# Measure-theoretic notions of prevalence

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# Abstract

Haar null sets (or shy sets) play an important role in studying properties of function spaces. In this thesis the concepts of transverse measures, left shy sets and right shy sets are studied in general Polish groups, and the basic theory for the left shy sets and right shy sets is established. In Banach spaces, comparisons of shy sets with other notions of small sets (e.g., Aronszajn null sets and Gaussian null sets in Phelps sense) are made, and examples of non-implications are given. Their thickness and preservations of various kinds of null sets under isomorphisms are investigated. A new description of shy sets is given and is used to study shy sets in finite dimensional spaces. The theory is applied to a number of specific function spaces. Some known typical properties are shown to be also prevalent.

Finally, the notions of left shy sets and right shy sets are applied to study properties of the space of homeomorphisms on $[0,1]$ that leave 0 and 1 fixed. Several interesting examples are given of non-shy sets which are null in the sense of some other notions. Examples of left-and-right shy sets are found which can be decomposed into continuum many disjoint, non-shy sets in $\mathcal{H}[0,1]$. This shows that the $\sigma$-ideal of shy sets of the non-Abelian, non-locally compact space $\mathcal{H}[0,1]$ does not satisfy the countable chain condition.

# Dedication

*To my Parents*

*and*

*Guixia, Tianyu and Jeanne*

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# Chapter 1

# Introduction

A glossary is made at the end of this thesis. The words are mainly from Real Analysis, Measure Theory, Topology and Functional Analysis. Some words (e.g., left shy and probe) are either new or in the literature and their definitions are given in the context of the thesis. Some terms (e.g., Banach space and first category) without explanations can be found in the text books [11] and [48].

Since 1899 when R. Baire [35, pp. 48] introduced the definitions of first category, second category of sets, and established the Baire category theorem, the Baire category theory has been widely used to characterize properties of sets, especially typical properties, and to show the existence of some pathological functions that are usually difficult to construct. Here a *typical property* is a property which holds for all points in a complete metric space except for a set of the first category. There are some restrictions in applications of the Baire category theory. For example, when we discuss a probabilistic result on the likelihood of a given property on a function space, the Baire category theory just provides us the topological structures; Lebesgue measure theory is not subsumed by the Baire category theory. Further in a Banach space it can be shown that there are Lipschitz functions which are not differentiable on a dense $G_\delta$ set, thus the topological category theory is not well suited for differentiability theory of Lipschitz functions. Frequently we need a notion of measure zero in a general setting, that can be used much as sets of the first category have been used to describe properties as typical.

In 1972 Christensen [12] first generalized the concept of sets of Haar measure zero on a locally compact space to an Abelian Polish group which is not necessarily locally compact. Such zero sets have been called *Haar zero sets, Haar null sets, Christensen null sets or shy sets* [12], [56], [8], [27]. We state it in the following definition.

*A universally measurable set $S$ in an Abelian Polish group $G$ is said to be a Haar zero set if there is a Borel probability measure $\mu$ on $G$ such that $\mu(S+x)=0$ for all $x \in G$.*

Christensen [14] showed that this concept of Haar zero sets is very successful in many respects, for example in extending Rademacher's classical theorem that a locally Lipschitz mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is differentiable almost everywhere to mappings between general Banach spaces. In 1980 T. Topsøe and J. Hoffman-Jørgensen [56] extended the notion of Haar zero by Christensen to general topological semigroups. In 1994 Brian R. Hunt, T. Sauer and J. A. Yorke [27] rediscovered the concept of Haar zero sets in Banach spaces. They called such sets shy sets. (In [27] all subsets of a Borel shy set are also termed shy.) In [27] the complement of a shy set is called *prevalent*, and a property in a metric space is said to be *prevalent* if it holds for all points except for a shy set, or it is said that *almost every* function satisfies such property. In [27] the notion of shy sets is used to study properties of certain classes of functions in function spaces. For example, it is easy to show that almost every sequence $\{a_i\}_{i=1}^{\infty}$ in $\ell^2$ has the property that $\sum_{i=1}^{\infty} a_i$ diverges. An attractive result by Hunt [28] is that almost every function in the space of continuous functions is nowhere differentiable. It has been popular to study typical properties of function spaces since Banach [3] and Mazurkiewicz [40] applied the Baire category theorem to give existence proofs of nowhere differentiable, continuous functions independently.

The study of prevalent properties of function spaces offers an interesting contrast to the typical properties. Usually, it is more difficult to show a property is prevalent than to show a property is typical. A typical property may not be a prevalent property and vice versa. This thesis is devoted to systematically studying shy sets, left shy sets and right shy sets in non-separable Banach spaces and general Polish groups, and investigating whether a typical property is prevalent, and showing that some known sets characterized by other notions are shy or non-shy.

In Chapter 2 we introduce various *transverse* notions, and the concepts of *left shy sets*, *right shy sets* in general Polish groups. The basic theory for the left shy sets and right shy sets is established, showing that, in a general Polish group, the countable union of left shy sets is again left shy, and open sets are neither left shy nor right shy. There are several extensions of the concept of shy sets to non-separable Banach spaces (see Christensen [14], Topsøe and Hoffman-Jørgenson [56], Hunt et al. [27], Borwein and Moors [8]). The latter three are equivalent. We compare shy sets, in a Banach space, with $s$-null sets (see Section 2.3), Aronszajn null sets [2], Preiss–Tišer null sets [46] and Gaussian null sets in Phelps sense [45], giving examples of non-implications. Also we discuss their thickness of various kinds of null sets, and their preservations under isomorphisms. In [18] Dougherty has classified non-shy sets into eight types, and mentioned some examples of non-implications, while we give examples in details for all non-implications, and exact characterizations for some non-shy sets.

In [27] Hunt, Sauer and Yorke introduced the definition of a probe. A finite dimensional subspace $P$ of a Banach space is called *a probe* for a set $T$ or its complement if Lebesgue measure supported on $P$ is transverse to a universally measurable set which contains the complement of $T$. To show a set is shy we often try to find some probe of this set. In Chapter 3 we use probes and elementary linear arguments to study some simple prevalent properties in function spaces. Also we use the dimensions of probes to give a new description of shy sets as follows. A shy set $S \subseteq X$ is said to be *m-dimensional* if $S$ has a $m$-dimensional probe but no $n$-dimensional probe for $n < m$, and $S$ is said to be *infinite-dimensional* if $S$ has no finite dimensional probes. We find that, in finite dimensional spaces, this new concept for shy sets is related to Kakeya problems and Besicovitch sets (see [21] and [22]), and so some results are obtained. In [12], [56] and [27] it is shown independently that compact sets are shy in Abelian Polish groups, Abelian semigroups and Banach spaces. A method using results from functional analysis is given to show that certain sets including compact sets are shy in Banach spaces.

In Chapter 4 several known typical properties are shown to be also prevalent. In [10] Bruckner and Petruska showed that, in the spaces $\mathcal{F} = b\mathcal{A}$, $b\mathcal{DB}^1$, $b\mathcal{B}^1$ of bounded approximately continuous functions, bounded Darboux Baire 1 functions and bounded Baire 1 functions equipped with the supremum norm, for any $\sigma$-finite Borel measure $\mu$ on $[0,1]$, the typical functions are discontinuous $\mu$ almost everywhere on $[0,1]$. We show that, in the spaces $\mathcal{F} = b\mathcal{A}$, $b\mathcal{DB}^1$, $b\mathcal{B}^1$, for any $\sigma$-finite Borel measure $\mu$ on $[0,1]$, the prevalent functions are also discontinuous $\mu$ almost everywhere on $[0,1]$. In [50] it was shown that, in the space $BSC[a,b]$ of bounded symmetrically continuous functions on $[a,b]$ equipped with supremum norm, the typical functions have $c$-dense sets of points of discontinuity. Here we show that the prevalent functions of $BSC[a,b]$ also have $c$-dense sets of points of discontinuity. On the space $C[0,1]$ of continuous functions we can also impose the operation of multiplication of functions so that $C[0,1]$ becomes a semigroup. In Chapter 4 we also study the multiplicatively shy sets in the space $C[0,1]$ of continuous functions, and discuss the relations of multiplicatively shy sets and additively shy sets. For example, we show that the set of continuous functions on $[0,1]$ with at least one zero is not additively shy, but is multiplicatively shy in $C[0,1]$.

We use $\mathcal{H}[0,1]$ to denote the space of homeomorphisms on $[0,1]$ that leave 0 and

1 fixed. In Chapter 5 we show that, in $\mathcal{H}[0,1]$, there exists

(1) a Borel probability measure $\mu$ that is both left transverse and right transverse to a Borel set $X$, but is not transverse to $X$.

(2) a Borel probability measure $\mu$ that is left transverse to a Borel set $X$, but is not right transverse to $X$.

(3) a Borel probability measure $\mu$ that is right transverse to a Borel set $X$, but is not left transverse to $X$.

We also find examples of left-and-right shy sets which can be decomposed into continuum many disjoint, non-shy sets in $\mathcal{H}[0,1]$. This answers the problem $(P_0)$ posed by Jan Mycielski in [41]: Does the existence of a Borel probability measure left transverse to a set $Y$ imply that the set $Y$ is shy in a non-locally compact, completely metrizable group? From this we conclude immediately that the $\sigma$-ideal of shy sets in $\mathcal{H}[0,1]$ does not satisfy the countable chain condition. In [25] and [26] Graf, Mauldin and Williams defined a Borel probability measure $P_a$ on $\mathcal{H}[0,1]$ from a probabilistic point of view, and studied whether some interesting sets are null under this measure $P_a$. In Chapter 5 we use our notions of shy sets, left shy sets and right shy sets to study some of the sets discussed by Graf, Mauldin and Williams. For example, Graf, Mauldin and Williams showed that, for any $m \in (0,1]$ and $l \in [1,+\infty)$, the sets $\{h \in \mathcal{H}[0,1] : h(x) \geq mx\}$ and $\{h \in \mathcal{H}[0,1] : h(x) \leq lx\}$ are null under the measure $P_a$. We show that these two sets are neither shy nor prevalent. Comparisons of prevalence results are made with some typical results, and some known results in [25] and [26].

# Chapter 2

# Shy sets

## 2.1 Introduction

Christensen [12], Topsøe and Hoffman-Jørgensen [56], and Hunt et al. [27] studied shy-sets in Abelian Polish groups, topological semigroups and Banach spaces independently. In this section we study left shy sets and right shy sets in general Polish groups, establishing the basic theory for left shy sets and right shy sets. We compare shy sets with other notions of null sets in Banach spaces. For example, in the Banach space of continuous functions on $[0, 1]$ that are zero at $x = 0$, the set of continuous, nowhere Hölder continuous functions with exponent $\alpha$, $0 < \alpha < 1/2$, has Wiener measure zero, but this set is prevalent (see Hunt [28]). Christensen [12] asked whether any collection of disjoint universally measurable non-shy sets must be countable. (This property of a $\sigma$-ideal is called *the countable chain condition*; see page 59.) Dougherty [18] answered this question in the negative by giving an example. Solecki [52] showed that a Polish group admitting an invariant metric satisfies the countable chain condition iff this group is locally compact.

## 2.2 The transverse notion

In all of our discussions in this section we suppose that $X$ is a linear topological space and we assume we have been given a set $S \subseteq X$ that is a universally measurable set in $X$. Some of the terminology applies as well to an Abelian topological group since, for some of the terminology, we use only the additive group structure to define the concepts. Most of our discussion in the sequel, however, will be in the setting of a Banach space, and, occasionally, in a non-Abelian Polish group.

Our goal is to define a measure-theoretic notion of smallness analogous to the topological notions of first category and a direct generalization of the notion of a set of Lebesgue measure zero in finite dimensional spaces.

All measures in the sequel are assumed to be defined on the Borel subsets of the space and can be extended to all universally measurable subsets.

**Definition 2.2.1** We say that a probability measure $\mu$ on $X$ is *transverse* to a set $S$ if

$$\mu(S + y) = 0$$

for all $y \in X$. Thus $\mu$ assigns zero measure to $S$ and to every translate of $S$.

Note that there is no interest in measures here that are not diffuse. If $\mu(\{x_0\}) > 0$ for some point $x_0 \in X$ then $\mu$ cannot be transverse to any non-empty set. In fact, if $\mu$ is transverse to some non-empty set $A$ then $\mu(A+y) = 0$ for all $y \in X$. Choose $y_0 \in A$ then $x_0 \in A - y_0 + x_0$. So $0 < \mu(\{x_0\}) \leq \mu(A - y_0 + x_0) = 0$. This is impossible.

Some authors (e.g., [8]) call the measure $\mu$ a *test-measure* for $S$ if $\mu$ is transverse to $S$. We could also adapt language from [27] and call $\mu$ a *probe* in the sense that it is used to test or prove or "probe" the measure-theoretic nature of the set $S$.

In many applications the construction of a transverse measure for a set $S$ (if there is one) can be done by a simpler device. Often a measure $\mu$ can be found that is supported on a finite dimensional subspace or on some simple compact set. For example if every line in the direction $x$ for some $x \in X$ meets $S$ in a set of one dimensional linear measure zero then a probability measure $\mu$ transverse to $S$ can be constructed by writing

$$\mu(E) = \lambda_1(\{t \in [0,1] : tx \in E\})$$

where $\lambda_1$ denotes one-dimensional Lebesgue measure.

**Lemma 2.2.2** *A set $S$ has the property that every line in the direction $x$ ($x \in X$) meets $S$ in a set of linear dimensional measure zero if and only if the probability measure*

$$\mu(E) = \lambda_1(\{t \in [0,1] : \ tx \in E\})$$

*is transverse to $S$.*

**Proof.** For any $y \in X$,

$$\mu(S+y) = \lambda_1(\{t \in [0,1] : \ tx \in S+y\}) = \lambda_1(\{t \in [0,1] : \ tx - y \in S\}).$$

Thus the result follows. ■

The above lemma leads to the following definition:

**Definition 2.2.3** An element $x \in X$ is said to be *transverse* to a set $S$ if

$$\{t \in \mathbb{R} : \ tx + y \in S\}$$

is Lebesgue measure zero for every $y \in X$.

Again we can say that $x$ is a *test* or a *probe* for $S$, using language that other authors have found convenient.

It is convenient also to express this fact in a variety of manners. Thus we say that the *subspace spanned by $x$ is transverse to $S$* or that *the interval*

$$[0,x] = \{tx : \ t \in [0,1]\}$$

*is transverse to* $S$. By extension of this a collection $\{x_1, x_2, \dots, x_k\}$ of linearly independent elements of $X$ is also said to be *transverse to* $S$ if

$$\{(t_1, t_2, \dots, t_k) \in \mathbb{R}^k : \ t_1x_1 + \cdots + t_kx_k + y \in S\}$$

is of $k$-dimensional Lebesgue measure zero. Also we have a similar result as in the one-dimensional case. The proof is similar.

**Lemma 2.2.4** *A set $S$ has the property that every translate of the k-dimensional subspace generated by the independent elements $\{x_1, \dots, x_k\}$ $(x_1, \dots, x_k \in X)$ meets $S$ in a set of k-dimensional Lebesgue measure zero if and only if the measure*

$$\mu(E) = \lambda_k(\{(t_1, t_2, \dots, t_k) \in [0, 1]^k : \ t_1x_1 + \cdots + t_kx_k \in E\})$$

*is transverse to $S$, where $\lambda_k$ denotes the k-dimensional Lebesgue measure.*

Occasionally linear arguments fail to produce a transverse measure and one needs to seek other compact sets on which the measure is supported. Let $F : [0, 1] \to X$ be a continuous function. Then $C = F([0, 1])$ is a compact set and we say $C$ is *transverse to* $S$ if

$$\{t \in [0, 1] : \ F(t) \in S + y\}$$

is Lebesgue measure zero for every $y \in X$. In this case a probability measure transverse to $S$ can be defined by

$$\mu(E) = \lambda_1(\{t \in [0, 1] : \ F(t) \in E\})$$

since $\mu(S + y) = 0$ for every $y \in X$. We will give some applications in Section 3.4, Section 4.7 and Section 5.3.

## 2.3 Null sets in a separable Banach space

The definitions in this section are generalizations of the notion of a set of Lebesgue measure zero in $n$-dimensional spaces to subsets of a separable Banach space.

We assume that $S$ is a universally measurable subset of a separable Banach space $X$. The following definition is a version from [12] in a separable Banach space. Its extensions to non-separable Banach spaces and Polish groups can be found in Section 2.12 and Section 2.9 respectively.

**Definition 2.3.1** A universally measurable subset $S$ of a separable Banach space $X$ is said to be a *Christensen null set* (shy) if there is a Borel probability measure on $X$ that is transverse to $S$.

This concept is our central concern throughout. The article [27] has popularized the term *shy* for Borel sets that are Christensen null and all their subsets and we shall make use of this term too. The complement of a shy set is said to be *prevalent* . In his original paper Christensen [12] called shy sets *Haar zero sets* and other authors have used the term *Haar null sets*.

The remaining definitions in this section are narrower than Definition 2.3.1. Each of the following classes of sets is a proper subset of the shy sets in general, but may coincide in special cases.

**Definition 2.3.2** A universally measurable set $S$ in a separable Banach space $X$ is said to be an *s-null set* if there is a partition of $S$ into universally measurable subsets

$$S = \bigcup_{i=1}^{\infty} S_i$$

such that for each $i$ there exists an element $e_i$ in $X$ that is transverse to $S_i$.

The following definition is from Aronszajn [2].

**Definition 2.3.3** A Borel set $S$ in a separable Banach space $X$ is said to be an *Aronszajn null set (Borel sense)* if for every sequence $\{e_1, e_2, e_3, \dots\}$ whose linear span is dense in $X$ there is a partition of $S$ into Borel subsets

$$S = \bigcup_{i=1}^{\infty} S_i$$

such that for each $i$ the element $e_i$ is transverse to $S_i$.

It is convenient for contrast to give a similar definition motivated by that of Aronszajn [2]. The original paper studies the Borel version of Definition 2.3.3—we give a universally measurable version as well for comparison.

**Definition 2.3.4** A set $S$ in a separable Banach space $X$ is said to be an *Aronszajn null set (universally measurable sense)* if for every sequence $\{e_1, e_2, e_3, \ldots\}$ whose linear span is dense in $X$ there is a partition of $S$ into universally measurable subsets

$$S = \bigcup_{i=1}^{\infty} S_i$$

such that for each $i$ the element $e_i$ is transverse to $S_i$.

Perhaps the narrowest version of a null set in this spirit is that from Preiss Tišer [46].

**Definition 2.3.5** A universally measurable set $S$ in a separable Banach space $X$ is said to be a *Preiss-Tišer null set* if every element of the space is transverse to $S$.

Definition 2.3.1, 2.3.2 and 2.3.5 all have similar forms in the Borel sense, i.e., replacing the universally measurable condition by the requirement that the sets are Borel. The above definitions in the universally measurable sense are parallel to the definitions in the Borel sense but they are different. A Christensen null set need not be Borel. The following simple example can show this.

**Example 2.3.6** In $\mathbb{R}$ there is a Christensen null set which is not Borel. We know that in $\mathbb{R}$ there is an analytic set $B$ which is not Borel (see [11, pp. 492]). So there is a Borel set $F \subseteq B$ such that $B \setminus F$ is Lebesgue measure zero. Thus the set $B \setminus F$ is universally measurable and so is Christensen null but it is not Borel from the following statement.

*In the space $\mathbb{R}^n$ a universally measurable set is Christensen null iff it has n-dimensional Lebesgue measure zero.*

The above result will be given and proved in the next section.

From the above it is easy to see that, in the real line $\mathbb{R}$, Preiss–Tišer null sets, Aronszajn null sets, $s$-null sets, and Christensen null sets are equivalent to Lebesgue measure zero sets that are universally measurable. However, in $\mathbb{R}^n$ ($n \geq 2$) the class of Preiss–Tišer null sets is much smaller than the class of Lebesgue measure zero sets that are universally measurable. This is because any proper subspace of $\mathbb{R}^n$ ($n \geq 2$) is Lebesgue measure zero but is not Preiss–Tišer null.

We now compare in general these different sets in the universally measurable sense. The following inclusions are obvious:

*Preiss–Tišer null* $\Rightarrow$ *Aronszajn null* $\Rightarrow$ *$s$-null* $\Rightarrow$ *Christensen null.*

The following are two examples to show that, in general, the first two inclusions are proper.

**Example 2.3.7** In the plane $\mathbb{R}^2$ the set $\{\lambda e_1 : \lambda \in \mathbb{R}\}$ is obviously Aronszajn null where $e_1 = (1, 0)$. However it is not Preiss–Tišer null since the element $e_1$ itself is not transverse to it.

**Example 2.3.8** There is an $s$-null set that is not Aronszajn null. In fact, in Example 2.4.4 the set $K$ is compact in the infinite dimensional separable Banach space $X$. So it is $s$-null (see Theorem 3.4.2) . Note the set $K$ is not Gaussian null in Phelps sense (see Gaussian null sets in Section 2.4 and Example 2.4.4). Since an Aronszajn null set must be a Gaussian null set in Phelps sense [45], thus the set $K$ is not Aronszajn null. Since an $s$-null set is Christensen null so the set $K$ is also an example of a Christensen null set that is not Aronszajn null.

*Remark.* In the plane $\mathbb{R}^2$ a Christensen null set is an $s$-null set (see Theorem 3.2.4). However we do not know whether there is a Christensen null set that is not $s$-null in $\mathbb{R}^n$ $(n > 2)$.

**PROBLEM 1** *In a separable Banach space is there a Christensen null set that is not s-null?*

## 2.4 Gaussian null sets

The main purpose of this section is to compare the null sets in Section 2.3 with two other kinds of small sets, Gaussian null sets in Phelps sense and Gaussian null sets in the ordinary sense. For convenience we require Gaussian measures defined on Borel sets. The following three definitions are reproduced from [45].

**Definition 2.4.1** A *non-degenerate Gaussian measure* $\mu$ on the real line $\mathbb{R}$ is one having the form

$$\mu(B) = (2\pi b)^{-1} \int_B \exp[(-2b)^{-1}(t-a)^2]dt,$$

where $B$ is a Borel subset of $\mathbb{R}$ and the constant $b$ is positive. The point $a \in \mathbb{R}$ is called the *mean* of $\mu$.

**Definition 2.4.2** A Borel probability measure $\lambda$ on a Banach space $X$ is said to be a *Gaussian measure* of *mean* $x_0$ if for each $f \in X^*$, $f \neq 0$, the measure $\mu = \lambda \circ f^{-1}$ is a non-degenerate Gaussian measure on the real line $\mathbb{R}$ as above, where $a = f(x_0)$.

**Definition 2.4.3** (**Phelps**) A Borel subset $B$ of a separable Banach space $X$ is called a *Gaussian null set* in Phelps sense if $\mu(B) = 0$ for every non-degenerate Gaussian measure $\mu$ on $X$.

Recall that a Borel subset $B$ of a separable Banach space $X$ is an ordinary Gaussian null set if there is a non-degenerate Gaussian measure $\mu$ such that $\mu(B) = 0$. Here we also say that the set $B$ is *Gaussian null in the ordinary sense.*

It is easy to see that a Gaussian null set in Phelps sense is Gaussian null in the ordinary sense. However the converse is not true. We know that there does not exist a positive $\sigma$-finite measure on $\ell_2$ (the space of square summable sequences) whose null sets are translation invariant (see Theorem 2.14.3 or [51, pp. 108]). So for any non-degenerate Gaussian measure $\mu$ there is a Borel set $S \subseteq \ell_2$ such that $\mu(S) = 0$ but $\mu$ is not transverse to $S$. Thus the set $S$ is not Gaussian null in Phelps sense.

We compare the two kinds of Gaussian null sets with the null sets Section 2.3. A translate of a non-degenerate Gaussian measure is again such a measure [34], so a Gaussian null set in Phelps sense is Christensen null. In [45] it is shown that a Aronszajn null set is Gaussian null. Thus we have the following implications:

*Aronszajn null* $\Rightarrow$ *Gaussian null in Phelps sense* $\Rightarrow$ *Christensen null.*

We know that, in $\mathbb{R}^n$, Gaussian measures are mutually absolutely continuous with respect to the n-dimensional Lebesgue measure and that a Borel set in $\mathbb{R}^n$ is Christensen null iff it is Lebesgue measure zero (see Theorem 2.7.5). Thus in $\mathbb{R}^n$ Gaussian null sets in two senses and Christensen null sets in Borel sense are all equivalent to the Borel sets of Lebesgue measure zero. The following example from [45] shows that the second inclusion is proper in an infinite dimensional separable Banach space.

**Example 2.4.4** In an infinite dimensional separable Banach space $X$ there is a compact set which is not Gaussian null in Phelps sense. Let $\{w_n\} \subseteq X$ have dense linear span and satisfy $\|w_n\| \to 0$, then its symmetric closed convex hull $K$ is compact but is not Gaussian null. In fact, we define $L : \ell_2 \to X$ by setting, for $x = \{x_n\} \in \ell_2$,

$$Lx = \sum 2^{-n} x_n w_n.$$

It is clear that $L$ is linear and has dense range. Let $U$ denote the unit ball of $\ell_2$. Then $LU \subseteq K$. Since $K$ is the closed convex hull of the compact set $\{\pm w_n\} \cup \{0\}$, it is compact. By the continuity of $L$ we know that if $\mu$ is any non-degenerate Gaussian measure on $\ell_2$ then $\lambda = \mu \circ L^{-1}$ is a non-degenerate Gaussian measure on $X$. Note $U \subseteq L^{-1}K$ and so $\lambda(K) = (\mu \circ L^{-1})(K) = \mu(L^{-1}K) \geq \mu(U)$. It is known [47] that any non-degenerate Gaussian measure on $\ell_2$ assigns positive measure to any non-empty open set. Thus $\lambda(K) > 0$ and $K$ is not Gaussian null in Phelps sense. However the set $K$ is $s$-null and of course Christensen null (see Theorem 3.4.2).

We now leave the following as an open problem.

**PROBLEM 2** *In an infinite dimensional separable Banach space is a Gaussian null set in Phelps sense necessarily Aronszajn null?*

## 2.5 Sets of Wiener measure in $C_0[0,1]$

In this section we summarize some material from Kuo [34] on the Wiener measure in the space $C_0[0,1]$. We compare the null sets, especially Christensen null sets with the null sets of zero Wiener measure on the space $C_0[0,1]$ of real-valued continuous functions $x(t)$ with $x(0) = 0$. $C_0[0,1]$ is a Banach space with the supremum norm. Let $\mathcal{R}$ denote the Borel $\sigma$-field of $C_0[0,1]$. A subset $I$ of $C_0[0,1]$ of the following form

$$I = \{x \in C_0[0,1] : (x(t_1), x(t_2), \cdots, x(t_n)) \in E\},$$

where $0 < t_1 < t_2 < \cdots < t_n \leq 1$ and $E$ is a Borel subset of $\mathbb{R}^n$, will be called a *cylinder set*.

**Definition 2.5.1** Let $I$ be a cylinder set. Define

$$w(I) = [(2\pi)^n t_1(t_2 - t_1)\cdots(t_n - t_{n-1})]^{-\frac{1}{2}} \int_E \exp\left\{-\frac{1}{2}\left[\frac{u_1^2}{t_1} + \right.\right.$$
$$\left.\left. + \frac{(u_2 - u_1)^2}{t_2 - t_1} + \cdots + \frac{(u_n - u_{n-1})^2}{t_n - t_{n-1}}\right]\right\} du_1 \cdots du_n.$$

The unique extension of $w$ to the Borel $\sigma$-field $\mathcal{R}$ is called the *Wiener measure* on $C_0[0,1]$.

*Remark.* The measure $w$ does indeed possess a unique extension to the $\sigma$-field $\mathcal{R}$ (see [34, Theorem 3.2, pp. 43]). When $n = 1$ this Wiener measure is a Gaussian measure of mean 0 on $C_0[0,1]$ (see Kuo [34, pp. 128]).

**Example 2.5.2** If $0 < t \leq 1$, then from the definition of $w$

$$w(\{x \in C_0[0,1] : \ a \leq x(t) \leq b\}) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}} dx.$$

**Example 2.5.3** Let $0 < s < t \leq 1$ be fixed and consider the random variable $x(t) - x(s)$. Let $E \subseteq \mathbb{R}^2$ be the set $E = \{(x,y) : \ a \leq x - y \leq b\}$. Then

$$\begin{aligned} w(\{x : \ a \leq x(t) - x(s) \leq b\}) &= w(\{x : \ (x(t), x(s)) \in E\}) \\ &= \frac{1}{\sqrt{(2\pi)^2 s(t-s)}} \int\int_E e^{-\frac{1}{2}\left\{\frac{v^2}{s} + \frac{(u-v)^2}{t-s}\right\}} du dv \\ &= \frac{1}{\sqrt{(2\pi)^2 s(t-s)}} \int_{-\infty}^{+\infty} \int_{v+a}^{v+b} e^{-\frac{1}{2}\left\{\frac{v^2}{s} + \frac{(u-v)^2}{t-s}\right\}} du dv. \end{aligned}$$

By making a change of variables and simplifying we have

$$w(\{x : \ a \leq x(t) - x(s) \leq b\}) = \frac{1}{\sqrt{2\pi s(t-s)}} \int_a^b e^{-\frac{\tau^2}{2(t-s)}} d\tau.$$

Now we discuss the Wiener measures of the following sets for $\alpha > 0$,

$$C_\alpha = \left\{ x \in C_0[0,1] : \begin{array}{l} \exists a = a(x) \text{ such that for} \\ \forall t, s \in [0,1], \ |x(t) - x(s)| \leq a|t-s|^\alpha \end{array} \right\}.$$

For convenience of discussion we use the following notations:

1 $S$ = dyadic rational numbers in $[0, 1]$.

2

$$H_\alpha[a] = \left\{ x \in C_0[0,1] : \begin{array}{c} \exists s_1, s_2 \in S \text{ such that} \\ |x(s_1) - x(s_2)| > a|s_1 - s_2|^\alpha. \end{array} \right\}.$$

3

$$H_\alpha = \left\{ x \in C_0[0,1] : \begin{array}{c} \forall a > 0, \exists s_1, s_2 \in S \text{ such that} \\ |x(s_1) - x(s_2)| > a|s_1 - s_2|^\alpha \end{array} \right\}.$$

The following theorem is reproduced from Kuo [34].

**Theorem 2.5.4**

$$(a).\ w(C_\alpha) = 1 \ \textit{if}\ 0 < \alpha < 1/2.$$

$$(b).\ w(C_\alpha) = 0 \ \textit{if}\ \alpha > 1/2.$$

**Proof.** (This is a sketch of a proof from Kuo [34].) (a). For $0 < \alpha < 1/2$, it is known that $\lim_{n\to\infty} w(H_\alpha[n]) = 0$ (see a proof in [34, pp. 42]). It is easy to see that

$$C_\alpha = \widetilde{H_\alpha} = \bigcup_{n=1}^{\infty} \widetilde{H_\alpha[n]} = \lim_{n\to\infty} \widetilde{H_\alpha[n]}$$

where $\widetilde{H_\alpha}$ denotes the complement of $H_\alpha$ and so on. Thus

$$w(C_\alpha) = \lim_{n\to\infty} w\left(\widetilde{H_\alpha[n]}\right) = 1 - \lim_{n\to\infty} w(H_\alpha[n])$$
$$= 1 - 0 = 1.$$

(b). For constants $\alpha > 1/2$ and $a > 0$, let

$$J_{\alpha,a,n} = \left\{ x \in C_0[0,1] : \left| x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right) \right| \le a \cdot \left(\frac{1}{2^n}\right)^\alpha \text{ for all } k = 1, 2, \cdots, 2^n \right\}.$$

Clearly for all $n$, $\widetilde{H_\alpha[a]} \subseteq J_{\alpha,a,n}$. Kuo [34] asserted that the random variables

$$x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right), k = 1, 2, \cdots, 2^n$$

are independent and each is normally distributed with mean 0 and variance $1/2^n$. Thus

$$w(J_{\alpha,a,n}) = \prod_{k=1}^{2^n} w\left(\left\{x \in C_0[0,1] : \left|x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right)\right| \leq a \cdot \left(\frac{1}{2^n}\right)^{\alpha}\right\}\right)$$

$$= \prod_{k=1}^{2^n} \int_{-a\cdot\left(\frac{1}{2^n}\right)^{\alpha}}^{a\cdot\left(\frac{1}{2^n}\right)^{\alpha}} \frac{1}{\sqrt{2\pi 2^{-n}}} e^{-\frac{\tau^2}{2\frac{1}{2^n}}} d\tau$$

$$= \prod_{k=1}^{2^n} \sqrt{\frac{2}{\pi}} \int_0^{a\cdot\left(\frac{1}{2^n}\right)^{\alpha-1/2}} e^{-\frac{s^2}{2}} ds.$$

Since $e^{-s^2/2} \leq 1$, so

$$w(J_{\alpha,a,n}) \leq \prod_{k=1}^{2^n} \left\{\sqrt{\frac{2}{\pi}} a \cdot \left(\frac{1}{2^n}\right)^{\alpha-1/2}\right\}$$

$$= \left\{\sqrt{\frac{2}{\pi}} a \cdot \left(\frac{1}{2}\right)^{\alpha-1/2}\right\}^{2^n}$$

$$= e^{2^n\left\{\ln\sqrt{\frac{2}{\pi}}a - (\alpha-1/2)n\ln 2\right\}} \to 0 \text{ as } n \to \infty.$$

Hence for each $a > 0$ and $\alpha > 1/2$, $\lim_{n\to\infty} w(J_{\alpha,a,n}) = 0$, and so $w\left(\widetilde{H_\alpha[a]}\right) = 0$. Therefore $w(C_\alpha) = \lim_{n\to\infty} w\left(\widetilde{H_\alpha[n]}\right) = 0$. ■

**Corollary 2.5.5** *In the space $C_0[0,1]$ the set of continuous, nowhere Hölder continuous functions with exponent $\alpha$, $0 < \alpha < 1/2$, has Wiener measure zero.*

**Proof.** For every continuous function $f$ which is Hölder continuous with exponent $\alpha$ $(0 < \alpha < 1/2)$, there exists a $M > 0$ such that for any $t, s \in [0,1]$, $|f(t) - f(s)| \leq M|t-s|^{\alpha}$ and so $f \in C_\alpha$. Using similar methods as for the set of continuous, nowhere differentiable functions in Section 4.2 it can be shown that the set of continuous, nowhere Hölder continuous functions with exponent $\alpha$ $(0 < \alpha < 1/2)$ is universally measurable. By Theorem 2.5.4, the set of continuous functions that are not somewhere Hölder continuous with exponent $\alpha$ $(0 < \alpha < 1/2)$ has Wiener measure zero. Thus the result follows. ■

This result contrasts sharply with that the set of continuous, nowhere Hölder continuous functions with exponent $0 < \alpha < 1/2$ is prevalent (see Section 4.2).

## 2.6 Non-measure-theoretic variants

Our concern will be with the various notions of null sets presented in Section 2.3, in particular, with the notion of a shy set (Christensen null set). All of these are based on measure-theoretic concepts. Many require that the intersection of a set with a line in some direction have Lebesgue measure zero. One could ask for this intersection to be smaller.

Following Aronszajn [2, pp. 156-157] let $\mathcal{B}$ be a class of Borel sets of the real line that forms a $\sigma$-ideal and is closed under translation and affine transformation. The natural choices for $\mathcal{B}$ are the $\sigma$-ideals of first category sets, or of countable sets, or of $\sigma$-porous sets. Many more choices are possible. We can say that a Borel set $S$ in a separable Banach space $X$ is $\mathcal{B}$-null if for every sequence $\{e_1, e_2, e_3, \dots\}$ whose linear span is dense in $X$ there is a partition of $S$ into Borel subsets

$$S = \bigcup_{i=1}^{\infty} S_i$$

such that for each $i$ the set of real numbers

$$\{t \in \mathbb{R} : x + te_i \in S\}$$

belongs to $\mathcal{B}$ for every $x \in X$.

Let us focus just on the case where $\mathcal{B}$ is the collection of all countable subsets of reals. Then we can define a class of null sets that are sharper than Aronszajn null sets and due to Zarantonello [62].

**Definition 2.6.1** A Borel set $S$ in a separable Banach space $X$ is said to be a *Zarantonello null set* if for every sequence $\{e_1, e_2, e_3, \dots\}$ whose linear span is dense in $X$

there is a partition of $S$ into Borel subsets

$$S = \bigcup_{i=1}^{\infty} S_i$$

such that for each $i$ every line in the direction of the element $e_i$ meets $S_i$ in a countable set.

Even in finite dimensional spaces the Zarantonello null sets are smaller than our other classes of null sets. For example take a collection of Cantor sets $\{C_1, C_2, \ldots, C_n\}$ and consider the product set

$$S = C_1 \times C_2 \times C_3 \times \cdots \times C_n$$

in $\mathbb{R}^n$. Such a set cannot be a Zarantonello null set for any choice of Cantor sets. In fact, on each $C_k$ we can construct a positive Borel measure $\mu_k$ without point mass, i.e., $\mu_k(\{x\}) = 0$ for any $x \in C_k$. Thus on the set $S$ the product measure $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ is also positive without point mass. But for any Zarantonello null set $A \subseteq \mathbb{R}^n$ we have $A = \bigcup_{i=1}^{n} A_i$ and $A_i \cap (x + \lambda e_i)$ is countable for every $x \in \mathbb{R}^n$ where $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the $i$th position. Thus $\mu_i(A_i \cap \lambda e_i) = 0$. By Fubini's theorem we have $\mu(A_i) = 0$ and thus $\mu(A) = 0$. So the set $S$ is not a Zarantonello null set.

This narrower notion of null sets plays a role in the differentiability structure of Lipschitz mappings on Banach spaces. For example a real-valued Lipschitz function on a separable Banach space is Gâteaux differentiable outside of a Borel set that is Aronszajn null. If the function is also convex then it is Gâteaux differentiable outside of a Borel set that is Zarantonello null (see Aronszajn [2, pp. 173]).

## 2.7 Basic theory

In this section we present the basic parts of the theory of Christensen null sets (shy sets) in a separable Banach space $X$ with an indication of proofs. Some of these extend to more general settings and some of the other notions of null sets share one or more of these properties. Some of the proofs here use the original methods of Christensen [12].

