathbf{z} \leq_e \mathbf{a}$, *either* $\mathbf{z} \leq_e \mathbf{b}$ *or there exists* $\mathbf{y} <_e \mathbf{a}$ *such that* $\mathbf{y} \quad \mathbf{z} = \mathbf{a}$ *and* $\mathbf{y} \leq_e \mathbf{b}$.

Hence the lower cone of $\mathbf{a}$ is split into degrees which are below $\mathbf{b}$ and degrees whose join with a degree below both $\mathbf{a}$ and $\mathbf{b}$, is $\mathbf{a}$.

§4.3 PROOF OF THEOREM 4.1.2

Due to the density of the Σ_2 e-degrees it suffices to show that for every pair of distinct *incomparable* Σ_2 e-degrees $\mathbf{a}$ and $\mathbf{b}$, $\{\mathbf{z}: \mathbf{z} <_e \mathbf{a}\} \neq \{\mathbf{z}: \mathbf{z} <_e \mathbf{b}\}$. Towards a contradiction, suppose degrees $\mathbf{a}$ and $\mathbf{b}$ are a counterexample. Let $A \in \mathbf{a}$ and $B \in \mathbf{b}$.

We first prove a general technical lemma (Lemma 4.3.1), which implies that if there exist Σ_2 approximations to A and $W_e(B)$ for $e \in \omega$, satisfying certain conditions, then there exists $C <_e A$ such that $C \not\leq_e B$. It only remains to show that such approximations exist. B must be non-r.e., so $K_B^0 \equiv_e B$ is non-r.e.. Therefore $G(K_B^0) <_e K_B^0 \equiv_e B$ from property 2 of G, the Gutteridge operator, which is described later. Hence $G(K_B^0) = W(A)$ for some e-operator W. We then use this fact and certain key properties of G to generate the desired approximations (Corollary 4.3.3).

As an additional application of Lemma 4.3.1, we prove Corollary 4.3.4.

Lemma 4.3.1. If $A \not\leq_e B$, $A \in \Sigma_2$ *and there exists a* Σ_2*-approximation* $\{A^s\}_{s\in\omega}$ *to* A *with infinitely many true stages* T, *and a strong array* $\{B_e^s\}_{e,s\in\omega}$ *such that for every* e, x, $\lim_{t\in T} B_e^s(x) = W_e(B)(x)$, *then there exists* $C <_e A$ *such that* $C \not\leq_e B$.

Proof. Assume A, B, $\{A^s\}_{s\in\omega}$ and $\{B_e^s\}_{e,s\in\omega}$ satisfy the conditions of the lemma. We construct an e-operator Θ such that $A \not\leq_e \Theta(A)$ and $\Theta(A) \not\leq_e B$. Letting $C = \Theta(A)$ yields the lemma.

We attempt to meet the following requirements, listed in order of priority.

0. $A \neq W_0(\Theta(A))$.
1. $\Theta(A)^{\{1\}} \neq W_0(B)^{\{1\}}$.
2. $A \neq W_1(\Theta(A))$.
3. $\Theta(A)^{\{3\}} \neq W_1(B)^{\{3\}}$.

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The proof is virtually identical to that of Lemma 2.2.1. In order to satisfy requirement r, we construct an e-operator Θ_r and set

$$\Theta = \bigcup_r \Theta_r.$$

Θ_r^s is the set of instructions $\langle F,x\rangle$ which have been enumerated into Θ_r by the end of stage s; $\{\Theta_r^s\}_{s\in\omega}$ is a recursive enumeration of Θ_r. Set

$$\Theta_{<r} = \bigcup_{q<r} \Theta_q$$

and $\Theta_{\geq r} = \Theta - \Theta_{<r}$. $\{\Theta^s\}_{s\in\omega}$, $\{\Theta_{<r}^s\}_{s\in\omega}$ and $\{\Theta_{\geq r}^s\}_{s\in\omega}$ denote the natural recursive enumerations of Θ, $\Theta_{<r}$ and $\Theta_{\geq r}$ respectively, generated by $\{\Theta_q^s\}_{s\in\omega}$. Using the technique of Proposition 1.4.3 we can redefine $\{A^s\}_{s\in\omega}$ and $\{B_e^s\}_{e,s\in\omega}$ so that in addition, for every r, $\{A^t\}_{(t)_0=r}$ is a Σ_2-approximation to A with infinitely many true stages, T.

Length of agreement functions.

$$l(e,s) = \mu z\ [z = s \quad \text{or} \quad A^s(z) \neq W_e^s(\Theta^s(A^s))(z)].$$
$$L(e,s) = \mu z\ [z = s \quad \text{or} \quad \Theta^s(A^s)^{\{2e+1\}}(z) \neq (B_e^s)^{\{2e+1\}}(z)].$$

We attend to requirement r at stages $s+1$, where $(s)_0 = r$. If $r = 2e$ then we arrange that for $z \in W_e^s(\Theta^s(A^s))\restriction l(e,s)$, $z \in$

$W_e^s(\theta^s(F^s \cup A^s \restriction z))$, where $F^s \subseteq A^s$ is finite and $\lim_{s \in T} F^s = F \subseteq A$ (F finite). This is done in such a way that if $l(e,s) \to \infty$ as s increases in T, then $z \in W_e(\theta(F \cup A \restriction z))$ for every $z \in W_e(\theta(A))$. If requirement r fails, $W_e(\theta(A)) = A$ is r.e., which is a contradiction. If $r = 2e+1$, then we code $A^s \restriction L(e,s)$ into $\theta^s(A^s)^{\{r\}}$. If requirement r fails, $A \equiv_e \theta(A)^{\{r\}} \leq_e \theta(A) = W_e(B)$, which contradicts $A \not\leq_e B$.

Construction.

Stage 0.

Do nothing.

Stage s+1.

Let $r = (s)_0$.

Case 1. $r = 2e$.

For every z, set

$$E_z^s = U(\theta_{<r}, A, \theta_{<r}^s(A^s), s) \cup A^s \restriction z.$$

For every z, x such that

1.1. $z \in W_e^s(\theta^s(A^s)) \restriction l(e,s) - W_e^s(\theta^s(E_z^s))$,

1.2. $x \in u(W_e, \theta(A), z, s) - \theta^s(E_z^s)$, enumerate $\langle E_z^s, x\rangle$ into θ_r. If z satisfies 1.1 and $E_z^s \subseteq A$, we say $s+1$ is *(r,z)-active*.

Case 2. $r = 2e+1$.

For every $z \in A^s \upharpoonright L(e,s)$, enumerate $\langle A^s, \langle z,r\rangle\rangle$ into θ_r.

End of construction.

Proposition 1. *For every r, $\theta_r(\omega) \subseteq \omega^{[\geq r]}$.*

Proof. It suffices to show that for every r, s, $\theta_r^s(\omega) \subseteq \omega^{[\geq r]}$. The proof is by induction on s. $\theta_r^0 = \emptyset$ for every r. Assume $\theta_r^s(\omega) \subseteq \omega^{[\geq r]}$ for every r. If $r \neq (s)_0$ then $\theta_r^{s+1} = \theta_r^s$. Suppose $r = (s)_0$ and $\langle F, x\rangle$ is enumerated into θ_r at stage $s+1$.

Case 1. $r \equiv 2e$.

Then $x \in u(W_e, \theta(A), z, s) - \theta^s(E_z^s)$ for some $z \in W_e^s(\theta^s(A^s))\upharpoonright l(e,s)$. Hence $x \in \theta^s(A^s) - \theta^s(E_z^s)$. Now $\theta_{<r}^s(A^s) \subseteq \theta_{<r}^s(E_z^s)$ (Proposition 2). Therefore $x \in \theta^s(A^s) - \theta_{<r}^s(A^s) = \theta_{\geq r}^s(A^s) \subseteq \theta_{\geq r}^s(\omega) \subseteq \omega^{[\geq r]}$ by the induction hypothesis.

Case 2. $r \equiv 2e+1$.

Then $x = \langle z,r\rangle \in \omega^{[\geq r]}$ for some z. $\square$

Proposition 2. *For every* r, z, *if* $r = (s)_0 = 2e$ *then,*

.1. $\theta^s_{<r}(A^s) \subseteq \theta^s_{<r}(E^s_z)$,

.2. *if* $z \in W^s_e(\theta^s(A^s))\restriction l(e,s)$ *then*

.1. $z \in W^s_e(\theta^{s+1}(E^s_z))$,

.2. *if* $s+1$ *is* (r,z)*-active then* $A^s\restriction z \subseteq A$, $\theta^s_{<r}(A^s) \subseteq \theta_{<r}(A)$ *and* $z \in W_e(\theta(A))$,

Proof. Assume $r = (s)_0 = 2e$. From the construction, $U(\theta_{<r}, A, \theta^s_{<r}(A^s), s) \subseteq E^s_z$, therefore 1 holds.

Suppose $z \in W^s_e(\theta^s(A^s))\restriction l(e,s)$. Then either $z \in W^s_e(\theta^s(E^s_z))$ or by the action taken at stage $s+1$, $u(W_e, \theta(A), z, s) \subseteq \theta^s(E^s_z) \cup \theta^{s+1}(E^s_z) \subseteq \theta^{s+1}(E^s_z)$. Therefore 2.1 holds.

Assume in addition that $s+1$ is (r,z)-active. Then $A^s\restriction z \subseteq E^s_z \subseteq A$. By 1, $\theta^s_{<r}(A^s) \subseteq \theta^s_{<r}(E^s_z) \subseteq \theta_{<r}(A)$ and by 2.1, $z \in W^s_e(\theta^{s+1}(E^s_z)) \subseteq W_e(\theta(A))$. □

Proposition 3. *For every* r,

.1. *requirement* r *is satisfied,*

.2. $\theta_r(A)$ *is finite.*

Proof. The proof is by induction on r. Assume 1–2 hold for every $r < q$. We show that 1–2 hold for $r = q$. It follows from the induction hypothesis that $U(\theta_{<q}, A, \theta_{<q}(A))$ is finite.

Case 1. $q = 2e$.

Then requirement q is "$A \neq W_e(\Theta(A))$". Suppose $A = W_e(\Theta(A))$. We define a recursive enumeration $\{A'^s\}_{s\in\omega}$ of a set A' as follows.

$$A'^0 = U(\Theta_{<q}, A, \Theta_{<q}(A)).$$

$$A'^{s+1} = A'^s \cup W_e^s(\Theta^s(A'^s)).$$

Claim 1. $A' = A$.

Proof. We can easily show that $A'^s \subseteq A$ for every s, by induction. Hence $A' \subseteq A$.

Since $W_e^s(\Theta^s(A'^s)) \subseteq A'^{s+1}$ for every s, $W_e(\Theta(A')) \subseteq A'$. We show that $A \restriction z \subseteq A'$ for every z, by induction. Hence $A \subseteq A'$.

Assume $A \restriction z \subseteq A'$. If $z \notin A$ then we are done.

Suppose $z \in A = W_e(\Theta(A))$. Choose $s \in T$ such that $(s)_0 = q$, $z \in W_e^s(\Theta^s(A^s)) \restriction l(e,s)$, $\Theta_{<q}^s(A^s) = \Theta_{<q}(A)$ and $U(\Theta_{<q}, A, \Theta_{<q}^s(A^s), s) = U(\Theta_{<q}, A, \Theta_{<q}(A))$. Then $E_z^s = U(\Theta_{<q}, A, \Theta_{<q}^s(A^s), s) \cup A^s \restriction z \subseteq U(\Theta_{<q}, A, \Theta_{<q}(A)) \cup A \restriction z \subseteq A'$ by the induction hypothesis. Now $z \in W_e^s(\Theta^{s+1}(E_z^s))$ (Proposition 2), so $z \in W_e(\Theta(A')) \subseteq A'$. Therefore $A \restriction (z+1) \subseteq A'$, and we are done. □

A is r.e. by Claim 1, which is a contradiction. Therefore $A \neq W_e(\Theta(A))$, so 1 holds.

