AN INQUIRY INTO HIGH SCHOOL STUDENTS' UNDERSTANDING OF LOGARITHMS

by

Tetyana Berezovski

M.Sc, Lviv State University, Ukraine, 1991

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

IN THE FACULTY

OF

EDUCATION

SIMON FRASER UNIVERSITY

Fall 2004

APPROVAL

NAME Tetyana (Tanya) Berezovski

DEGREE Master of Science

TITLE An Inquiry into High School Students' Understanding of Logarithms

EXAMINING COMMITTEE:

Chair Peter Liljedahl

Rina Zazkis, Professor
Senior Supervisor

Stephen Campbell, Assistant Professor
Member

Malgorzata Dubiel, Senior Lecturer, Department of Mathematics
Examiner

Date November 18, 2004

SIMON FRASER UNIVERSITY

PARTIAL COPYRIGHT LICENCE

Simon Fraser University

Ethics Approval

The author, whose name appears on the title page of this work, has obtained human research ethics approval from the Simon Fraser University Office of Research Ethics for the research described in this work, or has conducted the research as a member of a project or course approved by the Ethics Office.

A copy of the approval letter has been filed at the Theses Office of the University Library at the time of submission of this thesis or project.

The original application for ethics approval and letter of approval is filed with the Office of Research Ethics. Inquiries may be directed to that Office.

ABSTRACT

The mathematical concept of logarithms plays a crucial role in many aspects of human existence. This study aimed to analyze and describe the issues involved in high school students' understanding of logarithms and to highlight the most common difficulties students face as they develop their understanding.

Two general theoretical ideas guided this investigation: mathematical understanding and obstacles. The adapted version of Confrey's model for students' understanding of exponents was applied to investigate students' understanding of logarithms. As a result, a description of students' difficulties with logarithms, and suggestions of possible explanations of the sources of these difficulties were presented.

As for teaching practice, I focused on the initial introduction of the logarithms. In the traditional curriculum, logarithms are introduced as exponents. However, historically logarithms were developed completely independently of exponents. Further research will investigate the feasibility and the benefits of the historical approach for teaching.

DEDICATION

This work is dedicated to my husband, my children, and my friend Lynne:

My beloved husband has sacrificed the pleasures of his life for me, so I could work on this research.

My loving children have given me strength to be the person that I am.

My devoted fried Lynne has inspired me to continue this journey of my life.

ACKNOWLEDGEMENTS

I would like to thank Dr. Rina Zazkis for inspiring me to complete this thesis. Her knowledge, guidance, unlimited encouragement and positive feedback directed me in this research, and opened a whole path for future study.

I would also like to thank Dr. Stephen Campbell, who despite his business and family complications, found time to employ his expertise in the field and offered many helpful suggestions.

Gratitude is extended to my other instructors at Simon Fraser University, including Dr. Tom O'Shea, Dr. Len Berggren, Dr. Peter Borwein, and Dr. Peter Liljedahl.

Finally I want to thank my Faculty Associate Norm Ellis for his unconditional support in my perseverance to be a mathematics educator. His determination in directing me back to university will never be forgotten.

TABLE OF CONTENTS

APPROVAL ii
ABSRACT iii
DEDICATION iv
ACKNOWLEDGEMENTS v
TABLE OF CONTENTS vi
LIST OF TABLES viii
LIST OF FIGURES viii

CHAPTER I: IDENTIFYING THE PROBLEM AND CONSIDERING A SOLUTION

Rationale 1
Personal Motivation 3
About this Study: Beginning my Journey 4
Purpose of the Study 6
Thesis Organization 7

CHAPTER II: LOGARITHMS IN THE HISTORY OF MATHEMATICS AND IN THE CURRICULUM

The Historical Development of Logarithms 8
Genesis of the Napierian Logarithm 11
Curricular Outcomes 14
Importance of Logarithms 15
Conclusion 17

CHAPTER III: THEORETICAL CONSIDERATIONS

Introduction 18
Skemp's Theory of Understanding 18
Sierpinska's Theory of Understanding 21
Hiebert and Carpenter's Theory of Understanding 22
Obstacles to Understanding 24
Confrey's Model for Investigation of Student's Understanding of Exponential Functions 26
Conclusion 30

CHAPTER IV: METHODOLOGY

Participants and Setting 31
The General Structure of the Unit 33

CHAPTER V: ANALYSIS OF DATA

Tasks and Frameworks	44
System of the Interpretive Frameworks for Logarithms	46
Results and Analysis	48

CHAPTER VI: DISCUSSION AND PEDAGOGICAL IMPLICATIONS

Discussion of Obstacles	75
Significance of the Study	77
Synopsis of the Pedagogical Considerations	77
Limitations of the Study	79
Suggestions for Future Research	79
Reflections on my Personal Journey	80
Summary	81

APPENDICES

A.	Logarithms Quiz #1	83
B.	Logarithms Unit Test	84
C.	Activity: Multiplying on the Slide Rule	90

REFERENCES 92

LIST OF TABLES

Table 1 - Table of Values of Powers of 2. 37

Table 2 - Exponential verses Logarithmic Forms 38

Table 3 - Quantitative Summary of Task 1 49

Table 4 - Quantitative Summary of Task 2 52

Table 5 - Quantitative Summary of Task 3 56

Table 6 - Table of Strategies Students Used in Their Essays-Explanations 64

LIST OF FIGURES

Figure 1a. Participant's Response 1IC. 57

Figure 1b. Participant's Response 1IC. 57

Figure 2. Participant's Response 2IC. 58

Figure 3. Participant's Response 1C. 59

Figure 4. Participant's Response 2C. 60

Figure 5. Participant's Response 3C. 60

Figure 6. Essay #1. 65

Figure 7. Essay #2. 66

Figure 8. Essay #3. 67

Figure 9. Essay #4. 68

Figure 10. Essay #5. 69

CHAPTER I: IDENTIFYING THE PROBLEM AND CONSIDERING A SOLUTION

Rationale

An increasing concern in recent years with regards to the degree of students' learning in mathematics classes, has led to in-depth investigations of specific subject-matters. Mainly, these investigations have covered the three following areas: (1) analysis of student errors, (2) analysis of mathematical and semantic structures, and (3) synthesis of how people construct an understanding of mathematical concepts (Harel & Confrey, 1994).

Although an understanding of student errors and mathematical structures is very important, the learning entailed in how to teach more effectively depends on an understanding of how students can come to know mathematical concepts; or, more specifically, are able "to specify the operations involved in constructing a mathematical reality" (Steffe, 1988, p. 119).

Much work in mathematics education and in the research in mathematics education has been done on students' understanding of the fundamental operations of addition and multiplication. However, very limited attention was given to an investigation of students' understanding of the relationship between the two.

At the elementary level, the conceptual understanding of the distributive property which relates multiplication and addition was based upon the conceptual understanding of the relationship between additive and multiplicative structures (Campbell & Zazkis, 1994). A detailed analysis was undertaken of the difficulties in general, and the obstacles in particular, that students have to deal with when learning the distributive property. Also, the development of the pre-service elementary school teachers' understanding of the arithmetic sequences through building connections between additive and multiplicative structures was the topic of the research conducted by Zazkis & Liljedahl (2002).

At the secondary level, a conceptual understanding of exponential functions was based upon the conceptual understanding of exponents as repeated products of the same multipliers/bases (Confrey, 1994). Viewing multiplication as repeated addition, different mathematical concepts can be introduced in the school curriculum: the concept of ratio and proportion at the junior grade levels, and the concepts of exponential and logarithmic functions at the senior secondary levels.

A conceptual understanding of the relationship between additive and multiplicative structures is essential to the understanding the concept of logarithms. Somehow, such an important concept has not attracted researchers' attention in recent years. Research of students' understanding of logarithms is very limited. What are the obstacles that students experience when learning logarithms? Are there any similarities between the learning of the distributive property, the learning of exponents and the learning of logarithms? These questions led me to undertake a study on senior high school students' understanding of logarithms.

As a result, the rationale for this study is three-fold: to describe students' understanding of logarithms, highlight the most common difficulties that appear within the process of understanding of logarithms, and investigate how the knowledge of historical development of the concept of logarithms can influence students' understanding.

Personal Motivation

There are a few reasons that I am personally motivated to research how students construct an understanding of logarithms. Firstly, I suspect that the result-oriented approach on how to solve real-life word problems with logs, so widely practiced in our classrooms, limits students' understanding of the nature of this important concept. Students' understanding of logarithms is generally speaking, very poor. It deserves a detailed investigation by researchers. The fundamental assumption is that students have to understand mathematics, but in reality not many of them do. What are the main essences of students' understanding of logarithms?

Secondly, I believe that this limitation is due to the absence of the introductory phase in the content of logarithms. None of the most popular textbooks offer any historical background in regard to development of the concept. One of the reasons that the concept of logarithms is considered difficult could be attributed to the fact that slide rules and logarithmic tables have not been at school for many years, a whole generation of new teachers has never really experienced, understood or valued this sophisticated topic.

Thirdly, if students do not grasp the essence of the concept of logarithms, they are unable to use it as a cognitive tool in their mathematical thinking. However, "It is expected that students will solve exponential, logarithmic and trigonometric equations and identities. It is expected that students will represent and analyze exponential and logarithmic functions, using technology as appropriate,"(Integrated Resource Package 2000, p. 178,180) as outlined in the *Prescribed Learning Outcomes by the Ministry of Education of British Columbia*. In terms of the recommended learning resources, there is a wide variety of graphing calculator exploratory activities, and two textbooks: Mathematics 12, Western Canadian Edition, and MATHPOWER 12, Western Edition.

Finally, I chose the subject of logarithms simply from curiosity. There is so many "whys" related to this concept. It is important to mention that there are three categories of whys in the teaching of mathematics: the chronological, the logical, and the pedagogical (Jones, 1957). Why do students not like logs? Why do teachers not appreciate logs? Why is such an important concept turned into a routine of pushing the LOG button?

About this Study: Beginning my Journey

It is every teacher's desire to see that students understand and can apply the knowledge they gain to real-life. Logarithms are very important in many aspects of our existence. Today, they are used in the fields of cosmology, engineering, chemistry, finance, statistics, and others. Logarithms were used to design space telescopes, and make sense of the enormous ranges of data received back on earth (Marson, 2003).

I believe that many of those questions can be addressed if we were to take a historical perspective in teaching mathematics. "Connecting mathematics with its history makes learning the basic skills more interesting and motivates students to sustain their interest in mathematics" (Reimer L., Reimer W., 1995, p. 105).

In my study, I investigate which alternative activities and tools can be used by learners, and what their impact will be on students in terms of their understanding of the nature of logarithms. In the study of mathematical understanding, knowledge of the historical development of mathematical ideas provides us with another view of students' actions. Sierpinska (1994) draws an analogy between the historical development of mathematical concepts and concept development by individual. I believe students' understanding can be acquired and strengthened through a study of the historical development of the concept. It is not new that the history of mathematics is pedagogically important topic (Toumasis, 1993; Dennis, 2000). As an attempt to investigate students' understanding of logarithms, I selected and developed one of the tasks with the historical development of the concept in mind.

As a practicing teacher of high school mathematics I became aware of students' difficulties in learning the concept of logarithms. I have long reasoned if I taught the concept differently, then students would understand. Therefore, my initial intention was to find the curriculum that would uncover a variety of methods that could be helpful to teach logarithms and logarithmic function, in addition to the widely used "logarithm as the inverse of the exponent" approach. The material that relied on the historical development of logarithms was quite novel and looked very promising.

Having read and analyzed the historical development of logarithms I realized that searching for the best way of explaining logarithms became a secondary concern for me, while the students' understanding dominated my research. I began to question what it takes for students to understand logarithms? My belief is that the main goal of mathematics teaching is student understanding. I also thought that a student's understanding exists in his or her mind, and acknowledged that I cannot know exactly what is happening there. However, I hoped to collect the external evidences that could provide me with sufficient information in regard to student's understanding. I believed that students themselves would try to make sense of what they understand, and this could by the activities they perform in the problem solving.

Thinking of students' understanding led me to explore the meaning of this term by turning to experts and theories. I elaborate on this in Chapter III. From my observations and interpretations, I attempted to generate descriptions of students' understanding of logarithms and explored the extent to which their understanding could be influenced by the knowledge of the historical development of logarithms.

Purpose of the Study

The purpose of this study is to provide an account of high school students' understandings and misunderstandings of the concept of logarithms. This study could be understood as an investigation into the cognitive difficulties encountered when co-generating additive and multiplicative structures. (For example, many students will write that $\log(x+5) = \log x + \log 5$, or $\log(6x) = \log 6 \log x$).

The goal is to investigate the possible sources of the difficulties that affect students' understanding, and explore how it is possible to overcome or at least to reduce these difficulties. It is hoped that this research will provide useful information for those who teach the concept of logarithms, and design curriculum to be used in the teaching of logarithms.

This study was designed to investigate the way in which grade 12 students conceptualize logarithms. Specifically focused on students' understanding of logarithms, the study has been designed with the intent of providing some insight into how students develop mathematical meaning of this concept, how they generalize it, and how they employ it in problem-solving. In addition, this study was designed to investigate the main obstacles with which students have to deal when learning and applying logarithms.

Thesis Organization

The thesis is organized into six chapters. Chapter I is an introduction. The Rationale is mainly focused on the premises that preceded the topic of my study. Also, the purpose and the nature of the study are discussed. The specific areas of focus are defined.

Chapter II presents a discussion on "educational goals" and identifies the meaning of educational goals for learning logarithms. The historical development of logarithms is explored in this chapter. The relationship between the conceptual development and the curricular outcomes outlined in the Integrated Resource Package is explored.

Chapter III provides the considerations from the relevant literature.

Chapter IV details the methodology for the study. It includes a description of the participants, discussion of techniques and instruments of the research. It contains specific information regarding procedures, and data collection.

Chapter V presents the system of interpretive frameworks used in the data analysis, the results from working with the participants, and pedagogical ideas relevant to the specific questions.

In Chapter VI, a discussion of the obstacles in the students' understanding of logarithms is presented. The significance and limitations of the conducted study are highlighted. This chapter includes the synopsis of the pedagogical considerations, suggestions for future research, and my reflections on the experiences I had while working on this study. The chapter concludes with a brief summary.

CHAPTER II:
LOGARITHMS IN THE HISTORY OF MATHEMATICS AND IN THE CURRICULUM

The goals of education determine its contents and structure. To acknowledge educational goals in learning logarithms it is necessary to identify what logarithms are, how they came to be, and why they are important. For this matter I conferred with the historical development of logarithms. Then, I tried to establish the relationship between the conceptual content, identified thorough the lens of historical facts and events, and the present-day version of these goals that appear in the form of curricular outcomes. Specifically, I focused on the set of curricular outcomes identified in the Integrated Resource Package, published by Ministry of Education of British Columbia in 2000.

The Historical Development of Logarithms

The development of a concept by an individual does not necessarily follow the same path as the historical development. However, there is much to be gained from the knowledge of the historical development of a mathematical concept. In particular, in the study of mathematical understanding, knowledge of historical development of mathematical ideas provides us with another perspective on students' activities. Commonalities that occur in the way a student's understanding of a mathematical concept develops and the way it developed historically are, according to Siepinska (1994) citing Piaget and Garcia (1989) and Skarga (1989), attributable to commonalities in mechanisms of development and to preservation of the historical meaning of terminology.

