COMPARING PRICE MOVEMENTS OF OPTIONS AND THE UNDERLYING INDEX

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ABSTRACT

In theory, a call option and its underlying index should move in the same direction, while a put option and its underlying index should move in opposite directions. This property is referred to as the Empirical Monotonicity Property (EMP) when applied to time series of prices.

In this paper, we use daily call and put options' data to conduct empirical tests of the EMP, including three violation types. Further, we investigate the effect of grouping the option prices by their Black-Scholes implied volatility and by moneyness, and also the effect of using different quotes (bid, offer, and bid-offer midpoint). In addition to EMP, which depends on the signs of the price changes, we also test another theoretical constraint concerning the magnitude of these changes.

This is followed by a discussion of the possible causes for violations of the EMP. We use regression analysis to test whether volatility changes may be one of these causes. Lastly, we summarize the implications of our study to hedging strategies.

Keywords: Option pricing, Hedging, Monotonicity Property, Volatility

Subject Terms: Price Movements of Options and the Underlying Index

DEDICATION

此文献给我最亲爱的爸爸妈妈。感谢我最亲爱的爸爸妈妈。感谢你们给我提供到加拿大念书这个证明与提高自我的机会。你们的爱，智慧，理解和信任，一直以来支持着我完成学业，并指引我迎接新的人生挑战。我为有你们这样的父母而自豪。我爱你们。

This paper is dedicated to my parents. Thank you for providing me the opportunity to study in Canada. Thank you for your unconditional love, care, encouragement and support with my 4-year undergraduate study and 1-year graduate study. I love you.

I also would like to dedicate this paper to my boyfriend. Your love, support, encouragement, and understanding empowered me to go through this year.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

GLOSSARY

Monotonicity property (in the comparative-statics sense)	The **monotonicity property** is satisfied in the **comparative-statics sense** if c (t, S,…) is an increasing function of S. That is, increasing S, keeping time and all other variables constant, increases the call price c.
Monotonicity property (in a dynamic sense)	If the monotonicity property is satisfied in a **dynamic sense**, this has the following meaning: Suppose we compare times t and t+dt, and suppose the underlying asset price and the call price change to S+dS and c+dc, respectively. Then dS and dc have the same sign.
The empirical monotonicity property	The empirical monotonicity property (EMP) states that for a given time interval $[t, t + \Delta t]$ and for a given call (put) option and its underlying asset, Δc and ΔS have the same sign for the call option, and have opposite signs for the put option, over that time interval.
Call option's payoff at expiration (for a European call)	c = max (S – K, 0), where K is the strike price specified in the option.
Put option's payoff at expiration (for a European put)	p = max (K – S, 0), where K is the strike price specified in the option.

1. INTRODUCTION

The owner of a European call option has the right, but not the obligation, to buy an "underlying" asset at a predetermined price ("strike price") at a given time (option's expiration). Mathematically, the call's payoff at expiration is

$$c_T = max\left(S_T - K, 0\right), \tag{1}$$

where T is the expiration time, S_T is the underlying asset's price at time T, and K is the strike price specified in the contract. It is clear that as S_T increases, the value of the call option will increase as well.

The owner of a European put option has the right, but not the obligation, to sell an "underlying" asset at a predetermined price ("strike price") at a given time (option's expiration). Mathematically, the put's payoff at expiration is (with notation as above)

$$p_T = max\left(K - S_T, 0\right). \tag{2}$$

It is clear that as S_T increases, the value of the put option will decrease.

This is also true for the exercise value of American-style options, where exercise is allowed before the option's expiration. However, in this paper we limit ourselves to European options.

The question arises: Is it still true to say that, before expiration, increasing the underlying asset's price will increase the call price and will decrease the put price? (This question will be made more exact in the next section.) This property, if it is satisfied, is called the monotonicity property. This is the subject of this paper, where this property is tested empirically.

The monotonicity property is very important in the context of hedging. Consider, for example, a hedged position with a stock and a call. Different models (or the same model with different parameters) may specify different hedge ratios, but in all of them the hedge ratio (the number of shares per one written call) will be positive. Regardless of the model, such a hedge will definitely not work if the stock and the call move in opposite directions. As we shall see later, it is an empirical fact that such "violations" occur rather frequently.

The empirical monotonicity property (EMP) was tested on option prices from the Chicago Mercantile Exchange in Bakshi, Cao and Chen (2000), and on an International dataset in Pérignon (2006). ("Empirical" means investigating the monotonicity property of a time series of prices. This will be defined more exactly in the next section.) We conduct our test on option prices on the S&P 500 index from US exchanges. More specifically, we conduct our empirical analysis by using three different option prices: bid, offer and bid-offer midpoint prices, based on daily (end of the day) observations. We further categorize our data by different criteria, such as BS implied volatility and moneyness. Further details will be provided in subsequent sections. In addition, we use close prices for the corresponding underlying asset's prices. The daily observations that we use are admittedly less informative compared to tick-by-tick intra-day price changes, as in Bakshi, Cao and Chen (2000) and Pérignon (2006). At the same time, daily observations are less demanding to handle, while still being sufficient to provide insightful findings with acceptable accuracy.

Based on the selected data, call options' prices (more specifically, bid-offer midpoint prices) move in the opposite direction compared to the index (the underlying asset price) 15% of the time. In addition, the percentage of violations is the lowest when using bid-offer midpoint prices, and highest when using bid quotes (16.7463%). Similar results are found for put options, where theory suggests that the co-movement should be in opposite directions. With bid-offer midpoint prices, put options' prices and the underlying index move in the same direction 21% of the time. Again, the rate of violations is lowest for bid-offer midpoint prices and highest for the bid quotes (25.6742%). In addition, the violation rates from puts, ranging from 20.9702% to 25.6742%, are much higher than the violation rates from calls, ranging from 14.8903% to 16.7463%, in all three categories. We also perform tests by grouping data by BS implied volatility and moneyness.

After that we examine the causes of the violations of the EMP. Our findings suggest that BS implied volatility does not cause higher violation. In contrast, it looks like that BS implied volatility is negatively related to violation rates, i.e., options with higher BS implied volatilities actually have lower violation rates. In addition, bid/offer quotes and moneyness definitely affect violation rates. Furthermore, other underlying variables in BS model and some qualitative factors (i.e. market markers' activities) also attribute to violations.

The remainder of the paper is organized as follows. Section 2 elaborates on the meaning (actually meanings) of the monotonicity property. Section 3 is a literature review, relying on three main relevant articles. One article is theoretical, and among other things it proves the monotonicity property. The other two focus on testing the EMP. In section 4, we summarize the theory behind the monotonicity property. In section 5, we perform our own empirical tests of the EMP on both call and put options written on S&P500 index. An analysis of our results, with comparison to previous works, is provided in Section 6. Section 7 offers a summary of this paper and some implications for hedging activities.

2. MEANING OF THE MONOTONICITY PROPERTY

The monotonicity property can be defined in a theoretical sense and in an empirical sense. Let us start with the theoretical sense, applied to a call. Suppose we have a model which specifies the call price as a function c (t, S,…) of time (t) and the underlying asset's price (S), and possibly of other variables. There are actually two ways to define monotonicity.

- We say that the **monotonicity property** is satisfied in the **comparative-statics sense** if c (t, S,…) is an increasing function of S. That is, increasing S, keeping time and all other variables constant, increases the call price c. In other words, here the monotonicity property is defined via the partial derivative with respect to S.
- If the monotonicity property is satisfied in a **dynamic sense**, this has the following meaning: Suppose we compare times t and t+dt, and suppose the underlying asset price and the call price change to S+dS and c+dc, respectively. Then dS and dc have the same sign.

In textbooks, in the context of the Black-Scholes model, typically the comparative-statics interpretation is discussed. It is rather intuitive that it should also be satisfied in general, not only in the Black-Scholes model. To see this, we take time $t < T$ and two possible values of S_t, such as S''>S' (regardless whether they are greater than K or not). From the perspective of time t, as time moves forward to the expiration time, the distribution of the terminal price S_T depends on the starting point at time t. If we start from S", there is higher probability that $S_T > K$, and the distribution of S_T will be shifted to the right (compared to starting the price process from S'). Hence c (t, S_t) should be higher if S_t = S". It is also clear that a put will then be less valuable.

This logic depends on the assumption that the underlying asset's price process is suitably well behaved. However, in principle, this is not always the case. One can construct an example where starting the price process from S” (in the above example) will increase the likelihood of the stock going *down* and therefore the call price in this case will be *lower*.

Thus the monotonicity property (in the comparative-statics sense) is model-dependent. It is satisfied in the classical Black-Scholes model in Black and Scholes (1973), Merton (1973), as well as in most other option pricing models, for example, Cox and Ross (1976), Derman and Kani (1994) and Rubinstein (1994). All these models are based on the assumption that the underlying asset price is the single state variable which is the sole source of uncertainty. It follows that the option can be dynamically replicated by the underlying asset and the riskless rate, and thus its price must be equal to the value of the replicating portfolio, which is a function of t and S (in the above notation). It also follows that option prices must be perfectly correlated with each other and with the underlying asset, and that the monotonicity property is satisfied. We will elaborate on the theory in Section 4.

Next, based on Pérignon (2006), let us discuss the dynamic version of the property, in the context of a theoretical model. Mathematically, assuming that the price of the option is a function of the underlying asset price and time, Ito’s lemma gives

$$dc = c_t dt \ + \ c_s dS \ + \ \frac{1}{2} c_{ss} (dS)^2 \tag{3}$$

$$dp = p_t dt \ + \ p_s dS \ + \ \frac{1}{2} p_{ss} (dS)^2 \tag{4}$$

(Subscripts denote partial derivatives.) One can further assume that S is dependent on a single standard Brownian motion. That is,

$$dS(t)/S(t) \;=\; \mu(S(t),t)\;dt \;+\; \sigma(S(t),t)\;dZ(t) \tag{5}$$

where μ and σ are functions of $S(t)$ and t and where Z is a standard Brownian motion. Then $(dS)^2 = \sigma^2 S^2 dt$ in equations (3) and (4), which will become:

$$dc = (c_t + \tfrac{1}{2}\, c_{ss}\, \sigma^2 S^2)dt + c_s dS \tag{6}$$

$$dp = (p_t + \tfrac{1}{2}\, p_{ss}\, \sigma^2 S^2)dt + p_s dS \tag{7}$$

As a result, *assuming* that $c_S>0$ (that is, monotonicity in the comparative-statics sense is satisfied) and *assuming* that $(c_t + \frac{1}{2}\, c_{ss}\, \sigma^2 S^2)dt$ is negligible, it follows that dc and dS have the same sign. Similarly, dp and dS have opposite signs as long as $pS<0$ and $(p_t + \frac{1}{2}\, p_{ss}\, \sigma^2 S^2)dt$ is negligible.