Claim 2. *For every* z, *there are only finitely many* (q,z)*-active stages.*

Proof. Towards a contradiction, suppose z is a counterexample. Choose a (q,z)-active stage $t+1$ such that $U(\Theta_{<q},A,\Theta_{<q}(A),s)$ has reached a limit by stage t and for every $s \geq t$, $A\restriction z \subseteq A^s$. Then $\Theta^t_{<q}(A^t) = \Theta_{<q}(A)$ (Proposition 2), therefore $E^t_z = U(\Theta_{<q},A,\Theta_{<q}(A)) \cup A\restriction z$, and $z \in W^t_e(\Theta^{t+1}(E^t_z))$ (Proposition 2). Let $s+1 > t$ be a (q,z)-active stage. It follows from the definition of E^s_z and the choice of t that $E^t_z \subseteq E^s_z$. Therefore $z \in W^s_e(\Theta^s(E^s_z))$, which is a contradiction. □

By 1 we can choose a least y such that $A(y) \neq W_e(\Theta(A))(y)$. Choose a stage t such that

$$(\forall s \geq t)\ [A\restriction(y+1) \subseteq A^s \text{ and } W_e(\Theta(A))\restriction(y+1) \subseteq W^s_e(\Theta^s(A^s))].$$

Claim 3. *For every* $z > y$, *there are no* (q,z)*-active stages after stage* t.

Proof. Suppose not. Choose $z > y$ and a (q,z)-active stage $s+1 > t$. Note $l(e,s) > z > y$ and from the construction, $E^s_y \subseteq E^s_z \subseteq A$.

Case 1. $y \in A$.

Then $y \in W^s_e(\Theta^s(A^s))\restriction l(e,s)$. Therefore $y \in W_e(\Theta(A))$ (Proposition

2), which contradicts the choice of y.

Case 2. $y \in W_e(\Theta(A))$.

Then $y \in A^s$. Therefore $y \in A^s \upharpoonright z \subseteq A$ (Proposition 2), which contradicts the choice of y. □

From the construction $x \in \Theta_q(A)$ if and only if there exist z, t such that $\langle E_z^t, x\rangle$ is enumerated into Θ_q at (q,z)-active stage $t+1$. The set of stages t for which there exists z such that t is (q,z)-active is finite (Claims 2 and 3). Since only finitely many instructions are enumerated into Θ_q at each stage, 2 holds.

Case 2. $q = 2e+1$.

Then requirement q is "$\Theta(A)^{\{q\}} \neq W_e(B)^{\{q\}}$".

Claim 4. *For every* z, *if* $z \in \Theta_q^s(A^s)^{\{q\}}$ *then* $z \in A^s$.

Proof. Assume $z \in \Theta_q^s(A^s)^{\{q\}}$. Then there exists an instruction $\langle F, \langle z,q\rangle\rangle \in \Theta_q^s$ such that $F \subseteq A^s$. Suppose such an instruction is enumerated into Θ_q at stage $t+1 \leq s$. An inspection of the construction shows that $q = (t)_0$, $z \in A^t \upharpoonright L(e,t)$ and $F = A^t$. □

Suppose $\Theta(A)^{\{q\}} = W_e(B)^{\{q\}}$. Since $\Theta_{<q}(A)$ is finite, (by the induction hypothesis), $m = \max(\Theta_{<q}(A)^{\{q\}})$ is defined.

Claim 5. $\Theta(A)^{\{q\}}\restriction[>m] = A\restriction[>m]$.

Proof. Let $z > m$.

Suppose $z \in \Theta(A)^{\{q\}}$. Then $z \notin \Theta_{<q}(A)^{\{q\}}$, so $z \in \Theta_q(A)^{\{q\}}$ (Proposition 1). Choose $s \in T$ such that $z \in \Theta_q^s(A^s)^{\{q\}}$. Then $z \in A^s \subseteq A$ (Claim 4).

Suppose $z \in A$. Choose $s \in T$ such that $(s)_0 = q$ and $z \in A^s\restriction L(e,s)$. Then $A^s \subseteq A$ and $\langle A^s,\langle z,q\rangle\rangle$ is enumerated into Θ_q at stage $s+1$. Therefore $z \in \Theta(A)^{\{q\}}$. □

By Claim 5 $A \leq_e \Theta(A) = W_e(B) \leq_e B$, which is a contradiction. Therefore $\Theta(A)^{\{q\}} \neq W_e(B)^{\{q\}}$, so 1 holds.

By 1 we can choose a least y such that $\Theta(A)^{\{q\}}(y) \neq W_e(B)^{\{q\}}(y)$.

From the construction $x \in \Theta_q(A)$ if and only if there exist $s \in T$, z such that $(s)_0 = q$, $x = \langle z,q\rangle$ and $z \in A^s\restriction L(e,s)$. However $\lim_{s\in T} L(e,s) = y$, therefore $\Theta_q(A)$ is finite. □

Hence all requirements are satisfied. ∎

Definition 4.3.2 (Cooper). An s-*operator*, Ψ, is an e-operator such that for every $\langle F,x\rangle \in \Psi$, $\langle F,x\rangle = \langle\emptyset,\langle i,j\rangle\rangle$ or $\langle F,x\rangle = \langle\{i\},\langle i,j\rangle\rangle$, for some i, j.

The Gutteridge [1971] (pp. 42-46) operator, G, is an s-operator with the following properties:

1. $G(X)$ is r.e. implies $X \leq_T \emptyset'$.
2. $G(X) \equiv_e X$ implies X is r.e.
3. For every i there exists j such that $\langle\{i\},\langle i,j\rangle\rangle \in G$ and $\langle\emptyset,\langle i,j\rangle\rangle \notin G$.
4. For every i, $\{j\colon \langle\{i\},\langle i,j\rangle\rangle \in G\}$ is finite.

Corollary 4.3.3. *If* $A \in \Sigma_2$, $A \nleq_e B$ *and* $G(K_B^0) \leq_e A$ *then there exists* $C <_e A$ *such that* $C \nleq_e B$.

Proof. Assume A and B satisfy the hypothesis. Choose an e-operator W such that $W(A) = G(K_B^0)$. Let $\{A^s\}_{s\in\omega}$ be a Σ_2-approximation to A such that for every r, $\{A^t\}_{(t)_0=r}$ is a Σ_2-approximation to A with infinitely many true stages. Let $\{W^s\}_{s\in\omega}$ be a recursive enumeration of W.

Definition. For every e, let $B_e^s = \{x\colon \exists j\, [\langle\langle x,e\rangle,j\rangle \in W^s(A^s)$ and $\langle\langle x,e\rangle,j\rangle \notin G^s(\emptyset)]\}$.

B_e^s is our guess at $W_e(B)$ at stage s. This makes sense due to the fact that $W(A) = G(K^0_B)$, properties 3 and 4 of G, and the definition of K^0_B. Let T denote the set of true stages in $\{A^s\}_{s\in\omega}$. That B^s_e is defined in terms of A gives:

Proposition 1. *For every* e, x, $\lim_{s\in T} B^s_e(x) = W_e(B)(x)$.

Proof. Fix e and x. Choose a stage t such that all instructions $\langle F, \langle\langle x,e\rangle, j\rangle\rangle \in G$ are in G^t, and

$$(\forall s > t)\ \forall j\ [s \in T \Rightarrow W^s(A^s)(\langle\langle x,e\rangle, j\rangle) = G(K^0_B)(\langle\langle x,e\rangle, j\rangle)].$$

Assume $s \in T$ and $s > t$.

Case 1. $x \in W_e(B)$.

Then there exists j such that $\langle\langle x,e\rangle, j\rangle \in G(K^0_B)$ and $\langle\langle x,e\rangle, j\rangle \notin G(\emptyset)$. By the choice of t, $\langle\langle x,e\rangle, j\rangle \in W^s(A^s)$. Hence $x \in B^s_e$.

Case 2. $x \notin W_e(B)$.

Then for every j, $G(K^0_B)(\langle\langle x,e\rangle, j\rangle) = G(\emptyset)(\langle\langle x,e\rangle, j\rangle)$. By the choice of t, $W^s(A^s)(\langle\langle x,e\rangle, j\rangle) = G(K^0_B)(\langle\langle x,e\rangle, j\rangle)$ and $G(\emptyset)(\langle\langle x,e\rangle, j\rangle) = G^s(\emptyset)(\langle\langle x,e\rangle, j\rangle)$ for every j. Hence $x \notin B^s_e$. □

An application of Lemma 4.3.1 completes the proof. ■

Corollary 4.3.4. *For every pair of* Σ_2 *e-degrees* $\mathbf{a}$ *and* $\mathbf{b}$, *if* $\mathbf{a} \not\leq_e \mathbf{b}$ *and* $\mathbf{b}$ *is low then there exists* $\mathbf{c} <_e \mathbf{a}$ *such that* $\mathbf{c} \not\leq_e \mathbf{b}$.

Proof. Assume $\mathbf{a}$ and $\mathbf{b}$ satisfy the hypothesis, $A \in \mathbf{a}$ and $B \in \mathbf{b}$. Let $\{A^s\}_{s\in\omega}$ be a Σ_2-approximation to A with infinitely many true stages, and let $\{B^s\}_{s\in\omega}$ be a low approximation to B. Set $B_e^s = W_e^s(B^s)$. Then for every x, $\lim_s B_e^s(x) = W_e(B)(x)$. Applying Lemma 4.3.1 yields $C \leq_e A$ such that $C \not\leq_e B$. $\mathbf{c} = \deg_e C$ is the desired degree. ∎

CHAPTER V

EMBEDDING THE DIAMOND IN THE Σ_2 E-DEGREES

§5.1 INTRODUCTION

Lachlan [1966] has shown that it is not possible to embed the diamond lattice in the r.e. Turing degrees while preserving least and greatest elements, that is, there do not exist incomparable r.e. Turing degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$. Cooper [1984] has asked if the r.e. Turing degrees are elementarily equivalent to the enumeration degrees below $\mathbf{0}'_{\mathbf{e}}$.

Such an embedding is possible in the Σ_2 *enumeration degrees*, which implies a negative answer to Cooper's question.

Theorem 5.1. *There exist a pair of low e-degrees* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *such that* $\boldsymbol{a} \wedge \boldsymbol{b} = \mathbf{0}_{\boldsymbol{e}}$ *and* $\boldsymbol{a} \vee \boldsymbol{b} = \mathbf{0}'_{\boldsymbol{e}}$.

§5.2 PROOF OF THEOREM

We show that there exist low sets A and B such that

$$\forall e\ [W_e(A) = W_e(B) \Rightarrow W_e(A) \text{ is r.e.}]$$

and

$$\chi_K \leq_e A \oplus B.$$

Letting $\mathbf{a} = \deg_e A$ and $\mathbf{b} = \deg_e B$ yields the theorem.

Let $\{K^s\}_{s\in\omega}$ be a recursive enumeration of K such that

$$\forall s\ [\max K^s < s \text{ and } K^{3s} = K^{3s+1} = K^{3s+2}].$$

Without loss of generality we may assume that:

$$(\forall e)(\forall s)\ [W_e^{3s} = W_e^{3s+1} = W_e^{3s+2}].$$

We construct low approximations $\{A^s\}_{s\in\omega}$ and $\{B^s\}_{s\in\omega}$ to A and B respectively.

The choice of enumerations of K and W_e allows a construction which involves three different types of actions at each stage to be mimicked by one which involves a single type of action at each stage.

Definitions.

Length of agreement functions.

$L(e,s) = \mu z\ [z = s \text{ or } W_e^s(A^s)(z) \neq W_e^s(B^s)(z)]$.

$M(e,s) = \max\ \{L(e,t)\colon t \leq s\}$.

$M(e,s)$ is called a **maximum length of agreement function.**

The possible outcomes of the construction, with regard to equality between $W_e(A)$ and $W_e(B)$, for $e < n$, are indexed by elements of ${}^{<\omega}2$. That is, $\sigma \in {}^{<\omega}2$ corresponds to the outcome $W_e(A) = W_e(B)$ if $\sigma(e) = 1$ and $W_e(A) \neq W_e(B)$ if $\sigma(e) = 0$.

