# RANKING NON-REGULAR DESIGNS

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# Abstract

Fractional factorial designs are commonly used in industrial and experiments to identify factors affecting a response or process. The focus of this thesis is on two-level orthogonal designs, however, the methods we consider can be generalized to non-orthogonal designs. Orthogonal designs can be classified into two broad categories: *regular* designs, which have a simple aliasing structure, in that any two effects are orthogonal or fully aliased; and *non-regular* designs which have a complex aliasing structure, in that there exist effects that are neither orthogonal nor fully aliased. This thesis focuses on the study of non-regular designs.

In many industrial settings robust parameter designs are performed as a strategy for variance reduction. In these situations the experimenter is mainly interested in the estimation of control-by-noise interactions. For non-regular fractional factorial designs, the "goodness" of the design can be judged using the generalized aberration criteria. We extend the definitions of generalized aberration to emphasize the control-by-noise interactions. Theoretical results are used to show how one can construct the set of all non-isomorphic multi-factor designs from the existing set of all non-isomorphic designs. We then use the set of all non-isomorphic multi-factor designs to construct a catalog of generalized minimum aberration robust parameter designs.

Next, we focus attention on factorial designs and introduce the projection estimation capacity sequence and use this new criterion to select good non-regular designs. Two theoretical results are presented that will be practically useful when searching for good designs. Based on these results, a simple search procedure is implemented to find such designs. Catalogues of designs are constructed for 20, 24 and 28 runs.

Finally, we discuss topics for future research. Firstly, we show how projection estimation capacity can be modified and used to rank robust parameter designs. Secondly, we show how one could use the projection estimation capacity to select follow-up runs in a factorial experiment. The selection of additional runs is briefly discussed and it is shown how one can select follow-up runs to ensure the overall design is orthogonal.

# Dedication

To my parents and Carina Neumann.

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# Chapter 1

# Introduction

Designed experiments are widely used in many forms of scientific investigation as a systematic way to investigate the effect of a large number of factors on a process. An experimental design consists of a series of trials where the level settings of some of the factors are changed and the effect of this change is measured on the outcome of the process. The series of experimental trials aids the investigator in modelling the effect of the input variables on the output of the process.

Factorial experiments date back to the work of Fisher (1935) and Yates (1937), where they were used in agricultural and biological investigations. These investigations tended to take a long time to complete, and the primary interest was in the comparison of factors which increased crop yield. The time concerns of performing these experiments led to the development and use of fractional factorial designs (FFD's). The latter half of the century has seen an increased interest in factorial experiments, and in particular FFD's, for use in industrial investigations. (see Box, Hunter and Hunter, 1978). Industrial experiments are often costly to perform and the primary interest of the investigation is for process optimization.

In recent years, emphasis has been put on the use of designs for variation reduction, (see Taguchi, 1986, Welch, Yu, Kang and Sacks, 1990, and Shoemaker, Tsui and Wu, 1991). Taguchi (1986) advocated the use of Robust Parameter Designs (RPD's) as a strategy for making a process less sensitive to variation which is hard to control. The factors of interest in this case are divided into two broad categories: control factors, whose values can be fixed during the experiment and under normal operating conditions and noise factors, which are hard to control during normal operating conditions, but can be held fixed during the experiment. The goal of the experiment is to change the control factor settings to make the process robust to differing levels of the noise factors. Typically this can be done by observing the control-by-noise interactions.

In many investigations the experimenter is interested in testing the effect of a large number of factors on the process. Of particular concern in designing many investigations is the selection of a fraction of the runs from the full factorial design. When the cost of running the experiment is of particular concern it is advantageous to select the smallest subset of runs that still allows for the estimation of the effects of interest. There are often many design choices that can be used to accomplish these goals.

The most common and widely used design choice is a *regular* FFD (Box and Hunter, 1961 and Fries and Hunter, 1981). Regular FFD's are constructed by assigning additional factors to the higher-order interactions of a full factorial design. In this manner the experimenter gains the ability to consider additional factors by sacrificing the ability to estimate higher-order interactions. That is, the estimate of the main effect is made to be indistinguishable from the estimate of the interaction column to which it was assigned, in this case we say that the two effects are *fully aliased.* For a particular run size and a fixed number of factors there are a large number of possible FFD's that can be used. In this case it is important to select a "good" design among the set of possible designs. Box and Hunter (1961) introduced the concept of resolution to rank the many possible FFD's. Fries and Hunter (1980) introduced minimum aberration (MA) as a refinement of resolution.

In recent years, *non-regular* FFD's have received considerable attention as an alternative strategy to regular FFD's. Non-regular designs can be characterized as having main effects that are partially aliased with two factor interactions (2fi's). The most common and widely used non-regular designs are the Plackett-Burman designs (Plackett and Burman, 1946), which are usually advocated as main effects plans, i.e. assume all interactions are zero. However, in many experimental situations the validity of this assumption is questionable. When some interactions are assumed active, main effects will be aliased with a large number of two-factor interactions. This makes analysis more difficult as there are many models to choose from and very few of these models will be fully distinguishable. However, when only a few of the two-factor interactions are active, the iterative method proposed by Hamada and Wu (1992) showed that two-factor interactions could be identified and estimated with reasonable precision. Due to the success of the method employed by Hamada and Wu (1992), and subsequent analysis strategies for non-regular designs (see Box and Meyer, 1993 and Chipman, Hamada and Wu, 1997) non-regular designs have become more widely used in industrial settings. Non-regular orthogonal designs can be found for any run size that is a multiple of four, as opposed to regular designs which have run sizes which are a power of two. Thus, non-regular designs are sometimes advocated since they provide greater flexility in the choice of run-size or design and they can entertain more models. Recently, generalized resolution and generalized aberration (Tang and Deng, 1999) were introduced to extend the notions of resolution and MA to non-regular designs. These new criteria were used to construct sets of good non-regular designs.

When selecting a good design, the experimenter will usually search for the design among the class of all *non-isomorphic* designs. Two designs are said to be *isomorphic* if the second design can be obtained from the first design by relabelling the rows, relabelling the columns, and exchanging the levels of a column. Thus, the set of non-isomorphic designs represents the entire class of designs and can also be viewed as the smallest such set of all designs.

The objective of this thesis is to study two loosely related topics involving the selection of non-regular designs. First, we consider the selection of non-regular RPD's with generalized MA. As noted previously, in RPD's the experimenter places greater importance on the estimation of control-by-noise interactions. Secondly, we introduce the projection estimation capacity of a design and use this new criterion to select good designs for screening experiments. The term projection simply indicates that a subset of factors from the original design is considered. In essence, the projection estimation capacity sequentially maximizes the number of models that allow for the estimation of $k$ main effects and all the associated 2fi's between them. Thus, we are interested in studying 2fi's associated with a subset of factors from the original design. In Chapter

2, regular FFD's, regular RPD's and non-regular FFD's are introduced in detail. In Chapter 3, we introduce the non-regular RPD and use the new methodology to create a catalog of non-regular RPD's with MA. The projection estimation capacity is introduced in Chapter 4. In Chapter 5 we discuss future work, and extend the definition of projection estimation capacity to study RPD's.

# Chapter 2

# Factorial Designs

Cats is Dogs, and Rabbits is Dogs, and so's Parrots; but this 'ere Tortoise is an Insect, so there ain't no charge for it!

-Charles Keene, 1869

Two-level full factorial designs and two-level FFD's are commonly used in industrial (see Box, Hunter and Hunter, 1978) and agricultural (see Kempthorne, 1952) experiments as a systematic method to sift through a large number of factors that may affect the process. Selection of an appropriate two-level design usually balances the need for as much information about the factors and the experimenters' desire to perform as few runs as possible. Selection of a regular $2^{m-p}$ FFD is typically based on Resolution (see Box and Hunter, 1961) and Minimum Aberration (see Fries and Hunter, 1981). The next section introduces the full factorial design and motivates the need for a FFD. Next the FFD and ranking criterion for these designs are reviewed.

In recent years there has been increased attention on the use of non-regular designs in industrial experiments. Non-regular designs can be found for any run-size which is a multiple of four. In the last section we introduce the non-regular designs and the concept of generalized resolution and generalized aberration (Tang and Deng, 1999).

## 2.1 Two-Level Full Factorial Designs

In many industrial experiments it is of interest to test the effect of a large number of factors on a process. It is common in screening for a small number of important factors or effects to consider each of these factors at only two-levels. In general, if we consider $m$ factors then an experimental plan that considers all possible combinations of the $m$ factors would require $2^m$ runs. When the $2^m$ runs are performed in a completely random order the design is referred to as a full factorial, or just a factorial design. These designs are performed to determine which factors have a significant impact on the process. The significant factors can then be used to achieve an optimal setting for the process, or can be considered in more detail in a more focused follow-up experiment. In more complicated situations it may be desirable to set the level of a factor to minimize the effect of another factor. Discussion of analysis and estimation of effects is illustrated through the use of an example.

**Example 2.1** Box, Hunter and Hunter, (1978, pg. 307).
Consider the yield from a particular chemical reaction where it is believed that temperature (T), concentration (C) and catalyst (K) may affect the response, chemical yield, which is denoted by the vector $\mathbf{y}$.

An experiment was designed to test each of these factors at two levels, high $(+1)$ and low $(-1)$. If all combinations of the three factors are to be performed, the experiment would require $2^3 = 8$ runs. The set of level combinations for this experiment are shown in the *design matrix* X.

$$
\begin{array}{cc}
 & \begin{array}{ccc} C & K & T \end{array} \\
X = & \begin{pmatrix}
-1 & -1 & -1 \\
+1 & -1 & -1 \\
-1 & +1 & -1 \\
+1 & +1 & -1 \\
-1 & -1 & +1 \\
+1 & -1 & +1 \\
-1 & +1 & +1 \\
+1 & +1 & +1
\end{pmatrix}.
\end{array}
$$

The columns of the design matrix $X$ indicate the levels of the factors that are set for each run (row) of the experiment, and can be used to estimate the effect of these factors on the process. In some contexts the level settings in $X$ are replaced by 1 (high) and 0 (low) or the factor values that are to be performed. Note that the choice of symbols to represent the two levels of the factors is unimportant, provided that a consistent notation is used throughout the experiment. The $\pm 1$ coding is used for two reasons. Firstly, the $\pm 1$ coding can be used to easily obtain effect estimates from the experiment. Secondly one can obtain contrast coefficients for estimating the interaction between two variables by simply multiplying the level settings for each run of the design. For example, the main effect of temperature on the experiment can be calculated as

$$ME(T) = \frac{1}{2^{m-1}} X_T' \mathbf{y},$$

where $X_T$ is the column of the design matrix corresponding to the factor $T$.

If we wish to estimate the mean, the main effects and all possible interaction effects between the three factors, the *full factorial design matrix* would require $2^3 - 1 = 7$ columns for the effects and one additional column for the mean. If the mean is denoted as (1), the main effects are denoted as $C$, $K$, and $T$ and the interactions as $CK$, $CT$, $KT$ and $CKT$, the $2^3$ full factorial design matrix is

$$X = \begin{array}{c} \begin{array}{cccccccc} (1) & C & K & T & CK & CT & KT & CKT \end{array} \\ \left( \begin{array}{cccccccc} +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 \\ +1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 \\ +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \end{array} \right) \end{array}. \quad (2.1)$$

In general, for a $2^m$ full factorial design, we consider the normal linear model,

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad (2.2)$$

where $\mathbf{y}$ is a $2^m \times 1$ response vector, $X$ is the full factorial design matrix and $\epsilon_i \sim \mathrm{N}(0, \sigma^2)$, iid. All of the columns of $X$ are orthogonal, i.e. $X'X = 2^m I$, where $I$ is the $2^m \times 2^m$ identity matrix. From (2.2), an estimate of $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{n-1})'$ is

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (X'X)^{-1}X'\mathbf{y} \\ &= \frac{1}{2^m}X'\mathbf{y} \\ &= (\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_{n-1})', \end{aligned}$$

where $\hat{\beta}_0$ is an estimate of the mean and $\hat{\beta}_1, \ldots, \hat{\beta}_{n-1}$, is the usual ordinary least squares estimates of $(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_{n-1})$ and $n = 2^m$. Note that the factorial effects are calculated as twice the value of the ordinary least squares estimate. Using the value of twice the ordinary least squares estimate leads to an interpretation of a factorial effect being the average response at the high level minus the average response at the low level. For example, $ME(C) = \bar{y}(C+) - \bar{y}(C-)$, where $ME(C)$ denotes the main effect of factor $C$, $\bar{y}(C+)$ is the average of the response at the high level of $C$ and $\bar{y}(C-)$ is the average of the responses at the low level of $C$.

## 2.2 Regular Two-level Fractional Factorial Designs

In many experiments it is not possible, due to operational restrictions or cost, to run a full factorial experiment. In these situations it may be desirable to perform only a fraction of the runs from a full factorial experiment. When a fraction of runs is to be performed, it is usually desirable to select the subset of runs such that the resulting FFD is still orthogonal. The most common situation is to assign an additional $p$ factors to certain interaction columns of a $2^{m-p}$ full factorial design matrix. In this manner the first $m-p$ factors are assigned to the independent columns of the $2^{m-p}$ full factorial design matrix, and the additional $p$ factors are assigned to certain interactions formed by the first $m-p$ factors.

In order to illustrate the use of FFD's, we consider an example. Suppose a $2^{6-3}$ FFD is to be performed. Denoting the 6 factors by $A, B, C, D, E$ and $F$, we would first assign factors $A, B$ and $C$ to the columns of the $2^3$ full factorial design matrix in (2.1). The remaining three factors are assigned to selected interaction columns of the full factorial design matrix. For example, one possible assignment would be $D = AB$, $E = AC$ and $F = BC$. This implies that the level settings for $D, E$ and $F$ are determined by the interactions $AB, AC$ and $BC$, respectively. Letting $I$ denote a column of +1's, we have three relations defined by

$$I = ABD = ACE = BCF.$$

That is, when we multiply $X_A$ by $X_B$, we get $X_D$, the multiplication of $X_A$, $X_B$ and $X_D$ will yield a column of +1's, and similarly for the other two relations. The three relations above are referred to as *fractional generators*, or simply *generators* of the design. It is important to note that a $2^{m-p}$ FFD is completely determined by the generators.

The generators $D = AB$ and $E = AC$ together imply a third relation

$$DE = AABC,$$

but multiplying $X_A$ by $X_A$ will result in a column of +1's and the relation can be simplified to $DE = BC$, or equivalently $I = BCDE$. The multiplication of any of the generators will imply another relation in the group. The entire set of relations is referred to as the *Defining contrast subgroup* (DCS), and an element of the group is referred to as a *word*. In general, the DCS for a $2^{m-p}$ FFD will have $2^p$ words, including the mean $I$. Therefore, the DCS for the above example would be

$$I = ABD = ACE = BCF = DEF = BCDE = ACDF = ABEF.$$

A $2^{6-3}$ FFD can be used to explore the effect of six factors on the process without performing the 64 runs of the $2^6$ full factorial design. However, the reduction in run-size does not come without a cost. For example, the level settings of $D$ were determined by the $AB$ interaction from the full $2^3$ design. As a consequence the effect estimate of factor $D$ will be indistinguishable from the effect estimate of the $AB$ interaction. In this case we say that $D$ is *aliased* with $AB$. In addition, if we multiply every element in the DCS by factor $D$ we find that

$$D = AB = EF = ACF = BCE = ACDE = BCDF = ABDEF.$$

In order to estimate the effect of $D$, one must assume that the $AB$, $EF$, $BCE$, $ACF$, $ACDE$, $BCDF$ and $ABDEF$ interactions are all negligible. In general, we can compute a similar list of aliased effects for every factor in the experiment and we will refer to the list as an *alias string.*

Based on the above discussion, it is of interest to note that words of length three cause main effects to be aliased with 2fi's. Four letter words cause main effects to be aliased with three factor interactions, and 2fi's to be aliased with other 2fi's. Similarly, words of length five cause main effects to be aliased with four factor interactions and 2fi's to be aliased with three factor interactions. However, in many experimental settings interactions involving three or more factors are assumed to be negligible. For example, if there are words of length five or higher in the DCS for a FFD, then all main effects and 2fi's are aliased with three factor or higher-order interactions, which can be assumed negligible. In this sense, one can perform only a fraction of runs and still obtain all of the desired information about a process.

In many experimental situations, there are three empirical rules that are often used when choosing and subsequently analyzing FFD's.

1. **Effect Sparsity** (Box and Meyer, 1986). Only a few of the factorial effects are active.

2. **Effect Hierarchy** (Box, Hunter and Hunter, 1978, pg. 374). Main effects are more likely to be important than 2fi's and 2fi's are more likely to be important than three factor interactions and so forth. Effects of the same magnitude are equally likely to be significant.

3. **Effect Heredity** (Hamada and Wu, 1992). An interaction is more likely to be significant if at least one of its parents is significant.

The above rules help provide a justification for running a FFD. However, they can also be used to aid the experimenter in deciding how to assign the additional factors to the interaction columns of the full factorial design.

Box and Hunter (1961) introduced the concept of *resolution*, the length of the shortest word in the DCS, in order to distinguish between two competing designs. Based on the above discussion, it is clear that designs with higher resolution will be preferred to designs with smaller resolution, due to effect hierarchy. However, designs with the same resolution may not have the same properties. Fries and Hunter (1980) introduce the concept of *aberration* to distinguish between designs with the same resolution.

Let $A_i$ be the number of words of length $i$ in the DCS of a FFD and define its *Word Length Pattern* (WLP) to be

$$W = (A_1, A_2, \ldots, A_m).$$

The resolution is defined to be the smallest $i$ such that $A_i \neq 0$. The definition of minimum aberration (MA) (Fries and Hunter, 1980) can be written as:

**Definition 2.1** *(Minimum Aberration FFD's.)*
*For any two $2^{m-p}$ FFD's, $\mathcal{D}_1$ and $\mathcal{D}_2$, we say that $\mathcal{D}_1$ has less aberration than $\mathcal{D}_2$ if there exists an $r$ such that, $A_i(\mathcal{D}_1) = A_i(\mathcal{D}_2)$ for all $i \leq r-1$ and $A_r(\mathcal{D}_1) < A_r(\mathcal{D}_2)$. If no other design has less aberration than $\mathcal{D}_1$, then $\mathcal{D}_1$ is the* minimum aberration *FFD.*

The MA criterion provides a good general rule for selecting a FFD. As such, the criterion can be used to help decide how to assign additional factors to the interaction columns in a full factorial design. Typically, when all factors are treated equally and resources do not permit running the full factorial design, an experimenter will select an MA FFD.

The MA criterion has also been suitably adapted for more complex experimental situations. For example Bingham and Sitter (2003) and Zhu (2000) adapted the MA criterion for robust parameter designs, and Deng and Tang (1999) and Tang and Deng (1999) extended the MA notion to non-regular designs. Both of these situations will be discussed in the following sections.

### 2.2.1 Robust Parameter Designs

RPD is a strategy for running planned experiments with the goal of identifying the level settings of the *control factors* so that the system is robust to random variation in the *noise factor* settings. Control factors are variables whose values can be held fixed during the experiment and under normal operating conditions. They are often described as the factors of interest within the process. Noise factors are hard to control during normal operating conditions, but can be held fixed during the experiment. These factors are often external to the process, such as temperature, heat, light and humidity, or even customer usage factors. Generally, during the experiment the noise factors are varied systematically to represent the variation that would be expected during normal operating conditions.

Typically, a two-level FFD is performed and models for both the mean and variance are constructed. A standard method for identifying which factors impact the mean and variance is the *response model* approach where significant control-by-noise interactions are used to choose settings of control factors that make the process robust to variation in the noise factors (see, Welch, Yu, Kang and Sacks, 1990, and Shoemaker, Tsui and Wu, 1991). As a result, estimation of control-by-noise interactions is viewed as more important than, say, estimation of noise-by-noise interactions. In light of the asymmetric ranking of importance of effects of the same order, the selection of a good experiment plan is more complicated than standard settings where all effects of the same order are treated equally. So, for example, selecting a MA FFD (Fries and Hunter, 1980) may not be optimal.

A two-level FF RPD is denoted as $2^{(m_1+m_2)-(p_1+p_2)}$ , where $m_1$ is the number of control factors and $m_2$ is the number of noise factors, $p_1$ is the degree of fractionation for the control factors and $p_2$ is the degree of fractionation for the noise factors. It is appropriate to note that a fractional factorial (FF) RPD is not necessarily of the form $2^{m_1-p_1} \otimes 2^{m_2-p_2}$ where $\otimes$ is the Kronecker product and $2^{m_1-p_1}$ is a FFD for the control factors and $2^{m_2-p_2}$ is a FFD for the noise factors.

The design matrix for a FF RPD is identical to that of a FFD. The difference lies in the existence of two types of factors in the RPD. We can borrow results from FFD's and adapt them to FF RPD's.

In order to account for the asymmetric ranking of effects, Bingham and Sitter (2003) propose a new MA criterion for $2^{(m_1+m_2)-(p_1+p_2)}$ FF RPD's to reflect the increased interest in the control-by-noise interactions via changing the definition of word lengths in the DCS as presented in Table 2.1. Zhu (2000) proposed a different

Table 2.1: Word Lengths for RPD's Proposed by Bingham and Sitter (2003)

| Word Length | Words |
|---|---|
| 1 | $C, N$ |
| 1.5 | $CN$ |
| 2.0 | $CC, NN$ |
| 2.5 | $CCN, CNN$ |
| 3.0 | $CCC, CCNN$ |
| 3.5 | $CCCN, CNNN$ |
| 4.0 | $CCCC, NNN, CCCNN, CCNNN$ |
| 4.5 | $CCCCN, CNNNN$ |
| 5.0 | $CCCCC, NNNN, CCCNNN, CCNNNN, CCCCNN$ |
| 5.5 | $CCCCCN, CNNNNN$ |
| 6.0 | $CCCCCC, NNNNN, CCCNNNN, CCCCNNN,$ $CCCCCNN, CCNNNNN$ |

alternate word length definition for $2^{(m_1+m_2)-(p_1+p_2)}$ FF RPD's,

$$W(j,k) = \begin{cases} 1 & \text{If } \mathrm{Max}(j,k) = 1 \\ j & \text{If } j > k \text{ and } j > 1 \\ k + 1/2 & \text{If } j \leq k \text{ and } k \geq 2 \end{cases}, \tag{2.3}$$

where $j$ is the number of control factors and $k$ is the number of noise factors. Let $\mathcal{D}$ be a $2^{(m_1+m_2)-(p_1+p_2)}$ FF RPD and define the RPD WLP as

$$W_{RPD} = (B_{1.0}, B_{1.5}, \ldots, B_m),$$

where $B_q$ is the number of words of length $q$ according to the definition in Table 2.1 or alternatively (2.3). Using the RPD WLP, an MA FF RPD is defined as follows:

**Definition 2.2** *(Minimum Aberration FF RPD's.)*
*For any two $2^{(m_1+m_2)-(p_1+p_2)}$ FF RPD's, $\mathcal{D}_1$ and $\mathcal{D}_2$, we say that $\mathcal{D}_1$ has less aberration than $\mathcal{D}_2$ if there exists an $r$ such that, $B_i(\mathcal{D}_1) = B_i(\mathcal{D}_2)$ for all $i \leq r-1$ and $B_r(\mathcal{D}_1) < B_r(\mathcal{D}_2)$. If no other design has less aberration than $\mathcal{D}_1$, then $\mathcal{D}_1$ is the* minimum aberration *FF RPD.*

Although we focus attention on the alternative word lengths proposed by Bingham and Sitter (2003) and Zhu (2000) it is important to mention that the methodology outlined could be applied to different ranking schemes chosen by the experimenter.

## 2.3 Non-Regular Two-level Fractional Factorial Designs

In recent years, the class of *non-regular* designs has received considerable attention. A non-regular design can be characterized as an orthogonal array with $4n$ runs, where main effects are partially aliased with 2fi's. Non-regular designs are often selected from the class of Hadamard matrices, which include the well known Plackett-Burman designs (Plackett and Burman, 1946).

An advantage of non-regular designs, in addition to the flexibility of run-size, is that some of the factorial effects are only partially aliased. Consequently, it may be possible to jointly estimate more than one effect in a particular alias string. In the regular FFD case, effects are either fully aliased or independent, thus joint estimation of effects in an alias string is not possible. Several methods for analyzing non-regular designs are available (see Wu and Hamada, 2000, chapter 8).

Recently, Deng and Tang (1999) proposed $G$-aberration as a way to rank non-regular designs. Tang and Deng (1999) further proposed $G_2$-aberration as a relaxed version of $G$-aberration to rank non-regular FFD's. Fontana, Pistone and Rogantin (2000) introduced the indicator function as a useful tool for studying factorial designs. Ye (2003) further expanded this work and discovered how it could be used to rank designs according to the $G_2$-aberration criterion.