Finally, following Pérignon (2006), we can discuss the empirical version of the property. For a given time interval $[t, t + \Delta t]$ and for a given call option and its underlying asset, denote,

$$\Delta S = S(t + \Delta t) - S(t) \tag{8}$$

$$\Delta c = c(t + \Delta t) - c(t) \tag{9}$$

We say that the call satisfies the empirical monotonicity property (EMP) over that time interval if Δc and ΔS have the same sign.

The above theoretical discussion implies that the EMP is predicted to be satisfied over sufficiently small time intervals.

3. LITERATURE REVIEW

The three main papers on which this work is based are Bergman, Grundy, and Wiener (1996), for the theory, and Bakshi, Cao and Chen (2000) and Pérignon (2006) for the empirical work.

Every textbook on options includes a chapter on no-arbitrage relationships ("restrictions") of option prices, for example upper and lower bounds. This goes back to Merton (1973). These are general properties in the sense that they do not depend on any assumptions on the volatility of the underlying asset. Bergman, Grundy, and Wiener (1996) is a theoretical paper which adds more properties to the list. They also do not make any assumptions on the volatility, but they need assumptions on the general form of the price dynamics. For example, a single-state diffusion process is needed for some of the results.

BGW are interested in the partial derivatives (comparative statics) relative to all state variables. They are also interested in a second derivative (convexity) relative to the underlying price (S). For the purpose of our work, we are interested only in the part on monotonicity, where BGW investigate the first partial derivative relative to S. They actually have two proofs under two different sets of assumptions. In the next section, we will summarize one of them. It should be mentioned that BGW's analysis applies for a general derivative security whose payoff is a monotonic function of S, not only calls or puts. (See examples in the following section)

BGW also demonstrate that there exist theoretical counterexamples where the monotonicity property is not satisfied. This may happen if the volatility is stochastic or if the underlying process is not a diffusion (i.e. either discontinuous or non-Markovian). The intuition behind this was explained in the previous section.

In Bakshi, Cao and Chen (2000), they focused on testing the monotonicity property on both call and put options. They started with introducing the monotonicity property and then listed some testable predictions for both call and put options. Their analyses were based on intraday observations: (1) the S&P500 spot index; (2) lead-month S&P500 futures prices; and (3) bid-ask midpoint prices for S&P500 index options. The advantages of using intraday data include improved accuracy, and the ability to determine the optimal rebalancing frequency of hedging strategies in option trading. The use of bid-ask midpoint prices may help eliminate the impact of bid-ask bounces. The cash index data and the futures data were obtained from the Chicago Mercantile Exchange. The source of the option data was the Berkeley Option Database. The period under study was from March 1, 1994 to August 31, 1994, totalling 3.8 million observations on index calls and puts. They grouped their data by moneyness, as follow.

Table I – Grouping Criteria – Moneyness in Bakshi, Cao and Chen (2000)

	Criteria	
	Call Options	Put Options
ITM	$S/K \geq 1.03$	$K/S \geq 1.03$
ATM	$S/K \in (0.97, 1.03)$	$K/S \in (0.97, 1.03)$
OTM	$S/K \leq 0.97$	$K/S \leq 0.97$

BCC did not explain how the figures 0.97 and 1.03 were chosen. (See further discussion in Section 5.) In addition to moneyness, they also grouped the data by time to maturity. They categorized options with less than 60 days to maturity as short-term, with 60-180 days to maturities as medium-term, and with more than 180 days to maturity as long-term. Short- and medium-term constitute 80% (83%) of the entire hourly call (put) sample.

After explaining data collection and their methodology, they listed four types of violations for which the conducted test.

Table II – Violation types in Bakshi, Cao and Chen (2000)

	Call Options	**Put Options**
Type I violation	$\Delta S \times \Delta c < 0$	$\Delta S \times \Delta p > 0$
Type II violation	$\Delta S \neq 0$ but $\Delta c = 0$	$\Delta S \neq 0$ but $\Delta p = 0$
Type III violation	$\Delta S=0$ but $\Delta c \neq 0$	$\Delta S = 0$ but $\Delta p \neq 0$
Type IV violation	$\Delta c/\Delta S > 1$, $\Delta S \neq 0$	$\Delta p/\Delta S < -1$, $\Delta S \neq 0$

For comparison purposes, since we use only daily observations in our paper, we summarize their results for "1-day sampling interval" only. (Other sampling intervals include 30 minutes, 1 hour, 2 hours and 3 hours.)

BCC use two methods to represent the underlying index: The straightforward method is to use the index itself. An alternative method is to use the lead-month index futures. The results are summarized in the following table.

Table III – Violation Results in Bakshi, Cao and Chen (2000)

			Violations by calls		
	Type I (%)	Type II (%)	Type III (%)	Type IV (%)	Total (%)
Cash Index	9.1	3.6	0.00	11.5	24.2
Index Futures	7.2	3.5	0.00	7.7	18.4
			Violations by puts		
	Type I (%)	Type II (%)	Type III (%)	Type IV (%)	Total (%)
Cash Index	5.4	2.8	0.00	13.2	21.4
Index Futures	6.5	2.7	0.00	9.6	18.8

The above two tables show that, generally speaking, violation rates relative to the index futures price are fewer than from those relative to the cash index. In addition, Type IV violation occurred most frequently among the four types, while the second most frequent occurrence was Type I violation. Type III violation was rare for both call and put options since the S&P500 and its futures prices rarely stayed unchanged for one-day period. In this section, they also tested whether violations of the monotonicity property are related to violations of the put-call parity. The results showed that eliminating put-call parity violations would not overcome their empirical findings.

BCC further refined their analysis by investigating some possible causes for violations. Firstly they tested whether the occurrences of violations differed across moneyness and time to maturity. For ITM calls, the relationship to time-to-maturity was U-shaped, with the medium-term options showing least type I violations. In contrast, for ATM and OTM calls, the relationship was hump-shaped, with the medium-term options showing the highest type I violations. Why did medium-term ATM and OTM call prices move more frequently in opposite directions with the underlying asset compared to short-term and long-term options? BCC could not explain why the results from medium-term ITM calls were opposite to the results from medium-term ATM and OTM calls.

BCC also tested whether violations are related to market microstructure factors: (i) time of day; (ii) dollar bid-ask spread; (iii) number of quote revisions, and (iv) daily trading volume. They performed their analysis on hourly call-option prices. Results are summarized in the following table.

Table IV – Violation Results Grouped by Four Market Microstructure Factors in Bakshi, Cao and Chen (2000)

	Violation Rates	Time of Day	Dollar Bid-Ask Spread	No. of Quote Revisions	Trading Volume
Type I	Highest	11AM – 12 PM	3/16 – 1/4	310 – 755	0 – 14
	Lowest	10AM – 11 AM	≥3/4	<16	0
Type II	Highest	11AM – 12 PM	< 3/16	< 16	14 – 115
	Lowest	9AM – 10 AM	1/2 – 3/4	≥ 755	0
Type IV	Highest	1PM – 2 PM	≥3/4	≥755	0 – 14

	Lowest	10AM – 11 AM	< 3/16	67 – 310	≥ 935

Then lastly, in the same section, they investigated the magnitude of the violations. They grouped the data by moneyness and time to maturity, and tested whether the magnitude for each violation type satisfies an upper-bound constraint.

BCC's work was continued by Pérignon (2006). He investigated the monotonicity property but not the upper bound for the magnitude of the price change. The distinction that we made in Section 2 between static and dynamic monotonicity is based mainly on his work. Pérignon set four testing criteria for violations.

Table V – Violation Testing Criteria in Pérignon (2006)

	Violations for Call Options	Violations for Put Options
Type I	$\Delta S < 0, \Delta c > 0$	$\Delta S > 0, \Delta p > 0$
Type II	$\Delta S > 0, \Delta c < 0$	$\Delta S < 0, \Delta p < 0$

Pérignon performed empirical analysis on a dataset consisting of prices of five index options, written on the European (DJ EURO STOXX-50), French (CAC 40), German (DAX), Swiss (SMI) and British (FTSE) stock indices. For each of the five contracts, the data included all transaction prices in 2002, totaling 1.4 million observatioins of call and put option prices and more than 173 million traded contracts. The dataset also included the intra-day value of the underlying stock indices observed every 15 seconds for the DAX and every 60 seconds for other indices (Pérignon, 2006). A notable feature of the data was the use of transacton prices instead of using bid-ask midpoint prices, which reduces the sensitivity to bid-offer spread manipulation.

Pérignon presented resutls for different sampling interval: tick-by-tick, 30 minutes, 1 hour, 2 hours, 3 hours and 1 day. In addition, he greoups the data by moneyness and time to maturity.

Again, for comparison purposes, we summarize violation rates only by "1 day" and moneyness in the following table.

Table VI – Call Options Violation Testing Results in Pérignon (2006)

CALL	European		France		Germany		Switzerland		UK	
	I	II	I	II	I	II	I	II	I	II
1 day	3.0	10.1	1.8	9.8	0.6	6.6	2.0	9.5	1.0	7.9
OTM	10.1	10.5	14.1	14.0	6.2	6.5	8.7	9.1	6.0	6.3
ATM	9.5	10.1	13.4	14.7	5.9	6.4	9.9	10.6	6.3	6.9
ITM	8.1	7.9	17.6	15.7	4.3	4.3	12.2	9.2	5.8	3.8

For 1 day sampling interval, Type I violation occurs more frequently than Type II violation. Violation rates grouped by moneyness are summarized in the following table.

Table VII – Call Options Violation Testing Results by Type in Pérignon (2006)

	Type I	Type II
OTM	45.1	46.4
ATM	45	48.7
ITM	48	40.9

A similar summary can be made for put options:

Table VIII – Put Options Violation Testing Results in Pérignon (2006)

PUT	European		France		Germany		Switzerland		UK	
	I	II	I	II	I	II	I	II	I	II
1 day	3.3	8.0	2.9	7.9	0.6	5.7	4.1	7.6	0.9	8.6
OTM	9.1	8.9	14.2	14.2	6.4	6.4	9.3	8.7	5.0	5.0
ATM	10.0	9.9	14.7	15.1	6.3	6.1	11.5	10.6	5.2	6.8
ITM	10.4	10.3	16.0	17.0	4.6	4.7	9.8	9.1	5.4	4.5

Table IX – Put Options Violation Testing Results by Type in Pérignon (2006)

	Type I	Type II
OTM	44	43.2
ATM	47.7	48.5
ITM	46.2	45.6

The results in the above tables show that for OTM options, puts had lower violation rates for both Types; for ATM options, calls had lower violation rates for Type I, and similar violation rates for Type II; for ITM options, puts had lower violation rates for Type I and higher violation rates for Type II.

After that, Pérignon discussed three main causes for the violaitons. The first one was other underlying variables in the option pricing model. He concluded that violations of the EMP were mainly due to volatility shocks. The second cause was the bid-ask bounce. He concluded that Type I violation occurred more frequently when changes in option prices were computed between a bid price and an ask price. Type II violation occurred more frequently when changes in option prices were computed between an ask price and a bid price. The third cause was rational trading tactics, such as price/time priority and liquidity. He concluded that violations of the EMP were negatively related to (1) the relative changes in the underlying assets; (2) the relative changes in the underlying asset, and (3) the level of activity of the option contract. In addition, violations of the EMP were likely to happen right before the market closed and on Fridays.

