# BROADCASTING ON TORUS-LIKE CHORDAL RINGS

by

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# Abstract

A $d$-dimensional *toroidal mesh* can be constructed from a mesh (grid graph) by adding wrap-around connections in each of the $d$ dimensions. If the wrap-around connections are *skewed*, then the resulting graph often belongs to a class of Cayley graphs called *circulants*. For a triangular mesh the graph is a *chordal ring* of degree 3. We define a similar class of odd degree graphs, *odd-circulants*, which are constructed by deleting one edge from each vertex of a circulant graph. *Broadcasting* algorithms are developed using geometric and tabular techniques, assuming an all-ports (*shouting*) communications model with circuit-switched routing. For $N$ vertices and degree $\Delta$ the algorithms broadcast in $\log_{(\Delta+1)} N$ rounds (matching the lower bound). We consider only values for $N$ of the form $(\Delta + 1)^{nd}$ where $n$ and $d$ are integers. The main part of the thesis develops broadcasting schemes to reduce the total path length travelled by a message. The diameter of a graph is a lower bound on this total path length and, for values of $n$ greater than 1, graphs of even degree give total path lengths less than 1.2 times the diameter of the graph. For broadcasting on a $d$-dimensional toroidal mesh with $(\Delta + 1)^{nd}$ nodes we show that the ratio of total path length to diameter tends to 1 as $d$ increases, provided $n$ is greater than 1.

# Acknowledgments

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# Chapter 1

# Introduction

## 1.1 Motivation

An important part of massively parallel computers is the network of processor interconnections. Algorithms for such computers often assume the existence of efficient communications primitives to transfer information between processors in an orderly way. This thesis examines broadcasting (the transfer of a single message from one node to all other nodes) on a mesh network with wraparound connections.

By using wraparound connections the mesh forms a torus and the corresponding network graph is made vertex-transitive. As well as reducing the graph diameter, this symmetry generally simplifies broadcasting algorithms. The diameter of the torus graph is still large ($\sqrt{N}$) compared with the hypercube ($\log N$), but circuit switching techniques described in Section 1.3 can reduce the delays associated with a large diameter. Because of physical constraints on device packaging and the layout of connections, simple mesh networks have practical advantages over many other topologies.

An optimal broadcasting algorithm on a 2-dimensional square mesh torus is described in Peters and Syska [13], in terms of a recursive tiling. This thesis extends the tiling method to three dimensions and higher.

Tilings of the torus are difficult to visualise in higher dimensions and an alternative approach has been adopted. A different network has been chosen which is similar to the torus but which has allowed us to develop a simple and efficient broadcasting algorithm. The wraparound torus connections are skewed in such a way as to change the torus network graph to a circulant graph (which we will also refer to as a *skewed torus graph*). In essence we are choosing chord lengths for the circulant graph so as to approximate a torus mesh network.

This combination of the properties of a mesh with those of a circulant graph has already given useful results in fault tolerance (Bruck, Cypher and Ho [2]). Algorithms for $d$-dimensional skewed torus networks can be designed using a 1-dimensional circulant graph, which is much easier to visualise and can also be modified to have nodes of an odd degree. Several authors (see Section 1.2) have described the advantages of the hexagonal grid torus which is naturally skewed. Representing the hexagonal grid torus as a circulant graph suggests that the chord lengths (which differ by only 1) are badly chosen for some applications. Vertices of the three dimensional skewed torus have the same degree as those of a hexagonal grid torus, and when viewed as a circulant graph the skewed torus only differs from the hexagonal grid torus in having its chords spaced further apart.

Analysis of the skewed torus as a circulant graph leads to broadcasting algorithms which have total path lengths only slightly larger than the diameter of the graph. Only broadcasting algorithms have been considered in this thesis and the algorithms developed place tight restrictions on the number of vertices in the graph. This is not a property of the graphs but results from assumptions that, in every round of the broadcast, each informed node sends the message to the maximum number of uninformed vertices, and also that recursive techniques will be used. Further work is required to determine how well the skewed torus is suited to other network algorithms.

## 1.2 Background

The initial stimulus for this work comes from Peters and Syska [13], which describes an optimal broadcast algorithm on a square mesh torus in two dimensions, and shows its equivalence to recursive tiling. Their algorithm broadcasts to $N$ vertices in $\log_5 N$ rounds.

The same tiling has been used independently in Tsai and McKinley [16] for meshes without the torus connections by making minor adjustments at the edges. They also consider algorithms which use only 3 of the 4 ports available at each vertex, giving the effect of a degree 3 graph and requiring $\log_4 N$ rounds for a broadcast.

Some advantages of two dimensional hexagonal meshes are discussed in Chen, Shin and Kandlur [4] and also Bruck, Cypher and Ho [2]. Hexagonaal meshes have a particularly elegant tiling described in Vince [17]. Recently Deserable[7] has shown a Cayley graph representation of a hexagonal grid which they have named the Arrowhead Torus. This has a recursive structure which is useful for information dissemination.

Another variation is the error tolerant mesh in Bruck, Cypher and Ho [2] which is related to the skewed torus described in Chapter 3 of this thesis, and also uses the approach of embedding a mesh in a circulant graph. Bruck, Cypher and Ho [3] approach fault tolerant meshes by defining extra edges along redundant dimensions which are linearly dependent on the standard coordinate axes for the mesh.

Recently Tong and Padubidri [15] have analysed meshes formed using the diagonal connections between nodes. The results of Bruck, Cypher and Ching-Tien [3] show that these are often isomorphic to normal square meshes and in some cases they are isomorphic to skewed torus meshes.

General types of meshes are treated fully in Conway and Sloane [5].

Relations between various tilings in 2 dimensions are discussed in Senechal [14] and Grunbaum and Shephard [9].

This thesis views a skewed torus mesh as a circulant graph and thus generalises many of the above approaches.

## 1.3 Definitions and Properties

**Two-dimensional torus network** A $p \times q$ two-dimensional torus network consists of $pq$ nodes, each with a label $(i, j)$ where $i \in Z_p$ and $j \in Z_q$. Each node $(i, j)$ is connected to the four nodes

$$(i \pm 1 \bmod p, j \pm 1 \bmod q).$$

The torus network can be derived from a $p \times q$ square mesh network by connecting each node at the (rectangular) boundary of the mesh to the corresponding node on the opposite side of the boundary. The torus network can thus be regarded as embedded in the surface of a torus. In graph-theoretic terms, a one-dimensional torus network can be represented by a cycle and a two-dimensional torus network by the direct product of two cycles. The direct product $G \times H$ of graphs

$$G = (V_G, E_G) \text{ and } H = (V_H, E_H)$$

is the graph $(V, E)$ where $V$ is the cartesian product $V_G \times V_H$ and $E$ is the union

$$\{((g, h_1), (g, h_2)) | g \in V_G, (h_1, h_2) \in E_H\}$$

$$\cup \{((g_1, h), (g_2, h)) | h \in V_H, (g_1, g_2) \in E_G\}.$$

The torus mesh has a diameter of $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$. Often $p = q$ to give symmetry and to minimise the diameter for a torus of $p \times q$ nodes. For $n$ odd, an $n \times n$ 2-dimensional torus has a diameter of $n - 1$.

***d*-dimensional torus network** The two-dimensional torus network graph can be generalised to a $d$-dimensional torus graph defined as the direct product of $d$ cycles. Assuming each cycle contains n vertices, there are $n^d$ vertices and the diameter of the torus graph is $d\lfloor \frac{n}{2} \rfloor$. The graph can be regarded as being embedded in the $d$-dimensional surface of a torus in $(d+1)$ dimensions. The torus in $(d+1)$ dimensions is a $d$-dimensional cube with opposite $(d-1)$-dimensional faces identified.

(The torus is also known as the $k$-ary $n$-cube: for example, Linder and Harden [10].)

**Skewed torus network** When the graph derived from a square mesh network is regarded as being embedded in the surface of a torus, the torus is represented as a square with opposite sides identified, and with the edges of the graph drawn parallel to the sides of the square. It is possible to draw the graph so that edges slope relative to the square and connect up in different ways at opposite sides of the square. We will define this as a skewed torus connection. The new graph formed in this way is often a circulant graph, and this thesis concentrates on skewed torus networks represented as circulant graphs.

**Chordal ring graph** A chordal ring is a degree 3 graph with an even number of vertices, formed by adding one edge to each vertex of a cycle graph. If the vertices of the cycle are labeled 0 to $N-1$ around the cycle, where $N$ is even, the added edge incident on vertex $i$ is defined as incident at its other end on the vertices

$$\begin{cases} i + c \bmod N & \text{for even i} \\ i - c \bmod N & \text{for odd i} \end{cases}$$

where $c$ ( the **chord length**) is an odd integer less than or equal to $N/2$. When the cycle is drawn as a circle, the added edges form chords of the circle, giving the graph its name. We will refer to the cycle as the **main cycle** of the graph and the added edge at a vertex as the $c$-**chord** where $c$ is replaced by its actual length. For consistency, we will refer to an edge which is part of the main cycle as a **1-chord**. ( Chordal rings are sometimes defined differently in the literature, for example as being an alternative name for circulant graphs. )

**Circulant graph** A circulant is a graph with $N$ vertices labelled 0 to $N-1$. Its edges are defined by a set of chord lengths $\{c_0, c_1, c_2, \ldots\}$, with edges connecting vertex $i$ to vertices in the set

$$\{i \pm c_0, i \pm c_1, i \pm c_2, \ldots\} \pmod N.$$

It is an even-degree graph unless one of the chords is of length $N/2$. Its name is derived from its incidence matrix, which is a circulant matrix. (Some authors omit the negative edge from the definition, as one node's negative edge is another node's positive one.) We will restrict our attention to circulants for which $c_0 = 1$, and the edges corresponding to $c_0$ will form a Hamiltonian cycle corresponding to the main cycle of a chordal ring. As with the chordal ring, we will refer to this cycle as the **main cycle** and to $c_0$ as the **1-chord**.

**Odd-Circulant graph** Circulant graphs normally have even degree. To investigate graphs of odd degree we consider a generalisation of the chordal ring and circulant graphs, and define an odd-circulant graph by removing half of the chords of a particular length from a circulant graph. Even-numbered vertices have a negative chord removed and odd vertices have a positive chord of the same chord length removed. This partially removed chord length will be referred to as the **alternating chord** and the chords of other lengths as **symmetric chords**. As with the chordal ring, the number of vertices must be even and the removed chord must have an odd length and (for our algorithm to work) we choose it to be the longest or next to longest chord. Note that in some cases the graph is isomorphic to one where a shorter chord is the alternating chord.

**Circuit switched routing** Communication between distant vertices in a network can be improved by circuit switched routing hardware, whereby the source vertex sends a header containing the destination address through the network to 'build' a path. At each intermediate vertex on the path, input and output ports used by the header are connected. When the destination vertex receives the header, it returns an acknowledgement to the source vertex, establishing a direct connection between source and destination. The bytes of the message are then sent in pipeline fashion. Since the message is switched through intermediate vertices there is no need to buffer the entire message. The links of the path can be released as the last byte passes through each vertex, or by an acknowledgement from the destination vertex when the last byte is received.

**Wormhole routing** Circuit switched routing can be implemented without having an acknowledgement returned when the path has been established. Instead, the remaining bytes follow the header immediately in pipeline fashion and the last byte releases the switches as it passes through. This method is known as wormhole routing.

**Link-bounded** or **'shouting'** communication models allow a processor to use all its links simultaneously. In contrast, the processor-bounded (or *whispering*) model allows the use of only one link at any given time. We will consider only the link-bounded model.

## 1.4 Model of Computation

### 1.4.1 The Linear Cost Model

This thesis uses a linear cost model with link-bounded circuit-switched routing, which is the same model as in Peters and Syska [13]. The model assumes that the transmission time for a message has 3 components:

- a fixed startup time of $\alpha$ to initiate a message
- a switching time of $\delta$ for each vertex along the path
- a transmission time of $\tau$ for each character in the message, where $1/\tau$ is the bandwidth of the communication links.

To send a message of length L through $k$ vertices to its destination thus costs

$$\alpha + k\delta + L\tau.$$

Other factors which affect transmission times are assumed to have been absorbed into the constants $\alpha$, $\delta$, $\tau$ and their effect is small. For example, routers can 'switch through' several paths by connecting pairs of ports and there may be contention when these paths are being set up. However, there is no router contention after the paths have been established, and no buffering of messages, so the propagation time of a message from source to destination is not affected by the number of vertices through which it is switched or the number of other messages that are being switched through those vertices.

### 1.4.2 Broadcasting Assumptions

Only vertex-transitive networks will be considered, so it is sufficient to describe the broadcasting algorithm starting from any vertex. A broadcast is broken into rounds.

Initially only one vertex is informed. At each round, every informed vertex can send out the message on all its ports. (A 'shouting' model is assumed.) Rounds are not allowed to overlap: a round starts when all activity for the previous round has finished. Edges of the network graph are undirected but only half-duplex communications is used (in the context of broadcasting it would be perverse to send the same message along an edge in both directions at once). The paths used by the messages during any round must be edge-disjoint, but not necessarily vertex-disjoint.

The total time to complete a broadcast is the sum of the longest time for a message transmission in each round. Writing $k_i$ for the maximum path length in round $i$, if there are $R$ rounds with $K = \sum k_i$, then the total time to complete a broadcast is

$$(\alpha + L\tau)R + \delta K.$$

For $N$ vertices, each of degree $\Delta$, the minimum number of rounds is $\lceil \log_{\Delta+1} N \rceil$, and the minimum total path length must be at least the diameter of the graph.

In this thesis for simplicity we always choose $N$ to be a power of $\Delta + 1$ so that the lower bound on the number of rounds, $\log_{\Delta+1} N$, is an integer. The skewed torus network in $d$ dimensions is graph of even degree, with $\Delta = 2d$, and we will only consider values of $N = (2d+1)^{nd}$ where $n$ is an integer. Similarly for odd-circulant graphs with degree $\Delta = 2d - 1$ we will assume $N = (2d)^{nd}$.

The main algorithm described in Chapter 7 achieves the minimum number of rounds and has a total path length within a small multiple (less than 1.2 for graphs of even degree) of the diameter of the corresponding normal torus network. For the skewed torus network in $d$ dimensions, this ratio of the total path length to the normal torus diameter tends to 1 as $d$ increases provided $n$ is greater than 1. Note, however, that the number of vertices, $N = (2d+1)^{nd}$, increases extremely rapidly with $d$: when $n = 2$ a value of $d = 3$ gives 117649 vertices.

Path lengths for broadcasts on graphs of odd degree are similar, but there is no standard torus to compare with. Instead we compare the path length of the broadcast with an estimate of the diameter of the graph.

## 1.5 Outline of Thesis

Chapter 2 discusses the visualisation of a broadcasting algorithm on a mesh network in terms of spanning one dimension at a time and as a recursive tiling. It shows how the normal wraparound torus connections can be modified to make the corresponding graph into a circulant.

Chapter 3 derives a near optimal solution for the square mesh with skewed connections in 2 dimensions, and Chapter 4 extends this to 3 dimensions.

Chapter 5 gives a geometric solution to broadcasting on a triangular torus mesh, and derives constraints on the chord length in the corresponding chordal ring graph.

Chapter 6 extends the idea of circulant graphs to odd degree and solves the recursive broadcast problem for degree 5. A geometric interpretation is given as tiling a mixed triangular/square torus mesh in 3 dimensions.

In Chapters 7 and 8 these results are generalised to graphs of any degree. Although these results were inspired by geometric properties of the skewed torus-connected mesh, they are dealt with in terms of the corresponding circulant and chordal ring graphs. The results are summarised in Chapter 9.

Some associated work has been included in appendices:

Appendix A describes alternative approaches to tiling the normal torus-connected mesh in three dimensions.

Appendix B derives the maximum number of vertices for a fixed diameter of skewed torus mesh, and discusses some relationships between square, triangular and hexagonal meshes.

Appendix C describes how to decide if a tiling can form a torus.

Appendix D looks briefly at the practical layout of a $d$-dimensional torus network.

# Chapter 2

# Broadcasting Visualisations

## 2.1 Introduction

This chapter describes broadcasting in terms of

- Spanning one dimension at a time for a normal torus
- Tiling $n$-dimensional space for a normal torus
- Circulant graphs related to tilings for a skewed torus

For each method we describe a broadcast strategy with up to 125 vertices. The chapter ends by describing a tabular format for broadcast schemes on circulant graphs. The tabular format is treated in detail in Chapters 7 and 8.

## 2.2 Describing Paths

A broadcasting algorithm must decide which vertices to inform at each round and also find edge-disjoint paths through the network graph. These are to be chosen in such a way that $K$ (the total path length for all rounds) and $R$ (the number of rounds) are minimised. Correctness and optimality require that each vertex be informed exactly once.

In each round, each informed vertex sends to a fixed number of target vertices equal to the degree of the vertex. When developing algorithms, vertices will be numbered from 0 to $N-1$ and the paths will be described in two ways:

- The vertex numbers along the path, starting with the source vertex and ending with the target vertex. These vertex numbers will generally be given relative to the source vertex, so the source vertex will be thought of as vertex 0. Thus a path from source vertex $x$ is given by adding $x$ (modulo $N$) to each of the vertex numbers along the corresponding path from vertex 0.

- The number of steps along each coordinate axis of the geometric mesh representation, ignoring the order of steps. Thus a path will be expressed as a vector of dimension $d$ relative to the source vertex. (These vectors do not form a vector space because of the skewed connections.)

  We often ignore the underlying geometric model and describe paths in terms of chord lengths in a chordal ring, circulant or odd-circulant graph. Each chord length corresponds to a different coordinate axis. Each vertex of a circulant graph is incident on two chords for each dimension: the positive and negative directions along each axis. For the chordal ring or odd-circulant graph there will be one axis where the corresponding chord exists only in the positive or negative direction of the axis, depending on whether it is an odd- or even-numbered vertex.

## 2.3 Broadcasting to One Dimension at a Time

In the link-bounded model with vertices of degree $\Delta$, each vertex can inform at most $\Delta$ other vertices in one round, to give $\Delta+1$ informed vertices. Thus a broadcast can inform at most $(\Delta+1)^R$ vertices in $R$ rounds. We can regard the $(\Delta+1)^R$ vertices to be in $R$-dimensional space

$$Z_{\Delta+1} \times Z_{\Delta+1} \times \cdots \times Z_{\Delta+1},$$

where a vertex is represented by a vector $(x_1, x_2, \ldots, x_R)$. The natural broadcast scheme is to span one dimension at a time, as in Fig 2.1 where the lines represent paths rather than edges of the graph.

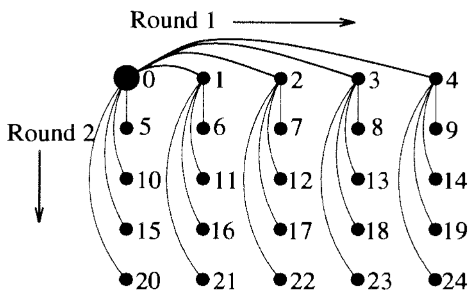


Figure 2.1: Spanning one dimension at a time

After $r$ rounds all vertices of the form

$$(x_1, x_2, \ldots, x_r, 0, 0, \ldots, 0)$$

have been informed, and in round $r + 1$ vertex

$$(a_1, a_2, \ldots, a_r, 0, 0, \ldots, 0)$$

sends to vertices

$$(a_1, a_2, \ldots, a_r, x, 0, \ldots, 0) \text{ for } 1 \leq x \leq \Delta$$

Thus for $\Delta + 1 = 5$, starting at vertex 0,
In round 1, vertex 0 informs 1 to 4
In round 2 vertices 0 to 4 inform 5 to 24
In round 3 vertices 0 to 24 inform 25 to 124 ...

Note that the rounds are independent, in the sense that the dimensions can be used in any order (see the discussion for circulant graphs in section 2.8.1). For example, the reverse order for 125 vertices gives the broadcast pattern
In round 1 vertex 0 informs 25, 50, 75, 100
In round 2 vertex 0 informs 5, 10, 15, 20 ; 25 informs $30, 35, 40, 45$ ; ...
In round 3, vertex 0 informs 1 to 4 ; 5 informs 6 to 10 ; ...

In these examples we have ignored details of the paths between vertices. The order of the rounds is important because it affects the choice of paths, and in a practical broadcast algorithm the order of rounds is the reverse of the visualisation of building a tile of increasing size. The correct order for a practical broadcast algorithm is to send the maximum distance in the first round and to send to the immediate neighbours in the last round (otherwise the last round has many vertices communicating via long paths which will guarantee congestion). The natural order for tiling is the reverse of this. The initial tile grows by sending to its immediate neighbours, and this is repeated recursively.

## 2.4 Broadcasting as Recursive Tiling

The visualisation of broadcasting as recursive tiling is useful to describe edge-disjoint paths of shortest length and also to show how the torus is formed. Section 2.4.1 describes the method in detail.

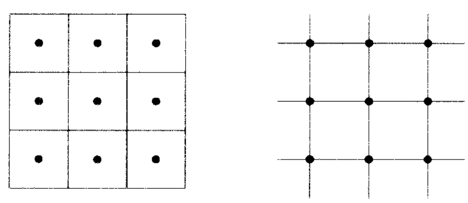

Figure 2.2: Standard square mesh shown as a tiling and a graph

We regard each square region as a vertex and each common edge as a communications link. Thus the 'wires' between processors are perpendicular to the tile edges. Generally a dot will be drawn at the centre of each tile as a reminder of the vertex that it represents.

A vertex and its four neighbours form a cross shape as in Figure 2.3a. In any round a vertex can inform its four neighbours, or any other four vertices via suitable circuit-switched paths.

The analysis in Appendix A shows that a good (and somewhat obvious) choice for the last round is for each vertex to send to its four immediate neighbours. This is assumed by Tsai and McKinley [16] and changes the problem to that of identifying a dominating set. Golomb and Welch [8] give a general method for defining a suitable dominating set in $d$ dimensions and conjecture that it is essentially unique.

The analysis for $d$ dimensions in Appendix A also suggests a first round when $2d+1$ is a prime. This first round is generated by a combination of $i$ steps along each dimension $i$, and can be represented by the vectors

$$k(1,2,3,...,d) \bmod (2d+1) \text{ for } k = 1,2\ldots 2d.$$

For example in three dimensions, the vectors $k(1,2,3) \pmod 7$ are

$$\begin{aligned} & (1,2,3) \\ (2,4,6) &= (2,-3,-1) \\ (3,6,9) &= (3,-1,2) \\ (4,8,12) &= (-3,1,-2) \\ (5,10,15) &= (-2,3,1) \\ (6,12,18) &= (-1,-2,-3) \pmod 7. \end{aligned}$$

Notice that the values for the coordinates are simply permuted, together with changes in sign.

### 2.4.1 Broadcasting to 25 Vertices

Initially one vertex, represented by a square tile, is informed. We will use $x, y$ coordinates for the tiles and assume the initial tile is at (0,0). Details of communication paths will be ignored until later. There are two rounds. In round 2 each informed tile informs its four neighbours, at relative coordinates

$$\{(0,\pm1),(\pm1,0)\},$$

giving cross-shapes as in Fig 2.3a. In round 1 the initial vertex (0,0) sends to the vertices

$$\{(+1,+2),(-2,+1),(-1,-2),(+2,-1)\}$$

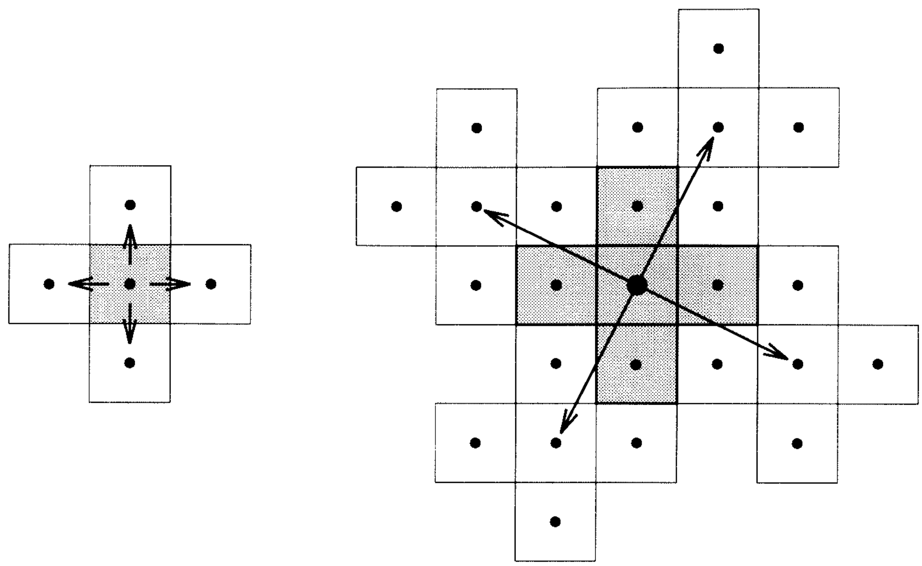

Figure 2.3: a) Round 2 Forms Cross Shapes... b) Round 1 Informs Each Cross

at the centres of the crosses. This is equivalent to adding 4 copies of the cross around the original one, as shown in Fig 2.3b. The rounds have been described in reverse order above, to emphasize how the tiling proceeds.

Fig 2.4 shows the rounds of a broadcast in the correct order. Round 1 informs four widely dispersed vertices (the centres of the surrounding crosses). In round 2 each informed vertex informs its neighbours. The broadcast is now complete, and we can join opposite edges of the final shape to form a torus.

Seen from this view, the broadcast strategy is to have the informed vertices evenly dispersed throughout the mesh after each round. Each informed vertex is then 'responsible' for a small region around itself. These regions are all the same shape in any specific round and can be considered separately. Provided the broadcast paths remain within a region for a round, we do not need to consider path contention between different regions. (This is not a necessary condition, but makes it easier to identify edge disjoint paths.)

The remainder of the thesis numbers the rounds of a broadcast in the correct order for a broadcast, but will often descibe them in some other order.

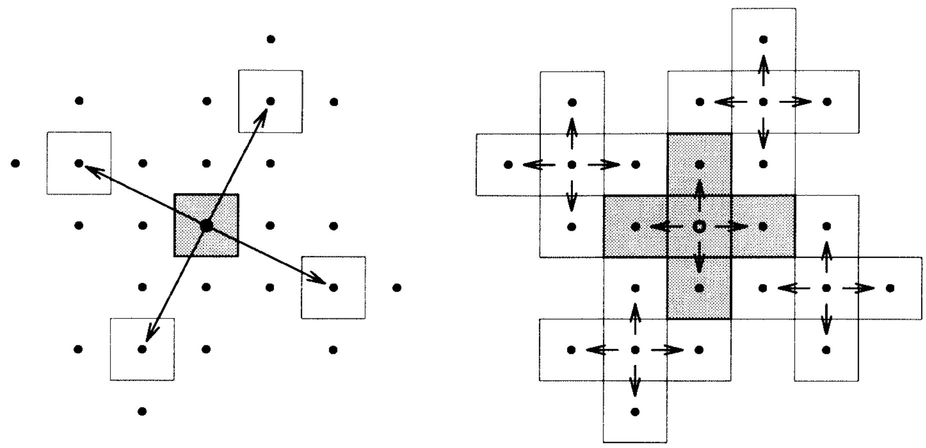

Figure 2.4: a) Round 1 disperses ... b) Round 2 fills in

### 2.4.2 Broadcasting to $5^{2n}$ Vertices

The tiling pattern described for 25 vertices can be repeated, but scaled up by a factor of 5 each time. Thus round 1 will inform vertices

$$\{(5^{n-1}, 2 \times 5^{n-1}), (-5^{n-1}, -2 \times 5^{n-1}), (2 \times 5^{n-1}, -5^{n-1}), (-2 \times 5^{n-1}, 5^{n-1})\}.$$

The shape of the combined tile will always be approximately a cross or a square. Figure 2.5 from Peters and Syska [13] shows how the pattern develops for $n = 4$. Note that the resulting shape will tile a normal torus only after an even number of rounds. (The method discussed in Appendix C shows that it can tile a skewed torus after an odd number of rounds.)

## 2.5 Relation to Circulant Graphs

This section looks at embedding a mesh network in a circulant graph and compares the resulting graph with that of a normal torus-connected mesh network.

Initially we will consider a $4 \times 4$ mesh. The tiles are numbered row by row as in the shaded area of Figure 2.6. The normal torus connections are indicated by surrounding the shaded area by copies of itself - it is easy to see from this diagram that, for example, the torus connection places vertex 0 next to vertices 3 and 12.