Let $\mathcal{D}$ be a $2^m$ full factorial design. A design point (run) of $\mathcal{D}$ is denoted by $\boldsymbol{x} = (x_1, \ldots, x_m)$, where the collection of all design points $\boldsymbol{x}$ of $\mathcal{D}$ are the solutions to

$$\{x_1^2 - 1 = 0, \ldots, x_m^2 - 1 = 0\}.$$

That is, the design points in $\mathcal{D}$ are the collection of all possible combinations of $\pm 1$ for the $m$ factors.

In general, a FFD $\mathcal{F}$ with $m$ factors will consist of a subset of $n \leq 2^m$ runs from $\mathcal{D}$. The indicator function for a FFD can now be introduced.

**Definition 2.3** *(Ye, 2003).*
*Let $\mathcal{D}$ be a $2^m$ full factorial design. The indicator function of a fraction $\mathcal{F}$ from $\mathcal{D}$ is*

$$F(\boldsymbol{x}) = \begin{cases} r_{\boldsymbol{x}} & if\ \boldsymbol{x} \in \mathcal{F} \\ 0 & if\ \boldsymbol{x} \in \mathcal{D} - \mathcal{F}, \end{cases}$$

*where $r_{\boldsymbol{x}}$ is the number of times the given run $\boldsymbol{x}$ appears in the design, and $\mathcal{D} - \mathcal{F}$ is the set of all points in $\mathcal{D}$ that are not contained in $\mathcal{F}$.*

For a given design, $\mathcal{D}$, and a run, $\boldsymbol{x} \in \mathcal{D}$, define a contrast

$$X_J(\boldsymbol{x}) = \prod_{j \in J} x_j \quad \text{on} \quad \mathcal{D},$$

where $J \in \mathcal{P}$ and $\mathcal{P}$ is the set of all subsets of $\{1, 2, \ldots, m\}$. Note that the empty set, $\emptyset$, is an element of $\mathcal{P}$.

It is well known that $\{X_J; J \in \mathcal{P}\}$ forms a basis of $R^{2^m}$, and thus, any polynomial function can be written as

$$F(\boldsymbol{x}) = \sum_{J \in \mathcal{P}} b_J X_J(\boldsymbol{x}),$$

where

$$b_J = \frac{1}{2^m} \sum_{\boldsymbol{x} \in \mathcal{F}} X_J(\boldsymbol{x}),$$

$b_\emptyset = n/2^m$ and $X_\emptyset = I$ corresponds to a column of ones. Since $x_i^2 = 1$ the above form is free of squared terms, which leads to the coefficients $b_J$ having unique solutions. It is important to note that the $b_J$ terms are a measure of the correlation of the effect $X_J$ with the mean $X_\emptyset$, therefore, these terms will be useful in terms of ranking designs.

We say that $X_J$ and $X_K$ are fully aliased if $b_{J \cup K} = b_\emptyset$. If the two effects are not fully aliased then $-1 < b_{J \cup K}/b_\emptyset < 1$. A design $\mathcal{F}$ is said to be a non-regular design if there exists at least one $b_J$ such that $-1 < b_J/b_\emptyset < 1$.

If $|b_J/b_\emptyset| = 1$ for all $b_J \neq 0$ the design is regular and the effect $X_J$ is said to be a *word* of the design. Note that, we say that the column $X_J$ is a word of the design, which is analogous to saying that the $J$ interaction is a word in the DCS. If $|b_J/b_\emptyset| \neq 1$ then $X_J$ is a *fractional word* (or simply a word) of the design $\mathcal{F}$.

**Example 2.2**
Consider the $2^{6-3}$ FFD from section 2.2. The indicator function of the design is

$$\begin{aligned} F(\boldsymbol{x}) &= \frac{1}{8}(1 + x_A x_B x_D)(1 + x_A x_C x_E)(1 + x_B x_C x_F) \\ &= \frac{1}{8}(1 + x_A x_B x_D + x_A x_C x_E + x_B x_C x_F + x_D x_E x_F \\ &\quad + x_A x_C x_D x_F + x_B x_C x_D x_E + x_A x_B x_E x_F). \end{aligned}$$

For a given run $\boldsymbol{x} = (x_A, x_B, x_C, x_D, x_E, x_F)$ the function $F(\boldsymbol{x})$ is zero if the run is not in the design and 1 if the run is in the design. Since $F(\boldsymbol{x})$ can be written in terms of basis elements $X_J$, $J \in \mathcal{P}$, the term $x_A x_B x_D = X_{ABD}(\boldsymbol{x})$ corresponds to the column $X_{ABD}$ where $b_{ABD} = 1/8$.

In addition, $|b_{ABD}/b_{\emptyset}| = 1$, which implies that $A$ is fully aliased with $BD$, $B$ is fully aliased with $AD$ and $D$ is fully aliased with $AB$.

### Example 2.3

Consider the two non-isomorphic 12-run designs with 6 factors, as shown in Table 2.2. The indicator functions for the two designs are respectively,

Table 2.2: 2 Non-Isomorphic 12-run Designs.

| Design 1 | | | | | | Design 2 | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | -1 |
| 1 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | -1 | -1 | -1 | 1 |
| 1 | -1 | 1 | -1 | 1 | 1 | 1 | -1 | 1 | -1 | -1 | 1 |
| 1 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | -1 |
| 1 | -1 | -1 | -1 | -1 | 1 | 1 | -1 | -1 | -1 | 1 | -1 |
| -1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | -1 | -1 | -1 |
| -1 | 1 | -1 | 1 | 1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 |
| -1 | 1 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | -1 | 1 | -1 |
| -1 | -1 | 1 | 1 | -1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 |
| -1 | -1 | 1 | -1 | 1 | -1 | -1 | -1 | 1 | -1 | 1 | 1 |
| -1 | -1 | -1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

$$\begin{aligned} F(\boldsymbol{x}) &= \frac{1}{16}(3 + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_5 \\ &\quad +x_1x_3x_5 - x_2x_3x_5 - x_1x_4x_5 + x_2x_4x_5 - x_3x_4x_5 \\ &\quad -x_1x_2x_6 + x_1x_3x_6 - x_2x_3x_6 - x_1x_4x_6 + x_2x_4x_6 \\ &\quad +x_3x_4x_6 + x_1x_5x_6 + x_2x_5x_6 + x_3x_5x_6 + x_4x_5x_6 \\ &\quad +x_1x_2x_3x_4 - x_1x_2x_3x_5 + x_1x_2x_3x_6 - x_1x_2x_4x_5 \\ &\quad +x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_1x_2x_4x_6 - x_1x_3x_4x_6 \\ &\quad -x_2x_3x_4x_6 - x_1x_2x_5x_6 + x_1x_3x_5x_6 + x_2x_3x_5x_6 \\ &\quad +x_1x_4x_5x_6 + x_2x_4x_5x_6 - x_3x_4x_5x_6 \\ &\quad +2x_1x_2x_3x_4x_5x_6) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} F(\boldsymbol{x}) &= \frac{1}{16}(3 + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_5 \\ &\quad +x_1x_3x_5 + x_2x_3x_5 + x_1x_4x_5 + x_2x_4x_5 + x_3x_4x_5 \\ &\quad +x_1x_2x_6 + x_1x_3x_6 - x_2x_3x_6 - x_1x_4x_6 + x_2x_4x_6 \\ &\quad -x_3x_4x_6 - x_1x_5x_6 - x_2x_5x_6 + x_3x_5x_6 + x_4x_5x_6 \\ &\quad +x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 \\ &\quad +x_2x_3x_4x_5 - x_1x_2x_3x_6 - x_1x_2x_4x_6 + x_1x_3x_4x_6 \\ &\quad +x_2x_3x_4x_6 + x_1x_2x_5x_6 - x_1x_3x_5x_6 + x_2x_3x_5x_6 \\ &\quad +x_1x_4x_5x_6 - x_2x_4x_5x_6 - x_3x_4x_5x_6 \\ &\quad -2x_1x_2x_3x_4x_5). \end{aligned} \tag{2.5}$$

The term $x_1x_2x_3$ in the indicator function represents the 123 interaction $X_{123}$, where $b_{123} = 1/16$. For ease of discussion we will simply refer to the term $x_1x_2x_3$ as a 3-factor word in the design. The two indicator functions are almost identical except for the 6-factor word in (2.4) and the 5-factor word in (2.5). Therefore, conventional wisdom would favour the first design over the second, since words of larger length are preferred.

Minimum $G$-aberration is a generalization of the MA criterion to non-regular designs (Deng and Tang, 1999). The $G$-aberration defined for a FFD can also be expressed in terms of the indicator function (Li, Lin and Ye, 2002).

**Definition 2.4** *Let $\mathcal{F}$ be a FFD with indicator function $F(\boldsymbol{x}) = \sum_{J\in\mathcal{P}} b_J X_J$. If $b_J \neq 0$, $X_J$ is a word of length $\|J\| + (1 - |b_J/b_\emptyset|)$, where $\|J\|$ is the number of letters of the word $X_J$, and $|b_J/b_\emptyset|$ is a measure of the degree of confounding for the word $X_J$.*

It can be shown that $|b_J/b_\emptyset| = l/t$ where $t = n/4$ for some $l = 1, \ldots, t-1$ (Deng and Tang, 1999). From this, an extended word length pattern of a design $\mathcal{F}$ can be defined.

**Definition 2.5** *Given a design $\mathcal{F}$ with indicator function $F(\boldsymbol{x}) = \sum_{J \in \mathcal{P}} b_J X_J$, where $X_J$ is a word of the design $\mathcal{F}$, let $f_{i+l/t}$ be the number of words of length $(i + l/t)$, where $l = 1, \ldots, t-1$. Then the extended word length pattern (EWLP) is*

$$(f_3, \ldots, f_{3+(t-1)/t}, \ldots, f_m, \ldots, f_{m+(t-1)/t}).$$

Based on the EWLP, Tang and Deng (1999) defined the generalized resolution as

**Definition 2.6** *Given a design $\mathcal{F}$ with EWLP*

$$(f_3, \ldots, f_{3+(t-1)/t}, \ldots, f_m, \ldots, f_{m+(t-1)/t}),$$

*the generalized resolution of $\mathcal{F}$ is the smallest $i + l/t$ such that $f_{i+l/t} \neq 0$.*

**Example 2.4**

Reconsider the designs in Table 2.2. The EWLP's will be of the form

$$[(f_3, \ldots, f_{3.67}), \ldots, (f_{6.0}, \ldots, f_{6.67})]$$

(in order to assist in readability we group words whose floor, largest integer less than or equal to this value, is the same within brackets).

Therefore the two EWLP's are $[(0,0,20), (0,0,15), (0,0,0), (0,1,0)]$ for Design 1 and $[(0,0,20), (0,0,15), (0,1,0), (0,0,0)]$ for Design 2 and we say that Design 1 has less $G-$aberration than Design 2, since $f_{5.33}(\mathcal{D}_2) < f_{5.33}(\mathcal{D}_2)$, and $f_r(\mathcal{D}_1) = f_r(\mathcal{D}_2)$ for all $r < 5.33$.

The EWLP can be viewed as a smooth transition from words of length $i$ to length $i+1$, starting with words with the maximal degree of confounding and ending with words with the least degree of confounding $(i + l/t \in [i, i+1))$.

In addition to ranking designs using $G$-aberration, Tang and Deng (1999) propose the use of $G_2$-aberration which is a relaxed version of $G$-aberration.

**Definition 2.7** *Given a design $\mathcal{F}$ with indicator function $F(\boldsymbol{x}) = \sum_{J \in \mathcal{P}} b_J X_J$, let*

$$\alpha_l(\mathcal{F}) = \sum_{\|J\|=l} \left( \frac{b_J}{b_\emptyset} \right)^2,$$

*where $||J||$ is the number of letters of the word $X_J$. The generalized word length pattern (GWLP) is then*

$$(\alpha_3(\mathcal{F}), \ldots, \alpha_m(\mathcal{F})).$$

Notice that the GWLP of a regular FFD is equivalent to the usual WLP. As shown in Ye (2003), $G_2$-aberration may be preferable since it has the additional property, well known for regular FFD's, that

$$\sum_{i=1}^{m} \alpha_i(\mathcal{F}) = \frac{2^m}{n} - 1$$

if $\mathcal{F}$ has no replicates.

Given two FFD's, $\mathcal{F}_1$ and $\mathcal{F}_2$, we say that $\mathcal{F}_1$ has less $G$-aberration than $\mathcal{F}_2$ if there exists an $r$ such that $f_q(\mathcal{F}_1) = f_q(\mathcal{F}_2)$ for all $q \leq r - 1$ and $f_r(\mathcal{F}_1) < f_r(\mathcal{F}_2)$ and less $G_2$-aberration if there exists a $t$ such that $\alpha_q(\mathcal{F}_1) = \alpha_q(\mathcal{F}_2)$, for all $q \leq t - 1$ and $\alpha_t(\mathcal{F}_1) < \alpha_t(\mathcal{F}_2)$. A design has minimum $G$- or $G_2$-aberration if there is no other design with less aberration.

Although the discussion has focused on orthogonal arrays, the indicator function and the definitions of $G$- and $G_2$-aberration can be applied to any design. In the following chapter we modify the definition of $G$- and $G_2$-aberration can be modified for FF RPD's.

# Chapter 3

# Non-Regular Robust Parameter Designs

In Chapter 2, regular FF RPD's were introduced and it was shown how these designs can be ranked using the MA criterion. However, an open question is how to rank non-regular FFD's for robust parameter experiments. In this chapter we consider the selection of optimal RPD's from two-level orthogonal arrays, where the effect estimates are either orthogonal, partially aliased, or fully aliased. That is, we develop new methodology for ranking both regular and non-regular FF RPD's.

The indicator variable approach is used to study non-regular RPD's and also develop methodology for selecting optimal FF RPD's. In section 3.2, a theoretical result is developed that implies one can use the class of all non-isomorphic two-level orthogonal arrays to find the MA FF RPD, based on either $G$- or $G_2$-aberration. Finally, we consider the selection of optimal 12-, 16- and 20-run FF RPD's. Theoretical results are developed that aid in finding MA FF RPD's from existing catalogs of designs. The methodology is demonstrated through the construction of a catalog of designs for 12, 16 and 20 runs.

## 3.1 Indicator Functions and Robust Parameter Designs

In this section, the indicator function approach (Ye, 2003) is adapted to consider both control and noise factors so that the properties of a RPD can be studied. Begin by considering a two-level full factorial RPD, $\mathcal{D}$, denoted as $2^{(m_1+m_2)}$, where $m_1$ refers to the number of control factors and $m_2$ refers to the number of noise factors. Let $\mathcal{D}_C$ be the $2^{m_1}$ full factorial design for the control factors and $\mathcal{D}_N$ be the $2^{m_2}$ full factorial design for the noise factors. Then $\mathcal{D} \equiv \mathcal{D}_C \otimes \mathcal{D}_N$ will be the $2^{m_1+m_2}$ full factorial RPD. A design point of $\mathcal{D}$ will be denoted by $\boldsymbol{x} = (x_1, \ldots, x_{m_1}, x_{m_1+1}, \ldots, x_{m_2})$, where the collection of all design points $\boldsymbol{x}$ of $\mathcal{D}$ are the solutions to $\{x_1^2 - 1 = 0, \ldots, x_{m_1}^2 - 1 = 0, x_{m_1+1}^2 - 1 = 0, \ldots, x_{m_1+m_2}^2 - 1 = 0\}$ (i.e., the points in $\mathcal{D}$ are the collection of all possible combinations of $\pm 1$ for the $m_1 + m_2$ factors).

In general, a FF RPD is any design $\mathcal{F}$ with $m_1 + m_2$ factors consisting of a subset of $n \leq 2^{(m_1+m_2)}$ runs from $\mathcal{D}$. It is appropriate to note that a FF RPD is not necessarily of the form $\mathcal{F}_C \otimes \mathcal{F}_N$ where $\mathcal{F}_C$ is a FFD with $m_1$ control factors and $\mathcal{F}_N$ is a FFD with $m_2$ noise factors. For the remainder of this chapter, we will restrict attention to two-level orthogonal designs, but the indicator function approach and ranking methodology apply to non-orthogonal designs as well.

The design matrix for a FFD is identical to that of a FF RPD. The difference lies in the existence of two types of factor in the RPD. Therefore, we can borrow results from FFD's and adapt them to FF RPD's. Consequently, we need only consider fractions of a $2^m = 2^{m_1+m_2}$ design unless it is important to distinguish the control and noise factors. Therefore, the indicator function for a FF RPD is identical to the indicator function of a FFD, see equation (2.4).

For a given design, $\mathcal{D}$, and a run, $\boldsymbol{x} \in \mathcal{D}$, define a contrast

$$X_L(\boldsymbol{x}) = \prod_{l \in L} x_l \quad \text{on} \quad \mathcal{D},$$

where $L \in \mathcal{P}$ and $\mathcal{P}$ is the set of all subsets of $\{1, 2, \ldots, m\}$. Further, define $\mathcal{P}_C$ to be the set of all subsets of $\{1, 2, \ldots, m_1\}$ and $\mathcal{P}_N$ to be the set of all subsets of

$\{1, 2, \ldots, m_2\}$, where an element of $\mathcal{P}$ is of the form $L \equiv J \cup K$ where $J \in \mathcal{P}_C$ and $K \in \mathcal{P}_N$. Note that the empty set, $\emptyset$, is an element of $\mathcal{P}$, $\mathcal{P}_C$ and $\mathcal{P}_N$. This adaptation of the indicator variable approach is useful for distinguishing between the control and noise factors and allows for the study and comparison of FF RPD's.

Similar to the development in section 2.3, $\{X_L; l \in \mathcal{P}\}$ forms a basis of $R^{2^m}$, and thus, any polynomial function can be written as

$$\begin{aligned} F(\boldsymbol{x}) &= \sum_{L \in \mathcal{P}} b_L X_L(\boldsymbol{x}) \\ &= \sum_{J \in \mathcal{P}_C} \sum_{K \in \mathcal{P}_N} b_{J \cup K} X_{J \cup K}(\boldsymbol{x}), \end{aligned}$$

where

$$b_L = (1/2^m) \sum_{\boldsymbol{x} \in \mathcal{F}} X_L(\boldsymbol{x}),$$

$b_\emptyset = n/2^m$ and $X_\emptyset$ corresponds to a column of ones.

Recall from Chapter 2 that $X_J$ and $X_K$ are fully aliased if $b_{J \cup K} = b_\emptyset$. If the two effects are not fully aliased then $-1 < b_{J \cup K}/b_\emptyset < 1$. If $|b_L/b_\emptyset| = 1$ for all $b_L \neq 0$ the design is regular and the effect $X_L$ is said to be a *word* of the design. This is analogous to saying that $X_L$ is a term (i.e. word) in the defining contrast sub-group. If $|b_L/b_\emptyset| \neq 1$ then $X_L$ is a *fractional word* (or simply a word) of the design $\mathcal{F}$.

Motivated by the non-regular FFD and the regular FF RPD, we now consider how to rank a non-regular FF RPD. Recall that Bingham and Sitter (2003) and Zhu (2000) propose alternative rankings for the words in a regular FF RPD. The same redefinitions of the word length can be applied to the fractional words of the non-regular FF RPD.

Recall, that the EWLP can be viewed as a smooth transition from words of length $i$ to length $i+1$, starting with words with the maximal degree of confounding and ending with words with the least confounding. Keeping this smooth transition in mind, we now define the EWLP of a FF RPD.

**Definition 3.1** *Let $\mathcal{F}$ be a RPD with indicator function*

$$F(\boldsymbol{x}) = \sum_{J \in \mathcal{P}_C} \sum_{K \in \mathcal{P}_N} b_{J \cup K} X_{J \cup K}.$$

*If $b_{J\cup K} \neq 0$ then $X_{J\cup K}$ is a word of the design $\mathcal{F}$ with word length $r+(1-|b_{J\cup K}/b_{\emptyset}|)/2$, where $|b_{J\cup K}/b_{\emptyset}|$ is a measure of the degree of confounding for the word $X_{J\cup K}$. Further let $g_{r+l/2t}$ be the number of words of length $r+l/2t$, where $r = 2.0, 2.5, 3.0, 3.5, \ldots$ according to Table 2.1 or $r = W(j,k)$ from (2.3), depending on the ranking chosen by the experimenter. Then the RPD EWLP is*

$$(g_{2.0}, \ldots, g_{2.0+(t-1)/2t}, \ldots, g_{m-1}, \ldots, g_{m+(t-1)/2t}).$$

Notice that the word-length in the RPD case is scaled by a fraction of one half to allow for a smooth transition from words of length $r$ to $r+1/2$.

**Example 3.1**

Consider the designs in Table 2.2 as RPD's with 1 control factor and 5 noise factors. If the first five columns of both designs are used for the noise factors and the remaining column for a control factor, using the definition of word length in Table 2.1, then the EWLP's will be of the form

$$[(g_{2.5}, \ldots, g_{2.83}), \ldots, (g_{6.0}, \ldots, g_{6.33})].$$

In order to save space the word length pattern begins with the first non-zero word.

For design 1, the EWLP is

$$[(0,0,10), (0,0,0), (0,0,10), (0,0,10), (0,0,0), (0,0,5), (0,1,0), (0,0,0)]$$

and for design 2, the EWLP is

$$[(0,0,10), (0,0,0), (0,0,10), (0,0,10), (0,0,0), (0,0,5), (0,0,0), (0,1,0)].$$

Observe that Design 2 has smaller RPD $G-$aberration than Design 1, in contrast to the non-RPD situation, where Design 1 had less $G$-aberration then Design 2. Now consider assigning the control factor to column 1 and the noise factors to the remaining 5 columns. The EWLP for Design 1 remains unchanged whereas the EWLP for Design 2 becomes

$$[(0,0,10), (0,0,0), (0,0,10), (0,0,10), (0,1,0), (0,0,5), (0,0,0), (0,0,0)].$$

There are two important things to take away from this example. Firstly, the MA FFD may not lead to an MA FF RPD, thus complete searches over the class of all non-isomorphic designs must be done to ensure that the MA FF RPD has been found. Secondly, the assignment of control and noise factors to the columns of the design can also result in designs with different properties.

Using a similar strategy to that of the EWLP, the GWLP for a FF RPD can be defined.

**Definition 3.2** *Given a RPD $\mathcal{F}$ with indicator function*

$$F(\boldsymbol{x}) = \sum_{J \in \mathcal{P}_C} \sum_{K \in \mathcal{P}_N} b_{J \cup K} X_{J \cup K},$$

*let $\gamma_r(\mathcal{F}) = \sum_{\mathcal{W}_r} (b_{J\cup K}/b_{\emptyset})^2$, where $\mathcal{W}_r$ is the set of all $J \cup K$ with length $r$ as in Table 2.1, or $r = W(j,k)$ from (2.3).The RPD GWLP is $(\gamma_{2.0}(\mathcal{F}), \ldots, \gamma_m(\mathcal{F}))$.*

Based on the RPD EWLP and following the path of Tang and Deng (1999) we define the generalized resolution of an FF RPD as follows:

**Definition 3.3** *Given an FF RPD $\mathcal{F}$ with EWLP*

$$(g_{2.0}, \ldots, g_{2.0+(t-1)/t}, \ldots, g_m, \ldots, g_{m+(t-1)/t}).$$

*the generalized resolution of $\mathcal{F}$ is the smallest $q + i/t$ such that $g_{q+i/t} \neq 0$.*

Non-regular FF RPD's can now be ranked using these generalizations of the existing methodologies. Given two FF RPD's, $\mathcal{F}_1$ and $\mathcal{F}_2$, we say that $\mathcal{F}_1$ has less *G*-aberration than $\mathcal{F}_2$ if there exists a $q$ such that $g_r(\mathcal{F}_1) = g_r(\mathcal{F}_2)$ for all $r \leq q-1$ and $g_q(\mathcal{F}_1) < g_q(\mathcal{F}_2)$ and less $G_2$-aberration if there exists a $t$ such that $\gamma_r(\mathcal{F}_1) = \gamma_r(\mathcal{F}_2)$, for all $r \leq t-1$ and $\gamma_t(\mathcal{F}_1) < \gamma_t(\mathcal{F}_2)$. A design has minimum *G*- or $G_2$-aberration if there is no other design with less aberration.

Based on these definitions, designs can be ranked and ordered. However, actually obtaining the MA FF RPD's is another matter. In the next section, we develop results that help find optimal FF RPD's from existing catalogs of two-level orthogonal arrays.