4. THE THEORY BEHIND THE MONOTONICITY PROPERTY

Recall that the property that the price of a call option is a monotonically increasing function of the value of its underlying asset is called the monotonicity property (in the comparative static sense). Likewise, for a put, it means that the price of a put option is a monotonically decreasing function of the value of its underlying asset. In what follows, c and p denote prices of European call and put, respectively. The same property can be defined for American options.

The monotonicity property depends on the assumption that the option price can be written as a function of time and underlying asset's price. Then monotonicity means that delta of the option, which is the partial derivative to the underlying asset's price (holding the time and other variables fixed), is positive for a call and negative for a put.

The monotonicity property holds not only for a call or a put option. Under certain condition, if the terminal payoff is an increasing function of the underlying asset's price, so is the value function before expiration. Likewise, if the terminal payoff is a decreasing function of the underlying asset's price, so is the value function before expiration. In other words, the monotonicity property is "inherited" from the payoff function to the value function. A call option or a put option is a special case.

In what follows, we will give a few examples with a general boundary condition. We will also provide proofs of the monotonicity property under certain conditions. This is based on Bergman, Grundy and Wiener (1996) and Bick (2008).

Suppose there is a derivative security which pays $q[S(T)]$ at time T, where $\{S(t);\ t \geq 0\}$ represents the price process of the underlying stock. Suppose the time- t value, for $t \leq T$, is $V(t, S(t))$, where $V(t, S)$ is a Calculus function of two variables.

Then the paper by Bergman, Grundy and Wiener (1996) states if q is an increasing function of S, then so is $V(t, S)$ for any t. That is, the partial derivative $V_s(t, S)$ is positive. If, instead, q is a decreasing function of S, then so is $V(t, S)$ for any t, so that the partial derivative $V_s(t, S)$ is negative.

4.1 Examples of the Monotonicity Property

Let us check a few examples in the classical BS model in Black and Scholes (1973), where the interest rate is a constant r and the stock price process is a geometric Brownian motion

$$dS(t)/S(t) = \mu\, dt + \sigma\, dZ(t), \tag{10}$$

where μ and σ are constants and where Z is a standard Brownian motion. Suppose the stock does not pay dividends.

- For a K-strike call, the payoff is $q(S) = max\,(S - K, 0)$, which is an increasing function of S. Here we have the well-known formula for delta

  $$c_S = N(d_1), \tag{11}$$

  where N is the standard normal cumulative distribution function and d_1 is a certain well-known expression. This is positive, thus c is an increasing function of S.

- Consider the payoff function $\mathrm{q(S) = S^a}$, where a is a positive constant. This is clearly an increasing function of S. Here we have for the time-t value:

  $$V(t, S) = S^a \times e^{\left(r+\frac{1}{2}\sigma^2 a\right)(a-1)(T-t)}. \tag{12}$$

(See McDonald (2006).) This is also an increasing function of S.

- Consider the payoff function $q(S) = ln(S)$, which is an increasing function of S. Here we have the time-t value:

$$V(t, S) = e^{-r(T-t)}[\, lnS + \left(r - \frac{1}{2}\sigma^2\right)(T-t)]. \tag{13}$$

(See Neuberger (1994).) This is also an increasing function of S.

4.2 Proof in the Case where σ Is Constant (i.e. Standard BS Model)

Proposition: In the Black-Scholes model, with constant volatility σ, consider a derivative security which pays $q[S(T)]$ at time T. Suppose $q(S)$ is an increasing (or a decreasing) function of S. Suppose the time-t value, for $t \leq T$, is of form $V(t, S(t))$. Then, for any t, V is an increasing (or decreasing, respectively) function of S.

Proof: As we know from Feynman-Kac Theorem

$$V = e^{-r(T-t)} E^*[q(S(T)|S(t) = S], \tag{14}$$

where E^* denotes expectation relative to the process

$$dS(t)/S(t) = rdt + \sigma\, dZ^*(t) \tag{15}$$

where Z^* is also a standard Brownian motion. (This is a new process for stock price S, although we use the same notation.). As we know, we can write $S(t)$ from (15) as

$$S(t) = e^{k+mt+\sigma Z^*(t)}, \tag{16}$$

where k is a constant (such that $S(0) = e^k$), and $m = r - \frac{1}{2}\sigma^2$. Clearly, (16) implies that

$$S(T) = S(t)\, e^{m(T-t)+\sigma[Z^*(T)-Z^*(t)]}. \tag{17}$$

We also know that, from time-t perspective, $Z^*(T) - Z^*(t) \sim N\,(0, \sigma^2\tau\,)$, where $\tau = T - t$.

Fix a time $t \leq T$. Also fix $Z^*(t) = z'$ and $S(t) = S'$ such that $S' = \exp(k + mt + \sigma z')$

Then

$$V(t, S') = e^{-r(T-t)} E^*[q(S(T)|S(t) = S']$$

$$= e^{-r(T-t)} E^*[q(S' e^{m\tau + \sigma[Z^*(T) - Z^*(t)]})|Z^*(t) = z']$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} q(S' e^{m\tau + \sigma x}) \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-x^2/2\sigma^2\tau}\, dx \tag{18}$$

Now suppose, instead of S', we take S'' such that $S'' > S'$. Then

$$S'' e^{m\tau + \sigma x} > S' e^{m\tau + \sigma x} \tag{19}$$

$$\Rightarrow q(S'' e^{m\tau + \sigma x}) > q(S' e^{m\tau + \sigma x}) \tag{20}$$

$$\Rightarrow V(t, S'') > V(t, S') \tag{21}$$

This is the desired result. The case where q is decreasing is similar. QED

4.3 Proof in the Case where σ Is Not Necessarily a Constant

Proposition: Suppose the risk-neutralized process of S from the Feynman-Kac theorem is of the form $f(t, Z^*(t))$ where Z^* is a standard Brownian motion and $f(t, z)$ is a monotonic function of the second variable. (Thus Eq. (16) is a special case.) Consider a derivative security which pays $q(S(T))$ at time T, and suppose $q(S)$ is a monotonic function of S. Suppose the time-t value, for $t \leq T$, is of the form $V(t, S(t))$. Then, for any t, V is a monotonic function of S in the same direction (increasing or decreasing) as q.

Proof: As we know from the famous Feynman-Kac theorem,

$$V = e^{-r(T-t)} E^*[q(S(T)|S(t) = S], \tag{22}$$

Where E^* denotes expectation relative to the process.

$$dS(t) = rS(t)\, dt + f_{Z^*}(t, Z^*(t))\, dZ^*(t), \tag{23}$$

And where Z^* is a standard Brownian motion. (The coefficient of $dZ^*$is a result of Ito's lemma)

We also know that, from time-t perspective, $Z^*(T) \sim N(Z^*(t), \sigma^2\tau)$, where $\tau = T - t$.

Fix a time $t \leq T$. Also, fix $Z^*(t) = z'$ and let $S(t) = S' = f(t, z')$. Then

$$V(t, S') = e^{-r(T-t)}E^*[q(S(T)|S(t) = S']$$

$$= e^{-r(T-t)}E^*[q(f(T, Z^*(T))|Z^*(t) = z']$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} q(f(T, y)) \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-(y-z')^2/2\sigma^2\tau} \, dy$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} q(f(T, x + z')) \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-x^2/2\sigma^2\tau} \, dx \qquad (24)$$

Where in the last equality we used a change of variable $x = y - z'$.

Now suppose we take $S'' > S'$, this means

$$S'' = f(t, z'') > f(t, z') = S' \qquad (25)$$

There are two "combinations" of changes of q and f:

<u>Case 1</u>: q and f are both increasing in S. Then equation (25) entails:

$$z'' > z' \qquad (26)$$

$$\Rightarrow f(T, x + z'') > f(T, x + z') \qquad (27)$$

$$\Rightarrow q(f(T, x + z'')) > q(f(T, x + z')) \qquad (28)$$

$$\Rightarrow V(t, S'') > V(t, S') \qquad (29)$$

<u>Case 2</u>: q is increasing in S and f is decreasing in S. Then equation (25) entails:

$$z'' < z' \qquad (30)$$

$$\Rightarrow f(T, x + z'') > f(T, x + z') \qquad (31)$$

$$\Rightarrow q(f(T, x + z'')) > q(f(T, x + z')) \qquad (32)$$

$$\Rightarrow V(t, S'') > V(t, S') \tag{33}$$

The other two cases are similar. QED

4.4 BGW Version of the Property (Bergman, Grundy and Wiener (1996))

Proposition: Suppose the risk-neutralized process of S from the Feynman-Kac theorem is of the form:

$$dS(t) = rS(t)dt + \sigma(t, S(t))\, S(t)\, dZ^*(t) \tag{34}$$

where Z^* is a standard Brownian motion. Consider a derivative security which pays $q(S(T))$ at time T, and suppose $q(S)$ is a monotonic function of S.Then, for any t, V is a monotonic function of S in the same direction (increasing or decreasing) as q.

BGW's proof (a simplified informal outline): Suppose, for simplicity, that we make the comparison at time 0. Suppose $S''(0) > S'(0) > 0$ are two given numbers, interpreted as possible starting points for the stock price path. Suppose we use a given realization (path) of $\{Z^*(t); t \in [0, T]\}$ to create the whole path for $\{S'(t); (t \in [0, T])\}$ and then the whole path $\{S''(t); (t \in [0, T])\}$. This is done using the above starting points and the rule (34). This construction has the property that:

$$S''(0) \geq S'(0) \implies S''(T) \geq S'(T) \tag{35}$$

Explanation: The rule (34) specifies that, wherever you are on the path, the next increment of S depends only on t and $S(t)$. This means that if there is intersection $S''(t) = S'(t)$ at some point time t, then the two trajectories must be equal at all points after that. In particular, S'' cannot go below S', and thus equation (35) must hold.

If q is an increasing function, one can obtain that $q(S''(T) \geq q(S'(T))$. So far we only discuss one realization of Z^*. If we look at all possible realizations, then $S'(T)$ and $S''(T)$ can be regarded as random variables, and we conclude that, with probability 1,

$$q(S''(T) \geq q\ (S'(T)) \tag{36}$$

$$\Longrightarrow [q(S''(T))] \geq\ E^*[q(S'(T))] \tag{37}$$

Or with a different notation,

$$E^*[q(S(T))|S(0) = S''(0)] \geq E^*[q(S(T))|S(0) = S'(0)] \tag{38}$$

As before, this entails that:

$$V\big(0, S''(0)\big) \geq V\big(0, S'(0)\big). \tag{39}$$

The case where q is decreasing is similar. QED

5. METHODOLOGY AND RESULTS

5.1 Data

We analyze the price dynamics of European style options written on the S&P500 in US exchanges from OptionMetrics. We perform our tests based on daily European option prices (bid and offer quotes) and their corresponding underlying (closing) asset prices from June 1, 2005 to May 31, 2006. Our dataset contains approximately 150 thousand observations of call and put options prices and over two thousand contracts (options ID's in database) traded.

