THREE CONTRIBUTIONS
TO THE THEORY OF RECURSIVELY ENUMERABLE CLASSES

by

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ABSTRACT

Priority arguments are applied to three problems in the theory of r.e. classes.

Chapter I: A conjecture of P. R. Young in A Theorem on Recursively Enumerable Classes and Splinters, PAMS 17,5 (1966), pp. 1050-1056, that an r.e. class can be constructed with any pre-assigned finite number of infinite r.e. subclasses, is answered in the affirmative.

Chapter II: Standard classes and indexable classes were introduced by A. H. Lachlan (cf. On the Indexing of Classes of Recursively Enumerable Sets, JSL 31 (1966), pp. 10-22). A class $\mathcal{C}$ of r.e. sets is called sequence enumerable if the r.e. $\mathcal{C}$ sequences can all be enumerated simultaneously, and subclass enumerable if the r.e. subclasses of $\mathcal{C}$ can all be enumerated simultaneously. It is shown that if $\mathcal{C}$ is a class of r.e. sets, $\mathcal{C}$ is standard $\Rightarrow$ $\mathcal{C}$ is sequence enumerable $\Rightarrow$ $\mathcal{C}$ is indexable $\Rightarrow$ $\mathcal{C}$ is subclass enumerable, but none of the implications can be reversed.

Chapter III: A partially ordered set $(\mathcal{P},\prec)$ is represented by the r.e. class $\mathcal{C}$ if $(\mathcal{P},\prec)$ is isomorphic to $(\mathcal{C},\subseteq)$. Sufficiently many p.o. sets are proved representable to verify a conjecture of A. H. Lachlan that representable p.o. sets and arbitrary p.o. sets are indistinguishable by elementary sentences.

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INTRODUCTION

Recursively enumerable classes were defined and first investigated by Rice [2]. For general information on recursive functions we refer the reader to the text of Kleene [1]. The "priority method" of Friedberg and Muchnik is discussed in Sacks [4].

Chapter 1 is to appear under the title "Infinite Subclasses of Recursively Enumerable Classes" in the Proceedings of the American Mathematical Society. The main results of Chapter 2 were announced at the American Mathematical Society Annual Meeting in Houston, January, 1967.

REVIEW OF BASIC DEFINITIONS. These are mostly standard in the literature.

N will denote the set of natural numbers, and N^n the cartesian product of N with itself n times.

We will take as primitive the notion of <u>n argument partial recursive (p.r.) function</u>. Intuitively, this is a function from a subset of N^n into N which can be effectively enumerated when considered as a set of ordered (n+1)-tuples. <u>Church's Thesis</u> identifies this intuitive idea with various provably equivalent formal definitions (see [1]). We will make free use of Church's Thesis.

<u>Definition 0.1</u> An <u>n argument recursive function</u> is an n argument p.r. function whose domain is N^n.

<u>Definition 0.2</u> $R \subseteq N^n$ is a <u>recursive subset of N^n</u> if the

characteristic function f of R defined by

$$f(x_1,\ldots,x_n) = \begin{cases} 0 \text{ if } (x_1,\ldots,x_n) \in R \\ 1 \text{ if } (x_1,\ldots,x_n) \notin R \end{cases}$$

is an n argument recursive function. That is, we can effectively decide membership of R.

Definition 0.3 $S \subseteq N^n$ is in Σ_m if

$(x_1,\ldots,x_n) \in S \Leftrightarrow (Q_1 y_1)\ldots(Q_m y_m)[(x_1,\ldots,x_n,y_1,\ldots,y_m) \in R]$

where R is a recursive subset of N^{n+m} and $Q_1,\ldots,Q_m$ are m alternating quantifiers of which the first is $\exists$.

$S \subseteq N^n$ is in Π_m ... (replace $\exists$ by $\forall$).

Definition 0.4 $S \subseteq N^n$ is a recursively enumerable (r.e.) subset of N^n if S is in Σ_1. That is, we can effectively enumerate S.

Definition 0.5 A sequence $\langle S_{x_1,\ldots,x_m} \mid x_1,\ldots,x_m \geq 0\rangle$ of r.e. subsets of N^n is an m-dimensional r.e. sequence of r.e. subsets of N^n if

$\{(x_1,\ldots,x_m,y_1,\ldots,y_n) \mid (y_1,\ldots,y_n) \in S_{x_1,\ldots,x_m}\}$ is an r.e. subset of N^{n+m}.

Definition 0.6 A sequence $\langle \varphi_{x_1,\ldots,x_m} \mid x_1,\ldots,x_m \geq 0\rangle$ of n argument p.r. functions is an m-dimensional r.e. sequence of n argument p.r. functions if the function φ defined by

$\varphi(x_1,\ldots,x_m,y_1,\ldots,y_n) \equiv \varphi_{x_1,\ldots,x_m}(y_1,\ldots,y_n)$ is an $(m+n)$ argument p.r. function.

<u>Definition 0.7</u> A class $\mathcal{C}$ of r.e. subsets of N^n is an <u>r.e. class of r.e. subsets of N^n</u> if either $\mathcal{C}$ is empty or there is a 1-dimensional r.e. sequence $<S_x \mid x \geq 0>$ of r.e. subsets of N^n s.t.* $\mathcal{C} = \{S_x \mid x \geq 0\}$. That is, not only can each member of $\mathcal{C}$ be effectively enumerated, but they can all be effectively enumerated simultaneously. Such a sequence as $<S_x \mid x \geq 0>$ is called a <u>recursive enumeration</u> of $\mathcal{C}$.

<u>Definition 0.8</u> A class $\mathcal{C}$ of n argument p.r. functions is an <u>r.e. class of n argument p.r. functions</u> ... (obtained from 0.6 as 0.7 is from 0.5).

FUNDAMENTAL THEOREM. (Church, Kleene, Post, Turing)
The class of all n argument p.r. functions is an r.e. class.
The class of all r.e. subsets of N^n is an r.e. class.

CONVENTION. Although it will be convenient to have the above definitions in their full generality, we make the following convention. Unless otherwise stated,

<u>recursive set</u> means recursive subset of N

<u>r.e. set</u> means r.e. subset of N

<u>r.e. sequence</u> means 1-dimensional r.e. sequence of r.e. sets.

<u>r.e. class</u> means r.e. class of r.e. sets.

* such that

CHAPTER 1

INFINITE RECURSIVELY ENUMERABLE SUBCLASSES

P.R. Young [7] has constructed an infinite r.e. class with no proper infinite r.e. subclasses, and has asked if infinite r.e. classes with m+1 infinite r.e. subclasses exist for every $m \geq 0$. It can further be asked what is the most general partially ordered set we can represent by the system of the infinite r.e. subclasses of such a class (under inclusion). These questions are answered by the theorem below. Our construction is based on a formulation of Young's due to A. H. Lachlan.

THEOREM. (a) Let $m \geq 1$, $n \geq 1$. Let $\{F_i \mid 1 \leq i \leq m+1\}$ be a class of subsets of $\{x \mid 1 \leq x \leq n\}$ closed to subsets and with

$$\bigcup \{F_i \mid 1 \leq i \leq m+1\} = \{x \mid 1 \leq x \leq n\}.$$

Then there is an infinite class $\mathcal{C}^*$ of r.e. sets and distinct r.e. sets $A_1, \ldots, A_n$ not in $\mathcal{C}^*$ such that

$$\mathcal{C} = \mathcal{C}^* \cup \{A_i \mid 1 \leq i \leq n\}$$

is an infinite r.e. class with infinite r.e. subclasses

$$\mathcal{C} - \{A_i \mid i \in F_j\} \ (1 \leq j \leq m+1).$$

(b) There is an infinite r.e. class with one infinite r.e. subclass.

CONVERSELY, any infinite r.e. class with finitely many infinite r.e. subclasses is of one of these forms.

PROOF. We prove the converse first. Let $\mathcal{C}$ be an infinite r.e. class with $m+1(m\geq 1)$ infinite r.e. subclasses. If $\mathcal{C}_1$ is an r.e. subclass and $X\in\mathcal{C}-\mathcal{C}_1$, then $\mathcal{C}_1\cup\{X\}$ is an r.e. subclass. It follows that each infinite r.e. subclass lacks only finitely many members of $\mathcal{C}$. Let $\mathcal{C}^*$ be the intersection of the infinite r.e. subclasses, then $\mathcal{C}-\mathcal{C}^*$ is finite, with members $A_1,\dots,A_n$ say $(n\geq 1)$. Now the infinite r.e. subclasses of $\mathcal{C}$ have the form

$$\mathcal{C}-\{A_i \mid i\in F_j\} \quad (1\leq j\leq m+1)$$

where the F_j are subsets of $\{x \mid 1\leq x\leq n\}$. These are closed to subsets, for if $F\subseteq F_j$

$$\mathcal{C}-\{A_i \mid i\in F\} = (\mathcal{C}-\{A_i \mid i\in F_j\})\cup\{A_i \mid i\in F_j-F\}$$

and the union of two r.e. classes is an r.e. class. Also, by definition of $A_1,\dots,A_n$,

$$\bigcup\{F_i \mid 1\leq i\leq m+1\} = \{x \mid 1\leq x\leq n\}.$$

This completes the proof of the converse.

Note that in (a) $m+1\leq 2^n$. We give a construction for the case $m+1<2^n$ and obtain as corollaries the case $m+1=2^n$ and (b), both of which Young has already proved.

Since $m+1<2^n$ there are subsets $G_1,\dots,G_l$ say of $\{x \mid 1\leq x\leq n\}$ different from all the F_i. For each $i,k(1\leq i\leq l, 1\leq k\leq m+1)$ there is a number

$$p(i,k)\in G_i-F_k,$$

since the F_i are closed to subsets. For each k with $i\leq k\leq m+1$ there will be a different variation of the construction. We will show that these variations in the construction all give rise to the same $\mathcal{C}^*$, $A_1,\ldots,A_n$.

At this point we make some informal remarks in an attempt to motivate the construction which follows. $\mathcal{C}$ will be enumerated in an r.e. sequence

$$\langle A_1,\ldots,A_n, V_0, V_1,\ldots\rangle.$$

To begin with the (x+1)-st member of the sequence is just $\{x\}$. There will be an increasing function r(i) such that the V_i different from each of $A_1,\ldots,A_n$ are $V_{r(0)}, V_{r(1)}\ldots$. These will be disjoint from each other and from each of $A_1,\ldots,A_n$. At step s of the construction we will work with a recursive approximation to r(i), namely r(i,s). We must ensure (see Lemma 1 below) that for all sufficiently large s, r(i,s) is constant. Call this Requirement r(i). Let W_e be the r.e. set with index e in some recursive enumeration of the r.e. sets. By "amalgamating" members of the sequence, we will satisfy what we can call Requirement W_e, which is : if W_e intersects infinitely many of $V_{r(0)}, V_{r(1)},\ldots$ it intersects them all and

$1 \le i \le l \Rightarrow (Ex)[x\epsilon W_e \& (z)(x\epsilon A_z \Leftrightarrow z\epsilon G_i)]$ (see Lemma 2 below).

Now if we take W_e to be $\bigcup \mathcal{C}_1$ where $\mathcal{C}_1$ is an infinite r.e. subclass we get for all i $V_{r(i)} \in \mathcal{C}_1$ and for all i with $1 \le i \le l$ there is $z\epsilon G_i$ with $A_z \in \mathcal{C}_1$. Thus the only possibilities for $\mathcal{C}_1$ are

$$\mathcal{C} - \{A_z \mid z \in F_k\} \quad (1 \le k \le m+1).$$

A_x, $r(x)$, and $V_{r(x)}$ will be independent of the variation used to get the sequence

$$\langle A_1, \ldots, A_n, V_0, V_1, \ldots \rangle,$$

but in variation k use of the function p will ensure that for no y,z do we have

$$V_y = A_z \text{ with } z \in F_k,$$

and thus an enumeration of $\mathcal{C} - \{A_z \mid z \in F_k\}$ is easily obtained.

In carrying out the construction, the following difficulty is encountered: an "amalgamation" which we wish to make satisfy Requirement W_e will cause $r(i,s+1) \neq r(i,s)$. That is to say, Requirement W_e and Requirement $r(i)$ conflict. We therefore assign priorities to our Requirements as follows

$$r(0),\ W_0,\ r(1),\ W_1, \ldots .$$

Thus, when the conflict above occurs, we make the amalgamation only if $e<i$. It is now possible to deal with this situation, where only finitely many Requirements conflict with a given Requirement.

VARIATION k OF THE CONSTRUCTION. Let $\langle W_e \mid e \ge 0 \rangle$ be an r.e. sequence enumerating all the r.e. sets. Let $P = \{(d,e) \mid d \in W_e\}$. The s-th pair will mean the s-th member of P to appear in an effective enumeration of P without repetitions. We define

$$W_e^s = \{d \mid (d,e) \text{ is among the first s pairs}\}.$$

We define for each $s \geq 0$, by induction on s. sets $A_1^s, \ldots, A_n^s$ and a sequence of sets $\langle V_0^s, V_1^s, \ldots \rangle$. Let

$$A_x^0 = \{x-1\} \ (1 \leq x \leq n)$$

$$V_x^0 = \{n+x\} \ (x \geq 0).$$

Assume for induction

(1) the V_x^s, A_x^s are all finite

(2) the A_x^s are all different, in fact $x-1 \in A_x^s - \bigcup\{A_y^s \mid y \neq x\}$

(3) only finitely many of the V_x^s can be the same

(4) if V_x^s is different from all the A_y^s then it is disjoint from them all

(5) of the V_x^s which differ from all the A_y^s, different ones are disjoint

(6) V_0^s is different from all the A_x^s ,

properties evidently possessed by the A_x^0, V_x^0 .

We can define a function $r(i,s)$ by

$$r(0,s) = 0$$

$$r(i+1,s) = \mu x[x > r(i,s) \ \& \ V_x^s \neq A_j^s \text{ for } 1 \leq j \leq n$$
$$\& \ V_x^s \neq V_{r(j,s)}^s \text{ for } 0 \leq j \leq i].$$

Suppose the (s+1) -st pair is (d,e) and there is a $j > e$ such that $d \in V_{r(j,s)}^s$. There is at most one such j by (5). Define

$$R(e,s,i,x) = x \in W_e^{s+1} \ \& \ (z) \ (x \in A_z^s \Leftrightarrow z \in G_i).$$

Case 1. There is i with $1 \leq i \leq l$ and $\sim(Ex)R(e,s,i,x)$. Let i be the least such number. Put

$$A_x^{s+1} = A_x^s \cup V_{r(j,s)}^s \qquad \text{if } x \in G_i$$

$$A_x^{s+1} = A_x^s \qquad \text{if } x \notin G_i$$

$$V_x^{s+1} = A_y^{s+1} \qquad \text{if } V_x^s = A_y^s \text{ for some } y \text{(there can be only one such } y \text{ by (2))}$$

$$V_x^{s+1} = A_{p(i,k)}^s \cup V_{r(j,s)}^s \qquad \text{if } V_x^s = V_{r(j,s)}^s$$

$$V_x^{s+1} = V_x^s \qquad \text{otherwise}$$

This is where variation k arises.

Case 2. Case 1 does not obtain but there is i with $i<j$ and $W_e^{s+1} \cap V_{r(i,s)}^s = \emptyset$. Let i be the least such number. Put

$$A_x^{s+1} = A_x^s \qquad \text{for all } x$$

$$V_x^{s+1} = V_{r(i,s)}^s \cup V_{r(j,s)}^s \qquad \text{if } V_x^s = V_{r(i,s)}^s \text{ or } V_{r(j,s)}^s$$

$$V_x^{s+1} = V_x^s \qquad \text{otherwise.}$$

If there is no $j>e$ such that $d \in V_{r(j,s)}^s$ or if neither Case 1 nor Case 2 occurs put

$$A_x^{s+1} = A_x^s \quad \text{for all } x$$

$$V_x^{s+1} = V_x^s \quad \text{for all } x.$$

We show that (1),..., (6) are preserved. We have $A_x^s \subseteq A_x^{s+1}$, $V_x^s \subseteq V_x^{s+1}$ (so that the A_x^s, V_x^s are all non-empty), $V_x^s = V_y^s \Rightarrow V_x^{s+1} = V_y^{s+1}$, $V_x^s = A_y^s \Rightarrow V_x^{s+1} = A_y^{s+1}$ for all x,y.
(1) and (3) are clear.

For (2) we need consider only Case 1, where the result follows by induction hypothesis (4) and the definition of r.