My choice to study the historical development of logarithms was motivated by the work of Smith and Confrey (1994). In this work, Smith and Confrey outline the historical development of the concept of logarithms and note the consistency of the development with students' actions (Confrey, 1991; Confrey & Smith, 1994; Confrey & Smith, 1995). These consistencies were observed during teaching interviews designed to investigate how students learn about exponential function. Since the development of the logarithmic function followed the development of the exponential function, Smith and Confrey (1994), investigated the historical development of logarithms in search for explanations for students actions. The authors explain how the early work of Archimedes and that of Napier form a consistent whole that illustrates the development of what they call the multiplicative world. In this world, the action is multiplication operating on the elements that are ratios. This is in contrast to the action of addition, in the additive world that acts on magnitudes. Four historical works preceded their conjectures.

1. *The development of arithmetic and geometric sequences.*

Since Euclid's time, numbers were used specifically for counting. They represented an arithmetic sequence {1, 2, 3, ...}. Euclid also introduced a geometric sequence as {81, 54, 36, 24, 16} that cannot be extended for integers. With time, the concept of number was expanded to rational and real numbers, and also a greater variety of number sequences. Even so, numbers and ratios were viewed as different entities for the next 1500 years.

2. *The juxtaposition of arithmetic and geometric series.*

Archimedes was one of the first scientists who placed arithmetic and geometric series side by side. He noticed that in such arrangement, multiplication in one series corresponds to addition in the other. However, this fact was simply considered interesting rather than scientifically useful.

A subsequent contribution to the field was done by Jobst Burgi, from Switzerland. He calculated and constructed tables with entries very close together.

Burgi printed his results only by 1620 (there is a speculation that the idea of logarithms had occurred to him in 1588). This was five years after Napier had published his *Descriptio*. Both men produced very similar work. The differences between their works were mainly in the terminology and the numerical values they used. Their fundamental principles were the same. The essence of the principle of logarithms was established in Burgi's *Arithmetische und geometrische Progress-Tabulen*. Burgi shall be regarded as an independent discoverer who lost credit for the invention because of Napier's priority in publication. In one respect his logarithms come closer to ours than do Napier's. However, the two systems share the disadvantage that a logarithm of a product or quotient is not the sum or difference of the logarithms.

3. *The development of continuous geometric worlds*

In fourteenth century Europe, the work on space, time, and motion was developed from the study and interpretation of Aristotle's work. Aristotle saw the velocity of an object changing additively when the force on the object changed multiplicatively. It led Thomas Bradwardine, followed by Nicole Oresme, to create the mathematics of the continuous multiplicative world (continuous ratios).

4. *The cogeneration of continuous additive and continuous geometric worlds.*

Napier merged two great ideas together: Burgi's cogeneration of arithmetic and geometric series, and the world of continued ratios. He created a model based on two particles, each traveling in a straight line, in such a way that additive change in position of one particle would correspond with proportional change in position of the second (MacDonald, 1966).

The key to Napier's work can be explained very simply. To keep the terms in a geometric sequence of integral powers of a given number close together, it is necessary to take as a given number something quite close to 1. He chose .9999999.

Napier did not think about a base for his system, but his tables were compiled through repeated multiplication, equivalent to powers of .9999999. Napier's intuitive understanding of the relationship between position and rate of change allowed him to construct the relationship between additive and multiplicative worlds. Even though Napier's goal was to simplify the computations, his work had the greatest contribution to the development of the number *e* and natural logarithms.

The historical development of the logarithms discussed above helped me develop tasks to investigate students' understanding, and a lens through which to view students' actions as they solved problems. In addition, the historical development highlighted the importance of representation in the creation of the logarithmic function. Finally, the literature on the logarithmic and exponential functions alerted me to various mental activities to look for as I observed the students solving problems.

Genesis of the Napierian Logarithm

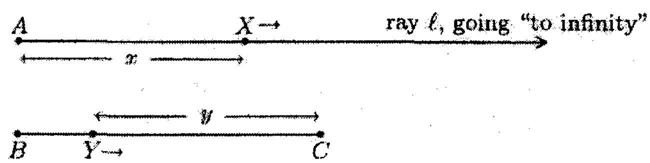
The power and importance of the logarithm lie in fact that it converts a product into a sum and a multiplication problem into an addition problem; and similarly, a quotient into a difference and a division problem into a subtraction problem. This obviously has computational significance. It was known during John Napier's time how a problem of multiplication can be changed into one of addition. For example, the following formulae convert products into sums:

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\sin x \cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$$

It seems likely that such facts would have helped Napier hit upon the idea of logarithms (Shirali, 2002).



Genesis of the Naperian logarithm

Consider a ray *l*, with A as the endpoint, and a line segment BC of unit length. Let X and Y start from A and B move along *l* and BC respectively, starting with the same initial speed; let X move at a constant speed, and let Y move at a speed proportional to the distance YC. This means that its speed decreases steadily as it approaches C, and it takes infinitely long to reach its destination. If the distance x represent *AX*, and y represent *YC*, then the relationship between x and y is a "Napierian Logarithm":

$$x = Naplog\, y$$

when x is 0, y is 1, and as x increases to infinity, y is decreases to 0.

The relationship of the Napierian Logarithmic function to one of the functions we know today is: $Naplog\, y = -\ln y$.

Napier publicized his invention in 1614, in a book titled *A Description of the Wonderful Law of Logarithms.* (Shirali, p.36) This contained a table of Napierian logarithms. The significance of the new invention was quickly seen. In 1543, Copernicus published his theory of the solar system. To proceed in his work, he needed to perform thousands of complicated and lengthy calculations. A similar difficulty faced Johannes Kepler (1571-1630), who completed an enormous number of arithmetical computations to obtain his famous laws of planetary motions. Napier's logarithms helped to solve this problem. The discovery of logarithms opened a whole era of discoveries in astronomy.

> The miraculous powers of modern calculation are due to three inventions: the Arabic notation, Decimal Fractions and **Logarithms**. The invention of logarithms in the first quarter of the seventeenth century was admirably timed, for Kepler was then examining planetary orbits, and Galileo had just turned the telescope to the stars. During the Renaissance German mathematicians had constructed trigonometric tables with great accuracy, but its greater precision enormously increased the work of the calculator. It is no exaggeration to say that the invention of logarithms "by shortening the labors doubled the life of the astronomer.
>
> Logarithms were invented by John Napier (1550-1617), Baron of Merchiston, in Scotland. It is one of the great curiosities of the history of science that **Napier constructed logarithms before exponents were used**. To be sure, Stifel and Stevin had made attempts to denote powers by indices, but this notation was not generally known - not even to T. Harriot, whose ALGEBRA appeared long after Napier's death. **That logarithms flow naturally from the exponential symbol was not observed until much later** (Cajori, 1919, cited in Shirali, 2002, p.37).

Even though, Napier's discovery was so important to the development of mathematics and science, it is our obligation to mention that it was John Briggs (1561-1631) who proposed to Napier, when they met in Scotland, that the definition of logarithm be modified. They agreed that the logarithm of 1 would be 0, and the logarithm of 10 would be 1. Thus was born the common logarithm (Boyer, 1968).

Curricular Outcomes

This study was designed with the expectation that any student enrolled in the Principles of Mathematics 12 course would be able to participate in the research project. The research design intentionally incorporated the curricular outcomes for logarithms, logarithmic expressions and logarithmic functions as stated in the 2000, British Columbia, Mathematics 12 curriculum guides.

The next general outcomes from *Principles of Math 12 Prescribed Learning Outcomes* that form two main organizers of this guide relevant to this study are:

C: Patterns and Relations (Variables and Equations)

It is expected that students will solve exponential, logarithmic and trigonometric equations and identities.

It is expected that students will:

C2. Solve and verify exponential and logarithmic equations and identities.

D: Patterns and Relations (Relations and Functions)

It is expected that students will represent and analyse exponential and logarithmic functions, using technology as appropriate.

It is expected that students will:

D2. change functions from exponential form to logarithmic form and vice versa;
D3. model, graph, and apply logarithmic functions to solve problems;
D4. explain the relationship between the laws of logarithms and the laws of exponents (IRP, 2000, p.A-38).

While learning that above listed outcomes, students and teachers identify particular goals. These are the only goals I identify as educational.

Educational goals describe how teachers and students conceive the ideal result that they try to achieve in the process of learning the specific concept. According to the expectations listed in the IRPs/ NTCM, I identified several educational goals of the process of learning logarithms:

1. Students are expected to know how to graph logarithmic function, and its main properties.
2. Students are expected to simplify logarithmic expressions using the main properties of logarithms.
3. Students are expected to solve logarithmic equations.
4. Students are expected to apply the knowledge of logarithms to solve real life problems.

Those goals determine the certain level of students' cognitive activities while they are learning logarithms.

Importance of Logarithms

Logarithms possess a rich mathematical content that has had value all the way from the time of their invention to the recent diversity of their applications. For example, in the eighteenth century, Ernst Weber (1795-1878) suggested that the sensitivity of senses decreases as the magnitude of the stimulus increases. Later, Gustav Fechner (1801-1887), formulated the following law:

The response of the senses varies as the logarithm of the stimulus.

In symbolic language, if Y denotes the sensation (the effect that a stimulus produces on our senses) produced by a stimulus X, then X and Y are related by a law in the form: $Y = \log_b kX$, where constants k and b depend on the particular situation at hand. Only a few years later, Fechner used this law to find the method of measuring sensation. This method shows that when a stimulus invades our senses, our body only takes in its logarithm and sends this logarithm to the brain to create a sensation (Shirali, 2002). So, *our body acts like a log table.*

As well as uses in the measuring of human senses, such as sound, light, taste, fragrance, etc., logarithms are very often employed to solve many difficult mathematical problems. For example, in the nineteenth century, Gauss found the answer to the question of *(Prime Number Theorem)* how many primes are in a thousand, million, billion, etc. His answer was in the form of the *approximate formula* that involved a logarithm:

$P(x) \approx \frac{x}{\ln x}$, where x is a positive number. Here are some examples:

$$P(2) = 1$$
$$P(10) = 4$$
$$P(100) = 25$$
$$P(1000) = 168$$
$$P(10{,}001{,}000) - P(10{,}000{,}000) = 61$$

Later, Legendre found a better fit for approximation: $P(x) \approx \frac{x}{\ln x - 1.08}$

In early times, students studied logarithms essentially to compute. Now with wide usage of powerful calculators, this need has practically vanished from our school curriculum. However logarithms possess a significance which has absolutely nothing to do with computations.

Conclusion

In learning mathematics, it is not necessary for an individual learner to follow the path of the historical development of a specific concept. Though, our teaching may be a complement to the analysis of the historical genesis of mathematical concepts.

It is important to mention that in the curriculum, logarithm is defined as an exponent, and a logarithmic function as an inverse of the exponential function. Also the topic of logarithms follows the topic of exponents. However, the analysis of the historical and conceptual development of logarithms suggests an alternative way of defining and introducing logarithms independently from exponents. Even though I have not tried to delve into this possibility yet, the historical development of the logarithms discussed in this chapter helped me develop tasks to investigate students' understanding. It also became a lens through which to view students' activities as they solved problems. In addition, the historical development highlighted the importance of representation in the creation of the logarithmic function, which is not part of the curriculum. Finally the literature on the logarithmic and exponential functions alerted me to various mental actions to look for as I observed the students solving problems.

CHAPTER III:
THEORETICAL CONSIDERATIONS

Introduction

The important mathematical concept of logarithms plays a crucial role in advanced mathematics courses, including calculus, differential equations, number theory and complex analysis. Unfortunately, this concept also gives students serious difficulties. In order to address the question of why students find this concept difficult, it seems practical to begin with an investigation on how students acquire an understanding of logarithms. My concern with the question of understanding has its sources in the practical problems of teaching high school mathematics, although an understanding in mathematics has been identified as the central problem in mathematics education (Sierpinska, 1994).

It is most important to know exactly what an *understanding* in mathematics is. For the answers I consulted the experts. The purpose of this work is to summarize and discuss my interpretation of an ongoing research in mathematics education about understanding. I use examples of logarithms to interpret and clarify the ideas expressed by researchers. The theories about understanding were chosen due to the influence they had on the development of my own thinking about understanding.

Skemp's Theory of Understanding

In 1976, Richard Skemp marked the beginning of a modern mathematics education research movement into the study of understanding. The now classical article entitled "Relational and Instrumental Understanding" sought to define and describe these two types of understanding and to explain why so many teachers felt that instrumental understanding was a type of understanding. Skemp credited Stieg Mellin-Olsen with the coining and definition of the terms. According to Skemp relational understanding is "knowing what to do and why," while instrumental understanding is "rules without reasons" (p. 152). Further reading reveals an expansion and revision of Skemp's categories of understanding.

Following the publication of Skemp's (1976) article in *Mathematics Teaching*, debate about both the definitions and categories of understanding Skemp identified was carried out in person and in print (Backhouse, 1978; Buxton, 1978; Byers & Herscovics, 1977; and Tall, 1978). This discussion prompted Skemp to revise his definitions of instrumental and relational understanding and to include a new type of understanding that he called formal understanding. Skemp (1987) elaborated on these new definitions attributed to Byers and Herscovics.

> *Instrumental understanding* is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
>
> *Relational understanding* is the ability to deduce specific rules or procedures from more general mathematical relationships.
>
> *Formal understanding* ...is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning. (Skemp, 1987, p. 166)

The language of "knowing" found in Skemp's (1976) original work regarding instrumental and relational understanding has now been replaced with "abilities." Hence, one can assume that the result of understanding is for Skemp linked to the abilities that it produces. The question remains, how does one acquire these abilities?

In his book *The Psychology of Learning Mathematics*, Skemp writes that "*To understand something means to assimilate it into an appropriate schema*" (Skemp, 1987, p. 29). We can unravel this sentence a bit if we know what *assimilate* and *schema* mean.

By *schema*, Skemp refers to a group of connected concepts, each of which has been formed by abstracting invariant properties from sensory motor input or from other concepts. The concepts are then connected by relations or transformations. A schema has two main functions. It integrates existing knowledge, and it is a mental tool for the acquisition of new knowledge. An example of how a schema works is given by Skemp.

> When we see some particular car, we automatically recognize it as a member of the class of private cars. But this class-concept is linked by our mental schemas with a vast number of other concepts, which are available to help us behave adaptively with respect to the many different situations in which a car can form a part. Suppose the car is for sale. Then all our motoring experience is brought to bear, reviews of performance may be recalled, questions to be asked (m.p.g.?) present themselves. (Skemp, 1976, p. 24)

This does not mean that schemas are only used when we have had some previous experience with a situation; they are also used in problem situations with which we have no experience. For example if one had never solved a logarithmic equation before, but had solved linear equations, various techniques and information about solving linear equations might come to mind as one tried to solve the problem. According to Skemp, "The more schemas we have available, the better our chance of coping with the unexpected" (Skemp, 1976, p. 24).

