VALIDATION OF NORMAL INVERSE GAUSSIAN DISTRIBUTION FOR SYNTHETIC CDO PRICING

by

Shirley Xin
Bachelor of Management, Nanjing Audit University 2008

Hui Wang
Bachelor of Economics, South-Central University for Nationalities, 2008

PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

In the Faculty
of
Business Administration
Financial Risk Management Program

SIMON FRASER UNIVERSITY
Summer 2010

Approval

Name: **Shirley Xin and Hui Wang**

Degree: **Master of Arts**

Title of Project: **Validation of Normal Inverse Gaussian Distribution for Synthetic CDO Pricing**

Supervisory Committee:

____

Peter Klein
Senior Supervisor
Professor of Finance

____

Jijun Niu
Second Reader
Assistant Professor of Finance

Date Approved: ____

Abstract

How to determine the default loss distribution of the whole credit portfolio is the most critical part for pricing CDOs. This paper follows Kalemanova et al (2007) and assesses the pricing efficiency of both one-factor Gaussian Copula model the Normal Inverse Gaussian (NIG) Copula model during the turbulent market condition by using data in 2008 and 2009. In addition, we test the price impact of the skewed NIG distribution by adjusting the value of the two parameters. The results show that NIG Copula performs much better than Gaussian Copula, and the introduction of the asymmetry factor in NIG distribution can further improve the modeling results.

Keywords: Synthetic CDO; One Factor Copula Model; Normal Inverse Gaussian distribution

Executive Summary

In this paper, we follow the philosophy in Kalemanova et al (2007) and assess the pricing efficiency of both Gaussian and Normal Inverse Gaussian Copula Model during the turbulent market condition in 2008 and 2009. Meanwhile we examine the price impact of the skewed NIG distribution by adjusting the value of the two parameters.

The first part of the paper shows a brief introduction of synthetic CDO pricing method and a review of the latest literature on standard model amendments.

The second part of the paper presents the modeling process of synthetic CDO pricing. Following the steps in Kalemanova et al (2007), we first show the general semi-analytic approach for synthetic CDO pricing, and the critical assumption of Large Homogeneous Portfolio (LHP) Model. Then in the section of One Factor Copula Model, we take Gaussian Copula for instance to derive the tranche expected loss. Normal Inverse Gaussian is discussed in the end as an alternative distribution assumption.

The third part of the paper describes the market data, and defines the value of all parameters embedded in each model. The comparison of the market data to the output of each model shows the empirical observation, which is partly against the conclusion in Kalemanova et al (2007). Based on this, the further testings on NIG (1) and NIG (2) are made to evaluate the pricing impact of the tail heaviness and Asymmetry of NIG distribution on the pricing efficiency.

The last part of the paper contains the conclusion based on all the models' performance.

Dedication

I wish to dedicate this paper to my dearest parents and all my friends who support me during my study at SFU.

Shirley Xin

I wish to dedicate this paper to my beloved parents and all my families for their endless love and support over these years of my studies.

Hui Wang

Acknowledgements

We would like to express our gratitude to all those people who have offered us supports and helps during this final project. First of all, we really appreciate Dr. Peter Klein for his profound guidance and valuable suggestions. We would also like to thank Dr. Jijun Niu for being the second reader on out project. Meanwhile, we would like to thank Dr. Phil Goddard, Shawn Hou, Wilson Mak and Edward Chu for their helps on Matlab coding.

Table of Contents

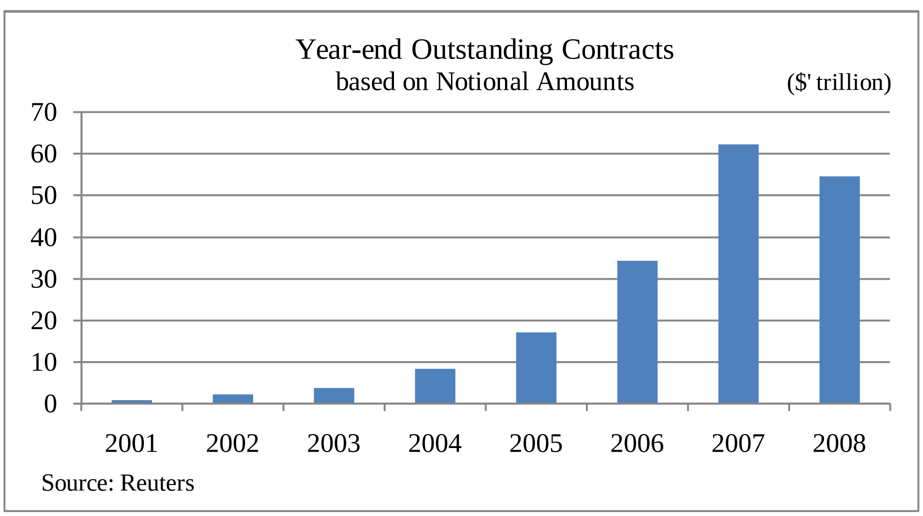
List of Figures

List of Tables

1 Introduction

In early 1990s when financial institutions such as banks were still the dominant players of the market, credit derivatives e.g. credit default swap (CDS) and collateralized debt obligation (CDO) were mainly used as a powerful risk management tool to mitigate the credit exposure associated with the lending transactions. Although it remained the ostensible purpose of creating this type of activities, gradually more and more hedge funds and individual asset managers saw the arbitrage opportunities and began to speculate. In 1999, the International Swap and Derivatives Association (ISDA) issued the Credit Derivatives Definition and standardized the CDS contract. Since then we could see a dramatic growth of the credit derivative market; the notional amount of the outstanding contracts almost doubled from $ 34.2 trillion at 2006 year-end to $62.3 trillion dollars at 2007 year-end[1], and among all the credit derivative products, index trades and synthetic CDOs together have taken up 46% of the market. (Refer to Appendix A: Figure 1.2)

Figure 1.1: Global Credit Derivative Market



[1] Source: Reuters: FACTBOX - Credit derivatives market in facts, figures. Feb.5, 2009. (http://www.reuters.com/article/idUSL544102220090205)

Unlike the value of financial instruments such as equity stocks and options, the value of a CDO is largely determined by credit risk. Based on different valuation criteria we could manually divide CDOs into two types, cash CDO and synthetic CDO. The underlying assets of cash CDO are usually fixed income securities with less liquidity. We cannot observe the debtor's default probability from market quotes; therefore the valuation process relies heavily on the rating agencies. A synthetic CDO is a portfolio of CDS. It is structured in a way that default losses on the portfolio are allocated to tranches[2], and the debtor's default probability of synthetic CDO could be implied from market quotes. This gives such instrument an apparent advantage over cash CDO. In this paper, we only consider the pricing model for synthetic CDOs.

From the quantitative perspective, how to account for the default loss distribution of the whole credit portfolio is critical for the valuation process. Sophisticated numerical techniques such as Monte Carlo simulation were used in order to fit the loose assumptions. Ever since Gaussian Copula, a semi-analytical approach of CDO pricing model, was introduced by Li (2000), it has been widely adopted by market practitioners. This formula allows people to build unbelievable structures into the market. Its simplicity and computational efficiency therefore made the model as the industrial standard.

However, the limitations of Gaussian Copula were hardly concerned. In late 2007, the financial market began to behave beyond the model's expectation, and high-risk credit derivatives which were widely held by all kinds of financial institutions in the U.S. subprime mortgage market quickly became worthless. Li's model was then blamed as a

[2] Hull, J., *Options, Futures, and Other Derivatives*, 7th edn. New Jersey: Pearson Prentice Hall, 2009, pp. 532.

“recipe for disaster”[3]: typically, (1) it does not consider the dynamic changes of the debtor’s credit situation when calculating the expected loss distribution of the portfolio, and (2) it cannot fit the fat tail feature of the loss distribution. Therefore, when we calculate the implied correlation using market quote of different tranches of the same CDO, we could observe the “correlation smile” phenomena.

In this paper we elaborate on an amended synthetic CDO pricing model based on a more fat-tailed distribution assumption called Normal Inverse Gaussian distribution. The structure of the paper is as follows: In Section One we conduct a brief literature review on synthetic CDO pricing models. Followed by a short summary of the general semi-analytical approach and the large homogeneous portfolio (LHP) assumption, we then present the pricing formulas using one factor Gaussian Copula and NIG Copula models respectively. The third section shows the estimated results as well as the comparative analysis of the two models. Conclusion is finally summarized in the last section.

1.1 Literature Review

Currently there are primarily three methods to amend the standard one-factor CDO pricing model. The first one is to extend the model by adding further stochastic factors. Andersen and Sidenius (2005) believed that the correlation between debtors was a stochastic process. It therefore used the random factor loading (RFL) model, and also tested the spread payment of different tranches using random recovery rate. The second one is to describe the correlation structure by using different copula functions, for example, Schönbucher and Schubert (2001), Laurent and Gregory (2005), Schloegl and

[3] Salmon, F, “Recipe for Disaster: The Formula That Killed Wall Street”, *Wired Magazine*, March 2009.

O'Kane (2005) used Student t-copula; Hull and White (2006) employed an implied copula approach, which was a backward computation of conditional default probability and hence the correlation structure using available market quote. The third one is to replace the normal distribution assumption in the standard copula with other fat-tailed distributions. Typical examples include double-t distribution in Hull and White (2004), the one-factor heavy-tailed copula in Wang, Rachev and Fabozzi (2007). Burtschell, Gregory and Laurent (2009) conducted a comparative analysis of the pricing efficiency of standard Gaussian Copula, Student t-Copula, Double t-Copula, Clayton Copula, Marshall-Olkin Copula and two RFL models. They concluded that the modeling outcome of Double t-Copula and two RFL models were closer to the market quote of synthetic CDO, and also solved the "correlation smile" better than other models.

However, due to the instability of the double-t distribution[4] under convolution[5], the pricing formula of synthetic CDO cannot be solved analytically. Instead, additional numerical methods have to be applied in order to calculate the quantiles of the distribution, i.e. the default thresholds. Recently, the generalized hyperbolic distribution (GH) has been introduced into CDO pricing models. Common forms include Normal Inverse Gaussian (NIG) distribution in Kalemanova, Schmid and Werner (2007), Variance Gamma distribution in Moosbrucker (2006), GH distribution in Eberlein and Frey (2007). Particularly NIG distribution, which was first introduced in the field of financial modeling by Brandorff-Nielsen (1997), has already been widely used in the industrial practice. Its two special characteristics made it very suitable for CDO pricing:

[4] It can also be called as bivariate t distribution.

[5] Convolution is a mathematical operation on two functions, producing a third function that is typically viewed as a modified version of one of the original functions.

(1) it can reflect the tail-heaviness and asymmetry using two parameters; and (2) it is stable under convolution so that numerical computation could be significantly reduced. Kalemanova, Schmid and Werner (2007) compared modeling results using Gaussian Copula, Double t-Copula with degrees of freedom of 3 and 4, NIG Copula with one and two free parameters. They then concluded that the standardized symmetric NIG distribution fit CDO second tranche exactly, while the skewed NIG distribution with two free parameters brought only a very slight improvement.

1.2 Purpose of the Paper

In this paper, we follow the philosophy in Kalemanova et al (2007) and define the pricing efficiency as the size of the absolute error between modeling outcome and actual market quote of all CDO tranches. The major purpose of this paper is to assess the pricing efficiency of both Gaussian Copula and NIG Copula Model during the turbulent market condition in 2008 and 2009, to see whether the conclusion in Kalemanova et al (2007) still holds. Furthermore, we aim to examine the price impact of the skewed NIG distribution by adjusting the value of the two free parameters.

2 Modeling

In this section, we present the modeling process of synthetic CDO pricing. Following the steps in Kalemanova et al (2007), we first show the general semi-analytic approach for synthetic CDO pricing, and the critical assumption of Large Homogeneous Portfolio (LHP) Model. Then in the section of One Factor Copula Model, we take Gaussian Copula for instance to derive the tranche expected loss. Normal Inverse Gaussian is discussed in the end as an alternative distribution assumption.