For (4) and (5), consider V_x^{s+1}, V_y^{s+1} different from each other and from all the A_z^{s+1}. Then V_x^s, V_y^s are different from each other and from all A_z^s, so by induction hypothesis (4), (5) they are disjoint from each other and from all the A_z^s. The desired conclusion is that V_x^{s+1}, V_y^{s+1} are disjoint from each other and from all the A_z^{s+1}. If Case 1 occurs, then neither V_x^s nor V_y^s is equal to $V_{r(j,s)}^s$, for $V_x^s = V_{r(j,s)}^s$ implies that $V_x^{s+1} = A_{p(i,k)}^s \cup V_{r(j,s)}^s = A_{p(i,k)}^{s+1}$. Thus $V_x^{s+1} = V_x^s$ and $V_y^{s+1} = V_y^s$. The two are therefore disjoint. Also V_x^s is disjoint from $V_{r(j,s)}^s$ and from A_z^s for all z, and so V_x^{s+1} is disjoint from A_z^{s+1} for all z. If Case 2 occurs at least one V_x^s, V_y^s is different from both $V_{r(i,s)}^s$ and $V_{r(j,s)}^s$, or $V_x^{s+1} = V_y^{s+1}$. If both have this property then $V_x^{s+1}=V_x^s$ and $V_y^{s+1} = V_y^s$ and the result follows since $A_z^{s+1} = A_z^s$ for all z. This leaves the case where say $V_x^s = V_{r(i,s)}^s$ and V_y^s differs from both $V_{r(i,s)}^s$ and $V_{r(j,s)}^s$, then $V_y^{s+1} = V_y^s$ and the result follows by induction hypotheses (4), (5).

(6) follows by a similar argument, using the fact that in the construction $j>0$.

Define $A_x = \bigcup \{A_x^s \mid s \geq 0\}$ $(1 \leq x \leq n)$

$$V_x = \bigcup \{V_x^s \mid s \geq 0\} \quad (x \geq 0).$$

We can find the members of A_x^s, V_x^s effectively from x, s so

$$\langle A_1,\ldots,A_n,\ V_0,V_1,\ldots\rangle$$

is an r.e. sequence enumerating an r.e. class $\mathcal{C}$. Let $\mathcal{C}^* = \mathcal{C} - \{A_1,\ldots,A_n\}$.

LEMMA 1. For each x and all sufficiently large (s.l.)s, $r(x,s)$ is a constant, $r(x)$ say.

PROOF OF LEMMA 1. By induction on x. $r(0,s) = 0$ for all s so $r(0) = 0$. We suppose the result holds for all $y \le x$ and we show it holds for $x+1$. There is an s_0 such that if $s \ge s_0$ $r(y,s) = r(y)$ for all $y \le x$. In Case 1 or Case 2

$$r(z,s+1) = r(z,s) \qquad \text{if } z<j$$
$$r(z,s+1) = r(z+1,s) \qquad \text{if } z \ge j,$$

for if we divide the members of the sequence

$$V_0^s\ ,\ V_1^s\ ,\ldots$$

which differ from each of $A_1^s,\ldots,A_n^s$ into equivalence classes under set equality, $V^s_{r(z,s)}$ is the first member of the $(z+1)$-st such class, and by (4) and (5) the only effect of either case on the computation of r is the loss of the original $(j+1)$-st class. So if $r(x+1,s+1) \ne r(x+1,s)$ with $s \ge s_0$, Case 1 or Case 2 occurs with $x+1 \ge j > e$ and by our induction hypothesis $x+1 = j$. If Case 1 occurs we have $R(e,t,i,d)$ for all $t>s$ by (4), because d does not belong to $V^t_{r(j,t)}$ for any j. Now $r(x+1,s+1) \ne r(x+1,s)$ can hold for only finitely many $s \ge s_0$ through Case 1 -- at most 1 times for each $e<x+1$, and through Case 2 -- at most $x+1$ times for each $e<x+1$ by our induction hypothesis. Thus

$r(x+1,s) = r(x+1)$, a constant, for all s.l.s. Q.E.D.

By (2) $A_1^s,\ldots,A_n^s$ are distinguished by the numbers $0,\ldots, n-1$ for all s, so $A_1,\ldots,A_n$ are.

We now wish to prove that the $V_{r(u)}$ are distinct, disjoint from each other and from all the A_u. Suppose $x \in V_{r(u)} \cap V_{r(v)}$ with $u \neq v$. Then for all s.l.s $x \in V^s_{r(u)} \cap V^s_{r(v)}$, so for all s.l.s $x \in V^s_{r(u,s)} \cap V^s_{r(v,s)}$ contradicting (5). Similarly $x \in V_{r(u)} \cap A_v$ contradicts (4). The $V_{r(u)}$ are therefore distinct since they are non-empty.

Now we show that for all x, either $V_x = V_{r(u)}$ for some u or $V_x = A_{p(u,k)}$ for some u. It follows by induction on s that if V_x^s is equal to one of the A_y^s then it is equal to $A^s_{p(u,k)}$ for some u. Suppose that there is no u such that $V_x = V_{r(u)}$. There is then a u such that $r(u)<x<r(u+1)$. Consider s s.l. that $r(v,s) = r(v)$ for all $v \leq u+1$. Then $V_x^s = A^s_{p(z,k)}$ for some z, in which case $V_x = A_{p(z,k)}$, or $V_x^s = V^s_{r(v,s)}$ for some $v \leq u$, in which case $V_x = V_{r(v)}$.

By induction on s, A_x^s, $r(x,s)$ and $V^s_{r(x,s)}$ are all independent of k. Thus A_x, $r(x)$ and $V_{r(x)}$ are all independent of k.

LEMMA 2. If W_e intersects infinitely many of $V_{r(0)}$, $V_{r(1)},\ldots,$ then

(7) $1 \leq i \leq l \Rightarrow (Ex)R(e,i,x)$, where we define

$$R(e,i,x) = x \in W_e \text{ and } (z)(x \in A_z \Leftrightarrow z \in G_i)$$

(8) $i \geq 0 \Rightarrow (Ex)[x \in W_e \cap V_{r(i)}]$.

PROOF OF LEMMA 2. First we have: if a, t are any given numbers there is $s>t$ and $j>a$ such that the $(s+1)$-st pair is (d,e) and $d \epsilon V^s_{r(j,s)}$. For there are infinitely many $y>a$ such that W_e intersects $V_{r(y)}$. Also these $V_{r(y)}$ are disjoint. So for infinitely many members d of W_e, $d \epsilon V_{r(y)}$ for some $y>a$. So there is $s>t$ such that (d,e) is the $(s+1)$-st pair and $d \epsilon V_{r(y)}$ with $y>a$. Now $d \epsilon V^s_{r(j,s)}$ for some j. Suppose $j<y$. Consider $u>s$ s.l. that $r(y,u) = r(y)$ and $d \epsilon V^u_{r(y)}$. Then $d \epsilon V^u_{r(y,u)} \cap V^u_{r(j,s)}$ so by (4) and (5) $V^u_{r(y,u)} = V^u_{r(j,s)}$, but $r(y,u)>r(j,u) \geq r(j,s)$ contradicting the definition of $r(y,u)$. Thus $j \geq y>a$.

Now suppose there is a least $i \leq l$ such that $\sim(Ex)R(e,i,x)$. For each $y<i$ let $d(y)$ be such that $R(e,y,d(y))$. Let t be s.l. that for each $y<i$ $R(e,t,y,d(y))$. Put $a = e$ and let s, j correspond to a, t as above. Since for each $y<i$ we have $R(e,s,y,d(y))$ and $\sim(Ex)R(e,s,i,x)$ Case 1 ensures that $R(e,s+1,i,d)$ and so $R(e,i,d)$, contradiction.

Suppose there is a least i such that $\sim(Ex)[x \epsilon W_e \cap V_{r(i)}]$. Let t be s.l. that $r(i,t) = r(i)$, for each $y<i$ $(Ex)[x \epsilon W^{t+1}_e \cap V^t_{r(y,t)}]$ and $r(y,t) = r(y)$ and for $1 \leq y \leq l$ $R(e,t,y,d(y))$ ($d(y)$ as above). Put $a = \max(e,i)$ and let s, j correspond to a, t as above. Then Case 2 ensures that $d \epsilon W^{s+1}_e \cap V^{s+1}_{r(i,s+1)}$ so $(Ex)[x \epsilon W_e \cap V_{r(i)}]$, contradiction. Q.E.D.

Recall that $\mathcal{C}$ has distinct members

$$A_1, \ldots, A_n, V_{r(0)}, V_{r(1)}, \ldots$$

and $\mathcal{C}^* = \{V_{r(0)}, V_{r(1)}, \ldots\}$. Suppose $\mathcal{C}_1$ is an infinite r.e. subclass. Put $\bigcup \mathcal{C}_1 = W_e$ (it is an r.e. set). W_e intersects infinitely many of the $V_{r(i)}$ so we can apply (7) and (8). By (8), since the $V_{r(i)}$ are disjoint from each other and from all the A_i, $V_{r(i)} \in \mathcal{C}_1$ for all i. So the only possibilities for $\mathcal{C}_1$ are

$$\mathcal{C} - \{A_z \mid z \in G_i\} \quad (1 \leq i \leq l)$$

$$\mathcal{C} - \{A_z \mid z \in F_k\} \quad (1 \leq k \leq m+1).$$

We discount the first possibility. For by (7),

$$x \in W_e \text{ and} (z)(x \in A_z \Leftrightarrow z \in G_i) \quad \text{for some } x.$$

$x \notin V_{r(j)}$ for any j so the only members of $\mathcal{C}$ which x belongs to are the $\{A_z \mid z \in G_i\}$. So one of these sets must be in $\mathcal{C}_1$.

We complete the proof by showing that

$\mathcal{C} - \{A_z \mid z \in F_k\}$ is r.e. $(1 \leq i \leq m+1)$. For consider the construction of $\mathcal{C}$ by variation k, in an r.e. sequence

$$\langle A_1, \ldots, A_n, V_0, V_1, \ldots \rangle$$

Since $V_x \neq$ any member of $\mathcal{C}^* \Rightarrow V_x = A_{p(i,k)}$ some i and $p(i,k) \notin F_k$, the r.e. sequence obtained by omitting the z-th member of the original one for each $z \in F_k$ enumerates the desired class.

(a) when $m+1 = 2^n$:

Let $n' = n+1$, $m' = 2^{n+1}-2$ and the F'_k $(1 \leq k \leq 2^{n+1}-1)$ be all the subsets of $\{x \mid 1 \leq x \leq n+1\}$ except the whole set. Let $\mathcal{C}$, $\mathcal{C}^*$ be constructed for m', n', F'_k as above. Define

$$\mathcal{C}_1 = \mathcal{C}^* \cup \{A_{n+1}\}.$$

Then $\mathcal{C}_1$ has no proper infinite r.e. subclasses (and this proves(b)) and $\bigcup \mathcal{C}_1 \subseteq \{x \mid x \geq n\}$. Define

$$\mathcal{C}_2 = \mathcal{C}_1 \cup \{\{i\} \mid 0 \leq i \leq n-1\}$$

and $\mathcal{C}_2$ is the required class. For given an infinite r.e. subclass $\mathcal{C}_3$ of $\mathcal{C}_2$, $\mathcal{C}_3 - \{\{i\} \mid 0 \leq i \leq n-1\}$ is an infinite r.e. subclass of $\mathcal{C}_1$ and therefore is $\mathcal{C}_1$. On the other hand any combination of the $\{i\}$ can be added to $\mathcal{C}_1$.

CHAPTER 2

STRONG ENUMERATION PROPERTIES OF R.E. CLASSES

The starting point for the investigation in this chapter is the work of Lachlan in §1 of [5] and §1 of [6]. We are concerned with certain "enumeration properties" of classes of r.e. sets which are stronger than mere recursive enumerability. After the definitions (2.1 through 2.6) we will be able to state our main theorem.

Definition 2.1 (Lachlan, [5]) Let $\langle W_x \mid x\geq 0\rangle$ be a standard enumeration of the r.e. sets, that is an r.e. sequence with the property that if $\langle S_x \mid x\geq 0\rangle$ is any r.e. sequence, there is a recursive function g with $S_x = W_{g(x)}$ for all x. Then a class $\mathcal{C}$ of r.e. sets is called standard if there is a recursive function f s.t. $\langle W_{f(x)} \mid x\geq 0\rangle$ is a recursive enumeration of $\mathcal{C}$, and s.t. $W_{f(x)} = W_x$ whenever $W_x \in \mathcal{C}$.

Equivalent Definition 2.2 A class $\mathcal{C}$ of r.e. sets is standard $\Leftrightarrow$ for every r.e. sequence $\langle S_x \mid x\geq 0\rangle$ there is an r.e. $\mathcal{C}$ sequence (that is, an r.e. sequence all of whose members belong to $\mathcal{C}$) $\langle T_x \mid x\geq 0\rangle$ s.t. $T_x = S_x$ whenever $S_x \in \mathcal{C}$.

PROOF OF EQUIVALENCE

First let $\mathcal{C}$ satisfy 2.1 and let $\langle S_x \mid x\geq 0\rangle$ be any r.e. sequence. Let g be a recursive function s.t. $S_x = W_{g(x)}$ for all x, and define $T_x = W_{fg(x)}$ for all x. Then $\langle T_x \mid x\geq 0\rangle$ is an r.e. sequence, $T_x \in \mathcal{C}$ for all x, and

$S_x \in \mathcal{C} \Rightarrow W_{g(x)} \in \mathcal{C} \Rightarrow W_{fg(x)} = W_{g(x)} \Rightarrow T_x = S_x$. Thus $\mathcal{C}$ satisfies 2.2. Now let $\mathcal{C}$ satisfy 2.2 and let $\langle T_x \mid x \geq 0 \rangle$ be an r.e. $\mathcal{C}$ sequence corresponding to the r.e. sequence $\langle W_x \mid x \geq 0 \rangle$, that is $W_x \in \mathcal{C} \Rightarrow T_x = W_x$. Let f be a recursive function s.t. $T_x = W_{f(x)}$ for all x. Then $\langle W_{f(x)} \mid x \geq 0 \rangle$ enumerates $\mathcal{C}$ and $W_x \in \mathcal{C} \Rightarrow W_{f(x)} = W_x$. Q.E.D.

Definition 2.3 A class $\mathcal{C}$ of r.e. sets is called sequence enumerable if the r.e. $\mathcal{C}$ sequences can be enumerated as the rows of a 2-dimensional r.e. $\mathcal{C}$ sequence.

Definition 2.4 (Lachlan, [6]) A class $\mathcal{C}$ of r.e. sets is called indexable if there is a recursive enumeration $\langle S_x \mid x \geq 0 \rangle$ of $\mathcal{C}$ s.t., if $\langle T_x \mid x \geq 0 \rangle$ is any r.e. $\mathcal{C}$ sequence then there exists a recursive function f with $T_x = S_{f(x)}$ for all x. The sequence $\langle S_x \rangle$ is called an indexing of $\mathcal{C}$.

Definition 2.5 A class $\mathcal{C}$ of r.e. sets is called subclass enumerable if there is a 2-dimensional r.e. $\mathcal{C}$ sequence whose rows enumerate all the non-empty r.e. subclasses of $\mathcal{C}$.

Equivalent Definition 2.6 A class $\mathcal{C}$ of r.e. sets is subclass enumerable $\Leftrightarrow$ there is a recursive enumeration $\langle S_x \mid x \geq 0 \rangle$ of $\mathcal{C}$ s.t., if $\mathcal{A}$ is an r.e. subclass of $\mathcal{C}$, then there exists an r.e. set W with $\mathcal{A} = \{S_x \mid x \in W\}$.

PROOF OF EQUIVALENCE

First let $\mathcal{C}$ satisfy 2.5 and let $\langle S_{xy} \mid x,y \geq 0 \rangle$ be a 2-dimensional r.e. $\mathcal{C}$ sequence whose rows enumerate all the non-empty r.e. subclasses of $\mathcal{C}$. Let τ be a 1-1 two argument recursive function with range N (e.g. the Cantor pairing function). Given a number, we can effectively find the pair that τ makes correspond to that number, and so it is possible to define an r.e. sequence $<S_x \mid x\geq 0>$ by

$$S_{\tau(x,y)} = S_{x,y}$$

Then $<S_x \mid x\geq 0>$ is a recursive enumeration of $\mathcal{C}$. Let $\mathcal{A}$ be an r.e. subclass of $\mathcal{C}$, and we show that there exists an r.e. set W with $\mathcal{A} = \{S_x \mid x\epsilon W\}$. If $\mathcal{A}$ is empty, take W to be empty. Suppose then that $\mathcal{A}$ is non-empty. Then there is x_0 with $\mathcal{A} =\{S_{x_0,y} \mid y\geq 0\} = \{S_{\tau(x_0,y)} \mid y\geq 0\}$ and we can take $W = \{\tau(x_0,y) \mid y\geq 0\}$.