**Theorem 2.7.1** *Christensen null sets have empty interior.*

**Proof.** Suppose a Christensen null set $S \subseteq X$ had non-empty interior. Then there are a non-empty ball $B \subseteq S$ and a Borel probability measure $\mu$ such that $\mu(S+x) = 0$ for any $x \in X$. So $B + x \subseteq S + x$ and $\mu(B + x) \leq \mu(S + x) = 0$ for any $x \in X$. Therefore $\mu(B + x) = 0$. Note that $X$ is separable. There is a sequence $\{x_i\} \subseteq X$ which is dense in $X$. So

$$X = \bigcup_{i=1}^{\infty}(B + x_i) \quad \text{and} \quad \mu(X) \leq \sum_{i=1}^{\infty} \mu(B + x_i) = 0.$$

This is a contradiction. Hence the result follows. ■

**Theorem 2.7.2** *A countable union of Christensen null sets is itself a Christensen null set.*

**Proof.** (This proof is reproduced from Christensen [12].) Let $S_n$ be a sequence of Christensen null sets and let $\mu_n$ be their corresponding transverse Borel probability measures. Through translating and normalizing and induction a sequence of Borel probability measures $\mu_n'$ can be found in a neighborhood of zero such that $\chi_{S_n} * \mu_n' = 0$ and $\|x - x * \mu_n'\| \leq 1/2^n$ where $x$ is the convolution of different $\mu_n'$, $i = 1, 2, \ldots, n-1$. Then $\mu = \mu_1' * \mu_2' * \cdots$ is well defined. Since $\mu = x_n * \mu_n' * y_n$ where $x_n = \mu_1' * \cdots * \mu_{n-1}'$ and $y_n = \mu_{n+1}' * \mu_{n+2}' * \cdots$, so $\chi_{S_n} * \mu = 0$ and therefore $\chi_S * \mu = 0$ where $S = \bigcup_{n=1}^{\infty} S_n$.
■

**Theorem 2.7.3** *Every translate of a Christensen null set is also a Christensen null set.*

**Proof.** This conclusion follows directly from the definition of Christensen null sets. ■

**Theorem 2.7.4** *In an infinite dimensional separable Banach space every compact set or a $\sigma$-compact set is Christensen null.*

The proof of this theorem will be given as Theorem 3.4.2 where it is used to illustrate some of the methods of the subject.

**Theorem 2.7.5** *In the space $\mathbb{R}^n$ a universally measurable set is Christensen null if and only if it is Lebesgue measure zero.*

**Proof.** Let $S \subseteq \mathbb{R}^n$ be a Christensen null set and $\mu$ be its transverse Borel probability measure. Then by Fubini's theorem we know

$$\int_{\mathbb{R}^n} \mu(S-y)d\lambda_1(y) = \int_{\mathbb{R}^n} \lambda_1(S-x)d\mu(x) = \lambda_1(S)\mu(\mathbb{R}^n).$$

Then the result follows easily. ■

These five properties show that the Christensen null sets can be expected to play a role in the study of infinite dimensional Banach spaces analogous to the role that sets of Lebesgue measure zero play in finite dimensional spaces.

From their definitions and the comparisons in Section 2.3 and Section 2.4 we see that Preiss-Tišer null sets, $s$-null sets, Aronszajn null sets and Gaussian null sets in Phelps sense all have empty interior, and are translation invariant in a separable Banach space. From the following theorem we see that they all also satisfy the countable union property (that is, the countable union of these sets of same kind is also a set of this kind) in a separable Banach space.

**Theorem 2.7.6** *Let $X$ be a separable Banach space, $S_n \subseteq X$ be a sequence of sets of one of classes of Preiss–Tišer null, s-null, Aronszajn null and Gaussian null in Phelps sense. Then $\bigcup S_n$ is also in the same class as $S_n$.*

**Proof.** (i). Let $S_n$ be Preiss–Tišer null sets. Then for every $x \in X$, $x$ is transverse to $S_n$, $n = 1, 2, \cdots$. Thus $\{t \in \mathbb{R} : tx + y \in S_n\}$ is Lebesgue measure zero for every $y \in X$. Note that

$$\left\{t \in \mathbb{R} : tx + y \in \bigcup_{n=1}^{\infty} S_n\right\} = \bigcup_{n=1}^{\infty} \{t \in \mathbb{R} : tx + y \in S_n\}.$$

Thus $\{t \in \mathbb{R} : tx + y \in \bigcup_{n=1}^{\infty} S_n\}$ is Lebesgue measure zero for every $y \in X$. So $\bigcup_{n=1}^{\infty} S_n$ is also Preiss–Tišer null.

(ii). Let $S_n$ be s-null sets. Then $S_n = \bigcup_{i=1}^{\infty} S_{ni}$ such that for each $i$ there exists an element $e_{ni} \in X$ that is transverse to $S_{ni}$. Thus the countable union of $S_n$ has a partition as

$$\bigcup_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} S_{ni}.$$

Thus $\bigcup_{n=1}^{\infty} S_n$ is also s-null.

(iii). Let $S_n$ be Aronszajn null sets. Then for every sequence $\{e_1, e_2, \cdots\}$ whose linear span is dense in $X$, there are partitions of $S_n$ into universally measurable sets

$$S_n = \bigcup_{i=1}^{\infty} S_{ni}$$

such that for each $i$, the element $e_i$ is transverse to $S_{ni}$. By the same argument as in (i) we know that $e_i$ is transverse to $\bigcup_{n=1}^{\infty} S_{ni}$. Thus the countable union

$$S = \bigcup_{n=1}^{\infty} S_n = \bigcup_{i=1}^{\infty} \left( \bigcup_{n=1}^{\infty} S_{ni} \right)$$

is Aronszajn null.

(iv). The result for Gaussian null sets in Phelps sense follows directly from the definition. ■

## 2.8 The completeness assumption

Throughout we have taken our space to be a Banach space. One might have thought the theory does not use the completeness in any fundamental way and that the same definitions would be useful in normed linear spaces.

A simple example shows that this is not the case. Let $\ell_f$ denote the subspace of $\ell_\infty$ composed of sequences that have only finitely many non-zero elements. Then $\ell_f$ is an infinite dimensional incomplete, normed linear space. It is clear that $\ell_f$ is separable since the set of sequences of rational numbers that have finitely many non-zero elements must be dense in $\ell_f$.

Write $S_n$ for the members of $\ell_f$ that have zero in all positions after the first $n$. Then $\ell_f = \bigcup_{n=1}^{\infty} S_n$. Each $S_n$ is a closed proper subspace of $\ell_f$ and so elementary arguments (cf. Section 3.3) show that each $S_n$ is shy in the space $\ell_f$. In this case a countable union of shy sets comprises the whole space $\ell_f$ and our theory loses one of its main features.

This same flaw would be apparent in any space that could be expressed as a countable union of proper, closed subspaces.

Thus, throughout, the theory will be developed in Banach spaces or, more generally, in completely metrizable topological groups. The same feature applies, of course, to category arguments.

## 2.9 Haar null sets in Polish groups

The theory of Christensen null sets was originally expressed in Abelian Polish groups and motivated by an attempt to generalize to non-locally compact groups the notion of sets of Haar measure zero. Since a non-locally compact group does not have a Haar measure some different approach is needed.

In order that many theorems from Harmonic analysis carry over to the case of non-locally compact Abelian Polish groups and the new concept "Haar zero" coincides with the Haar measure on locally compact Abelian Polish groups the following definition was introduced in [12].

**Definition 2.9.1** Let $G$ be an Abelian Polish group. A universally measurable set $S \subseteq G$ is a *Haar zero set* if there is a Borel probability measure $\mu$ on $G$ such that $\mu(S + x) = 0$ for any $x \in G$.

This definition satisfies our aim, and has proved useful in the differentiability theory for Lipschitz mappings between Banach spaces. We know that an infinite dimensional separable Banach space is also an Abelian Polish group. So the theory in the setting of Abelian Polish groups fits the theory for infinite dimensional separable Banach spaces. Theorem 2.7.1, Theorem 2.7.2 and Theorem 2.7.3 remain valid in this new setting. Theorem 2.7.4 also remains valid but the proof needs modification. We state here and give a brief proof by using the ideas from [12]. [1]

**Theorem 2.9.2** *In a non-locally compact Abelian Polish group $G$ a compact set or a $\sigma$- compact set is Haar zero.*

**Proof.** (This proof is sketched from [12].) We will show that for any universally measurable set $A$ that is not Haar zero we have

$$F(A, A) = \{g \in G : \ (g + A) \cap A \text{ is not Haar zero } \}$$

is a neighborhood of zero element. If this is already proved, suppose there was a compact set $S$ which is not Haar zero. Then the set $S - S$ would be both a neighborhood of zero element and a compact set. This contradicts that $G$ is not locally compact. Thus the set $S$ is Haar zero.

[1] A similar result for "total-porosity" in an infinite dimensional Banach space was given in [1].

Now we show that $F(A, A)$ is a neighborhood of zero for a non-Haar zero set $A$. Suppose this were not the case. Then we may choose a sequence $g_n$ in $G$ not belonging to $F(A, A)$ such that $d(x, x + g_n) \leq 1/2^n$ where $x = \sum_{i=1}^{n-1} g_i$ and $d$ is an invariant metric on $G$ compatible with the topology. Set

$$A' = A \setminus \left(\bigcup_n (g_n + A) \cap A\right).$$

Then $A'$ is not Haar zero. Now we define a mapping $\theta$ from the Cantor group $K = \{0, 1\}^N$ to $G$ by

$$\theta(x) = \sum_{n=1}^{\infty} x(n) g_n.$$

Since $A'$ is not Haar zero then there exists $g \in G$ such that $\theta^{-1}(g + A')$ has non-zero Haar measure in $K$. Then $\theta^{-1}(g + A') - \theta^{-1}(g + A') = U$ is a neighborhood in $K$. Then for large $v$, $\epsilon_v \in U$ where $\epsilon_v = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in the $v$-th position. So $\theta(\epsilon_v) = g_v \in A' - A'$ and hence $(g_v + A') \cap A' \neq \emptyset$. This is a contradiction. Therefore $F(A, A)$ is a neighborhood of zero. ■

The theory can also be developed in the setting of a non-Abelian Polish group. Christensen in [14, pp. 123] indicated the extension of Christensen null concept to such setting by using the two-sided invariance, and also pointed out that such extension does not fit any non-separable metric group. Here we will introduce four definitions in a completely metrizable separable group $G$, also called Polish groups.

Since we are not assuming that the group operation is commutative we shall write the operation as a multiplication. In cases where we explicitly assume an Abelian group we revert to additive notation.

**Definition 2.9.3** **(Christensen)** A universally measurable set $X \subseteq G$ is called *shy* if there exists a Borel probability measure $\mu$ such that $\mu(gXh) = 0$ for all $g, h \in G$. We also say that $\mu$ is *transverse* to $X$.

**Definition 2.9.4** A universally measurable set $X \subseteq G$ is called *left shy* if there exists a Borel probability measure $\mu$ such that $\mu(gX) = 0$ for all $g \in G$. We also say that $\mu$ is *left transverse* to $X$.

**Definition 2.9.5** A universally measurable set $X \subseteq G$ is called *right shy* if there exists a Borel probability measure $\mu$ such that $\mu(Xh) = 0$ for all $h \in G$. We also say that $\mu$ is *right transverse* to $X$.

**Definition 2.9.6** A universally measurable set $X \subseteq G$ is called *left-and-right shy* if there exists a Borel probability measure $\mu$ such that $\mu(Xh) = 0$ and $\mu(hX) = 0$ for all $h \in G$. We also say that $\mu$ is *left-and-right transverse* to $X$.

From these definitions we can easily see the following implications:

*shy* $\Rightarrow$ *left-and-right shy* $\Rightarrow$ *left shy (right shy).*

Note that left-and-right shy is formally stronger than left shy and right shy. We show they are equivalent. This following theorem seems to be not in the current literature.

**Theorem 2.9.7** *If $X \subseteq G$ is left shy and right shy, then $X$ is left-and-right shy.*

**Proof.** Since $X \subseteq G$ is left shy and right shy, then there exist Borel probability measures $\mu_1$ and $\mu_2$ such that

$$\mu_1(Xh) = \mu_2(hX) = 0 \qquad (\forall h \in G).$$

Now we define a new Borel probability measure $\nu$ by

$$\nu(B) = \mu_1 \times \mu_2(\{(f,g) \in G \times G : \; fg \in B\})$$

for all universally measurable sets $B \subseteq G$. By Fubini's theorem,

$$\nu(B) = \mu_1 \times \mu_2(\{(f,g) \in G \times G : \; f \in Bg^{-1}\}) = \int_G \mu_1(Bg^{-1}) d\mu_2(g)$$

and

$$\nu(B) = \mu_1 \times \mu_2(\{(f,g) \in G \times G : \; g \in f^{-1}B\}) = \int_G \mu_2(f^{-1}B)d\mu_1(f).$$

Thus

$$\nu(Xh) = \int_G \mu_1(Xhg^{-1})d\mu_2(g) = 0$$

and

$$\nu(hX) = \int_G \mu_2(f^{-1}hX)d\mu_1(f) = 0.$$

So $\nu$ is left-and-right transverse to $X$. The result follows. ■

We will give several examples in Chapter 5 to show that, on the Polish group $\mathcal{H}[0,1]$ (see Chapter 5), there exists

(1) a probability measure that is both left transverse and right transverse to a Borel set $X \subseteq \mathcal{H}[0,1]$, but is not transverse to $X$.

(2) a probability measure that is left transverse to a Borel set $X \subseteq \mathcal{H}[0,1]$, but is not right transverse to $X$.

(3) a probability measure that is right transverse to a Borel set $X \subseteq \mathcal{H}[0,1]$, but is not left transverse to $X$.

Also in the Polish group $G$ we will give examples of left-and-right shy sets which can be decomposed into continuum many non-shy sets.

Jan Mycielski [41] showed that the family $\mathcal{S}$ of shy sets is closed under finite union in an arbitrary completely metrizable group and also closed under countable unions in any Polish group. In the following theorems we will develop the corresponding properties for left shy sets and right shy sets and some further properties. The methods are well known although the details may not be in the literature.

**Theorem 2.9.8** *If $Y_1$ and $Y_2$ are left shy sets. Then $Y_1 \cup Y_2$ is also a left shy set.*

**Proof.** Let $\mu_1$ and $\mu_2$ be the probability measures on $G$ left transverse to $Y_1$ and $Y_2$ respectively. We define

$$m(X) = \mu_1 \times \mu_2(\{(x, y) \in G^2 : \; x^{-1}y, \; y^{-1}x \in X\})$$

for universally measurable sets $X \subseteq G$. Then $m$ is a probability measure. For any $g \in G$,

$$0 \leq m(gY_1) \leq \int_G \mu_1(ygY_1)\mu_2(dy) = 0$$

and

$$0 \leq m(gY_2) \leq \int_G \mu_2(xgY_2)\mu_1(dx) = 0.$$

Thus $m(gY_1) = m(gY_2) = 0$ and hence $m(g(Y_1 \cup Y_2)) = 0$. ∎

The separability is needed in the next two theorems, but was not used in the preceding one.

**Theorem 2.9.9** *If $G$ is a Polish group and $Y_1, Y_2, \ldots$ are left shy, then $\bigcup_{i=1}^{\infty} Y_i$ is also left shy.*

**Proof.** Let $m_j$ be a probability measure left transverse to $Y_j$. Since $G$ is separable there exists a compact set $C_j$ with diameter $\leq 1/2^j$ and $m_j(C_j) > 0$. Without loss of generality we can assume that $m_j(C_j) = 1$ and that the unity of $G$ belongs to $C_j$. Since diameter of $C_j \leq 1/2^j$, the infinite product $g_1 g_2 \cdots$, for $g_j \in C_j$ converges in the sense of group multiplication. Let $m^{\Pi}$ be the product measure of the measures $m_j$ in the product space $\prod C_j$. We define $m(X) =$

$$m^{\Pi}\left\{(g_1, g_2, \cdots) \in \prod C_j : \; (g_1 \cdots g_{i-1} g_{i+1} g_{i+2} \cdots) g_i \in X, \; i = 1, 2, \ldots\right\}.$$

It is easy to see that $m$ is a probability measure. We will show that $m$ is left transverse to all $Y_i$. Let $\prod^i C_j = C_1 \times \cdots \times C_{i-1} \times C_{i+1} \times C_{i+2} \times \cdots$ and $m_i^{\Pi}$ be the product measure of $m_1, \ldots, m_{i-1}, m_{i+1}, m_{i+2}, \ldots$ in $\prod' C_j$. Since $m_i$ is left transverse to $Y_i$, then for any $h \in G$ we have

$$m(hY_i) = \int_{\prod' C_j} m_i((g_1 \cdots g_{i-1} g_{i+1} \cdots)^{-1} h Y_i) m_i^{\Pi}(dg) = 0.$$

Thus

$$m\left(h\left(\bigcup_{i=1}^{\infty} Y_i\right)\right) \leq \sum_{i=1}^{\infty} m(hY_i) = 0.$$

So the result follows. ■

By similar arguments we can have the following.

**Theorem 2.9.10** *If $G$ is a Polish group and $Y_1, Y_2, \ldots$ are right shy, then $\bigcup_{i=1}^{\infty} Y_i$ is also right shy.*

**Corollary 2.9.11** *If $G$ is a Polish group and $Y_1, Y_2, \ldots$ are left-and-right shy, then $\bigcup_{i=1}^{\infty} Y_i$ is also left-and-right shy.*

**Theorem 2.9.12** *If $G$ is a Polish group then any non-empty open set is neither left shy nor right shy.*

**Proof.** It is known that every metrizable topological group must have a left invariant metric and a right invariant metric (see [30, pp. 58]). Thus we need only show the results for a Polish group with a compatible right invariant metric $\rho$.

Let $S$ be a non-empty open set in $G$ and $B(f, r) \subseteq S$ is a non-empty ball. Suppose $S$ is right shy. We need find a contradiction. Since $G$ is separable, let $\{f_i\} \subseteq G$ be a countable set dense in $G$. Then

$$G \subseteq \bigcup_{i=1}^{\infty} B(f, r) f^{-1} f_i$$

where

$$B(f, r) f^{-1} f_i = \{h f^{-1} f_i : \ h \in B(f, r)\}.$$

In fact, it is easy to see that the above inclusion holds if

$$B(f,r)f^{-1}f_i = B(f_i,r).$$

In fact, for $g \in B(f,r)f^{-1}f_i$, then $g = hf^{-1}f_i$ and $\rho(h,f) < r$. By the right invariance of $\rho$ we have

$$\rho(g,f_i) = \rho(hf^{-1}f_i,f_i) = \rho(hf^{-1},\epsilon) = \rho(h,f) < r$$

where $\epsilon$ is the unit element of $G$. So $g \in B(f_i,r)$ and hence $B(f,r)f^{-1}f_i \subseteq B(f_i,r)$. For $g \in B(f_i,r)$,

$$\rho(gf_i^{-1}f,f) = \rho(gf_i^{-1},\epsilon) = \rho(g,f_i) < r.$$

So $gf_i^{-1}f \in B(f,r)$ and $g \in B(f,r)f^{-1}f_i$. Thus $B(f_i,r) \subseteq B(f,r)f^{-1}f_i$.

Since $S$ is right shy, so there exists a Borel probability measure $\mu$ such that $\mu(Sg) = 0$ for all $g \in G$. Thus

$$\mu(G) \leq \sum_{i=1}^{\infty} \mu(B(f,r)f^{-1}f_i) \leq \sum_{i=1}^{\infty} \mu(Sf^{-1}f_i) = 0.$$

This contradicts $\mu(G) = 1$ and so $S$ is not right shy.

By similar arguments we can show that $S$ is not left shy. The theorem follows.

■

In the following we give some basic properties of left shy sets. The right shy sets have the same properties. Their proofs are similar.

**Theorem 2.9.13** *If $G$ is a Polish group, then left shy sets have empty interior.*

**Proof.** From the above theorem we know that every open set is not left shy. So any set with non-empty interior is not left shy. That is, a left shy set has empty interior.

■

**Theorem 2.9.14** *If $G$ is a Polish group, then any left translate or right translate of a left shy set is also left shy.*

**Proof.** Let $S$ be a left shy set and $\mu$ be its transverse Borel probability measure. Then for any $x \in G$, $\mu(xS) = 0$. For every $y \in G$, by the associativity, $\mu(x(yS)) = \mu((xy)S) = 0$ for all $x \in G$, and so $yS$ is left shy. For every $h \in G$, we define a Borel probability $\mu_1$ by $\mu_1(A) = \mu(Ah)$ for any universally measurable set $A \subseteq G$. Then for any $x \in G$,

$$\mu_1(x(Sh)) = \mu_1((xS)h) = \mu(xS) = 0$$

by the transversality of $\mu$ to $S$. Thus $\mu_1$ is transverse to $Sh$ and $Sh$ is left shy. ■

**Theorem 2.9.15** *Let $G$ be a Polish group. Then*

*(i) A set $S \subseteq G$ is shy if and only if*

$$S^{-1} = \{h^{-1} \in G : h \in S\}$$

*is shy.*

*(ii) A set $S \subseteq G$ is left shy if and only if $S^{-1}$ is right shy.*

*(iii) A set $S \subseteq G$ is right shy if and only if $S^{-1}$ is left shy.*

**Proof.** It is easy to see that a set $S$ is Borel or universally measurable iff $S^{-1}$ is Borel or universally measurable respectively. Since $(S^{-1})^{-1} = S$ and $(ii), (iii)$ coincide, we need only show the necessities of $(i)$ and $(ii)$.

(i). Suppose that $S$ is shy. Then for all $f, g \in G$ there exists a Borel probability measure $\mu$ such that $\mu(fSg) = 0$. We define a Borel probability measure $\widetilde{\mu}$ by

$$\widetilde{\mu}(X) = \mu(\{h^{-1} \in G : h \in X\}).$$

Then for all $f, g \in G$,

$$\widetilde{\mu}(fS^{-1}g) = \widetilde{\mu}((f^{-1})^{-1}S^{-1}(g^{-1})^{-1}) = \widetilde{\mu}((g^{-1}Sf^{-1})^{-1}) = \mu(g^{-1}Sf^{-1}) = 0.$$

Therefore $S^{-1}$ is shy.

(ii). Suppose that $S$ is left shy. Then for all $f \in G$, there exists a Borel probability measure $\mu$ such that $\mu(fS) = 0$. Use $\widetilde{\mu}$ as in $(i)$ we have

$$\widetilde{\mu}(S^{-1}f) = \widetilde{\mu}(S^{-1}(f^{-1})^{-1}) = \widetilde{\mu}((f^{-1}S)^{-1}) = \mu(f^{-1}S) = 0.$$

■

Examples(i), (ii) and (iii) mentioned in this section will show that a measure that proves a set in a non-Abelian Polish group is shy on one side, does not prove that it is shy on the other side. Indeed even if that measure proves that it is shy on both sides it does not prove that it is shy since that requires more. Specifically even if $\mu(gX) = \mu(Xh) = 0$ for all $g$, $h$ in the group it does not follow that $\mu(gXh) = 0$ for all $g$ and $h$. But there may yet exist some other measure for which this is the case and that is the source of the problem.

In a locally compact Polish group this problem does not arise because of the following theorem, proved in Mycielski [41, Theorem 1, pp. 31].

**Theorem 2.9.16** *In a locally compact Polish group, if there exists a Borel probability measure that is left (right) transverse to a set $S$ then $S$ is a set of Haar measure zero (for any Haar measure on the group).*

In Chapter 5, we will give examples of left-and-right shy sets without being shy in the non-locally compact Polish group.

## 2.10 Preservations of null sets under isomorphisms

In this section we will discuss the preservation of our null sets under isomorphisms. Here a mapping $T$ from a Banach space $X$ to a Banach space $Y$ is an isomorphism if $T$ is one-one, linear and continuous.

**Theorem 2.10.1** *Let $X$ be an infinite dimensional separable Banach space and $T$ be an isomorphism on $X$. Let $S$ be one of sets of Gaussian null, Gaussian null in Phelps sense, Christensen null, s-null, Aronszajn null and Preiss–Tišer null. Then $T(S)$ is also a null set of the same kind.*

**Proof.** Since $T$ is one-one and continuous, by Theorem 11.16 in [11], if a set $S \subseteq X$ is Borel or universally measurable then $T(S)$ is Borel or universally measurable respectively. We now need only show the assertion for each kind of Borel null sets.

(i). Let $S$ be a Gaussian null set in the ordinary sense. Then there is a Gaussian measure $\mu$ such that $\mu(S) = 0$. We define a measure on Borel sets by $\widetilde{\mu}(E) = \mu(T^{-1}E)$. Since for any $f \in X^*$, $\widetilde{\mu} \circ f^{-1} = (\mu \circ T^{-1}) \circ f^{-1} = \mu \circ (fT)^{-1}$ and $fT \in X^*$, thus $\widetilde{\mu}$ is also a non-degenerate Gaussian measure. Since $\widetilde{\mu}(T(S)) = \mu(S) = 0$, the set $T(S)$ is also Gaussian null in the ordinary sense.

(ii). Let $S$ be a Gaussian null set in Phelps sense. From the proof of (i) it is easy to see that $\mu \circ T$ is a Gaussian measure iff $\mu$ is a Gaussian measure. Thus $T(S)$ is Gaussian null in Phelps sense.

(iii). Let $S$ be a Christensen null set. Then there is a Borel probability measure $\mu$ such that for every $x \in X$, $\mu(S + x) = 0$. As in (i) we define a measure $\widetilde{\mu}$ on Borel sets. Then for every $y \in X$, there is some $x \in X$ such that $y = Tx$ since $T$ is one-one. Thus

$$\widetilde{\mu}(T(S) + y) = \widetilde{\mu}(T(S) + Tx) = \widetilde{\mu}(T(S + x)) = \mu(S + x) = 0.$$

and hence $T(S)$ is Christensen null.

(iv). Let $S$ be an $s$-null set. Then there is a sequence $\{\epsilon_i\}$ such that $S = \bigcup S_i$ and $S_i \cap (x + \mathbb{R}\epsilon_i)$ is Lebesgue measure zero for every $x \in X$. Here $\mathbb{R}\epsilon_i$ is the one dimensional space spanned by $\epsilon_i$. It is easy to see that $T(S) = \bigcup T(S_i)$. For any $y \in X$, there is some $x \in X$ such that $y = Tx$. Note

$$T[S_i \cap (x + \mathbb{R}\epsilon_i)] = T(S_i) \cap (Tx + \mathbb{R}T\epsilon_i) = T(S_i) \cap (y + \mathbb{R}T\epsilon_i).$$

We can define a probability measure on universally measurable sets by $\widetilde{\mu}(E) = \lambda_1(T^{-1}E)$. Thus

$$\widetilde{\mu}(T(S_i) \cap (y + \mathbb{R}Te_i)) = 0$$

for any $y \in X$ and hence $T(S)$ is $s$-null.

(v). By using the same method as in (iv) we can get the results for Preiss–Tišer null sets and Aronszajn null sets.

■

The statements are true if $T$ is an isomorphism from an infinite dimensional separable Banach space to an infinite dimensional separable Banach space. We can also extend one of the assertions to Abelian Polish groups in the following.

**Theorem 2.10.2** *Let $G_1$ and $G_2$ be Abelian Polish groups. Let $T$ be a homeomorphism from $G_1$ to $G_2$ such that*

$$T(x_1 + x_2) = T(x_1) \dotplus T(x_2)$$

*for all $x_1, x_2 \in G_1$, where $+$ and $\dotplus$ denote the operations on $G_1$ and $G_2$ respectively. Then a set $S \subseteq G_1$ is shy in $G_1$ if and only if $T(S)$ is shy in $G_2$.*

**Proof.** Since $T$ is a homeomorphism, both $T$ and the inverse $T^{-1}$ of $T$ map Borel sets and universally measurable sets into Borel sets and universally measurable sets respectively. $T^{-1}$ also satisfies

$$T^{-1}(y_1 \dotplus y_2) = T^{-1}(y_1) + T^{-1}(y_2)$$

for all $y_1, y_2 \in G_2$. This can be seen from the assumption by applying $T$ to its both sides. So we only need to show the necessity.

If $S \subseteq G_1$ is shy in $G_1$, then there is a Borel probability measure $\mu$ such that for all $x \in G_1$, $\mu(S + x) = 0$. Now we define a Borel probability measure $\widetilde{\mu}$ on $G_2$ by

$$\widetilde{\mu}(P) = \mu(T^{-1}(P))$$

for all universally measurable sets $P \subseteq G_2$. Then

$$\widetilde{\mu}(G_2) = \mu(T^{-1}(G_2)) = \mu(G_1) = 1.$$

So $\widetilde{\mu}$ is a Borel probability measure. We now check that $\widetilde{\mu}$ is transverse to $T(S)$. For any $y \in G_2$ there is some $x \in G_1$ such that $y = T(x)$. Then

$$\widetilde{\mu}(T(S) \dot{+} y) = \widetilde{\mu}(T(S) \dot{+} T(x)) = \widetilde{\mu}(T(S+x)) = \mu(S+x) = 0.$$

Thus $T(S)$ is shy in $G_2$ and the result follows. ■

## 2.11 Measurability issues

Our definition of a shy set (i.e., Christensen null set) requires that the set be universally measurable. This requirement is essential in view of the following theorem of Bogachev [5] cited by Preiss [46].

**Theorem 2.11.1 (Bogachev)** *Let $X$ be an infinite dimensional separable Banach space and let $\{e_i\}$ be a sequence of elements whose linear span is dense in $X$. Then there is a decomposition of $X$ into a sequence of sets $X_i$ such that the intersection of $X_i$ with any line in the direction $e_i$ has linear measure zero.*

From this theorem, if we do not require the universal measurability in the definition of shy sets, every separable Banach space would be the countable union of shy sets. So we are unable to omit the reference to the universal measurability of the sets in Section 2.3 as regards shy sets or $s$-null sets. If we remove the requirement of universal measurability in the definition of a shy set, the same problem arises in Polish groups. The following argument of Dougherty [18] shows this. Let $G$ be an uncountable Polish group. Let $\preceq$ be a well-ordering of $G$ in minimal order type. Then $\preceq$ and its complement are subsets of $G^2$. Let $\mu = \mu_1 \times \mu_2$ and $\mu' = \mu_2 \times \mu_1$ where $\mu_1$ is an atomless measure on $G$ and $\mu_2$ is a measure concentrated on a single point of $G$. If the continuum hypothesis holds then $\mu$ and $\mu'$ are Borel measures transverse to $\preceq$ and its complement respectively.[2] This violates the additivity of shy sets. Thus the universal measurability cannot be omitted.

There are four possible ways of requiring measurability in defining a shy set $S$.

(a) $S$ is Borel.

(b) $S$ is universally measurable.

(a') $S$ is Borel or a subset of a Borel shy set.

(b') $S$ is universally measurable or a subset of a universally measurable shy set.

The case (a) is too narrow. It does not include some sets we often meet. The case (b) is our choice. It has been proved very useful in many respects. The case (a') is the choice of Hunt et al. [27]. This choice is not compatible with our choice in case (b). If we follow Hunt et al. [27] and call a set shy only if it is contained in a Borel shy set then we arrive at a different theory. The following argument, also from Dougherty [18], shows this. Let $X$ be an uncountable Polish group with an invariant metric which is not locally compact. If the continuum hypothesis holds, then there is a subset $S \subseteq X$ such that $S \cap A$ has cardinality less than $2^{\aleph_0}$ whenever $A$ is a $\sigma$-compact set but $S \cap A$ has cardinality $2^{\aleph_0}$ whenever $A$ is a Borel set which is not included in a $\sigma$-compact set (see [24]). Since any Borel probability measure on $X$ is based on $\sigma$-compact subsets of $X$ and the continuum hypothesis implies that sets of cardinality less than $2^{\aleph_0}$ have measure zero under any atomless measure, then the set $S$ is universally measurable and shy. However, any Borel set which includes $S$ must be the complement of a $\sigma$-compact set and therefore prevalent. Thus this set $S$ cannot be contained in any Borel shy set.

The case (b') is more general than the cases (a), (b) and (a'). A subset of a universally measurable shy set need not be universally measurable. For example, on

[2]This can be shown under weaker hypothesis (e.g., Martin's axiom).

the real line, there exist Bernstein sets, sets such that the sets and their complements intersect every perfect set. Choose a Bernstein subset of a Cantor set of measure zero. Then this set is Lebesgue measure zero, but is not measurable with respect to all finite Borel measures on the real line. Such a set would be shy according to the case (b'), but not shy according to cases (a) and (b). The set $S$ in last paragraph is shy according to the case (b'), but is not shy according to the case (a').

There are other interesting questions that can be posed and which are related to this discussion. In [18] Dougherty asked whether an analytic shy set must always be included in a Borel shy set and whether any analytic non-shy set must include a Borel non-shy set. Solecki [52] answered the first question affirmatively and the second question negatively in the following.

**Theorem 2.11.2 (Solecki)** *Let $G$ be an Abelian Polish group, then*

*(i) If $A \subseteq G$ is analytic and Haar null then there exists a Borel set $B \subseteq G$ which is Haar null and $A \subseteq B$.*

*(ii) Assume that $G$ is not locally compact and admits an invariant metric then there exists an analytic set $A$ such that $A \triangle B$ is Haar null for no co-analytic set $B$.*

## 2.12 Extension to non-separable Abelian groups

The non-separable case requires some attention to the measure theoretic details. Here is a simple example showing what can go wrong.

**Example 2.12.1** Consider the non-separable Banach space $\ell_\infty$, the space of all bounded sequences with the supremum norm. The cube

$$C_n = \{(x_1, x_2, x_3, \dots) : |x_i| < n, \ i = 1, 2, \dots\}$$

admits a transverse Borel probability measure.

Take the measure $\mu$ defined on $\ell_\infty$ as follows

$$\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n \times \cdots$$

where

$$\mu_i = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$

for Borel sets $B \subseteq \mathbb{R}, i = 1, 2, \ldots$.

Then $\mu(C_n) = 0$ for all $n$ and indeed, $\mu(C_n + x) = 0$ for all $x \in \ell_\infty$. Thus each set $C_n$ has a Borel probability measure transverse to it. But

$$\ell_\infty = \bigcup_{n=1}^{\infty} C_n,$$

so that the whole space is a union of countably many sets that we might have been inclined to call "shy".

One of the features of finite Borel measures on Polish spaces that is lacking for non-separable spaces is the close connection with the compact sets. An analysis of the proofs in Example 2.12.1 shows that this property of Borel measures on Polish spaces is what is missing in general. We will see in the next section that the measure $\mu$ we constructed in the last example is not "tight" even though it is a Borel probability measure.

One way around this is to restrict attention to measures that have this property. The simplest approach, followed in Hunt et al. [27], is to consider just those Borel probability measures with compact support. In a Polish space the shy sets would not change. For if there is a Borel probability measure transverse to a set $S$ then there must also exist a Borel probability measure with compact support that is also transverse to that set. We will see this from the following theorem.

**Theorem 2.12.2** *Any probability measure $\mu$ on a Polish space $G$ is tight.*

**Proof.** (This proof is a sketch of some of the main ideas from [44, pp. 28-30].) Since $G$ is separable, for each natural number $n$ we can find countably many balls $B_{nj}$ of radius $1/n$ such that $G = \bigcup_j B_{nj}$. So $G = \bigcup_j \bar{B}_{nj}$ where $\bar{B}_{nj}$ is the closure of $B_{nj}$. For arbitrary $\epsilon > 0$ there exists an integer $k_n$ such that

$$\mu\left(\bigcup_{j=1}^{k_n} \bar{B}_{nj}\right) > 1 - \frac{\epsilon}{2^n}.$$

Let

$$X_n = \bigcup_{j=1}^{k_n} \bar{B}_{nj} \text{ and } K = \bigcap_{n=1}^{\infty} X_n.$$

Then

$$\mu(G \setminus K) = \sum_n \mu(G \setminus X_n) < \sum_n \epsilon/2^n = \epsilon.$$

We now need only show that $K$ is compact. Let $\{x_n\} \subseteq K$ be an infinite sequence then there is $n_1 \leq k_1$ such that $K \cap \bar{B}_{1n_1} = K_1$ contains infinite points of $\{x_n\}$. Note $K_1 \subseteq \bigcup_{j=1}^{k_2} \bar{B}_{2j}$ . There is an integer $n_2 \leq k_2$ such that $K_1 \cap \bar{B}_{2n_2}$ contains infinite points of $\{x_n\}$. Induction yields a sequence of sets $\{K_n\}$ such that $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$ and each $K_n$ contains infinite points of $\{x_n\}$. Note the diameter of $K_n$ is less than $2/n$ and $G$ is complete. So $\bigcap_n K_n$ is a singleton. Thus there is a subsequence of $\{x_n\}$ which converges to that single point. Therefore $K$ is compact and the result follows. ■

Another development is given in Borwein and Moors [8] for shy sets (Haar null sets) in non-separable Abelian topological groups. Their method overcomes the short-comings that an open set may be shy by, again, restricting the class of measures to which the definition of shy sets is allowed to apply (see the Glossary for the definition of Radon measure).

**Definition 2.12.3 (Borwein-Moors)** Let $G$ be a completely metrizable Abelian topological group. A universally Radon measurable set $A \subseteq G$ is called a *Haar null* *set* if there exists a Radon probability measure $p$ on $G$ such that $p(g + A) = 0$ for each $g \in G$.

From Theorem 2.12.2 we know that in an infinite dimensional separable Banach space the definition of Borel Christensen null sets coincides with the definition of Borel shy sets in [27], and also coincides with the above definition of Borwein and Moors. In a non-separable Banach space, by Theorem 2.12.2, the definition of Borel shy sets defined by Hunt et al. [27] still coincides with the definition of Borel Haar null sets by Borwein and Moors [8], but the definitions of Borel shy sets defined in [27] and [8] are different from Christensen Definition 2.9.3. Example 2.12.1 illustrates this.

In the next section we will discuss a further extension to completely metrizable topological semigroups and give the main properties for the corresponding Haar null sets.

## 2.13 Extension to semigroups

Much of the material developed so far can be extended to certain topological semigroups. An account appears in Topsøe and Hoffman-Jørgenson [56, pp. 373–378].

We suppose that we are given a topological (usually completely metrizable) semigroup $G$ for which the operations $x \to ax$ and $x \to xa$ are continuous. Topsøe and Hoffman-Jørgenson call such semigroups "separately continuous semigroups". We assume always that $G$ has a unit element, denoted as 1. Thus $1x = x1 = x$ for all $x \in G$.

A Radon probability measure $\mu$ defined on $G$ is said to be *transverse* to a universally Radon measurable set $S$ in $G$ if the set

$$\{z \in G : \; xzy \in S\}$$

has $\mu$-measure zero for every $x, y \in G$ and the element 1 is in the support of $\mu$.

Here are the details needed to see that this definition extends the notion of sets of Haar measure zero in locally compact groups. Let $G$ be a locally compact group with right Haar measure $\rho$, $\mu$ be a Radon probability measure transverse to a Radon universally measurable set $S \subseteq G$. Then by Fubini's theorem we have

$$0 = \int_G \mu(x^{-1}S)\rho(dx) = \int_G \int_G 1_S(xy)\mu(dy)\rho(dx) = \int_G \rho(Sy^{-1})\mu(dy).$$

Since $\rho(Sy^{-1})$ is continuous with respect to $y \in G$ and $1 \in \operatorname{supp}\mu$ so $\rho(S) = 0$. Thus $S$ is Haar measure zero.