We define a recursive set C_σ for every $\sigma \in {}^{<\omega}2$ by induction on $\mathrm{lh}(\sigma)$.

$C_\emptyset = \{s\colon s \equiv 0 \bmod 3\}$.

$\mathrm{lh}(\sigma) > 0$:

$C_{\sigma^-\hat{}(1)} = \{s\colon s \in C_{\sigma^-}$ and $(\forall t < s)\ [t \in C_{\sigma^-} \Rightarrow$

$M(e(\sigma),t) < M(e(\sigma),s)]\}$;

$C_{\sigma^-\hat{}(0)} = C_{\sigma^-} - C_{\sigma^-\hat{}(1)}$.

Intuitively, C_σ may be viewed as the set of stages at which there is evidence that σ corresponds to the true outcome of the construction. We denote $C_\sigma \restriction (s+1)$ by C_σ^s. From the definition of C_σ it is clear that C_σ^s is fully determined by the sequences $\{A^t\}_{t \leq s}$ and $\{B^t\}_{t \leq s}$.

Since we are interested only in those e for which $W_e(A) = W_e(B)$, we restrict our attention to the following subset of ${}^{<\omega}2$:

$$\mathcal{S} = \{\sigma : \sigma \in {}^{<\omega}2 - \{\emptyset\} \text{ and } \sigma(e(\sigma)) = 1\}.$$

We define an element $c(\sigma, s)$ and a finite set $E(\sigma, s)$ for each $\sigma \in \mathcal{S}$.

$$c(\sigma,s) = \begin{cases} \max C_\sigma^s, & \text{if } C_\sigma^s \neq \emptyset, \\ \uparrow & \text{otherwise.} \end{cases}$$

$$E(\sigma,s) = \begin{cases} W_{e(\sigma)}^{c(\sigma,s)}(A^{c(\sigma,s)}) \restriction L(e(\sigma), c(\sigma,s)), & \text{if } c(\sigma,s)\downarrow, \\ \emptyset, & \text{otherwise.} \end{cases}$$

So $c(\sigma,s)$ is the greatest stage up to s, at which there is evidence that σ is the true outcome of the construction, and $E(\sigma,s)$ is $W_{e(\sigma)}(A)$ as it appears at stage $c(\sigma,s)$, below its point of disagreement with $W_{e(\sigma)}(B)$.

Proposition 1. *For every* $\sigma, \tau \in {}^{<\omega}2$,

.1. $\tau \subseteq \sigma \Rightarrow C_\sigma \subseteq C_\tau$,

.2. $s \in C_\sigma \cap C_\tau \Rightarrow \sigma$ *and* τ *are compatible.*

Proposition 2. *For every* n, *if* $s \equiv 0 \bmod 3$ *then there is a unique* $\sigma \in {}^{n}2$ *such that* $s \in C_\sigma$.

Proof. The two propositions follow from the observation that $C_\emptyset = \{s: s \equiv 0 \bmod 3\}$ and for every σ, $C_\sigma = C_{\sigma 1} \sqcup C_{\sigma 0}$. □

$\lesssim$ denotes the following partial order on ${}^{<\omega}2$:

$$\sigma \lesssim \tau \iff \sigma = \tau \text{ or } [(\sigma \cap \tau)1 \subseteq \sigma \text{ and } \tau \not\subseteq \sigma].$$

For $s \equiv 0 \bmod 3$, $\sigma(n,s)$ denotes the unique element of ${}^{n}2$ such that $s \in C^s_{\sigma(n,s)}$ (Proposition 2), and $\sigma_n = \mu\sigma\,(\exists^\infty s\,[\sigma = \sigma(n,s)])$. Note that if $\{A^s\}_{s\in\omega}$ and $\{B^s\}_{s\in\omega}$ are low approximations, then σ_n corresponds to the true outcome of the construction.

$$\mathcal{S}_n = \{\sigma \in \mathcal{S}: e(\sigma) = n\},\quad \mathcal{S}_{\le n} = \bigcup_{k \le n} \mathcal{S}_k \text{ and } \mathcal{S}_{>n} = \mathcal{S} - \mathcal{S}_{\le n}.$$

Requirements.

We attempt to satisfy the following requirements, listed in order of priority.

N: $\forall e\ [e \in \overline{K} \Leftrightarrow e \in A \cap B]$.

Q_0: If $\sigma_1 \in \mathcal{S}$ then

$$\exists u\ (\forall t > u)\ (\forall s > t)\ [E(\sigma_1, t) \subseteq W_0^s(A^s) \text{ or } E(\sigma_1, t) \subseteq W_0^s(B^s)].$$

P_0: $\exists^\infty s\ [k \in W_e^s(A^s)] \Rightarrow k \in W_e(A)$, where $0 = \langle k, e\rangle$.

Q_1: If $\sigma_2 \in \mathcal{S}$ then

$$\exists u\ (\forall t > u)\ (\forall s > t)\ [E(\sigma_2, t) \subseteq W_1^s(A^s) \text{ or } E(\sigma_2, t) \subseteq W_1^s(B^s)].$$

P_1: $\exists^\infty s\ [k \in W_e^s(B^s)] \Rightarrow k \in W_e(B)$, where $0 = \langle k, e\rangle$.

Q_2: If $\sigma_3 \in \mathcal{S}$ then

$$\exists u\ (\forall t > u)\ (\forall s > t)\ [E(\sigma_3, t) \subseteq W_2^s(A^s) \text{ or } E(\sigma_3, t) \subseteq W_2^s(B^s)].$$

P_2: $\exists^\infty s\ [k \in W_e^s(A^s)] \Rightarrow k \in W_e(A)$, where $1 = \langle k, e\rangle$.

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If these requirements are met then the theorem is proved. Assume that all requirements N, P_n and Q_n are satisfied. $\overline{K} \leq_e A \oplus B$ from requirement N. Now $\chi_K \equiv_e \overline{K}$ since K is r.e. Hence $\chi_K \leq_e A \oplus B$. $\{A^s\}_{s\in\omega}$ and $\{B^s\}_{s\in\omega}$ are low approximations, from requirements P_n. Thus A and B are low. Suppose $W_e(A) = W_e(B)$. Then $\sigma_{e+1} \in \mathcal{S}$ and from requirement Q_e we can choose u such that

$$(*) \qquad (\forall t > u)\ (\forall s > t)\ [E(\sigma_{e+1}, t) \subseteq W_e^s(A^s) \text{ or } E(\sigma_{e+1}, t) \subseteq W_e^s(B^s)].$$

Let

$$Y = \bigcup_{t>u} E(\sigma_{e+1}, t).$$

From $*$ and the lowness of the approximations, $Y \subseteq W_e(A) \cup W_e(B) = W_e(A)$. $W_e(A) = W_e(B)$, the lowness of the approximations and C_{σ_e} infinite imply that $W_e(A) \subseteq Y$. Therefore $W_e(A) = Y$, and Y is r.e. since $C_{\sigma_{e+1}}$ is recursive.

Use function.

Let V be an e-operator.

$$u'(V,X,k) = \begin{cases} \uparrow & \text{if } k \notin V(X), \\ D_z & \text{where } z = \mu i\ [k \in V(D_i) \text{ and } D_i \subseteq X], \text{ otherwise.} \end{cases}$$

At each stage every $\sigma \in \mathcal{S}$ is assigned a *status*, *ON* or *OFF*. The status of σ indicates whether the strategy for ensuring that $W_{e(\sigma)}(A)$ is r.e., (in the case that $\sigma_{\mathrm{lh}(\sigma)} = \sigma$ and $W_{e(\sigma)}(A) = W_{e(\sigma)}(B)$), is active or not. Henceforth we will call this the strategy *associated with* σ. When we refer to the status of σ *at stage* s we mean the status of σ *at the end of stage* s. If σ is not explicitly assigned a status at stage s+1, then it is the same as at stage s.

Restraint functions.

$$R(p,s) = \begin{cases} \max\{[\max u'(W_e^s, X^s, k)]+1,\ p+1\}, & \text{if } u'(W_e^s, X^s, k)\downarrow, \\ p+1, & \text{otherwise,} \end{cases}$$

where $p = 2\langle k,e\rangle$ and $X = A$, or $p = 2\langle k,e\rangle+1$ and $X = B$.

$R(p,s)$ is associated with requirement P_p. If $u'(W_e^s, X^s, k)\downarrow$ and $(K \cap A^s \cap B^s)\restriction R(p,s) = \emptyset$ then unless a set of lesser canonical index, (respecting higher priority restraints), is found for putting k in $W_e(X)$, the aim is not to disturb X below $R(p,s)$ after stage s.

$r(\sigma,A,s)$ and $r(\sigma,B,s)$ are defined in the construction, for every $\sigma \in \mathcal{S}$. For $X = A, B$, we agree that if σ is *OFF* at stage s then $r(\sigma,X,s) = \emptyset$, and if σ is *ON* at stage s+1 and $r(\sigma,X,s+1)$ is not explicitly defined then $r(\sigma,X,s+1) = r(\sigma,X,s)$.

$r(\sigma,A,s)$ and $r(\sigma,B,s)$ are associated with requirement $Q_{e(\sigma)}$. $r(\sigma,X,s) \subseteq X^s$ for $X = A, B$. If σ is *ON* at stage s then either $E(\sigma,s-1) \subseteq W^s_{e(\sigma)}(A^s) \cap W^s_{e(\sigma)}(B^s)$ or $E(\sigma,s-1) \subseteq W^s_{e(\sigma)}(r(\sigma,X,s))$ where X is either A or B.

In the construction certain stages are designated *σ-active* for one or more $\sigma \in \mathcal{S}$. If s is σ-active then one of the sets A or B is *marked* at stage s.

At a σ-active stage the strategy associated with σ is potentially threatened and active measures are taken to preserve it.

$$a(\sigma,s) = \begin{cases} \text{the greatest } \sigma\text{-active stage } \leq s, & \text{if one exists,} \\ s, & \text{otherwise.} \end{cases}$$

Construction.

Stage 0.

$A^0 = B^0 = \omega$. All $\sigma \in \mathcal{S}$ are *OFF*.

Stage s+1 (s ≡ 0 mod 3).

Turn all $\sigma \in \mathcal{S}$ such that $\sigma > \sigma(s,s)$ *OFF*.

Turn every $\sigma \in \mathcal{S}$ such that $\sigma \subseteq \sigma(s,s)$ *ON* and set

$$r(\sigma,X,s+1) = \emptyset \quad \text{for} \quad X = A,\ B.$$

Stage s+1 ($s \equiv 1 \bmod 3$).

If there exists $p \leq s$ such that

1. $p = 2\langle k,e\rangle$ and there exists D such that

.1. $k \in W_e^{s+1}(D)$,

.2. $(\forall m < p)\ [D \restriction R(m,s) \subseteq A^s]$,

.3. $(\forall \sigma \in \mathcal{S}_{\leq p})\ [D \cap r(\sigma,B,s) \subseteq A^s]$,

.4. $u'(W_e^s,A^s,k)\uparrow$ or the canonical index of D is strictly less than the canonical index of $u'(W_e^s,A^s,k)$,

or

2. $p = 2\langle k,e\rangle+1$ and there exists D such that

.1. $k \in W_e^{s+1}(D)$,

.2. $(\forall m < p)\ [D \restriction R(m,s) \subseteq B^s]$,

.3. $(\forall \sigma \in \mathcal{S}_{\leq p})\ [D \cap r(\sigma,A,s) \subseteq B^s]$,

.4. $u'(W_e^s,B^s,k)\uparrow$ or the canonical index of D is strictly less than the canonical index of $u'(W_e^s,B^s,k)$

then let p^* be the least such. Turn all $\sigma \in \mathcal{S}_{>p^*}$ OFF.

Case 1. $p^* = 2\langle k,e\rangle$.