## 3.2 Non-Isomorphic Robust Parameter Designs

In this section, a method for finding the minimum $G$-aberration and $G_2$-aberration FF RPD is discussed. In order to ensure that the globally optimal design has been achieved an exhaustive search over all non-isomorphic RPD's must be performed. However, it is possible to take advantage of existing algorithms and lists of non-isomorphic non-regular two-level orthogonal arrays (see, Sun, Li and Ye, 2003) to find the class of non-isomorphic FF RPD's. We specifically consider the 12-, 16- and 20-run orthogonal arrays since these are the most common non-regular two-level designs.

Let $\mathcal{C}$ be the set of all $n \times m$ two level orthogonal arrays for some fixed $n = 4t, t = 1, 2, \ldots$ and $m \leq (n-1)$, and let $m_1 + m_2 = m$, where $m_1$ is the number of control factors and $m_2$ is the number of noise factors. Also assume without loss of generality that the first $m_1$ columns of any design in $\mathcal{C}$ are used for the control factors.

**Definition 3.4** *$C_1 \in \mathcal{C}$ is FF isomorphic to a design $C_2 \in \mathcal{C}$ if design $C_2$ can be obtained from $C_1$ by performing any combination of:*

- *row exchanges*
- *column exchanges*
- *column level exchanges*

*where a column level exchange consists of changing the high level to a low level and the low level to a high level within a column of the design.*

**Definition 3.5** *$C_1 \in \mathcal{C}$ is multi-factor (MF) isomorphic to a design $C_2 \in \mathcal{C}$ if design $C_2$ can be obtained from $C_1$ by performing any combination of:*

- *row exchanges*
- *column level exchanges*
- *column exchanges on the first $m_1$ columns and column exchanges on the remaining $m_2$ columns.*

The column exchange for FF isomorphism can be viewed as a two-step process where the first step is a column exchange of the first $m_1$ columns with the last $m_2$ columns, and the second step is a column exchange among the first $m_1$ columns and a column exchange among the remaining $m_2$ columns.

**Result 3.1** *If two designs $C_1$ and $C_2$ are MF isomorphic, they are FF isomorphic.*

**Proof:** The result follows directly from the fact that the set of allowable operations in the definition of MF isomorphism is a subset of the allowable operations in the definition of FF isomorphism. □

**Result 3.2** *It follows that any two designs $C_1$ and $C_2$ that are FF non-isomorphic are MF non-isomorphic.*

Based on these two results we are now ready to discuss how to find the set of MF non-isomorphic designs from the set of all FF non-isomorphic designs.

**Theorem 3.1** *Let $\mathcal{B}$ denote the set of all FF non-isomorphic $n \times m$ FFD's and $\mathcal{A}$ be the set of all MF non-isomorphic $n \times (m_1 + m_2)$ FF RPD's, where $m = m_1 + m_2$. If $\mathcal{A}^*$ is the set of all FF RPD's obtained by taking all possible column exchanges of the first $m_1$ columns with the remaining $m_2$ columns of every design in $\mathcal{B}$ then $\mathcal{A} \subseteq \mathcal{A}^*$.*

**Proof:** Assume there exists an $A \in \mathcal{A}$ such that $A \notin \mathcal{A}^*$. If the distinction between control and noise factors is ignored this implies that $A \notin \mathcal{B}$ which is a contradiction since $\mathcal{B}$ is the class of all FF non-isomorphic designs. □

This theorem is important since it allows us to take the set of all non-isomorphic two-level orthogonal arrays, which can be found using the algorithm in Sun, Li and Ye (2003) or from the set of 12-, 16- and 20-run non-isomorphic designs obtained from these authors, and obtain the set of all non-isomorphic RPD's by simply performing every possible exchange of the first $m_1$ columns with the remaining $m_2$ columns. This set can then be used to find the minimum $G$-aberration or $G_2$-aberration FF RPD.

## 3.3 Selection of Robust Parameter Designs

### 3.3.1 12-Run Plackett-Burman Design

In this section, three theorems are presented. Discussion is based on the Plackett-Burman design since every 12-run orthogonal array is isomorphic to the Plackett-Burman design, or can be found from a subset of columns from the Plackett-Burman design. The first two theorems together imply that one need only take the MA 12-run Plackett-Burman design with $m$ factors and consider all possible assignments of $m_1$ as control factors and $m_2$ as noise factors to obtain an MA RPD (except for the single case where $m_1 = 1$, $m_2 = 5$ and one uses the word-length definition in Table 2.1), whether using $G$- or $G_2$-aberration. The last theorem argues that for $m \leq 5$ any $m_1$, $m_2$ combination and either definition of word length (Table 2.1 or (2.3)), any such assignment will do.

**Theorem 3.2** *Let $\mathcal{A}$ be a 12-run RPD minimum aberration ($G$ or $G_2$) two-level orthogonal design according to the definition in Bingham and Sitter (2003), with $m_1$ control factors and $m_2$ noise factors. If the $m_1$ control factors and the $m_2$ noise factors are treated equally then $\mathcal{A}$ will be a MA FFD, except for the single case $m_1 = 1, m_2 = 5$.*

**Proof:** There is exactly one FF non-isomorphic 12-run design for $m = 2, 3, 4, 7, 8, 9, 10, 11$ columns. For each of these cases the theorem holds, since, if the factors are treated equally, every RPD will reduce to this non-isomorphic choice. For the case of 5 and 6 factors there are two FF non-isomorphic designs. Consider the case of 5 factors where the indicator functions for the MA FFD and the one other FF non-isomorphic design are, respectively,

$$
\begin{aligned}
F(\boldsymbol{x}) \quad = \quad & \frac{1}{8}(3 - x_1x_2x_3 - x_1x_2x_4 + x_1x_3x_4 - x_2x_3x_4 - x_1x_2x_5 \\
& -x_1x_3x_5 + x_1x_4x_5 - x_2x_3x_5 + x_2x_4x_5 - x_3x_4x_5 \\
& -x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 \\
& -x_2x_3x_4x_5) \qquad (3.1)
\end{aligned}
$$

and

$$\begin{aligned} F(\boldsymbol{x}) &= \frac{1}{8}(3 - x_1x_2x_3 - x_1x_2x_4 + x_1x_3x_4 - x_2x_3x_4 + x_1x_2x_5 \\ &\quad -x_1x_3x_5 + x_2x_3x_5 - x_1x_4x_5 + x_2x_4x_5 - x_3x_4x_5 \\ &\quad -x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 - x_1x_3x_4x_5 \\ &\quad +x_2x_3x_4x_5 - 2x_1x_2x_3x_4x_5). \end{aligned} \tag{3.2}$$

Consider (3.1) and notice that all ten possible 3-factor words and all five possible 4-factor words appear, represented in the design by the terms $x_ix_jx_k$ and $x_ix_jx_kx_l$, respectively, and have identical absolute coefficient values for each of these terms. This implies that any labelling of $m_1$ factors as control factors and $m_2$ factors as noise factors will result in the same RPD indicator function. Now consider (3.2) and notice it is identical to (3.1) except for the additional term $x_1x_2x_3x_4x_5$. Using a similar argument as above, any labelling of $m_1$ control factors and $m_2$ noise factors will result in the same RPD indicator function. Since the first fifteen terms in each indicator function are identical, the RPD EWLP and the RPD GWLP for each design will be the same for any choice of $m_1$ control and $m_2$ noise factors, except for the additional five factor term in (3.2). This additional term results in the MA FFD in (3.1) having less RPD aberration than (3.2).

Consider the case of 6 factors where the indicator function of the MA FFD and the one other FF non-isomorphic design, shown in Table 2.2, are represented by (2.4) and (2.5) respectively. Both designs contain all twenty 3-factor words and all fifteen 4-factor words each with the same absolute value of the coefficient. Thus, any labelling of these factors to control and noise will result in the same RPD indicator function, except for the contribution of the term $x_1x_2x_3x_4x_5x_6$ in (2.4) and the term $x_1x_2x_3x_4x_5$ in (2.5). Looking at the ranking in Table 2.1, we see that any word with five factors always has a smaller rank than a word with six factors, provided $m_1 \neq 1$. Therefore using the ranking scheme in Bingham and Sitter (2003) it must be the case that (2.4) has less aberration than (2.5) regardless of the assignment of $m_1$ control factors and $m_2$ noise factors to the columns of the design, provided $m_1 \neq 1$. $\square$

**Theorem 3.3** *There exists a 12-run RPD MA (G or $G_2$) orthogonal design $\mathcal{A}$ according to the definition in Zhu (2000), with $m_1$ control factors and $m_2$ noise factors such that, if the $m_1$ control factors and the $m_2$ noise factors are treated equally, $\mathcal{A}$ will be an MA FFD.*

**Proof:** There is exactly one FF non-isomorphic 12-run design for $m = 2, 3, 4, 7, 8, 9, 10, 11$ columns. For each of these cases the theorem holds, since, if the factors are treated equally, every RPD will reduce to this non-isomorphic choice. For the case of 5 and 6 factors there are two FF non-isomorphic designs. For the case of 5-factors the proof is the same as the proof in Theorem 3.2.

Considering the case of 6 factors, (2.4) is the MA FFD. It must be shown that (2.4) will be one of the MA RPD's using the definition in Zhu (2000). As shown in Theorem 3.2 the only terms that have to be considered in (2.4) and (2.5) are the five factor word in (2.5) and the six factor word in (2.4). Looking at the ranking scheme proposed by Zhu (2000), and given in (2.3), one of the two RPD indicator function choices from (2.5) will result in the 5 factor word having a rank smaller than the rank of the 6 factor word from (2.4). The other case from (2.5) will result in the 5 factor word having the same rank as the 6 factor word from (2.4). For example, consider the case of 3 control and 3 noise factors, the 6 factor word from (2.4) would have $W(3, 3) = 3.5$ from (2.3). The 5 factor word in (2.5) will have $W(3, 2) = 3$ or $W(2, 3) = 3.5$. Thus the two 6 factor FF non-isomorphic designs will lead to both (2.4) and (2.5) being minimum aberration. $\square$

The above two theorems show that the MA FFD will always result in a MA RPD for a particular labelling of $m_1$ control factors and $m_2$ noise factors (except for the case $m_1 = 1$, $m_2 = 5$ using the word lengths in Table 1). This result can be strengthened for $m \leq 5$.

**Theorem 3.4** *If $\mathcal{A}$ is a MA $12 \times m$ FFD with $m \leq 5$ any assignment of $m_1$ factors as control and $m_2$ factors as noise will result in an MA (G or $G_2$) RPD according to the definition in Bingham and Sitter (2003) and Zhu (2000).*

**Proof:** Consider the case of 5-factors. Using the same argument as in Theorem 3.2, the result follows. For the case of 2-factors the design is equivalent to 3 replicates of a full factorial with 2 factors and hence the theorem holds. For the case $m = 3$, the 3-factor word is the only term in the indicator function. Hence, any labelling of control and noise factors results in the same RPD indicator function. For the case $m = 4$, the indicator function contains all four 3-factor words and the 4-factor word and all coefficients have the same absolute value, hence using the same argument as for the case of 5 factors the theorem holds. □

**Result 3.3** *If $\mathcal{A}$ is the Minimum G-aberration FF RPD according to the definition in Bingham and Sitter (2003) or Zhu (2000) then $\mathcal{A}$ will be the minimum $G_2$-aberration RPD.*

**Proof:** A complete enumeration, aided by the above results, of all 12-run orthogonal arrays was performed to verify that the minimum $G-$ and minimum $G_2-$aberration designs are the same. □

Table A.1 and Table A.2 present the best $G$-aberration designs for 7 through 11 factors for the definitions in Bingham and Sitter (2003) and Zhu (2000). Table A.2 only presents the cases where the design differs from those in Table A.1. Since there is only one non-isomorphic design for 7 through 11, it is sufficient to show the columns used for control factors or the columns used for the noise factors. If $m_1 \leq m_2$ ($m_1 > m_2$), the $m_1$ control ($m_2$ noise) factor columns will be shown. Since any subset of 7 or greater columns from the 12-run Plackett-Burman results in the same design, the first $s = 7, \ldots, 11$ columns have been selected to represent the design. The definition of word-length in Zhu (2000) usually leads to the same designs as those found using the definition of word-length in Bingham and Sitter (2003) and only differ for assignments of 3 through 5 control factors.

### 3.3.2 16-Run Orthogonal Arrays

In this section, the results found from a complete search of all 16-run orthogonal arrays using Theorem 3.1 are discussed. There are five non-isomorphic 16-run orthogonal arrays, one of which is a regular FFD and the remaining are non-regular. These five designs can be found in Hall (1961) (see also Wu and Hamada, 2000). As in Hall (1961) these are labelled as designs $I$, $II$, $III$, $IV$ and $V$, where $I$ denotes the regular design. Tables A.3 and A.4 contain the MA FF RPD's based on the definitions in Bingham and Sitter (2003) and Zhu (2000), respectively. The design columns represent which non-isomorphic design is used and the first $m_1$ columns are used as control factors and the remaining $m_2$ columns are used for the noise factors. As shown in Tang and Deng (1999), the only possible correlation of words in a 16-run design are 0 or 1/2. Thus the EWLP is presented as $[(g_{2.0}, g_{2.25}), (g_{2.5}, g_{2.75}), \ldots]$.

There are a few interesting things to notice from the tables. If the definition of word-length in Bingham and Sitter (2003) is used, all designs are regular FFD's for 7 factors or less and all designs are non-regular for 9 or more factors. Whereas using the definition in Zhu (2000) all designs are non-regular FFD's for 7 or more columns and overall for only 3 cases are the best designs regular. Secondly, the two different critera tended to locate and rank different designs as minimum $G$-aberration.

When ranking the designs using $G_2$-aberration the best designs can always be found from the regular design, except for the case of 8 factors where both definitions of word-length found the best design from design $III$. Complete tables of 16-run $G_2$-aberration designs can be obtained from the author upon request.

### 3.3.3 20-Run Orthogonal Arrays

Based on the catalog of non-isomorphic orthogonal arrays obtained from Sun, Li, and Ye (2003), a catalog of MA RPD's was obtained using the definitions of $G$- and $G_2$-aberration and application of Theorem 3.1. In order to save space, these have been omitted from the text, however, complete tables of 20-run $G$- and $G_2$-aberration designs can be obtained from the author upon request. There are a few interesting features of note from this search. Firstly, a small number of minimum $G$- or $G_2$-aberration designs, based on the definition in Bingham and Sitter (2003) and Zhu (2000) were the same as the minimum $G$- or $G_2$-aberration FFD. The majority of minimum $G_2$-aberration RPD's were the same as the minimum $G$-aberration RPD's based on both definitions of word length. For all designs with 5 factors or less, the $G$- and $G_2$-aberration designs were equivalent using both definitions of word length.

## 3.4 Discussion

In this chapter, the definitions of the indicator function, EWLP and GWLP are extended to non-regular FF RPD's and are used to rank such designs. The extension developed is specifically applied to the case of non-regular FF RPD's, however the results in section 3 hold for any experiment where there are two different factor types. It was shown that different assignments of control and noise factors to the columns of a design may result in RPD's with different properties. The key result in Theorem 3.1 allows the use of existing tables of non-isomorphic non-regular FFD's to more easily obtain the minimum $G$- and $G_2$-aberration designs. The tabled results for 12, 16 and 20 runs focus only on the word-length definitions suggested by Bingham and Sitter (2003) and Zhu (2000) but the methodology outlined could be applied to other ranking schemes chosen by an experimenter. For 12-run designs, three theorems were presented which showed how to quickly find the minimum $G$- or $G_2$-aberration designs.

# Chapter 4

# Projection Properties of Non-Regular Designs

In many industrial applications screening experiments are performed at the initial stages of the experimental process to test the significance of a large number of main effects and some 2fi's. Typically, the experimenter chooses a design with a relatively small number of runs that will allow for the estimation of a large number of main effects and some 2fi's, assuming that only a few of the main effects are active. The difficulty with most experimental situations can be viewed as two-fold: often the experimenter has no prior knowledge of which effects are important, thus it is desirable to select a design that allows for joint estimation of all main effects and the associated 2fi's, and cost usually limits the number of experimental trials that can be performed.

A regular $2^{m-p}$ resolution $V$ design is often the best design choice and is selected based on two important properties. Firstly, the design projects onto a full factorial in any subset of four factors (Box and Hunter, 1961). That is, any subset of size four from the $m$ factors will form a $2^4$ full factorial design. This geometric projection result has an important statistical consequence in that any model containing four factors and the associated interactions can be estimated. Secondly, a resolution $V$ design allows for the joint estimation of all main effects and 2fi's. However, if the design is not Resolution $V$, it will have poor projection properties. For example, a resolution $IV$ design has projections onto four factors that do not allow for the estimation of all main effects and all 2fi's.

Unlike regular designs, non-regular designs often enjoy some very attractive projection properties. Cheng (1995) showed that if a non-regular design has a run-size that is not a multiple of eight, then any projection onto four factors allows estimation of all main effects and all 2fi's. Bulutoglu and Cheng (2003) showed that the same projection holds provided the design is generated cyclically (Paley construction).

Motivated by the above work, in this chapter we formulate a criterion of projection estimation capacity that can be systematically used to select designs with good projection properties. We are able to find designs that closely resemble regular resolution $V$ designs, in terms of projection estimation properties.

## 4.1 Projection Estimation Properties

The focus of this chapter is on two-level orthogonal arrays, with $n$ runs and $m$ factors. In the discussion that follows we assume that the $n$ runs are performed in a completely random order, and for ease of discussion we will refer to an $n \times m$ two-level orthogonal array as an $n \times m$ design. Although discussion is focused on orthogonal arrays run in a completely random order the methodology discussed can be applied to non-orthogonal arrays, and arrays with more than two-levels. The criterion can also be suitably modified for RPD's and designs with restriction on randomization. Some of these concepts are further elaborated on in Chapter 5.

### 4.1.1 Projection Estimation Capacity

Given an $n \times m$ design $\mathcal{D}$, we say that model can be estimated if all of the effects can be jointly estimated under $\mathcal{D}$.

**Definition 4.1** *Given $\mathcal{D}$, an $n \times m$ array, let $\rho_k(D)$ be the number of estimable models containing $k$ main effects and their associated 2fi's. Further define*

$$p_k(D) = \frac{\rho_k(D)}{\binom{m}{k}},$$

*and call $(p_1, p_2, \ldots, p_m)$ the* Projection Estimation Capacity *(PEC) sequence of $\mathcal{D}$.*

Since it is desirable to have $p_k(D)$ as large as possible we will use the PEC to rank designs. We propose a ranking strategy that sequentially maximizes the PEC sequence.

**Definition 4.2** *Given two $n \times m$ designs, $\mathcal{D}_1$ and $\mathcal{D}_2$, $\mathcal{D}_1$ is more desirable than $\mathcal{D}_2$ if there exists an integer $k$ such that $p_j(\mathcal{D}_1) = p_j(\mathcal{D}_2)$ for $j = 1, \ldots, k$ and $p_k(\mathcal{D}_1) > p_k(\mathcal{D}_2)$. If no other design is more desirable than $\mathcal{D}_1$ we will say that $\mathcal{D}_1$ is the* Maximum Projection Estimation Capacity *(MPEC) design.*

Designs with resolution $V$ or higher are the only experimental plans that have $p_i = 1$ for all $i = 1, \ldots, m$. However, if an experimenter believes in factor sparsity then only $k \leq m$ of the factors are assumed to be active. Assuming effect heredity and effect hierarchy it is not unreasonable to assume that the 2fi's among the set of $k$ active factors may be important. Thus a design with $p_i = 1$ for $i = 1, \ldots, k$ would provide almost the same information as a resolution $V$ design. When it is not possible to estimate all models with $k$ main effects and the associated 2fi's, then the design that allows for maximum number of estimable models would be preferred. Thus, the MPEC $n \times m$ orthogonal array is the design that most closely resembles a resolution $V$ design.

The projection properties of non-regular designs have been discussed in many articles. For example Lin and Draper (1992) and Box and Bisgaard (1993) showed that some small run Plackett-Burman designs when projected onto three factors contain a complete $2^3$ design and a half-replicate of the $2^3$ design. Wang and Wu (1995) studied the projections of certain $n \times m$ orthogonal arrays onto 4 and 5 factors. When every projection allowed for the estimation of all main effects and 2fi's they said that the design had a *hidden projection property.* Li, Deng and Tang (2004) studied the projection properties of 20-run orthogonal arrays and ranked designs based on the number of estimable models containing 5 main effects and the associated 2fi's and the average D-efficiency of these designs. Although related, the previous work has not considered a systematic approach for ranking designs based on their desirable projection properties. In order to aid in the selection of $n \times m$ arrays, we begin by considering some properties of the PEC sequence.

For orthogonal arrays of strength 2, obviously we have $p_1 = p_2 = 1$, and trivially, $p_3 = 1$ if and only if there are no words of length 3. In addition $p_i = 0$ for $i = k, \ldots, m$ where $k + \binom{k}{2} \geq n - 1$. That is, any model containing more parameters than degrees of freedom is non-estimable. We will focus attention on situations where $p_1 = p_2 = p_3 = 1$.

**Lemma 4.1** *(Cheng, 1995 and Bulutoglu and Cheng, 2003)*
*Given an $n \times m$ orthogonal array, if $n$ is not a multiple of eight or $n > 8$ and the design is generated cyclically (Paley construction), then $p_4 = 1$.*

In addition to the projection estimation onto four factors, some results have been shown for projections onto five factors. The following lemma by Cheng (1998) extends the result of Diamond (1995) on the fold-over of the 12-run Plackett Burman design.

**Lemma 4.2** *(Cheng, 1998)*
*Given an $n \times m$ orthogonal array of strength three, if $n$ is not multiple of 16 then $p_5 = 1$.*

In order for a regular design to have $p_3 = 1$ the design is necessarily resolution $IV$. In addition, exact expressions for $p_4$ and $p_5$ have been found for resolution $IV$ designs:

**Lemma 4.3** *If $\mathcal{D}$ is a $2^{m-p}_{IV}$ design with WLP $(a_4, \ldots, a_m)$ then*

$$p_4 = 1 - \frac{a_4}{\binom{m}{4}} \quad \text{and} \quad p_5 = 1 - \frac{(m-4)a_4}{\binom{m}{5}}.$$

**Proof:**

Since $\mathcal{D}$ is a resolution $IV$ design, there are two possibilities when projecting onto 4 factors: i) the 4 factors do not form a word of length 4 in the DCS; or ii) the four factors form a word in the DCS. In case i), the model containing the 4 main effects and their 2fi's is obviously estimable. In case ii), the model containing the 4 main effects and 2fi's will not be estimable, since some of the 2fi's involving the 4 factors will be aliased with other 2fi's involving the 4 factors. In case ii), there are $a_4$ subsets of 4 factors that appear as words in the DCS, hence $\rho_4 = \binom{m}{4} - a_4$ and

$$p_4 = 1 - \frac{a_4}{\binom{m}{4}}.$$

If $\mathcal{D}$ is projected onto 5 factors there are two possibilities of interest for the set of 5 factors: i) No set of 4 of the 5 factors form a 4 letter word in the DCS ; and ii) one set of 4 of the 5 factors form a 4 letter word in the DCS. Note, if the set of 5 factors form more than one word of length 4 in the DCS this would result in a design with resolution less than $IV$. In case i) the model containing 5 main effects and their associated 2fi's will be obviously estimable. In case ii) the model containing 5 main effects and their associated 2fi's will not be estimable since some of the 2fi's are aliased with other 2fi's.

In order to count the number of non-estimable models for case ii), consider a given subset of four factors that form a word in the DCS and add a fifth column to form a projection onto five factors. The addition of any of the remaining $m-4$ factors to the existing four factors will lead to a model which cannot be estimated. Therefore there is a total of $(m-4)a_4$ non-estimable models. Hence $\rho_5 = \binom{m}{5} - (m-4)a_4$ and

$$p_5 = 1 - \frac{(m-4)a_4}{\binom{m}{5}}.$$

□

Contained in the first two lemmas, there is a strong message that non-regular designs provide ample opportunity for finding designs with good projection properties. In fact, we have found designs of 28 runs that have $p_5 = 1$ (see section 4.3). While the third lemma, suggests that regular designs will not have the same desirable projection properties.

The PEC has many advantages over the traditional method for ranking designs introduced in Chapter 2. The first advantage is that the same criterion can be used to rank both regular and non-regular designs. The PEC can also be used to judge if the design will provide sufficient information about main effects and 2fi's for the process under consideration. Lastly, the PEC allows an experimenter to easily compare designs with different run sizes and different number of factors. In order to illustrate these points we consider two examples.

