A COMPARATIVE STUDY OF PROBLEM SOLVING BEHAVIOURS OF AVERAGE COLLEGE ALGEBRA STUDENTS WORKING ALONE AND IN DYADS.

by
Lin Hammill
B. Sc., Simon Fraser University, 1987
M. Sc., Simon Fraser University, 1989

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Author: ______________________
(signature)

____
(name)

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(date)

APPROVAL

NAME: Lin Hammill

DEGREE: Doctor of Philosophy

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EXAMINING COMMITTEE:

Chair: Dr. Carl Schwarz

Harvey Gerber
Senior supervisor
Associate Professor of Mathematics

Tom O'Shea
Associate Professor of Education

Katherine Heinrich
Professor of Mathematics

Sandy Dawson
Internal External Examiner
Professor of Education

Susan Pirie
External Examiner
Professor of Education
University of British Columbia

Date Approved: October 29, 1996

Abstract

The aim of the present study was to investigate the actions of college algebra students attempting to solve non-routine problems, either alone or in dyads. Problem solving interviews with the individual subjects and with pairs, each one hour long, were videotaped and transcripts made from the tapes were analyzed. Subjects also completed exit interviews.

In exit interviews all but one subject stated a strong preference for working alone. However, they all cited the provision of a second point of view as being the major benefit of working with a partner. The literature would also lead one to expect that working in dyads would require that the students attempt to construct an agreed upon representation of the problem and then decide upon the approach to be taken to solve it. However, this is not what happened. There was little discussion of the structure of the problem and almost no analysis of proposed strategies. Constructive controversy was almost entirely absent.

Nevertheless, pairs were much more successful in solving the problems. This increased success arose from four factors: an increase in persistence, the more able partner leading the pair, an increased opportunity for oral rehearsal, and, to a lesser extent, the correction of minor errors. The particular character of any problem session depended on both the academic and social interactions of the partners and five categories of pairs emerged from the study: socializers, tutor/pupil pairs, partners, individuals and hostile pairs.

The students, whether working alone or in pairs, exhibited a wide variety of mathematical skills and strategies in their attempts to solve the problems. Despite this, they were not successful in solving many of the problems. Several factors contributed to their lack of success. They were generally so fixated upon finding an answer that little effort was put into analyzing the structure of the problem or generating and comparing various strategies. Another factor in their lack of success was that while the problems given them often required a structural approach, the students were generally working at an operational level for this material.

DEDICATION

In the memory of

my mother,
Florence Mildred Grosskleg,

and

my grandmother,
Helen Grace Hammill.

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CHAPTER I NATURE AND PURPOSE OF THE STUDY

INTRODUCTION

Two important themes in research in mathematics education which have emerged in recent years are problem solving and small group processes. The 1980 *Agenda for Action* of the National Council of Teachers of Mathematics (NCTM) stated that "problem solving must be the focus of school mathematics in the 1980's." [p.1] They reiterated this view in 1989 by saying that "problem solving should be the central focus of the mathematics curriculum." [NCTM 1989, p. 23] In their *Standards for Introductory College Mathematics before Calculus,* [Cohen, 1995] the American Mathematical Association of Two-Year Colleges (AMATYC) gives as their first standard for intellectual development, "Students will engage in substantial mathematical problem solving." [p.10] The NCTM 1991 *Professional Standards for Teaching Mathematics* promotes cooperative work as a means to develop students' mathematical power. Similarly, the National Research Council, in *Everybody Counts* [1989], said that students "must learn to work cooperatively in small teams to solve problems as well as to argue convincingly for their approach amid conflicting ideas and strategies." [p.61] The second of AMATYC's standards for pedagogy is "Mathematics faculty will foster interactive learning through student writing, reading, speaking, and collaborative activities so that students can learn to work effectively in groups and communicate about mathematics both orally and in writing." [Cohen, 1995, p. 16]

Cooperative work is becoming increasingly popular in teaching mathematics, especially in teaching problem solving. Slavin [1987] has said, "The Age of Cooperation is approaching." However, as the literature review will show, although both problem solving and cooperative work have been extensively researched, both processes are very complex and neither is fully understood. Thus, I propose to explore the problem solving skills used by average college algebra students working singly and in dyads. In this study, I analyze and compare the problem solving processes exhibited by these students as they work alone and in pairs. A better understanding of these processes has implications for the understanding of the process of mathematical problem solving itself and for the place of group work in problem solving instruction.

RATIONALE

Central to any question regarding problem solving is understanding what actually goes on when a student or group of students attempt to solve a non-routine problem. While much is already known, there are important gaps in the literature. Silver [1985b] has discussed "raw" heuristics used by fifth and sixth grade students, and Sowder [1988] has produced a list of inappropriate strategies used by sixth and eighth grade students. However, most of the studies done of college students have been of the treatment/comparison type or have concentrated their analysis on regulatory behaviour. As well, many of them have concentrated on very able, rather than average, students. It would be valuable to know just what skills and untaught, and sometimes unintended, strategies average college students use in the attempt to solve a difficult problem.

Collaborative, small group processes in general, and in mathematics education in particular, are not well understood and there is no well developed theory. In their *Overview of Research on Cooperative Learning related to Mathematics*, Davidson and Kroll [1991] have called for research into just what occurs during cooperative learning, stating that, "To date, a relatively small percent of the studies have attempted to study the interactions that take place during cooperative work to determine how various academic, social, or psychological effects are produced." [p.363] Silver [1985b] sees the study of small groups in mathematical problem solving as important for two reasons: (1) Small groups are commonly advocated in the popular literature and yet we know little about their effects and (2) small groups are a way to study externalized internal dialogue, providing some insight into thinking processes during problem solving. This study will add to available knowledge in this area by exploring the contrasts and similarities between problem solving behaviour exhibited by students working alone and in pairs.

Good, Mulryan and McCaslin [1992] have said, "Problems of learning are complex, and we need more process studies that illustrate how groups of students attempt to reduce ambiguity and risk when faced with difficult problems requiring creative thought". [p. 193]

RESEARCH QUESTIONS

The research is intended to address the following interrelated questions:

1. In attempting to solve non-routine problems what basic skills and particular strategies do average college algebra students use?

2. Is there a difference in quantity or type between the skills and strategies used by

students working alone and by students working in pairs?

3. What factors might account for any differences in the problem solving process as seen in individuals and in pairs?

OVERVIEW

In order to address these questions, I recruited several college students to participate in problem solving sessions. The students were studying at the college algebra and precalculus levels and were generally average to slightly above average in their mathematics achievement. I videotaped them while they attempted to solve nonroutine mathematics problems, first working alone and then later in pairs. A detailed framework of analysis was developed as I reviewed the taped sessions. Skills, strategies, beliefs and pair interactions were analyzed in detail and with reference to the research literature. As themes emerged from the analysis, the framework was modified. In the end, a detailed picture developed of what actually occurred while these average college students worked on mathematics problems.

CHAPTER 2 LITERATURE REVIEW

To provide a better understanding of the ways in which the proposed research has been designed, the review of literature will begin with a summary of research about problem solving in general and then link that with a summary of research about classroom culture and small group processes as they relate to mathematical problem solving. A short discussion of Sfard's concepts of operational and structural understanding will help to provide a understanding of strategy choices students make when solving problems.

WHAT IS A PROBLEM?

"Problems have occupied a central place in the school mathematics curriculum since antiquity, but problem solving has not." [Stanic & Kilpatrick, 1988, p.1] As problem solving comes to the fore in discussions of mathematics education, the multiple interpretations of the term *problem* have become apparent. Many of the problems referred to by Stanic and Kilpatrick and which they illustrate by examples taken from various text books, are routine exercises which simply require the student to substitute the given data into an already familiar solution pattern or to follow a previously taught algorithm. Halmos [1980], on the other hand, says that, "The major part of every meaningful life is the solution of problems" [p.523] and that, "what mathematics really consists of is problems and solutions." [p.519] Halmos' position, and that illustrated by Stanic and Kilpatrick's examples, are at opposite ends of the spectrum of definitions given to the term *problem*, which stretches from "routine exercises" to "the heart of mathematics." Perhaps the most useful distinction is that made by Schoenfeld who distinguishes between exercises, which are routine for the solver, and problems, in which the solver "does not have ready access to a (more or less) prepackaged means of solution." [1985b, p. 54]

Good, Mulryan and McCaslin [1992] view problem solving as "adaptive learning in a social setting." [p.173] They propose a three part psychological definition of problem-solving: "(1) maintaining the intention to learn (2) while enacting alternative task strategies (3) in the face of uncertainty." [p.173] They argue that this definition integrates motivation, affect, and cognition. Like Schoenfeld, they place the solver in the centre of their definition. Brown [1984 & Brown and Walter 1990] also puts the learner at the centre, but would replace problem solving with the concept of a "situation". Giving the student a situation to investigate rather than a problem to solve leads to problem posing by the student. [See also Silver 1994] This idea will be revisited in the next section of this review.

Stanic and Kilpatrick [1988] have identified three themes which characterize the place of problem solving in the mathematics curriculum:

(i) Problem solving as context,

- (a) as justification,
- (b) as motivation,
- (c) as recreation,
- (d) as a vehicle to introduce a skill or concept,
- (e) as practice,

(ii) Problem solving as a skill, and

(iii) Problem solving as art. [pp. 13-15]

The results of a survey of problem solving courses reported by Schoenfeld [1983b] appear to fit into themes (i) and (ii) rather than (iii). Almost all problems presented in mathematics textbooks are traditionally of themes (i) and (ii) as well. Stanic and Kilpatrick see the last theme, problem solving as art, as a deeper and more comprehensive view which has emerged from the work of George Polya.

Problem solving researchers have been greatly influenced by the work of Polya, whose 1945 *How to Solve It* has become a much cited classic. The theme of problem solving as art is clear in his preface, "Having tasted the pleasure in mathematics he will not forget it easily and then there is a good chance that mathematics will become something for him: a hobby, or a tool of his profession, or his profession, or a great ambition." [1973, pp.v-vi] Polya emphasized that mathematics consists mostly of observations and experiments, of building mental pictures, of guessing and trying to feel what is true, and then of putting forth and testing hypotheses. He presented a four part framework for problem solving and then used specific examples to introduce a dictionary of heuristics, or rules of thumb, that can be used to assist in solving mathematical problems. I will return to these heuristics below.

THEORETICAL FRAMEWORKS

Four types of theoretical frameworks for problem solving research will be discussed in this section: Frameworks based on the process, frameworks based on the cognitive and non-cognitive resources used, a framework which is a combination of these, and a framework based on schema acquisition and rule automation.

Frameworks based on process vary in the number of steps identified. Many of them can be traced back to John Dewey's 1933 basic plan, given by Noddings [1985] as:

1. Undergoing a feeling of lack -- identifying a problematic situation.
2. Defining the problem.
3. Engaging in means-ends analysis; devising a plan.
4. Executing; carrying out the plan.
5. Undergoing or living through the consequences.
6. Evaluating: looking back to assess whether the result satisfies the initial conditions; looking ahead to generalization of both methods and results. [p. 346]

This framework differs from those which follow by including both the posing of a problem (steps 1 and 2) and undergoing the consequences of it (step 5). This framework is really about a situation to investigate [see Brown 1984] rather than a problem to solve.

Polya's four step plan collapses Dewey's first two steps into one and eliminates his step five. Thus Polya obtains:

1. Understanding the problem.
2. Devising a plan.
3. Carrying out the plan.
4. Looking back. [1973, pp.xvi-xvii]

This framework will no longer fit a situation to be investigated but is more reflective of the procedure for solving (non-routine) textbook or instructor posed problems. Polya emphasises the importance of carrying out step one before beginning steps two and three, and that step four is essential since "(s)ome of the best effects may be lost if the student fails to reexamine and to *reconsider* the completed solution." [p.6]

Mason, Burton and Stacey [1982] collapse the framework even further to only three steps:

1. Entry: read, formulate the question precisely and decide what to do.
2. Attack: implement plans.
3. Review: check, reflect and extend.

Another three step procedure is given by Noddings [1985] as typical of an approach based on routine story problems and an "observables only" approach to theory:

1. Translating words to mathematical expressions.
2. Executing; that is, calculating.
3. Checking results in initial equations. [p. 347]

If we accept Schoenfeld's definition of problem (see above) then this clearly impoverished procedure cannot even be considered, for it will only be applicable to routine textbook exercises.

Noddings has created a four step framework based on ideas from cognitive psychology and the work of Mayer, Silver and others. In her plan, Polya's first two steps are collapsed into one step, representation, while Dewey's step five is included to create:

1. Creation of a representation.
2. Executing a plan based on the representation.
3. Undergoing the consequences.
4. Evaluating the results. [1985, p.349]

She believes that step three is crucially important in order to avoid the deadly artificiality of school problems. However, Dewey's plan was devised for real world situations and it is difficult to see what exactly Noddings means by undergoing the consequences in classroom situations. She proposes debriefing sessions at the end of problem solving periods. Correct answers are handed out and the students are encouraged to discuss what they may have done wrong and how they could get the correct answer. Her step four involves checking and evaluating solutions with reference to the problem and the student's representation of it.

Garofalo and Lester [1985] suggest another four step framework, similar to Polya's but with each step more broadly defined:

1. Orientation: Strategic behaviour to assess and understand a problem.
2. Organization: Planning of behaviour and choice of actions.
3. Execution: Regulation of behaviour to conform to plans.
4. Verification: Evaluation of decisions made and of outcomes of executed plans. [p.171]

They emphasize that there are both cognitive and metacognitive (see below) behaviours at each stage and that their framework makes this clear. The stages where metacognitive behaviour occurs most often will vary with the problem situation.

Frameworks based on resources necessary for solving problems attempt to categorize these resources in various ways. Resnick and Ford [1981] categorize knowledge into two classes: (1) algorithmic routines; (2) and strategies for assessing knowledge, detecting relationships and choosing paths of action. They identify three aspects of problem solving strategies: "(1) how the problems are represented; (2) how features of the task environment interact with an individual's knowledge; and (3) how problems are analyzed and knowledge structures are searched to bring initially unrelated information to bear on a task." [p.214] However, this analysis deals only with the cognitive aspects of problem solving.

Throughout the 1980's researchers became more aware of the crucial part played by non-cognitive and metacognitive factors in mathematical problem solving, leading them to create theoretical frameworks which incorporate these factors. Schoenfeld, in 1983, asserted that the cognitive behaviours of problem solvers are embedded in, and are shaped by, social and metacognitive factors. The problem solver's beliefs about the task, about the social environment of the task, and about herself or himself in relationship to the task and the environment, Schoenfeld said, are as important as any cognitive factors [1983a]. He suggested three separate categories of analysis, later [1985b] modified to four:

1. Resources: Mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand.
2. Heuristics: Strategies and techniques for making progress on unfamiliar problems; rules of thumb for effective problem solving.
3. Control: Global decisions regarding the selection and implementation of resources and strategies.
4. Belief Systems: One's "mathematical world view", the set of (not necessarily conscious) determinants of an individual's behaviour. [p.15]

Schoenfeld argues strongly for the crucial importance of the last two categories for researchers in mathematical problem solving. I will examine the last three categories in more detail in the following section.

Another theoretical framework, which is complimentary to, rather than opposed to, Schoenfeld's, has been developed by Perkins and Simmons (1988) as a model for knowledge in science, mathematics, and computer programming. They identify four categories that distinguish important types of knowledge, and these they call frames of knowledge.

1. The content frame: facts, definitions, and algorithms of the subject matter along with content-oriented metacognitive knowledge such as strategies for monitoring the execution of an algorithm, memorization and recall strategies.
2. The problem solving frame: specific and general problem solving strategies and beliefs about problem solving; processes to keep organized during problem solving.
3. The epistemic frame: specific and general norms and strategies regarding claims of validity within the domain.
4. The inquiry frame: specific and general beliefs and strategies to extend and challenge the knowledge within a domain. [p. 305]

Perkins and Simmons say that their model is "orthogonal" to Schoenfeld's, that Schoenfeld's model addresses the form of knowledge while their own addresses "what the knowledge in question concerns." [p.314] Each of Schoenfeld's four categories would appear in each of their four frames, although possibly in varying proportions. Perkins and

Simmons address ways in which each frame could be faulty and patterns of misunderstanding that cross all frames. One of their major concerns is that most instruction concerns only the first two frames. All four, they assert, need to be taught and taught in relation to each other. When Perkins and Simmons' knowledge framework is viewed in conjunction with Schoenfeld's four part problem solving analysis, a richer picture of the problem solving process becomes available.

Through several years of teaching remedial mathematics at the college level, Clement and Konold [1989] developed a classification of basic problem solving skills that includes both cognitive and non-cognitive skills and classifies these as either general or stage-specific.

I. Stage-Specific skills

- A. Comprehending and representing
  1. Viewing representation as a solution step
  2. Finding the goal and the givens
  3. Drawing and modifying diagrams
- B. Planning, Assembling and Implementing a Solution
  1. Breaking the problem into parts (setting subgoals)
  2. Organizing chains of operations or inferences in multistep problems
- C. Verifying the Solution
  1. Viewing verification as a solution step
  2. Assessing the reasonableness of the answer in terms of initial estimates

II. General Skills and Attitudes

- A. Alternately Generating and Evaluating Ideas (as opposed to recalling algorithms)
- B. Striving for Precision in the Use of:
  1. Inferences
  2. Verbal expressions
  3. Symbols and diagrams

4. Algorithms

C. Monitoring Progress

1. Making written records to keep track of and organize solution elements and partial results
2. Using confusion as a signal to rethink part of the solution
3. Proceeding slowly in the expectation of making and needing to correct errors [p.27]

They assert that this classification reflects the skills and attitudes actually possessed by their students, making it a more useful tool for analyzing problem solving activity than either Schoenfeld's framework or Polya's heuristics. This framework has the benefit of including cognitive skills, monitoring behaviour and beliefs, as well as a classification of the stages of the problem solving process (which is analogous to Schoenfeld's original three step plan).

Sweller [1989 & 1990] and Owen and Sweller [1989] suggest a theoretical framework for mathematical learning and problem solving which has a very different perspective than do those already discussed. Their framework is based on rule-automation, schema acquisition, and the domain specificity of problem solving skills. Their theory has six points: (1) Problem solving skill is dependent upon domain specific knowledge; (2) this knowledge base largely consists of schema and automated rules; (3) strategies chosen are generally dependent on available schema; (4) means-ends analysis, although an efficient problem solving strategy, interferes with schema acquisition; (5) learning is facilitated when means-ends analysis is avoided by the use of goal free problems and worked examples; (6) in order to reduce cognitive load and allow for schema acquisition, the format of instructional materials must minimize the need for learners to integrate disparate sources of information. [Sweller 1989, p.457] Although the framework is theoretically detailed and they are able to produce a great deal of evidence to support it [see also Sweller, Mawer & Ward 1983 and Owen & Sweller 1985], their theory has the major drawback of dealing only with relatively routine exercises where activated schema save time and effort. Problem solving, as studied by Polya or Schoenfeld, involves acting even when schema are not available. Sweller's attention, however, is entirely on schema acquisition, to the point where he would eliminate some problem solving strategies (see point 4 above) and minimize exposure to an important problem solving skill (see point 6). The conflict here is really due to different conceptions of problems and problem solving.

COGNITIVE AND NON-COGNITIVE RESOURCES

In this section I will first discuss problem solving strategies and heuristics, followed by metacognition and finally metacognitive knowledge, control and belief.

Heuristics

The use of problem solving heuristics did not begin with Polya, but since the appearance of *How to Solve It* in 1945, Polya's heuristics have been a focus for those teaching and researching problem solving. His *Short Dictionary of Heuristic* takes up almost 200 pages of the 1973 edition of *How to Solve It*, and includes such entries as: Did you use all the data?, Draw a figure, Generalization, Induction and mathematical induction. Expert problem solvers immediately recognize strategies that they commonly use. Numerous problem solving courses have been taught using Polya's heuristics and numerous studies of their effectiveness have been undertaken with mixed results. Lucas

[1974], with university students, and Kantowski [1977], with grade nine students, obtained small positive effects. However, by 1979, Begle, in a survey of research to that date, could only say that a lot of effort had gone into studying heuristic instruction with no clear results [as cited in Schoenfeld 1992].