2.1 General Semi-analytic Approach for Synthetic CDO Pricing

Basically, the purpose of pricing a synthetic CDO is to determine the fair value of each tranche in the same structure. The protection buyer of CDO tranche pays periodic spread payments to the protection seller at pre-settled payment dates, and the spread payment is determined by the outstanding notional principal of each tranche. In case of a default event, the compensation will be paid on the loss due to default event to the protection buyer.

Suppose that $t_0, t_1, ..., t_n$ are payment dates with $t_0 = 0$ and $t_n = T$. K_1 and K_2 are the attachment and detachment points of the tranche, which means this tranche is responsible to cover the portfolio loss from K_1 to K_2. With the assumption that interest rate, r, is constant, the discount factor is:

$$B(t_0, t_i) = e^{-r(t_i - t_0)}. \tag{1}$$

Moreover, the percentage loss of the tranche K_1 to K_2 is denoted as $L_{(K_1,K_2)}(t_i)$, and under the risk neutral condition, the expected tranche loss at time t is $EL_{(K_1,K_2)}(t_i)$.

The premium leg of the tranche is the sum of the present value of expected periodic payments at each payment date:

$$\text{Premium Leg} = \sum_{i=1}^{n} \Delta t_i \cdot S \cdot \left[1 - EL_{(K_1,K_2)}(t_i)\right] \cdot B(t_0, t_i) \ , \qquad (2)$$

where $\Delta t_i = t_i - t_{i-1}$, and S is the breakeven tranche spread.

The protection leg refers to the difference between the residual principles between time t_i and t_{i-1}. In reality, the compensation will be made immediately after the default happened. For simplicity, we assume that the compensation is paid only on the payment date. [6] Therefore, the protection leg is:

$$\begin{aligned} \text{Protection Leg} &= E\left[\int_{t_0}^{t_n} B(0,t_i) dL_{(K_1,K_2)}(t_i)\right] \\ &\approx \sum_{i=1}^{n} E\left[B(0,t_i)\left(L_{(K_1,K_2)}(t_i) - L_{(K_1,K_2)}(t_{i-1})\right)\right] \\ &= \sum_{i=1}^{n} \left(EL_{(K_1,K_2)}(t_i) - EL_{(K_1,K_2)}(t_{i-1})\right) \cdot B(0,t_i) \ . \end{aligned} \qquad (3)$$

The breakeven spread on the tranche occurs when the present value of the payments (Premium Leg) equals the present value of the payoffs (Protection Leg) or [7]

$$\text{Premium Leg} = \text{Protection Leg} \ . \qquad (4)$$

---

[6] Kalemanova et al. “The Normal Inverse Gaussian Distribution for Synthetic CDO Pricing”, *Journal of Derivatives*, Spring 2007, pp. 5.

[7] Note (2), pp. 534-535.

Therefore the breakeven spread of the tranche covering the portfolio loss of K_1 to K_2 is:

$$S = \frac{\sum_{i=1}^{n} \left[EL_{(K_1,K_2)}(t_i) - EL_{(K_1,K_2)}(t_{i-1})\right] \cdot B(t_0,t_i)}{\sum_{i=1}^{n} \Delta t_i \cdot \left[1 - EL_{(K_1,K_2)}(t_i)\right] \cdot B(t_0,t_i)} \,. \tag{5}$$

In order to calculate tranche spread in Eq.(5), we need the series of tranche expected loss at each the payment date.

When the portfolio suffer a loss of $L(t_i)$, the corresponding loss of the tranche K_1 to K_2 is:

$$\begin{aligned} L_{(K_1,K_2)}(t_i) &= \frac{1}{K_2 - K_1} \cdot max\{min(L(t_i), K_2) - K_1, 0\} \\ &= \begin{cases} 0 & \text{if } 0 \leqslant L(t_i) < K_1, \\ \frac{L(t_i)-K_1}{K_2-K_1} & \text{if } K_1 \leqslant L(t_i) \leqslant K_2, \\ 1 & \text{if } K_2 < L(t_i) \leqslant 1 \end{cases} \end{aligned} \tag{6}$$

If the continuous portfolio loss distribution function, $F(t,x)$, is given, then the expected loss on the tranche is:

$$\begin{aligned} EL_{(K_1,K_2)}(t) &= \frac{1}{K_2 - K_1} \int_{K_1}^{1} (min(x, K_2) - K_1) dF(t,x) \\ &= \frac{1}{K_2 - K_1} \left(\int_{K_1}^{1} (x - K_1) dF(t,x) - \int_{K_2}^{1} (x - K_2) dF(t,x) \right) \,. \end{aligned} \tag{7}$$

Thus, the central problem in the pricing of a CDO tranche is to derive the loss distribution of the reference portfolio.[8]

2.2 Large Homogeneous Portfolio Assumption

During the estimation process of the continuous portfolio loss distribution function, it is critical to assume that the reference portfolio is composed by infinite number of

[8] Note (4), pp. 6.

homogeneous assets with the same pairwise default correlation. Then the portfolio has follow traits:

a) The unsystematic risk of the reference portfolio is diversified away because of the infinite number of the underlying assets.

b) All the underlying assets in the portfolio are equally weighted and share the same spread and recovery rate.

c) The default dependence structure follows the One Factor Copula Model, whose form is characterized by different distribution assumptions.

d) The pairwise default correlation is constant for the whole structure, so that the correlation can be implied from equity tranche to price other mezzanine tranches and senior tranches.

2.3 One Factor Gaussian Copula Model

The LHP approach is based on a One Factor Gaussian Copula Model of correlated defaults.[9] It has been approved that One Factor Copula Model is useful in describing the joint default probability among different credit entities. One Factor Gaussian Copula Model was introduced by Li (1999, 2000), and then was developed as the market standard model. The following section shows how to derive the loss distribution of the reference portfolio by using this model.

[9] Note (4), pp. 7.

We assume that the portfolio is composed of n equally weighted underlying assets. As the default indicator, the asset return of the *i*-th instrument can be expressed as follows:

$$X_i(t) = a_i Z_c(t) + \sqrt{1 - a_i^2} Z_i(t) \ , \tag{8}$$

where for $t = 1$ to n, $Z_c(t)$ is the common factor, and $Z_i(t)$ is the individual factor. Both the common factor and individual factor are independent random variables which follow Gaussian distribution. The covariance of $X_i(t)$ and $X_j(t)$ is $a_i a_j$. Meanwhile, due to the stability of Gaussian distribution under convolution, $X_i(t)$ follows Gaussian distribution as well. Then the default threshold can be derived efficiently:

$$D_i(t) = \Phi^{-1}(\pi_i(t)) \ , \tag{9}$$

where $\pi_i(t)$ is the default probability of *i*-th asset before time t. If $X_i(t) \le D_i(t)$, this will lead to a default event on this asset.

Hence the default probability of *i*-th asset is:

$$\begin{aligned} P(\tau \le t) = P[X_i(t) \le D_i(t)] &= P\left[Z_i(t) \le \frac{D_i(t) - a_i Z_c(t)}{\sqrt{1 - a_i^2}}\right] \\ &= \Phi\left(\frac{D_i(t) - a_i Z_c(t)}{\sqrt{1 - a_i^2}}\right) . \end{aligned} \tag{10}$$

LHP assumes that all the underlying assets are homogeneous, so that $D_i(t) = D(t)$, and $a_i = a$. Hence the conditional default probability becomes:

$$P[\tau \le t | Z_c(t)] = \Phi\left(\frac{D(t) - a Z_c(t)}{\sqrt{1 - a^2}}\right) . \tag{11}$$

For the recovery rate $R = 0$, if k of n underlying instruments default, the conditional default probability of $\frac{k}{n}$ percentage loss of the portfolio is:

$$P\left[L(t) = \frac{k}{n}\middle| Z_c(t)\right] = \binom{n}{k} \Phi\left(\frac{D(t) - aZ_c(t)}{\sqrt{1-a^2}}\right)^k \left[1 - \Phi\left(\frac{D(t) - aZ_c(t)}{\sqrt{1-a^2}}\right)\right]^{n-k} . \quad (12)$$

The unconditional default probability of $\frac{k}{n}$ percentage loss of the portfolio therefore is:

$$P\left[L(t) = \frac{k}{n}\right] = \int_{-\infty}^{+\infty} \binom{n}{k} \Phi\left(\frac{D(t) - au}{\sqrt{1-a^2}}\right)^k \left[1 - \Phi\left(\frac{D(t) - au}{\sqrt{1-a^2}}\right)\right]^{n-k} d\Phi(u) \ , \quad (13)$$

where $\Phi(u)$ is the conditional distribution function of $Z_c(t)$.

The cumulative default probability that the portfolio percentage loss is less than x, for $x \in [0, 1]$ is:

$$F_n(t, x) = \sum_{k=0}^{[nx]} P\left[L(t) = \frac{k}{n}\right] \ . \quad (14)$$

Eq.(14) can be rearranged by substitute $S = \Phi\left(\frac{D(t) - a\mu}{\sqrt{1-a^2}}\right)$.:

$$F_n(t, x) = -\int_0^1 \sum_{k=0}^{[nx]} \binom{n}{k} S^k (1 - S)^{n-k} d\Phi\left(\frac{D(t) - \sqrt{1-a^2}\Phi^{-1}(S)}{a}\right) \ . \quad (15)$$

Under LHP assumptions, n will runs into infinite, and then the cumulative default probability with infinite underlying CDSs is:

$$F_\infty(t, x) = \lim_{n \to +\infty} \left[-\int_0^1 \sum_{k=0}^{[nx]} \binom{n}{k} S^k (1 - S)^{n-k} d\Phi\left(\frac{D(t) - \sqrt{1-a^2}\Phi^{-1}(S)}{a}\right)\right] \ . \quad (16)$$

Since

$$\lim_{n\to+\infty}\sum_{k=0}^{[nx]}\binom{n}{k}S^k(1-S)^{n-k}=\begin{cases}0, & \text{if } x<S\\ 1, & \text{if } x>S\end{cases} \tag{17}$$

the cumulative distribution function of the portfolio loss is given:

$$\begin{aligned} F_\infty(t,x) &= -\int_0^x d\Phi\left(\frac{D(t)-\sqrt{1-a^2}\Phi^{-1}(S)}{a}\right) \\ &= 1-\Phi\left(\frac{D(t)-\sqrt{1-a^2}\Phi^{-1}(x)}{a}\right) \\ &= \Phi\left(\frac{\sqrt{1-a^2}\Phi^{-1}(x)-D(t)}{a}\right) \ . \end{aligned} \tag{18}$$

The expected loss of tranche based on One Factor Gaussian Copula Model can be computed analytically:

$$EL_{(K_1,K_2)}(t)=\frac{\Phi_2(-\Phi^{-1}(K_1),D(t),\rho)-\Phi_2(-\Phi^{-1}(K_2),D(t),\rho)}{K_2-K_1}\ , \tag{19}$$

where $\Phi_2()$ is the bivariate normal distribution, and ρ is the covariance matrix:

$$\rho=\begin{bmatrix}1 & -\sqrt{1-a^2}\\ -\sqrt{1-a^2} & 1\end{bmatrix}. \tag{20}$$

Fortunately, this algorithm can be implemented easily using the Matlab build-in function MVNCDF( ). Sample code is demonstrated in Appendix C (I).

In the case of non-zero recovery rate, K can be substituted by $\frac{K}{1-R}$, so that the tranche expected loss becomes:

$$EL^R_{(K_1,K_2)}(t)=EL_{(\frac{K_1}{1-R},\frac{K_2}{1-R})}(t). \tag{21}$$

2.4 Alternative Distribution Assumptions

One advantage of One Factor Copula Model is that the common factor and individual factor can capture any distribution assumptions to develop different characters of default dependence structure. Since Gaussian distribution overlooks the tail heaviness trait of the financial market, some other distributions with fat tails could be introduced in order to fit this characteristic of the financial market.