**Example 4.1**

Suppose an experimenter wishes to test the effect of 7 factors on a process, where the primary interest is on the estimation of models containing main effects and 2fi's. They begin by considering two choices.

1. $\mathcal{D}_1$ is the MA $2^{7-1}$ resolution $VII$ 64-run FFD.
2. $\mathcal{D}_2$ is the MA $2^{7-2}$ resolution $IV$ 32-run FFD.

The PEC for $\mathcal{D}_1$ is $(1,1,1,1,1,1,1)$, since $\mathcal{D}_1$ is resolution $VII$. Since $\mathcal{D}_2$ is a resolution $IV$ design we apply the results of Lemma 4.3 to find $p_4 = 0.971$ and $p_5 = 0.857$. The full PEC sequence was found to be $(1,1,1,0.971,0.857,0.571,0)$.

Comparing the two designs, $\mathcal{D}_1$ would definitely be preferable in terms of PEC, to $\mathcal{D}_2$, since any model from $\mathcal{D}_1$ containing $k$ main effects and 2fi's is estimable. On the other hand $\mathcal{D}_2$ does not allow for the estimation of all models containing four factors. If run-size economy is a concern then $\mathcal{D}_1$ with 64-runs is a fairly large experiment, but sacrificing 32 runs leads to a design with worse PEC. However, Lemma 4.1 suggests that a non-regular design could be used and it would allow for the estimation of all models containing 4 factors.

**Example 4.2**

Consider, $\mathcal{D}_3$, a $20 \times 7$ orthogonal array, with design matrix

$$X = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

The PEC is $(1, 1, 1, 1, 1, 0, 0)$. Note that using Lemma 4.1, $p_4 = 1$. In addition there are insufficient degrees of freedom to estimate models containing more than 5 factors and the associated 2fi's, hence $p_6 = p_7 = 0$.

Comparing the three designs, we notice that $\mathcal{D}_3$ has a superior PEC than $\mathcal{D}_2$ except for the entry $p_6$ which is greater than zero for $\mathcal{D}_2$. However, if factor sparsity is assumed true it is unlikely that more than 5 of 7 factors will be active. Thus it could be argued that $\mathcal{D}_3$ will provide the same information as $\mathcal{D}_1$ in most situations, and will provide more information than $\mathcal{D}_2$. In addition $\mathcal{D}_3$ has a much smaller run-size than both $\mathcal{D}_1$ and $\mathcal{D}_2$ making it a particularly attractive choice for many experimental situations. In section 4.2, we will see that non-regular designs usually offer a viable alternative to running a resolution $V$ design.

### 4.1.2 Useful Theoretical Results

Ideally, if one were to consider the selection of a MPEC design the search would start from the class of all $n \times m$ non-isomorphic orthogonal arrays. In general, the class of all non-isomorphic $n \times m$ designs is unavailable; however, as an alternative one could start with the class of non-isomorphic Hadamard matrices and search for the MPEC design that is a projection of a Hadamard matrix. This approach will not yield every $n \times m$ design (see Li, Deng, and Tang, 2004) but it still provides a large class of designs to consider. For run-sizes larger than twenty this search can still be computationally infeasible unless some simplifications are made. The new ranking criterion lends itself well to reducing computations based on some theoretical results.

**Theorem 4.1** *Given an $n \times m$ array $p_j \geq p_k$ for $j < k$.*

**Proof:** Let $\eta_i$ be the number of non-estimable models containing $i$ main effects and their associated 2fi's. Then $\rho_i + \eta_i = \binom{m}{i}$.

Consider, a two stage projection where we first project the $n \times m$ array onto a subset of $k$ factors, and calculate $\rho_k$ and $\eta_k$, then project this set of $n \times k$ arrays onto $j$ factors and calculate $\rho_j$ and $\eta_j$ from $\rho_k$ and $\eta_k$. Notice that projecting in a two stage fashion will yield $\binom{m}{k}\binom{k}{j}$ designs, whereas projecting in one stage yields $\binom{m}{j}$ designs. Now, each of the $\binom{m}{j}$ subsets of size $j$, say $A_i$, $i = 1, \ldots, \binom{m}{j}$ appears a multiple number of times when selected in two stages, where the multiple is the number of sets of size $k$ that contain $A_i$. For a fixed $A_i$ there are $m - j$ elements remaining and we must select $k - j$ of these elements, thus each set $A_i$ appears $\binom{m-j}{k-j}$ times.

Thus each $n \times j$ design found from the one-stage projection appears $\binom{m-j}{k-j}$ times. Secondly, it is important to note that if the model containing $k$ factors is estimable then the model containing $j$ factors is estimable. Lastly, if the model containing $k$ factors is non-estimable it may be possible to estimate some of the models containing

$j$ factors. Combining these results

$$\rho_j\binom{m-j}{k-j} \geq \rho_k\binom{k}{j}.$$

That is, we scale $\rho_k$ up by the number of projections of $k$ factors onto $j$ factors, and $\rho_j$ up to account for the $\binom{m-j}{k-j}$ copies of each $\binom{m}{j}$ design. The inequality in the relationship accounts for the fact that some of the non-estimable models containing $k$ factors may be estimable when projecting onto $j$ factors.

Finally,

$$\begin{aligned} p_k &= \frac{\rho_k}{\binom{m}{k}} = \frac{\rho_k\binom{k}{j}}{\binom{m}{k}\binom{k}{j}} \\ &\leq \frac{\rho_j\binom{m-j}{k-j}}{\binom{m}{k}\binom{k}{j}} = \frac{\rho_j}{\binom{m}{j}} \\ &= p_j. \end{aligned}$$

□

As a consequence, if there exists an $n \times m$ array with $p_k = 1$, then $p_j = 1$ for all $j < k$. A second result that will be useful in searching designs is stated and proved below.

**Theorem 4.2** *Given an $n \times (m+p)$ array $\mathcal{D}_2$ with $p \geq 1$. There exists a projection of $\mathcal{D}_2$ onto $\mathcal{D}_1$, an $n \times m$ array consisting of a subset of columns from $\mathcal{D}_2$ such that $p_j(\mathcal{D}_1) \geq p_j(\mathcal{D}_2)$.*

**Proof:** Similar to the proof of Theorem 4.1, we consider projecting $\mathcal{D}_2$ in two stages. At the first stage, construct the $\binom{m+p}{m} = t$, $n \times m$ designs $B_1, \ldots, B_t$, and at stage two project each of these designs onto $j$ factors. This yields a total of $\binom{m+p}{m}\binom{m}{j}$, $n \times j$ designs, where each $n \times j$ design will be repeated $\binom{m+p-j}{m-j}$ times.

For each $B_i$, calculate $\rho_j(B_i)$. Then

$$\sum_{i=1}^{t}\rho_j(B_i) = \rho_j(\mathcal{D}_2)\binom{m+p-j}{m-j}.$$

That is, the total number of estimable projections found from a two-stage projection is the same as the number of estimable projections from a one-stage projection scaled up by the number of times each $n \times j$ design is repeated in the two-stage procedure.

This leads to

$$\begin{aligned} p_j(\mathcal{D}_2) &= \frac{\rho_j(\mathcal{D}_2)}{\binom{m+p}{j}} = \frac{\rho_j(\mathcal{D}_2)\binom{m+p-j}{m-j}}{\binom{m+p}{j}\binom{m+p-j}{m-j}} \\ &= \frac{\sum_{i=1}^{t} \rho_j(B_i)}{\binom{m+p}{m}\binom{m}{j}} = \frac{\sum_{i=1}^{t} p_j(B_i)}{\binom{m+p}{m}} \\ &= \bar{p_j}, \end{aligned}$$

where $\bar{p_j}$ is the average value of $p_j(B_i)$ for $i = 1, \ldots t$.

Let $\mathcal{D}_1$ be the $n \times m$ projection design, $B_i$ such that $p_j(B_i)$ is a maximum then,

$$p_j(\mathcal{D}_1) = max(p_j(B_i)) \geq p_j(\mathcal{D}_2).$$

□

**Corollary 4.1** *Given an $n \times m$ array $\mathcal{D}_1$, and an $n \times (m+p)$ array $\mathcal{D}_2$ with $p \geq 1$, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are the maximum projection estimation capacity designs, then $p_j(\mathcal{D}_1) \geq p_j(\mathcal{D}_2)$.*

**Proof:** From Theorem 4.2 there exists an $n \times m$ projection design $B_i$ consisting of $m$ columns from $\mathcal{D}_2$ such that $p_j(B_i) \geq p_j(\mathcal{D}_2)$. Since $\mathcal{D}_1$ is the MPEC design

$$p_j(\mathcal{D}_1) \geq max(p_j(B_i)) \geq p_j(\mathcal{D}_2).$$

□

It is worth mentioning that these theorems are extremely general and will apply to both orthogonal and non-orthogonal arrays. In addition, the above theorems will prove useful when searching for designs with larger run sizes. For example, consider the selection of a $24 \times 12$ orthogonal array. Since no complete catalogue of non-isomorphic designs is available, one would search for a design that is a projection of a Hadamard matrix. There are 60 non-isomorphic 24-run Hadamard matrices and approximately 1.3 million projections onto 12 factors. Thus a complete search of all projections would require searching more than 81 million designs. Based on the above theorems, we propose a simple search procedure to reduce the number of computations. At step one of the search, compute the projection estimation capacity sequence for each non-isomorphic $n \times (n-1)$ Hadamard matrix, and find the top twenty designs among this class. At step two, project the top twenty designs onto $n-2$ columns, compute the projection estimation vector and find the top twenty designs. Sequentially applying this search will allow one to quickly find a design with $n-k$ factors. For example, if the methodology was applied to find a design with 12 columns it would require searching 6120 designs and will not only yield a good design with 12 factors but a table of designs for 23 through 12 factors. In addition, if the experimenter were to find a $n \times (n-k)$ design with $p_i = 1$ then any of the projections of this design onto a smaller number of factors will also have $p_i = 1$. This procedure may not yield the optimal design which is a projection of a Hadamard matrix, but will hopefully provide a "good" design.

## 4.2 Maximum Projection Estimation Capacity Designs with 20 runs

The previous section introduced the concept of a MPEC design. In this section, 20-run MPEC designs are tabled for use by the practitioner, and other issues in design selection are discussed.

### 4.2.1 Complete Catalog of 20-run Designs

The new ranking method for non-regular designs is first applied to the complete catalogue of non-isomorphic $20 \times m$ orthogonal designs found by Sun, Li and Ye (2003). In the case of twenty runs, there are enough degrees of freedom to permit estimation of models that contain at most five main effects and their associated 2fi's. Applying Lemma 4.1, $p_4 = 1$ and thus the PEC sequence is $(1, 1, 1, 1, p_5, 0, \ldots, 0)$. Therefore, designs are ranked on the basis of $p_5$ and to save space we do not present the full PEC sequence. Table B.1 presents the top 5 designs ranked by their PEC sequence. If the design can be found from a projection of a Hadamard matrix, the Hadamard matrix is given and the columns used to construct the design are provided. In the case of 20 runs, there are three non-isomorphic Hadamard matrices (available on Neil Sloan's webpage) and are denoted as Had.20.$k$, for $k = 1, 2, 3$. When the design cannot be found from the projection of a Hadamard matrix, an integer code for the runs is provided. Consider a run which consists of $+1$'s and $-1$'s. Changing the $-1$'s to 0's gives rise to a binary sequence which will be represented in base ten. For example, a run $(-1, -1, -1, +1, +1)$ is represented by the integer 3 and a run $(+1, +1, -1, -1, +1, -1)$ is represented by the integer 50. For an $n \times m$ array, the integer codes of the runs will be between 0 and $2^m - 1$. For convenience the integer codes of the runs are presented in ascending order.

Table 4.1 provides a summary of the results found by searching all non-isomorphic $20 \times k$ arrays for all $k \geq 6$. Notice from Table 4.1 that there exists a $20 \times 6$ and a

Table 4.1: MPEC designs with 20 runs.

| $k$ | Maximum $p_5$ | Minimum $p_5$ | Number of Designs with the maximum $p_5$ |
|---|---|---|---|
| 6 | 1 | 0 | 11 |
| 7 | 1 | 0 | 2 |
| 8 | 0.929 | 0 | 2 |
| 9 | 0.873 | 0 | 6 |
| 10 | 0.857 | 0.429 | 2 |
| 11 | 0.848 | 0.584 | 1 |
| 12 | 0.826 | 0.622 | 1 |
| 13 | 0.821 | 0.648 | 1 |
| 14 | 0.797 | 0.663 | 1 |
| 15 | 0.785 | 0.67 | 1 |
| 16 | 0.777 | 0.675 | 1 |
| 17 | 0.773 | 0.676 | 1 |
| 18 | 0.765 | 0.676 | 1 |
| 19 | 0.761 | 0.676 | 1 |

$20 \times 7$ array that allow for the joint estimation of any model containing up to 5 main effects and their associated 2fi's. See Example 4.2 for a discussion of the $20 \times 7$ case.

To further illustrate, suppose there are 14 factors. As an alternative, the experimenter could choose the 32-run MA $2_{IV}^{14-9}$ design (see, Wu and Hamada, 2000, pg. 195), where 62 percent of all models containing 5 factors and the associated 2fi's can be estimated. By comparison, 80 percent of all models containing 5 main effects and the associated 2fi's can be estimated for the $20 \times 14$ design in Table 4.1. In this case, dropping 12 runs actually results in a design with superior PEC. In fact, notice that for designs with 14 or more factors more than 76 percent of all models containing 5 factors and the associated 2fi's can be jointly estimated.

### 4.2.2 Other Issues in Design Selection

In this section we focus attention on the 11, $20 \times 6$ arrays in Table 4.1 with $p_5 = 1$. One logical question to ask is how would one can distinguish between these 11 arrays. One possibility is to choose the array with minimum $G$-aberration (See section 2.3). A second possibility would be to choose the design with the highest average efficiency. Using a strategy similar to Sun (1993) we compute the efficiency of all estimable designs and average over all possible models.

**Definition 4.3** *Given an $n \times m$ orthogonal array, let $\mathcal{F}$ be the class of models containing $k$ main effects and their associated 2fi's, let*

$$d_k = \frac{\sum_{\mathcal{F}}(\det(X'X/n))^{1/p}}{\binom{m}{k}},$$

*where $p$ is the number of parameters in the model (i.e. $p = 1 + k + \binom{k}{2}$), and call $(d_1, \ldots, d_m)$ the* projection information capacity *(PIC).*

We now take a closer look at the non-isomorphic $20 \times 6$ arrays. In addition to computing $p_5$ and $d_5$, we also rank the designs according to $G$-aberration. There are 75 non-isomorphic $20 \times 6$ orthogonal arrays. Table 4.2 presents the top 11 designs based on $p_5$, the associated $d_5$ values and (the last column) the rank of the 11 designs with $p_5 = 1$ according to $G$-aberration from the entire set of 75 designs.

The first thing to notice is that the top three designs correspond to the designs with smallest $G$-aberration and also have the highest efficiency. The remaining designs,

Table 4.2: Top 11, $20 \times 6$ orthogonal arrays.

| $k$ | $p_5$ | $d_5$ | $G$-aberration Rank |
|---|---|---|---|
| 6 | 1.0 | 0.8614 | 1 |
| 6 | 1.0 | 0.8602 | 2 |
| 6 | 1.0 | 0.8248 | 3 |
| 6 | 1.0 | 0.8237 | 5 |
| 6 | 1.0 | 0.8054 | 14 |
| 6 | 1.0 | 0.8042 | 17 |
| 6 | 1.0 | 0.7871 | 8 |
| 6 | 1.0 | 0.7677 | 20 |
| 6 | 1.0 | 0.7494 | 10 |
| 6 | 1.0 | 0.7494 | 11 |
| 6 | 1.0 | 0.7494 | 39 |

although having worse $G$-aberration than the top three designs, have high efficiency. Thus, in order to distinguish between competing designs it would be useful to take the design with the highest rank according to either $D$-efficiency or minimum $G$-aberration. In order to reduce the computational burden of the search procedure we choose to use $D$-efficiency as the secondary sorting criterion. Notice that the value of $d_k$ can be easily calculated when finding $p_k$, that is $p_k$ is found by checking if the determinant is zero for each projection onto $k$ factors, and thus the determinant must be calculated anyways. Whereas, computing the $G$-aberration would require finding the EWLP for each design in the search, which is more computationally intensive.

### 4.2.3 20-run Designs Using the Search Procedure

For evaluative purposes, the search method proposed in section 4.1.2 is modified and applied to the 20-run designs. Since there are only three non-isomorphic 20-run Hadamard matrices, we first project each of these onto 18 factors and compute $p_5$ and $d_5$ for each projection. The top twenty designs with different values of $p_5$ and $d_5$ are selected and then used to find a good 17 factor design. This method is sequentially applied to find designs with 6 through 18 factors.

In addition to retaining the top twenty designs in the search procedure, we also performed the same procedure retaining the top 10, 30 and 40 designs. In order to save space, these results have been omitted, but it is important to note that retaining only the top 10 designs resulted in designs with smaller values of $p_5$ and led to a large number of designs that were not the MPEC design. In addition, retaining the top 30 and top 40 designs resulted in similar results to those found using only the top 20 designs.

Table 4.3 presents the minimum value of $p_5$ obtained by this search and the difference between the search result and the true value of $p_5$ found from the complete search (Table 4.1).

Table 4.3: 20-run PEC designs, using the search procedure

| $k$ | $p_5$ Search | $p_5$ True | difference |
|---|---|---|---|
| 6 | 1 | 1 | 0 |
| 7 | 1 | 1 | 0 |
| 8 | 0.929 | 0.929 | 0 |
| 9 | 0.873 | 0.873 | 0 |
| 10 | 0.857 | 0.857 | 0 |
| 11 | 0.835 | 0.848 | 0.013 |
| 12 | 0.826 | 0.826 | 0 |
| 13 | 0.821 | 0.821 | 0 |
| 14 | 0.797 | 0.797 | 0 |
| 15 | 0.785 | 0.785 | 0 |
| 16 | 0.777 | 0.777 | 0 |
| 17 | 0.773 | 0.773 | 0 |
| 18 | 0.765 | 0.765 | 0 |

As can be seen, the search method works exceedingly well. With the encouraging results from 20-runs we now focus our attention on the 24- and 28-run designs, where no complete search is yet possible, due to the large number of possible designs.

## 4.3 Projection Estimation Capacity Designs with 24 and 28 Runs

In the case of 24 and 28 runs there are enough degrees of freedom to permit estimation of models containing at most six main effects and their associated 2fi's. In the case of 24 runs, there are 60 non-isomorphic 24-run Hadamard matrices (which are readily available from Neil Sloan's webpage). Since $n$ is a multiple of eight, applying Lemma 4.1, $p_4 = 1$ if the design in generated cyclically. This is true for the Plackett-Burman design but is not true for the majority of designs in the catalog. The PEC sequence is therefore of the form $(1, 1, 1, p_4, p_5, p_6, 0, \ldots, 0)$ and in order to save space only the values $(p_4, p_5, p_6)$ are presented. Table B.2 presents a catalog of the top three designs for $m = 6, \ldots, 23$. Designs for $m = 6, \ldots, 22$ were found using the simple search procedure proposed in section 4.2.3. The catalog was constructed by taking the top 20 designs with different values of $(p_4, p_5, p_6, d_4, d_5, d_6)$ at step $k + 1$ and searching all projections of these designs onto $k$ factors. Using the same notation as Neil Sloan we denote the non-isomorphic Hadamard matrices, as Had.24.$k$,for $k = 1, \ldots, 60$ and give the columns used to construct designs for $m = 6, \ldots, 22$ factors.

As can be seen in Table B.2, the best Hadamard matrix with 23 factors (23.1) has a value of $p_4 = 1$ and $p_5 = 0.919$, indicating that all models containing four main effects and the associate 2fi's are estimable and that approximately 92 percent of all projections onto 5 factors will allow for the estimation of 5 main effects and their associated 2fi's. Consider 19 factors. The best $24 \times 19$ design has a value of $p_5 = 0.925$, while the best $20 \times 19$ design has $p_5 = 0.761$. Thus, the additional 4 runs provides a significant improvement over the competing 20-run case.

It is interesting to note that in Table B.2 the values of $p_6$ decrease from a maximum value of approximately 40 percent for the $24 \times 19$ array to as small as zero for the $24 \times 12$ array. The decrease is a result of the simple search procedure that was implemented. Consider a $24 \times (k+1)$ design with a PEC sequence $(p_4, p_5, p_6)$ and find all of the $\binom{k+1}{k}$ projections onto a $24 \times k$ design and examine the sequence $(p'_4, p'_5, p'_6)$, for each $24 \times k$ design. Theorem 4.2 guarantees that there exists at least one $24 \times k$ design where $p'_j \geq p_j$, for $j = 4, 5,$ and 6, however the theorem does not guarantee that remaining values in the sequence will all be larger for the chosen $24 \times k$ design. Hence, it is possible that the top twenty $24 \times k$ projection designs with the largest value of $p'_4$ will have smaller values of $p'_5$ and $p'_6$ than the $24 \times (k+1)$ design. Note, that a consequence of Theorem 4.2 guarantees that every $p'_j$ found from the $24 \times k$ projection designs will equal 1 if $p_j = 1$ for the $24 \times (k+1)$ design. As can be seen in Table B.2, the search finds the largest value of $p_4 = 1$ and then tries to maximize $p_5$ while sacrificing information on $p_6$. Finally once $p_5 = 1$ we see that the values of $p_6$ start increasing.

In order to help alleviate this problem, we consider three modifications of the search procedure. First, we consider simply taking the top 40 designs at each step instead of the top 20 designs. Table B.3 presents the designs which differ on PEC sequences from those found in Table B.2. Comparing Table B.2 and Table B.3 the best $24 \times 9$ array found by searching the top 40 designs had a much larger value of $p_6$. In general retaining the 40 designs did not lead to a significant improvement in the PEC sequence. Second, we consider taking a weighted average $p = (3p_4 + 2p_5 + p_6)/6$ and using the top twenty designs with the largest value of $p$ at step $k+1$ to find the best designs at step $k$. Table B.4 presents the catalog of designs found using the weighted average search. As can be seen, using a simple weighted average tends to find designs with slightly smaller values of $p_4$ and $p_5$ and much larger values of $p_6$. Thus, if the experimenter is interested in models with 6 main effects and their associated 2fi's these designs will provide a much better alternative to the previous designs. Lastly, we considered taking the top twenty designs that sequentially maximized the sequence $(p_6, p_5, p_4)$. Using the result in Theorem 4.1, we know that a design with a large value of $p_6$ will necessarily have larger values of $p_4$ and $p_5$, and from Theorem 4.2 we know that the value of $p_6$ will increase as we move from step $k+1$ to step $k$ in the search. As opposed to sacrificing information on $p_6$ to increase $p_5$ we use a search procedure that will always increase $p_6$ and ensure that $p_4$ and $p_5$ are also as large as possible. Table B.5 presents the catalog of designs based on searching the top twenty designs with the largest values of $p_6$. Comparing Table B.2 and Table B.5 it is interesting to note that for limited sacrifice on the estimation of models containing 4 and 5 factors designs with much larger values of $p_6$ have been found.

Comparing the results from each search we notice that a different set of designs was found for each procedure proposed. The experimenter has a choice when deciding which design to run. If estimation of models containing only 4 and 5 factors is of utmost importance then the designs in Table B.2 and Table B.3 would be the best choice of experimental plan. However, if there is some concern that the possible models will contain 6 factors then the designs in Table B.4 and Table B.5 will provide a good choice of design.

In the case of 28-runs, there are 487 non-isomorphic Hadamard matrices, which can be found on Neil Sloan's webpage. Applying Lemma 4.1, $p_4 = 1$ since 28 is not a multiple of eight, and we consider maximizing the sequence $(p_5, p_6)$. Table B.6 presents a catalog of designs based on searching the top 20 designs and maximizing the sequence $(p_5, p_6)$.

The results in this case are even more encouraging than those for the 20- and 24-run cases. First there exists a $28 \times 27$ design with $p_5 = 1$ and the other designs allow for approximately 99.8 percent of all models containing 5 factors and the associated 2fi's to be estimated. In addition, for the best $28 \times 27$ design, approximately 97 percent of all models containing 6 factors and the associated 2fi's can be estimated. Also, for limited sacrifice in terms of the estimation of 5 factor models this can be improved to as high as 98.1 percent.