Schoenfeld began teaching problem solving courses using Polya's heuristics in the late 1970's, but he realized that the heuristics were descriptive rather than prescriptive and needed to be much more detailed to be effective. He analyzed the most frequently used heuristics in order to characterize them in sufficient detail and to provide the appropriate amount and kind of training in their use. This seemed very successful until he began to look at videotapes of students actually solving problems. What he saw was not the systematic and creative use of heuristics that he expected. Instead, the students failed to consider alternatives or to monitor their activities, often spending most of a session on a "wild goose chase." [Schoenfeld 1985a & 1987a] Schoenfeld went on to study the importance to the problem solving process of metacognition and beliefs (discussed below).

In an investigation of fifth and sixth grade students, Silver [1985b] noted the existence of untaught (or at least not intentionally taught) heuristics such as the tendency to draw a diagram, examine special cases or generalize from specific cases. These he called "raw" heuristics. There were significant differences in the heuristics shown by different students. Silver speculated that the existence of these "raw" heuristics may be crucial to the success or lack of success of research into teaching of heuristic processes. If Polya's heuristics are descriptive, as Schoenfeld has said, then the "raw" heuristics which Silver saw may be simply a step in a natural process of acquiring heuristics.

While useful "raw" heuristics may appear in many students, inappropriate or erroneous strategies also appear. Sowder [1988] observed that students who correctly solve routine story problems may be using strategies which are of little value. He provides a representative list of strategies for sixth and eighth grade students: (1) Find the numbers and add (or subtract, whatever is your favourite operation), (2) guess at the operation to be used, (3) look at the numbers and they will tell you the operation, (4) try all the operators and choose the most reasonable answer, (5) look for key words, (6) decide if the answer is to be larger or smaller then the givens and then choose the operation accordingly (e.g. multiplication makes bigger), (7) choose the operator whose meaning fits the story. Strategy seven, he says, is rarely seen. Bell, Greer, Grimson and Mangan [1989] obtained similar results. Unfortunately, no equivalent list of strategies actually in use has been created for non-routine problems or for more advanced students.

The heuristic most studied is "Can you think of a similar problem?" Sweller [1989], Owen and Sweller [1985 & 1989] and Sweller, Mawer and Ward [1983] began their investigations of schema acquisition by looking at expert-novice comparisons. They found that experts exhibited a better memory than novices, classified problems by the underlying mathematical structure rather than the surface structure, and, more often than the novices, worked forward rather than using a means-ends analysis. This led them to their studies of schema acquisition and rule automation and the development of the theory discussed in the last section. Studying problem solving from a schema based theory led Reed and Bolstad [1991] to compare student learning of algebraic rate problems through the use of examples, or rules, or a combination of both. They found that the combination of both examples and rules was most successful. Unfortunately, all of these studies suffer from the use of routine textbook exercises, often single step problems, and so tell us little about the place of schema acquisition in non-routine problem solving. More interesting are studies carried out on problem classification. Krutetskii's [1976] long term studies carried out in the Soviet Union showed that capable students seemed to grasp the pattern of a problem whole, while average students were able to classify problems into types "only after appropriate analytic-synthetic orienting activity." [p.232] Similar results were obtained by Silver [1979], Schoenfeld and Herrmann [1982], Gliner [1989] and Ross [1989], all of whom showed that experts classified problems based on the underlying mathematical structure, while novices tended to classify them by surface structure. Schoenfeld and Herrmann additionally showed that, after a course in problem solving, students' classifications were closer to those of the experts.

Metacognition

Since about 1960, the phenomenon of consciousness has been gaining favour with researchers and theorists, and more recently there has been an increased interest in the consciousness of consciousness and, with that, an interest in what is generally called metacognition [Kilpatrick 1985]. The term metacognition, however, has various interpretations, even within the scope of mathematics education. Schoenfeld [1987b] identified three categories of behaviour that are seen as within the scope of metacognition; Knowledge of one's own thought processes, control and regulation of one's thought processes, and beliefs and intuition. By grouping knowledge and belief together, Garofalo and Lester [1985] developed two categories which contain the same phenomena as

Schoenfeld's three. They further divided metacognitive knowledge and beliefs into knowledge and beliefs about person (oneself and other), about the task at hand (its scope and requirements), and about strategies. Metacognitive regulation includes planning one's course of action, evaluating outcomes, and monitoring the implementation of all of these. Garofalo, Lester, and Schoenfeld all emphasized that beliefs have a strong influence upon what knowledge is used and what control enacted. Other researchers [McLeod 1989 & Campione, Brown and Connell 1988] separated beliefs from metacognition and so considered only two categories of metacognitive behaviour, knowledge of cognition and executive or regulatory processes. Campione, Brown and Connell saw the first category as including "conscious and stable knowledge about cognition, about themselves as learners, about the resources they have available to them and about the structure of knowledge in the domains in which they work" [p.94], while the second category included self-regulation, monitoring and organization. Whether beliefs are considered as part of metacognition or not, it is clear that beliefs are very important and that often it is very difficult to disentangle effects due to beliefs from those due to knowledge or control.

Metacognitive Knowledge

There is little research bearing directly on students' knowledge of their cognition during mathematical problem solving. Broekman and Susÿn-van Zade [1992] gave a puzzling problem to adults to solve and found several different strategies in use. Most subjects, though, had great difficulty in explaining why they chose the strategy they did, and in explaining the methods they used. In their *Agenda for Metacognitive Research in the Next Decade*, Garner and Alexander [1989] placed metacognitive knowledge first.

They suggested that this is very important since self reporting is often used to determine the cognitive activities of adults and children. Fortunato, Hecht, Tittle and Alvarez [1991] suggested another reason for increased research into metacognitive knowledge. They suggest that classroom discussions of strategy choice and task knowledge can be used as an aid in developing students' metacognitive awareness and control.

Regulation

That students are very weak in the area of metacognitive regulation, that is, control and monitoring, has been demonstrated by Garofalo and Lester [1985] and especially by Schoenfeld [1985b, 1987a&b, 1988b, 1989b, 1992]. Schoenfeld videotaped both students and experts as they worked on non-routine problems and then analyzed the resulting protocols. He produced charts which showed how long an individual stayed at each of six levels (read, analyze, explore, plan, implement, and verify) during a session. While students spent almost their entire time at a single level, explore or implement, the experts spent time on all levels and made many more transitions between levels. In particular, they spent more time analyzing, planning, and verifying. While students would often spend an entire problem solving session on a single "wild goose chase", the experts monitored their progress and took corrective action if they did not appear to be making progress after a reasonable length of time. It was often, Schoenfeld concluded, simply this lack of monitoring and control that caused the students not to succeed. Goos and Galbraith [1996] found very similar result in their study of two sixteen year olds working on nonroutine mathematics problems. The biggest limitation of all these studies is that the subjects were almost all very able mathematics students. One would expect, though, that the results could only be worse if less able students were studied.

Beliefs

Student's metacognitive skills are poor, Schoenfeld said. "Their perception is that their minds are essentially autonomous with regard to problem solving: they just do 'what comes to mind'." [1985a p. 372] And so the students' beliefs about their own minds are seen to be very important to their problem solving behaviour. While contextual factors, control, beliefs, attitudes and affect all interact during problem solving, Lester, Garofalo and Kroll [1989] conjectured that beliefs may play the dominant, even overpowering role. Beliefs about the task at hand, about mathematics itself, about schooling, about oneself and one's relationship to each of these, all affect how one approaches a mathematical problem, and what cognitive and metacognitive resources one makes use of. Students, Schoenfeld asserted, develop their beliefs about mathematics from their experiences in the classroom and these beliefs have a powerful influence on their behaviour [1992]. The beliefs students learn in the classroom are often very negative. Typical of these are: Math problems have one and only one right answer; there is only one correct way to solve any math problem; ordinary students shouldn't expect to understand math, rather they should memorize; mathematics is done alone; assigned problems can be solved in five minutes or less by any student who has studied the material; school math has nothing to do with the rest of the world; proof is irrelevant to discovery and invention [p.359]. Davis [1989] asserted that the typical student's understanding of his or her job as a student was just as negative: They are to come to school, come on time, be quiet, do what they are told, do it in the way they are told to do it, and stay out of trouble. While Schoenfeld's list described students' beliefs about the content to be learned, Davis' list described students' beliefs about the relationship between themselves, the content to be learned, and the teacher. Students' beliefs about themselves can also be crucially important. McLeod [1985] linked such beliefs with metacognitive control, stating that, "(o)ne's locus of control, then, is a system of beliefs about whether the rewards and successes of life ... can be attributed to causes that are internal or external." [p.275] This is confirmed by Dweck's study of motivation [1986]. She contrasted the entity theory of intelligence with the incremental theory and showed that belief in the former led to performance goals rather than learning goals. Learning goals led to a mastery orientation, while performance goals could lead to avoidance of challenge, low persistence and learned helplessness. All these beliefs, about mathematics, about the classroom, and about themselves, shape the students' problem solving behaviour. To the extent that beliefs are learned in the classroom, it is only through change in classroom practice that a change in beliefs will come about.

CLASSROOM CULTURE

Schoenfeld [1988a] reported on a well taught grade ten geometry class. The class was well organized, the presentation was clear, and the students did well on standardized tests. However, the students developed a fragmented view of the subject matter and perspectives regarding mathematics itself that were likely to impede their future mathematical growth. Elsholz and Elsholz (1989) reported on a kindergarten pupil who had already learned that the classroom has different rules than the rest of the world. This child had learned that when you divide five items between 2 children, each receives two and a half. While this worked well for cookies, his partner was quite dismayed when he cut the fifth balloon in half. These two examples are part of an emerging trend in research that views mathematics learning as inherently social, and places the cognitive and metacognitive processes solidly in a context. Cobb [1986] asserted that all "cognition is necessarily contextually bounded." [p.2] Actions that may seem irrational (cutting a perfectly good balloon in half) usually turn out to be rational when considered in their full context.

The view that cognition is a social phenomenon leads to a view of education as socialization rather than instruction. This view of mathematics education leads to classrooms where there is discussion and debate, socially shared problem solving, and a shift from presentation to discovery and from product to process. This, Resnick [1988] called "teaching mathematics as an ill structured discipline" and at its heart "lies the proposal that talk about mathematical ideas should become a much more central part of students' mathematics experience than it is now." [p.53] Lampert's grade five mathematics class contained just these kinds of debates and discussions [1990]. Students were not told how to solve problems and were expected to answer questions about their assumptions and strategies. Problems were used to engage students in making conjectures and testing those conjectures. Lampert made the comparison between how mathematics is experienced by students and how it is known by mathematicians and asserted that central to this comparison is intellectual authority. In her class, authority shifted around and was shared through the centrality of debates and discussions.

Apprenticeship among tailors in Liberia served Lave, Smith and Butler [1988] as a model for cognitive apprenticeship, which focused on day by day engagement in learning and doing. The strength of the apprenticeship model is its view of learning as a process where the line between teaching and content disappears. Brown, Collins and Duguid [1989], basing their ideas on the work of Vygotsky, Leontiev and others, argued that "(t)he activity in which knowledge is developed and deployed ... is not separable from or ancillary to learning and cognition. Nor is it neutral. Rather, it is an integral part of what is learned." [p.32] This theory they called "situated cognition" and they linked it with the educational approach of cognitive apprenticeship. They saw cognitive apprenticeship as enculturating students through activity and social interaction. Learning, they asserted, "advances through collaborative social interaction and the social construction of knowledge." [p.40]

Teaching mathematics as an ill structured discipline, cognitive apprenticeship, and situated cognition are all part of an emerging trend (for example, Alibert 1988, Baxter 1993, The Cognition and Technology Group at Vanderbilt 1990, Davis 1989, 1987, Rogers 1990) that draws upon constructivism and the work of Vygotsky. Mathematics education is viewed as a complex whole in which content cannot be separated from teaching, and learning is seen as social, interactive, and constructive rather than absorptive. The emphasis is on activity, on doing mathematics. Davis [1989] called this "experiential education" and said that it is more effective because learning a culture is more important than learning dead facts. A major theme that appears in these studies is a linkage between epistemology and pedagogy. Mathematics itself is seen as a collaborative, sense making activity and, from this, it follows that the mathematics classroom should reflect this view.

SMALL GROUP PROCESSES

Collaborative activities emerge as a major theme in the studies discussed in the last section. Collaboration is seen as natural to mathematics and, therefore, as crucial to mathematics education. Schoenfeld [1989b] believes that small groups are the point where students enter the world of mathematical discourse; the point where they begin to enter the community of mathematicians. In practice, collaboration in the classroom often appears in the form of small groups working together on a problem or a project, or group members helping each other while working on individual worksheets. There are three major questions to be addressed abou' collaborative small group processes in the classroom: What are the outcomes desired, why should small groups be used to achieve these outcomes, and how can collaborative work be structured to achieve these outcomes?

There are several non-academic reasons for promoting small group work. Cohen [1994] cited cooperative learning as a strategy to promote positive social behaviour and interracial acceptance as well as a way to manage heterogeneity in diverse classrooms. Sapon-Shevin and Schniedewind [1990] said that communicating, sharing, and finding common goals are central values in education, which can be realized through cooperative learning.

However, increased academic achievement is the most cited reason for the use of small groups in mathematics education. There is much evidence that small groups can increase achievement, especially, but not entirely, with regard to basic skills. [Davidson 1985, Dees 1991, Good, Mulryan and McCaslin 1992, Hart 1993, Johnson & Johnson 1985, Slavin 1989/90, Treisman 1992]. Kromrey and Purdom [1995] asserted that cooperative learning allows for active and meaningful learning and promotes long term retention. Noddings [1989] identified two general academic purposes for the use of small groups in mathematics education: to strengthen learning outcomes, especially basic skills, and to contribute to the development of higher order thinking. The conflict in purposes is based on philosophical differences, with those citing the first pointing to the motivational structure of small groups and those citing the latter approaching small group processes from a Dewey-Vygotskian perspective.

Foremost among current researchers in the first group is Slavin, who calls his method STAD or Student teams-achievement division [1987]. His small, mixed ability groups work as teams, competing for points and recognition with other teams in the class, in such away that even lower ability students can contribute to their team. He has reported many positive results for achievement gain in basic skills [Slavin 1987, 1989/90] and he noted that this achievement depends upon the existence of both group goals and individual accountability.

Many researchers in the second philosophical group reject Slavin's team model due to the competition inherent in it and due to their commitment to the development of higher order thinking [Good, Mulryan and McCaslin 1992]. Collaborative groups, rather than competitive teams, were the subject of a study by Phelps and Damen [1989] of grade four mathematics students. They found significantly greater gains in understanding of basic concepts amongst collaborative pairs. This was in contrast to simple skill achievement where they found the collaboration ineffective. Much of the theory used to explain collaborative small group processes is based on the work of Vygotsky, Luria and Leontiev [Schoenfeld 1987b & Good Mulrayan and McCaslin 1992]. In Vygotsky's psychology, the individual and the social are seen as interactive elements of a single system. By working in collaboration with a peer or a teacher, a learner may be able to function at a higher level than he or she could achieve working alone. This level, above the student's actual development but where he or she is able to function, Vygotsky calls the "zone of proximal development" and it is here that higher order thinking is learned [Cole 1985].

Four processes, identified by Good, Mulryan and McCaslin, [1992] that might account for the success of small groups in enhancing higher level thinking are: (1) The exchange of reasoning strategies, (2) constructive controversy, (3) the need to verbalize one's cognitive processes, (4) the encouragement of one's peers. Bossert [1988/89] also noted four factors that could account for the success of cooperative methods: (1) Stimulation of higher order thinking, (2) constructive controversy promoting problem solving skills, (3) increased opportunities to rehearse information orally, (4) peer encouragement and involvement leading to friendship, acceptance and an increase in cognitive processing skills. Rosenthal's [1995] study of advanced university mathematics classes led him also to four ways in which small groups could prove beneficial: (1)

Students are better able to learn and retain concepts when they are actively involved, (2) students can learn from each other and from teaching each other, (3) students get practice in working and communicating with others, (4) students sense a warmer, more welcoming and more caring atmosphere. This last factor, he said, may be especially helpful to women students.

Dees [1985] asserted that the benefits of cooperative work in complex tasks such as concept learning and problem solving may derive from three factors: (1) Working cooperatively forces students to attend to the problem at hand, (2) discussing the problem leads to clarification for both the speaker and listener, (3) working together increases confidence. Later [1991], she conjectured that it is dealing with controversy that may be responsible for improvement in higher level thinking. The cognitive rehearsal of one's own position and the attempt to understand others' positions may result in a high level of mastery. However, she notes [1985] that students need instruction on how to work together.

Noddings [1985] postulated three factors that may be important in small group processes: (1) When students encounter challenge and disbelief this may lead them to examine their own beliefs and strategies more closely, (2) the collective may supply background information that the individual may not have, (3) students, in taking charge of their own actions, may internalize orderly approaches to problematic situations. Phelps and Damon [1989] conjectured that the major reason for the success of their pairs of fourth grade students in learning basic concepts was due in great measure to the necessity for partners to "publicly recapitulate their own emerging understanding of the task" forcing them to "bring to consciousness the ideas that they are just beginning to grasp intuitively." [p.645] Webb [1991] studied verbal interactions in small groups studying mathematics. She conjectured three features of optimal group work that make it potentially effective for learning mathematics: (1) Immediate feedback and explanation, (2) the use of language that fellow students understand, and (3) a shared understanding of difficulties. Stacey [1992] cited three possible reasons for use of small groups in problem solving: (1) The opportunity for pooling ideas, (2) the need to explain and express ideas clearly, (3) the reduction of anxiety.

Thus a summary list of important mediating factors that might account for the success of small groups is:

1. An increased focus on the task at hand.
2. Increased opportunities to rehearse information orally leading to greater integration of the information.
3. Constructive controversy, in which students encounter challenge and disbelief, in which they challenge others and then use discussion to examine beliefs and strategies more closely.
4. The pooling of ideas and strategies and background information.
5. Reduction of anxiety and corresponding increase in confidence.
6. Encouragement from peers, a warmer, welcoming and supportive atmosphere.

These factors are still in the nature of conjectures, as the internal workings of small groups are still not well understood. This is partly due to the focus of much research being, until recently, on the product, achievement, rather than on the process, and it is partly due to the complexity of the factors involved. However there have been findings of interest with regard to factors 1, 2, and 3. Webb [1991] reviewed and analyzed research regarding verbal interactions in small groups in mathematics classrooms. She found that the most consistent indicator of success is the giving of detailed explanations. The mechanism accounting for this may, she asserted, be that the helper must clarify and organize his or her thinking, often giving explanations in new or different ways. Webb also found that, while off-task discussion was negatively correlated with achievement, a simple count of interactions was not a good predictor. However, Cohen [1994] found that a count of interactions was a good predictor of achievement. The difference in the two results may be due to the different nature of the tasks involved. Cohen's were inherently group tasks while Webb's tasks could have been accomplished individually.

Zook and Di Vesta [1989] studied students of an educational psychology class working at mathematics problems. Those who were required to supply overt verbalization before making decisions required more time but made fewer errors and worked forward on more problems. These studies indicate that rehearsal is a factor in the success of groups. Studies of constructive controversy are rarer. Smith, Johnson, and Johnson [1981] compared grade six students studying controversial social subjects in conditions requiring controversy, consensus, or individual thought and found that the controversy condition led to higher achievement and retention, a greater search for information and a more accurate understanding of two perspectives.

The underlying mechanisms that could lead to the success of small group collaboration are clearly very complex, involving not only cognitive and metacognitive factors but psychological and social factors as well. It is not surprising, then, that while results have been generally positive, there have been some mixed results.

Good, Reys, Grouws and Mulryan [1990], through classroom observation, found that small, mixed ability work groups displayed both strengths and weaknesses. Active learning, interesting activities, and an enhanced opportunity for mathematical thinking contrasted with curriculum discontinuity, inadequate pacing and student passivity. Cooperative group work is usually expected to increase the engagement of students. However, as Salomon and Globerson [1989] report, this is not always the case. In small reading/writing work groups they found several negative social-psychological effects. *Free riders* are less able members of a group who leave the work to the more able. *The sucker effect* takes place when a more able member of a group puts in less work in order that others not take advantage of him or her. In *the status differential effect* we see higher status members dominating the group. *Ganging up on the task* involves expending effort to avoid actually doing the task. They note that research in this area is scant so that empirically based recommendations are not available.