Now let $\mathcal{C}$ satisfy 2.6, and let $<S_x \mid x\geq 0>$ have the property of 2.6. It is clear that we can modify a recursive enumeration of all the r.e. sets to get a recursive enumeration of all the non-empty r.e. sets $<V_x \mid x\geq 0>$. There is a two argument recursive function $r(x,y)$ s.t. the range of $\lambda y r(x,y)$ is V_x: we can let $r(x,0)$, $r(x,1)$,... enumerate the members of V_x as they appear in some simultaneous enumeration of $<V_x>$, "marking time" with repetitions to cover the case V_x is empty. Now let $S_{xy} =_{df} S_{r(x,y)}$ and $<S_{x,y} \mid x,y\geq 0>$ is a 2 dimensional r.e. $\mathcal{C}$ sequence with the property required by 2.5. For if $\mathcal{A}$ is a non-empty r.e. subclass of $\mathcal{C}$ there is x_0 with

$$\mathcal{A} = \{S_x \mid x\epsilon V_{x_0}\} = \{S_{r(x_0,x)} \mid x\geq 0\} = \{S_{x_0,x} \mid x\geq 0\}. \text{ Q.E.D.}$$

MAIN THEOREM. If $\mathcal{C}$ is a class of r.e. sets, $\mathcal{C}$ is standard $\Rightarrow \mathcal{C}$ is sequence enumerable $\Rightarrow \mathcal{C}$ is indexable $\Rightarrow \mathcal{C}$ is subclass enumerable, but none of the implications can be reversed.

We will not complete the proof of this theorem until the end of the chapter. THEOREM 2.1 proves the three implications, and the three counterexamples we need are provided by EXAMPLES 2.2, 2.1 and 2.4. THEOREM 2.2 shows that three results on standard classes from [5] which were pointed out in [6] to fail for indexable classes, do generalise to sequence enumerable classes. THEOREM 2.3 generalises to subclass enumerable classes a closure condition proved in [6] for indexable classes, and EXAMPLE 2.3 shows that this condition does not characterize subclass enumerable classes.

THEOREM 2.1 If $\mathcal{C}$ is a class of r.e. sets, $\mathcal{C}$ is standard $\Rightarrow \mathcal{C}$ is sequence enumerable $\Rightarrow \mathcal{C}$ is indexable $\Rightarrow \mathcal{C}$ is subclass enumerable.

PROOF. First implication. This already follows from Example 4 of [5]. We will give a proof from Definition 2.2. Since we can effectively arrange a 2 dimensional r.e. sequence as a single r.e. sequence (as in the first half of the proof of equivalence of Definitions 2.5 and 2.6), 2 dimensional r.e. sequences also have the property of Definition 2.2. Namely, to every 2 dimensional r.e. sequence $\langle S_{x,y} \mid x,y \geq 0\rangle$, there corresponds a 2 dimensional r.e. $\mathcal{C}$ sequence $\langle T_{x,y} \mid x,y\geq 0\rangle$ s.t. $S_{x,y} \in \mathcal{C} \Rightarrow T_{x,y} = S_{x,y}$. Now define $\langle S_{x,y}\rangle$ as follows. Let $\langle P_x \mid x\geq 0\rangle$ be a recursive enumeration of all the r.e. subsets of N^2 and take $S_{x,y} = \{z \mid (z,y) \in P_x\}$. If $\langle U_x \mid x\geq 0\rangle$ is any r.e. sequence there is x_0 s.t. $P_{x_0} = \{(z,y) \mid z\in U_y\}$ and so $U_y = \{z \mid (z,y) \in P_{x_0}\} = S_{x_0,y}$. Thus all r.e. sequences are enumerated as rows of $\langle S_{x,y}\rangle$ (<u>the class of all r.e. sets is sequence enumerable</u>). The corresponding $\mathcal{C}$ sequence $\langle T_{x,y}\rangle$ then enumerates by its rows all the r.e. $\mathcal{C}$ sequences.

Q.E.D.

Second implication. Let $\mathcal{C}$ be a sequence enumerable class of r.e. sets and let $\langle S_{x,y} \mid x,y\geq 0\rangle$ be a 2 dimensional r.e. $\mathcal{C}$ sequence whose rows enumerate all the r.e. $\mathcal{C}$ sequences. Form an r.e. $\mathcal{C}$ sequence $\langle S_x \mid x\geq 0\rangle$ asin the first half of the proof of equivalence of Definitions 2.5 and 2.6. We show that $\langle S_x \mid x\geq 0\rangle$ is an indexing of $\mathcal{C}$. For let $\langle T_x \mid x\geq 0\rangle$ be any r.e. $\mathcal{C}$ sequence. Then there is x_0 s.t. $T_x = S_{x_0,x}$ for all x. Define a recursive function f by $f(x) = \tau(x_0,x)$. Then for all x, $T_x = S_{x_0,x} = S_{\tau(x_0,x)} = S_{f(x)}$.

Q.E.D.

Third implication. Let $\langle S_x \mid x\geq 0\rangle$ be an indexing of the indexable class $\mathcal{C}$ and we show that $\langle S_x\rangle$ has the property of Definition 2.6. Thus let $\mathcal{A}$ be a non-empty r.e. subclass of $\mathcal{C}$ (if $\mathcal{A}$ is empty we take W to be empty). Then $\mathcal{A}$ is enumerated by an r.e. sequence and this must have the form $\langle S_{f(x)} \mid x\geq 0\rangle$ for some recursive function f. Taking W to be the range of f, W is r.e. and $\mathcal{A} = \{S_x \mid x \in W\}$.

Q.E.D.

The example given [6] to distinguish indexable classes from standard classes suffices to distinguish indexable classes from sequence enumerable classes.

EXAMPLE 2.1. The class $\mathcal{C} = \{\{0\},\{1\}\}$ is indexable but not sequence enumerable.

PROOF. Any enumeration of $\mathcal{C}$ is an indexing. $\mathcal{C}$ is not sequence enumerable by a diagonal argument.

Q.E.D.

<u>Definition 2.7</u> (Rice, [2]) A sequence $\langle F_x \mid x \geq 0 \rangle$ of finite sets is a <u>strongly r.e. sequence of finite sets</u> if there is a recursive function f s.t.

(1) either $f(x) = 1$ or $f(x)$ has the form $p_0^{a_0+1} \cdot p_1^{a_1+1} \cdot \ldots \cdot p_n^{a_n+1}$, where p_n is the $(n+1)$ st prime ($p_0 = 2$), and

(2) if $f(x) = 1$, F_x = the empty set, and if $f(x) = p_0^{a_0+1} \cdot p_1^{a_1+1} \cdot \ldots \cdot p_n^{a_n+1}$, $F_x = \{a_0, a_1, \ldots, a_n\}$.

That is, given x we can write down the members of F_x, "once and for all".

THEOREM 2.2 Suppose that $\mathcal{C}$ is a class of r.e. sets, $\langle F_x \mid x \geq 0 \rangle$ is a strongly r.e. sequence of finite sets s.t. each member of $\mathcal{C}$ contains F_x for some x, and $\langle T_x \mid x \geq 0 \rangle$ is an r.e. $\mathcal{C}$ sequence s.t. for each x, $F_x \not\subseteq T_x$. Then $\mathcal{C}$ is not sequence enumerable.

PROOF. We show that if $\langle S_{x,y} \mid x,y\geq 0\rangle$ is a 2 dimensional r.e. $\mathcal{C}$ sequence, during any simultaneous enumeration of $\langle S_{x,y}\rangle$ we can construct an r.e. $\mathcal{C}$ sequence $\langle U_x \mid x\geq 0\rangle$ s.t. for each x, $S_{xx} \neq U_x$, so that $\langle U_x\rangle$ is not a row of $\langle S_{x,y}\rangle$. To obtain U_x: enumerate S_{xx} and $F_0, F_1, \ldots$ simultaneously. Since $S_{xx} \in \mathcal{C}$, by the property of the sequence $\langle F_x\rangle$ we will find a y s.t. $F_y \subseteq S_{xx}$. Take $U_x = T_y$, and we have $F_y \not\subseteq U_x$, so that $U_x \neq S_{xx}$. It is clear that $\langle U_x\rangle$ can be made an r.e. sequence.

Q.E.D.

Definition 2.8 (Rice, [2]) A class $\mathcal{C}$ of r.e. sets is called completely recursively enumerable (c.r.e.) if there is a strongly r.e. sequence $\langle F_x \mid x\geq 0\rangle$ s.t. for any r.e. set W,

$W \in \mathcal{C} \Leftrightarrow$ there is an x s.t. $F_x \subseteq W$.

NOTE. This is not Rice's original definition. That 2.8 is equivalent to the original definition is Rice's "Key array conjecture", proved by Myhill and Shepherdson in [3].

Generalising Lemma 1.1 of [5], we have

COROLLARY 2.1. Let $\mathcal{C}$ be a sequence enumerable class of r.e. sets and $\mathcal{M}$, $\mathcal{L}$ be c.r.e. classes s.t. $\mathcal{M} \cup \mathcal{L} \supseteq \mathcal{C}$, then either $(\mathcal{M} - \mathcal{L}) \cap \mathcal{C}$ or $(\mathcal{L} - \mathcal{M}) \cap \mathcal{C}$ is empty.

PROOF. Let $\mathcal{C}$ be a class of r.e. sets, and let $\mathcal{M}$, $\mathcal{L}$ be c.r.e classes s.t. $\mathcal{M} \cup \mathcal{L} \supseteq \mathcal{C}$. Suppose that $A \in (\mathcal{M} - \mathcal{L}) \cap \mathcal{C}$, $B \in (\mathcal{L} - \mathcal{M}) \cap \mathcal{C}$ and we show that $\mathcal{C}$ is not sequence enumerable. Let $\langle G_x \mid x \geq 0\rangle$ and $\langle H_x \mid x \geq 0\rangle$ be strongly r.e. sequences of finite sets s.t. for any r.e. set W,

$W \in \mathcal{M} \Leftrightarrow$ there is an x s.t. $G_x \subseteq W$

$W \in \mathcal{L} \Leftrightarrow$ there is an x s.t. $H_x \subseteq W$.

Define a strongly r.e. sequence $\langle F_x \mid x \geq 0\rangle$ and an r.e. $\mathcal{C}$ sequence $\langle T_x \mid x \geq 0\rangle$ by

$$F_{2x} = G_x \quad , \; F_{2x+1} = H_x$$

$$T_{2x} = B \quad , \; T_{2x+1} = A \; .$$

Since $\mathcal{M} \cup \mathcal{L} \supseteq \mathcal{C}$, each member of $\mathcal{C}$ contains F_x for some x . Also, for each x $F_x \not\subseteq T_x$, for otherwise it would follow that either $B \in \mathcal{M}$ or $A \in \mathcal{L}$. Thus by the theorem, $\mathcal{C}$ is not sequence enumerable. Q.E.D.

COROLLARY 2.2. If a sequence enumerable class of r.e. sets contains a finite set, then it has a least member.

PROOF. That this property follows from the property of Corollary 1 is Corollary 1.3 of [5]. For a direct proof from the Theorem, let a class $\mathcal{C}$ of r.e. sets contain a finite set, then it contains a minimal finite set F. If F is not least, let A be in $\mathcal{C}$ s.t. $A \not\supseteq F$. Define

$$F_0 = F, \; F_{x+1} = \{c_x\}$$

where $c_0, c_1, c_2, \ldots$ is the complement of F in increasing order. Also define

$$T_0 = A, \; T_{x+1} = F.$$

Then $<F_x \mid x \geq 0>$ is a strongly r.e. sequence, and $<T_x \mid x \geq 0>$ is an r.e. $\mathcal{C}$ sequence s.t. for each x, $F_x \not\subseteq T_x$. Also each member of $\mathcal{C}$ contains F_x for some x. For if $W \in \mathcal{C}$ and W does not contain F_{x+1} for an x, $W \subseteq F$ and in fact $W = F$ since F is minimal, so that W contains F_0.

Q.E.D.

A finite class of r.e. sets is standard if and only if it has a least member (Theorem 1.5 of [2]). This also carries over to sequence enumerable classes. We need only prove

COROLLARY 2.3. If a finite class has no least member, it is not sequence enumerable.

PROOF. Let $\mathcal{C}$ be $\{A_0, A_1, \ldots, A_n\}$, and suppose $\mathcal{C}$ has no least member. It follows that for each j $(0 \leq j \leq n)$
$A_j - A_0 \cap \ldots \cap A_n$ is non-empty. For $0 \leq j \leq n$, define

$F_j = \{x_j\}$, where $x_j \in A_j - A_0 \cap \ldots \cap A_n$.

$T_j = A_i$, where i is chosen so that $x_j \notin A_i$.

Now applying the Theorem, $\mathcal{C}$ is not sequence enumerable. (The sequences here are finite, but it makes no difference to the conclusion of the Theorem).

Q.E.D.

As pointed out in [6], none of the three above Corollaries holds for indexable classes, the class $\{\{0\},\{1\}\}$ providing a counterexample in each case. Our next task is to distinguish sequence enumerable classes from standard classes. Our example of a class which is sequence enumerable but not standard will have a least member (the empty set); thus the example answers in the affirmative the conjecture on page 15 of [6] that there is an indexable class which is not standard but which has a least member.

EXAMPLE 2.2. There is a class of r.e. sets which is sequence enumerable but not standard.

REMARK. The class $\mathcal{C}$ which we construct has the empty set as a member; otherwise it consists of singletons and pairs.

PROOF. As shown in proving the first implication of Theorem 2.1, the class of all r.e. sets is sequence enumerable. It easily follows that the class of all r.e. sets with less than three members is sequence enumerable. Thus let $\langle W_{x,y} \mid x,y \leq 0 \rangle$ be a 2 dimensional r.e. sequence of r.e. sets with less than three members, such that every r.e. sequence of r.e. sets with less than three members occurs as a row. By Definitions 2.2 and 2.3 it will be sufficient to produce an r.e. class $\mathcal{C}$, an r.e. sequence $\langle S_x \mid x \geq 0 \rangle$ and a 2 dimensional r.e. sequence $\langle T_{x,y} \mid x,y \geq 0 \rangle$ such that

A. the empty set belongs to $\mathcal{C}$ and $\mathcal{C}$ contains no set with more than two members

B. $(x)(y)\ (T_{x,y} \in \mathcal{C})$

C(x). $(\exists y)(W_{x,y} \notin \mathcal{C}\ .\vee.\ S_y \in \mathcal{C}$ and $S_y \neq W_{x,y})$ for each x

D(x). $(\exists y)((z)(W_{x,z} \in \mathcal{C}) \Rightarrow (z)(W_{x,z} = T_{y,z}))$ for each x .

We can regard ourselves as trying to satisfy the double infinity of conditions C(0), D(0), C(1), D(1),... with this order of priorities, subject to the constraints A and B, during an enumeration of $< W_{x,y} \mid x,y \geq 0>$. Before giving the detailed construction we explain briefly how it works. To satisfy C(x) say, choose a number y and an ordered triple (p,q,r). We will ensure that

$W_{x,y} \notin \mathcal{C}$ .v. $S_y \in \mathcal{C}$ and $S_y \neq W_{x,y}$. Put p in S_y and $\{p\}$ in $\mathcal{C}$. At some stage we may have $W_{x,y} = \{p\}$ (<u>Event 1</u>). Then we remove $\{p\}$ from $\mathcal{C}$ by converting it to $\{p,q\}$. Later we may have $W_{x,y} = \{p,q\}$ (<u>Event 2</u>). Then we put r in S_y and $\{p,r\}$ in $\mathcal{C}$. It is clear how by using different y and disjoint triples for different x we can satisfy C(x) for each x. Suppose however we are trying simultaneously to satisfy D(i) say, by choosing a number j and making

$$(z)\ (W_{i,z} \in \mathcal{C}) \Rightarrow (z)\ (W_{i,z} = T_{j,z}).$$

Before Event 1 occurs, we may have $W_{i,0} = \{p\}$. We therefore put p in $T_{j,0}$. If then Event 1 occurs, because of B we put q in $T_{j,0}$. While $W_{i,0}$ is still $\{p\}$ Event 2 may occur. Perhaps $W_{i,0}$ is $\{p,r\}$ and so we may be in trouble if we put $\{p,r\}$ in $\mathcal{C}$. To get over this difficulty we act roughly as follows. If $x \leq i$ (i.e. C(x) has priority over D(i)) we put $\{p,r\}$ in $\mathcal{C}$ and choose a new number j' to use for D(i). This time we will leave $T_{j',0}$ empty unless it turns out that $W_{i,0}$ is $\{p,q\}$ or $\{p,r\}$ and we will then make

$T_{j',0} = W_{i,0}$. If $x>i$ (i.e. $D(i)$ has priority over $C(x)$) we put r in S_y but do not put $\{p,r\}$ in $\mathcal{C}$. Instead we choose a new value of y and a new triple (p',q',r') for use with $C(x)$. Now is say $W_{i,1} = \{p'\}$ there is no need to put p' in $T_{j,1}$ unless $W_{i,0}$ becomes $\{p,q\}$ (because $\{p\} \notin \mathcal{C}$), and then we can complete our first attempt at satisfying $C(x)$ by putting $\{p,r\}$ in $\mathcal{C}$. The <u>y-th triple</u> is $(3y,\ 3y+1,\ 3y+2)$. This triple may be used in conjunction with S_y to satisfy a condition $C(x)$ as above. Define $F_0 = \{0\}$, $F_1 = \{1,2\}$, $F_2 = \{3,4,5\}, \ldots$. Then we will have for each x

$$W_{x,y} \notin \mathcal{C} \ .\vee.\ S_y \in \mathcal{C} \text{ and } S_y \neq W_{x,y}$$

where $y \in F_x$. Note that F_x has $x+1$ members and there are x conditions $D(i)$ with higher priority than $C(x)$. The full construction now follows.