As Skemp points out his definition of understanding is not based on finding *the* appropriate schema, but *an* appropriate schema. This explains why students may think that they understand a concept when they do not. Suppose for example that a student thinks that the notation $logx$ means $log \cdot x$. The student may believe s/he understands the notation; it is assimilated into his/her schema for multiplication. However, we know that this assimilation will be damaging to the student's understanding of the concept of function. All is not lost, however. A student can *reconstruct* their schema if s/he encounters situations for which his/her existing schemas are not adequate. Skemp notes this is not an easy or a comfortable process because of the strength of existing schema.

"If situations are then encountered for which they are not adequate, this stability of the schemas becomes an obstacle to adaptability." (Skemp, 1976, p. 27)

Skemp's view on understanding (Skemp,1976) is a very brief introduction to his theory of learning mathematics. Knowing how and knowing what are the by-products of learning, while understanding is part of the schema building process.

Sierpinska's Theory of Understanding

"The central problem of education is not so much the description and categorization of the processes of development of knowledge, as the intervention into these processes" (Sierpinska, 1994, p. 121). However, description and categorization should precede the intervention, the way the diagnosis precedes the treatment.
Sierpinska's (1994) theory of understanding is based on the idea that understanding is "the act of grasping meaning" (1994, p. 27). Epistemological obstacles are a major feature of understanding in Sierpinska's theory. Obstacles to the historical development of mathematical ideas form a basis for conjecturing which mathematical concepts might be obstacles for students. Students overcome obstacles by what Sierpinska calls reorganizations. "Every next stage starts with a reorganization, at another level, of ways of understanding constructed at the previous stage, the understandings of the early stages become integrated into those of the highest levels" (Sierpinska, 1994, p. 122). These reorganizations result in modification of the students theories about a mathematical concept.

Epistemological obstacles, "barriers to changes in frame of mind" (Sierpinska, 1994, p. 121) become opportunities for the occurrence of an act of understanding. Sierpinska provides the following justification for her focus on the epistemic subject:

> If we want to speak about understanding of some mathematical topic in normative terms this notion of sujet épistémique comes in handy. To be exact, it is not the way 'a certain concrete Gauss' has developed his understanding between one work and another that will give us some guidance as to what acts of understanding have to be experienced or what epistemological obstacles have to be overcome in today's students. We

> have to know how a notion has developed over large periods of time, and in what conditions (questions, problems, paradoxes) were the great breakthroughs in this development brought about. This, and not historical facts about exactly who did what and when, can be instructive in designing our teaching and facilitating understanding processes in our students. (p. 40)

For example, suppose a student has constructed the graph of $y = \log x$ and the graph of $y = \ln x$ on his calculator. Having only seen two examples of the logarithmic function s/he abstracted the idea that the logarithmic functions is an increasing function. This may prove to be an obstacle when the student tries to determine the representation of $y = \log_{1/2} x$.

Hiebert and Carpenter's Theory of Understanding

Hiebert and Carpenter (1992) proposed a cognitive science perspective on students' understanding of mathematics. Their theory of understanding is based on the assumptions that "knowledge is represented internally and these internal representations are structural" (Hiebert, Carpenter, 1992, p. 66), that there is a relationship between internal representations and external ones, and that internal representations are connected. They further explain that internal representations and connections can be inferred from analyzing a student's external representations and connections. The basis for Hiebert and Carpenter's definition of understanding is the existence of internal representations and connections. "Mathematics is understood if its mental representation is part of a network of representations" (Hiebert, Carpenter, 1992, p. 67).

What then are these mental representations and connections? Although we do not know how a student represents mathematical ideas or concepts internally; we can suppose, according to Hiebert and Carpenter, that these representations are influenced by external representations (physical materials, pictures, symbols, etc.) in problem situations the student is asked to solve. A student may solve problems with representations both in and out of school. Both types of experiences help them form networks of representations. Hiebert and Carpenter contend these mental representations are needed to "think about mathematical ideas."

Hiebert and Carpenter propose two metaphors for these networks of representations. The first is that networks are structured like vertical hierarchies. Representations are details of other more overarching representation. Hence, if a student has a mental representation of function, in terms of a vertical hierarchy, an associated representation would be a linear function. The second metaphor is that networks are structured like webs. Representations of information form nodes connected to other nodes. Connections, according to Hiebert and Carpenter, are formed in one of two ways: by noting similarities and differences, and by inclusion. Once similarities and differences are noted, a student can connect his or her mental representation of the idea to existing structures.

Heibert and Carpenter explain the growth of understanding in terms of adjoining to, and reorganization of, existing networks. Adjoining may occur when a student becomes aware of a mathematical idea for the first time. In an attempt to make sense of the idea, the student searches for ways it might be related to existing mental representations. One result of this process is the connection of new ideas to mental representations not related to them. For example, consider the addition of logarithms: log 4 + log 5. A student might connect this representation to his knowledge of the distributive property. This connection will result in the following calculation: log 4 + log

5 = log 9. Hence the idea is adjoined, however the connection is not useful. This connection can be modified through a process Hiebert and Carpenter call reorganization. Reorganization can occur when a student reflects on his or her thinking and is aware of an inconsistency. For example, if a student subsequently sees log 4 + log 5 = log 20, he or she may have cause for reorganization. The new information is not consistent with current mental representations for adding logarithms.

Due to the importance placed on the communication and understanding of mathematics in both school and society, Hiebert and Carpenter explain how written symbols can be understood by students. If a symbol is to carry some meaning, then it "must be represented internally as a mathematical object" (Hiebert, Carpenter, 1992, p. 72). Therefore, in order for a student to understand log 4, for example, he or she must have an internal representation for the mathematical object. Without the internal representation the symbol log 4 has no meaning and cannot be understood.

Obstacles to Understanding

Each of the theories of understanding contains the idea of obstacle and modifications in the face of obstacles. Skemp notes that a student may encounter a situation for which his or her schemas are not adequate. In this situation "this stability of the schemas becomes an obstacle to adaptability" (Skemp, 1976, p. 27) and the schemas must be reconstructed (modified) "before the new situation can be understood" (Skemp, 1976, p. 27). Naturally there is no guarantee that a student will successfully reconstruct his schema. Skemp notes that if an effort at reconstruction fails, then "the new experience can no longer be successfully interpreted and adaptive behavior breaks down — the individual cannot cope" (Skemp, 1976, p. 27).

As for Sierpinska, she sees mental representation as a possible source of obstacles. For example, suppose a student has only worked with logarithms with bases greater than 1. This may prove to be an obstacle when the student is asked to compare two numbers: $\log_{1/2} 3$ and $\log_{1/2} 5$. S/he may conclude that the latter is larger considering only the whole numbers and ignoring the base.

Sierpinska (1994) notes that overcoming obstacles is not the only way that understanding can be produced. Rather she judges acts of understanding that "consist in overcoming an obstacle" (p. 124) to be more important than any other acts of understanding. Overcoming obstacles may be part of a participant's development of understanding the logarithmic function.

The broader interpretation of obstacles can be attributed to the historical development of knowledge in a discipline, as well as to the knowledge of an individual learner. In this framework, to learn means to overcome difficulty.

In the literature, understanding is described as assimilation, an act, and a network/ web. Although each characterization is different, all of them were developed to explain and predict students' behaviour. Representations and connections are incorporated in every theory. External representations and connections reveal information about students' internal representations. Therefore, in my study, I collected and analyzed an external data that provided me with an opportunity to hypothesize about students' understanding.

Confrey's Model for Investigation of Student's Understanding of Exponential Functions

"Exponential functions represent an exceedingly rich and varied landscape for examining ways in which students construct their understanding of mathematical concepts" (Confrey, 1991, p.125). The following model was developed to investigate how students construct the meaning of the concept of exponential function. This was done through a set of five interpretive frameworks which have been useful in modeling the students understanding of exponents, exponential expressions, and exponential functions. These five interpretive frameworks are roughly ordered from the simplest insights to the more complex.

In the following, I introduce my interpretation of the frameworks identified by Confrey (1991) and exemplify each framework in terms of student's actions within these frameworks. Further, I investigate the possibility of adapting each framework to address students' work with logarithms. Additionally, I provide examples of specific tasks that may identify the main aspects of students' understanding of logarithms.

Framework One: Exponents and Exponential Expressions as Numbers.

In this framework the author explores the student's understanding of the negative and fractional exponents. The student explains that 9^2 equals nine times itself (repeated multiplication). However, he fails to connect the meaning of exponent when he attempts to extend it into the case of zero, negative, or fractional exponents. The student also experiences enormous difficulty when trying to place $1/3$, 9^3, 9^{-3}, 2, 9^0, 10, -3 and -1 on the number line. His responses to this problem indicate that *exponents and exponential expressions do NOT behave as" numbers."*

In the traditional curriculum, the concept of logarithm is presented as an inverse of the exponent. For example: the student correctly answers what $\log_3 9$ is, by using the definition ($3^2 = 9 \Rightarrow \log_3 9 = 2$). But he fails to answer what $\log_3 1/9$, or $\log_3 1$ are.

The student's understanding of logarithms as numbers could also be assessed by the questions: compare $\log_3 9$ and $\log_{1/3}(1/9)$, or simplify: $\log_3 54 - \log_3 8 + \log_3 4$.

Framework Two: Exponential Expressions and Local Operational Meaning.

The main issue explored in the second interpretive framework is the student's understanding of an operational character of exponents. Within this framework, the researcher examines the student's attempt to create an operational meaning for negative exponents by viewing them as "opposite" of a positive exponent, *operationally* opposite, as opposed to being opposite in magnitude. The student constructs *Local Operational Meaning.* Then, the student tries to gain insights into the isomorphic relationship between multiplicative structure of exponential expressions and the additive structure of the exponents. Finally, the operational meaning of the exponents is extended to the system of multiplicative relationships (1/100, 1/10, 1, 10, 100 ...).

The isomorphism between multiplicative and additive structures is easily recognized with logarithms. For example: $\log_3 54 - \log_3 8 + \log_3 4 = \log_3(54 \div 8 \times 4)$. The logarithm reduces the operational complexity. Consequently, the operational meaning of logarithms can be extended to the concept of *common logarithms,* additive relationships of the system ($\log 1/100, \log 1/10, \log 1, \log 10, \log 100 \ldots$)

Framework Three: Exponents as Systematically Operational

The aim of this framework is to gain an understanding of how the student systemizes/generalizes the operational character of exponents. The main question is "What does it mean to develop insight into the system of relationships between exponents and exponential expressions?" Those relationships can be described as a set of rules for translating back and forth, as an isomorphic relationship which stresses the operational character, or a function showing the one-to-one correspondence in the isomorphism.

In the case of logarithms, this particular framework can help us investigate how students acquire an understanding of the systematic operational relations with logarithms. In addition to the understanding of what happens when calculating the logarithm of the negative number or zero, students have to be able to operate with logarithms in two ways. For example, simplify $\log_3 90 - \log_3 10$; and expand $\log_c a^2 b$. By doing these types of exercises, students begin to recognize the isomorphism between the additive and multiplicative structures, and between the subtractive and divisive structures. In this way we can investigate how students construct the operational meaning of logarithms.

Framework Four: Exponents as Counters

In this framework the researcher investigates which obstacles undermine the student's understanding of exponents as counters, and the roots of those obstacles. The student agrees that exponents express repeated multiplication, but does he understand repeated multiplication? "...if we are to understand the multiplicative aspects of exponential functions, we must understand the concept of repeated multiplication." (Confrey 1991, p.142) The student is asked to illustrate with a diagram 2x3x4 and 3^4.

When thinking of logarithms as the inverse of the exponents, logarithms can be viewed in the light of repeated addition. For example: log2 + log2 + log2 = log(2 x 2 x 2) = log 2^3, on the other hand, log2 + log2 + log2 = 3 x log2.

Perhaps, this particular framework can be modified to investigate students understanding of the logarithmic laws, but in the context of this study *Logarithms as Counters* are not investigated.

Framework Five: Exponents as Functions

In this framework, two main perspectives are guiding the researcher's investigation of students' understanding of *exponential functions*. One perspective is the operational basis of the exponential functions. Does the student understand what a function is? How does a student relate the definition of the exponent to the exponential function (range, domain, max/min points, and intersections)? What are different representations of the exponential function, and how are they related?

Another perspective of the proposed framework is to examine how a student applies the knowledge of the concept of exponents to solve different types of problems, cross-curricular problems in particular. The student is presented with a problem: A computer originally cost $4,000. Each year it is worth 90% of the previous year's value.

1. How much is it worth after 1 year?
2. How much does it cost after 3 years?
3. Ten years?
4. Does it become worthless?
5. When is it worth less than $1000?
6. Write the equation giving its value after *t* years.
7. Draw the graph.

As for logarithms, it seems like this framework can be adapted without major modifications. The conceptual understanding of logarithmic function can be determined from how students construct graphs from the given equations ($y = \log_{1/2} x$, and $y = \log_2 x$), or how they derive the algebraic representation of the function from the given graph.

Since logarithmic functions are useful in modeling phenomena across many fields, including statistics, economics, science, physics and biology, the ability to use them requires special attention. How students employ logarithms to solve real life problems will reveal their understanding and the difficulties they experience.

When reviewing Jere Confrey's article *The Concept of Exponential Functions: A Student's Perspective*, it seemed natural to reflect on the students' understanding of logarithms among the frameworks she proposed. With a minor adjustment, the above model can be used in this study. The details on the modified version of these frameworks are introduced in Chapter V.

Conclusion

Understanding is described in the literature as assimilation, or an act. Although each of the characterizations is different, both of them were developed to explain and predict students' behavior. In addition, each of them includes as both representations and connections either as understanding is developed, or as the understanding itself. The literature on representation cited in this chapter helped me realize the student's external representations and connections could be used as evidence of their internal theories. Hence, I noted that the representations I am looking for and at in this study are external. From these, and the student's conception and application of a mathematical concept, I can hypothesize the internal theories that the student is using to make sense.

CHAPTER IV: METHODOLOGY

This chapter introduces the participants of the study, and describes the data collection process in detail. The discussion on the instruments of the study follows. Also, some ideas that could lead to a more efficient pedagogical design for learning logarithms are presented. The conceptual flow of the material as it was delivered to the participants is shown with examples.

Participants and Setting

Participants in this research are 27 secondary school students who were enrolled in the Principles of Mathematics 12 course at Holy Cross Secondary School in Surrey, British Columbia. They were 17 or 18 years old. Many of the students knew each other well, as they were in the same school for four years, and the entire school population includes only 750 students. Generally, the achievement level of these students will range from middle to high. Many of them had chosen to enroll in this course because of their future plan to attend university programs which require a Mathematics 12 credit. Typically, these students are very motivated and could be considered as "the cream of the crop". However, only a few of them, if any, would be considered mathematically gifted by the definition of contemporary standards.

The purpose of this study is to investigate secondary school students' understanding of logarithms. This study is an attempt to determine how this understanding is acquired, what difficulties the concept of logarithms presents to students, and what possible ways there are to overcome these difficulties.