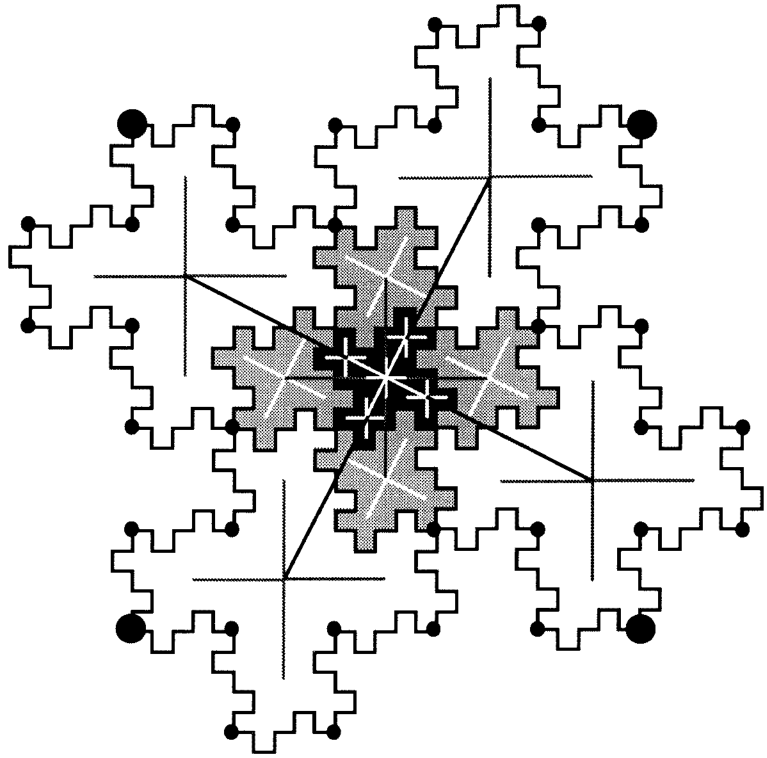

Figure 2.5: Broadcast Pattern for $5^4$ Vertices

Appendix C discusses this way of representing torus connections, which is useful for more complex shapes. It is also mentioned in Coxeter and Moser [6], page 24.

Figure 2.7a shows the $4 \times 4$ mesh network graph redrawn so that the vertices lie on a circle. Solid lines represent the mesh connections and the normal torus connections are shown as dotted lines. Drawn in this way the mesh looks very similar to a circulant graph and only minor adjustments (Figure 2.7b) are needed to convert it to a circulant graph. Vertex 0 is connected to vertex 15 instead of 3; vertex 4 to 3 instead of 7; vertex 8 to 7 instead of 11; and vertex 12 to 11 instead of 4.

In terms of the tiling diagram (Figure 2.6) these adjustments are equivalent to sliding the copies of the $4 \times 4$ square as in Figure 2.8a.

We will describe the graph with wraparound connections illustrated by Figure 2.8 as a skewed torus. This is consistent with the definition in Chapter 1, which is repeated here for convenience:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 | 4 | 5 | 6 | 7 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 8 | 9 | 10 | 11 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 12 | 13 | 14 | 15 | 12 | 13 | 14 | 15 |
| 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 | 4 | 5 | 6 | 7 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 8 | 9 | 10 | 11 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 12 | 13 | 14 | 15 | 12 | 13 | 14 | 15 |
| 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 | 4 | 5 | 6 | 7 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 8 | 9 | 10 | 11 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 12 | 13 | 14 | 15 | 12 | 13 | 14 | 15 |

Figure 2.6: $4 \times 4$ tiling, showing normal torus connections

When the graph derived from a square mesh network is regarded as being embedded in the surface of a torus, the torus is represented as a square with opposite sides identified, and with the edges of the graph parallel to the sides of the square. It is possible to draw a modified graph so that edges of the mesh slope relative to the square and connect up in different ways at opposite sides of the square. We will define this as a skewed torus connection. This definition extends into 3 dimensions and higher and the 3-dimensional skewed torus will be discussed in Chapter 4.

Note that a mesh network can form a skewed torus if and only if the tiling representation can tile the plane by translation (reflection and rotation are not allowed), and this also extends to higher dimensions. E.g. if a 3-dimensional shape can tile space by translation, then the shape can form a torus. This follows from a consideration of the automorphism group of the tiling, described in Senechal [14] and Grunbaum and Shephard [9].

Although the circulant graph in Figure 2.7b is vertex-transitive, there is an asymmetry between the $x$ and $y$ directions of the tiling representation in Figure 2.8a: a

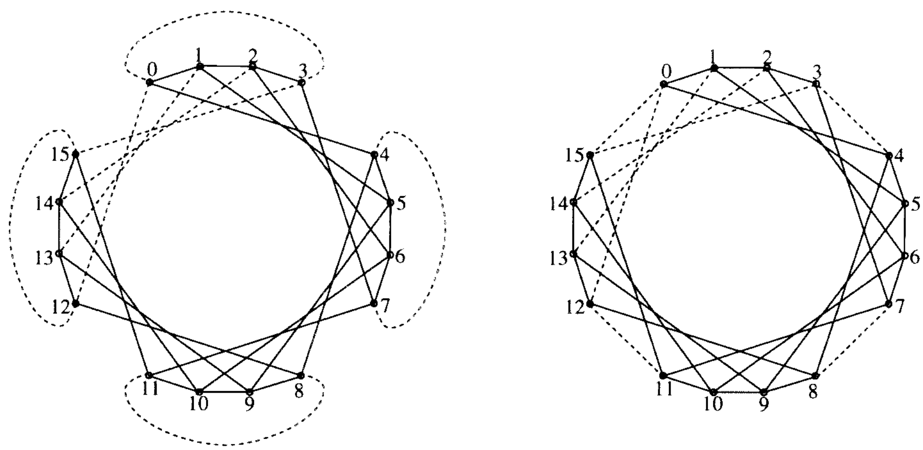


Figure 2.7: a) Redrawn 4 × 4 mesh ... b) modified to form a circulant graph

path in the $x$-direction gives a Hamiltonian cycle; the $y$-direction does not. We cannot skew in both directions at once. In terms of the circulant graph it means that following a path consisting only of the chords of length 4 does not give a Hamiltonian cycle.

This asymmetry can be removed by adding a vertex as shown in Figure 2.8b. Note that this extra vertex corresponds to the redundant vertex used for resilience in Bruck, Cypher and Ho [2]. Because the graph is vertex-transitive we can regard any vertex from Figure 2.8b as the 'extra' one and the torus can be cut so that the remaining vertices form the 4 × 4 square mesh. This suggests other opportunities for designing fault tolerant meshes which are beyond the scope of this thesis.

Up to this point we have always started by considering a mesh and deriving a circulant graph. It is instructive to derive the mesh tiling from an arbitrary circulant graph of degree 4. We can always regard the tiles as a strip of squares which represent the main cycle of the circulant graph, and such a strip can always be wound into a torus as shown in Figure 2.9.

The chord lengths of the circulant graph correspond to the number of squares in one loop around the torus. Thus a short chord length corresponds to a tightly wound torus. This construction can be extended in a variety of ways to give representations

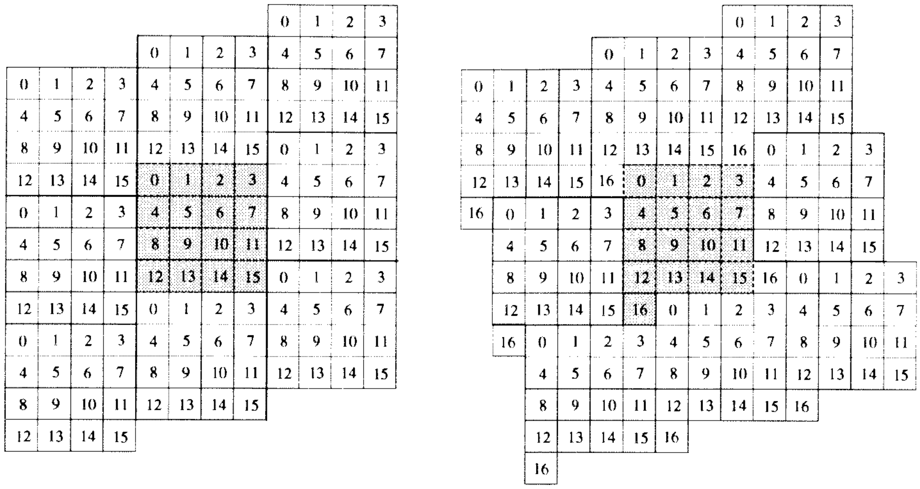

Figure 2.8: a) The skewed 4 × 4 tiling b) A 17 vertex tiling skewed both ways

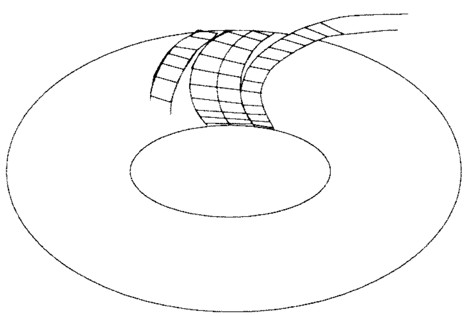

Figure 2.9: Winding the torus from a strip of squares

for other constructions, such as tiling with triangles or hexagons and tiling in 3 dimensions. It also gives a canonical form for the diagram of skewed torus connections. We have already shown (Figure 2.6) how to surround the square with copies of itself and use this representation to show which tiles will be adjacent when the square is wrapped into the corresponding torus. If we represent the circulant by a row of squares numbered 0 to 15 and repeat this row of squares shifted by the circulant chord length each time, we get a canonical representation of the skewed torus. Any combination of tiles 0 to 15 from this representation which tiles the plane by translation represents the same skewed torus or circulant graph. We will use this approach for degree 3 graphs in Chapter 5.

## 2.6 Broadcasting on a Skewed Torus in Two Dimensions

This case is treated in detail since

- it can be compared with the optimal solution known for the normal torus (Peters and Syska);
- it is easy to visualise; and
- it illustrates all the methods used in higher dimensions.

## 2.7 Five Vertices

This trivial case forms the basis for all others. The chords can only be $\pm 2$.

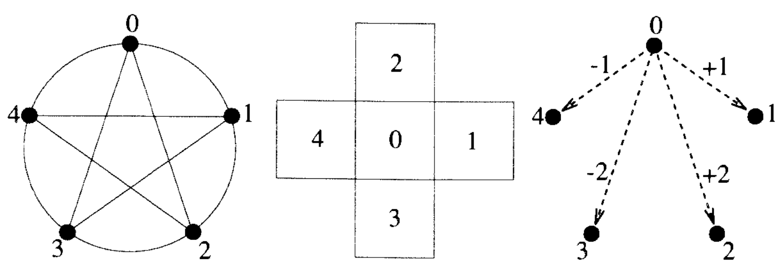


Figure 2.10: Graph, tiling and paths for broadcasting

The cross shape can be made into a skewed torus (see Appendix C).

## 2.8 Twenty-five Vertices

We start by using the same method as for 5 vertices, and apply it twice, increasing the scale by a factor of 5 for round 1.

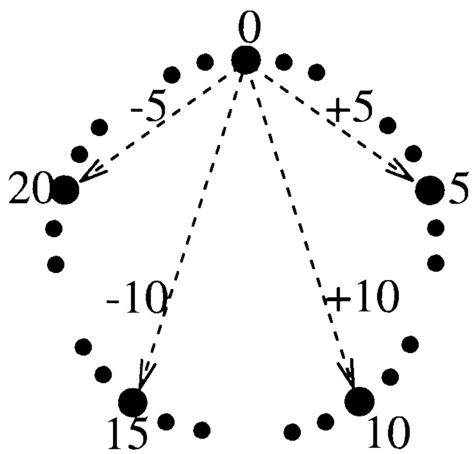


Figure 2.11: Paths for round 1 with 25 vertices

Initially assume a chord length of 2 (a very tightly wound torus) and that the targets in the last round are vertices $\pm 1, \pm 2$. Thus, scaling by 5, the target vertices in the first round must be vertices $\pm 5, \pm 10$, as in Fig 2.11 where the target vertices are shown enlarged. We can extend this method so that round 1 always divides the vertices into 5 'tiles' and broadcasts to the centres of the tiles. The next round takes each tile and divides it into 5 smaller tiles, until in the last round the tile representing each target consists of only 1 vertex.

### 2.8.1 Varying the Chord Length

A chord length of 2 would be inefficient for large networks, so we must choose a different chord length. In general we will choose a chord length of approximately $\sqrt{N}$ to represent a skewed torus similar to the normal $n \times n$ torus. This value gives a graph of minimum diameter for a normal torus with $n^2$ vertices. For a skewed torus it is possible to achieve a smaller diameter, as described in Appendix B.3, but in this thesis we restrict ourselves to changing the normal torus as little as possible.

Since in the last round of a broadcast the targets of an informed vertex are its immediate neighbours, changing the chord length changes the vertices each informed vertex sends to in the last round.

Figure 2.12 illustrates why increasing the chord length by an appropriate value does not alter the set of all destination vertices.

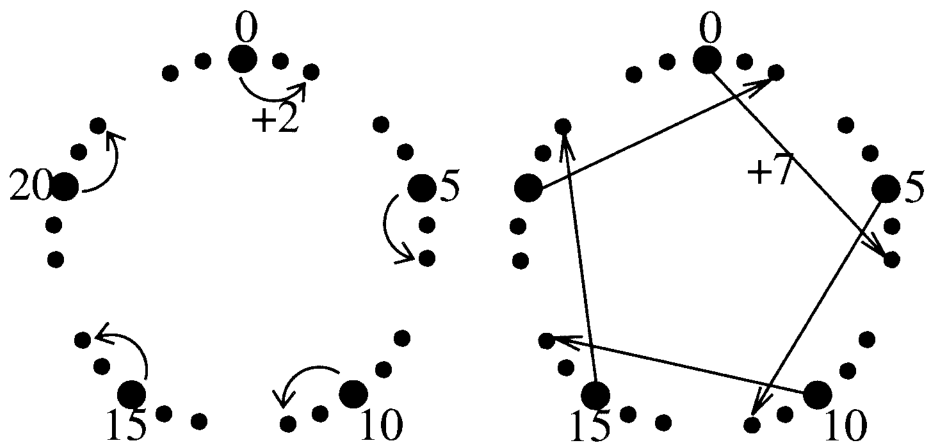


Figure 2.12: Adding 5 to all targets informs the same vertices

In the last round, the vertices informed via the +2 chord are

$$0+2, 5+2, 10+2, 15+2, 20+2 = 2, 7, 12, 17, 22.$$

The same set of vertices are informed if we add any multiple of 5 to the chord length. For example, with a chord length of 7

$$0+7, 5+7, 10+7, 15+7, 20+7 = 7, 12, 17, 22, 2 \pmod{25}.$$

Similarly, the $-2$ chord could be replaced by a $-7$ chord. In general, for a graph of $5^n$ vertices, the distance between informed vertices at the start of round $r$ is $5^{n-r+1}$ and we can add any multiple of $5^{n-r+1}$ to the chord length.

Changing the order of rounds can destroy the rotational symmetry on which we base the option to change chord lengths. For example, we could start with vertex 0 sending to vertices

$$-2, -1, 1, 2$$

or even, by adding multiples of 5, to vertices

$$-2, -11, 1, 7.$$

To show that changing the chord length does not interfere with the ability to change the order of rounds, we will derive a general expression for the set of vertices informed by the broadcast and argue that changing the order of rounds has no effect because of the commutative properties of addition.

We assume that in the round that has target vertices of

$$\pm 5^{n-r}, \pm 2 \times 5^{n-r}$$

that for each target number $i$ where $-2 \leq i \leq 2$
we can add an arbitrary multiple $f(r,i)$ of $5^{n-r+1}$.

Note that $f(r,i)$ must be the same for each vertex that is sending the message in that round, but can be different for different target numbers or rounds. In this round vertex 0 will inform the set of vertices

$$\{[5^{n-r}i + 5^{n-r+1}f(r,i)] \bmod 5^n | -2 \leq i \leq 2\}.$$

The set of all vertices $X_n$ informed after $n$ rounds is then

$$X_n = \{\sum_{r=1}^{n}[5^{n-r}i + 5^{n-r+1}f(r,i)] \bmod 5^n | -2 \leq i \leq 2\}$$

Since addition is commutative, the order of summation is irrelevant. Thus the order of rounds does not affect the set of vertices reached. The order defined by $r = 1$ to $r = n$ which we initially assumed gives rotational symmetry for the infomed vertices. From the rotational symmetry we deduce that the arbitrary multiple of $5^{n-r+1}$ can be ignored for each target vertex in each round.

If we return to the tiling representation and number the tiles to correspond to a chord length of 7, then tiles 5, 10, 15 and 20 appear in a familiar pattern shown in Fig 2.13. This pattern is the same as for the normal torus, so for 25 vertices the same pattern can be wrapped into a skewed torus.

Other chord lengths possible are 12, -8, or -3 (by adding multiples of 5 to 2). The minus signs can be eliminated by reflecting the diagram (which is a null operation because of symmetry).

If we choose a Hamiltonian cycle consisting of chords of length 3, then a path of 8 chords of length 3 brings us to vertex 24, which is equivalent to vertex $-1 \bmod 25$ and

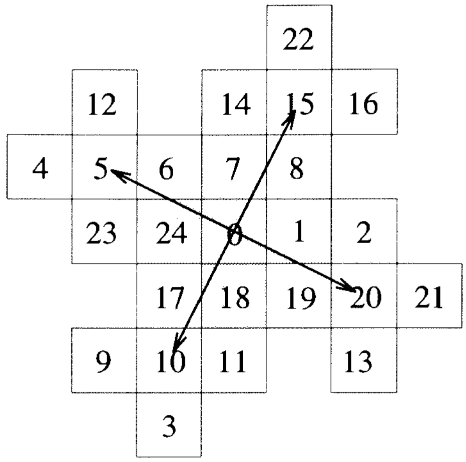


Figure 2.13: The pattern for tiling a skewed torus

thus adjacent to vertex 0. Thus if we renumber the vertices along this Hamiltonian cycle, vertex 0 will be adjacent to vertex 8. This shows that chord lengths of 3 and 8 give isomorphic graphs. In a similar way, since $12 \times 2 = 24$, chord lengths of 2 and 12 give isomorphic graphs.

### 2.8.2 Terminology

Here we define the basic terms used for describing broadcasts in subsequent chapters. In this chapter we have dealt with two-dimensional meshes which can be represented by graphs with vertices of degree 4. For meshes in $d$ dimensions the corresponding graphs will have vertices of degree $2d$.

**Base vertices** are those which are already informed at the start of the round being considered.

**Target vertices** are those to be informed during the round. Normally only paths in the positive direction will be described since the negative direction can be treated symmetrically by reflection. Each base vertex informs $d$ target vertices in each direction, and we will refer to the target vertices as target number $\pm 1$ to target number $\pm d$ where the sign of the target number indicates its direction.

**Edges** will be referred to as chords, giving the full chord length e.g. $(2d+1)^2+3$ or the chord length modulo $(2d+1)$, e.g. the *3-chord*. The 'chords' in the main cycle of the circulant have length 1 and are referred to as 1-chords. This notation will not be ambiguous for the chord lengths which are actually chosen.

**Path length** is the number of chords traversed to get from a base vertex to a target vertex.

**Forwards** or **backwards** will be used to refer to positive and negative chords from a vertex respectively.

### 2.8.3 Describing Paths With Tables

A vertex will be referred to by the difference between its vertex number and that of the nearest base vertex. Thus we effectively describe the algorithm in the region around vertex 0. Paths for each round of a broadcast will be described by a $d \times d$ table. Row $i$ describes how many of each chord length are needed for a path from the base vertex to target $i$. Negative values indicate chords in the reverse direction. This table can be regarded as a matrix and, if the positive chord lengths are written as a column vector, then multiplication by the matrix results in a column vector of the positive target vertices informed by vertex 0 in that round. The order of chords in a path is described separately.

Paths to the negative target numbers are given by reversing the sign of all numbers in the body of the table.

For example, the 25 vertex broadcast can be represented as the table below.

| $d=2$ | Chord Lengths 1, 7 | | | |
|---|---|---|---|---|
| | Round 1 | | Round 2 | |
| Target | Chord No. | | Chord No. | |
| Number | 1 | 2 | 1 | 2 |
| 1 | 1 | 0 | −2 | 1 |
| 2 | 0 | 1 | −1 | −2 |

Matrix multiplication gives the (positive) target vertices informed by round 2 to be

$$\begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -15 \end{pmatrix}.$$

Note that $-15 = 10 \bmod 25$ as required.

### 2.8.4 Broadcasting to 125 Vertices

For an approximately square shape when the torus is unrolled, we take a chord length close to $\sqrt{125} \approx 11$. The chord length must be 2 (or $-2$) modulo 5, so that the neighbours of vertex 0 are $\pm 1, \pm 2$ modulo 5 for the last round of the broadcast. Suitable values for the chord length are 12 or 13. As before, we start by assuming the target vertices in the three rounds will be

$$\begin{array}{lll} \pm 25, \pm 50 & (\bmod\ 125) & \text{in round 1} \\ \pm 5, \pm 10 & (\bmod\ 25) & \text{in round 2} \\ \pm 1, \pm 2 & (\bmod\ 5) & \text{in round 3.} \end{array}$$

Although this has the same numeric values in the last 2 rounds as the 25 vertex broadcast, the tiling changes because of the different chord length. If we assume a chord length of 12, the diameter of the graph is 9 and the 3 rounds can be represented by:

| $d = 2$ | Chord Lengths 1, 12 | | | | | |
|---|---|---|---|---|---|---|
| | Round 3 | | Round 2 | | Round 1 | |
| Target | Chord No. | | Chord No. | | Chord No. | |
| Number | 1 | 2 | 1 | 2 | 1 | 2 |
| 1 | 1 | 0 | 5 | 0 | 1 | 2 |
| 2 | 0 | 1 | $-2$ | 1 | 2 | 4 |

The longest paths in the three rounds are of length $(1+0)$, $(5+0)$, and $(2+4)$ respectively, giving a total path length of 12 whereas the diameter is only 9. We can adjust the total path to be optimal, but just get the known tiling with adjustments at the edges to allow a skewed wraparound. Instead, in the next chapter we extend the method to any number of rounds and in Chapter 7 we show that the result is very close to optimal.

A chord length of 13 gives similar results to the chord length of 12, and can be described by the table below.

| $d=2$ | Chord Lengths 1, 13 | | | | | |
|---|---|---|---|---|---|---|
| | Round 3 | | Round 2 | | Round 1 | |
| Target | Chord No. | | Chord No. | | Chord No. | |
| Number | 1 | 2 | 1 | 2 | 1 | 2 |
| 1 | 1 | 0 | 5 | 0 | −1 | 2 |
| 2 | 0 | 1 | −3 | 1 | −2 | 4 |

# Chapter 3

# The Two Dimensional Skewed Torus

## 3.1 Introduction

This chapter deals with broadcasting on a skewed torus network in terms of a circulant graph. The broadcasting algorithm developed in this way is then visualised as a tiling. As in Peters and Syska [13], it is based on a $5^n \times 5^n$ mesh containing $N = 5^{2n}$ vertices.

For the skewed torus we take the chord length to be $2 + \sqrt{N} = 2 + 5^n$ to give a graph with a diameter of $\sqrt{N} - 2$. We will develop a broadcast scheme requiring the minimum number of rounds ($2n$) and with a total path length only 15% greater than the diameter of the graph. (Note that this is not claimed to be the best chord length or the best possible broadcast algorithm. The diameter for a skewed torus can be as low as $\sqrt{N/2}$ with a chord length of $\sqrt{2N}$, as shown in Appendix B. )

### 3.1.1 Initial Selection of Vertices For Each Round

We can regard the $5^{2n}$ vertices to be in $2n$-dimensional space

$$Z_5 \times Z_5 \times \cdots \times Z_5,$$

and the natural broadcast scheme is to span one dimension at a time. We start by assuming the base vertices will broadcast to the target vertices shown in the table below for each round. Later we will adjust some of these numbers to take advantage of rotational symmetry, and the table indicates this in modulo notation. For example, in round $2n$ vertex 0 sends to vertices $-2$, $-1$, 1 and 2 $\pmod 5$ and we will later choose these to be vertices $-(2+\sqrt{N}), -1, 1$ and $2+\sqrt{N}$ (which is valid since $\sqrt{N}$ is a multiple of 5).

| Round | Targets | |
|---|---|---|
| $2n$ | $\pm 1, \pm 2$ | $\pmod 5$ |
| $2n-1$ | $\pm 5, \pm 10$ | $\pmod{25}$ |
| $2n-2$ | $\pm 25, \pm 50$ | $\pmod{125}$ |
| ... | | |
| 1 | $\pm 5^{2n-1}, \pm 2 \times 5^{2n-1}$ | $\pmod{5^{2n}}$ |

This construction gives the optimum number of rounds for a broadcast, but does not guarantee the existence of edge disjoint paths in any round. Techniques for selecting paths are described in Chapter 7 but their existence can also be deduced informally by considering the tiling representation of the broadcast which is described below.

The broadcast can be viewed as a two dimensional recursive tiling with vertices described in terms of $xy$ coordinates, where the $x$ direction represents the main cycle of the circulant the $y$ direction represents the chords. In this case the tile at coordinates $(x, y)$ represents vertex number $x + (2+\sqrt{N})y$. Consider first the shape of the tiles corresponding to the table below where the rotational symmetry is not used.

| Round | Targets |
|---|---|
| $2n$ | $\pm 1, \pm 2$ |
| $2n-1$ | $\pm 5, \pm 10$ |
| $2n-2$ | $\pm 25, \pm 50$ |
| ... | |
| 1 | $\pm 5^{2n-1}, \pm 2 \times 5^{2n-1}$ |

If we do not use rotational symmetry to adjust the target vertex numbers, the tiling model of the broadcast (which begins with round $2n$) will first inform vertices $-2$ to $+2$ in round $2n$ and then vertices $-12$ to $+12$ in round $2n-1$, creating a long thin shape along the $x$ axis. Rounds $2n$ to $n+1$ will continue to copy the shape in the $x$ direction and then rounds $n$ to 1 will copy it sideways in the $y$ direction. Intuitively each round of a tiling method should fit five copies of the shape from the previous round closely together to give a new shape of tile. This closeness corresponds to an efficient use of communications paths, and the path lengths should start at 1 (adjacent tiles in round $2n$) and increase as the tile grows (maximal dispersion of the message in round 1). In terms of the broadcast, the longest paths are used in round 1 when there is only one source vertex; the shortest paths are used in the last round when every vertex is either a source or a target. (The total number of edges used in round $r$ when a path is of length $p$ is $5^{r-1}p$, because there are $5^{r-1}$ source vertices.)

The shape of the tile can be improved by using rotational symmetry to adjust target vertex numbers for the last $n$ rounds and by interleaving them with the first $n$ rounds, so that the shape grows simultaneously in the $x$ and $y$ directions. In the last round each informed vertex sends to its immediate neighbours, $\pm 1$ and $\pm(\sqrt{N}+2)$, which can be regarded as single steps along the $x$- and $y$-axes to form a cross shape. In the next to the last round an informed vertex will send to the target vertices $\pm\sqrt{N}$, $\pm 2\sqrt{N}$ which can be represented by $xy$ coordinates $(-2,1),(2,-1),(-4,2),(4,-2)$ and give short path lengths of 3, 3, 6 and 6 respectively. Lower numbered rounds scale up these last two rounds, giving the target vertices for each round shown in the table below. Note that the rounds have been renumbered: the left half of the table corresponds to the original rounds $2n$ to $n+1$, which have now become the even numbered rounds from $2n$ to 2.

| Round | Targets | Round | Targets |
|---|---|---|---|
| $2n$ | $\pm 1, \pm(\sqrt{N}+2)$ | $2n-1$ | $\pm 5^n, \pm 2\times 5^n$ |
| $2n-2$ | $\pm 5, \pm 5(\sqrt{N}+2)$ | $2n-3$ | $\pm 5^{n+1}, \pm 2\times 5^{n+1}$ |
| $2n-4$ | $\pm 25, \pm 25(\sqrt{N}+2)$ | $2n-5$ | $\pm 5^{n+2}, \pm 2\times 5^{n+2}$ |
| ... | | | |
| 2 | $\pm 5^{n-1}, \pm 5^{n-1}(\sqrt{N}+2)$ | 1 | $\pm 5^{2n-1}, \pm 2\times 5^{2n-1}$ |

Since the rounds can be in any order, interleaving them to take advantage of the chord length in this way is valid.

We now specify the paths for the last 2 rounds in terms of the chord length of $2 + 5^n$, using the tabular form from the previous chapter. Note that negative targets are omitted as the entries would be the same but with the signs reversed.

| $d = 2$ | Chord Lengths 1, $5^n + 2$ | | | |
|---|---|---|---|---|
| | Round 2n | | Round 2n-1 | |
| Target | Chord No. | | Chord No. | |
| Number | 1 | 2 | 1 | 2 |
| 1 | 1 | 0 | $-2$ | 1 |
| 2 | 0 | 1 | $-4$ | 2 |

As a check, matrix multiplication gives the (positive) target vertices informed by round 2n-1 to be

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 5^n + 2 \end{pmatrix} = \begin{pmatrix} 5^n \\ 2 \times 5^n \end{pmatrix}.$$

The total path length for this pair of rounds is $1 + 6 = 7$.