CONSTRUCTION.

<u>Definition of $\langle S_x \mid x \geq 0\rangle$</u>

$$S_y =_{df} \begin{cases} \{3y\} & \text{if } W_{x,y} \neq \{3y,3y+1\} \\ \{3y,3y+2\} & \text{if } W_{x,y} = \{3y,3y+1\}\ , \end{cases}$$

where $y \in F_x$. This defines an r.e. sequence.

<u>Definition of $\mathcal{C}$ and $\langle T_{x,y} \mid x,y \geq 0\rangle$.</u>

We stipulate in the first place that the empty set belongs to $\mathcal{C}$. The other members of $\mathcal{C}$ and the members of the T sequence are obtained in a construction step s of which follows step s (s = 0,1,2,...) of an enumeration of $\langle W_{x,y} \mid x,y\geq 0\rangle$. $W^s_{x,y}$, $T^s_{x,y}$ and $\mathcal{C}^s$ refer to the situation after step s of the enumeration but before step s of the construction. We set up a framework for the process by attaching numbers (respectively, ordered pairs) to even (odd) values of s in such a way that every number (ordered pair) is attached to infinitely many values of s. For each condition C(x) or D(x), after a certain step there will always be a number associated with the condition. The associate of C(x) will belong to F_x and the associate of D(x) will correspond to a row of the T sequence. The associate of a condition may change. We define

$Q(y,i,s) = (\exists j)(\exists z)(j$ is associated with $D(i)$.&. $W^s_{i,z}$ is either $\{3y\}$ or $\{3y,3y+2\}$.&. $T^s_{j,z}$ is either $\{3y\}$ or $\{3y,3y+1\})$.

Instructions for a step s attached to x

If C(x) has no associate, associate with C(x) the least member of F_x. Let y be the associate of C(x). Do nothing more unless one of the following mutually exclusive cases holds.

1. $\{3y,3y+2\} \in \mathcal{C}^s$. Do nothing.
2. $\{3y,3y+2\} \notin \mathcal{C}^s$ and there is $y_1 < y$ in F_x s.t. $\sim(\exists i)_{i<x}\, Q(y_1,i,s)$. Then for the least such y_1, put $\{3y_1,3y_1+2\}$ in $\mathcal{C}$ and change the associate of C(x) to y_1.

3. Neither 1. nor 2. holds, $\{3y,3y+1\} \in \mathcal{C}^s$, $W^s_{x,y} = \{3y,3y+1\}$ and $\sim(\exists i)_{i<x}\, Q(y,i,s)$. Then put $\{3y,3y+2\}$ in $\mathcal{C}$.

4. Neither 1. nor 2. holds, $\{3y,3y+1\} \in \mathcal{C}^s$, $W^s_{x,y} = \{3y,3y+1\}$ and $(\exists i)_{i<x}\, Q(y,i,s)$. Then if $y + 1 \in F_x$ change the associate of $C(x)$ to $y + 1$, and otherwise do nothing.

5. Neither 1. nor 2. holds, $\{3y,3y+1\} \notin \mathcal{C}^s$, $\{3y\} \in \mathcal{C}^s$ and $W^s_{x,y} = \{3y\}$. Then convert any occurrence of $\{3y\}$ in $\mathcal{C}^s$ to $\{3y,3y+1\}$.

6. Neither 1. nor 2. holds, $\{3y,3y+1\} \notin \mathcal{C}^s$, $\{3y\} \notin \mathcal{C}^s$ and $W^s_{x,y} \neq \{3y\}$. Then put $\{3y\}$ in $\mathcal{C}$.

<u>Instructions for a step s attached to (i,k)</u>

If $D(i)$ has no associate, associate with $D(i)$ the least number which has not yet been an associate. Let j be the associate of $D(i)$. For each previous associate j_1 of i, if there is u, z s.t. $T^s_{j_1,z} = \{3u\}$ and $\{3u\} \notin \mathcal{C}^s$, put $3u+1$ in $T_{j_1,z}$. Do nothing more unless one of the following mutually exclusive cases holds.

7. There is y s.t. $W^s_{i,k} = \{3y\}$, $T^s_{j,k}$ is empty, $\{3y\} \in \mathcal{C}^s$, $y \in F_x$, and it is not the case that $x>i$ and $(\exists y_1)(y_1 \neq y \;\&\; y_1 \in F_x \;\&\; Q(y_1,i,s))$.
Then put $3y$ in $T_{j,k}$.

8. There is y s.t. $W^s_{i,k} = \{3y\}$, $T^s_{j,k} = \{3y\}$ and $\{3y\} \notin \mathcal{C}^s$.
Then put $3y+1$ in $T_{j,k}$.

9. There is y s.t. $W^s_{i,k} = \{3y,3y+1\}$, $\{3y,3y+1\} \in \mathcal{C}^s$ and $T^s_{j,k}$ is empty or is $\{3y\}$. Then put $3y,3y+1$ in $T_{j,k}$.

10. There is y s.t. $W^s_{i,k} = \{3y,u\}$ with $u \neq 3y,3y+1,3y+2$, $\{3y\} \notin \mathcal{C}^s$ and $T^s_{j,k} = \{3y\}$. Then put $3y+1$ in $T_{j,k}$.

11. There is y s.t. $W^s_{i,k} = \{3y,3y+2\}$, $\{3y,3y+2\} \in \mathcal{C}^s$ and $T^s_{j,k}$ is empty or is $\{3y\}$. Then put $3y,3y+2$ in $T_{j,k}$.

12. There is y s.t. $W^s_{i,k} = \{3y,3y+2\}$, $\{3y,3y+2\} \in \mathcal{C}^s$ and $T^s_{j,k} = \{3y,3y+1\}$. Then choose a new associate for i.

13. There is y s.t. $W^s_{i,k} = \{3y,3y+2\}$, $\{3y,3y+2\} \notin \mathcal{C}^s$, $\{3y\} \notin \mathcal{C}^s$ and $T^s_{j,k} = \{3y\}$. Then put $3y+1$ in $T_{j,k}$

This completes the construction.

Remark. To connect the program with the preceding discussion. At a step s attached to x we try to satisfy condition $C(x)$, using the step s associate of $C(x)$, and at a step s attached to (i,k) we try to satisfy condition $D(i)$ by making element (j,k) of the T sequence equal to element (i,k) of the W sequence if the latter is in $\mathcal{C}$, where j is the step s associate of $D(i)$. 6. is the first stage of working on $C(x)$, and 5. corresponds to Event 1. 3. corresponds to Event 2. 2. is the case where we return to a previous attempt to satisfy $C(x)$. If $i < x$, $y \in F_x$ and $Q(y,i,s)$, this means that we cannot complete the attempt with y to satisfy $C(x)$, because of a conflict with $D(i)$. That accounts for the appearance of Q in 2., 3., and 4. In 4. we choose a new triple for $C(x)$ if possible. We are eventually able to satisfy $C(x)$ because of the restriction in 7., which ensures that each $D(i)$ with $i<x$ can block only one attempt to satisfy $C(x)$, whereas we have $x+1$ attempts at our disposal (corresponding to the members of F_x); we make these attempts with the first member of F_x, the second member of F_x, ... (see 4.) and either one of them works out or we will be able to return to a previous attempt by 2.

The proof consists of a sequence of six lemmas.

LEMMA 1. For each x, the associate of $C(x)$ is eventually constant.

PROOF. Because of 1., if the associate of $C(x)$ ever changes by 2. it never changes again. If the associate of $C(x)$ never changes by 2., it can change only by 4., and this happens finitely often since F_x is finite.

Q.E.D.

LEMMA 2. For any y, if $\{3y,3y+2\}$ is put into $\mathcal{C}$ at step v and $\{3y\} \in \mathcal{C}^u$, then $u < v$.

PROOF. Suppose that $u \geq v$, then by 1.,2., and 3. $\{3y\} \in \mathcal{C}^v$. But also $\{3y,3y+1\} \in \mathcal{C}^v$, for otherwise 2. occurs at v (3. cannot) and 4. must have occurred at $w < v$ while y was an associate of $C(x)$ and so $\{3y,3y+1\} \in \mathcal{C}^w$ which implies $\{3y,3y+1\} \in \mathcal{C}^v$, contradicition. It follows that 5. occurred at a step $t < v$ and 6. is impossible between t and v, contradiction.

Q.E.D.

LEMMA 3. For each i, the associate of D(i) is eventually constant.

PROOF. The associate of D(i) can change at step s attached to (i,k) for k = 0,1,2,... by occurrences of 12. Suppose this happens with $y \in F_x$ and $x > i$ and we derive a contradiction. Let $t < s$ be the step at which $3y+1$ was put into $T_{j,k}$. At t 8. or 13. must have occurred, because $W^s_{i,k} = \{3y, 3y+2\}$ eliminates the possibilities 9. and 10. at t. $T^t_{j,k} = \{3y\}$ in either case so there is $u < t$ s.t. 7. occurred at u. Let v be the step at which $\{3y, 3y+2\}$ is put into $\mathcal{C}$. Then $v < s$. Also $v > u$ by Lemma 2, since $\{3y\} \in \mathcal{C}^{u+1}$. We have $\sim Q(y,i,v)$ since 2. or 3. occurs at v. But after step u we have $T^{u+1}_{j,k} = \{3y\}$, $W^{u+1}_{i,k} = \{3y\}$, and by step s we have $T^s_{j,k} = \{3y, 3y+1\}$ and $W^s_{i,k} = \{3y, 3y+2\}$. Thus at step v we must have $Q(y,i,v)$ by the definition of Q. Contradiction.

Thus the y in occurrences of 12. belongs to F_x with $x \leq i$. Consider s s.l. that for each $x \leq i$ and each $y \in F_x$ the membership of $\{3y\}$, $\{3y, 3y+1\}$ and $\{3y, 3y+2\}$ in $\mathcal{C}$ has been permanently fixed; there is such an s by 5. and 6. Then we show 12. can happen for at most one such s which will prove the Lemma.

Suppose there are two such s, s_0 and s_1, s.t. the associate of D(i) changes to j_0 at s_0 and then to j_1 at $s_1 > s_0$. Let s_1 be attached to (i,k), $W^{s_1}_{i,k} = \{3y, 3y+2\}$,

$T^{s_1}_{j_0,k} = \{3y, 3y+1\}$, $y \in F_x$ and $x \leq i$. There are then t, u with $s_0 < u < t < s_1$, u and t are attached to (i,k), 7. occurs at u, and 8. or 13. occurs at t. Thus $\{3y\} \in \mathcal{C}^u$ but $\{3y\} \notin \mathcal{C}^t$, which contradicts our assumption about s_0.

Q.E.D.

LEMMA 4. $(j)(k)\ T_{j,k} \in \mathcal{C}$ (i.e. B. holds).

PROOF. Let j,k be given. If $T_{j,k}$ is empty then $T_{j,k} \in \mathcal{C}$. First suppose $T_{j,k}$ gets a member by the "previous associate" case, i.e. there is s, i, u s.t. j is a previous associate of D(i), $T^s_{j,k} = \{3u\}$ and $\{3u\} \notin \mathcal{C}^s$. Thus there is t < s s.t. $\{3u\} \in \mathcal{C}^t$ and 5. occurred between t and s so $\{3u, 3u+1\} \in \mathcal{C}$ and $T_{j,k} \in \mathcal{C}$. Thus we can assume that $T_{j,k}$ acquires members only at steps s attached to (i,k) when j is the associate of i. $T_{j,k}$ can never havemore than 2 members since at the end of any step $T_{j,k}$ has at most 2 members. If $T_{j,k}$ has one member then $T_{j,k} = \{3y\}$ for some y, and $3y \in W_{i,k}$ so if $\{3y\} \notin \mathcal{C}$, for s.l. s $\{3y\} \notin \mathcal{C}^s$ and $\{3y, 3y+1\} \in \mathcal{C}^s$ and one of cases 8., 9., 10., 11., or 13. yields a contradiction. If $T_{j,k}$ has 2 members let s be the step when it gets its second member or both its members if it gets them both at once. Consideration of the possibilities 8., 9., 10., 11. and 13 at s, using in 8., 10., and 13. the fact that $T^s_{j,k} = \{3y\}$ and $\{3y\} \notin \mathcal{C}^s$ implies that $\{3y, 3y+1\} \in \mathcal{C}$, gives $T_{j,k} \in \mathcal{C}$.

Q.E.D.

LEMMA 5. Let y be the final associate of C(x). Then $W_{x,y} \notin \mathcal{C}$ .v. $S_y \in \mathcal{C}$ & $S_y \neq W_{x,y}$ (so that C(x) holds).

PROOF. We note that the associate of C(x) starts at the least member of F_x and possibly increases by occurrences of 4. By 2. it may decrease, but then by 1. it remains fixed.

First case. $\{3y,3y+2\} \in \mathcal{C}$. If this happens by 3. then $W_{x,y} = \{3y,3y+1\}$. If it happens by 2. we also have $W_{x,y} = \{3y,3y+1\}$ becuase of the condition in 4. Thus by the definition of S_y, $S_y = \{3y,3y+2\}$ and

$$S_y \in \mathcal{C} \quad \& \quad S_y \neq W_{x,y}.$$

Second case. $\{3y,3y+2\} \notin \mathcal{C}$ and y is not the greatest member of F_x.

a) $\{3y,3y+1\} \in \mathcal{C}$ & $W_{x,y} = \{3y,3y+1\}$. Then for s.l. s 3. or 4. occurs and we get a contradiction.

b) $\{3y,3y+1\} \in \mathcal{C}$ & $W_{x,y} \neq \{3y,3y+1\}$. Then by 5. $3y \in W_{x,y}$ so if $W_{x,y} \in \mathcal{C}$ then $W_{x,y} = \{3y\}$, which is impossible by 5. and 6.

c) $\{3y,3y+1\} \notin \mathcal{C}$ & $\{3y\} \in \mathcal{C}$. If $W_{x,y} \in \mathcal{C}$ then $S_y = \{3y\} \in \mathcal{C}$. If $W_{x,y} = \{3y\}$, 5. gives a contradiction.

d) $\{3y,3y+1\} \notin \mathcal{C}$ & $\{3y\} \notin \mathcal{C}$. Then 6. can never occur so $3y \in W_{x,y}$ and $W_{x,y} \notin \mathcal{C}$.