Stacey [1992] also found negative effects from group work. In a written test of problem solving she noted that pairs and triples did no better than individuals. In order to try to uncover the reasons for this she videotaped small groups of seventh, eighth and ninth grade students solving three non-routine problems. All groups had produced many possible strategies and all but one produced at least one correct strategy. However, in many cases a correct strategy was bypassed in favour of a simpler but erroneous one. Chosen strategies were usually easier to carry out and easier to understand than those not chosen. Groups that persisted with an incorrect strategy showed a marked absence of checking behaviour. These observations seem to indicate that collaborative groups cannot be counted on to provide external monitoring and control while a student is learning to internalize this behaviour.

Although discussion is seen as central to cooperative small group work, some researchers have been disappointed in the level of discourse observed within small groups working on mathematical tasks. Cohen [1994] found that if students are not taught differently they operate on the most concrete level. Pirie and Schwarzenberger [1988] defined mathematical discussion as purposeful talk on a mathematical subject in which there are genuine pupil contributions and interactions. However in a longitudinal study of classroom discussion they found few instances which fitted their definition. More often one pupil would talk while others showed no signs of reaction so that there was no real interaction. In other cases, the goals of the talk were not well defined, so that while it was interactive, it was not truly purposeful.

In her 1994 conceptual review of small group research, Cohen [Cohen, 1994] proposed that the variability of results suggests that the theoretical advantages of cooperative learning may actually only be obtained under certain conditions. She reviewed cooperative learning in general, not specifically in the context of mathematics education, and she focused on the character of interactions and their relationship to achievement. The problem of motivating members to work as a group was seen to be of critical importance and might be addressed by including both goal and resource interdependence. Goal interdependence exists when each student can only achieve his or her goal if all other members also achieve their goals. Slavin's [1987] STAD is a good example of this. Resource independence exists when a student can only achieve his or her goal if others provide needed resources. Jigsaw is an example of resource interdependence [Aronson et al, 1978]. However, these two factors might not always result in the sought for interaction. The type of task and amount and type of structuring of the interaction is also critical. Cohen found that a key distinction is whether the task is a true group task or a problem that could as easily be done by an individual. As well, she found that for relatively low level outcomes a limited and highly structured interaction was adequate and often superior, while for higher order thinking skills the interaction must be less constrained and more elaborate.

Another factor in the performance of small groups is the makeup of the groups. Here both social and cognitive aspects must be considered. Peterson, Janicki, and Swing [1981] and Swing and Peterson [1982] found that, in mixed ability groups, both high and low ability students benefitted from time spent explaining, while medium ability students in these groups spent less time explaining and benefited less. Webb's findings [1982, 1991] support these results, indicating that medium ability students may do better in homogeneous groups while high and low ability students do better in heterogeneous groups. In her review of the research, Noddings [1989] noted that most researchers using small groups in mathematics have chosen mixed ability groups to study, and she speculates that both high and low ability students might perform well in homogeneous groups if the tasks were designed appropriately. Webb [1991] found that in mixed ability groups with a smaller range of abilities, for example, medium and high ability or medium and low ability, all students tended to be active participants. Perceived status of the individuals in a group also affects the interactions within the group. Cohen [1984, 1994] found that differences in perceived expertise, in attractiveness or popularity, and in race all affect the nature and extent of interactions within a group. Webb [1991] noted that there is little research that examines the role of personality factors in mathematics groups. She found mixed results with respect to gender. In mixed groups with equal numbers of boys and girls, they did not differ in their interactions but in groups with a majority of boys, girls were less likely than boys to receive answers to their questions. In groups with a majority of girls, the girls tended to direct questions to the boy who often ignored their requests. Boys outperformed girls in both unbalanced groups but not in the balanced group. Hoyles said that group work cannot be seen as a panacea, as their "effects may depend on so many elusive and subtle conditions" [1985, p.212].

Noddings [1985] has said that, in theory, cooperative small groups provide a learning environment that is useful for developing problem solving abilities. However, despite much research evidence supporting the use of small groups, there is still much that is not well understood about what happens when small groups are used to facilitate mathematical problem solving. Indeed, Good, Mulryan and McCaslin state that their major task, in their analysis of research on small group process in mathematics [1992], is to argue for more research, especially process-oriented and interview research. [p.167-168]

OPERATIONAL AND STRUCTURAL UNDERSTANDING

Anna Sfard has, for several years, been developing a theoretical framework that involves a duality of understanding in mathematics. She believes that mathematical notions can be conceived of in two complementary ways: structurally, as objects; and operationally, as processes. Applying Sfard's framework to the actions of the students helps to make understandable their approaches to the problems and their choices of strategies. Abstract mathematical notions, Sfard argues [Sfard, 1991], can be conceived of in two distinct but complementary ways. They can be conceived of structurally; that is as objects that can somehow be "seen" in the mind's eye and manipulated as wholes. They can also be conceived of operationally; as processes, or as algorithms, operations, or actions. Thus, a function can be seen structurally as a set of ordered pairs or operationally, as a computational process. Structural conceptions can be characterized as static, instantaneous, and integrative while operational conceptions are dynamic, sequential, and detailed. While Sfard acknowledges that the division of mathematical concepts into two categories similar to hers is not new [see, for example Hiebert and Lefevre, 1986], she notes that her theory is different in two important ways: firstly, it combines both philosophical and psychological perspectives and, secondly, it is conceived as a complementary duality rather than as a dichotomy [Sfard, 1991, pp. 7-9].

While she asserts that her two ways of understanding mathematical concepts are not opposed and are in fact "inseparable, though dramatically different, facets of the same thing" [1991, p. 9], she sees operational conceptions as generally preceding structural ones. This is a particulary important point which she justifies both historically and psychologically. The transition from an operational understanding of a concept to a structural one involves three stages [1991, p. 18-19] and can be lengthy, painful and uncertain. The first of these stages is *interiorization*, in which the learner becomes acquainted with a process which will eventually lead to the new concept. The second stage is *condensation*, in which lengthy processes are squeezed into more manageable units. The learner becomes more able to think of the process as a whole and perhaps combine it with other processes within a larger procedure. The final stage is *reification*, where the learner is finally able to see the notion as an object. This is an ontological shift and is generally a sudden leap. This difficult transition, from operational to structural understanding, needs to happen over and over again during the learning of mathematics. As procedures become reified they become the objects of another set of procedures, which then will also need to be reified. It is these transitions in the process of understanding which may prove to be of crucial importance. Freudenthal [1991, p.98] refers to the importance of the discontinuities in the learning process; the jumps where the operational matter on one level becomes the subject matter of a higher level.

Pirie and Kieren [1992] have criticized Sfard's model on the grounds that she portrays the growth of understanding as linear. They have proposed a model of concept formation [1989, 1992] which involves eight levels and is essentially recursive, with the learner moving backward as well as forward between the levels. While their model offers a detailed picture of the growth of understanding, I will not undertake a full review of it here since the present study is not centrally concerned with how understanding comes about but rather involves the structure of the students' understanding.

Sfard asserts that the idea of duality, as opposed to dichotomy, is central to her model. While theoretically it would be possible to approach all of mathematics operationally, structural understanding has great advantages over operational in "that it is more integrative, more economical, and manipulable, more amenable to holistic treatment." [Sfard, 1994, p.53] Operational conceptions are sequential and each step in the procedure must be remembered in correct order if the procedure is to be carried out. This can create a very heavy cognitive load and can lead to a feeling of only local understanding. Structural conceptions, on the other hand, are holistic, can be understood in terms of metaphors or visual images [1994, p. 53], and allow for parallel, rather than sequential, processing. Fuzzy images can be unfolded to reveal the details when they are needed. This allows for a more global understanding.

Sfard is careful to note that, although it appears that operational understanding is more easily acquired than structural, and that structural understanding offers important advantages, these conceptions are really complementary, and both are necessary parts of mathematical knowledge. She says, "almost any mathematical activity may be seen as an intricate interplay between the operational and structural versions of the same mathematical ideas: when a complex problem is being tackled, the solver would repeatedly switch from one approach to the other in order to use his knowledge as proficiently as possible." [Sfard, 1991, p.28] There are also times when a structural conception may precede its operational counterpart. Geometric ideas, Sfard notes, may be an example of this. The visual image of a circle will certainly precede the operational idea, that is, the algorithm for creating a circle. [Sfard 1991, p.10] It may also be that professional mathematicians are able to read definitions and reify the concepts defined without the interiorization and condensation phases [Sfard, 1994]. What does appear clear, from her work and the investigations of others, is that, for school and college mathematics students, a procedural understanding, especially with regard to algebra, is primary and the transition to a structural understanding is inherently difficult and problematic.

Carolyn Kieran, in her survey of research on the teaching and learning of school algebra, [Kieran, 1992] uses Sfard's structural-procedural duality as an organizing theme. The major problematic transition she considers is that from arithmetic to algebra, which requires that the student learn to operate on algebraic expressions rather than on numbers. "Until a student is able to conceive of an algebraic expression as a mathematical object rather than a process, algebraic manipulation can be a source of conflict." [Kieran, 1992, p.393] This is confirmed by Herscovics and Linchevski who, in a study of grade seven students, explored an important cognitive gap between arithmetic and algebra. They characterized it as the "inability to operate with or on the unknown" [Herscovics and Linchevski, 1994, p.75]. Lee and Wheeler's [1989] study of grade 10 students points to a dissociation between algebra and arithmetic which suggests that the students lack a structural conception of algebra. While the students do see algebra as different from arithmetic, they appear to view it simply as a set of procedures to be carried out on letters rather than numbers. That the student's versions of these rules are sometimes different from those they know for arithmetic, clearly indicates that they do not see algebra as generalized arithmetic. A structural conception of algebra is particularly important in problem solving, where the construction of equations involves the ability to represent numerical relationships symbolicly. As part of the transition to algebra, students must make the change from an arithmetic to an algebraic approach to solving word problems. The arithmetic approach involves "working backwards using a linear, sequential approach involving a string of inverse operations" [Kieran, 1992, p.393], while an algebraic approach requires the student to think in terms of forward operations and of relationships amongst numerical quantities. Lesh, Post, and Behr describe this as the need to first describe and then calculate [Lesh, Post and Behr, 1987, p.657].

SUMMARY

This review of the research literature has led from problem solving as a cognitive process, through metacognition and belief systems, to a view of mathematics education, and so also of mathematical problem solving, as embedded in its social context. This is a view in which the content, in this case problem solving, cannot be separated from the way in which it is taught. An increasingly common way for problem solving to be taught is through collaborative groups. Both of these themes, problem solving and small group processes, have been extensively researched. However, the process of learning to solve non-routine mathematical problems is still not well understood. As more of the factors involved are studied, the complexity of the issues involved becomes apparent. Small group processes, which explicitly involve social and psychological factors, are even less well understood. Although results of studies of small group interactions seem generally positive, the literature has shown that there can be negative effects as well. The concepts of operational and structural understanding may contribute to an understanding of the problem solving process for both individuals and pairs.

CHAPTER III METHODOLOGY

METHODOLOGY

Schoenfeld has said that "any particular approach to studying intellectual behavior is likely to illuminate some aspects of that behavior, to obscure other aspects, and to distort some beyond recognition." [Schoenfeld, 1985b, p.283] Thus, the choice of methodology will depend upon the behaviour to be studied and the intended focus of the study. The field of mathematics education, lying as it does at the crossroads of many established fields, encompasses a wide variety of research methodologies. Although research methodologies can be adopted from disciplines as diverse as psychology, sociology, epistemology and cognitive science, mathematics education does have its own aims and its own specific problems. Howe and Eisenhart state that mathematics education is a field of study rather than a discipline and as such adopts methods from overlapping disciplines. Thus, methodologies multiply and, in the end, "a methodology must be judged by how well it informs research purposes." [Howe and Eisenhart, 1990, pp.4-5]

The major division in methodologies is between quantitative and qualitative methods. Each method uses different techniques, is based on a different paradigm and holds different assumptions about the world. Four major differences have been identified by Firestone [1987]. (1) Assumptions about the world differ, with quantitative research accepting a positivist philosophy, while qualitative research embraces a phenomenological paradigm. (2) Purposes are different, with quantitative research seeking to explain changes primarily through objective measurement, while qualitative research is concerned with understanding from the actors' perspective. (3) The approach to research differs, with quantitative research being usually experimental or correlational, while the prototypical qualitative study is an ethnographic study. (4) The role of the researcher differs. Quantitative researchers generally aim at detachment, while the qualitative researcher may become immersed in the study. [Firestone, 1987] These two major research methodologies are not opposed but rather, give different kinds of information and can be used together to give a more complete picture of the phenomena under study. While a quantitative study will assess the magnitude of a relationship or a change more precisely, qualitative studies are stronger in depiction of detail, in description of detail, and in attention to the point of view of those being studied [Firestone, 1987].

In the present study, it is the process of finding a solution to a problem that is under investigation. This gives rise to a qualitative approach since it involves detailed, close up observation of the process, rather than the final product. Krutetskii [1976] found that qualitative study was particularly useful in studying students' individual differences in the process of problem solving. Marshall and Rossman [1989] assert that qualitative research methods are appropriate when the research questions are exploratory, explanatory or descriptive. In the present study the questions are exploratory and descriptive, and so the qualitative paradigm is appropriate.

The present study relies heavily on verbal reports as data. Genest and Turk [1981] identified four methods of obtaining verbal reports: The continuous monologue, often electronically recorded; random sampling of thoughts, often in response to a signal or bell; event recording, in which the subject is asked to report whenever a particular cognitive event takes place; and various reconstructive procedures. Ericson and Simon [1980] developed a three part classification. The first category is talk/think aloud, in which the subject is asked to report everything they are thinking at the same time as they carry out a task. The second category is concurrent probing in which the subject is asked to report on specific aspects of their cognition. This requires intermediate processing such as scanning or analyzing before reporting. The third category is retrospective probing, in which the subject is asked to recall cognitive events. The present study used primarily think aloud methodology.

I have chosen to adopt a non-interventionist strategy. In order to document the whole problem solving process in as naturalistic a way as possible, it was necessary to allow each session to proceed without intervention. I wished to avoid any training effect that interviewer comments or questions might have had on the session, or on future sessions. That this non-interventionist strategy does have limitations is pointed out by Schoenfeld [1985b]. At times it may serve simply to document phenomena without shedding light on their workings. However, choosing a methodology is a matter of weighing trade offs and in the present case non interference is required in order to investigate the independent problem solving processes of the students. I wish to investigate what they do without guidance or assistance, to see what strategies they choose and what skills they bring to bear on the problem. Any intervention by an interviewer could easily redirect the attention of the student and so disrupt the problem solving process.

The use of verbal reports as data is not new. Introspection and retrospection have a long history but by the middle of this century they came into disrepute as they were shown to be unreliable. At the other end of the spectrum was behavioursim, which tried to be scientifically pure. By the 1970's the dominance of behavioursim was waning as protocol analysis proved useful in artificial intelligence research and as Piaget's work showed that the clinical interview could provide reliable and interesting results. The work of Soviet researchers also began to make an impact. The limitations of pure empirical studies became apparent and exploratory methodologies, such as the clinical interview and think-aloud protocols became commoner. [Schoenfeld, 1985b, Ginsburg et al, 1983] When one's interest is in eliciting cognitive activities in an unbiased fashion, a naturalistic form of enquiry would seem ideal. However, as Ginsburg et al [1983] point out, "naturalistic observation is usually not practical as a technique and must be replaced by the protocol methods." [p. 17] Standardized testing is also of limited value when the aims of research are exploration and description of a complex phenomenon such as problem solving.

However, there are several limitations that must be considered with regard to using verbal reports as data; reactivity of the subject to the experimental environment, incompleteness of verbal reports, inconsistency with other observations, idiosyncrasy, and the influence of researcher bias on interpretations [Ericson and Simon, 1980, Genest and Turk, 1981]. Reactivity will arise when the research setting is essentially atypical for the subjects. Schoenfeld [1985b] sought to reduce reactivity by having his subjects work in pairs, where dialogue is more natural and performance stress is lowered. However, for the present study, this presents a difficulty; we wish to compare the problem solving process when students work alone with the process when they work in pairs. However, in their review, Ericson and Simon [1980] assert that appropriate types of talk aloud instructions, specifically asking for verbalization without explanation, do not seem to interfere with performance. Asking the subjects to verbalize their reasoning did have an effect; better solutions to problems were obtained but more time was taken. Thus, reactivity, in the present study, although not eliminated, was reduced by careful instructions.

Incompleteness of think aloud verbal reports can stem from two sources; unavailability of information to the subject, and failure to report all information. For example, processes which are so often repeated as to become automated are less often and less fully reported, and heavy cognitive loads produce less, or less complete, verbalization [Ericson and Simon, 1980]. While the incompleteness of reports may make some information unavailable, it does not invalidate the information which is obtained.

Ericson and Simon [1980] reported that the inconsistency of verbal reports with other data can stem from two sources: The retrieval of information which, while not identical to the information sought, is related to it; and the generalization and filling out of incomplete memories. Furthermore, they asserted, this inconsistency is not generally found in concurrent reporting. If verbal reports are accompanied by other reports of behaviour, it becomes possible to check the consistency of the reports with the other data sources. In the present study information is obtained from several distinct sources; verbatim transcripts of think aloud problem sessions, the written work produced during these sessions, longer term workbook problems and informal exit interviews.

The last two limitations are less significant. While individual verbal reports will be idiosyncratic, multiple subjects can be used to lessen the importance of this factor. In this study there were 14 subjects working on a variety of problems. Researcher subjectivity may be present in any form of research, and the significance of the data will need to be judged with respect to the researcher's implicit or explicit theoretical assumptions.

Schoenfeld has noted that evidence from think-aloud protocols may reasonably be considered suspect, serving to illustrate a perspective but perhaps not to document it in a rigorous fashion [1985b]. Schoenfeld states that, "Issues regarding the validity and generality of verbal methods are, however, singularly complex and subtle." [1985c, p.174] Any method of gathering and analyzing verbal data will illuminate some aspects of the problem solving process and obscure others. There are trade offs to be made. Ginsburg, Kossan, Swartz and Swanson [1983] concur that there are serious questions to be answered if researchers are to accept verbal protocol methods (both think-aloud sessions and clinical interviews) as legitimate research methodologies. However, they say, "the fact remains that, over a wide range of conditions and situations, people are reasonably good at telling what they believe, want, and expect." [pp. 26-27] They believe that it is reasonable to rely on subjects' reporting of some of their cognitive processes. Thus, they "believe that introspective reports can provide useful information and protocol methods have a place in research." [p.27]

Nevertheless, Ginsburg et al [1983] do identify some issues of concern. The first is that only in those domains to which the subjects have access can their reports of mental states and processes be expected to be accurate. However, Ginsburg et al report that it is an empirical fact that subjects can accurately report on aspects of their activities in some areas, and mathematics is the example they use. The second issue they identify is selectivity. Subjects must select a level or aspect of the phenomenon to describe. "What a subject reports will always involve selectivity and interpretation. Introspective descriptions are not representations of an unconceptualized mental given, but, of necessity, reflect the subject's skills and habits of categorization." [p. 29] However, they point out, there is an unbounded number of descriptions; a complete characterization is thus not possible with any methodology. Researchers who use a non-interventionist think aloud procedure must be especially careful in that they must rely on the context and task structure to inform the subjects of the level of report expected. Another issue is that of report interference, that is, the concern that reporting on mental states and process might in itself change those states or processes. This has been at least partially addressed by Ericson and Simon [1980] as discussed previously. Ginsburg et al assert that if we allow for "the possibility of error, there seems to be no reason to reject all process reports out of hand." [pp.30-31] Ambiguity is a fourth issue of concern. Potential ambiguity of subjects' responses can be a feature of protocol methods. Non-interventionist techniques such as the think aloud process of the present study can be particularly prone to problems of ambiguity. Thus its effects must be taken into account when analyzing the verbal protocols produced. Despite these concerns Ginsburg et al conclude that "[t]o evaluate the fruitfulness of verbal data would be to see what its payoff has been or is likely to be. And in the case of research on mathematical thinking, we believe the payoff has already been significant." [p.35]

Qualitative methodology is an appropriate methodology for the present study with its aims of discovery and description of complex phenomena. The particular choice of a noninterventionist, think aloud problem session is justified by the nature of the phenomena being studied.