Paths in previous rounds can be the same, but scaled by increasing powers of 5. Thus the scaling factor in rounds 1 and 2 is $5^{n-1}$ and in rounds $2g - 1, 2g$ is $5^{n-g}$. The total path length will be a geometric sum

$$(1 + 6)(1 + 5 + 25 ... + 5^{n-1}) = 7(5^n - 1)/4$$

which is about 7/4 times the diameter. The next section looks at ways of reducing the total path length.

### 3.1.2 Reducing Path Lengths

If a path contains a large number of short steps along the main cycle, they can be replaced by a long chord. This gives a shorter path when the number of steps along the main cycle is more than half the length of the chord. The longest path along the main cycle is 4 times the scaling factor of $5^{n-g}$ in round $2g - 1$. This is greater than half the chord length when

$$4 \times 5^{n-g} > (5^n + 2)/2,$$

which is only when $n - g = n - 1$, the first pair of rounds! (We could also consider replacing large enough multiples of $\sqrt{N} + 2$ by paths along the main cycle, but this is never feasible.)

The chord which replaces the steps along the main cycle is in the opposite direction to other chords in the path, so the number of chords decreases by one instead of increasing. Thus the target vertex with the longest path in the first round can be improved from (in vector notation)

$$(-4 \times 5^{n-1}, 2 \times 5^{n-1})$$

$$= (-5^n + 5^{n-1}, 2 \times 5^{n-1})$$

to

$$(5^{n-1} + 2, 2 \times 5^{n-1} - 1).$$

Note that the vector representation is not unique.

The length of the first path is reduced in this way from $6 \times 5^{n-1}$ to

$$(5^{n-1} + 2) + 2 \times 5^{n-1} - 1 = 3 \times 5^{n-1} + 1.$$

The total path length is reduced by $3 \times 5^{n-1} - 1$ to

$$7(5^n - 1)/4 - 3 \times 5^{n-1} + 1 = (23 \times 5^{n-1} - 3)/4,$$

which is within a factor of 1.15 of the diameter of the normal torus mesh.

So the first round for a broadcast on a skewed torus with $n = 2$ (number of vertices, $N = 625$) and a chord length of $(5^2 + 2)$ has target vertices

$$\pm 125, \pm 250$$

These target vertices can be represented in coordinate form as

$$\pm(-10, 5), \pm(7, 9)$$

since

$$125 = \pm(-10 \times 1 + 5 \times 27), 250 = \pm(7 \times 1 + 9 \times 27).$$

The table describing all four rounds of the broadcast is shown below.

| $n = 2, N = 625$ | Chord Lengths 1, 27 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Round 4 modulo $5^1$ | | Round 3 modulo $5^3$ | | Round 2 modulo $5^2$ | | Round 1 modulo $5^4$ | |
| Target Number | Chord 1 | Chord 2 | Chord 1 | Chord 2 | Chord 1 | Chord 2 | Chord 1 | Chord 2 |
| 1 | 1 | 0 | −2 | 1 | 5 | 0 | −10 | 5 |
| 2 | 0 | 1 | −4 | 2 | 0 | 5 | 7 | 9 |

The path lengths for the four rounds are 1, 6, 5, and 16 giving a total of 28 compared with the previous result of $(1 + 6)(1 + 5) = 42$. This should be compared with a diameter of 23 for the skewed torus and 24 for the normal torus.

For larger $n$, rounds $2n - 2, 2n - 4, \ldots$ can also be adjusted to give shorter paths for target number 1 by replacing $5^{n-g} - 1$ of the 1-chords by $(5^{n-g} - 1)/2$ of the 2-chords. This gives a more compact shape for the tile shown in Figure 3.2 but does not reduce the maximum path length for the round because of target number 2. Our final broadcast for $N = 5^{2n}$ vertices can thus be represented by the tables below. Note that the table showing the general formula descibes all rounds except round 1.

| Last 4 Rounds | Chord Lengths 1, $5^n + 2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $g = n, n - 1$ | Round $2n$ modulo $5^1$ | | Round $2n - 1$ modulo $5^{n+1}$ | | Round $2n - 2$ modulo $5^2$ | | Round $2n - 3$ modulo $5^{n+2}$ | |
| Target Number | Chord 1 | Chord 2 | Chord 1 | Chord 2 | Chord 1 | Chord 2 | Chord 1 | Chord 2 |
| 1 | 1 | 0 | −2 | 1 | 1 | 2 | −10 | 5 |
| 2 | 0 | 1 | −4 | 2 | 0 | 5 | −20 | 10 |

| General Formula | Chord Lengths 1, $5^n + 2$ | | | |
|---|---|---|---|---|
| | Round $2g$ modulo $5^{n-g+1}$ | | Round $2g - 1$ modulo $5^{2n-g+1}$ | |
| Target Number | Chord 1 | Chord 2 | Chord 1 | Chord 2 |
| 1 | 1 | $(5^{n-g} - 1)/2$ | $-2 \times 5^{n-g}$ | $5^{n-g}$ |
| 2 | 0 | $5^{n-g}$ | $-4 \times 5^{n-g}$ | $2 \times 5^{n-g}$ |

| First 2 Rounds | Chord Lengths 1, $5^n+2$ | | | |
|---|---|---|---|---|
| when $g=1$ | Round 2 modulo $5^n$ | | Round 1 modulo $5^{2n}$ | |
| Target Number | Chord 1 | Chord 2 | Chord 1 | Chord 2 |
| 1 | 1 | $(5^{n-1}-1)/2$ | $-2\times 5^{n-1}$ | $5^{n-1}$ |
| 2 | 0 | $5^{n-1}$ | $5^{n-1}+2$ | $2\times 5^{n-1}-1$ |

Fig 3.1 to Fig 3.3 show the tiling representation of the broadcast for $n=2$ for rounds 1,2 and 3. The shading allows the shape of the tile to be seen. It is clear from Figure 3.3 that the tiling is not as compact as the original one for the normal torus.

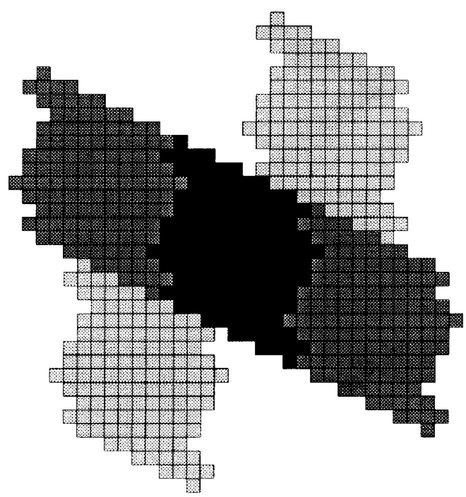

Figure 3.1: Round 1 tiles for 625 vertices

Round 4 is represented by any one of the five crosses shown shaded in Figure 3.3. Edge disjoint paths for each round can be found by inspection. For larger values of $n$ paths can be scaled up together with the tiles.

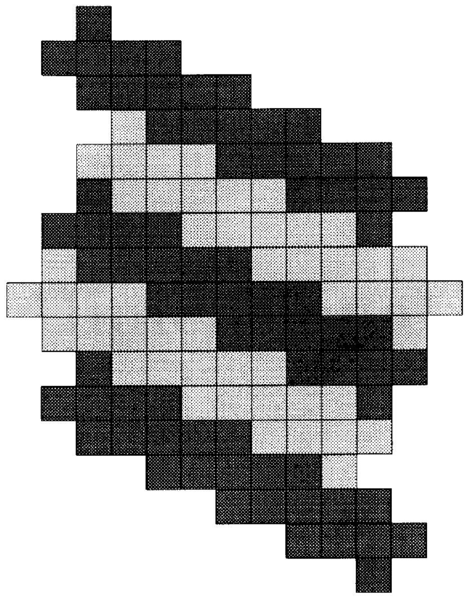

Figure 3.2: Round 2 tiles for 625 vertices

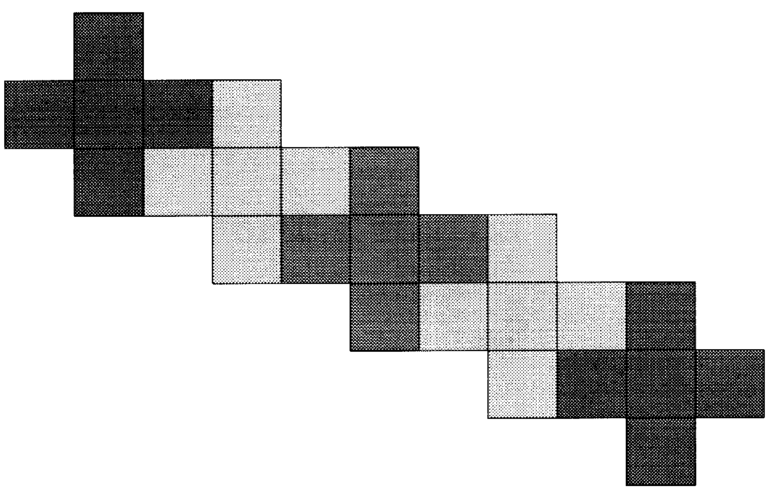

Figure 3.3: Round 3 tiles for 625 vertices

# Chapter 4

# The Three Dimensional Skewed Torus

## 4.1 Introduction

The three dimensional torus is a cube with opposite faces connected. This section deals with algorithms for the three dimensional *skewed* torus using the same general techniques as for two dimensions. More factors can be adjusted in three dimensions to find an optimal solution, but none has been found to improve on the basic result of a total path length which is less than 1.2 times the diameter of the corresponding normal torus.

## 4.2 Choosing Chord Lengths for the Circulant Graph

There are many ways to choose the chord length for a circulant graph to represent a skewed torus, but we consider only those which are suitable for the type of algorithm we have adopted. Note that changing the number of nodes slightly and choosing a different chord length can give a graph with a significantly smaller diameter, but we have restricted this thesis to a skewed torus formed by minor adjustments to the normal torus with $7^{3n}$ vertices, where the normal torus can be regarded as having chords of lengths $7^n$ and $7^{2n}$. Thus for the skewed torus we choose chord lengths of $7^n + 2$ and $7^{2n} + 3$ which will allow a broadcasting algorithm similar to that for two dimensions. Other adjustments to chord lengths and paths which were considered include

- to choose the signs in $7^n \pm 2, 7^{2n} \pm 3$;
- to swap the 2 and the 3, giving chord lengths $7^n \pm 3$ , $7^{2n} \pm 2$;
- to adjust the long chord by multiples of the short chord, e.g. $7^{2n} + 7^n - 3$; and
- replace multiple short chords with a long chord.

In higher dimensions the possible combinations increase exponentially, but a small set is sufficient to give a good bound on path length and only the last of these adjustments was in fact used.

## 4.3 Standard Skewed Construction for $N = 7^{3n}$ vertices

A normal three dimensional $7^n \times 7^n \times 7^n$ mesh network embedded in a circulant graph would have chord lengths of $7^n$ and $7^{2n}$. For the skewed torus we will take chord lengths of $(7^n + 2)$ and $(7^{2n} + 3)$ . Negative targets are treated symmetrically, so we will omit the $\pm$ sign from tables and deal only with positive target vertices. In a similar way to two dimensions, we start by assuming the positive target vertices for each round are given by the table below.

| Round | Target Vertices | |
|---|---|---|
| $3n$ | $1, 2, 3$ | (mod 7) |
| $3n - 1$ | $7, 14, 21$ | (mod 49) |
| $3n - 2$ | $49, 98, 147$ | (mod 343) |
| ... | | |
| 1 | $7^{3n-1}, 2 \times 7^{3n-1}, 3 \times 7^{3n-1}$ | (mod $7^{3n}$) |

Following the method used for two dimensions, in three dimensions we interleave the rounds in groups of three and renumber them, so the target vertices for the last three rounds will be

| Round | Targets | |
|---|---|---|
| $3n$ | $1, 2, 3$ | (mod 7) |
| $3n-1$ | $7^n, 2 \times 7^n, 3 \times 7^n$ | (mod $7^{n+1}$) |
| $3n-2$ | $7^{2n}, 2 \times 7^{2n}, 3 \times 7^{2n}$ | (mod $7^{2n+1}$) |

Each group of three rounds scales the previous group by 7. In the last round each vertex sends to its immediate neighbours and the target vertices for the last three rounds can be represented by the table below.

| Last 3 Rounds | Chord Lengths 1, $7^n+2$, $7^{2n}+3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round $3n$ (mod 7) Chord No. | | | Round $3n-1$ (mod $7^{n+1}$) Chord No. | | | Round $3n-2$ (mod $7^{2n+1}$) Chord No. | | |
| Target Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | 0 | 0 | 1 | 1 | −1 | −3 | 0 | 1 |
| 2 | 0 | 1 | 0 | −1 | 2 | −1 | −6 | 0 | 2 |
| 3 | 0 | 0 | 1 | 0 | 3 | −2 | −9 | 0 | 3 |

For example, in round $3n-1$ the vertices informed will be

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & -1 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 7^n+2 \\ 7^{2n}+3 \end{pmatrix} = \begin{pmatrix} 7^n - 7^{2n} \\ 2 \times 7^n - 7^{2n} \\ 3 \times 7^n - 2 \times 7^{2n} \end{pmatrix} = \begin{pmatrix} 7^n \\ 2 \times 7^n \\ 3 \times 7^n \end{pmatrix} \bmod 7^{n+1}.$$

In the last three rounds each informed node is effectively broadcasting to a $7 \times 7 \times 7$ skewed torus. The total path length for these last three rounds is $12 + 5 + 1 = 18$ although the diameter of a normal $7 \times 7 \times 7$ torus network is only 9.

This broadcast scheme for a group of three rounds can be repeated with a scaling factor of 7, and the values adjusted by using a chord of length $7^n + 2$ to replace two chords of length 1, or of length $7^{2n} + 3$ to replace three chords of length 1.

Thus round $3n-3$ has targets 7, 14 and 21 which can be replaced by

$$1+2(7^{2n}+3),(7^n+2)+4(7^{2n}+3) \text{ and } 7(7^{2n}+3).$$

Powers of 7 greater than $7^1$ can be ignored because of rotational symmetry.

Similarly, round $3n-4$ has target vertices

$$7^{n+1}, 2\times 7^{n+1} \text{ and } 3\times 7^{n+1}$$

which can replaced by

$$-2+7(7^n+2)-4(7^{2n}+3),$$
$$-1+14(7^n+2)-9(7^{2n}+3), \text{ and}$$
$$21(7^n+2)-14(7^{2n}+3).$$

Powers of 7 greater than $7^{n+1}$ can be ignored because of rotational symmetry.

It is convenient to think of $7^n+2$ being separated into a 'useful' term of $7^n$ and an 'error' term of 2. Thus for round $3n-4$, $7^n$ is a 'useful' chord length for target vertices $7^{n+1}$ , $2\times 7^{n+1}$ and $3\times 7^{n+1}$. Since the actual chord length is $7^n+2$ ,we must correct the 'error' of $2\times 7$, $4\times 7$ and $6\times 7$ with the other chords.

In round $3n-5$, with target vertices $7^{2n+1}$, $2\times 7^{2n+1}$ and $3\times 7^{2n+1}$, the $7^{2n}+3$ chord provides the 'useful' term of $7^{2n}$.

To give integral values for entries in the tables, note that $(7^k-1)/3$ is an integer for any $k$, and

$$\begin{aligned}
7^k &= 1+3\times(7^k-1)/3,\\
2\times 7^k &= 2+3\times 2(7^k-1)/3,\\
4\times 7^k &= 1+3\times(1+4[7^k-1]/3), \text{ and}\\
6\times 7^k &= 3\times 2(7^k-1)
\end{aligned}$$

(where in the last line the $3\times$ comes from the length of the 3-chord).

Using $g$ to number the groups of rounds gives the general results for $g>1$ shown in the tables below.

| General Formula 1 | Chord Lengths 1, $7^n+2$, $7^{2n}+3$ | | |
|---|---|---|---|
| | Round $3g$ working mod$7^{n-g+1}$ | | |
| Target | Chord No. | | |
| Number | 1 | 2 | 3 |
| 1 | 1 | 0 | $(7^{n-g}-1)/3$ |
| 2 | 0 | 1 | $2(7^{n-g}-1)/3$ |
| 3 | 0 | 0 | $7^{n-g}$ |

| General Formula 2 | Chord Lengths 1, $7^n+2$, $7^{2n}+3$ | | |
|---|---|---|---|
| | Round $3g-1$ working mod$7^{2n-g+1}$ | | |
| Target | Chord No. | | |
| Number | 1 | 2 | 3 |
| 1 | $-2$ | $7^{n-g}$ | $-2(7^{n-g}-1)/3$ |
| 2 | $-1$ | $2\times 7^{n-g}$ | $-1-4(7^{n-g}-1)/3$ |
| 3 | 0 | $3\times 7^{n-g}$ | $-2\times 7^{n-g}$ |

| General Formula 3 | Chord Lengths 1, $7^n+2$, $7^{2n}+3$ | | |
|---|---|---|---|
| | Round $3g-2$ working mod$7^{3n-g+1}$ | | |
| Target | Chord No. | | |
| Number | 1 | 2 | 3 |
| 1 | $-3\times 7^{n-g}$ | 0 | $7^{n-g}$ |
| 2 | $-6\times 7^{n-g}$ | 0 | $2\times 7^{n-g}$ |
| 3 | $-9\times 7^{n-g}$ | 0 | $3\times 7^{n-g}$ |

Since the broadcast uses all edges from each informed vertex, some path must begin with a 2-chord. The table for General Formula 3 does not include any 2-chords, and so we must adjust one of the paths to correct this. We can include a 2-chord at the start of the path to target number 1 or 2, so that the maximum path length (to target 3) for this round is not increased. That path must also include a 2-chord in the opposite direction to nullify the effect of the initial 2-chord.

Finally, the first group of rounds can be adjusted to give shorter paths. When $g=1$, we find that $-9\times 7^{n-g}$ exceeds $7^n$ and we can replace $(7^n+2)$ of the 1-chords with a single 2-chord.

Round 3 cannot be adjusted to give shorter paths, so General Formula 1 with $g = 1$ reduces to the table shown below.

| Round 3 | Chord Lengths 1, $7^n + 2$, $7^{2n} + 3$ | | |
|---|---|---|---|
| | Working mod$7^n$ | | |
| Target | Chord No. | | |
| Number | 1 | 2 | 3 |
| 1 | 1 | 0 | $(7^{n-1} - 1)/3$ |
| 2 | 0 | 1 | $2(7^{n-1} - 1)/3$ |
| 3 | 0 | 0 | $7^{n-1}$ |

In round 2 the chords of length $7^n + 2$ gives 'errors' of $4 \times 7^{n-1}$ to target number 2, and $6 \times 7^{n-1}$ to target number 3. These can be partially corrected by using one *less* of the $7^n + 2$ chords, and then correcting the remaining 'error' with the other chords. By doing this, the paths for round 2 are as shown in the table below.

| Round 2 | Chord Lengths 1, $7^n + 2$, $7^{2n} + 3$ | | |
|---|---|---|---|
| | Working mod$7^{2n}$ | | |
| Target | Chord No. | | |
| Number | 1 | 2 | 3 |
| 1 | $-2$ | $7^{n-1}$ | $-2(7^{n-1} - 1)/3$ |
| 2 | 2 | $2 \times 7^{n-1} - 1$ | $7^{n-1}$ |
| 3 | 0 | $3 \times 7^{n-1} - 1$ | $(7^{n-1} + 2)/3$ |

In round 1 a single $7^n + 2$ chord is again used to correct a large part of the 'error'.

| Round 1 | Chord Lengths 1, $7^n + 2$, $7^{2n} + 3$ | | |
|---|---|---|---|
| | Working mod$7^{3n}$ | | |
| Target | Chord No. | | |
| Number | 1 | 2 | 3 |
| 1 | $-3 \times 7^{n-1}$ | 0 | $7^{n-1}$ |
| 2 | $7^{n-1} + 2$ | $-1$ | $2 \times 7^{n-1}$ |
| 3 | $-2 \times 7^{n-1} + 2$ | $-1$ | $3 \times 7^{n-1}$ |

We can check round 1 by

$$\begin{pmatrix} -3\times 7^{n-1} & 0 & 7^{n-1} \\ 7^{n-1}+2 & -1 & 2\times 7^{n-1} \\ -2\times 7^{n-1}+2 & -1 & 3\times 7^{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 7^n+2 \\ 7^{2n}+3 \end{pmatrix}$$

$$= \begin{pmatrix} -3\times 7^{n-1}+7^{3n-1}+3\times 7^{n-1} \\ 7^{n-1}+2-7^n-2+6\times 7^{n-1}+2\times 7^{3n-1} \\ -2\times 7^{n-1}+2-7^n-2+3\times 7^{3n-1}+9\times 7^{n-1} \end{pmatrix} = \begin{pmatrix} 7^{3n-1} \\ 2\times 7^{3n-1} \\ 3\times 7^{3n-1} \end{pmatrix}.$$

The total path length for the first 3 rounds is thus

$$\begin{cases} 7^{n-1} & \text{for round 3} \\ (3\times 7^{n-1}-1)+(7^{n-1}+2)/3 & \text{for round 2} \\ 2\times 7^{n-1}-2+3\times 7^{n-1}+1 & \text{for round 1} \end{cases}$$

giving a total path length for the first 3 rounds of

$$28\times 7^{n-1}/3-4/3.$$

The total path length for all subsequent rounds is

$$\begin{aligned} 18(1+7+7^2\cdots) &= 18[(7^{n-1}-1)/(7-1)] \\ &= 3(7^{n-1}-1) \end{aligned}$$

and the total path length for the whole broadcast is

$$\begin{aligned} 3(7^{n-1}-1)+(28\times 7^{n-1}/3-4/3) &= 37\times 7^{n-1}/3-13/3 \\ &= 12.33\times 7^{n-1}-4.33. \end{aligned}$$

The diameter of the corresponding $7^n\times 7^n\times 7^n$ torus is $3(7^n-1)/2$, and the broadcast algorithm gives a total path length of less than $(37/21)/(3/2) = 1.174$ times the diameter of the normal torus.

This is worse than it might seem: the paths for round $3n-2$ are four times as long as the theoretical best, but paths for the last round dominate the final total. Ideally we would prefer to find a general method to improve the later rounds. In three dimensions and higher, geometric intuition fails and we have not pursued this further. Instead, in Chapter 7, the basic method is extended to $d$ dimensions and the ratio of actual to optimal total path length is shown to converge to 1 as $d$ increases.

# Chapter 5

# Tiling With Triangles

## 5.1 Introduction

Broadcasting on triangular meshes can be represented by triangular tilings. Little work has been done on meshes or torus meshes of odd degree. Tsai and McKinley [16] give some constructions for broadcasts which use only 3 out of 4 available ports on a normal degree 4 mesh. As in previous chapters we will represent the torus connected mesh as a cycle graph with chords, but it will be a chordal ring rather than a circulant graph. Broadcasting on such graphs (chordal rings) is dealt with in Arden and Lee [1]. Arden and Lee assert that their formulae for the diameters of chordal rings imply that the largest chordal ring of diameter 5 has only 34 vertices, but in this chapter we derive a chordal ring geometrically which has 38 vertices and is of diameter 5. This has been shown previously by Morillo, Comellas and Fiol [11].

## 5.2 The Diameter of a Triangular Mesh

We will look at two possible torus connected triangular meshes:

1. a simple conversion of a square mesh, with diameter $\approx 0.94\sqrt{N}$, and
2. the maximum nodes for a given diameter, with diameter $\approx 0.8\sqrt{N}$.

In later sections we use a chordal ring with chord length $\sqrt{N} + 3$ (chosen for the algorithm we develop). This has a diameter of approximately $\sqrt{N}$.

### 5.2.1 Converting a Square Mesh to Triangular

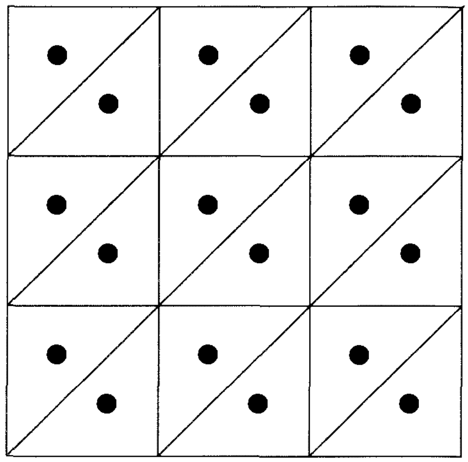

Figure 5.1: Converting a Square Mesh to Triangular

We start with a normal $n \times n$ square tiling. By dividing each square diagonally into 2 triangles (Figure 5.1), we get a $2n \times 2n$ triangular mesh of $2n^2$ triangles. If the square mesh is connected as a normal torus, then the resulting triangular mesh will have a diameter of approximately $4n/3$. This diameter has been derived by considering the propagation of a message outward from the centre. (See Figure 5.2 for n=9, which shows the distance of each vertex from the centre.)

Triangles within a distance $n$ of the centre form approximately a hexagon (shown shaded), which leaves two (roughly) triangular regions at opposite corners of the square. Because of the wraparound connections each of these triangular regions is surrounded by triangles which are at a distance $n$ from the centre of the square. The furthest triangle from the centre of the square must be at the centre of the one of the two triangular regions, and by symmetry will lie on the diagonal of the square. If we think of information flowing out from the centre of the square, this diagonal fills twice as fast from the corner of the square (by wraparound connections at the sides) as it fills from the original path outwards from the centre. The last triangle reached

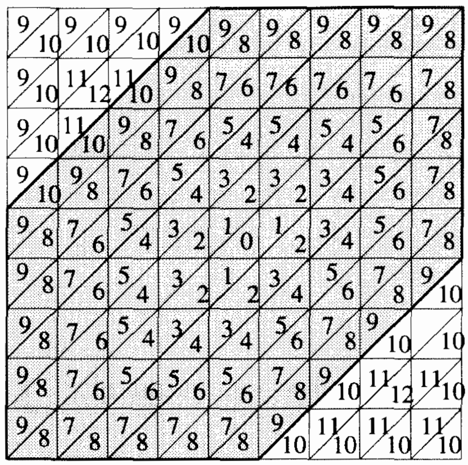

Figure 5.2: Distances of Vertices from the Centre of the Square

will thus be two thirds the distance from the corner of the square to the base of the triangular region, which is $2/3 \times n/2 = n/3$. The path length from the centre of the square to the furthest triangle is thus $n + n/3$. So for a triangular mesh of $N = 2n^2$ vertices, $n = \sqrt{N/2}$ and the diameter when the square is connected as a normal torus is

$$\frac{4}{3}\sqrt{N/2} = \sqrt{\frac{8}{9}N} \approx 0.94\sqrt{N}.$$

This is surprisingly close to the diameter of a normal square mesh torus.

### 5.2.2 Minimum Diameter for a Triangular Torus Mesh

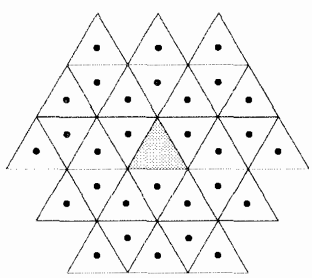

Figure 5.3: Vertices within a distance 4 of the centre

We proceed by finding the maximum number of vertices within a fixed distance of a central vertex, and derive a lower bound for the diameter. The tiling for the maximum number of vertices is adjusted to give a skewed torus shape and this gives the maximum number of vertices for a triangular torus of given diameter (see Morillo, Comellas and Fiol [11] for a proof).

Fig 5.3 shows the 31 vertices within a distance of 4 from the central shaded vertex. Consideration of how the shape grows when we increase the selected distance from the centre gives the maximum number of vertices $N$ within a distance $D$ of the centre to be

$$1 + 3 + 6 + 9 + \cdots = 1 + 3D(D+1)/2.$$

We derive from this a lower bound for D

$$\begin{aligned} N &= 1 + 3D(D+1)/2 \\ (D+0.5)^2 &> D(D+1) = 2(N-1)/3 \\ D &> \sqrt{2(N-1)/3} - 0.5 \end{aligned}$$

The resulting shape in Fig 5.3 cannot tile the plane because the numbers of upright and inverted triangles differ. Thus it cannot form a torus.

Figure 5.4 shows extra vertices, marked x, added to give a shape (reminiscent of the outine of a kettle) that tiles the plane and can therefore form a torus. It has a diameter of 5 and represents 38 vertices. Increasing the diameter by 1 to an even value gives a kettle which is 1 strip of triangles wider in the $x$ direction. Increasing the diameter to the next odd value makes the kettle 2 strips taller in the $y$ direction and 1 strip wider in the $x$ direction. The spout is always on the middle row and a horizontal Hamiltonian path in the $x$ direction starting on the middle row wraps round to the top row of the kettle and then repeats progressively 1 row lower down beneath the middle and top rows.