Data collection relies on the following sources: written questionnaires, and in-class discussions. The written questionnaires consisted of two quizzes, one unit test, and an essay assignment that were administered during and upon completion of the unit on logarithms. The assessment tasks were constructed to test the standard curriculum material. Additionally, they included several non-standard questions that required understanding of the concept rather than just performance of a learned algorithm or technique. Specific questions and tasks for the data collection instruments were designed before they were given to the students. The written questionnaires were placed in Appendix A.

The unit *Logarithms and Exponents* was taught as a part of the Principles of Mathematics 12 course. The duration of this unit was 14 lessons. As this is approximately 17% of the course, it is the second longest unit. The concept of logarithms was introduced to the students as the major part of this unit. The first part of this unit deals mainly with the algebraic representations of exponents and logarithms, their definitions, main laws and applications. The second part introduces the geometrical interpretation of the concepts and a relationship between the graphs of the exponential and logarithmic functions; thereafter, it broadens students' conceptual understanding on the subject of number e and natural logarithms.

Every lesson was carefully preplanned, based on the assumption that students have some prerequisite knowledge. The prerequisite knowledge of this unit is the students' ability to evaluate expressions involving exponents and evaluate exponents involving positive, negative, zero, and fractional exponents. In what follows I provide details of the teaching sequence.

The general structure of the unit

Lesson 1: Real-Life Samples of Exponential Curves

Lesson 2: Definition of Logarithm, Common Logs

Lesson 3: The Laws of Logarithms

Lesson 4: Change of Base Law

Lesson 5: Solving Exponential Equations

Lesson 6: Solving Logarithmic Equations and Identities

Lesson 7: Modeling Real-Life Situations: half-life, double time, compounded interest

Lesson 8: Modeling Real-Life Situations: Earthquakes, Loudness, Acidity, Radioactivity

Lesson 9: Analyzing the Graphs of Exponential Function

Lesson 10: Analyzing the Graphs of Logarithmic Function

Lesson 11: Number *e*, Natural Logarithms

Lesson 12: Continuous Growth and Decay Problems

Lesson 13: The Unit Review

Lesson 14: The Unit Test

Lesson #1:

In the first meeting, students were working with word problems on exponential growth and exponential decay. The following question was posed to the students:

> You invested \$1000 at 8% compounded annually. What amount will you get back at the end of the first year, the second year, then the nth year? Write the expression that represents the amount at the end of the nth year.

It was assumed that the students' previous knowledge of geometrical growth which was introduced in grade 10 would help them to solve the problem. The students' responses were:

After 1 year, the amount is \$1000 + 8% of \$1000,

Or, amount = 1000(1.08) = \$1080

After 2 years: amount = 1000(1.08)(1.08) = \$1166.40

After n years, amount = $1000(1.08)^n$

Next, the students were asked to find the amount at the end of the 9th year. Without difficulty students substituted n with 9, and calculated:

Amount $= 1000(1.08)^9 = \$1999$

The second type of problem concerned exponential decay. For example,

> *Several layers of glass are stacked together. Each layer reduces the light passing through it by 5%. Write the equation that represents the percent, P, of light passing through n layers as a function of n.*

Once again, students started with simpler questions:

Let the initial intensity of a light be 100%, then

After one layer: P(1) = 100% - 5% of 100% = 100% - .05 x 100% =100%(0.95) =95%

After two layers: P(2) = 100%(0.95)(0.95)

After n layers: $P(n) = 100(0.95)^n$

Next, students were asked to find the intensity of the light remaining after 10 layers. Without hesitation, students found that $P(10) = 100\%(0.95)^{10} = 59.87\%$.

In the discussion, students also pointed out that in the first question the amount increases, and in the second, the amount decreases. Then, when I asked them why this happens, students responded that the number in the brackets (BASE) in the first question is greater than one, and in the second - is less than one.

Another discussion followed the previous one. In this case, I changed the previously discussed questions which appear above. The students were asked in how many years their invested amount would double? The purpose of this question was to invite students to look at the problem from another perspective in a familiar environment, and to introduce a new concept. I anticipated that students would be able to arrive at the correct algebraic representation of the problem, and the majority of them met this expectation. Students proposed the following steps:

$2 \times 1000 = 1000(1.08)^n$, and tried to solve

$2000 = 1000(1.08)^n$

$2 = (1.08)^n$

At this point, some students (6 out of 20) decided to solve for n by dividing both sides by 1.08, but they were corrected by the other students who pointed that $(1.08)^n$ is an exponential expression.

All students agreed that this equation cannot be solved, or they did not know how to solve it. To reinforce their attention to the problem, I asked them to try to find after how many layers of glass that only half of the initial intensity of a light would remain. Once again, students faced a problem in solving the equation: $50 = 100(0.95)^n$, or $0.5 = (0.95)^n$. However, I still had a couple of students who wanted to divide by 0.95. This was an indication to me that those students were mistaking exponential expression for the operation of multiplication. Moreover, when I asked the students who "solved" the equation to explain how they found the answer, one of the students said that he *distributed n* into the brackets, and as a result he was be able to solve the equation. Since this discussion was open to every student in the class, many students volunteered to clarify the situation, by showing that in both the equations the exponential expressions would be expended as follows:

$(1.08)^{n} = 1.08 \times 1.08 \times 1.08 \times \ldots \times 1.08$ - n times

$(0.95)^{n} = 0.95 \times 0.95 \times 0.95 \times \ldots \times 0.95$ - n times

Again, everybody agreed that they didn't know how to solve the exponential equations described above. Then, students were asked to compare the next two equations, and attempted the solutions to the equations:

1) $2^{3x-1} = 16$ and 2) $2^{x} = 10$

The purpose of this activity was to show the students that they can solve some exponential equations, and to prepare common ground for the introduction of logarithms.

Lesson #2:

During this lesson, students were introduced to logarithms. The reason for studying logarithms was to develop a method for solving exponential equations that were encountered in the previous lesson. The students were asked to construct a table of values of the powers of 2 and the corresponding exponents (example is provided below). After they had completed this activity, they were asked several questions of the following nature: *What is an exponent of 2 that will produce 32?* (The purpose of this particular question was to verify if students understood the connection of powers to exponents, since it is a reverse sequence to the one they used to form the table: exponents $\Rightarrow$ powers.)

Can you solve the question $2^x = 10$ *by using the table? What is your best answer? How did you arrive at it?* (The purpose of this question was two-fold: to illustrate that there are other numbers that students did not include in their tables; and, to make students reflect on how the exponents operate.)

Table 1: Table of Values of Powers of 2.

x	2^x
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256

Next, students were asked to create several questions regarding the information displayed in the table. Students were challenged by the necessity to be explicit if the question was about an unknown exponent. Then, the teacher generalized the problem by saying that similar tables can be constructed for any base (we limit this study only to positive and not 1). Students were asked if they could generate tables for 1.086 or 3/7. The teacher summarized all the investigations into the following form: $a = b^x, b > 0, b \neq 1$. When the importance of knowing what an exponent is was established, the teacher introduced logarithm, and defined it as follows: $\log_b a = x, a > 0, b > 0, b \neq 1$. Next, the relationship between the exponential and logarithmic forms was explored. Students were engaged in different activities, such as:

Determine each logarithm.

a) $\log_5 25$ *b)* $\log_7 \sqrt{7}$ *c)* $\log_{123} 1$ *d)* $\log_a a$ *e)* $\log_2 \frac{1}{8}$ *f)* $\log_2(-4)$

Table 2: Exponential verses Logarithmic Forms

Complete the following table.

EXPONENT FORM	LOGARITHM FORM
$3^4 = 81$	
$7^{1/2} = \sqrt{7}$	
	$\log_2 32 = 5$
	$\log_8 8 = 1$

Students learned how to change functions from exponential form to logarithmic form and vice versa.

Lesson #3, 4:

The lesson began with an introduction of the common logarithms. The Laws of Logarithms (base 10) were discussed. The Multiplication Law, if x and y are positive real numbers, then $\log xy = \log x + \log y$, was proven. Students were asked to prove the Division Law, if x and y are positive real numbers, then $\log x/y = \log x - \log y$, and the Power Law, if x is positive real number and p is any real number, then $\log x^p = p \log x$. The Laws were generalized for the base a, and the Change of Base Law, if x is positive real number not equal to 1, and y is a positive real number, then $\log_x y = \log y / \log x$, was discussed. Students learned how to use the graphing calculator to calculate logarithms.

Lesson #5,6:

Students learned how to solve two types of exponential equations (without logs and with logs). Students learned how to solve logarithmic equations and verify logarithmic identities. The restrictions on the set of solutions were discussed.

After this meeting the students were assessed on the definition of logarithm, the relationship between logs and exponents, the laws of logarithms, verifying the log identities, and solving exponential and logarithmic equations. Students were not allowed to use the graphing calculator. A complete quiz is included in Appendix A.

Here are some questions:

- Rewrite in exponential form: $\log_2 64 = 6$
- Convert into logarithmic form: $3^{-3} = 1/27$
- Find the value of the expression $\log_3 54 - \log_3 8 + \log_3 4$
- Solve: $\log_{12}(3-x) + \log_{12}(2-x) = 1$
- State the values of x for which the given identity is true: $\log_5(x+1) + \log_5(x-4) = \log_5(x^2 - 3x - 4)$

Lesson #7,8:

First, as a prerequisite to these lessons, students should be able to understand how changing the logarithm of a number is related to a change in the number itself. For example, students should be able to solve each of the following equations:

$\log 10 = x$

$\log 100 = x$

$\log 1000 = x$

$\log x = 4$

$\log x = 5$

How is the change in the logarithm of the number related to the change in the number itself?

An application of exponents and logarithms was the main focus of the next couple of lessons. Students learned how to create exponential equations to model the real-life situations, and then use logarithms to solve these equations. Several types of word problems introduced in class included:

- Exponential growth problem (population increase, *double time*)
- Exponential decay (reduction of the level of radiation, *half-life*)
- Compounded interest (annually, semi-annually, monthly, weekly, daily)
- Magnitudes of earthquakes (*ten-fold*)

After those lessons students were given another quiz. Previously, in the first quiz students had been tested on their understanding of the basic properties of logarithms and exponents. Subsequently, at this point, they were assessed on their ability to apply the previously developed knowledge to solve real-life problems.

An example:

In 1995, the population of Calgary was 828,500, and was increasing at the rate of 2.2% per year.

a. When will the population double?

b. Rewrite the equation using base 2.

Lesson #9, 10:

During these classes students learn about exponential and logarithmic *functions.* The graphical representations of these functions along with their properties are important conceptual components. On one hand, students discuss the relationship between the definition, algebraic and graphical representations of logarithmic function. On the other, they reinforce their understanding of the inverse relationship between the exponential and logarithmic functions. The main properties of the exponential and logarithmic functions such as their graphical representations, domains, ranges, asymptotes, and intercepts were analyzed. Transformations of these functions; for example, how to sketch the graph of the function $y = 3\log(x-2)$ using the graph of the function $y = \log x$ without graphing calculators, were discussed in detail. It is anticipated that students who are able to graph and analyze the functions with and without technology, should be able to find the equations from the given graphs; and, when given the equations, graph the functions and identify the domain, range, asymptotes, and intercepts of the graph.

Lesson #11, 12:

During these meetings, students learned about number e and the *natural logarithm*. The number e was introduced with the compound interest problem:

> *Suppose you invest $1 in a bank that pays you 100% interest per annum. If the compounding is done n times a year, then the sum of money at the end of the year is* $(1+1/n)^n$. *What happens to this expression as n increases?*

The natural logarithm was defined as the logarithm with the base e. Students learned how to develop a general formula for continuous growth and decay problems, and how to solve these problems by using natural logarithms and technology.

Lesson #13:

Before this class, as a review activity, students were assigned a short essay. They were asked to explain to their younger schoolmates the meaning of logarithm using language comprehensible by the chosen age group. The responses to this task demonstrated how students understood the basics of the concept of logarithms. This assignment was part of data collection.

During this meeting, students reviewed the skills and concepts they learned during the whole unit *Exponents and Logarithms*, highlighting the key points. Then, they worked independently on the chapter-check section. The questions that caused them difficulty were discussed in class. Part of the in-class discussion was included in this study.

Lesson #14: UNIT TEST

The tasks for the final unit test were carefully pre-selected so the students' responses would reveal the answers to the following questions:

- Do the students understand how exponential functions can represent growth or decay?
- Can the students solve exponential equations?
- Can the students find the inverse of an exponential function?
- Can students find the domain, range, asymptotes, and the intersects of logarithmic function?
- Do the students understand and apply logarithms?
- Can the students solve equations involving logarithms?
- Do the students understand and apply the concept of continuous growth or decay?

The Unit Test included 20 multiple choice questions, and 5 written response questions. Some questions were designed to examine the standard curriculum material, and very often could be solved by following a learned algorithm or technique. Other questions required a deeper conceptual understanding, as in a non-standard question used in the unit test (Which number is larger 25^{625} or 26^{620}? Explain). The complete Unit Test is included in Appendix B.

CHAPTER V: ANALYSIS OF DATA

Tasks and Frameworks

In this chapter I try to provide an explanation of the meaning of the concept of logarithms as it was constructed by the students. Strategies and tools used in the students' approaches helped to shed light on their understanding. The amount of data was overwhelming. The guiding principle in choosing the tasks was in the variety of the approaches students took, rather than number of right or wrong answers. The data was analyzed by looking for common trends in participants' responses. Data selected for the analysis consisted of a subset of the following six tasks:

1. Simplify the following expression: $\log_3 54 - \log_3 8 + \log_3 4$.
2. Solve: $\log_{12}(3-x) + \log_{12}(2-x) = 1$.
3. Which number is larger 25^{625} or 26^{620}? Explain.
4. Give the domain of the relation $\log_x(y-2) = \log_x(4-x)$.
5. In short essay format, the students were asked to explain to younger schoolmates "What is a Logarithm?"
6. In-class discussion of the question: find the exact value of $5\log_3 9$.

The above tasks were chosen as they describe a variety of ideas and methods for gathering evidence of students' understanding of logarithms. They focus on the critical and significant aspects of this concept.

The analysis of these tasks was done through four interpretive frameworks which have been useful in modeling the students' understanding of logarithms.

Framework One: Logarithms and Logarithmic Expressions as Numbers

Framework Two: Logarithmic Expressions and Local Operational Meaning.

Framework Three: Logarithms as Systematically Operational.

Framework Four: Logarithms as Functions.

The presented set of interpretive frameworks is the result of my adaptation of Confrey's model for investigation of student's understanding of exponents. In Chapter III, section *Confrey's Model for Investigation of Student's Understanding of Exponential Functions*, I presented my interpretation of the frameworks identified by Confrey (1991) in the article *The Concept of Exponential Functions: A Student's Perspective*. There, to remind the reader, she proposed a set of five frameworks:

Framework One: Exponents and Exponential Expressions as Numbers.

Framework Two: Exponential Expressions and Local Operational Meaning.

Framework Three: Exponents as Systematically Operational

Framework Four: Exponents as Counters

Framework Five: Exponents as Functions

When analyzing the proposed frameworks, it seemed natural to think about the students' understanding of logarithms. All but one framework were actually modified for this study. When reflecting on *Framework Four: Exponents as Counters*, I found it difficult to view logarithms as counters. I could think of some special examples, but could not generalize them. Hence, the proposed system, created for the purpose of interpreting the meaning of logarithms as it is constructed by students, consists of four, rather then five, interpretive frameworks.