Third case. $\{3y,3y+2\} \notin \mathcal{C}$ and y is the greatest member of F_x. The second case argument is valid except in a). Let s be s.l. attached to x s.t. $\{3y,3y+1\} \in \mathcal{C}^s$ and

$\{3y,3y+1\} = W^s_{x,y}$. We show that 2. or 3. occurs at s, giving a contradiction. For suppose that

$$(y_1)_{y_1 \in F_x} (\exists i)_{i<x}\, Q(y_1,i,s).$$

Then since F_x has x+1 members and there are only x numbers < x, there are y_1, y_2 in F_x s.t. for some $i < x$, $Q(y_1,i,s)$ and $Q(y_2,i,s)$. Let j be the associate of D(i) at s and let k_1, k_2 be s.t. $W^s_{i,k_1} = \{3y_1\}$ or $\{3y_1,3y_1+2\}$, $W^s_{i,k_2} = \{3y_2\}$ or $\{3y_2,3y_2+2\}$, $T^s_{j,k_1} = \{3y_1\}$ or $\{3y_1,3y_1+1\}$, and $T^s_{j,k_2} = \{3y_2\}$ or $\{3y_2,3y_2+1\}$. There exist u_1, u_2 with (say) $u_1 < u_2$ and both < s s.t. T_{j,k_1}, T_{j,k_2} first acquire members at u_1, u_2 respectively. At u_1, 7. must occur with $y = y_1$, $k = k_1$, and at u_2 7. must occur with $y=y_2$, $k=k_2$. For the only other possibilities, in the case u_1 for instance, are 9. and 11.; 9. is impossible because then $W^{u_1}_{i,k_1} = \{3y_1,3y_1+1\}$ which implies $W^s_{i,k_1} = \{3y_1,3y_1+1\}$ instead of $\{3y_1\}$ or $\{3y_1,3y_1+2\}$, and 11. is impossible because then $T^{u_1+1}_{j,k_1} = \{3y_1,3y_1+2\}$ which implies $T^s_{j,k_1} = \{3y_1,3y_1+2\}$ instead of $\{3y_1\}$ or $\{3y_1,3y_1+1\}$. Since 7. occurs at u_2, we have $\sim Q(y_1,i,u_2)$. But since 7. occurs at u_1 with $y = y_1$ $k = k_1$, we have $W^{u_1}_{i,k_1} = \{3y_1\}$ and $T^{u_1+1}_{j,k_1} = \{3y_1\}$, and combining this with $W^s_{i,k_1} = \{3y_1\}$ or $\{3y_1,3y_1+2\}$, and $T^s_{j,k_1} = \{3y_1\}$ or $\{3y_1,3y_1+1\}$, we must have $Q(y_1,i,u_2)$. Contradiction.

Q.E.D.

LEMMA 6. Let j be the final associate of $D(i)$. Then $(z)(W_{i,z} \in \mathcal{C}) \Rightarrow (z)(W_{i,z} = T_{j,z})$ (so that $D(i)$ holds).

PROOF. Suppose that $(z)\ (W_{i,z} \in \mathcal{C})$. Consider $W_{i,k}$.

<u>First case.</u> $W_{i,k}$ is empty. Then $T_{j,k}$ is empty.

<u>Second case.</u> $W_{i,k}$ has one member. Then $W_{i,k} = \{3y\}$ for some y, $y \in F_x$ say, and $\{3y\} \in \mathcal{C}$. $T_{j,k}$ can acquire members only by 7. and 8. Suppose 8. occurs at a step t, then there is a step $s < t$ s.t. 7. occurs at s; then by 5. and 6. and the fact that $\{3y\} \in \mathcal{C}$, $\{3y\} \in \mathcal{C}^s$ implies that $\{3y\} \in \mathcal{C}^t$, which is impossible. Thus 8. can never occur, and to show that $T_{j,k} = \{3y\}$ we have only to show that 7. sometime occurs. Thus we must show that if s is chosen sufficiently large, and $x > i$, then for each $y_1 \neq y$ in F_x, $\sim Q(y_1,i,s)$.

For no $y_1 \neq y$ in F_x can we have $\{3y_1\} \notin \mathcal{C}$. Otherwise suppose (say) $y_1 < y$ and $\{3y_1\}$, $\{3y\} \in \mathcal{C}$. Then because y is sometime an associate of $C(x)$, 4. must occur with y_1 for y, and so $\{3y_1,3y_1+1\} \in \mathcal{C}$ which implies $\{3y_1\} \notin \mathcal{C}$ by 5. and 6., contradiction. There is therefore s_0 with $\{3y_1\} \notin \mathcal{C}^t$ for all $t \geq s_0$ and all $y_1 \neq y$ in F_x, but $\{3y\} \in \mathcal{C}^{s_0}$ and $W^{s_0}_{i,k} = \{3y\}$. Further we can suppose that j is the associate of $D(i)$ at s_0. For finitely many z, making up a set Z, we may have $W^{s_0}_{i,z} = \{3y_1\}$ or $\{3y_1,3y_1+2\}$ and $T^{s_0}_{j,z} = \{3y_1\}$ or $\{3y_1,3y_1+1\}$ with $y_1 \neq y$ in F_x. $W_{i,z} \in \mathcal{C}$, so $W_{i,z} = \{3y_1,3y_1+1\}$ or $\{3y_1,3y_1+2\}$. If

$W_{i,z} = \{3y_1, 3y_1+1\}$ 9. will ensure that eventually $T_{j,z}$ will be $\{3y_1, 3y_1+1\}$, and if $W_{i,z} = \{3y_1, 3y_1+2\}$, 11. will ensure that $T_{j,z}$ will become $\{3y_1, 3y_1+2\}$ ($T_{j,z}$ can never be $\{3y_1, 3y_1+1\}$ because then 12. would occur and j would not be the final associate of D(i)). Thus let $s \geq s_0$ be s.l. that for each $z \in Z$ we do not have $W^s_{i,z} = \{3y_1\}$ or $\{3y_1, 3y_1+2\}$ and $T^s_{j,z} = \{3y_1\}$ or $\{3y_1, 3y_1+1\}$ with $y_1 \neq y$ in F_x.

Now if there is $y_1 \neq y$ in F_x s.t. $Q(y_1, i, s)$, there is z, which must $\notin Z$, with $W^s_{i,z} = \{3y_1\}$ or $\{3y_1, 3y_1+2\}$ and $T^s_{j,z} = \{3y_1\}$ or $\{3y_1, 3y_1+1\}$. Since $z \notin Z$, $T^{s_0}_{j,z}$ is empty. But $\{3y_1\} \notin \mathcal{C}^t$ for any $t \geq s_0$, and so $T^s_{j,z}$ is in fact $\{3y_1, 3y_1+1\}$, both members being acquired by 9. at a step t with $s_0 \leq t < s$, and this is impossible because $W^t_{i,z}$ cannot be $\{3y_1, 3y_1+1\}$.

<u>Third case</u>. $W_{i,k}$ has two members and $W_{i,k} = \{3y, 3y+1\}$ for some y. Then the only cases which can affect $T_{j,k}$ are 7., 8., and 9., and 9. will ensure that $T_{j,k} = \{3y, 3y+1\}$.

<u>Fourth case.</u> $W_{i,k}$ has two members and $W_{i,k} = \{3y, 3y+2\}$ for some y. Then 11. will ensure that $T_{j,k} = \{3y, 3y+2\}$. 12. cannot occur since j is the final associate of i.

Q.E.D.

A. is clearly satisfied, and so Lemmas 4, 5, and 6 complete the proof.

In [5] it was shown that if a standard class contains an increasing r.e. sequence, then it contains its limit (that is, its union). In [6] this closure condition was shown to be a property of indexable classes also. The Theorem below (or, rather, its Corollary, 2.4) generalises the result to subclass enumerable classes.

THEOREM 2.3. Let $\langle T_x \mid x \geq 0\rangle$ be an r.e. sequence of r.e. sets s.t. $T = \bigcup\{T_x \mid x \geq 0\}$ is infinite and there is a strongly r.e. sequence of finite sets $\langle U_x \mid x \geq 0\rangle$ with

$U_x \subseteq T_x$, $U_x \subseteq U_{x+1}$, and $\bigcup\{U_x \mid x \geq 0\} = T$. Then any subclass enumerable class containing T_x for each x also contains T.

PROOF. Let $\mathcal{C}$ be a class of r.e. sets containing T_x for each x but not containing T. Let $\langle S_{i,j} \mid i,j \geq 0\rangle$ be a 2-dimensional r.e. sequence of members of $\mathcal{C}$, and denote by $\mathcal{C}_i$ the r.e. subclass of $\mathcal{C}$ given by the i-th row. We show we can enumerate an r.e. sequence $\langle R_x \mid x \geq 0\rangle$ s.t.

1) $R_x \in \mathcal{C}$ for all x

2) If $\langle R_x \mid x \geq 0\rangle$ enumerates $\mathcal{C}^*$, then $\mathcal{C}^* \neq \mathcal{C}_i$ for all i.

Then $\mathcal{C}$ cannot be subclass enumerable.

DISCUSSION.

<u>To satisfy the $\mathcal{C}_0$ condition:</u> Let $S^s_{i,j}$ be the approximation to $S_{i,j}$ after step s of a recursive enumeration of $\langle S_{i,j}\rangle$. We will make each member of the R sequence a member of the T sequence. If $S_{0,0}$ is a T_x, then for all s $S^s_{0,0} \subseteq T$ and

so there is y s.t. $S_{0,0} \subseteq U_y$ (since $\bigcup\{U_x \mid x \geq 0\} = T$). Thus after step s of $S_{0,0}$ we can wait till we find such a y. If we do, then there is a largest x s.t. $U_x \subseteq S^s_{0,0}$. We then ensure that for each n, R_n is T_z with z sufficiently large that $U_{x+1} \subseteq T_z$ -- namely $z \geq x+1$. If at a later step we have to increase x we can do so since for any finite subset F of T_z, there is x' with $F \subseteq U_{x'}$. If $S_{0,0}$ is fact T_a, say, and the largest x with $U_x \subseteq T_a$ is x(a), then $U_{x(a)+1}$ will be in R_n for each n and $S_{0,0} \notin \mathcal{C}^*$. On the other hand, if $S_{0,0}$ is not a T_x but is contained in T, in order to make us go wrong by containing U_x for each x it must be T, which is impossible since $T \notin \mathcal{C}$.

If we try this for all $\mathcal{C}_x$ (with $S_{0,0}$, $S_{1,0}$, $S_{2,0}, \ldots$) we will be liable to injure the $R_x \in \mathcal{C}$ conditions (by making $R_x = T$). Thus we set up a priority ordering

$\mathcal{C}^* \neq \mathcal{C}_0$, $R_0 \in \mathcal{C}$, $\mathcal{C}^* \neq \mathcal{C}_1$, $R_1 \in \mathcal{C}, \ldots$

To satisfy $\mathcal{C}_1$, we restrict only R_n with $n \geq 1$; we therefore have to get <u>two</u> members of $\mathcal{C}_1$ not in $\mathcal{C}^* - R_0$ (so that they cannot both be R_1 and $\mathcal{C}_1 - \mathcal{C}^*$ is non-empty). We look for two members of $\mathcal{C}_1$ differing on the largest U_x contained in them. We also add the condition that $\mathcal{C}^*$ is infinite for the case where two such members cannot be found. And so on, for $\mathcal{C}_2, \mathcal{C}_3, \ldots$.

CONSTRUCTION. Define a 2-dimensional r.e. sequence $\langle F_{i,j} \mid i,j \geq 0\rangle$ by

$$F_{i,j} = \{x \mid (\exists y)(\exists z)(U_x \subseteq S^y_{i,j} \subseteq U_z)\}.$$

Now $F_{i,j}$ is finite for all i,j. $S_{i,j} \neq T$ since $T \notin \mathcal{C}$, i.e. $(T-S_{i,j}) \cup (S_{i,j}-T)$ is non-empty. If $T - S_{i,j}$ is non-empty. since $T = \bigcup\{U_x \mid x \geq 0\}$, $U_x \subseteq S_{i,j}$ is false for all s.l. x and $F_{i,j}$ is finite. On the other hand, if $S_{i,j}-T$ is non-empty, there is y with $S^y_{i,j} - T$ non-empty, and so $S^y_{i,j} \subseteq U_z$ is false for all z. Thus if $x \in F_{i,j}$ we have $U_x \subseteq S^{y_1}_{i,j} \subseteq U_{z_1}$ for some y_1, z_1, and so $y_1 < y$ and $U_x \subseteq S^y_{i,j}$. This can hold for only finitely many x since $\bigcup\{U_x \mid x \geq 0\}$ is infinite and so in any case $F_{i,j}$ is finite.

Let $F^s_{i,j}$ be the set of numbers in $F_{i,j}$ after step s of a recursive enumeration of $<F_{i,j}>$. Define a recursive function c(i,s) as follows.

If there are numbers $j_0, j_1, \ldots, j_i$ s.t.

$$F^s_{i,j_0}, F^s_{i,j_1}, \ldots, F^s_{i,j_i}$$

are distinct and non-empty, put

$$c(i,s) = \text{the largest member of } F^s_{i,j_0} \cup \ldots \cup F^s_{i,j_i},$$ where j_i is chosen to be as small as possible.

Otherwise we put

$$c(i,s) = 0.$$

We claim that for each i, c(i,s) attains a largest value.

CASE 1. There are numbers $j_0, \ldots, j_i$ s.t. $F_{i,j_0}, \ldots, F_{i,j_i}$ are distinct and non-empty. In this case if j_i is chosen as small as possible, then for all s s.l. that

$$j \le j_i \Rightarrow F^s_{i,j} = F_{i,j}$$

we will have

$c(i,s)$ = the largest member of $F_{i,j_0} \cup \ldots \cup F_{i,j_i}$.

Thus $c(i,s)$ is bounded since it converges.

CASE 2. Otherwise, in which case $\{F_{i,j} \mid i,j \ge 0\}$ is finite, $\bigcup\{F_{i,j} \mid i,j \ge 0\}$ has a largest member n, and $c(i,s) \le n$ for all s.

If $\bigcup\{F_{i,j} \mid i,j \ge 0\}$ is empty, $c(i,s) = 0$ for all s. So in any case $c(i,s)$ attains a largest value.

Thus there is a function f and an r.e. sequence $\langle R_x \rangle$ given by $R_x = T_{f(x)}$ satisfying the conditions

(0) $f(n+1) > f(n)$ for all n

(1) $f(n) >$ the largest value of $c(0,s)$ for all n

(2) $f(n) >$ the largest value of $c(1,s)$ for all $n \ge 1$

....

(i+1) $f(n) >$ the largest value of $c(i,s)$ for all $n \ge i$

....

For we can define a recursive function $f(i,s)$ and a 2-dimensional strongly r.e. sequence $\langle R^s_i \rangle$ as follows by simultaneous induction on i and s.

$f(i,0) = 0$ $\qquad\qquad$ R^0_i = the empty set

Let

$t(i,s+1) = \max\{c(0,s+1)+1,\ldots,c(i,s+1)+1, f(i-1,s+1)+1\}$

if $i > 0$, and

$t(i,s+1) = c(0,s+1)+1$ if $i = 0$.

$$f(i,s+1) = \begin{cases} f(i,s) \text{ if } t(i,s+1) \leq f(i,s) \\ \mu x[x \geq t(i,s+1) \;\&\; R_i^s \subseteq U_x] \text{ otherwise} \end{cases}$$

$$R_i^{s+1} = R_i^s \cup T_{f(i,s+1)}^{s+1}$$

where $\langle T_x^s \rangle$ is a recursive enumeration of $\langle T_x \rangle$.

Note that the definition of $f(i,s+1)$ is possible because $T \subseteq \bigcup\{U_x \mid x \geq 0\}$. Define $R_i = \bigcup\{R_i^s \mid s \geq 0\}$. $\mathrm{Lim}_s\, f(i,s) = f(i)$ say exists because for each i, $c(i,s)$ attains a largest value. $R_i = T_{f(i)}$ and conditions (0), (1),... hold.

$\langle R_x \rangle$ enumerates $\mathcal{C}^* \subseteq \mathcal{C}$. The proof is completed by showing that $\mathcal{C}^* \neq \mathcal{C}_i$ for each i.

We need consider only the case where each member of $\mathcal{C}_i$ is in the sequence $\langle T_x \rangle$. Thus for each i,j, $F_{i,j} = \{y \mid U_y \subseteq S_{i,j}\}$ because $T \subseteq \{U_x \mid x \geq 0\}$. If Case 2 occurs, either there is no y s.t. $U_y \subseteq$ a member of $\mathcal{C}_i$ or there is a largest such y. But by condition (0) for each y there is a member of $\mathcal{C}^*$ containing U_y. So $\mathcal{C}_i \neq \mathcal{C}^*$. If Case 1 occurs, the i+1 sets $S_{i,j_0},\ldots,S_{i,j_i}$ are all different, and if $n \geq i$, condition (i+1) yields $f(n) >$ the largest value of $c(i,s) \geq$ the largest member of $F_{i,j_0} \cup \ldots \cup F_{i,j_i}$, so $U_{f(n)} \subseteq R_n$ but $U_{f(n)} \not\subseteq S_{i,j_m}$ for

$0 \le m \le i$. Thus R_n is different from each of $S_{i,j_0},\ldots,S_{i,j_i}$ for all $n \ge i$ and it follows that at least one of $S_{i,j_0},\ldots,S_{i,j_i} \in \mathcal{C}_i - \mathcal{C}^*$ and $\mathcal{C}_i \ne \mathcal{C}^*$.

Q.E.D.

COROLLARY 2.4. If a subclass enumerable class contains an increasing r.e. sequence then it also contains its limit.