The original definition of Haar null sets in Topsøe and Hoffman-Jørgenson [56, pp. 374] just requires the existence of a $\tau$-smooth probability measure. The class of $\tau$-smooth probability measures is larger than the class of Radon probability measures. However the definition of a Haar null set by requiring the existence of a $\tau$-smooth probability measure transverse to a universally Radon measurable set coincides with the definition of a Haar null set by requiring the existence of a Radon probability measure transverse to the set. In fact let $\mu$ be a $\tau$-smooth probability measure transverse to a universally Radon measurable set $S$. Then the support of $\mu$:

$$\operatorname{supp}\mu = \{x \in G : \mu(U) > 0, \forall U \text{ a neighborhood of } x\}$$

is closed and contained in every closed $F$ with $\mu(G \setminus F) = 0$. So $G \setminus F \subseteq G \setminus \operatorname{supp}\mu$. Thus by the definition of $\tau$-smoothness of $\mu$ we have $\mu(G \setminus \operatorname{supp}\mu) = 0$. Note that a probability measure admits at most countably many disjoint sets of positive measure. Thus $\operatorname{supp}\mu$ is separable. We define a measure $\bar{\mu}$ by

$$\bar{\mu}(A) = \mu(A \cap \operatorname{supp}\mu)$$

for a universally Radon measurable set $A \subseteq G$. It is easy to see that $\bar{\mu}$ is transverse to the set $S$. Since $\operatorname{supp}\mu$ is closed and separable, by Theorem 2.12.2, the measure $\mu$ is tight on $\operatorname{supp}\mu$. Note $\mu(G \setminus \operatorname{supp}\mu) = 0$. So $\bar{\mu}$ is a Radon probability measure on $G$ which is transverse to $S$.

From the above discussion it is easy to see that in a non-separable Abelian topological group the definition of Haar null sets by Borwein and Moors [8] is equivalent to the definition of Haar null sets by Topsøe and Hoffman-Jørgenson [56].

The following are the main properties of Haar null sets on completely metrizable semigroups. See Topsøe and Hoffman-Jørgenson [56, pp. 373-378] and Borwein and Moors [8] for proofs.

**Theorem 2.13.1** *Let $G$ be a completely metrizable topological semigroup, then*

*(i) If $A$ is a Haar null set then $xAy$ is also Haar null for any $x, y \in G$.*

*(ii) A Haar null set has no interior.*

*(iii) The union of countably many Haar null sets is also Haar null.*

*(iv) If $G$ is an Abelian group and $A \subseteq G$ is not Haar null then the unit element 0 is an interior point of $A - A$.*

*(v) If $G$ is an Abelian group, then every compact set of $G$ is Haar null.*

**PROBLEM 3** *In a non-locally compact, non-Abelian Polish group, are compact sets left shy or right shy?*

If we impose the multiplication as an operation on the space $C[a, b]$ of continuous functions with supremum norm it will become an Abelian Polish semigroup with unit element for which the operation $f \to fg$ is continuous. We will discuss the "multiplicatively shy" sets in the space $C[a, b]$ and their relations to the "additively shy" sets in Chapter 4.

## 2.14 Why not measure zero sets of a single measure

Our null sets in all cases were required to have the property of translation invariance. One might ask whether we could not have achieved this more directly by taking the measure zero sets for an appropriate measure.

There are many characterizations possible for the sets of Lebesgue measure zero in a finite dimensional space. The simplest one to conceive is merely that these are the null sets for a single measure (Lebesgue measure) that happens to be translation invariant. This gives immediately a class of sets that is translation invariant.

In a compact group there is a unique translation invariant probability measure (Haar measure) which plays the same role. In a non-compact but locally compact, Abelian topological group again Haar measure can be used. The measure is unique up to multiples and translation invariant and so the sets that are of zero Haar measure play the role that we require. The following theorem and its proof are sketched from [14].

**Theorem 2.14.1** *Let $G$ be a locally compact, Abelian topological group. Then a set $S \subseteq G$ is Haar null iff $S$ is of zero measure for any Haar measure on $G$.*

**Proof.** Let $h$ denote a Haar measure on $G$. Since $h$ is $\sigma$-finite, we can use Fubini's theorem to show that for any Borel probability measure $\mu$ on $G$,

$$\int \left( \int \chi_S(xy) h(dx) \right) \mu(dy) = \int \left( \int \chi_S(xy) \mu(dy) \right) h(dx).$$

That is,

$$\int h(Sy^{-1}) \mu(dy) = \int \mu(Sx^{-1}) h(dx)$$

where $\chi_S$ is the characteristic function of the set $S$. If $S$ is Haar null and $\mu$ is a test-measure for $S$, then the right hand and the left hand of this equation are zero.

Since $h$ is translation invariant, $S$ is zero for the Haar measure $h$. Conversely, if $S$ is zero for the Haar measure, we can obtain a test-measure $\mu$ by choosing $\mu$ with density with respect to the Haar measure. ∎

In a locally compact non-Abelian group there is a right Haar measure and a left Haar measure. One might worry that the measure zero sets are only invariant on one side but that is not the case. The measure zero sets for a right Haar measure again serve as our class of sets invariant under the group operations (see [41]).

Why have we been unable to pursue the same course in an infinite dimensional Banach space? The first problem is that there is no nontrivial translation invariant finite or $\sigma$-finite measure on such a space.

**Theorem 2.14.2** *There is no nontrivial translation invariant finite or $\sigma$-finite measure on an infinite dimensional Banach space.*

**Proof.** The proof of this theorem in detail is very complicated and long (see [61, pp. 138-143]). For our purpose we only show that there is no translation invariant Radon probability measure $\mu$ on an infinite dimensional Banach space.

Suppose that there were a translation invariant Radon probability measure $\mu$ on an infinite dimensional Banach space $X$. Then there are a compact set $K$ and an open ball $B(x, \epsilon)$ contained in $X$ such that $0 < \mu(K) < \infty$ and $\mu(B(x, \epsilon)) < \infty$. Since the space $X$ is infinite dimensional, we can construct an infinite sequence of disjoint open balls $\{B(y, \epsilon/4)\}$ which are contained in $B(x, \epsilon)$. Since $\mu$ is translation invariant, so the $\mu$ measures of all such balls $B(y, \epsilon/4)$ are zero. Note

$$K \subseteq \bigcup_{z \in K} B(z, \epsilon/4).$$

By the compactness of $K$ and the translation invariance of $\mu$ there are finite many $z_i$ such that

$$K \subseteq \bigcup_{i=1}^{m} B(z_i, \epsilon/4) \quad \text{and} \quad \mu(K) \le \sum_{i=1}^{m} \mu(B(z_i, \epsilon/4)) = 0.$$

This is a contradiction. Thus the result follows. ■

The above theorem can be generalized to any infinite dimensional topological space (see a proof in [61]).

In order to describe our class of null sets we need not necessarily require that the measure be translation invariant, only that the set of measure zero sets for that measure be translation invariant. Such a measure is said to be *quasi-invariant*. For example the Gaussian measures in $\mathbb{R}^n$ are quasi-invariant and, indeed, the measure zero sets of such a measure are precisely the Lebesgue measure zero sets. But again this is not possible in an infinite dimensional space. In fact the following theorem shows that it would never be possible.

**Theorem 2.14.3** *Let $X$ be a locally convex, linear, infinite dimensional topological vector space. Then there is no nontrivial $\sigma$-finite quasi-invariant measure defined on the Borel subsets of $X$.*

**Proof.** (This proof is reproduced from [61].) By using the remark following Theorem 2.14.2 we need only show that if a nontrivial quasi-invariant measure $\mu$ exists then a nontrivial translation invariant measure exists.

We use $\mathcal{B}$ to denote the class of Borel sets and define a map $T$ on $X \times X$ by $(x, y) \to (xy, y)$. Then $T$ is an automorphism on $(X \times X, \mathcal{B} \times \mathcal{B})$. By the Radon-Nikodym theorem there is a function $f(x, y) > 0$ such that for every $F \in \mathcal{B} \times \mathcal{B}$,

$$(\mu \times \mu)(T(F)) = \int_F f(x, y) d\mu(y).$$

Consider the following three maps on $X \times X \times X$ as follows:

$$T_1 : (x, y, z) \to (xy, y, z), \quad T_2 : (x, y, z) \to (x, yz, z),$$

$$T_3 : (x, y, z) \to (xz, y, z).$$

We can easily see that $T_1 \circ T_2 = T_3 \circ T_2 \circ T_1$. Then from the uniqueness of the density function we get that for every $(x, y, z) \in X \times X \times X$,

$$f(x, yz)f(y, z) = f(xy, z)f(y, z)f(x, y).$$

Thus there is some $x_0 \in X$ such that $f(x_0, yz) = f(x_0y, z)f(x_0, y)$. Set $f(x_0, x_0^{-1}y) = g(y)$. Replacing $x_0y$ by $y$ we have that for every $(y, z) \in X \times X$, $f(y, z) = g(yz)/g(y)$. Putting $d\nu = g^{-1}d\mu$. Then $\nu$ is translation invariant. In fact, for any $F \in \mathcal{B} \times \mathcal{B}$ we have

$$(\nu \times \mu)(T(F)) = \int_{T(F)} g^{-1}(x)d\mu(x)d\mu(y) = \int_F g^{-1}(xy)f(x, y)d\mu(x)d\mu(y)$$

$$= \int_F g^{-1}(xy)\frac{g(xy)}{g(x)}d\mu(x)d\mu(y) = \int_F g^{-1}d\mu(x)d\mu(y) = (\nu \times \mu)(F).$$

Thus for $F = A \times X \in \mathcal{B} \times \mathcal{B}$ we have

$$\int_B \nu(Ay)d\mu(y) = \int_B \nu(A)d\mu(y) = \nu(A)\mu(B).$$

From the uniqueness of the density function we have $\nu(Ay) = \nu(A)$ for any $y \in B$. Since $B \in \mathcal{B}$ is arbitrary we have that for any $y \in X$, $\nu(Ay) = \nu(A)$. ■

In Section 2.4 we saw that on an infinite dimensional separable Banach space a Gaussian null set in Phelps sense is null for every non-degenerate Gaussian measure. However, in general a Borel shy set cannot be null for every $\sigma$-finite Borel measure. In fact all Borel measures live on shy sets!

**Theorem 2.14.4** *(Aronszajn [2]) Given any $\sigma$-finite Borel measure $\mu$ on an infinite dimensional separable Banach space $X$ there is a Borel s-null set $S$ so that $\mu$ is concentrated on $S$, i.e., $\mu(B) = \mu(S \cap B)$ for all Borel sets $B$.*

Immediately from the above theorem we have the following

**Corollary 2.14.5** *Given any $\sigma$-finite Borel measure $\mu$ on an infinite dimensional separable Banach space $X$, for any Borel set $A \subseteq X$ there is a Borel s-null set $B$ and a Borel $\mu$ null set $U$ such that $A = B \cup U$. In particular $X$ can be decomposed into a $\mu$ null set and a Borel s-null set.*

For a proof of the above theorem see [2, pp. 155-156]. The theorem provides another proof that the Borel shy sets cannot be described as the null sets for any one $\sigma$-finite Borel measure.

## 2.15 Classification of non-shy sets

Hunt et al. [27] gave a number of variants of non-shy sets. Dougherty [18] gave more refined characterizations of non-shy sets. Let $S$ be a universally measurable subset of a Polish group. In the assertions below $\epsilon$ will vary over positive real numbers, $\mu$ over probability measures on $G$, and $t$ over translation functions $g \to g_1 g g_2$. The following eight properties from Dougherty [18] define different classes, in general, of non-shy sets.

| | | |
|---|---|---|
| (1) | $\exists\mu\forall t \;\; \mu(t(S)) = 1$ | [*prevalent*] |
| (2) | $\forall\epsilon\exists\mu\forall t \;\; \mu(t(S)) > 1 - \epsilon$ | [*lower density 1*] |
| (3) | $\exists\epsilon\exists\mu\forall t \;\; \mu(t(S)) > \epsilon$ | [*positive lower density*] |
| (4) | $\exists\mu\forall t \;\; \mu(t(S)) > 0$ | [*observable*] |
| (1') | $\forall\mu\exists t \;\; \mu(t(S)) = 1$ | [*ubiquitous*] |
| (2') | $\forall\epsilon\forall\mu\exists t \;\; \mu(t(S)) > 1 - \epsilon$ | [*upper density 1*] |
| (3') | $\exists\epsilon\forall\mu\exists t \;\; \mu(t(S)) > \epsilon$ | [*positive upper density*] |
| (4') | $\forall\mu\exists t \;\; \mu(t(S)) > 0$ | [*non-shy*]. |

In [18] Dougherty mentioned the results and examples in the following (i), (ii), (iii) and (iv) but did not give proofs or explanations. Here we take this opportunity to verify these results and examples. Furthermore, we find non-implications (3) $\not\Rightarrow$

(2') and (1') $\not\Rightarrow$ (4), and give examples for them in (v) and (vi). From (i) to (vi) we will get a clear picture of relations among the eight classes of non-shy sets.

(i). Some implications. From the definitions we see that the implications (j) $\rightarrow$ (k) and (j') $\rightarrow$ (k') for $j < k$ are trivial. We now show that (j) $\rightarrow$ (j') for each j. Let $S$ satisfy (1) then there exists a Borel probability measure $\mu_0$ such that $\mu_0(t(S)) = 1$ for each $t$. By Fubini's theorem for any Borel probability measure $\mu$ on $G$,

$$\begin{aligned}\mu * \mu_0(Sg_2) &= \int_G \mu(Sg_2 g^{-1})\mu_0(dg) \\ &= \int_G \int_G 1_S(g_1 g g_2^{-1})\mu_0(dg)\mu(dg_1) \\ &= \int_G \mu_0(g_1^{-1} S g_2)\mu(dg_1) = 1.\end{aligned}$$

Thus there must exist $g_2, g \in G$ such that $\mu(Sg_2g^{-1}) = 1$, and (1') holds. Similar arguments can show that (j) $\rightarrow$ (j'), j= 2,3,4.

(ii). (4) $\not\Rightarrow$ (3'). For the interval $S_1 = [0,1]$ and the Gaussian measure $\mu_1$, defined by

$$\mu_1(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$

for all Borel sets $B \subseteq \mathbb{R}$, the set $S_1$ satisfies (4) but does not satisfy (3'). In fact, for every $\epsilon > 0$, we construct a function

$$f(x) = \begin{cases} -(\epsilon/4)x + \epsilon/2, & 0 \le x \le 1 \\ \epsilon/4, & 1 \le x \le (2-2\epsilon)/\epsilon \\ 0, & x \ge (2-2\epsilon)/\epsilon \end{cases}$$

and extend it evenly to $(-\infty, 0)$. Then the measure $\mu_2$ defined by

$$\mu_2(B) = \int_B f(x) dx$$

is a Borel probability measure, but for any $x \in \mathbb{R}$, $\mu_2(x + S_1) < \epsilon$.

(iii). (2) $\not\Rightarrow$ (1'). The set $S_2 = (-\infty, 0) \cup (1, \infty)$ satisfies (2) since for any $\epsilon > 0$ the measure $\mu_2$ in (ii) satisfies $\mu_2(x + \tilde{S}_2) < \epsilon$ for each $x \in \mathbb{R}$. Here $\tilde{S}_2$ is the complement of $S_2$. Note that $\mathbb{R} = (x + S_2) \cup (x + \widetilde{S}_2)$ and $(x + S_2) \cap (x + \widetilde{S}_2) = \phi$. So for the Gaussian measure $\mu_1$ as in (ii) and any $x \in \mathbb{R}$, $\mu_1(x + S_2) < 1$. Thus $S_2$ does not satisfy (1').

(iv). (2') $\not\Rightarrow$ (3). The set $S_3$ of positive real numbers satisfies (2') since for any $\epsilon > 0$ and any Borel probability measure $\mu$ on $\mathbb{R}$ we can choose $x$ small enough so that $\mu(x + S_3) > 1 - \epsilon$. On the other hand, for any $\epsilon > 0$ and Borel probability measure $\mu$ we can choose $x$ large enough so that $\mu(x + S_3) < \epsilon$. Thus $S_3$ does not satisfy (3).

(v). (3) $\not\Rightarrow$ (2'). In $\mathbb{R}$, let

$$S_4 = \bigcup_{n=-\infty}^{\infty} (2n, 2n + 1).$$

For the Gaussian measure $\mu_1$ as in (ii) and every $x \in \mathbb{R}$, it is easy to see that

$$\mu_1(x + S_4) \geq \int_{1/2}^{1} e^{-\frac{x^2}{2}} dx \geq e^{-1/2}$$

and

$$\mu_1(x + S_4) \leq 1 - \int_{1/2}^{1} e^{-\frac{x^2}{2}} dx \leq 1 - e^{-1/2} < 1.$$

So $S_4$ satisfies (3) but does not satisfy (2').

(vi). (1') $\not\Rightarrow$ (4). We will show that the sets $S_1(A)$, $S_1(A^+)$ and $S_1(A^-)$ in Theorem 2.15.1 are all examples for this non-implication.

From the above (i) to (vi) we can make the following conclusions. (2) $\not\Rightarrow$ (1') implies (2) $\not\Rightarrow$ (1) and (2') $\not\Rightarrow$ (1'); (2') $\not\Rightarrow$ (3) implies (2') $\not\Rightarrow$ (2); (3) $\not\Rightarrow$ (2') implies (3) $\not\Rightarrow$ (2) and (3') $\not\Rightarrow$ (2'); (4) $\not\Rightarrow$ (3') implies (4) $\not\Rightarrow$ (3); (1') $\not\Rightarrow$ (4) implies (1') $\not\Rightarrow$ (1) and (4') $\not\Rightarrow$ (4). Here we indicate all relations among (1)–(4) and (1')–(4') in a table as follows. We use $\rightarrow$ to denote $\Rightarrow$ and $\nrightarrow$ to denote $\not\Rightarrow$.

We now study the following sets to justify (vi). Let $A$ be a non-empty set of natural numbers,

$$S_1(A) = \left\{ s \in \mathbb{R}^{\mathbb{N}} : \begin{array}{l} \exists N > 0 \text{ such that if } n \geq N, \\ s(n) > 0 \text{ for } n \in A \text{ and } s(n) < 0 \text{ for } n \notin A \end{array} \right\}.$$

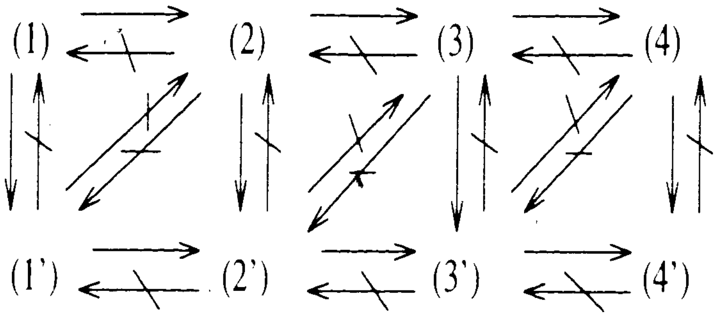


Figure 2.1: The relations of non-shy sets in general

$$S_1(A^+) = \{s \in \mathbb{R}^{\mathbb{N}} : \exists N > 0 \text{ such that if } n \geq N,\ s(n) > 0 \text{ for } n \in A\}$$

and

$$S_1(A^-) = \{s \in \mathbb{R}^{\mathbb{N}} : \exists N > 0 \text{ such that if } n \geq N,\ s(n) < 0 \text{ for } n \in A\}.$$

Note that

$$S_1(A) = \bigcup_{m=1}^{\infty} S_{1m}(A),\ \ S_1(A^+) = \bigcup_{m=1}^{\infty} S_{1m}(A^+) \text{ and } S_1(A^-) = \bigcup_{m=1}^{\infty} S_{1m}(A^-)$$

where

$$S_{1m}(A) = \{s \in \mathbb{R}^{\mathbb{N}} : \text{ if } n \geq m,\ s(n) > 0 \text{ for } n \in A \text{ and } s(n) < 0 \text{ for } n \notin A\},$$

$$S_{1m}(A^+) = \{s \in \mathbb{R}^{\mathbb{N}} : \text{ if } n \geq m,\ s(n) > 0 \text{ for } n \in A\}$$

and

$$S_{1m}(A^-) = \{s \in \mathbb{R}^{\mathbb{N}} : \text{ if } n \geq m,\ s(n) < 0 \text{ for } n \notin A\}.$$

By using the methods as in Theorem 3.5.4 we can show that $S_{1m}(A), S_{1m}(A^+)$ and $S_{1m}(A^-)$ are Borel sets. So the sets $S_1(A)$, $S_1(A^+)$, $S_1(A^-)$ and their complements are Borel sets. In the following theorem we show that all $S_1(A)$, $S_1(A^+)$, $S_1(A^-)$ and their complements are ubiquitous but not observable.

**Theorem 2.15.1** *Let $A$ be a non-empty set of natural numbers. Then the sets $S_1(A)$, $S_1(A^+)$ and $S_1(A^-)$ and their complements are all ubiquitous Borel sets but not observable. (That is, they satisfy (1') but not (4).)*

**Proof.** Case 1. The set $A$ is an infinite set. For a given Borel probability measure $\mu$ on $\mathbb{R}^{\mathbb{N}}$ we can choose a number $b_n$ for each $n$ such that

$$\mu(\{s \in \mathbb{R}^{\mathbb{N}}: \ |s(n)| > b_n\}) < 2^{-n-1}.$$

Then

$$\mu(\{s \in \mathbb{R}^{\mathbb{N}}: \ |s(n)| \leq b_n \text{ for all } n \geq N\}) > 1 - 2^{-N}.$$

Let

$$S = \bigcup_{N=1}^{\infty} \{s \in \mathbb{R}^{\mathbb{N}}: \ |s(n)| \leq b_n \text{ for all } n \geq N\}.$$

Then $\mu(S) = 1$. Define elements $t,\ t^+,\ t^- \in \mathbb{R}^{\mathbb{N}}$ as follows.

$$t(n) = \begin{cases} |b_n| + 1, & n \in A \\ -|b_n| - 1, & n \notin A, \end{cases}$$

$t^+(n) = |b_n| + 1$ for $n \in A$ and $t^-(n) = -|b_n| - 1$ for $n \in A$. Then $S + t \subseteq S_1(A)$, $S + t^+ \subseteq S_1(A^+)$ and $S + t^- \subseteq S_1(A^-)$. Thus $S_1(A)$, $S_1(A^+)$ and $S_1(A^-)$ are ubiquitous. That is, they satisfy (1'). Now we consider their complements.

$$\widetilde{S_1(A)} = \left\{ s \in \mathbb{R}^{\mathbb{N}}: \begin{array}{l} \forall N > 0,\ \exists n_0 \in A,\ n_0 \geq N \text{ such that } s(n_0) \leq 0, \\ \text{or } \ \exists m_0 \notin A,\ m_0 \geq N \text{ such that } s(m_0) \geq 0 \end{array} \right\},$$

$$\widetilde{S_1(A^+)} = \{s \in \mathbb{R}^{\mathbb{N}}: \ \forall N > 0,\ \exists n_0 \in A,\ n_0 \geq N \text{ such that } s(n_0) \leq 0\},$$

$$\widetilde{S_1(A^-)} = \{s \in \mathbb{R}^{\mathbb{N}}: \ \forall N > 0,\ \exists n_0 \in A,\ n_0 \geq N \text{ such that } s(n_0) \geq 0\}.$$

Now we define $t_1, t_1^- \in \mathbb{R}^{\mathbb{N}}$ to satisfy the following.

$$t_1(n) = \begin{cases} -|b_n| - 1, & n \in A \\ |b_n| + 1, & n \notin A, \end{cases}$$

and $t_1^-(n) = |b_n| + 1$ for $n \in A$. Then clearly we have

$$t_1 + S \subseteq \widetilde{S_1(A)},\ t_1 + S \subseteq \widetilde{S_1(A^+)} \text{ and } \ t_1^- + S \subseteq \widetilde{S_1(A^-)}.$$

Thus $\widetilde{S_1(A)}$, $\widetilde{S_1(A^+)}$ and $\widetilde{S_1(A^-)}$ are ubiquitous too. That is, they also satisfy (1'). From the definitions of (4) and (1') it is easy to see that a set satisfies (4) iff the complement of this set does not satisfy (1'). Therefore the sets $S_1(A)$, $S_1(A^+)$ and $S_1(A^-)$ and their complements are ubiquitous but not observable.

Case 2. The set $A$ is a non-empty finite set. For the set $S_1(A)$, by checking every step of the proof in Case 1, it is easy to see the conclusion still remains valid. For the sets $S_1(A^+)$ and $S_1(A^-)$ we choose an infinite set $B \subseteq \mathbb{N} \setminus A$ and define

$$C_1 = \{s \in \mathbb{R}^{\mathbb{N}} : \ s(n) > 0 \text{ for } n \in A \cup B\}$$

and

$$C_2 = \{s \in \mathbb{R}^{\mathbb{N}} : \ s(n) < 0 \text{ for } n \in A \cup B\}.$$

Then

$$C_1 \subseteq S_1(A^+),\ C_1 \subseteq \widetilde{S_1(A^-)} \text{ and } \ C_2 \subseteq S_1(A^-), C_2 \subseteq \widetilde{S_1(A^+)}.$$

By Case 1 we know that $C_1$ and $C_2$ are ubiquitous. Thus $S_1(A^+)$, $S_1(A^-)$, $\widetilde{S_1(A^+)}$ and $\widetilde{S_1(A^-)}$ are also ubiquitous. Thus the sets $S_1(A), S_1(A^+)$ and $S_1(A^-)$ and their complements are ubiquitous but not observable. ■

Dougherty [18] showed that the set

$$S(A) = \{s \in \mathbb{R}^{\mathbb{N}} : \ s(n) > 0 \text{ for } n \in A \text{ and } s(n) < 0 \text{ for } n \notin A\},$$

is upper density 1. Here we sketch his proof and show more from Theorem 2.15.1.

**Corollary 2.15.2** *$S(A)$ is upper density 1 but not observable.*

**Proof.** For a given Borel probability measure $\mu$ on $\mathbb{R}^{\mathbb{N}}$ and $\epsilon > 0$ we can choose $b_n$ for each $n$ such that

$$\mu(\{s \in \mathbb{R}^{\mathbb{N}} : \ |s(n)| > b_n\}) < \epsilon.$$

Thus $\mu(\{s \in \mathbb{R}^{\mathbb{N}} : |s(n)| \leq b_n\}) > 1 - \epsilon$ and hence $S(A)$ is upper density 1. Note that $S(A) \subseteq S_1(A)$. From Theorem 2.15.1 we will see that $S_1(A)$ is not observable and therefore $S(A)$ is not observable. ■

For the remainder of this section we will try to describe in $\mathbb{R}^n$ $(n \geq 1)$ the classes of sets in (1)-(4) and (1')-(4'). In $\mathbb{R}^n$ the set $S$ in (1) has full Lebesgue measure. This is proved in Theorem 2.7.5. Now we look at the sets satisfying (1'). (1') is equivalent to that $\forall\mu\exists t$ $\mu(t(\widetilde{S})) = 0$ where $\widetilde{S}$ is the complement of $S$. Thus take $\mu$ to be a Gaussian measure and so there is a $t$ such that $\mu(t(\widetilde{S})) = 0$. Since the Lebesgue measure and any Gaussian measure are mutually absolutely continuous, so $\lambda_n(t(\widetilde{S})) = 0$ and $\lambda_n(\widetilde{S}) = 0$ where $\lambda_n$ is the $n$-dimensional Lebesgue measure. Thus the set $S$ also has full Lebesgue measure and hence all sets in (1) and (1') are equivalent to having full Lebesgue measure.

If a set $S$ satisfies (4'), then, for every Gaussian measure $\mu$, there exists a $t \in \mathbb{R}^n$ such that $\mu(t(S)) > 0$. By the mutually absolute continuity of $\lambda_n$ and $\mu$ we have $\lambda_n(t(S)) = \lambda_n(S) > 0$. Again by the mutually absolute continuity of $\lambda_n$ and $\mu$, for every Gaussian measure and $t \in \mathbb{R}^n$, we have $\mu(t(S)) > 0$. So the set $S$ satisfies (4). Since (4) $\Rightarrow$ (4') so all sets in (4) and (4') are equivalent to having positive Lebesgue measure.

In $\mathbb{R}^n$, for every set $S$ satisfying (3') it is easy to see $\lambda_n(S) > 0$. If not, $\lambda_n(S+x) = 0$ for any $x \in \mathbb{R}^n$. Thus for any Gaussian measure $\mu$, $\mu(x + S) = 0$ which contradicts (3'). Now we can claim $\lambda_n(S) = \infty$. If not then $0 < \lambda_n(S) < \infty$. For any $\epsilon > 0$ we can construct a function $F(x)$ by defining

$$F(x) = \begin{cases} -\frac{\epsilon}{2\lambda_n(S)}|x| + \frac{\epsilon}{\lambda_n(S)}, & 0 \leq |x| \leq \lambda_n(S) \\ \frac{\epsilon}{2\lambda_n(S)}, & \lambda_n(S) \leq |x| \leq \frac{3}{2}\lambda_n(S) \\ 0, & \frac{3}{2}\lambda_n(S) \leq |x| \leq \frac{2-2\epsilon}{\epsilon}\lambda_n(S). \end{cases}$$

Then the probability measure $\mu$ induced by $F(x)$,

$$\mu(B) = \int_B F(x)dx$$

satisfies that for any $x \in \mathbb{R}^n$, $\mu(x + S) < \epsilon$. This contradicts (3'). Thus for any set $S$ satisfying (3') we have $\lambda_n(S) = \infty$. Therefore for any set $S$ satisfying one of (2), (3), (2') and (3') we have $\lambda_n(S) = \infty$. However we currently have no exact characterizations for sets $S$ in (2), (3), (2') and (3').

Based on the above discussions and (i)-(v) we can obtain the relations among (1)-(4) and (1')-(4') as follows.

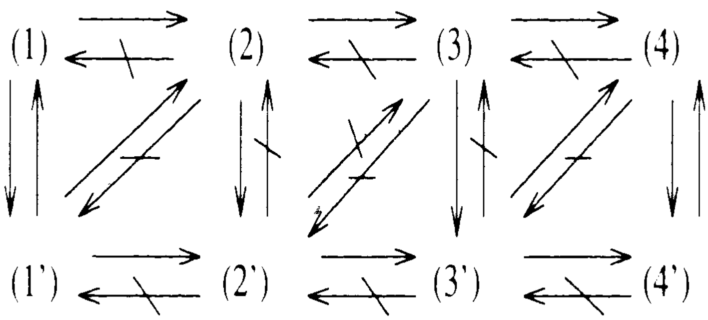


Figure 2.2: The relations of non-shy sets in $\mathbb{R}^n$

## 2.16 Prevalent versus typical

Any finite dimensional space can be decomposed into a set of Lebesgue measure zero and a first category set. The proof is entirely elementary. One shows that there is a dense open set (say one that contains all points with rational coordinates) of arbitrarily small Lebesgue measure. An appropriate intersection of dense open sets gives a measure zero, dense set of type $G_\delta$.

In our more general setting we do not have a notion of measure, just a notion of measure zero. Thus such an argument does not work, although one expects a similar decomposition should be available. The following theorem of Preiss and Tišer is remarkable because the measure zero part of the decomposition is in the strongest terms and the first category part is given to be $\sigma$-porous (see [54] for $\sigma$-porous). $\sigma$-porous sets form a smaller class than the sets of the first category.

**Theorem 2.16.1 (Preiss–Tišer)** *An infinite dimensional separable Banach space can be decomposed into a Preiss–Tišer null set and a set that is a countable union of closed porous sets.*

This result is of some interest in the study of derivatives on Banach spaces. From Aronszajn [2] we know that the set of points of Gâteaux non-differentiability of a real-valued Lipschitz function on an infinite dimensional separable Banach space $X$ is Aronszajn null. However the Fréchet differentiability of a real-valued Lipschitz function on an infinite dimensional separable Banach space is completely different. One of the Preiss–Tišer results [46, pp. 222, Proposition 1] says that, on an infinite dimensional separable Banach space, there is a real-valued Lipschitz function which is Fréchet non-differentiable on a given countable union of closed porous sets. Thus from the above theorem there is a real-valued Lipschitz function which is Fréchet differentiable only on a subset of a Preiss–Tišer null set in an infinite dimensional separable Banach space.

The decomposition in the above theorem is not possible in a finite dimensional Banach space because both a closed porous set and a Preiss–Tišer null set are Lebesgue measure zero.

A natural question is to ask whether such a decomposition, even a weaker one, is possible in a non-separable Banach space. We leave it as a problem at the end of this section. In the following we consider the same problem in a Polish group. Note that, however, a discrete group could not have such decomposition. Also the discrete group cannot be decomposed into a shy set and a first category set. Any compact or locally compact Polish group with a diffuse Haar measure can be decomposed into a shy set and a first category set by using the same proof that works for $\mathbb{R}^n$. In the following we give a sufficient condition for a general Polish group to be decomposed into a shy set and a first category set.

**Theorem 2.16.2** *Let $G$ be an Abelian Polish group which permits a Borel probability measure $\mu$ such that there is a constant $b < 1$ so that*

$$\mu(B(x, r)) \leq b\mu(B(x, 2r))$$

*for every $x \in X$ and $r \leq 1/2$. Then $G$ can be decomposed into a shy set and a first category set.*

**Proof.** Since $G$ is separable, let $\{x_i\}$ be a sequence of points that is dense in $G$. Let

$$X_n = \bigcup_{i=n}^{\infty} B\left(x_i, \frac{1}{2^i}\right) \text{ and } X = \bigcap_{n=1}^{\infty} X_n.$$

Then $X$ is a dense $G_\delta$ set and so $G \setminus X$ is of the first category. By the assumption for any $x \in G$,

$$\begin{aligned} \mu(X + x) \leq \mu(X_n + x) &\leq \sum_{i=n}^{\infty} \mu\left(B\left(x_i + x, \frac{1}{2^i}\right)\right) \\ &\leq \sum_{i=n}^{\infty} b^i \mu(B(x_i + x, 1)) \leq \sum_{i=n}^{\infty} b^i \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus the set $X$ is shy in $G$ and the decomposition $G = X \cup (G \setminus X)$ is our desire.

■

*Remark.* Such a theorem is true for finite dimensional spaces since Lebesgue measures can replace $\mu$ in the proof of the above theorem. However we do not know whether an infinite dimensional Banach space permits a Borel probability measure satisfying the condition in the above theorem.

**PROBLEM 4** *Does every Polish group permit a decomposition into a shy set and a set of the first category?*

## 2.17 Fubini's theorem

Fubini's theorem is one of the most important theorems in measure theory. It is natural to ask whether there is a version of Fubini's theorem that holds for Haar zero sets. Christensen [12] gave the following example to show that a full version of Fubini's theorem does not exist.

**Example 2.17.1** Let $H$ be a separable infinite dimensional Hilbert space and let $T$ be the unit circle in the complex plane. There exists in the product group $H \times T$ a Borel measurable set $A$ such that

(i) For every $h \in H$, the section $A(h) = \{t \in T : (h,t) \in A\}$ has Haar measure one in $T$.

(ii) For every $t \in T$, the section $A(t)$ is a Haar zero set in $H$.

(iii) The complement of $A$ is a Haar zero set in the product group $H \times T$.

**Proof.** (This proof is reproduced from [12].) Since an infinite dimensional separable Hilbert space is isometric to the space $L_2$ of square integrable functions. We assume $H = L_2$, let

$$A = \{(f,t) : \text{ the Fourier series of } f \text{ converges at } t\}.$$

Then from the famous result of Carleson that the Fourier series for any $L_2$ function is almost everywhere convergent, (i) follows. Note that for every $t \in T$, $A(t)$ is a closed linear proper subspace of $H$. So it is Haar zero and (ii) follows. Clearly the product measure of the one point measure in $H$ with mass 1 at zero and the Haar measure in $T$ is transverse to the complement of $A$. ■

From the above example we see that if a full version of Fubini's theorem holds, then (i) and (ii) imply that $A$ is Haar zero. This contradicts (iii). However, as indicated in [12], a weaker version of Fubini's theorem does hold. See Borwein and Moors [8, Theorem 2.3] for details.

## 2.18 Countable chain condition

In this section we discuss a property called *the countable chain condition* of the $\sigma$-ideal of shy sets in some spaces.

**Definition 2.18.1** Let $G$ be a Polish group. An ideal $\mathcal{F}$ of subsets of $G$ is said to satisfy *the countable chain condition* if each disjoint family of universally measurable sets in $G$ that do not belong to $\mathcal{F}$ is at most countable.

We also say that the group $G$ does not satisfy the countable chain condition if the ideal $\mathcal{F}$ does not satisfy the countable chain condition. A completely metrizable space is separable if and only if every family of pairwise disjoint non-empty open sets of this space is countable (see [20]). Thus the above definition is only for Polish groups.

In 1972 Christensen [12] asked whether any family of disjoint universally measurable non-shy sets in a Polish group must be countable. This is obviously true in finite dimensional spaces. Dougherty [18] answered this problem affirmatively in some Polish groups.

Theorem 3.5.4 exhibits a collection $\{S(A) : A \subseteq \mathbb{N}\}$ that forms a family of pairwise disjoint non-shy sets in an Abelian Polish group that is uncountable. The following theorem and its proof are reproduced from Dougherty [18].

**Theorem 2.18.2** *The Abelian Polish group $\mathbb{R}^{\mathbb{N}}$ does not satisfy the countable chain condition.*

**Proof.** Specifically the collection

$$\{S(A) : A \subseteq \mathbb{N}\}$$

from Theorem 3.5.4 consists of $2^{\aleph_0}$ disjoint non-shy elements. So $\mathbb{R}^{\mathbb{N}}$ does not satisfy the countable chain condition. ∎

Mycielski observed that if $A$ and $B$ are sets of natural numbers whose symmetric difference is infinite, then not only are the sets $S(A)$ and $S(B)$ disjoint, but also any translate of $S(A)$ intersects $S(B)$ in a shy set (see [18]). Thus we can get $2^{\aleph_0}$ non-shy sets of $\mathbb{R}^{\mathbb{N}}$ which mutually have this strong disjointness property, by taking $S(A)$ for $2^{\aleph_0}$ sets $A \subseteq \mathbb{N}$ which have infinite symmetric difference with each other.

In Abelian Polish groups Dougherty [18] obtained the following general result by unusual methods.

**Theorem 2.18.3 (Dougherty)** *Let $G$ be a Polish group which has an invariant metric and is not locally compact. Suppose that there exist a neighborhood $U$ of the identity and a dense subgroup $G_*$ of $G$ such that, for any finitely generated subgroup $F$ of $G_*$, $\overline{F \cap U}$ is compact. Then the ideal of shy sets of $G$ does not satisfy the countable chain condition.*

By using a result of Kechris [29] that in a Polish group there is a Borel selector $s : G/H \to G$ for the cosets of $H$ where $H$ is a closed subgroup of $G$, Dougherty [18] showed the following result.

**Theorem 2.18.4** *Suppose $(G, +)$ is an Abelian Polish group and $H$ is a closed subgroup of $G$. If the ideal of shy subsets of $H$ does not satisfy the countable chain condition, then the ideal of shy subsets of $G$ does not satisfy the countable chain condition.*

After obtaining Theorem 2.18.3, Dougherty [18] conjectured that a Polish group $G$ satisfies the countable chain condition only if $G$ is locally compact. Later Solecki [52] answered this problem perfectly in the following.