System of the Interpretive Frameworks for Logarithms

Framework One: Logarithms and Logarithmic Expressions as Numbers

In this framework I try to delve into students' understanding of logarithms as numbers. Students have to be able to connect the meaning of logarithms when attempting to solve specific questions, such as:

Calculate: $\log_3 27$;
Find: $\log_{687} 1$;
Compare: $-\log_2 7$ and $\log_2 1/7$;
Simplify: $\log_3 54 - \log_3 8 + \log_3 4$.

It is expected that students' responses to these problems will indicate that *logarithms and logarithmic expressions DO or DO NOT behave as "numbers"* in their understanding.

Framework Two: Logarithmic Expressions and Local Operational Meaning.

The main issue explored in the second interpretive framework is the students' understanding of an operational character of logarithms. The isomorphism between multiplicative and additive structures is easily recognized with logarithms. For example, $\log_3 54 - \log_3 8 + \log_3 4 = \log_3(54 \div 8 \times 4)$. The logarithm reduces the operational complexity. Consequently, the operational meaning of logarithms can be extended to the concept of the *common logarithms*, additive relationships of the system ($\log 1/100$, $\log 1/10$, $\log 1$, $\log 10$, $\log 100$ ...)

Framework Three: Logarithms as Systematically Operational.

This particular framework helps in investigating how students acquire an understanding of the systematic operational relations with logarithms. Students have to be able to operate with logarithms in two ways. For example, simplify $\log_3 90 - \log_3 10$; and expand $\log_c a^2 b$. By doing these types of exercises students began to recognize the isomorphism between the additive and multiplicative structures, and between the subtractive and divisive structures. This way we can investigate how students construct the operational meaning of logarithms.

Framework Four: Logarithms as Functions.

In this framework, two main perspectives are guiding the researcher's investigation of students understanding of *logarithmic functions*. One perspective is the operational basis of the logarithmic functions. Does the student understand what a function is? How does the student relate the definition of the logarithm to the logarithmic function (range, domain, asymptotes, and intersections)? What are different representations of the logarithmic function, and how they are related? Another perspective is to examine how students apply the knowledge of logarithms to solve different types of problems.

When applying the above presented set of four interpretive frameworks, it has to be understood that they are not isolated schemas for the analysis. Very often they overlap. When using them as a set, they may become a powerful instrument in highlighting different aspects in one standard approach.

Results and Analysis

In this section I provide a summary of students' solutions, and then analyze them according to the proposed frameworks. Thereafter, I share pedagogical ideas that address the problem identified in students' work. The information in this section is organized in the following order: every educational task chosen for the analysis is followed by a quantitative summary of students' attempts, detailed analysis, and concludes with a feasible pedagogical suggestion.

Task 1:

Simplify the following expression: $\log_3 54 - \log_3 8 + \log_3 4$.

This question was a part of Quiz #1given at the end of Lesson #6 (see page 39). It was assumed that this question, used as an assessment instrument, would reveal information about students' understanding of the basic properties of logarithms.

Note the following abbreviations:

C - Correct solution **IC** - Incorrect solution **PC** - Partially Correct solution

Table 3: Quantitative Summary of Task 1

C/IC/PC	Examples of Solutions	Number of students presenting this solution
1C	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3 \frac{54}{8} \times 4 = \frac{\log 27}{\log 3} = 3$	9
2C	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3\left(\frac{54}{8}\right) + \log 4 = \log_3(6.75) + \log_3 4$ $= \log_3(6.75 \times 4) = \log_3(27) = 3$	3
IC	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3(54 - 8 + 4) = \log_3 50 = 3.5609$	3
PC	$54^3 \div 8^3 \times 4^3 = 19683$	2

Analysis

While the majority of students showed no difficulties in answering the question (12 out of 17), there were some students (5 out of 17) who indicated full or partial misunderstanding of the concept.

PC) $54^3 \div 8^3 \times 4^3 = 19683$ (2 students)

This answer can be considered partially correct. Firstly, this response shows students' misinterpretation of logs and exponents. Base 3 ended up as an exponent 3. We can speculate about the reason this happened. It could be just a simple lack of attention, or, more obviously, students didn't understand the definition of logarithm. In class, logs were defined as exponents. Students connected their answers to exponents, but in an inappropriate manner. ($\log_3 54 \Rightarrow 54^3$) This indicates that students did not understand the meaning of exponents. Since this understanding is a pre-requisite knowledge to the learning of logarithms, these students had failed to understand logarithms.

Secondly, the participants understood how logarithms operate because they correctly changed the operations of subtraction into division, and addition into multiplication. However, this type of understanding is only instrumental. As it was emphasized in class, logs allow changing from more complex arithmetic operations to simpler ones (log*ab* = log*a*+log*b*). Though students indicated it in the reverse order correctly, their responses may be due to a lack of sufficient attention in the classroom.

It is hard to speculate on the further analysis of this specific error. However, the second typical mistake that appeared in the same assignment fits a better profile for future interpretations. It is much richer in terms of the content.

IC) $\log_3 54 - \log_3 8 + \log_3 4 = \log_3(54 - 8 + 4) = \log_3 50 = 3.5609$ (3 students)

This response can be interpreted as follows. It is obvious that students used the distributive property to group the whole numbers. It provided them with freedom for the next action. Now they could combine the numbers. It is clear that students did not view $\log_3 54, \log_3 8, \log_3 4$ as numbers. It seems that since logarithms appear symbolically as "log" of arithmetic numbers, that students' familiar knowledge of those numbers guided them in their actions. In these instances, students do not grasp the fact that *logarithms behave as numbers*.

According to the second framework, students' actions indicate that they did not understand the *operational character of logarithms*. It seems that students did not remember that the logarithm *increases an operational complexity*. If the bases are the same, when the logarithmic expressions are added, the operation of logs is multiplication.

From their actions, we can say that students extended the "abbreviation" meaning of logarithms into an operational meaning.

According to the Skemp's theory of understanding, students believe they understood the notation of logarithms ($\log_3 54 = \log_3 \times 54$). This was assimilated into their schema for multiplication. It is damaging to the students' understanding, but it can be *reconstructed.*

According to Hiebert & Carpenter, students connected this representation to their knowledge of the distributive property. This connection can be modified through a process called *reorganization.*

According to Sierpinska, students have grasped the "abbreviation" meaning as the operational meaning. This became an obstacle that can be overcome.

Pedagogical Considerations

The traditional curriculum does not place enough emphasis on logarithms as numbers. Students have encountered a variety of obstacles in the process. To overcome these obstacles students could complete the following task: Organize in increasing order the following set of numbers {-log2, log5, log1/2, log1, -log3/4}.

Students have to establish an operational character of logarithms. What is a log of one and the fraction? For example, log7 and -log7 are operationally opposite. However, their inputs are not opposite in magnitude because $-\log 7 = \log 1/7$ and $-\log 7 \neq \log(-7)$. It is critically important for students to be aware that the logarithms cannot be added when operation on the expression is addition. ($\log 4 + \log 5 \neq \log 9$)

Task 2:

Solve: $\log_{12}(3-x)+\log_{12}(2-x)=1$

This question was a part of the first written assignment Quiz #1 (17 students participated). This task was intended to assess students' understanding of the laws of logarithms, and the definition of logarithm with emphasis on the set of the permissible values. It was expected that students would be able not only to solve the problem, but also to verify the correctness of their answers.

Table 4: Quantitative Summary of Task 2

C/IC/PC	**Examples of Solutions**	**number of students presenting this solution**
C	$\log_{12}(3-x)+\log_{12}(2-x)=1$ $\log_{12}(3-x)+\log_{12}(2-x)=\log_{12}12$ $(3-x)(2-x)=12$ $x^2-5x-6=0$ $x=-1$ $x=6(\textit{rejected})$	5
IC	$\log_{12}(3-x)+\log_{12}(2-x)=1$ $3\log 12-x\log 12+2\log 12-x\log 12=\log 1$	4
1PC	$\log_{12}(3-x)+\log_{12}(2-x)=1$ $\log_{12}(3-x)+\log_{12}(2-x)=\log_{12}12$ $(3-x)(2-x)=12$ $x^2-5x-6=0$ $x=-1$ $x=6$	6

C/IC/PC	Examples of Solutions	number of students presenting this solution
2PC	$\log_{12}(3-x)+\log_{12}(2-x)=1$ $(3-x)+(2-x)=12$ $-2x=12-5$ $x=-\frac{7}{2}$	2

Analysis

Out of 17 students who participated in this activity, only 5 solved this problem correctly. 6 students found the answers, but did not notice that one of the solutions is an extraneous root. Their answers can be considered as partly correct. The rest of the group produced incorrect solutions to the presented problem.

The result 1PC) indicates that there was a significant group of students (6 out of 17) who had good algebra skills. They knew how to solve the equation. They correctly applied the law of multiplication, and converted the left side into a logarithm with an appropriate base. However, their final step of "accepting the roots" indicates that these participants did not understand the definition of logarithm. None of them even tried to substitute 6 into the given equation, and check the legitimacy of this solution. Perhaps, students decided that because 6 is a positive number, a check would be unnecessary. However, if following this line of reasoning, they should have checked at least -1 because it is negative, but they did not. This illustrates that these students are not equipped with the necessary problem-solving techniques. After finding the solutions, the students did not reflect on the problem.

The result 2PC) produced by 2 out of 17 students, can be considered partially correct. The missing step of converting 1 into the logarithm of 12 is absolutely correct. However, the right side (3-x) + (2-x) can be interpreted as cancellation of L-O-G in front of every term. Once again, students did not treat logarithms as numbers.

The result IC) generated by 4 out of 17 students indicates complete misunderstanding of the concept of logarithms; therefore, these students cannot comprehend how logarithms operate. There is an evidence of applying the distributive law, what is in this case is a useless tool derived from the past experiences with whole numbers.

After approximately 30% of class-time, according to frameworks one, two, and three, the majority of the students had not acquired an understanding of the local, and consequently, systematic *operational meaning* of logarithms. However, as the follow-up analysis of the after-quiz review showed, by the end of the unit, some more of the students were able to grasp the operational meaning of logs.

Pedagogical Considerations

The students' failure to verify the appropriateness of the roots of the equation could have been caused by:

1. students' misunderstanding of the mathematical enterprise as a whole, with a specific set of rules and definitions that must be obeyed or satisfied;
2. students have chosen the result-oriented approach, which is so widely employed by the current curriculum, where goal satisfies the means.

To analyze the difficulties students experienced while acquiring the operational meaning of logarithms, it is important to compare their similar experiences from the past. For example, in the case of the whole number system, or integers, or fractions, it appears that the operational meanings of those numbers were developed in a certain order. First, students learned about the numbers, compared them, and then learned how to add or subtract. Later they learned how those numbers could be multiplied or divided. However, when learning logarithms, the first rule after the definition is the "Law of Logarithms for Multiplication" (Addison-Wesley, p.81), or "Logarithm of a Product" (MATHPOWER 12, p.103). Both titles represent the following law: $\log_a xy = \log_a x + \log_a y, a > 0, a \neq 1$. The words in the titles *multiplication* and *product* obviously emphasized the historical reason for the invention of logarithms, when the same rule, but in the reverse order, would produce the rule for the *addition* of logarithms. Similarly, the "Law of Logarithms for Division" (Addison-Wesley, p.82) or the "Logarithm of a Quotient"(MATHPOWER 12, p.103) could be rephrased as the *subtraction* of logs rule. These changes would place participants into a familiar situation of working with a system of numbers, and stress that logarithms are numbers. Under the circumstances just mentioned, on one hand, the students' prior experience with other number systems becomes an obstacle in learning logarithms. On another, the historical development of logarithms created difficulty for students' understanding of the operational meaning of the concept. Using a format familiar to students, identifying rules for addition and subtraction of logarithms with the same base, and emphasizing that the product of logs does not equal the log of the product would help students' understanding.

Task 3:

Which number is larger 25^{625} or 26^{620}? Explain.

This question was a part of the final unit test (27 participants). At the time of the test, all required material was covered. To answer this non-standard question, students required a conceptual understanding of logarithms rather than memorization of a learned algorithm or technique. It was expected that the strategies students applied to solve this problem would reveal their level of understanding of the concept.

Table 5: Quantitative Summary of Task 3

C/IC/PC	Examples of Solutions	number of students presenting this solution
1C	26^{620} is larger because the logarithm of this number is larger	7
2C	$26^{620} = \left(25^{1.012}\right)^{620} = 25^{632.6}$ $25^{632.6} > 25^{625}$	1
1IC	25^{625} is larger because it has larger exponent	9
2IC	25^{625} is larger as it can be written as $5^{2^{5^4}}$	1
3IC	"25^{625} *is bigger, the bigger the* ***power*** *the larger the answer.* ***Powers*** *are more significant than bases.*"(Exact words)	1

8 students gave the right answer; however, only 3 of them presented mathematically sound solutions. Others followed their intuition, or just made a lucky guess. 19 out of 27 students answered incorrectly.

Analysis

Which is bigger 25^{625} or 26^{620} ? Explain.

25^{625} is bigger because it has a bigger exponent

Figure 1a. Participant's Response 1IC.

Which is bigger 25^{625} or 26^{620} ? Explain.

25^{625} is bigger because if you have 25 to the power of 625 and you keep on (x) the 25 by itself for a more number of times it will end up being bigger eventhough the 26 is bigger than the 25 but the 620 is smaller.

Figure 1b. Participant's Response 1IC.

The participants' responses coded 1IC) presented above were the most common. 9 out of 27 students produced exactly the same explanation of their choice. They believed that a larger exponent will determine a larger number. Their guess was based on the premise that the exponent indicates the number of self-multipliers of 25 or 26. Students assumed that the "longer" product is the larger. (One of the explicit results in this regard is posted above). These participants did not connect this problem to the concept of logarithm whatsoever.

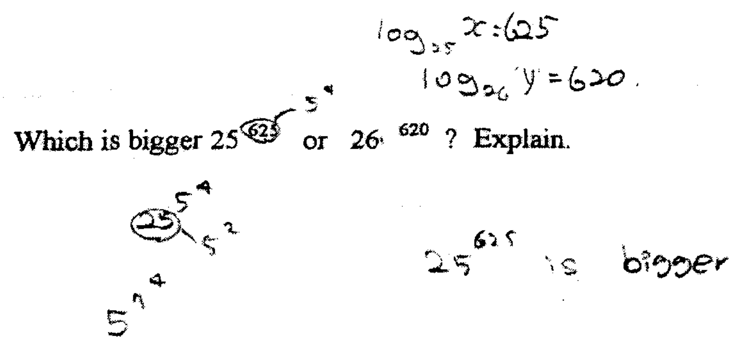


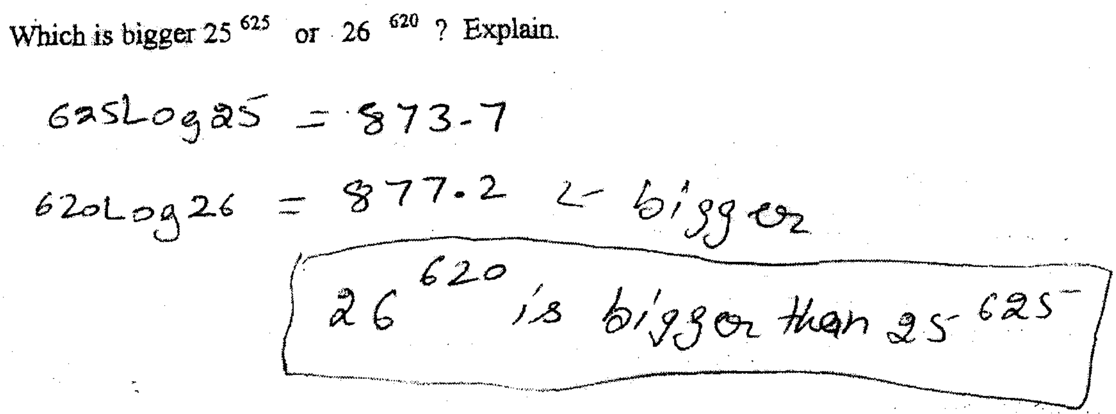
Figure 2. Participant's Response 2IC.