PILOT STUDIES

To investigate the possibilities and limitations of the intended study, its setting, the interview procedures, and the appropriateness of the problems to be used, two pilot studies were conducted prior to the main study.

The first pilot study

The first pilot study was conducted in the fall semester of 1993. I was teaching a Math 190 course, Mathematics for Elementary School Teachers, at Kwantlen College's Richmond campus. As part of their course work, students were asked to work in pairs on an opened ended problem. It was expected that they would work on the problem all semester, keeping a record of their work and their ideas and feelings about it, in a notebook kept for that purpose. At the end of the semester the notebooks were collected to be evaluated and at that time I asked for volunteers who would allow me to use their notebooks as part of my research. Two pairs and one individual (her partner had withdrawn from the course) volunteered.

The purpose of this pilot study was threefold; to see if such long term problem solving could shed light on the problem solving process, to determine if the students would record their thoughts and feelings as well as their work, and to ascertain if the problems were rich enough to elicit substantial work but not so difficult as to be intimidating.

All the volunteers, and indeed everyone in the class, put a substantial amount of effort into solving the problems, and the volunteers all made substantial progress on their problems. The most notable difference between these long term problem sessions and the short, one hour sessions of the second pilot study, was the amount of effort that was put into keeping clear and detailed records of all work done. Notebooks were neat and organized, reflecting, I believe, both the lack of time pressure involved and the necessity for keeping clear records when a problem was to be returned to in a few days or a week's time. Colour coding was used by one pair to help them see patterns in their geometric problem. Another student neatly cut out and pasted in the drawings that were the essence of her solution attempt. One student listed the supplies she thought she would need to do the problem and then carefully listed possible strategies: "The methods we intend to use to come to our solution are: 1, Guess and test, 2, look for a pattern, 3, draw a picture, 4, draw a diagram, 5, use direct reasoning, 6, identify subgoals." The students had been told not to expect to solve their problems quickly, and the problems themselves were open ended so they were approached with a different attitude than were the problems given in the second pilot study. Although the students expressed some frustration, they were generally relaxed, orderly, and willing to follow up an idea about which they were not entirely sure.

Although the students did not write a great deal about their feelings in the notebooks, they did sometimes record their frustration, puzzlement, or disappointment when a idea turned out not to be useful. The student who worked alone began to paste stickers with sayings such as "Yes!" and "I'm proud of you!" into her notebook when she had completed a days work or had come to an interesting result. While none of the students wrote any substantial entries about their feelings, or attempted to analyze how they felt, many did make brief entries about their feelings of frustration or triumph.

All the problems given elicited a substantial amount of work and at least some progress was made by all the students. None of the problems appeared to intimidate the students with its apparent difficulty and none was so easy that the students were able to "complete" it before the semester was over.

I concluded that the long term problem notebooks could supply a perspective on the problem solving procedures of college students that might be different from that seen from short term problem sessions alone. It also appeared that the notebooks might provide some information about the feelings and attitudes of the students as they worked on the problems. Further, the problem set appeared to be appropriate.

The second pilot study

The second pilot study also took place during the fall of 1993. Six students, working at the precalculus level in mathematics at Kwantlen College's Richmond campus, took part. The study consisted of a videotaped problem solving session with each student. There were four main purposes to this pilot study: to test the thinking aloud procedure for the problem sessions, to evaluate the appropriateness of the problems chosen, to determine such details as the best placement of the camera, the size of the work paper and so on, and, finally, to develop a framework for analysis of the sessions.

Each interview lasted approximately 50 minutes and the students were given three problems to attempt to solve during that time. The problems were given one at a time and up to 15 minutes was allowed for each problem. The students were instructed to try to think out loud, that is, to say aloud what they were thinking without explaining what they were doing. They were told to talk aloud as though they were talking to themselves as they worked. They were further told that they could ask me for formulae that they did not remember but, otherwise, they were expected to work on their own. Several sheets of paper, pens, and a calculator were placed on the table.

It quickly became apparent that several changes were necessary in the mechanics of the problem session. In order to be able to follow the videotaped sessions, large sheets of paper needed to be substituted for the smaller sheets and the pens replaced by felt pens in a variety of colours. A ruler was also supplied. Audio taping was added to the video taping in order to insure that quieter voices were recorded, and the interview room was changed to a quieter location. Since all the subjects asked to know if their solutions were correct and what the correct solutions were, I decided to tell them the results at the end of each complete session and also to explain to them how to solve the problems they had been unable to solve. Although in the main study, where students took part in more than one session, this could lead to some training effect, I decided that this reassurance was a necessity as it appeared very important to the students.

None of the students in the pilot study showed any real discomfort with the think aloud procedure. They occasionally had to be reminded to think aloud and one student spoke very quietly, mumbling a great deal of what she said. I found that, if I had them start each session by reading the question aloud, this prompted them to speak as they worked. At first I found it difficult to tell when a student had actually finished with a problem, rather than simply become frustrated. I modified my instructions to include a statement that if they finished before the time was up they were to explicitly tell me that they were done.

Few of the problems were actually solved correctly. One student solved two of three problems, two more solved one each, and the remaining three solved none of their questions. However, none of the problems was so difficult that the students were unable to make any headway, and most elicited a serious solution attempt. Too many problems of the "brain teaser" type had been included. The students recognized them as of this type and then tried to find the "trick" rather than trying to solve the problem. One problem in particular led to misinterpretations and a great deal of consternation. Another problem was dropped as it was very difficult to follow the solution attempts. It involved counting paths through a grid and the students simply traced paths with their fingers. This was very difficult to follow on the videotape. It was decided to include in the main study more problems that required algebraic modelling, as these led to richer problem solving sessions involving the handling of variables and the construction of equations. Since the students in the main study were to be college precalculus and algebra students, these were the type of problems they were studying. It was also decided to included more geometric problems, as these also led to richer problem solving sessions.

One of the main tasks of this second pilot study was to develop a framework for the analysis of the problem session. The framework of Clement and Konold (see chapter two) was developed during their work with remedial college students and so was an appropriate starting place for the present study. All tapes of the pilot study were reviewed with this framework in mind, and it worked well as a starting point but needed to be filled in with more detailed questions. The framework involves both cognitive and noncognitive skills and it divides these into two main categories: stage specific skills and general skills. I retained Cement and Konold's stage specific skills, just adding more specific questions. However, I found that for my purposes, it was necessary to modify the subcategories of the general skills category. The original three categories of generating and evaluating ideas, striving for precision, and monitoring progress were modified to four; strategy selection, precision, monitoring, and belief and affect. I made this modification as I was specifically interested in the strategies that the students used and how they chose them, as well as how their beliefs and emotions affected the problem solving process. From this pilot study I was able to develop specific questions to ask under each category and I used this elaborated framework as a starting point for the main study. (See the main study for the final framework.)

This second pilot study provided me with some insight into the students' problem solving process. The subjects were all volunteers and so I had expected that they would be relatively comfortable with mathematical problems. This turned out to be so. Only one student exhibited a great deal of frustration and this student would not even attempt one of the problems given her. The problem involved deciding how to fold a sheet of paper so as to obtain the box with largest volume. She read the problem and immediately gave up, saying that she had never been able to do problems that involved spatial relationships. Another student became quite frustrated on one particular problem. It was an algebraic problem of a type he had seen in class and so was familiar to him. He knew that he ought to be able to do the problem and was very frustrated when he was unable to remember how. "I can't remember this simple problem and I have a major math test coming up," he said. There was little record keeping by any of the students. They drew diagrams and graphs and wrote down calculations and final answers but made no attempt at systematically recording their work. There was also a noticeable lack of planning. Schoenfeld has noted this in his studies of problem solving activities by more advanced university students [1985b] and so this was expected. "Now, why have I done that?" one student asked after completing an unnecessary calculation. Perhaps the most noticeable characteristic exhibited by these algebra students was their inability to use variables appropriately and to construct algebraic models. Variables were never defined and were often used as a kind of shorthand to translate information from the problem rather than as representing some quantity. One student used letters as subscripts on numbers to indicate where the numbers came from but never saw that she could use a letter to construct an equation which would represent the same process. This lack of fluency in the use of variables led me to include more algebraic problems in the main study.

THE MAIN STUDY

The subjects

Subjects were recruited from Kwantlen College's Richmond and Surrey campuses. Kwantlen College is a two year community college with four campuses serving the southern and eastern suburbs of Vancouver. The Surrey and Richmond campuses offer university transfer courses, business courses, and other two year career programs. While many students come directly from high school, a significant number are mature students returning to school after an absence of up to several years. Many students are married and may have children, and many work full or part time while attending college. A significant number of these students require preparatory or remedial mathematics courses and between one third and one half of the mathematics department's offerings are at this level. The usual sequence of courses at this level begins with Math 092, Fundamental Mathematics, followed by Math 093, Intermediate Algebra, and Math 112, College Mathematics (precalculus), with students entering at different levels depending upon their backgrounds and their results on an assessment test.

It was decided to conduct the study with students at the Math 093 and Math 112 level. Students at the fundamental level, Math 092, were considered unsuitable as their exposure to algebra and geometry was minimal, and so a different set of problems would have been needed. Thus, subjects were recruited by announcements given in Math 093 and Math 112 classes during the spring semester of 1994 (see Appendix A). This was initially done on the Richmond campus only, but when two students from the Surrey campus volunteered they were included also. I also accepted students from Math 190, Mathematics for Elementary School Teachers, and Math 115, Elementary Statistics, as these two courses are considered to be at the same level as Math 112. The announcements emphasized that I was looking for average students rather than just the best students. Subjects were told that they would take part in three video taped problem solving sessions, would complete a longer workbook problem, and would participate in a short interview after all the problem sessions were completed. They were offered a stipend of $10 for each problem session and $20 for completing the whole study, submitting the completed workbook and completing the exit interview. Fourteen subjects were recruited in this manner. (See Appendix B for subject consent forms.)

The problems

Eighteen problem were used for the videotaped problem sessions. It was hoped that all of these should be true problems, in Schoenfeld's sense, that is, problems for which the solver does not possess a more or less complete algorithm. [1985b, p.54] On the other hand, it was also necessary to choose problems for which the subjects did have the necessary algebraic and geometric knowledge and skills. This meant, for example, avoiding the use of trigonometry, as the students in Math 093 were often not introduced to trigonometry until near the end of their course. Also, formal geometry was avoided, as most of the students would have had little, if any, exposure to it. Within this context, I wanted to choose problems that would elicit a wide range of strategies and skills from the participants, but which one might reasonably expect could be completed within the 15 minute time period which was allowed. It was not intended that the subjects should be able to solve all the problems with ease. Rather the problems were intended to be difficult enough to possibly produce some frustration, while not so difficult that the subjects would be unable to make any headway. Most of the problems were at the level of difficulty of the more difficult problems the students might see in their college mathematics text books.

Since I wished to compare the problem solving activities of the pairs and the individuals it was necessary to ensure that the students did not see similar problems in more than one session. If they had the problems would no longer be true problems for them. Thus it was necessary to have a variety of problems. Guided by the results of the second pilot study, I choose eighteen problems in three broad categories, choosing one problem from each category for each session. The three categories were: Familiar problems, generally algebraic in nature, similar to, although generally more difficult than, most of the applications problems the students might see in their text books; problems of geometry and analytic geometry; and unfamiliar problems, generally logic and counting problems of a type most of the students would not have seen before. It was hoped that the familiar problems would focus on the subjects' strategies for constructing algebraic models and their skills in the use of variables. There were two problems of analytic geometry and four focusing on more general geometric ideas. These problems were intended to focus on geometric and spatial reasoning, as well as the use and modification of diagrams. There was also an algebraic component to several of these geometric problems. The unfamiliar problem class was included to ascertain how the subjects would apply their skills to novel situations. The problems were chosen not just to focus on specific strategies but also because they were rich enough to elicit a variety of general problem solving behaviours. They were generally multi-stepped problems that required planning as well as calculation. The complete text of all problems is included in Appendix C.

Problem sessions

Problem sessions were conducted in an interview room on the Richmond campus. Students were seated at a table and furnished with large sheets of paper, felt pens in various colours, a calculator, and a ruler. A video camera was placed across from the subject or behind pairs of subjects and focused on the paper in front of them. An audio tape recorder was also placed on the table.

Each session lasted approximately 50 minutes. Most sessions consisted of three problems and fifteen minutes was allowed for each problem. A few students finished their problems so quickly that they were given a fourth problem. While a fifteen minute time constraint may seem artificial, it is consistent with the situation in which the students generally find themselves in their classrooms. Whether working on problems during class or while wrting an exam, the students usually face relatively rigid time constraints. Thus, fifteen minutes per problem is consistent with what they might expect under classroom conditions.

Individual problem sessions

During individual problem sessions students were seated at the interview table with pens and paper in front of them and the video camera focused on the paper. The students worked on three problems, one from each category, during each session. They were given one problem at a time with up to fifteen minutes to work on that problem. Before the interview began the problems had been divided into the three groups and then one problem had been randomly selected from each group. They were then presented in random order.

The students were instructed to "think aloud" as they worked through the problems. They were told not to explain what they were doing so much as to speak aloud as though they were talking to themselves while they worked. They were asked to begin each problem by reading it aloud and, if they finished before the time was up, to tell me clearly that they were finished with that problem. They were further told that they might ask me for formulae that they did not remember (such as the area of a circle) but were otherwise to work alone. At the end of each complete session subjects were told which problems they had answered correctly and were given a solution outline for any problem they had not solved.

Most students seemed able to follow the think aloud protocol with reasonable comfort, although many had to be reminded at times to think aloud. One, Carl, was extremely nervous during the first problem of his session and he noted that it was affecting his concentration. However, he seemed able to relax after that and did not display any further discomfort. Another student, Candy, found it extremely difficult to work on her own. I remained in the room during the sessions and she repeatedly turned to me for assistance. As a result, it required two 50 minute sessions to obtain 3 independently completed problems. It was planned that each subject would take part in just one individual problem solving session. Candy, however, had two individual sessions since her partners did not turn up at the appointed time.

Paired problem sessions

After all of the individual problem sessions were completed dyads were formed and subjects were asked to work together on problems. The physical set up for these interviews was similar to that for the individual sessions. Subjects were seated, one on the end and one on the side of the table, and the paper, pens, calculator and ruler were place diagonally between them. The camera was again focused on the paper and an audio recorder was also used. They were asked to begin each session by reading the problem and then to think aloud as they worked. No directions were given as to how they should work together, who should read the problem, who should write and so on. This was left entirely up to them.

Problems were chosen by first eliminating any problems that had been attempted by either partner, and then randomly choosing one from within each category. Once again, they were presented in random order.

Most subjects took part in two paired problem sessions, the first held about two weeks after their individual sessions and the second about two weeks later. Pairings were made based on availability for appointment times. No effort was made to match partners by ability, course level, sex or personality. Only two pairs knew each other before the study, one pair only slightly and the other pair were friends. As a result most problem sessions began with my introducing the partners. There were no problems with the think aloud protocol for pairs and, as one might expect, none had to be reminded to think aloud in the more natural situation of talking to a partner.

Workbooks

When they volunteered to be part of the study each subject was given an open ended problem and a notebook in which to record their work. They were given the following written directions:

"You have been given a problem to work on over the next two months. You should plan on working on this problem for about one hour a week, more if you wish. Please do all your work in the workbook and do not erase or tear out anything that you do, even if you later decide that it was not getting you anywhere. Please date each entry that you make in the book. If you do any work on separate pieces of paper, please date them and attach them to the book.

Record not just your working steps but also your guesses and ideas, even if you do not follow them up. Also record your thoughts and feelings about the problem solving process as you go along. Anything that seems at all relevant can be recorded.

The problem you have been given is a complex problem. It may be quite difficult or there may be many steps to it. You may not be able to solve it during the two month time limit. Do your best. If, before the semester is over, you are sure you have solved the problem completely, then attempt to generalize it, to go beyond the original question to other related questions. I expect that you will work on the problem alone. However if you do not understand the problem or find that you are completely stuck you may ask me about it."

No attempt was made to monitor the progress that the subjects were making on the workbooks during the time that they had them. The notebooks were returned to me after the final pair interview, generally during the exit interview. It was hoped that the workbooks would provide a different perspective on the process of problem solving, one in which time constaints did not play a part, and where the problem did not have any single right answer or set finishing point.

Exit Interview

At the end of the study, interviews were conducted with each subject. These audio taped interviews were informal and open ended. Although I had prepared a set of general questions, I allowed the subjects to direct the interview in whatever directions they wished. The questions I had prepared were:

1. Please give me a short history of your study of mathematics. I am interested in how much mathematics you have studied in high school and in college and why you made the choices you did.

2. What have been the most important influences on your attitude towards mathematics and on your achievement in mathematics?

3. What do you think is central to achievement in mathematics?

4. What are your feelings about solving mathematical problems?

5. When studying mathematics, do you usually work alone or with another person or a group? Why?

6. During this study you were asked to work on problems alone and in pairs. Do you have any comments on the similarities or differences between the two experiences?

7. Do you have any additional comments you wish to make?

These questions were used as quidelines only.

METHOD OF DATA ANALYSIS

Marshall and Rossman state that "data analysis is the process of bringing order, structure, and meaning to the mass of collected data." [1989, p.112] The data for this study comes from five sources; the individual task-centred interviews, the paired task centred interviews, the workbooks, the exit interviews, and field notes made during and after each interview.

I began the analysis by creating complete verbatim transcripts of all the problem sessions and of the exit interviews. Then I reviewed the individual and pair video tapes and transcripts with the aid of the analytic framework I had created during the second pilot study. In these early stages of analysis I was simply trying to make sense of the problem sessions; to become familiar with exactly what had happened during each session. At the same time I was assessing and expanding my framework as common themes and patterns emerged. While I kept the same basic outline as I had developed during the pilot study, I refined the questions, making them more detailed and complete, and I added complete sections relating to Sfard's structural/operational duality and to pair interactions. The final analytical framework is as follows.

STAGE SPECIFIC SKILLS

I Comprehension

1. Does the subject view understanding the problem as part of the solution process?

2. Does the subject draw or modify diagrams, where appropriate?

3. Does the subject note the goals and given information, noting all the conditions of the problem?

4. Does the subject differentiate between mathematically relevant and irrelevant details?

5. Does the subject make appropriate or inappropriate assumptions?

II Planning, Assembling and Implementing a Solution

1. Does the subject explore the problem (using examples, extreme cases and so on)? Are the results of the exploration used appropriately?

2. Does the subject make a systematic analysis of the problem, organizing chains of inference?

3. Does the analysis or exploration lead to a plan or directly to a solution?

4. Does the subject create (implicitly or explicitly) a plan? Is the plan appropriate to the problem? Is it carried out? Completely or in part?

5. Does the subject identify goals and subgoals, breaking the problem into parts?

6. Are diagrams used or modified?

7. Does the subject rely on general principles?

8. Does the subject attempt to carry out an algorithm? Is it appropriate? Was it carried out correctly?

9. Are operations and calculations carried out correctly?

III Verification

1. Does the subject treat verification as part of the solution process?

2. Does the subject view verification as something within his or her grasp or as an ultimately external process?

3. During verification, does the subject:

   (i) check calculations,

   (ii) assess the reasonableness of his/her answer in the context of the original question,

   (iii) verify the logical validity of the solution method?

4. Does the subject's confidence in his/her solution affect the process of verification?

GENERAL SKILLS AND ATTITUDES

I Strategy Selection

1. What general and specific strategies are used or considered for use?

2. Does the subject evaluate strategies before implementation?

3. What criteria does the subject use to select a strategy?

4. Does the subject switch strategies? What criteria are used in the decision to switch? Is the switch useful?

II Precision

1. Does the subject strive for precision in the use of:

   (i) inferences,

   (ii) verbal expressions,

   (iii) symbols,

   (iv) diagrams,

   (v) algorithms?

2. How does precision or lack of precision affect the solution attempt?

III Monitoring

1. Does the subject write down or otherwise record the information from the problem statement?
2. Does the subject keep written records to organize his/her solution steps?
3. Does the subject stop and reread or reflect on the problem periodically?
4. Does the subject monitor his or her progress?
5. Does the subject monitor his or her mental state?
6. Does the subject proceed at a rate appropriate to his or her competence?