In the following section, we replace the Gaussian distribution with the Normal Inverse Gaussian (NIG) distribution, and construct a different One Factor Copula Model. With the general semi-analytic approach and LHP assumptions, we can compare the default dependence of NIG Copula with that of Gaussian Copula and determine whether NIG distribution provides a greater improvement on pricing CDO tranches.[10]

2.4.1 The Main Properties of the NIG Distribution

Normal Inverse Gaussian distribution is generated by the Normal distribution and Inverse Gaussian (IG) distribution. It is a special case of the generalized hyperbolic distribution (GH) and it has four real parameters to control the properties.

To show the process of deriving Normal Inverse Gaussian distribution, we need a random variable Y. If $Y \sim IG(\alpha, \beta)$, which is called Inverse Gaussian distribution, and $\alpha > 0,\ \beta > 0$, its density function is:

[10] This improvement is measured by the absolute error between the modeling outcome and the market quote of CDO tranches.

$$f_{IG}(y;\alpha,\beta) = \begin{cases} \frac{\alpha}{\sqrt{2\pi\beta}} y^{-\frac{3}{2}} \cdot exp\left(-\frac{(\alpha-\beta y)^2}{2\beta y}\right) & , \text{ if } y > 0 \\ 0 & , \text{ if } y \leq 0 \end{cases} \tag{22}$$

If a random variable $X \sim NIG(\alpha,\beta,\mu,\delta)$,

$$X|Y = y \sim N(\mu + \beta y, y)$$
$$Y \sim IG(\delta\gamma, \gamma^2) \text{ , with } \gamma = \sqrt{\alpha^2 - \beta^2}$$

Then the density function, f_{NIG}, is

$$f(x;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{\pi\sqrt{\delta^2 + (x-\mu)^2}} \cdot e^{\delta\gamma + \beta(x-\mu)} \text{ ,} \tag{23}$$

where α is the tail heaviness, β is the asymmetry parameter, μ is the location, δ is the scale parameter, and

$$\gamma = \sqrt{\alpha^2 - \beta^2} \text{ ,}$$
$$K_1(w) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}w(t+t^{-1})}dt \text{ .}$$

$K_1(w)$ is the modified Bessel function.

The main properties of NIG distribution are:

$$X \sim NIG(\alpha,\beta,\mu,\delta) \implies cX \sim NIG(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta) \tag{24}$$

and in case of two independent variables: $X \sim NIG(\alpha,\beta,\mu_1,\delta_1)$ and $Y \sim NIG(\alpha,\beta,\mu_2,\delta_2)$,

$$X + Y \sim NIG(\alpha,\beta,\mu_1 + \mu_2, \delta_1 + \delta_2) \text{ .} \tag{25}$$

The mean, variance, skewness, and kurtosis of this density function are:

$$\text{mean} = \mu + \delta\frac{\beta}{\gamma} \qquad \text{variance} = \delta\frac{\alpha^2}{\gamma^3}$$

$$\text{skewness} = 3\frac{\beta}{\alpha\sqrt{\delta\gamma}} \qquad \text{kurtosis} = 3 + 3\left(1 + 4\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\delta\gamma} .$$

2.4.2 The NIG Copula Model with LHP Assumption

To form NIG Copula Model, the asset return, $X_i(t)$, is required to follow the standard NIG distribution, which has the zero mean and unit variance. The common factor and individual factor in Copula Model are of the following forms:

$$Z_c \sim NIG\left(\alpha, \beta, -\frac{\beta\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\right) , \tag{26}$$

$$Z_i \sim NIG\left(\frac{\sqrt{1-a^2}}{a}\alpha, \frac{\sqrt{1-a^2}}{a}\beta, -\frac{\sqrt{1-a^2}}{a}\frac{\beta\gamma^2}{\alpha^2}, \frac{\sqrt{1-a^2}}{a}\frac{\gamma^3}{\alpha^2}\right) . \tag{27}$$

Then we have:

$$aZ_c \sim NIG(\frac{\alpha}{a}, \frac{\beta}{a}, -a\frac{\beta\gamma^2}{\alpha^2}, a\frac{\gamma^3}{\alpha^2}) , \tag{28}$$

$$\sqrt{1-a^2}Z_i(t) \sim NIG(\frac{\alpha}{a}, \frac{\beta}{a}, -\frac{1-a^2}{a}\frac{\beta\gamma^2}{\alpha^2}, \frac{1-a^2}{a}\frac{\gamma^3}{\alpha^2}) , \tag{29}$$

Since

$$X_i(t) = aZ_c(t) + \sqrt{1-a^2}Z_i(t) , \tag{30}$$

and with the property of NIG distribution under convolution (refer to Eq.(24) and Eq.(25)), $X_i(t)$ is derived as:

$$X_i \sim NIG\left(\frac{\alpha}{a}, \frac{\beta}{a}, -\frac{1}{a}\frac{\beta\gamma^2}{\alpha^2}, \frac{1}{a}\frac{\gamma^3}{\alpha^2}\right) . \tag{31}$$

whose mean and variance are:

$$E(X_i) = 0$$
$$Var(X_i) = 1$$

The default threshold based on NIG distribution is derived from:

$$D(t) = F^{-1}_{NIG(\frac{1}{a})}(\pi) \ , \tag{32}$$

where π is the default probability of each underlying instrument.

The default event on the *i*-th asset occurs when $X_i(t) \leq D(t)$ and the default probability of *i*-th asset is:

$$P[\tau \leq t] = P[X_i(t) \leq D(t)] \tag{33}$$

Following the similar steps of the deviation of Gaussian Copula Model, the portfolio loss distribution of large homogeneous portfolio is implied as:

$$F_\infty(t,x) = 1 - F_{NIG(1)}\left(\frac{D(t) - \sqrt{1-a^2}F^{-1}_{NIG(\frac{\sqrt{1-a^2}}{a})}(x)}{a}\right) \ , \tag{34}$$

where $F_{NIG(s)}(x)$ denotes the distribution function for $x \sim F_{NIG}\left(x; s\alpha, s\beta, -s\frac{\beta\gamma^2}{\alpha^2}, s\frac{\gamma^3}{\alpha^2}\right)$. x is the portfolio loss, and $x \in [0,1]$.

Based on Eq.(7), the expected loss can be rearranged as:

$$EL_{(K_1,K_2)}(t) = \frac{1}{K_2 - K_1}\int_{K_1}^{K_2}(x - K_1)dF_\infty(t,x) + (1 - F_\infty(t,K_2)) \ , \tag{35}$$

and the integration part can be calculated as follows:

$$\int_{K_1}^{K_2}(x-K_1)f_\infty(t,x)dx$$
$$= \int_{lb}^{ub}\left(F_{NIG(\frac{\sqrt{1-a^2}}{a})}(y)-K_1\right)f_{NIG(1)}\left(\frac{D(t)-\sqrt{1-a^2}y}{a}\right)\cdot\frac{\sqrt{1-a^2}}{a}dy\ , \tag{36}$$

where

$$lb = F^{-1}_{NIG(\frac{\sqrt{1-a^2}}{a})}(K_1)\ ,\ ub = F^{-1}_{NIG(\frac{\sqrt{1-a^2}}{a})}(K_2)\ .$$

The implication of this algorithm in Matlab is demonstrated in Appendix C (II).

3 Market Data

In this paper, 5-year Dow Jones CDX.NA.IG index is employed in order to compare the effects of different distribution assumptions on test results. CDX family contains North American and Emerging Market companies. It is a standardized index, which is composed by 125 equally weighted CDSs of investment grade entity. The successive tranches have attachment or detachment points at 0%, 3%, 7%, 10%, 15%, and 30%. All the underlying CDSs are assumed to share the same recovery rate and default probability, which are consistent with the index.

The ninth series of Dow Jones CDX.NA.IG index has the effective date on 21-September-2007 and the maturity date on 20-December-2012, and rolls every 6 months in March and September. The market quote of this CDS index portfolio is 156.5 basis points on 22-September-2008 and is 271.0 basis points on 20-March-2009.

If we assume it is a continuously paid default swap spread, under constant default intensity model, we could get the following relationship between the index spread, S, and the hazard rate, λ, from time 0 to T:

$$S(0,T)\int_0^T e^{-\lambda(0,T)\cdot t}b(0,T)dt = (1-R)\lambda(0,T)\int_0^T e^{-\lambda(0,T)\cdot t}b(0,t)dt \ ,$$

$$\lambda(0,T) = \frac{S(0,T)}{1-R} \ , \qquad (37)$$

where $b(0,t)$ is the risk-free discount factor. Hence the default probability, π, could then be calculated using the follow formula:

$$\pi = 1 - e^{-\lambda t} \tag{38}$$

The default threshold, D, based on Gaussian Copula or NIG Copula can then be calculated from Eq.(9) or Eq.(32). K_1, K_2 in Eq.(7) are attachment and detachment points of each tranche tested. The constant recovery rate, 40%, is the known number of 5-year Dow Jones CDX.NA.IG index and is also used in Kalemanova et al (2007).

With the assumption of constant default correlation for each tranche of the same CDS index portfolio, the pairwise correlation,ρ, becomes the only estimated parameter in the Gaussian Copula, which in this case is implied from the spread of equity tranche[11]. For simplicity, in this paper, we only calculate the implied compound correlation.[12]

Besides the parwise correlation, NIG Copula requires two more parameters, α, the tail-heaviness factor, and β, the asymmetry factors. The value determination of these two parameters would certainly affect the shape and the moments of NIG distribution, and hence is critical during the modeling process.

During the test, the value of α and β estimated in Kalemanova et al (2007) is initially employed in order to test whether or not it could return the fair prices of the CDX tranches. Then we make further amendments to adjust the value of NIG distribution parameters, and check whether there is a significant improvement on the pricing efficiency of NIG Copula.

---

[11] The market quote of equity tranche is different from that of the others. The market quote of equity tranche is in terms of percentage of outstanding notional principal. And the periodic payments on equity tranche equal to outstanding notional principal times the sum of the spread and plus 500 basis points.

[12] For a tranche (K_1, K_2), this is the value of the correlation, ρ, that leads to the spread calculated from the model being the same as the market quote of the tranche spread. (See Note (2), pp. 539.)

In NIG (1), besides the pairwise correlation ρ, we only have one free parameter α, and $\beta = 0$, which means this is a standard symmetric NIG distribution. To test the effect of the skewness of NIG distribution on the pricing efficiency of the model, NIG (2) is introduced. NIG (2) frees the second parameter β, therefore is a more generalized skewed NIG distribution.

3.1 Modeling Outcome and Comparative Analysis

3.1.1 Validation of the Estimation in Kalemanova et al (2007)

As we can see, in Appendix A: Figure 3.1 the expected loss of the equity tranche (in term of percentage loss of the reference portfolio) on 22-September-2008 increases through time and becomes smooth when the time is close to maturity date. The increasing trend of the expected loss is consistent with the corresponding default probability of each underlying CDS (refer to Appendix A: Figure 3.2). However, the Gaussian Copula overlooked the fat tail trait of the financial market. The expected loss based on Gaussian Copula is higher than those based on NIG Copulas at the beginning, after all the expected losses intersect when time lies between 1.75 years to 2 years, the expected loss based on Gaussian Copula is among the lowest comparing with the two NIG Copulas. With the increase of the tail heaviness, the expected loss becomes higher when time is close to maturity. This phenomenon of the expected loss and corresponding default probability on 22-September-2008 is similar to that on 20-March-2009 (refer to Appendix A: Figure 3.3 and Figure 3.4). Appendix B: Table 3.5 to Table 3.7 present further detailed data of the default probability of each underlying CDS in the reference portfolio, the default threshold, and the expected loss of the equity tranche.

Table 3.1 shows the comparison of market quotes on 22-September-2008 to the outputs of each model. In general, the Gaussian Copula and the two NIG Copulas all overprice the mezzanine tranches, and underprice the senior tranche. Comparing with the Gaussian Copula, the two NIG Copula models return more accepted test result based on the size of the absolute error. The introduction of β in NIG (2) doesn't bring so much improvement to the test result; it only shows a slightly better fit on the second tranche. Despite the fact that the pricing efficiency of NIG copula is improved for the whole structure of the CDS index portfolio, we find 89.91% of the error in NIG (1) is resulted from the second tranche, and that in NIG (2) is 76.65%. Using the same parameter values on NIG (1) and NIG (2) in Kalemanova et al (2007), the overprice on second tranche is completely against their conclusion, which is that NIG Copula could exactly match the market quote of the second tranche.