5.2 Violation Testing Criteria

In order to test the validity of the EMP, it is relatively easy to count the number of incidences when Δc and ΔS do not have the same sign, and when Δp and ΔS do not have the opposite signs, using our selected dataset of option and underlying asset prices. We use lower-case c and p to denote prices of European-style calls and puts, respectively.

Three types of violations are tested for call options and three analogous types are tested for put options.

Violations for Call Options:

$$Type\ I: \Delta S < 0, \Delta c > 0$$

$$Type\ II: \Delta S > 0, \Delta c < 0$$

$$Type\ III: \Delta S \neq 0, \Delta c = 0$$

Violations for Put Options:

$$Type\ IV: \Delta S > 0, \Delta p > 0$$

$$Type\ V: \Delta S < 0, \Delta p < 0$$

$$Type\ VI: \Delta S \neq 0, \Delta p = 0$$

Here Δc and Δp are defined as changes in option prices in two consecutive business days:

$$\Delta c \;=\; c_{t+1} - c_t \tag{40a}$$

$$\Delta p \;=\; p_{t+1} - p_t \tag{40b}$$

Our violation types are similar, but not identical, to the ones in Bakshi, Cao and Chen (2000) and Pérignon (2006). In Pérignon's work, Type III and VI (in our notation) are not covered. BCC's Type I violation (see our Section 3) is split in our work into two types of different severity, as will be explained below. Their Type VI violation is actually not a violation of monotonicity but of the upper (lower) bound for Δc (Δp), which we discuss later in the paper. Unlike BCC, we do not include the cases "$\Delta S = 0$ but $\Delta c \neq 0$" or "$\Delta S = 0$ but $\Delta p \neq 0$" (their Type III) because they are problematic to interpret. Recall that in equations (6) and (7) above, $(c_t + \frac{1}{2}\, c_{ss}\, \sigma^2 S^2)dt$ and $(p_t + \frac{1}{2}\, p_{ss}\, \sigma^2 S^2)dt$ were neglected. However, if dS=0, then these "negligible" terms become important in affecting the sign of the option price change.

Among the three types of violations for a call, type I is the most serious violation for the following reason. It is well known that theta (Θ), which is the rate of change of the value of the option with the passage of time, is always negative for a call option. As a result, as time passes by, Θ will induce the call price to decrease in value. Thus, if ΔS < 0, there are two causes for Δc to decrease over the time interval. However in violation of Type I, Δc is still positive. Thus a positive Δc value in this situation violates the EMP "severely," thus it requires more attention in order to identify the possible cause of the violation.

The analysis for put options is more complex compared to call options. Theta for a put option may have either a positive or a negative sign. Theta will be negative when the put option is deep in the money (ITM), and will be positive when the put option is deep out of the money (OTM). As a result, a deep OTM put is expected to go down in price (like a call) as time passes, and it is also expected to go down if $\Delta S>0$, hence Type IV violation is "severe." In contrast, a deep ITM put is expected to go up as time passes, and it is also expected to go up if $\Delta S<0$, hence Type V violation is "severe."

5.3 Data Analyzing Process and Methodology

There are three steps in our data analysis process. Firstly we conduct the analysis by using three different options prices: bid, offer and bid-offer midpoint prices. Secondly we group our data by implied volatility: one group is with all the options whose BS implied volatility is less than 100%, and the other one contains all options whose BS implied volatility is greater than or equal to 100%. It seems intuitive that higher volatility of the underlying asset increases the probability of a violation. Thus, we would like to test whether the second group (i.e. the group which includes options with BS implied volatility greater than or equal to one) will produce higher violation rates.

Thirdly, we analyze our data based the moneyness of the options: in the money (ITM) and out of the money (OTM). We do not analyze the situation when the options are at the money (ATM) since there are too few data (i.e. only one contract, traded for three times) to analyze.

The customary definition of moneyness, as in Hull (2006), is as follows

Table X – Moneyness defined

	Criteria	
	Call Options	Put Options
ITM	$S_T > K$	$S_T < K$
ATM	$S_T = K$	$S_T = K$
OTM	$S_T < K$	$S_T > K$

As it was pointed out in Section 3, the definition used in Bakshi, Cao and Chen (2000), and later in Pérignon (2006), is as follows: A call option is ITM if $S/K \geq 1.03$, ATM if $S/K \in (0.97, 1.03)$, and OTM if $S/K \leq 0.97$. An analogous definition is applied to puts. They do not explain the motivation for such a definition, and the choice of the number 1.03. It may be related to moneyness relative to the futures price. In our work we employ the customary definition as in Hull's book.

The following is the detailed methodology used in this paper.

(i) **<u>For Δc and Δp</u>**: We use three different quotes: bid, offer and bid-offer midpoint quotes, to calculate the changes in option prices. Although we would like to use the observed transaction prices as Δc and Δp, it is unavailable for the daily data in the database. As a substitute, we investigate whether bid and offer quotes may provide additional insights.

(ii) **<u>For ΔS</u>**: There are two prices available: one is the close prices, and the other is the average of lowest and highest price in each trading day. We choose to use "closing" asset prices instead of average prices, in order to better "match" the timing of observed option prices and the corresponding underlying asset prices.

(iii) We count the number of Δc x ΔS, which has negative signs, for call options. In addition, we further classify whether the violation is a type I or Type II violation. Similarly, we count the number of Δp x ΔS, which has positive signs, for put options. In addition, we further classify whether the violation is a type IV or Type V violation. For options whose prices changes equal to zero, i.e., Δc = 0, or Δp = 0, we count the number of occurrences, whose corresponding ΔS is different from zero.

(iv) We classify data by BS implied volatility (IV): one group with options' implied volatility smaller than one, the other with implied volatility greater than or equal to one, and then we follow the same procedure to identify the violation rates within each group.

(v) Lastly, we categorize options by moneyness, ITM and OTM, in order to test how moneyness affects violation rates (or whether there is any influence).

5.4 Results

5.4.1 Testing the violation rates for call options without any grouping

Table XI – General Results: Violation Rates for Call Options

Call Options	Total Number	Number of Complying	Number of Violations	Percentage of Violations
With bid prices	77,426	64,460	12,966	16.75%
With offer prices	77,426	64,841	12,585	16.25%
With bid-offer midpoint	77,426	65,897	11,529	14.89%

Table XII – Violation Rates for Call Options by Violation Types

		Bid	Offer	Midpoint
# of violations	Type I	1981	2469	2132
	Type II	5376	5621	5611
	Type III	5609	4495	3786
	Total #	77426	77426	77426
Violation Rates	Type I	2.56%	3.19%	2.75%
	Type II	6.94%	7.26%	7.25%
	Type III	7.24%	5.81%	4.89%
Total		16.75%	16.25%	14.89%

5.4.2 For call options, testing the violation rates after grouping options by BS implied volatility: IV <1 and IV ≥1

Table XIII – Violation Rates for Call Options, Grouped by Implied Volatility

Call Options	Implied Volatility (IV)	Total Number	Number of Complying	Number of Violations	Percentage of Violations
With bid prices	IV < 1	66,009	54,249	11,760	17.82%
	IV≥1	779	745	34	4.36%
With offer prices	IV < 1	66,009	54,644	11,365	17.22%
	IV≥1	779	737	42	5.39%
With bid-offer midpoint	IV < 1	66,009	55,617	10,392	15.74%
	IV≥1	779	746	33	4.24%

5.4.3 Violation Rates for Call Options Grouped by Moneyness

Table XIV – Violation Rates for Call Options, Grouped by Moneyness[1]

	ITM			OTM		
	Bid	Offer	Midpoint	Bid	Offer	Midpoint
# of violations	3931	4162	3779	9035	8422	7750
Total Number	49570	49570	49570	27854	27854	27854
Violation Rates	7.93%	8.40%	7.62%	32.44%	30.24%	27.82%
# of violations						
Type I	1194	1486	1237	787	983	895
Type II	2439	2387	2287	2937	3234	3324
Type III	298	289	255	5311	4205	3531
Violation Rates						
Type I	2.41%	3.00%	2.50%	2.83%	3.53%	3.21%
Type II	4.92%	4.82%	4.61%	10.54%	11.61%	11.93%
Type III	0.60%	0.58%	0.51%	19.07%	15.10%	12.68%

From above four tables, three conclusions emerge.

(1) Violations rates are lowest when using bid-offer midpoint quotes as the changes of option prices. When we use bid, offer and bid-offer midpoint prices for computing changes in

[1] We only analyze the situations of ITM and OTM, but not ATM, since ATM is a very special case according to our definition and there is not enough data to perform the analysis.

option prices, violation Type III, Type II and again Type II have the highest frequency, respectively. It is interesting to observe that Type I violation, which is the most severe in terms of being contradictory to the theory, is the one which is less frequent.

(2) By categorizing data by BS implied volatility, options with BS implied volatilities greater than or equal to one actually have lower violation rates. In addition, for options with implied volatilities less than one, using bid-offer midpoint prices produces the lowest violation rates. The same conclusion can be made for options with BS implied volatilities greater than or equal to one. The above results indicate that BS implied volatility is negatively related to violations rates. Assuming that implied volatility is highly correlated with "true" volatility, this seems counterintuitive.

(3) Violation rates are much lower for ITM options than for OTM options. Therefore, moneyness can be considered as a factor affecting violation rates. For ITM options, Type II violation (i.e., $\Delta S<0$, but $\Delta c>0$) are most frequent among all three violation types, and in fact more than half of the violations are of Type II. This is followed by Type I violations, which constitute one third of the violation rates. For OTM options, Type III, when $\Delta S \neq 0$ but $\Delta c = 0$, counts for almost half the violations. This makes sense, because at this level of the index the call price is low. Because the option price can only change in multiples of the tick size, it does not respond to small changes in the underlying index. Again, Type II violations constitute one third of the violations. Even though for OTM options, Type I "weights" the least among three types of violations, the percentage is actually very similar to the one in ITM options.

Our results are not identical to those in Bakshi, Cao and Chen (2000) and in Pérignon (2006). In BCC, they compare the violation rates across moneyness and maturity for call options only. In the case of a cash index and ITM call options with all maturities (i.e., short, medium or long term to maturity), they have higher rate of Type I violations compared to what we find. However, for OTM calls, their violation rates for type I are lower, while for type II they are higher, They also experiment with using index futures prices instead of the cash index, which we do not do. For ITM calls, for both type I and type II violations, BCC's violation rates of medium term calls are lower than our rates using offer prices only, while are still higher than our rates using bid and bid-offer midpoint prices. For OTM calls, for type I violations, violation rates for all maturities are higher than our results, regardless of which quotes of option prices are used. However, violation rates for all maturities are lower than our results, regardless of which option quotes are used.