IV Belief and Affect

1. What is the students' general attitude to mathematics? To problem solving?
2. What specific beliefs appear either explicitly or implicitly?
3. How does the subject react to confusion and frustration? Is there persistence in the face of frustration?

4. Does the subject attempt to get an answer at any cost?

5. Does the subject rely on, or wait for, inspiration?

6. Does the subject show confidence in his or her problem solving procedure and solution?

7. Are there indications that the subject views mathematical problem solving from within the context of a classroom culture?

V Structural/operational strategies

1. Does the subject choose structural or operational methods?

2. Is there an apparent reason for this choice?

PAIR INTERACTIONS

1. How cooperative is the pair?

2. Is there a clearly dominant partner? What appears to be the reason for that dominance?

3. Are partners willing to openly challenge each other?

4. Do partners support each other? Do they support each other even when they do not appear to understand?

5. Do both partners generate ideas and suggest strategies?

6. Do both partners evaluate suggested strategies?

7. How is a decision about strategy selection made?

8. How is the decision that they have finished the problem made?

9. Is there evidence that one partner's persistence keeps them both on task? Is there

evidence that one partner's confusion or frustration creates confusion or frustration in the other?

10. Do they monitor each other in:

    (i) exploration and analysis,

    (ii) planning,

    (iii) calculation?

11. Do they discuss their mental states, beliefs and attitudes?

I then reviewed each transcript using this final framework. For each problem session I created a separate file in which I summarized the problem solving process and then answered each applicable question in the analytic framework.

I next reviewed the exit interviews and basic information sheets filled out by each subject. I first summarized the interviews based on the main interview questions. Then, I returned to them to analyze their content with respect to how they might relate to the questions in the analytic framework. Here, I especially concentrated on the categories of belief and affect, and pair interactions.

The workbooks were not approached seriously by many of the students and so the results were disappointing. However, I did review each notebook in the light of applicable items from the analytic framework.

At this point, I had a file for each subject containing the analysis of his or her information sheet, exit interview, and workbook as well as any relevant information from my field notes. I also had a file for each problem session, both individual and pair. I then summarized these files under the categories listed in the framework. As the theme of a structural/operational duality emerged from this analysis, I reviewed the files again and added that category to my framework.

The data anlysis methods used are part of a well established tradition in qualitative research, in which themes and categories of analysis emerge during the process of analysis. Marshall and Rossman state that "data collection and analysis go hand in hand, to promote the emergence of sustantive theory grounded in empirical data." [1989, p.113] They go on to suggest five modes into which qualitative data analysis falls: organizing the data; generating categories, themes, and patterns; testing the emergent hypothesis against the data; searching for alternate explanations; and writing the report. Glaser and Strauss discuss the constant comparison method of data analysis which is "concerned with generating and plausibly suggesting (but not provisionally testing) many categories, properties, and hypotheses about general problems." [1967, p.104] In the constant comparison method the researcher may be guided by initial concepts and hypotheses but these may be changed or discarded as data is collected and analyzed. McMillan and Schumacher discuss inductive analysis which "means the patterns, themes and categories of analysis emerge from the data rather than being imposed on the pata prior to data collection" [1989, p415]. A constant redesigning of categories of analysis is thus a well known technique in qualitative research.

TRUSTWORTHINESS

Qualitative research has often been attacked as sloppy, unsophisticated and subjective. This has led to an ongoing debate regarding the trustworthiness of qualitative research. The initial debate regarding the legitimacy of qualitative research in education was in terms of a choice between the entrenched quantitative methodology and the new qualitative methods. [Howe and Eisenhart, 1990] Lincoln and Guba [1985] refined this debate by distinguishing between research methods and epistemologies. They noted that quantitative research is based on a positivist or naive realist philosophy while qualitative research is based on a phenomenological approach which seeks to understand actions from the actors' perspectives. They explained that different research paradigms require different criteria for trustworthiness. However this does not mean that there are no canons that stand as criteria for qualitative research. They pose four questions:

1. How can one establish confidence in the truth of the findings in a particular inquiry?

2. How applicable are the findings of a particular enquiry to another setting or another group of people?

3. How can one be reasonably sure whether the findings of an inquiry would be repeated if the study were conducted with the same or similar participants in the same or a similar setting?

3. How can one be sure that the findings of an inquiry are determined by the inquiry itself and are not the product of the researcher's biases or interests?

[p. 290]

In the conventional paradigm these criteria are referred to in terms of internal validity, external validity, reliability and objectivity. They replace these with four constructs for qualitative research; credibility, transferability, dependability, and confirmability.

To ensure credibility, it is necessary to demonstrate that the inquiry was conducted in such a way that conclusions can be drawn in confidence. Reviewing a number of relevant studies [Clement and Konold, 1989, Ginsburg, 1981, Posner and Gertzog, 1982, Schoenfeld, 1985b] led to the development of the methodology in the present study so that the data collection and analytic methods fit into a well established tradition in research in mathematics education. As well, the appropriateness of the data collection and analysis techniques was confirmed by the pilot studies. Results of the present study are based on several sources of data; individual and pair interviews, workbooks, and personal interviews. Taking data from these differing sources produces a triangulation that enhances the credibility of the results. Additionally, all claims in the analysis are supported by data from one or more of the sources listed above.

Transferability refers to the generalizability of the study to other populations, settings and treatment arrangements. To ensure transferability the researcher must provide a sufficiently rich description to enable a person interested in making a transfer to reach a conclusion about the advisability of so doing. The purpose of this chapter has been to give as complete a picture as possible of the subjects and the setting, as well as the data collection techniques used.

Dependability refers to the extent to which other researchers, using the same data, would come to the same results. Accessibility of the data and procedures leads to dependability. In the present study, dependability is enhanced by the use of mechanically recorded data (video and audio taped interviews) and low inference descriptors (verbatim interview transcripts), as well as clear descriptions of data collection and analysis methods.

Confirmability refers to whether the finding of the study could be confirmed by another. This can be enhanced by making explicit several important aspects of the design; the role of the researcher, selection of subjects, social context, data collection and analysis techniques, and analytical premises. This has been done in the present and subsequent chapters.

Howe and Eisenhart [1990] see Lincoln and Guba as representing only one position in the debate about the trustworthiness of qualitative studies. Others [Erickson, 1986, Goetz & LeCompte 1984] focused instead on the particulars of the various research methodologies rather than on epistemology. Howe and Eisenhart take a position supporting this second position. They claim that "a variety of specific standards are legitimate, because standards must be linked to the different - and legitimate - disciplines, interests, purposes, and expertise that fall under the rubric of qualitative research." [p.3] They propose five standards for qualitative research, four of which relate to trustworthiness. (The fifth relates to ethics and value.)

First, there should be a fit between research questions and data collection and analysis techniques. The present study is intended to be exploratory and thus the techniques used are exploratory in nature. Collection techniques include think aloud problem sessions, open ended work book problems and exit interviews. None of these presupposes the behaviour that may be witnessed. Data analysis is similarly exploratory in nature with the analytic framework evolving as the analysis proceeds.

Second, techniques must be competently applied. This chapter has given a detailed description of the techniques used, allowing the reader to judge the competency of their application. The legitimacy of the techniques used has been discussed elsewhere in this chapter. Data from the analysis are embedded in the results reported in chapter four.

Third, studies must be judged against a background of existent knowledge. The literature review, chapter two, has placed this study clearly in the context of current research on the relevant topics.

Fourth is overall warrant. This encompasses the first three standards but also requires that conclusions are those drawn after respected theoretical explanations have been tentatively applied to the data. This study draws on results and theory from many current researchers in the field and attempts to build on existent theory. Chapters four and five not only report the findings but attempt to place them in a theoretical framework.

Howe and Eisenhart [1990] argue that "standards must be anchored wholly within the process of enquiry" [p.3] and that "legitimate research methodologies may and should proliferate." [p.4] They argue that research, quantitative or qualitative, should be judged in terms of its success in addressing educational problems.

SUMMARY

This chapter has provided a description of the methodology of this study and its theoretical justification. As well it has included a full description of the methods of data analysis applied and how these methods were developed. The construction of the analytic framework was continuous throughout the study, being developed and added to as new themes emerged.

CHAPTER IV FINDINGS

SUMMARY OF DATA

Demographic description of participants

Participants were recruited from Kwantlen College, Richmond and Surrey campuses. There were fourteen participants, eight women and six men. Two subjects did not complete the full study, one for medical reasons and the other for lack of interest. The mean age of the subjects was 27.5, with the oldest being 44 and the youngest 18. Nine subjects were full time students, four worked full time and 5 more had part time jobs. Four were married, one man and three women, and three women were parents, two of these were raising children alone. The subjects' intentions in attending college were varied. Ten planned on transferring to university, six in science and engineering, one in education and three had not decided on a major as yet. One subject was in a college diploma program and one planned on transferring to the British Columbia Institute of Technology. Two students were interested in personal enrichment. Most of the subjects were currently enrolled in oneof two of the college's three preparation level courses: Math 093, Intermediate Algebra with Trigonometry; and Math 112, College Mathematics. Math 093 is a prerequisite for Math 112 which is the college's precalculus course. One student had recently completed Math 190, Mathematics for Elementary School Teachers, and one subject had completed Math 120, Calculus, several years previously. Two had also completed Math 115, Introductory Statistics. This set of students is quite representative of precalculus level mathematics students at the college, both in the diversity of mathematical backgrounds and in their personal demographics.

Results of problem sessions

Each problem was scored as correct, incorrect or incomplete. A problem was considered incomplete if either the student quit before the time was up or the student ran out of time. Additionally, incomplete and incorrect problems were also evaluated to see if substantial progress towards a solution had been achieved. Substantial progress was considered to have been made if the student had been able to develop a strategy or plan that could lead to a solution and had sttempted to put that plan into effect. For example, when Carl attempted problem 4, the spider and the fly, he drew a diagram of the room as though it were a box opened out and then attempted to find the shortest straight line route between the spider and the fly. He forgot to consider one of the possible routes and so did not find the solution. However, he was classified as having made substantial progress on the problem.

The fourteen individuals did three problems each, except for two who each completed 4 problems. Of these 44 problems, 9 were done correctly, 24 were incorrect, 11 were incomplete and on 7 substantial progress had been made. The eleven pairs did three problems each except for one pair which completed four. Of these 34 problems, 18 were correct, 8 incorrect, 8 incomplete and on 3 substantial progress had been made. Thus the percentage correct for individuals was 20.5, while the percentage correct for pairs was 53. Table 1 gives the results organized by problem. Table 2 gives the results of individual problem sessions for each student and Table 3 gives the results for pairs.

Table 1 Results of problem sessions by problem

↓Problem Result→	C	X	I	S	Problem ↓	C	X	I	S
1 Single	0	2	1	0	10 Single	2	1	1	0
Pair	1	0	0	0	Pair	2	0	0	0
2 Single	0	2	2	2	11 Single	0	3	0	0
Pair	0	0	1	0	Pair	1	1	0	1
3 Single	2	2	0	0	12 Single	0	0	0	0
Pair	1	0	0	0	Pair	2	0	0	0
4 Single	0	7	0	1	13 Single	0	2	2	2
Pair	0	2	0	0	Pair	0	0	0	0
5 Single	0	1	1	1	14 Single	0	0	1	0
Pair	0	0	2	1	Pair	3	0	0	0
6 Single	2	1	2	1	15 Single	0	0	0	0
Pair	0	0	0	0	Pair	1	0	2	0
7 Single	0	0	1	0	16 Single	2	1	0	0
Pair	3	0	1	0	Pair	1	0	1	0
8 Single	1	0	0	0	17 Single	0	2	0	0
Pair	1	1	0	0	Pair	1	0	0	0
9 Single	0	0	0	0	18 Single	0	0	0	0
Pair	1	0	1	0	Pair	0	3	0	1

C: correct, X: incorrect, I: incomplete, S: substantial progress made (these are also counted in incorrect or incomplete category)

Table 2 Results of individual problem sessions

Name	Question	Result	Name	Question	Result
Diane	13	I	Tanya	6	X
	1	X		17	X
	4	X		11	X
Carl	3	X		4	X
	16	C	Simon	6	I S
	4	X S		17	X
Randy	3	C		4	X
	1	I	Cecil	3	C
	4	X		2	I
Karen	13	X		10	X
	5	X S	Shelly	6	I
	11	X		2	X S
Candy	14	I		8	C
	7	I	Janet	13	X S
	4	X		2	I
Karla	6	C		11	X
	5	I	Carol	3	X
	10	C		16	X
Sam	13	I S		10	I
	16	C		1	X
	10	C	Kevin	6	C
				2	X S
				4	X

C: correct, X: incorrect, I: incomplete, S: substantial progress made.

Table 3 Results of pair problem sessions

Names	Problem	Result	Names	Problem	Result
Sam & Simon	7	C	Kevin & Cecil	7	C
	2	I		5	I S
	12	C		8	X
Karen and Karla	7	I	Janet & Carl	14	C
	18	X		18	X S
	4	X		8	C
Carol & Shelly	14	C	Diane & Sam	15	I
	17	C		18	X
	11	X S		11	C
Carl & Randy	7	C	Carol & Shelly (#2)	15	C
	5	I		18	X
	9	C		4	X
Karen & Candy	15	I	Kevin & Janet	3	C
	16	C		1	C
	10	C		10	C
Karla & Diane	14	C		12	C
	16	I			
	9	I			

C: correct, X: incorrect, I: incomplete, S: substantial progress made.

Results of workbooks

Eleven of the twelve students who remained in the study turned in notebooks at the end, but only three of these had done any substantial amount of work on their problems. One completed only half a page of work and several did only two or three pages in the approximately two months that they had the notebooks. Most of the students apologized, saying that they simply had too many more pressing projects to work on. All the questions contained several parts and were open ended, asking for generalizations and extensions, and as such cannot be evaluated as simply correct or incorrect. Four students were not able to answer even the first question posed on their problem sheet. Of the seven who could complete at least the first step, three were able to make at least some progress on the following questions, and three more made substantial progress in answering the extension problems. Carl and Kevin put in significant amounts of work on their problems. Carl worked on problem 2, map colouring, and was able to determine minimum numbers for several configurations although, of course, without proof, and he seemed to feel no need for proof. Kevin worked on problem 5, the secret numbers and quickly saw that each figure led to a system of simultaneous equations. He then spent a great deal of time trying to find general solution methods for these systems. He was successful for the triangle and made some progress for larger systems. It is significant that he realized that, although trial and error might work, it would not lead him to a generalizable solution and so he avoided it.

The poor effort put into the notebooks by most of the students limited their significance to the study. While some students may have been intimidated by the open ended nature of the problems, it is equally likely that they were simply too busy with assignments, labs, and exams to spend the time required to understand and make progress on the problems. This is in contrast with the students in the pilot study who completed notebook problems as part of their assigned course work. In the pilot study, all students put substantial effort into the problems.

PORTRAITS OF REPRESENTATIVE PARTICIPANTS

In this section I will give a brief description of four of the participants in the study. The information used to form these portraits was obtained from a brief information sheet each volunteer was asked to fill out, and from exit interviews conducted at the end of the study.

Kevin was 21 years old at the time of the study, and was taking four college courses while working part time. His aim in attending college was to transfer to university, but he was not as yet sure of his intended field of study. Although he had completed Math 12 two years previously, Kevin was enrolled in Math 093, College Algebra, where he had been placed by the college placement test. He was finding the course fairly easy, but thought that it was important that he "have it solid" before going on. Kevin said that the most important influences on his attitude toward mathematics were his teacher and his own motivation, which he saw as springing from his career goals. Of central importance to achievement in mathematics is, Kevin thought, the ability to remain constantly focused. Kevin said that he found problems, especially those with practical applications, far more interesting than sets of routine questions. Kevin was one of only two students who put substantial work into their workbooks, and he made significant progress on the problem assigned, number 5, the secret numbers. Kevin arrived at all his interviews on time and remained focused on the problems throughout each session. Kevin was the leader in his problem session with Janet, but in his second pair session deferred to Cecil who was several years older. Overall, he gave the impression of a cooperative, serious and able student.

At eighteen, Candy was the youngest subject in the study. She entered college directly from high school, and the semester in which the study took place was her first at the college. She had completed Math 12 but was enrolled in Math 112, precalculus, because she couldn't just "jump into calculus and know everything." Candy planned on transferring to university in the sciences, perhaps in bio-resource engineering. The most important factor shaping her attitude to, and achievement in, mathematics was, she said, her teachers and the ways in which they taught. Her favourite teacher, who had taught her grade eleven math class, explained everything and wrote everything down on overheads. "He'd slap them down and we had to learn to write very fast." She believed that it was important to ask questions in math class, and to have the questions answered completely. Candy wanted to have everything explained to her, and believed that that was how she learned best. She was finding that her college math course kept a very fast pace. She thought problem solving was a new way of thinking which "no one can, like, think like that right off the bat. They have to learn a new method." Candy completed two individual problem sessions, rather than the usual single session, as one of her partners failed to turn up. During both sessions, she found it very difficult to work on her own.

She constantly looked to me for direction and to help her out when she got stuck, so that the two sessions only provided three truly independently worked problems, two of which she was unable to complete. She made no real effort to solve her workbook problem, writing less than half a page in over two months. In her one pair session, Candy was cooperative but was easily distracted by extraneous details.

Carl was 31 years old at the time of the study and a full time student at the college, planning to transfer to medical radiography at B.C.I.T. Carl attended a private secondary school in Vancouver, from which he graduated in 1981, having completed Math 12. However B.C.I.T. required that he have Math 12 or equivalent within the past five years, so he was enrolled in the college's Math 112 at the time of the study. Carl believed that the most important influence on his attitude to, and achievement in, mathematics had been his experiences in school. Part way through grade nine, he was promoted from one mathematics stream to another, and he found it very difficult to catch up. Even so, he received good marks which he felt he did not deserve, and which did not encourage him to work harder. He missed portions of his high school courses due to participation in the school band, and he felt that, even when in class, he did not put in sufficient effort. He believed that he had returned to math, and to his studies in general, with a much more mature attitude. Carl was one of the two students who put substantial work into his workbook problem, number 2, the map colouring problem, and he too made substantial progress. Puzzles, he said, had always intrigued him, and when he found an interesting problem he liked to follow it up. Although often very nervous, Carl was an enthusiastic and cooperative student. He often stopped by my office to discuss extensions or generalizations of problems he was doing in class.

At the age of 28, Diane was a single parent with one school-aged child. At the time of the study, she was taking three college courses with the intention of transferring to university to study occupational therapy. She had completed Math 12 in 1982, and then had taken Math 093 during her first semester in college, one year before this study took place. After that, she enrolled in Math 112 and received a C; she was retaking the course to obtain a higher mark. She had always had difficulty with math, and had recently been tested and found to be below average in spatial ability. As she said, "When someone says stand a swimming pool on end and flip it over I go ieeegh!" She said that attitude is the most important factor in achievement in mathematics. Especially important is one's willingness to persevere. "You've got to keep going," she said. "And I think that's really the whole thing. 'Cause I, for me, the easiest thing to do is just to give up, to walk away from it if I can't figure it out. So if I keep doing it I, eventually, I might get it." She found problem solving difficult but once she could "get a handle on it" she found that she often enjoyed it. Diane was finding going to school along with caring for her child to be very stressful, even overwhelming at times, and said that, as the semester neared its end, she was having fantasies of running away to somewhere else. Nevertheless, she turned up to all of her appointments on time and worked seriously on the problems given her. Her second session with a partner turned out to be very stressful, as there was a great deal of hostility between the two of them. Diane attributed this to the different communication styles of men and women. Diane did minimal work on her workbook problem.

No student in the study was typical but these four could be said to be representative of the diversity in the group and in the college student body itself. The one clear difference between the study group and the general student body was that all participants were Canadian born, and all had English as a first language. This does not reflect the student body of the college.

DESCRIPTIONS OF PROBLEM SOLUTIONS

The eighteen problems used for the video taped problem sessions were divided into three general categories: familiar problems, generally algebra problems or what the students see as word problems in their textbooks; geometric problems and problems of analytic geometry; and logic and counting problems which would generally be unfamiliar to the students. In this section I will provide brief descriptions of the kinds of solutions the students provided to each problem. The full text of each problem, with solution, is provided in Appendix C.