Table 3.1: The Market Quote and Modeling Outcome of Each Tranche
22-September-2008 (bp)

	Market Quote[13]	Gaussian	NIG(1)	NIG(2)
CDX Index	156.5000			
0-3%	65.7950%	65.7950%	65.7950%	65.7950%
3-7%	869.5000	1886.7908	1703.6568	1614.6424
7-10%	395.5100	724.2516	462.3443	549.6528
10-15%	187.5550	250.3531	190.4495	245.5106
15-30%	91.7650	22.3734	67.9534	76.8759
Absolute Error		**1478.2222**	**927.6972**	**972.1300**
Rho		0.110107	0.189630	0.199591
Alpha			0.4794	0.6020
Beta			0	-0.1605

[13] Source from: Bloomberg.

On 20-March-2009, the test result is consistent with that on 22-September-2008. (Refer to Table 3.2) In general, the two NIG Copulas show great improvements on the absolute error compared with Gaussian Copula, but still significantly overprice the second tranche to the corresponding market quotes. The absolute error on second tranche takes up to 63.85% of the total absolute error for NIG (1), and it makes up 59.55% of the total absolute error. Again this result challenges the conclusion made in Kalemanova et al (2007).

Table 3.2: The Market Quote and Modeling Outcome of Each Tranche
20-March-2009 (bp)

	Market Quote	Gaussian	NIG(1)	NIG(2)
CDX Index	271.0000			
0-3%	79.9550%	79.955%	79.955%	79.955%
3-7%	50.7000	2958.0301	2477.4382	2309.6472
7-10%	19.6600	1656.3466	1201.3093	1184.8680
10-15%	591.0500	960.0521	640.5660	732.2856
15-30%	139.2750	280.9085	282.0227	367.1109
absolute error		5054.6524	3800.6511	3793.2267
rho		0.219201	0.379211	0.394263
alpha			0.4794	0.6020
beta			0	-0.1605

Furthermore, in terms of the total absolute error, there is a significant degradation on the pricing efficiency of both Gaussian Copula and NIG Copula on 20-March-2009; the total absolute error of Gaussian Copula is 3.4 times of that on 22-September-2008, and errors of the two NIG Copulas almost quadrupled. This is a good indication that during the financial crunch in 2008 and 2009, the market has reacted way beyond the models' estimation capacity. The industry is now in urgent need of further amendment.

3.1.2 Further Testing

One reasonable guess about the malfunction of the two NIG Copula models is that the estimated parameter value in Kalemanova et al (2007) could no longer capture the characteristics of the volatile market. In order to assess the price impact of the two free parameters, α and β, we further compare testing results with the actual market quotes by changing their values[14].

Table 3.3: The Market Quote and The Test Result of NIG(1)
22-September-2008

	Market Quote	NIG(1)	NIG(1) Test 1	NIG(1) Test 2
CDX Index	156.5000			
0-3%	65.7950%	65.7950%	65.7950%	65.7950%
3-7%	869.5000	1703.6568	1746.0453	1596.9507
7-10%	395.5100	462.3443	500.3086	376.9297
10-15%	187.5550	190.4495	197.6258	172.9966
15-30%	91.7650	67.9534	62.6488	78.4021
Absolute Error		**927.6972**	**1020.5309**	**773.9524**
Rho		0.189630	0.174187	0.227033
Alpha		0.4794	0.5500	0.3500
Beta		0	0	0

Table 3.3 presents our further testings of NIG (1) on 22-September-2008. We find the absolute error increases when adding tail-heaviness. Meanwhile, NIG (1) Test 2 shows a

[14] The determination of the value of α and β is based on trial.

significant improvement of the whole structure; the absolute error dropped 153.74 bp when α changed from 0.4794 to 0.35. The second tranche is now less overpriced, while the senior tranche is less underpriced.

Table 3.4: The Market Quote and The Test Result of NIG(2)

22-September-2008

	Market Quote	NIG(2)	NIG(2) Test 1	NIG(2) Test 2	NIG(2) Test 3	NIG(2) Test 4
CDX Index	156.5000					
0-3%	65.7950%	65.7950%	65.7950%	65.7950%	65.7950%	65.7950%
3-7%	869.5000	1614.6424	1525.3812	1412.6082	1339.5984	1456.9597
7-10%	395.5100	549.6528	492.1974	423.0968	387.6944	467.6698
10-15%	187.5550	245.5106	238.1101	229.3854	225.1476	233.8689
15-30%	91.7650	76.8759	94.2907	117.4674	129.3901	96.4300
Absolute Error		**972.1300**	**805.6494**	**638.2276**	**553.1317**	**710.5984**
rho		0.199591	0.231102	0.267960	0.287730	0.248468
alpha		0.6020	0.5000	0.4000	0.4000	0.7020
beta		-0.1605	-0.1605	-0.1605	-0.2000	-0.3605

Table 3.4 shows the test result of NIG (2) on 22-September-2008. We still could observe the modeling improvement by decreasing the value of α only. The absolute error dropped 333.9 bp by changing it from 0.6020 to 0.4, primarily due to the improvement of result for second tranche. Then based on the results we get from NIG(2) Test 2, Test 3 shows a further improvement of the model by increasing the absolute value of the asymmetry factor β, i.e. adding more left skewness of this NIG distribution. Although this time the senior tranche is more overvalued, it is cancelled out by the amelioration in other tranches. Interestingly, NIG (2) Test 4 amends the model from another direction. If we want to add more fat-tail feature into the model, we need to add more left skewness of the distribution.

This might be a reasonable clue showing that there is a non linear relationship between the spread payment of the tranches and the parameter value of α and β.

There is a same phenomenon in test result of both NIG (1) and NIG (2): decreasing the value of α improves the modeling results of NIG Copula. In Table 3.3, the adjustment of α from 0.4794 to 0.3500 causes a decrease in absolute error from 927.6972 bp to 773.9524 bp. And in Table 3.4, by making β fixed at -0.1605, we could observe that the change of α from 0.6020 to 0.4000 leads to the absolute error decrease dramatically from 972.1300 bp to 638.2276 bp.

The difference between the market data used in this paper and that in Kalemanova et al (2007) can be one possible explanation to the phenomenon that less fat tail feature improves the result of both NIG (1) Copula and NIG (2) Copula. In Kalemanova et al (2007), 5-year iTraxx Euro index is applied as the market data. This index is is composed by 125 equally weighted CDSs of investment grade entities in Europe, rather than in North America. And it is possible that Dow Jones CDX.NA.IG index portfolio has less fat tail feature than that of iTraxx Euro index.

4 Conclusion

The main purpose of this paper is to compare the effect of NIG distribution to that of Gaussian distribution on Synthetic CDO Pricing, and to further test the pricing efficiency of One Factor Copula Model based on the skewed NIG distribution.

Basically, NIG Copula brings the fat tail trait into the model, and therefore it produces better result to fit the market quote than Gaussian Copula. The standard NIG distribution captured by One Factor Copula Model has two parameters to define its tail heaviness and symmetry. Moreover, there is one advantage of NIG distribution: due to the stability of NIG distribution under convolution, the computation of default threshold is time efficient.

In this paper, we employ the parameter value of NIG distribution estimated in Kalemanova et al (2007), and get an unfavorable result on the second tranche. This differs from their conclusion. By making further adjustments on the two parameters, we observe that decreasing the tail heaviness and increasing the left skewness of NIG (2) lead to a significant improvement to the test result. Therefore, the key issue about using NIG Copula is how to estimate the tail heaviness and asymmetry of the NIG distribution.

For further development of this paper, there are at least two points should be considered: 1) how to determine the optimal value of the tail heaviness and asymmetry of the NIG distribution to fit the market quote; 2) is the result sensitive to the value of the recovery rate.

Reference:

[1] Andersen, L., and J. Sidenius, “Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings”, *Journal of Credit Risk*, 2005, 1(1): 29-70.

[2] Barndorff-Nielsen, O., “Normal Inverse Gaussian Distributions and Stochastic Volatility Modelling”, *Sandinavian Journal of Statistics*, 1997, 24: 1-13.

[3] Burtschell X., J. Gregory and J. Laurent, “A Comparative Analysis of CDO Pricing Models”, Working paper, 2009, ISFA Actuarial School, University of Lyon and BNP Paribas.

[4] Duffie, D., and K. Singleton, *Credit Risk: Pricing, Measurement and Management*. New Jersey: Princeton University Press, 2003.

[5] Eberlein, E., and R. Frey, “Advanced Credit Portfolio Modeling and CDO Pricing”, Working paper, 2007, University of Freiburg.

[6] Hull, J., *Options, Futures, and Other Derivatives*, 7^{th} edn. New Jersey: Pearson Prentice Hall, 2009, pp. 517-544.

[7] Hull, J., and A. White, “Valuation of a CDO and an n^{th} to Default CDS Without Monte Carlo Simulation”, *Journal of Derivatives*, 2004, 12(2): 8-23.

[8] Hull, J., and A. White, “Valuation Credit Derivatives Using an Implied Copula Approach”, *Journal of Derivatives*, 2006, 14(2): 8-28.

[9] Kalemanova, A., B. Schmid and R. Werner, “The Normal Inverse Gaussian Distribution for Synthetic CDO Pricing”, *Journal of Derivatives*, 2007.

[10] Laurent, J., and J. Gregory, “Basket Default Swaps, CDOs and Factor Copulas”, *Journal of Risk*, 2005, 7(4): 8-23.

[11] Li, D., “On Default Correlation: A Copula Approach”, *Journal of Fixed Income*, 2000, 9: 43-54.

[12] Moosbrucher, T., “Pricing CDOs with Correlated Variance Gamma Distributions”, Working Paper, 2006, Center for Financial Research, University of Cologne.

[13] O’Kane, D., and S. Turnbull, “Valuation of Credit Default Swaps”, *Lehman Brothers Quantitative Credit Research*, Vol. 2003-Q1/Q2.

[14] Schloegl, L., and D. O’Kane, “A Note on the Large Homogeneous Portfolio Approximation with the Student t Copula”, *Finance and Stochastic,* 2005, 9(4): 577-584.

[15] Schönbucher, P., *Credit Derivatives Pricing Models*. New York: Wiley, 2003.

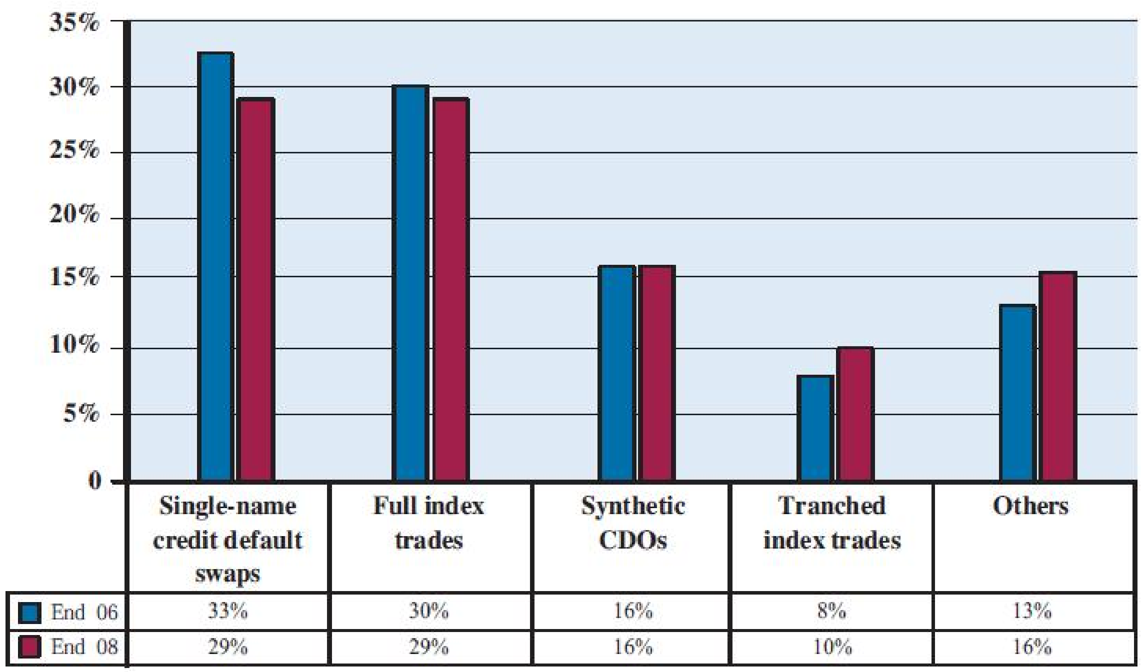
[16] Schönbucher, P., and D. Schubert, “Copula Dependent Default Risk in Intensity Models”, Working paper, 2001, Bonn University.