Choose D with least canonical index satisfying 1.1-1.4. Set

$A^{s+1} = A^s \cup D$ and $B^{s+1} = B^s$.

Case 2. $p^* = 2\langle k,e\rangle+1$.

Choose D with least canonical index satisfying 2.1-2.4. Set $B^{s+1} = B^s \cup D$ and $A^{s+1} = A^s$.

If no such p exists, do nothing.

Stage s+1 (s ≡ 2 mod 3).

Let $F^s = A^s \cap B^s \cap K^{s+1}$.

If there exists $p \leq s$ such that

$$F^s \restriction R(p,s) \neq \emptyset$$

then let p' be the least such. Otherwise set $p' = s$.

Turn all $\sigma \in \mathcal{S}_{>p'}$ *OFF*.

Case 1. $(\forall\sigma \in \mathcal{S}_{\leq p'})\ [r(\sigma,A,s) \cap F^s = r(\sigma,B,s) \cap F^s = \emptyset]$.

Subcase 1.1. $p' \equiv 0 \bmod 2$.

Set $B^{s+1} = B^s - F^s$ and $A^{s+1} = A^s$. Mark A.

For every $\sigma \in \mathcal{S}$ such that σ is *ON at the end of this stage* and $\sigma \subseteq \sigma(s-2,s-2)$, set

$$r(\sigma,A,s+1) = U(W_{e(\sigma)},A,E(\sigma,s),s+1)$$

and $r(\sigma,B,s+1) = \emptyset$. We call $s+1$ a σ-active stage.

Subcase 1.2. Otherwise.

Same as for Subcase 1.1 with A and B interchanged.

Case 2. Otherwise.

Choose $\sigma' \in \mathcal{S}_{\leq p}$, first $\leq$-minimal and then of *greatest length* such that

$$r(\sigma',A,s) \cap F^s \neq \emptyset \quad \text{or} \quad r(\sigma',B,s) \cap F^s \neq \emptyset.$$

Turn all $\sigma > \sigma'$ OFF.

Subcase 2.1. A is marked at stage $\alpha(\sigma',s)$.

Set

$$A^{s+1} = \bigcup_{\alpha(\sigma',s)\leq t\leq s} A^t \quad \text{and} \quad B^{s+1} = B^s - (A^{s+1} \cap K^{s+1}).$$

Mark A.

For every $\sigma \in \mathcal{S}$ such that σ is ON *at the end of this stage* and $\sigma \subseteq \sigma'$, set

$$r(\sigma,A,s+1) = U(W_{e(\sigma)},A,E(\sigma,s),s+1)$$

and $r(\sigma,B,s+1) = \emptyset$. We call $s+1$ a σ-active stage.

Subcase 2.2. Otherwise.

Same as for Subcase 2.1 with A and B interchanged.

In either Case 1 or Case 2 if there exists $p \lesssim p'$ such that

$$R(p,s+1) \neq R(p,s) \quad \text{or}$$
$$A^{s+1} \restriction R(p,s) \neq A^{s} \restriction R(p,s) \quad \text{or} \quad B^{s+1} \restriction R(p,s) \neq B^{s} \restriction R(p,s)$$

then let p'' be the least such. Otherwise set $p'' = p'$. Turn all $\sigma \in \mathcal{G}_{>p''}$ *OFF*.

If A^{s+1} or B^{s+1} is not explicitly defined then it is the same as A^{s} or B^{s} respectively.

End of construction.

Proposition 3. $\bar{K} \subseteq A \cap B$.

Proof. $A^{0} = B^{0} = \omega$ and the only elements which are removed from A or B at a stage s+1 are elements of K^{s+1}. □

The construction reflects our intuitive notion of C_{σ} and the priorities of the various requirements. If $\sigma, \tau \in \mathcal{G}$ and $\sigma < \tau$ then the strategy associated with σ is given prioirity over that associated with τ. This is due to the definition of σ_{n}.

At stage s ($s \equiv 0 \bmod 3$), a new candidate for σ_p appears for each $p \leq s$, namely $\sigma(p,s)$. So at stage $s+1$ we abandon old candidates $\sigma \in \mathcal{S}$ where $\sigma > \sigma(p,s)$, an action which reflects the definition of σ_p. If $\sigma(p,s) \in \mathcal{S}$ then $E(\sigma(p,s),s) \subseteq W_{p-1}^{s+1}(A^{s+1}) \cap W_{p-1}^{s+1}(B^{s+1})$ (Proposition 4.1), so $r(\sigma(p,s),X,s+1)$ can be set to $\emptyset$ for $X = A, B$. This allows greater opportunity to meet requirements P_n at the following stage, as there are no restraints imposed on A or B by the strategies associated with $\sigma \in \mathcal{S}$ where $\sigma \subseteq \sigma(s,s)$. As far as these strategies are concerned, we have the freedom to remove elements of K from either A or B.

At stage $s+1$ ($s \equiv 1 \bmod 3$), we attempt to meet requirements P_n. Suppose $p^* = 2\langle k,e\rangle$. The fact that only elements of K are ever removed from either A or B, and all such elements must be removed if we are to meet requirement N, necessitate conditions 1.2 and 1.3. Though there may be infinitely many t such that $u'(W_e^t, A^t, k)\downarrow$, each such set may contain an element of K which must later be removed from A for the sake of a higher priority requirement. To overcome this difficulty the trick is to attempt to put k in $W_e(A)$ at a stage when the restraints imposed by higher priority requirements are minimal and $A \cap B$ is disjoint from K on or below these restraints. This makes condition 1.4 necessary.

At stage $s+1$ ($s \equiv 2 \bmod 3$), we attempt to meet requirement N, while doing the least amount of damage to our strategy for meeting the other requirements. We also pursue our strategy for meeting requirements Q_n. If Case 1 holds then the removal of F^s from A or

B does not disturb the strategy associated with requirement Q_p where $p \leq p'$ or with P_p where $p < p'$. If Subcase 1.1 holds then meeting requirement $P_{p'}$ involves attempting to put k in $W_e(A)$ where $p' = 2\langle k,e\rangle$. Hence removal of F^s from B assures that if $k \in W_e^s(A)$ then $k \in W_e^{s+1}(A^{s+1})$. But then for $\sigma \in \mathcal{S}$ such that $\sigma \subseteq \sigma(s-2,s-2)$, $E(\sigma,s) = E(\sigma,s-2)$ is no longer necessarily contained in $W_{e(\sigma)}^{s+1}(A^{s+1}) \cap W_{e(\sigma)}^{s+1}(B^{s+1})$. However since $A^{s-2} \subseteq A^{s+1}$ (Proposition 4.1), $E(\sigma,s) \subseteq W_{e(\sigma)}^{s+1}(A^{s+1})$. Therefore $E(\sigma,s) \subseteq W_{e(\sigma)}^{s+1}(r(\sigma,A,s+1))$, which is as desired.

If Case 2 holds then we consider the minimal $\sigma \in \mathcal{S}_{\leq p'}$ of *greatest length* such that the strategy associated with σ is threatened, and we attempt to preserve the strategy associated with every $\tau \subseteq \sigma$ such that $\tau \in \mathcal{S}$. The length condition is necessary if we hope to satisfy all the requirements Q_n. The simplest remedy is to choose a set X, either A or B, and arrange that for every $\tau \subseteq \sigma$ such that $\tau \in \mathcal{S}$, $E(\tau,s) \subseteq W_{e(\tau)}^{s+1}(r(\tau,X,s+1))$, and in choosing X to backtrack and choose the same set which was used at the last stage at which similar adjustments had to be made. There are two advantages to this approach. Firstly, it safeguards us against the following situation: there are $\tau', \tau'' \subseteq \sigma$ with $\tau', \tau'' \in \mathcal{S}$, $E(\tau',s) \subseteq W_{e(\tau')}^{s+1}(r(\tau',A,s+1))$, $E(\tau'',s) \subseteq W_{e(\tau'')}^{s+1}(r(\tau'',B,s+1))$ and there is a stage $t > s$ ($t \equiv 2 \bmod 3$) such that $r(\tau',A,t) = r(\tau',A,s+1)$, $r(\tau'',B,t) = r(\tau'',B,s+1)$ and $K^{t+1} \cap r(\tau',A,t) \cap r(\tau'',B,t) \neq \emptyset$. Now $E(\tau,t) = E(\tau,t-1)$ for $\tau = \tau'$, τ''. So here the removal of F^t from either A or B may disrupt the strategy associated with τ' or τ'', and if this situation was repeated infinitely often, there is no guarantee that we could even satisfy the requirements Q_0 and Q_1. Secondly, if $\sigma \in \mathcal{S}$ and $\sigma < \sigma_{lh(\sigma)}$ then *backtracking* allows the restraints associated with σ to settle down. A danger inherent in this approach is that we may choose the same set *too often* and force one of the sets A or B to be ω. But this difficulty can be overcome by noting that at stages $t \in C_\sigma$, $E(\tau,t) \subseteq W^t_{e(\tau)}(A^t) \cap W^t_{e(\tau)}(B^t)$ for every $\tau \subseteq \sigma$, with $\tau \in \mathcal{S}$. So if C_σ is infinite, then there are infinitely many opportunities to *switch* sets. If $R(p'',s+1) \neq R(p'',s)$ or A or B is disturbed below $R(p'',s)$ then the strategy associated with requirement Q_p is abandoned for $p > p''$.

The choice of enumeration of K and an inspection of the construction yield the next four propositions.

Proposition 4. *For every* s,

.1. $X^{3s} = X^{3s+1} \subseteq X^{3s+2}$ *for* $X = A, B$,

.2. $A^{3s+2} \subseteq A^{3s+3}$ *or* $B^{3s+2} \subseteq B^{3s+3}$,

.3. $A^{3s} \subseteq A^{3s+3}$ *or* $B^{3s} \subseteq B^{3s+3}$,

.4. $K^t \cap A^t \cap B^t = \emptyset$ *for* $t = 3s, 3s+1$,

Proposition 5. *For every* $\sigma \in \mathcal{S}$, *if* σ *is OFF at stage* s *and ON at stage* $s+1$ *then* $s \equiv 0 \bmod 3$ *and* $\sigma \subseteq \sigma(s,s)$.

Proposition 6. *For every* $\sigma \in \mathcal{S}$, *if* σ *is ON at stage* $s+1$ *then* $lh(\sigma) \leq s$.

Proposition 7. *For every* $\sigma \in \mathcal{S}$, *if* $\sigma > \sigma(3s,3s)$ *then* σ *is OFF at stages* $3s+1$ *to* $3s+3$.

Proposition 8. *For every* $\sigma, \tau \in \mathcal{S}$, *if* σ *is ON at stage* $s+1$ *and* $\tau \subseteq \sigma$ *then* τ *is ON at stage* $s+1$.

Proof. Suppose not. Choose $\sigma, \tau \in \mathcal{S}$ with $\tau \subseteq \sigma$, and a least s such that σ is *ON* at stage s+1 and τ is *OFF*. We arrive at a contradiction by showing that τ is *ON* at stage s+1.

Case 1. $s \equiv 0 \bmod 3$.

Since σ is *ON* at stage s+1, $\sigma < \sigma(s,s)$ or $\sigma \subseteq \sigma(s,s)$ (Propositions 6 and 7). Therefore $\tau < \sigma(s,s)$ or $\tau \subseteq \sigma(s,s)$. If $\tau \subseteq \sigma(s,s)$ then τ is turned *ON* at stage s+1.

Assume $\tau < \sigma(s,s)$. Then $\sigma < \sigma(s,s)$. σ is not turned *ON* at stage s+1, therefore σ must be *ON* at stage s. So τ must be *ON* at stage s (by our choice of s), and τ is not turned *OFF* at stage s+1.

Case 2. $s \equiv 1 \bmod 3$.

As in the previous case where $\tau < \sigma(s,s)$, σ and τ must be *ON* at stage s. Since σ is not turned *OFF* at stage s+1, either $p^{*}\uparrow$

or $e(\tau) < e(\sigma) \leq p^*$. In either case τ is not turned *OFF* at stage s+1.