**Theorem 2.18.5** *Let $G$ be a Polish group admitting an invariant metric. Then each family of universally measurable or, equivalently, closed, pairwise disjoint sets which are not Haar null is countable iff $G$ is locally compact.*

In Chapter 5, we show that the $\sigma$-ideal of shy sets in the non-Abelian, non-locally compact space $\mathcal{H}[0,1]$ of automorphisms does not satisfy the countable chain condition. So we conjecture that the above Solecki's result is true for non-Abelian Polish groups. We will leave it as an open problem in Chapter 5.

Theorem 2.18.3 implies that an infinite dimensional separable Banach space does not satisfy the countable chain condition. However, the proof of Theorem 2.18.3 is in general. We pose the following problem.

**PROBLEM 5** *Using techniques in Banach space theory, show that any infinite dimensional separable space does not satisfy the countable chain condition.*

## 2.19 Thickness of non-shy sets

In this section we will discuss a property known as *thickness*. The definition is from [56].

**Definition 2.19.1** A subset $A$ of a topological semigroup $G$ is *thick* if the unit element is an interior point of $AA^{-1}$.

In 1929 Ostrowski [42] showed the well-known fact that every subset of the real line with positive Lebesgue measure is thick. That is, the non-shy sets in the real line are thick. Of course the converse does not hold. For example, the Cantor set in the real line is shy but it is thick (see [56, pp. 367-368, Theorem 2.4.2]). It was Christensen who first showed that non-shy sets are thick in any non-locally compact Abelian Polish group (see our Theorem 2.9.2 for a proof). Hoffman-Jørgensen [56] extended it to completely metrizable, Abelian topological semigroups in the following form. We have stated this as Theorem 2.13.1 (iv). For reference we repeat it here, using our new terminology.

**Theorem 2.19.2** *Let $G$ be a completely metrizable, Abelian topological semigroup. Let $A$ be a universally Radon measurable subset of $G$. If $A$ is not a Haar null set, then $A$ is thick.*

We discuss now, in a Banach space, the thickness of non-Preiss–Tišer null sets, non-Aronszajn null sets, non-$s$-null sets, non-Gaussian null sets in Phelps sense and non-Gaussian null sets in the ordinary sense.

In $\mathbb{R}$, all these sets are equivalent to the sets of positive Lebesgue measure and so they are thick. In $\mathbb{R}^n$ $(n > 1)$, each straight line is non-Preiss–Tišer null but not thick. However, in $\mathbb{R}^n$ $(n > 1)$, non-Aronszajn null sets, non-$s$-null sets, non-Gaussian null sets in Phelps sense are equivalent to the sets of positive Lebesgue measure. So they all are thick.

In an infinite dimensional separable Banach space we know that, from Example 2.3.8, there is a compact set $K$ which is not Gaussian null in Phelps sense. Note that $K - K$ is also a compact set and it cannot be a neighborhood of the zero element in an infinite dimensional space. Thus, from the comparison in Section 2.3 and Section 2.4 non-Aronszajn null sets, non-Preiss–Tišer null sets and non-Gaussian null sets in Phelps sense may not be thick in an infinite dimensional separable Banach space. Non-Gaussian null sets in the ordinary sense are thick (see [56, pp. 372-373] for a proof).

Recall (Problem 1) that we do not yet know whether all Christensen null sets are $s$-null. If we cannot answer this problem, perhaps we can answer the following problem.

**PROBLEM 6** *In an infinite dimensional separable Banach space, are all non-$s$-null sets thick?*

# Chapter 3

# Probes

## 3.1 Introduction

To show that a set $S$ in a Banach space $X$ is shy (i.e., a Christensen null set) we need first to establish that $S$ is universally measurable (perhaps by showing that it is a Borel set or an analytic set in $X$) and then to exhibit an appropriate probability measure on $X$ that is transverse to $S$. It is the finding of this testing measure or probe that requires some techniques.

Proving that a set is prevalent amounts merely to showing that the complement of the set is shy and so this does not introduce any new problems. However, showing that a set is non-shy will require showing that *every* possible measure fails to be a testing measure or probe for the set. We will see that, occasionally, this is not so hard and requires only a few measure-theoretic observations.

Where possible, rather than constructing a testing measure, we would prefer to use Lemma 2.2.2 and Definition 2.2.1 and exhibit an element $x \in X$ transverse to $S$. This merely picks a direction in the space so that all lines in that direction intersect $S$ in a set of linear measure zero. It is of some intrinsic interest to know if this is possible. Asserting only that the set $S$ is shy says much less.

Failing that we might show that there is a decomposition $S = \bigcup_{i=1}^{\infty} S_i$ and find elements $x_i$ transverse to $S_i$. In the language of Definition 2.3.2 we would say that the set $S$ is $s$-null. Again there is some intrinsic interest in knowing whether this is possible.

Where these ideas fail we may hope to find a finite dimensional subspace transverse to $S$. There would be some intrinsic interest in knowing the least dimension that could be selected when this technique works.

Where linear arguments fail (as they do in a variety of situations) perhaps we can construct a curve (a continuous image of an interval) that is transverse to $S$. Again knowing that the set $S$ admits a curve all of whose translates intersect $S$ in a set of measure zero (i.e., measure zero along the curve) is of some intrinsic interest.

In this chapter we survey some of the methods that have been used to establish that a given set is shy or non-shy and provide some concrete examples to illustrate the methods.

## 3.2 Finite dimensional probes

The language of probes in [27] was introduced to have a convenient way of expressing the techniques. Often to show that a set $A$ is shy one finds a subspace (usually one or two dimensional) that proves that the set $A$ is shy. That subspace was called a *probe* for the complement of $A$ (see Hunt et al. [27]). Here we extend this to call the subspace or the measure supported on it also a probe for the set $A$.

A subspace $P$ of $X$ that is a $n$-dimensional probe for a shy set $S \subseteq X$ is also a $m$-dimensional probe for the shy set $S$ if $m > n$. In fact we can find $m-n$ linear independent elements $\{x_1, \ldots, x_{m-n}\}$ such that the dimension of the span of $(\{x_1, \ldots, x_{m-n}\} \cup P)$ is $m$ provided $X$ has at least $m$ linear independent elements.

Thus it is necessary to give a new definition to describe shy sets.

**Definition 3.2.1** A shy set $S \subseteq X$ is said to be *m-dimensionally shy* if $S$ has a $m$ dimensional probe but no $n$ dimensional probes for $n < m$, and $S$ is said to be *infinite-dimensionally shy* if $S$ has no finite dimensional probes. The empty set is said to be *0-dimensionally shy.*

**Theorem 3.2.2** *In $\mathbb{R}^2$ there exist 2-dimensionally shy sets.*

**Proof.** In 1928 Besicovitch [4] constructed a set of $E \subseteq \mathbb{R}^2$ of Lebesgue measure zero which includes line segments of length 1 in every orientation (see also Proposition 12.2 in [22, pp. 163]). Thus any Borel set containing the set $E$ and having Lebesgue measure zero is a 2-dimensionally shy set. ■

**Theorem 3.2.3** *In $\mathbb{R}^n$ $(n > 2)$ any shy set is at most 2-dimensional.*

**Proof.** Let $F$ be a $n$-dimensional Lebesgue measure zero set of $\mathbb{R}^n$. By Theorem 7.13 [21, pp. 106] there is a 2-dimensional subspace $P$ of $\mathbb{R}^n$ such that every translate of $P$ intersects $F$ in a set of $k$-dimensional measure zero. Thus $F$ is an at most 2-dimensionally shy set and hence $F$ is an at most 2-dimensionally shy set. ■

**Theorem 3.2.4** *Any shy set in $\mathbb{R}^2$ space can be decomposed into two at most 1-dimensionally shy sets.*

**Proof.** By Theorem 2.7.5, a set $A \subseteq \mathbb{R}^2$ is shy iff $A$ is 2-dimensional Lebesgue measure zero. Let $A$ be Lebesgue measure zero in $\mathbb{R}^2$. Then $\lambda_2(A) = 0$ and by Fubini's theorem we have a Lebesgue one-dimensional measure zero set $N \subseteq \mathbb{R}$ such that for all $x_2 \in \mathbb{R} \setminus N$, $\lambda_1(A_{x_2}) = 0$ where $A_{x_2}$ is the section of $A$ at $x_2$ (the second variable). Let

$$B = \{(x_1, x_2) \in A : \; x_2 \notin N\}$$

and

$$C = \{(x_1, x_2) \in A : x_2 \in N\}.$$

Then $A = B \cup C$. We shall show that $B$ and $C$ are at most 1-dimensional shy sets. For the set $B$ we choose $\epsilon_1 = (1, 0)$. Thus for any $c = (c_1, c_2) \in \mathbb{R}^2$,

$$\{\alpha_1 \in \mathbb{R} : c + \alpha_1 \epsilon_1 \in B\} = A_{c_2}$$

is 1-dimensional Lebesgue measure zero when $c_2 \in \mathbb{R} \setminus N$, and is empty if $c_2 \in N$. Anyway it is 1-dimensional Lebesgue measure zero. Thus the set $B$ is an at most 1-dimensionally shy set. For the set $C$ choose $\epsilon = (0, 1)$, then for any $c = (c_1, c_2) \in \mathbb{R}^2$ the set $\{\alpha \in \mathbb{R} : c + \alpha\epsilon \in C\}$ is one-dimensional Lebesgue measure zero. Thus the set $C$ is an at most 1-dimensionally shy set. Hence the result follows. ∎

**PROBLEM 7** *In $\mathbb{R}^n$ $(n > 2)$, can any 2-dimensionally shy set be decomposed into two at most 1-dimensionally shy sets?*

**PROBLEM 8** *Let $X$ be an infinite dimensional separable Banach space. Does there exist a Borel shy set $S \subseteq X$ that is not an n-dimensionally shy set for any n?*

## 3.3 Elementary linear arguments

In many simple situations a crude linear argument suffices to show a set is shy. We illustrate with some elementary examples, mostly from the literature.

Certainly a proper subspace of a Banach space $X$ is shy if it is universally measurable. In fact if $S \subseteq X$ is a proper subspace then every element $x$ of $X \setminus S$ is transverse to $S$.

Choose $x \notin S$. For any $y \in S$, let $G = \{\lambda \in \mathbb{R} : y + \lambda x \in S\}$. Then $G$ is a singleton. If not, there exist $\lambda_1$ and $\lambda_2$, $\lambda_1 \neq \lambda_2$, such that $y + \lambda_1 x \in S$ and $y + \lambda_2 x \in S$.

Since $S$ is a linear space, then

$$\frac{1}{\lambda_2 - \lambda_1}[(y + \lambda_2 x) - (y + \lambda_1 x)] = x \in S.$$

This contradicts the choice of $x$. The result follows.

**Example 3.3.1** The set $S$ of differentiable functions is a proper linear subspace of $C[0,1]$. S. Mazurkiewicz in [40] showed that the set $S$ is not Borel, but is co-analytic (see also [11, pp. 503]). So the set $S$ is universally measurable and therefore is shy.

Note, however, that in general a linear subspace need not be a Borel set nor need it be universally measurable. [1] The following example is from Christensen [12].

**Example 3.3.2** Let $E$ be a separable Fréchet space and let $a_i$, $i \in I$, be an algebraic basis of $E$ and $b_i$ the coefficient functionals. Then $b_i^{-1}(0)$ are proper linear subspaces of $E$. If $b_i^{-1}(0)$ are universally measurable for some $i$'s, these sets are shy sets. In fact, however, the subspaces $b_i^{-1}(0)$ are universally measurable for at most finitely many $i \in I$. If not, there is a sequence $\{i_j\}$ of $I$ such that each $b_{i_j}^{-1}(0)$ is universally measurable. Set $L_n = \bigcup_{j \geq n} b_{i_j}^{-1}(0)$. Then $L_n$ is a universally measurable proper linear subspace. So $L_n$ is shy. Since $a_i$, $i \in I$ is an algebraic basis of $E$, so $E$ is the union of $L_n$'s and hence $E$ is shy, which is impossible.

For many examples of sets $S$ that have a probe $x$, the set

$$\{t \in \mathbb{R} : \; tx + y \in S\}$$

that we are required to show to be Lebesgue measure zero for every $y \in X$ is, in fact, a singleton or empty set. Thus every line in the direction $x$ intersects $S$ in at most one point. In many of the simplest applications this is the case and the measure-theoretic arguments reduce to simple computations.

[1] Assuming Martin's axiom, Talagrand in [53] showed every separable infinite-dimensional Banach space has a hyperplane that is universally of measure zero without being closed.

In any example that illustrates a shy set it is of intrinsic interest to know that the set is not merely shy, but has very small intersections with all lines in some particular direction.

**Example 3.3.3** As an example (from [27, pp. 226]) consider the set $S$ of all convergent series in the space $\ell_p$ for $1 < p \leq \infty$. This is a closed, proper subspace and so, trivially, it is shy. For a specific transverse element we can take the element $x = \{1, 1/2, 1/3, \dots\}$. It belongs to $\ell_p$ but diverges. The set $S$ intersects each line in the direction $x$ in at most one element.

The same argument applies to a hyperplane provided it is universally measurable. It is not always the case that hyperplanes are universally measurable except in concrete examples as shown in Example 3.3.2.

**Example 3.3.4** As another example (from [27, pp. 226]) the set $S$ of functions $f$ in $L_1[0,1]$ for which $\int_0^1 f(t)\,dt = 0$. This is evidently a closed hyperplane in $L_1[0,1]$ and so, trivially, shy. For a specific transverse element take $f_0 = 1$. It is easy to see that the set $S$ intersects each line in the direction $f_0$ at most one element.

These same arguments do not need much linearity in the problem. For example if $S$ is a universally measurable proper subset of a Banach space $X$ that satisfies

(i) $y_1 - y_2 \in S$ if $y_1, y_2 \in S$,

(ii) $ty \notin S$ for Lebesgue a.e. $t \in \mathbb{R}$ if $y \notin S$,

then $S$ must be shy in $X$. Indeed, let $y \notin X \setminus S$ and let

$$S_0 = \{t \in \mathbb{R} : \; x + ty \in S\}$$

for any $x \in X$. If $S$ is empty there is nothing to prove. If $S$ is not empty, choose $t_1 \in S_0$. Then by (ii), $(x + ty) - (x + t_1 y) = (t - t_1)y \in S$ only for a Lebesgue measure zero set of numbers of $t - t_1$. Thus $\lambda_1(S_0) = 0$ and the set $S$ is shy.

**Example 3.3.5** As a further example to illustrate simple methods of this type consider the Banach space $C[0,1]$ of continuous functions equipped with the supremum norm. Let $S$ denote the set of all continuous functions that are monotone on some closed subinterval of $[0,1]$. Write, for any closed subinterval $I \subseteq [0,1]$, $S(I)$ for the continuous functions that are monotone on $I$. Then

$$S = \bigcup S(I)$$

where the union is taken over the countable collection of all subintervals $I$ of $[0,1]$ with rational endpoints.

It is easy to see that each $S(I)$ is closed and is shy. To see that it is closed we just need note that the uniform limit of a Cauchy sequence of monotonic and continuous functions is also monotonic and continuous. So $S(I)$ is closed.

To see that $S(I)$ is shy take any element $g \in C[0,1]$ that is not a.e. differentiable on $I$. Then $g$ is transverse to $S(I)$. In fact the set

$$\{t \in \mathbb{R} : f + tg \in S(I)\}$$

can contain at most one point for any $f \in C[0,1]$. If not, there are distinct $t_1, t_2 \in \mathbb{R}$ such that $f + t_1 g$, $f + t_2 g \in S(I)$. Then $f + t_1 g$, $f + t_2 g \in S(I)$ are a.e differentiable on $I$ and so is $(f + t_1 g) - (f + t_2 g) = (t_1 - t_2)g$ on $I$. This contradicts the choice of $g$.

Consequently we have shown that $S$ is shy, indeed that $S$ is an $s$-null set in the sense of Definition 2.3.2. In particular, we can express this observation in the following language:

**Theorem 3.3.6** *The set of functions in $C[0,1]$ that are monotone in no interval forms a prevalent set.*

For more theorems of this type in the function space $C[a,b]$ see Sections 4.2 and 4.3.

In a Banach space, a universally measurable convex set that does not contain a line segment in some direction is easily seen to be shy. If a convex set $S$ does not contain a line segment in some direction $x$, then $S$ intersects the line in at most one point. So the element $x$ is transverse to $S$ and $S$ is shy. Thus convex sets that do not span the whole space are, in general, shy. It is of some interest to find out whether closed, convex spanning sets in certain spaces are non-shy. This will be discussed further in Section 3.5.

## 3.4 Compact sets

A number of arguments can be used to show that a compact (or a $\sigma$-compact) subset of an infinite dimensional Banach space must be shy. See, for example, the original article [12] of Christensen where the argument uses the fact that $A - A$ is a neighborhood of the zero element for any universally measurable non-shy set $A$. If a compact set $A$ were non-shy then the set $A - A$ is compact and also a neighborhood of the zero element. This is impossible in an infinite dimensional Banach space.

**Example 3.4.1** For an elementary illustration of a concrete example, here is how to show that the compact Hilbert cube in $\ell_2$ is shy:

We use

$$I^\infty = \{x \in \ell_2 : |x_n| \leq n^{-1}\}$$

to denote the compact Hilbert cube in $\ell_2$. Choose $y = \{n^{-2/3}\}$. Then $y \in \ell_2 \setminus I^\infty$. It is easy to see that $y$ is transverse to $I^\infty$. So $I^\infty$ is shy in $\ell_2$.

Hunt et al. gave an interesting proof in [27, pp. 225] that uses a category argument to exhibit that every compact set in an infinite dimensional Banach space admits a transverse element, indeed that the set of transverse elements is residual. This is of some general interest since it allows us to obtain transverse elements without the necessity to construct one in advance. This is also of some intrinsic interest since we see that "most" directions in the space are transverse to the set. In the following we give a similar but simpler proof suggested by J. Borwein to show every compact set admits a transverse element, an argument which also show that the transverse elements are residual.

**Theorem 3.4.2** *A compact subset of an infinite dimensional Banach space is shy.*

**Proof.** Let $V$ be an infinite dimensional Banach space and $S \subseteq V$ be a compact set. Then, if the linear span of $S$ is denoted by Span$S$, we have

$$\text{Span}S = \bigcup_{n=1}^{\infty} S_n$$

where

$$S_n = \{\alpha x + \beta y : \ \alpha, \beta \in [-n, n], \ x, y \in S\}.$$

Since $S$ is compact, all sets $S_n$ are compact. Thus Span$S$ is $\sigma$-compact and first category. Hence $V \setminus \text{Span}S \neq \emptyset$. Therefore $S$ is transverse to every element $x \in V \setminus \text{Span}S$ since the line $\lambda x$ intersects $S$ in at most one element. Thus $S$ is shy. ■

**Corollary 3.4.3** *A $\sigma$-compact set in an infinite dimensional Banach space is s-null.*

The above method neither holds for Gaussian null sets nor for Aronszajn null sets since the fact that there is a Borel probability measure is transverse to the one-dimensional probe of the set $V \setminus S$ does not guarantee $\mu(S) = 0$ for all non-degenerate Gaussian measures $\mu$. In fact there exist compact sets which are not Gaussian null sets in Phelps sense (see Example 2.3.8). From the fact that a translate of a non-degenerate Gaussian measure is also a non-degenerate Gaussian measure we know that a Gaussian null set in Phelps sense is also a shy set. Thus the class of shy sets is wider than the class of Gaussian null sets in Phelps sense. Recall that an Aronszajn null set is also a Gaussian null set in Phelps sense. Thus the class of shy sets is also wider than the class of Aronszajn null sets.

In non-locally compact groups with invariant metrics compact sets are shy (see [18]). However we do not know whether the invariant metrics are needed for this statement.

**PROBLEM 9** *In a general non-locally compact Polish group without invariant metrics, are compact sets shy?*

In the following we will give an interesting method to show that certain sets in an infinite dimensional Banach space are shy. The following theorem and lemma that we will display were introduced by Aronszajn [2] for separable Banach spaces. In fact they remain valid for non-separable Banach spaces. We reproduce the theorem from [16, pp. 219] (see [16] for a proof), and give a simpler proof, suggested by J. Borwein, of the lemma.

**Theorem 3.4.4** **(Josefson-Nissenzweig)** *Let $X$ be an infinite dimensional Banach space. Then there is a sequence $\{u_n\} \subseteq X^*$, $\|u_n\| = 1$ such that $\{u_n\}$ converges to 0 in the weak$^*$-topology.*

**Lemma 3.4.5** *Let $X$ be a Banach space. Suppose $u_n$ are elements of $X^*$ with norm 1 and $u_n \to 0$ in weak$^*$-topology, while $S$ is a bounded set such that $u_n(x) \to 0$ uniformly for $x \in S$. Then there is an element $a \in X$ transverse to $S$, indeed*

$$\{t \in \mathbb{R} : \ x + ta \in S\}$$

*contains at most one point for every $x \in X$.*

**Proof.** Since $u_n \to 0$ uniformly on $S$, so $\delta_S^*(u_n) \equiv \sup_{x \in S} u_n(x) \to 0$. Thus $\delta_A^*(u_n) \to 0$ where $A \equiv \overline{\text{aco}}S$ is the convex hull of $S$. We claim that Span$S \subseteq \bigcup_{n=1}^{\infty}(nA) \neq X$ where Span$S$ is the linear span of $S$. We need only show that $\bigcup_{n=1}^{\infty}(nA) \neq X$. If

$\bigcup_{n=1}^{\infty}(nA) = X$, then by the Baire category theorem, the interior of the set $A$ is not empty and hence $\delta_A^*(u_n) \nrightarrow 0$. Thus there exists an element $a \in X \setminus \mathrm{Span}S$ that is transverse to the set $S$. ■

From Theorem 3.4.4 and Lemma 3.4.5 we can show that certain sets in an infinite dimensional Banach space are shy. We write this as a theorem as follows.

**Theorem 3.4.6** *Let $X$ be an infinite dimensional Banach space, and $\{u_n\} \subseteq X^*$ be a sequence as in Theorem 3.4.4. If a set $S$ is universally measurable and bounded such that $u_n(x) \to 0$ uniformly for $x \in S$. Then the set $\overline{aco}S$ is shy where $\overline{aco}S$ is the convex hull of $S$.*

Using the above theorem we can obtain Theorem 3.4.2 immediately. Let $S$ be a compact set in an infinite dimensional Banach space $X$. Then $S$ is bounded and closed. By Theorem 3.4.4 there is a sequence $\{u_n\} \subseteq X^*$, $\|u_n\| = 1$ such that $u_n \to 0$ weakly. Standard arguments show that the bounded sequence $\{u_n\}$ converges to 0 weakly implies that $u_n(x) \to 0$ uniformly for $x \in S$. From the above theorem the set $S$ is shy.

## 3.5 Measure-theoretic arguments

To show that a set is non-shy "requires" showing that there are no transverse elements, indeed that there are no transverse probability measures at all.

The following simple fact from the theory of Borel measures on Polish spaces is very useful in showing a set to be non-shy. See Theorem 2.12.2 for a proof.

> *Every probability measure defined on the Borel subsets of a Polish space assigns positive measure to some compact set.*

This then gives an immediate proof of the following:

**Lemma 3.5.1** *A set $S$ in a separable Banach space $X$ that contains a translate of every compact subset is non-shy.*

**Proof.** Suppose that the set $S$ were shy then there is a Borel probability measure $\mu$ that is transverse to $S$. That is, $\mu(S+x) = 0$ for each $x \in X$. According to the above basic fact and assumption there are a compact set $K$ and $x \in X$ such that $\mu(K) > 0$ and $K + x \subseteq S$. Then $K \subseteq S - x$ and $0 < \mu(K) \leq \mu(S - x) = 0$. This is a contradiction. ■

For example (from [6]), the positive cone $C$ of the space $c_0$ of null sequences is non-shy. (It is clear that $C$ is nowhere dense in $c_0$.) In fact, every compact set $K$ in $c_0$ is contained in a set $\{y \in c_0 : \ |y_n| \leq x_n, \ n = 1, 2, \ldots\}$ for a certain $x \in c_0$. So $K + x \subseteq C$. By Lemma 3.5.1, the positive cone C is non-shy. Borwein and Fitzpatrick [7] use Lemma 3.5.1 along with a characterization of reflexive separable Banach spaces to justify the following.

**Theorem 3.5.2** *In any non-reflexive separable Banach space there is a nowhere dense closed convex subset that is not shy.*

**Proof.** A separable Banach space $E$ is not reflexive if and only if there exists a closed convex subset $C$ of $E$ with empty interior that contains some translate of each compact set in $E$ (see [37]). Thus, the result follows from the last lemma. ■

In contrast to the above theorem Borwein and Moors [8] and Matsoušková in [36] showed that in any super-reflexive space a nowhere dense closed convex set is shy.

Here is another example, from Dougherty [18], of an argument that exploits this same measure-theoretic fact. Let $\mathbb{R}^{\mathbb{N}}$ denote the space of all sequences of real numbers furnished with the metric

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}.$$

This is an Abelian Polish group, complete in this metric but non-locally compact. For any $A \subseteq \mathbb{N}$, let

$$S(A) \equiv \{s \in \mathbb{R}^{\mathbb{N}} : \; s(n) < 0 \text{ for } n \notin A \text{ and } s(n) > 0 \text{ for } n \in A\}.$$

We reproduce the following lemma from [18] and give a proof.

**Lemma 3.5.3** *For any $A \subseteq \mathbb{N}$, the set $S(A)$ contains a translate of every compact subset of $\mathbb{R}^{\mathbb{N}}$.*

**Proof.** For any compact set $C$ the set $C_i = \{x(i) \in \mathbb{R} : \; x \in C\}$ is a bounded closed set in $\mathbb{R}$, $i = 1, 2, \ldots$. Let $(a_i, b_i)$ be intervals such that $C_i \subseteq (a_i, b_i)$. We define $t \in \mathbb{R}^{\mathbb{N}}$ by

$$t(i) = \begin{cases} |b_i| + 1, & \text{if } i \in A, \\ -|a_i| - 1, & \text{if } i \notin A. \end{cases}$$

Then it is easy to see that $C + t \subseteq S(A)$. ■

From this lemma we conclude immediately that the set $S(A)$ in the Abelian Polish group $\mathbb{R}^{\mathbb{N}}$ cannot be shy.

**Theorem 3.5.4** *For any $A \subseteq \mathbb{N}$ the set $S(A)$ is a Borel set that is not shy in $\mathbb{R}^{\mathbb{N}}$.*

**Proof.** From Lemma 3.5.1 and Lemma 3.5.3 we need only show that the set $S(A)$ is a Borel set. In $\mathbb{R}^{\mathbb{N}}$ any sequence $s_n \to s$ implies $s(i) \to s(i)$ for each $i \in \mathbb{N}$. Thus for each pair of integers $(i, n)$, the sets

$$A_{in} = \left\{ s \in \mathbb{R}^{\mathbb{N}} : \; s(i) \geq \frac{1}{n} \right\} \text{ and } B_{in} = \left\{ s \in \mathbb{R}^{\mathbb{N}} : \; s(i) \leq -\frac{1}{n} \right\}$$

are closed sets. Note that

$$S(A) = \left( \bigcap_{i \in A} \bigcup_{n} A_{in} \right) \bigcap \left( \bigcap_{i \notin A} \bigcup_{n} B_{in} \right).$$

Thus $S(A)$ is a Borel set. ■

Theorem 3.5.4 exhibits an interesting feature of shy sets that is not shared by the Lebesgue measure zero sets in a finite dimensional space. The collection $\{S(A) : A \subseteq \mathbb{N}\}$ forms a family of pairwise disjoint non-shy sets that is uncountable. See Section 2.18 for a further discussion of this. The search for families of uncountably many pairwise disjoint non-shy sets in non-Abelian Polish groups is our primary goal in Chapter 5.

# Chapter 4

# Prevalent properties in some function spaces

## 4.1 Introduction

During the middle of nineteen century, many famous mathematicians tried to prove the differentiability of continuous functions. Frustrated, some of them gave examples of continuous functions that are not differentiable on a dense set or further on the set of irrationals. It was K. Weierstrass who first constructed a convincing example of a continuous function that has no point of differentiability in 1875. After Weierstrass many examples of continuous nowhere differentiable functions were discovered by other mathematicians. In 1931 S. Banach and S. Mazurkiewicz separately gave similar existence proofs by using the Baire Category theorem in separate papers [3] and [40] respectively. Since then Baire category arguments have been widely used to prove the existence of functions which are difficult to visualize. It is now a common practice to show that certain classes of functions are "typical" in spaces of functions by showing that they form residual subsets in those spaces.

The same program can be carried out using the measure-theoretic notion of prevalence, rather than the topological notion of category. The earliest such result is probably that of N. Wiener [60] showing that the nowhere differentiable functions are full measure in the space $C[0,1]$ of continuous functions $x(t)$ with $x(0) = 0$ in the sense of the Wiener measure. In 1994 Hunt et al. [27] applied the notion of shy sets (rediscovering the Haar zero sets of Christensen) to the same kind of problems. In this chapter we will show that certain classes of functions in various function spaces are prevalent.

## 4.2 Continuous, nowhere differentiable functions

In this section we will sketch some of the ideas from the paper of Hunt [28]. Later sections in this chapter are devoted to similar problems in some specific function spaces.

Let $S$ denote the set of all continuous functions on $[0,1]$ that are somewhere differentiable, i.e., for which there is at least one point of differentiability. It is proved in [28] that $S$ is shy in the space $C[0,1]$. In fact the author does not address the measurability issue but shows that $S$ is contained in a shy Borel set. (The definition used in that paper for a shy set is exactly this, that an arbitrary set is shy if it is a subset of a Borel shy set. Our definition requires us to prove that the set is universally measurable if not a Borel set.)

Let us show how to check that $S$ is universally measurable. A continuous function $f$ has a finite derivative at some point $x \in [0,1]$ if and only if for every positive integer $n$ there is a positive integer $m$ such that if $0 < |h_1|, |h_2| < 1/m$ and $x+h_1,\ x+h_2 \in [0,1]$ then

$$(*) \qquad \left|\frac{f(x+h_1)-f(x)}{h_1} - \frac{f(x+h_2)-f(x)}{h_2}\right| \le \frac{1}{n}.$$

For each pair $(n, m)$, let

$$E(n, m) = \{(f, x) \in C[0,1] \times [0,1] : (*) \text{ holds } \}.$$

Then $C[0,1] \backslash S$ is the projection into $C[0,1]$ of $\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E(n, m)$. Note that $E(n, m)$ is closed. So $C[0,1] \backslash S$ is analytic [11] and therefore the set $S$ is universally measurable (see [11]). Furthermore, $S$ is not Borel (see [38] and [39] for details).

We now turn to the method of proof that the set $S$ of somewhere differentiable functions is shy. As pointed out in [28] there is no element transverse to $S$. Thus the simplest of the methods is not available here. To see this, let $g$ be a continuous function. For $f(x) = -xg(x)$ and every $\lambda$, $f + \lambda g$ is differentiable at $x = \lambda$. Hence there is no element transverse to $S$. The reason here is that a linear combination of nowhere differentiable functions can be differentiable.

In [28] it is shown that there is a two-dimensional subspace of $C[0,1]$ that is transverse to $S$. We sketch parts of the proof here. In [28] Hunt used two continuous nowhere differentiable functions

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos 2^k \pi x \text{ and } h(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin 2^k \pi x.$$

Through complicated and elegant computations he showed that there exists a constant $c > 0$ such that for any $\alpha, \beta \in \mathbb{R}$ and any closed interval $I \subseteq [0,1]$ with length $\epsilon \leq 1/2$,

$$\max_I(\alpha g + \beta h) - \min_I(\alpha g + \beta h) \geq \frac{c\sqrt{\alpha^2 + \beta^2}}{(\log \epsilon)^2}.$$

By using this result he showed that the set

$$G = \{(\alpha, \beta) \in \mathbb{R}^2 : f + \alpha g + \beta h \text{ is Lipschitz at some } x \in [0,1]\}$$

is two-dimensional Lebesgue measure zero for any $f \in C[0,1]$. It is easy to show that the set $G$ is of type $F_\sigma$ and that a function differentiable at a point $x$ is Lipschitz at $x$. So the set $S$ is contained in $G$. In [28] Hunt showed $\lambda_2(G) = 0$ as follows.

Let $N \geq 2$ be an integer and split $[0, 1]$ into $N$ closed intervals $I$ of length $\epsilon = 1/N$. Split $G$ into $\bigcup_I \bigcup_{n=1}^{\infty} S_n$ where

$$S_n = \{(\alpha, \beta) \in G: \ f + \alpha g + \beta h \text{ is } n\text{-Lipschitz at some point } x \in I\}.$$

For any pair of points $(\alpha_1, \beta_1)$, $(\alpha_2, \beta_2) \in S_n$, $|f_i(x) - f_i(x_i)| \leq n|x - x_i| \leq n\epsilon$ on denoting $f_i = f + \alpha_i g + \beta_i h$. One checks that

$$\max_I (f_1 - f_2) - \min_I (f_1 - f_2) \leq 4n\epsilon.$$

Therefore, by using the result we first mentioned above we have

$$\sqrt{(\alpha_1 - \alpha_2)^2 + (\beta_2 - \beta_1)^2} \leq \frac{4n}{c}\epsilon(\log \epsilon)^2.$$

Trivially the result follows.

Hunt's methods show that the somewhere differentiable functions form a shy set that is 2-dimensional in the language that we introduced in Section 3.2. It would be interesting to know if more can be said about this set. We leave this again as an open problem.

**PROBLEM 10** *Is the set of somewhere differentiable functions in the space $C[0, 1]$ s-null?*

## 4.3 Prevalent properties in $C[a, b]$

To illustrate some of our methods we shall prove some of the simpler prevalence results in the space $C[a, b]$ of continuous functions equipped with the supremum norm.

**Theorem 4.3.1** *For $c \in [a, b]$ and $C \in \mathbb{R}$, let*

$$S_c = \{f \in C[a, b] : \ f(c) = C\}.$$

*Then $S_c$ is closed, nowhere dense and shy.*

**Proof.** It is easy to see that $S_c$ is closed and nowhere dense in $C[a,b]$. Take a non-zero constant function $g = d \neq C$. Then

$$\{\lambda \in [0,1] : f + \lambda g \in S_c\}$$

can contain no more than one element for any $f \in C[a,b]$. If not, there are $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 \neq \lambda_2$, $f + \lambda_1 g \in S_c$ and $f + \lambda_2 g \in S_c$. Then $f(c) + \lambda_1 d = f(c) + \lambda_2 d$. So $(\lambda_2 - \lambda_1)d = 0$ which contradicts $d \neq 0$. Therefore the one-dimensional Lebesgue measure is transverse to $S_c$ and $S_c$ is shy. ∎

A continuous function is called *nowhere monotonic* on an interval $[a,b]$ if it is not monotonic on any subinterval of $[a,b]$.

A continuous function is called *nowhere monotonic type* on an interval $[a,b]$ if for any $\gamma \in \mathbb{R}$ the function $f(x) - \gamma x$ is not monotonic on any subinterval of $[a,b]$.

It is well known (see, e.g., [11, pp. 461–464]) that nowhere monotonicity and being nowhere monotonic type are typical properties in the space $C[a,b]$. In this section we will show directly that these two properties are also prevalent properties in $C[a,b]$. The following theorem follows easily from the result of Hunt sketched in Section 4.2. The proof, here, is more elementary.

**Theorem 4.3.2** *The prevalent function $f \in C[a,b]$ is of nowhere monotonic type.*

**Proof.** Given any interval $I$, let

$$G(I) = \{f \in C[a,b] : f \text{ is of monotonic type on } I\},$$

and let

$$G_n(I) = \left\{ f \in C[0,1] : \begin{array}{l} \text{there exists a } \gamma \in [-n,n] \text{ such that} \\ f - \gamma x \text{ is monotonic on } I \end{array} \right\}.$$

Then

$$G(I) = \bigcup_{n=1}^{\infty} G_n(I).$$

We show that $G_n(I)$ is a closed set and therefore $G(I)$ is a Borel set. For any Cauchy sequence $\{f_i\} \subseteq G_n(I)$ there exists a function $f \in C[0,1]$ such that $f_i \to f$ uniformly. Then there exist $\gamma_i \in [-n, n]$ such that $f_i(x) - \gamma_i x$ are monotonic on $I$. Then we can choose a subsequence $\{\gamma_{i_j}\}$ of $\{\gamma_i\}$ such that $\gamma_{i_j} \to \gamma \in [-n, n]$. It is easy to see that $f - \gamma x$ is monotonic on $I$. Thus $G_n(I)$ is closed.

Now we show that $G_n(I)$ is a shy set. Choose a function $g \in C[a, b]$ that is nowhere differentiable on $I$. For any $f \in C[a, b]$, let

$$G_{n,g} = \{\lambda \in \mathbb{R} : \ f + \lambda g \in G_n\}.$$

Then $G_{n,g}$ is a singleton or empty set. If not there exist $\lambda_1, \lambda_2 \in G_{n,g}$, $\lambda_1 \neq \lambda_2$. Then there exist $\gamma_1$ and $\gamma_2$ such that $f + \lambda_1 g - \gamma_1 x$ and $f + \lambda_2 g - \gamma_2 x$ are monotonic on $I$. Therefore

$$(f + \lambda_1 g - \gamma_1 x) - (f + \lambda_2 g - \gamma_2 x) = (\lambda_1 - \lambda_2)g + (\gamma_2 - \gamma_1)x$$

is differentiable almost everywhere on $I$. This contradicts our assumption that the function $g$ is nowhere differentiable. Thus $G_{n,g}$ is a singleton or empty set and hence $G_n(I)$ is a shy set. It follows that the set $G_n(I)$ is a shy set.

Let $\{I_k\}$ be an enumeration of all intervals whose endpoints are rational numbers. Then each $G(I_k)$ is shy and the union is also shy. Hence the result follows. ■

**Corollary 4.3.3** *The prevalent function $f \in C[a, b]$ is nowhere monotonic.*

Note that we have proved this earlier. As a corollary our methods actually show more.

**Corollary 4.3.4** *The set of functions which are somewhere monotonic is s-null in $C[a, b]$.*

## 4.4 Prevalent properties in $b\mathcal{A}$, $b\mathcal{DB}^1$, $b\mathcal{B}^1$

Following Bruckner [9], Bruckner and Petruska [10] we use $b\mathcal{A}$, $b\mathcal{DB}^1$, $b\mathcal{B}^1$ to denote the spaces of bounded approximately continuous functions, bounded Darboux Baire 1 functions and bounded Baire 1 functions defined on $[0,1]$ respectively, all of which are equipped with supremum norms. All these spaces are Banach spaces and form a strictly increasing system of closed subspaces (see [9], [10]). In [10] it was shown that for a given arbitrary Borel measure $\mu$ on $[0,1]$ the typical function in $\mathcal{F} = b\mathcal{A}$, $b\mathcal{DB}^1$, or $b\mathcal{B}^1$ is discontinuous $\mu$ almost everywhere. In this section we will show that such typical properties in these three spaces are also prevalent properties for any $\sigma$-finite Borel measure.