[17] Wang, D., S. Rachev and F. Fabozzi, “Pricing of Credit Default Index Swap Tranches with One-Factor Heavy-Tailed Copula Models”, Working paper, 2007, University of California, Santa Barbara.

Appendix A

Figure 1.2: Credit Derivative Product Range

Type	2000	2002	2004	2006
Basket products	6.0%	6.0%	4.0%	1.8%
Credit linked notes	10.0%	8.0%	6.0%	3.1%
Credit spread options	5.0%	5.0%	2.0%	1.3%
Equity linked credit products	n/a	n/a	1.0%	0.4%
Full index trades	n/a	n/a	9.0%	30.1%
Single-name credit default swaps	38.0%	45.0%	51.0%	32.9%
Swaptions	n/a	n/a	1.0%	0.8%
Synthetic CDOs – full capital	n/a	n/a	6.0%	3.7%
Synthetic CDOs – partial capital	n/a	n/a	10.0%	12.6%
Tranched index trades	n/a	n/a	2.0%	7.6%
Others	41.0%	36.0%	8.0%	5.7%



Source: Barrett, Ross and John Ewan. Credit Derivatives Report 2006. British Bankers' Association.

Figure 3.1: The Expected Losses on Equity Tranche
(Implied from the market quote on 22-September-2008)

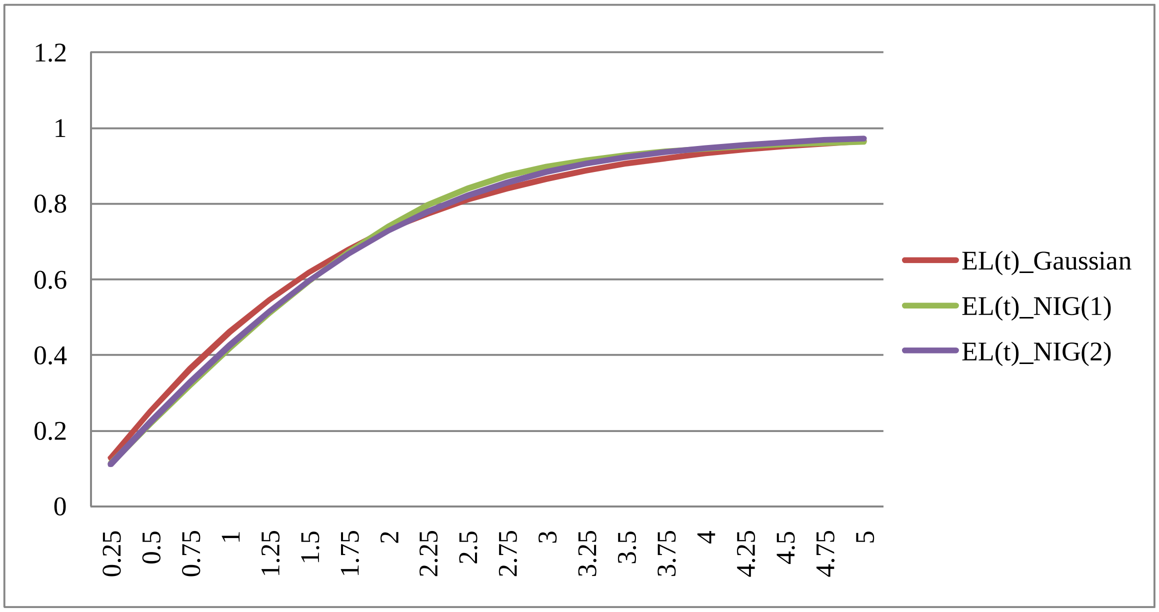


Figure 3.2: The Default Probability of Each Underlying CDS in CDX index
(Implied by the market quote on 22-September-2008)

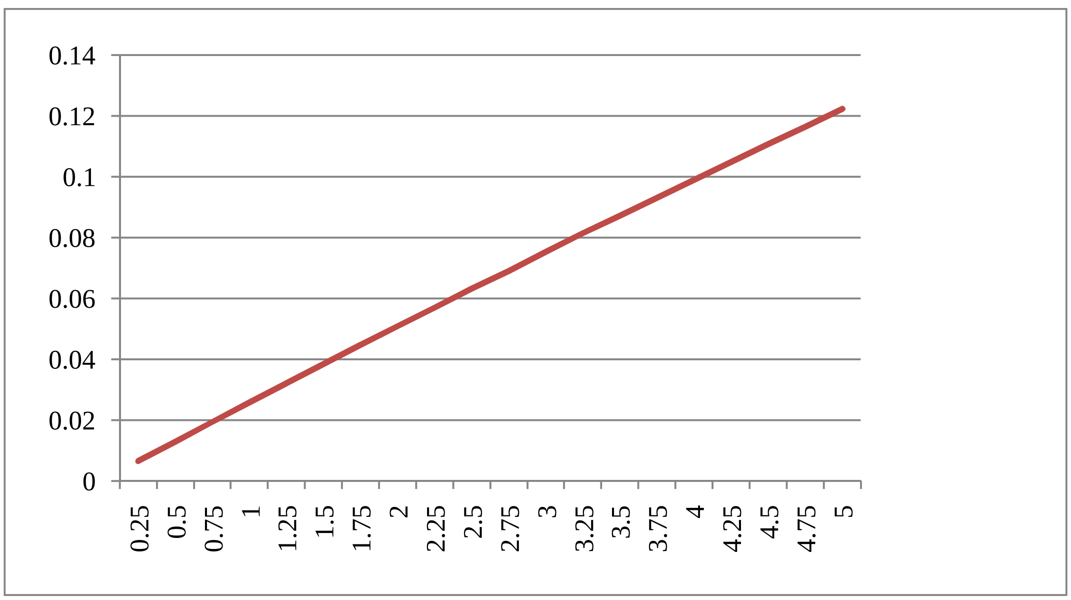


Figure 3.3: The Expected Losses on Equity Tranche
(Implied from the market quote on 20-March-2009)

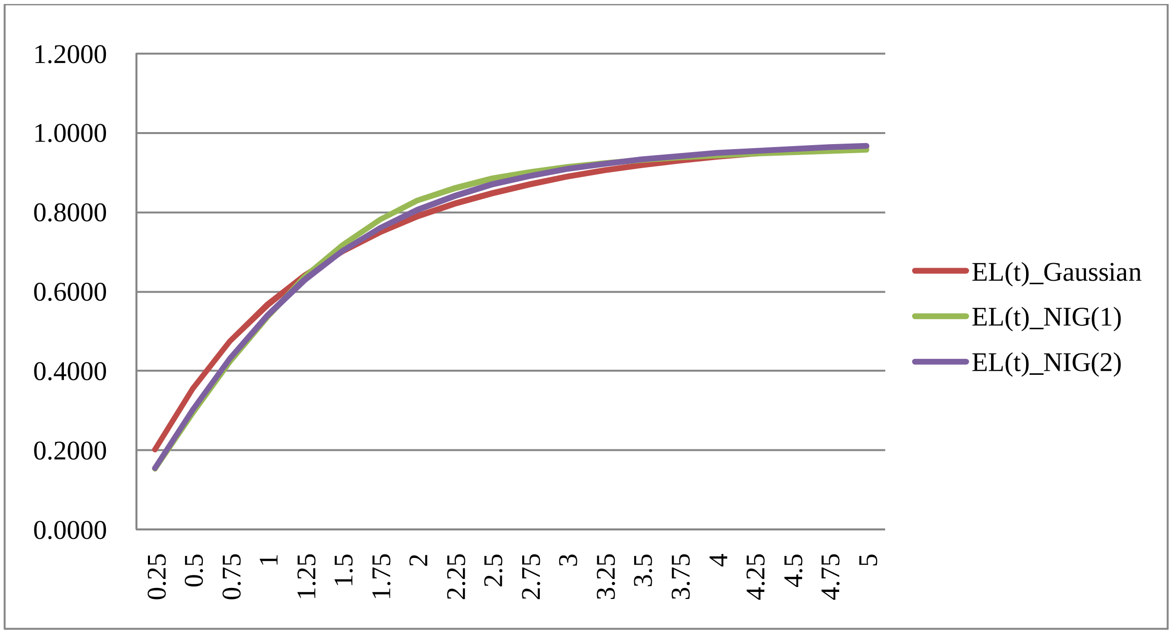


Figure 3.4: The Default Probability of Each Underlying CDS in CDX index
(Implied by the market quote on 20-March-2009)

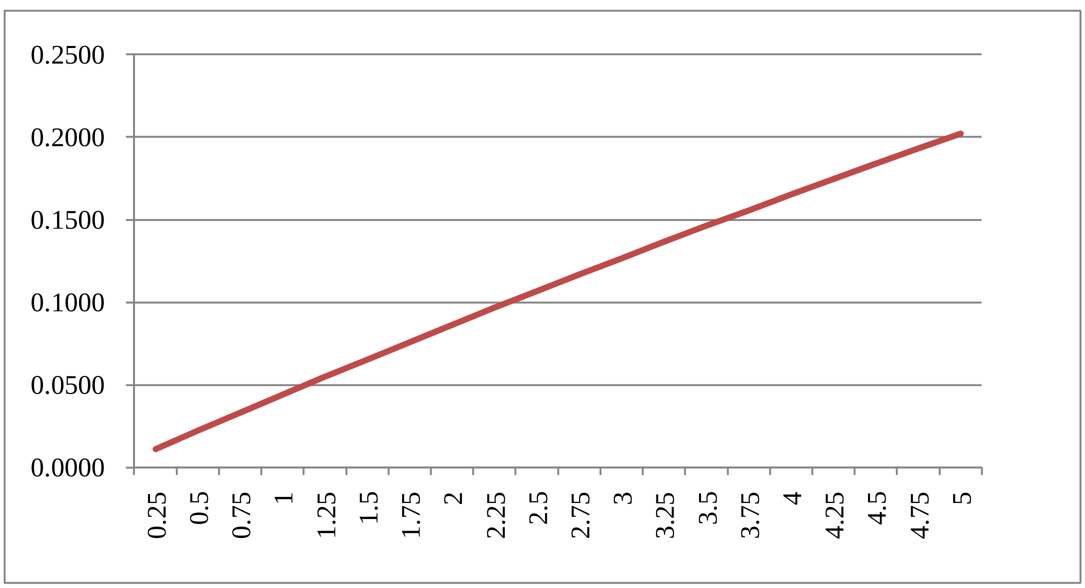


Figure 4: The Portfolio Loss Distribution from LHP Model
(Based on market quote on 22-September-2008)