In general for even $D$ this construction requires an extra $3D/2+1$ vertices to form a skewed torus of diameter $D+1$. So for even values of $D$ there exists a skewed torus of $N$ vertices with a diameter of $D+1$ where

$$\begin{aligned} N &= (1 + 3D(D+1)/2) + (3D/2 + 1) \\ &= (3D^2 + 6D + 4)/2 \end{aligned}$$

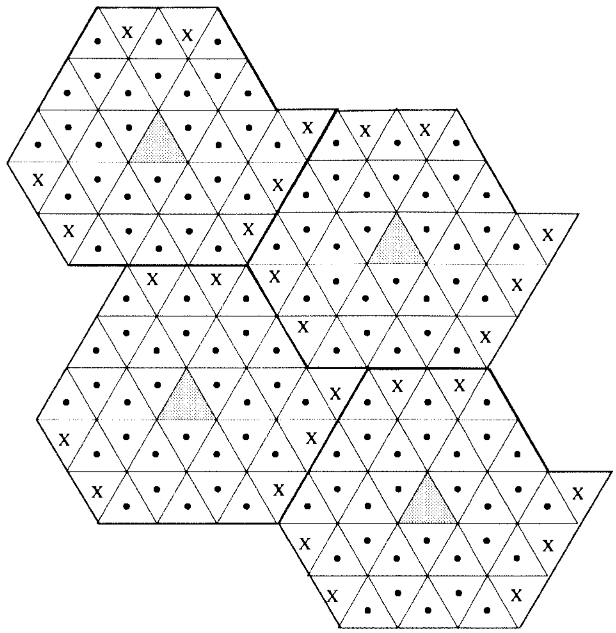

Figure 5.4: Maximum vertices in a diameter 5 torus

$$\begin{aligned} &= (3(D+1)^2+1)/2 \\ D+1 &= \sqrt{(2N-1)/3} \\ &\approx 0.8\sqrt{N}. \end{aligned}$$

Representing the torus as a chordal ring (see section 5.3.1), the chord length will be $3(D+1) \approx \sqrt{6N}$.

Similar calculations show that for an odd value of $D$, there exists a skewed torus of diameter $D+1$ with $3(D+1)^2/2 - D - 1$ vertices and the chord length of the corresponding chordal ring is $3(D+1)+1$.

Thus the minimum diameter of a triangular mesh skewed torus is about $0.8\sqrt{N}$. We will show later that, for any odd diameter, the skewed torus with the maximum number of nodes also has maximum symmetry.

## 5.3 Broadcasting on a Degree 3 Torus Mesh

The degree 3 mesh can be represented as a chordal ring graph or as a tiling. The tile representing a vertex is a triangle and together with its 3 neighbours it forms a larger triangle. Thus the obvious recursive broadcast algorithm builds triangles of increasing size. The total path length for a broadcast to N vertices based on this tiling would be

$$1 + 3 + 5 + 11 + 21 + \cdots + (2^{r+1} + (-1)^r)/3 = \lfloor \frac{4}{3}\sqrt{N} - 1 \rfloor .$$

Unfortunately a triangle cannot be wrapped symmetrically into a torus. (The plane cannot be tiled by translated triangles - half of them need to be rotated.) Various modifications to the basic recursion are possible. For example, the last round of the broadcast could group 4 triangles into a parallelogram, which does roll up into a torus. This would increase the total path to $\lfloor \frac{5}{3}\sqrt{N} - 1 \rfloor$. Alternatively, we could make any other round form a parallelogram.

### 5.3.1 Representation as a Chordal Ring

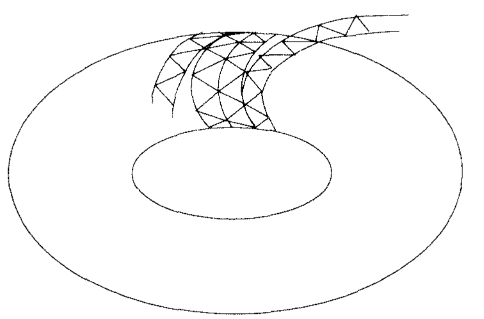

Figure 5.5: Winding a torus from triangles

If we consider the torus to be wound from a long strip of $N$ triangles as shown in Figure 5.5 then:

- for the ends of the strip to join up, N must be even.
- the chord length must be odd, so that adjacent loops have the triangle bases aligned.

- any odd chord length less than $N$ is possible, and without loss of generality we can assume the chord is no greater than $N/2$. (It may seem that the torus represented by a chord length of $N-3$ is physically very different from one with a chord length of 3, but turning one such torus inside out gives the other.)
- connections between 2 adjacent loops alternate to either side.

The strip of triangles can be identified with the main cycle of a chordal ring and numbered in sequence. Placing multiple copies of the strip of triangles for $N = 16$

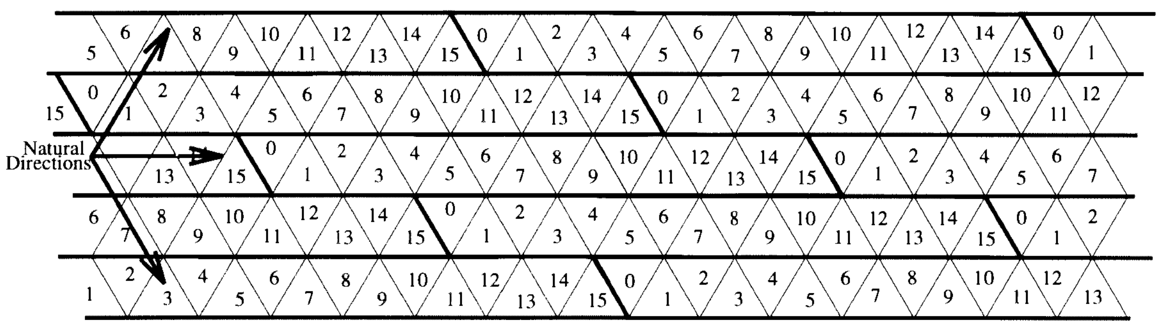


Figure 5.6: Multiple copies of a strip of 16 triangles

next to each other, as in Figure 5.6, demonstrates which triangles are adjacent to each other when wound into the torus represented by a chord length of 5.

Just as a square mesh has two natural directions corresponding to the positive $x$ and $y$ coordinate axes, so the triangular mesh has three natural directions shown by the arrows in Figure 5.6. When the torus is drawn as a flat mesh these directions define natural straight-line paths. For a chordal ring with a chord length of $c$, the corresponding natural paths are the main cycle and the two paths which alternate 1-chords with c-chords. The natural paths can be represented as

$$+1, +1, +1, \cdots$$

$$+c, +1, +c, +1, \cdots$$

$$-1, +c, -1, +c, \cdots$$

or the reverse of any of these paths.

### Properties of the Three Natural Paths

It is useful to choose chord lengths for the corresponding graph which increase the symmetry of the graph. We will show that for the graphs we are considering, where the number of vertices is a power of four, the natural paths are not equivalent. In particular, at least one of the natural paths is not Hamiltonian and therefore the graph cannot be redrawn with that path as the main cycle. For completeness we derive here the conditions under which the natural paths of a chordal ring are not only Hamiltonian but also have the property that when the graph is redrawn with a different natural path as the main cycle, the chord lengths of the chordal ring are unchanged.

For a chord length of $c$, where even numbered vertices have a chord in the positive direction, suppose we follow the natural path from vertex number 0 of

$$+c, +1, +c, +1 \cdots$$

or the reverse path

$$-1, -c, -1, -c, -1 \cdots$$

until it returns to vertex number 0. Since $c$ is odd for a chordal ring, $+c$ will always start on an even numbered vertex and $+1$ on an odd numbered vertex. When we take this path as the main cycle and renumber the vertices accordingly, vertex adjacencies must be preserved and so vertex number 1 must be renumbered as vertex number 1, $-1$ or $C$ (where $C$ is the new chord length) since the original vertex number 1 remains adjacent to vertex number 0. For the chord length to remain unchanged either vertex number 1 or vertex number $-1$ must be renumbered as vertex $c$, which means it is $c$ steps along the path which forms the new main cycle. Since $c$ is odd and chord lengths alternate, a path of length $c$ must both begin and end with the same chord length.

Thus one possibility is to choose $c$ so that a path of length $c$ from vertex number 0 of the form

$$+c, +1, +c, +1, \cdots + c$$

which, combining terms into pairs, leads to vertex number

$$(c+1)(c-1)/2+c \pmod N$$

where this is vertex number 1 or $-1$. For vertex number $-1$ the next step along the path would lead to vertex number 0, and imply that $c+1=N$, and thus $c$ is effectively 1. Thus we must have

$$(c+1)(c-1)/2+c=1 \pmod N$$

Alternatively, considering a path of length $c$ from vertex number 0 of the form

$$-1,-c,-1,-c\cdots-1,$$

which, combining terms in pairs, leads to vertex number

$$(-c-1)(c-1)/2-1 \pmod N,$$

and we can choose $c$ so that this is vertex number 1 or $-1$ where for vertex $-1$ we get $c-1=N$ and this case can be ignored. Thus we must have

$$(-c-1)(c-1)/2-1=1 \pmod N.$$

Rearranging terms for these two cases gives

$$(c+1)^2/2=2 \pmod N \text{ or } (c^2-1)/2=-2 \pmod N$$

as the condition for the natural path $+c,+1,\cdots$ (or its reverse) to give an unchanged chord length when it is viewed as the main cycle.

Similarly for the other natural path

$$+c,-1,+c,-1\cdots \text{ or } +1,-c,+1,-c,+1\cdots$$

the condition is

$$(c^2-1)/2=-2 \pmod N \text{ or } (c-1)^2/2=-2 \pmod N.$$

For both of the natural paths to give an unchanged chord length we need to consider the four possible pairs of equations which will satisfy one of the equations for each of the paths. The only non-trivial solution is

$$\begin{aligned}(c-1)^2/2 &= -2 \pmod{N}\\ (c^2+3)/2 &= 0 \pmod{N}.\end{aligned}$$

Putting $c = 2a + 1$ gives suitable values of N to be given by

$$N/2 = (a(a+1)+1)/k \text{ for integers } a, k, N/2.$$

Clearly $a(a+1)$ is even and so $N/2$ is odd. Thus $N$ cannot be a multiple of four.

Thus the three natural directions are equivalent, in the sense that we can regard any of them as the main cycle without changing the chord length of the chordal graph representation, when $(c^2+3)/2 = 0 \pmod{N}$. Taking values for $c$ of $3, 5, 7, 9, \ldots$ gives suitable values of $N$ to be

$$N = 6, 14, 26, 42, \ldots.$$

Note that $N = 6, c = 3$ gives a chordal ring which is also a circulant.

Earlier in this chapter the maximum number $N$ of vertices in a skewed torus with a diameter $D+1$ was shown to be

$$(3(D+1)^2+1)/2$$

when $D$ is even.

The natural paths in these maximal graphs are equivalent, in the above sense, as shown by taking $c = 3(D+1)$ and $N = (c^2+3)/6 = (3(D+1)^2+1)/2$.

The remainder of this chapter deals with chordal rings where the number of vertices is a power of four.

### 5.3.2 Chordal Rings With 16 Vertices

A chordal ring graph with 16 vertices and a chord length of 5 is shown in Fig 5.7. It assumes even-numbered vertices have a positive, clockwise chord. The natural

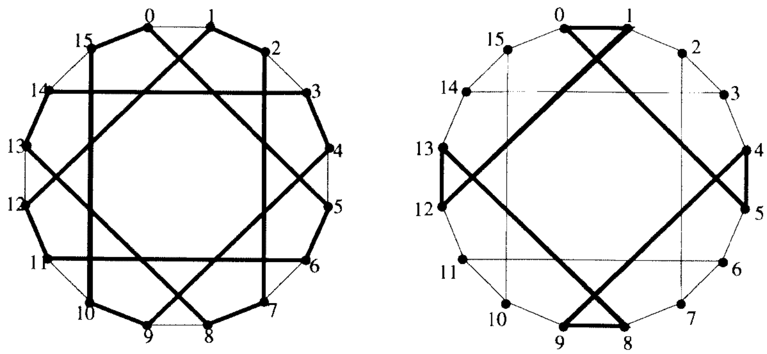


Figure 5.7: Chordal ring showing natural paths in bold

paths through vertex 0 (except the obvious main cycle) are shown with bold edges. To minimise the number of rounds in a broadcast we have assumed the number of vertices is a power of 4. For $N = 4^n$ it is possible to have Hamiltonian cycles in 2 of the natural directions and a pair of interlaced loops in the third direction. On the chordal ring diagram, the natural paths can be represented as

- the main cycle: $+1, +1, +1, \ldots$
- using negative chords: $+1, -c, +1, -c, \ldots$
- using positive chords: $+c, +1, +c, +1, \ldots$

The main cycle is clearly a Hamiltonian cycle. Positive chords give a Hamiltonian cycle if and only if the highest common factor of $N$ and $c+1$ is 2. Negative chords give a Hamiltonian cycle if and only if the highest common factor of $N$ and $c-1$ is 2.

As c is odd, both $c-1$ and $c+1$ are even. They differ by 2 and so one of them must equal 2 modulo 4. Since $N$ is a power of four,

$$c - 1 = 2 \text{ or } c + 1 = 2 \pmod{N}.$$

Therefore exactly two of the three cycles must be Hamiltonian. When choosing $c$ we can arrange for the non-Hamiltonian cycle to include half the vertices by taking $c = 3$ or 5 modulo 8, so that $c-1$ and $c+1$ are 2 and 4 modulo $N$.

The fairly trivial case of 16 vertices with a broadcast of only 2 rounds will define the tile shape used in all tessellations. If we try to proceed as for even degree graphs with a chord length of $c$, we will need $c$ to be both odd and even. It must be odd because the graph is a chordal ring, and it must equal 2 modulo 4 for the algorithm to work.

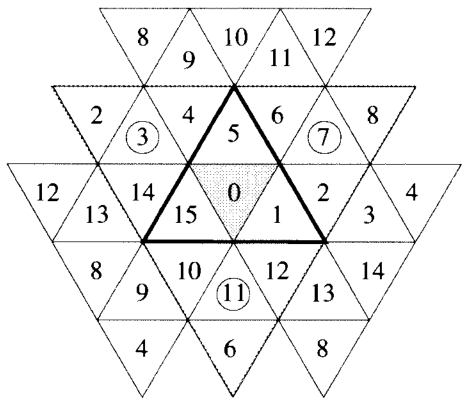


Figure 5.8: Tiles numbered modulo 16

To solve this problem and find short disjoint paths and a shape which tiles the plane, we proceed geometrically. Fig 5.8 shows, for a chord length of 5, the tiles numbered modulo 16. The last round sends to $+1$, $-1$ and $+5$ (a 'large triangle'). We need to add 3 nearby 'large triangles' to span all integers modulo 16. The resulting shape must tile the plane by translation if it is to form a torus.

The 3 adjacent large triangles with centres 3, 7 and 11 duplicate vertices 2 and 6. If we take 2 adjacent large triangles they must be those with centres 3 and 11, because both 2 and 6 are in the triangle with centre 7. The third triangle must then have centre 8 to span all integers modulo 16. Thus the base vertices for round 2 of the broadcast will be 0, 3, 8 and 11. This gives 4 possible shapes for the 16-triangle tile, shown in Figure 5.9. All of these shapes tile the plane and can therefore be rolled into a torus. (All 4 tiles give the same torus but cut along different edges.)

### 5.3.3 The Tabular Method for $N = 4^n$ Vertices

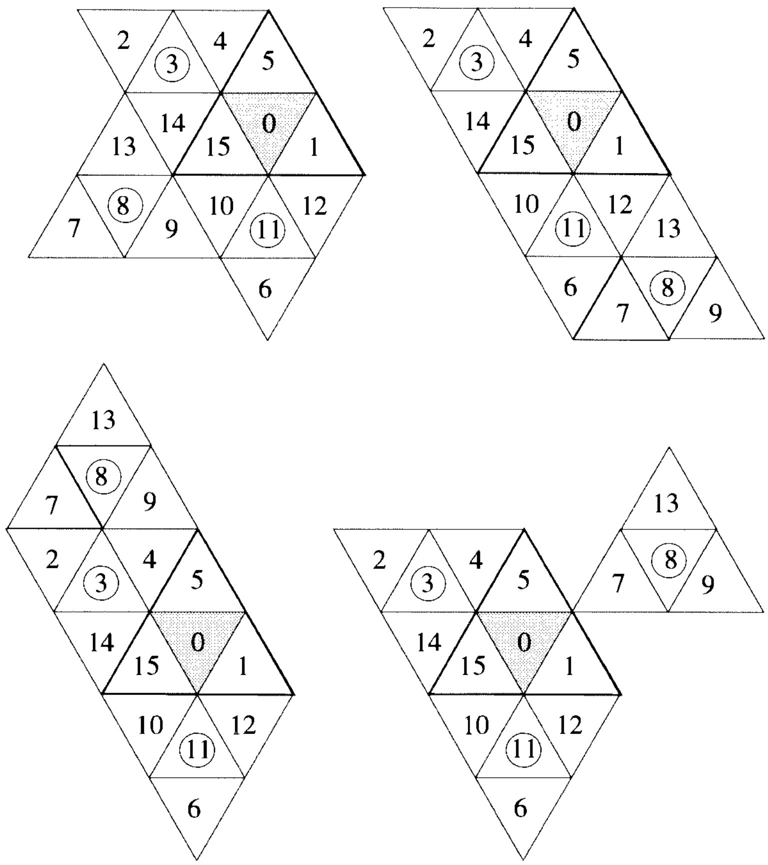


Figure 5.9: Shapes which tile the torus

Figure 5.10 shows the chordal ring representations of two possible broadcasting patterns (with chord lengths of 5 and 3 respectively) for round 2 (the last round) on 16 vertices. The broadcasting patterns both consist of 2 groups of 8 vertices. We will use one of the two patterns for the last round when we extend the broadcasting to larger graphs. Since the pattern repeats after 8 vertices we can add any multiple of 8 to the target vertex numbers in the last round of the broadcast in larger graphs. Since the target vertices in the last round are always adjacent to the base vertex this implies the chord length will be of the form $8k+3$ or $8k+5$. Note that this is precisely the condition required to make the non-Hamiltonian natural path include half of the vertices.

When extending the broadcasting algorithm to larger graphs we will not inform

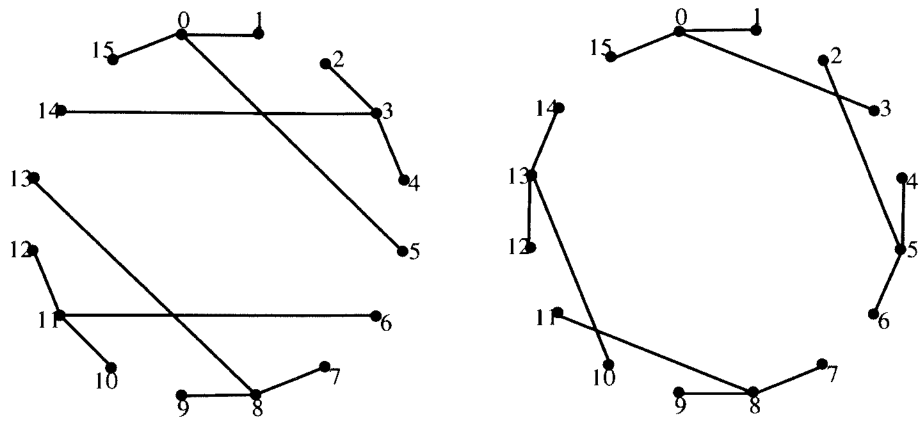


Figure 5.10: Degree three: two different last rounds for 16 vertices

odd-numbered vertices until the last few rounds, so as to avoid complications with the alternating chord direction. Round $n$ will, as always, inform adjacent vertices. Round $n-1$ will inform an equal number of odd and even vertices, either {0, 3, 8, 11} or {0, 5, 8, 13} (mod 16). Rounds 1 to $n-2$ will inform only even vertices. This will involve a slight change order we choose for rounds - we will interleave all rounds except the last two.

Assuming a chord length of $\sqrt{N}+3$ (= 3 mod 8 for $N > 16$, so natural paths will have their maximum cycle length) the method of interleaving rounds which was introduced in Chapter 3 can be adapted to give the tables below. Note that in round $n-1$ the target vertex numbers are $\pm\sqrt{N}$ and $2\sqrt{N}$ which are all even numbers. In round $n-2$ the target vertex numbers are adjusted from $\{-4, 4, 8\}$ to $\{-3, 5, 8\}$ (mod 16) as is required for round $n-1$. Round $n-2$ is no longer symmetric and an extra row has been added to the table for target number $-1$. To avoid confusion rounds $n-1$ and $n-2$ have not been renumbered.

| | Chord Lengths 1, $\sqrt{N}+3$ | | | |
|---|---|---|---|---|
| | Round $n$ | | Round $n-1$ | |
| Target | Chord No. | | Chord No. | |
| Number | 1 | 2 | 1 | 2 |
| 1 | 1 | 0 | $-3$ | 1 |
| 2 | 0 | 1 | $-6$ | 2 |

| | Chord Lengths 1, $\sqrt{N}+3$ | | | |
|---|---|---|---|---|
| | Round $n-2$ | | Round $n-3$ | |
| Target | Chord No. | | Chord No. | |
| Number | 1 | 2 | 1 | 2 |
| 1 | 2 | 1 | −12 | 4 |
| 2 | 2 | 2 | −24 | 8 |
| −1 | −3 | 0 | 12 | −4 |

| | Chord Lengths 1, $\sqrt{N}+3$ | | | |
|---|---|---|---|---|
| | Round 2 | | Round 1 | |
| Target | Chord No. | | Chord No. | |
| Number | 1 | 2 | 1 | 2 |
| 1 | $\sqrt{N}/16$ | $\sqrt{N}/16$ | $-3\sqrt{N}/4$ | $\sqrt{N}/4$ |
| 2 | $\sqrt{N}/8$ | $\sqrt{N}/8$ | $-3\sqrt{N}/2$ | $\sqrt{N}/2$ |

The actual broadcast must finish with the rounds we have numbered round $n-2$ and round $n$, so that no odd vertices are informed until the penultimate round. Thus we swap rounds $n-1$ and $n-2$.

The total path length for this broadcast is $3\sqrt{N}-3$. Adjustments to the last round can reduce it to $2\sqrt{N}-7$, but in the next section we give a tiling construction which reduces the total path length to approximately $\frac{4}{3}\sqrt{N}$.

## 5.4 A Recursive Tiling Broadcast to $4^n$ Vertices

A shorter total path length can be achieved by tiling techniques. The successive tilings for a broadcast using the 'maple leaf' shape from Figure 5.9 is shown in Figure 5.11. The shape at the start of each round is shown shaded and the final shape is roughly a rhombus. Note that the tiles are added alternately on the left and right to reduce the path lengths, so the initial tile is one third of the way along the shorter diagonal of the rhombus. The total path length for the broadcast is approximately $\frac{4}{3}\sqrt{N}$ (the distance from the initial tile to the furthest corner) which is about $\sqrt{2}$

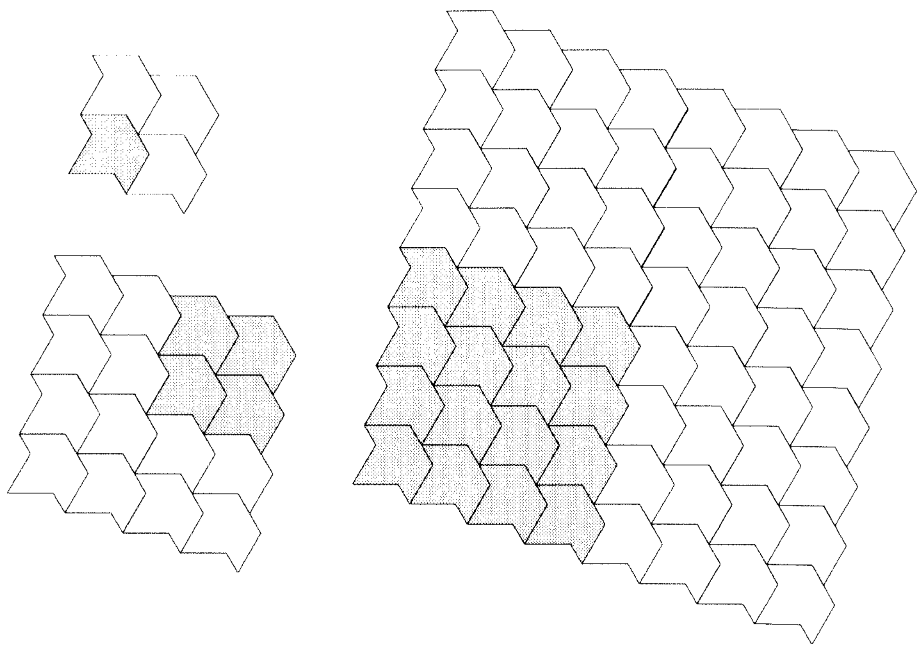

Figure 5.11: Recursive tiling with $4^3, 4^4, 4^5$ triangles in groups of 16

times the diameter of the mesh formed by dividing square tiles diagonally to convert a normal square mesh torus to a triangular one.

Rolling the final shape into a torus differs from the normal broadcast pattern. There are $2^{n-2}$ maple leaves along each edge of the rhombus, each leaf having 3 triangles along the edge. Thus there are $3 \times 2^{n-2}$ triangles along each edge of the rhombus. If opposite edges of the rhombus are connected in the obvious way to form a torus, each horizontal strip of triangles will join up with itself to form a cycle after traversing the tile 3 times. Thus each cycle would use 3 triangles from the edge of the rhombus, and so there must be $3 \times 2^{n-2}/3 = 2^{n-2}$ cycles. The total number of triangles is $2^{2n}$ and so each cycle must contain $2^{n+2}$ triangles.

An extra skew is introduced to ensure that a horizontal strip of triangles wraps round to an adjacent strip and forms a Hamiltonian cycle. This is always possible as the sawtooth pattern along the edge of the rhombus repeats every 3 triangles, the horizontal strip would normally be wrapped round to the strip $2^{n-2}$ rows higher and $2^{n-2} = \pm 1 \pmod 3$. The width of the rhombus doubles at each round of the broadcast,so each pair of rounds of the broadcast multiplies the width fourfold. As a result, a path across the rhombus is increased by 3 times the width of the original rhombus by a pair of rounds. Because of the serrated edge, the width of the rhombus (i.e. the number of triangles in a row) is not constant, but the sum of 3 adjacent rows is constant. The 3 rows correspond to the cycle mentioned above which contains $2^{n+2}$ triangles. The corresponding chord length for the chordal ring graph is thus

$$\begin{cases} 5+16+64+\cdots+\sqrt{N}=(4/3)\sqrt{N}-1/3 & \text{for } n \text{ even} \\ 11+32+128+\cdots+\sqrt{N}=(4/3)\sqrt{N}+1/3 & \text{for } n \text{ odd} . \end{cases}$$

The chord length is always 3 or 5 mod 8, so the natural paths have the maximum possible cycle lengths.

The diameter of the chordal ring cannot be greater than the number of chords to reach half way round the ring $((3/8)\sqrt{N})$ plus half the chord length $((2/3)\sqrt{N})$ , provided the latter is larger (forward chords exist only at alternate vertices). Thus a rough upper bound on the diameter is

$$(2/3+3/8)\sqrt{N} \approx 1.04\sqrt{N}$$

and the ratio of the total path to the diameter is at least 1.28.

Chordal rings with diameters as low as $\sqrt{(2N-1)/3}$ have been described in this chapter, giving an upper bound for the ratio of total path length to diameter of $\sqrt{8/3} \approx 1.6$.

# Chapter 6

# Graphs of Degree Five

## 6.1 Introduction

Degree 5 meshes can be treated essentially in the same way as even-degree meshes. The degree 5 mesh can be regarded as a 3-dimensional mesh consisting of many connected layers of a 2-dimensional triangular mesh. It can also be viewed as layers of a 2-dimensional square mesh where only alternate vertices in adjacent layers are connected. Wraparound connections convert the mesh to a skewed torus which can also be represented as a circulant-like graph. This chapter derives a broadcast scheme from a circulant-like graph representation and then discusses geometrical interpretations and variations for degree 5. There are no 'standard' degree 5 meshes to base this work on. It is not clear what values should be taken for chord lengths and diameter, and degree 5 gives more choices than can be investigated in detail.

## 6.2 Minimum Diameter for Degree Five Torus Mesh

The number $N_D$ of vertices within a distance $D$ from a given vertex is a sum over the 2-dimensional layers of the mesh, where the distance from the given vertex to the layer reduces the distance which can be reached within the layer. We consider the

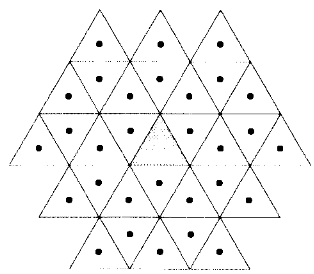

Figure 6.1: A layer which has grown for 4 steps

effect of increasing $D$ one step at a time, forming a 3-dimensional region with vertices at the boundary of the region being a distance $D$ from the centre. There are two separate cases depending on the orientation of the triangles in adjacent layers. (This depends on whether the chord length is even or odd when the graph is regarded as a circulant, as described in the next section.)