The result 2IC) was unique in its own way. The participant shows an attempt to connect the posed problem to the concept of logarithms. The first step of converting 625 into $\log_{25} x$, and 620 into $\log_{26} y$ illustrates his understanding of the definition of logarithm as an exponent. The student changed exponential expressions into logarithmic expressions accurately. However, it seems that he or she never went beyond the basics. When the first idea was abandoned by the student, and he/she tried to investigate another one. Exponents were brought in. 625 attracts the participant's attention, since it is a power of 5, and the base of the first power is five squared. The student tried to combine both in an inappropriate manner. His choice of the answer is not justified.

3IC) One of the students wrote:

"25^{625} *is bigger, the bigger the* ***power*** *the larger the answer.* ***Powers*** *are more significant than bases.* "(Exact words)

It seems that the meaning of the word POWER became a barrier to the participant's understanding of exponents in the first place. It was defined to the student that a logarithm is an exponent. However, if the student associates the word "*exponent*" with the word "*power*", but does not clearly distinguish the difference between "8 is a power of 2" and "2 raised to the *power* of 3". It created great confusion which becomes an obstacle in understanding logarithms.



<u>Figure 3.</u> Participant's Response 1C.

7 out of 27 students produced the answer 1C). They indicated the correct larger number; however, they exhibited only a procedural understanding. The idea of comparing logarithms of the numbers is very sound if using the base of the common logarithm. Indeed, the function of the common logarithm is an increasing function, and the larger input number has the greater value. However, in this particular case, students did not show an understanding of that property. It seems that they were satisfied to find two numbers which students could compare, and the result of this comparison led them to the conclusion. Luckily, one of the participants produced a completed and mathematically sound solution:

Which is bigger 25^{625} or 26^{620} ? Explain.

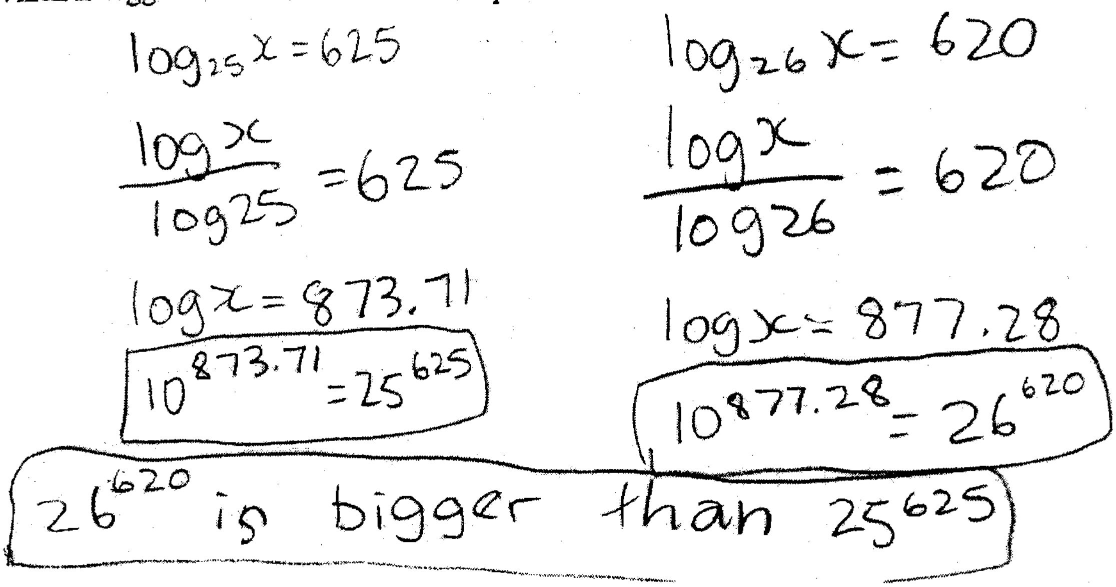


Figure 4. Participant's Response 2C.

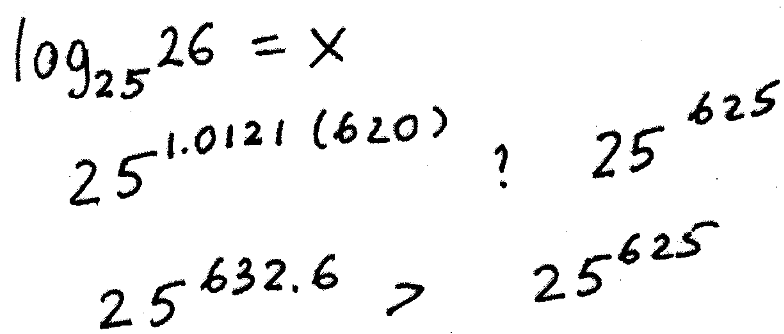


Figure 5. Participant's Response 3C.

2C) and 3C) are absolutely correct solutions each produced by only one student. At this academic level, the participant had exhibited the deepest possible understanding of the concept. These solutions illustrate that the students understand that any real number can be presented in the form of a logarithm.

Pedagogical Considerations

Analysis of the result 1IC) may suggest that students demonstrated misunderstanding of the concept of exponents that is a pre-requisite for learning logarithms. They illustrated only a procedural understanding of exponents, which is far too limited upon which to build the knowledge of logarithms.

Additionally, it seems that the choice of 625 as an exponent misled the participant. Coincidently, the selected numbers from the problem 25 and 625 were the powers of 5. If student would apply the change of base law to $\log_{25} x$ and $\log_{26} y$, he/she would solve the problem in no time, but the problem contained the destructor that played a fatal role in a "weak" student's thinking.

When the majority of the participants (15 out of 27) failed to connect this problem to logarithms, they also indicated a very poor (if any) understanding of exponents. Not having the prerequisite knowledge became a barrier in the understanding of logarithms. Nevertheless, since only 3 students from this cohort failed the unit test, I might assume that the rest of the group could have memorized and manipulated the basic properties of logarithms.

According to frameworks one and four, a large part of the participants failed to understand that any real number can be presented as a logarithm. The meaning "any" comes from the understanding of logarithmic function's set of values or range. To stress students' attention on "any" I would include the following activities in the teaching repertoire.

<u>Activity 1:</u> Present the following numbers: $\{1, 3, -2, 1/2, 0.1, \pi\}$ as logarithms with the base 2.

<u>Activity 2:</u> Find four different logarithmic representations of number 4.

For example, $4 = \log_2 16$

How many logarithmic representations does one real number have?

Task 4:

Give the domain of the relation $\log_x(y-2)=\log_x(4-x)$

Choose one of the following:

a. $x>0, x\neq 1$
b. $x>4$
c. $0<x<4$
d. $0<x<4, x\neq 1$
e. $-4<x<4$

This question was included in the multiple choice section of the unit test. The purpose of this question was to evaluate students' ability to relate the definition of logarithm to the logarithmic function; specifically, how the definition will influence the domain of the function represented algebraically.

Quantitative Summary of Task 4:

a. $x>0, x\neq 1$	3 students
b. $x>4$	6 students
c. $0<x<4$	6 students
d. $0<x<4, x\neq 1$	10 students
e. $-4<x<4$	2 students

Analysis

The correct answer to this question is **d**. 10 out of 27 participants chose **d**. They correctly combined the restrictions on the base x, $x>0, x\neq 1$ and the power (4-x)>0. The participants who chose **a** or **c** as an answer exhibited only partial understanding: in **a** the emphasis was placed on the base only, and in **c** the value of one for the base was not excluded. In order to obtain the results **b** or **e**, participants knew that they had to verify the domain of the function. However, poor algebra skills prevented them from getting the correct answer.

Pedagogical Consideration

The benefit of having a multiple-choice section in the test is a guaranteed, unbiased assessment. Although, on occasion students tend to guess the answer without solving the question, the answer doesn't reveal the level of students' understanding of the particular concept. However, the question chosen for the analysis had a very good set of answer-choices. In order to bridge the question with a correct solution, a student had to be able to recognize many important properties of the logarithmic function.

Task 5:

In short essay format, students were asked to explain to their younger schoolmates "What is a Logarithm?"

This task was intended to assess students' understanding of the definition and the meaning of logarithm. It was expected that students would be explicit in their explanation of what a logarithm is. Students were not limited to any specific strategies they might use in their explanations. It was anticipated that some of them would make up their own examples of problems involving logarithms. Students were asked to use an appropriate language, suitable for younger students. This question was a part of the review session before the final unit test. 17 participants wrote the essay.

In the analysis of student essays I find that 8 out of 17 students tried to explain what a logarithm is. 4 out of 17 students explained what a logarithm does, 5 out of 17 students tried to explain both what it is and how it works. I provide a summary of what was attempted to by each student in the following table.

Table 6: Table of Strategies Students Used in Their Essays-Explanations

Student's essay #	"log is an exponent"	Formulas/log laws	Own Examples	Historical references	Story telling	Incorrect examples
1.	X	X	X			
2.	X	X	X	X		
3.				X	X	X
4.	X		X			
5.	X				X	
6.	X	X	X			
7.	X	X	X			
8.	X	X	X			X
9.	X	X	X		X	
10.	X					X
11.			X		X	
12.		X	X			
13.			X		X	
14.			X		X	
15.		X			X	X
16.			X	X		
17.		X	X			

In the following, I analyze several essays in detail. These essays are chosen to illustrate the variety in students' responses.

What is log?

Log was made based upon 10. There might be some other numbers that you might be using, but for now I am going to use 10 as a base.

Long ago, people wanted to count the numbers and write it out. However, a number like this for example, 1000000000000000000000, it is going to take them very long time to write and read. Then they decided to use exponents. By using the exponents, you can easily express that big number into a simple number like 10^{21}. Moreover, number with 100 zeros can be simply expressed like this, 10^{100}.

This is the base of the logarithms. Well, let's now look at the actual log. Here is the number, 1000000. To express this number into an exponential form, it would be written like this, 10^6. In this case, you say 6 is the log of a million. In specifically, the 10 was the base, therefore you could say (6.) is log of a million with a base of 10. To express this in mathmatical way, you write, $6 = \log_{10} 1000000$, which is came from $10^6 = 1000\,000$.

Figure 6. Essay #1.

Analysis

This response can be interpreted as follows. In the first half, the student tried to create a prelude for the definition of logarithms. By using an example, s/he illustrated how scientific notation of the large number simplifies the number notation (from using 21 digits to using only 4, 100000000000000000000 = 10^{20}). In the second half, the student defined logarithm as an exponent, and used an example: $10^6 = \log 1000000$.

On a basic level, the student's work can be considered appropriate; however, it lacks examples involving fractions, zero, or negative numbers. Incorporation of those in the discussed explanation would illustrate the fact that any real number can be presented as a logarithm. Moreover, the same number could have multiple log representations (such as $2 = \{\log 100,\ \log_3 9,\ \log_7 49,\ \log_{2/3} 4/9,\ -\log_5 1/25\}$).

From the above work, it can be justified that the student understands the definition of logarithm. Since s/he limited the examples used in the essay to the common logarithms only, a concern is raised about the level of his/her understanding.

A log is nothing more than an exponent. They are a simple way of expressing numbers in terms of a single base. Most logs (Common logs) are done with a base 10. But we can use logs with different bases such as 2 or 25. They come in handy when we need to add logs together in which case we just multiply the numbers after the log only if the base is the same. We can use logs to find the value of an exponent when it is a variable.

Figure 7. Essay #2.

Analysis

Even though this essay is one of the shortest written, it is very explicit. The student illustrated that s/he understands not only what a logarithm is, and what it is used for, but also one of the operational properties of logarithms. However, this explanation is the summary of the student's perception of logs. This essay does not fulfill the expectation of explaining the meaning of logarithms to the younger students. Perhaps, if the student tried to complete the task as it was assigned, s/he would reveal the level of his/her understanding of the concept of logarithms.

A logarithm function is the inverse of an exponential function. If x is b to the power of y, $x = b^y$, then y is the logarithm of x in the base *b* (meaning *y* is the power we have to raise *b* to, in order to get x), and you can write write $\log_b x = y$. For example, $\log_{10}100 = 2$ (since $10^2=100$) and $\log_2 8 = 3$ (since $2^3=8$). Logarithm functions act equally but oppositely to exponential functions when multiplying and dividing. You add the powers when multiplying and subtract them when you divide, *Exponents* – $5^3 \times 5^7 = 5^{10}$; *Logarithm* – $\text{Log}_5 3 + \text{Log}_5 7 = \text{Log}_5 21$.

Figure 8. Essay #3.

Analysis

This response reflects the most common "traditional" way of introducing logarithms. It begins with a general statement that a logarithm is an inverse of the exponent, and is supported by two examples: $\log_{10} 100 = 2$ and $\log_2 8 = 3$. However, the third sentence of the paragraph contains the phrase "logarithmic functions act *equally* but *oppositely* to exponential functions" that is not explicit about the operational meaning of logarithmic function. The opening statement is about a logarithmic FUNCTION and exponential FUNCTION, but the examples in the last statement express the relationship of "equality" between the exponential and logarithmic expressions. There is no clear connection between expressions and functions.

This essay lacks consistency in explanation. It looks like the author possesses only procedural understanding. S/he has memorized the rules, and exhibited how the procedures work. Nonetheless, his/her explanation of logarithms leads the listener to believe that logarithm is nothing more than another little trick that can be used when dealing with a specific situation within the field of mathematics.

Logarithms were invented by 2 people, John Napier in 1614, and Joost Burgi in 1620. Napier used an algebraic approach, while Burgi used geometrical formulas. However, logarithms were recognized as exponents by 1694. Logarithms were invented to simplify mathematical calculations. The only difference between exponents and "logs" is inverse. Logarithms are the inverse of exponents. For example, 5^(-2) = 1/25, this is the exponent. But, 5^(1/25) = -2 is the inverse, and the LOG form. The base remains the same, but the exponent and the final answer are reversed. There are many forms of logarithms, but the 2 most popular are common and natural logarithms.

Figure 9. Essay #4.