Variations between our results and results in Bakshi, Cao, and Chen (2000) can be explained by the different dataset used. They use hourly data, whereas we use daily observations. It can be argued that daily data is "more smooth" and less volatile, hence they are expected to exhibit less EMP violations

Our results are different compared to Pérignon's results. For ITM calls, for type I violations, his violation rates are lowest for options on the German index, and they are still higher than our violation rates. In contrast, for type II violations, German-data and UK-data violation rates are lower than our violation rates (while violation rates from the other three indices are still higher compared to our results). For OTM calls, analysis for type I violations is the same as for ITM calls as above. However, analyses for type II violations are different. Violation rates for the French index are high, and actually have higher violation rates than our results. Again, higher violation rates (compared to our results) can be explained by using intra-day data compared to daily data. Lower violation rates (compared to our results) can be regarded as the results of market-specific factors.

5.4.4 Testing the violation rates for put options without any grouping

Table XV – General Results for Put Options

Put Options	Total Number	Number of Complying	Number of Violations	Percentage of Violations
With bid prices	77,572	57,656	19,916	25.67%
With offer prices	77,572	59,697	17,875	23.04%
With bid-offer midpoint	77,572	61,305	16,267	20.97%

Table XVI – Violation Rates for Put Options by Violation Types

		Bid	Offer	Midpoint
# of violations	Type IV	3055	3682	3570
	Type V	4751	5105	5229
	Type VI	12111	9088	7468
	Total #	77572	77572	77572
Violations Rates	Type IV	3.94%	4.75%	4.60%
	Type V	6.12%	6.58%	6.74%
	Type VI	15.61%	11.72%	9.63%
	Total	25.68%	23.04%	20.97%

5.4.5 For put options, testing the violation rates after grouping options by BS implied volatility: IV< 1and IV ≥1

Table XVII – Violation Rates for Put Options, Grouped by Implied Volatility

Put Options	Implied Volatility (IV)	Total Number	Number of Complying	Number of Violations	Percentage of Violations
With bid prices	IV<1	58,651	41,670	16,981	28.95%
	IV≥1	209	17	192	91.87%
With offer prices	IV<1	58,651	43,407	15,244	25.95%
	IV≥1	209	62	147	70.33%
With bid-offer midpoint	IV<1	58,636	44,952	13,684	23.34%
	IV≥1	209	63	146	69.86%

5.4.6 Violation Rates for Put Options Grouped by Moneyness.

Table XVIII – Violation Rates for Put Options, Grouped by Moneyness[2]

	ITM			OTM		
	Bid	Offer	Midpoint	Bid	Offer	Midpoint
# of violations	2184	2118	2051	17732	15756	14215
Total Number	27875	27875	27875	49695	49695	49695
Violation Rates	7.84%	7.60%	7.36%	35.68%	31.71%	28.60%
# of violations						
Type IV	898	990	928	2156	2691	2641
Type V	1081	929	946	3670	4176	4283
Type VI	205	199	177	11906	8889	7291
Violation Rates						
Type IV	3.22%	3.55%	3.33%	4.34%	5.42%	5.31%
Type V	3.88%	3.33%	3.39%	7.39%	8.40%	8.62%
Type VI	0.74%	0.71%	0.64%	23.96%	17.89%	14.67%

From four result-tables above, we observe that put options produce very different results than call options. Firstly, the violation rates from put options are much higher. Secondly, in contrast to the results from call options, BS implied volatility is positively related to the violation rates. In fact, violation rates from options whose BS implied volatility is less than one are only one third of the violation rates from options whose implied volatility is greater than or equal to one. This result agrees with the intuition that higher volatility of the underlying asset price may cause higher violation rates for the option prices.

Results from table XVIII are similar to the results from call options: violations rates are much lower from ITM options than from OTM options. Thus, moneyness is one of the factors affecting violations. For ITM options, Type IV and Type V weight similarly: both count for almost half of the violations. Recall that for ITM put options, Type V violation is most severe, in

[2] We only analyze the situations of ITM and OTM, but not ATM, since ATM is a very special case according to our definition and there is not enough data to perform the analysis.

terms of being in contradiction to the theory. Interestingly, it is more frequent compared to Type I violation for calls. In other words, violations for ITM put options have a higher probability of disrupting a hedge, compared to a call.

In contrast, for OTM puts, Type IV violation is “severe,” as we saw. The results are analogous to the ones from OTM Call options. Also, Type VI violation rates count up to more than half of the violations rates among all three types. The logic (tick size movements) is similar to the one for a call. Comparing again the two “severe” cases, violation rates for Type IV, in spite of being lowest among the three types, are still higher than Type I violation rates for calls.

In Bakshi, Cao, and Chen (2000) only analyze call options, hence we cannot make comparisons with their results. In Pérignon (2006), for ITM puts, for both type IV and type V violations, violations rates (from all five underlying indices) are higher than our results. However, analyses for OTM puts are different. For type I violation, his violation rates are lowest for the UK data, and they are lower than our violation rates. For Type II violation, violation rates in Germany and UK are lower than our results. These conclusions are very similar to the ones for call options.

5.4.7 Comparing the magnitude of the price changes Δc (or Δp) and ΔS

In previous sections, our focus was on the sign of delta, the partial derivative with respect to S. Another well-known property is that the absolute value of delta is below one. In the Black-Scholes model, this is clear from the explicit formula for delta. See graphs (below) from (Hull 2006)). In the general case, this is proved in (Bergman, Grundy and Wiener (1996). Thus, in general, the delta of a call option is between 0 and 1, and the delta of a put option is between -1 and 0.

In our time series of option prices, we test whether 0 < Δc/ΔS < 1 and -1 < Δp/ΔS < 0 (given ΔS≠0) holds for call and put options, respectively. In other words, there is a violation if Δc/ΔS is either less than or equal to 0 (i.e. a violation of the EMP), or greater than or equal to 1 (i.e. an "upper-bound violation" for Δc/ΔS). Likewise, there is a violation if Δp/ΔS is either less than or equal to -1 (i.e. a "lower-bound violation" for Δp/ΔS), or greater than or equal to 0 (i.e. a violation of the EMP).

In the results tables listed below, the number of violations for Δc/ΔS is the number of observations where it is either less than or equal to 0 or greater than or equal to 1. For Δp/ΔS, this is the number of observations where it is either less than equal to -1 or greater than equal to 0. The total number is the number of ITM and OTM options.

Figure I – call's delta in the BS model as a function of the stock price

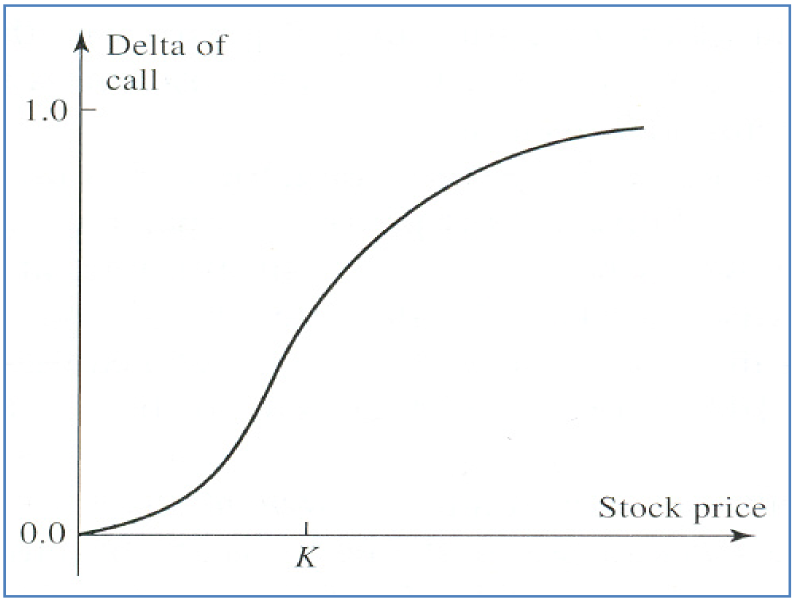

Figure II – put's delta in the BS model as a function of the stock price

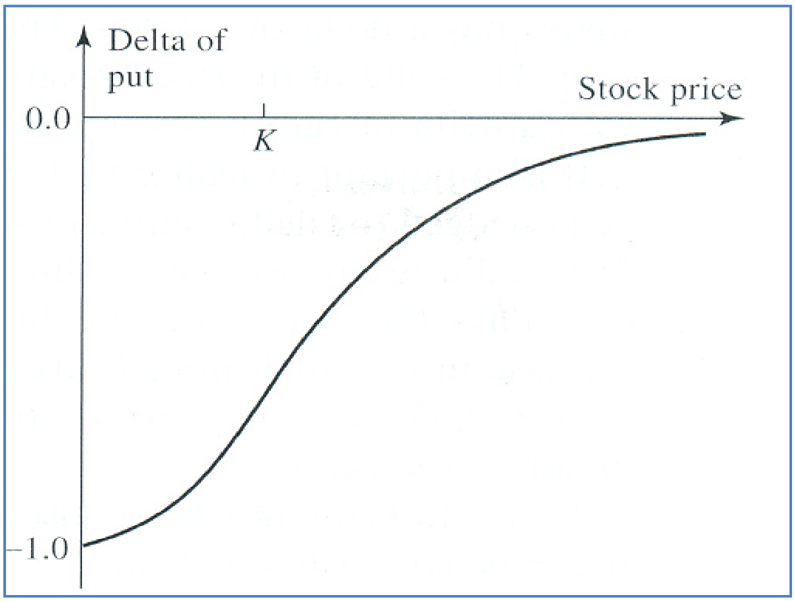

Table XIX – Violation Rates for the Two-sided Inequality for Call Options

	ITM			OTM		
	Bid	Offer	Midpoint	Bid	Offer	Midpoint
# of violations	19031	18692	18393	9692	9131	8338
Total Number	49570	49570	49570	27854	27854	27854
Violation Rates	38.39%	37.71%	37.11%	34.80%	32.78%	29.93%

Table XX – Violation Rates for the Two-sided Inequality for Put Options

	ITM			OTM		
	Bid	Offer	Midpoint	Bid	Offer	Midpoint
# of violations	9716	9896	9674	18822	16972	15258
Total Number	27875	27875	27875	49695	49695	49695
Violation Rates	34.86%	35.50%	34.70%	37.87%	34.15%	30.70%

A violation means that at least one of the inequalities is violated. The above results indicate the following. All violation rates from call options and put options are higher than the results from table XIV and XX, because these violation rates are violations for two-sided inequalities. For calls, violation rates are for violations of the EMP and violations of the upper-bound of the inequality. For puts, violation rates are for violations of the EMP and violations of the lower-bound of the inequality. Thus, if we use the violation rates in tables XIV and XX, minus the violation rates in tables XXII and XXIII, the results are the violation rates of upper-bound (for calls) and of lower-bound (for puts) only. The results are listed in the tables below.