Regarding each layer as a triangular mesh where all layers of triangles are oriented in the same direction, each layer will grow independently, forming the shape shown in Figure 6.1. For example, in the first step the initial vertex will inform a vertex in each of the two adjacent layers and 3 adjacent vertices in its own layer. Since there are layers on both sides of the initial vertex, after D steps there will be $2D+1$ layers, and using the results of the previous chapter, the number of vertices in a layer will grow by $1, 3, 6, 9\ldots$ in successive steps. There will be two layers growing by each of these values at each step, except for the single initial layer which grows by $3D$ in step $D$.

This gives a recurrence relation:

$$N_D = N_{D-1} + 2(1+3+6+\cdots+3(D-1)] + 3D = N_{D-1} + 3D^2 + 2.$$

Solving the recurrence relation with $N_0 = 1$ gives

$$N_D = D^3 + 3D^2/2 + 5D/2 + 1$$

and thus

$$D^3 < N_D < (D+1)^3,$$

leading to a lower bound for the diameter $D$ for the degree 5 mesh of

$$\lfloor \sqrt[3]{N} \rfloor.$$

When adjacent triangles in different layers are oriented in opposite directions, it is easier to consider each layer to be a square mesh with a checkerboard pattern, and connections between layers to exist only when a black square is above a white one . Now each layer grows in a diamond shape containing $k^2 + (k+1)^2$ squares and new layers are formed one at a time on alternate sides of the original layer. A new layer formed at step $k$ starts with $k^2$ squares. The number of vertices $N_D$ at a distance $\leq D$ is then

$$\begin{cases} (D+1)^2 + D^2 + \frac{D}{6}(7D^2+2) - (\frac{D}{2})^2 & \text{for even D} \\ (D+1)^2 + D^2 + \frac{D}{6}(7D^2 - 3D - 1) - 1/2 + (\frac{D+1}{2})^2 & \text{for odd D.} \end{cases}$$

In both cases this is roughly $7D^3/6$, so a lower bound on the diameter is

$$\lfloor \sqrt[3]{\frac{6N}{7}} \rfloor \approx 0.95\sqrt[3]{N}.$$

## 6.3 Broadcasting on $6^n$ Vertices

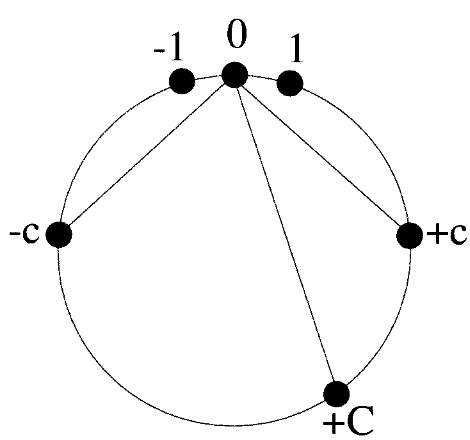


Figure 6.2: Representation of a degree 5 mesh

The degree 5 mesh will be represented by a degree 6 circulant with an edge removed from each vertex to give an alternating chord which has a positive direction on even vertices and a negative direction on odd vertices. These edges will alternate in direction in the same way as for a chordal ring. The other chord will be described as the symmetric chord. A broadcast with n rounds will inform $N = 6^n$ vertices and we can represent the graph with chords $\pm c, +C$ as in Fig 6.2, where as usual only the chords from vertex 0 are shown. As with the degree 3 chordal ring graph there must be an even number of vertices and the asymmetric chord must have an odd length. For 36 vertices, choose chord lengths of $+2, -2, +3$. Thus a vertex can inform the five vertices closest to it on the circle. Round 2 informs vertices $\pm 1, \pm 2, +3$ and round 1 informs $\pm 6, \pm 12, +18$. This extends naturally to $6^n$ vertices in $n$ rounds. To reduce the total path length we add suitable multiples of 6 to the chord lengths. For example, $-2$ could be changed to $-2 + 6 = 4$. By analogy with the normal three dimensional torus we look for suitable chord lengths close to powers of $\sqrt[3]{3} \approx 3.3$ and this suggests chord lengths of 3 and 10. For example, we could choose $-2 + 6 = 4$, $2 - 6 = -4$ and $3 + 6 = 9$. Experience with graphs of degree 3 suggests that the chord lengths should be larger to compensate for the alternating chord. For example, choosing chord lengths of $\pm 4, +15$ reduces the diameter from 6 to 4. The rounds are then

| | Chord Lengths 1, 4, 15 | | | | | |
|---|---|---|---|---|---|---|
| | Round 2 | | | Round 1 | | |
| Target | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | 0 | 0 | 2 | 1 | 0 |
| 2 | 0 | 1 | 0 | 0 | 3 | 0 |
| 3 | 0 | 0 | 1 | −1 | 1 | 1 |

Note that the total path length is 4, which is equal to the diameter. For larger graphs we would try to make chord lengths odd, as a path which includes more than one asymmetric chord must have odd-length chords separating them.

So the paths for round 1 can be described by chords

$+1, +1, +4$ to vertex 6

$+4, +4, +4$ to vertex 12

$+15, +4, -1$ to vertex 18

$-1, -1, -4$ to vertex -6, and

$-4, -4, -4$ to vertex -12.

These paths for round 1 are shown in Fig 6.3. Note that they are edge disjoint. The paths can be scaled up directly for $N = 6^n$ vertices, but it would lead to a total path length of

$$1,+ 4(1 + 6 + 6^2 + \cdots + 6^{n-2}) = 1 + 4(6^{n-1} - 1)/5 \approx 0.8 \times 6^{n-1} \approx N/8.$$

For large $N$ this compares badly with the diameter which is of the order $\sqrt[3]{N}$.

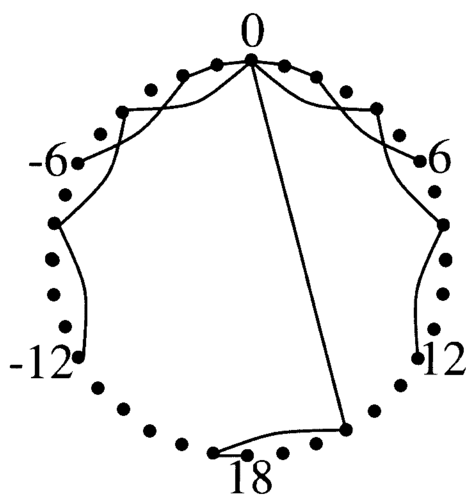


Figure 6.3: Broadcast round 1 for 36 vertices

## 6.4 Geometrical Interpretation

We can again think of winding the torus from a long strip of tiles, but this time continuing to form many layers with adjacent layers connected. The outer layer is then twisted round (in the fourth dimension) and connected to the inner layer. This is analogous to the way a skewed cylinder is twisted round and then the ends joined to form a 2 dimensional skewed torus. Alternatively we can think of taking a very long, slightly tapered cylinder and pushing one end into the other so it spirals round inside itself, and then connect the layers. There are two ways to wind the torus from a long strip.

- As already stated, form a skewed torus from a strip of triangles but, instead of joining the ends, continue to wind many layers on the torus. Join each vertex to the one in the layer above and in the layer below. (The inside layer wraps round to the outside layer.) This corresponds with having one short chord (with an odd length) and two long chords from each vertex. When the long chords have an odd length, triangles in adjacent layers will be oriented in opposite directions.

- Start with a strip of squares coloured alternately black and white and wind a multilayer torus as for the strip of triangles. In even layers connect white squares to the layer below and black ones to the layer above. (Clearly the number of squares in each layer must be odd so that a white square is always above a black square.) This corresponds with having two short chords and one long chord at each vertex. If the short chords have an odd length then each layer will have a checkerboard pattern.

Since these two representations are of the same graph, they must be equivalent. For example, the 36 vertex graph with chords $\pm 4, +15$ is isomorphic with the graph having chords $\pm 16, +9$ and these will correspond to the two ways to wind the torus.

Natural paths for the broadcast are most easily visualised by considering each layer of the torus to be a triangular mesh. There are then three natural paths within a layer and one natural path perpendicular to the layers. We can also consider natural paths consisting of alternating short and long chords, which is possible when the length of the symmetric chord is odd. In this case the triangular layers alternate in direction and are harder to visualise.

If we regard the torus as being wound from a strip of squares, with many layers, then there are two obvious paths in each layer. Paths perpendicular to the layers are not possible as the connections allow only one of the perpendicular paths for each square. Each time we go down a layer there are four directions to choose from. Two of these directions correspond to the natural paths around a triangular mesh (Chapter 5). The other 2 directions seem equally natural, and lead to another square with a downward path when the symmetric chord has an odd length. They correspond to using a mixture of short and long chords.

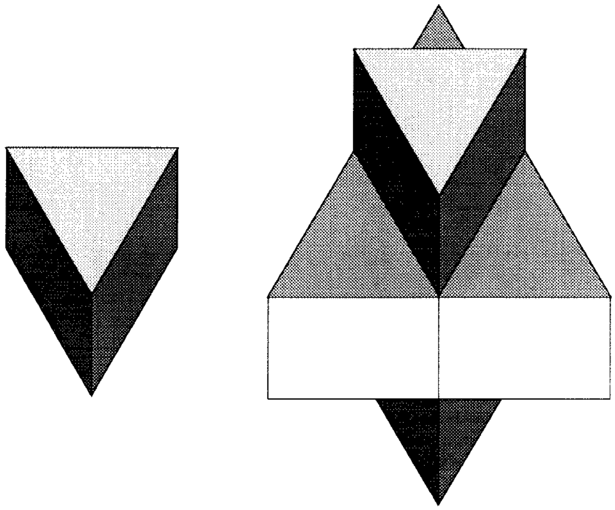

Figure 6.4: Initial tile and immediate neighbours

The three-dimensional geometry makes it reasonable to consider three rounds before scaling up. The six-vertex tile representing the last round of the broadcast has two corresponding forms. It can be a 'large' triangle (formed from four triangular prisms corresponding to vertices) with two triangular projections (Figure 6.4) or it can be a three-dimensional cross of cubes with one cube removed. Note that the triangular projections will face in opposite directions if the symmetric chords have an odd length. The growth of this tile with successive rounds of the broadcast is difficult to visualise, as shown in Figure 6.5, and we will consider only the tabular construction. For example, taking $6^3$ vertices with chords of length $6 + 2$ and $6^2 + 3$ the table describing the rounds of the broadcasting algorithm is

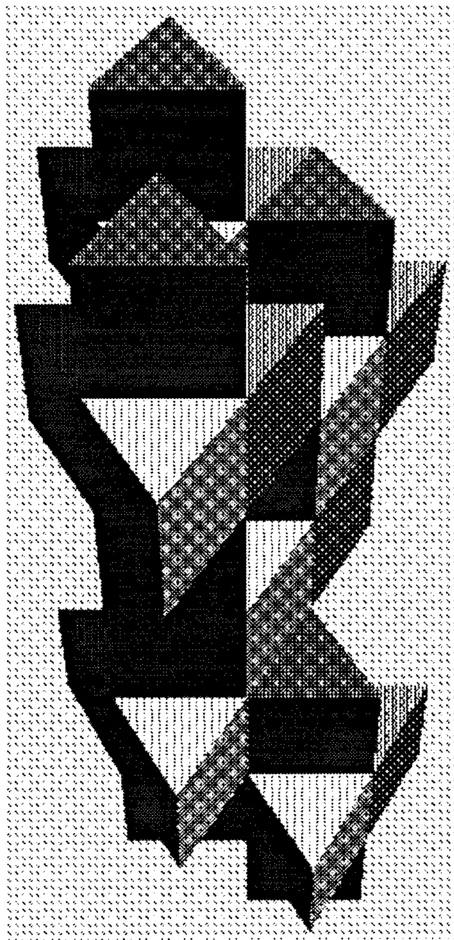

Figure 6.5: Tiling visualisation for 36 vertices

| $6^3$ Vertices | Chord Lengths 1, 8, 39 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 | | | Round 2 | | | Round 1 | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | 0 | 0 | −2 | 1 | 0 | −3 | 0 | 1 |
| 2 | 0 | 1 | 0 | −4 | 2 | 0 | −6 | 0 | 2 |
| 3 | 0 | 0 | 1 | −6 | 3 | 0 | −9 | 0 | 3 |

Since round two is calculated modulo $6^2$ and round one modulo $6^3$ longer chords can be used to reduce the path lengths, giving

| $6^3$ Vertices | Chord Lengths 1, 8, 39 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 | | | Round 2 | | | Round 1 | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | 0 | 0 | 1 | 1 | −1 | −3 | 0 | 1 |
| 2 | 0 | 1 | 0 | −1 | 2 | −1 | 2 | −1 | 2 |
| 3 | 0 | 0 | 1 | −1 | 2 | 1 | −1 | −1 | 3 |

Paths corresponding directly to round 1 of this table do not however exist. Multiple chords of length 39 must be separated by other chords of odd length. In round 1 this increases the longest path length from 5 to 7.

This broadcasting algorithm can then be extended to $6^{3n}$ vertices by repeating the final 3 rounds scaled up by a factor of 6. Again, longer chords are used whenever possible to shorten the maximum path length in each round. For $6^6$ vertices with chord lengths of $6^2 + 2$, $6^4 + 3$ the rounds are shown in the next two tables.

| $6^6$ Vertices | Chord Lengths 1, $6^2 + 2$, $6^4 + 3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 6 | | | Round 5 | | | Round 4 | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | 0 | 0 | 1 | 1 | −1 | −3 | 0 | 1 |
| 2 | 0 | 1 | 0 | −1 | 2 | −1 | −6 | 0 | 2 |
| 3 | 0 | 0 | 1 | 0 | 3 | −2 | −9 | 0 | 3 |

| $6^6$ Vertices | Chord Lengths 1, $6^2 + 2$, $6^4 + 3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 | | | Round 2 | | | Round 1 | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 6 | 0 | 0 | 6 | 6 | −6 | −18 | 0 | 6 |
| 2 | 0 | 6 | 0 | −6 | 12 | −6 | −36 | 0 | 12 |
| 3 | 0 | 0 | 6 | 0 | 18 | −12 | −54 | 0 | 18 |

By replacing short chords by long chords the 3 rounds shown in the last table can then be shortened to

| $6^6$ Vertices | Chord Lengths 1, $6^2+2$, $6^4+3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 | | | Round 2 | | | Round 1 | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 0 | 0 | 2 | 0 | 6 | −4 | −18 | 0 | 6 |
| 2 | 0 | 0 | 4 | 2 | 11 | 4 | 2 | -1 | 12 |
| 3 | 0 | 0 | 6 | −1 | 17 | 1 | −16 | -1 | 18 |

Further adjustments are required to ensure that multiple chords of length 39 can alternate with steps along the main cycle and that each round uses each edge out of an informed node.

## 6.5 Path Lengths

The total path length for $6^3$ vertices is

$$1 + 4 + (5 + 2) = 12,$$

and for $6^6$ vertices is

$$1 + (5 + 1) + 12 + (6 + 3) + 19 + (35 + 2) = 84.$$

These compare badly with lower bounds on the diameter of 6 and 36 respectively, and reflect the lack of geometric intuition in arranging the triangular shapes. The general case is treated in Chapter 8.

# Chapter 7

# Broadcasting on Circulant Graphs

## 7.1 Introduction

This chapter describes optimal algorithms for broadcasting on circulant graphs of degree $2d$ with $(2d+1)^d$ or $(2d+1)^{nd}$ vertices which represent skewed torus networks in $d$ dimensions. It uses the tabular method developed in earlier chapters.

Note that there are many cases covered:

- graphs with $(2d+1)^d$ or $(2d+1)^{nd}$ vertices
- examples and methods of construction for tables
- deciding the order of edges in a path
- calculation of path lengths
- special cases for the first round or first group of rounds
- special cases for the first round or first group of rounds
- differences for odd and even rounds or target numbers.

A summary is included here as a guide to the results.

Section 7.2 deals with graphs of $(2d+1)^d$ vertices, and mainly describes techniques for constructing tables and deciding the order of edges in a path. The path length for round 1 is calculated and, by assuming results from Section 7.3 for the other rounds, Section 7.2 derives an upper bound on the total path length of about $3d^2/2$.

Section 7.3 deals with graphs of $(2d+1)^{nd}$ vertices, and is mainly concerned with calculating path lengths. In summary:

- the last $d$ rounds have a combined path length of $d(5d-3)/2$.
- middle rounds simply scale up the last $d$ rounds. so that the combined path length for the last $d(n-1)$ rounds is $S(5d-3)/4$ where $S$ is the maximum scaling factor.
- alternatives for constructing the middle rounds exist, but do not affect the total path length.
- the first $d$ rounds are grouped in pairs to simplify the path lengths,
  - round $d$ has a path length of $S$.
  - rounds $d-1$, $d-2$ combined are $S(2d+1)+d+1-(2+1/d)$.
  - rounds $d-3$, $d-4$ combined are $S(2d+1)+d+3$.
  - rounds $d-5$, $d-6$ combined are $S(2d+1)+d+5$.
  - ...
- the total path length is less than $1+\frac{3}{4d}+\frac{3}{4S}$.

## 7.2 Broadcast on a Circulant of $(2d+1)^d$ Vertices

In this section we consider circulants of the form

$$C((2d+1)^d, \{1, (2d+1)+2, (2d+1)^2+3, \ldots, (2d+1)^{(d-1)}+d\}),$$

where $(2d+1)^d$ is the number of vertices and the chord lengths are

$$\pm 1, \pm[(2d+1)+2], \ldots, \pm[(2d+1)^{(d-1)}+d].$$

The number of vertices has been selected to model a d-dimensional square mesh torus, and the chord lengths approximate those of the mesh, so as to give a diameter close to that of the normal torus for comparison. Note that the chord lengths give a complete set of residues modulo $(2d+1)$ if we include 0 . The inclusion of 0 will be assumed for simplicity - it represents transmission from a vertex to itself and can be ignored. We will take the residues to be from $-d$ to $d$ rather than from 0 to $2d$ and use symmetry to deal with $\pm x$ as a single case.

Only small values of $d$ give a reasonable number of vertices. For example, $d = 5$ gives $11^5 = 161051$ vertices, but we will assume $d$ can take any value.

### 7.2.1 $N = (2d+1)^d$: Example for $d = 3, N = 7^3$

Note that the table for the last round is always diagonal as a vertex informs its immediate neighbours last. Chord lengths are 1,9 and 52.

| $d=3$ | Chord Lengths 1, $7+2$, $7^2+3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 | | | Round 2 | | | Round 1 | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | 0 | 0 | 1 | 1 | $-1$ | $-3$ | 0 | 1 |
| 2 | 0 | 1 | 0 | $-1$ | 2 | $-1$ | 3 | $-1$ | 2 |
| 3 | 0 | 0 | 1 | 0 | 3 | $-2$ | 0 | $-1$ | 3 |

Matrix multiplication gives the vertices informed by round 1 (relative to each base vertex) to be

$$\begin{pmatrix} -3 & 0 & 1 \\ 3 & -1 & 2 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 9 \\ 52 \end{pmatrix} = \begin{pmatrix} 49 \\ 98 \\ 147 \end{pmatrix}.$$

Vertices $-49, -98, -147$ will also be informed during round 1. The negative targets are treated symmetrically as shown in the table below.

| $d = 3$ | Negative Targets Treated Symmetrically | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 | | | Round 2 | | | Round 1 | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| $-1$ | $-1$ | 0 | 0 | $-1$ | $-1$ | 1 | 3 | 0 | $-1$ |
| $-2$ | 0 | $-1$ | 0 | 1 | $-2$ | 1 | $-3$ | 1 | $-2$ |
| $-3$ | 0 | 0 | $-1$ | 0 | $-3$ | 2 | 0 | 1 | $-3$ |

### 7.2.2 The General Strategy of Broadcasting

#### Use of Rotational Symmetry

Every base vertex obeys the same algorithm and, as base vertices are always equally spaced, rotational symmetry allows us to ignore multiples of the difference between base vertex numbers in any round. Thus if base vertex numbers are $(2d + 1)$ apart, then adding $(2d + 1) + 5$ to the base vertex numbers will give the same set of vertices as adding 5. Paths starting out along equal positive and negative chords are mirror images of each other, so only one needs to be described.

#### Order of Chords in a Path

Each path in a round starts (obviously) with one of the chords incident on a base vertex. In each round the distances to target vertices are approximately multiples of one of the chords, which we will refer to as the **main chord** for that round. After th first chord of a path, the path follows the appropriate main chord for the round until it is near to its designated target vertex. At this stage, each path has reached a vertex near a different target and there is no danger of edge contention, so the path to the target vertex can be completed with the remaining chords in any order.

#### Informing All Vertices

The number of rounds required for $(2d + 1)^d$ vertices is $d$. The first round starts at base vertex 0 and informs $2d$ target vertices chosen so that all $2d + 1$ vertices are equally spaced around the main cycle. Thus after the first round the informed vertices are $k(2d+1)^{d-1}$ for $k = -d$ to $+d$. The paths to the target vertices are very similar. After the first chord, the path follows several main chords of length $(2d+1)^{d-1} + d$. Note that it overshoots each of the target vertices by a multiple of $d$ plus the length of the first chord in the path. This overshoot is corrected using the two shortest chords, of lengths 1 and $2d+1$, as described in detail in the next section.

The second round then informs all other multiples of $(2d+1)^{d-2}$, with paths consisting mainly of the $(2d+1)^{d-2} + (d-1)$ chord. Although a vertex informs widely dispersed vertices, by rotational symmetry we can consider each base vertex to inform the nearest $d$ target vertices on either side of itself, with a difference $(2d+1)^{d-2}$ between target vertex numbers. This process continues until the final round informs all multiples of $(2d+1)^0$ which is clearly all vertices. Again, by rotational symmetry, it is as if each base vertex informed the $d$ nearest vertices on either side of itself in the final round.

### 7.2.3 Detailed Paths

There are $(2d+1)^d$ vertices and broadcasting requires $d$ rounds. Note that until the last round there are relatively few informed vertices. (For round 1 and dimension $d = 3$ the round starts with only 1 vertex out of the 343 informed). This sparseness makes edge-disjoint paths plentiful, but it is correspondingly difficult to select the 'best' set of paths algorithmically.

#### Round $d$, the Last Round

The base vertices are every multiple of $(2d+1)$. Target vertices are all other vertices. Each base vertex informs its immediate neighbours. By rotational symmetry we can ignore multiples of $(2d+1)$, so the chords are effectively $-d$ to $+d$ and we can regard each vertex as informing the d vertices on either side of it. There are $2d$ uninformed vertices between base vertices, so round $d$ correctly completes the broadcast with a path length of 1.

**Round $d-1$**

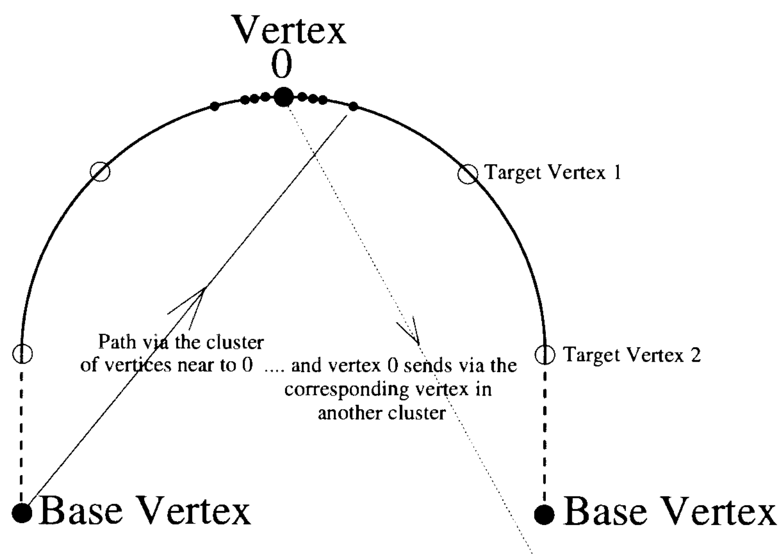


Figure 7.1: Cluster of vertices after the first chord in the paths for round $d-1$

The base vertices are all multiples of $(2d+1)^2$. Target vertices are all other multiples of $(2d+1)$. Each path must start with immediate neighbours, and this has the effect of reaching vertices numbered plus or minus ($1, 2d+3, 3, 4, \ldots d$) relative to each base vertex, as in Figure 7.1. (Note that the rotational symmetry does not apply to $2d+3$ because in round $d-1$ we work modulo $(2d+1)^2$.) The description of paths is simpler if we take a mixture of positive and negative values covering all absolute values ($1, 3, 4, \ldots d$ and $2d+3$). The opposite sign is treated symmetrically by reflection. Select even values to be negative. If $d$ is odd, also take the value of 1 to be negative. Figure 7.1 shows the vertices reached if we take one step along every path for round $d-1$. Because of rotational symmetry the paths go through vertices which are clustered in the same pattern around each base vertex.

**Edge-disjoint Paths for Round $d-1$**

**Vertex** $2d+3$ takes a 1-chord forward and a 3-chord backward to the target vertex $2d+1$. (If $d=2$ the 3-chord does not exist, but two 1-chords backward are possible.)

**Vertex** $-2x$ takes $x$ of the 2-chord steps to arrive at the target vertex $x(2d+1)$.

**If $d$ is even** vertex 1 takes $d/2$ 2-chords and then one $d$-chord forward to the target vertex $(d/2+1)(2d+1)$.

**If $d$ is odd** vertex $-1$ takes $(d+1)/2$ 2-chords forward and then one $d$-chord backward to the target vertex $(d+1)/2 \times (2d+1)$.

**Remaining odd vertices** $(2y+1)$ take $(d-y)$ 2-chords forward to the target vertex $(d-y+1)(2d+1)$.

As required we have informed vertices

$$(2d+1), x(2d+1), (\lfloor \frac{d}{2} \rfloor + 1)(2d+1), \text{ and } (d-y+1)(2d+1),$$

where $x$ ranges from 2 to $\lfloor \frac{d}{2} \rfloor$ and y ranges from 1 to $\lfloor \frac{(d-1)}{2} \rfloor$.

For $d > 3$ the maximum path length is $d$. For $d = 2$ or 3 the maximum path length is $d+1$.

### Paths are Edge-disjoint

The initial edge in each path is the same as a path in round $d$, but on a restricted set of base vertices. All initial edges start on a base vertex equal to 0 mod $(2d+1)^2$ and end on a target vertex which is non-zero $\pmod{2d+1}$. Hence the edges are disjoint. No subsequent edges are incident on base vertices. The parts of the paths along 2-chords start at a cluster of vertices from $-d$ to $+d$ around a base vertex and follow 'parallel' 2-chords. After $i$ 2-chords the paths are incident on a cluster from $-d+2i$ to $d+2i$ around vertex $i(2d+1)$ relative to each base vertex. At each step one path ends on the correct target vertex and the cluster moves on with one less path. The clusters of vertices do not overlap each other since

$$i(2d+1) + d + 2i < (i+1)(2d+1) - d + 2(i+1)$$

and they do not overlap clusters around the next base vertex because the clusters only extend to the $d(2d+1)$ target vertex and base vertex numbers differ by $(2d+1)^2$ . All paths along the 2-chords terminate at the target vertices except for paths starting at 1 and $2d + 3$ relative to the base vertex. The final edges of these two paths must be disjoint, as no other edge is incident on the same target vertex. The only other edge used is from $(2d + 3)$ to $(2d + 4)$ relative to each base vertex (and the mirror image).