Case 3. $s \equiv 2 \bmod 3$.

As in the previous case σ and τ must be *ON* at stage s. At stage s+1, $e(\tau) < e(\sigma) \leq p''$ since σ is not turned *OFF* at stage s+1. Now either Case 1 holds, or $\sigma \succ \sigma'\!\downarrow$ since σ is not turned *OFF* at stage s+1, hence $\tau \succ \sigma'$. In either case τ is not turned *OFF* at stage s+1. □

Corollary 9. *For every* $\sigma, \tau \in \mathcal{S}$, *if* $s+1$ *is* σ*-active and* $\tau \subseteq \sigma$ *then* $s+1$ *is* τ*-active.*

Proof. This follows from the previous proposition and an inspection of the construction. □

Proposition 10. *For every* $\sigma \in \mathcal{S}$, *if* $\sigma \subseteq \sigma(3s,3s)$ *then*

.1. $r(\sigma,X,3s+1) = r(\sigma,X,3s+2) = \emptyset$ *for* $X = A, B$,

.2. $E(\sigma,3s) = E(\sigma,t)$ *for* $t = 3s+1, 3s+2$,

.3. $E(\sigma,3s) \subseteq W^t_{e(\sigma)}(A^t) \cap W^t_{e(\sigma)}(B^t)$ *for* $t = 3s, 3s+1, 3s+2$.

Proof. Assume $\sigma \in \mathcal{S}$ and $\sigma \subseteq \sigma(3s,3s)$. Then $\sigma(lh(\sigma),3s) = \sigma$ (Proposition 1.1), $c(\sigma,3s) = 3s$ and $E(\sigma,3s) = W^{3s}_{e(\sigma)}(A^{3s})\restriction L(e(\sigma),3s) = W^{3s}_{e(\sigma)}(B^{3s})\restriction L(e(\sigma),3s)$. 1 follows from an inspection of the construction. $C_\sigma \subseteq C_\emptyset = \{s\colon s \equiv 0 \bmod 3\}$ gives 2, and 3 follows from

Proposition 4.1. □

Definition. For every $\sigma \in \mathcal{S}$,

$$O(\sigma,s) = \{u\colon \forall t\ [(u \leq t \leq s) \Rightarrow \sigma \text{ is ON at stage } t]\}.$$

$$o(\sigma,s) = \begin{cases} [\min O(\sigma,s)]\dot{-}1, & \text{if } O(\sigma,s) \neq \emptyset, \\ \uparrow & \text{otherwise.} \end{cases}$$

Remark. Since all σ are *OFF* at stage 0, if $o(\sigma,s)\downarrow$ then $o(\sigma,s) = [\min O(\sigma,s)]-1$. So σ is *OFF* at stage $o(\sigma,s)$.

Proposition 11. *If* $o(\sigma,s)\downarrow$, *then* $o(\sigma,s) \equiv 0 \bmod 3$ *and* $\sigma \subseteq \sigma(o(\sigma,s),o(\sigma,s))$.

Proof. This is an easy corollary to Proposition 5 and the previous Remark. □

Proposition 12. *For every* $\sigma \in \mathcal{S}$, *if* $s+1$ *is* σ*-active then*

.1. $s \equiv 2 \bmod 3$,

.2. σ *is ON at stage* $s+1$,

.3. $r(\sigma,X,s+1) \subseteq X^{s+1}$ *for* $X = A, B$,

.4. $r(\sigma,A,s+1) \cap B^{s+1} \cap K^{s+1} = r(\sigma,B,s+1) \cap A^{s+1} \cap K^{s+1} = \emptyset$.

Proof. Assume $\sigma \in \mathcal{S}$ and s+1 is σ-active. An inspection of the construction yields 1-3. 4 follows from 3 and Proposition 4.4. □

Proposition 13. *For every* $\sigma, \tau \in \mathcal{S}$, *if* s+1 *is* σ*-active and* $\tau > \sigma$ *then* τ *is OFF at stage* s+1.

Proof. Assume $\sigma, \tau \in \mathcal{S}$, s+1 is σ-active and $\tau > \sigma$. Then $s \equiv 2 \bmod 3$ (Proposition 12.1). If Case 1 holds at stage s+1 then $\sigma \subseteq \sigma(s-2,s-2)$, so τ is *OFF* at stage s+1 (Proposition 7). If Case 2 holds then $\sigma \subseteq \sigma'$; since $\tau > \sigma$, $\tau > \sigma'$, therefore τ is turned *OFF* at stage s+1. □

Proposition 14. *For every* $\sigma \in \mathcal{S}$, *if* $\sigma \subseteq \sigma(3s,3s)$ *and* σ *is ON at stage* 3s+3 *then*

.1. 3s+3 *is* σ*-active,*

.2. $E(\sigma,3s+2) \subseteq W^{3s+3}_{e(\sigma)}(r(\sigma,X,3s+3))$, *where* X *is the set marked at stage* 3s+3.

Proof. Assume $\sigma \in \mathcal{S}$, $\sigma \subseteq \sigma(3s,3s)$ and σ is *ON* at stage 3s+3. Then σ is *ON* at stages 3s+1 and 3s+2 (Proposition 5) and $E(\sigma,3s+2) = E(\sigma,3s) \subseteq W^{3s+2}_{e(\sigma)}(A^{3s+2}) \cap W^{3s+2}_{e(\sigma)}(B^{3s+2})$ (Propositions 10.2 and 10.3).

Suppose Case 1 holds at stage 3s+3. Then 1 clearly holds. If Subcase 1.1 holds then A is marked at stage 3s+3 and $A^{3s+3} = A^{3s+2}$, so $E(\sigma,3s+2) \subseteq W^{3s+3}_{e(\sigma)}(A^{3s+3})$. Hence $E(\sigma,3s+2) \subseteq W^{3s+3}_{e(\sigma)}(r(\sigma,A,3s+3))$ by definition of $r(\sigma,A,3s+3)$. If Subcase 1.2 holds we get a similar result.

Suppose Case 2 holds at stage $3s+3$. Then $r(\sigma',A,3s+2) \neq \emptyset$ or $r(\sigma',B,3s+2) \neq \emptyset$, by definition of σ', so σ' must be *ON* at stage $3s+2$. Since $\sigma \subseteq \sigma(3s,3s)$, $\sigma' \not\subseteq \sigma$ and $\sigma' \nmid \sigma$ (Propositions 10.1 and 7). $\sigma \nmid \sigma'$ since σ is not turned *OFF* at stage $3s+3$. Therefore $\sigma \subset \sigma'$. So 1 clearly holds. If Subcase 2.1 holds then A is marked at stage $3s+3$ and $A^{3s+2} \subseteq A^{3s+3}$, so $E(\sigma,3s+2) \subseteq W^{3s+3}_{e(\sigma)}(r(\sigma,A,3s+3))$. If Subcase 2.2 holds we get a similar result. □

Corollary 15. *For every* $\sigma \in \mathcal{S}$, *if* $r(\sigma,A,s) \neq \emptyset$ *or* $r(\sigma,B,s) \neq \emptyset$ *then* $O(\sigma,s)$ *contains a* σ*-active stage.*

Proof. Assume $r(\sigma,A,s) \neq \emptyset$ or $r(\sigma,B,s) \neq \emptyset$ for some $\sigma \in \mathcal{S}$. Then σ must be *ON* at stage s. Hence $O(\sigma,s) \neq \emptyset$, so $o(\sigma,s)\downarrow$. Now $s \geq o(\sigma,s)+3$ (Propositions 11 and 10.1), therefore $o(\sigma,s)+3 \in O(\sigma,s)$ is σ-active (Propositions 11 and 14.1). □

Proposition 16. *For every* $\sigma \in \mathcal{S}$, *if* σ *is ON at stage* $s+1$ *then*

.1. $r(\sigma,X,s+1) \subseteq X^{s+1}$ *for* $X = A, B$,

.2. $r(\sigma,A,s+1) \cap B^{s+1} \cap K^{s+1} = r(\sigma,B,s+1) \cap A^{s+1} \cap K^{s+1} = \emptyset$,

.3. *if* $\neg[s \equiv 0 \bmod 3$ *and* $\sigma \subseteq \sigma(s,s)$ *or* $s \equiv 1 \bmod 3$ *and* $\sigma \subseteq \sigma(s-1,s-1)]$ *then*

  .1. $a(\sigma,s+1)$ *is a* σ-*active stage in* $O(\sigma,s+1)$,

  .2. $r(\sigma,X,s+1) = r(\sigma,X,a(\sigma,s+1))$ *for* $X = A, B$,

  .3. $r(\sigma,A,s+1) \cap B^{s+1} = r(\sigma,A,s+1) \cap B^{a(\sigma,s+1)}$ *and* $r(\sigma,B,s+1) \cap A^{s+1} = r(\sigma,B,s+1) \cap A^{a(\sigma,s+1)}$,

  .4. $E(\sigma,s) \subseteq W^{s+1}_{e(\sigma)}(r(\sigma,X,s+1))$, *where* X *is the set marked at stage* $a(\sigma,s+1)$.

Proof. Fix $\sigma \in \mathcal{S}$. The proof is by induction on s. Assume 1–3 hold for every $s < t$. We show that 1–3 hold for $s = t$. Assume σ is ON at stage $t+1$. If $|O(\sigma,t+1)| \leq 3$ then 1–3 follow from Propositions 11, 10.1, 14 and 12. Suppose $|O(\sigma,t+1)| > 3$.

If $t \equiv 0 \bmod 3$ and $\sigma \subseteq \sigma(t,t)$, or $t \equiv 1 \bmod 3$ and $\sigma \subseteq \sigma(t-1,t-1)$ then 1 and 2 follow from Proposition 10.1 and 3 holds vacuously. So assume that $\neg[t \equiv 0 \bmod 3$ and $\sigma \subseteq \sigma(t,t)$ or $t \equiv 1 \bmod 3$ and $\sigma \subseteq \sigma(t-1,t-1)]$. If $t \equiv 2 \bmod 3$ and $\sigma \subseteq \sigma(t-2,t-2)$ then 1–3 follow from Propositions 14 and 12, so assume that $\neg[t \equiv 2 \bmod 3$ and $\sigma \subseteq \sigma(t-2,t-2)]$.

Then by the induction hypothesis the following hold:

1'. $r(\sigma,X,t) \subseteq X^t$ for $X = A, B$.

2'. $r(\sigma,A,t) \cap B^t \cap K^t = r(\sigma,B,t) \cap A^t \cap K^t = \emptyset$.

3'. .1. $a(\sigma,t)$ is a σ-active stage in $O(\sigma,t)$.

.2. $r(\sigma,X,t) = r(\sigma,X,a(\sigma,t))$ for $X = A, B$.

.3. $r(\sigma,A,t) \cap B^t = r(\sigma,A,t) \cap B^{a(\sigma,t)}$ and

$r(\sigma,B,t) \cap A^t = r(\sigma,B,t) \cap A^{a(\sigma,t)}$.

.4. $E(\sigma,t-1) \subseteq W^t_{e(\sigma)}(r(\sigma,X,t))$, where X is the set marked at stage $a(\sigma,t)$.

$O(\sigma,t+1) = O(\sigma,t) \cup \{t+1\}$ and 3'.1 imply 3.1. Since $\sigma \not\subseteq \sigma(t,t)$, $E(\sigma,t) = E(\sigma,t-1)$.

Case 1. $t \equiv 0 \bmod 3$.

Then $X^{t+1} = X^t$ for $X = A, B$ (Proposition 4.1). Since $\sigma \not\subseteq \sigma(t,t)$ and σ is not turned *OFF* at stage $t+1$, $r(\sigma,X,t+1) = r(\sigma,X,t)$ for $X = A, B$. So 1 follows from 1'. 2 follows from 1 and Proposition 4.4. Since $t \not\equiv 2 \bmod 3$, $t+1$ cannot be σ-active, so $a(\sigma,t+1) = a(\sigma,t)$. 3.2–3.4 follow from 3'.2–3'.4.