Analysis

Even though this essay contains the amazing historical fact that logarithms were invented separately from exponents, and only much later the relationship between them was established, it illustrates complete conceptual misunderstanding of logarithms. The student tries to connect logs and exponents through the inverse, yet s/he cannot substantiate it with correct examples.

The inverse statement to $5^{-2} = 1/25$ is presented as $5^{1/25} = -2$. Obviously, the student struggles with an understanding of the exponents as well as the inverse relationship. What s/he does is a simple manipulation of three numbers. Probably the only fact s/he remembers is that the base always remains as a base regardless of the form: exponential or logarithmic.

The simple explanation of Logarithms means the exponent. For example, if you were given a question, Log3^9=2. We know that 3^2=9, which means that three squared is nine. The equation would simply be, "The log to the base of 3 of 9 is 2." The way that I think of it in my opinion is that log is the exponent. The key concept to Logarithms is attempting to achieve the same base of both numbers and then attempting to get the exponents and solving for x. There are different basic concepts that need to be understood before moving on to the next level of Logarithms. Firstly, when you are multiplying two bases then you add the two exponents that are provided. For example, logb^(mn) = Logbm + LogBn or Log (4*5)= Log4 + Log 5. Then secondly, when you are dividing your bases; then you subtract the two exponents. For example, Logb (m\n)= Logbm- Log bn or Log16^2= Log16/Log2. When there is an exponent in front of the base, which is logged, then bring that exponent in front of the Log. For example, Logb M^P = P Logbm.

Figure 10. Essay #5.

Analysis

Prior to the analysis of this piece some notations used by the student must be interpreted. The notation: Log3^9=2 must be read as "the log with a base 3 of 9 equals 2", so the sign "^" is used just for separation of the base of the logarithm and the number under the log. This interpretation is gathered from the context provided.

The logarithm is defined here as an exponent; that is correct. However, it is hard to explain the underlined sentence in the essay. It sounds like the student tried to describe when and how logarithms are used, but it is very unclear. It gives us some doubt that s/he understands the properties of logarithms. Moreover, the student uses the word "base" in the meaning of "power". This can be an indication that s/he does not possess prerequisite knowledge of exponents. Some logarithmic laws are illustrated with examples, others are not. However, the *change of base* law is demonstrated with an incorrect example: $\log_{16} 2 \neq \frac{\log 16}{\log 2}$.

The student has included many properties of logarithms in his/her essay; many of them are presented by using variables. However, none of them contains the restrictions that usually follow the formulas, and some of them were illustrated with incorrect examples. Considering the reasons discussed above, it can be concluded that this particular author could not construct the understanding of logarithms because of the lack of the prerequisite knowledge of exponents. Some memorization of the facts is evident, but not an understanding of those facts.

Pedagogical Considerations

The proposed task was not an ordinary assignment students would work on in their mathematics class. It was the first time they were engaged in writing essays on a mathematical topic. Even though many students found it challenging, they agreed that it was beneficial for them. The choice of this type of an assignment was motivated by the experiences I had as a student.

In the former Soviet classrooms, students were asked to *think aloud* when working on the problem. I did it from elementary level through high school (In my opinion students should be trained how to "think aloud"). I cannot recall exactly if I did it in every subject, but for sure I have done it in mathematics (algebra and geometry), and in physics classes. Even though "thinking aloud" is a very beneficial task for assessing students' understanding, it has some disadvantages. The most important one is that a teacher can ask only a few students to participate due to the lack of time and intensity of the course, while the essay form involves the entire class.

In terms of the outcomes, I have expected more than I received. On a basic level it served the purpose: the essays revealed whether students understood the concept of logarithms or not. However, beyond this it was difficult to justify to what extent the understanding occurred. Only very few works exposed some specifics about participants' understanding.

For example, in essay #3 the student tried to communicate the meaning of logarithm as a number (framework one), by defining logarithm as an exponent. Then, s/he continued with an explanation of the operational meaning of logarithms (frameworks two and three). Finally, s/he attempted to introduce the logarithmic function (framework four). Although not many attempts could be considered successful in terms of the completing the assigned task, it was apparent that the student made a sincere effort to include bits of everything s/he knew about logarithms.

The huge benefit of including this task in the teaching repertoire is that it brings to light the student's individuality, his/her personal view of the concept. It personalizes student's understanding. This is what should be at heart of qualitative research.

Task 6:

In-class discussion of the question: find the exact value of $5\log_3 9$.

In the previously discussed question students were asked to explain their understanding in writing. During the next activity, participants are invited to express their understanding orally. In terms of the whole class involvement and interaction between the students in the class, in-class discussion has a lot to offer. The following episode is from the discussion that occurred in the class during the review session prior to the unit test. It is followed by a thorough analysis.

Teacher: Can you find the exact value of $5\log_3 9$?

Ryan: The answer is 15.

Teacher: How did you get this answer, can you explain?

Ryan: 3 times 3 equal 9, so 3 times 5 equal 15.

Teacher: I see, so what you are saying is that $\log_3 9$ equals 3? Did I understand you correctly?

Several: It is wrong.

Teacher: What is wrong?

Ryan: I know, the log is 2 not 3. I just didn't write it down.

Teacher: Would anybody explain why does $\log_3 9$ equal 2?

Bob: Because it equals $\log 9$ divided by log 3, and it equals 2 (answer was given using calculator).

Teacher: Why does $\log_3 9$ equal $\log 9$ divided by log 3?

Bob: Because of change of base law.

Sharon: Somehow I got 1.2756.

Bob: O, you just forgot the bracket after 9.

Sharon: Why do you need that bracket?

Bob: If you miss it, then you are finding a logarithm of 9 over log3, not a quotient of two logs.

Sharon: I see it works now. Thanks.

Ryan: But it is not fair, you've used a calculator?

Teacher: It is a legitimate solution, but did anybody get the same answer without a calculator?

Becky: I did. I can show it.

The teacher looked at Becky's work, and asked her to explain it to the class.

Becky: First, I took 5 under the log, so it became $\log_3 9^5$. Then, I knew I have to find an exponent of 3 that equals 9 to the power of 5. Basically, I solved the equation $3^x = 9^5$. Converting 9 to 3 squared, I got $x = 10$.

Analysis

The dialogue between Bob and Sharon could be interpreted according to framework one - logarithm as a number and framework three - systematic operational meaning of logarithms. While Bob applied the change of base law without hesitation and presented one logarithm as a quotient of the two common logarithms, he still exhibits only instrumental understanding. He failed to apply the definition of the logarithm directly. Sharon obviously didn't remember the change of base law as she computed not the quotient of the logs, but log of the quotient with her calculator. This was pointed out by Bob in his reply. The questions she asked and her behavior are the evidence that she doesn't possess even an instrumental understanding of logarithms.

It could be noticed that Ryan mistakenly connected logarithm with a base, not with an exponent. In his first choice, he picked 3, because in his view, 9 is the product of identical numbers. In the second attempt, he realized that 9 is 3^2 and that he incorrectly picked the base instead of the exponent in the first case. He was not confident in his understanding of the definition of the logarithm.

Becky's solution shows her understanding of an operational meaning of logarithms, and a good prerequisite knowledge of exponents. Her solution was very explicit and satisfying to her classmates. However, the immediate "trivial" solution is missed. Becky failed to understand the definition of logarithm as an exponent, otherwise she would have found the answer without more ado.

Pedagogical Considerations

I chose this piece for the discussion, because the teacher's input to the students' explanation was very limited. The problem was solved by the students, and explained by the students without the teacher's supervision. Nevertheless, it revealed much more information about students' understanding of logarithms from the qualitative prospective.

The presented discussion illustrated how students can profit from explaining solutions to the whole class. For example, Ryan's first response would remain unchanged, if he chose not to participate in the discussion. Sharon benefited from listening to Bob's explanation, and making a correction in her own solution. Bob and Becky were challenged with explanation of their methods.

I think many will agree that the need to communicate mathematical ideas promotes meaningful learning and understanding of the particular concept as the major goal for students and for teachers.

CHAPTER VI: DISCUSSION AND PEDAGOGICAL IMPLICATIONS

This chapter is devoted to the discussion of the students' understanding of logarithms as it relates to the present study. Then, the significance and limitations of the conducted study are highlighted. The chapter includes some pedagogical ideas, suggestions for further research and recommendations for classroom instructions. The chapter concludes with the reflections on my experiences as a researcher and an educator during this study.

Discussion of Obstacles

The overall result of the analysis of this study indicates that many participants had experienced difficulties with logarithms. In the following, I will try to summarize those difficulties and possibly identify their sources.

The notion of an epistemological obstacle was introduced by the French philosopher Bachelard in his studies of conditions guiding the evolution of scientific thinking. These are obstacles that are encountered in the historical development of knowledge in a discipline. According to Sierpinska (1994), epistemological obstacles are "barriers to changes in the frame of mind." According to Hershkovich (1989), cognitive obstacles are obstacles in the acquisition or development of individual knowledge by the learner. For the purpose of the present discussion the distinction between these different kinds of obstacles will not be made.

As the results showed, many participants struggle with the idea of accepting logarithmical quantities as numbers. It is an indication of a cognitive-epistemological obstacle in the development of understanding the concept of logarithms. Another type of cognitive-epistemological obstacle is an acknowledgment that the logarithm of the product equals the sum of the logarithms of multipliers. This order of presenting a rule comes from the exponential rule, and doesn't emphasize the additive property of logarithms. This cognitive-epistemological obstacle was reiterated as an obstacle experienced by the individual participants of the study. As beliefs and ways of thinking are influenced by prior experiences, the most common cognitive-epistemological obstacle experienced by the participants is based on experiences with whole numbers, integers, etc. It causes a difficulty when the experience with the distributive property of whole numbers influences the understanding of the operational meaning of logarithms.

In the previous paragraph, I described how students' difficulties could be attributed to the cognitive-epistemological obstacles. However, another possible source of difficulties is in the instructional content, which has examples that illustrate how logarithms can be used in real-life situations. The problems used in our textbooks illustrate how exponential function can be used to model a real-life problem, and then logs are only used to solve an exponential equation. It gives logarithms a secondary role. Also, the standard pedagogical approach of introducing the laws of logarithms (see page 38) could be reinforcing the obstacle in the understanding of the operational meaning of logs, rather than trying to overcome it. My belief is that some problems discussed previously can be avoided, or at least minimized, by using the historical materials in developing the concept of logarithms in secondary school mathematics. My suggestions in this regard are discussed in the following section.

Significance of the Study

The present study provides, for the first time, a system of four interpretive frameworks which were used to model the students' understanding of logarithms. The results of the study present a description of students' difficulties with logarithms, and also suggest possible explanations of the sources of these difficulties. The description of the students' difficulties as they can be attributed to the cognitive-epistemological obstacles is discussed. However, the nature of the relationship between the epistemological and cognitive obstacles could be explored by more focused research. The results of this research may be valuable toward the creation of a more efficient pedagogical design.

Synopsis of the Pedagogical Considerations

The pedagogical ideas that I have unraveled from this study go far beyond the concept of logarithms. I found that some students have very weak algebra skills, while others do not understand exponents. The lack of prerequisite knowledge undermines their learning process before it begins. But these questions were not a part of this study. However, several other sources directly related to the context of this study were identified. They provoked immediate difficulties in students' understanding of logarithms. I believe these relevant sources were identified. Suggestions for dealing with them were presented.

It was always important for the teacher to be consistent with the terminology used in the mathematics class. If the word "exponent" is dominant in the teacher's vocabulary, the meaning of this word should be made clear to the students. Shifting from "exponent" to "power" creates confusion for students, as a word "power" has dual meaning in mathematics. One of them is an "exponent"; another is an "exponential expression" (such as 6^{21}). It becomes an obstacle when students change the logarithmic form into the exponential form and vise-versa. It leads students to the simple manipulation of three terms: exponent, base, and power.

There are two distinct causes of the challenges students experience in the development of their understanding of logarithms. The first cause is rooted in the concept of logarithms itself. It consists of the issues related to the curriculum and the historical development of logarithms. Regarding the material offered by the most reliable textbooks, I would suggest some reorganization. Perhaps, explaining the most common laws of logarithms in order right to left would help students to understand that logarithms are numbers (for more details see page 55).

Traditionally, the logarithm is defined as an exponent. Even though it is true, the educators have to be aware of the fact that historically, logarithms were developed completely independently from exponents. So, if Napier could discover logarithms without exponents, probably there is another way they could be explained. For those who are interested in expanding their teaching repertoire, I suggest an activity that involves a basic slide rule (see Appendix C).

The second origin of difficulties in understanding directly relates to the pedagogical tool and instruments the teacher uses to introduce logarithms. For the purpose of this study, I found it very effective to use the non-standard activities. Students have to be able to communicate their understanding to themselves and to others. While multiple-choice questions limit students' communication, open-ended tasks completed verbally or in writing are better suited for assessment of understanding. In the previous chapter I have presented detailed analysis of some activities of this kind.

Limitations of the Study

All participants of this study were students of an independent school. In this particular group, the socio-economic status is unlikely to have differed from that of a public school classroom, with the possible exception of parental involvement in school life. In the case of this particular independent school, parents are compelled to volunteer. However, I feel that an individual student's case study would have been a worthwhile asset to further insight of the obstacles encountered. Regretfully, the pressures of student time availability did not permit these case studies. Also, it would be beneficial to administer a questionnaire to assess students' prerequisite knowledge prior to beginning of the unit. A compilation of interviews and opinions gathered from my fellow colleagues in the field might also be beneficial and of interest.

Suggestions for Future Research

I was surprised and fascinated by the results uncovered in this research. New ideas arose by the end of the work I did with my students which would now be interesting to pursue in future research. One possible investigation might explore how the educational goals of teachers and students differ and how they may be similar. A comparison of teacher's and students' goals might reveal some pedagogical obstacles. Another topic to explore might be a detailed look at the obstacles and semantics of students' understanding of logarithms. The historical development of logs in the classroom is an entire topic of study itself, and it worth serious consideration in terms of discovering whether or not the old tools and techniques could be useful in helping with understanding of logs. We have to ask ourselves whether the withdrawal of the logarithmic tables and slide rules was a wise move.

It is my sincere hope that others will utilize the proposed set of interpretive frameworks which were developed for this study. I found they were extremely useful in conducting this study.

Reflections on my Personal Journey

> What are the significant products of research in mathematics educations? I propose two simple answers: 1. The most significant products are the transformations in the being of the researchers. 2. The second most significant products are stimuli to other researchers and teachers to test out conjectures for themselves in their own context. (Mason, 1998, p. 357).

Throughout the process of writing and completing this study, my own conception of logarithms was enriched and transformed. The whole experience of evaluating my teaching practice through the lens of the students' understanding of logarithms was extremely beneficial in terms of my professional growth.

> Our minds are such sensitive instruments, and as we strive to examine our own understandings, the character of those understandings will inevitably shift; this is an essence of the process of reflection and construction (Confrey, 1991, p.129.)

One thing which became clear to me is that students construct their personal understandings of the mathematical concepts they encounter. In order to be successful in dealing with students' understanding, the teacher has to perceive a problem from the student's point of view. The questions are how to merge the student's understanding with the teacher's understanding successfully, and whether the teacher's understanding of the concept of logarithms is deep enough. By doing this research I tried to answer these questions for myself. I am by nature a person who is always 'hungry' for new knowledge, and is anxious to discover something useful and helpful.