The familiar problems included problems 3, 6, 7, 13, 14, and 15. Although classified as of a familiar type, with the exception of problem 6, the automatic washer, the students were not expected to have algorithms for any of these problems. This turned out to be the case and the students had a great deal of difficulty with most of the problems in this category.

Problem 3, the shopping trip, was solved by 3 of 5 who attempted it. In each of these cases it was done by trial and error, beginning with guesses of $10 or $20. In one of the incorrect attempts, the student constructed and solved an incorrect equation and in the other the student used a single step arithmetic operation.

Problem 6, the automatic washer, was approached algorithmically by four of the five who attempted to solve it. In two cases the algorithm was not completely remembered. The one student who had no algorithm for this problem attempted to construct an equation but quickly became very confused, using and changing variables several times. She was unable to construct a useful equation.

None of the students who attempted to solve problem 7, the ski trip, had an available algorithm, and all attempted to construct and solve an equation. It is notable that no student began with the relationships in the problem and tried to construct an algebraic model of them. Rather, all began by naming variables and then constructing expressions for the various quantities in the problem. Although they were all able to create the necessary algebraic expressions, in only two attempts were they then able to construct a correct equation from these expressions. In one case the problem was answered by guess and check.

The students found problems 13, the tanks in the desert, and 15, the commuter, particularly difficult. There were no correct solutions to problem 13 and only one to problem 15. In problem 13, diagrams were drawn but were not accurate, failing to consider the movement of the tanks. This led to wrong assumptions, and consequently, to a great deal of confusion. Only one student attempted to construct an equation and he was clearly trying to implement an inaccurately remembered algorithm. Constructing an equation was seen as the solution method for problem 15, but none of the students was able to produce an appropriate equation. All subjects confused time and distance, and generally failed to shift their points of view from the commuter to the husband. Both of these problems led to a great deal of confusion and frustration.

While all those who attempted problem 14, the cistern, found the problem statement intimidating, three of four were able to solve it by breaking it into parts. The only part in which they subsequently experienced any difficulty was the last hour, for which they could not directly calculate the result.

The geometric problems were numbers 4, 8, 9, 10, 11 and 12. These problems required knowledge of basic geometry and analytic geometry including Pythagoras' theorem and the condition for perpendicularity of lines in the plane. None required trigonometry, a topic which students in Math 093 might not yet have studied.

Problem 4, the spider and the fly, was not solved by any student in the study, although one did make substantial progress. The general approach to the problem was to draw and calculate several routes, until one appeared shortest. There was no attempt to set up any kind of decision criteria, although some did attempt to justify their choice as being the most direct. The student who made substantial progress flattened out the box and then considered only straight lines. Unfortunately, he missed one of the possible lines.

Problems 8 and 9, the triangle and the tangent circle, were both begun by graphing the appropriate lines. The only difficulty in problem 8 was in finding the intersection point. One pair knew how to do this and another did it by guess and check. The third pair was unable to find it and instead attempted to use trigonometry, but made serious errors. Only two pairs attempted problem 9, and one pair was able to use the perpendicularity condition and special triangles to solve it very quickly. The other pair forgot to consider perpendicularity and made an erroneous assumption leading them into some confusion.

Problem 10, the four circles, was solved in four of six attempts. One student seemed to see the complete solution immediately, while the others took varying lengths of time and modifications of the diagram to find the idea. One student's diagram was so inaccurate that, although he had a viable solution method, he obtained an incorrect answer. One simply gave up very quickly.

Problem 11, the two circles, gave the students considerably more difficulty. Here it was necessary to assign variables to the radii and construct and simplify an algebraic expression, in contrast to problem 10, where the answer could be calculated directly. Only in two of five solution attempts were letters used to represent the unknown radii, and one of these was correct while the other pair made substantial progress towards a solution. Two of the others made the erroneous assumption that the radius of the inner circle was equal to 1. The other one attempted to measure the lengths from the diagram and then use them in inappropriate formulae.

Problem 12, the folded paper, was solved by both pairs who attempted it, but in each case they solved it using trigonometry. This allowed them to solve it in an operational, forward calculating method. To solve it without the use of trigonometry requires one to name variables and work with algebraic expressions.

The remaining problems, 1, 2, 5, 16, 17, and 18, were logic and counting problems. It was thought that these problems would likely be unfamiliar to the students.

Two, numbers 16 and 17, required counting, one, number 18, required a knowledge of factorization, one, number 5, would likely have resulted in an infinite series (although a closed answer was possible without the techniques of calculus), one, number 2, required an explanation, and the last one, number 1, simply required a logical analysis.

Only one of four attempts to solve problem 1, the sleepy passenger, was successful. The successful student found the answer almost immediately. To obtain the correct answer, one has to reason backward to see that the time slept is two thirds of the second half of the journey. Two of those who obtained a wrong answer calculated forward, and incorrectly obtained one quarter. The final student made the incorrect assumption that the passenger could not travel while sleeping.

There were no successful solutions to problem 2, division by nine. This problem required the students to provide an explanation for the rule they were given and it was clear that they did not understand how one could do this. They all began with several examples and then most continued by looking for a pattern. Two went beyond this to consider the effect on the remainders obtained when dividing by nine, of nine being one less than ten . One of the two came very close to providing an acceptable explanation.

All of the students who attempted to solve problem 5, the squares, constructed the first few terms in an infinite series. For two of these, the first three terms of the series were correct. They had broken the problem down into steps and added the additional fraction shaded at each step. Two students made errors in this process. One student abandoned this attempt and attempted to fit an exponential or logarithmic function to the problem, apparently recalling the iterative nature of some of the interest problems he had seen in class.

Three of the five attempts to solve problem 16, making change, were successful. In these cases, the students began by listing several examples, then decided upon some organizational scheme for their combinations. One student attempted to count without such a scheme, and obtained an incorrect answer. The final pair did not try to directly count the combinations, but attempted to set up a system of simultaneous equations. It is clear that they did not really consider how this might give them an answer, but were simply trying something that looked familiar.

The three who attempted problem 17, handshakes, each used a different method. One simply did a one step calculation, with inadequate analysis. One tried to use tree diagrams which, although used for counting, were inappropriate for this problem. The third solution was correct, and was achieved by drawing a diagram to illustrate the situation. It appeared that this student had a ready made algorithm for the problem.

There were no correct solutions to problem 18, factorial, although one pair made substantial progress. One pair simply guessed, and one pair spent most of their time trying to find patterns in the incorrectly interpreted calculator output. The others considered the factors that would contribute zeros to the product, but none was careful enough in this analysis to obtain the correct answer.

OPERATIONAL/STRUCTURAL ANALYSIS

Sfard's operational-structural model adds a great deal to the understanding of the solution attempts displayed by many of the students in the present study. Many of the problems presented to the students required a structural, describe first, calculate later, approach, while most of the students approached the problems operationally. Even some of the apparently arbitrary strategy choices can be understood now as the attempts of students who cannot see the mathematical structure of a problem to, nevertheless, find a solution.

The students in the study showed an overwhelming preference for operational solutions over structural ones. This becomes quite clear when one considers those problems which allow for solutions of either type. All those who solved the shopping trip problem, number 3, did so by trial and error, a method which in this case involved only direct calculations. Only one subject, Carl, attempted initially to construct an equation for this problem, and his solution attempt was unsuccessful as he did not take into account the iterative nature of the problem. None of those who solved this problem showed any awareness that they could replace their trial numbers with a variable and so construct an equation. This is particularly notable in the case of Karen, who had to make several trials before arriving at the correct answer. It is, however, interesting to note that few of the students were entirely happy with a trial and error solution, and in one case the student never even submitted his answer, but spent the rest of the time allotment attempting to fit an equation to his solution. "Okay. 17.50. The answer is 17.50. But we've got to figure a way to do it without guessing." [Randy] It was very clear that they all believed that there was an algebraic method of solution, and that it was preferred. However, they had no idea how to obtain it.

Another problem which allowed for both an operational and a structural approach was problem 12, the folded paper. This problem could be solved in an operational mode with the use of trigonometry, or with a structural approach that avoided trigonometry and used only Pythagoras' theorem. This second method, however, required the solver to use a variable and to work forward, creating and solving an equation in which the variable appeared twice. Kieran has noted, following Filloy and Rojano, that it is with equations of this type that algebra students must make the transition from arithmetic to algebraic thinking, that is, from an operational to a structural approach [Kieran, 1992, p.393]. Both of the pairs who solved this problem (Simon and Sam, and Kevin and Janet) did so by the trigonometric, operational method. In the case of the second pair, Janet knew no trigonometry, and therefore Kevin had to provide a concise explanation of trigonometry before he could go ahead and solve the problem. Despite this, and despite having been told that trigonometry was not needed, this was the approach they choose.

Problem 14, the cistern, while having a long and apparently quite complex problem statement, can be solved almost entirely by operational means. Only during the final step, calculating the fraction of an hour after 3 p.m. for the cistern to fill, did any of the pairs who attempted it have any difficulty at all. And it is only for this final step that anything other than a direct calculation need be considered. Karen and Diane had a particularly difficult time trying to construct an equation for this final step:

K 1/8 of job is done

D We need to set up an equation.

K in 1/6 of t.

D t is equal to time, ok.

K 1/8 of job needs to be done in 1/6 time. ... Hum, hum, hum, hum, hum. Yeah we need an equation here. [unclear]with respect of.

D Well 1/8 of t. Would it be 1/8 of t? ...

K Well, not really 1/8 of time. We need 1/8 of the job, because the job here.

D Right.

K We need, it has to be done in 1/6 of the time, er, in 1/6 of. Well, this is done. It does 1/6.

D 1/6

K of the job.

D Yes.

K So we need 1/6 times what equals 1/8.

[Karen and Diane] They did finally construct and solve an equation which gave them the correct answer to the problem, but even then they had little confidence in it and did not see its place in the structure of the problem. This became clear when, in attempting to verify their answer, they, without realizing it, redid calculations that they had already done. The other two pairs which solved this problem (Carl and Janet, and Shelly and Carol) calculated this last step directly, never using algebra at all. "Alright, so, the end of the third hour we have, ah, 3/24 left to fill up. And if we go another full hour we'll have filled it another 4/24. ... So it's 3/4 of an hour." [Shelly and Carol] This problem, because of its length and apparent complexity, initially intimidated all who attempted it. It was, however, successfully solved in three of the four sessions when it was presented (the one individual who attempted it, Candy, gave up very early on), I believe, because the students were able to use an operational approach.

While for the cistern problem an operational approach was appropriate, in problem 13, the column of tanks, a structural approach, involving construction of an algebraic model of the problem, was necessary. None of the four students who attempted this problem was able to solve it, and two of them made no attempt at all to use a variable. One of those who did use a variable, Sam, was clearly attempting to reconstruct a poorly remembered algorithm. The other, Janet, only considered it briefly and, not being able to see how to use letters, abandoned the idea.

J Try something totally different. If I take speed plus speed times, one times is the same time .3 time .3 equals. That's time 1, time 2 equals distance 1 plus distance 2. But what would that tell me if I did that? That would tell me the total distance. I already know the total distance. I can just add these two 'cause that would be what 22, 32, 37 and a half km. ...[Janet]

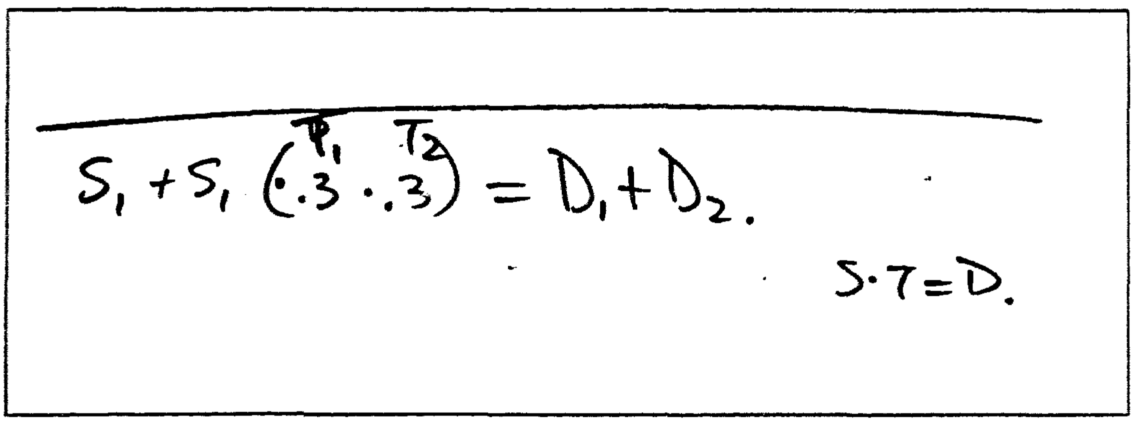

Figure 1 Janet, problem 13

Janet was not considering the mathematical structure of the problem, but was simply trying to use her formal understanding that rate times time will give distance. Since she already knew the distance, she could not see how this would be of any help and she made no further attempt to use a variable. For most of her session, Janet tried to find an answer by calculating whichever times or distances could be calculated from the given numerical information. The other two students who attempted this problem took a similar approach, making no attempt at all to construct an equation. They drew diagrams, created tables, and calculated whatever quantities could be directly calculated, but did not appear to even consider the possibility of constructing an algebraic model, of "describing first, calculating after."

Three of the five attempts to solve problem 7, the ski trip, were successful. One of these was done by guess and check and the other two were completed with the use of equations which were constructed with difficulty. In no case did the student or students begin with the relationships amongst the quantities in the problem and use these as the basis for an algebraic model. Rather, they named one or more variables and then began to see what quantities could be constructed with these variables. Only then did they attempt to put these expressions into some sort of relationship to each other. This was generally a very unsure process and confidence in the result was proportionately low. The pair, Kevin and Cecil, who solved the problem by guess and check had created an erroneous equation in just this manner, but were too intimidated by its complexity to try to solve it. The students who were unable to solve the problem had approached it in a similar manner. One of these pairs, Karen and Karla, was able, with some difficulty, to create correct expressions for all the important quantities in the problem but then was unable to construct an equation from them. Although they clearly knew the relationship between the original fare and the reduced fare, they were unable to use this information to construct an equation. This was the difficulty for all those who attempted to solve this problem.

The two problems involving areas of circles, problems 10 and 11, clearly demonstrate the difference between a structural and an operational solution. Both problems involve adding one or more lines to the existing drawing, then calculating two areas, and finally subtracting one area from the other. However, in problem 10, the four circles, the solution can be obtained by direct calculation, while in problem 11, two circles, it is necessary to assign variables to unknown lengths, use Pythagoras' theorem, and simplify the resulting expression to remove the variables. Problem 10 was solved in four of six attempts, while problem 11 was solved correctly in only one of five attempts.

All but one of the students who attempted problem 10, the four circles, constructed the appropriate lines quite quickly after seeing the problem. In one case, Sam's solution, this led to an immediate solution, as though Sam had instantly seen the solution whole. The others spent some time fitting the pieces together and one, Cecil, got confused by his inaccurate diagram, counted the small pieces incorrectly, and so came up with an incorrect answer. However, all but Carol realized that they needed only to calculate the area of the constructed square and then subtract the areas of the sectors from it. At this point the problem became simply operational, and most were able to carry out the necessary arithmetic operations.

The two circle problem involved drawing in the one radius and then subtracting the area of one circle from the other. However, in this case the radii of both circles were unknown, so that it was necessary to construct an algebraic expression for the area and then simplify that expression. All but one of those who attempted this problem drew in the required line and realized that they must subtract one area from the other, but only the two pairs, Shelly and Carol, and Sam and Diane, went on to use letters to represent unknown lengths. Even then, both these pairs had a great deal of difficulty with the resulting algebra. While they seemed to have a clear idea of the geometry involved, each step in the algebra appeared isolated and was not understood as part of an overall plan. Sam and Diane drew in several more lines and chose variables in such a way that their algebraic simplification became quite complicated. They were, however, able to solve the problem in the end. Shelly and Carol chose more appropriate variables, calling one radius a and the other c, but then wrote both a in terms of c and c in terms of a. Substituting both expressions into the expression for area, they forgot to square the radii. They kept clear records and appeared to understand each step but seemed unable to see the overall picture and, in the end, were unable to solve the problem:

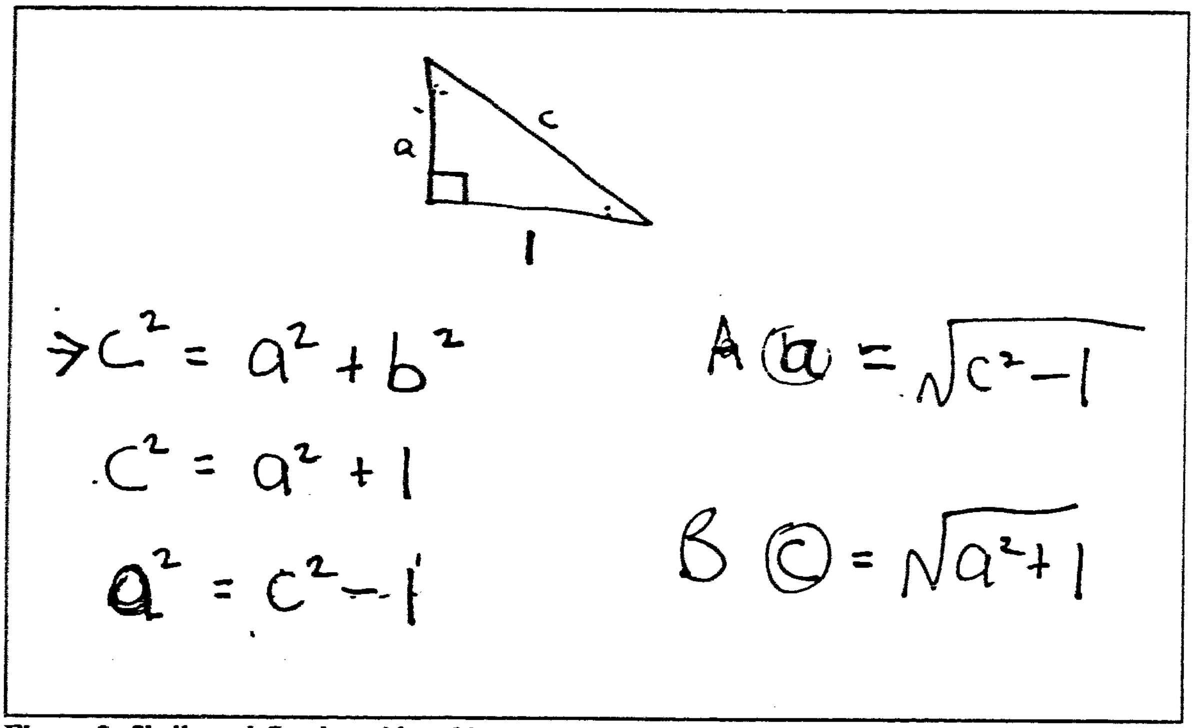

Figure 2 Shelly and Carol, problem 11

S That's the area.

$$A = \pi\left(\sqrt{a^2+1}\right) - \pi\left(\sqrt{c^2-1}\right)$$

Figure 3 Shelly and Carol, problem 11

Now because there's a pi squared on both of these ... we can do

$$A = \pi\left(\sqrt{a^2+1} - \sqrt{c^2-1}\right)$$

Figure 4 Shelly and Carol, problem 11

I don't know if it makes any difference. Opps. And that's our area ... That's as good as I can get it. I don't know about you. (laugh)

C` Yeah, ah, I don't know.

S There's got to be some other way. But, I mean, at least we, algebraically we can get it to that point. And I can't. I don't know about, how, whether you've got any other ideas.

C Nope.

[Shelly and Carol] Shelly and Carol, as well as two of the other students who attempted this problem, originally made the assumption that the radius of the inner circle was equal to one, as it seemed to be on the diagram. Tanya did this and when, at the end of the session, before being given the answeis to the problems, she was asked if she could solve this problem without this assumption, she was certain that the problem could not be solved in that case. Karla began the problem without assuming that this radius was of length one but later, when at an apparent dead end, decided to explore this possibility and then never returned to the more general problem. This simplifying assumption transforms the problem into one that can be solved by operational means only, and a correct answer was obtained in this manner, although the solutions were deemed incorrect. Janet, the other student to attempt this problem, did not draw in the required radius, but made several attempts to measure different lengths on the drawing until she believed that she had obtained the width of the annular region. She then tried to use this for the radius in the area formula for a circle. Her procedure does not seem entirely unreasonable if one assumes that she was simply looking for some operational way to find an answer. The procedure for finding an area which she seemed to be carrying out, was to find a length and use it in an area formula to calculate the required quantity.