Hence the paths are edge-disjoint. The number of steps along each chord is shown for $d = 2, 3, 10$ and 11 in tables below, with extra columns to show the target vertex calculation and to indicate which is the first chord in each path. Note that $d = 2$ does not follow the pattern exactly.

| Round 1: $d = 2$, vertices = 25, chords = 1, 7 | | | | | |
|---|---|---|---|---|---|
| Target Vertex | Chord No. 1 | 2 | Result | Mod 25 | First Chord |
| 1 | $-2$ | 1 | $-2 + 7 = 5$ | 5 | 2 |
| 2 | $-1$ | 2 | 1+14=15 | $-10$ | 1 |

| Round 2: $d = 3$, vertices = 343, chords = 1, 9, 52 | | | | | | |
|---|---|---|---|---|---|---|
| Target Vertex | Chord No. 1 | 2 | 3 | Result | Mod 49 | First Chord |
| 1 | 1 | 1 | $-1$ | $1 + 9 - 52 = -42$ | 7 | 2 |
| 2 | $-1$ | 2 | $-1$ | $-1 + 18 - 52 = -35$ | 14 | 1 |
| 3 | 0 | 2 | 1 | 18+52=70 | 21 | 3 |

For sparse tables the zeroes have been omitted.

| Round 9: $d$=10, chords = $1, 23, 444, \ldots$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | | | First |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Chord |
| 1 | 1 | 1 | −1 | | | | | | | | 2 |
| 2 | | 2 | | −1 | | | | | | | 4 |
| 3 | | 3 | | | | −1 | | | | | 6 |
| 4 | | 4 | | | | | | −1 | | | 8 |
| 5 | | 5 | | | | | | | | −1 | 10 |
| 6 | 1 | 5 | | | | | | | | 1 | 1 |
| 7 | | 6 | | | | | | | 1 | | 9 |
| 8 | | 7 | | | | | 1 | | | | 7 |
| 9 | | 8 | | | 1 | | | | | | 5 |
| 10 | | 9 | 1 | | | | | | | | 3 |

| Round 10: d=11, chords = $1, 25, 532, \ldots$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | | | | First |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | Chord |
| 1 | 1 | 1 | −1 | | | | | | | | | 2 |
| 2 | | 2 | | −1 | | | | | | | | 4 |
| 3 | | 3 | | | | −1 | | | | | | 6 |
| 4 | | 4 | | | | | | −1 | | | | 8 |
| 5 | | 5 | | | | | | | | −1 | | 10 |
| 6 | −1 | 6 | | | | | | | | | −1 | 1 |
| 7 | | 6 | | | | | | | | | 1 | 11 |
| 8 | | 7 | | | | | | | 1 | | | 9 |
| 9 | | 8 | | | | | 1 | | | | | 7 |
| 10 | | 9 | | | 1 | | | | | | | 5 |
| 11 | | 10 | 1 | | | | | | | | | 3 |

**Round $d+1-r$, for $1 < r < d$**

(Note that this includes round $d-1$ which has just been treated in detail.) The base vertices are all multiples of $(2d+1)^r$ and so by rotational symmetry we can work modulo $(2d+1)^r$. Target vertices are multiples of $(2d+1)^{r-1}$, so the main chord used will be the $r$-chord of length $(2d+1)^{r-1}+r$. (We have chosen to make $r$ number the rounds in reverse order so as to simplify expressions.) The overshoots to be corrected (where we take the first chord of the path to be part of the correction) are plus or minus $r, 2r, \ldots, dr$ and we will try to correct these with the 2-chord, which has length $(2d+1)+2 = 2d+3$.

On the following pages we give examples for round $d-2$ (where $r=3$) in dimensions $d = 5, 10$ and 11.

| $d=5$, round 3, chords $= 1, 13, 124, 1335, 14646$ | | | | | | |
|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | First |
| Vertex | 1 | 2 | 3 | 4 | 5 | Chord |
| 1 | 1 | | 1 | −1 | | 3 |
| 2 | −1 | | 2 | | −1 | 5 |
| 3 | 1 | | 3 | 1 | | 4 |
| 4 | 1 | −1 | 4 | | | 1 |
| 5 | -2 | −1 | 5 | | | 2 |

(Here we have added an extra column to indicate which is the first chord in each path.)

| $d = 10$, round 8 chords $= 1, 23, 444, \ldots$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | | | First |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Chord |
| 1 | 1 | | 1 | −1 | | | | | | | 3 |
| 2 | | | 2 | | | −1 | | | | | 6 |
| 3 | | | 3 | | | | | | −1 | | 9 |
| 4 | | | 4 | | | | | 1 | | −2 | 10 |
| 5 | | −1 | 5 | | | | | 1 | | | 8 |
| 6 | | −1 | 6 | | 1 | | | | | | 5 |
| 7 | 2 | −1 | 7 | | | | | | | | 2 |
| 8 | −1 | −1 | 8 | | | | | | | | 1 |
| 9 | | −1 | 9 | −1 | | | | | | | 4 |
| 10 | | −1 | 10 | | | | −1 | | | | 7 |

| $d = 11$, round 9 chords $1, 25, 532, \ldots$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | | | | First |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | Chord |
| 1 | 1 | | 1 | −1 | | | | | | | | 3 |
| 2 | | | 2 | | | −1 | | | | | | 6 |
| 3 | | | 3 | | | | | | −1 | | | 9 |
| 4 | −1 | | 4 | | | | | | | | −1 | 11 |
| 5 | | −1 | 5 | | | | | | | 1 | | 10 |
| 6 | | −1 | 6 | | | | 1 | | | | | 7 |
| 7 | | −1 | 7 | 1 | | | | | | | | 4 |
| 8 | 1 | −1 | 8 | | | | | | | | | 1 |
| 9 | −2 | −1 | 9 | | | | | | | | | 2 |
| 10 | | −1 | 10 | | −1 | | | | | | | 5 |
| 11 | | −1 | 11 | | | | | −1 | | | | 8 |

**Round 1**

In round 1 we are working modulo $(2d+1)^d$ and the main chord is of length $(2d+1)^{d-1}+d$ giving overshoots of up to $d^2$. As a result only the 1-chord and 2-chord can be used to correct the overshoot: any other chord would be too large to correct the overshoot and too small to be ignored by the modulo $(2d+1)^d$ arithmetic. The best choice of paths is now essentially unique, as can be seen in the tables below for $d = 10, 11$: the number of $d - chords$ is equal to the target number; the overshoot is $d$ times the target number; and this overshoot is corrected by using as many 2-chords as possible. The remaining overshoot is corrected by 1-chords. Note that each of the chords must be the first chord of some path, and where necessary we will mark the first chord of a path with an '*' in the table. This represents a penalty of 2 steps in the path because the unwanted first chord must eventually be cancelled out with a reverse chord of the same length.

| Round 1: $d = 10$, chords $= 1, 23, 444, \dots$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | | |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | -10 | | | | | | | | | 1 |
| 2 | 3 | -1 | | | | | | * | | 2 |
| 3 | -7 | -1 | * | | | | | | | 3 |
| 4 | 6 | -2 | | * | | | | | | 4 |
| 5 | -4 | -2 | | | * | | | | | 5 |
| 6 | 9 | -3 | | | | * | | | | 6 |
| 7 | -1 | -3 | | | | | * | | | 7 |
| 8 | -11 | -3 | | | | | | | | 8 |
| 9 | 2 | -4 | | | | | | | * | 9 |
| 10 | -8 | -4 | | | | | | | | 10 |

| Round 1: $d = 11$ chords $1, 25, 532, \ldots$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | | | |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | −11 | | | | | | | | | | 1 |
| 2 | 3 | −1 | | | | | | * | | | 2 |
| 3 | −8 | −1 | * | | | | | | | | 3 |
| 4 | 6 | −2 | | * | | | | | | | 4 |
| 5 | −5 | −2 | | | * | | | | | | 5 |
| 6 | 9 | −3 | | | | * | | | | | 6 |
| 7 | −2 | −3 | | | | | * | | | | 7 |
| 8 | −13 | −3 | | | | | | | | | 8 |
| 9 | 1 | −4 | | | | | | | * | | 9 |
| 10 | −10 | −4 | | | | | | | | * | 10 |
| 11 | 4 | −5 | | | | | | | | | 11 |

In general the path length for round 1 is $2d + 2$, plus a possible extra length of 2 to ensure that every edge out of an informed vertex is the start of one of the paths. For small $d$ we will show that this extra length can be transfered to a shorter path and will not affect the maximum path length. The examples assumed initially that chord $i$ is the start of the path to target $i$, marked with an '*' on the main diagonal of the table where this increases the path by 2. This gave a path length of $2d + 4$ to target number 8. The maximum path length was reduced to $2d + 2$ by swapping chord 8 to be the first chord of the path to target 2 and chord 2 to be the first chord of the path to target 8. This is shown by moving the '*' up from the row for target 8 to the row for target 2. For $d = 11$ target 10 should be adjusted in a similar way, swapping with either target 1 or target 10.

### Creating the Tables for Round $d + 1 - r$

In practice, since d is small, it is easy to plan paths for round $d + 1 - r$, $1 < r < d$, using a table and filling entries as described below.

- Label columns with the chord number from 1 to d. Label rows with the target vertex number from 1 to d. Thus cell $(i, j)$ contains the number of $j$-chords in the path to target vertex $i$ (where target vertex $i$ is $i(2d+1)^{r-1}$). Note that column $r$ is always $1, 2, \ldots, d$ except for the last round which has 1's on the diagonal as a vertex sends to all its neighbours.

- Row $i$ requires a correction of $-ir$. Start with the last row and work upwards so that the longest paths get first attention. For each row try to correct the overshoot with as few chords as possible, trying in the order shown
  - The 2-chord (of length $2d+3$);
  - Chords $d, d-1, \ldots, r+1$ (shorter chords cannot use the rotational symmetry);
  - The 1-chord.

- Adjust rows $d$ to $r-1$ so each of them uses chord 1 or a chord number in the range $d$ to $r$. If necessary, add two extra chords in each row to achieve this. The aim is to match targets (rows) 1 to $d$ with chords 1 to $d$ (since each path from a given base vertex must begin with a different chord). At the same time we can balance path lengths by giving distant vertices priority. Complete the matching by adding pairs of equal chords in opposite directions - these are indicated by entering '*' in the cell. The first chord of a path now determines the target vertex according to this matching.

The results of this procedure are shown below for $d = 9$. Only the positive target vertices are considered, the paths for the negative target vertices are given by reflection.

| Round 9 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Target Vertex | Chord Number 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | | | | | | | | |
| 2 | | 1 | | | | | | | |
| 3 | | | 1 | | | | | | |
| 4 | | | | 1 | | | | | |
| 5 | | | | | 1 | | | | |
| 6 | | | | | | 1 | | | |
| 7 | | | | | | | 1 | | |
| 8 | | | | | | | | 1 | |
| 9 | | | | | | | | | 1 |

| Round 8 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Chord Number 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | 1 | -1 | | | | | | |
| 2 | | 2 | | -1 | | | | | |
| 3 | | 3 | | | | -1 | | | |
| 4 | | 4 | | | | | | -1 | |
| 5 | -1 | 5 | | | | | | | -1 |
| 6 | | 5 | | | | | | | 1 |
| 7 | | 6 | | | | | 1 | | |
| 8 | | 7 | | | 1 | | | | |
| 9 | | 8 | 1 | | | | | | |

| Round 7 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Target Vertex | Chord Number 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | | 1 | -1 | | | | | |
| 2 | -1 | | 2 | | -1 | | | | |
| 3 | -1 | | 3 | | | | | -1 | |
| 4 | | -1 | 4 | 1 | | | | | 1 |
| 5 | -1 | -1 | 5 | | | | 1 | | |
| 6 | -1 | -1 | 6 | 1 | | | | | |
| 7 | | -1 | 7 | | | | | | |
| 8 | 1 | -1 | 8 | -1 | | | | | |
| 9 | | -1 | 9 | | | -1 | | | |

| Round 6 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Chord Number 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | | * | 1 | -1 | | | | |
| 2 | | | | 2 | | | | -1 | |
| 3 | | -1 | | 3 | | | | | 1 |
| 4 | | -1 | | 4 | 1 | | | | |
| 5 | 1 | -1 | | 5 | | | | | |
| 6 | | -1 | | 6 | | | | | |
| 7 | | -1 | | 7 | | | -1 | | |
| 8 | 1 | -2 | | 8 | | | | | 1 |
| 9 | | -2 | | 9 | | 1 | | | |

| Round 5 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | | * | | 1 | -1 | | | |
| 2 | -1 | | | | 2 | | | | -1 |
| 3 | | -1 | | | 3 | 1 | | | |
| 4 | 1 | -1 | | | 4 | | | | |
| 5 | | | | * | 5 | | -1 | | -2 |
| 6 | | -1 | | | 6 | | | | -1 |
| 7 | | -2 | | | 7 | | 1 | | |
| 8 | | -2 | | | 8 | -1 | | 1 | |
| 9 | | -2 | | | 9 | 1 | | | -1 |

| Round 4 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Chord Number | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | | * | | | 1 | -1 | | |
| 2 | | -1 | | * | | 2 | | | 1 |
| 3 | | | | | * | 3 | | | -2 |
| 4 | | | | | | 4 | | -3 | |
| 5 | -1 | -1 | | | | 5 | | -1 | |
| 6 | -1 | -2 | | | | 6 | 1 | | |
| 7 | | -2 | | | | 7 | | | |
| 8 | 1 | -2 | | | | 8 | -1 | | |
| 9 | | -3 | | | | 9 | | | 1 |

| Round 3 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | | * | | | | 1 | -1 | |
| 2 | -1 | | | * | | | 2 | 1 | |
| 3 | | -1 | | | * | | 3 | | |
| 4 | 1 | -1 | | | | * | 4 | -1 | |
| 5 | -2 | -2 | | | | | 5 | | 1 |
| 6 | | -2 | | | | | 6 | | |
| 7 | 1 | -2 | | | | | 7 | -1 | |
| 8 | -1 | -3 | | | | | 8 | 1 | |
| 9 | | -3 | | | | | 9 | | |

| Round 2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Chord Number | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | | * | | | | | 1 | -1 |
| 2 | 2 | | | * | | | | 2 | -2 |
| 3 | -3 | -1 | | | * | | | 3 | |
| 4 | -2 | -1 | | | | * | | 4 | -1 |
| 5 | 2 | -2 | | | | | * | 5 | |
| 6 | 3 | -2 | | | | | | 6 | -1 |
| 7 | -2 | -3 | | | | | | 7 | 1 |
| 8 | -1 | -3 | | | | | | 8 | |
| 9 | | -3 | | | | | | 9 | -1 |

| Round 1 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | -9 | | * | | | | | | 1 |
| 2 | 3 | -1 | | * | | | | | 2 |
| 3 | -6 | -1 | | | * | | | | 3 |
| 4 | 6 | -2 | | | | * | | | 4 |
| 5 | -3 | -2 | | | | | * | | 5 |
| 6 | 9 | -3 | | | | | | * | 6 |
| 7 | | -3 | | | | | | | 7 |
| 8 | -9 | -3 | | | | | | | 8 |
| 9 | 3 | -4 | | | | | | | 9 |

The total path length for this example is

$$20 + 13 + 13 + 13 + 13 + 12 + 11 + 9 + 1 = 105$$

compared with a diameter of 81 for the corresponding $19 \times 19 \times \cdots \times 19$ normal torus.

### 7.2.4 $N = (2d+1)^d$: Path Length of Round 1

We show that the maximum path length in round 1 is $2d + 4$. The overshoots to be corrected with the 2-chord (of length $2d + 3$) are $d, 2d, \ldots, d^2$. The longest path is likely to be for correcting $d^2$, and we show this is in fact generally true. Note that a path for target number $x$ gives rise to a path for target number $x - 2$ by removing two $d$-chords and a reverse 2-chord then adding three 1-chords. The new path is the same length, but contains more 1-chords. We then have the possibility of creating a shorter path by replacing many 1-chords with a 2-chord. As before, we use the main chord of a path to reach a vertex near the target vertex and then correct the overshoot with the remaining chords. This ensures that the paths are edge-disjoint.

- For even $d$, $0 \leq i < d/2$, (ignoring details of special cases when $d$ is small)

$$\begin{cases} (d-2i)d = (\frac{d}{2} - 1 - i)(2d+3) + (\frac{d}{2} + 3 + 3i) & \text{target } d - 2i \\ (d-1-2i)d = (\frac{d}{2} - 1 - i)(2d+3) + (-\frac{d}{2} + 3 + 3i) & \text{target } d - 1 - 2i. \end{cases}$$

When $i < d/6$, the path lengths for these are at most

$$\begin{cases} (d-2i)+(\frac{d}{2}-1-i)+(\frac{d}{2}+3+3i)=2d+2 & \text{target } d-2i \\ (d-1-2i)+(\frac{d}{2}-1-i)+(\frac{d}{2}-3-3i)=2d-5-6i & \text{target } d-1-2i. \end{cases}$$

Paths will be shorter when $i \geq d/6$. For example,

$$(\frac{d}{2}+3+3i) \geq d+3 > (2d+3)/2+1,$$

so a single $(2d+3)$ chord would give a shorter path than $(\frac{d}{2}+3+3i)$ 1-chords. The maximum path length will be longer if the first chord is not productive. When $d$ is so large that

$$(\frac{d}{2}+3+3i) < d+1 \text{ for } i < 4,$$

for example $d = 24$, there are 4 paths of length $2d+2$ which cannot be shortened. We must also take into account the correction for the first chord of each path, since all first chords except the 1-chord, 2-chord and $d$-chord entail a reverse chord of the same length. This adds an unproductive 2 chords to the path length, and one of the four paths of length $2d+2$ must be extended to $2d+4$.

- For odd $d$, a similar analysis also gives a maximum path length of $2d+4$ since

$$\begin{cases} (d-2i)d=(\frac{d-1}{2}-i)(2d+3)-(d-3-6i)/2 & \text{target } d-2i \\ (d-1-2i)d=(\frac{d-1}{2}-1-i)(2d+3)+(d+9+6i)/2 & \text{target } d-1-2i \end{cases}$$

and when $i < d/6$, the path lengths for these are at most

$$\begin{cases} (d-2i)+(\frac{d-1}{2}-i)+(d-3-6i)/2=2d-2 & \text{target } d-2i \\ (d-1-2i)+(\frac{d-1}{2}-1-i)+(d+9+6i)/2=2d+2 & \text{target } d-1-2i. \end{cases}$$

### 7.2.5 $N = (2d+1)^d$: Upper and Lower Bounds on Total Path Length

In Section 7.3 we will show that paths in round $d+1-r$ $(1 < r < d)$ are at most $d+r$ in length, even when the 2-chord is not used for correcting the overshoot, and also that the path length for round 1 is at most $2d+4$, so the total path length for the broadcast is at most

$$\begin{cases} 1 & \text{for round } d \\ d & \text{for round } d-1 \\ \sum_{r=3}^{r=d-1}[d+r] & \text{for rounds } d-2 \text{ to } 2 \\ 2d+4 & \text{for round } 1. \end{cases}$$

Thus the total path length is

$$1+3d+4+(d-3)(3d+2)/2 = 3d^2/2 - d/2 + 2.$$

So for $d > 3$ the total path is at most 1.5 times the diameter of the corresponding normal torus-connected mesh. A tighter limit might be achieved by using the 2-chords to correct the overshoot. For large $d$, the 2-chord can reduce most of the overshoot of $rd$ in round $d-r+1$ in about $r/2$ steps leaving a further correction of at most $d+1$. Since the 2-chord gives the largest correction possible, a lower bound on the path length this algorithm can achieve is

$$1+d+\sum_{r=3}^{r=d-1}[d+r/2+1]+(2d+4) = 5+3d+(d-3)(2d+(d+2)/2)/2 \approx 5d^2/2.$$

This is about 1.25 times the diameter. The hand-optimised example for $d=9$ achieved a ratio of 1.3.

## 7.3 Circulants With $(2d+1)^{dn}$ Vertices

In this section we consider broadcasting on circulants of the form

$$C((2d+1)^{dn}, \{1, (2d+1)^n+2, (2d+1)^{2n}+3, \ldots, (2d+1)^{(d-1)n}+d\}).$$

This is intended to model the recursive application $n$ times of the method in Section 7.2 in $d$ dimensions. The total number of rounds is $dn$, and this is done as $n$ groups of $d$. Each group of $d$ rounds uses each of the $d$ chord lengths in turn as the main chord, and the next group of $d$ rounds repeats the same pattern scaled down by a factor of $2d+1$. The 2-chord is now large compared with $d^2$ and is less useful for overshoot correction. Most overshoot correction is done with the $d$-chord. General methods for selecting the entries in the tables for the broadcast are given later in this chapter.

We will first give an example for $d = 3, n = 3, N = 7^9$ and then calculate an upper bound on the total path length for the general case.

### 7.3.1 An Example for $d = 3, n = 3, N = 7^9$

The tables below describe a broadcast for a circulant graph of $7^9$ vertices with chord lengths of 1, $7^3$+2 and $7^6$+3.

| | Scaling Factor = 1 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 9 modulo $7^1$ | | | Round 8 modulo $7^4$ | | | Round 7 modulo $7^7$ | | |
| | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | | | 1 | 1 | -1 | -3 | * | 1 |
| 2 | | 1 | | -1 | 2 | -1 | -6 | | 2 |
| 3 | | | 1 | | 3 | -2 | -9 | | 3 |

| | Scaling Factor = 7 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 6 modulo $7^2$ | | | Round 5 modulo $7^5$ | | | Round 4 modulo $7^8$ | | |
| | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | * | 2 | 1 | 7 | -5 | -21 | * | 7 |
| 2 | -1 | | 5 | -1 | 14 | -9 | -42 | | 14 |
| 3 | | | 7 | | 21 | -14 | -63 | | 21 |

| | Scaling Factor = 49 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 modulo $7^3$ | | | Round 2 modulo $7^6$ | | | Round 1 modulo $7^9$ | | |
| | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | * | 16 | 1 | 49 | -33 | -147 | | 49 |
| 2 | -1 | | 33 | -1 | 97 | 50 | 51 | -1 | 98 |
| 3 | | | 49 | | 146 | 17 | 96 | -1 | 147 |

The diameter of the corresponding $343\times343\times343$ normal torus mesh is $3\times171 = 513$. The total path length for the broadcast is

$$244 + 163 + 49 + 84 + 35 + 7 + 12 + 5 + 1 = 598 \approx 1.18 \times 513.$$

Except for the last round, the difference between base vertex numbers is large enough to guarantee disjoint paths, and the last round consists trivially of the disjoint paths of length 1 to all immediate neighbours.

### 7.3.2 Notation for Calculation of the Total Path Length

In this section we will use the following notation:

$i$ is the target number.

$j$ is the round number within a group.

$g$ is the group of rounds, where $0 \leq g < n-1$, so that $dn+1-(dg+j)$ is the round number.

$s$ is the scaling factor for successive groups of rounds, and is equal to $(2d+1)^g$.

$S$ is the scaling factor used in the first round, equal to $(2d+1)^{n-1}$.

Note that $g$ and $j$ number the rounds in reverse order so as to simplify expressions.

### 7.3.3 The Path Length for the Last $d$ Rounds

The scaling factor $s$ is equal to 1 and we number this group of rounds with $g = 0$ and j varying from 1 for round $dn$ to $j = d$ for round $d(n-1)+1$. In round $dn+1-j$ the main chord for paths is of length $(2d+1)^{(j-1)n}+j$ and target $i$ is vertex $i(2d+1)^{(j-1)n}$. Choosing a path consisting of $i$ main chords for target $i$ in round $dn+1-j$ gives an overshoot of $ij$ for target $i$, for $i = 1$ to $d$. The overshoots can be corrected to give a path length of at most $d+j$ for $1 < j < d$ as described below. Rounds corresponding to $j = 1, d$ and $d-1$ are considered separately from rounds for $j = 2$ to $d-2$.

$j = 1$ This is the last round, where the paths are trivially of length 1 to adjacent nodes.

$j = d$ we can use only the 1-chord to correct the overshoot of $d^2$, giving a worst path length of $d + d^2$ to target number $d$. The first chord to target number $i$ can be the $i$-chord without affecting the worst path length.

$j = d - 1$ The overshoot can be expressed as

$$ij = i(d-1) = (i-1)d + (d-i)$$

which gives a correction path consisting of $(i-1)$ $d$-chords and $(d-i)$ 1-chords. Thus a total of $= d - 1$ chords can correct the overshoot. Together with the $i$ main chords this gives a path length of $d + i - 1$ and since $j = d - 1$ this path length is at most $d + j$. We must also allows for the 2 unproductive steps which are sometimes required for the first chord of the path and its reverse. For the first $d - 2$ targets , we have $i < d - 1$ and the path length is at most $d + j - 2$. If we use the 1-chord and d-chord as the first chords for paths to the last two targets, no path length will exceed $d + j$.

$1 < j \leq d - 2$ For any round in this range, the strategy depends on whether the target number $i$ is less than, equal to, or greater than $j$.

- For target $i > j$ use entries on the main diagonal of the table to correct the overshoot of $ij$ with $j$ $i$-chords, which together with the $i$ main chords gives a path length of $i + j$. As $d$ is the largest value for $i$, the path length $i + j$ is at most $d + j$. For this range of values for $i$ we can use the $i$-chord as the first chord of the path without increasing the path length.

- For target $i = j$ we have

$$ij = j^2 = (j-2)(j+1) + (j+2).$$

So we can use $j - 2$ $(j+1)$-chords and a $(j+2)$-chord, giving a path length of $2j - 1 < d + j$ with the $j$-chord as the first chord of the path.

- For target $i < j$, shorter overshoot corrections can be constructed from the $i = j$ case by replacing two $(j+1)$-chords by a $(j+2)$-chord. Since we start with $j-2$ $(j+1)$-chords, this is possible until

$$i = j - \lfloor (j-2)/2 \rfloor = \lfloor (j+1)/2 \rfloor .$$

  If we continue beyond this (a negative number of chords is interpreted as chords in the opposite direction) the correction will increase by 3 chords for each smaller target number until it is $2j$ for target 1, since the overshoot for this target is $j$ and

$$j = (-j)(j+1) + j(j+2).$$

  As $j$ is at most $d-2$, $2j$ is at most $d+j-2$ and even this poor construction guarantees a maximum path length of $d+j$ chords to any target vertex.

So for the last $d$ rounds the total path will be

$$\begin{aligned} 1 + (d + d^2) + \sum_{j=2}^{j=d-1} (d+j) &= 1 + d + d^2 + (d-2)(3d+1)/2 \\ &= d(5d-3)/2. \end{aligned}$$

### 7.3.4 Path Length for Rounds $d+1$ to $d(n-1)$

The last $d$ rounds are then scaled up by a factor $(2d+1)$ for each group of $d$ rounds, and since

$$\sum_{g=0}^{g=n-2} (2d+1)^g = ((2d+1)^{n-1} - 1)/2d$$

the total path length for the last $d(n-1)$ rounds will be at most

$$\begin{aligned} [d(5d-3)/2][(2d+1)^{n-1} - 1]/2d &< (5d-3)(2d+1)^{n-1}/4 \\ &= S(5d-3)/4. \end{aligned}$$

Note that we can easily derive new paths rather than scaling up the paths found for the last $d$ rounds. The new paths are described below, but do not reduce the bound on maximum path length. Paths can be shortened if we use $d$-chords to reduce the scaled up number of 1-chords. Because of the scaling factor $s = (2d+1)^g$, the largest numbered target vertex requires the longest path when $s > 1$ and $j < d$. The next lower target vertex requires $s$ fewer steps along the main chords, which decreases the overshoot by $js$. This decrease in overshoot by $js$ can be corrected by

$$\frac{j}{d}(s-1) \; d\text{-chords, a } (j+1)\text{-chord and a 1-chord,}$$

giving a total of

$$\begin{aligned} 2j+2 &< 2d+1 \\ &\leq s \text{ extra chords for the correction.} \end{aligned}$$

Since there are $s$ fewer steps along the main chord, we can always construct a shorter path to a smaller numbered target vertex when $s > 1$, $j < d$. (We have already found the results for $s = 1$ and for $j = d$.) Note that planning detailed paths is simplified because we know $d$ is a factor of $s - 1$. A correction of $xs - x$ can be done with $\frac{x(s-1)}{d}$ $d$-chords and the remaining correction of $x$ done in the same way as for $s = 1$.