Based on the results from the 24-run searches we saw that simply taking the top twenty designs and sequentially maximizing the PEC may lead to designs with poor estimation of models containing 6 factors. Since we are only maximizing the sequence $(p_5, p_6)$ and both of these values are initially large we expect that the search above will perform reasonably well. However, we notice in Table B.6 that for a limited sacrifice in $p_5$ a larger value of $p_6$ can be found for 25 or more factors. In the case of 24 or fewer factors only the designs with $p_5 = 1$ are found. Table B.7 presents the catalog of designs found from searching the top twenty designs that maximized the sequence $(p_6, p_5)$. Comparing the two tables it is interesting to notice that the designs with less than 22 factors in Table B.7 are superior to those in Table B.6. In addition for designs with 23 or more factors, a very limited sacrifice on the estimation of models containing 5 factors leads to designs with larger values of $p_6$.

The above results suggest that the simple search procedure that was implemented will not perform as well when the PEC sequence has more than one non-zero value of $p$. Development of more sophisticated algorithms to search for designs will be left for future research.

## 4.4 Projection Estimation of Regular Designs

In this section we show how to find $p_3$ and $p_4$ for a regular resolution $III$ design.

**Lemma 4.4** *If $\mathcal{D}$ is a $2^{m-p}_{III}$ design with WLP $(a_3, \ldots, a_m)$ then*

$$p_3 = 1 - \frac{a_3}{\binom{m}{3}} \quad \text{and} \quad p_4 = 1 - \frac{(m-3)a_3 + a_4}{\binom{m}{4}}.$$

**Proof:**

If $\mathcal{D}$ is projected onto 3 factors then there are two possibilities that can arise: i) the 3 factors do not form a word of length 3 in the DCS; or ii) the 3 factors form a word in the DCS. In case i), the model containing the 3 main effects and their 2fi's will be estimable. In case ii), the model containing 3 main effects and 2fi's will not be estimable, since some of main effects will be aliased with 2fi's. In case ii), there are $a_3$ subsets of 3 factors that appear as words in the DCS, hence $\rho_3 = \binom{m}{3} - a_3$ and

$$p_3 = 1 - \frac{a_3}{\binom{m}{3}}.$$

If $\mathcal{D}$ is projected onto 4 factors there are four possibilities for the set of 4 factors: i) no set of 3 of the 4 factors form a word of length 3 in the DCS; ii) the 4 factors do not form a word of length 4 in the DCS; iii) one set of 3 of the 4 factors form a 3 letter word in the DCS, and iv) the 4 factors form a word in the DCS.

In cases i) and ii), the model containing 4 main effects and the 2fi's will be estimable.

In case iii) the model containing 4 main effects and their 2fi's will not be estimable since some of the main effects will be aliased with 2fi's. In order to count the number of non-estimable models for case iii), consider a given subset of 3 factors that form a word in the DCS and add a fourth column to form a projection onto four factors.

The addition of any of the remaining $m-3$ factors to the existing 3 factors will lead to a model which cannot be estimated. Therefore there is a total of $(m-3)a_3$ non-estimable models. In case iv), the model will not be estimable since 2fi's will be aliased with other 2fi's, and there are a total of $a_4$ non-estimable models. Note that if the three factors form a word of length three then the four factors cannot also form a word of length 4. Hence $\rho_4 = \binom{m}{4} - (m-3)a_3 + a_4$ and

$$p_4 = 1 - \frac{(m-3)a_3 + a_4}{\binom{m}{4}}.$$

□

When a resolution $III$ design is projected onto five factors, there are five possibilities: i) no set of 3 or 4 or all 5 of the factors form a word in the DCS; ii) the 5 factors form a word in the DCS with 5 or more factors; iii) at least one set of 3 of the 5 factors form a single 3 letter words in the DCS; iv) at least one set of 4 of the 5 factors form a word in the DCS, and v) the 5 factors form 2 word in the DCS each of length 3 and one word of length 4. In cases i) and ii), the models will be estimable, in cases iii), iv) and v), the models will not be estimable. If we try and use a similar argument to the proof of Lemma 4.4 to count the number of non-estimable models in cases iii), iv) and v), we discover that the number of length-three words and the number of length-four words will provid insufficient information. In case v), only the model containing the five factors will be non-estimable, and in case iii) there are $(m-4)(m-3)$ non-estimable models that will arise from a single word of length 3. Similarly, in case iv) their are $(m-4)$ non-estimable models that will arise from a single word of length 4. In this situation, knowledge on the number of three and four letter words and the aliasing structure will be required. A complete treatment of this problem is left for future research.

## 4.5 Discussion

In this chapter we introduced the PEC sequence and demonstrated how it can be used to find good 2-level orthogonal arrays. Two theorems were presented which showed properties of the PEC for any design (orthogonal or not). In addition, the definition of the PEC sequence does not require that the design be a 2-level orthogonal array. With suitable modification one could apply the PEC criterion to find good three-level designs, good mixed-level designs, or designs of higher-order. A modification of the PEC to rank RPD's will be discussed in the next chapter.

# Chapter 5

# Future Work

## 5.1 Projection Estimation Capacity for Robust Parameter Designs

In this section we consider how one might extend the notion of projection estimation capacity to robust parameter designs. Recall, for a factorial design the projection estimation capacity sequentially maximizes the number of estimable models containing $k$ main effects and their associated 2fi's. When the design is regular, the model containing $k$ main effects and the associated 2fi's will be estimable if the $k$ main effects and 2fi's can be estimated clear of other effects when 3 and higher-order interactions are assumed negligible. In the case of non-regular designs, the estimation of main effects and 2fi's will not be clear of other interactions since main effects are partially aliased with 2fi's. However, the partial aliasing can still result in a model that is estimable. In order to further distinguish between competing designs, the design with the largest PIC is selected. When the design is regular, $d_k = 1$ for all $k = 1, \ldots, m$ and in the case of non-regular designs $d_k \neq 1$. The non-regular design with the largest value of $d_k$ will be the design that most closely resembles a regular design.

In order to extend PEC to RPD's, one must decide on a class of important models. Although there are many possible choices for an important model, the selected model should allow for the estimation of control-by-noise interactions. Recall that the control-by-noise interactions will be used to determine how to adjust the value of the control factors so that the process is robust to random variations in the noise factors. Therefore, any ranking criterion proposed by the experimenter, should incorporate this importance of control-by-noise interactions.

One possible choice of important models is those that allow for the joint estimation of control main effects, noise main effects and control-by-noise interactions. Consider a $n \times (m_1 + m_2)$ RPD with $m_1$ control factors and $m_2$ noise factors, and let $\nu_{k_1,k_2}$ be the number of models that allow for the estimation of $k_1$ control main effects, $k_2$ noise main effects and thirr $k_1 \times k_2$ associted control-by-noise interactions, and

$$p_{k_1,k_2} = \frac{\nu_{k_1,k_2}}{\binom{m_1}{k_1}\binom{m_2}{k_2}},$$

the corresponding proportion. The difficulty with this definition is deciding on an appropriate ranking criterion. For example, consider a model containing 3 control factors and 2 noise factors and an alternative model containing 2 control factors and 3 noise factors. Both models have $1+2+3+6 = 12$ parameters but represent different model choices. Investigating model differences at this level would be difficult. When no prior information on the process is available, the experimenter is looking for the design that will provide the most amount of information on all models of interest. Thus one possibility is to consider

$$p_k^* = \frac{\sum_{\{(i_1,i_2)|i_1\geq 1, i_2\geq 1, i_1+i_2=k\}} \nu_{i_1,i_2}}{\sum_{\{(i_1,i_2)|i_1\geq 1, i_2\geq 1, i_1+i_2=k\}} \binom{m_1}{i_1}\binom{m_2}{i_2}},$$

where $\binom{m_1}{0} = 1$ and construct the sequence $(p_2^*, \ldots, p_{m_1+m_2}^*)$. That is, consider the proportion of all estimable models containing at least one control-by-noise interaction for a fixed number of main effects. In order to further distinguish between competing designs one could use the PIC as a secondary criterion.

Although the above sequence tries to capture the models of interest, it may not provide enough information in some settings. For example, in many RPD's the experimenter is not only interested in estimating control-by-noise interactions, but would like some information on the control-by-control interactions as well. In order to accommodate this concern, one could consider refining the search criterion by considering the proportion of projections that allow for the estimation of the main effects and their associated 2fi's provided that the model contains at least one control-by-noise interaction. That is let $\rho'_{k_1,k_2}$ be the number of models that allow for the estimation of $k_1$ control main effects, $k_2$ noise main effects, $k_1k_2$ control-by-noise interactions, $\binom{k_1}{2}$ control-by-control interactions and $\binom{k_2}{2}$ noise-by-noise interactions, and let

$$p'_k = \frac{\sum_{\{(i_1,i_2)|i_1\geq 1, i_2\geq 1, i_1+i_2=k\}} \rho'_{i_1,i_2}}{\sum_{\{(i_1,i_2)|i_1\geq 1, i_2\geq 1, i_1+i_2=k\}} \binom{m_1}{i_1}\binom{m_2}{i_2}}.$$

In order to select a good RPD, one would first maximize the values of $p^*_k$, then maximize the associated $d_k$ values and finally maximize the values of $p'_k$. That is, we wish to find the most efficient design that allows for the estimation of all models containing control-by-noise interactions, and then from this set of designs select the design that provides the most amount of information about the other effects of interest. As seen in Chapter 3, the assignment of control and noise factors to the columns of the design can lead to RPD's with different properties. The same will be true for the definitions and the ranking criterion proposed here. Thus a complete treatment of the subject would require searching the complete catalog of non-isomorphic designs and trying every possible relabelling of control and noise factors.

## 5.2 Larger-Scale Experiments and Non-orthogonal Arrays.

In this section, we briefly discuss a few results related to designing larger scale experiments and finding non-orthogonal arrays with good projection properties. In section 4.2, we discussed and implemented a search procedure for finding good designs with 24 and 28 runs. In order to select good designs, one must start from the class of all non-isomorphic arrays. However, for larger scale experiments and non-orthogonal arrays the class of all non-isomorphic arrays is not readily available. Thus, we consider augmenting an already existing array with a set of additional runs to find good designs for larger-scale experiments.

**Theorem 5.1** *Let $\mathcal{D}$ be an $n \times m$ array with PEC sequence $(p_1, \ldots, p_m)$. If a set of $k$ runs is added to $\mathcal{D}$, the resulting $(n + j) \times m$ array, $\mathcal{D}^*$, will have a PEC sequence $(p_1^*, \ldots, p_m^*)$ where $p_k^* \geq p_k$ for all $k = 1, \ldots, m$.*

**Proof:** Recall that if an $n \times m$ array $X$ has rank $r$ then any $(n + j) \times m$ array $X^*$ that contains the $n$ rows of $X$ will have a rank that is greater than or equal to $r$.

Consider design $\mathcal{D}$ and $\mathcal{D}^*$. Take any projection onto $k$ factors, and construct the array with columns: all +1's, the $k$ main effects and the associated $\binom{k}{2}$ possible 2fi's between the $k$ main effects, and call the resulting designs $X$ and $X^*$, where $X^*$ will always contain the runs in $X$. If the rank of $X$ is larger than $r = 1 + k + \binom{k}{2}$ the model will be estimable. Therefore all estimable models from $\mathcal{D}$ will also be estimable from $\mathcal{D}^*$, since the rank of $X^*$ for any given projection will be larger than the rank of $X$. If the rank of $X$ is less than $r$, the model will not be estimable, however the rank of $X^*$ may be larger than $r$, thus implying that the number of estimable models from $\mathcal{D}^*$ will be greater than or equal to the number of estimable models from $\mathcal{D}$. Hence, $p_k^* \geq p_k$ for $k = 1, \ldots, m$. □

The above theorem provides a justification for adding runs to an existing design to find designs with larger runs sizes. If the experimenter is not concerned with adding runs so that the resulting array is orthogonal, then any additional set of runs will allow more models to be estimated. For example, to find a 30-run two-level design with 24 factors, we could start with the best $28 \times 24$ run orthogonal array and add two additional runs. There are a total of $2^{24}$ possible runs that could be added and hence $\binom{2^{24}}{2}$ possible sets of size two. In order to reduce the computation of searching all possible sets of size two, one could randomly select any two possible runs and compute the PEC. Repeating this a large number of times and selecting the design with largest PEC will result in a "good" design.

Adding runs in an orthogonal manner is slightly more difficult. It may not be possible to add 4 runs to an existing orthogonal array so that resulting array is still orthogonal. However, if a set of orthogonal runs is added to the existing $n \times m$ design then the resulting $(n + 4r) \times m$ array will be orthogonal. One possible way to ensure the additional runs are orthogonal is to select the smallest $r$ such that $4r \geq m$.

This procedure will generally result in arrays with much larger run sizes, however the additional runs will always lead to an improvement of the PEC.

# Chapter 6

# Conclusion

FFD's are commonly used in industrial and agricultural experiments to identify factors affecting a process or response. In many applications, the experimenter wishes to test a large number of main effects and still have the ability to detect two factor interactions. In these cases, non-regular designs often allow for the estimation of more models and have a smaller run size than the competing regular FFD's. The aim of this thesis was to study non-regular designs and show how they can be used in many practical situations as an alternative to regular FFD's.

In review Chapter 2, FFD's were introduced as an alternative to full factorial designs. FFD's are employed in many situations since they have much smaller run sizes than the full factorial design. We demonstrated how the aliasing structure of a regular FFD can be used to rank designs via resolution and aberration. Next we considered non-regular FFD's and discussed how to rank these design. We saw that a non-regular design is not defined in terms of a set of defining relations and instead used the indicator function to characterize the properties of such designs. More specifically, we introduced generalized resolution, $G$-aberration and the relaxed version $G_2$-aberration as possible methods for ranking non-regular designs.

In Chapter 3, we considered a modification of the indicator function that allowed for the study of multi-factor non-regular designs. The indicator was then used as a tool to study RPD's. We showed how one could extend the notions of $G$- and $G_2$-aberration to rank non-regular FF RPD's. A theorem was presented that showed how one could obtain the set of all non-isomorphic FF RPD's by considering every possible relabelling of control and noise factors to the columns of the designs in the set of non-isomorphic FFD's and thus build upon previous work on FFD's. Finally, tables of generalized MA FF RPD's were found for 12- and 16-run orthogonal arrays.

In Chapter 4, the PEC sequence was introduced and used to rank non-regular designs. We considered how a simple search procedure could be implemented to find designs with larger run sizes. In particular, the search procedure was used to find a catalog of good designs for orthogonal arrays with 24 and 28 runs. In addition, we illustrated how one could find the percentage of estimable models for a regular design from the number of words of length three and four in the defining contrast subgroup.

Finally, in Chapter 5 we discussed two topics for future research. First, we focused on possible ways of extending PEC to rank RPD's. We saw that this extension will depended on the types of models the experimenter wishes to estimate. We proposed one extension, where we select the most efficient designs that allow for the estimation of models containing all main effects and 2fi's from the set of designs that provide maximal information on models containing control main effects, noise main effects and control-by-noise interactions. Secondly, we showed how the PEC sequence could be used to find designs with large run sizes where no catalog of designs is readily available via adding additional runs.

# Appendix A

# Non-Regular MA FF RPD's

## A.1 12-run MA FF RPD's

Table A.1: $G$- and $G_2$-Aberration* for 12-Run Designs Using Bingham-Sitter's Definition

| $s_1 + s_2$ | Columns** | RPD EWLP |
|---|---|---|
| $1+6$ | $(1)$ | $[(0,0,15),(0,0,0),(0,0,20),(0,0,20),(0,2,0),(0,0,15)]$ |
| $2+5$ | $(1,2)$ | $[(0,0,25),(0,0,10),(0,0,20),(0,1,10),(0,2,0),(0,1,5)]$ |
| $3+4$ | $(1,2,3)$ | $[(0,0,30),(0,0,19),(0,0,16),(0,2,4),(0,1,0),(0,1,1)]$ |
| $4+3$ | $(4,5,7)$ | $[(0,0,30),(0,0,22),(0,0,16),(0,2,2),(0,1,0),(0,1,0)]$ |
| $5+2$ | $(5,7)$ | $[(0,0,25),(0,0,20),(0,0,20),(0,1,5),(0,2,0),(0,1,0)]$ |
| $6+1$ | $(7)$ | $[(0,0,15),(0,0,20),(0,0,20),(0,0,15),(0,2,0),(0,1,0)]$ |
| $1+7$ | $(1)$ | $[(0,0,21),(0,0,0),(0,0,35),(0,0,35),(0,5,0),(0,0,35)]$ |
| $2+6$ | $(1,7)$ | $[(0,0,36),(0,0,15),(0,0,40),(0,2,20),(0,6,0),(0,3,15)]$ |
| $3+5$ | $(1,2,3)$ | $[(0,0,45),(0,0,31),(0,0,35),(0,5,10),(0,3,0),(0,4,5)]$ |
| $4+4$ | $(1,2,3,6)$ | $[(0,0,48),(0,0,40),(0,0,32),(0,6,5),(0,2,0),(0,4,1)]$ |
| $5+3$ | $(5,7,8)$ | $[(0,0,45),(0,0,40),(0,0,35),(0,5,6),(0,3,0),(0,4,0)]$ |
| $6+2$ | $(6,8)$ | $[(0,0,36),(0,0,35),(0,0,40),(0,2,15),(0,6,0),(0,3,0)]$ |
| $7+1$ | $(8)$ | $[(0,0,21),(0,0,35),(0,0,35),(0,0,35),(0,5,0),(0,3,0)]$ |

* $G-$ and $G_2$-aberration designs are the same.

**If $s_1 \leq s_2$ ($s_1 > s_2$), the $s_1$ control ($s_2$ noise) factor columns are given.

(Table A.1 Continued)

| $s_1 + s_2$ | Columns** | RPD EWLP |
|---|---|---|
| $1+8$ | (1) | $[(0,0,28),(0,0,0),(0,0,56),(0,0,56),(0,10,0),(0,0,70)]$ |
| $2+7$ | (1,2) | $[(0,0,49),(0,0,21),(0,0,70),(0,5,35),(0,10,0),(0,5,35)]$ |
| $3+6$ | (1,2,8) | $[(0,0,63),(0,0,46),(0,0,66),(0,9,20),(0,9,0),(0,11,15)]$ |
| $4+5$ | (1,2,3,6) | $[(0,0,70),(0,0,64),(0,0,60),(0,14,11),(0,4,0),(0,11,5)]$ |
| $5+4$ | (6,7,8,9) | $[(0,0,70),(0,0,70),(0,0,60),(0,14,9),(0,4,0),(0,11,1)]$ |
| $6+3$ | (6,8,9) | $[(0,0,63),(0,0,65),(0,0,66),(0,9,16),(0,9,0),(0,11,0)]$ |
| $7+2$ | (8,9) | $[(0,0,49),(0,0,56),(0,0,70),(0,5,35),(0,10,0),(0,8,0)]$ |
| $8+1$ | (9) | $[(0,0,28),(0,0,56),(0,0,56),(0,0,70),(0,10,0),(0,8,0)]$ |
| $1+9$ | (1) | $[(0,0,36),(0,0,0),(0,0,84),(0,0,84),(0,18,0),(0,0,126)]$ |
| $2+8$ | (1,2) | $[(0,0,64),(0,0,28),(0,0,112),(0,8,56),(0,20,0),(0,10,70)]$ |
| $3+7$ | (1,2,3) | $[(0,0,84),(0,0,64),(0,0,112),(0,18,35),(0,15,0),(0,20,35)]$ |
| $4+6$ | (1,2,3,9,) | $[(0,0,96),(0,0,94),(0,0,104),(0,24,21),(0,12,0),(0,29,15)]$ |
| $5+5$ | (1,2,3,4,5) | $[(0,0,100),(0,0,110),(0,0,100),(0,28,15),(0,8,0),(0,28,5)]$ |
| $6+4$ | (6,8,9,10) | $[(0,0,96),(0,0,110),(0,0,104),(0,24,19),(0,12,0),(0,29,1)]$ |
| $7+3$ | (8,9,10) | $[(0,0,84),(0,0,98),(0,0,112),(0,18,36),(0,15,0),(0,23,0)]$ |
| $8+2$ | (9,10) | $[(0,0,64),(0,0,84),(0,0,112),(0,8,70),(0,20,0),(0,18,0)]$ |
| $9+1$ | (10) | $[(0,0,36),(0,0,84),(0,0,84),(0,0,126),(0,18,0),(0,18,0)]$ |
| $1+10$ | (1) | $[(0,0,45),(0,0,0),(0,0,120),(0,0,120),(0,30,0),(0,0,210)]$ |
| $2+9$ | (1,2) | $[(0,0,81),(0,0,36),(0,0,168),(0,12,84),(0,36,0),(0,18,126)]$ |
| $3+8$ | (1,2,3) | $[(0,0,108),(0,0,85),(0,0,176),(0,28,56),(0,30,0),(0,38,70)]$ |
| $4+7$ | (1,2,3,4,5) | $[(0,0,126),(0,0,130),(0,0,168),(0,42,36),(0,21,0),(0,53,35)]$ |
| $5+6$ | (1,2,3,4,5,6) | $[(0,0,135),(0,0,160),(0,0,160),(0,50,25),(0,15,0),(0,60,15)]$ |
| $6+5$ | (7,8,9,10,11) | $[(0,0,135),(0,0,170),(0,0,160),(0,50,25),(0,15,0),(0,61,5)]$ |
| $7+4$ | (8,9,10,11) | $[(0,0,126),(0,0,161),(0,0,168),(0,42,39),(0,21,0),(0,56,1)]$ |
| $8+3$ | (9,10,11) | $[(0,0,108),(0,0,140),(0,0,176),(0,28,71),(0,30,0),(0,46,0)]$ |
| $9+2$ | (10,11) | $[(0,0,81),(0,0,120),(0,0,168),(0,12,126),(0,36,0),(0,36,0)]$ |
| $10+1$ | (11) | $[(0,0,45),(0,0,120),(0,0,120),(0,0,210),(0,30,0),(0,36,0)]$ |

* $G-$ and $G_2$-aberration designs are the same.

**If $s_1 \leq s_2$ ($s_1 > s_2$), the $s_1$ control ($s_2$ noise) factor columns are given.

Table A.2: $G$- and $G_2$-Aberration* for 12-Run Designs, Zhu's Defintion

| $s_1 + s_2$ | Columns** | RPD EWLP |
|---|---|---|
| $4+3$ | $(1,2,3,7)$ | $[(0,0,18),(0,0,30),(0,0,16),(0,2,5),(0,2,2),(0,0,2)]$ |
| $3+5$ | $(1,2,4)$ | $[(0,0,15),(0,0,60),(0,1,6),(0,7,40),(0,0,0),(0,3,10)]$ |
| $4+4$ | $(1,2,3,5)$ | $[(0,0,24),(0,0,60),(0,2,20),(0,7,20),(0,2,5),(0,1,6)]$ |
| $3+6$ | $(1,2,3)$ | $[(0,0,18),(0,0,90),(0,2,7),(0,12,80),(0,0,0),(0,12,30)]$ |
| $4+5$ | $(1,2,3,5)$ | $[(0,0,30),(0,0,100),(0,4,24),(0,16,50),(0,3,11),(0,6,30)]$ |
| $5+4$ | $(1,2,3,4,6)$ | $[(0,0,40),(0,0,90),(0,6,50),(0,12,24),(0,10,25),(0,2,16)]$ |
| $4+6$ | $(1,2,3,4)$ | $[(0,0,36),(0,0,150),(0,8,28),(0,30,100),(0,3,21),(0,20,90)]$ |
| $5+5$ | $(1,2,3,4,6)$ | $[(0,0,50),(0,0,150),(0,10,60),(0,30,60),(0,15,55),(0,10,80)]$ |

* $G-$ and $G_2$-aberration designs are the same.
**If $s_1 \leq s_2$ ($s_1 > s_2$), the $s_1$ control ($s_2$ noise) factor columns are given.

## A.2 16 run MA FF RPD's

Table A.3: $G$-Aberration for 16-Run Designs Using Bingham and Sitter's Definition

| $s_1 + s_2$ | Columns* | RPD EWLP |
|---|---|---|
| $1+4$ | $I(8,1,2,4,7)$ | $[(0,0),(0,0),(0,0),(0,0),(0,0),(1,0)]$ |
| $2+3$ | $I(4,8,1,2,3)$ | $[(0,0),(0,0),(0,0),(1,0)]$ |
| $3+2$ | $I(1,2,4,8,15)$ | $[(0,0),(0,0),(0,0),(1,0)]$ |
| $4+1$ | $I(1,2,4,8,15)$ | $[(0,0),(0,0),(0,0),(0,0),(1,0)]$ |
| $1+5$ | $I(8,1,2,3,4,5)$ | $[(0,0),(0,0),(0,0),(2,0),(0,0),(1,0)]$ |
| $2+4$ | $I(4,8,1,2,3,13)$ | $[(0,0),(1,0),(0,0),(2,0),(0,0),(0,0)]$ |
| $3+3$ | $I(4,8,13,1,2,3)$ | $[(0,0),(0,0),(1,0),(2,0),(0,0),(0,0)]$ |
| $4+2$ | $I(1,2,4,8,7,11)$ | $[(0,0),(1,0),(2,0),(0,0),(0,0),(0,0)]$ |
| $5+1$ | $I(1,2,4,7,8,11)$ | $[(0,0),(0,0),(2,0),(1,0),(0,0),(0,0)]$ |

*The roman numeral indicates which of Hall's designs. The first $s_1$ listed columns are assigned to control factors and the remaining $s_2$ columns to noise factors.