Summary

In this study I have found it necessary to perceive students activities as an evidence of their understanding of logarithms. This concept appeared to be assimilated, connected and grasped by many of the participants in various ways. It was also evident that some students were unable to understand the concept, as they lack the prerequisite knowledge required to understand logs as numbers and therefore do not develop an operational meaning of logarithms. Other students had misunderstandings which can be modified. Some considerations in this regard are proposed above; however, this requires further investigations that were not the part of this study.

One of the main accomplishments of this research was the development of the interpretive frameworks for the appreciation of Confrey's (1991) system of interpretive frameworks and modification/adaptation of this system for investigating students' understanding of logarithms. Also, the common difficulties in the students' understanding of this phenomenon were discovered. Possible sources of these difficulties were analyzed and some pedagogical suggestions to deal with these difficulties were proposed. Both the significance of the study and its limitations were reviewed. Finally, possible directions for future research were identified.

APPENDICES

Appendix A

LOGARITHMS
Quiz #1

Show your work clearly, writing dawn all the entries in your calculator if used.

1. Write in the exponential form: $\log_2 64 = 6$;

2. Present in the logarithmic form: $3^{-3} = \frac{1}{27}$;

3. Simplify:

   a. $\log_2 40 - \log_2 5$;

   b. $\log_3 54 - \log_3 8 + \log_3 4$;

4. State the values of x for which the following identity is true:
   $\log_5(x+1) + \log_5(x-4) = \log_5(x^2 - 3x - 4)$;

5. Solve: $4^{2x}(8^{x+3}) = 32^{4-x}$;

6. Solve: $4 \times 3^{3x} = 9^{x+1}$;

7. Solve: $\log_{12}(3-x) + \log_{12}(2-x) = 1$;

BONUS QUESTION:
Solve for a: $(2\log b)^2 + 8(\log a)(\log b) = 0$

Appendix B

LOGARITHMS UNIT TEST

Name ______________________ Date ___________

PART I: MULTIPLE CHOICE.

1. Write $y = 3^x$ in logarithmic form.

a) $x = \log_3 y$ b) $y = \log_x 3$ c) $y = \log_3 x$

d) $x = \log_y 3$ e) $3 = \log_x y$

2. Which of the following expressions is equivalent to $y = \log_a x$?

a) $y = a^x$ b) $x = a^y$ c) $x = y^a$ d) $a = x^y$ e) $a = y^x$

3. The expression $\log_{\frac{1}{b}} \frac{1}{x}$ is equivalent to

a) $-\log_{\frac{1}{x}} \frac{1}{b}$ b) $-\log_b x$ c) $\log_b x$

d) $-\log_x b$ e) $\log_x b$

4. What is the domain of the function $f(x) = \log_8(x+2)$?

a) all real numbers b) real numbers > -2

c) real numbers > 0 d) real numbers > 2

e) integers > 8

5. Give the domain of the relation $\log_x(y-2) = \log_x(4-x)$.

a) $x > 0,\ x \neq 1$ b) $x > 4$

c) $0 < x < 4$ d) $0 < x < 4,\ x \neq 1$

e) $-4 < x < 4$

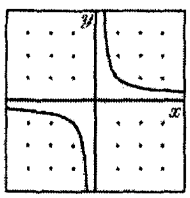
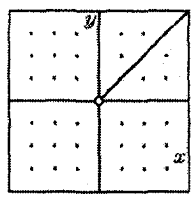
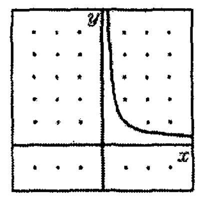
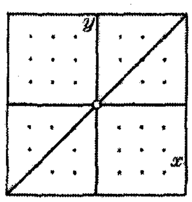
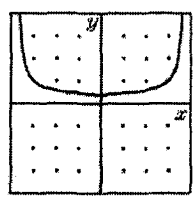
6. What is the equation of the asymptote of $f(x) = \log_{\frac{1}{3}}(x-1)$?

a) $x = 1$ b) $x = 0$ c) $x = \frac{1}{3}$ d) $y = 0$ e) $y = 1$

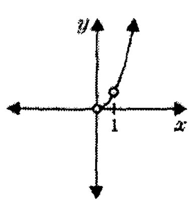
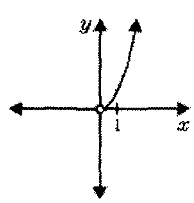
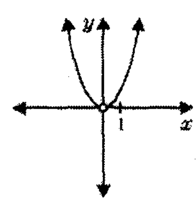
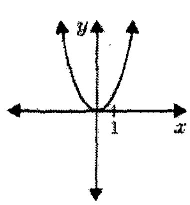
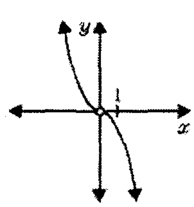
7. Which of the following is an asymptote for the graph of $y = 2^{x-1} + 3$?

a) $x = 1$ b) $x = 0$ c) $y = 3$ d) $y = 0$ e) $y = 1$

8. Which graph best represents the equation $\log y = -\log x$?

a) 
b) 
c) 
d) 
e) 

9. Which of the following is the graph of $\log y = 2\log x$?

a) 
b) 
c) 
d) 
e) 

10. Evaluate: $\log_5 5^{-15}$

a) -15 b) -5 c) $\frac{1}{15}$ d) $\frac{1}{5}$ e) 5

11. Find the exact value of $3\log_2 4$.

a) $\frac{3}{2}$ b) 6 c) 7 d) 4 e) 5

12. Write $\log_3 17$ using base ten logarithms.

a) $\log 8.5$ b) $\log 37$ c) $\log 51$ d) $\dfrac{\log 17}{\log 3}$ e) $\dfrac{\log 3}{\log 17}$

13. Simplify: $(\log_x y)(\log_y x)$

a) 1 b) 0 c) $\log_x 2y$

d) $xy^{(x+y)}$ e) $\log_{xy}(x+y)$

14. Express $4\log_5 p - \log_5 q$ in the form $\log_5$ ___.

a) $\log_5 \dfrac{p^4}{q}$ b) $\log_5 \dfrac{4p}{q}$ c) $\log_5 4pq$

d) $\log_5 (pq)^4$ e) $\log_5 (p+q)$

15. Solve: $\log_2(x-3) + \log_2(x+1) = 5$

a) $-7,\ -5$ b) $-7,\ 5$ c) $-5,\ 7$ d) 7 e) 5

16. What is the result when $\log \dfrac{A\sqrt{B}}{C}$ is expanded?

a) $\log A - \frac{1}{2}\log B + \log C$ b) $\log A - \frac{1}{2}\log B - \log C$

c) $\log A + \frac{1}{2}\log B - \log C$ d) $\log A + 2\log B - \log C$

e) $\log A - 2\log B - \log C$

17. Given that $\log_2 3 = x$, $\log_2 5 = y$, and $\log_2 7 = z$ express $\log_2 \frac{15}{7}$ in terms of x, y, and z.

a) $1 + x - z$ b) $x + y - z$ c) $2^x \cdot 2^y \div 2^z$

d) $\dfrac{xy}{z}$ e) $xy - z$

18. For each increase of 1 unit in magnitude of an earthquake, there is a ten-fold increase in the amount of energy released. Use the Richter scale given to calculate how many times more energy was released by the Alaska earthquake of 1964 than by the Columbia earthquake of 1983.

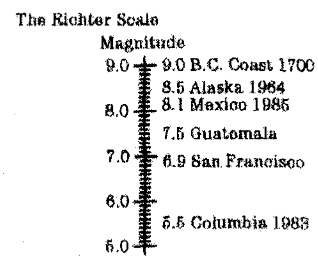
a) 3 times more

b) 30 times more

c) 100 times more

d) 1000 times more

e) 10 000 times more



19. A population of bacteria can multiply fivefold in 48 h. If there are 1000 bacteria now, how many will there be in 144 h?

a) 5000 b) 25 000 c) 75 000 d) 125 000 e) 144 000

20. Solve for x: $(\log x)^2 = a^2$

a) $\pm a$ b) a c) $\frac{1}{10}a$

d) 10^a e) 10^a or 10^{-a}

PART II: WRITTEN RESPONSE.

1. Given $y = \log_5(x+6) + 2$
   a. Construct a graph on the grid provided.
   b. Find the equations of asymptote/s, if any.
   c. Identify x-intercepts (algebraically).
   d. Identify y-intercepts (algebraically).
   e. Label the graph.

2. You have \$ 10,000 now and you wish it to grow to \$15,000 in two years. What should be the interest rate compounded quarterly for this to happen?

3. Happy Town has 5 million people and is increasing yearly by 3%. Lonely Town has 7 million people and is decreasing yearly by 8%.

   a. When will the two towns have the same population? Solve algebraically.

   b. Compare how big the population of Happy Town will be after 10 years if it is growing yearly versus continuously.

4. The half-life of Cobalt-57 is 270 days. A hospital buys 30 mg of this substance.

   a. Write a function describing how much of the substance will remain after *n* days.

   b. How much Cobalt will remain after 2 years?

   c. Rewrite the equation to reflect the exponential decay daily. Use 4 decimal places.

5. What is a bigger number 25^{625} or 26^{620}? Explain.

Appendix C

Activity: Multiplying on the Slide Rule

1. Fold and cut notebook paper in halves. Fold each half vertically.

2. Number each half starting at the lowest line close to the edge with the following set of numbers {1, 2, 4, 8, 16, 32, 64, 128, etc.} Look at the diagram provided below.

C 2048	2048 **D**
1024	1024
512	512
256	256
128	128
64	64
32	32
16	16
8	8
4	4
2	2
1	1

3. Label these strips "C" and "D."

4. Circle "1" on your C-scale, and slide it opposite "4" on the D-scale. Notice that numbers on C, times 4 , now give corresponding answers on D.

5. Find the following products by using you slide rule:

8x64 = ; 4x16 = ;

16x8 = ; 4x128 = ;

6. Explain the steps you take to multiply on this slide rule.

7. Now, express each number as a power with a base 2.

C 2048	2048 D
1024	1024
512	512
256	256
128	128
64	64
32	32
16	16
8	$8=2^3$
$2^2=4$	$4=2^2$
$2^1=2$	$2=2^1$
$2^0=1$	$1=2^0$

8. What is happening to the exponents (LOGS) when you are multiplying the numbers?

9. Try to explain in steps how to divide by using this slide rule. Make up some examples.

Suggestion: try to repeat this activity in base 10.

REFERENCES

Alexander R., Kelly B. (1999). *Mathematics 12, Western Canadian Edition.* Addison Wesley Longman Ltd.: Toronto.

Boyer, C. (1968) *A History of Mathematics*, New York, Wiley and Sons.

British Columbia Ministry of Education (2000). *Principles of Mathematics 12: Integrated Resource Package 2000.* Victoria, BC: Government Printing Office.

Cajori, F. (1919). *A History of Mathematics.* MacMillan Press. Campbell. S. R. & Zazkis, R. (1994). The distributive flaws: Latent conceptual gaps in preservice teachers' understanding of the property relating multiplication to addition. Proceedings of the Conference for *Psychology of Mathematics Education - North American Chapter.* D. Kirshener (Ed), Baton Rouge, Louisiana.

Confrey, J. (1991). The concept of exponential functions: A student's perspective. *Epistemological foundations of mathematical experience*, In L. Steffe (Ed.), 124-159. New York: Springer-Verlag.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *Multiplicative reasoning in the learning of mathematics* (pp. 291-330). Albany: State University of New York Press.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-64.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*, 66-86. Dennis, D. (2000). The Role of Historical Studies in Mathematics and Sciences; Educational Research. In D. Lesh & a. Kelly (Eds.), *Research Design in Mathematics and Science Education.* Mahwah, NJ: Lawrence Erlbaum.

Harel, G. & Confrey, J. (Eds.). (1994). *The Development of Multiplicative Reasoning in the Learning of Mathematics.* Albany: State University of New York Press.

Hershkovics, N. (1989). Cognitive obstacles encountered in the Learning of Algebra. In S. Wagneer and C. Kieran (Eds.). *Research issues in the learning and teaching of Algebra.* (pp.60-86). Lawrence Erlbaum Associates, Reston, VA.

Hiebert, J., & T. Carpenter (1992). Learning and teaching with understanding. In D. A.Jones, P. (1957). The History of Mathematics as a Teaching Tool. *Mathematics Teacher, 50,* 59-64.

Grouws, D. (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Mason, J. (1998). Researching for the inside in mathematics education. In A. Sierpinska and J, Kilpatric (Eds.), Mathematics education as a research domaon: A search for identity. Kluwer Academic Publishers, Dortrecht, pp. 357-377.

MacDonald, William, trans. (1966) Mirifici Logarithmorum Canonis Constructio by John Napier (1615). *The Construction of Logarithms.* London: Dawson of Pall Mall.

Marson, R. (2003). *Far Out Math 43*. Canby, OR: TOPS Learning Systems.

National Council of Teachers of Mathematics. (1969). *Historical Topics for the Mathematics Classroom.* Washington, DC: Author.

Piaget, J. & Garcia, R. (1989). *Psychogenesis and the History of Science*. Ney York, Columbia University Press.

Reimer, L. & Reimer, W. (1995). Connecting Mathematics with Its History: A Powerful, Practical Linkage. In P.A. House Ed., Connecting Mathematics Across the Curriculum. 1995. Reston, VA: NTCM

Shirali, S. (2002). *Mathematical Marvels : A Primer on Logarithms*. Hyderabad, India: Universities Press.

Sierpinska, A. (1990). Some remarks on understanding in mathematics. *For the Learning of Mathematics*, *10*(3), 24-36, 41.

Sierpinska, A. (1994). *Understanding in mathematics*. Washington, DC: Falmer Press.

Skarga, B. (1989). *Granice historycznosci*, Warsaw, Poland, Panstwowy Instytut Wydawniczy.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20-26.

Skemp, R. (1987). *Psychology of learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum.

Smith, E., & Confrey, J. (1994). Multiplicative structures and the development of logarithms: What was lost by the invention of function? In G. Harel & J. Confrey (Eds.), *Multiplicative reasoning in the learning of mathematics* (pp. 331 - 360). Albany: State University of New York Press.

Steffe, L. (1988). Number Concepts and Operations in the Middle Grades. In J. Hiebert & M. Behr (Eds.), *Children's Construction of Number Sequences and Multiplying Schemes*. (pp. 119 - 140). Hillsdale, NJ: Lawrence Erlbaum.

Toumasis, C. (1993). Teaching logarithms via their history. *School Science and Mathematics, 93* (8), 428-434.

Wyman, D. (2000). *MATHPOWER 12, Western Edition*. McGraw-Hill Ryerson: Toronto.

Zazkis, R. & Campbell, S. R. (1996). Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education, 27*(5), 540-63.

Zazkis, R. & Liljedahl, P. (2002). Arithmetic sequence as a bridge among conceptual fields. *Canadian Journal of Science, Mathematics and Technology Education, 2*(1). 91-118.