**Theorem 4.4.1** *Let $\mu$ be a $\sigma$-finite Borel measure on $[0,1]$. The prevalent function in $\mathcal{F} = b\mathcal{A}$, $b\mathcal{DB}^1$, or $b\mathcal{B}^1$ is discontinuous $\mu$ almost everywhere on $[0,1]$.*

**Proof.** Let

$$S = \{f \in \mathcal{F} : \ f \text{ is continuous on a set } F_X,\ \mu(F_X) > 0\}.$$

We show first that the set $S$ is a Borel set. Note

$$S = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$$

where

$$A_{\frac{1}{n}} = \left\{ f \in \mathcal{F} : \ \mu(C_f) \geq \frac{1}{n} \right\}$$

and $C_f$ is the set of continuity points of $f$. Let $\{f_m\}$ be a Cauchy sequence in $A_{\frac{1}{n}}$, then $f_m \to f \in \mathcal{F}$. Let

$$C = \bigcap_{N=1}^{\infty} \bigcup_{m=N}^{\infty} C_{f_m}.$$

Then $C \subseteq C_f$. Note

$$\mu(C) = \lim_{N\to\infty} \mu\Big(\bigcup_{m=N}^{\infty} C_{f_m}\Big) \geq \liminf_{m\to\infty} \mu(C_{f_m}) \geq \frac{1}{n}.$$

So $f \in A_{\frac{1}{n}}$. Thus $A_{\frac{1}{n}}$ is closed and the set $S$ is a Borel set.

We now show that the set $S$ is a shy set. It is well known that there is a function $g \in \mathcal{F}$ which is discontinuous $\mu$ almost everywhere on $[0, 1]$. See [10, pp. 331, Theorem 2.4]. We will use this function $g$ as a probe. For any given function $f \in \mathcal{F}$, let

$$S_g = \{\lambda \in \mathbb{R} : \ f + \lambda g \in S\}.$$

We claim that $S_g$ is Lebesgue measure zero. For distinct $\lambda_1, \lambda_2 \in S_g$, if $\mu(F_{\lambda_1} \cap F_{\lambda_2}) > 0$, then

$$(f + \lambda_1 g) - (f + \lambda_2 g) = (\lambda_1 - \lambda_2)g$$

would be continuous on $F_{\lambda_1} \cap F_{\lambda_2}$. This contradicts the choice of the function $g$. Thus for distinct $\lambda_1$ and $\lambda_2$ the corresponding sets $F_{\lambda_1}$ and $F_{\lambda_2}$ satisfy $\mu(F_{\lambda_1} \cap F_{\lambda_2}) = 0$. Since $\mu$ is $\sigma$-finite on $[0, 1]$ then $[0, 1] = \bigcup_{i=1}^{\infty} X_i$ where $\mu(X_i) < \infty$ and $X_i \cap X_j = \emptyset$, $i \neq j$. Let

$$S_{mn} = \left\{\lambda \in S_g : \ \mu(F_\lambda \cap X_m) \geq \frac{1}{n}\right\}.$$

Then $S_{mn}$ is finite. If not, there exist countably many $\lambda_i \in S_{mn}$ such that

$$\infty = \sum_{i=1}^{\infty} \mu(F_{\lambda_i} \cap X_m) = \mu(X_m \cap (\bigcup_{i=1}^{\infty} F_{\lambda_i})) \leq \mu(X_m) < \infty.$$

This is a contradiction. Hence $S_{mn}$ is finite and $S_g = \bigcup_{m,n=1}^{\infty} S_{mn}$ is at most countable. Thus $S_g$ is Lebesgue measure zero and the result follows. ∎

In the proof of Theorem 4.4.1 we did not use any special property of functions in $b\mathcal{A}$, $b\mathcal{DB}^1$, $b\mathcal{B}^1$ except that in all these classes there are functions that are discontinuous $\mu$ almost everywhere on $[0, 1]$. Thus we can extend Theorem 4.4.1 in a general form as follows (see [32] for a typical version).

**Theorem 4.4.2** *Let $\mu$ be a $\sigma$-finite Borel measure on $[0, 1]$. Let $\mathcal{F}$ be a linear space of bounded functions $f : [0, 1] \to \mathbb{R}$ with supremum metric. Suppose that there is a function $f \in \mathcal{F}$ that is discontinuous $\mu$ almost everywhere on $[0, 1]$. Then the prevalent function in $\mathcal{F}$ is discontinuous $\mu$ almost everywhere on $[0, 1]$.*

## 4.5 Prevalent properties in $D[a,b]$

In this section we use $D[a,b]$ to denote the set of differentiable functions $f$ whose derivatives are bounded and $f(a) = 0$ and furnished with the metric (for $f, g \in D[a,b]$)

$$\rho(f,g) = \sup_{x\in[a,b]} |f'(x) - g'(x)|.$$

Then $D[a,b]$ is a Banach space. We study prevalent properties in this Banach space.

**Theorem 4.5.1** *Both the prevalent function and the typical function $f \in D[a,b]$ are monotonic on some subinterval of $[a,b]$.*

**Proof.** Let

$$DZ = \{f \in D[a,b] : f'(x) = 0 \text{ on a dense set of } [a,b]\}.$$

Then $DZ$ is a closed linear subspace of $D[a,b]$. In fact, let $f, g \in DZ$, then the sets

$$\{x \in [a,b] : f'(x) = 0\} \text{ and } \{x \in [a,b] : g'(x) = 0\}$$

are of type $G_\delta$ since $f'$ and $g'$ are Baire 1 functions. But the intersection of two dense sets of type $G_\delta$ is also dense, so

$$\{x \in [a,b] : \alpha_1 f'(x) + \alpha_2 g'(x) = 0\} \supseteq \{x \in [a,b] : f'(x) = 0\} \cap \{x \in [a,b] : g'(x) = 0\}$$

is dense for any $\alpha_1, \alpha_2 \in \mathbb{R}$. Thus $\alpha_1 f + \alpha_2 g \in DZ$. Similarly we can show $DZ$ is closed. Suppose $\{f_n\}$ is a Cauchy sequence in $DZ$ such that $f_n' \to f'$ uniformly where $f \in D[a,b]$. Again since $f_n$ are all Baire 1 functions, the sets $\{x \in [a,b] : f_n'(x) = 0\}$ are dense and of type $G_\delta$. So the set

$$\bigcap_{n=1}^{\infty} \{x \in [a,b] : f_n'(x) = 0\} \subseteq \{x \in [a,b] : f'(x) = 0\}$$

is dense and of type $G_\delta$. Thus $f \in DZ$ and hence $DZ$ is a closed linear space. It is easy to see that $DZ$ is a proper subspace of $D[a,b]$. For example the function $x - a \in D[a,b]$ but $f \notin DZ$. Thus from the results in Section 3.3 $DZ$ is shy.

For any differentiable, nowhere monotonic function $f$, the derivative $f'$ is a Baire 1 function and so the set $\{x \in [a, b] : f'(x) = 0\}$ is a $G_\delta$ set. Further since $f$ is nowhere monotonic then $\{x \in [a, b] : f'(x) = 0\}$ is dense in $[a, b]$. Therefore the set

$$G = \{f \in D[a, b] : f \text{ is nowhere monotonic on } [a, b]\}$$

is a subset of $DZ$. In fact the set $G$ is of type $G_\delta$ in $D[a, b]$. Let $I$ be an open subinterval of $[a, b]$, and

$$F(I) = \{f \in D[a, b] : f \text{ is monotonic on } I\}.$$

For any $f \in F(I)$, $f'(x) \geq 0$ on the entire $I$ or $f'(x) \leq 0$ on the entire $I$. So due to the metric defined on $D[a, b]$ it is easy to see that $F(I)$ is a closed set. Hence the set of functions in $D[a, b]$ that are somewhere monotonic is the union of all those $F(I)$ over intervals $I$ with rational endpoints. So it is of type $F_\sigma$ and therefore $G$ is of type $G_\delta$ in $D[a, b]$. The result follows. ∎

We know that, for any $\sigma$-finite Borel measure $\mu$, the derivative of the typical function $f \in D[a, b]$ is discontinuous $\mu$ almost everywhere on $[a, b]$ (see [10]). By applying Theorem 4.4.2 we obtain the following theorem.

**Theorem 4.5.2** *Let $\mu$ be a $\sigma$-finite Borel measure on $[a, b]$. Then the derivative of the prevalent function $f \in D[a, b]$ is discontinuous $\mu$ almost everywhere on $[a, b]$.*

If we do not use the fact that for any $\sigma$-finite Borel measure $\mu$ there exists a function $g \in D[a, b]$ whose derivative is discontinuous $\mu$ almost everywhere, we can easily obtain a simpler result as follows.

**Theorem 4.5.3** *The derivative of the prevalent function $f \in D[a, b]$ is discontinuous on a dense set of $[a, b]$.*

**Proof.** Let $I$ be a subinterval of $[a, b]$ and let

$$A(I) = \{f \in D[a, b] : f' \text{ is continuous on } I\}.$$

Then from the definition of the metric on $D[a,b]$ it is easy to see that $A(I)$ is closed. Also for any $f, g \in A(I)$ and $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in A(I)$. So $A(I)$ is a closed linear subspace of $D[a,b]$. We will see that $A(I)$ is also proper in $D[a,b]$. Take $c \in I$. The function

$$f(x) = \begin{cases} (x-c)^2 \sin \frac{1}{x-c} & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases}$$

has as its derivative the function

$$f'(x) = \begin{cases} 2(x-c)\sin \frac{1}{x-c} - \cos \frac{1}{x-c} & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases}$$

which is discontinuous at $x = c$. Thus $A(I)$ is a closed linear proper subspace of $D[a,b]$. So $A(I)$ is shy. The set $S$ of functions in $D[a,b]$ whose derivatives are continuous on some subinterval is the union of the $A(I)$ taken over all subintervals $I$ with rational endpoints. Hence the set $S$ is shy (as a countable union) and the theorem follows.

■

## 4.6 Prevalent properties in $BSC[a,b]$

The space $BSC[a,b]$ of bounded symmetrically continuous functions equipped with the supremum norm is a complete space (see [55]). From [50] we know that the set of functions $f \in BSC[a,b]$, which have a $c$-dense sets of points of discontinuity, is residual. In this section we show that such a set is also prevalent. Here we say that a set is *c-dense* in a metric space $(X, \rho)$ if it has continuum many points in every non-empty open set.

In [33] Pavel Kostyrko showed the following theorem.

**Theorem 4.6.1** *Let $(X, \rho)$ be a metric space. Let $F$ be a linear space of bounded functions $f : X \to \mathbb{R}$ furnished with the supremum norm $\|f\| = \sup_{x\in X}\{|f(x)|\}$. Suppose that in $F$ there exists a function $h$ such that its set $D(h)$ of points of discontinuity* *is uncountable. Then*

$$G = \{f \in F : \ D(f) \text{ is uncountable}\}$$

*is an open residual set in* $(F, d)$, $d(f, g) = \|f - g\|$.

By modifying the methods in [50] we can get a stronger result in separable metric spaces.

**Theorem 4.6.2** *Let* $(X, \rho)$ *be a separable metric space. Let* $F$ *be a complete metric linear space of bounded functions* $f : X \longrightarrow \mathbb{R}$ *furnished with supremum norm* $\|f\| = \sup_{x \in X}\{|f(x)|\}$. *Suppose that there is a function* $h$ *such that its set* $D(h)$ *of points of discontinuity is c-dense in* $(X, \rho)$. *Then*

$$G = \{f \in F : \ D(f) \text{ is c-dense}\}$$

*is a dense residual* $G_\delta$ *set in* $(F, d)$, *where* $d(f, g) = \|f - g\|$.

**Proof.** Given a non-empty set $O$, we can show that

$$A(O) = \{f \in F : \ D(f) \cap O \text{ is of power } c\}$$

is a dense open set by using the methods in [50]. In fact, let $\{f_n\} \subseteq F \setminus A(O)$ be a convergent sequence. Then there is a function $f \in F$ such that $f_n \longrightarrow f$ uniformly. Let $\epsilon_n$ denote the set $D(f_n) \cap O$. Then $\epsilon_n$ is at most countable and so the union $\bigcup_{n=1}^{\infty} \epsilon_n$ is at most countable. We know that $f$ is continuous at each point $x \in O \setminus \bigcup_{n=1}^{\infty} \epsilon_n$, so $f \in F \setminus A(O)$. Hence $F \setminus A(O)$ is closed and $A(O)$ is open.

Now we show that $A(O)$ is dense in $F$. For every ball $B(f, \epsilon) \subseteq F$, if $f \in A(O)$ there is nothing to prove. We assume $f \in F \setminus A(O)$, then $f$ has at most countably many points of discontinuity in $O$. From the assumption there is a function $h \in F$ such that $h$ has a $c$-dense set of points of discontinuity in $O$. Let $M$ be a constant such that $|h(x)| \leq M$ for all $x \in X$ and set

$$g = f + \frac{\epsilon}{2M} h.$$

Then $g \in F$ is discontinuous in continuum many points of $O$ and

$$d(g, f) = d\left(f + \frac{\epsilon}{2M}h, f\right) = d\left(\frac{\epsilon}{2M}h, 0\right) < \epsilon.$$

Thus $g \in A(O) \bigcap B(f, \epsilon)$ and hence $A(O)$ is dense.

Let $\{x_i\}$ be a dense countable subset of $X$, then

$$G = \bigcap_{i=1}^{\infty} \bigcap_{m=1}^{\infty} A(B(x_i, 1/m))$$

is a dense $G_\delta$ set where $B(x_i, 1/m)$ is the open ball centered at $x_i$ and with radius $1/m$. Thus $G$ is a dense residual $G_\delta$ set in $F$. ■

**Corollary 4.6.3** *The typical functions in $R[a, b]$, the space of Riemann integrable functions furnished with the supremum norm, have c-dense sets of points of discontinuity.*

**Proof.** The set of points of discontinuity of any bounded symmetrically continuous function is Lebesgue measure zero [55, pp. 27, Theorem 2.3]. Thus such a function is Riemann integrable by the well known fact that a bounded Lebesgue measurable function is Riemann integrable iff its set of points of discontinuity is Lebesgue measure zero [11]. Tran in [57] has constructed a symmetrically continuous function whose set of points of discontinuity is $c$-dense. Hence the result follows. ■

The following theorem gives a similar form of Theorem 4.6.2 in the sense of prevalence.

**Theorem 4.6.4** *Let $(X, \rho)$ be a separable metric space, $\mu$ be a $\sigma$-finite Borel measure that is non-zero on every open set in $(X, \rho)$. Let $F$ be a complete metric linear space of bounded functions $f : X \to \mathbb{R}$ furnished with the supremum norm $\|f\| = \sup_{x \in X}\{|f(x)|\}$. Suppose that in $F$ there exists a function $h$ such that its set of points of discontinuity is c-dense in $(X, \rho)$. Then the prevalent function $f \in F$ has a c-dense set of points of discontinuity.*

**Proof.** Let

$$G = \{f \in F : D(f) \text{ is } c\text{-dense in } X\}.$$

By Theorem 4.6.2 the set $G$ and its complement are Borel sets. We need only show that for every $f \in F$ the following set

$$S = \left\{ \lambda \in \mathbb{R} : \begin{array}{l} f + \lambda h \text{ is discontinuous at most countably many} \\ \text{points in some non-empty open set } O_\lambda \subseteq (X, \rho) \end{array} \right\}.$$

is a Lebesgue measure zero set. For every $\lambda \in S$ there exists a non-empty open set $O_\lambda \subseteq (X, \rho)$ such that $f + \lambda h$ is discontinuous at most countably many points in $O_\lambda$. If there are two distinct $\lambda_1$ and $\lambda_2$ such that $\mu(O_{\lambda_1} \cap O_{\lambda_2}) > 0$, then both $f + \lambda_1 h$ and $f + \lambda_2 h$ have at most countably many points of discontinuity in the non-empty open set $O_{\lambda_1} \cap O_{\lambda_2}$. Therefore $(\lambda_1 - \lambda_2)h$ has at most countably many points of discontinuity on such non-empty open set. This contradicts the property of function $h$. Hence for distinct $\lambda_1$ and $\lambda_2$ their corresponding non-empty open sets $O_{\lambda_1}$ and $O_{\lambda_2}$ satisfy $\mu(O_{\lambda_1} \cap O_{\lambda_2}) = 0$. Since $\mu$ is $\sigma$-finite on $(X, \rho)$ then $X = \bigcup_1^\infty X_i$, where $\mu(X_i) < \infty$ and $X_i \cap X_j = \emptyset$, $i \neq j$. Let

$$S_{nm} = \left\{ \lambda \in S : \mu(O_\lambda \cap X_m) \geq \frac{1}{n} \right\}$$

Then $S_{nm}$ is finite. If not, there exist countably many $\lambda_i$ in $S_{nm}$ such that

$$\infty = \sum_{i=1}^{\infty} \mu(O_{\lambda_i} \cap X_m) = \mu\left( X_m \bigcap \left( \bigcup_{i=1}^{\infty} O_{\lambda_i} \right) \right) \leq \mu(X_m) < \infty.$$

This is a contradiction. Thus $S_{nm}$ is finite and $S = \bigcup_{n,m=1}^{\infty} S_{nm}$ is at most countable. So the set $S$ is a Lebesgue measure zero set. The span of $h$ is an one-dimensional probe for the set $G$ and the result follows. ■

**Corollary 4.6.5** *The prevalent function $f \in BSC_n[a,b]$, the space of bounded $n$-th symmetrically continuous functions furnished with the supremum norm, has a $c$-dense set of points of discontinuity.*

**Proof.** The space $BSC_n[a,b]$ is a complete metric space (see [33]). By Tran's results [57] there exist functions $h_1$ and $h_2$ in $BSC_1[a,b]$ and $BSC_2[a,b]$ respectively such that $h_1$ and $h_2$ have $c$-dense sets of points of discontinuity on $[a,b]$. Also note that $BSC_1[a,b] = BSC[a,b] \subseteq BSC_{2k-1}[a,b]$ and $BSC_2[a,b] \subseteq BSC_{2k}[a,b]$ (see [33] for details). Thus the result follows from Theorem 4.6.4. ∎

Applying Theorem 4.6.4 and Tran's results [57] we obtain immediately a prevalent property in the space $R[a,b]$, which is also a typical property in $R[a,b]$ as shown in Corollary 4.6.3.

**Corollary 4.6.6** *The prevalent function $f \in R[a,b]$ has a c-dense set of points of discontinuity.*

In the following theorem we display another prevalent property in the space $BSC[a,b]$.

For $X \subseteq \mathbb{R}$ a function $f : X \to \mathbb{R}$ is called *countably continuous* if there is a countable cover $\{X_n : n \in \mathbb{N}\}$ of $X$ (by arbitrary sets) such that each restriction $f|X_n$ is continuous.

By constructing an example of a non-countably continuous function Krzysztof Ciesielski [15] answered a question of Lee Larson: whether every symmetrically continuous function is countably continuous. We will show that non-countable continuity is both a typical property and a prevalent property in the space $BSC[a,b]$.

**Theorem 4.6.7** *Let $F = \{f \in BSC[a,b] : f \text{ is countably continuous}\}$. Then $F$ is a closed, nowhere dense, linear subspace of $BSC[a,b]$.*

**Proof.** First we show that $F$ is a linear space. For $f_1, f_2 \in F$, there are countable covers $\{X_n^1 : n \in \mathbb{N}\}$ and $\{X_n^2 : n \in \mathbb{N}\}$ of $[a,b]$ such that the restrictions $f_i|X_n^i$ are continuous. It is easy to see that

$$[a,b] \subseteq \bigcup_{i,j}(X_i^1 \cap X_j^2).$$

So $\{X_i^1 \cap X_j^2 : i,j \in \mathbb{N}\}$ is a countable cover of $[a,b]$. Also the restrictions $f|X_i^1 \cap X_j^2$ are continuous. Thus for any $\alpha, \beta \in \mathbb{R}$, $\alpha f_1 + \beta f_2 \in F$.

We now show that $F$ is closed. Let $\{f_n\} \subseteq F$ be a convergent sequence. Then there is a function $f \in BSC[a,b]$ such that $f_n \to f$ uniformly. For each $f_n$ there exists a countable cover $\{X_i^n\}$ of $[a,b]$ such that $f_n|X_i^n$ is continuous. Since $[a,b] \subseteq \bigcup_{i=1}^{\infty} X_i^n$, so

$$[a,b] \subseteq \bigcap_{n=1}^{\infty}(\bigcup_{i=1}^{\infty} X_i^n) = \bigcup_{i=1}^{\infty}(\bigcap_{n=1}^{\infty} X_i^n).$$

Note that for any $m$ the restriction $f_m|\bigcap_{n=1}^{\infty} X_i^n$ is continuous. Thus the restriction $f|\bigcap_{n=1}^{\infty} X_i^n$ is continuous since $f$ is the uniform limit of $f_n$. Therefore $f \in F$ and $F$ is closed.

We now show that $F$ is nowhere dense in $BSC[a,b]$. For any ball $B(f,\epsilon)$, if $f \notin F$ there is nothing to prove. If $f \in F$, then $f$ is countably continuous. Choose a function $g \in BSC[a,b]$ that is not countably continuous (see Ciesielski's construction in [15]). Then $f + (g/M)\epsilon/2 \in B(f,\epsilon)$ where $M = \sup_{x\in[a,b]} |g(x)|$. Obviously, $f + (g/M)\epsilon/2$ is also not countably continuous. If not, $f + (g/M)\epsilon/2 - f = (g/M)\epsilon/2$ is countably continuous since $F$ is linear. Since $F$ is closed, it is nowhere dense. The result follows.

■

**Corollary 4.6.8** *Both the prevalent function $f \in BSC[a,b]$ and the typical function are not countably continuous.*

**Proof.** In [15] Krzysztof Ciesielski constructed a function $f \in BSC[a,b]$ that is not countably continuous. Thus the set $F$ in Theorem 4.6.7 is a closed, nowhere dense, proper and linear subspace of $BSC[a,b]$. Therefore the result follows. ■

## 4.7 Multiplicatively shy sets in $C[0,1]$

In the space $C[0,1]$ of continuous functions on $[0,1]$ there is another algebraic operation of importance — multiplication. $C[0,1]$ is a Banach algebra with multiplication $fg$ of elements $f, g \in C[0,1]$ defined in the pointwise sense. Under this operation $C[0,1]$ is a Polish semigroup with unit. (The unit is the function $f(x) \equiv 1$.) This allows for two distinct notions of shy sets.

Let $S \subseteq C[0,1]$ be universally measurable. Then $S$ is said to be an *additively shy set* if there is a Borel probability measure $\mu$ so that

$$\mu(S+f) = 0 \qquad (\forall f \in C[0,1]).$$

Also $S$ is said to be a *multiplicatively shy set* if there is a Borel probability measure containing the unit element in its support so that

$$\mu(fS) = 0 \qquad (\forall f \in C[0,1]).$$

In this section we propose to investigate this definition rather briefly. The ideas arise from purely formal considerations and may or may not be useful in applications.

**Theorem 4.7.1** *Let $M$ denote the functions in $C[0,1]$ that are somewhere monotonic. Then $M$ is both additively shy and multiplicatively shy.*

**Proof.** We have already showed that $M$ is additively shy (see Example 3.3.5). We now show that $M$ is multiplicatively shy. For $I \subseteq [0,1]$ we use $M(I)$ to denote the set of functions in $C[0,1]$ that are monotonic on $I$. Take a function $g$ nowhere differentiable with $0 < g(x) \leq 1$. Write

$$F(t) = g(x)^t \qquad (0 \leq t \leq 1).$$

Define a Borel probability measure $\mu$ by

$$\mu(X) = \lambda_1(\{t \in [0,1] : F(t) \in X\}).$$

Then the support of $\mu$ contains the unit element $f(x) \equiv 1$. In fact, for every ball $B(1, \epsilon)$, there exist $0 < \delta < 1$ and $0 < \eta < 1$ such that $g(x) \geq \delta$ for all $x \in [0, 1]$ and $1 - \epsilon < \delta^t \leq 1$ for all $t > \eta$. Thus $\mu(B(1, \epsilon)) \geq 1 - \eta > 0$ and hence $1 \in \text{supp}\mu$. For any $f \in C[0, 1]$, consider the set

$$T = \{t \in [0, 1] : F(t) \in fM(I)\}.$$

We claim that $T$ contains at most one element. In fact, if not, there are $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and $m_1, m_2 \in M(I)$ such that

$$g(x)^{t_1} = fm_1 \text{ and } g(x)^{t_2} = fm_2.$$

Thus

$$g(x)^{t_2 - t_1} = \frac{m_2}{m_1}.$$

This is impossible since the left side of the equality is nowhere differentiable on $I$ but the right side is almost differentiable on $I$. Thus $\lambda_1(T) = 0$ and $M(I)$ is shy. Note

$$M = \bigcup_I M(I)$$

where the union is taken over all subintervals of $[0, 1]$ with rational endpoints. Therefore $M$ is multiplicatively shy. ■

**Theorem 4.7.2** *Let $Z$ be the set of continuous functions on $[0, 1]$ with at least one zero. Then $Z$ is not additively shy, but is multiplicatively shy in $C[0, 1]$.*

**Proof.** It is easy to see that $Z$ is closed. In fact, for any Cauchy sequence $\{f_n\} \subseteq Z$ there exists a function $f \in C[0, 1]$ such that $f_n \to f$ uniformly. For each $n$, there exists $x_n \in [0, 1]$ such that $f_n(x_n) = 0$. Thus there exists a subsequence $x_{n_j} \to x_0 \in [0, 1]$. Let $j \to \infty$ in $f_{n_j}(x_{n_j}) = 0$ we have $f(x_0) = 0$. Therefore $f \in Z$ and $Z$ is closed. Now we show the assertion in two steps.

(i). Write $F : [1,2] \to C[0,1]$ by $F(t) = t$. Then for any $f \in C[0,1]$,

$$\{t \in [1,2] : \ F(t) \in fZ\} = \emptyset.$$

So we can construct a Borel probability measure $\mu$ with support containing the unit by

$$\mu(X) = \lambda_1(\{t \in [1,2] : \ F(t) \in X\}).$$

Obviously $\mu$ is transverse to the set $Z$. Thus $Z$ is multiplicatively shy.

(ii). We show that $Z$ is not additively shy. Suppose $Z$ were additively shy. Note that every translation of $Z$ is also additively shy, and

$$C[0,1] = \bigcup_{r \in \mathbb{Q}} (Z + r)$$

where $\mathbb{Q}$ is the set of rational numbers in $\mathbb{R}$. That is, $C[0,1]$ is the countable union of countably many additively shy sets. This is a contradiction. Thus $Z$ is not additively shy. ■

Both the set $Z$ in the above theorem and $C[0,1] \setminus Z$ are upper density 1 under the additive operation (see Section 2.15 for the definition of upper density 1). In fact, for every $\epsilon > 0$ and every Borel probability measure $\mu$, there exists a compact set $K \subseteq C[0,1]$ such that $\mu(K) > 1 - \epsilon$ since $\mu$ is tight on $C[0,1]$. Choose functions $g, h \in C[0,1]$ satisfying

$$g(0) = 1 + \max_{f \in K} \max_{x \in [0,1]} |f(x)|, \ g(1) = -1 - \max_{f \in K} \max_{x \in [0,1]} |f(x)|$$

and

$$h(x) \equiv 1 + \max_{f \in K} \max_{x \in [0,1]} |f(x)|$$

on $[0,1]$. Then

$$g + K \subseteq Z \text{ and } h + K \subseteq C[0,1] \setminus Z.$$

Thus

$$K \subseteq Z - g \text{ and } K \subseteq C[0,1] \setminus Z - h.$$

Therefore both $Z$ and $C[0,1] \setminus Z$ are upper density 1.

**Corollary 4.7.3** *The set $S_c$ in Theorem 4.3.1 is also multiplicatively shy.*

**Proof.** For any function $f \in S_c$, $f - C$ has at least one zero point. By Theorem 4.7.2 $S_c - C$ is multiplicatively shy, and so is $S_c$. ■

As we have seen, the non-invertible elements in the multiplicative semigroup $C[0,1]$ generate some curious results as regards multiplicatively shy sets. One, possibly better, way to investigate these notions would be to restrict our attention to the set $C^+[0,1]$ of strictly positive valued functions in $C[0,1]$..

**Lemma 4.7.4** *With the operation of multiplication $C^+[0,1]$ becomes an Abelian Polish group.*

**Proof.** We only need show that $C^+[0,1]$ is Polish. On it we impose an invariant metric $d$ so that

$$d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)| + \max_{x\in[0,1]} \left| \frac{1}{f(x)} - \frac{1}{g(x)} \right|.$$

Then the space $C^+[0,1]$ is complete under the metric $d$. In fact, for any Cauchy sequence $\{f_n\} \subseteq C^+[0,1]$ and a fixed point $x \in [0,1]$, $\{f_n(x)\}$ and $\{1/f_n(x)\}$ are Cauchy sequences in the uniform metric. So the set $\{1/f_n(x)\}$ is bounded and there exists a function $f(x) > 0$ such that

$$f_n(x) \to f(x) \text{ and } \frac{1}{f_n(x)} \to \frac{1}{f(x)}.$$

Therefore a function $f$ is generated such that $f_n \to f$ and $1/f_n \to 1/f$ pointwise. Note $\{f_n\}$ is a Cauchy sequence under the metric $d$. So $f_n \to f$ and $1/f_n \to 1/f$

uniformly. Thus $f$ is continuous and $f \in C^+[0,1]$. The separability of $C^+[0,1]$ is clear and hence $C^+[0,1]$ is Polish. ■

The shy sets in the space $C^+[0,1]$ could be called again multiplicatively shy sets.

**Theorem 4.7.5** *A set $S \subseteq C^+[0,1]$ is multiplicatively shy in $C^+[0,1]$ if and only if*

$$\ln S = \{\ln f : f \in S\}$$

*is additively shy in $C[0,1]$.*

**Proof.** Note that the function $f(x) = \ln x$ is a one-one, continuous mapping from $C^+[0,1]$ onto $C[0,1]$ and satisfies

$$\ln(g_1 g_2) = \ln g_1 + \ln g_2 \qquad (\forall g_1, g_2 \in C^+[0,1]).$$

Thus by Theorem 2.10.2 the result follows. ■

**Corollary 4.7.6** *The prevalent function $f \in C^+[0,1]$ is nowhere differentiable.*

**Proof.** Let $S$ denote the set of functions in $C[0,1]$ that are differentiable at some point of $[0,1]$ and let $S^+$ denote the set of functions in $C^+[0,1]$ that are differentiable at some point of $[0,1]$. Then $S = \ln S^+$. Since the set $S$ is universally measurable and shy in $C[0,1]$ (see Section 4.2), by Theorem 4.7.5, $S^+$ is multiplicatively shy. ■

*Remark.* We can construct a mapping $F : [0,1] \times [0,1] \to C^+[0,1]$ by

$$F(t_1, t_2) = [\exp(g(x))]^{t_1} [\exp(h(x))]^{t_2}$$

where $g(x)$ and $h(x)$ are as in Section 4.2. The Borel probability measure $\mu$ defined by

$$\mu(X) = \lambda_2(\{(t_1, t_2) \in [0,1] \times [0,1] : F(t_1, t_2) \in X\}),$$

with support containing the unit, is transverse to $S^+$.

Similarly, we have the following results.

**Corollary 4.7.7** *The prevalent function $f \in C^+[0,1]$ is nowhere of monotonic type.*

**Corollary 4.7.8** *The set* $\exp Z$ *is not multiplicatively shy in $C^+[0,1]$ where $Z$ is as in Theorem 4.7.2.*

# Chapter 5

# Space of automorphisms

## 5.1 Introduction

The space $\mathcal{H}[0,1]$ is defined as the set of all homeomorphisms $h : [0,1] \to [0,1]$ that fix $h(0) = 0$ and $h(1) = 1$. Note that these are exactly the strictly increasing continuous functions leaving the endpoints fixed. This is a subspace of the complete metric space $C[0,1]$. We shall show that $\mathcal{H}[0,1]$ admits a complete metric. There are discussions on the typical properties of functions in the complete metric space $\mathcal{H}[0,1]$ in [11], for example.

We can impose some algebraic structure on $\mathcal{H}[0,1]$ in several ways. The most natural way is to consider the group operation, defined as composition of functions. In this chapter we shall study some kinds of prevalence and give some examples of non-shy sets by using the arguments we developed in Chapter 2 and Chapter 3.

For a different measure-theoretic study in the space $\mathcal{H}[0,1]$, see Graf, Mauldin and Williams [25] and [26]. We follow ideas of Mauldin and Ulam (see [58]), who have defined a probability measure $P$ on $\mathcal{H}[0,1]$ that is "natural" from a probabilistic point of view. Roughly, to generate a homeomorphism $h \in \mathcal{H}[0,1]$ randomly with respect to the uniform distribution over $[0,1]$ one chooses $h(1/2)$ from $(0,1)$ at random with respect to the uniform distribution, then one chooses $h(1/4)$ from $(0, h(1/2))$ and $h(3/4)$ from $(h(1/2), 1)$ at random again with respect to the uniform distribution. This continues defining $h$ on all dyadic rational numbers. With probability 1, $h$ is strictly increasing and uniformly continuous on dyadic rational numbers and so defines a member $h \in \mathcal{H}[0,1]$. A measure $P$ is defined on $\mathcal{H}[0,1]$ to reflect these notions (see [25] for details). In this chapter that say the measure $P$ means such a measure. It has the property that the expected values for $h(t)$ is $t$, that is,

$$\int_{\mathcal{H}[0,1]} h(t) dP(h) = t.$$

A number of useful properties are shown in [25] and [26]. For example, $P$ almost every $h \in \mathcal{H}[0,1]$ touches the line $y = x$ on the interval $(0,1)$. Note, however, that their study is of a different nature than our study here. The group structure plays no significant role in the notion of "prevalence" relative to the measure $P$. The measure $P$ is determined in a probabilistic manner and inside or outside composition by elements of $\mathcal{H}[0,1]$ changes that determination. Thus that study can only be used to contrast with our study here. In this chapter we use $P$ to denote such a measure. We find that some sets studied in [25] and [26] are very successful as examples to answer the following open problems.

Jan Mycielski [41] asked whether, in a non-Abelian Polish group, the existence of a Borel probability measure left transverse to a set $Y$ implies the set $Y$ is shy. We shall answer this problem negatively in $\mathcal{H}[0,1]$ by showing that every set $\{h \in \mathcal{H}[0,1] : h'(0) = \alpha\}$ $(\alpha > 0)$ is left-and-right shy without being shy.

Slawomir Solecki showed that a Polish group admitting an invariant metric satisfies the countable chain condition if and only if this Polish group is locally compact. The problem is left whether all non-locally compact, non-Abelian Polish groups or some of them do not satisfy the countable chain condition. $\mathcal{H}[0,1]$ is a non-locally compact, non-Abelian Polish group with no invariant metric that makes it complete. We shall show that this group does not satisfy the countable chain condition.

The organization of this chapter is as follows. In Section 5.2, we show that the group $\mathcal{H}[0,1]$ is a non-locally compact, non-Abelian Polish group without invariant metrics. In Section 5.3, 5.4, 5.5, 5.8, we use compact curves to construct Borel probability measures to show that certain sets are shy, or left shy, or right shy. In particular, we show that, in $\mathcal{H}[0,1]$, there exists

(1) a Borel probability measure $\mu$ that is both left transverse and right transverse to a Borel set $X$, but is not transverse to $X$.

(2) a Borel probability measure $\mu$ that is left transverse to a Borel set $X$, but is not right transverse to $X$.

In Section 5.6, we give a characterization of compact sets in $\mathcal{H}[0,1]$ and a compact set argument to show that a set is non-shy, or non-left shy, or non-right shy. In Section 5.7, 5.9, 5.10, 5.11, 5.12, we use the compact set argument of Section 5.6 to show that certain sets are non-shy, or non-left shy, or non-right shy. Specifically, we show that some sets of the first category are non-shy, some left-and-right shy sets are non-shy, and some typical properties are not prevalent. Finally, in Section 5.13, using the results in Section 5.12, we show that the non-Abelian Polish group $\mathcal{H}[0,1]$ does not satisfy the countable chain condition. In Section 5.14, several open problems are given.

## 5.2 Space of automorphisms

We shall first prove that $\mathcal{H}[0,1]$ is a Polish group that is neither Abelian nor locally compact.

**Lemma 5.2.1** *The space $\mathcal{H}[0,1]$ is a non-locally compact, non-Abelian Polish group.*

**Proof.** $\mathcal{H}[0, 1]$ is a $G_\delta$ set in a complete space (see [11, pp. 468]). Indeed, it is topologically complete with respect to the metric,

$$\sigma(g, h) = \rho(g, h) + \rho(g^{-1}, h^{-1}),$$

which is topologically equivalent to the uniform metric $\rho$. (This is mentioned in Oxtoby [43].) The composition of two functions in $\mathcal{H}[0, 1]$ will usually not commute. For example, $f(x) = \sin\left(\frac{\pi}{2}x\right)$ and $g(x) = x^2 \in \mathcal{H}[0, 1]$,

$$f \circ g(x) = \sin\left(\frac{\pi}{2}x^2\right) \text{ but } g \circ f(x) = \left(\sin\frac{\pi}{2}x\right)^2.$$

So $\mathcal{H}[0, 1]$ is not Abelian.

Now we show that the Polish group $\mathcal{H}[0, 1]$ is not locally compact. For any function $f \in \mathcal{H}[0, 1]$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $\sigma(f, g) < \epsilon$ for all $g \in \mathcal{H}[0, 1]$, $\rho(f, g) < \delta$. Thus we can find a closed rectangle contained in $B(f, \epsilon)$ such that its sides are parallel to axes and two of its vertices are on the graph of $f$. Let $a$ and $b$ be $x$-coordinates of the two vertices with $a < b$. Choose $a < c < b$ and define a sequence of functions $\{f_n\}$ for large $n$. Precisely, let $f_n(x) = f(x)$ when $0 \le x \le a$; $f_n(x) = f(x)$ when $b \le x \le 1$; then $f_n$ is the segment function connecting $(a, f(a))$ and $(c, f(b) - 1/n)$ when $a \le x \le c$; and $f_n$ is the segment function connecting $(c, f(b) - 1/n)$ and $(b, f(b))$ when $c \le x \le b$. Then $\{f_n\}$ and all its subsequences fail to converge in $\mathcal{H}[0, 1]$, and so $\mathcal{H}[0, 1]$ is not locally compact. It is clear that $\mathcal{H}[0, 1]$ is separable since it is a subspace of the space $C[0, 1]$. Thus $\mathcal{H}[0, 1]$ is Polish. ■

The sequence $\{f_n\} \subseteq B(f, \epsilon)$ in the above proof also fails to converge in the uniform metric. So the uniform metric $\rho$ does not give a complete metric for the set of all homeomorphisms. Also we can claim that the metric $\rho$ is not invariant. In fact, choose functions $x$, $x^2$, $\sin\left(\frac{\pi}{2}x\right) \in \mathcal{H}[0, 1]$ and let $f(x) = x^2$. It is easy to see that the function $x - \sin\left(\frac{\pi}{2}x\right)$ attains maximum at $\theta_0$ on $[0, 1]$ where $0 < \theta_0 < 1$ and

$\cos\left(\frac{\pi}{2}\theta_0\right) = 2/\pi$. By computation $\theta_0 \approx 0.560664$. Note

$$f(x) - f\left(\sin\frac{\pi}{2}x\right) = \left(x - \sin\frac{\pi}{2}x\right)\left(x + \sin\frac{\pi}{2}x\right) \quad \text{and} \quad \theta_0 + \sin\frac{\pi}{2}\theta_0 \approx 1.331842.$$

Thus

$$\rho\left(x, \sin\frac{\pi}{2}x\right) < \rho\left(f(x), f\left(\sin\frac{\pi}{2}x\right)\right).$$

Therefore the uniform metric $\rho$ is not (left) invariant. It is, trivially, right invariant since for any $f, g, h \in \mathcal{H}[0,1]$,

$$\max_{0\leq x\leq 1} |f(h(x)) - g(h(x))| = \max_{0\leq x\leq 1} |f(x) - g(x)|,$$

that is $\rho(f\circ h, g\circ h) = \rho(f,g)$.