(Table A.3 continued)

| $s_1+s_2$ | Columns* | RPD EWLP |
|---|---|---|
| $1+6$ | $I(8,1,2,3,4,5,6)$ | $[(0,0),(0,0),(0,0),(4,0),(0,0),(3,0)]$ |
| $2+5$ | $I(8,14,1,2,3,4,5)$ | $[(0,0),(2,0),(0,0),(4,0),(0,0),(1,0)]$ |
| $3+4$ | $I(1,2,4,7,8,11,13)$ | $[(0,0),(3,0),(4,0),(0,0),(0,0),(0,0)]$ |
| $4+3$ | $I(1,2,4,8,7,11,13)$ | $[(0,0),(3,0),(4,0),(0,0),(0,0),(0,0)]$ |
| $5+2$ | $I(1,2,4,7,8,11,13)$ | $[(0,0),(2,0),(4,0),(1,0),(0,0),(0,0)]$ |
| $6+1$ | $I(1,2,4,7,8,11,13)$ | $[(0,0),(0,0),(4,0),(3,0),(0,0),(0,0)]$ |
| $1+7$ | $I(8,1,2,3,4,5,6,7)$ | $[(0,0),(0,0),(0,0),(7,0),(0,0),(7,0)]$ |
| $2+6$ | $III(1,10,2,4,7,8,12,14)$ | $[(0,0),(0,0),(1,12),(1,12),(0,12),(0,12)]$ |
| $3+5$ | $I(1,2,4,7,8,11,13,14)$ | $[(0,0),(6,0),(7,0),(0,0),(0,0),(1,0)]$ |
| $4+4$ | $I(1,2,4,8,7,11,13,14)$ | $[(0,0),(6,0),(8,0),(0,0),(0,0),(0,0)]$ |
| $5+3$ | $I(1,2,4,7,8,11,13,14)$ | $[(0,0),(6,0),(7,0),(1,0),(0,0),(0,0)]$ |
| $6+2$ | $III(2,4,7,8,12,14,1,10)$ | $[(0,0),(0,6),(1,12),(1,12),(0,12),(0,6)]$ |
| $7+1$ | $I(1,2,4,7,8,11,13,14)$ | $[(0,0),(0,0),(7,0),(7,0),(0,0),(0,0)]$ |
| $1+8$ | $II(8,1,2,3,4,5,6,7,12)$ | $[(0,4),(0,0),(0,12),(7,0),(0,16),(7,0)]$ |
| $2+7$ | $II(8,12,1,2,3,4,5,6,7)$ | $[(0,4),(0,12),(0,0),(7,16),(0,0),(7,16)]$ |
| $3+6$ | $II(8,11,12,1,2,4,5,6,7)$ | $[(0,8),(3,16),(0,0),(8,16),(0,0),(3,16)]$ |
| $4+5$ | $II(8,11,12,15,1,2,4,5,6)$ | $[(0,12),(4,24),(0,0),(7,16),(0,0),(1,8)]$ |
| $5+4$ | $II(4,5,8,10,12,6,7,9,11)$ | $[(0,12),(6,4),(8,0),(0,24),(0,8),(0,0)]$ |
| $6+3$ | $II(1,2,4,5,6,7,8,11,12)$ | $[(0,8),(7,16),(0,0),(7,16),(0,0),(0,16)]$ |
| $7+2$ | $II(1,2,3,4,5,6,7,8,12)$ | $[(0,4),(7,12),(0,0),(7,16),(0,0),(0,16)]$ |
| $8+1$ | $II(4,5,6,7,8,9,10,12,11)$ | $[(0,4),(0,12),(7,0),(7,0),(0,16),(0,16)]$ |
| $1+9$ | $III(10,1,2,3,4,5,8,9,12,13)$ | $[(0,8),(0,0),(0,24),(4,16),(0,32),(6,16)]$ |
| $2+8$ | $III(8,10,1,2,3,4,5,6,7,12)$ | $[(0,12),(0,12),(0,24),(7,16),(0,32),(7,16)]$ |
| $3+7$ | $III(8,10,12,1,2,3,4,5,6,7)$ | $[(0,12),(0,36),(0,0),(7,48),(0,0),(7,48)]$ |
| $4+6$ | $II(8,11,12,15,1,2,4,5,6,7)$ | $[(0,16),(6,32),(0,0),(13,32),(0,0),(3,32)]$ |
| $5+5$ | $IV(6,8,10,12,14,1,2,3,4,5)$ | $[(0,24),(0,36),(0,28),(2,58),(0,2),(1,44)]$ |
| $6+4$ | $II(1,2,4,5,6,7,8,11,12,15)$ | $[(0,16),(10,32),(0,0),(11,32),(0,0),(1,32)]$ |
| $7+3$ | $III(1,2,3,4,5,6,7,8,10,12)$ | $[(0,12),(7,36),(0,0),(7,48),(0,0),(0,48)]$ |
| $8+2$ | $II(1,2,4,5,6,7,8,12,11,15)$ | $[(0,12),(5,12),(6,16),(3,16),(8,16),(0,16)]$ |
| $9+1$ | $III(2,3,4,5,8,9,10,12,13,11)$ | $[(0,8),(0,24),(4,16),(6,16),(0,32),(0,32)]$ |
| $1+10$ | $V(4,1,2,3,8,9,10,11,12,13,14)$ | $[(0,12),(0,0),(0,36),(1,36),(0,52),(7,36)]$ |
| $2+9$ | $IV(10,12,1,2,3,4,5,6,7,8,9)$ | $[(0,20),(0,12),(0,48),(4,36),(0,64),(6,44)]$ |
| $3+8$ | $V(8,9,10,1,2,3,4,5,6,7,12)$ | $[(0,24),(0,36),(0,36),(7,52),(0,48),(7,60)]$ |
| $4+7$ | $V(8,9,10,12,1,2,3,4,5,6,7)$ | $[(0,24),(0,72),(0,0),(7,96),(0,4),(7,108)]$ |
| $5+6$ | $III(8,9,10,11,12,2,3,4,5,6,7)$ | $[(0,32),(6,64),(0,0),(13,64),(0,16),(3,96)]$ |
| $6+5$ | $III(2,3,4,5,6,7,8,9,10,11,12)$ | $[(0,32),(10,64),(0,0),(11,64),(0,16),(1,96)]$ |
| $7+4$ | $III(1,2,3,4,5,6,7,8,10,12,14)$ | $[(0,24),(7,72),(0,0),(7,96),(0,0),(1,96)]$ |
| $8+3$ | $V(1,2,3,4,5,6,7,8,9,10,12)$ | $[(0,24),(7,36),(0,36),(7,52),(0,48),(0,60)]$ |
| $9+2$ | $III(1,2,4,8,9,10,11,12,13,7,14)$ | $[(0,20),(3,28),(5,36),(3,40),(7,44),(0,48)]$ |
| $10+1$ | $V(1,2,4,8,9,10,11,12,13,14,7)$ | $[(0,12),(0,36),(1,36),(7,36),(0,52),(0,60)]$ |
| $1+11$ | $V(4,1,2,3,8,9,10,11,12,13,14,15)$ | $[(0,16),(0,0),(0,48),(1,48),(0,80),(14,48)]$ |
| $2+10$ | $V(8,12,1,2,3,4,5,6,7,9,10,11)$ | $[(0,28),(0,12),(1,72),(7,40),(0,100),(7,88)]$ |
| $3+9$ | $IV(10,12,14,1,2,3,4,5,6,7,8,9)$ | $[(0,36),(0,36),(0,76),(4,80),(0,96),(6,112)]$ |
| $4+8$ | $V(8,9,10,11,1,2,3,4,5,6,7,12)$ | $[(0,40),(0,72),(0,48),(8,112),(0,64),(7,144)]$ |
| $5+7$ | $V(8,9,10,11,12,1,2,3,4,5,6,7)$ | $[(0,40),(0,120),(0,0),(8,160),(0,16),(7,208)]$ |
| $6+6$ | $III(2,3,8,10,12,15,4,5,9,11,13,14)$ | $[(0,48),(7,40),(16,32),(0,152),(0,48),(14,128)]$ |

(Table A.3 continued)

| $s_1 + s_2$ | Columns* | RPD EWLP |
|---|---|---|
| $7+5$ | $V(1,2,3,4,5,6,7,8,9,10,11,12)$ | $[(0,40),(7,120),(0,0),(7,160),(0,16),(1,208)]$ |
| $8+4$ | $V(1,2,3,4,5,6,7,12,8,9,10,11)$ | $[(0,40),(7,72),(0,48),(7,112),(0,64),(1,144)]$ |
| $9+3$ | $IV(2,3,4,5,6,8,9,10,12,7,11,13)$ | $[(0,36),(3,51),(9,48),(3,91),(0,99),(0,123)]$ |
| $10+2$ | $V(1,2,4,8,9,10,11,12,13,14,7,15)$ | $[(0,28),(0,48),(8,48),(7,64),(0,104),(0,108)]$ |
| $11+1$ | $V(1,2,4,8,9,10,11,12,13,14,15,7)$ | $[(0,16),(0,48),(1,48),(14,48),(0,80),(0,112)]$ |
| $1+12$ | $IV(12,1,2,3,4,5,6,7,8,9,10,11,14)$ | $[(0,20),(0,0),(0,68),(5,48),(0,112),(10,112)]$ |
| $2+11$ | $IV(12,14,1,2,3,4,5,6,7,8,9,10,11)$ | $[(0,36),(0,20),(0,96),(5,64),(0,160),(10,128)]$ |
| $3+10$ | $V(8,9,10,1,2,3,4,5,6,7,11,12,13)$ | $[(0,48),(1,36),(2,108),(7,100),(0,152),(7,204)]$ |
| $4+9$ | $V(8,9,10,12,1,2,3,4,5,6,7,11,13)$ | $[(0,56),(1,72),(2,96),(7,144),(0,132),(7,252)]$ |
| $5+8$ | $V(8,9,10,11,12,1,2,3,4,5,6,7,13)$ | $[(0,60),(0,120),(2,60),(8,192),(0,96),(7,304)]$ |
| $6+7$ | $V(8,9,10,11,12,13,1,2,3,4,5,6,7)$ | $[(0,60),(0,180),(0,0),(10,240),(0,48),(7,384)]$ |
| $7+6$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13)$ | $[(0,60),(7,180),(0,0),(7,240),(0,48),(3,384)]$ |
| $8+5$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13)$ | $[(0,60),(7,120),(2,60),(7,192),(0,96),(1,304)]$ |
| $9+4$ | $V(1,2,3,4,5,6,7,8,10,9,11,12,13)$ | $[(0,56),(8,76),(2,96),(7,152),(0,132),(0,256)]$ |
| $10+3$ | $IV(2,3,4,5,6,8,9,10,12,14,7,11,13)$ | $[(0,48),(3,72),(9,76),(3,142),(0,147),(0,213)]$ |
| $11+2$ | $IV(2,3,4,5,6,7,8,9,10,12,14,11,13)$ | $[(0,36),(1,64),(8,72),(6,112),(0,152),(0,176)]$ |
| $12+1$ | $IV(2,3,4,5,6,7,8,9,10,11,12,14,13)$ | $[(0,20),(0,68),(5,48),(10,112),(0,112),(0,176)]$ |
| $1+13$ | $IV(14,1,2,3,4,5,6,7,8,9,10,11,12,13)$ | $[(0,24),(0,0),(0,88),(6,64),(0,160),(15,160)]$ |
| $2+12$ | $IV(2,4,3,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,44),(1,16),(10,96),(0,116),(0,224),(10,224)]$ |
| $3+11$ | $V(8,9,10,1,2,3,4,5,6,7,11,12,13,14)$ | $[(0,60),(3,36),(4,144),(7,144),(0,228),(7,336)]$ |
| $4+10$ | $V(8,9,10,12,1,2,3,4,5,6,7,11,13,14)$ | $[(0,72),(3,72),(4,144),(7,204),(0,208),(7,432)]$ |
| $5+9$ | $V(8,9,10,11,12,1,2,3,4,5,6,7,13,14)$ | $[(0,80),(2,120),(4,120),(8,260),(0,176),(7,508)]$ |
| $6+8$ | $V(8,9,10,11,12,13,1,2,3,4,5,6,7,14)$ | $[(0,84),(0,180),(4,72),(10,304),(0,144),(7,576)]$ |
| $7+7$ | $V(8,9,10,11,12,13,14,1,2,3,4,5,6,7)$ | $[(0,84),(0,252),(0,0),(14,336),(0,112),(7,672)]$ |
| $8+6$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14)$ | $[(0,84),(7,180),(4,72),(7,304),(0,144),(3,576)]$ |
| $9+5$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14)$ | $[(0,80),(9,124),(4,120),(7,268),(0,176),(1,512)]$ |
| $10+4$ | $IV(2,3,4,5,6,8,10,11,12,14,7,9,13,15)$ | $[(0,72),(6,98),(12,112),(3,258),(0,190),(0,450)]$ |
| $11+3$ | $IV(2,3,4,5,6,7,8,9,10,12,14,11,13,15)$ | $[(0,60),(3,88),(12,112),(6,196),(0,228),(0,364)]$ |
| $12+2$ | $IV(2,3,4,5,6,7,8,9,10,11,12,14,13,15)$ | $[(0,44),(1,84),(10,96),(10,160),(0,224),(0,288)]$ |
| $13+1$ | $IV(2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,24),(0,88),(6,64),(15,160),(0,160),(0,288)]$ |
| $1+14$ | $V(8,1,2,3,4,5,6,7,9,10,11,12,13,14,15)$ | $[(0,28),(0,0),(7,84),(7,84),(0,224),(14,252)]$ |
| $2+13$ | $V(8,9,1,2,3,4,5,6,7,10,11,12,13,14,15)$ | $[(0,52),(3,12),(8,144),(7,124),(0,320),(10,340)]$ |
| $3+12$ | $V(8,9,10,1,2,3,4,5,6,7,11,12,13,14,15)$ | $[(0,72),(6,36),(7,180),(7,200),(0,336),(8,504)]$ |
| $4+11$ | $V(8,9,10,12,1,2,3,4,5,6,7,11,13,14,15)$ | $[(0,88),(6,72),(8,192),(7,288),(0,308),(7,684)]$ |
| $5+10$ | $V(8,9,10,11,12,1,2,3,4,5,6,7,13,14,15)$ | $[(0,100),(6,120),(7,180),(8,364),(0,272),(7,820)]$ |
| $6+9$ | $V(8,9,10,11,12,13,1,2,3,4,5,6,7,14,15)$ | $[(0,108),(3,180),(8,144),(10,420),(0,240),(7,924)]$ |
| $7+8$ | $V(8,9,10,11,12,13,14,1,2,3,4,5,6,7,15)$ | $[(0,112),(0,252),(7,84),(14,448),(0,224),(7,1008)]$ |
| $8+7$ | $V(8,9,10,11,12,13,14,15,1,2,3,4,5,6,7)$ | $[(0,112),(0,336),(0,0),(21,448),(0,224),(7,1120)]$ |
| $9+6$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,108),(10,184),(8,144),(7,428),(0,240),(3,928)]$ |
| $10+5$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,100),(13,132),(7,180),(7,388),(0,272),(1,832)]$ |
| $11+4$ | $V(1,2,3,4,5,6,7,8,9,10,12,11,13,14,15)$ | $[(0,88),(13,96),(8,192),(7,336),(0,308),(0,712)]$ |
| $12+3$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,72),(13,76),(7,180),(8,280),(0,336),(0,560)]$ |
| $13+2$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,52),(10,72),(8,144),(10,244),(0,320),(0,448)]$ |
| $14+1$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,28),(7,84),(7,84),(14,252),(0,224),(0,448)]$ |

Table A.4: $G$-Aberration for 16 Run Designs, Zhu's Definition

| $s_1 + s_2$ | Columns* | RPD EWLP |
|---|---|---|
| $1+4$ | $I(8,1,2,4,7)$ | $[(0,0),(0,0),(0,0),(0,0),(0,0),(1,0)]$ |
| $2+3$ | $II(1,2,4,8,12)$ | $[(0,0),(0,0),(0,0),(0,4)]$ |
| $3+2$ | $II(4,8,12,1,2)$ | $[(0,0),(0,0),(0,4)]$ |
| $4+1$ | $I(1,2,4,7,8)$ | $[(0,0),(0,0),(0,0),(0,0),(1,0)]$ |
| $1+5$ | $III(1,2,4,8,10,12)$ | $[(0,0),(0,0),(0,0),(0,8),(0,0),(0,4)]$ |
| $2+4$ | $III(1,2,4,8,10,12)$ | $[(0,0),(0,4),(0,0),(0,8),(0,0),(0,0)]$ |
| $3+3$ | $I(1,2,3,4,8,12)$ | $[(0,0),(0,0),(1,0),(2,0))]$ |
| $4+2$ | $II(4,5,6,7,8,12)$ | $[(0,0),(0,4),(0,4),(0,0),(1,0))]$ |
| $5+1$ | $III(2,4,8,10,12,1)$ | $[(0,0),(0,0),(0,8),(0,0),(0,4),(0,0)]$ |
| $1+6$ | $IV(1,2,4,6,8,10,12)$ | $[(0,0),(0,0),(0,0),(0,16),(0,0),(0,12)]$ |
| $2+5$ | $IV(1,2,4,6,8,10,12)$ | $[(0,0),(0,8),(0,0),(0,16),(0,0),(0,4)]$ |
| $3+4$ | $III(1,2,3,4,8,10,12)$ | $[(0,0),(0,8),(1,0),(0,16),(0,0),(0,0)]$ |
| $4+3$ | $III(2,3,4,5,8,10,12)$ | $[(0,0),(0,16),(0,8),(0,0),(1,0))]$ |
| $5+2$ | $II(1,4,5,6,7,8,12)$ | $[(0,0),(0,8),(2,4),(0,0),(1,4),(0,0)]$ |
| $6+1$ | $IV(2,4,6,8,10,12,1)$ | $[(0,0),(0,0),(0,16),(0,0),(0,12),(0,0)]$ |
| $1+7$ | $III(10,1,2,4,7,8,12,14)$ | $[(0,0),(0,0),(0,0),(0,12),(0,0),(2,24)]$ |
| $2+6$ | $III(1,10,2,4,7,8,12,14)$ | $[(0,0),(0,0),(0,0),(2,24),(0,0),(0,24)]$ |
| $3+5$ | $III(1,2,4,7,8,10,12,14)$ | $[(0,0),(0,10),(2,2),(0,24),(0,0),(0,12)]$ |
| $4+4$ | $III(1,2,4,10,7,8,12,14)$ | $[(0,0),(0,14),(1,8),(0,24),(1,2),(0,0)]$ |
| $5+3$ | $III(1,2,4,7,10,8,12,14)$ | $[(0,0),(0,24),(0,18),(0,0),(1,6),(0,0)]$ |
| $6+2$ | $III(2,4,7,8,12,14,1,10)$ | $[(0,0),(0,0),(2,24),(0,0),(0,24),(0,0)]$ |
| $7+1$ | $III(1,2,4,7,8,12,14,10)$ | $[(0,0),(0,0),(0,12),(0,0),(2,24),(0,0)]$ |
| $1+8$ | $V(4,1,2,3,8,9,10,12,14)$ | $[(0,0),(0,4),(0,0),(1,36),(0,0),(1,40)]$ |
| $2+7$ | $V(2,4,1,8,9,10,11,12,14)$ | $[(0,0),(0,24),(0,0),(0,36),(0,0),(3,24)]$ |
| $3+6$ | $IV(1,2,3,4,6,8,10,12,14)$ | $[(0,0),(0,24),(1,0),(0,64),(0,0),(0,24)]$ |
| $4+5$ | $V(1,2,4,7,8,9,10,11,12)$ | $[(0,0),(0,56),(0,24),(0,0),(1,0),(2,32)]$ |
| $5+4$ | $III(1,2,3,4,5,8,10,12,15)$ | $[(0,0),(0,48),(2,32),(0,0),(1,16),(3,0)]$ |
| $6+3$ | $III(2,3,4,5,6,7,8,10,12)$ | $[(0,0),(0,36),(4,24),(0,0),(3,24),(0,0)]$ |
| $7+2$ | $II(1,2,3,4,5,6,7,8,12)$ | $[(0,0),(0,16),(7,16),(0,0),(7,16),(0,0)]$ |
| $8+1$ | $V(1,2,3,8,9,10,12,14,4)$ | $[(0,4),(0,0),(1,36),(0,0),(1,40),(0,0)]$ |
| $1+9$ | $V(4,1,2,3,8,9,10,11,12,14)$ | $[(0,0),(0,8),(0,0),(1,52),(0,0),(3,60)]$ |
| $2+8$ | $V(1,2,4,8,9,10,11,12,13,14)$ | $[(0,0),(0,36),(0,0),(0,48),(0,0),(7,48)]$ |
| $3+7$ | $V(1,2,3,4,8,9,10,12,13,15)$ | $[(0,0),(0,48),(1,0),(0,72),(0,0),(6,48)]$ |
| $4+6$ | $V(1,2,4,7,8,9,10,11,12,13)$ | $[(0,0),(0,84),(0,36),(0,0),(1,0),(6,96)]$ |
| $5+5$ | $V(1,2,3,4,5,8,9,10,12,15)$ | $[(0,0),(0,84),(2,48),(0,0),(1,24),(4,60)]$ |
| $6+4$ | $III(2,3,4,5,6,7,8,10,12,15)$ | $[(0,0),(0,72),(4,48),(0,0),(3,48),(7,0)]$ |
| $7+3$ | $III(1,2,3,4,5,6,7,8,10,12)$ | $[(0,0),(0,48),(7,48),(0,0),(7,48),(0,0)]$ |
| $8+2$ | $III(1,2,3,4,5,6,7,8,10,12)$ | $[(0,8),(0,16),(7,40),(0,0),(7,48),(0,0)]$ |
| $9+1$ | $V(1,2,3,8,9,10,11,12,14,4)$ | $[(0,8),(0,0),(1,52),(0,0),(3,60),(0,0)]$ |