Table XXI – Violations Rates for the Upper-Bound of Δc/ΔS

	ITM			OTM		
	Bid	Offer	Midpoint	Bid	Offer	Midpoint
# of Delta violations	15100	14530	14614	657	709	588
Total Number	49570	49570	49570	27854	27854	27854
Violation Rates	30.46%	29.31%	29.48%	2.36%	2.55%	2.11%

Table XXII – Violations Rates for the Lower-Bound of Δp/ΔS

	ITM			OTM		
	Bid	Offer	Midpoint	Bid	Offer	Midpoint
# of Delta violations	7532	7778	14614	1090	1216	1043
Total Number	27875	27875	27875	49695	49695	49695
Violation Rates	27.02%	27.90%	27.35%	2.19%	2.45%	2.10%

Recall that violation rates of the EMP for ITM calls and for ITM puts are lower than those for OTM calls and OTM puts, respectively. Here we see that violation rates of the upper-bound constraint (for calls) and of the lower-bound constraint (for puts) yield the opposite comparison. In fact, violation rates of the upper-bound for ITM calls and ITM puts are over ten times higher compared to OTM calls and OTM puts, respectively, regardless of which option quotation is used. These results conform to our expectations for the following reasons. The deltas of an OTM call is small, thus empirically it is unlikely for Δc to be greater than ΔS. In contrast, for deep ITM calls, since the delta values are close to one, Δc should be closer to ΔS, and market imperfections may cause it to be higher. Likewise, for put options, since deep ITM puts have delta values close to -1, it is rather likely that Δp will exceed ΔS in absolute value.

To sum up: It makes sense theoretically, and we have confirmed empirically, that the frequency of the upper/lower bounds violations is influenced by moneyness. It is higher for ITM options and lower for OTM options.

6. CAUSES OF VIOLATIONS

EMP violations may have various causes, such as other underlying variables in the pricing model, different quotes used in the measurement of price differences (i.e. bid, offer or their average, combined in different ways for the purpose of taking differences), and some microstructure factors.

6.1 Are Violations Caused by Changes in Other Underlying Variables

Firstly, we address the question whether violations are caused by changes in other underlying variables. For example, it is known that value of an option will increase as the volatility of the underlying asset increases, keeping other variables fixed. With a similar interpretation, the value of a call will decay as it approaches expiration.

In the Black-Scholes model, the value of an option (V) can be written as

$$V = V(S, \sigma, t, K, \delta, r) \qquad (41)$$

S: current value of the underlying asset

σ: the volatility of the underlying asset

t: passage of time

K: the strike price

δ: the dividend yield

r: the risk free interest rate

The Taylor expansion of V gives

$$\Delta V = Delta \times \Delta S + Vega \times \Delta\sigma + Theta \times \Delta t + Diva \times \Delta\delta + Rho \times \Delta r + \frac{1}{2} Gamma \times (\Delta S)^2 \qquad (42)$$

Where the coefficients (the "Greeks") are the first partial derivatives relative to the corresponding variables (except for Gamma, which is a second derivative)

6.2 Single-Variable Linear Regression

From equation (42), one will notice that an empirical test of the relationship between ΔV and ΔS ignores the influence of other variables. In this section, we study the empirical effect of adding σ as a variable. We do it in two steps: First, we regress ΔV on Delta×ΔS, then we regress two variables, ΔV on Delta×ΔS and Vega×Δσ. We would like to test whether adding one more variable could improve the explanatory power, which would identify a source of the violations of the EMP.

Following Bakshi, Cao and Chen (2000), our first regression (simple linear regression – SLR) is

$$\Delta V = \beta_0 + \beta_1 \times Delta \times \Delta S, \tag{43}$$

where Deltas are obtained from the OptionMetrics database. They are calculated by using the Black-Scholes model, and thus this test is model-specific. The logic behind this is that Delta×ΔS is the change in the option price predicted by the BS model, neglecting the effect of other variables. Thus in theory β_1 should be 1. Even if the "true" model is different, multiplying ΔS by the BS Delta is likely to improve the fit of the regression, compared to using ΔS alone.

We eliminate those option prices for which deltas are unavailable. We further categorize our dataset by moneyness of the options and calculate Δc and Δp by using three different quotes: bid, offer and bid-offer midpoint prices. The results are presented in the following tables.

Table XXIII – SLR Results: ITM Call Options with Bid Quotes

Regression Statistics	
Multiple R	0.9715
R Square	0.9439
Adjusted R Square	0.9439
Standard Error	1.4875
Observations	39397

	df	SS	MS	F	Significance F
Regression	1	1E+06	1E+06	662238	0
Residual	39395	87170	2.2127		
Total	39396	2E+06			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	0.0403	0.0075	5.3637	8E-08	0.0255	0.055
Delta×ΔS	0.9616	0.0012	813.78	0	0.9592	0.9639

Table XXIV – SLR Results: ITM Call Options with Offer Quotes

Regression Statistics	
Multiple R	0.9724
R Square	0.9456
Adjusted R Square	0.9456
Standard Error	1.4729
Observations	39397

	df	SS	MS	F	Significance F
Regression	1	1E+06	1E+06	684468	0
Residual	39395	85468	2.1695		
Total	39396	2E+06			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	0.0308	0.0074	4.1374	4E-05	0.0162	0.0453
Delta×ΔS	0.968	0.0012	827.33	0	0.9657	0.9703

Table XXV – SLR Results: ITM Call Options with AVG Quotes

Regression Statistics	
Multiple R	0.9733
R Square	0.9474
Adjusted R Square	0.9474
Standard Error	1.4425
Observations	39397

	df	SS	MS	F	Significance F
Regression	1	1E+06	1E+06	708957	0
Residual	39395	81970	2.0807		
Total	39396	2E+06			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	0.0355	0.0073	4.878	1E-06	0.0212	0.0498
Delta×ΔS	0.9648	0.0011	842	0	0.9625	0.967

Table XXVI – SLR Results: OTM Call Options with Bid Quotes

Regression Statistics	
Multiple R	0.943
R Square	0.8892
Adjusted R Square	0.8892
Standard Error	0.7065
Observations	27388

	df	SS	MS	F	Significance F
Regression	1	109701	109701	219788	0
Residual	27386	13669	0.4991		
Total	27387	123370			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.193	0.0043	-45.31	0	-0.202	-0.185
Delta×ΔS	0.8825	0.0019	468.82	0	0.8789	0.8862

Table XXVII – SLR Results: OTM Call Options with Offer Quotes

Regression Statistics	
Multiple R	0.9454
R Square	0.8937
Adjusted R Square	0.8937
Standard Error	0.7141
Observations	27388

	df	SS	MS	F	Significance F
Regression	1	117406	117406	230227	0
Residual	27386	13966	0.51		
Total	27387	131371			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.213	0.0043	-49.42	0	-0.222	-0.205
Delta×ΔS	0.913	0.0019	479.82	0	0.9093	0.9167

Table XXVIII – SLR Results: OTM Call Options with AVG Quotes

Regression Statistics	
Multiple R	0.9518
R Square	0.9059
Adjusted R Square	0.9059
Standard Error	0.6562
Observations	27388

	df	SS	MS	F	Significance F
Regression	1	113521	113521	263606	0
Residual	27386	11794	0.4306		
Total	27387	125314			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.203	0.004	-51.28	0	-0.211	-0.196
Delta×ΔS	0.8978	0.0017	513.43	0	0.8943	0.9012

Table XXIX – SLR Results: ITM Put Options with Bid Quotes

Regression Statistics	
Multiple R	0.9687
R Square	0.9383
Adjusted R Square	0.9383
Standard Error	1.3879
Observations	10880

	df	SS	MS	F	Significance F
Regression	1	318661	318661	165434	0
Residual	10878	20953	1.9262		
Total	10879	339615			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.113	0.0133	-8.475	3E-17	-0.139	-0.087
Delta×ΔS	1.0943	0.0027	406.74	0	1.089	1.0996

Table XXX – SLR Results: ITM Put Options with Offer Quotes

Regression Statistics	
Multiple R	0.9691
R Square	0.9391
Adjusted R Square	0.9391
Standard Error	1.3913
Observations	10880

	df	SS	MS	F	Significance F
Regression	1	324590	324590	167689	0
Residual	10878	21056	1.9357		
Total	10879	345646			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.132	0.0133	-9.913	5E-23	-0.159	-0.106
Delta×ΔS	1.1044	0.0027	409.5	0	1.0991	1.1097

Table XXXI – SLR Results: ITM Put Options with AVG Quotes

Regression Statistics	
Multiple R	0.9707
R Square	0.9423
Adjusted R Square	0.9422
Standard Error	1.3461
Observations	10880

	df	SS	MS	F	Significance F
Regression	1	321619	321619	177498	0
Residual	10878	19711	1.812		
Total	10879	341329			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.123	0.0129	-9.492	3E-21	-0.148	-0.097
Delta×ΔS	1.0994	0.0026	421.3	0	1.0943	1.1045

Table XXXII – SLR Results: OTM Put Options with Bid Quotes

Regression Statistics	
Multiple R	0.9416
R Square	0.8866
Adjusted R Square	0.8866
Standard Error	0.4861
Observations	47962

	df	SS	MS	F	Significance F
Regression	1	88637	88637	375085	0
Residual	47960	11333	0.2363		
Total	47961	99970			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.12	0.0022	-54.02	0	-0.124	-0.116
Delta×ΔS	1.0991	0.0018	612.44	0	1.0956	1.1027

Table XXXIII – SLR Results: OTM Put Options with Offer Quotes

Regression Statistics	
Multiple R	0.9371
R Square	0.8781
Adjusted R Square	0.8781
Standard Error	0.518
Observations	47962

	df	SS	MS	F	Significance F
Regression	1	92663	92663	345372	0
Residual	47960	12868	0.2683		
Total	47961	105530			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.133	0.0024	-56.34	0	-0.138	-0.129
Delta×ΔS	1.1238	0.0019	587.68	0	1.1201	1.1276

Table XXXIV– SLR Results: OTM Put Options with AVG Quotes

Regression Statistics	
Multiple R	0.9518
R Square	0.9059
Adjusted R Square	0.9059
Standard Error	0.4432
Observations	47962

	df	SS	MS	F	Significance F
Regression	1	90638	90638	461510	0
Residual	47960	9419.1	0.1964		
Total	47961	100058			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.127	0.002	-62.56	0	-0.131	-0.123
Delta×ΔS	1.1115	0.0016	679.34	0	1.1083	1.1147

The following observations and conclusions can be made from above tables. The regression confirms that the EMP is satisfied empirically on the average. Generally speaking, all models have strong explanatory power in explaining changes in options prices, since all models have relatively high R^2 values. For call options, models with ITM options have higher R^2 value than models with OTM options. The logic for that was explained in a previous section: For low-priced options, the constraint is to be priced in multiples of the tick size. Both the intercept term and the independent variable, Delta$\times\Delta$S, are significant at 5% significance level which indicate that we can reject the null hypothesis that β_1 is equal to zero. Furthermore, the β_1's from regressions for calls are positive, which empirically proves that on the average call option prices comply with the EMP. Further analysis can be done by setting different null hypotheses, such as that β_1 is greater than zero (i.e. using a one-sided test) and/or β_1 is equal to one. One more point worth mentioning: In Bakshi, Cao and Chen (2000), they adjusted the standard errors by White's heteroscedasticity-consistent estimator, in order to address "non-constant volatility". We do not do that here. This can be the topic of further investigation.