One might ask whether a different equivalent metric on $\mathcal{H}[0,1]$ might be found that is invariant. Our next theorem is cited in Christensen [12] where it is attributed to Dieudonné, who mentioned it in [17] as an example but did not give a detailed proof. It follows from this theorem that there is no invariant metric on this space, since it is known that an invariant metric on a Polish group must make it complete (see [31, (2.3)]).

**Theorem 5.2.2** *There is no left invariant metric on the Polish group $\mathcal{H}[0,1]$ that makes the group complete.*

**Proof.** We know that every metrizable group must have a left (right) invariant metric (see [30, pp. 58]). For a Polish group, if it has an invariant metric then it is complete in this metric (see [31, (2.3)]).

Suppose that $\mathcal{H}[0,1]$ is complete in a left invariant metric $d$. Now consider a sequence $\{u_n\}$:

$$u_n(x) = \begin{cases} \frac{2}{n}x, & 0 \leq x \leq \frac{1}{2} \\ 1 + (2 - \frac{2}{n})(x-1), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then $u_n \in \mathcal{H}[0,1]$ and

$$u_n^{-1}(x) = \begin{cases} \frac{n}{2}x, & 0 \le x \le \frac{1}{n} \\ \frac{1}{2-\frac{2}{n}}(x-1)+1, & \frac{1}{n} \le x \le 1. \end{cases}$$

Since the uniform metric $\rho$ is topologically equivalent to the metric $\sigma$ as in Lemma 5.2.1, by Theorem 9.24 in [11, pp. 388], for every $u \in \mathcal{H}[0,1]$ and $\epsilon > 0$ there is a $\delta > 0$ such that for all $v \in \mathcal{H}[0,1]$,

$$\rho(u,v) < \delta \Rightarrow d(u,v) < \epsilon \text{ and } d(u,v) < \delta \Rightarrow \rho(u,v) < \epsilon.$$

Since $\rho$ is right invariant,

$$\rho(u_n, u_m) = \rho(u_n u_m^{-1}, x) = \left| \frac{1}{n} - \frac{1}{m} \right|.$$

Then there exists a $N > 0$ such that $\rho(u_n u_m^{-1}, x) < \delta$ if $n, m > N$. Thus $d(u_n u_m^{-1}, x) < \epsilon$ if $n, m > N$, and so $\{u_n\}$ is a Cauchy sequence in the $d$ metric but it cannot converge in the $d$ metric. In fact, if $\{u_n\}$ converges to an element $u \in \mathcal{H}[0,1]$, then $\{u_n^{-1}\}$ would converge to $u^{-1}$ in the $d$ metric since the $d$ is left invariant. Thus again by Theorem 9.24 in [11, pp. 388], $\{u_n^{-1}\}$ would converge to $u^{-1}$ in $\rho$ metric (uniformly). But $\{u_n^{-1}\}$ converges pointwise to the function

$$u_0^{-1}(x) = \begin{cases} 0, & x = 0 \\ \frac{x+1}{2}, & 0 < x \le 1 \end{cases}$$

which is discontinuous at $x = 0$. This is a contradiction. Thus there is no left invariant metric on $\mathcal{H}[0,1]$ that makes it complete. The result follows. ■

## 5.3 Some examples of prevalent properties in $\mathcal{H}[0,1]$

In this section we will give some prevalence results by considering the following probability measure. Define a mapping $F : [1/2, 1] \to \mathcal{H}[0,1]$ by $F(t) = x^t$. It is easy to see that $F$ is continuous, and so $F([1/2,1])$ is a compact set. As in Section 2.2 we define a probability measure $\mu$ by

$$\mu(X) = 2\lambda_1(\{t \in [1/2,1] : F(t) \in X\}). \tag{5.1}$$

We use this example of a measure to show that a measure may be left transverse to a set without being right transverse, and that a measure may be left-and-right transverse without being transverse.

We begin with a simple theorem in order to contrast it with the more surprising results that follow.

**Theorem 5.3.1** *For $a, b \in (0,1)$, let $S_{a,b}$ denote the set of functions $h \in \mathcal{H}[0,1]$ such that $h(a) = b$. Then the probability measure $\mu$ defined by 5.1 is transverse to $S_{a,b}$.*

**Proof.** It is easy to see that the set $S_{a,b}$ is closed and nowhere dense in $\mathcal{H}[0,1]$. In particular $S_{a,b}$ is a Borel set.

Now for any $g, h \in \mathcal{H}[0,1]$, consider the set

$$T = \{t \in [1/2,1] : g \circ F(t) \circ h \in S_{a,b}\}.$$

If $t \in T$, then on $I$, $g(h^t(a)) = b$ and hence $h^t(a) = g^{-1}(b)$ where $g^{-1}$ is the inverse function of $g$. So the set $T$ can contain at most one element and hence the probability measure $\mu$ is transverse to $S_{a,b}$. ■

**Corollary 5.3.2** *The set $S_{a,b}$ is closed, nowhere dense, and shy.*

In particular, from this theorem we see that for a given function $f$ the set of elements $h \in \mathcal{H}[0,1]$ such that $h(x) = f(x)$ for all $x$ in some subinterval of $[0,1]$ must be both first category and shy. The reason is that this set can be expressed as the countable union of the sets $S_{a,f(a)}$ in the theorem taken over all rational numbers $a$.

In contrast to this simple result we shall show how a Borel probability measure may be left transverse or right transverse or left-and-right transverse and yet not be transverse to a set. We start with an elementary example that illustrates the distinctness of left/right transverse notions in $\mathcal{H}[0,1]$.

**Theorem 5.3.3** *Given an interval $I$ with the closure $\widetilde{I} \subseteq (0,1)$, let $G(I)$ denote the set of functions in the space $\mathcal{H}[0,1]$ that are linear on $I$. Then the probability measure $\mu$ defined by 5.1 is left-and-right transverse to $G(I)$, but is not transverse to $G(I)$.*

**Proof.** First we show that $G(I)$ is a Borel set. It is easy to see that

$$G(I) = \{f \in \mathcal{H}[0,1] : f \text{ is linear and slop} f > 0 \text{ on } I\}$$

where slop$f$ is the slope of $f$ on $I$. Let

$$F_n = \left\{f \in \mathcal{H}[0,1] : f \text{ is linear and slop} f \geq \frac{1}{n} \text{ on } I\right\}.$$

Then

$$G(I) = \bigcup_{n=1}^{\infty} F_n.$$

We will show that $F_n$ is closed. For any Cauchy sequence $\{f_k\} \subseteq F_n$ there exists $f \in \mathcal{H}[0,1]$ such that $f_k \to f$ uniformly. On $I$ we denote $f_k(x) = \alpha_k x + \beta_k$ where $\alpha_k \geq 1/n$. Choose two distinct points $x_1, x_2 \in I \setminus \{0\}$. Then

$$f_k(x_1) - f_k(x_2) = (\alpha_k x_1 + \beta_k) - (\alpha_k x_2 + \beta_k) = \alpha_k(x_1 - x_2) \to f(x_1) - f(x_2).$$

So $\alpha_k \to \alpha \in \mathbb{R}$ and therefore $\beta_k \to \beta \in \mathbb{R}$. Then $\alpha_k x + \beta_k \to \alpha x + \beta = f(x)$ on $I$. Since all $\alpha_k \geq 1/n$ so $\alpha \geq 1/n$ and $f \in F_n$. Thus each $F_n$ is closed and the set $G(I)$ is a Borel set. Further, we can show that $F_n$ is nowhere dense and hence $G(I)$ is of the first category. In fact, since the complete metric $\sigma$ on $\mathcal{H}[0,1]$ is equivalent to the uniform metric $\rho$, for any non-empty open ball $B(f,\epsilon) \subseteq \mathcal{H}[0,1]$, there exists a $\delta > 0$ such that for any $h \in \mathcal{H}[0,1]$, $\rho(f,h) < \delta$ we have $\sigma(f,h) < \epsilon$. It is easy to construct a non-linear function $g \in \mathcal{H}[0,1]$ such that $\rho(f,g) < \delta$. Then $g \in B(f,\epsilon)$ but $g \notin F_n$. So $F_n$ is nowhere dense and $G(I)$ is of the first category.

For any $g \in \mathcal{H}[0,1]$, we use the Borel probability measure $\mu$ defined by 5.1 and consider the set

$$T_1 = \{t \in [1/2, 1] : \ g \circ F(t) \in G(I)\}.$$

If $t \in T_1$ then on $I$,

$$g(x^t) = \alpha(t)x + \beta(t)$$

where $\alpha(t) > 0$. Let $y = x^t$, then $g(y) = \alpha(t)y^{1/t} + \beta(t)$ for $y$ on a subinterval of $[0,1]$. Suppose $\lambda_1(T_1) > 0$. Then the functions $y = x^t$ $(t \in T_1)$ map $I$ into uncountably many subintervals of $[0,1]$. Thus there must exist $t_1, t_2 \in T_1$, $t_1 < t_2$ such that for $y$ on some interval $J$,

$$\alpha(t_1)y^{\frac{1}{t_1}} + \beta(t_1) = \alpha(t_2)y^{\frac{1}{t_2}} + \beta(t_2).$$

Then by differentiating both sides of the equality above with respect to $y \in J$ we have

$$y^{\frac{1}{t_1} - \frac{1}{t_2}} = \frac{t_1\alpha(t_2)}{t_2\alpha(t_1)}.$$

This is impossible for all $y \in J$. So $\lambda_1(T_1) = 0$ and hence $\mu$ is left transverse to $G(I)$.

We now show that $\mu$ is also right transverse to $G(I)$. For any $h \in \mathcal{H}[0,1]$, consider the set

$$T_2 = \{t \in [1/2, 1] : \ F(t) \circ h \in G(I)\}.$$

If $t \in T_2$ then on $I$,

$$[h(x)]^t = \alpha(t)x + \beta(t)$$

where $\alpha(t) > 0$. Differentiating both sides with respect to $x$ on $I$ we have

$$t[h(x)]^{t-1}h'(x) = \alpha(t).$$

From the above equality we can know that the set $T_2$ has at most two elements. If not, there exist $t_1, t_2 \in T_2$, $t_1 < t_2$ such that on $I$

$$t_1[h(x)]^{t_1-1}h'(x) = \alpha(t_1) \text{ and } t_2[h(x)]^{t_2-1}h'(x) = \alpha(t_2).$$

Then

$$h(x)^{t_2-t_1} = \frac{t_1\alpha(t_2)}{t_2\alpha(t_1)}.$$

This is impossible since the function $h(x)$ is strictly increasing. Thus $\lambda_1(T_2) = 0$ and hence $\mu$ is right transverse to $G(I)$.

We now show that the probability measure $\mu$ is not transverse to $G(I)$. Choose $g, h \in \mathcal{H}[0,1]$ such that $g(x) = 1 + \alpha \ln x$ and $h(x) = e^{x-1}$ on $I$ where $\alpha$ is a positive constant depending on $I$. Then on $[1/2, 1]$ the function

$$g \circ F(t) \circ h(x) = g(h^t(x)) = \alpha t x + 1 - \alpha t$$

is linear for any $t \in [1/2, 1]$. Thus

$$\mu(g^{-1}G(I)h^{-1}) = 2\lambda_1(\{t \in [1/2, 1] : g \circ F(t) \circ h \in G(I)\}) = 1 \neq 0.$$

Therefore $\mu$ is not transverse to $G(I)$. ■

**Corollary 5.3.4** *The set of functions in the space $\mathcal{H}[0,1]$ that are somewhere linear (i.e., linear in at least one subinterval of $[0,1]$) is a left shy and right shy set, and is of the first category in $\mathcal{H}[0,1]$.*

**Proof.** For any interval $I$ with endpoints in $(0,1)$, $G(I)$ is left shy, right shy and first category from the above theorem. The set of functions in $\mathcal{H}[0,1]$ that are somewhere linear is the union of all $G(I)$ over all subintervals with rational endpoints in $(0,1)$, and so the result follows. ■

**Corollary 5.3.5** *The set of piecewise linear functions in $\mathcal{H}[0,1]$ is left-and-right shy, and first category.*

**Proof.** Let $S$ denote the set of piecewise linear functions in $\mathcal{H}[0,1]$. From Theorem 5.3.3 we know that the set $S$ is contained in a set that is left-and-right shy. So we need only show that $S$ is Borel. Let

$$S_{m,n,l,p} = \left\{ f \in \mathcal{H}[0,1] : \begin{array}{l} f(x) = \sum_{i=1}^{m} (\alpha_i x + \beta_i)\chi_{[a_i,b_i]}(x),\ 1/l \leq \alpha_i \leq l, \\ -p \leq \beta_i \leq p \text{ for any partition } \{[a_i,b_i]\}_{i=1}^{m} \text{ of } [0,1] \\ \text{with } \min_{1\leq i\leq m} |b_i - a_i| \geq 1/n \end{array} \right\}.$$

It can be checked that

$$S = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{l=1}^{\infty} \bigcup_{p=1}^{\infty} S_{m,n,l,p}.$$

We will show that the sets $S_{m,n,l,p}$ are closed and nowhere dense. For any Cauchy sequence $\{f_k\} \subseteq S_{m,n,l,p}$ there exists $f \in \mathcal{H}[0,1]$ such that $f_k \to f$ uniformly. Then there exist $\alpha_i^k$, $\beta_i^k$ and partitions $\{[a_i^k, b_i^k]\}_{i=1}^{m}$ of $[0,1]$ with $\min_{1\leq i\leq m} |b_i^k - a_i^k| \geq 1/n$ such that

$$f_k(x) = \sum_{i=1}^{m} (\alpha_i^k x + \beta_i^k)\chi_{[a_i^k,b_i^k]}(x)$$

and $1/l \leq \alpha_i^k \leq l$, $-p \leq \beta_i^k \leq p$, $i = 1, \cdots, m$. By standard arguments we can choose a subsequence $\{k_j\}$ of $\mathbb{N}$ such that

$$\alpha_i^{k_j} \to \alpha_i \in [1/l, l], \quad \beta_i^{k_j} \to \beta_i \in [-p, p]$$

and

$$a_i^{k_j} \to a_i \in [0,1], \quad b_i^{k_j} \to b_i \in [0,1].$$

It is easy to see that $\{[a_i, b_i]\}_{i=1}^{m}$ is a partition of $[0,1]$ and

$$f(x) = \sum_{i=1}^{m} (\alpha_i x + \beta_i)\chi_{[a_i,b_i]}(x) \in S_{m,n,l,p}.$$

Thus $S_{m,n,l,p}$ is closed. Similar arguments as for $F_n$ in Theorem 5.3.3 show that $S_{m,n,l,p}$ is nowhere dense. Thus $S$ is Borel and the result follows. ■

*Remark.* For any closed interval $I \subseteq (0,1)$ and any function $f \in \mathcal{H}[0,1]$, choose $g(x) = 1 + \alpha \ln x$ and $h(x) = f^{-1}(e^{x-1})$ on $I$ where $\alpha$ is a positive constant depending on $I$. Then

$$g \circ f^t \circ h(x) = g(f^t(h(x))) = \alpha t x + 1 - \alpha t$$

is linear for any $t \in [1/2, 1]$. So we cannot expect to choose some function $f \in \mathcal{H}[0, 1]$ such that the compact curve $F(t) = f^t$ ($t \in [1/2, 1]$) would be transverse to the set of somewhere linear functions in $\mathcal{H}[0, 1]$.

There still remains the following problem. The measure which proves that the sets $G(I)$ are left shy and right shy does not prove that they are shy. Some other probe would be needed. In fact, the existence of a left-and-right transverse measure does not imply the existence of a transverse measure as we shall show in Theorem 5.12.3.

**PROBLEM 11** *Let $S$ be the set of functions in the space $\mathcal{H}[0, 1]$ that are somewhere linear on $[0, 1]$. Is the set $S$ shy?*

## 5.4 Right transverse does not imply left transverse

In this section we will show that a Borel probability measure right transverse to a set need not be left transverse to the set $Y$. We will use again the compact curve $F(t)$ and the Borel probability measure $\mu$ defined by 5.1 to verify this in the following two theorems.

**Theorem 5.4.1** *Given an interval $I$ with the closure $\bar{I} \subseteq (0, 1)$, let $G(I)$ be the set of functions in the space $\mathcal{H}[0, 1]$ that are of the form $\alpha \ln x + \beta$ on $I$ where $\alpha$ ($> 0$), $\beta$ are constants depending on the corresponding functions. Then the probability measure $\mu$ defined by 5.1 is right transverse to $G(I)$ but is not left transverse to $G(I)$.*

**Proof.** Similar arguments as in the proof of Theorem 5.3.3 show that $G(I)$ is a Borel set and is of the first category. For any $h \in \mathcal{H}[0, 1]$ consider the set

$$T_h = \{t \in [1/2, 1] : F(t) \circ h \in G(I)\}.$$

We claim that $T_3$ contains at most one element. If not, there exist $t_1, t_2 \in T_3$, $t_1 < t_2$ such that $F(t_1) \circ h$, $F(t_2) \circ h \in G(I)$. Then on $I$

$$[h(x)]^{t_1} = \alpha(t_1) \ln x + \beta(t_1),$$

$$[h(x)]^{t_2} = \alpha(t_2) \ln x + \beta(t_2).$$

By differentiating both sides of the above two equalities with respect to $x$, it follows that on $I$,

$$[h(x)]^{t_2 - t_1} = \frac{t_1 \alpha(t_2)}{t_2 \alpha(t_1)}.$$

This is impossible since $h(x)$ is strictly increasing on $I$. So $\lambda_1(T_3) = 0$ and the probability measure $\mu$ defined by 5.1 is right transverse to $G(I)$.

We now show that $\mu$ is not left transverse to $G(I)$. Choose a function $g \in \mathcal{H}[0,1]$ such that on $I$, $g(x) = 1 + \alpha \ln x$. Then for any $t \in [1/2, 1]$,

$$g \circ F(t)(x) = g(x^t) = 1 + \alpha t \ln x.$$

Thus $g \circ F(t) \in G(I)$ and so

$$\mu(g^{-1}G(I)) = 2\lambda_1(\{t \in [1/2, 1] : \ g \circ F(t) \in G(I)\}) = 1 \neq 0.$$

Hence $\mu$ is not left transverse to $G(I)$. ■

Similar arguments as for Corollary 5.3.4 show the following corollary.

**Corollary 5.4.2** *The set of functions in the space $\mathcal{H}[0,1]$ that are somewhere of the form $\alpha \ln x + \beta$ is right shy and first category where $\alpha > 0$, $\beta$ are constants depending on the corresponding functions.*

If a Borel probability measure $\mu_1$ is right transverse but not left transverse to a set $S$ contained in a Polish group $G$, then the measure $\mu_2$, defined by $\mu_2(X) \equiv \mu_1(X^{-1})$, is left transverse but not right transverse to the set $S^{-1}$. Thus from Theorem 5.4.1 we can obtain a Borel probability measure that is left transverse to a set but not right transverse to this set. Here we supply a further concrete instance of a measure that is left transverse but not right transverse to a set.

**Theorem 5.4.3** *Given an interval $I$ with the closure $\bar{I} \subseteq (0,1)$ and an $\alpha > 0$, let $G(I)$ denote the set of functions in the space $\mathcal{H}[0,1]$ that are of the form $(1+\alpha \ln x)^\beta$ on $I$ where $\beta > 0$ is a constant depending on the corresponding functions. Then the probability measure $\mu$ defined by 5.1 is left transverse to $G(I)$ but not right transverse to $G(I)$.*

**Proof.** To show that $G(I)$ is a Borel set we modify part of the proof of Theorem 5.3.3. Note that

$$G(I) = \left\{ f \in \mathcal{H}[0,1] : \ f = (1+\alpha \ln x)^{\beta_f} \text{ on } I \right\}.$$

Let

$$F_n = \{ f \in G(I) : \ 1/n \leq \beta_f \leq n \}.$$

Then

$$G(I) = \bigcup_{n=1}^{\infty} F_n.$$

We will show that $F_n$ is closed and nowhere dense. That $F_n$ is nowhere dense in $\mathcal{H}[0,1]$ can be shown in a similar way as in Theorem 5.3.3. We now show that each $F_n$ is closed. For any Cauchy sequence $\{f_k\} \subseteq F_n$, there exists $f \in \mathcal{H}[0,1]$ such that $f_k \to f$ uniformly, and $1/n \leq \beta_{f_k} \leq n$. So we can find a subsequence $\{k_j\}$ of $\mathbb{N}$ such that $\beta_{f_{k_j}} \to \beta \in \mathbb{R}$. Thus $1/n \leq \beta \leq n$ and

$$f = (1+\alpha \ln x)^\beta$$

on $I$ and hence $f \in F_n$. Therefore $F_n$ is closed and it follows that the set $G(I)$ is Borel.

Now we show that the probability measure $\mu$ defined by 5.1 is left transverse to $G(I)$. For any $g \in \mathcal{H}[0,1]$ consider the set

$$T_4 = \{t \in [1/2, 1] : \ g \circ F(t) \in G(I)\}.$$

If $t \in T_4$, then on $I$,

$$g(x^t) = (1 + \alpha \ln x)^{\beta(t)}.$$

Let $y = x^t$ then

$$g(y) = \left(1 + \frac{\alpha}{t} \ln y\right)^{\beta(t)}.$$

Suppose $\lambda_1(T_4) \neq 0$. Then the functions $y = x^t$ $(t \in T_4)$ map $I$ into uncountably many intervals contained in $[0,1]$. Thus there exist $t_1, t_2 \in T_4$, $t_1 < t_2$ such that for $y$ on some interval $J \subseteq [0,1]$,

$$\left(1 + \frac{\alpha}{t_1} \ln y\right)^{\beta(t_1)} = \left(1 + \frac{\alpha}{t_2} \ln y\right)^{\beta(t_2)}.$$

Then on $J$,

$$\frac{\ln\left(1 + \frac{\alpha}{t_1} \ln y\right)}{\ln\left(1 + \frac{\alpha}{t_2} \ln y\right)} = \frac{\beta(t_2)}{\beta(t_1)}.$$

By differentiating both sides of the last equality with respect to $y$, it is easy to see that on $J$,

$$\frac{\ln\left(1 + \frac{\alpha}{t_1} \ln y\right)}{\ln\left(1 + \frac{\alpha}{t_2} \ln y\right)} = \frac{t_2 + \alpha \ln y}{t_1 + \alpha \ln y}.$$

From the last two equalities we have

$$\frac{t_2 + \alpha \ln y}{t_1 + \alpha \ln y} = \frac{\beta(t_2)}{\beta(t_1)}.$$

Again by differentiating both sides with respect to $y \in J$, it is easy to see that on $J$, $t_2 - t_1 = 0$. This contradicts that $t_2 > t_1$. Thus $\lambda_1(T_4) = 0$ and hence the probability $\mu$ is left transverse to $G(I)$.

We conclude by showing that the probability measure $\mu$ is not right transverse to $G(I)$. Choose $h \in \mathcal{H}[0,1]$ such that $h(x) = 1 + \alpha \ln x$ on $I$. Then for any $t \in [1/2, 1]$,

$$F(t) \circ h(x) = (1 + \alpha \ln x)^t.$$

So $F(t) \circ h \in G(I)$ and

$$\mu(G(I)h^{-1}) = 2\lambda_1(\{t \in [1/2, 1] : \ F(t) \circ h \in G(I)\}) = 1.$$

Hence $\mu$ is not right transverse to $G(I)$. ∎

Again similar arguments as for Corollary 5.3.4 show the following corollary.

**Corollary 5.4.4** *For any $\alpha > 0$, the set of functions in the space $\mathcal{H}[0,1]$ that are of the form $(1 + \alpha \ln x)^\beta$ somewhere is left shy and first category, where $\beta > 0$ is a constant depending the corresponding functions.*

## 5.5 A set to which the measure $\mu$ is left-and-right transverse

In the following let us look at one more example of left-and-right shy sets in $\mathcal{H}[0,1]$.

**Theorem 5.5.1** *Let $0 < \epsilon < M < \infty$, and*

$$S_{\epsilon,M} = \{h \in \mathcal{H}[0,1] : \ \epsilon|x-y| \leq |h(x) - h(y)| \leq M|x-y|, \ \forall x, y \in [0,1]\}.$$

*Then the Borel probability measure $\mu$ defined by 5.1 is left-and-right transverse to $S_{\epsilon,M}$.*

**Proof.** For any Cauchy sequence $\{f_n\} \subseteq S_{\epsilon,M}$ there is a function $f \in \mathcal{H}[0,1]$ such that $f_n \to f$ uniformly. Note that

$$\epsilon|x-y| \leq |f_n(x) - f_n(y)| \leq M|x-y|, \qquad \forall x, y \in [0,1].$$

Then

$$\epsilon|x-y| \leq |f(x)-f(y)| \leq M|x-y|, \qquad \forall x, y \in [0,1].$$

Therefore $f \in S_{\epsilon,M}$ and $S_{\epsilon,M}$ is closed. Since the complete metric $\sigma$ on $\mathcal{H}[0,1]$ is equivalent to the uniform metric, thus for any non-empty open ball $B(f,\epsilon) \subseteq \mathcal{H}[0,1]$, there exists a $\delta > 0$ such that for any $h \in \mathcal{H}[0,1]$, $\rho(f,h) < \delta$ we have $\sigma(f,h) < \epsilon$. So we can construct a function $g \in \mathcal{H}[0,1]$ such that $\rho(f,g) < \delta$ and $g(x) = \epsilon/2x$ for sufficiently small $x \in (0,1)$. Then $g \in B(f,\epsilon)$ but $g \notin S_{\epsilon,M}$. So $S_{\epsilon,M}$ is nowhere dense.

We now show that $S_{\epsilon,M}$ is left shy and right shy. For any $g \in \mathcal{H}[0,1]$, consider the set

$$T_1 = \{t \in [1/2, 1] : \ g \circ F(t) \in S_{\epsilon,M}\}.$$

If $t \in T_1$ then

$$\epsilon|x-y| \leq |g(x^t)-g(y^t)| \leq M|x-y|, \qquad \forall x, y \in [0,1].$$

Thus

$$\epsilon|z_1^{\frac{1}{t}} - z_2^{\frac{1}{t}}| \leq |g(z_1)-g(z_2)| \leq M|z_1^{\frac{1}{t}} - z_2^{\frac{1}{t}}|. \qquad \forall z_1, z_2 \in [0,1].$$

Since the function $g$ is differentiable almost everywhere, we have

$$\epsilon\frac{1}{t}z^{\frac{1}{t}-1} \leq g'(z) \leq M\frac{1}{t}z^{\frac{1}{t}-1}$$

for almost every $z \in [0,1]$. Suppose that $T_1$ contains two or more elements. Then there exist $t_1, t_2 \in [1/2, 1]$, $t_1 < t_2$ such that

$$\epsilon\frac{1}{t_1}z^{\frac{1}{t_1}-1} \leq g'(z) \leq M\frac{1}{t_1}z^{\frac{1}{t_1}-1}$$

and

$$\epsilon\frac{1}{t_2}z^{\frac{1}{t_2}-1} \leq g'(z) \leq M\frac{1}{t_2}z^{\frac{1}{t_2}-1}$$

for almost every $z \in [0,1]$. Thus

$$1 \geq \frac{\epsilon\frac{1}{t_2}z^{\frac{1}{t_2}-1}}{M\frac{1}{t_1}z^{\frac{1}{t_1}-1}} = \geq \frac{\epsilon t_1}{Mt_2}z^{\frac{1}{t_2}-\frac{1}{t_1}}$$

for almost every $z \in [0,1]$. Since $1/t_2 - 1/t_1 < 0$, the above inequality is impossible. Thus $T_1$ contains at most one element and hence $\lambda_1(T_1) = 0$. It follows that the Borel probability measure $\mu$ defined by 5.1 is left transverse to $S_{\epsilon,M}$.

We now show that $\mu$ is also right transverse to $S_{\epsilon,M}$. For any $g \in \mathcal{H}[0,1]$, consider the set

$$T_2 = \{t \in [1/2, 1] : \ F(t) \circ g \in S_{\epsilon,M}\}.$$

If $t \in T_2$, then

$$\epsilon|x - y| \leq |g^t(x) - g^t(y)| \leq M|x - y|.$$

Since $g$ is differentiable almost everywhere,

$$\epsilon \leq |tg^{t-1}(x)g'(x)| \leq M$$

for almost every $x \in [0,1]$. If $T_2$ did contain two or more elements, there exist $t_1, t_2 \in [0,1]$, $t_1 < t_2$ such that

$$\epsilon \leq |t_1 g^{t_1-1}(x)g'(x)| \leq M$$

and

$$\epsilon \leq |t_2 g^{t_2-1}(x)g'(x)| \leq M$$

for almost every $x \in [0,1]$. Thus

$$\frac{\epsilon}{M} \leq |\frac{t_1}{t_2} g^{t_1-t_2}(x)| \leq \frac{M}{\epsilon}$$

for almost every $x \in [0,1]$. This is impossible since $g(x) \to 0$ as $x \to 0$. So $\lambda_1(T_2) = 0$ and the probability measure $\mu$ is right transverse to $S_{\epsilon,M}$. By Theorem 2.9.7 the result follows. ■

**Corollary 5.5.2** *The set S*

$$\left\{f \in \mathcal{H}[0,1] : \ 0 < |\frac{f(x) - f(y)}{x - y}| < \infty, \ \forall x, y \in [0,1], \ x \neq y\right\}$$

*is left-and-right shy, and first category in* $\mathcal{H}[0,1]$.

**Proof.** Since the set $S = \bigcup_{n=1}^{\infty} S_{1/n,n}$, the result follows. ■

**PROBLEM 12** *Is the Borel probability measure $\mu$ defined by 5.1 transverse to the set $S_{\epsilon,M}$ in Theorem 5.5.1?*

## 5.6 A compact set argument

In this section we will give a characterization of compact sets in $\mathcal{H}[0,1]$. By applying such a characterization of compact sets we shall find examples of sets in $\mathcal{H}[0,1]$ that are neither left shy nor right shy. Some of these results contrast sharply with those in [25] and [26]. The following lemma contains a simple method that can be used to show the non-shyness of some sets.

**Lemma 5.6.1** *If a set $S$ in $\mathcal{H}[0,1]$ contains a two-sided translate (left, or right) of every compact set, then $S$ is not shy (left shy, or right shy).*

**Proof.** For any Borel probability measure $\mu$ there is a compact set $K \subseteq \mathcal{H}[0,1]$ such that $\mu(K) > 0$ (see Section 3.5). Since there functions $g, h \in \mathcal{H}[0,1]$ such that $g \circ k \circ h \in S$ for all $k \in K$, then $K \subseteq g^{-1}Sh^{-1}$ and so $\mu(g^{-1}Sh^{-1}) \geq \mu(K) > 0$. Thus $S$ is not shy. The non-left shy result and the non-right shy result can be shown in the same way. ■

We first give a characterization of compact sets in $\mathcal{H}[0,1]$ by applying the Arzelá-Ascoli theorem to the space $\mathcal{H}[0,1]$.

**Theorem 5.6.2** *A set $K \subseteq \mathcal{H}[0,1]$ is compact iff*

*(i) $K$ is closed,*

*(ii) $K$ is equicontinuous, and*

*(iii) for every non-empty closed set* $K_0 \subseteq K$,

$$\inf_{h \in K_0} h(x) \text{ and } \sup_{h \in K_0} h(x)$$

*are both in* $\mathcal{H}[0,1]$.

**Proof.** Let $K \subseteq \mathcal{H}[0,1]$ be a compact set. Since $\mathcal{H}[0,1]$ is Hausdorff, (i) follows. By Theorem 9.58 in [11] $K$ is totally bounded. Let $\epsilon > 0$, and let $f_1, \cdots, f_n$ be an $\epsilon/3$-net in $K$. For every $f \in K$, there exists $j \leq n$ such that

$$\sigma(f, f_j) = \rho(f, f_j) + \rho(f^{-1}, f_j^{-1}) < \frac{1}{3}\epsilon$$

where $\rho$ is the the uniform metric. Thus for any $x, y \in [0,1]$,

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)|$$
$$< \epsilon/3 + |f_j(x) - f_j(y)| + \epsilon/3.$$

Since each $f_j$ is uniformly continuous on $[0,1]$, there exists a $\delta > 0$ such that $|f_i(x) - f_i(y)| < \epsilon/3$ for all $1 \leq i \leq n$ if $|x - y| < \delta$. Therefore $|f(x) - f(y)| < \epsilon$, and hence (ii) is true.

We now show that (iii) is true. For any non-empty closed set $K_0 \subseteq K$, let

$$g(x) = \sup_{f \in K_0} f(x) \text{ and } h(x) = \inf_{f \in K_0} f(x).$$

It is easy to see that both $g$ and $h$ leave $x = 0, 1$ fixed. Since $K_0 \subseteq K$ is closed, so it is compact. For any $x_1, x_2 \in [0,1]$, $x_1 < x_2$, there exist $f_1, f_2 \in K_0$ such that $f_1(x_1) = g(x_1)$ and $f_2(x_2) = h(x_2)$. Since $f_1, f_2$ are strictly increasing,

$$g(x_1) = f_1(x_1) < f_1(x_2) \leq \sup_{f \in K_0} f(x_2) = g(x_2)$$

and

$$h(x_1) = \inf_{f \in K_0} f(x_1) \leq f_2(x_1) < f_2(x_2) = h(x_2).$$

Thus $g(x)$ and $h(x)$ are strictly increasing functions. Since $K$ is equicontinuous from (ii), for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in K$, $|f(x) - f(y)| < \epsilon/2$ if $|x - y| < \delta$. So for all $x, y \in [0, 1]$, $|x - y| < \delta$ and $f \in K_0$,

$$\begin{aligned} f(x) &\leq f(y) + |f(x) - f(y)| \\ &\leq \sup_{k \in K_0} k(y) + \sup_{k \in K_0} |k(x) - k(y)|. \end{aligned}$$

Thus

$$\sup_{k \in K_0} k(x) - \sup_{k \in K_0} k(y) \leq \sup_{k \in K_0} |k(x) - k(y)| \leq \frac{1}{2}\epsilon.$$

That is, $g(x) - g(y) < \epsilon$. Changing the positions of $x$ and $y$ yields $g(y) - g(x) < \epsilon$. Therefore $|g(x) - g(y)| < \epsilon$ and so $g$ is continuous. For the continuity of $h$, note that for all $x, y \in [0, 1]$, $|x - y| < \delta$ and $f \in K_0$,

$$f(x) \leq f(y) + \sup_{k \in K_0} |k(x) - k(y)| \leq f(y) + \frac{1}{2}\epsilon.$$

So

$$\inf_{k \in K_0} k(x) \leq f(y) + \frac{1}{2}\epsilon,$$

and hence by the arbitrariness of $f \in K_0$,

$$\inf_{k \in K_0} k(x) \leq \inf_{k \in K_0} k(y) + \frac{1}{2}\epsilon.$$

That is, $h(x) - h(y) \leq \frac{1}{2}\epsilon < \epsilon$. Changing the positions of $x$ and $y$ yields $h(y) - h(x) < \epsilon$. Thus $|h(x) - h(y)| < \epsilon$ and hence $h(x)$ is continuous. Therefore (iii) is true.

To show the sufficiency of the conditions (i), (ii) and (iii), let $\{f_n\}$ be any infinite sequence in $K$. Since $K$ is equicontinuous and uniformly bounded, $\{f_n\}$ has a uniformly convergent subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ by the Arzelá-Ascoli Theorem. We claim $f \in \mathcal{H}[0, 1]$ and the theorem is proved. Suppose $f \notin \mathcal{H}[0, 1]$. Then $f$ must be constant on some subinterval $[a, b] \subseteq [0, 1]$. Fix $c \in (a, b)$, and consider the set

$$P_1 = \{f_{n_k} : f_{n_k}(c) \leq f(c)\}.$$

This set is then closed. If it is non-empty and infinite, then for all $x \in (a, c)$,

$$\sup_{f_{n_k} \in P_1} f_{n_k}(x) = f(c),$$

which violates the condition (iii). Thus $P_1$ is finite. Similarly the set

$$P_2 = \{f_{n_k} : f_{n_k}(c) \geq f(c)\}$$

is also closed. If it is non-empty and infinite, then for all $x \in (c, b)$,

$$\inf_{f_{n_k} \in P_2} f_{n_k}(x) = f(c),$$

which also violates the condition (iii). Thus $P_2$ is finite. But both $P_1$ and $P_2$ cannot be finite, and so we have a contradiction. Thus $f \in \mathcal{H}[0, 1]$ and so $K$ is compact. ■

## 5.7 Examples of non-left shy, non-right shy sets

In this section we apply Theorem 5.6.2 to show that some sets discussed in [25] and [26] are neither left shy nor right shy.

**Theorem 5.7.1** *For any function $q(x) \in \mathcal{H}[0, 1]$, let*

$$G_1 = \left\{ h \in \mathcal{H}[0, 1] : \lim_{x \to 0^+} \frac{h(x)}{q(x)} = 0 \right\}.$$

*Then $G_1$ is a Borel set that is neither left shy nor right shy.*

**Proof.** We first show that $G_1$ is a Borel set. Note

$$\begin{aligned} G_1 &= \left\{ h \in \mathcal{H}[0, 1] : \lim_{n \to \infty} \frac{h(2^{-n})}{q(2^{-n})} = 0 \right\} \\ &= \bigcap_{p=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ h \in \mathcal{H}[0, 1] : \frac{h(2^{-n})}{q(2^{-n})} < \frac{1}{p} \right\}. \end{aligned}$$

Since all $h$ are continuous and the metric on $\mathcal{H}[0,1]$ is equivalent to the uniform metric, the sets

$$\left\{h \in \mathcal{H}[0,1] : \frac{h(2^{-n})}{q(2^{-n})} < \frac{1}{p}\right\}$$

are open. Hence $G_1$ is a Borel set.