(Table A.4 continued)

| $s_1+s_2$ | Columns* | RPD EWLP |
|---|---|---|
| $1+10$ | $V(4,1,2,3,8,9,10,11,12,13,14)$ | $[(0,0),(0,12),(0,0),(1,72),(0,0),(7,88)]$ |
| $2+9$ | $V(1,2,4,8,9,10,11,12,13,14,15)$ | $[(0,0),(0,48),(0,0),(0,64),(0,0),(14,96)]$ |
| $3+8$ | $V(1,2,3,4,8,9,10,11,12,13,14)$ | $[(0,0),(0,72),(1,0),(0,96),(0,0),(14,96)]$ |
| $4+7$ | $V(1,2,4,7,8,9,10,11,12,13,14)$ | $[(0,0),(0,120),(0,48),(0,0),(1,0),(14,224)]$ |
| $5+6$ | $V(1,2,3,4,5,8,9,10,11,12,15)$ | $[(0,0),(0,128),(2,68),(0,0),(1,36),(12,184)]$ |
| $6+5$ | $V(1,2,3,4,5,6,8,9,10,13,14)$ | $[(0,0),(0,120),(4,88),(0,0),(3,72),(8,112)]$ |
| $7+4$ | $V(1,2,3,4,5,6,7,8,9,10,12)$ | $[(0,0),(0,96),(7,96),(0,0),(7,96),(0,48)]$ |
| $8+3$ | $V(1,2,3,4,5,6,7,8,9,10,12)$ | $[(0,12),(0,48),(7,84),(0,16),(7,112),(0,0)]$ |
| $9+2$ | $IV(1,2,3,4,5,6,7,8,9,10,12)$ | $[(0,16),(0,16),(4,84),(0,0),(6,108),(0,0)]$ |
| $10+1$ | $V(1,2,3,8,9,10,11,12,13,14,4)$ | $[(0,12),(0,0),(1,72),(0,0),(7,88),(0,0)]$ |
| $1+11$ | $V(4,1,2,3,8,9,10,11,12,13,14,15)$ | $[(0,0),(0,16),(0,0),(1,96),(0,0),(14,128)]$ |
| $2+10$ | $V(1,4,2,3,8,9,10,11,12,13,14,15)$ | $[(0,0),(1,48),(0,0),(0,128),(0,0),(14,144)]$ |
| $3+9$ | $V(1,2,3,4,8,9,10,11,12,13,14,15)$ | $[(0,0),(0,96),(1,0),(0,128),(0,0),(28,192)]$ |
| $4+8$ | $V(1,2,4,7,8,9,10,11,12,13,14,15)$ | $[(0,0),(0,160),(0,64),(0,0),(1,0),(28,448)]$ |
| $5+7$ | $V(1,2,3,4,5,8,9,10,11,12,13,14)$ | $[(0,0),(0,180),(2,96),(0,0),(1,48),(28,432)]$ |
| $6+6$ | $V(1,2,3,4,5,6,8,9,10,11,12,15)$ | $[(0,0),(0,180),(4,136),(0,0),(3,104),(24,336)]$ |
| $7+5$ | $V(1,2,3,4,5,6,7,8,9,10,11,12)$ | $[(0,0),(0,160),(7,160),(0,0),(7,160),(15,192)]$ |
| $8+4$ | $V(1,2,3,4,5,6,7,12,8,9,10,11)$ | $[(0,16),(0,96),(7,144),(0,64),(7,224),(15,0)]$ |
| $9+3$ | $IV(1,2,3,4,5,6,7,8,9,10,12,14)$ | $[(0,24),(0,48),(4,148),(0,20),(14,208),(0,0)]$ |
| $10+2$ | $V(1,2,3,4,5,6,7,8,9,10,11,12)$ | $[(0,24),(0,16),(8,112),(0,0),(7,188),(0,0)]$ |
| $11+1$ | $V(1,2,3,8,9,10,11,12,13,14,15,4)$ | $[(0,16),(0,0),(1,96),(0,0),(14,128),(0,0)]$ |
| $1+12$ | $IV(12,1,2,3,4,5,6,7,8,9,10,11,14)$ | $[(0,0),(0,20),(0,0),(5,116),(0,0),(10,224)]$ |
| $2+11$ | $V(2,4,1,3,5,8,9,10,11,12,13,14,15)$ | $[(0,0),(3,48),(0,0),(0,192),(0,0),(14,240)]$ |
| $3+10$ | $V(1,2,3,4,5,8,9,10,11,12,13,14,15)$ | $[(0,0),(2,96),(1,0),(0,256),(0,0),(28,288)]$ |
| $4+9$ | $IV(1,2,3,12,4,5,6,7,8,9,10,11,14)$ | $[(0,6),(8,112),(1,48),(0,260),(0,34),(12,456)]$ |
| $5+8$ | $V(1,2,3,4,5,8,9,10,11,12,13,14,15)$ | $[(0,0),(0,240),(2,128),(0,0),(1,64),(56,864)]$ |
| $6+7$ | $V(1,2,3,4,5,6,8,9,10,11,12,13,14)$ | $[(0,0),(0,252),(4,192),(0,0),(3,144),(56,784)]$ |
| $7+6$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13)$ | $[(0,0),(0,240),(7,240),(0,0),(7,240),(45,576)]$ |
| $8+5$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13)$ | $[(0,20),(0,160),(7,220),(2,128),(21,368),(15,192)]$ |
| $9+4$ | $V(1,2,3,4,5,6,7,8,10,9,11,12,13)$ | $[(0,32),(1,96),(7,216),(2,96),(21,392),(0,64)]$ |
| $10+3$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13)$ | $[(0,36),(1,48),(8,200),(1,32),(14,368),(0,0)]$ |
| $11+2$ | $IV(1,2,3,4,5,6,7,8,9,10,11,12,14)$ | $[(0,32),(0,24),(5,160),(0,0),(10,288),(0,0)]$ |
| $12+1$ | $IV(1,2,3,4,5,6,7,8,9,10,11,12,14)$ | $[(0,20),(0,0),(5,116),(0,0),(10,224),(0,0)]$ |
| $1+13$ | $IV(14,1,2,3,4,5,6,7,8,9,10,11,12,13)$ | $[(0,0),(0,24),(0,0),(6,152),(0,0),(15,320)]$ |
| $2+12$ | $IV(1,14,2,3,4,5,6,7,8,9,10,11,12,13)$ | $[(0,0),(6,48),(0,0),(0,256),(0,0),(15,384)]$ |
| $3+11$ | $V(1,2,3,4,5,6,8,9,10,11,12,13,14,15)$ | $[(0,0),(6,96),(1,0),(0,384),(0,0),(28,480)]$ |
| $4+10$ | $IV(1,2,3,14,4,5,6,7,8,9,10,11,12,13)$ | $[(0,8),(10,128),(1,56),(0,416),(0,48),(20,704)]$ |
| $5+9$ | $IV(1,2,3,10,11,4,5,6,7,8,9,12,13,14)$ | $[(0,12),(12,168),(2,124),(0,440),(1,156),(24,888)]$ |
| $6+8$ | $V(1,2,3,4,5,6,8,9,10,11,12,13,14,15)$ | $[(0,0),(0,336),(4,256),(0,0),(3,192),(112,1568)]$ |
| $7+7$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14)$ | $[(0,0),(0,336),(7,336),(0,0),(7,336),(105,1344)]$ |
| $8+6$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14)$ | $[(0,24),(0,240),(7,312),(4,256),(35,592),(45,576)]$ |
| $9+5$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14)$ | $[(0,40),(2,160),(7,316),(4,256),(35,684),(15,272)]$ |
| $10+4$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14)$ | $[(0,48),(3,96),(8,312),(3,144),(28,676),(0,96)]$ |
| $11+3$ | $V(1,2,3,4,5,6,7,8,9,10,12,11,13,14)$ | $[(0,48),(3,48),(10,276),(1,48),(14,592),(0,0)]$ |
| $12+2$ | $IV(2,3,4,5,6,7,8,9,10,11,12,14,11pt,15)$ | $[(0,40),(1,20),(10,212),(0,0),(10,448),(0,0)]$ |
| $13+1$ | $IV(1,2,3,4,5,6,7,8,9,10,11,12,13,14)$ | $[(0,24),(0,0),(6,152),(0,0),(15,320),(0,0)]$ |

(Table A.4 continued)

| $s_1 + s_2$ | Columns* | RPD EWLP |
|---|---|---|
| $1+14$ | $V(8,1,2,3,4,5,6,7,9,10,11,12,13,14,15)$ | $[(0,0),(0,28),(0,0),(14,168),(0,0),(14,476)]$ |
| $2+13$ | $IV(2,4,1,3,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,4),(3,60),(0,0),(15,268),(0,0),(10,660)]$ |
| $3+12$ | $IV(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,0),(12,96),(1,0),(0,512),(0,0),(30,768)]$ |
| $4+11$ | $V(1,2,3,8,4,5,6,7,9,10,11,12,13,14,15)$ | $[(0,12),(12,136),(1,60),(7,600),(7,40),(16,1168)]$ |
| $5+10$ | $IV(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,16),(15,192),(2,144),(0,704),(1,224),(40,1376)]$ |
| $6+9$ | $V(1,2,3,8,9,10,4,5,6,7,11,12,13,14,15)$ | $[(0,30),(18,192),(2,232),(6,752),(12,432),(76,1556)]$ |
| $7+8$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,0),(0,448),(7,448),(0,0),(7,448),(210,2688)]$ |
| $8+7$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,28),(0,336),(7,420),(7,448),(56,896),(105,1344)]$ |
| $9+6$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,48),(3,240),(7,436),(8,512),(63,1100),(45,816)]$ |
| $10+5$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,60),(6,160),(8,448),(6,384),(49,1164),(15,432)]$ |
| $11+4$ | $V(1,2,3,4,5,6,7,8,9,10,12,11,13,14,15)$ | $[(0,64),(6,96),(11,432),(4,192),(35,1088),(1,144)]$ |
| $12+3$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,60),(6,48),(13,364),(1,64),(15,896),(0,0)]$ |
| $13+2$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,48),(3,16),(15,268),(0,0),(10,660),(0,0)]$ |
| $14+1$ | $V(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ | $[(0,28),(0,0),(14,168),(0,0),(14,476),(0,0)]$ |

# Appendix B

# Projection Estimation Capacity Designs

## B.1 20-run MPEC Orthogonal Arrays

Table B.1: 20-Run MPEC Designs

| Design | $p_5$ | Runs/Had.20.$k$\{Columns\} |
|---|---|---|
| 6.1 | 1 | $\{0,1,7,11,12,19,20,26,29,30,34,39,40,45,46,48,53,54,57,59\}$ |
| 6.2 | 1 | $\{0,1,6,11,12,19,21,26,28,31,34,39,41,45,46,48,53,54,56,59\}$ |
| 6.3 | 1 | 1 $\{1,2,4,5,9,17\}$ |
| 6.4 | 1 | 1 $\{1,2,4,5,17,18\}$ |
| 6.5 | 1 | 1 $\{1,2,3,4,6,10\}$ |
| 7.1 | 1 | 1 $\{1,2,4,5,13,17,18\}$ |
| 7.2 | 1 | 1 $\{1,2,4,6,8,10,19\}$ |
| 7.3 | 0.952 | 1 $\{1,2,4,5,10,11,16\}$ |
| 7.4 | 0.952 | 1 $\{1,3,4,6,7,11,16\}$ |
| 7.5 | 0.952 | 1 $\{1,2,3,5,11,16,18\}$ |

Design $k.j$, is the $j$ best $20 \times k$ Orthogonal array

(Table B.1 continued.)

| Design | $p_5$ | Runs/Columns |
|---|---|---|
| 8.1 | 0.929 | 1 $\{1, 3, 4, 6, 7, 13, 14, 15\}$ |
| 8.2 | 0.929 | $\{0, 3, 14, 59, 60, 85, 89, 101, 106, 118, 150, 157, 167,$ $169, 176, 192, 207, 218, 236, 243\}$ |
| 8.3 | 0.911 | 2 $\{1, 2, 4, 8, 9, 10, 11, 16\}$ |
| 8.4 | 0.911 | 2 $\{1, 2, 4, 5, 9, 11, 12, 19\}$ |
| 8.5 | 0.892 | 2 $\{1, 2, 3, 4, 7, 9, 10, 12\}$ |
| 9.1 | 0.873 | 2$\{1, 2, 3, 4, 7, 9, 10, 12, 18\}$ |
| 9.2 | 0.865 | 1$\{1, 2, 3, 5, 6, 11, 16, 17, 18\}$ |
| 9.3 | 0.865 | 1$\{1, 3, 4, 5, 8, 9, 13, 17, 18\}$ |
| 9.4 | 0.865 | 2$\{1, 2, 3, 4, 8, 9, 11, 18, 19\}$ |
| 9.5 | 0.865 | 2$\{1, 2, 3, 4, 8, 9, 10, 11, 12\}$ |
| 10.1 | 0.857 | 2$\{1, 2, 3, 4, 8, 9, 10, 12, 16, 19\}$ |
| 10.2 | 0.857 | 2 $\{1, 3, 4, 6, 7, 11, 13, 14, 18, 19\}$ |
| 10.3 | 0.849 | 2$\{1, 2, 3, 4, 6, 7, 11, 13, 14, 18\}$ |
| 10.4 | 0.845 | 2$\{1, 2, 3, 4, 7, 9, 10, 12, 16, 18\}$ |
| 10.5 | 0.841 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 12, 16\}$ |
| 11.1 | 0.848 | 2$\{1, 2, 3, 4, 6, 7, 11, 13, 14, 18, 19\}$ |
| 11.2 | 0.835 | 2$\{1, 2, 3, 4, 8, 9, 10, 11, 12, 16, 19\}$ |
| 11.3 | 0.833 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 16\}$ |
| 11.4 | 0.829 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 12, 16, 18\}$ |
| 11.5 | 0.829 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 18\}$ |
| 12.1 | 0.825 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 12, 16, 18, 19\}$ |
| 12.2 | 0.821 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 16, 18\}$ |
| 12.3 | 0.818 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 18, 19\}$ |
| 12.4 | 0.815 | 2$\{1, 2, 3, 4, 5, 6, 7, 11, 13, 14, 18, 19\}$ |
| 12.5 | 0.808 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 15, 16\}$ |
| 13.1 | 0.820 | 2$\{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 16, 18, 19\}$ |
| 13.2 | 0.799 | 2$\{1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 15, 16, 17\}$ |
| 13.3 | 0.798 | 2$\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 16, 18\}$ |
| 13.4 | 0.797 | 2$\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 16, 17\}$ |
| 13.5 | 0.795 | 2$\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 16, 17, 18\}$ |

(Table B.1 continued.)

| Design | $p_5$ | Runs/Columns |
|---|---|---|
| 14.1 | 0.797 | 2{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 16, 18, 19} |
| 14.2 | 0.796 | 2{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 16, 17, 18} |
| 14.3 | 0.787 | 2{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 15, 16, 17} |
| 14.4 | 0.787 | 2{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15, 16, 19} |
| 14.5 | 0.786 | 2{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 16, 17} |
| 15.1 | 0.785 | 2{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 16, 17, 18} |
| 15.2 | 0.784 | 2{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 15, 16, 17} |
| 15.3 | 0.780 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17} |
| 15.4 | 0.779 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 17, 18} |
| 15.5 | 0.779 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 18, 19} |
| 16.1 | 0.777 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18} |
| 16.2 | 0.773 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17} |
| 16.3 | 0.773 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 18, 19} |
| 16.4 | 0.772 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17} |
| 16.5 | 0.771 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 18, 19} |
| 17.1 | 0.773 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19} |
| 17.2 | 0.769 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17} |
| 17.3 | 0.763 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18} |
| 17.4 | 0.760 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19} |
| 17.5 | 0.738 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18} |
| 18.1 | 0.765 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18} |
| 18.2 | 0.760 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19} |
| 18.3 | 0.735 | 2{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19} |
| 18.4 | 0.708 | 1{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19} |
| 18.5 | 0.690 | 1{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18} |
| 19.1 | 0.761 | 2 |
| 19.2 | 0.706 | 1 |
| 19.3 | 0.676 | 3 |

## B.2 24-run PEC Orthogonal Arrays

Table B.2: 24-run PEC designs, Top 20 Search

| Design | $(p_4, p_5, p_6)$ | Had.24.$k\{Columns\}$ |
|---|---|---|
| 6.1 | (1, 1, 1.000) | 58{2, 5, 9, 14, 16, 20} |
| 6.2 | (1, 1, 1.000) | 58{2, 14, 16, 17, 20, 23} |
| 6.3 | (1, 1, 1.000) | 58{2, 5, 14, 16, 17, 20} |
| 7.1 | (1, 1, 1.000) | 58{2, 5, 14, 16, 17, 20, 23} |
| 7.2 | (1, 1, 1.000) | 58{2, 8, 9, 13, 16, 17, 23} |
| 7.3 | (1, 1, 1.000) | 58{3, 8, 9, 13, 16, 17, 23} |
| 8.1 | (1, 1, 0.786) | 58{2, 3, 13, 16, 17, 19, 20, 23} |
| 8.2 | (1, 1, 0.750) | 58{2, 5, 9, 14, 16, 17, 20, 23} |
| 8.3 | (1, 1, 0.750) | 58{2, 3, 14, 16, 17, 19, 20, 23} |
| 9.1 | (1, 1, 0.000) | 58{13, 14, 15, 16, 17, 18, 19, 20, 21} |
| 9.2 | (1, 0.992, 0.655) | 58{2, 3, 13, 14, 16, 17, 19, 20, 23} |
| 9.3 | (1, 0.992, 0.643) | 58{2, 3, 5, 13, 14, 16, 19, 20, 23} |
| 10.1 | (1, 1, 0.000) | 58{13, 14, 15, 16, 17, 18, 19, 20, 21, 22} |
| 10.2 | (1, 0.988, 0.362) | 58{2, 13, 14, 15, 16, 17, 18, 19, 20, 21} |
| 10.3 | (1, 0.988, 0.314) | 58{2, 13, 14, 15, 16, 18, 19, 20, 22, 23} |
| 11.1 | (1, 1, 0.000) | 58{13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 11.2 | (1, 0.987, 0.312) | 58{2, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22} |
| 11.3 | (1, 0.987, 0.294) | 58{2, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23} |
| 12.1 | (1, 1, 0.000) | 58{13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 12.2 | (1, 0.987, 0.273) | 58{2, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 12.3 | (1, 0.967, 0.536) | 58{2, 3, 5, 9, 10, 13, 14, 16, 17, 19, 20, 23} |
| 13.1 | (1, 0.988, 0.252) | 58{2, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 13.2 | (1, 0.965, 0.366) | 58{2, 4, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 13.3 | (1, 0.956, 0.513) | 58{2, 3, 5, 9, 10, 11, 13, 14, 16, 17, 19, 20, 23} |
| 14.1 | (1, 0.969, 0.350) | 58{2, 4, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 14.2 | (1, 0.948, 0.396) | 58{2, 4, 5, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 14.3 | (1, 0.948, 0.394) | 58{2, 4, 6, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24} |
| 15.1 | (1, 0.952, 0.387) | 58{2, 4, 5, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 15.2 | (1, 0.952, 0.386) | 58{2, 4, 6, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 15.3 | (1, 0.937, 0.406) | 58{2, 4, 5, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24} |
| 16.1 | (1, 0.940, 0.402) | 58{2, 4, 5, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 16.2 | (1, 0.940, 0.401) | 58{2, 4, 6, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 16.3 | (1, 0.940, 0.401) | 58{2, 4, 5, 6, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |

Design $k.j$, is the $j$ best $24 \times k$ Orthogonal array, found using the search method.

(Table B.2 continued.)

| Design | $(p_4, p_5, p_6)$ | Had.24.$k\{Columns\}$ |
|---|---|---|
| 17.1 | $(1, 0.932, 0.407)$ | 58{2, 4, 5, 6, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 17.2 | $(1, 0.932, 0.406)$ | 58{2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 17.3 | $(1, 0.932, 0.406)$ | 58{2, 3, 4, 6, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 18.1 | $(1, 0.928, 0.408)$ | 58{2, 4, 5, 6, 7, 9, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 2324} |
| 18.2 | $(1, 0.928, 0.408)$ | 58{2, 3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 2324} |
| 18.3 | $(1, 0.928, 0.408)$ | 58{2, 3, 4, 6, 7, 9, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 19.1 | $(1, 0.925, 0.409)$ | 58{2, 3, 4, 5, 6, 7, 9, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 19.2 | $(1, 0.925, 0.408)$ | 58{2, 3, 4, 5, 6, 7, 8, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 19.3 | $(1, 0.925, 0.408)$ | 58{2, 3, 5, 6, 7, 8, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 20.1 | $(1, 0.923, 0.408)$ | 58{2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 20.2 | $(1, 0.923, 0.408)$ | 58{2, 3, 4, 5, 6, 7, 8, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 20.3 | $(1, 0.921, 0.406)$ | 58{2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 21.1 | $(1, 0.922, 0.407)$ | 58{2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 21.2 | $(1, 0.919, 0.402)$ | 58{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 21.3 | $(1, 0.915, 0.394)$ | 58{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22} |
| 22.1 | $(1, 0.920, 0.404)$ | 58{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 22.2 | $(1, 0.917, 0.398)$ | 58{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 22.3 | $(1, 0.902, 0.439)$ | 60{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 23.1 | $(1, 0.919, 0.401)$ | 58 |
| 23.2 | $(1, 0.902, 0.439)$ | 60 |
| 23.3 | $(0.997, 0.940, 0.549)$ | 57 |

Table B.3: 24-run PEC designs, Top 40 Search

| Design | $(p_4, p_5, p_6)$ | Had.24.$k\{Columns\}$ |
|---|---|---|
| 8.1 | $(1, 1, 0.786)$ | 58{2, 3, 13, 16, 17, 19, 20, 23} |
| 8.2 | $(1, 1, 0.786)$ | 58{2, 5, 8, 9, 14, 17, 20, 23} |
| 8.3 | $(1, 1, 0.786)$ | 58{3, 5, 6, 9, 13, 14, 16, 22} |
| 9.1 | $(1, 1, 0.667)$ | 58{3, 5, 6, 7, 13, 16, 17, 18, 22} |
| 9.2 | $(1, 1, 0.000)$ | 58{13, 14, 15, 16, 17, 18, 19, 20, 21} |
| 9.3 | $(1, 0.992, 0.655)$ | 58{2, 3, 13, 14, 16, 17, 19, 20, 23} |
| 10.1 | $(1, 1, 0.000)$ | 58{13, 14, 15, 16, 17, 18, 19, 20, 21, 22} |
| 10.2 | $(1, 0.996, 0.619)$ | 58{3, 5, 6, 7, 8, 13, 16, 17, 18, 22} |
| 10.3 | $(1, 0.988, 0.362)$ | 58{2, 13, 14, 15, 16, 17, 18, 19, 20, 21} |

Designs with different PEC than those found using the top 20

Table B.4: 24-run PEC designs, $3p_4 + 2p_5 + p_6$ Weighted Average Search

| Design | $(p_4, p_5, p_6)$ | Had.24.$k${*Columns*} |
|---|---|---|
| 6.1 | (1, 1, 1.000) | 52{5, 6, 8, 10, 11, 15} |
| 6.2 | (1, 1, 1.000) | 52{5, 6, 8, 10, 11, 20} |
| 6.3 | (1, 1, 1.000) | 52{5, 6, 8, 10, 15, 20} |
| 7.1 | (1, 1, 1.000) | 52{5, 6, 8, 10, 11, 15, 20} |
| 7.2 | (1, 1, 1.000) | 52{5, 6, 8, 10, 11, 15, 24} |
| 7.3 | (1, 1, 1.000) | 52{5, 6, 8, 10, 11, 20, 24} |
| 8.1 | (1, 1, 0.929) | 52{5, 6, 8, 10, 11, 15, 20, 24} |
| 8.2 | (1, 1, 0.929) | 52{2, 6, 8, 10, 12, 15, 19, 24} |
| 8.3 | (1, 1, 0.929) | 52{4, 5, 7, 8, 11, 20, 23, 24} |
| 9.1 | (1, 0.992, 0.845) | 52{5, 6, 8, 10, 11, 15, 20, 23, 24} |
| 9.2 | (1, 0.992, 0.845) | 52{2, 6, 8, 10, 12, 15, 19, 20, 24} |
| 9.3 | (1, 0.992, 0.845) | 52{4, 5, 7, 8, 11, 19, 20, 23, 24} |
| 10.1 | (1, 0.984, 0.786) | 52{5, 6, 8, 10, 11, 15, 19, 20, 23, 24} |
| 10.2 | (1, 0.984, 0.786) | 52{2, 6, 8, 10, 12, 15, 19, 20, 23, 24} |
| 10.3 | (1, 0.984, 0.786) | 52{4, 5, 7, 8, 11, 15, 19, 20, 23, 24} |
| 11.1 | (1, 0.978, 0.708) | 52{5, 6, 8, 10, 11, 15, 16, 19, 20, 23, 24} |
| 11.2 | (1, 0.978, 0.708) | 52{2, 6, 8, 10, 12, 15, 17, 19, 20, 23, 24} |
| 11.3 | (1, 0.978, 0.708) | 52{4, 5, 7, 8, 11, 15, 19, 20, 21, 23, 24} |
| 12.1 | (1, 0.974, 0.689) | 52{2, 5, 6, 8, 10, 11, 15, 16, 19, 20, 23, 24} |
| 12.2 | (1, 0.974, 0.689) | 52{2, 6, 7, 8, 10, 12, 15, 17, 19, 20, 23, 24} |
| 12.3 | (1, 0.974, 0.689) | 52{4, 5, 7, 8, 10, 11, 15, 19, 20, 21, 23, 24} |
| 13.1 | (1, 0.970, 0.677) | 52{2, 5, 6, 7, 8, 10, 11, 15, 16, 19, 20, 23, 24} |
| 13.2 | (1, 0.970, 0.677) | 52{2, 6, 7, 8, 10, 11, 12, 15, 17, 19, 20, 23, 24} |
| 13.3 | (1, 0.970, 0.677) | 52{2, 4, 5, 7, 8, 10, 11, 15, 19, 20, 21, 23, 24} |
| 14.1 | (1, 0.966, 0.658) | 52{2, 5, 6, 7, 8, 10, 11, 15, 16, 17, 19, 20, 23, 24} |
| 14.2 | (1, 0.966, 0.658) | 52{2, 6, 7, 8, 10, 11, 12, 15, 17, 18, 19, 20, 23, 24} |
| 14.3 | (1, 0.966, 0.658) | 52{2, 4, 5, 7, 8, 10, 11, 15, 16, 19, 20, 21, 23, 24} |
| 15.1 | (0.999, 0.960, 0.649) | 52{2, 5, 6, 7, 8, 10, 11, 15, 16, 17, 18, 19, 20, 23, 24} |
| 15.2 | (0.999, 0.960, 0.649) | 52{2, 6, 7, 8, 10, 11, 12, 15, 17, 18, 19, 20, 22, 23, 24} |
| 15.3 | (0.999, 0.960, 0.649) | 52{2, 4, 5, 7, 8, 10, 11, 15, 16, 17, 19, 20, 21, 23, 24} |
| 16.1 | (0.999, 0.959, 0.634) | 52{2, 5, 6, 7, 8, 10, 11, 12, 15, 16, 17, 18, 19, 20, 23, 24} |
| 16.2 | (0.999, 0.959, 0.634) | 52{2, 6, 7, 8, 9, 10, 11, 12, 15, 17, 18, 19, 20, 22, 23, 24} |
| 16.3 | (0.999, 0.959, 0.634) | 52{2, 4, 5, 6, 7, 8, 10, 11, 15, 16, 17, 19, 20, 21, 23, 24} |
| 17.1 | (0.999, 0.956, 0.626) | 52{2, 4, 5, 6, 7, 8, 10, 11, 12, 15, 16, 17, 18, 19, 20, 23, 24} |
| 17.2 | (0.999, 0.956, 0.626) | 52{2, 5, 6, 7, 8, 9, 10, 11, 12, 15, 17, 18, 19, 20, 22, 23, 24} |
| 17.3 | (0.999, 0.956, 0.626) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 19, 20, 21, 23, 24} |
| 18.1 | (0.999, 0.954, 0.617) | 52{2, 4, 5, 6, 7, 8, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 23, 24} |
| 18.2 | (0.999, 0.954, 0.617) | 52{2, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 22, 23, 24} |
| 18.3 | (0.999, 0.954, 0.617) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 19, 20, 21, 22, 23, 24} |

Design $k.j$, is the $j$ best $24 \times k$ Orthogonal array, found using the search method.