Case 2. $t \equiv 1 \bmod 3$.

Since σ is not turned *OFF* at stage $t+1$, $r(\sigma,X,t+1) = r(\sigma,X,t)$ for $X = A, B$. So 1 follows from 1' and Proposition 4.1. As in Case 1, $a(\sigma,t+1) = a(\sigma,t)$. So 3.2 and 3.4 follow from 3'.2 and 3'.4.

Now $K^{t+1} = K^t$ by the choice of enumeration of K. If $p^*\uparrow$ at stage $t+1$ then $X^{t+1} = X^t$ for $X = A, B$, so 2 and 3.3 follow from 2' and 3'.3.

Otherwise $e(\sigma) \leq p^*$, since σ is not turned *OFF* at stage $t+1$. Assume Case 1 holds at stage $t+1$. Then $A^{t+1} = A^t \cup D$ and $B^{t+1} = B^t$ where D satisfies 1.1–1.4 (at stage $t+1$). Now $r(\sigma,B,t+1) \cap A^{t+1} = (r(\sigma,B,t) \cap A^t) \cup (r(\sigma,B,t) \cap D) = r(\sigma,B,t) \cap A^t$ by 1.3. So 3.3 follows from 3'.3. $r(\sigma,A,t+1) \cap B^{t+1} \cap K^{t+1} = r(\sigma,A,t) \cap B^t \cap K^t$ and $r(\sigma,B,t+1) \cap A^{t+1} \cap K^{t+1} = r(\sigma,B,t) \cap A^t \cap K^t$ from above. So 2 follows from 2'. If Case 2 holds we get a similar result.

Case 3. $t \equiv 2 \bmod 3$.

Since σ is not turned *OFF* at stage $t+1$, $e(\sigma) \leq p'' \leq p'$.

If we can show that 1 holds then 2 follows from Proposition 4.4.

Assume Case 1 holds at stage $t+1$. Since $\sigma \not\subseteq \sigma(t-2,t-2)$, $t+1$ cannot be σ-active, so $a(\sigma,t+1) = a(\sigma,t)$. Since σ is not turned *OFF* at stage $t+1$, $r(\sigma,X,t+1) = r(\sigma,X,t)$ for $X = A, B$. So 3.2 and 3.4 follow from 3'.2 and 3'.4. $e(\sigma) \leq p'$ implies $r(\sigma,A,t) \cap F^t = r(\sigma,B,t) \cap F^t = \emptyset$. So 1 and 3.3 follow from the definition of A^{s+1} and B^{s+1}, 1' and 3'.3.

Assume Case 2 holds at stage $t+1$. Then $r(\sigma',A,t) \neq \emptyset$ or $r(\sigma',B,t) \neq \emptyset$, so $a(\sigma',t)$ is a σ'-active stage in $O(\sigma',t)$ (Corollary 15). Since σ is not turned *OFF* at stage $t+1$, $\sigma \nsucc \sigma'$, so $\sigma \subseteq \sigma'$, $\sigma < \sigma'$ or $\sigma' \subset \sigma$.

Assume $\sigma \subseteq \sigma'$. Then $t+1$ is σ-active and 1, 3.2 and 3.3

clearly hold. Now $O(\sigma',t) \subseteq O(\sigma,t)$ (Proposition 8), so $\alpha(\sigma',t) \in O(\sigma,t)$ and $\alpha(\sigma',t)$ is σ-active (Corollary 9).

Assume Subcase 2.1 holds at stage $t+1$. Now either $E(\sigma,t) = E(\sigma,\alpha(\sigma',t)-1)$ or $\alpha(\sigma',t) \leq c(\sigma,t) \leq t$. In the former case $E(\sigma,t) \subseteq W_{e(\sigma)}^{\alpha(\sigma',t)}(r(\sigma,A,\alpha(\sigma',t)))$ and $r(\sigma,A,\alpha(\sigma',t)) \subseteq A^{\alpha(\sigma',t)}$, by the induction hypothesis; $A^{\alpha(\sigma',t)} \subseteq A^{t+1}$, by definition of A^{t+1}, so $E(\sigma,t) \subseteq W_{e(\sigma)}^{t+1}(A^{t+1})$, hence $E(\sigma,t) \subseteq W_{e(\sigma)}^{t+1}(r(\sigma,A,t+1))$ by definition of $r(\sigma,A,t+1)$. In the latter case $E(\sigma,t) \subseteq W_{e(\sigma)}^{c(\sigma,t)}(A^{c(\sigma,t)})$; since $\alpha(\sigma',t) \leq c(\sigma,t) \leq t$, $A^{c(\sigma,t)} \subseteq A^{t+1}$, by definition of A^{t+1}, and the rest goes as before. If Subcase 2.2 holds we get a similar result.

Now assume that $\sigma < \sigma'$ or $\sigma' \subset \sigma$. Then $t+1$ is not σ-active, so $\alpha(\sigma,t+1) = \alpha(\sigma,t)$. Since σ is not turned OFF at stage $t+1$, $r(\sigma,X,t+1) = r(\sigma,X,t)$ for $X = A, B$. So 3.2 and 3.4 follow from 3'.2 and 3'.4. $\alpha(\sigma,t) \leq \alpha(\sigma',t)$ (Proposition 13 and Corollary 9), and $\alpha(\sigma',t) \in O(\sigma,t)$, since $\alpha(\sigma,t) \in O(\sigma,t)$.

Assume Subcase 2.1 holds at stage $t+1$. For every u such that $\alpha(\sigma',t) \leq u+1 \leq t$, $\neg[u \equiv 0 \bmod 3$ and $\sigma \subseteq \sigma(u,u)$ or $u \equiv 1 \bmod 3$ and $\sigma \subseteq \sigma(u-1,u-1)]$. Otherwise $u+3$ or $u+2$ would be a σ-active stage less than or equal to $t+1$ (Proposition 14) and no such stage exists. So by the induction hypothesis, for all such u, $r(\sigma,X,u+1) = r(\sigma,X,\alpha(\sigma,u+1)) = r(\sigma,X,\alpha(\sigma,t)) = r(\sigma,X,t)$ for $X = A, B$, $r(\sigma,A,u+1) \cap B^{u+1} = r(\sigma,A,u+1) \cap B^{\alpha(\sigma,u+1)} = r(\sigma,A,t) \cap B^{\alpha(\sigma,t)} = r(\sigma,A,t) \cap B^{t}$ and $r(\sigma,B,u+1) \cap A^{u+1} = r(\sigma,B,u+1) \cap A^{\alpha(\sigma,u+1)} = r(\sigma,B,t) \cap A^{\alpha(\sigma,t)} = r(\sigma,B,t) \cap A^{t}$.

Then $r(\sigma,B,t+1) \cap A^{t+1} = r(\sigma,B,t) \cap \left[\bigcup_{a(\sigma',t)\leq u\leq t} A^u\right] =$

$\bigcup_{a(\sigma',t)\leq u\leq t} (r(\sigma,B,t) \cap A^u) = \bigcup_{a(\sigma',t)\leq u\leq t} (r(\sigma,B,u) \cap A^u) =$

$r(\sigma,B,t) \cap A^t$. Note that $r(\sigma,A,t) \cap F^t = r(\sigma,B,t) \cap F^t = \emptyset$ by the choice of σ'. So $r(\sigma,A,t+1) \cap B^{t+1} = r(\sigma,A,t) \cap [B^t - (A^{t+1} \cap K^{t+1})]$ $= r(\sigma,A,t) \cap B^t$ since $r(\sigma,A,t) \cap B^t \cap A^{t+1} \cap K^{t+1} \subseteq$ $r(\sigma,A,t) \cap B^t \cap K^{t+1} = r(\sigma,A,t) \cap F^t = \emptyset$ by 1'. So 3.3 follows from 3'.3.

We have already noted that $r(\sigma,X,t+1) = r(\sigma,X,t)$ for $X = A, B$. Now $A^t \subseteq A^{t+1}$ from stage $t+1$ and from above, $r(\sigma,B,t) \cap (B^t - B^{t+1}) = r(\sigma,B,t) \cap (B^t \cap A^{t+1} \cap K^{t+1}) =$ $(r(\sigma,B,t+1) \cap A^{t+1}) \cap B^t \cap K^{t+1} = (r(\sigma,B,t) \cap A^t) \cap B^t \cap K^{t+1} =$ $r(\sigma,B,t) \cap F^t = \emptyset$. So 1 follows from 1'. If Subcase 2.2 holds we get a similar result. □

Proposition 17. *For every $\sigma \in \mathcal{S}$, if σ is ON at stage $s+1$ then for every $p < e(\sigma)$, $R(p,s+1) = R(p,o(\sigma,s+1))$, $A^{s+1}\restriction R(p,s+1) = A^{o(\sigma,s+1)}\restriction R(p,s+1)$ and $B^{s+1}\restriction R(p,s+1) = B^{o(\sigma,s+1)}\restriction R(p,s+1)$.*

Proof. Fix $\sigma \in \mathcal{S}$ and $p < e(\sigma)$. Let $p = 2\langle k,e\rangle$ or $p = 2\langle k,e\rangle+1$. The proof is by induction on s. Assume σ is ON at stage s+1. If $|O(\sigma,s+1)| = 1$ then $s = o(\sigma,s+1) \equiv 0 \bmod 3$ (Proposition 11) and the result follows from Proposition 4.1 and the fact that $W_e^{s+1} = W_e^s$. So assume that $|O(\sigma,s+1)| > 1$, $R(p,s) = R(p,o(\sigma,s))$, $A^s\restriction R(p,s) = A^{o(\sigma,s)}\restriction R(p,s)$ and $B^s\restriction R(p,s) = B^{o(\sigma,s)}\restriction R(p,s)$.

Now $o(\sigma, s+1) = o(\sigma, s)$, so by the induction hypothesis it suffices to show that $R(p, s+1) = R(p, s)$ and $X^{s+1} \upharpoonright R(p, s) = X^{s} \upharpoonright R(p, s)$ for $X = A, B$.

Case 1. $s \equiv 0 \mod 3$.

$W_e^{s+1} = W_e^s$ and Proposition 4.1 imply the result.

Case 2. $s \equiv 1 \mod 3$.

If $p^* \uparrow$ at stage $s+1$ then the situation is similar to that in Case 1.

Otherwise $p < e(\sigma) \leq p^*$, since σ is not turned *OFF* at stage $s+1$. Assume Case 1 holds. Then $A^{s+1} \upharpoonright R(p, s) = A^{s} \upharpoonright R(p, s)$ (by 1.2 at stage $s+1$) and $B^{s+1} = B^{s}$. $R(p, s+1) = R(p, s)$ since $W_e^{s+1} = W_e^s$. If Case 2 holds we get a similar result.

Case 3. $s \equiv 2 \mod 3$.

Since σ is not turned *OFF* at stage $s+1$, $e(\sigma) \leq p''$ and the result holds by definition of p''. □

Proposition 18. *For every* p,

.1. *if* $\sigma_{p+1} \in \mathcal{S}$ *then*

$\exists u\ (\forall t > u)\ (\forall s > t)\ [E(\sigma_{p+1},t) \subseteq W_p^s(A^s)$ *or* $E(\sigma_{p+1},t) \subseteq W_p^s(B^s)]$,

.2. *if* $\sigma_{p+1} \notin \mathcal{S}$ *then*

$(\exists^{<\infty}\sigma \in \mathcal{S})\ [\sigma_p\hat{\ }(1) \subseteq \sigma$ *and* $\exists s\ [r(\sigma,A,s) \neq \emptyset$ *or* $r(\sigma,B,s) \neq \emptyset]]$,

.3. *if* $\sigma_{p+1} \notin \mathcal{S}$ *then for every* $\sigma \in \mathcal{S}$ *such that* $\sigma_p\hat{\ }(1) \subseteq \sigma$

.1. $\lim_s r(\sigma,X,s)$ *exists for* $X = A,\ B$,

.2. $\lim_s r(\sigma,B,s) \cap A^s$ *and* $\lim_s r(\sigma,A,s) \cap B^s$ *exist*,

.3. $\lim_s r(\sigma,B,s) \cap A^s$ *and* $\lim_s r(\sigma,A,s) \cap B^s$ *are disjoint from* K,

.4. *if* $p = 2\langle k,e\rangle$ *then* $k \notin W_e^s(A^s)$ *for sufficiently large* s *or* $\lim_s u'(W_e^s,A^s,k)$ *exists and*

if $p = 2\langle k,e\rangle+1$ *then* $k \notin W_e^s(B^s)$ *for sufficiently large* s *or* $\lim_s u'(W_e^s,B^s,k)$ *exists*,

.5. .1. $\lim_s R(p,s)$ *exists*,

.2. $\lim_s X^s{\upharpoonright}R(p,s)$ *exists for* $X = A,\ B$,

.3. $(A \cap B \cap K){\upharpoonright}(\lim_s R(p,s)) = \emptyset$.