### 7.3.5 $N = (2d+1)^{nd}$: General Examples for Rounds $d+1$ to $d(n-1)$

It is easy to write down a general scheme which meets the $s(d+j)$ limit when $s > 1$ and $1 < j < d-1$. For these case we can reach close to target $i$ with a path consisting of $is$ main chords ($j$-chords), giving an overshoot of $ijs$. A large part of the overshoot can be corrected with $ij(s-1)/d$ $d$-chords, leaving $ij$ still to be corrected. The remaining overshoot of $ij$ requires $i$ $(j+1)$-chords and $i$ 1-chords in the opposite direction. The results of this are shown in the table below. Note that the last row has been simplified: an overshoot of $djs$ can be corrected by $js$ $d$-chords. Columns containing only zeros have been omitted and only the first and last two rows of the table are shown.

| General scheme for round $dn+1-(dg+j)$, scaling factor $s=(2d+1)^g$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | |
| Vertex | 1 | 2 | ... | $j$ | $j+1$ | ... | $d$ |
| 1 | 1 | | | $s$ | $-1$ | | $-j(s-1)/d$ |
| 2 | 2 | | | $2s$ | $-2$ | | $-2j(s-1)/d$ |
| ... | | | | | | | |
| $d-1$ | $d-1$ | | | $(d-1)s$ | $-(d-1)$ | | $-(d-1)j(s-1)/d$ |
| $d$ | | | | $ds$ | | | $-js$ |

Specific cases can be improved by inspection to give shorter paths for some of the targets (although the maximum path length is not reduced). The remaining overshoot of $ij$ can always be corrected with extra $d$-chords to within a distance $d/2$ of the target vertex. The table below shows what is possible when $j$ is a factor of $d$. In this case the remaining overshoot of $ij$ is often a multiple of $d$ and can be corrected entirely with $d$-chords.

| $d=10$, $j=5$, chords $=1,23,444,\ldots$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | | |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 1 | * | | | $s$ | $-1$ | | | | $-5(s-1)/10$ |
| 2 | | | * | | $2s$ | | | | | $-s$ |
| 3 | 1 | | | * | $3s$ | $-1$ | | | | $-5(3s-1)/10$ |
| 4 | | | | | $4s$ | | * | | | $-2s$ |
| 5 | 1 | | | | $5s$ | $-1$ | | * | | $-5(5s-1)/10$ |
| 6 | | | | | $6s$ | | | | * | $-3s$ |
| 7 | 1 | | | | $7s$ | $-1$ | | | | $-5(7s-1)/10$ |
| 8 | | | | | $8s$ | | | | | $-4s$ |
| 9 | 1 | | | | $9s$ | $-1$ | | | | $-5(9s-1)/10$ |
| 10 | | | | | 10s | | | | | $-5s$ |

Using $d$-chords to reduce the remaining overshoot down to $d/2$ does not always give the shortest path. For example, in the following table, the remaining overshoot is left at 54 which can be corrected with 6 9-chords. If it were reduced to 4 with 5 extra 10-chords, it would then need 4 1-chords to complete the path. (The 4-chord cannot be used because it is smaller than rotational symmetry allows).

| $d = 10$, $j = 6$, chords $= 1, 23, 444, \ldots$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Target | Chord Number | | | | | | | | | |
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 1 | * | | | | $s$ | $-1$ | | | $-6(s-1)/10$ |
| 2 | $-2$ | | * | | | $2s$ | | | | $-1-12(s-1)/10$ |
| 3 | | | | * | | $3s$ | | | $-2$ | $-18(s-1)/10$ |
| 4 | | | | | | $4s$ | | $-3$ | | $-24(s-1)/10$ |
| 5 | | | | | * | $5s$ | | | | $-3s$ |
| 6 | | | | | | $6s$ | | | -4 | $-36(s-1)/10$ |
| 7 | | | | | | $7s$ | $-2$ | $-1$ | | $-2-42(s-1)/10$ |
| 8 | | | | | | $8s$ | | | $-2$ | $-3-48(s-1)/10$ |
| 9 | | | | | | $9s$ | | | $-6$ | $-54(s-1)/10$ |
| 10 | | | | | | $10s$ | | | | -6s |

### 7.3.6 $N = (2d+1)^{nd}$: Path Lengths for Rounds 1 to $d$

The scaling factor for this first group of rounds is $S = (2d+1)^{n-1}$ and the length of the 2-chord can be expressed as $(2d+1)S + 2$ . The corrections required for target $i$ in round $d+1-j$ are

$$ijS \text{ for } i, j = 1 \text{ to } d.$$

Whenever $ij > d$ we can use the 2-chord of length $(2d+1)S + 2$ to give a shorter correction path. As before, target vertex number $d$ will always have the longest path. In round $d+1-j$ target number $d$ will require a correction of $djS$. There are six cases considered below, depending on the values of $j$ and $d$.

1. For $j = 1$ (round $d$) the path is simply $S$ $d$-chords and the path length is $S$.

2. For $j = 2$ (round $d-1$) we express the overshoot as

$$2dS = [(2d+1)S + 2] - [(S-1)/d]d - 3.$$

Here the main chord $(2d+1)S+2$ is used as the correction, so the number of main chords is reduced from $dS$ to $dS-1$ with a remaining correction of $(S-1)/d$ $d$-chords and three 1-chords. The path length is thus

$$dS - 1 + (S-1)/d + 3 = S(d+1/d) + 2 - 1/d.$$

3. For $j = d$ (round 1), the overshoot for even $d$ is

$$d^2S = (d/2)[(2d+1)S+2] - d(S+2)/2,$$

and including the $dS$ main chords the path length is

$$dS + d/2 + d(S+2)/2 = 3d(S+1)/2.$$

4. For $j = d$ (round 1), the overshoot for odd $d$ is

$$d^2S = [(d-1)/2][(2d+1)S+2] + (d+1)S/2 - (d-1),$$

and including the $dS$ main chords the path length is

$$dS + (d-1)/2 + (d+1)S/2 - (d-1) = (3d+1)S/2 - (d-1)/2.$$

5. For $2 < j < d$, j even

$$djS = (j/2)((2d+1)S+2) - [j(S-1)/(2d)]d - 3j/2,$$

and including the $dS$ main chords the path length is

$$dS + j/2 + j(S-1)/(2d) + 3j/2.$$

6. For $2 < j < d$, j odd

$$djS = ((j-1)/2)((2d+1)S+2) + [(2d-j+1)(S-1)/(2d)]d + (2d-3j+3)/2,$$

and including the $dS$ main chords the path length is

$$dS + (j-1)/2 + (2d-j+1)(S-1)/(2d) + (2d-3j+3)/2.$$

The combined path length for rounds corresponding to $j$ and $(j+1)$ for even values of $j$ greater than 2 simplifies to

$$S(2d+1)+d+j-1$$

and for $j=2$, rounds $d-1$ and $d-2$, the combined path length is

$$S(2d+1)+d+1-(2+1/d).$$

Combined path lengths in pairs this way will reduce the number of cases which need to be considered in calculating the total path length.

### 7.3.7 $N=(2d+1)^{nd}$: Total Path Length for $dn$ Rounds

For even $d$, the above results give a path length for the first $d$ rounds of

$$\begin{aligned}
& S-(2+1/d)+3\frac{d}{2}S+3\frac{d}{2} \\
& +\sum_{x=1}^{x=\frac{d-2}{2}} S(2d+1)+d+(2x-1) \\
= & \ S+\frac{(d-2)}{2}[S(2d+1)+d]+\frac{d-2}{2}^2-(2+1/d)+3\frac{d}{2}S+3\frac{d}{2} \\
= & \ Sd^2+3d^2/4-\frac{d}{2}-1-1/d.
\end{aligned}$$

For odd $d$, the path length for the first $d$ rounds is

$$\begin{aligned}
& S-(2+\frac{1}{d})+S(d+\frac{1}{2}-\frac{1}{2d})+(2d-5/2+\frac{1}{2d})+(S(3d+1)/2-\frac{d}{2}+1/2) \\
& +\sum_{x=1}^{x=\frac{d-3}{2}} S(2d+1)+d+(2x-1) \\
= & \ S+\frac{(d-3)}{2}[S(2d+1)+d]+\frac{d-3}{2}^2+S(5\frac{d}{2}+1-\frac{1}{2d})+(3\frac{d}{2}-4-\frac{1}{2d}) \\
= & \ S(d^2-\frac{1}{2}-\frac{1}{2d})+3d^2/4-3\frac{d}{2}-7/4-1/(2d).
\end{aligned}$$

In both cases, ignoring small negative terms we get a path length which is less than $(S+3/4)d^2$. Thus the total path length including the first $(n-1)d$ rounds is less than

$$(5d-3)S/4+(S+3/4)d^2 < S(d^2+\frac{5}{4}d)+\frac{3}{4}d^2.$$

The diameter of a comparable normal torus connected mesh is

$$d[(2d+1)S-1]/2 = S(d^2+\frac{d}{2})-\frac{d}{2}.$$

Thus for $(2d+1)^{dn}$ vertices the ratio of the total path length for broadcasting on the a skewed torus to the diameter of a normal torus mesh is less than

$$1+\frac{3}{4d}+\frac{3}{4S}$$

and approaches 1 for increasing dimension $d$ when $n>1$.

# Chapter 8

# Odd-Circulant Graphs

Circulant graphs have even degree unless one of the chord lengths equals half the number of vertices (which is not dealt with in this thesis). To investigate other graphs of odd degree we consider a generalisation of the chordal ring and circulant graphs. Define an odd-circulant graph by removing half of the edges of a particular odd chord length from a circulant graph. The even vertices have a negative edge removed and the odd vertices have a positive edge of the same chord length removed. As with the chordal ring, the number of vertices must be even and the removed chord must have an odd length and also be (for our algorithm) the longest or next to longest chord. Broadcasting algorithms similar to those for circulants will be developed for odd-circulant graphs, taking into account that the required chord may not exist with the correct sign.

We will continue to regard $d$ as the dimension of the geometric visualisation, but the degree of the corresponding graph is now $2d-1$ and the number of vertices informed after $r$ rounds of the broadcast is $(2d)^r$.

Chapter 5 shows the results for degree 3 vertices in detail, using tiling techniques in 2 dimensions. In Chapter 6 we have considered a degree 5 odd-circulant with $6^3$ vertices in terms of a 3-dimensional mesh. We will extend the broadcasting algorithm to a degree 5 circulant of $6^{3n}$ vertices and also consider higher degree odd-circulants of degree $(2d-1)$ with $N=(2d)^d$ vertices. The total path length we derive for a broadcast on an odd-circulant with $(2d)^d$ vertices is very similar to that for a circulant.

The case $N = (2d)^{dn}$ vertices has not been attempted.

## 8.1 Broadcasting on an Odd-Circulant

The general strategy is the same as for a normal circulant with minor changes to ensure the sign of the $d$-chord is correct for the chosen path.

- The asymmetric chord must be of odd length, so for even $d$ we cannot have a $d$-chord of length $d \pmod{2d}$. The details of broadcasting therefore differ for even $d$, and in the last round it is necessary to increase the vertex number of alternate base vertices by one. (As in Chapter 5, the last round of broadcasting needs to group base vertices in pairs if $(2d-1) = 3 \bmod 4$.) Alternatively, the $(d-1)$-chord can be made the asymetric one. This leaves a gap in the immediate neighbours of a vertex, and the second last round must still be adjusted to make the gaps of adjacent nodes interlock.
- There must also be an odd length chord between two $d$-chords in a path. To make this easier, the length of the 2-chord is made odd when $d > 3$.
- The exact reflection of a path containing $d$-chords is not possible. (The chords used can be the same, but with opposite signs and a slightly different order). Forward paths can only use the $d$-chord from even vertices; backward paths from odd vertices.

The total path length for $N = (2d)^d$ vertices is shown to be at most $(3d^2 + d)/2$.

### 8.1.1 When the Dimension $d$ is Even

The chord lengths are chosen to be

$$\begin{aligned} &\pm\ 1, \\ &\pm\ [2d+3], \\ &\pm\ [(2d)^2+2], \end{aligned}$$

$$\begin{aligned}
\pm\ & [(2d)^3 + 4],\\
\pm\ & [(2d)^4 + 5],\\
\pm\ & \cdots,\\
\pm\ & [(2d)^i + (i+1)],\\
\pm\ & \cdots,\\
\pm\ & [(2d)^{d-2} + (d-1)],\\
& [(2d)^{d-1} + (d+1)]
\end{aligned}$$

where the sign of the $d$-chord is negative for odd vertices. Note that the length of the $d$-chord has been increased to $(2d)^{d-1} + (d+1)$ so it will be odd. Some caution is needed when correcting overshoots as the $d$ chord corrects by $d+1$, the 2-chord corrects by 3 or $2d+3$, and the 3-chord corrects by 2.

Details of the even case have been omitted as it is similar to that for odd values of $d$ described below, but requires a minor adjustment in the last two rounds. In round $d-1$ the target vertices are adjusted to be

$$(2d+1), 4d, (6d+1), 8d\ldots \text{ and } (-2d+1), 4d, (-6d+1), 8d\ldots.$$

### 8.1.2 When the Dimension $d$ is Odd

For odd dimensions greater than three, the chord lengths are chosen to be

$$\begin{aligned}
\pm\ & 1,\\
\pm\ & [2d+3],\\
\pm\ & [(2d)^2 + 2],\\
\pm\ & [(2d)^3 + 4],\\
\pm\ & [(2d)^4 + 5],\\
\pm\ & \cdots,\\
\pm\ & [(2d)^i + (i+1)],\\
\pm\ & \cdots,\\
\pm\ & [(2d)^{d-2} + (d-1)],
\end{aligned}$$

$$[(2d)^{d-1} + d]$$

where the sign of the $d$-chord is negative for odd vertices. The 2-chord and 3-chord have been adjusted to make the 2-chord an odd length. When $d = 3$ (degree 5 vertices) this would give chord lengths

$$\pm 1, \pm[2d+3], [(2d)^2 + 2]$$

which is not possible, as the asymmetric chord must have an odd length. We deal with degree 5 first.

## 8.2 Degree 5, $d = 3$, $N = 6^{3n}$

We will write the round number as

$$3n + 1 - 3g - j \text{ where } 0 \le g < n, 0 < j \le 3$$

to simplify expressions. The rounds for $N = 6^{3n}$ vertices can be expressed in the usual way.

| Last 3 Rounds | Chord Lengths 1, $6^n+2$, $6^{2n}+3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round $3n$ | | | Round $3n-1$ | | | Round $3n-2$ | | |
| | mod6 | | | mod$6^{n+1}$ | | | mod$6^{2n+1}$ | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | 0 | 0 | 1 | 1 | −1 | −3 | 0 | 1 |
| 2 | 0 | 1 | 0 | −1 | 2 | −1 | −6 | 0 | 2 |
| 3 | 0 | 0 | 1 | −3 | 3 | −1 | −9 | 0 | 3 |

| General | Chord Lengths 1, $6^n+2$, $6^{2n}+3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round $3(n-g)$ mod$6^{g+1}$ Chord No. | | | Round $3(n-g)-1$ mod$6^{n+g+1}$ Chord No. | | | Round 3(n-g)-2 mod$6^{2n+g+1}$ Chord No. | | |
| Target Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | $6^g/2-2$ | 1 | $-6^g/2$ | $6^g$ | $-6^g/2$ | $-3\times 6^g$ | 0 | $6^g$ |
| 2 | 1 | $6^g-2$ | 1 | $-6^g$ | $2\times 6^g$ | $-6^g$ | $-6\times 6^g$ | 0 | $2\times 6^g$ |
| 3 | 1 | $\frac{3}{2}\times 6^g-2$ | 1 | $-\frac{3}{2}\times 6^g$ | $3\times 6^g$ | $-\frac{3}{2}\times 6^g$ | $-9\times 6^g$ | 0 | $3\times 6^g$ |

Writing $S$ for $6^{n-1}$ we obtain

| First 3 | $S=6^{n-1}$, Chord Lengths 1, $6^n+2$, $6^{2n}+3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 mod$6^n$ Chord No. | | | Round 2 mod$6^{2n}$ Chord No. | | | Round 1 mod$6^{3n}$ Chord No. | | |
| Target Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | $S/2-2$ | 1 | $-S/2$ | $S$ | $-S/2$ | $-3S$ | 0 | $S$ |
| 2 | 1 | $S-2$ | 1 | $S/2+2$ | $2S-1$ | $S/2$ | 2 | $-1$ | $2S$ |
| 3 | 1 | $\frac{3}{2}S-2$ | 1 | $-1$ | $3S-1$ | 1 | $-3S+2$ | $-1$ | $3S$ |

The total path length for the degree 5 broadcast is given by the following table.

| | Round | | | | Total |
|---|---|---|---|---|---|
| $g$ | $3(n-g)$ | $3(n-g)-1$ | $3(n-g)-2$ | | for Row |
| 0 | 1 | 7 | 12 | = | 20 (= 19.5 + 0.5) |
| 1 | 9 | 36 | 72 | = | 117 (= 19.5 × 6) |
| ... | | | | | |
| r | $\frac{3}{2}\times 6^g$ | $6\times 6^g$ | $12\times 6^g$ | = | $19.5\times 6^g$ |
| ... | | | | | |
| n-1 | $\frac{3}{2}\times 6^{n-1}$ | $3\times 6^{n-1}+1$ | $6\times 6^{n-1}+1$ | = | $10.5\times 6^{n-1}+2$ |
| Total | $0.5+19.5\sum_{g=0}^{n-2}6^g+10.5\times 6^{n-1}+2$ | | | = | $2.4\times 6^n-1.4$ |

There is no 'standard' torus to compare this result with. However the diameter of the graph is roughly $1.5\times 6^n$, so the ratio of total path length to diameter is 1.6.

A better choice of chord lengths that was considered was $\{\pm 1, \pm 3, 5\}$, so that the short and long chords could alternate, but when investigated it gave a greater total path length, as calculated below. This choice of chord lengths required vertices to be paired in the same way as for degree 3 chordal rings. Vertex 0 sent to vertices 1, 3, 5 and vertex 7 sent to vertices 2, 4, 6 (i.e. $-5, -3, -1$ relative to vertex 7). Ignoring this and other minor adjustments in the final rounds gives the following tables to describe the broadcast.

| Last 3 Rounds | Chord Lengths 1, $6^n+3$, $6^{2n}+5$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round $3n$ mod6 Chord No. | | | Round $3n-1$ mod$6^{n+1}$ Chord No. | | | Round $3n-2$ mod$6^{2n+1}$ Chord No. | | |
| Target Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 1 | 0 | 0 | 2 | 1 | $-1$ | $-5$ | 0 | 1 |
| 2 | 0 | 1 | 0 | $-1$ | 2 | $-1$ | $-10$ | 0 | 2 |
| 3 | 0 | 0 | 1 | 1 | 3 | $-2$ | $-15$ | 0 | 3 |

| Gen. | Chord Lengths 1, $6^n+3$, $6^{2n}+5$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round $3(n-g)$ mod$6^{g+1}$ Chord No. | | | Round $3(n-g)-1$ mod$6^{n+g+1}$ Chord No. | | | Round $3(n-g)-2$ mod$6^{2n+g+1}$ Chord No. | | |
| Target No. | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 0 | $6^g/8$ | $6^g/8$ | $-3$ | $6^g$ | $-0.6(6^g-1)$ | $-5\times 6^g$ | 0 | $6^g$ |
| 2 | 0 | $6^g/4$ | $6^g/4$ | $-1$ | $2\times 6^g$ | $-0.2(6^{g+1}-1)$ | $-10\times 6^g$ | 0 | $2\times 6^g$ |
| 3 | 0 | $\frac{3}{8}6^g$ | $\frac{3}{8}6^g$ | 1 | $3\times 6^g$ | $-1.8(6^g+1/9)$ | $-15\times 6^g$ | 0 | $3\times 6^g$ |

| First 3 | $S = 6^{n-1}$, Chord Lengths 1, $6^n + 3$, $6^{2n} + 5$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Round 3 | | | Round 2 | | | Round 1 | | |
| | $\bmod 6^n$ | | | $\bmod 6^{2n}$ | | | $\bmod 6^{3n}$ | | |
| Target | Chord No. | | | Chord No. | | | Chord No. | | |
| Number | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 0 | $S/8$ | $S/8$ | $-3$ | $S$ | $-0.6(S-1)$ | $S+3$ | $-1$ | $S$ |
| 2 | 0 | $S/4$ | $S/4$ | 3 | $2S-1$ | 0 | $2S+6$ | $-2$ | $2S$ |
| 3 | 0 | $\frac{3}{8}S$ | $\frac{3}{8}S$ | 0 | $3S-1$ | $-0.6(S-1)$ | $-3S+6$ | $-2$ | $3S$ |

This gives a total broadcast path length of about $2.525 \times 6^n$, which is worse than the previous result. Other possibilities include increasing the odd chord length to $2 \times 6^{2n} + 5$, but this has not been investigated.

## 8.3 Degrees $9, 13, 17 \ldots$

This section deals with odd dimensions greater than $d = 3$ for odd-circulant graphs with $N = (2d)^d$ vertices. For each round number we will give paths for the broadcast and calculate the maximum path length.

### Round 1 for Degrees $9, 13, 17 \ldots$

As for the normal circulant, we will consider the chord length to be separated into a 'useful' part and an 'error' term or overshoot. The target vertices in round 1 are multiples of $(2d)^{d-1}$ and so round 1 will use a main chord of length $(2d)^{d-1} + d$ where $d$ is regarded as the overshoot for each main chord in a path. The overshoots to be corrected for paths to target numbers 1, 2, 3, $\ldots, d$ are $d, 2d, 3d, \ldots, d^2$ respectively and they must be corrected with chords of length 1 and $2d + 3$. These overshoots and chord lengths are the same as for round one of the broadcast on a normal circulant, so the maximum path length must be at least the value $2d + 4$ which was calculated for the normal circulant. The maximum path length on an odd-circulant might be worse because there are fewer chords available than in the corresponding circulant.

We can show that this is not the case in round 1 by finding how many chords of odd length will be in any path.

Since at most $d$ of the chords in the path to a target vertex are main chords, the rest must be of length 1 or $2d+3$. Hence $d+2$ of the chords in a path for broadcasting on an odd circulant in round 1 will be of length 1 or $2d+3$ and the requirement for odd-length chords between $d$-chords can be satisfied.

In round 1 there is only one base vertex. The first chord of paths, except for paths starting with a 1-chord, 2-chord or $d$-chord, will spread paths at least $(2d)^2 = 4d^2$ apart and prevent edge contention (since the largest overshoot is only $d^2$). The last chord in these paths will be the reverse of the first chord to cancel its effect. Paths starting with a 1-chord, 2-chord or $d$-chord will pass close to target vertices and must be planned carefully.

As an example, consider round one for $d = 5, 2d = 10, N = 100,000$, a degree 9 graph with 100,000 vertices which requires 5 rounds for the broadcast. The table showing how many of each chord length is required for each target vertex is shown below. As before, the entries of '*' for chords 3 and 4 indicate that the path begins with a step along the chord, but this is cancelled out further along the path by a step along an equal chord in the opposite direction.

| Round 1, $d=5$ chords $= 1, 13, 102, 1004, 10005$ | | | | | |
|---|---|---|---|---|---|
| Target | Chord Number | | | | |
| Vertex | 1 | 2 | 3 | 4 | 5 |
| 1 | -5 | | | | 1 |
| 2 | 3 | -1 | | * | 2 |
| 3 | -2 | -1 | * | | 3 |
| 4 | 6 | -2 | | | 4 |
| 5 | 1 | -2 | | | 5 |

Using this table we can construct a list of vertices along each path (relative to the source vertex). Each path is shown as a column in the table below. The longest path is of length 12, to reach target number 4.

| Round 1: Vertex Numbers Along Paths | | | | | |
|---|---|---|---|---|---|
| Path | To Target Number | | | | |
| Length | 1 | 2 | 3 | 4 | 5 |
| 1 | −1 | 1,004 | 102 | −13 | 10,005 |
| 2 | −2 | 11,009 | 10,107 | −26 | 9,992 |
| 3 | 10,003 | 10,996 | 10,094 | 9,979 | 19,997 |
| 4 | 10,002 | 21,001 | 20,099 | 9,980 | 19,984 |
| 5 | 10,001 | 21,002 | 20,098 | 19,985 | 29,989 |
| 6 | 10,000 | 21,003 | 30,103 | 19,986 | 29,990 |
| 7 | | 21,004 | 30,102 | 29,991 | 39,995 |
| 8 | | 20,000 | 30,000 | 29,992 | 39,996 |
| 9 | | | | 39,997 | 50,001 |
| 10 | | | | 39,998 | 50,000 |
| 11 | | | | 39,999 | |
| 12 | | | | 40,000 | |

The paths to negative targets are slightly different as the backward d-chords (of length 10,005) are available only at odd vertices. Note that there are only 4 backward paths.

| Round 1: Vertex Numbers Along Negative Paths | | | | | |
|---|---|---|---|---|---|
| Path | To Target Number | | | | |
| Length | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ |
| 1 | 1 | $-1,004$ | $-102$ | 13 | does |
| 2 | $-10,004$ | $-991$ | $-101$ | $-9,992$ | not |
| 3 | $-10,003$ | $-10,996$ | $-10,106$ | $-9,979$ | exist |
| 4 | $-10,002$ | $-10,997$ | $-10,093$ | $-19,984$ | |
| 5 | $-10,001$ | $-21,002$ | $-20,098$ | $-19,985$ | |
| 6 | $-10,000$ | $-21,003$ | $-20,097$ | $-29,990$ | |
| 7 | | $-21,004$ | $-30,102$ | $-29,991$ | |
| 8 | | $-20,000$ | $-30,000$ | $-39,996$ | |
| 9 | | | | $-39,997$ | |
| 10 | | | | $-39,998$ | |
| 11 | | | | $-39,999$ | |
| 12 | | | | $-40,000$ | |

**Round 2 for Degrees** $9, 13, 17 \ldots$

The target vertices are multiples of $(2d)^{d-2}$ and so round 2 will use a main chord of length $(2d)^{d-2} + d - 1$, where $d - 1$ is the overshoot for each main chord in a path. The overshoots to be corrected for paths to target numbers $1, 2, 3, \ldots, d$ are $d - 1, 2(d - 1), 3(d - 1), \ldots, d(d - 1)$ respectively, and they must be corrected with chords of length 1 and $2d + 3$.

Odd target $(2i + 1)$ has an overshoot of

$$(2i + 1)(d - 1) = 2id + d - 2i - 1$$

and, since

$$(2i + 1)(d - 1) = i(2d + 3) + d - 1 - 5i$$

and

$$-3(d - 1)/2 < d - 1 - 5i < d - 1 \text{ for } i = 0 \text{ to } (d - 1)/2,$$

the overshoot can be corrected by

$i$ chords of length $(2d+3)$, up to one $d$-chord, and up to $(d-1)/2$ 1-chords.

So the path length to odd targets is at most

$$(2i+1)+i+1+(d-1)/2 = 3i+(d+3)/2 \leq 2d.$$

Similarly since

$$2i(d-1) = (i-1)(2d+3)+2d+3-5i$$

and

$$(11-d)/2 < 2d+3-5i < 2d-2 \text{ for } i=1 \text{ to } (d-1)/2,$$

the overshoot for an even target $2i$ can be corrected by

$(i-1)$ chords of length $(2d+3)$, up to 2 $d$-chords, and up to $(d-1)/2$ 1-chords.

So the path length to even targets is at most

$$2i+(i-1)+2+(d-1)/2 = 3i+(d+1)/2 \leq 2d-1.$$

In both cases the maximum path length can be taken as $2d$.

### Rounds 3 to $d-2$ for Degrees $9, 13, 17\ldots$

For round number $d+1-r$, where $3 \leq r \leq d-2$, target vertices are multiples of $(2d)^{r-1}$ and so round $d+1-r$ will use a main chord of length $(2d)^{r-1}+r$, where $r$ is the overshoot for each main chord in a path. The overshoots to be corrected for paths to target numbers $1, 2, 3, \ldots d$ are $r, 2r, 3r, \ldots, dr$ respectively, and they must be corrected with chords of length 1 and $2d+3$, together with the $d$-chord,$(d-1)$-chord,..., $(r+1)$-chord.
If $r$ is odd, correct the overshoot of $dr$ by using $r$ $d$-chords. If $r$ is even (and thus not equal to $d-2$), correct $dr$ with

$r/2-1$ $d$-chords, a $(d-1)$-chord, $r/2$ $(d-2)$-chords and an $(r+1)$-chord.

**Round $d-2$ for Degrees $9, 13, 17\ldots$**

The overshoot is $2d$ for target vertex $d$. Correct with two $(d-2)$-chords and a 4-chord. If $d = 5$, use two 4-chords and two 1-chords.

**Round $d-1$ for Degrees $9, 13, 17\ldots$**

The overshoot will be $3, 6, \ldots, 3d$. Correct the larger overshoots (those greater than $d$) by using one less main chord and suitable other chords. For example, an overshoot of $3d$ can be corrected by a $2d+3$ main chord and a $(d-3)$-chord. The maximum path length will be $d$, just as for round $d-1$ of a broadcast on a circulant graph.

**Round $d$ for Degrees $9, 13, 17\ldots$**

The last round is always to adjacent nodes, with a path length of 1.

**Total Path Length for Degrees $9, 13, 17\ldots$**

Thus the total path length for a broadcast for odd dimension $d$ (i.e. odd-circulant graphs of degrees $9, 13, 17\ldots$) is less than

$$\begin{aligned} & 1 + d + d + 4 + d + 5 + \cdots + d + (d-1) + 2d + 2d + 4 \\ = \;& 1 + (d-1)(3d+4)/2 \\ \approx \;& 3d^2/2. \end{aligned}$$

This result is very similar to the results for a normal circulant.

### 8.3.1 Comparison of the Total Path Length to the Diameter

For $N = (2d+1)^d$ vertices and the chord lengths approximated by powers of $2d+1$, the longest path to a vertex can contain at most $d$ chords of any length. Otherwise a longer chord could be used to reduce the path length. If the path includes more alternating chords than compensating odd chords, just increase the number of odd chords with pairs which cancel out. As there are $d$ chord lengths we can approximate the diameter by $d^2$. The slight deviations of chord lengths from powers of $2d + 1$ are assumed to have only a small affect on the diameter.

Thus the ratio of total path length to diameter is approximately 1.5.

# Chapter 9

# Conclusions

In this thesis we have shown how a torus can be modified to form a circulant graph. Assuming an all links communications model with circuit-switched routing we derived broadcasting algorithms which are optimal with respect to the number of rounds and almost optimal in the total path length a message travels. This has been extended to graphs of any odd degree, based on the chordal ring structure which we introduced to describe a triangular mesh network.