PROOF. Let the subclass enumerable class $\mathcal{C}$ contain the increasing r.e. sequence $\langle T_x \mid x \ge 0\rangle$. If $T = \bigcup\{T_x \mid x \ge 0\}$ is finite, $T = T_x$ for some x and so $T \in \mathcal{C}$. Otherwise define

$$U_x = T_0^x \cup T_1^x \cup \ldots \cup T_x^x$$

where T_x^s is the set of numbers in T_x by the end of step s of a recursive enumeration of $\langle T_x\rangle$ and the conditions of the theorem are satisfied.

Q.E.D.

COROLLARY 2.5. A simple example of an r.e. class which is closed to r.e. limits in the sense of Corollary 2.4 but is not subclass enumerable is $\{N-\{x\} \mid x \ge 0\}$.

PROOF. Define $U_x = \{n \mid n < x\}$, $T_x = N-\{x\}$ and it follows that any subclass enumerable class containing $\{N-\{x\} \mid x \ge 0\}$ must also contain N.

Q.E.D.

The following example shows that THEOREM 2.3 does not characterize subclass enumerable classes.

EXAMPLE 2.3. There exists a non-empty r.e. class $\mathcal{C}$, consisting of singletons and pairs, which is not subclass enumerable.

PROOF. Let $\sigma(i,j) = 2\,\tau(i,j)$, where τ is the Cantor pairing function, so that σ is an effective one-one correspondance between ordered pairs of natural numbers and the even numbers. Let $\mathcal{C}^*$ be the class of r.e. sets of cardinality one or two.

Take $\mathcal{C} = \mathcal{C}^* - \{\{\sigma(i,j)\} \mid (\exists y)(y \in R_j \ \&\ \sigma(i,j) \in W_{i,y})\}$ where $\langle R_j \mid j \geq 0\rangle$ is a recursive enumeration of the r.e. sets and $\langle W_{i,j} \mid i,j \geq 0\rangle$ is a 2-dimensional r.e. sequence whose rows include all the r.e. sequences.

Suppose that $\mathcal{C}$ is subclass enumerable, then by Definition 2.6 there is an i s.t. $\langle W_{i,y} \mid y \geq 0\rangle$ is a recursive enumeration of $\mathcal{C}$, and if $\mathcal{A}$ is an r.e. subclass of $\mathcal{C}$ there is a j with $\mathcal{A} = \{W_{i,y} \mid y \in R_j\}$.

Let $\mathcal{A}$ be the r.e. class enumerated by the r.e. sequence $\langle U_j \mid j \geq 0\rangle$ defined as follows:

CASE (1j). $\sim(\exists y)(y \in R_j \ \&\ \sigma(i,j) \in W_{i,y})$.

Then put $U_j = \{\sigma(i,j)\}$. $U_j \in \mathcal{C}$.

CASE (2j). $(\exists y)(y \in R_j \ \&\ \sigma(i,j) \in W_{i,y})$.

Fix such a y, say y_0. $\{\sigma(i,j)\} \notin \mathcal{C}$, and so $W_{i,y_0} = \{\sigma(i,j), n\}$, with $n \neq \sigma(i,j)$. Put $U_j = \{\sigma(i,j), m\}$, where m is an odd number different from n.

$\mathcal{A} \subseteq \mathcal{C}$, and so there is a j with $\mathcal{A} = \{W_{i,y} \mid y \in R_j\}$.

In CASE (1j), $\{\sigma(i,j)\} \in \mathcal{A} - \{W_{i,y} \mid y \in R_j\}$, contradiction. In CASE (2j), take y_0, n, m as above. We have $\{\sigma(i,j),n\} \in \{W_{i,y} \mid y \in R_j\} - \mathcal{A}$, giving a contradiction. For if $\{\sigma(i,j),n\} \in \mathcal{A}$, $\{\sigma(i,j),n\} = U_k$ say. Since n is odd and $\sigma(i,j)$ is even, $\sigma(i,j) = \sigma(i,k)$ and therefore $j = k$ because σ is one-one. This contradicts $m \neq n$.

Q.E.D.

Our final example completes the proof of the Main Theorem.

EXMPLE 2.4. There is a class of r.e. sets which is subclass enumerable but not indexable.

REMARK. The class $\mathcal{C}$ which we construct consists of singletons and pairs, and contains all the pairs.

PROOF. Let $\langle W_{x,y} \mid x,y \geq 0\rangle$ be a 2-dimensional r.e. sequence of r.e. sets with less than three members including every r.e. sequence of r.e. sets with less than three members among its rows, and let $\langle \varphi_x \mid x \geq 0\rangle$ be a recursive enumeration of all the one-argument p.r. functions. Put $\varphi(x,y) =_{df} \varphi_x(y)$. Let $(i(n),j(n))$ be the (n+1)st ordered pair of natural numbers in some effective numbering, for example we can define the functions i and j by $\tau(i(n),j(n)) = n$, where τ is the Cantor pairing function.

DISCUSSION. To make $\mathcal{C}$ non-indexable, for each p we want to prevent the r.e. sequence $\langle W_{p,x} \mid x \geq 0\rangle$ from being an indexing of $\mathcal{C}$, that is (by Definition 2.4), we should be able to construct an r.e. sequence $<U_x \mid x \geq 0>$ s.t. if $W_{p,x} \in \mathcal{C}$ for all x, then $U_x \in \mathcal{C}$ for all x, and for each q there is an x with $\varphi(q,x)$ undefined or $\varphi(q,x)$ defined but $W_{p,\varphi(q,x)} \neq U_x$. Let us assign a position β in the U sequence to the p.r. function φ_q, and a singleton $\{\alpha\}$ to the pair (p,q). Compute $\varphi(q,\beta)$ and if it is defined enumerate $W_{p,\varphi(q,\beta)}$. Put $\{\alpha\}$ in $\mathcal{C}$ and α in U_β. As long as $\varphi(q,\beta)$ is undefined or $W_{p,\varphi(q,\beta)} \neq \{\alpha\}$ we can leave it at that. But if we ever have $\varphi(q,\beta)$ defined and $W_{p,\varphi(q,\beta)}=\{\alpha\}$ we remove $\{\alpha\}$ from $\mathcal{C}$ by converting it to a pair; if subsequently it turns out that $W_{p,\varphi(q,\beta)}$ is a pair we can make U_β a different pair. In general $(p,q) = (i(n),j(n))$ and by using distinct α for different n, and distinct β for different q, $\mathcal{C}$ can be made non-indexable.

Simultaneously, however, we must make $\mathcal{C}$ subclass enumerable, that is (by Definition 2.5) we have to enumerate r.e. classes $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \ldots$ with $\mathcal{K}_l \subseteq \mathcal{C}$ s.t. for each k, if $W_{k,x} \in \mathcal{C}$ for all x, then there is l with $\{W_{k,x} \mid x \geq 0\} = \mathcal{K}_l$. A singleton $\{\alpha\}$ used as above can give us trouble, for if $W_{k,x}$ is apparently $\{\alpha\}$ while $\{\alpha\} \in \mathcal{C}$, we put $\{\alpha\}$ in $\mathcal{K}_l$, but then if $\{\alpha\}$ is removed from $\mathcal{C}$ we must make the $\{\alpha\}$ in $\mathcal{K}_l$ a pair, and $W_{k,x}$ may in fact be a different pair.

We assign priorities to our requirements, the $(i(n),j(n))$ non-indexability requirement taking priority over the row k subclass enumerability requirement if and only if $n \le k$. If $n \le k$ and a conflict arises, we choose a new l to use with k, converting each member of the old $\mathcal{X}_l$ to a pair. Define

$F_0 = \{0\}$, $F_1 = \{1,2\}$, $F_2 = \{3,4,5\}, \ldots$. We use as our α for the $(i(n),j(n))$ requirement a member of F_n. We show that we can eventually find an $\alpha \in F_n$ which is "safe" as regards possible conflicts with a row k requirement with $k < n$.

CONSTRUCTION. Let $W^s_{x,y}$, $\varphi(x,y,s)$ refer to the situation after step $s(=0,1,2,\ldots)$ of a simultaneous enumeration of $\langle W_{x,y} \mid x,y \ge 0\rangle$ and computation of $\langle \varphi_x(y) \mid x,y \ge 0\rangle$.

There is $x \in F_n$ s.t. $P(n,x) =_{df}$ for each $k < n$,

$$x \notin \bigcup\{W_{k,z} \mid z \ge 0\} \;.\vee.\; (\exists y)(\exists z)(y \ne x \;\&\; W_{k,z} = \{x,y\}) \;.\vee.$$
$$(\exists y)(\exists z)(y \in F_n \;\&\; y \ne x \;\&\; W_{k,z} = \{y\}).$$

For if $\sim P(n,x)$ for each $x \in F_n$, for each $x \in F_n$ let $k(x)$ be the least number $<n$ s.t.

$(\exists z)(W_{k(x),z} = \{x\} \;\&\; (y)(y \in F_n \;\&\; y \ne x \Rightarrow \sim(\exists z)(W_{k(x),z} = \{y\}))$, and $k(x)$ is a one-one function with $n+1$ elements in its domain and n elements in its range, which is impossible. There is therefore a binary recursive function $\gamma(n,x)$ s.t.

$$(n)(x)(\gamma(n,x) \in F_n) \;\&\; (n)(\exists y)(x)(x \ge y \Rightarrow \gamma(n,x) = \gamma(n,y) \;.\&.\; P(n,\gamma(n,y))).$$

This is true because $P(n,x)$ has the form

$(k)_{k<n}\ (\exists y)(z)\ Q(n,x,k,y,z)$

where Q is a recursive relation. $\mathrm{Lim}_x \gamma(n,x)$ is the "safe" value of $\alpha \in F_n$ referred to above.

At step s of the definition of $\mathcal{C}$, we also define numbers $\alpha(n,s)$, $\beta(n,s)$ for each $n \leq s$, working successively on $n=0$, $n=1,\ldots,$ $n=s$. We stipulate that all pairs are to be in $\mathcal{C}$, and we enumerate them separately.

Instructions for step s.

There is one case if n=s.

$\alpha(n,s) = \gamma(n,0)$

$\beta(n,s)$ = the least number which has not yet been a value of $\beta(m,t)$ with $i(m) = i(n)$.

Put $\{\alpha(n,s)\}$ into $\mathcal{C}$.

There are three cases if $n < s$.

Suppose $\alpha(n,s-1) = \gamma(n,m)$.

Case 1 $\varphi(j(n), \beta(n,s-1))$ is defined by step s and

$\alpha(n,s-1) \in W^s_{i(n),\varphi(j(n),\beta(n,s-1))}$.

Then we put

$\alpha(n,s) = \gamma(n,m)$

$\beta(n,s) = \beta(n,s-1)$.

Also we remove $\{\alpha(n,s)\}$ from $\mathcal{C}$, by converting any occurrence of it to a pair.

Case 2 Case 1 does not occur.

Then we put

$\alpha(n,s) = \gamma(n,m+1)$

2.1 If $\alpha(n,s) = \alpha(n,s-1)$ put $\beta(n,s) = \beta(n,s-1)$.

2.2 If $\alpha(n,s) \neq \alpha(n,s-1)$ put $\beta(n,s) =$ the least number which has not yet been a value of $\beta(m,t)$ with $i(m) = i(n)$, and remove $\{\alpha(n,s-1)\}$ from $\mathcal{C}$ and put $\{\alpha(n,s)\}$ in.

NOTE. For each n there are two possibilities:

<u>For some s, Case 1 occurs.</u> Then for all $t \geq s-1$,

$\alpha(n,t) = \alpha(n,s-1) = \alpha(n)$ say.

$\beta(n,t) = \beta(n,s-1) = \beta(n)$ say.

Also if $x \in F_n$, $\{x\} \notin \mathcal{C}$.

<u>Case 1 never occurs.</u> Then for all s.l. s $\alpha(n,s)$, $\beta(n,s)$ are constants $\alpha(n)$, $\beta(h)$ as before. This time $\alpha(n) = \mathrm{Lim}_x\ \gamma(n,x)$ and so $P(n,\alpha(n))$.

Also if $x \in F_n$, $\{x\} \in \mathcal{C} \Leftrightarrow x = \alpha(n)$.

<u>$\mathcal{C}$ is not indexable.</u>

Suppose otherwise then there is p s.t. the r.e. sequence $\langle W_{p,x} \mid x \geq 0\rangle$ is an indexing of $\mathcal{C}$. Define an r.e. sequence $\langle U_x \mid x \geq 0\rangle$ as follows. Let $i^{-1}(p)$ be $n_0, n_1, n_2, \ldots$ in increasing order. For every number x there is a unique k s.t. x is chosen as $\beta(n_k,s)$ for some s and we have $\beta(n_k,t) = x$ for all $t \geq s$ unless Case 2.2 ever happens and then x is never a value of $\beta(n_k,t)$ again. $\alpha(n_k,s)$ is put into U_x. Consider the succeeding steps t at which $\beta(n_k,t) = x$. If Case 1 ever happens we know $\alpha(n_k,s) = \alpha(n_k)$, $\beta(n_k,s) = \beta(n_k)$, $\{\alpha(n_k)\} \notin \mathcal{C}$ and so $W_{p,\varphi(j(n_k),\beta(n_k))} = \{\alpha(n_k), y\}$ with $y \neq \alpha(n_k)$. We then put $z \neq y$, $\alpha(n_k)$ into U_x. If Case 2.2 ever happens we convert U_x to a pair.

For each x, $U_x \in \mathcal{C}$. For U_x is a singleton only if neither Case 1 nor Case 2.2 ever happens and then $\{\alpha(n_k,s)\} = \{\alpha(n_k)\} \in \mathcal{C}$.

Also $\langle U_x \mid x \geq 0\rangle$ is not reducible to $\langle W_{p,x} \mid x \geq 0\rangle$. For if the (q+1)-st p.r. function is total choose k s.t. $j(n_k) = q$. Let s be the first step at which $\beta(n_k,s) = \beta(n_k)$. Then $\alpha(n_k,s) = \alpha(n_k)$ also. Take $x = \beta(n_k)$ in the above definition of U_x. Case 2.2 can never occur. If Case 1 ever occurs $U_{\beta(n_k)}$ is a different pair from $W_{p,\varphi(q,\beta(n_k))}$, and otherwise $\alpha(n_k) \in U_{\beta(n_k)} - W_{p,\varphi(q,\beta(n_k))}$ so in either case $U_{\beta(n_k)} \neq W_{p,\varphi(q,\beta(n_k))}$ and the (q+1)-st p.r. function cannot reduce $\langle U_x (x \geq 0 \rangle$ to $\langle W_{p,x} \mid x \geq 0\rangle$, which is a contradiction.

<u>$\mathcal{C}$ is subclass enumerable.</u>

We enumerate a sequence of r.e. classes

$\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \ldots$

s.t. 1. $\mathcal{K}_l \subseteq \mathcal{C}$ for all l

2. For each k, if $(x)(W_{k,x} \in \mathcal{C})$ then there is l with $\mathcal{K}_l = \{W_{k,x} \mid x \geq 0\}$.

Follow a plan by which for each ordered pair (k,x) we consider the set $W_{k,x}$ at infinitely many steps of the construction of $\mathcal{C}$. For each k there may be a step at which we <u>associate</u>

one of the classes $\mathcal{K}_l$ with k. $\mathcal{K}_l$ remains associated with k unless it is <u>eliminated</u> in which case $\mathcal{K}_l$ is to be made to consist of a class of pairs. Later we may choose a new $\mathcal{K}_l$ to associate with k.

Step s corresponding to (k,x):

<u>Case (a)</u> $W^s_{k,x}$ is empty. Do nothing.

<u>Case (b)</u> $W^s_{k,x}$ is a singleton $\{y\}$, $y \in F_n$ say.

<u>$n \le k$</u> If $\{y\} \in \mathcal{C}$ at step s associate a value of l with k if one does not exist and put $\{y\}$ in $\mathcal{K}_l$. We note that this occurrence of $\{y\}$ in $\mathcal{K}_l$ came from $W_{k,x}$.

If $\{y\} \notin \mathcal{C}$ at step s, l is associated with k and $\{y\} \in \mathcal{K}_l$, eliminate $\mathcal{K}_l$.

<u>$n > k$</u> If $\{y\} \in \mathcal{C}$ at step s and $(\exists u)(\exists v)(u \neq y \ \&\ W^s_{k,v} = \{y,u\})$, associate a value of l with k if one does not exist and put $\{y\}$ in $\mathcal{K}_l$. Note that this occurrence of $\{y\}$ in $\mathcal{K}_l$ came from $W_{k,x}$.