We now show that $G_1$ is neither left shy nor right shy. For any compact set $K$, by Theorem 5.6.2 there exists a function $g \in \mathcal{H}[0,1]$ such that $k(x) \leq g(x)$ for all $k \in K$ and all $x \in [0,1]$. Choose

$$f(x) = q^2(g^{-1}(x)) \in \mathcal{H}[0,1].$$

Then for all $k \in K$,

$$f(k(x)) \leq f(g(x)) = q^2(g^{-1}(g(x))) = q^2(x).$$

Thus

$$\limsup_{x \to 0^+} \frac{f(k(x))}{q(x)} = 0,$$

and so $f \circ k \in G_1$. By Lemma 5.6.1, $G_1$ is not left shy. To show that $G_1$ is not right shy, choose

$$f(x) = g^{-1}(q^2(x)) \in \mathcal{H}[0,1].$$

Then for all $k \in K$,

$$k(f(x)) \leq g(f(x)) = g(g^{-1}(q^2(x))) = q^2(x).$$

Thus

$$\limsup_{x \to 0^+} \frac{k(f(x))}{q(x)} = 0,$$

and so $k \circ f \in G_1$. By Lemma 5.6.1, $G_1$ is not left shy. ∎

**Corollary 5.7.2** *Let*

$$S_1 = \{h \in \mathcal{H}[0,1] : h'(0) = 0\}.$$

*Then $S_1$ is neither left shy nor right shy.*

**Proof.** Let $q(x) = x$ in Theorem 5.7.1. Then $G_1 = S_1$ and the result follows. ■

**Theorem 5.7.3** *For any function $q(x) \in \mathcal{H}[0,1]$, let*

$$G_2 = \left\{ h \in \mathcal{H}[0,1] : \lim_{x \to 0^+} \frac{h(x)}{q(x)} = +\infty \right\}.$$

*Then $G_2$ is a Borel set that is neither left shy nor right shy.*

**Proof.** We first show that $G_2$ is a Borel set. Note

$$\begin{aligned} G_2 &= \left\{ h \in \mathcal{H}[0,1] : \lim_{n \to \infty} \frac{h(2^{-n})}{q(2^{-n})} = +\infty \right\} \\ &= \bigcap_{p=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ h \in \mathcal{H}[0,1] : \frac{h(2^{-n})}{q(2^{-n})} > p \right\}. \end{aligned}$$

Since all $h$ in $\mathcal{H}[0,1]$ are continuous and the metric on $\mathcal{H}[0,1]$ is equivalent to the uniform metric, so the sets

$$\left\{ h \in \mathcal{H}[0,1] : \frac{h(2^{-n})}{q(2^{-n})} > p \right\}$$

are open and hence $S_2$ is a Borel set.

We now show that $S_2$ is neither left shy nor right shy. For any compact set $K$, by Theorem 5.6.2, there exists a function $h \in \mathcal{H}[0,1]$ such that $k(x) \geq h(x)$ for all $k \in K$ and all $x \in [0,1]$. Choose

$$f(x) = q^{1/2}(h^{-1}(x)) \in \mathcal{H}[0,1].$$

Then for all $k \in K$,

$$f(k(x)) \geq f(h(x)) = q^{1/2}(h^{-1}(h(x))) = q^{1/2}(x).$$

Thus

$$\liminf_{x \to 0^+} \frac{f(k(x))}{q(x)} \geq \lim_{x \to 0^+} q^{-1/2}(x) = +\infty.$$

and so $f \circ k \in G_2$. By Lemma 5.6.1, $G_2$ is not left shy. To show $G_2$ is not right shy, choose

$$f(x) = h^{-1}(q^{1/2}(x)) \in \mathcal{H}[0,1].$$

Then for all $k \in K$,

$$k(f(x)) \geq h(f(x)) = h(h^{-1}(q^{1/2}(x))) = q^{1/2}(x).$$

Thus

$$\liminf_{x \to 0^+} \frac{k(f(x))}{q(x)} \geq \lim_{x \to 0^+} q^{-1/2}(x) = +\infty,$$

and so $k \circ f \in S_2$. By Lemma 5.6.1 again $S_2$ is not right shy. ■

**Corollary 5.7.4** *Let*

$$S_2 = \{h \in \mathcal{H}[0,1] : \ h'(0) = +\infty\}.$$

*Then $S_2$ is neither left shy nor right shy.*

**Proof.** Let $q(x) = x$ in Theorem 5.7.3. Then $G_2 = S_2$ and the result follows. ■

From Theorem 5.7.1, Theorem 5.7.3, Corollary 5.7.2 and Corollary 5.7.4 immediately we have the following result.

**Corollary 5.7.5** *The sets $G_1$, $G_2$, $S_1$ and $S_2$ in Theorem 5.7.1, Theorem 5.7.3, Corollary 5.7.2 and Corollary 5.7.4 are all neither shy nor prevalent.*

**Proof.** From Theorem 5.7.1 and Theorem 5.7.3, both $G_1$ and $G_2$ are neither left shy nor right shy. So they all are not shy. Since each one of both $G_1$ and $G_2$ is contained in the complement of the other, so both $G_1$ and $G_2$ are not prevalent. Similarly, we can show that both $S_1$ and $S_2$ are neither shy nor prevalent. ■

In [25] and [26], Graf, Mauldin and Williams showed that $P_a(S_1) = 1$ where $P_a$ is the right-average of the measure $P$ as in the introduction part of this chapter, that is

$$P_a(B) = \int_{\mathcal{H}[0,1]} P(\{h \in \mathcal{H}[0,1] : \ h \circ g^{-1} \in B\}) dP(g)$$

for Borel sets $B \in \mathcal{H}[0,1]$. In the following part of this chapter we always use $P_2$ to denote the right-average. However, we showed in Corollary 5.7.5 that $S_1$ is neither shy nor prevalent. From this point we know that both $P$ and $P_2$ are neither left transverse nor right transverse to the complement of $S_1$.

## 5.8 Examples of left-and-right shy sets

In order to compare with the results in last section we show here the following theorem.

**Theorem 5.8.1** *For any function $q(x) \in \mathcal{H}[0,1]$, let*

$$S = \left\{ h \in \mathcal{H}[0,1] : \ 0 < \liminf_{x \to 0^+} \frac{h(x)}{q(x)} \le \limsup_{x \to 0^+} \frac{h(x)}{q(x)} < +\infty \right\}.$$

*Then $S$ is a Borel set that is left-and-right shy in $\mathcal{H}[0,1]$.*

**Proof.** We first show that $S$ is a Borel set. Note

$$S = \bigcup_{p=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ h \in \mathcal{H}[0,1] : \ \frac{1}{p} < \frac{h(2^{-n})}{q(2^{-n})} < p \right\}.$$

Since all $h \in \mathcal{H}[0,1]$ are continuous and the metric on $\mathcal{H}[0,1]$ is equivalent to the uniform metric, all sets

$$\left\{ h \in \mathcal{H}[0,1] : \ \frac{1}{p} < \frac{h(2^{-n})}{q(2^{-n})} < p \right\}$$

are open, and so the set $S$ is a Borel set.

We now consider the set

$$S_{\alpha,\beta} = \left\{ h \in \mathcal{H}[0,1] : \ \alpha < \liminf_{x \to 0^+} \frac{h(x)}{q(x)} \le \limsup_{x \to 0^+} \frac{h(x)}{q(x)} < \beta \right\}$$

for all $\alpha, \beta > 0$, $\alpha < \beta$. Once we show that the set $S_{\alpha,\beta}$ is left shy and right shy, then

$$S = \bigcup_{n=1}^{\infty} S_{1/n,n}$$

is also left shy and right shy. We now show that the Borel probability measure $\mu$ defined by 5.1 is right transverse to $S_{\alpha,\beta}$ for any $\alpha, \beta > 0$, $\alpha < \beta$. For any $h \in \mathcal{H}[0,1]$, consider the set

$$R = \{t \in [1/2, 1] : \; F(t) \circ h \in S_{\alpha,\beta}\}.$$

If $t_1 \in R$, then for sufficiently small $x \in [0,1]$,

$$\alpha < \frac{h^{t_1}(x)}{q(x)} < \beta.$$

This means that, for any $t_2 \neq t_1$,

$$\frac{h^{t_2}(x)}{q(x)} = \frac{h^{t_1}(x)}{q(x)} h^{t_2-t_1}(x) \to 0 \text{ or } \infty$$

as $x \to 0^+$. Consequently $t_2 \notin R$. Thus $R$ contains at most one element and $\mu$ is right transverse to $S_{\alpha,\beta}$.

To prove that $S$ is left shy we choose a mapping $F_1 : [1/2, 1] \to \mathcal{H}[0,1]$ by $F_1(t) = q^t(x)$ and define a Borel probability measure $\mu_1$ by

$$\mu_1(X) = 2\lambda_1(\{t \in [1/2, 1] : \; F_1(t) \in X\}).$$

For any $h \in \mathcal{H}[0,1]$, consider the set

$$L = \{t \in [1/2, 1] : \; h \circ F(t) \in S_{\alpha,\beta}\}.$$

If $s \in L$, then for sufficiently small $x \in [0,1]$,

$$\alpha < \frac{h(q^s(x))}{q(x)} < \beta.$$

That is, $\alpha q(x) < h(q^s(x)) < \beta q(x)$ for sufficiently small $x \in (0,1)$. Let $y = q^s(x)$. Then

$$\alpha y^{\frac{1}{s}} \leq h(y) \leq \beta y^{\frac{1}{s}}$$

for sufficiently small $y \in (0,1)$. Suppose that $L$ contains two or more elements. Then there exist $s_1, s_2 \in [1/2, 1]$, $s_1 < s_2$ such that

$$\alpha y^{\frac{1}{s_1}} \leq h(y) \leq \beta y^{\frac{1}{s_1}}$$

and

$$\alpha y^{\frac{1}{s_2}} \le h(y) \le \beta y^{\frac{1}{s_2}}$$

for sufficiently small $y \in (0, 1)$. Thus

$$\frac{\alpha y^{\frac{1}{s_2}}}{\beta y^{\frac{1}{s_1}}} = \frac{\alpha}{\beta} y^{\frac{1}{s_2} - \frac{1}{s_1}} < 1$$

for sufficiently small $y \in (0, 1)$. This is impossible since $1/s_2 - 1/s_1 < 0$. Thus the set $L$ contains at most one element and hence the probability $\mu_1$ is left transverse to $S_{\gamma,\alpha}$. By Theorem 2.9.7 $S_{\gamma,\alpha}$ is left-and-right shy, and so the set $S$ is also left-and-right shy. ■

Let $q(x) = x$. Immediately we have the following.

**Corollary 5.8.2** *For any $\alpha$, $0 < \alpha < \infty$, the set*

$$\{h \in \mathcal{H}[0, 1] : \ h'(0) = \alpha\}$$

*is left-and-right shy in $\mathcal{H}[0, 1]$.*

This conclusion contrasts sharply with Corollary 5.7.2 and Corollary 5.7.4. In Section 5.12 we will see that the set $\{h \in \mathcal{H}[0, 1] : \ h'(0) = \alpha\}$ is not shy in $\mathcal{H}[0, 1]$.

As a contrast to the results in Theorem 5.7.1 and Theorem 5.7.3 we show that the sets $G_1$ in Theorem 5.7.1 and $G_2$ in Theorem 5.7.3 are of the first category in $\mathcal{H}[0, 1]$. It follows, too, that the sets $S_1$ in Corollary 5.7.2 and $S_2$ in Corollary 5.7.4 are also of the first category in $\mathcal{H}[0, 1]$.

**Theorem 5.8.3** *The sets $G_1$ in Theorem 5.7.1 and $G_2$ in Theorem 5.7.3 are of the first category in $\mathcal{H}[0, 1]$.*

**Proof.** Note that

$$G_1 = \bigcap_{p=3}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ h \in \mathcal{H}[0, 1] : \ \frac{h(2^{-n})}{q(2^{-n})} \le \frac{1}{p} \right\}.$$

The sets

$$\left\{h \in \mathcal{H}[0,1] : \frac{h(2^{-n})}{q(2^{-n})} \leq \frac{1}{p}\right\}$$

are closed and, hence, so is

$$N = \bigcap_{n=m}^{\infty} \{h \in \mathcal{H}[0,1] : \frac{h(2^{-n})}{q(2^{-n})} \leq \frac{1}{p}\}$$

for any $m \geq 1$. We claim that $N$ is nowhere dense in $\mathcal{H}[0,1]$. It follows that $G_1$ is first category. Since the complete metric $\sigma$ on $\mathcal{H}[0,1]$ is equivalent to the uniform metric, thus for any non-empty open ball $B(f,\epsilon) \subseteq \mathcal{H}[0,1]$, there exists a $\delta > 0$ such that for all $h \in \mathcal{H}[0,1]$, $\rho(f,h) < \delta$ we have $\sigma(f,h) < \epsilon$. If $f \notin N$ there is nothing to do. If $f \in N$, then we can construct a function $g \in \mathcal{H}[0,1]$ such that $\rho(f,g) < \delta$ and $g(x) = (2/p)q(x)$ for sufficiently small $x \in (0,1)$. Then $g \in B(f,\epsilon)$ but $g \notin N$. Thus $N$ is closed, nowhere dense.

For the set $G_2$, we know

$$G_2 = \bigcap_{p=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{h \in \mathcal{H}[0,1] : \frac{h(2^{-n})}{q(2^{-n})} \geq p\right\}.$$

Similar arguments as for $G_1$ show that, for any $m$, the set

$$\bigcap_{n=m}^{\infty} \left\{h \in \mathcal{H}[0,1] : \frac{h(2^{-n})}{q(2^{-n})} \geq p\right\}$$

is closed and nowhere dense in $\mathcal{H}[0,1]$. Therefore $G_2$ is of the first category. ∎

## 5.9 Non-shy sets that are of the first category

In the following theorems we will give further sharp contrasts between our measure-theoretic notion and category, and also between our results and results in [25] and [26]. The geometric pictures of the following sets are clear.

**Theorem 5.9.1** *For any function $q(x) \in \mathcal{H}[0,1]$, let*

$$S_> = \{h \in \mathcal{H}[0,1] : h(x) \geq q(x)\}$$

*and*

$$S_< = \{h \in \mathcal{H}[0,1] : h(x) \leq q(x)\}$$

*Then both $S_>$ and $S_<$ are closed, nowhere dense sets that are neither left shy nor right shy.*

**Proof.** It is easy to see that both $S_>$ and $S_<$ are closed. Since the complete metric $\sigma$ on $\mathcal{H}[0,1]$ is equivalent to the uniform metric $\rho$, for any non-empty open ball $B(f,\epsilon) \subseteq \mathcal{H}[0,1]$, there exists a $\delta > 0$ such that for all $h \in \mathcal{H}[0,1]$, $\rho(f,h) < \delta$ we have $\sigma(f,h) < \epsilon$. If $f \notin S_>$, there is nothing to prove. If $f \in S_>$, we can construct a function $g \in \mathcal{H}[0,1]$ such that $\rho(f,g) < \delta$ and $g(x) = q^2(x)$ for sufficiently small $x \in (0,1)$. Then $g \in B(f,\epsilon)$ but $g \notin S_>$. Thus $S_>$ is closed and nowhere dense in $\mathcal{H}[0,1]$. Similar arguments show that $S_<$ is also closed and nowhere dense in $\mathcal{H}[0,1]$.

We now show that both are neither left shy nor right shy. For any compact set $K$, by Theorem 5.6.2, there exist $g, h \in \mathcal{H}[0,1]$ such that $h(x) \leq k(x) \leq g(x)$ for all $k \in K$ and all $x \in [0,1]$. Choose functions

$$f_1(x) = q(h^{-1}(x)),\ f_2(x) = h^{-1}(q(x)),\ f_3(x) = q(g^{-1}(x))$$

and

$$f_4(x) = g^{-1}(q(x)) \in \mathcal{H}[0,1].$$

Then, for any $k \in K$ and all $x \in [0,1]$,

$$(f_1 \circ k)(x) = f_1(k(x)) \geq f_1(h(x)) = q(x),$$

$$(k \circ f_2)(x) = k(f_2(x)) \geq h(f_2(x)) = q(x),$$

$$(f_3 \circ k)(x) = f_3(k(x)) \leq f_3(g(x)) = q(x)$$

and

$$(k \circ f_4)(x) = k(f_4(x)) \leq g(f_4(x)) = q(x).$$

Thus $f_1 \circ k$, $k \circ f_2 \in S_>$ and $f_3 \circ k$, $k \circ f_4 \in S_<$. Therefore, by Lemma 5.6.1, both $S_>$ and $S_<$ are neither left shy nor right shy. ■

**Theorem 5.9.2** *For any $m \in (0,1]$, $l \in [1, +\infty)$ and $q(x) \in \mathcal{H}[0,1]$, the sets*

$$S_{m,>} = \{h \in \mathcal{H}[0,1] : \; h(x) \geq mq(x)\}$$

*and*

$$S_{l,<} = \{h \in \mathcal{H}[0,1] : \; h(x) \leq lq(x)\}$$

*are closed, nowhere dense sets that are neither shy nor prevalent.*

**Proof.** It is easy to see that all sets $S_{m,>}$ and $S_{l,<}$ are closed. Since the complete metric $\sigma$ on $\mathcal{H}[0,1]$ is equivalent to the uniform metric $\rho$, for any non-empty open ball $B(f,\epsilon) \subseteq \mathcal{H}[0,1]$, there exists a $\delta > 0$ such that for all $h \in \mathcal{H}[0,1]$, $\rho(f,h) < \delta$ we have $\sigma(f,h) < \epsilon$. If $f \notin S_{m,>}$, there is nothing to prove. If $f \in S_{m,>}$, we can construct a function $g \in \mathcal{H}[0,1]$ such that $\rho(f,g) < \delta$ and $g(x) = mq^2(x)$ for sufficiently small $x \in (0,1)$. Then $g \in B(f,\epsilon)$ but $g \notin S_{m,>}$. Thus $S_{m,>}$ is closed and nowhere dense in $\mathcal{H}[0,1]$. Similar arguments show that $S_{l,<}$ is also closed and nowhere dense in $\mathcal{H}[0,1]$.

We now show that $S_{m,>}$ and $S_{l,<}$ are neither shy nor prevalent. For $m = l = 1$, by Theorem 5.9.1, both $S_{1,>} = S_>$ and $S_{l,<} = S_<$ are neither left shy nor right shy. Since

$$S_> \cap S_< = \{h \in \mathcal{H}[0.1] : \; h(x) = q(x)\}$$

is a singleton and hence a shy set, so both $S_{1,>}$ and $S_{1,<}$ are neither shy nor prevalent. For $0 < m \leq 1$ and $1 \leq l < +\infty$, since $S_{m,>} \supseteq S_{1,>}$ and $S_{l,<} \supseteq S_{1,<}$, so $S_{m,>}$ and $S_{l,<}$ are not shy. On the other hand,

$$S_{m,>} \subseteq \mathcal{H}[0,1] \setminus G_1 \text{ and } S_{l,<} \subseteq \mathcal{H}[0,1] \setminus G_2$$

where $G_1$ and $G_2$ are same as in Theorem 5.7.1 and Theorem 5.7.3 respectively. By Corollary 5.7.5, $G_1$ and $G_2$ are not shy in $\mathcal{H}[0,1]$. Thus $S_{m,>}$ and $S_{l,<}$ are not prevalent in $\mathcal{H}[0,1]$. ■

In [26] it was shown that for every $m \in (0,1]$ and $l \in [1,+\infty)$,

$$P_a(\{h \in \mathcal{H}[0,1] : \ h(x) \geq mx\}) = 0$$

and

$$P_a(\{h \in \mathcal{H}[0,1] : \ h(x) \leq lx\}) = 0$$

where $P_a$ is the right-average of the measure $P$. However, according the above theorem, these two sets are neither shy nor prevalent and yet they are closed, nowhere dense sets in $\mathcal{H}[0,1]$.

## 5.10 Non-prevalent properties that are typical

In this section we give several non-prevalent properties that are typical. Again we use the sets discussed in [25] and [26] and compare our results with the corresponding results in [25] and [26].

**Theorem 5.10.1** *For any function $q(x) \in \mathcal{H}[0,1]$, let*

$$C_1 = \left\{h \in \mathcal{H}[0,1] : \ \liminf_{x\to 0^+} \frac{h(x)}{q(x)} = 0\right\}$$

*and*

$$C_2 = \left\{h \in \mathcal{H}[0,1] : \ \limsup_{x\to 0^+} \frac{h(x)}{q(x)} = +\infty\right\}.$$

*Then both $C_1$ and $C_2$ are neither shy nor prevalent.*

**Proof.** Note $C_1 \supseteq G_1$ and $C_2 \supseteq G_2$ where $G_1$ and $G_2$ are as in Theorem 5.7.1 and Theorem 5.7.3. By Corollary 5.7.5 both $G_1$ and $G_2$ are not shy, and so both $C_1$ and

$C_2$ are not shy. We now show that their complements $\widetilde{C_1}$ and $\widetilde{C_2}$ are also not shy. Let $Q$ be the set of rational numbers in $(0,1)$. Then

$$\widetilde{C_1} = \bigcup_{r\in Q}\{h \in \mathcal{H}[0,1] : \; h(x) \geq rq(x)\}$$

and

$$\widetilde{C_2} = \bigcup_{r\in Q}\{h \in \mathcal{H}[0,1] : \; h(x) \leq 1/rq(x)\}.$$

In fact, let $h \in \mathcal{H}[0,1]$ such that $\liminf_{x\to 0^+} \frac{h(x)}{q(x)} > c > 0$. Then there exists a $\delta > 0$ such that for all $x \in (0,\delta)$, $h(x) > cq(x)$. Let $r \in Q$ such that $r < \min(h(\delta), c)$. Then $h(x) \geq rq(x)$ for all $x \in [0,1]$. Thus

$$\widetilde{C_1} \subseteq \bigcup_{r\in Q}\{h \in \mathcal{H}[0,1] : \; h(x) \geq rq(x)\}.$$

The converse is obvious, and so the first equality of the above two holds. For the second equality, let $h \in \mathcal{H}[0,1]$ such that $\limsup_{x\to 0^+} \frac{h(x)}{q(x)} < M < +\infty$. Then there exists a $\delta > 0$ such that for all $x \in (0,\delta)$, $h(x) < Mq(x)$. Let $r \in Q$ such that $r < \min(1/M, q(\delta))$. Then $h(x) \leq 1/rq(x)$ for all $x \in [0,1]$. Thus

$$\widetilde{C_2} \subseteq \bigcup_{r\in Q}\{h \in \mathcal{H}[0,1] : \; h(x) \leq 1/rq(x)\}.$$

The converse is obvious and so the second equality holds. Thus $C_1$ and $C_2$ are Borel sets. By Theorem 5.9.2, $\widetilde{C_1}$ and $\widetilde{C_2}$ are not shy, and hence the results follow. ■

Let $q(x) = x$ in the above theorem. Then the sets

$$\left\{h \in \mathcal{H}[0,1] : \liminf_{x\to 0^+} \frac{h(x)}{x} = 0\right\} \text{ and } \left\{h \in \mathcal{H}[0,1] : \limsup_{x\to 0^+} \frac{h(x)}{x} = +\infty\right\}$$

are neither shy nor prevalent. By way of contrast, note that in [25] and [26] it was shown that these two sets have $P_a$ measures 1 (see [26, Theorem 5.10]). From the following theorem we see that these two sets are dense $G_\delta$ sets in $\mathcal{H}[0,1]$.

**Theorem 5.10.2** *The sets $C_1$ and $C_2$ in Theorem 5.10.1 are dense $G_\delta$ sets in $\mathcal{H}[0,1]$.*

**Proof.** From the proof of Theorem 5.10.1 we know

$$\widetilde{C_1} = \bigcup_{r \in Q} \{h \in \mathcal{H}[0,1] :\ h(x) \geq rq(x)\}$$

and

$$\widetilde{C_2} = \bigcup_{r \in Q} \{h \in \mathcal{H}[0,1] :\ h(x) \leq 1/rq(x)\}$$

where $Q$ is the set of rational numbers in $(0,1)$. By Theorem 5.9.2 we know that for any $r \in Q$, the sets

$$\{h \in \mathcal{H}[0,1] :\ h(x) \geq rq(x)\}$$

and

$$\{h \in \mathcal{H}[0,1] :\ h(x) \leq 1/rq(x)\}$$

are closed, nowhere dense sets. Thus $C_1$ and $C_2$ are dense $G_\delta$ sets in $\mathcal{H}[0,1]$. ■

**Theorem 5.10.3** *For any $q(x) \in \mathcal{H}[0,1]$, the typical function $h \in \mathcal{H}[0,1]$ satisfies*

$$\liminf_{x \to 0^+} \frac{h(x)}{q(x)} = 0 \text{ and } \limsup_{x \to 0^+} \frac{h(x)}{q(x)} = +\infty.$$

*Remark.* Although the sets $G_1$ and $G_2$ in Theorem 5.7.1 and Theorem 5.7.3 are also neither shy nor prevalent, that $C_1$ and $C_2$ in Theorem 5.10.1 are dense $G_\delta$ sets contrasts sharply with the fact that $G_1$ and $G_2$ are of the first category.

## 5.11 Corresponding results at $x = 1$

We now carry these results over to the right hand endpoint of the interval $[0,1]$. Since the group operation on $\mathcal{H}[0,1]$ is the composition of functions, we do not know whether the mapping $T : h \to \bar{h}$ by $\bar{h}(x) = 1 - h(1-x)$ changes the shyness of sets. We will leave it as an open problem. Here we will deal with some sets involving the behavior of functions at $x = 1$ individually.

**Theorem 5.11.1** *For any function $q(x) \in \mathcal{H}[0,1]$, let*

$$S_3 = \left\{ h \in \mathcal{H}[0,1] : \lim_{x \to 1^-} \frac{1-h(x)}{1-q(x)} = 0 \right\}$$

*and*

$$S_4 = \left\{ h \in \mathcal{H}[0,1] : \lim_{x \to 1^-} \frac{1-h(x)}{1-q(x)} = +\infty \right\}.$$

*Then both $S_3$ and $S_4$ are Borel sets that are neither left shy nor right shy in $\mathcal{H}[0,1]$.*

**Proof.** Again we show that both $S_3$ and $S_4$ are Borel sets. Note that

$$\begin{aligned} S_3 &= \left\{ h \in \mathcal{H}[0,1] : \lim_{n \to \infty} \frac{1-h(1-2^{-n})}{1-q(1-2^{-n})} = 0 \right\} \\ &= \bigcap_{p=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ h \in \mathcal{H}[0,1] : \frac{1-h(1-2^{-n})}{1-q(1-2^{-n})} < \frac{1}{p} \right\} \end{aligned}$$

and

$$\begin{aligned} S_4 &= \left\{ h \in \mathcal{H}[0,1] : \lim_{n \to \infty} \frac{1-h(1-2^{-n})}{1-q(1-2^{-n})} = +\infty \right\} \\ &= \bigcap_{p=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ h \in \mathcal{H}[0,1] : \frac{1-h(1-2^{-n})}{1-q(1-2^{-n})} > p \right\} \end{aligned}$$

Since all $h \in \mathcal{H}[0,1]$ are continuous and the metric on $\mathcal{H}[0,1]$ is equivalent to the uniform metric, then

$$\left\{ h \in \mathcal{H}[0,1] : \frac{1-h(1-2^{-n})}{1-q(1-2^{-n})} < \frac{1}{p} \right\}$$

and

$$\left\{ h \in \mathcal{H}[0,1] : \frac{1-h(1-2^{-n})}{1-q(1-2^{-n})} > p \right\}$$

are all open sets. Thus both $S_3$ and $S_4$ are Borel sets.

We now show that both $S_3$ and $S_4$ are neither left shy nor right shy. By Theorem 5.6.2, for any compact set $K$, there exist functions $g, h \in \mathcal{H}[0,1]$ such that $h(x) \leq k(x) \leq g(x)$ for all $k \in K$ and all $x \in [0,1]$.

For the set $S_3$, choose

$$f_1(x) = 1 - (1 - q(h^{-1}(x)))^2, \quad f_2(x) = h^{-1}(1 - (1 - q(x))^2).$$

Then for all $k \in K$,

$$f_1(k(x)) \geq f_1(h(x)) = 1 - (1 - q(x))^2$$

and

$$k(f_2(x)) \geq h(f_2(x)) = 1 - (1 - q(x))^2.$$

So

$$\frac{1 - f_1(k(x))}{1 - q(x)} \leq \frac{(1 - q(x))^2}{1 - q(x)} = 1 - q(x)$$

and

$$\frac{1 - k(f_2(x))}{1 - q(x)} \leq 1 - q(x).$$

Hence

$$\limsup_{x \to 1^-} \frac{1 - f_1(k(x))}{1 - q(x)} = \limsup_{x \to 1^-} \frac{1 - k(f_2(x))}{1 - q(x)} = 0.$$

Thus $f_1 \circ k$, $k \circ f_2 \in S_3$. By Lemma 5.6.1 $S_3$ is neither left shy nor right shy.

For the set $S_4$, choose

$$f_1(x) = 1 - (1 - q(g^{-1}(x)))^{1/2} \text{ and } f_2(x) = g^{-1}(1 - (1 - q(x))^{1/2}).$$

Then for all $k \in K$,

$$f_1(k(x)) \leq f_1(g(x)) = 1 - (1 - q(x))^{1/2},$$

$$k(f_2(x)) \leq g(f_2(x)) = 1 - (1 - q(x))^{1/2}.$$

So

$$\frac{1 - f_1(k(x))}{1 - q(x)} \geq \frac{1 - [1 - (1 - q(x))^{1/2}]}{1 - q(x)} = (1 - q(x))^{-\frac{1}{2}}$$

and

$$\frac{1 - k(f_2(x))}{1 - q(x)} \geq (1 - g(x))^{-\frac{1}{2}}.$$

Thus

$$\liminf_{x\to 1^-} \frac{1-f_1(k(x))}{1-q(x)} = \liminf_{x\to 1^-} \frac{1-k(f_2(x))}{1-q(x)} = +\infty.$$

Hence $f_1 \circ k$, $k \circ f_2 \in S_4$. By Lemma 5.6.1 $S_4$ is neither left shy nor right shy. ■

Using similar arguments as in Corollary 5.7.5 we obtain the following.

**Corollary 5.11.2** *The sets $S_3$ and $S_4$ in Theorem 5.11.1 are neither shy nor prevalent.*

Similarly as for Theorem 5.8.1 we show the following theorem. Note that the transverse curves are different.

**Theorem 5.11.3** *For any function $q(x) \in \mathcal{H}[0,1]$, let*

$$G = \left\{ h \in \mathcal{H}[0,1] :\ 0 < \liminf_{x\to 1^-} \frac{1-h(x)}{1-q(x)} \le \limsup_{x\to 1^-} \frac{1-h(x)}{1-q(x)} < +\infty \right\}.$$

*Then $G$ is a Borel set that is left-and-right shy in $\mathcal{H}[0,1]$.*

**Proof.** Similarly as for $S$ in Theorem 5.8.1 we can show that the set $G$ is a Borel set. We now show that the set

$$S_{\alpha,\beta} = \left\{ h \in \mathcal{H}[0,1] :\ \alpha < \liminf_{x\to 1^-} \frac{1-h(x)}{1-q(x)} \le \limsup_{x\to 1^-} \frac{1-h(x)}{1-q(x)} < \beta \right\}$$

is left shy and right shy for any $\alpha, \beta > 0$, $\alpha < \beta$.

Write $F_2 : [1/2, 1] \to \mathcal{H}[0,1]$ by

$$F_2(t) = 1 - (1 - q(x))^t$$

and define a Borel probability measure $\mu_2$ by

$$\mu_2(X) = 2\lambda_1(\{t \in [1/2, 1] :\ F_2(t) \in X\}).$$

For any $h \in \mathcal{H}[0,1]$, consider the following set:

$$L = \{t \in [1/2, 1] :\ h \circ F_2(t) \in S_{\alpha,\beta}\}.$$

If $t \in L$, then

$$\alpha < \frac{1 - h(1 - (1 - q(x))^t)}{1 - q(x)} < \beta$$

for all $x < 1$ sufficiently close to 1. Set $y = 1 - (1 - q(x))^t$. Then

$$\alpha(1 - y)^{\frac{1}{t}} < 1 - h(y) < \beta(1 - y)^{\frac{1}{t}}$$

for all $y < 1$ sufficiently close to 1. We claim that $L$ contains at most one element. If not, there exist $t_1, t_2 \in L$, $t_1 < t_2$ such that

$$\alpha(1 - y)^{\frac{1}{t_1}} < 1 - h(y) < \beta(1 - y)^{\frac{1}{t_1}}$$

and

$$\alpha(1 - y)^{\frac{1}{t_2}} < 1 - h(y) < \beta(1 - y)^{\frac{1}{t_2}}$$

for all $y < 1$ sufficiently close to 1. Thus

$$1 \leq \frac{\beta(1 - y)^{\frac{1}{t_1}}}{\alpha(1 - y)^{\frac{1}{t_2}}} = \frac{\beta}{\alpha}(1 - y)^{\frac{1}{t_1} - \frac{1}{t_2}}$$

for all $y < 1$ sufficiently close to 1, which is impossible since $1/t_1 - 1/t_2 > 0$. Thus $L$ is a singleton or empty set. So the set $S_{\alpha,\beta}$ is left shy.

To prove $S_{\alpha,\beta}$ is right shy, we choose $F_3 : [1/2, 1] \to \mathcal{H}[0, 1]$ by

$$F_3(t) = 1 - (1 - x)^t$$

and define a Borel probability measure $\mu_3$ by

$$\mu_3(X) = 2\lambda_1(\{t \in [1/2, 1] : F_3(t) \in X\}).$$

For any $h \in \mathcal{H}[0, 1]$, consider the set

$$R = \{t \in [1/2, 1] : F_3(t) \circ h \in S_{\alpha,\beta}\}.$$

If $t \in R$, then

$$\alpha < \frac{1 - [1 - (1 - h(x))^t]}{1 - q(x)} < \beta$$

for all $x < 1$ sufficiently close to 1. That is,

$$\alpha^{\frac{1}{t}}(1-q(x))^{\frac{1}{t}} < 1-h(x) < \beta^{\frac{1}{t}}(1-q(x))^{\frac{1}{t}}$$

for all $x < 1$ sufficiently close to 1. We claim that $R$ contains at most one element. If not, there exist $t_1, t_2 \in R$, $t_1 < t_2$ such that

$$\alpha^{\frac{1}{t_1}}(1-q(x))^{\frac{1}{t_1}} < 1-h(x) < \beta^{\frac{1}{t_1}}(1-q(x))^{\frac{1}{t_1}}$$

and

$$\alpha^{\frac{1}{t_2}}(1-q(x))^{\frac{1}{t_2}} < 1-h(x) < \beta^{\frac{1}{t_2}}(1-q(x))^{\frac{1}{t_2}}$$

for all $x < 1$ sufficiently close to 1. Thus

$$1 \leq \frac{\beta^{\frac{1}{t_1}}(1-q(x))^{\frac{1}{t_1}}}{\beta^{\frac{1}{t_2}}(1-q(x))^{\frac{1}{t_2}}} = \frac{\beta^{\frac{1}{t_1}}}{\alpha^{\frac{1}{t_2}}}(1-q(x))^{\frac{1}{t_1}-\frac{1}{t_2}}$$

for all $x < 1$ sufficiently close to 1, which is impossible since $1/t_1 - 1/t_2 > 0$. Thus $R$ is a singleton or empty set, and hence $S_{\alpha,\beta}$ is right shy. By Theorem 2.9.7, the set $S_{\alpha,\beta}$ is left-and-right shy in $\mathcal{H}[0,1]$. Thus

$$G = \bigcup_{n=1}^{\infty} S_{1/n,n}$$

is left-and-right shy. ■

Let $q(x) = x$. Immediately we have the following.

**Corollary 5.11.4** *For any $\alpha$, $0 < \alpha < +\infty$, the set*

$$\{h \in \mathcal{H}[0,1]: \ h'(1) = \alpha\}$$

*is left-and-right shy.*

**Theorem 5.11.5** *For any function $q(x) \in \mathcal{H}[0,1]$, let*

$$C_3 = \left\{h \in \mathcal{H}[0,1]: \ \liminf_{x\to 1^-} \frac{1-h(x)}{1-q(x)} = 0\right\}$$

*and*

$$C_4 = \left\{ h \in \mathcal{H}[0,1] : \limsup_{x \to 1^-} \frac{1 - h(x)}{1 - q(x)} = +\infty \right\}.$$

*Then both $C_3$ and $C_4$ are neither shy nor prevalent.*

**Proof.** Since $C_3 \supseteq S_3$ and $C_4 \supseteq S_4$ where $S_3$ and $S_4$ are as in Theorem 5.11.1, by Theorem 5.11.1, both $C_3$ and $C_4$ are not shy. Similarly as for Theorem 5.10.1 we can show

$$\widetilde{C_3} = \bigcup_{r \in Q} \{ h \in \mathcal{H}[0,1] : 1 - h(x) \geq r(1 - q(x)) \}$$

and

$$\widetilde{C_4} = \bigcup_{r \in Q} \{ h \in \mathcal{H}[0,1] : 1 - h(x) \leq 1/r(1 - q(x)) \}.$$

Thus $C_3$ and $C_4$ are Borel sets. Note

$$\begin{aligned} \widetilde{C_3} &\supseteq \{ h \in \mathcal{H}[0,1] : 1 - h(x) \geq 1 - q(x) \} \\ &= \{ h \in \mathcal{H}[0,1] : h(x) \leq q(x) \} \end{aligned}$$

and

$$\begin{aligned} \widetilde{C_4} &\supseteq \{ h \in \mathcal{H}[0,1] : 1 - h(x) \leq 1 - q(x) \} \\ &= \{ h \in \mathcal{H}[0,1] : h(x) \geq q(x) \}. \end{aligned}$$

By Theorem 5.10.1, $\widetilde{C_3}$ and $\widetilde{C_4}$ are not shy and hence $C_3$ and $C_4$ are not prevalent. ■

In contrast, in [25] and [26], it was shown that both $C_3$ and $C_4$ have $P_2$ measures 1 when $q(x) = x$. Similar arguments as for Theorem 5.8.3 show that both the sets $S_3$ and $S_4$ in Theorem 5.11.1 are of the first category. Also similar arguments as for Theorem 5.9.2 show that for any rational number $r \in (0,1)$, the sets

$$\{ h \in \mathcal{H}[0,1] : 1 - h(x) \geq r(1 - q(x)) \}$$

and

$$\{h \in \mathcal{H}[0,1]: \ 1 - h(x) \leq 1/r(1 - q(x))\}$$

are closed, nowhere dense sets in $\mathcal{H}[0,1]$. Thus we can have the following.