Proof. The proof is by induction on p. Assume 1-5 hold for every $p < q$. We show that 1-5 hold for $p = q$. By the induction hypothesis we can choose a stage $w > q$ such that for every $q' < q$ the following hold:

3'. If $\sigma_{q'+1} \notin \mathcal{S}$ then for every $\sigma \in \mathcal{S}$ such that $\sigma_{q'}\hat{\ }(1) \subseteq \sigma$, $r(\sigma,A,s)$, $r(\sigma,B,s)$, $r(\sigma,B,s) \cap A^s$ and $r(\sigma,A,s) \cap B^s$ have reached a limit by stage w and the limits of the latter two are disjoint from K.

4'. If $q' = 2\langle k,e\rangle$ then $u'(W_e^s,A^s,k)$ has reached a limit by stage w or for every $s > w$, $k \notin W_e^s(A^s)$ and if $q' = 2\langle k,e\rangle+1$ then $u'(W_e^s,B^s,k)$ has reached a limit by stage w or for every $s > w$, $k \notin W_e^s(B^s)$.

5'. $R(q',s)$, $A^s\restriction R(q',s)$ and $B^s\restriction R(q',s)$ have reached a limit by stage w and $(A \cap B \cap K)\restriction(\lim_s R(q',s)) = \emptyset$.

1. First assume that $\sigma_{q+1} \in \mathcal{S}$. Choose a stage $v > w$ such that σ_{q+1} is ON at stage v and for every $t \geq v$, $\sigma(t,t) \nless \sigma_{q+1}$.

Claim 1. *For every $t \geq v$, σ_{q+1} is ON at stage t.*

Proof. The proof is by induction on t. σ_{q+1} is ON at stage v. Towards a contradiction, suppose that σ_{q+1} is ON at stage $t \geq v$ and OFF at stage $t+1$. Note that $e(\sigma_{q+1}) = q$.

Case 1. $t \equiv 0 \bmod 3$.

From the construction $\sigma(t,t) < \sigma_{q+1}$ which contradicts the choice of v.

Case 2. $t \equiv 1 \bmod 3$.

Then $p^{*}\downarrow < q$ at stage $t+1$. This contradicts 4' for $q' = p^{*}$, by the action taken at stage $t+1$.

Case 3. $t \equiv 2 \bmod 3$.

Then at stage $t+1$, $p' < q$, $p'' < q \leq p'$, or Case 2 holds and $\sigma' < \sigma_{q+1}$. Suppose $p' < q$. Then $p' < q < w < t$. But this contradicts 5' for $q' = p'$, by definition of p'. By the same argument $\neg [p'' < q \leq p']$. So Case 2 must hold with $\sigma' < \sigma_{q+1}$. Now $\sigma_{q+1} = \sigma_q\hat{}(1)$ since $\sigma_{q+1} \in \mathcal{S}$. Therefore $\sigma' < \sigma_{q+1}$ implies $\sigma' < \sigma_q$. Hence for some $q' < q$, $\sigma_{q'+1} \notin \mathcal{S}$ and $\sigma_{q'}\hat{}(1) \subseteq \sigma'$. But $r(\sigma',A,s) \cap F^s \neq \emptyset$ or $r(\sigma',B,s) \cap F^s \neq \emptyset$, which contradicts 3'. ◻

Claim 2. $(\forall t \geq \upsilon)\ (\forall s \geq t)\ [E(\sigma_{q+1},t) \subseteq E(\sigma_{q+1},s)]$.

Proof. Fix $t \geq \upsilon$. The proof is by induction on $s \geq t$. For $s = t$ the result is clear. Assume $E(\sigma_{q+1},t) \subseteq E(\sigma_{q+1},s)$ for some $s \geq t$. By the induction hypothesis it suffices to show that $E(\sigma_{q+1},s) \subseteq E(\sigma_{q+1},s+1)$. If $s+1 \notin C_{\sigma_{q+1}}$ then $E(\sigma_{q+1},s+1) = E(\sigma_{q+1},s)$. So assume that $s+1 \in C_{\sigma_{q+1}}$. Let $s' = c(\sigma_{q+1},s)$. Then $L(q,s+1) > L(q,s')$, $E(\sigma_{q+1},s) = W_q^{s'}(A^{s'}) \restriction L(q,s')$ and $E(\sigma_{q+1},s+1) = W_q^{s+1}(A^{s+1}) \restriction L(q,s+1) = W_q^{s+1}(B^{s+1}) \restriction L(q,s+1)$. Now $E(\sigma_{q+1},s) \subseteq W_q^{s+1}(A^{s+1})$ or $E(\sigma_{q+1},s) \subseteq W_q^{s+1}(B^{s+1})$ (Propositions 16.3.4 and 16.1). So $E(\sigma_{q+1},s) \subseteq E(\sigma_{q+1},s+1)$. ◻

1 follows from Claims 1 and 2 and Propositions 10.2, 10.3, 16.3.4 and 16.1.

2-3. Now assume that $\sigma_{q+1} \notin \mathcal{S}$. Then we can choose a stage $\upsilon > w$ such that for every $s \geq \upsilon$, if $s \equiv 0 \bmod 3$ then $\sigma_q\hat{}(1) \not\subseteq \sigma(s,s)$. By Propositions 6 and 5 no σ with $lh(\sigma) > \upsilon$ and $\sigma_q\hat{}(1) \subseteq \sigma$ is ever turned *ON*, whence for all such σ, $r(\sigma,A,s) = r(\sigma,B,s) = \emptyset$ for every s. Hence 2 holds.

Also note that if $\sigma \in \mathcal{S}$ and $\sigma_q\hat{}(1) \subseteq \sigma$ then the status of σ reaches a limit since if σ is turned *OFF* after stage υ then σ remains *OFF*. So in addition we may assume that for every $\sigma \in \mathcal{S}$ with $\sigma_q\hat{}(1) \subseteq \sigma$ the status of σ has reached a limit by stage υ.

If for every $\sigma \in \mathcal{S}$ such that $\sigma_q\hat{}(1) \subseteq \sigma$ and σ is *ON* at stage υ there are only finitely many σ-active stages then $\lim_s \alpha(\sigma,s)$ exists and 3 follows from Propositions 16.3.2, 16.3.3 and 16.2.

Suppose not. Choose $\sigma \in \mathcal{S}$ first $\leq$-minimal and then of greatest length such that $\sigma_q\hat{}(1) \subseteq \sigma$, σ is *ON* at stage υ and there are infinitely many σ-active stages. Choose $u > \upsilon$ such that $u+1$ is σ-active and for every $t > u$ and every $\tau \supset \sigma$, t is not τ-active. Note that if $\sigma'\downarrow$ at stage $t > u$ and $\sigma \subseteq \sigma'$, then t is σ'-active since σ' is *ON* at stage t, by the choice of υ.

Remark 1. If $t > u$ and t is σ-active then Case 2 must hold at stage t since $\sigma \not\subseteq \sigma(t-3,t-3)$; also $\sigma' = \sigma$, by the choice of u.

Assume that A is marked at stage $u+1$.

Claim 3. *For every* $t \geq u$,

.1. $r(\sigma,X,\alpha(\sigma,t+1)) = r(\sigma,X,u+1)$ for $X = A, B$,

.2. A *is marked at stage* $\alpha(\sigma,t+1)$.

Proof. The proof is by induction on t. The result is clear for $t = u$. An inspection of the construction shows that since A is marked at stage $u+1$, $r(\sigma,B,u+1) = \emptyset$. Assume that for some $t > u$ the following hold:

1'. $r(\sigma,X,\alpha(\sigma,t)) = r(\sigma,X,u+1)$ for $X = A, B$.

2'. A is marked at stage $\alpha(\sigma,t)$.

If $t+1$ is not σ-active then $\alpha(\sigma,t+1) = \alpha(\sigma,t)$ and 1-2 follow from 1'-2'.

So assume that $t+1$ is σ-active. 2 follows from Remark 1, an inspection of the construction and 2'. Since $\alpha(\sigma,t) \geq u+1$, $E(\sigma,t) = E(\sigma,\alpha(\sigma,t)-1)$ by the choice of v. Now $E(\sigma,\alpha(\sigma,t)-1) \subseteq W_{e(\sigma)}^{\alpha(\sigma,t)}(r(\sigma,A,\alpha(\sigma,t)))$ (Proposition 16.3.4), $r(\sigma,A,\alpha(\sigma,t)) =_{dfn} U(W_{e(\sigma)},A,E(\sigma,\alpha(\sigma,t)-1),\alpha(\sigma,t))$ and $r(\sigma,A,t') = r(\sigma,A,\alpha(\sigma,t)) \subseteq A^{t'}$ for every t' such that $\alpha(\sigma,t) \leq t' \leq t$ (Propositions 16.3.2 and 16.1). From the construction $A^t \subseteq A^{t+1}$, hence $r(\sigma,A,t+1) =_{dfn}$

$U(W_{e(\sigma)}, A, E(\sigma,t), t+1) = r(\sigma, A, \alpha(\sigma,t))$, while $r(\sigma,B,t+1) = \emptyset$. 1 follows from 1'. □

So for every $t \geq u$, $r(\sigma,X,t+1) = r(\sigma,X,u+1)$ for $X = A, B$ (Claim 3.1 and Proposition 16.3.2). Hence we can choose $u' > u$ such that $u'+1$ is σ-active, $r(\sigma,A,u'+1) \cap K^{u'+1} = r(\sigma,A,u'+1) \cap K$ and $r(\sigma,A,u'+1) \cap B^{u'+1} \cap K^{u'+1} = \emptyset$ (Proposition 16.2).

Claim 4. *For every* $t \geq u'$, $\alpha(\sigma,t+1) = u'+1$.

Proof. The proof is by induction on t. The result is clear for $t = u'$. Assume that $\alpha(\sigma,t) = u'+1$ for some $t > u'$.

If $t+1$ is not σ-active then $\alpha(\sigma,t+1) = \alpha(\sigma,t) = u'+1$. So assume that $t+1$ is σ-active. Then by Remark 1, $\sigma' = \sigma$ so $r(\sigma,A,t) \cap F^t \neq \emptyset$ or $r(\sigma,B,t) \cap F^t \neq \emptyset$. We have shown that $r(\sigma,B,t) = \emptyset$, and $r(\sigma,A,t) \cap F^t = r(\sigma,A,t) \cap A^t \cap B^t \cap K^{t+1} \subseteq r(\sigma,A,t) \cap B^t \cap K^{t+1} = r(\sigma,A,u'+1) \cap B^{u'+1} \cap K^{t+1}$ by the induction hypothesis and Propositions 16.3.2 and 16.3.3. So $r(\sigma,A,t) \cap F^t = \emptyset$ by the choice of u', which is a contradiction. Therefore $t+1$ is not σ-active. □

Claim 4 contradicts the assumption that there are infinitely many σ-active stages. If B is marked at stage $u+1$ we get a similar result. So for every $\sigma \in \mathcal{S}$ such that $\sigma_q\hat{}(1) \subseteq \sigma$ and σ is ON at stage v there are only finitely many σ-active stages. So 3 holds.