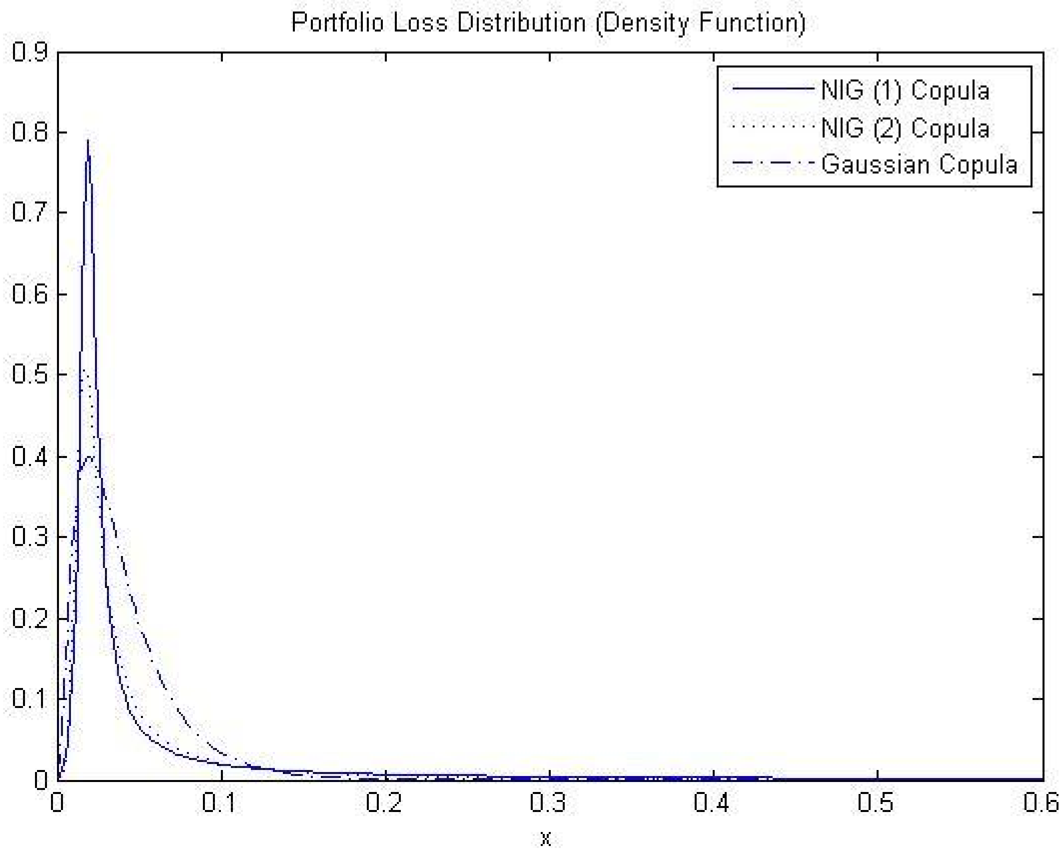
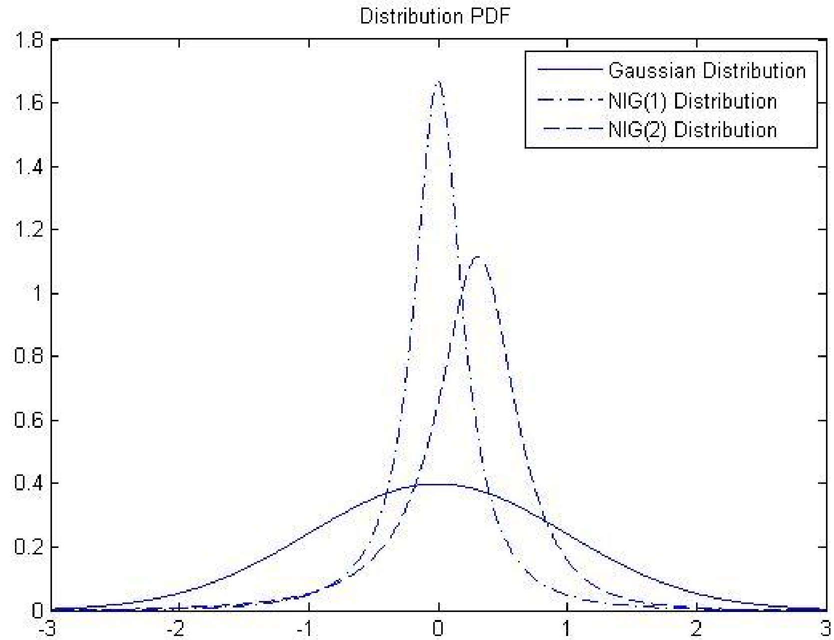
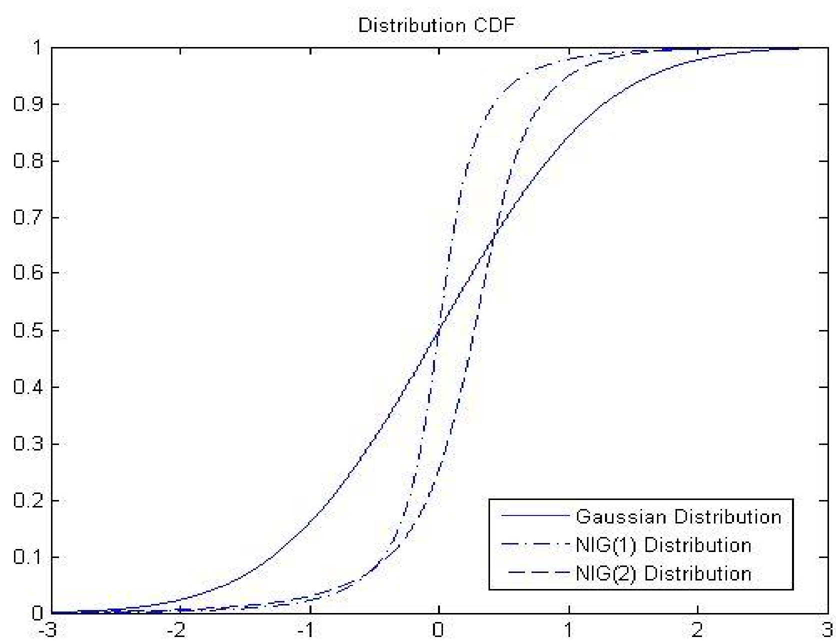


Figure 5: The Distribution of Asset Return

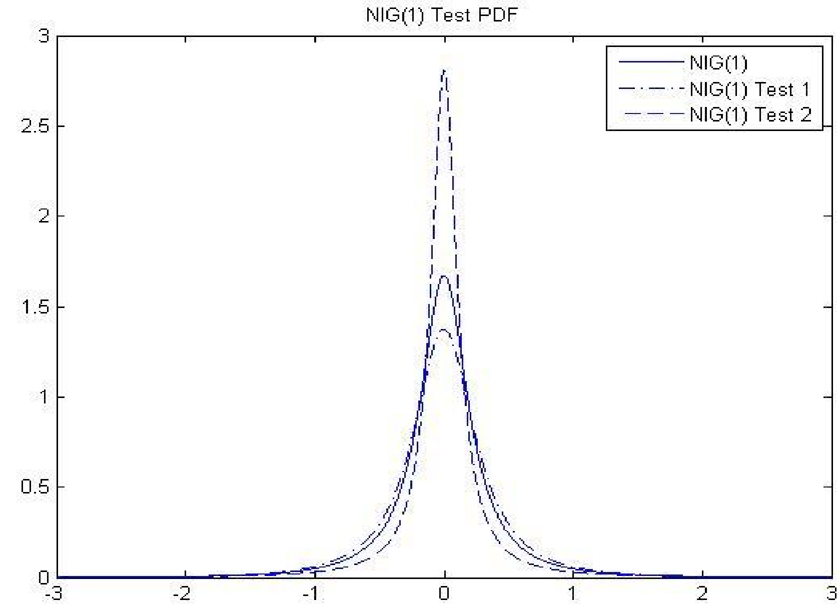
(Based on market quote on 22-September-2008)



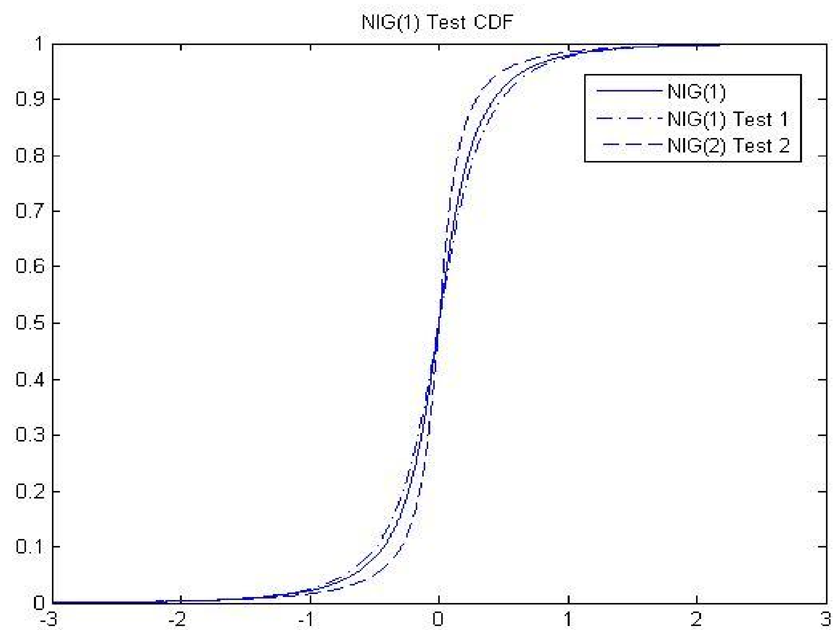
(a) The Probability Density Function of Gaussian and NIG



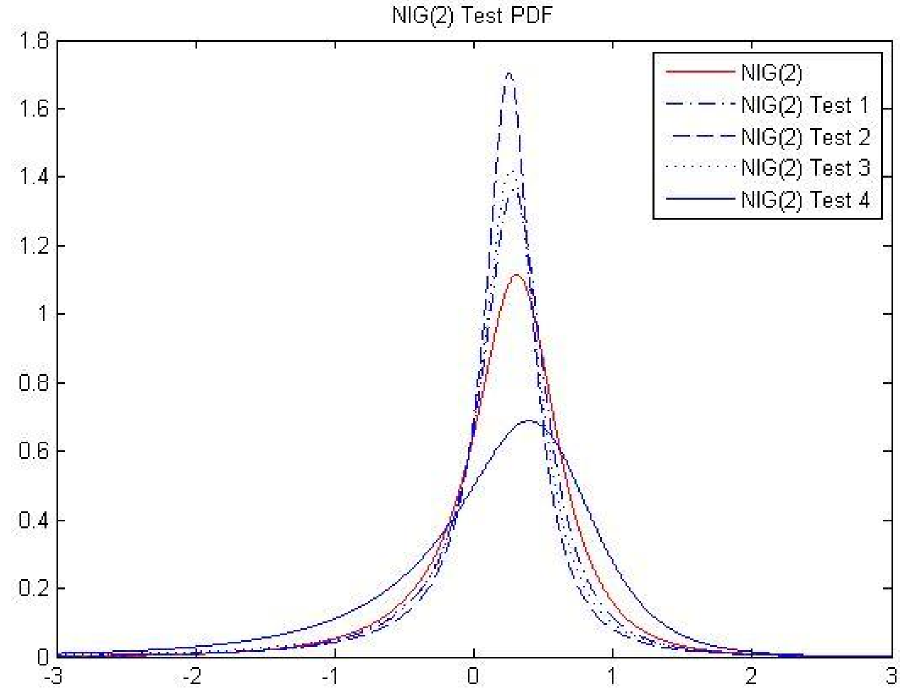
(b) The Cumulative Distribution Function of Gaussian and NIG