<u>Case (c)</u> $W^s_{k,x}$ is a pair $\{y,z\}$. Associate a value l with k if one does not exist and put $\{y,z\}$ in $\mathcal{K}_l$. If $\{y\}$ or $\{z\}$ is in $\mathcal{K}_l - \mathcal{C}$ at this time, convert any occurrence of $\{y\}$ or $\{z\}$ in $\mathcal{K}_l$ to $\{y,z\}$; otherwise just convert any occurrence of $\{y\}$ or $\{z\}$ in $\mathcal{K}_l$ which came from $W_{k,x}$ to $\{y,z\}$.

<u>1. is satisfied.</u>

Otherwise there are l,k,x,s,y s.t. l is associated with k and is never eliminated, s corresponds to (k,x), Case (b) occurs at step s putting $\{y\}$ in $\mathcal{K}_l$ for good, and $\{y\} \notin \mathcal{C}$.

First suppose $y \in F_n$ with $n \leq k$. If $W_{k,x} \neq \{y\}$ Case (c) gives a contradiction, and if $W_{k,x} = \{y\}$ Case (b) gives a contradiction since $\mathcal{K}_l$ is eliminated. Next suppose $y \in F_n$ with $n > k$. Then Case (c) gives a contradiction at a step corresponding to the (k,v) s.t. $W^s_{k,v} = \{y,u\}$ with $u \neq y$.

2. is satisfied.

Fix k and suppose that $(x)(W_{k,x} \in \mathcal{C})$. First we show that at infinitely many steps there is some value of l associated with k. This is clear unless $\{W_{k,x} \mid x \geq 0\}$ consists entirely of singletons $\{y\}$ with $y \in F_n$, $n > k$. With such an n Case 1 never occurs (because $\{y\} \in \mathcal{C}$) and $y = \alpha(n)$, $P(n,y)$. This implies that there is $z \in F_n$, $z \neq y$, $\{z\} \in \{W_{k,x} \mid x \geq 0\}$ which is impossible since at most one singleton from F_n can be in $\mathcal{C}$.

Suppose k is associated with infinitely many values of l. Then since $\bigcup\{F_n \mid n \leq k\}$ is finite there is a fixed y s.t. the $\mathcal{K}_l$ associated with k is eliminated infinitely often through Case (b) with this y. It follows that $\{y\} \notin \mathcal{C}$ and thus $\{y\}$ is eventually not in $\mathcal{C}$ and so for s.l. l associated with k we never have $\{y\} \in \mathcal{K}_l$. Contradiction.

Let l be the final associate of k. $\mathcal{K}_l = \{W_{k,x} \mid x \geq 0\}$ as far as pairs go. If $\{y\} = W_{k,x}$ and $\{y\} \notin \mathcal{K}_l$, $y \in F_n$ with $n > k$, $\{y\}$ is eventually in $\mathcal{C}$, and $\sim(\exists u)(\exists v)(u \neq y \ \& \ W_{k,v} = \{y,u\})$. But $y = \alpha(n)$, $P(n,y)$. Thus there is $z \in F_n$, $z \neq y$ s.t. $\{z\} \in \{W_{k,x} \mid x \geq 0\}$ which is impossible. Also Case (c) ensures that if $\{y\} \in \mathcal{K}_1$, $\{y\} = W_{k,x}$ for some x. Thus $\mathcal{K}_1 = \{W_{k,x} \mid x \geq 0\}$.

Q.E.D.

CHAPTER 3

PARTIALLY ORDERED SETS REPRESENTABLE BY R.E. CLASSES.

A partially ordered (p.o.) set $(\mathcal{P},\prec)$ is <u>represented by</u> the r.e. class $\mathcal{C}$ if $(\mathcal{P},\prec)$ is order isomorphic to $(\mathcal{C},\subseteq)$, that is to the p.o. set consisting of $\mathcal{C}$ ordered by the inclusion relation. $(\mathcal{P},\prec)$ is <u>representable</u> if it is represented by some r.e. class.

A. H. Lachlan has conjectured that p.o. sets and representable p.o. sets are indistinguishable by elementary sentences, and has pointed out that to prove this it is sufficient to show that all p.o. sets of a certain type (see the Proposition preceding Theorem 3.4) are representable. In this chapter the conjecture is verified.

Let $(\mathcal{P},\prec)$ be a countable p.o. set, and σ a function from N onto $\mathcal{P}$. Define a class $\mathcal{C}$ of subsets of N by the sequence of sets I(0), I(1), I(2),... with $I(x) =_{df} \{y \mid \sigma(y) \prec \sigma(x)\}$. We have

$$\sigma(x) \prec \sigma(y) \Leftrightarrow I(x) \subseteq I(y).$$

First suppose $\sigma(x) \prec \sigma(y)$ and let $z \in I(x)$. Then $\sigma(z) \prec \sigma(x)$, and so $\sigma(z) \prec \sigma(y)$ and $z \in I(y)$. Next suppose $I(x) \subseteq I(y)$. Then since $\sigma(x) \prec \sigma(x)$, $x \in I(x)$ and so $x \in I(y)$ and $\sigma(x) \prec \sigma(y)$.

Thus $\sigma(x)$ goes-to-$I(x)$ is a well defined order isomorphism from $(\mathcal{P},\prec)$ onto $(\mathcal{C},\subseteq)$.

If the relation R defined by $xRy =_{df} \sigma(x) \preccurlyeq \sigma(y)$ is an $\exists$ relation, then $\mathcal{C}$ is an r.e. class representing $(\mathcal{P},\preccurlyeq)$.

THEOREM 3.1 If $(\mathcal{P},\preccurlyeq)$ is a p.o. set s.t.

(i) there is a function σ:N onto $\mathcal{P}$ with the relation R defined by $xRy =_{df} \sigma(x) \preccurlyeq \sigma(y)$ an $\exists\forall \cap \forall\exists$ relation, and

(ii) $(\mathcal{P},\preccurlyeq)$ has a greatest member,

then $(\mathcal{P},\preccurlyeq)$ is representable by an r.e. class.

CONSTRUCTION As in the remarks preceding the statement of the theorem, we represent $\sigma(x)$ by a set which encodes the initial segment of $(\mathcal{P},\preccurlyeq)$ determined by $\sigma(x)$. Difficulties in doing this are overcome by assigning priorities to our requirements.

By hypothesis (i), there is a recursive function $c(x,y,s)$ with range $\subseteq \{0,1\}$ s.t. for all ordered pairs (x,y)

$\sigma(x) \preccurlyeq \sigma(y) \Rightarrow c(x,y,s) = 0$ for all s.l. s

$\sigma(x) \not\preccurlyeq \sigma(y) \Rightarrow c(x,y,s) = 1$ for all s.l. s .

We can assume that $c(x,x,s) = 0$ for all x,s.

In steps $s = 1,2,3,\ldots$ we will construct an r.e. sequence $\langle T(u) \mid u \geq 0\rangle$ enumerating an r.e. class $\mathcal{C}$. We will also enumerate an r.e. set A.

Each x will eventually have an <u>associate</u>. The associate of x may change, but it will do so only finitely often. Denote the <u>final associate</u> of x by a(x). We intend that T(a(x)) represents $\sigma(x)$. If $u \in A$, we intend that T(u) represents the greatest member of $(\mathcal{P},\preccurlyeq)$ given by hypothesis (ii).

Let $<G(x) \mid x \geq 0>$ be a recursive sequence of disjoint infinite sets. There will always be a greater-than-x tag defined in $G(x)$, initially taken to be the least member of $G(x)$. The x-tag may change, but again it will do so only finitely often. If $g(x)$ is the final greater-than-x tag, we intend that for all y

$$g(x) \in T(a(y)) \Leftrightarrow \sigma(x) \prec \sigma(y).$$

We follow a procedure by which we return to each ordered pair of natural numbers at infinitely many steps.

Instructions for an (x,y) step s

First, if y has no associate, associate with y the least number which has not yet been an associate. Let a_1 be the associate of y, and let g be the x-tag.

Case 1 $c(x,y,s) = 0$

Then put g in $T(a_1)$.

Case 2 $c(x,y,s) = 1$

Then do nothing unless $g \in T(a_1)$, in which event there are two subcases.

We know $x \neq y$.

Subcase 2(a) $x > y$

Then choose a new x-tag, taking it to be the least member of $G(x)$ greater than g. Set up a subprocedure by which g is put into $T(u)$ for each u.

Subcase 2(b) $x < y$

Then choose a new associate a_2 for y, where a_2 is the least number which has not yet been an associate. Put a_1 in A, and set up a subprocedure by which $T(a_1)$ becomes $\bigcup \mathcal{C}$.

This completes the construction.

Define $Q(x,y) = \{s \mid s \text{ is an } (x,y) \text{ step and } 2(a) \text{ or } 2(b) \text{ occurs at } s\}$

We prove the following

LEMMA $Q(x,y)$ is finite.

PROOF OF LEMMA. We show that if $s_1 < s_2$ and $s_1, s_2 \in Q(x,y)$ then there is s_3 s.t. $s_1 < s_3 < s_2$ and $c(x,y,s_3) = 0$. Since $s \in Q(x,y)$ implies that $c(x,y,s) = 1$, the assumption that $Q(x,y)$ is infinite will yield a contradiction, since $c(x,y,s)$ converges. Suppose then that $s_1 < s_2$ and $s_1, s_2 \in Q(x,y)$.

For $s \geq s_1$, denote the x-tag just after step s by $g(x,s)$, the associate of y just after step s by $a(y,s)$, and the set of numbers in $T(u)$ just after step s by $T(u,s)$.

Whether 2(a) or 2(b) occurs at s, we have

$g(x,s_1) \notin T(a(y,s_1),s_1)$.

However, since $s_2 \in Q(x,y)$,

$g(x,s_2-1) \in T(a(y,s_2-1),s_2-1)$.

It is therefore possible to define a step s_3 to be the least step with $s_1 < s_3 < s_2$ s.t.

$g(x,s_3) \in T(a(y,s_3),s_3)$, and we have

$g(x,s_3-1) \notin T(a(y,s_3-1),s_3-1)$.

$g(x,s_3) = g(x,s_3-1)$, for otherwise Subcase 2(a) shows that

$g(x,s_3) \not\in T(u,s_3)$ for any u. Also $a(y,s_3) = a(y,s_3-1)$, because otherwise by Subcase 2(b), $g(x,s_3) \not\in T(a(y,s_3),s_3)$, since the only numbers that can possibly be in $T(a(y,s_3),s_3)$ are discarded tags from the subprocedure of 2(a). The subprocedure of 2(b) does not affect the argument since $a(y,s_3)$ cannot be in A at s_3. Thus we have

$$g(x,s_3) \in T(a(y,s_3),s_3) - T(a(y,s_3),s_3-1)$$

and the only way this can happen is by Case 1. Since the G(x) are disjoint and associates of different numbers are different, s_3 is an (x,y) step. Thus $c(x,y,s_3) = 0$ and the Lemma is proved.

It is now easy to show that the x-tag changes only finitely often. For

the x-tag changes at s

$\Rightarrow s \in Q(x,y)$ for some $y < x$

and $\bigcup_{y<x} Q(x,y)$ is finite.

Also, the associate of y changes only finitely often. For

the associate of y changes at s

$\Rightarrow s \in Q(x,y)$ for some $x < y$

and $\bigcup_{x<y} Q(x,y)$ is finite.

Next we show that

$$g(x) \in T(a(y)) \Leftrightarrow \sigma(x) \prec \sigma(y)$$

First suppose $\sigma(x) \prec \sigma(y)$ and consider an (x,y) step s s.t. that $c(x,y,s) = 0$, the x-tag is g(x) and the associate of y is $a(y)$. Case 1 occurs and $g(x) \in T(a(y))$. Suppose then $g(x) \in T(a(y))$ and we obtain a contradiction from $\sigma(x) \nprec \sigma(y)$. Consider an (x,y) step s s.l. that $c(x,y,s) = 1$, the x-tag is fixed at $g(x)$, the associate of y is fixed at $a(y)$, and $g(x) \in T(a(y))$. One of Subcases 2(a), 2(b) occurs, and either the x-tag is changed or the associate of y is changed, which is impossible.

We have now

$$T(a(x)) \subseteq T(a(y)) \Leftrightarrow \sigma(x) \prec \sigma(y).$$

For if $T(a(x)) \subseteq T(a(y))$ and $\sigma(x) \nprec \sigma(y)$, $g(x) \in T(a(x))$ but $g(x) \notin T(a(y))$. And if $\sigma(x) \prec \sigma(y)$, we need consider only members of $T(a(x))$ of the form $g(z)$, for discarded associates are in $T(u)$ for all u (the subprocedure of 2(a)). If $g(z) \in T(a(x))$ then $\sigma(z) \prec \sigma(x)$, so $\sigma(z) \prec \sigma(y)$ and $g(z) \in T(a(y))$.

Thus ρ defined by

$$\rho(\sigma(x)) = T(a(x))$$

is a well defined order isomorphism from $(\mathcal{P},\prec)$ into $(\mathcal{C},\subseteq)$. It remains to show ρ is onto. If u is not of the form $a(x)$ for any x, then $u \in A$ and $T(u) = \bigcup\mathcal{C}$, by the subprocedure of 2(b). Let $\sigma(x_0)$ be the greatest member of $(\mathcal{P},\prec)$. For all x, $\sigma(x) \prec \sigma(x_0)$, and so $g(x) \in T(a(x_0))$. Also all discarded tags are in $T(a(x_0))$. Thus

$$T(u) = \bigcup\mathcal{C} = T(a(x_0)) = \rho(\sigma(x_0))$$

and ρ is onto.

Q.E.D.

THEOREM 3.2 If $(\mathcal{P}, \prec)$ is a p.o.set s.t.

(i) there is a function $\sigma : N$ onto $\mathcal{P}$ with the relation R defined by $xRy =_{df} \sigma(x) \prec \sigma(y)$ an $\exists\forall \cap \forall\exists$ relation, and

(ii) $(\mathcal{P}, \prec)$ is effectively a directed set, that is to say there is a binary recursive function u s.t.

$$\sigma(u(x,y)) \succcurlyeq \sigma(x) \ \& \ \sigma(u(x,y)) \succcurlyeq \sigma(y),$$

then $(\mathcal{P}, \prec)$ is representable by an r.e. class.

CONSTRUCTION This is of course a generalisation of THEOREM 3.1, and we just indicate briefly how to adapt the above construction. The existance of a greatest member was used in Subcase 2(b). By putting a_1 in A we really made a_1 an associate of x_0, where $\sigma(x_0)$ is the greatest member. Suppose the z-tag $\in T(a_1)$ when we do this, then we are safe because we know that $\sigma(z) \prec \sigma(x_0)$. In the present case, using the function u we can effectively find a y_1 s.t. for all z with the z-tag in $T(a_1)$, $\sigma(z) \prec \sigma(y_1)$, and we make a_1 an associate of y_1. The only problem is : a_1, now associated with y_1, could be transferred again to be an associate of y_2 (although not by conflict with the same x as before), then again to y_3, and so on. The isomorphism might not be <u>onto</u> $\mathcal{C}$. This is taken care of by bringing the requirement that a_1 is a final associate of something into the priority scheme. That is, in Case 2 we are faced with either injuring the "x-tag finally fixed" requirement, or injuring both the "y has a final associate" and the "a_1 is a final associate" requirements. Thus we make the conditions for 2(a), 2(b) $x \geq \min\{y, a_1\}$, $x < \min\{y, a_1\}$ respectively, rather than $x > y$, $x < y$.

Q.E.D.

THEOREM 3.3 If $(\theta, \prec)$ is a p.o. set s.t.
(i) there is a function $\sigma : N$ onto θ with the relation R defined by $xRy =_{df} \sigma(x) \prec \sigma(y)$ an $\exists\forall \cap \forall\exists$ relation, and
(ii) given a finite set $F \subseteq N$, we can effectively find a number $u(F)$ s.t. if $\{\sigma(x) \mid x \in F\}$ has an upper bound in $(\theta, \prec)$, then $\sigma u(F)$ is such an upper bound, then $(\theta, \prec)$ is representable by an r.e. class.

CONSTRUCTION

We assume that if F is a singleton $\{x\}$, then $u(F) = x$.
Let the recursive function c arising from (i) be defined as in THEOREM 3.1. Define for finite $F \subseteq N$

$$d(F,s) = \overline{sg} \prod_{x \in F} \overline{sg}\, c(x, u(F), s).$$

Then we can effectively find $d(F,s)$ given F and s, d takes values 0 and 1, and

$\{\sigma(x) \mid x \in F\}$ has an upper bound in $(\theta, \prec) \Rightarrow d(F,s) = 0$ for all s.l.s

$\{\sigma(x) \mid x \in F\}$ has no upper bound in $(\theta, \prec) \Rightarrow d(F,s) = 1$ for all s.l.s

The idea is to use the function $d(F,s)$ to satisfy an additional requirement: for each finite set F, if $\{\sigma(x) \mid x \in F\}$ has no upper bound, then eventually

$\{t \mid t$ is an x-tag and $x \in F\}$ is not contained in any $T(e)$. Then we can hope to use the construction of THEOREM 3.2 "in the limit."