4-5. Assume $q = 2\langle k,e\rangle$. Choose a stage $v > w$ such that for every $\sigma \in \mathcal{S}$ such that $\sigma < \sigma_{q+1}$, $r(\sigma,A,s)$, $r(\sigma,B,s)$, $r(\sigma,B,s) \cap A^s$ and $r(\sigma,A,s) \cap B^s$ have reached a limit by stage v and the limits of the latter two are disjoint from K. Suppose there exists D' such that

6'. $k \in W_e(D')$,

7'. $(\forall\sigma \in \mathcal{S})\ [\sigma < \sigma_{q+1} \Rightarrow D' \cap r(\sigma,B,v) \subseteq A^v \cap r(\sigma,B,v)]$,

8'. $(\forall q' < q)\ [D'{\restriction}R(q',v) \subseteq A^v{\restriction}R(q',v)]$.

Choose such a D' with least canonical index and a stage $u > v$ such that $u \equiv 1 \bmod 3$, $\sigma_{q+1} \subseteq \sigma(u-1,u-1)$, $k \in W_e^{u+1}(D')$ and $K^u{\restriction}(\max\ \{(\max D')+1,\ q+1\}) = K{\restriction}(\max\ \{(\max D')+1,\ q+1\})$.

Remark 2. For every $t > u$, if $D' \subseteq A^t$ then $u(W_e^t,A^t,k){\downarrow} = D'$. This follows from the definition of D' and the choice of w, v and u.

Now for every $\sigma \in \mathcal{S}_{\leq q}$ if $\sigma < \sigma_{q+1}$ or $\sigma \subseteq \sigma_{q+1}$ then $r(\sigma,X,u) = \emptyset$ for $X = A,\ B$ (Propositions 7 and 10.1). Examining stage $u+1$, we see that $u'(W_e^{u+1},A^{u+1},k){\downarrow} = D'$.

Claim 5. $u'(W_e^{u+2},A^{u+2},k) = D'$.

Proof. Since $D' \subseteq A^{u+1}{\restriction}R(q,u+1)$, by Remark 2 it suffices to show that $A^{u+1}{\restriction}R(q,u+1) \subseteq A^{u+2}$. $p' \geq q$ at stage $u+2$ by the choice of w.

If Case 1 holds at stage $u+2$ and $p' = q$ then $A^{u+2} = A^{u+1}$; if $p' > q$ then $F^{u+1}\restriction R(q,u+1) = \emptyset$, so $A^{u+1}\restriction R(q,u+1) \subseteq A^{u+2}$, by definition of A^{u+2}.

Suppose Case 2 holds at stage $u+2$. If Subcase 2.1 holds at stage $u+2$ then clearly $A^{u+1} \subseteq A^{u+2}$.

So assume that Subcase 2.2 holds. Since $r(\sigma',A,u+1) \cap F^{u+1} \neq \emptyset$ or $r(\sigma',B,u+1) \cap F^{u+1} \neq \emptyset$, $\sigma' \succ \sigma(u-1,u-1)$ (Proposition 7) and $\sigma' \not\subseteq \sigma(u-1,u-1)$ (Proposition 10.1). $\sigma_{q+1} \subseteq \sigma(u-1,u-1)$ implies $\sigma' \succ \sigma_{q+1}$ and $\sigma' \not\subseteq \sigma_{q+1}$. $\sigma' \prec \sigma_{q+1}$ by the choice of v. Therefore $\sigma_{q+1} \subset \sigma'$. Now $q < e(\sigma')$ and $a(\sigma',u+1)$ is a σ'-active stage in $O(\sigma',u+1)$ (Corollary 15). Then $B^{u+2} = \bigcup_{a(\sigma',u+1)\leq t\leq u+1} B^t$ and $A^{u+2} = A^{u+1} - (B^{u+2} \cap K^{u+2})$. $B^{u+2}\restriction R(q,u+1) = \bigcup_{a(\sigma',u+1)\leq t\leq u+1} (B^t\restriction R(q,u+1)) = B^{u+1}\restriction R(q,u+1)$ by Proposition 17, since $q < e(\sigma')$ and $o(\sigma',u+1) < a(\sigma',u+1)$. So $(A^{u+1} \cap B^{u+2} \cap K^{u+2})\restriction R(q,u+1) = (A^{u+1} \cap B^{u+1} \cap K^{u+2})\restriction R(q,u+1) = F^{u+1}\restriction R(q,u+1) = \emptyset$ since $q < e(\sigma') \leq p'$. Therefore $A^{u+1}\restriction R(q,u+1) \subseteq A^{u+2}$. □

Remark 3. Claim 5 implies that $R(q,u+2) = R(q,u+1)$. So $(A^{u+2} \cap B^{u+2} \cap K)\restriction R(q,u+2) = \emptyset$ by the choice of u and Proposition 4.4.

Claim 6. *For every* $t > u$,

1. $R(q,t+1) = R(q,u+2)$,
2. $A^{t+1}\upharpoonright R(q,t+1) = A^{u+2}\upharpoonright R(q,t+1)$ *and* $B^{t+1}\upharpoonright R(q,t+1) = B^{u+2}\upharpoonright R(q,t+1)$.

Proof. The proof is by induction on t. The result is clear for $t = u+1$. Assume that 1–2 hold for every t such that $u < t < z$. We show that 1–2 hold for $t = z$. $D' \subseteq A^{z}\upharpoonright R(q,z)$ by the induction hypothesis. By Remark 2 and the induction hypothesis it suffices to show that

$$(\dagger) \qquad A^{z+1}\upharpoonright R(q,z) = A^{z}\upharpoonright R(q,z) \text{ and } B^{z+1}\upharpoonright R(q,z) = B^{z}\upharpoonright R(q,z).$$

Case 1. $z \equiv 0 \bmod 3$.

This follows from Proposition 4.1.

Case 2. $z \equiv 1 \bmod 3$.

If $p^{*}\uparrow$ at stage $z+1$ then $\dagger$ is clear. Otherwise $p^{*} > q$ by the choice of w, u and D', so $\dagger$ holds by the choice of D at stage $z+1$.

Case 3. $z \equiv 2 \bmod 3$.

$F^{z}\upharpoonright R(q,z) = \emptyset$ by Remark 3 and the induction hypothesis. If Case 1 holds at stage $z+1$ then $\dagger$ follows from the definition of A^{z+1} and B^{z+1}.

So assume that Case 2 holds at stage $z+1$. Then

$r(\sigma',A,z) \cap F^z \neq \emptyset$ or $r(\sigma',B,z) \cap F^z \neq \emptyset$, so $\alpha(\sigma',z)$ is a σ'-active stage in $O(\sigma',z)$ (Corollary 15) and $\sigma' \nless \sigma_{q+1}$ by the choice of υ. Therefore $\sigma' > \sigma_{q+1}$, $\sigma' \subseteq \sigma_{q+1}$ or $\sigma_{q+1} \subset \sigma'$.

Assume that $\sigma' > \sigma_{q+1}$ or $\sigma' \subseteq \sigma_{q+1}$. If $\sigma' > \sigma_{q+1}$ then $\sigma' > \sigma(u-1,u-1)$, therefore $u+2 \leq o(\sigma',z) < \alpha(\sigma',z) \leq z$ (Proposition 7). If $\sigma' \subseteq \sigma_{q+1}$ then $\sigma' \subseteq \sigma(u-1,u-1)$; now either σ' is OFF at stage $u+2$, whence $u+2 \leq o(\sigma',z) < \alpha(\sigma',z) \leq z$, or $u+2$ is σ'-active (Proposition 14.1), whence $u+2 \leq \alpha(\sigma',z) \leq z$. So for every z' with $\alpha(\sigma',z) \leq z' \leq z$,

$$(\ddagger) \quad \begin{aligned} &R(q,z') = R(q,u+2) = R(q,z), \quad A^{z'} \upharpoonright R(q,z') = A^{u+2} \upharpoonright R(q,z') = \\ &A^{z} \upharpoonright R(q,z) \quad \text{and} \quad B^{z'} \upharpoonright R(q,z') = B^{u+2} \upharpoonright R(q,z') = B^{z} \upharpoonright R(q,z) \end{aligned}$$

by the induction hypothesis.

Assume that $\sigma_{q+1} \subset \sigma'$. Then $q < e(\sigma')$. So for every z' such that $\alpha(\sigma',z) \leq z' \leq z$, $R(q,z') = R(q,o(\sigma',z')) = R(q,o(\sigma',z)) = R(q,z)$, $A^{z'} \upharpoonright R(q,z') = A^{o(\sigma',z')} \upharpoonright R(q,z') = A^{o(\sigma',z)} \upharpoonright R(q,z) = A^{z} \upharpoonright R(q,z)$ and $B^{z'} \upharpoonright R(q,z') = B^{o(\sigma',z')} \upharpoonright R(q,z') = B^{o(\sigma',z)} \upharpoonright R(q,z) = B^{z} \upharpoonright R(q,z)$, since $o(\sigma',z') = o(\sigma',z) < \alpha(\sigma',z)$ (Proposition 17). So again for every z' with $\alpha(\sigma',z) \leq z' \leq z$, $\ddagger$ holds.

Assume that Subcase 2.2 holds at stage $z+1$. Then $B^{z+1} \upharpoonright R(q,z) = \bigcup_{\alpha(\sigma',z) \leq z' \leq z} (B^{z'} \upharpoonright R(q,z)) = B^{z} \upharpoonright R(q,z)$, from $\ddagger$. $A^{z+1} \upharpoonright R(q,z) = (A^{z} - (B^{z+1} \cap K^{z+1})) \upharpoonright R(q,z) = A^{z} \upharpoonright R(q,z) - (B^{z} \upharpoonright R(q,z) \cap K^{z+1} \upharpoonright R(q,z)) = A^{z} \upharpoonright R(q,z)$ since $(A^{z} \cap B^{z} \cap K^{z+1}) \upharpoonright R(q,z) = F^{z} \upharpoonright R(q,z) = \emptyset$, as noted earlier. If Subcase 2.1 holds, we get a similar result. □

If no D' satisfies 6'–8' then $k \notin W_e^t(A^t)$ for every $t \geq \upsilon$ by the choice of υ and w. So $R(q,t) = q+1$ for every $t \geq \upsilon$ and we can choose a stage $u > \upsilon$ such that $u \equiv 1 \bmod 3$, $\sigma_{q+1} \subseteq \sigma(u-1,u-1)$ and $(A^{u+2} \cap B^{u+2} \cap K) \restriction R(q,u+2) = \emptyset$. Claim 6 also holds for this choice of u by a similar proof.

If $q = 2\langle k,e\rangle+1$ we get a similar result. 4 and 5 follow from Claim 6 and the choice of u (and D'). □

Requirements N, Q_n and P_n are satisfied (Propositions 3 and 18).

APPENDIX

OPEN QUESTIONS

It is straightforward to show that an answer to the following question, combined with the results in this thesis would suffice to decide all sentences of the form $(\forall x_1)(\forall x_2)(\exists y)\alpha(x_1,x_2,y)$, ($\alpha$ quantifier-free) in $\mathrm{Th}(\mathcal{D}_e^+(\Sigma_2))$:

1. Do there exist incomparable Σ_2 e-degrees **a** and **b** such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e'$ and for every $\mathbf{z} <_e \mathbf{a}$, $\mathbf{z} \leq_e \mathbf{b}$?

Digressing from the Σ_2 e-degrees, one of the most intriguing "open" questions about the e-degrees is:

2. Are the e-degrees dense?

Case's [1971] result that no total e-degree is minimal relativises to show that no total e-degree is a minimal cover. Gutteridge [1971] showed that no total e-degree has a minimal cover and that any non-total e-degree has at most countably many minimal covers. Cooper [1982] claimed to have constructed a minimal cover, however the result remains to be published, suggesting that the question is still an open one.

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