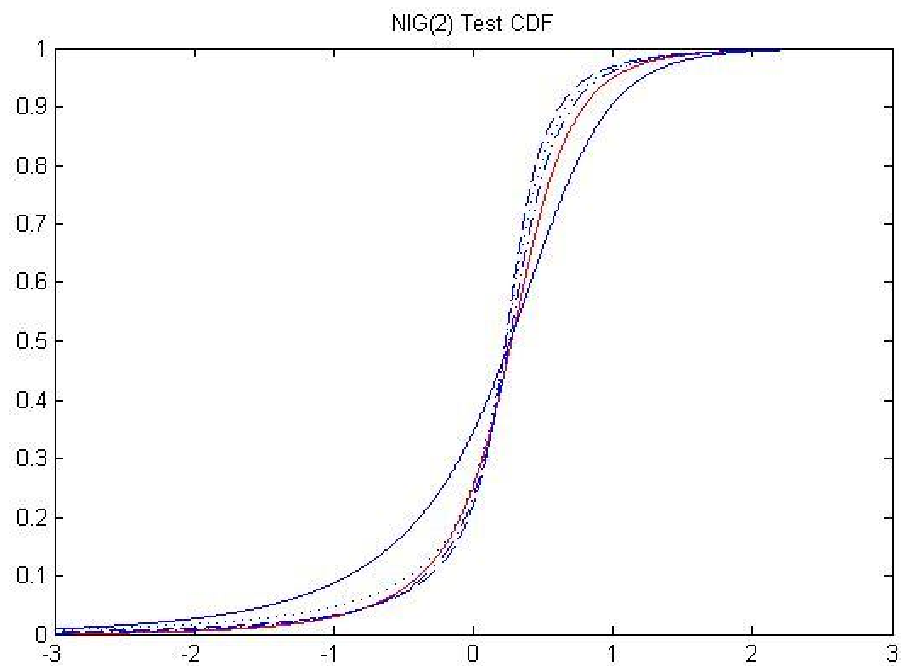
(c) The Probability Density Function of NIG (1)



(d) The Cumulative Distribution Function of NIG (1)



(e) The Probability Density Function of NIG (2)



(f) The Cumulative Distribution Function of NIG (2)

Appendix B:

Table 3.5: The time series data of Default probability

(Implied by the market quote on 22-September-2008 and 20-March-2009)

t	22-Sep-08	20-Mar-09
0.25	0.0065	0.0112
0.5	0.0130	0.0223
0.75	0.0194	0.0333
1	0.0257	0.0442
1.25	0.0321	0.0549
1.5	0.0384	0.0655
1.75	0.0446	0.0760
2	0.0508	0.0864
2.25	0.0570	0.0966
2.5	0.0631	0.1068
2.75	0.0692	0.1168
3	0.0753	0.1267
3.25	0.0813	0.1365
3.5	0.0872	0.1462
3.75	0.0932	0.1558
4	0.0991	0.1653
4.25	0.1049	0.1747
4.5	0.1107	0.1839
4.75	0.1165	0.1931
5	0.1223	0.2021

Table 3.6: The Time Series Data of Default Threshold
based on Gaussian Copula and NIG Copula

(Implied by the market quote on 22-September-2008 and 20-March-2009)

	22-Sep-08			20-Mar-09		
t	D(t)_Gaussian	D(t)_NIG(1)	D(t)_NIG(2)	D(t)_Gaussian	D(t)_NIG(1)	D(t)_NIG(2)
0.25	-2.4838	-2.9555	-3.2848	-2.2826	-2.7179	-3.0420
0.5	-2.2275	-2.4864	-2.7474	-2.0078	-2.1621	-2.4038
0.75	-2.0669	-2.2182	-2.4392	-1.8343	-1.8554	-2.0496
1	-1.9474	-2.0309	-2.2236	-1.7043	-1.6468	-1.8077
1.25	-1.8511	-1.8874	-2.0581	-1.5991	-1.4904	-1.6258
1.5	-1.7699	-1.7713	-1.9242	-1.5101	-1.3661	-1.4810
1.75	-1.6994	-1.6739	-1.8118	-1.4325	-1.2636	-1.3614
2	-1.6369	-1.5902	-1.7150	-1.3634	-1.1767	-1.2599
2.25	-1.5805	-1.5168	-1.6302	-1.3010	-1.1015	-1.1720
2.5	-1.5290	-1.4514	-1.5548	-1.2439	-1.0354	-1.0947
2.75	-1.4816	-1.3926	-1.4869	-1.1911	-0.9765	-1.0259
3	-1.4376	-1.3392	-1.4251	-1.1420	-0.9234	-0.9639
3.25	-1.3965	-1.2902	-1.3685	-1.0961	-0.8752	-0.9075
3.5	-1.3579	-1.2450	-1.3164	-1.0528	-0.8311	-0.8560
3.75	-1.3214	-1.2031	-1.2679	-1.0118	-0.7905	-0.8086
4	-1.2868	-1.1639	-1.2228	-0.9730	-0.7528	-0.7647
4.25	-1.2539	-1.1273	-1.1805	-0.9359	-0.7177	-0.7239
4.5	-1.2226	-1.0928	-1.1408	-0.9005	-0.6849	-0.6857
4.75	-1.1925	-1.0602	-1.1033	-0.8666	-0.6541	-0.6499
5	-1.1637	-1.0294	-1.0678	-0.8340	-0.6250	-0.6161

Table 3.7: The Time Series Data of Expected Loss on Equity Tranche based on Gaussian Copula and NIG Copula

(Implied by the market quote on 22-September-2008 and 20-March-2009)

	22-Sep-08			20-Mar-09		
t	EL(t)_Gaussian	EL(t)_NIG(1)	EL(t)_NIG(2)	EL(t)_Gaussian	EL(t)_NIG(1)	EL(t)_NIG(2)
0.25	0.1294	0.1126	0.1123	0.2017	0.1537	0.1559
0.5	0.2524	0.2197	0.2235	0.3556	0.2944	0.3026
0.75	0.3639	0.3221	0.3288	0.4743	0.4225	0.4313
1	0.4621	0.4195	0.4268	0.5675	0.5370	0.5401
1.25	0.5468	0.5110	0.5167	0.6416	0.6359	0.6301
1.5	0.6190	0.5961	0.5967	0.7012	0.7177	0.7030
1.75	0.6802	0.6731	0.6674	0.7496	0.7813	0.7610
2	0.7316	0.7405	0.7283	0.7892	0.8281	0.8066
2.25	0.7749	0.7968	0.7796	0.8218	0.8613	0.8422
2.5	0.8111	0.8411	0.8220	0.8488	0.8849	0.8699
2.75	0.8414	0.8742	0.8564	0.8713	0.9021	0.8917
3	0.8668	0.8981	0.8840	0.8901	0.9148	0.9088
3.25	0.8880	0.9155	0.9059	0.9059	0.9246	0.9224
3.5	0.9058	0.9284	0.9232	0.9192	0.9323	0.9333
3.75	0.9206	0.9381	0.9369	0.9305	0.9385	0.9421
4	0.9331	0.9457	0.9477	0.9401	0.9436	0.9493
4.25	0.9436	0.9517	0.9563	0.9482	0.9479	0.9553
4.5	0.9523	0.9566	0.9632	0.9552	0.9515	0.9603
4.75	0.9597	0.9607	0.9687	0.9611	0.9546	0.9645
5	0.9659	0.9640	0.9732	0.9662	0.9573	0.9680

Appendix C: Matlab Code

(I) Gaussian Copula

1) EL_NormSDist.m

```
function EL_bivn=EL_NormSDist(K1,K2,R,C,a)

% This function is developed to measure the expected loss of synthetic CDO
% using LHP approach, which is based on one factor Gaussian copula model of
% correlated defaults.
% Author: Shirley Xin, Arvin Wang
% Segal Graduate School of Business, Simon Fraser University
% Date: July 18, 2010

X1=-norminv(K1/(1-R),0,1);
X2=-norminv(K2/(1-R),0,1);
X3=C;

BiVar1=[X1;X3];
BiVar2=[X2;X3];
Mu=[0;0];
Sigma=[1 -sqrt(1-a^2);-sqrt(1-a^2) 1];

Phi1=mvncdf(BiVar1,Mu,Sigma);
Phi2=mvncdf(BiVar2,Mu,Sigma);

EL_bivn=(Phi1-Phi2)/((K2/(1-R))-(K1/(1-R)));

end
```

2) Spread_Gaussian.m

```
clc
clear all
format long

% This Script is developed to calculate the spreadpayment of synthetic CDO
% tranches, based on one factor Gaussian Copula
% Author: Shirley Xin, Arvin Wang
% Segal Graduate School of Business, Simon Fraser University
% Date: July 18, 2010

load CDXSpreadAvg.mat
```

```
% Set parameters
K1=0;
K2=0.03;
R=0.4;
Mat=5;
a=sqrt(0.219201);
rf=0.01670135;
dt=0.25;

% Calculate the discount factor
DiscFact=ones(21,1);
for i=1:1:Mat*4
    DiscFact(i+1)=exp(-rf*i*0.25);
end

% Calculate the expected loss of the tranche
EL=zeros(21,1);
for i=1:1:Mat*4
    EL(i+1)=EL_NormSDist(K1,K2,R,C_gaussian_d2(i),a);
end

% Protection Leg
ProtectLeg=sum(diff(EL).*DiscFact(2:end));
% Premium Leg
PremiumLeg=sum((1-EL(2:end)).*DiscFact(2:end)*dt);
% Spread payment
SpreadPayment=ProtectLeg/PremiumLeg
```

(II) Matlab Code for NIG Copula

*The Matlab toolbox of NIG distribution is available at Matlab Center File Exchange

1) intfunc.m

```
function y=intfunc(x,K1,R,C,a,alpha,beta,df)

% Function developed in order to calculate the Expected Loss
%   y=intfunc(x,K1,R,C,a)
%      x = Unknown parameter
%      K1 = Attachment point of the tranche
%      R = Recovery
%      C = default threshold
%      a = sqrt(rho), where rho is the pairwise correlation of default
%
% Author: Shirley Xin, Arvin Wang
% Segal Graduate School of Business, Simon Fraser University
% Date: July 21, 2010

s=sqrt(1-a^2)/a;
gamma=sqrt(alpha^2-beta^2);
mu=beta*gamma^2/alpha^2;
delta=gamma^3/alpha^2;

temp1=nigcdf(x,s*alpha,s*beta,-s*mu,s*delta)-(K1/(1-R));
temp2=nigpdf((C-sqrt(1-a^2)*x)/a,df*alpha,df*beta,-df*mu,df*delta)*(sqrt(1-a^2)/a);

y=temp1.*temp2;

end
```

2) Spread_NIG.m

```
clc
clear all
format long

% This Script is developed to calculate the spreadpayment of synthetic CDO
% tranches, based on one factor NIG Copula
% Author: Shirley Xin, Arvin Wang
% Segal Graduate School of Business, Simon Fraser University
% Date: July 20, 2010

% NIG toolbox developed by Kalemanova et al is available at Matlab Center
```

```
% File Exchange.

load CDXSpreadAvg.mat

% Set parameter values
K1=1.000000000000000e-074;  % attachment point of the tranche
K2=0.03;                    % detachment point of the tranche
R=0.4;                      % recovery rate
Mat=5;                      % time to maturity
a=sqrt(0.394263);           % a=sqrt(rho), where rho is default correlation
rf=0.01670135;              % 5yr government zero rate
dt=0.25;

alpha=0.6020;   % tail heavyness
beta=-0.1605;        % asymmetry parameter
df=2;           % NIG(df)
gamma=sqrt(alpha^2-beta^2);
s=sqrt(1-a^2)/a;

mu=beta*gamma^2/alpha^2;
delta=gamma^3/alpha^2;

% Calculate the discount factor & Default threshold
DiscFact=ones(21,1);
C_NIG=zeros(20,1);
for i=1:1:Mat*4
   DiscFact(i+1)=exp(-rf*i*0.25);
   C_NIG(i)=niginv(DefProb_d2(i),alpha/a,beta/a,-(1/a)*mu,(1/a)*delta);
end

% Determine the expected loss EL(t)
lowerbound=niginv(K1/(1-R),s*alpha,s*beta,-s*mu,s*delta);
upperbound=niginv(K2/(1-R),s*alpha,s*beta,-s*mu,s*delta);

FtK2=zeros(20,1);
intFtK1=zeros(20,1);
EL=zeros(21,1);
for i=1:1:Mat*4
   FtK2(i)=1-nigcdf((C_NIG(i)-sqrt(1-a^2)*upperbound)/a,df*alpha,df*beta,-df*mu,df*delta);
   intFtK1(i)=quad(@(x)intfunc(x,K1,R,C_NIG(i),a,alpha,beta,df),lowerbound,upperbound);
   EL(i+1)=((1-R)/(K2-K1))*intFtK1(i)+(1-FtK2(i));
end

% Protection Leg
ProtectLeg=sum(diff(EL).*DiscFact(2:end));
% Premium Leg
PremiumLeg=sum((1-EL(2:end)).*DiscFact(2:end)*dt);
% Spread payment
SpreadPayment=ProtectLeg/PremiumLeg
```