## 9.1 Summary of Results

The number of rounds is always optimal. The tables below show the ratio of total path length to the graph diameter for each case of the broadcasting algorithms.

### 9.1.1 Normal Torus Topology

| Degree of Graph | Number of Vertices | Diameter of Graph | Total Path Length | Path Ratio |
|---|---|---|---|---|
| 4 | $5^{2n}$ | $5^n - 1$ | $5^n - 1$ | 1 |
| 6 | $7^{3n}$ | $3(7^n - 1)/2$ | $11(7^n - 1)/6$ | 1.22 |

### 9.1.2 Skewed Torus Topology

For the skewed torus represented as a circulant we have based chord lengths on those for normal torus connections, making the chord lengths close to powers of $\sqrt[d]{N}$ for $N$ vertices in $d$ dimensions. For odd-circulants we have chosen chord lengths in an analogous manner to be close to powers of $\sqrt[d]{N}$.

For even degree graphs, the total path length for a broadcast is compared with the diameter of the corresponding normal torus mesh. This is reaonable because we are deliberately choosing chord lengths close to those of a mesh embedded in a circulant graph. For example, in two dimensions the minimal skewing of the torus reduces its diameter by 1. For graphs of odd degree, the diameter is also approximated by ignoring the affect of using chord lengths slightly different from powers of $\sqrt[d]{N}$ and is assumed to be $d^2$.

The most general case for even degree graphs, shown in the last row of the table, applies the basic algorithm $n$ times with a decreasing scale factor. For brevity the diameter and path length are given in terms of the initial scale factor $S = (2d+1)^{n-1}$. The results in this row are valid for $n > 1$.

| Vertex Degree | No.of Vertices | Diameter of Graph | Total Path Length | Path Ratio |
|---|---|---|---|---|
| 3 | $4^n$ | $\approx 0.96 \times 2^n$ | $1.33 \times 2^n$ | $\approx 1.4$ |
| 4 | $5^{2n}$ | $5^n - 2$ | $(23 \times 5^{n-1} - 3)/4$ | 1.15 |
| 5 | $6^{3n}$ | $\approx 1.5 \times 6^n$ | $2.4 \times 6^n$ | $\approx 1.6$ |
| 6 | $7^{3n}$ | $3(7^n - 1)/2 - 3$ | $37 \times 7^{n-1}/3 - 13/3$ | 1.17 |
| $2d - 1$ | $(2d)^d$ | $\approx d^2$ | $\approx 1.5 \times d^2$ | $\approx 1.5$ |
| $2d$ | $(2d+1)^d$ | $d^2$ | $< 1.5 \times d^2$ | $< 1.5$ |
| $2d$ | $(2d+1)^{dn}$, $n > 1$ | $Sd^2 + (S-1)d/2$ | $< S(d^2 + \frac{5}{4}d) + \frac{3}{4}d^2$ | $1 + \frac{3}{4d}$ |

## 9.2 Further Research

In this thesis we assume an optimal number of rounds, which constrains the number of vertices to a power of $\Delta + 1$. Alternatively we can choose the maximum number of vertices for a given diameter and concentrate on minimising the distance a message travels.

One of our aims in modifying the torus to form a circulant graph is to introduce algebraic techniques and develop general solutions for broadcasting and other problems such as gossiping. Recently Deserable[7] has shown a Cayley graph representation of a hexagonal grid. This has a recursive structure which is useful for information dissemination.

The matrix approach of Vince [17] may also yield general solutions.

# Appendix A

# Broadcasting on a Normal Torus

## A.1 Introduction

This appendix gives a variety of broadcast algorithms for a normal three-dimensional torus and discusses possible extensions to higher dimensions. The vertices are labelled in $d$ dimensions by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_d)$. As before we assume a square mesh in $d$ dimensions with wraparound connections to form a normal torus network with $N$ vertices. The broadcast will require $\log_{2d+1} N$ rounds and our objective is to find simple patterns leading to algorithms which minimise the total path length.

Recent work by Park and Choi [12] gives a general solution in $d$ dimensions based on spanning one dimension at a time, with no attempt to minimise path lengths.

## A.2 Guiding Principles

Only one of any pair of adjacent vertices can be informed before last round, otherwise there would be only $4d-2$ arcs out of the combined pair of adjacent vertices, instead of $4d$, and the number of rounds could not then be optimal. (Thus it seems sensible to send to all $2d$ adjacent vertices on the last broadcasting round. This corresponds to the tessellation of space by cross shapes.)

Informing adjacent vertices can be avoided in early rounds by sending to a restricted set of vertices until the last round. For example, vertices $\mathbf{x}$ defined by the sum

$$\sum w_i x_i = 0 \pmod{2d+1},$$

with $0 < w_i < d$. A vertex adjacent to $\mathbf{x}$ has a single $x_i$ changed by 1, so the above sum for the adjacent vertex will be $\pm w_i \neq 0$ which proves it is not in the restricted set.

The vertices which are not in the restricted set are distinct in the last round if and only if

$$w_i = \pm w_j \Rightarrow i = j,$$

in which case the $w_i$ must take all values from 1 to $d$. This is the condition given in Golomb and Welch [8] and they conjecture that it is the only solution for tiling $d$-space with a $d$-dimensional cross.

The diameter of a $d$-dimensional torus is $d^2$ and the maximum path length of a broadcast cannot be less than the diameter. A total path length equal to the diameter cannot be achieved for $d = 4r$ or $4r + 3$ on a d-dimensional $(2d+1) \times (2d+1) \times \ldots \times (2d+1)$ torus using $w_i = i$ (i.e. tessellation with cross shapes). The centres of some crosses are distance $d^2$ from the origin, and all centres become informed by the broadcast at the end of round $d - 1$. So the total path length excluding round $d$ is at least $d^2$. The last round has a path length of 1 to adjacent vertices, so the total path length must be at least $d^2 + 1$. For example, consider vertices

$$(d, -d, -d, d, d, -d, -d, d, \ldots) \text{ or } (-d, -d, d, d, -d, -d, d, d, -d, -d, d, \ldots).$$

These are at a distance $d^2$ from the origin and since $(d - 2d - 3d + 4d) \cdots = 0$ and $(-d - 2d + 3d) + \cdots = 0$ they satisfy

$$\sum i x_i = 0 \pmod{2d+1}.$$

Note that the optimum total path length *is* achieved in 4 dimensions on a 4-dimensional $3 \times 3 \times 3 \times 3$ torus using the construction in Golomb and Welch [8]. This seems to be a contradiction, as recursion will extend the construction to a $9 \times 9 \times 9 \times 9$ torus. The $3 \times 3 \times 3 \times 3$ torus has a diameter and a broadcast path length of $4 \times 1 = 4$. Scaling this up by a factor of 3 to broadcast to a $9 \times 9 \times 9 \times 9$ torus gives a total path length of $3 \times 4 + 4 = 16$ which is equal to the diameter of the torus. We must conclude that the uniqueness conjecture by Golomb and Welch [8] does not hold in 4 dimensions.

## A.3 Broadcasting Algorithms on the Torus

Here we describe two algorithms for the three dimensional torus mesh and mention several approaches which have been explored for extending the work to $d$ dimensions.

### A.3.1 Maximal Dispersion Broadcasting

In three dimensions a reasonable first round in broadcasting is sending to six vertices maximally dispersed through the torus. Each of the seven informed vertices could then send to six vertices maximally dispersed in a sphere-like region around it. This process would end with each vertex sending to its six immediate neighbours. Geometric intuition suggests this method would always work. Path conflicts would be avoided in general, and there seem no shortage of connections. The problem is to find a solution with a simple pattern which can be extended to higher dimensions.

#### The First Round of Maximum Dispersion

We will demonstrate a first round to inform $2d + 1$ symmetrically dispersed vertices when $2d + 1$ is prime. In three dimensions consider the vertices centred at

$$(1k, 2k, 3k) \bmod 7 \text{ for } k = 0, 1, \ldots 6.$$

These are valid centres for cross shaped tiles as

$$1 \times 1k + 2 \times 2k + 3 \times 3k = 14k = 0 \bmod 7.$$

We will prove that any two of these vertices are at a distance 6 apart. Regarding the coordinates as vectors, let the difference between 2 vertices $(1k, 2k, 3k)$ and $(1j, 2j, 3j)$ be $(a, b, c)$ , where $a = k - j; b = 2k - 2j; c = 3k - 3j$. Clearly, $a, b$ and $c$ must all be different and also non-zero modulo 7, otherwise $r(k - j) = 0 \pmod 7$, where $0 < r \leq 5$. Thus the absolute values of $a, b, c$ must be 1,2,3 in some order and will sum to 6. A similar argument in $d$ dimensions gives $2d + 1$ centres at a distance $d(d+1)/2$ apart.

Thus for a $7 \times 7 \times 7$ torus in three dimensions we will choose a first round which informs vertices centred at

$$(1k, 2k, 3k) \bmod 7 \text{ for } k = 1, \ldots 6.$$

The last round informs immediate neighbours, so we need only find the second round. There are many solutions for this, which can be found by a systematic search of all possibilities.

### Round 2 of Maximum Dispersion in 3 Dimensions

The final round of broadcasting is for each informed vertex to send to its six neighbours to form cross shapes. Using the algebraic formula from Golomb and Welch [8] the centres of the crosses are shown in the table below. For brevity, since all numbers are a single digit, we will write 104 to represent a cross centred at (1,0,4) etc.

| | | | | | | |
|---|---|---|---|---|---|---|
| 000 | 104 | 201 | 305 | 402 | 506 | 603 |
| 015 | 112 | 216 | 313 | 410 | 514 | 611 |
| 023 | 120 | 224 | 321 | 425 | 522 | 626 |
| 031 | 135 | 232 | 336 | 433 | 530 | 634 |
| 046 | 143 | 240 | 344 | 441 | 545 | 642 |
| 054 | 151 | 255 | 352 | 456 | 553 | 650 |
| 062 | 166 | 263 | 360 | 464 | 561 | 665 |

The second round of the broadcast must inform all these vertices using paths which are as short as possible. The seven maximally dispersed centres which are informed after the first round, {000, 135, 263, 321, 456, 514, 642 } are highlighted in the table below. They will each send to six vertices in the second round, preferably at distances of 3, but in some cases 4. To help select vertices close to (0, 0, 0) for round 2, the centres at a distances of 3 or 4 from (0, 0, 0) are shown subscripted in the table by 3 or 4 respectively.

| | | | | | | |
|---|---|---|---|---|---|---|
| **000** | $104_4$ | $201_3$ | 305 | 402 | $506_3$ | 603 |
| $015_3$ | $112_4$ | $216_4$ | 313 | $410_4$ | **514** | $611_3$ |
| 023 | $120_3$ | 224 | **321** | 425 | 522 | $626_4$ |
| $031_4$ | **135** | 232 | 336 | 433 | 530 | 634 |
| $046_4$ | 143 | 240 | 344 | 441 | 545 | **642** |
| 054 | $151_4$ | 255 | 352 | **456** | 553 | $650_3$ |
| $062_3$ | $166_3$ | **263** | $360_4$ | 464 | $561_4$ | $665_4$ |

From the subscripted vertices, we will choose six which vertex (0, 0, 0) will send to in the second round of the broadcast. For a broadcast which can be represented by a recursive tiling, each of the seven informed vertices sends to the same set of vertices relative to its own position. Therefore each vertex we choose for (0, 0, 0) implies the choice of a vertex for each of the other six informed vertices. This will progressively eliminate other possible targets and reduce the number of vertices remaining to choose from. Thus choosing vertex 112 eliminates vertices $112 + 456 = 561$, $112 + 514 = 626$ and the four others obtained by translating vertex 112 by each of the maximally dispersed vertices {000, 135, 263, 321, 456, 514, 642}

Selecting vertices in this way gives the set of vertices
{112, 120, 166, 665, 650, 611}
to be informed by (0, 0, 0) in round 2 of the broadcast. By tabulating this set translated modulo 7 by the other six informed vertices, it is possible to check that all valid centres for the crosses are accounted for. The following table has a column for each of the seven informed nodes with its list of target nodes below it.

| Maximally Dispersed Centres | | | | | | |
|---|---|---|---|---|---|---|
| 000 | 135 | 263 | 321 | 456 | 514 | 642 |
| 112 | 240 | 305 | 433 | 561 | 626 | 054 |
| 120 | 255 | 313 | 441 | 506 | 634 | 062 |
| 166 | 224 | 352 | 410 | 545 | 603 | 031 |
| 665 | 023 | 151 | 216 | 344 | 402 | 530 |
| 650 | 015 | 143 | 201 | 336 | 464 | 522 |
| 611 | 046 | 104 | 232 | 360 | 425 | 553 |

Each row in this table can be generated from any of its elements by adding that element to each of {000, 135, 263, 321, 456, 514, 642}. Because of this, in round 2 of the broadcast, 000 can send to any one element from each of the other 6 rows. This gives $7 \times 7 \times 7 \times 7 \times 7 \times 7$ possible ways to perform the second round of the broadcast, of which $3 \times 3 \times 4 \times 3 \times 3 \times 4$ have a maximum path length of 4.

The total path length for the broadcast is thus 6+4+1 = 11. The theoretical minimum is 9, giving a ratio of 1.22 between total path length and diameter.

### A.3.2 Knight's Move in Three Dimensions

This is the original unpublished solution by Dr. J. Peters as an extension of his work on visualising broadcasting as a tiling in 2 dimensions. In terms of the tiling on a $7 \times 7 \times 7$ torus

- Round 3 of the broadcast informs immediate neighbours, represented by cross-shaped tiles;
- Round 2 uses Knight's moves to form a sloping plane from seven of the crosses; and
- Round 1 stacks seven sloping planes to give a shape which can be wrapped into a 3-dimensional torus.

Round 2 consists of moves of two steps along one coordinate axis and one step along another axis. This corresponds with the knight's move in the game of chess. Expressing (0, 1, -2) as 015 modulo 7 in the same way as before, the vertices forming a sloping plane which are informed in round 2 by vertex (0, 0, 0) are
{ 015, 062, 120, 201, 506, 650 }. The first round stacks seven of these planes together, translating the planes by

$$\{(1,-1,-1),(2,-2,-2),(3,-3,-3),(-3,3,3),(-2,2,2),(-1,1,1)\}.$$

Expressing (1,-1,-1) in the form 166 modulo 7 as before, gives a more concise representation

{ 166, 255, 344, 433, 522, 611 },

and the following table lists all vertices informed after round 2. The first column of the table is a list of the vertices informed after round 1 which form the centres of the sloping planes. The first row shows the Knight's moves for round 2 which form the sloping plane with centre (0, 0, 0). Each row of the table corresponds to a different sloping plane.

| Knight's Move Centres | | | | | | |
|---|---|---|---|---|---|---|
| 000 | 166 | 255 | 344 | 433 | 522 | 611 |
| 015 | 104 | 263 | 352 | 441 | 530 | 626 |
| 062 | 151 | 240 | 336 | 425 | 514 | 603 |
| 120 | 216 | 305 | 464 | 553 | 642 | 031 |
| 201 | 360 | 456 | 545 | 634 | 023 | 112 |
| 506 | 665 | 054 | 143 | 232 | 321 | 410 |
| 650 | 046 | 135 | 224 | 313 | 402 | 561 |

Here round 1 of the broadcast has a path length of 3, 6 or 9 . Note that there is no edge contention between paths in round 2. The vertices on the six paths from the origin to the centres of the crosses forming a plane are

path 1: vertices 000 100 160 166 (note that 6 is equivalent to -1)
path 2: vertices 000 060 050 056 055 155 255
path 3: vertices 000 006 005 004 064 054 044 144 244 344
path 4: vertices 000 600 610 611
path 5: 000 010 020 021 022 622 522
path 6: 000 001 002 003 013 023 033 633 533 433
This spreads the message to the 7 sloping planes.

In round 2 each path can remain within a single plane. This is not strictly necessary, but makes it easy to see that there is no edge contention between different paths. To remain within a plane, each Knight's move in round 2 should start with two steps along one of the coordinate axes and then can take a single step to the target vertex. For example, paths from vertex 000 should be as shown by the rows of the table below.

| Path Number | Vertices on Path For Round 2 | | | |
|---|---|---|---|---|
| 1 | 000 | 100 | 200 | 201 |
| 2 | 000 | 010 | 020 | 120 |
| 3 | 000 | 001 | 002 | 062 |
| 4 | 000 | 600 | 500 | 506 |
| 5 | 000 | 060 | 050 | 650 |
| 6 | 000 | 006 | 005 | 015 |

Paths from other centres are the same, translated by the coordinates of the centre. The maximum path length for the knight's move broadcast is thus 9+3+1 = 13.

### The Knight's Move in $d$ Dimensions

Consider a generalised knight's move in $d$ dimensions which consists of $p$ steps along one coordinate axis and $q$ steps along another where $p, q$ and $2d+1$ are co-prime, $1 \leq p < q \leq d$, . (For the normal knight's move, $p = 1$ and $q = 2$ and these conditions can be satisfied in any number of dimensions.) Let the centres of the crosses be $a_1 a_2 \ldots a_d$, with $\sum i a_i = 0 \bmod 2d+1$ so the crosses will tile space.

For fixed $p, q$ there are exactly $2d$ centres of crosses of the form $00\ldots\pm p\ldots\pm q\ldots 00$ where all but 2 coordinates are zero. For $p = 2, q = 1$ they are

$$\begin{cases} a_i = 2, a_{2i} = -1 & \text{for } 2i \leq d \\ a_i = 2, a_{2d-2i+1} = 1 & \text{for } 2i > d \end{cases}$$

together with their mirror images in the origin. Thus the second to last step in the construction can always be of this form.

This gives a wide range of possible rounds by varying $p$ and $q$ and seems a promising approach to a general solution for $d$ dimensions.

# Appendix B

# Skewed Torus Representation

## B.1 Introduction

This appendix looks at the relationships among square, hexagonal and triangular tilings and how they can be wrapped into a torus. The close relationships among them is surprising. See Senechal [14] for a fuller treatment of this in 2 dimensions.

## B.2 The 2-Dimensional Square Mesh

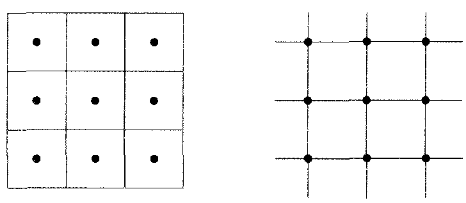

Figure B.1: Standard square mesh

This is the normal mesh, each vertex being referenced by its $x, y$ coordinates. We assume the number of rows and columns are the same, so as to minimise the diameter and to give symmetry between the $x$ and $y$ directions. The torus-connected (or wraparound) mesh is formed by connecting the top and bottom vertices and likewise connecting the side vertices.

## B.3 Maximum Vertices For a Given Diameter

We wish to select parameters to minimise the diameter of a graph with a fixed number of vertices, and this is equivalent to maximising the number of vertices for a fixed diameter. The vertices at the corners of a square mesh are twice as far from the centre as the vertices at the mid-points of the sides. This asymmetry can be eliminated by extending the mesh to include all vertices for which the distance $|x| + |y| < D$ for some constant $D$ (Figure B.2).

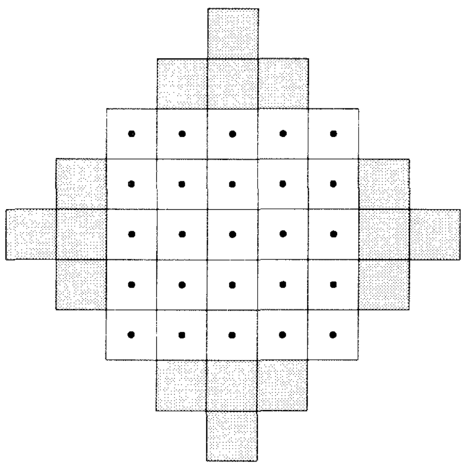

Figure B.2: Tiles for which $|x| + |y| \leq 4$

This almost doubles the number of vertices (to $n^2+(n-1)^2$ ) without increasing the diameter. The resulting shape is (roughly) a diamond, or a square rotated through 45°. Alternatively we can rotate the shape back to the original orientation and consider the tiles to be rotated through 45° as in Fig B.3. This leads to a checkerboard lattice, which can also be considered as hexagonal by joining vertices horizontally or vertically. (Figure B.4). Note that the hexagon-connected network is triangulated. The edges of the corresponding tiling form the dual graph, which is hexagonal. This distinction was not necessary for the square mesh, where the graph is self-dual. In the same way, a triangular tiling corresponds to a hexagonal network graph of degree 3. Degree 3 networks are discussed in Chapter 5.

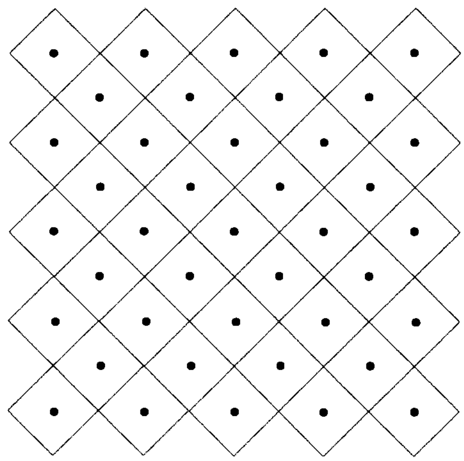

Figure B.3: Diamond rotated back to a square

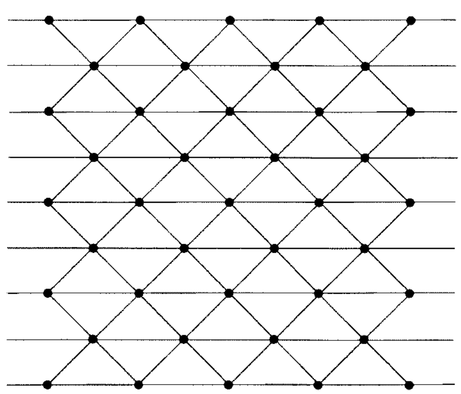

Figure B.4: Hexagonal connection derived from the diamond

# Appendix C

# Checking if a Shape can Form a Torus

## C.1 Wrapping an Irregular Shape Into a Torus

Can an irregular shape such as Fig C.1 wrap into a torus? To help visualise how the

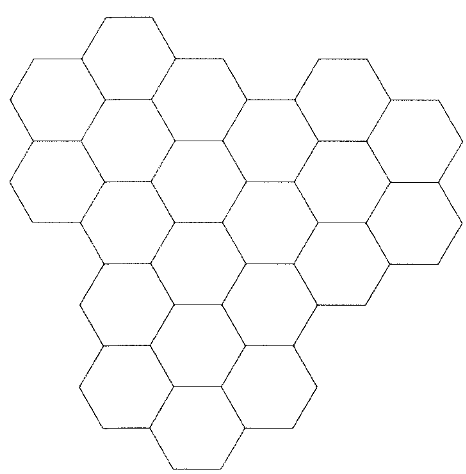

Figure C.1: Can this shape wrap into a torus?

edges of the shape fit together when it is wrapped as a torus, we will draw the shape surrounded by copies of itself. Any shape which tiles the plane by translation defines contact points which give the connections required to form a torus (see Senechal [14]). First consider a simple $2 \times 3$ rectangle. The original is shown shaded: Only the shaded

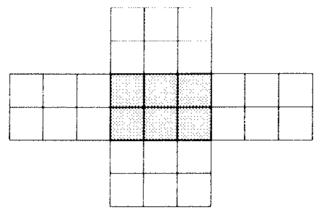

Figure C.2: Wrapping a rectangle on a torus

rectangle exists, the other rectangles are all identified with it. Thus the top of the shaded rectangle is connected to the bottom of a copy, the right is connected to the left of a copy and so on, in such a way as to represent the torus connections.

This method of representing torus connections is more useful for harder shapes such as the cross shown in Fig C.3. From this diagram it is easy to see that

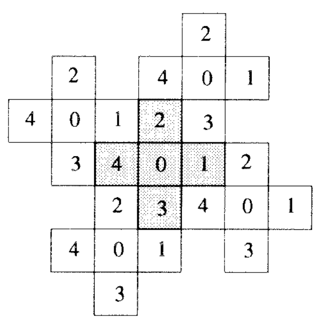


Figure C.3: Wrapping a cross on a torus

- the bottom of tile 4 leads to the top of 2;
- the left of 4 leads to the right of 3; and
- the top of tile 4 leads to the bottom of 1.

For the diamond it is possible to see there is only one torus connection, shown in Fig C.4a. Notice that any row (or column or diagonal) of the tiling corresponds to

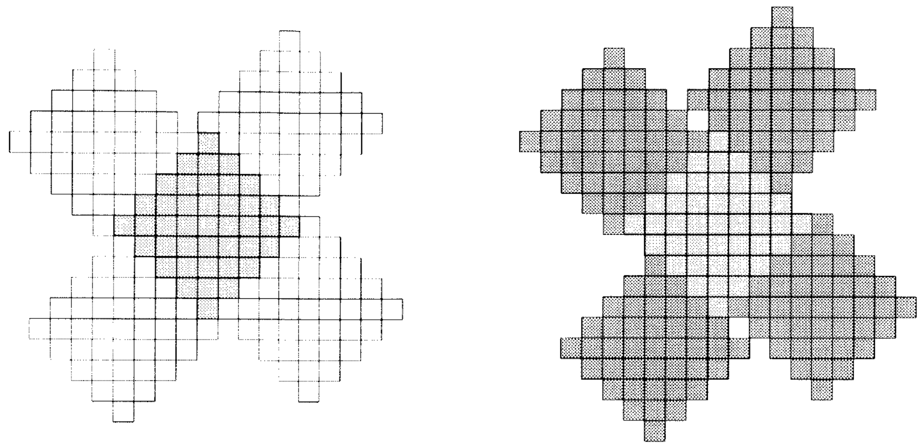

Figure C.4: a) Diamond tiling b) An impossible tiling

a Hamiltonian path through the diamond. Attempting to connect the edges in a more skewed fashion leads to Fig C.4b where it is impossible to fill the holes in the tiling, and therefore impossible to form a torus. Alternatively, to achieve a specified connection pattern, the holes in the tiling show where to add vertices to the diamond to complete the tiling.

## C.2 An Answer to the Initial Question

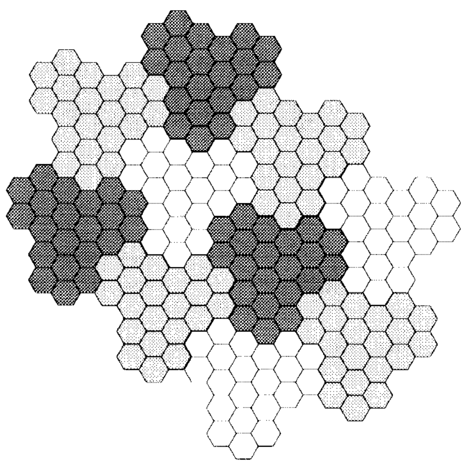

Figure C.5: Yes - this shape can wrap into a torus.

# Appendix D

# Practical Torus Layouts

Although this thesis is not concerned with fabrication details, note that there is a standard arrangement to avoid long connections by spacing the vertices evenly around a flattened loop (Figure D.1). Figure D.2 shows how this is applied in 2 dimensions.

Figure D.1: Spacing vertices evenly around a flattened loop

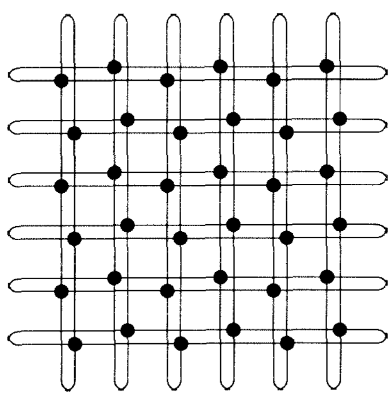

Figure D.2: Spacing vertices evenly on a 2-d torus

The same method can be extended to higher dimensions. It also extends to all patterns of mesh by flattening the torus as shown in Figure D.3. Thus a cycle forming

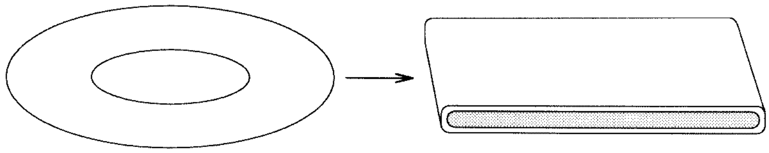

Figure D.3: Flattening a torus to give 2-d cycles

a spiral around the torus would be projected as shown in Figure D.4.

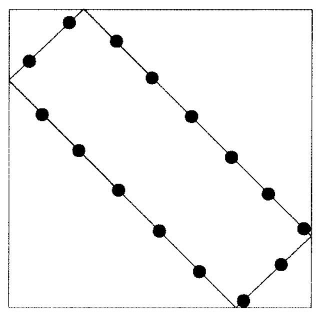

Figure D.4: A spiral path on a flattened torus

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