The contrast in success rates and frustration levels between these two problems is a further illustration of the students' preference for, and competence in, problems that can be solved by direct, operational means as compared to problems that require a structural understanding. It is also interesting to note that most of the students were able to immediately see the geometric structure of both problems, and yet none were able to see the overall algebraic structure of the second problem, adding support to Sfard's observation that geometry may be more commonly understood structurally. [Sfard, 1991. p.10]

We have seen that, where possible, subjects chose an operational over a structural approach to the problems. Problems that could be solved by either approach were generally solved by an operational, direct calculation approach; in these cases a structural approach was never even considered. Subjects were often unsuccessful at solving problems that required a describe first, calculate later, structural approach.

STAGE SPECIFIC SKILLS

Stage specific skills are those skills involved in a particular stage of the solution attempt. Following Clement and Konold [Clement and Konold, 1989] I consider three stages: (I) comprehension, (II) planning, assembling and implementing a solution, and (III) verification. I will discuss the findings from both individual and pair sessions within these categories.

Comprehension

Most students appeared to view understanding the problem as an integral part of the solution process. They had been asked to read the problem statement aloud and most reread it a second or even a third time, more slowly and often with long pauses. They did not, however, generally make explicit note of the givens and the goal. The exception to this was problem 6, the washer, in which several subjects explicitly noted that the time for the cold water and for the cold and hot water together had been given, and the time for the hot water was needed. In most cases, they transferred the information from the problem sheet to their working paper. This was done with the aid of a diagram or drawing whenever possible. The drawings were usually quite simple, and often used only to organize information from the problem statement. However, they were occasionally more elaborate than was necessary for the problem. The drawing below created by Candy, when she and Karla were starting problem 15, the commuter, contains no useful information.

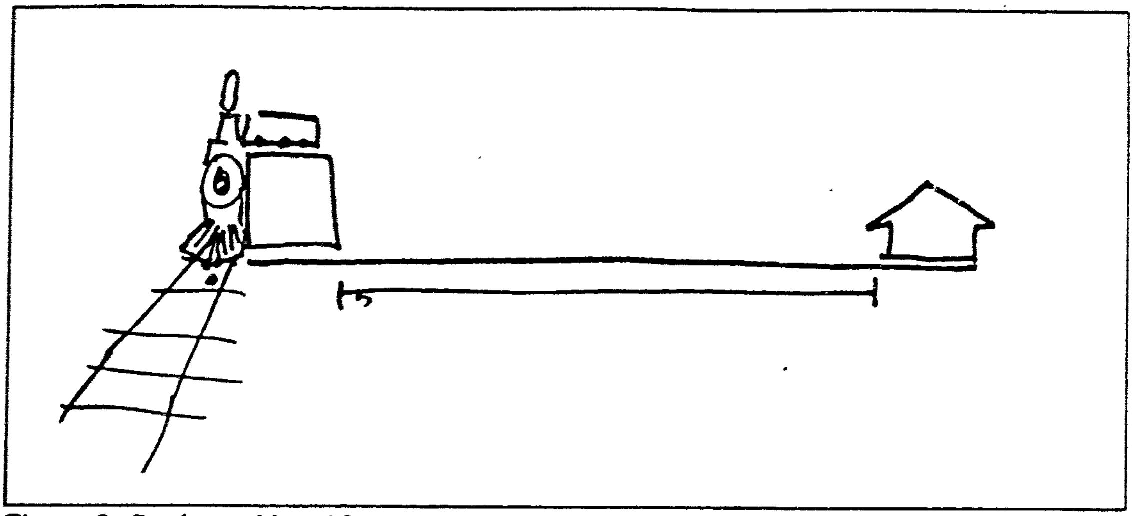

Figure 5 Candy, problem 15

One student, Karen, highlighted important information on the question sheet. Almost all students redrew, on their working sheet, any diagram that was given on the question sheet. On the two questions, problems 8 and 9, where equations of lines were involved, all subjects immediately graphed the lines and then returned to the problem statement before going further. It was clear, in these cases, that they saw the construction of a graph as integral to really understanding the problem.

However, at least two students were willing to attempt to solve a problem which they clearly did not understand. Janet worked with Carl on problem 18, factorial. She was not familiar with the idea of the factorial and read the two examples carefully before turning to the definition of n!. "I'm just trying to get through that n one," she said, "'cause that would be the key, wouldn't it? To figure out that one?" However, she made no effort to do this, despite the fact that it was clear that Carl had seen factorials before and she could call on him. Instead, she wanted to use the definition as an equation to be solved. She had clearly misread the multiplication symbols as x's and, seeing an equation, abandoned her attempt to come to grips with the definition of factorial. Another student, Karla, was clear that she did not understand the key point of problem 5, squares. She said "I think there's something I'm not quite picking up. ... If the process continues forever? See, I don't, don't really see how that process can continue forever. ... I think the key is, if the process continues forever." She then attempted to answer the question, without having clarified what she clearly knew to be the key idea. These two students were willing to attempt to solve a problem when they clearly knew that they did not understant concepts central to the problem and so it appears that they did not see understanding the problem as crucial to solving the problem.

To solve the problems given, it was necessary to sort relevant from irrelevant details, and to make certain assumptions. Often, the students made the wrong assumptions. They assumed things that were not necessarily true and, more rarely, did not make other assumptions that were necessary to the solution of the problem. Occasionally, they were concerned with details that were irrelevant mathematically.

The two problems involving circles, problems 10 and 11, each involve assumptions about radii. To solve problem 10, it is necessary to assume that all the radii are equal and the circles are tangent. Although these assumptions are not given in the problem statement, all subjects who attempted this problem made these assumptions, generally without explicitly saying that they were doing so. Subjects had much more difficulty with problem 11. Here one cannot assume that the lines are in the proportions in which the appear on the diagram. However, this convention was not clear to all the students. In all but two solution attempts, the assumption was made that the radius of the inner circle was equal in length to the tangent line given, as it appeared on the diagram. In two of three solutions where this assumption was made, it was made explicitly and the subjects were aware that they might be incorrect. This was in sharp contrast to problem 10, where the (correct) assumption was generally made unconsciously and no doubt at all was expressed.

Other problems also required conventional assumptions and, in several cases, the subjects made these assumptions explicit. In problem 4, the spider and the fly, it was necessary to assume that the fly does not move. Randy was concerned enough about this convention that he presented two answers, one valid if the fly could not move and the other if it could. Shelly noted that to solve problem 6, the washer, it was necessary to assume that the flow of hot water did not affect the flow of cold water, and vice versa, even when both were operating at the same time. In solving problem 16, making change, Carl asked for confirmation that all coins of any denomination were to be considered identical, a necessary convention. Karen and Karla explicitly discussed assumptions to be made in solving problem 7, the ski trip. They noted that they needed to assume that everyone pays the same fare, and that the club does not make any profit on the trip. Karla and Candy noted that, to solve problem 15, the commuter, it is necessary to assume that the commuter has not telephoned her husband to tell him of the change in trains.

Incorrect assumptions led some students into difficulties. In his analysis of problem 1, the sleepy passenger, Randy made the erroneous assumption that the passenger cannot travel while sleeping. This made the problem impossible to solve, and led Randy into great confusion. However, he at no time reconsidered this assumption. Attempting to solve problem 13, the tanks in the dessert, led both Diane and Karen, but not Janet or Sam, to assume that the times for the two parts of the messenger's journey were equal. This led both Diane and Karen to contradictory calculations but, like Randy, they never questioned their assumption. Carl assumed that in problem 3, the shopping trip, the amount lent was the same each time. Although it was possible to check this assumption, he made no attempt to do so. Cecil, on the other hand, having made the assumption that the rule for division by nine (problem 2) only worked for numbers under 100, checked this assumption and found that he was wrong. In this problem it was easy to check, something not generally true.

Planning, Assembling and Implementing a Solution

Little planning was evident in most problem solving sessions. Solutions were more generally attempted through exploration and analysis, or the application of a known algorithm. A notable exception to this was Kevin's explicit plan to solve problem 12, the folded paper, by the use of trigonometry. Since his partner, Janet, knew no trigonometry, he had first to explain what trigonometry could tell them about a triangle. He went on from there to present a complete solution plan.

K I was thinking, if we could find that. This is 90, or is it? I'm assuming it is. Well, it is, 'cause it's the corner. Okay. This is 90 degrees. This is 15. ... See, if we could find this then we could subtract. The whole. This whole angle here is 180, right?

J Right, yup.

K Okay, so we could subtract. So what's left is 90. These two things, these two angles here

J Are going to be 90.

K Add up to 90.

J Yeah.

K So, if we figure out this one, subtract it from 90 to find out this one

J Um hum.

K And then this is also a right angle, so we can figure out this side.

J From this?

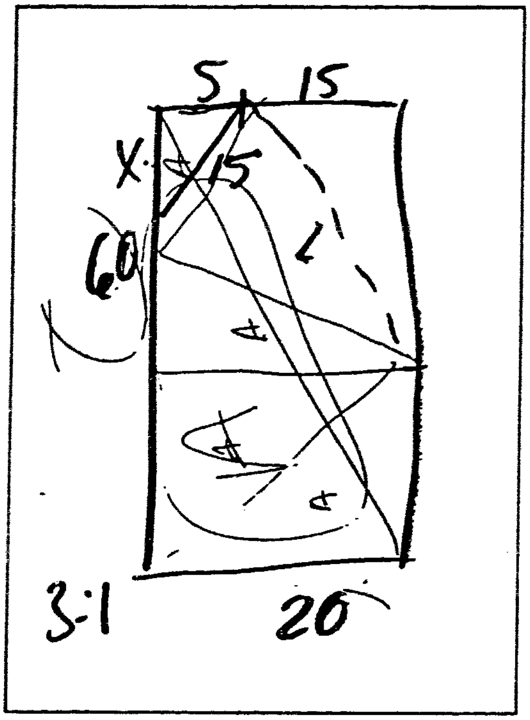

Figure 6 Kevin and Janet, problem 12

K From this, from using one of the things [trigonometric functions], I can't remember which one, you can figure out, if you have this side 20 and this angle, you can figure out what this side is.

J Um, cool! Okay.

K Shall I try that?

J Go for it.

[Kevin and Janet] It is clear that Kevin had not figured out every detail but was confident of his ability to carry out his plan. This was the only example in the study of such extensive explicit planning. There were, however, several examples where implicit planning was apparent. Both Carl, working with Janet, and Shelly, working alone, appeared to have devised complete plans for solving problem 8, the triangle, with only the details to be filled in. There appeared to be no stage in which they constructed their plans. Rather, they seem to have seen them whole once they had understood the problem sufficiently well. This was the case also for Sam's solution to the four circles problem, number 10, and for Carl, working with Randy on problem 9, the tangent circle. This is consistent with results of Krutetskii [1976] in which he found that capable students were often able to grasp a problem whole. All of these problems are geometric, supporting Sfard's [1991] conjecture that a structural orientation may be easier to achieve in geometry than in algebra.

Problems were seldom broken down into parts, with goals and subgoals. Problem 15, the cistern, lent itself to being divided up by time with each hour being calculated separately. This was the technique used by all who solved it. Problem 8, the triangle, was also generally broken down into 3 or 4 steps: graph the lines and find the length of the base; find the intersection point; and find the area. This was never done explicitly but was always done implicitly. The trial and error solutions to problem 3, the shopping trip, were also done step by step, usually by systematically taking the midpoint between the two previous guesses at each step. Again, this was not explicitly planned. Kevin's plan for the solution to number 12, the folded paper, provided him with clear steps to follow.

As stated above, this extensive planning occurred only in this one case.

By exploration I mean use of examples or of extreme cases to further one's understanding or analysis of the problem. Certain problems were far more likely to provoke exploratory activity than others. Exploration was used extensively in solution attempts for problem 3, the shopping trip. In each case, the use of examples led to a trial and error solution to the problem. Most students who attempted problem 2, division by 9, did so by trying numerous examples to confirm that the rule given actually worked, and then by looking for patterns in the result. Kevin went further and tried examples using 7 and 8 in place of 9, to see if there was a similar rule for these numbers. Problem 18, factorial, also provoked exploration by example in most of those who attempted it. However, the limitations of the calculator soon ended the exploration for those who understood how to read it's output. Karen and Karla erroneously assumed that all digits that did not appear on the calculator screen in scientific notation were zeros. As a result they spent most of their session looking for patterns in this output. All of those who attempted problem 9, making change, began by constructing several possible combinations. Most soon realized that they needed some organizational scheme if they hoped to find all the possibilities. In a similar manner, most who attempted problem 4, the spider and the fly, began by tracing several routes. However, in this case this strategy did not lead to any organizational plan for the routes, or any decision criteria for the shortest route. Rather, most simply tried several routes until one appeared to be the shortest. Some then attempted to justify their (erroneous) decisions on the basis that the chosen route appeared to be the most direct. In their attempt to clarify the situation in the commuter problem, number 15, Karla and Candy tried an example using particular times of arrival. However, they did not analyze their example in sufficient detail for it to be of help to them. Explorations, then, were of value in finding a solution to some problems, but not for others.

Many problems led to a great deal of time being spent on analysis. However, the analysis was often disorganized and unsystematic. Chains of inference were usually limited to two steps at most, and ideas and results were seldom recorded clearly enough to be of use. The most successful analyses were those involving geometric problems. Sam and Diane approached the two circle problem, number 11, by modifying the drawing extensively and analyzing the resulting figures. They added a rectangle with diagonals and analyzed the relationships of the line segments in that diagram. This was not directly helpful but they were finally able to use the complicated diagram to solve the problem. Similarly, both Karen and Cecil made extensive use of diagrams in analyzing problem 10, the four circles. Karen progressively simplified the problem of which area to subtract from which other area, through a series of seven different modifications of the given diagram. Cecil made fewer diagrams and his drawings were very imprecise, so that, while he had a viable idea he did not obtain a correct solution.

Most of the time spent trying to solve problem 15, the commuter, was spent in an attempt to analyze the situation, using drawings to represent the route travelled. These analyses were generally unsystematic and hampered by a tendency to confuse time and distance, or to confuse clock time and elapsed time. All those who attempted this problem showed much confusion and frustration, and were hindered by concentrating their analyses on what happened to the commuter, while ignoring what happened to the husband. Only Shelly, working with Carol, seemed able to shift her focus from the goal, the commuter's walking time, to an analysis of the husband's activity. Those who attempted to analyze problem 13, the tanks in the desert, also found themselves confused. Karla and Diane were both hampered in their analyses by the assumption that the time the messenger spent travelling to the end of the column would be equal to the time spent returning to the front of the column. But, even those who did not make this assumption, had difficulty with this problem, in part because they did not draw a sufficiently precise diagram to aid them.

Problem 2, division by nine, proved to be a puzzle for all those who tried it. They did not understand how to tell why something was true, and so most spent all of their time trying examples and looking for patterns. Shelly and Kevin were exceptions. Each attempted to analyze the problem based on remainders and the fact that nine was one less than ten. While neither was entirely successful in providing a clear explanation, their realization that the key lay in nine being one less than ten allowed them to come closer to providing an explanation for the phenomenon rather than just a description of it.

Several problems were generally approached with insufficient analysis. This was true for some of those who attempted to solve problem 1, the sleepy passenger. A time line was generally drawn and an answer given almost immediately afterwards. However, in several cases the answer of one quarter was obtained by a single erroneous inference. Problem 17, shaking hands, also generally received insufficient analysis, leading to a variety of simple, single step solutions. Only one pair solved this question correctly, and in this case it was clear that one of the partners had an available algorithm for problems of this type. Although most of those who attempted problem 18, factorial, spent a significant amount of time on their solutions, most of that time was spent on examples, rather than on analysis. In the end, all came to a consideration of factors which would produce zeros, but spent little time on this analysis. No one obtained the correct solution.

Diagrams were commonly used and could be crucial to the solution. Besides the geometric examples described above, drawings were also created for several problems. Shelly produces a simple drawing to help her to count the number of handshakes in problem 17. See figure 7. Most of those who attempted problem 14, the cistern, or problem 6, the automatic washer, drew pictures to help them organize the information. Similarly, Janet drew pictures of the items to be purchased in problem 3, and labelled each with its cost.

General principles were seldom called upon in attempting to solve the problems. Exceptions to this occur with problems 4, the spider and the fly, and problem 12, the folded paper. In problem 4, several students tried to use the principle that the length of a hypotenuse of a right triangle is shorter than the sum of the lengths of its sides. Kevin, and Shelly and Carol relied on this principle, and were shocked when it did not lead them to a shorter route. In the same problem, Carl appeared to have been guided by the principle that the shortest route between two points is a straight line in his decision to redraw the room flattened out in order to be able to draw straight lines. Kevin explicitly used the general principle that, using trigonometry, one need only know one (non-right) angle and one side of a right triangle to be able to find all the other measurements on the

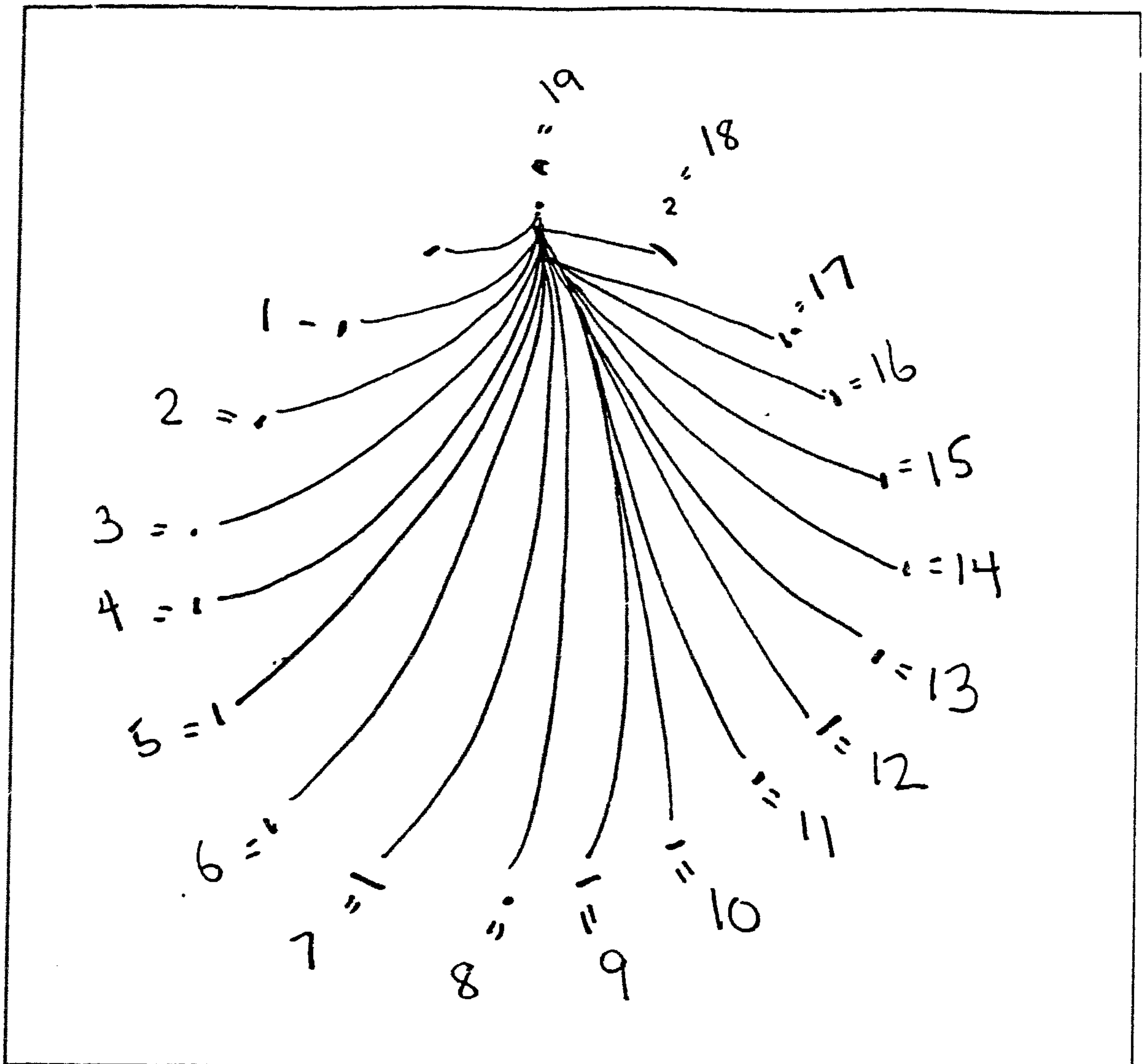

Figure 7 Shelly and Carol, problem 17

triangle, to guide his solution to problem 12.