(III) Calculation of Absolute Error

1) Abs_error.m

```
function absError=Abs_error(para,df,DefProb,quote)

% This Script is developed to calculate the absolute error bewteen market
% quote and NIG(x) modeling outcome
% Author: Shirley Xin, Arvin Wang
% Segal Graduate School of Business, Simon Fraser University
% Date: July 31, 2010

% Parameter value
% para=[alpha; beta; rho];

% Calculate the modeling outcome
SpreadPmt=zeros(5,1);
SpreadPmt(1)=Spread_func(1.000000000000000e-074,0.03,para(1),para(2),para(3),df,DefProb);
SpreadPmt(2)=Spread_func(0.03,0.07,para(1),para(2),para(3),df,DefProb);
SpreadPmt(3)=Spread_func(0.07,0.1,para(1),para(2),para(3),df,DefProb);
SpreadPmt(4)=Spread_func(0.1,0.15,para(1),para(2),para(3),df,DefProb);
SpreadPmt(5)=Spread_func(0.15,0.3,para(1),para(2),para(3),df,DefProb);

% Compare to the market quote
% Equity_Error=abs(quote(1)-SpreadPmt(1));
absError=sum(abs(SpreadPmt-quote));

% error=[Equity_Error;Abs_Error];

end
```

Appendix D: CDS Index Member List

Index: MARKIT CDX.NA.IG.9* 12/12

Spread Ticker: IBOXUG59
RED Code: 2165BYCG8

Effective Date: 09/21/07
Maturity Date: 12/20/12

Settlement Currency: USD
Fixed Rate: 40% per annum
Fixed Day Count Fraction: Actual / 360
Fixed Rate Payer Payment Dates: Each March 20, June 20, September 20 and December 20, commencing on September 21, 2007
Credit Events: Bankruptcy, failure to pay, restructuring

Name	Weight	Equity Ticker	Corp Ticker	5 Yr CDS Ticker
ACE Ltd	0.8	ACE US	ACE	CACE1U5
Aetna Inc	0.8	AET US	AET	CAET1U5
Rio Tinto Alcan Inc	0.8	AL CN	RIOLN	CAL1U5
Alcoa Inc	0.8	AA US	AA	CAA1U5
Altria Group Inc	0.8	MO US	MO	CMO1U5
American Electric Power Co Inc	0.8	AEP US	AEP	CAEP1U5
American Express Co	0.8	AXP US	AXP	CAXP1U5
American International Group Inc	0.8	AIG US	AIG	CAIG1U5
Amgen Inc	0.8	AMGN US	AMGN	CAMG1U5
Anadarko Petroleum Corp	0.8	APC US	APC	CAPC1U5
Arrow Electronics Inc	0.8	ARW US	ARW	CARW1U5
AT&T Inc	0.8	T US	T	CSBC1U5
AT&T Mobility LLC	0.8	24004Z US	T	CCNG1U5
AutoZone Inc	0.8	AZO US	AZO	CAZO1U5
Baxter International Inc	0.8	BAX US	BAX	CBAX1U5
Belo Corp	0.8	BLC US	BLC	CBLC1U5
Boeing Capital Corp	0.8	8891Z US	BA	CBACC1U5
Bristol-Myers Squibb Co	0.8	BMY US	BMY	CBMY1U5
Burlington Northern Santa Fe LLC	0.8	BNI US	BRK	CBNI1U5

Campbell Soup Co	0.8	CPB US	CPB	CCPB1U5
Capital One Bank USA NA	0.8	8125Z US	COF	CCOF1U5
Cardinal Health Inc	0.8	CAH US	CAH	CCAH1U5
Carnival Corp	0.8	CCL US	CCL	CCCL1U5
Caterpillar Inc	0.8	CAT US	CAT	CCAT1U5
CBS Corp	0.8	CBS US	CBS	CVIA1U5
Centex Corp	0.8	CTX US	PHM	CCTX1U5
CenturyLink Inc	0.8	CTL US	CTL	CCTL1U5
CIGNA Corp	0.8	CI US	CI	CCI1U5
CIT Group Inc/Old	0	CITGQ US	CIT	CCITG1U5
Comcast Cable Communications LLC	0.8	15659Z US	CMCSA	CCCC1U5
Computer Sciences Corp	0.8	CSC US	CSC	CCCS1U5
ConAgra Foods Inc	0.8	CAG US	CAG	CCAG1U5
ConocoPhillips	0.8	COP US	COP	CCOC1U5
Constellation Energy Group Inc	0.8	CEG US	CEG	CCEG1U5
Countrywide Home Loans Inc	0.8	8191Z US	BAC	CCCR1U5
COX Communications Inc	0.8	COX US	COXENT	CCOX1U5
CSX Corp	0.8	CSX US	CSX	CCSX1U5
CVS Caremark Corp	0.8	CVS US	CVS	CCVS1U5
Darden Restaurants Inc	0.8	DRI US	DRI	CDRI1U5
Deere & Co	0.8	DE US	DE	CDE1U5
Devon Energy Corp	0.8	DVN US	DVN	CDVN1U5
Dominion Resources Inc/VA	0.8	D US	D	CDR1U5
Duke Energy Corp	0.8	DUK US	DUK	CDUK1U5
EI du Pont de Nemours & Co	0.8	DD US	DD	CDD1U5
Eastman Chemical Co	0.8	EMN US	EMN	CEMN1U5
Embarq Corp	0.8	EQ US	CTL	CX361172
Federal Home Loan Mortgage Corp	0	FMCC US	FHLMC	CFHLM1U5
Federal National Mortgage Association	0	FNMA US	FNMA	CFNMA1U5
FirstEnergy Corp	0.8	FE US	FE	CFE1U5
Fortune Brands Inc	0.8	FO US	FO	CFO1U5
Gannett Co Inc	0.8	GCI US	GCI	CGCI1U5
General Electric Capital Corp	0.8	GELK US	GE	CGECC1U5
General Mills Inc	0.8	GIS US	GIS	CGIS1U5

Goodrich Corp	0.8	GR US	GR	CGR1U5
Halliburton Co	0.8	HAL US	HAL	CHAL1U5
Hewlett-Packard Co	0.8	HPQ US	HPQ	CHWP1U5
Honeywell International Inc	0.8	HON US	HON	CHON1U5
IAC/InterActiveCorp	0.8	IACI US	IACI	CX354186
Ingersoll-Rand Co	0.8		IR	CT761912
International Business Machines Corp	0.8	IBM US	IBM	CIBM1U5
International Lease Finance Corp	0.8	ILFC US	AIG	CILFC1U5
International Paper Co	0.8	IP US	IP	CIP1U5
iStar Financial Inc	0.8	SFI US	SFI	CT351304
JC Penney Co Inc	0.8	JCP US	JCP	CJCP1U5
Jones Apparel Group Inc	0.8	JNY US	JNY	CJNY1U5
Kraft Foods Inc	0.8	KFT US	KFT	CKFT1U5
Lennar Corp	0.8	LEN US	LEN	CLEN1U5
Ltd Brands Inc	0.8	LTD US	LTD	CLTD1U5
Liz Claiborne Inc	0.8	LIZ US	LIZ	CLIZ1U5
Lockheed Martin Corp	0.8	LMT US	LMT	CLMT1U5
Loews Corp	0.8	L US	L	CLTR1U5
Macy's Inc	0.8	M US	M	CFD1U5
Marriott International Inc/DE	0.8	MAR US	MAR	CMAR1U5
Marsh & McLennan Cos Inc	0.8	MMC US	MMC	CMMC1U5
MBIA Insurance Corp	0.8	16302Z US	MBI	CMBIN1U5
McDonald's Corp	0.8	MCD US	MCD	CMCD1U5
McKesson Corp	0.8	MCK US	MCK	CMCK1U5
MeadWestvaco Corp	0.8	MWV US	MWV	CMWV1U5
MetLife Inc	0.8	MET US	MET	CMET1U5
Motorola Inc	0.8	MOT US	MOT	CMOT1U5
National Rural Utilities Cooperative Finance Corp	0.8	2381A US	NRUC	CNRUC1U5
Newell Rubbermaid Inc	0.8	NWL US	NWL	CNWL1U5
News America Inc	0.8	14408Z US	NWSA	CNCP1U5
Nordstrom Inc	0.8	JWN US	JWN	CJWN1U5
Norfolk Southern Corp	0.8	NSC US	NSC	CNSC1U5
Northrop Grumman Corp	0.8	NOC US	NOC	CNOC1U5
Omnicom Group Inc	0.8	OMC US	OMC	COMC1U5

Progress Energy Inc	0.8	PGN US	PGN	CPGN1U5
Pulte Group Inc	0.8	PHM US	PHM	CPHM1U5
Quest Diagnostics Inc/DE	0.8	DGX US	DGX	CDGX1U5
RR Donnelley & Sons Co	0.8	RRD US	RRD	CX359760
Radian Group Inc	0.8	RDN US	RDN	CRDN1U5
Raytheon Co	0.8	RTN US	RTN	CRTN1U5
Rohm and Haas Co	0.8	ROH US	DOW	CROH1U5
Safeway Inc	0.8	SWY US	SWY	CSWY1U5
Sara Lee Corp	0.8	SLE US	SLE	CSLE1U5
Sempra Energy	0.8	SRE US	SRE	CSRE1U5
Simon Property Group LP	0.8	12968Z US	SPG	CSPG1U5
Southwest Airlines Co	0.8	LUV US	LUV	CLUV1U5
Sprint Nextel Corp	0.8	S US	S	CT357422
Starwood Hotels & Resorts Worldwide Inc	0.8	HOT US	HOT	CHOT1U5
Target Corp	0.8	TGT US	TGT	CTGT1U5
Textron Financial Corp	0.8	3339Z US	TXT	CTXTF1U5
Allstate Corp/The	0.8	ALL US	ALL	CALL1U5
Chubb Corp	0.8	CB US	CB	CCB1U5
Dow Chemical Co/The	0.8	DOW US	DOW	CDOW1U5
Hartford Financial Services Group Inc	0.8	HIG US	HIG	CHIG1U5
Home Depot Inc	0.8	HD US	HD	CHD1U5
Kroger Co/The	0.8	KR US	KR	CKR1U5
Sherwin-Williams Co/The	0.8	SHW US	SHW	CSHW1U5
Walt Disney Co/The	0.8	DIS US	DIS	CDIS1U5
Time Warner Inc	0.8	TWX US	TWX	CAOL1U5
Toll Brothers Inc	0.8	TOL US	TOL	CTOL1U5
Transocean Inc	0.8	3196976Z US	RIG	CRIG1U5
Union Pacific Corp	0.8	UNP US	UNP	CUNP1U5
Universal Health Services Inc	0.8	UHS US	UHS	CT357677
Valero Energy Corp	0.8	VLO US	VLO	CVLO1U5
Verizon Communications Inc	0.8	VZ US	VZ	CVZGF1U5
Wal-Mart Stores Inc	0.8	WMT US	WMT	CWMT1U5
Washington Mutual Inc	0	WAMUQ US	WM	CWM1U5
Wells Fargo & Co	0.8	WFC US	WFC	CWFC1U5

Weyerhaeuser Co	0.8	WY US	WY	CWY1U5
Whirlpool Corp	0.8	WHR US	WHR	CWHR1U5
Wyeth	0.8	WYE US	PFE	CAHP1U5
XL Group Plc	0.8	XL US	XL	CXL1U5