**Theorem 5.11.6** *For any $q(x) \in \mathcal{H}[0,1]$, the typical function $h \in \mathcal{H}[0,1]$ satisfies*

$$\liminf_{x \to 1^{-1}} \frac{1 - h(x)}{1 - q(x)} = 0 \text{ and } \limsup_{x \to 1^-} \frac{1 - h(x)}{1 - q(x)} = +\infty.$$

**PROBLEM 13** *Let $T : \mathcal{H}[0,1] \to \mathcal{H}[0,1]$ by $Th(x) = 1 - h(1-x)$, is it true that $S \subseteq \mathcal{H}[0,1]$ is shy or left/right shy iff $T(S)$ is shy or left/right shy?*

## 5.12 A non-shy set that is left-and-right shy

Jan Mycielski in [41] posed a problem, denoted by $(P_0)$, whether the existence of a Borel probability measure left transverse to a set $Y$ implies that $Y$ is shy in a non-locally compact, completely metrizable group. In this section we give examples of non-shy sets that are left-and-right shy in $\mathcal{H}[0,1]$ and so answer the problem $(P_0)$ negatively. These examples also allow us to conclude immediately that the $\sigma$-ideal of shy sets in $\mathcal{H}[0,1]$ does not satisfy the countable chain condition.

Corollary 5.8.2 and Corollary 5.11.4 show that the sets

$$\{h \in \mathcal{H}[0,1]: \ h'(0) = \alpha\}$$

and

$$\{h \in \mathcal{H}[0,1]: \ h'(1) = \alpha\}$$

are left-and-right shy sets. We shall show that these two sets are non-shy sets in $\mathcal{H}[0,1]$. The first two lemmas are basic facts about monotonic functions.

**Lemma 5.12.1** *For any $0 \leq \alpha \leq +\infty$, if $f \in \mathcal{H}[0,1]$ and satisfies*

$$\frac{f(c_n)}{c_n} \to \alpha$$

*for a decreasing sequence $\{c_n\}$ satisfying $c_n/c_{n+1} \to 1$ and $c_n \to 0^+$ as $n \to \infty$, then $f'(0) = \alpha$.*

**Proof.** For any $x < c_1$, there exists a $c_n$ such that $c_{n+1} \leq x \leq c_n$. Then $f(c_{n+1}) \leq f(x) \leq f(c_n)$ and so

$$\frac{f(c_{n+1})}{c_n} \leq \frac{f(x)}{x} \leq \frac{f(c_n)}{c_{n+1}}.$$

Note that

$$\frac{f(c_{n+1})}{c_n} = \frac{f(c_{n+1})}{c_{n+1}} \cdot \frac{c_{n+1}}{c_n}$$

and

$$\frac{f(c_n)}{c_{n+1}} = \frac{f(c_n)}{c_n} \cdot \frac{c_n}{c_{n+1}}.$$

Thus by the conditions we have

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \alpha.$$

That is, $f'(0) = \alpha$. ■

**Lemma 5.12.2** *For any $0 \leq \beta \leq +\infty$, if $f \in \mathcal{H}[0,1]$ and satisfies*

$$\frac{1 - f(d_n)}{1 - d_n} \to \beta$$

*for an increasing sequence $\{d_n\}$ satisfying $(1 - d_n)/(1 - d_{n+1}) \to 1$ and $d_n \to 1^-$ as $n \to \infty$, then $f'(1) = \beta$.*

**Proof.** For any $x > d_1$, there exists a $d_n$ such that $d_n \leq x \leq d_{n+1}$. Then $f(d_n) \leq f(x) \leq f(d_{n+1})$ and so

$$\frac{1 - f(d_{n+1})}{1 - d_n} \leq \frac{1 - f(x)}{1 - x} \leq \frac{1 - f(d_n)}{1 - d_{n+1}}.$$

Note that

$$\frac{1-f(d_{n+1})}{1-d_n} = \frac{1-f(d_{n+1})}{1-d_{n+1}} \cdot \frac{1-d_{n+1}}{1-d_n}$$

and

$$\frac{1-f(d_n)}{1-d_{n+1}} = \frac{1-f(d_n)}{1-d_n} \cdot \frac{1-d_n}{1-d_{n+1}}.$$

Thus from the conditions we have

$$\lim_{x\to 1^-} \frac{1-f(x)}{1-x} = \beta.$$

That is, $f'(1) = \beta$. ∎

**Theorem 5.12.3** *Let $0 \le \alpha,\ \beta \le +\infty$, and*

$$D_{\alpha,\beta} = \{h \in \mathcal{H}[0,1] :\ h'(0) = \alpha \text{ and } h'(1) = \beta\}.$$

*Then $D_{\alpha,\beta}$ is not shy.*

**Proof.** To show that $D_{\alpha,\beta}$ is not shy, we shall show that, for any compact set $K \subseteq \mathcal{H}[0,1]$, there exist functions $g, h \in \mathcal{H}[0,1]$ such that $g \circ h \circ k \in D_{\alpha,\beta}$ for any $h \in K$. First we choose a decreasing sequence $\{c_n\} \subseteq (0,1/4)$ such that all the intervals $[c_n, c_n + 1/nc_n]$ are pairwise disjoint, $c_n \to 0^+$ and $c_{n+1}/c_n \to 1$ as $n \to \infty$. Second we choose an increasing sequence $\{d_n\} \subseteq (3/4,1)$ such that all the intervals $[d_n, d_n + (1-d_n)^2 d_n]$ are contained in $(3/4,1)$, pairwise disjoint, $d_n \to 1$ and $(1-d_{n+1})/(1-d_n) \to 1$ as $n \to \infty$. For example, $c_n = 1/n$, $d_n = 1 - 1/n$ $(n > 5)$ satisfy the requirements. This can be verified by simple computations. By Theorem 5.6.2 there exist functions $f_1, f_2 \in \mathcal{H}[0,1]$ such that

$$f_1(x) \le h(x) \le f_2(x)$$

for all $h \in K$ and all $x \in [0,1]$. Now we choose a sequence of line segments $\{I_n\}$ contained in $(0,1/4) \times (0,1/2)$ and a sequence of line segments $\{J_n\} \subseteq (3/4,1) \times (1/2,1)$, as in the figure below, such that

(i) for any $n$, the lower end of $I_n$ is above the upper end of $I_{n+1}$, and the lower end of $J_{n+1}$ is above the upper end of $J_n$.

(ii) the corresponding point $x_n^1$ of $I_n$ tends to 0 from right, and the corresponding point $x_n^2$ of $J_n$ tends to 1 from left.

(iii) for any $n$, the line segment connecting $(x_n^1, f_1(x_n^1))$ and $(x_n^1, f_2(x_n^1))$ is contained in $I_n$, and the line segment connecting $(x_n^2, f_1(x_n^2))$ and $(x_n^2, f_2(x_n^2))$ is contained in $J_n$.

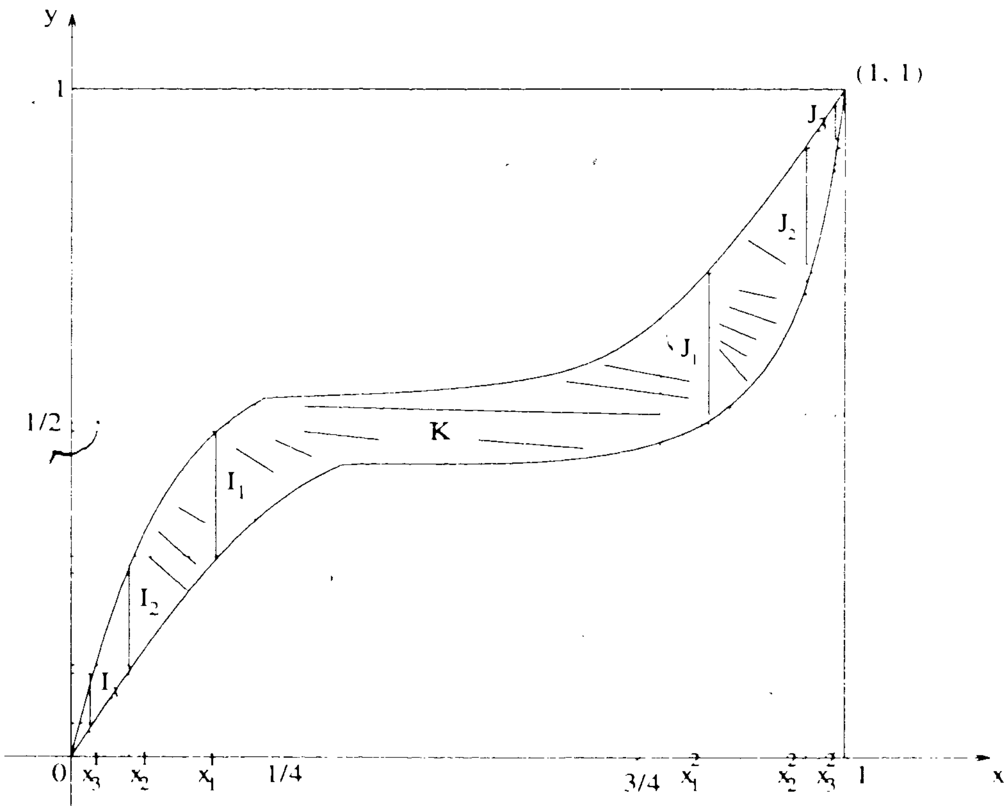


Figure 5.1: The construction of the automorphisms $g$ and $k$

We now construct functions $g, k \in \mathcal{H}[0,1]$ in nine cases, and show that $g \circ h \circ k \in D_{\alpha,\beta}$ for any $h \in K$ for each case.

Case 1: $0 < \alpha,\ \beta < +\infty$. We require that $4c_1\alpha < 1$ and $2\beta(1-d_1) < 1$ so that, for any $n$,

$$[\alpha c_n, \alpha(1+1/n)c_n] \subseteq (0, 1/2)$$

and

$$[1-\beta(1-d_n), 1-\beta(1-(1+(1-d_n)^2)d_n)] \subseteq (1/2, 1).$$

From the choice of $\{c_n\}$ and $\{d_n\}$ it is easy to verify that all these intervals are pairwise disjoint. So we construct a function $g \in \mathcal{H}[0,1]$ such that

$$g(I_n) = [\alpha c_n, \alpha(1+1/n)c_n]$$

and

$$g(J_n) = [1-\beta(1-d_n), 1-\beta(1-(1+(1-d_n)^2)d_n)].$$

We construct a function $k \in \mathcal{H}[0,1]$ such that $k(c_n) = x_n^1$ and $k(d_n) = x_n^2$. Then for any $h \in K$,

$$\begin{aligned} \frac{g(h(k(c_n)))}{c_n} &= \frac{g(h(x_n^1))}{c_n} \\ &\leq \frac{\alpha(1+1/n)c_n}{c_n} = \alpha(1+1/n). \end{aligned}$$

and

$$\frac{g(h(x_n^1))}{c_n} \geq \frac{\alpha c_n}{c_n} = \alpha.$$

Thus

$$\lim_{n\to\infty} \frac{g(h(k(c_n)))}{c_n} = \alpha.$$

So by Lemma 5.12.1, $(g \circ h \circ k)'(0) = \alpha$. On the other hand, for any $h \in \mathcal{H}[0,1]$,

$$\begin{aligned} \frac{1-g(h(k(d_n)))}{1-d_n} &= \frac{1-g(h(x_n^2))}{1-d_n} \\ &\leq \frac{1-(1-\beta(1-d_n))}{1-d_n} = \beta \end{aligned}$$

and

$$\begin{aligned} \frac{1-g(h(x_n^2))}{1-d_n} &\geq \frac{1-[1-\beta(1-(1+(1-d_n)^2)d_n)]}{1-d_n} \\ &= \beta(1-d_n(1-d_n)). \end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{1 - g(h(k(d_n)))}{1 - d_n} = \beta.$$

So by Lemma 5.12.2, $(g \circ h \circ k)'(1) = \beta$. Therefore $g \circ h \circ k \in D_{\alpha,\beta}$.

Case 2: $\alpha = 0$, $\beta = 0$. We claim that, for any $n$,

$$[c_n^2, (1 + 1/n)^2 c_n^2] \subseteq (0, 1/2)$$

are pairwise disjoint, and

$$[1 - (1 - d_n)^2, 1 - (1 - d_n)^2(1 - d_n(1 - d_n))^2] \subseteq (3/4, 1)$$

are also pairwise disjoint. Since $[c_n, (1 + 1/n)c_n]$ are pairwise disjoint, the first part of the claim is obvious. For the second part of the claim, if there were some $m$ such that

$$1 - (1 - d_m)^2 < 1 - (1 - d_{m+1})^2 \leq 1 - (1 - d_m)^2(1 - d_m(1 - d_m))^2,$$

then $1 - d_{m+1} < 1 - d_m$ and

$$\begin{aligned} 1 - d_{m+1} &\geq (1 - d_m)(1 - d_m(1 - d_m)) \\ &= 1 - d_m - d_m(1 - d_m)^2. \end{aligned}$$

That is,

$$d_m < d_{m+1} \leq d_m + d_m(1 - d_m)^2,$$

which contradicts that $[d_n, d_n + (1 - d_n)^2 d_n]$ are pairwise disjoint. Since $d_n > 3/4$, so $1 - (1 - d_n)^2 > 3/4$ and the claim follows. Now we construct a function $g \in \mathcal{H}[0, 1]$ such that

$$g(I_n) = [c_n^2, (1 + 1/n)^2 c_n^2]$$

and

$$g(J_n) = [1 - (1 - d_n)^2, 1 - (1 - d_n)^2(1 - d_n(1 - d_n))^2].$$

We also construct a function $k \in \mathcal{H}[0,1]$ such that $k(c_n) = x_n^1$ and $k(d_n) = x_n^2$. Then for any $h \in K$,

$$\frac{g(h(k(c_n)))}{c_n} = \frac{g(h(x_n^1))}{c_n} \leq \frac{(1+1/n)^2 c_n^2}{c_n} = (1+1/n)^2 c_n$$

and

$$\begin{aligned}\frac{1-g(h(k(d_n)))}{1-d_n} &= \frac{1-g(h(x_n^2))}{1-d_n} \\ &\leq \frac{1-[1-(1-d_n)^2(1-d_n(1-d_n))^2]}{1-d_n} \\ &= (1-d_n)(1-d_n(1-d_n))^2.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{g(h(k(c_n)))}{c_n} = 0$$

and

$$\lim_{n\to\infty} \frac{1-g(h(k(d_n)))}{1-d_n} = 0.$$

By Lemma 5.12.1 and Lemma 5.12.2, $(g\circ h\circ k)'(0) = 0$ and $(g\circ h\circ k)'(1) = 0$. Therefore $g\circ h\circ k \in D_{\alpha,\beta}$ for any $h \in K$.

Case 3: $\alpha = +\infty$, $\beta = +\infty$. We require $16c_1 < 1$. Then we claim that, for any $n$,

$$[c_n^{1/2}, (1+1/n)^{1/2} c_n^{1/2}] \subseteq (0, 1/2)$$

are pairwise disjoint, and

$$[1-(1-d_n)^{1/2}, 1-(1-d_n)^{1/2}(1-d_n(1-d_n))^{1/2}] \subseteq (1/2, 1)$$

are also disjoint. Since $[c_n, (1+1/n)c_n]$ are pairwise disjoint, the first part of the claim is true. For the second part of the claim, if there were some $m$ such that

$$1-(1-d_m)^{1/2} < 1-(1-d_{m+1})^{1/2} \leq 1-(1-d_m)^{1/2}(1-d_m(1-d_m))^{1/2},$$

then $1-d_{m+1}<1-d_m$ and

$$1-d_{m+1}\geq(1-d_m)(1-d_m(1-d_m)),$$

which, as same as in Case 2, yields a contradiction. Thus the claim is true. Now we construct a function $g\in\mathcal{H}[0,1]$ such that

$$g(I_n)=[c_n^{1/2},(1+1/n)^{1/2}c_n^{1/2}]$$

and

$$g(J_n)=[1-(1-d_n)^{1/2},1-(1-d_n)^{1/2}(1-d_n(1-d_n))^{1/2}].$$

We also construct a function $k\in\mathcal{H}[0,1]$ such that $k(c_n)=x_n^1$ and $k(d_n)=x_n^2$. Then for any $h\in K$,

$$\frac{g(h(k(c_n)))}{c_n}=\frac{g(h(x_n^1))}{c_n}\geq\frac{c_n^{1/2}}{c_n}=c_n^{-1/2}$$

and

$$\begin{aligned}\frac{1-g(h(k(d_n)))}{1-d_n}&=\frac{1-g(h(x_n^2))}{1-d_n}\\&\geq\frac{1-[1-(1-d_n)^{1/2}(1-d_n(1-d_n))^{1/2}]}{1-d_n}\\&=\frac{(1-d_n(1-d_n))^{1/2}}{(1-d_n))^{1/2}}.\end{aligned}$$

Thus

$$\lim_{n\to\infty}\frac{g(h(k(c_n)))}{c_n}=+\infty$$

and

$$\lim_{n\to\infty}\frac{1-g(h(k(d_n)))}{1-d_n}=+\infty.$$

By Lemma 5.12.1 and Lemma 5.12.2, $(g\circ h\circ k)'(0)=+\infty$ and $(g\circ h\circ k)'(1)=+\infty$. Therefore $g\circ h\circ k\in D_{\alpha,\beta}$ for any $h\in K$.

Case 4: $\alpha=0$, $0<\beta<+\infty$. From Case 1 and Case 2 we require $2\beta(1-d_1)<1$ and construct a function $g\in\mathcal{H}[0,1]$ such that

$$g(I_n)=[c_n^2,(1+1/n)^2c_n^2]$$

and

$$g(J_n) = [1 - \beta(1 - d_n), 1 - \beta(1 - (1 + (1 - d_n)^2)d_n)].$$

Also we construct a function $k \in \mathcal{H}[0,1]$ such that $k(c_n) = x_n^1$ and $k(d_n) = x_n^2$. Then from Case 1 and Case 2 we know that for any $h \in K$, $(g \circ h \circ k)'(0) = 0$ and $(g \circ h \circ k)'(1) = \beta$. Therefore $g \circ h \circ k \in D_{\alpha,\beta}$ for any $h \in K$.

Similarly as Case 4 we can use the corresponding parts of Case 1, Case 2 and Case 3 to construct the corresponding functions $g \in \mathcal{H}[0,1]$ and $k \in \mathcal{H}[0,1]$ so that for any $h \in K$, $g \circ h \circ k$ belongs to the corresponding $D_{\alpha,\beta}$ for Case $0 < \alpha < +\infty$, $\beta = 0$, Case $0 < \alpha < +\infty, \beta = +\infty$, Case $\alpha = 0$, $\beta = +\infty$, Case $\alpha = +\infty$, $0 < \beta < +\infty$ and Case $\alpha = +\infty$, $\beta = 0$ respectively. The proof is finished. ■

**Corollary 5.12.4** *The sets $S$ in Theorem 5.8.1 and $G$ in Theorem 5.11.3 are not shy.*

**Proof.** Since the set $D_{\alpha,\beta}$ $(0 < \alpha < \infty)$ in Theorem 5.12.3 is contained in both $S$ and $G$. Thus the result follows. ■

From Corollary 5.8.2 and Corollary 5.11.4 and Theorem 5.12.3 we know that the set $D_{\alpha,\beta}$ is left-and-right shy, but is not shy. This negatively answers the problem $(P_0)$ posed by Jan Mycielski [41].

## 5.13 $\mathcal{H}[0,1]$ does not satisfy the countable chain condition

From Theorem 5.12.3 we immediately have the following theorem for the non-locally, non-Abelian Polish group $\mathcal{H}[0,1]$.

**Theorem 5.13.1** *The $\sigma$-ideal of shy sets in $\mathcal{H}[0,1]$ does not satisfy the countable chain condition. Furthermore, there exist sets that are left-and-right shy, and which* *can be decomposed into continuum many disjoint, non-shy sets in* $\mathcal{H}[0,1]$.

**Proof.** For each $\alpha$ $(0 < \alpha < \infty)$, by Theorem 5.12.3, the set $D_\alpha \equiv \{h \in \mathcal{H}[0,1] : h'(0) = \alpha\}$ is non-shy. For distinct $\alpha_1$, $\alpha_2$ $(0 < \alpha_1, \alpha_2 < \infty)$, $D_{\alpha_1}$ and $D_{\alpha_2}$ are disjoint. So $\mathcal{H}[0,1]$ contains continuum many disjoint, non-shy sets $D_\alpha$, $0 < \alpha < \infty$. The result follows. ■

## 5.14 Functions with infinitely many fixed points

In this section we study the set of functions in $\mathcal{H}[0,1]$ that have infinitely many fixed points.

**Theorem 5.14.1** *The set of functions in* $\mathcal{H}[0,1]$ *which cross the line* $y = x$ *in* $(0,1)$ *infinitely many times is neither shy nor prevalent.*

**Proof.** Let

$$F = \left\{ h \in \mathcal{H}[0,1] : \begin{array}{l} h \text{ crosses } y = x \text{ infinitely many} \\ \text{times in } (0,1) \end{array} \right\}.$$

We first show that $F$ is a Borel set. Let

$$F_{n,p,q} = \left\{ h \in \mathcal{H}[0,1] : \begin{array}{l} \exists x_1, \cdots, x_{4n+4} \in (0,1) \text{ such that } x_{i+1} - x_i \geq 1/p \text{ for } i \\ = 1, \cdots, 4n+3, \ f(x) - x \geq 1/q \text{ if } x_{4i+1} \leq x \leq x_{4i+2} \\ \text{and } f(x) - x \leq -1/q \text{ if } x_{4i+3} \leq x \leq x_{4i+4} \text{ for } i = \\ 0, \cdots, n \end{array} \right\}.$$

It is easy to verify that $F_{n,p,q}$ are closed sets and the functions in $F_{n,p,q}$ cross $y = x$ in $(0,1)$ at least $2n+1$ times. Note

$$F = \bigcap_{n=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} F_{n,p,q}.$$

So $F$ is a Borel set.

Since $F \subseteq \mathcal{H}[0,1] \setminus \{h \in \mathcal{H}[0,1] : h(x) \leq x^2\}$, and by Theorem 5.9.1 the set $\{h \in \mathcal{H}[0,1] : h(x) \leq x^2\}$ is neither left shy nor right shy, so is the complement of $F$. We now show that $F$ is neither left shy nor right shy. For any compact set $K \subseteq \mathcal{H}[0,1]$ we use the ideas in Theorem 5.12.3 to choose $I_n$ and $x_n^1$. Choose a monotonic sequence $\{y_n\} \subseteq (0,1)$ and functions $g$, $k \in \mathcal{H}[0,1]$ such that $k(y_n) = x_n^1$ and for any $h \in K$, $g(h(x_{2n}^1)) > x_{2n}^1$, $g(h(x_{2n+1}^1)) < x_{2n+1}^1$, $h(k(y_{2n})) > y_{2n}$ and $h(k(y_{2n+1})) < y_{2n+1}$. Then $g \circ h$, $h \circ k \in F$. Therefore $F$ is neither left shy nor right shy in $\mathcal{H}[0,1]$. Hence $F$ is neither shy nor prevalent. ∎

**Theorem 5.14.2** *For any $m \in (0,1]$ and function $q(x) \in \mathcal{H}[0,1]$, let*

$$S = \{h \in \mathcal{H}[0,1] : \exists x \in (0,1) \text{ such that } h(x) = mq(x)\}.$$

*Then $S$ is neither shy nor prevalent.*

**Proof.** Let

$$S_n = \{h \in \mathcal{H}[0,1] : \exists x \in [1/n, 1-1/n] \text{ such that } h(x) = mq(x)\}.$$

Then

$$S = \bigcup_{n=2}^{\infty} S_n.$$

We now show that $S_n$ is a Borel set. In fact, for any Cauchy sequence $\{f_k\} \subseteq S_n$, there exist $x_k \in [1/n, 1-1/n]$ and $f \in \mathcal{H}[0,1]$ such that $f_k(x_k) = mq(x_k)$ and $f_k \to f$ uniformly. Thus there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \to x_0 \in [1/n, 1-1/n]$ and $q(x_{k_j}) \to q(x_0)$. So $f(x_0) = mq(x_0)$ and hence $f \in S_n$. $S_n$ is closed and hence the set $S$ is a Borel set. Since

$$S \subseteq \mathcal{H}[0,1] \setminus \{h \in \mathcal{H}[0,1] : h(x) \geq mq^{1/2}(x)\},$$

and by Theorem 5.9.2

$$\{h \in \mathcal{H}[0,1] : h(x) \geq mq^{1/2}(x)\}$$

is not shy, then $S$ is not prevalent. Since the set $F$ in Theorem 5.14.1 is a subset of $S$, by Theorem 5.14.1 the set $S$ is not shy. ■

In [25] and [26] it is shown that the set $S$ has $P$ measure 1 when $q(x) = x$ in the above theorem. In the above theorem we have shown that the set of all $h \in \mathcal{H}[0,1]$ such that $h(x) = q(x)$ for some $x \in (0,1)$ is not shy. As we have remarked earlier (see page 105), however, for any $q(x) \in \mathcal{H}[0,1]$, the set of all $h \in \mathcal{H}[0,1]$ such that $h(x) = q(x)$ for all $x$ in some subinterval of $(0,1)$ is shy.

## 5.15 Concluding remarks

The theory of shy sets in an Abelian Polish group seems to be completely understood ([12], [13], [14], [56], [27], [18], [52], [8]). Only partial results are available, however, for non-Abelian (non-locally compact) completely metrizable groups ([18], [41], [52], and some results in this thesis).

It had been an open problem (see Mycielski [41]) whether in a non-Abelian Polish group there could exist sets left shy and/or right shy that are not shy. We have answered this in the Polish group $\mathcal{H}[0,1]$ by proving the existence of non-shy Borel sets that are left-and-right shy. In addition we showed that the Polish group $\mathcal{H}[0,1]$ does not satisfy the countable chain condition.

Solecki [52] showed that the $\sigma$-ideal of shy sets in an Polish group admitting an invariant metric satisfies the countable chain condition iff this group is locally compact. The following problem is still open.

**PROBLEM 14** *Is it true that the $\sigma$-ideal of shy sets in a non-Abelian Polish group satisfies the countable chain condition iff the group is locally compact?*

The results in this chapter suggest the following problems. We conclude the thesis with these problems.

**PROBLEM 15** *Does there exist a left shy (right shy) set in $\mathcal{H}[0,1]$ that is not right shy (left shy)?*

**PROBLEM 16** *Does there exist a shy (left/right shy) set in $\mathcal{H}[0,1]$ that is of the second category.*

**PROBLEM 17** *Does the $\sigma$-ideal of left shy (right shy) sets in $\mathcal{H}[0,1]$ satisfy the countable chain condition?*

**PROBLEM 18** *In every non-Abelian, non-locally compact, completely metrizable group, are there sets that are left-and-right shy, but not shy?*

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# Glossary

**additively shy set** See page 93.

**almost every** See page 10.

**algebraic basis** A subset $E$ is an *algebraic basis* of a **Fréchet space** $F$ if every element in $F$ can be expressed as a unique linear combination of finitely many elements in $E$.

**analytic** Let $X$ be a complete, separable metric space. A set $E \subseteq X$ is said to be *analytic* if $E$ is the projection of a closed subset $C$ of $X \times \mathbb{N}^{\mathbb{N}}$ to $X$ where $\mathbb{N}^{\mathbb{N}}$ is the product of countably many copies of the space $\mathbb{N}$ of natural numbers and equipped with the usual metric

$$\rho(\mathbf{m}, \mathbf{n}) = \sum_{i=1}^{\infty} \frac{|m_i - n_i|}{2^i(1 + |m_i - n_i|)}$$

for all $\mathbf{m} = (m_1, m_2, m_3, \cdots)$, $\mathbf{n} = (n_1, n_2, n_3, \cdots) \in \mathbb{N}^{\mathbb{N}}$.

**approximately continuous** A function $f$ is said to be *approximately continuous* on an interval $[a, b]$ if for every $x \in [a, b]$, there exists a set $E$ with $x$ as its density point such that

$$\lim_{y \to x,\ y \in E} f(y) = f(x).$$

**Aronszajn null** See Definition 2.3.3 and Definition 2.3.4.

**atomless measure** A measure $\mu$ is an *atomless measure* if for every $x \in X$, $\mu(\{x\}) = 0$.

**automorphism** A **homeomorphism** $h$ of an interval $[a, b]$ that satisfies $h(a) = a$, $h(b) = b$ is called an *automorphism* of $[a, b]$.

**Baire 1 function** A function is said to be a *Baire 1 function* if it can be written as the pointwise limit of a sequence of continuous functions.

**Banach algebra** A *Banach algebra* $A$ is a Banach space that is also an algebra and satisfies that $\|xy\| \leq \|x\| \cdot \|y\|$ for all $x, y \in A$.

**Besicovitch set** A *Besicovitch set* is a plane set which contains a unit segment in every direction.

**Borel measure** A measure $\mu$ on a **topological space** $X$ is called a *Borel measure* if $\mu$ is defined on all **Borel sets** of $X$. In this thesis we consider $\mu$ defined on all **universally measurable sets**.

**Borel probability measure** A Borel measure on a **topological space** $X$ is called a *Borel probability measure* if $\mu(X) = 1$.

**Borel selector** Let $G$ be a **Polish group** and $H \subseteq G$ be a closed subgroup. Then a mapping $s : G/H \to G$ is a *Borel selector* for the cosets of $H$ if it is Borel measurable and $s(xH) \in xH$.

**Borel sets** If $X$ is a **topological space**, the smallest $\sigma$-algebra containing the closed sets is called the class of *Borel sets*.

**left-and-right shy** See Definition 2.9.6.

**Cantor group** The *Cantor group* is the set of all sequences of 0's and 1's equipped with the metric

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

and the usual group structure $xy = (x_1y_1, \cdots, x_ny_n)$.

**c-dense** A set $S \subseteq \mathbb{R}$ is *c-dense* in an interval $I$ if the intersection of $S$ with every subinterval of $I$ contains continuum many points. Generally, a set $S$ of a metric space is *c-dense* in an open set $O$ if the intersection of $S$ with every non-empty open subset of $O$ contains continuum many points.

**Christensen null** See Definition 2.3.1, Definition 2.9.1 and Definition 2.12.3.

**co-analytic** A set is *co-analytic* if it is the complement of an analytic set.

**compact** A subset $K$ of a metric space is called *compact* if every open covering of $K$ has a finite subcovering.

**completely metrizable** A **topological group** is called *completely metrizable* if its topology can be derived from a complete metric on it.

**continuum hypothesis** The assumption that $c = \aleph_1$ is called the *continuum hypothesis*, where $c$ is the cardinal of the set $\mathbb{R}$ and $\aleph_1$ is the next cardinal after the cardinal of an infinite countable set.

**convolution** A *convolution* of two measures $\mu$ and $\nu$ on a **topological group** $G$ is defined by

$$\mu * \nu(A) = \mu \times \nu(\{(x, y) : xy^{-1} \in A\})$$

for every **universally measurable** set $A$. A *convolution* of a measure $\mu$ with a characteristic function of a set $A$ is defined by

$$\chi_A * \mu(x) = \int_G \chi_A(xy)\mu(dy).$$

**countable chain condition** See Definition 2.18.1.

**countably continuous** See page 91.

**Darboux function** A real-valued function defined on an interval $[a, b]$ is said to be a *Darboux function* on $[a, b]$ if for each $x, y \in [a, b]$ with $x < y$ and $c$ between $f(x)$ and $f(y)$ there is $z \in (x, y)$ with $f(z) = c$.

**discrete group** A *discrete group* is the group with the discrete topology.

**Fréchet differentiable** A mapping $f$ from a Banach space to a Banach space $Y$ is *Fréchet differentiable* at $x \in X$ if

$$\lim_{t \to 0} \frac{f(x + th) - f(x)}{t}$$

exists for every $h \in X$ and the limit is uniform for $\|h\| \leq 1$.

**Fréchet space** A completely metrizable, locally convex space is called a *Fréchet space*

**Gâteaux differentiable** A mapping $f$ from a Banach space $X$ to a Banach space $Y$ is directionally *Gâteaux differentiable* at $x \in X$ if

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

exists for every $h \in X$. Further, the mapping $f$ is *Gâteaux differentiable* at $x \in X$ if there exists an element $g \in X^*$ such that

$$g(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

for every $h \in X$.

**Gaussian measure** See Definition 2.4.2.

**Gaussian null in the ordinary sense** A set $S$ in a Banach space is said *Gaussian null in the ordinary sense* if there exists a Gaussian measure $\mu$ such that $\mu(S) = 0$.

**Gaussian null in Phelps sense** See Definition 2.4.3.

**Haar measure** An invariant measure on a locally compact group in called a *Haar measure*.

**Haar null, Haar zero** See Definition 2:3.1, the comments following Definition 2.3.1, Definition 2.9.1, Definition 2.9.3 and Definition 2.12.3.

**Hölder continuous** A function $f$, defined on an interval $[a, b]$, is said to be *Hölder continuous* at $x \in [a, b]$ if there exist $M$, $\delta > 0$ such that for all $y \in [a, b]$, $|y - x| < \delta$, $|f(y) - f(x)| \leq M|x - y|$.

**Hölder continuous with exponent $\alpha$** A function $f$, defined on an interval $[a, b]$, is said to be *Hölder continuous with exponent* $\alpha$ at $x \in [a, b]$ if there exist $M, \delta > 0$ such that for all $y \in [a, b]$, $|y - x| < \delta$, $|f(y) - f(x)| \leq M|x - y|^{\alpha}$.

**homeomorphism** A bijection $h$ from a metric space $X$ to a metric space $Y$ is called a *homeomorphism* if both $h$ and $h^{-1}$ are continuous.

**hyperplane** A *hyperplane* in a Banach space $B$ is a set of the form $\{x \in B : f(x) = \alpha\}$ where $f$ is a non-zero linear functional on $B$ and $\alpha$ is a real number.

**isomorphism** A mapping $T$ from a Banach space $X$ onto a Banach space $Y$ is an *isomorphism* if $T$ is one-one, linear, continuous.

**left Haar measure** A Haar measure is a *left Haar measure* if it is left invariant.

**left shy** See Definition 2.9.4.

**left transverse** See Definition 2.9.4.

**Lipschitz (continuous)** A function $f$, defined on $[0,1]$, is *Lipschitz* at a point $x \in [0,1]$ if there is a constant $M > 0$ such that for all $y \in [0,1]$, $|f(x) - f(y)| \leq M|x - y|$. Once this inequality holds we say that $f$ is *$M$-Lipschitz*.

**locally finite** A Borel measure is *locally finite* if every point $x \in X$ has a neighborhood $U$ with $\mu(U) < \infty$.

**lower density 1** See page 48.

**$m$-dimensionally shy** See Definition 3.2.1.

**metrizable** A **topological space** $(X, \mathcal{T})$ is *metrizable* if there is a metric $d$ on it so that $\mathcal{T}$ is the topology of $(X, d)$.

**multiplicatively shy** See page 93.

**nowhere monotonic** See page 81.

**nowhere monotonic type** See page 81.

**$n$-th symmetrically continuous** A function $f$ is said to be *$n$-th symmetrically continuous* at a point $x$ of an open interval $(a, b) \subseteq \mathbb{R}$ if

$$\lim_{h \to 0} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n - 2i)h) = 0.$$

**observable** See page 48.

**Polish group** A *Polish group* is a separable completely metrizable group.

**Polish space** A *Polish space* is a separable completely metrizable space.

**positive cone** In an ordered vector space $E$, the set $\{x \in E : x \geq 0\}$ is called the *positive cone*.

**positive lower density** See page 48.

**positive upper density** See page 48.

**Preiss–Tišer null** See Definition 2.3.5.

**prevalent** See page 10.

**probe** See page 64.

**Radon measure** A Borel measure $\mu$ on a Hausdorff space is called a *Radon measure* if $\mu$ is locally finite and **tight**.

**Radon probability measure** A *Radon probability measure* $\mu$ on a Hausdorff space $X$ is called a *Radon probability measure* if it is a Radon measure with $\mu(X) = 1$.

**right Haar measure** A Haar measure is a *right Haar measure* if it is right invariant.

**right shy** See Definition 2.9.5.

**right transverse** See Definition 2.9.5.

**semigroup** A *semigroup* is simply a set $G$ with an associative operation: $G \times G \to G$. In this thesis we always assume that our semigroup has a unit element.

**shy** See Definition 2.3.1, Definition 2.9.3 and Definition 2.12.3.

**$s$-null** See Definition 2.3.2.

**spanning set** In a Banach space $X$, a set $C$ is a *spanning set* if the whole space $X$ is the affine hull of $C$. Here the affine hull of a set $A$ is the set

$$\left\{ \sum_{i=1}^{m} \alpha_i x_i : \sum_{i=1}^{m} \alpha_i = 1,\ x_i \in A,\ m \in \mathbb{N} \right\}.$$

**super-reflexive** A Banach space $E$ is said to be *super-reflexive* if each Banach space which is finitely representable in it is reflexive. We say a Banach space $E_1$ is finitely representable in $E$ if for each finite dimensional subspace $L \subseteq E_1$ and $\alpha > 0$ there exists an embedding $i : L \to E$ such that $\alpha^{-1}\|x\| \leq \|ix\| \leq \alpha\|x\|$ $(x \in L)$.

**support of a measure** The *support of a measure* $\mu$ on a metric space $X$ is defined by

$$\mathrm{supp}(\mu) = \{x \in X : \mu(U) > 0, \forall U \text{ a neighborhood of } x\}.$$

**symmetrically continuous** A function $f$ is said to be *symmetrically continuous* at a point $x$ of an interval $(a, b) \subseteq \mathbb{R}$ if

$$\lim_{h \to 0}[f(x+h) - f(x-h)] = 0.$$

**$\tau$-smooth** A Borel probability measure $\mu$ is *$\tau$-smooth* if $\mu(U) = \sup_\alpha \mu(U_\alpha)$ for every family of open sets $\{U_\alpha\}$ filtering up to the open set $U$.

**thick** See Definition 2.19.1.

**tight** A Borel measure $\mu$ is called *tight* if $\mu(B) = \sup \mu(K)$ for all Borel sets $B$, where the sup is taken over all compact sets $K \subseteq B$.

**topological group** A *topological group* is a group $(G, \cdot)$ endowed with a topology such that $(x, y) \to xy^{-1}$ is continuous from $G \times G$ to $G$.

**topological semigroup** A *topological semigroup* is a completely metrizable semigroup for which the operations $x \to ax$ and $x \to xa$ are continuous.

**topological space** A *topological space* is a pair $(X, \mathcal{T})$, where $X$ is a set and $\mathcal{T}$ is a collection of subsets of $X$ such that $\emptyset, X \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary unions and finite intersections.

**transverse** See Section 2.2.

**typical function** If a set of functions with some property is a residual set in a function space, then every function in this set is called a *typical function*

**typical property** See page 1.

**ubiquitous** See page 48.

**universally measurable** A set of a topological space $X$ is *universally measurable* if it belongs the $\mu$-completion of each finite Borel measure $\mu$ on $X$.

**universally Radon measurable** A set of a topological space $X$ is *universally Radon measurable* if it belongs to the $\mu$-completion of each finite Radon measure $\mu$ on $X$.

**upper density 1** See page 48.

**well-ordering** A strict linear ordering $<$ on a set $X$ is called a *well-ordering* for $X$ if every non-empty subset of $X$ contains a first element.

**Wiener measure** See Definition 2.5.1.

**Zarantonello null** See Definition 2.6.1.