Similar conclusions can be made for put options. In fact, this is worth further clarification. Note that our independent variable is Delta$\times\Delta$S, where Delta is negative for a put. Thus a positive β_1 means complying with the EMP. A change in the put price is in the opposite direction of the change in corresponding index is already reflected in the fact that delta is negative. Thus β_1 is theoretically expected to be positive. Also note that, according to Black-Scholes model, β_1 is expected to have value of one. Indeed, our regression results show that all twelve β_1's have values close to one. On average, calls and puts comply with the EMP according to the regression results.

6.3 A Two-Variable Regression

Since the above regression equation actually explains the validity of the EMP, not the violations of the EMP, we add one more independent variable, volatility, into the above regression equation. However, $\Delta\sigma$ as one of the inputs is unavailable. Even though we do have BS implied volatility in the dataset, we cannot use it in our regression, since, by definition, the Black-Scholes formula is correct when one uses the implied volatility. In what follows, we denote

$$H = \frac{\ln(High) - \ln(Low)}{Average\ of\ \ln(High) and\ \ln(Low)}.$$

Here high and low are the highest and lowest prices of the underlying asset during a trading day, respectively. We use H as a proxy to the volatility σ, and thus ΔH will be used as a proxy for $\Delta\sigma$. In other words, our analysis focuses on the effects from Delta×ΔS, and Vega×ΔH. Vega is taken from the database OptionMetrics. It is calculated from the Black-Scholes model. The logic for using the BS Vega is similar to the one for using the BS Delta, as explained in the previous subsection.

We run the following regression

$$\Delta V = \beta_0 + \beta_1 \times \text{Delta} \times \Delta S + \beta_2 \times \text{Vega} \times \Delta H. \qquad (44)$$

The results are presented in the following order: (1) call options: ITM with bid, offer and bid-offer midpoint prices; OTM with bid, offer and bid-offer midpoint prices; (2) put options: ITM with bid, offer and bid-offer midpoint prices; OTM with bid, offer and bid-offer midpoint prices.

Table XXXV – Results from Equation (44): ITM Call Options with Bid Quotes

Regression Statistics	
Multiple R	0.9721
R Square	0.945
Adjusted R Square	0.945
Standard Error	1.4759
Observations	38793

	df	SS	MS	F	Significance F
Regression	2	1E+06	725264	332967	0
Residual	38790	84492	2.1782		
Total	38792	2E+06			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	0.0382	0.0075	5.094	4E-07	0.0235	0.0529
Delta×ΔS	0.9599	0.0012	812.38	0	0.9576	0.9622
Vega×ΔH	-1.148	0.0526	-21.84	4E-105	-1.252	-1.045

Table XXXVI– Results from Equation (44): ITM Call Options with Offer Quotes

Regression Statistics	
Multiple R	0.9728
R Square	0.9464
Adjusted R Square	0.9464
Standard Error	1.4646
Observations	38793

	df	SS	MS	F	Significance F
Regression	2	1E+06	734670	342517	0
Residual	38790	83201	2.1449		
Total	38792	2E+06			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	0.0291	0.0074	3.9082	9E-05	0.0145	0.0437
Delta×ΔS	0.9666	0.0012	824.39	0	0.9643	0.9689
Vega×ΔH	-0.907	0.0522	-17.39	2E-67	-1.01	-0.805

Table XXXVII – Results from Equation (44): ITM Call Options with AVG Quotes

Regression Statistics	
Multiple R	0.9738
R Square	0.9483
Adjusted R Square	0.9483
Standard Error	1.432
Observations	38793

	df	SS	MS	F	Significance F
Regression	2	1E+06	729953	355987	0
Residual	38790	79539	2.0505		
Total	38792	2E+06			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	0.0337	0.0073	4.6237	4E-06	0.0194	0.0479
Delta×ΔS	0.9633	0.0011	840.22	0	0.961	0.9655
Vega×ΔH	-1.028	0.051	-20.15	8E-90	-1.128	-0.928

Table XXXVIII – Results from Equation (44): OTM Call Options with Bid Quotes

Regression Statistics	
Multiple R	0.9448
R Square	0.8926
Adjusted R Square	0.8926
Standard Error	0.6975
Observations	26864

	df	SS	MS	F	Significance F
Regression	2	108668	54334	111677	0
Residual	26861	13069	0.4865		
Total	26863	121736			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.192	0.0043	-45.18	0	-0.201	-0.184
Delta×ΔS	0.8785	0.0019	466.5	0	0.8748	0.8822
Vega×ΔH	-0.453	0.0207	-21.86	5E-105	-0.494	-0.413

Table XXXIX– Results from Equation (44): OTM Call Options with Offer Quotes

Regression Statistics	
Multiple R	0.9465
R Square	0.8959
Adjusted R Square	0.8959
Standard Error	0.7086
Observations	26864

	df	SS	MS	F	Significance F
Regression	2	116117	58058	115632	0
Residual	26861	13487	0.5021		
Total	26863	129604			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.212	0.0043	-48.93	0	-0.22	-0.203
Delta×ΔS	0.9104	0.0019	475.88	0	0.9066	0.9141
Vega×ΔH	-0.304	0.0211	-14.42	6E-47	-0.345	-0.262

Table XL – Results from Equation (44): OTM Call Options with AVG Quotes

Regression Statistics	
Multiple R	0.9533
R Square	0.9088
Adjusted R Square	0.9088
Standard Error	0.648
Observations	26864

	df	SS	MS	F	Significance F
Regression	2	112354	56177	133795	0
Residual	26861	11278	0.4199		
Total	26863	123632			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.202	0.004		0	-0.21	-0.194
Delta×ΔS	0.8944	0.0017		0	0.891	0.8979
Vega×ΔH	-0.378	0.0193		2E-85	-0.416	-0.341

Table XLI – Results from Equation (44): ITM Put Options with Bid Quotes

Regression Statistics	
Multiple R	0.9688
R Square	0.9385
Adjusted R Square	0.9385
Standard Error	1.3822
Observations	10829

	df	SS	MS	F	Significance F
Regression	2	315802	157901	82651	0
Residual	10826	20683	1.9104		
Total	10828	336485			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.107	0.0133	-8.055	9E-16	-0.133	-0.081
Delta×ΔS	1.0907	0.0027	404.38	0	1.0855	1.096
Vega×ΔH	0.5007	0.0548	9.1319	8E-20	0.3932	0.6081

Table XLII – Results from Equation (44): ITM Put Options with Offer Quotes

Regression Statistics	
Multiple R	0.9697
R Square	0.9404
Adjusted R Square	0.9404
Standard Error	1.3705
Observations	10829

	df	SS	MS	F	Significance F
Regression	2	320664	160332	85357	0
Residual	10826	20335	1.8784		
Total	10828	340999			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.118	0.0132	-8.94	5E-19	-0.144	-0.092
Delta×ΔS	1.0986	0.0027	410.75	0	1.0933	1.1038
Vega×ΔH	0.6043	0.0544	11.116	1E-28	0.4977	0.7108

Table XLIII – Results from Equation (44): ITM Put Options with bid-offer midpoint prices

Regression Statistics	
Multiple R	0.971
R Square	0.9429
Adjusted R Square	0.9429
Standard Error	1.3339
Observations	10829

	df	SS	MS	F	Significance F
Regression	2	318227	159113	89423	0
Residual	10826	19263	1.7793		
Total	10828	337490			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.112	0.0128	-8.766	2E-18	-0.138	-0.087
Delta×ΔS	1.0947	0.0026	420.52	0	1.0896	1.0998
Vega×ΔH	0.5525	0.0529	10.442	2E-25	0.4488	0.6562

Table XLIV – Results from Equation (44): OTM Put Options with Bid Quotes

Regression Statistics	
Multiple R	0.9418
R Square	0.887
Adjusted R Square	0.887
Standard Error	0.4831
Observations	47774

	df	SS	MS	F	Significance F
Regression	2	87487	43744	187441	0
Residual	47771	11148	0.2334		
Total	47773	98636			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.119	0.0022	-53.62	0	-0.123	-0.114
Delta×ΔS	1.0943	0.0018	608.89	0	1.0908	1.0978
Vega×ΔH	0.2177	0.0168	12.932	3E-38	0.1847	0.2507

Table XLV – Results from Equation (44): OTM Put Options with Offer Quotes

Regression Statistics	
Multiple R	0.9389
R Square	0.8816
Adjusted R Square	0.8816
Standard Error	0.5052
Observations	47774

	df	SS	MS	F	Significance F
Regression	2	90773	45387	177815	0
Residual	47771	12193	0.2552		
Total	47773	102967			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.128	0.0023	-55.47	0	-0.133	-0.124
Delta×ΔS	1.113	0.0019	592.18	0	1.1093	1.1167
Vega×ΔH	0.3589	0.0176	20.39	5E-92	0.3244	0.3934

Table XLVI – Results from Equation (44): OTM Put Options with bid-offer midpoint prices

Regression Statistics	
Multiple R	0.9527
R Square	0.9077
Adjusted R Square	0.9077
Standard Error	0.4356
Observations	47774

	df	SS	MS	F	Significance F
Regression	2	89119	44559	234789	0
Residual	47771	9066.2	0.1898		
Total	47773	98185			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
β_0	-0.123	0.002	-61.89	0	-0.127	-0.119
Delta×ΔS	1.1037	0.0016	680.98	0	1.1005	1.1068
Vega×ΔH	0.2883	0.0152	18.994	4E-80	0.2585	0.318

The following observations can be made from tables above. Generally speaking, R^2s are not improved very much by adding one more independent variable. In each model, all intercepts and independent variables are significant at 5% significance level (again, we can reject the null hypothesis that β_1 and β_2 are different from zero). Similarly, to the SLR, further analyses can be done by setting different null hypotheses: β_1 is greater than zero, and/or β_1 is equal to one; and β_2 is greater than zero. For call options, some coefficients of Vega$\times\Delta$H terms are negative, which indicate that volatility of the underlying asset can be considered as a cause of violations of the EMP for call options. For example, β_2 values from regressions for OTM call options have negative signs. For put options, the β_2 values from regressions for all ITM and OTM puts have positive signs as expected.

One point worth mentioning is that, according to the Black-Scholes model, we expect the coefficient of Vega$\times\Delta\sigma$ to be close to one. However, since we use ΔH as a proxy of $\Delta\sigma$, the coefficients of Vega$\times\Delta$H are not expected to be one anymore. Only the sign matters. To sum up, those results indicate that changes in the volatility of the underlying asset can be considered as a cause of the violations of the EMP. Further investigation is needed in order to explain these results in depth.