In steps $s = 1,2,3,\ldots$ we construct an r.e. sequence $\langle T(e) \mid e \geq 0\rangle$ enumerating an r.e. class $\mathcal{C}$.

For each y there will be a step after which y will always have a finite set of <u>associates.</u> An associate of y may be <u>transferred</u> to become an associate of $y_1 \neq y$, and a new associate of y will be chosen. We will ensure that y acquires a <u>final associate</u> which it never loses. If a_1 is a final associate of y, we intend that $T(a_1)$ represents $\sigma(y)$.

To make the isomorphism onto, each natural number e will become an associate and be transferred only finitely often, that is it will become a final associate.

Let $\langle G(x) \mid x \geq 0\rangle$ be a recursive sequence of disjoint infinite sets. There will always be a <u>greater-than-x-tag</u> defined in $G(x)$, initially chosen to be the least member of $G(x)$. The x-tag may change, but it will do so only finitely often. If $g(x)$ is the final greater-than-x tag, we intend that for all y, and all final associates a_1 of y,

$$g(x) \in T(a_1) \Leftrightarrow \sigma(x) \prec \sigma(y).$$

We follow a procedure by which we return to each ordered pair of natural numbers at infinitely many steps.

<u>Instructions for an (x,y) step s</u>

There are five operations to be performed in turn.

1) If y has no associate, associate with y the least number which has not yet been an associate.

2) Effectively find a number $\bar{s} \geq s$ s.t. for each set E with $\{t \mid t \text{ is a } z\text{-tag and } z \in E\} \subseteq T(e)$, where e is an associate of y, we do not have both $c(z,y,\bar{s}) = 0$ for each $z \in E \cup \{x\}$ and $d(E \cup \{x\}, \bar{s}) = 1$.

3) For each z_0 s.t. there is a set F and a number e with $\{t \mid t \text{ is a } z\text{-tag and } z \in F\} \subseteq T(e)$, $d(F,\bar{s}) = 1$ and $z_0 = \max F$, choose a new z_0-tag, taking it to be the least member of $G(z_0)$ greater than the old z_0-tag. Set up a subprocedure by which the old z_0-tag is put in every member of $\mathcal{C}$.

4) (a) For each z_0 s.t. there is an associate e of y with the z_0-tag $\in T(e)$, $c(z_0,y,\bar{s}) = 1$, and $z_0 \geq \min\{y,e\}$ choose a new z_0-tag,..., of $\mathcal{C}$. [as in 3)].

(b) For each e s.t. e is an associate of y and there is z_0 with the z_0-tag $\in T(e)$, $c(z,y,\bar{s}) = 1$, and $z_0 < \min\{y,e\}$ transfer e to be an associate of

$u(\{z \mid z < e \text{ and the } z\text{-tag} \in T(e)\})$.

If y loses at least one associate by (b), associate with y the least number which has not yet been an associate.

5) If $c(x,y,\bar{s}) = 0$ put the x-tag in $T(e)$ for every e associated with y.

This completes the construction.

<u>Lemma 1</u> For each finite set F, if $\{\sigma(x) \mid x \in F\}$ has no upper bound in $(\mathcal{P},\prec)$, then eventually we do not have

$\{t \mid t$ is an x-tag and $x \in F\} \subseteq T(e)$ for any e.

<u>Proof</u> Let s_0 be the step at which $d(F,s)$ is permanently fixed at 1. Then just after operation 3) of step s_0, we will have $\{t \mid t$ is a z-tag and $z \in F\}$ not $\subseteq T(e)$ for any e, for $d(F,\overline{s}_0) = 1$. We claim that this holds forever after. Supposing otherwise, the first time it becomes false is after operation 5) of a step $s \geq s_0$. Let s be an (x,y) step, then there is e associated with y at operation 5) of step s and a non-empty finite set E with $F = E \cup \{x\}$, $x \notin E$, $\{t \mid t$ is a z-tag and $z \in E\} \subseteq T(e)$, and $c(x,y,\overline{s}) = 0$. The same situation holds at operation 2) of step s.
$d(E \cup \{x\}, \overline{s}) = 1$ because $\overline{s} \geq s \geq s_0$. Suppose $z \in E$, then $c(z,y,\overline{s}) = 0$, because if $c(z,y,\overline{s}) = 1$, operation 4) would ensure that the z-tag $\notin T(e)$ at operation 5). Thus $c(z,y,\overline{s}) = 0$ for each $z \in E \cup \{x\}$ and $d(E \cup \{x\}, \overline{s}) = 1$, which contradicts the definition of $\overline{s}$.

Q.E.D.

<u>Lemma 2</u> Each e is a final associate.

<u>Proof</u> e certainly becomes an associate, by 1) or the last part of 4). We must show that e cannot be transferred infinitely often. Suppose otherwise, then we can choose s_0 as large as we please so that e is transferred by operation 4) (b) at step s_0. Let s_0 be sufficiently large that for each finite set F with max F < e and each z < e, $c(z, u(F), s_0)$ is at its final value. We can also

assume that for each F with $\max F < e$, $\{t \mid t$ is a z-tag and $z \in F\} \subseteq T(e)$ implies that $\sigma(z) \prec \sigma\, u\,(F)$ for each $z \in F$. (Lemma 1). Let $F_0 = \{z \mid z < e$ and the z-tag $\in T(e)$ at operation 4) (b) of $s_0\}$. Then e is transferred to $u(F_0)$. Let $s_1 > s_0$ be the next step at which e is transferred. Then there is $z_0 < e$ with the z_0-tag in $T(e)$ at operation 4) (b) of s_1, and $c(z_0, u(F_0), \overline{s}_1) = 1$. $\overline{s}_1 \geq s_1 > s_0$, and so by definition of s_0, $c(z_0, u(F_0), s) = 1$ for all $s \geq s_0$. Thus $T(e)$ cannot have acquired the z_0-tag (by operation 5)) between s_0 and s_1, and the z_0-tag is already in $T(e)$ at s_0, that is $z_0 \in F_0$. By definition of s_0 we therefore have $\sigma(z_0) \prec \sigma\, u(F_0)$, and $c(z_0, u(F_0), s_0) = 0$, which is a contradiction.

Q.E.D.

Lemma 3 Each y has a final associate.

Proof By 1) and the last clause of 4) (b) there is a step after which y always has some associate. Suppose the Lemma is false then there is s_0 sufficiently large that for each $x < y$, $c(x,y,s_0)$ has reached its final value, and such that y loses an associate by operation 4) (b) of step s_0. By the last clause of 4) (b) y acquires a new associate e which has not yet been an associate, so that $T(e)$ can contain only discarded tags. Let e be transferred from y at $s_1 > s_0$. Then there is $z_0 < e$ with the z_0-tag in

$T(e)$ at operation 4) (b) of s_1 and $c(z,y,\overline{s}_1) = 1$. $\overline{s}_1 \geq s_1 > s_0$ and so by definition of s_0, $c(z_0,y,s) = 1$ for all $s \geq s_0$. But the z_0-tag must have been put in $T(e)$ by operation 5) of some step between s_0 and s_1. This is a contradiction.

Q.E.D.

<u>Lemma 4</u> The z_0-tag changes only finitely often.

<u>Proof</u> The z_0-tag can change by 3) or by 4) (a). Let s_0 be sufficiently large that for each finite set F with $\max F = z_0$, $d(F,s_0)$ has reached its final value, and if that value is 1, it is impossible for $\{t \mid t \text{ is a } z\text{-tag and } z \in F\}$ to be contained in any $T(e)$. (Lemma 1). Then after s_0, the z_0-tag can no longer be changed by 3).

Suppose the Lemma is false and the z_0-tag is changed infinitely often by 4) (a). Let

$$Y = \{y \mid y \leq z_0 \text{ or } y \text{ sometime has an associate } \leq z_0\}.$$

Y is finite by Lemma 2, and so there is a fixed $y \in Y$ and infinitely many steps s s.t. for some x, s is an (x,y) step and the z_0-tag is changed at operation 4)(a) of s. Let s_1 be such a step, chosen sufficiently large that $c(z_0,y,s_1)$ has reached its final value, namely 1, and for each $e \leq z_0$, e has become a final associate (Lemma 2). Let s_2 be the next such step. Just after step s_1 the z_0-tag is not in $T(e)$ for any associate e of y, but for some associate e of y at step s_2, the z_0-tag is in $T(e)$. This cannot have come about by operation 5), by definition of s_1. Thus there was a transfer of e to y at some step s_3 between s_1 and s_2, with the z_0-tag in T(e) at operation 4) (b) of s_3. By definition of s_1, $e > z_0$. Let $F_0 = \{z \mid z < e$ and the z-tag $\in T(e)$ at operation 4)(b) of $s_3\}$. We have $y = u(F_0)$ and $z_0 \in F_0$. Thus

$$d(F_0,\overline{s}_3) = \overline{s}g \prod_{z \in F_0} \overline{s}g\, c(z,u(F_0),\overline{s}_3) = 1 \text{ since } c(z_0,y,\overline{s}_3) = 1.$$

But $\{t \mid t$ is a z-tag and $z \in F_0\} \subseteq T(e)$ at operation 3) of s_3, and so operation 3) ensures that this cannot be the case at operation 4)(b). This is a contradiction.

Q.E.D.

<u>Lemma 5</u> Let e be a final associate of y. Let g be the final x-tag. Then

$$g \in T(e) \Leftrightarrow \sigma(x) \prec \sigma(y).$$

<u>Proof</u> Suppose first $\sigma(x) \prec \sigma(y)$ and let s_0 be an (x,y) step sufficiently large that e is permanently an associate of y, the x-tag is fixed at g, and $c(x,y,\overline{s}_0) = 0$. Then $g \in T(e)$ by operation 5).

Suppose then that $g \in T(e)$ but $\sigma(x) \not\prec \sigma(y)$. Let s_0 be an (x,y) step sufficiently large that e is permanently an associate of y, the x-tag is fixed at g, $g \in T(e)$ and $c(x,y,\overline{s}_0) = 1$. Then operation 4) will contradict one of the conditions on s_0.

Q.E.D.

<u>Proof of THEOREM</u> Let a(y) be the least final associate

of y. We have $T(a(x)) \subseteq T(a(y)) \Leftrightarrow \sigma(x) \prec \sigma(y)$. ($\Leftarrow$) Let $g \in T(a(x))$. We can suppose g is a final tag, of z say. Then by Lemma 5 $\sigma(z) \prec \sigma(x)$ and so $\sigma(x) \prec \sigma(y)$ and Lemma 5 gives $g \in T(a(y))$.
($\Rightarrow$) Suppose $T(a(x)) \subseteq T(a(y))$ but $\sigma(x) \not\prec \sigma(y)$. Then if g is the final x-tag, by Lemma 5 $g \in T(a(x)) - T(a(y))$, contradiction.

Thus ρ defined by $\rho(\sigma(x)) = T(a(x))$ is a well defined order isomorphism from $(\mathcal{P}, \prec)$ into $(\mathcal{C}, \subseteq)$. Also ρ is onto by Lemmas 2 and 5.

Q.E.D.

Let $\mathcal{L}$ be the first order language consisting of a single binary predicate constant Q and individual constants $\underline{0}$, $\underline{1}$, $\underline{2}$, ... Let S_0, S_1, S_2,... be a fixed effective enumeration of the sentences in $\mathcal{L}$.

We say a p.o. set $(\mathcal{P}, \prec)$ <u>has property</u> (E) if there is a function σ: N onto $\mathcal{P}$ s.t. if $(\mathcal{P}, \prec)$ is made a model for $\mathcal{L}$ by interpreting Q as $\prec$ and $\underline{0}$, $\underline{1}$, $\underline{2}$, ... as $\sigma(0)$, $\sigma(1)$, $\sigma(2)$, ... we have
(i) $\{x \mid S_x \text{ holds in } (\mathcal{P}, \prec)\}$ is recursive in $\underset{\sim}{0}'$
(ii) there is a singulary recursive function f s.t. if S_x holds in $(\mathcal{P}, \prec)$ and S_x has the form $(\exists y)A(y)$, then $A(f(x))$ holds in $(\mathcal{P}, \prec)$.

A. H. Lachlan pointed out that if X, a sentence involving only the binary predicate constant Q, is satisfied by a p.o. set with Q interpreted as the $\prec$ relation, then we have the following

PROPOSITION X holds in a p.o. set with property (E), and that it follows that if every p.o. set with property (E) is representable by an r.e. class, then X is satisfied by an r.e. class with Q interpreted as the inclusion relation. This was the motivation for THEOREM 3.3.

THEOREM 3.4 If $(\mathcal{P},\prec)$ has property (E), then $(\mathcal{P},\prec)$ is representable by an r.e. class.

PROOF This follows from THEOREM 3.3.

Q.E.D.

COROLLARY P.o. sets and representable p.o. sets are indistinguishable by elementary sentences.

PROOF From THEOREM 3.4 and the PROPOSITION.

Q.E.D.

OUTLINE PROOF OF PROPOSITION

The proposition follows from an analysis of the proof of the completeness theorem (cf THEOREM 35 on page 394 of Kleen, [1]).

Let P be the sentence

$$(\forall x)Q(x,x) \ \& \ (\forall x)(\forall y)(\forall z)(Q(x,y) \ \& \ Q(y,z).\Rightarrow Q(x,z)).$$

Then the sentence P & X is consistent.

Let $Y_0 = \{P \ \& \ x\}$. Define an increasing sequence of finite sets of sentences

$$Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$$

as follows.

Define recursive f by

$f(0) = \mu y(\underline{y}$ does not occur in $S_0)$

$f(x+1) = \mu y(\underline{y}$ does not occur in $S_0, S_1, \ldots, S_{x+1}$ and y is not equal to any of $f(0), f(1), \ldots, f(x))$

To get Y_{x+1} from Y_x: -

Is $Y_x \cup \{S_x\}$ consistent? Answer using $\underset{\sim}{0}'$.

YES. Then Y_{x+1} is Y_x along with S_x, and also in the case that S_x has the form $(\exists y)A(y)$, the sentence $A(f(x))$.

NO. Then Y_{x+1} is Y_x along with $\sim S_x$.

Define $Y^* = \bigcup_{n=0}^{\infty} Y_x$. Note that Y_{x+1} is consistent if Y_x is (using the fact that $f(x)$ cannot occur in Y_x or S_x) and so Y^* is consistent. Also Y^* is clearly complete in $\mathcal{L}$.

We have

$\{x \mid S_x \in Y^*\}$ recursive in $\underset{\sim}{0}'$, and

$S_x \in Y^*$ and S_x has the form $(\exists y)A(y)$ implies that $A(f(x)) \in Y^*$.

Define a binary relation $\overline{Q}$ on the domain N of the natural numbers by

$\overline{Q}(m,n) \Leftrightarrow_{df} Q(\underline{m},\underline{n}) \in Y^*$.

This gives a way of making $(N,\overline{Q})$ a model of $\mathcal{L}$ and by formula induction we can show that for each x

S_x holds in $(N,\overline{Q}) \Leftrightarrow S_x \in Y^*$.

<u>Properties of $(N,\overline{Q})$</u>

Since $P \in Y^*$, P holds in $(N,\overline{Q})$ and so $\overline{Q}$ is a reflexive, transitive relation on N.

$\{x \mid S_x$ holds in N, $\overline{Q}\}$ is recursive in $\underset{\sim}{0}'$.

If S_x holds in $(N,\overline{Q})$ and S_x has the form $(\exists y)A(y)$, then $A(f(x))$ holds in $(N,\overline{Q})$.

X holds in $(N,\overline{Q})$.

Define a system $(\mathcal{P},\prec)$ by letting $\mathcal{P}$ be the equivalence classes on N of the equivalence relation

$$xEy \Leftrightarrow_{df} \overline{Q}(x,y) \ \& \ \overline{Q}(y,x),$$

and taking

$$x/_E \prec y/_E \Leftrightarrow_{df} \overline{Q}(x,y).$$

$\prec$ is well defined and $(\mathcal{P},\prec)$ is a partially ordered set. Define σ: N onto $\mathcal{P}$ by $\sigma(x) = x/_E$. We have

S_x holds in $(\mathcal{P},\prec) \Leftrightarrow S_x$ holds in $(N,\overline{Q})$ since σ is a homomorphism.

Thus $(\mathcal{P},\prec)$ has property (E) and X holds in $(\mathcal{P},\prec)$.

Q.E.D.

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