(Table B.4 continued)

| Design | $(p_4, p_5, p_6)$ | Had.24.$k\{Columns\}$ |
|---|---|---|
| 19.1 | (0.998, 0.952, 0.608) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 23, 24} |
| 19.2 | (0.998, 0.952, 0.608) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 22, 23, 24} |
| 19.3 | (0.998, 0.952, 0.608) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 19, 20, 21, 22, 23, 24} |
| 20.1 | (0.998, 0.952, 0.608) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 20.2 | (0.998, 0.952, 0.608) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 20.3 | (0.998, 0.952, 0.608) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 21.1 | (0.999, 0.946, 0.584) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 21.2 | (0.999, 0.946, 0.584) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 21.3 | (0.999, 0.945, 0.581) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 22.1 | (0.999, 0.944, 0.568) | 52{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 22.2 | (0.999, 0.944, 0.561) | 51{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 22.3 | (0.999, 0.942, 0.558) | 56{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 23.1 | (0.997, 0.938, 0.554) | 56 |
| 23.1 | (0.997, 0.940, 0.549) | 57 |
| 23.3 | (0.997, 0.937, 0.551) | 55 |

Table B.5: 24-run PEC designs, Maximizing $(p_6, p_5, p_4)$

| Design | $(p_4, p_5, p_6)$ | Had.24.$k\{Columns\}$ |
|---|---|---|
| 6.1 | (1, 1, 1.000) | 42{4, 5, 8, 9, 20, 23} |
| 6.2 | (1, 1, 1.000) | 42{3, 14, 18, 20, 21, 23} |
| 6.3 | (1, 1, 1.000) | 42{4, 8, 11, 16, 20, 21} |
| 7.1 | (1, 1, 1.000) | 42{4, 5, 8, 9, 16, 20, 23} |
| 7.2 | (1, 1, 1.000) | 42{4, 5, 8, 9, 14, 16, 20} |
| 7.3 | (1, 1, 1.000) | 42{3, 5, 14, 18, 20, 21, 23} |
| 8.1 | (1, 1, 0.964) | 42{4, 8, 9, 11, 16, 18, 20, 21} |
| 8.2 | (1, 1, 0.964) | 42{3, 4, 5, 14, 18, 20, 21, 23} |
| 8.3 | (1, 1, 0.964) | 42{4, 8, 11, 13, 16, 19, 20, 21} |
| 9.1 | (1, 1, 0.881) | 42{4, 5, 8, 9, 11, 19, 20, 21, 23} |
| 9.2 | (1, 1, 0.869) | 42{3, 4, 5, 9, 14, 18, 20, 21, 23} |
| 9.3 | (1, 1, 0.869) | 42{2, 3, 5, 8, 9, 14, 16, 20, 23} |
| 10.1 | (1, 0.988, 0.814) | 42{3, 4, 5, 9, 14, 18, 19, 20, 21, 23} |
| 10.2 | (1, 0.988, 0.800) | 42{4, 5, 6, 8, 11, 13, 16, 19, 20, 21} |
| 10.3 | (1, 1, 0.795) | 42{3, 4, 5, 9, 13, 14, 18, 20, 21, 23} |
| 11.1 | (0.996, 0.976, 0.753) | 42{3, 4, 5, 9, 13, 14, 18, 19, 20, 21, 23} |
| 11.2 | (0.996, 0.971, 0.753) | 42{3, 4, 5, 8, 9, 14, 18, 19, 20, 21, 23} |
| 11.3 | (1, 0.982, 0.747) | 42{3, 4, 5, 8, 9, 11, 18, 19, 20, 21, 23} |
| 12.1 | (0.993, 0.957, 0.705) | 42{3, 4, 5, 8, 9, 13, 14, 18, 19, 20, 21, 23} |
| 12.2 | (0.997, 0.959, 0.703) | 42{2, 3, 4, 5, 8, 9, 14, 18, 19, 20, 21, 23} |
| 12.3 | (0.997, 0.965, 0.702) | 42{4, 5, 6, 11, 13, 14, 16, 17, 19, 20, 21, 23} |
| 13.1 | (0.998, 0.962, 0.674) | 42{2, 3, 4, 5, 8, 9, 14, 16, 18, 19, 20, 21, 23} |
| 13.2 | (1, 0.967, 0.670) | 42{2, 3, 4, 5, 8, 9, 14, 16, 17, 18, 20, 21, 23} |
| 13.3 | (0.998, 0.961, 0.670) | 42{4, 5, 6, 8, 11, 13, 14, 16, 17, 19, 20, 21, 23} |

Design $k.j$, is the $j$ best $24 \times k$ Orthogonal array, found using the search method.

(Table B.5 continued.)

| Design | $(p_4, p_5, p_6)$ | Had.24.$k\{Columns\}$ |
|---|---|---|
| 14.1 | (0.997, 0.955, 0.651) | 42{4, 5, 6, 8, 11, 13, 14, 16, 17, 18, 19, 20, 21, 23} |
| 14.2 | (0.998, 0.957, 0.650) | 42{4, 5, 6, 8, 9, 11, 13, 14, 16, 18, 19, 20, 21, 23} |
| 14.3 | (0.996, 0.951, 0.650) | 42{3, 4, 5, 6, 8, 9, 11, 13, 14, 18, 19, 20, 21, 23} |
| 15.1 | (0.997, 0.950, 0.634) | 42{3, 4, 5, 6, 8, 9, 11, 13, 14, 16, 18, 19, 20, 21, 23} |
| 15.2 | (0.997, 0.950, 0.631) | 42{4, 5, 6, 8, 9, 11, 13, 14, 16, 17, 18, 19, 20, 21, 23} |
| 15.3 | (0.998, 0.951, 0.628) | 42{2, 3, 4, 5, 6, 8, 9, 11, 14, 16, 17, 19, 20, 21, 23} |
| 16.1 | (0.996, 0.946, 0.620) | 42{3, 4, 5, 6, 8, 9, 11, 13, 14, 16, 17, 18, 19, 20, 21, 23} |
| 16.2 | (0.997, 0.947, 0.617) | 42{2, 3, 4, 5, 6, 8, 9, 11, 13, 14, 16, 17, 19, 20, 21, 23} |
| 16.3 | (0.996, 0.945, 0.616) | 42{2, 3, 5, 6, 8, 9, 11, 13, 14, 16, 17, 18, 19, 20, 21, 23} |
| 17.1 | (0.997, 0.946, 0.607) | 42{2, 3, 4, 5, 6, 8, 9, 11, 13, 14, 16, 17, 18, 19, 20, 21, 23} |
| 17.2 | (0.997, 0.946, 0.602) | 42{3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 16, 17, 18, 19, 20, 21, 23} |
| 17.3 | (0.998, 0.947, 0.601) | 42{2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 16, 17, 18, 19, 20, 21, 23} |
| 18.1 | (0.998, 0.945, 0.593) | 42{2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 16, 17, 18, 19, 20, 21, 23} |
| 18.2 | (0.998, 0.945, 0.590) | 41{3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 18, 20, 22, 23, 24} |
| 18.3 | (0.998, 0.946, 0.590) | 42{2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 20, 21, 22, 23} |
| 19.1 | (0.998, 0.945, 0.583) | 42{2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23} |
| 19.2 | (0.997, 0.942, 0.581) | 42{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23} |
| 19.3 | (0.998, 0.943, 0.581) | 41{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 18, 20, 21, 23, 24} |
| 20.1 | (0.998, 0.943, 0.577) | 41{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 18, 20, 21, 22, 23, 24} |
| 20.2 | (0.998, 0.942, 0.574) | 42{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23} |
| 20.3 | (0.995, 0.938, 0.570) | 41{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 21.1 | (0.995, 0.937, 0.565) | 41{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24} |
| 21.2 | (0.998, 0.941, 0.564) | 42{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23} |
| 21.3 | (0.998, 0.941, 0.564) | 41{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24} |
| 22.1 | (0.996, 0.938, 0.561) | 41{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24} |
| 22.2 | (0.993, 0.931, 0.560) | 41{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24} |
| 22.3 | (0.996, 0.937, 0.559) | 56{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24} |
| 23.1 | (0.994, 0.933, 0.556) | 41 |
| 23.2 | (0.997, 0.938, 0.554) | 56 |
| 23.3 | (0.994, 0.933, 0.553) | 42 |

## B.3 28-Run PEC Orthogonal Arrays

Table B.6: 28-run PEC designs, Top 20 search

| Design | $(p_5, p_6)$ | Had.28.$k$\{*Columns*\} |
|---|---|---|
| 6.1 | (1, 1.000) | 487{3, 7, 10, 13, 14, 22} |
| 6.2 | (1, 1.000) | 487{7, 8, 9, 12, 13, 17} |
| 6.3 | (1, 1.000) | 487{6, 8, 9, 12, 13, 17} |
| 7.1 | (1, 1.000) | 487{6, 7, 8, 9, 12, 13, 17} |
| 7.2 | (1, 1.000) | 487{3, 6, 9, 10, 12, 17, 21} |
| 7.3 | (1, 1.000) | 487{6, 9, 10, 12, 13, 17, 21} |
| 8.1 | (1, 1.000) | 487{3, 7, 10, 13, 14, 15, 21, 22} |
| 8.2 | (1, 1.000) | 487{3, 6, 7, 8, 9, 12, 13, 17} |
| 8.3 | (1, 1.000) | 487{3, 6, 9, 10, 12, 13, 17, 21} |
| 9.1 | (1, 1.000) | 487{3, 6, 7, 9, 10, 12, 13, 17, 21} |
| 9.2 | (1, 1.000) | 487{3, 6, 7, 8, 9, 10, 12, 18, 21} |
| 9.3 | (1, 1.000) | 487{3, 6, 7, 8, 9, 10, 12, 13, 17} |
| 10.1 | (1, 1.000) | 487{3, 6, 7, 8, 9, 10, 12, 13, 17, 21} |
| 10.2 | (1, 1.000) | 487{3, 6, 7, 8, 9, 10, 12, 17, 18, 21} |
| 10.3 | (1, 1.000) | 487{3, 6, 7, 8, 9, 10, 12, 13, 17, 18} |
| 11.1 | (1, 1.000) | 487{3, 6, 7, 8, 9, 10, 12, 13, 17, 18, 21} |
| 11.2 | (1, 0.998) | 487{2, 3, 6, 7, 8, 9, 12, 13, 17, 18, 21} |
| 11.3 | (1, 0.998) | 487{2, 3, 6, 7, 8, 9, 10, 12, 13, 18, 21} |
| 12.1 | (1, 0.997) | 487{2, 3, 6, 7, 8, 9, 10, 12, 13, 17, 18, 21} |
| 12.2 | (1, 0.996) | 487{2, 3, 6, 7, 8, 9, 10, 12, 13, 17, 18, 19} |
| 12.3 | (1, 0.995) | 487{3, 5, 7, 8, 9, 10, 13, 14, 15, 18, 21, 24} |
| 13.1 | (1, 0.996) | 487{2, 3, 6, 7, 8, 9, 10, 12, 13, 17, 18, 19, 21} |
| 13.2 | (1, 0.992) | 487{2, 3, 6, 7, 8, 10, 12, 13, 17, 18, 19, 21, 26} |
| 13.3 | (1, 0.991) | 487{3, 5, 7, 8, 9, 10, 13, 14, 15, 18, 21, 22, 24} |
| 14.1 | (1, 0.996) | 487{2, 3, 6, 7, 8, 9, 10, 12, 13, 17, 18, 19, 21, 26} |
| 14.2 | (1, 0.988) | 487{2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 17, 18, 19, 21} |
| 14.3 | (1, 0.988) | 487{2, 3, 5, 6, 7, 8, 9, 10, 12, 17, 18, 19, 21, 26} |
| 15.1 | (1, 0.987) | 487{2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 17, 18, 19, 21, 26} |
| 15.2 | (1, 0.985) | 487{2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 18, 20, 23, 26} |
| 15.3 | (1, 0.984) | 487{2, 3, 5, 7, 9, 10, 11, 12, 13, 14, 15, 18, 21, 22, 24} |
| 16.1 | (1, 0.982) | 487{2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 17, 18, 19, 21, 26} |
| 16.2 | (1, 0.982) | 487{2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 19, 21, 26} |
| 16.3 | (1, 0.982) | 487{2, 3, 5, 7, 9, 10, 11, 12, 13, 14, 15, 18, 19, 21, 22, 24} |

Design $k.j$, is the $j$ best $28 \times k$ Orthogonal array, found using the search method.

(Table B.6 continued.)

| Design | $(p_5, p_6)$ | Had.28.$k\{Columns\}$ |
|---|---|---|
| 17.1 | (1, 0.980) | 487{2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 21, 22, 24} |
| 17.2 | (1, 0.980) | 487{2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 21, 22, 24} |
| 17.3 | (1, 0.979) | 487{2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 21, 22, 23} |
| 18.1 | (1, 0.979) | 487{2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 21, 22, 24} |
| 18.2 | (1, 0.978) | 487{2, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 21, 22, 23, 24} |
| 18.3 | (1, 0.978) | 487{2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 21, 22, 23, 24} |
| 19.1 | (1, 0.977) | 487{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 21, 22, 23, 24} |
| 19.2 | (1, 0.977) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 20, 23, 24, 26} |
| 19.3 | (1, 0.977) | 487{2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24} |
| 20.1 | (1, 0.976) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 23, 24, 26} |
| 20.2 | (1, 0.976) | 487{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 24, 26} |
| 20.3 | (1, 0.976) | 487{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24} |
| 21.1 | (1, 0.975) | 487{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24, 26} |
| 21.2 | (1, 0.975) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 23, 24, 26} |
| 21.3 | (1, 0.975) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 21, 22, 23, 24, 26} |
| 22.1 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25} |
| 22.2 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24, 26} |
| 22.3 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24, 25} |
| 23.1 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25} |
| 23.2 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 26} |
| 23.3 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 20, 21, 22, 23, 24, 25, 26} |
| 24.1 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26} |
| 24.2 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26} |
| 24.3 | (1, 0.974) | 487{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25} |
| 25.1 | (1, 0.974) | 487 {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26} |
| 25.2 | (0.999, 0.983) | 376{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27} |
| 25.3 | (0.999, 0.983) | 376{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27} |
| 26.1 | (1, 0.974) | 487 {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27} |
| 26.2 | (0.999, 0.983) | 376{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27} |
| 26.3 | (0.999, 0.983) | 480{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27} |
| 27.1 | (1, 0.974) | 487 |
| 27.2 | (0.999, 0.981) | 376 |
| 27.3 | (0.999, 0.980) | 377 |

Table B.7: 28-run PEC designs, Maximizing $(p_6, p_5)$

| Design | $(p_5, p_6)$ | Had.28.$k\{Columns\}$ |
|---|---|---|
| 6.1 | (1, 1) | 376{4, 8, 10, 20, 21, 25} |
| 6.2 | (1, 1) | 376{5, 10, 16, 19, 25, 27} |
| 6.3 | (1, 1) | 376{2, 8, 10, 16, 25, 27} |
| 7.1 | (1, 1) | 376{2, 5, 8, 10, 16, 25, 27} |
| 7.2 | (1, 1) | 376{3, 5, 16, 19, 25, 26, 27} |
| 7.3 | (1, 1) | 376{4, 5, 6, 16, 18, 21, 25} |
| 8.1 | (1, 1) | 376{4, 5, 6, 8, 16, 18, 21, 25} |
| 8.2 | (1, 1) | 376{2, 5, 16, 19, 22, 25, 26, 27} |
| 8.3 | (1, 1) | 376{5, 8, 10, 16, 20, 21, 25, 27} |
| 9.1 | (1, 1) | 376{2, 3, 5, 10, 16, 19, 25, 26, 27} |
| 9.2 | (1, 1) | 376{4, 5, 8, 10, 16, 20, 21, 25, 27} |
| 9.3 | (1, 1) | 376{3, 5, 7, 10, 16, 19, 25, 26, 27} |
| 10.1 | (1, 1) | 376{2, 3, 5, 10, 16, 19, 22, 25, 26, 27} |
| 10.2 | (1, 1) | 376{3, 4, 5, 6, 8, 15, 16, 18, 21, 25} |
| 10.3 | (1, 1) | 376{3, 5, 7, 10, 16, 19, 22, 25, 26, 27} |
| 11.1 | (1, 1) | 376{2, 3, 5, 7, 10, 16, 19, 22, 25, 26, 27} |
| 11.2 | (1, 1) | 376{2, 4, 5, 8, 10, 11, 16, 20, 21, 25, 27} |
| 11.3 | (1, 1) | 376{3, 4, 5, 6, 8, 12, 15, 16, 18, 21, 25, } |
| 12.1 | (1, 1) | 376{3, 4, 5, 6, 7, 8, 12, 15, 16, 18, 21, 25} |
| 12.2 | (1, 1) | 376{2, 3, 5, 7, 10, 16, 19, 22, 24, 25, 26, 27} |
| 12.3 | (1, 1) | 376{5, 7, 10, 11, 19, 20, 22, 23, 24, 25, 26, 27} |
| 13.1 | (1, 0.999) | 376{3, 5, 7, 10, 11, 19, 20, 22, 23, 24, 25, 26, 27} |
| 13.2 | (1, 0.999) | 376{2, 3, 4, 5, 6, 7, 8, 12, 15, 16, 23, 25, 27} |
| 13.3 | (1, 0.999) | 376{2, 3, 4, 5, 6, 7, 8, 12, 15, 16, 18, 21, 25} |
| 14.1 | (1, 0.998) | 376{2, 3, 4, 5, 6, 7, 8, 12, 15, 16, 18, 21, 25, 27} |
| 14.2 | (1, 0.998) | 376{2, 4, 5, 7, 10, 11, 14, 19, 20, 21, 22, 23, 24, 25} |
| 14.3 | (1, 0.997) | 376{2, 3, 4, 5, 7, 8, 12, 15, 16, 18, 21, 23, 25, 27} |
| 15.1 | (1, 0.996) | 376{2, 4, 5, 7, 10, 11, 14, 19, 20, 21, 22, 23, 24, 26, 27} |
| 15.2 | (1, 0.996) | 376{2, 4, 5, 7, 10, 11, 14, 19, 20, 21, 22, 23, 24, 25, 27} |
| 15.3 | (1, 0.996) | 376{2, 3, 4, 5, 6, 7, 8, 12, 15, 16, 18, 21, 23, 25, 27} |
| 16.1 | (1, 0.995) | 376{2, 4, 5, 7, 10, 11, 14, 19, 20, 21, 22, 23, 24, 25, 26, 27} |
| 16.2 | (1, 0.994) | 376{2, 4, 7, 8, 10, 11, 14, 19, 20, 21, 22, 23, 24, 25, 26, 27} |
| 16.3 | (1, 0.994) | 376{2, 4, 5, 7, 8, 10, 11, 14, 19, 20, 21, 22, 23, 24, 26, 27} |
| 17.1 | (1, 0.994) | 376{2, 4, 5, 7, 8, 10, 11, 14, 19, 20, 21, 22, 23, 24, 25, 26, 27} |
| 17.2 | (1, 0.994) | 376{2, 3, 4, 5, 7, 10, 11, 14, 19, 20, 21, 22, 23, 24, 25, 26, 27} |
| 17.3 | (1, 0.993) | 376{2, 3, 4, 5, 6, 7, 10, 11, 14, 19, 20, 21, 22, 23, 24, 25, 27} |

Design $k.j$, is the $j$ best $28 \times k$ Orthogonal array, found using the search method.

(Table B.7 continued.)

| Design | $(p_5, p_6)$ | Had.28.$k\{Columns\}$ |
|---|---|---|
| 18.1 | $(1, 0.993)$ | $376\{2,3,4,5,6,7,10,11,14,19,20,21,22,23,24,25,26,27\}$ |
| 18.2 | $(1, 0.992)$ | $376\{2,3,4,5,7,8,10,11,14,19,20,21,22,23,24,25,26,27\}$ |
| 18.3 | $(1, 0.992)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,15,16,17,19,22,23\}$ |
| 19.1 | $(1, 0.990)$ | $376\{2,3,4,5,6,7,10,11,14,16,19,20,21,22,23,24,25,26,27\}$ |
| 19.2 | $(1, 0.990)$ | $376\{2,3,4,5,6,7,8,10,11,14,19,20,21,22,23,24,25,26,27\}$ |
| 19.3 | $(1, 0.989)$ | $376\{2,3,4,5,6,7,10,12,14,15,16,17,19,20,21,23,24,26,27\}$ |
| 20.1 | $(1, 0.988)$ | $376\{2,3,4,5,6,7,8,10,11,14,16,19,20,21,22,23,24,25,26,27\}$ |
| 20.2 | $(1, 0.988)$ | $376\{2,3,4,5,6,7,10,11,14,15,16,19,20,21,22,23,24,25,26,27\}$ |
| 20.3 | $(1, 0.988)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,18,19,22,24,25\}$ |
| 21.1 | $(1, 0.987)$ | $377\{2,3,4,5,7,8,9,10,11,12,13,14,15,16,17,18,19,22,23,24,25\}$ |
| 21.2 | $(1, 0.987)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,18,19,22,23,24,25\}$ |
| 21.3 | $(1, 0.987)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,19,22,23,24,25\}$ |
| 22.1 | $(1, 0.987)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,22,23,24,25\}$ |
| 22.2 | $(1, 0.986)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,18,19,20,22,23,24,25\}$ |
| 22.3 | $(0.999, 0.986)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,17,18,19,20,22,23,24,25\}$ |
| 23.1 | $(0.999, 0.985)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,22,23,24,25\}$ |
| 23.2 | $(0.999, 0.985)$ | $377\{2,3,5,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,27\}$ |
| 23.3 | $(0.999, 0.985)$ | $377\{2,3,5,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26\}$ |
| 24.1 | $(0.999, 0.985)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25\}$ |
| 24.2 | $(0.999, 0.984)$ | $377\{2,3,5,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27\}$ |
| 24.3 | $(0.999, 0.984)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,18,19,20,21,22,23,24,25,26,27\}$ |
| 25.1 | $(0.999, 0.983)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,18,19,20,21,22,23,24,25,26,27\}$ |
| 25.2 | $(0.999, 0.983)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26\}$ |
| 25.3 | $(0.999, 0.983)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,22,23,24,25,26,27\}$ |
| 26.1 | $(0.999, 0.983)$ | $377\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27\}$ |
| 26.2 | $(0.999, 0.983)$ | $376\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27\}$ |
| 26.3 | $(0.999, 0.982)$ | $480\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27\}$ |
| 27.1 | $(0.999, 0.981)$ | 376 |
| 27.2 | $(0.999, 0.980)$ | 377 |
| 27.3 | $(0.999, 0.980)$ | 480 |

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