6.4 Are Violations Caused by Different Quotes?

From all results tables in Section 5, we can conclude that the choice of using different quotes, such as bid, or offer, or their average, will affect the changes in option's value, and thus will affect violation rates as well. For call options, violation rates are the highest when using bid prices and lowest when using bid- offer midpoints. Similar results are found for put options.

If we compare "horizontally", for example, we compare all type I violations by using different option prices and grouping options by moneyness, then the following conclusions can be made. For call options, violation rates are the highest when using offer quotes, for both ITM and OTM options. Type II violations do not have as clear pattern as Type I violations. Generally speaking, violation rates of Type II are very similar among three "option prices" inputs. However, if we also take moneyness into account, violation rates are the highest by using bid quotes for ITM options, and by bid-offer midpoints for OTM options. There is an apparent but different pattern for Type III violations: violations rates are the highest by using bid quotes regardless of moneyness of the options. For put options, the same results could be inferred from all the tables listed above. Thus, for all options, we can conclude that the choice of the quotation method used in the research affects the findings regarding the violation rate of the EMP.

In addition, we can test how different the results are by using bid – offer and offer – bid quotes when we calculate Δc or Δp. The results are listed below.

Table XLVII – Violation Rates for Call Options by Using (Bid – Offer) and (Offer –Bid) Quotes

	ITM		OTM	
	Bid(t+1) – Offer(t)	**Offer(t+1) – Bid(t)**	Bid(t+1) – Offer(t)	**Offer(t+1) – Bid(t)**
# of violations	9768	**6381**	12007	**8518**
Total Number	49570	**49570**	27854	**27854**
Violation Rates	19.71%	**12.87%**	43.11%	**30.58%**
# of violations				
Type I	46	**5979**	2	**7867**
Type II	9372	**38**	11729	**138**
Type III	350	**364**	276	**513**
Violation Rates				
Type I	0.09%	**12.06%**	0.01%	**28.24%**
Type II	18.91%	**0.08%**	42.11%	**0.50%**
Type III	0.71%	**0.73%**	0.99%	**1.84%**

Table XLVIII – Violation Rates for Put Options by Using (Bid – Offer) and (Offer –Bid) Quotes

	ITM		**OTM**	
	Bid(t+1) – Offer(t)	**Offer(t+1) – Bid(t)**	Bid(t+1) – Offer(t)	**Offer(t+1) – Bid(t)**
# of violations	4310	**5336**	18054	**22595**
Total Number	27875	**27875**	49695	**49695**
Violation Rates	15.46%	**19.14%**	36.33%	**45.47%**
# of violations				
Type IV	8	**5083**	10	**21495**
Type V	4092	**44**	17650	**77**
Type VI	210	**209**	394	**1023**
Violation Rates				
Type IV	0.03%	**18.24%**	0.06%	**43.25%**
Type V	14.68%	**0.16%**	35.52%	**0.15%**
Type VI	0.75%	**0.75%**	0.79%	**2.06%**

The choice of using either (bid – offer) or (offer – bid) prices is closely related to hedging strategies from the perspective of a one-day hedger, as these are the quotes which are relevant for hedging effectiveness. For both ITM and OTM call options, violation rates are lower by using (offer – bid) prices. However, the use of (offer – bid) prices will cause higher Type I violations, which is the "most severe" type from a theoretical point of view. Likewise, for both ITM and OTM put options, violation rates are lower by using (bid – offer) prices. Here the use of (offer – bid) prices will cause higher Type IV violations.

Let us elaborate on the interpretation for hedging. Suppose an investor A has a long position in the index which she wishes to hedge for one business day with a short position in a call. On the call position, the profit would be computed from the call prices as follows: – (offer(t+1) – bid(t)). Thus, A will want to choose ITM calls which have lower violation rates of 12.87%, compared to a violation rate of 30.5809% from OTM calls. In contrast, suppose an investor B has a short position in the index which he wishes to hedge for one business day with a long position in a call. Then, on the option component, his profit will be (bid(t+1) – offer(t)) prices. Thus, B will want to choose ITM calls, which have lower violation rates of 19.71%, compared to higher violation rates of 36.33% from OTM calls. Similar examples can be made for put options. In conclusion, the above results can be considered as guidance for which option to choose for the purpose of hedging.

6.5 Are Violations Caused by Market Microstructure Factors?

So far, our causes of violations are all quantitative factors. However, there are some qualitative factors as well, such as trading mechanisms (market microstructure) not reflected in our models. We summarize here two additional causes: market makers' behaviour and rational trading tactics.

As pointed out by Bakshi, Cao and Chen (2000), in the exchanges where options on S&P500 are traded, there are designated market makers. They exist in order to ensure the continual implementation of an "auto-quote" computer program. In other words, they control some parameters in the computer systems. More specifically, they can adjust (either widening or shrinking) the bid-offer spread, according to different options' characteristics, such as moneyness and maturity, and various market conditions. On one hand, these activities may help to improve the efficiency and liquidity of the market. On the other hand, they influence market's prices beyond the stock prices' "natural" movements by adding additional source of uncertainty. As a result, this influence may cause the violations of the EMP.

According to Perignon (2006), another possible cause of violations of the EMP is the rational trading tactics by traders, trying to unload positions at certain times of the day or the week. Options with less liquidity may induce traders, who are eager to get their transactions done before the market closes, to lower their limit selling price to be a little bit below the offer price observed in the market. For example, suppose trader A wants to sell call options, and the bid and offer quotes in the market are $99 and $100 respectively. Suppose the most recent trade was at $100, and the investor is willing to sell at $99.5. During the time while A is waiting to sell his/her options, the underlying asset's prices rises by a small amount that does not cause option's price to change. If A still desires to sell his/her options, A sells his/her options actually at $99.5. Thus, one can observe an increase in the underlying asset's price, but a decrease in the option's value. This is a violation of the EMP. Liquidity plays an important role in rational trading tactics, and this might be one of the causes of violations of the EMP.

7. SUMMARY AND IMPLICATIONS FOR HEDGING

In this work, we study the co-movement of option prices and the underlying index. In theory, a call option and the index should always move in the same direction, while a put option and the index should move in opposite directions. When applied to time series of prices, this is called the Empirical Monotonicity Property (EMP). In addition, the change in the option price should be, in absolute value, below the change in the underlying index.

After summarizing the definitions and the theoretical results, we conduct empirical tests of the above properties. We use call and put options on the S&P 500 index from June 1, 2005 to May 31, 2006 from US exchanges. Our tests follow previous work by Bakshi, Cao and Chen (2000) and Pérignon (2006). While they used intraday transaction data, for our study we had only daily data. To compensate for that, we extend the tests in different directions, using bid and offer prices.

We test three violation types for call options and three violation types for put options. For example, for a call, Type I violation means $\Delta S < 0$, and $\Delta c > 0$, Type II means $\Delta S > 0$, and $\Delta c < 0$ and Type III means $\Delta S \neq 0$, and $\Delta c = 0$. We also divide the data into subgroups in different ways. First, without any grouping, we obtain that call options' prices move in opposite direction to the underlying index 14.89% of the time (using bid-offer midpoint prices for option prices). The violation rates increase if we use either bid or offer prices. Among all three violation types, Type II violation has the highest occurrence rate, followed by Type III violation. Likewise, put options' prices move in the same direction of the underlying asset 20.97% of the time, using bid-offer midpoint as above. Again, the violation rates increase if we use either bid or offer prices.

Among all three violation types, Type VI violation (i.e. $\Delta S \neq 0$, and $\Delta p = 0$) has the highest rate, followed by Type V violation ($\Delta S > 0$, and $\Delta p > 0$).

We also test the violation rates while grouping the option prices by their BS implied volatility (IV). For call options, we find that violation rates are 4.24% for calls with IV<1 while it is 15.74% for calls with IV≥1. In contrast, for put options, violation rates are higher for options whose BS implied volatilities are high. It was 23.33% for IV < 1 and 69.86% for IV≥1. For call or put options, we obtained these results by using bid-offer midpoint prices. Similar results can be found by using either bid and offer prices.

We also test the violation rates while grouping the option prices by their moneyness. For call options, our results indicate that violation rates are much lower for ITM calls (7.62%), compared to OTM calls (27.82%), both with bid-offer midpoint prices. Similar results can be found for put options. The violation rates are much lower for ITM puts (7.36%), compared to OTM puts (28.60%).

Lastly, we compare the magnitude of the price changes Δc (or Δp) and ΔS. For example, there is an "upper-bound violation" for Δc if $\Delta c/\Delta S \geq 1$, and there is a "lower-bound violation" for Δp if $\Delta p/\Delta S \leq -1$. The results are very similar for call and put prices. The violation rates are lowest when we use bid-offer midpoint prices. In addition, violation rates are lower for ITM calls and puts, compared with OTM calls and puts, respectively.

After testing the violation rates for calls and puts, we attempt to find some possible causes for the violations of the EMP. We first conduct single linear regression in order to confirm that the relationship between ΔS and ΔV follows the EMP, ignoring other variables. Then we perform a multi-variable regression in order to test whether violations are caused by changes in other underlying variables. In our case we only do it with volatility of the underlying asset, based on a proxy. Our results indicate that changes in volatility may be one of the causes of violations of the EMP. We also find that violations are influenced by different option prices quotes, such as bid, or offer, or bid-offer midpoint prices. In addition, we find that violation rates also differ by using (bid – offer) and (offer –bid) quotes (grouping by moneyness). This may of interest to a one-period hedger, who, for example, may buy the option at the offer price and sell it at the bid price. Lastly, we summarize possible explanations from Bakshi, Cao and Chen (2000) and Pérignon (2006), relating EMP violations to activities of market makers and traders' rational trading tactics, respectively.

The extent of EMP violation is important in the context of hedging effectiveness. In the Black-Scholes model, or any other model, a typical hedge for a stock position involves shorting a call (or going long on a put), with the understanding that the call and the stock will move up or down in tandem (Bakshi, Cao and Chen (2000)).

However, the EMP does not always hold. In this paper we find that 14.89% of call price movements and 20.97% of put price movements are in the "wrong" direction compared to the index. Such a high rate of violation of the EMP will severely affect the effectiveness of the hedging strategies and will lead to unexpected losses. In a theoretical world, hedging is continuous, while in practice this means frequent position rebalancing. In addition to transaction costs, frequent trading may mean that there will be a certain percentage of EMP violations, which will compound the hedging losses.

Thus our violation rates analysis by moneyness and types of violations may be used as guidance for the purpose of hedging. For example, if one wishes to hedge a long position in the index, this can be done by going short on a call or going long on a put. The choice between these two possible hedges may be affected by our knowledge of the violation rates in each case. The same is true for the choice between OTM options and ITM options. Higher violation rates from OTM options may reduce the effectiveness of hedging.

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