Algorithms were used frequently. One of the most frequent algorithmic approaches appeared in all problems that involved time, rate (speed), and quantity (distance). Here it was common to use the formula r × t = d, or to create a table giving rates, times, and distances for two or more objects. This occurred even when it was not appropriate, such as in problem 13, the tanks in the desert, and problem 15, the commuter.

Sam attempted to use a different algorithm for the tank problem. While his choice of using relative speeds was appropriate to the problem, he remembered the algorithm incorrectly, and was unable to see or correct his error. Most of those who attempted problem 6, the automatic washer, did so algorithmically. For Karen this was successful, as she clearly knew the algorithm well, but both Simon and Tanya could not remember their methods correctly. This led to some frustration for Simon. "Damn it," he said "I've done these before." Shelly and Simon also approached problem 17, the handshakes, algorithmically. Shelly used an appropriate and correctly remembered algorithm while Simon used tree diagrams, a technique not appropriate to this question but used in other counting problems that he had seen.

Problems requiring an algebraic solution were not well done. Students usually recognized that they needed to construct and solve an equation, but lacked the skills to carry out the appropriate analysis. In no case did the students begin with a relationship central to the problem, and then attempt to construct an algebraic model of it. Instead, they named variables, constructed what quantities they could with those variables, and then attempted to set these expressions equal to something. Carl and Randy had constructed the expression 520/(x+5) to represent (correctly) the price of the trip per skier in problem 7, and then they wondered, "Ah, this equals something, right? I think that the problem is we don't have anything that it equals to." In the same problem Karen and Diane, were able to construct all of the expressions needed but were unable to put them together into a valid equation. Simon and Sam were able to arrive at a valid equation for this problem by using two variables, and then constructing two expressions, each equal to 520. They were then able to solve the resulting system of equations. In general, however, most of the students were left in great confusion when they needed to construct an equation. They named and renamed variables, calculated quantities and constructed expressions without being able to put together an equation which could lead them to a solution. After having failed in one attempt to construct an equation for problem 6, the automatic washer, Shelly tried a second time:

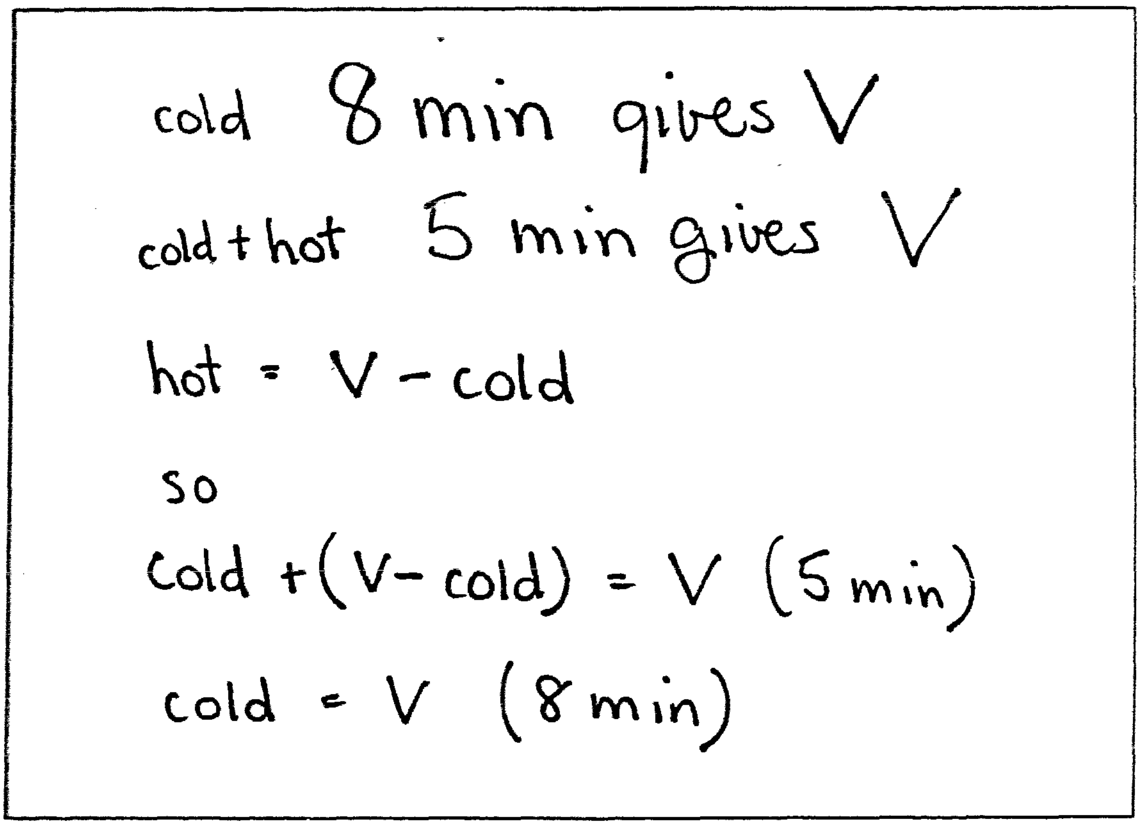

Figure 8 Shelly, problem 6

S Okay, I'll try this. Eight minutes gives volume for cold. For cold and hot, 5 minutes gives the same volume. And so hot is just, total volume minus the cold. So cold plus Y minus cold gives the volume. And that takes 5 minutes and cold gives the volume. Takes eight minutes. (Pause) I don't know what I would do now.

[Shelly] Shelly saw that she was not getting anywhere here, but she did not see what else

she could do.

There were very few errors in the mechanics of arithmetic, algebra or geometry. Almost all calculation errors that were made were noticed and corrected. Several students did not know the formulae for the area and circumference of a circle but only one, Janet, was unable to use the formula correctly once it was given to her. Kevin made several major errors in trigonometry while working on the triangle problem with Cecil. However, he made no such errors when he worked with Janet on the folded paper problem. Serious errors of manipulative algebra were made by Simon in his attempt to solve problem 6. While Kevin and Cecil made no algebraic errors in attempting to solve their equation in problem 7, they failed to simplify and the resulting equation became so complicated that both were greatly intimidated and ceased to attempt to solve it, settling instead for a trial and error solution.

Verification

Ten of fourteen individuals and eight of eleven pairs made some attempt at verification, but none were consistent, generally checking only one or sometimes two of their problems. Reasonableness of the solution, correctness of calculations and logical validity of the solution were all checked at various times but never were all three checked on the same problem. When Shelly and Carol had solved problem 17, handshakes, Shelly checked the reasonableness and logical validity of the answer and then wanted to check the calculations. Carol objected vehemently to this, possibly as this had been her only contribution to the solution, and Shelly did not insist.

On several problems, the solution method was seen whole by the student and then carried out with ease. This was true of Janet and Kevin, and Sam on the four circle problem; of Shelly and of Carl on the triangle; and of Carl and Randy on the tangent circles. In each of these cases, no verification attempt was made. It was as though the logic of the problem was so clear that they had no doubt whatsoever. In each of these cases the calculations were also very simple. There also was no attempt to verify any of the solutions obtained by trial and error. In one case, Karla and Candy solving problem 16, the same answer was obtained in two different ways and this was taken to be "proof" that it must be correct.

The subjects did not generally view verification as part of the solution process and they used it only some of the time. However, there were several instances where verification was integral to the process. In two problems, the shopping trip and the spider and the fly, Carl made estimations of quantities before calculating them. Unfortunately, in the case where his answer lay outside of the estimated interval, he never rechecked this quantity. In only two cases was the logical validity of an equation checked before the equation was solved. Both involved problem 7, the ski trip, and in one case, that of Karen and Karla, the equation was incorrect, but they were unable to see this. When they solved the equation, they recognized that their answer was unreasonable, but they had used up all of their time.

Karla noted that it was necessary to verify that the radius equalled one in problem 11, the two circles:

K Wait a sec, here. How would I know that this line, going from the centre to the end of the inner circle would be equal to the line that is perpendicular to the original one? There's got to be some rule that tells us something like that. If I take for granted that that one's equal, just to see what happens, if I took for granted that was equal. We put one unit.

[Karla] Karla then marked one beside the questioned line segment and went on the solve the problem using that assumption, although she clearly knew that she must justify it. Later in the solution attempt, she appeared to have forgotten entirely that this was an assumption. Kevin was doubtful of several of his calculations during his solution of problem 12 but he too never returned to check them. During their attempt to solve problem 8, the triangle, Cecil and Kevin became very unhappy that their calculations did not match the drawing they had made. They began to redraw the graph, this time to scale and then abandoned this attempt, apparently because it appeared to be too much work. Karen and Diane were similarly disturbed by their graph of the tangent circle problem, number 9, but they redrew their graph and corrected their error. Unfortunately, they went on to make further errors. For most of these students, finding an answer to the question was central. Verification was used only occasionally.

There was no simple link between confidence in a solution and verification. Tanya was confident that all four of her erroneous answers were correct and she attempted no verification at all, while Janet checked the reasonableness of two of the three problems she did, discovered in both cases that the answers obtained could not be correct, but submitted them anyway. Karen and Karla spent most of their time on problem 18, factorial, on a wild goose chase, and only saw another approach to the problem in the last few minutes of the allowed time. When they obtained a (incorrect) solution with this new approach, they were so relieved that they submitted it immediately, with no reconsideration at all.

GENERAL SKILLS AND ATTITUDES

General skills refers to skills that may be used in any phase of the problem solution, and include strategies and strategy selection, precision, and monitoring of the process. Attitudes include beliefs about mathematics in general and about problem solving in particular, and reactions to confusion or frustration.

Strategies and Strategy Selection

The students exhibited a wide variety of specific strategies during their attempts to solve the problems. What follows is an inventory of the strategies used with some examples of their use.

Draw and label a picture. In problems which involved a considerable amount of information or confusing information, most of the subjects immediately drew some kind of picture and labelled it with the information from the problem statement. In some cases, further information was added as it was discovered. Candy's drawing of the cistern, problem 15, is more elaborate than most (see figure 9).

Redraw a given diagram without modification. In almost every case where a diagram was given as part of the problem statement it was immediately redrawn. In some cases the diagram was not modified or used any further. This was done by Karen in her attempt to solve problem 5, squares, and by Tanya who redrew the diagram given in problem 4, the spider and fly, but made no further use of her drawing, choosing instead to trace her routes on the original diagram. Several other students redrew the diagram from the spider and the fly and then used it only to trace out possible routes.

Draw a new diagram or modify an existing diagram. In all of the geometric

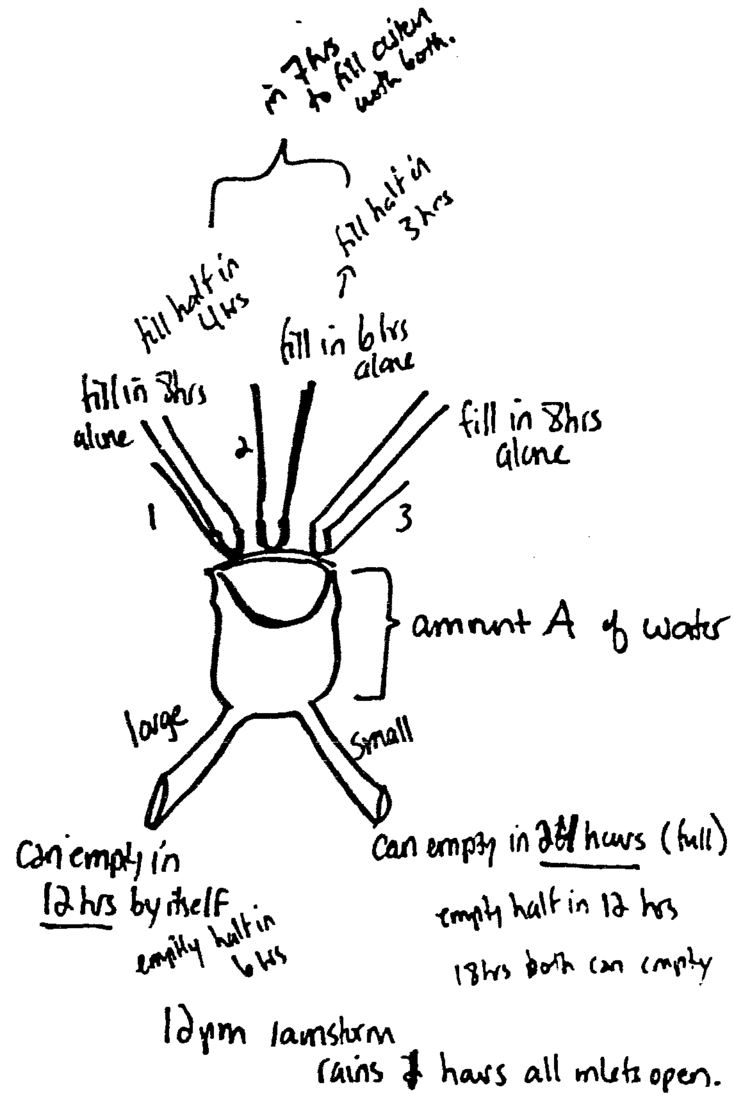

Figure 9 Candy, problem 15

problems the diagrams were drawn, or redrawn, and modified. In problems 8 and 9, the triangle and the tangent circle, subjects began their solution attempts by graphing the given line(s). The diagrams from problems 10 and 11, the circle problems, were always redrawn and modified to some extent, often extensively.

Draw a line to represent distance and/or time. All of those who attempted problem 1, the sleepy passenger, drew a line to represent the passenger's journey. A similar strategy was used by those working on the commuter, problem 15. In this case it was often unclear whether the line was meant to represent time or distance and the students often confused these dimensions of the problem.

Use trigonometry when right triangles are involved. This strategy was used by all who solved problem 12, folded paper, despite the fact that trigonometry is not needed to solve this problem. Kevin and Cecil attempted, unsuccessfully, to use it to solve the triangle problem.

Consider symmetry. Symmetry was used to limit the possible routes for most of those who worked on the spider and the fly, problem 4. Symmetry was also considered by Karla in her analysis of problem 11, the two circles.

Use Pythagoras' theorem. Pythagoras' theorem was used during solution attempts for the two circles and for the spider and the fly. However it was not generally used in the solutions to problem 12, the folded paper, where trigonometric approaches were preferred.

Measure a given diagram. Although she was warned that the diagram of the two circles in problem 11 was not to scale, Janet chose to measure it and use the measurements in her solution attempt. She also attempted to measure the position of the intersection point in the triangle problem, but was prevented by her partner, Carl.

Break a figure into triangles. Janet suggested breaking up the sheet of paper into various triangles when she was working with Kevin on the folded paper problem. Carol would have liked to use triangles to solve the four circles problem, but did not do so because the shapes were all curved and so not quite triangles.

Remember the algorithm. This appears to be a preferred strategy, used whenever an algorithm is available. When the algorithm was appropriate and clearly remembered, this strategy provided quick solutions. However in many cases the algorithm was not appropriate to the problem. Simon attempted to use a tree diagram to solve problem 17, handshakes, possibly because this was a counting problem and tree diagrams are used to count. In other cases, the algorithm was appropriate, but was not applied correctly, or was not well remembered. Sam attempted to use relative speeds to solve problem 13, the tanks in the desert, but he added the reciprocals of times, rather than the times.

Try a similar problem. This technique was used by several students in their attempts to solve problem 2, division by nine. They tried examples using 7 or 8 to ascertain if a similar rule held for these numbers. Simon attempted to confirm his answer to the handshakes problem by trying to solve the same problem for 3 rather than 20 people.

Make an assumption. Several students consciously made assumptions in order to make a problem simpler. This was true for Karla, and for Shelly and Carol, in their attempts to solve the two circles problem. We have seen that they assumed, incorrectly, that the radius of the inner circle was of length one unit.

Make an estimate. One student, Carl, made use of estimates during the solution process. In one case, he did this in order to limit the range of his answer, and, in the other, in order to see if it would be worth the time needed to carry out an exact calculation.

Use trial and error, or guess and check. These techniques were used on several problems, including problem 3, the shopping trip, and problem 7, the ski trip. However, most of those who found solutions this way did not consider this technique to be legitimate. Randy never submitted his trial and error solution to the shopping trip, but tried to find an equation that would give him the same answer. Shelly, and Kevin and Janet, however, showed no reluctance to accept trial and error solutions.

Make a table. This was generally done for problems such as number 6, the automatic washer, and number 13, the tanks in the desert, which involve time, rate and amount (distance).

Look for a pattern. This was the general technique used in attempts to solve problem 2, division by nine, and problem 5, the squares, and was also used for problem 18, factorial.

Break the problem into steps. This was tried only on the problems which clearly required it; the cistern, squares, and, to lesser extent, the triangle and the folded paper.

Rely on a general principle. General principles were occasionally used to guide analysis on problems such as the folded paper and, especially, the spider and the fly.

Use a variable, an equation or a formula. This was a preferred strategy and was used whenever possible, even when it was not applicable. Several subjects assumed that there were always formulae available, even if they did not know them. Cecil was starting his attempt to solve the four circle problem, when he said, "This is much trickier than the last one, definitely. (pause) I'm thinking that this is a trick question, um. If you know the area of each circle there must be a formula for the space in between." Similarly, Carl assumed that there must be some sort of function involved in the solution to problem 5, the squares. "I'm sure it's a log function. I'm, I'm just not sure exactly how to apply it." [Carl and Randy] Simon and Sam found problem 2, division by nine, very confusing. They knew that they were being asked for a proof but had no idea how to go about constructing one. "How would they make a proof in calculus or maybe a proof in algebra or whatever?" Simon said, "They always start with an equation. An equation with letters and symbols and ...." He went on to try to construct such an equation.

Eliminate most of the information and do a single step calculation. Carol applied this strategy to two of her problems and attempted to use it again when working with Shelly. "I hate these kind of questions," she said. "They have so much superfluous information in there I never know what to extract."

Calculate everything you can. Most of those who attempted problem 13, the tanks in the dessert, began by calculating everything that they could in the apparent hope that that would somehow lead them to an answer. Janet did this as well in her attempt to solve problem 11, the two circles.

Guess or guess which operation to use. Shelly and Carol were very clear that they had no idea how to solve problem 18, factorial, and were simply guessing. Others, faced with two or more possible operations to perform, sometimes simply guessed which one to do.

Write a potentially infinite series. This was attempted by Karla when trying to solve problem 5, squares.

Use a physical model. When Janet became confused about the relative movements of the tanks and the messenger in problem 13, she picked up two of the felt pens on the table and used them to model the situation.

Use a coordinate system. Simon placed the room in the spider and the fly in an x-y-z coordinate system. However, he made no real use of it.

The students, then, have a broad range of strategies available, most of them potentially quite useful if appropriately applied. However, they lacked skill in deciding which strategy to apply. There was generally little or no consideration of alternate strategies, and no evaluation of strategies. Most often, they followed the first strategy to come to mind until it was clear that it was leading nowhere, or until they became sufficiently confused. As there was little overt strategy evaluation, it was difficult to determine on what basis a particular strategy was chosen. However, a few criteria could be determined.

Algorithm. If an algorithm was known it was the first choice. If an algorithm failed to provide an answer or provided an unreasonable answer, it was generally abandoned.

Ease of use. Ease of use was another important quality in strategy choice. Shelly, for example, chose guess and check to find the intersection point in problem 8, the triangle, because it looked easier.

Leads to immediate partial results. A strategy which led to an immediate partial result was also preferred. This is clearly seen in the choice of trigonometry to solve problem 12, the folded paper. Trigonometry generated a series of partial results, while the use of Pythagoras' theorem did not provide a numerical answer until the last step.

Not a legitimate method. Trial and error and guess and check were rejected or questioned by several students because they were not seen to be legitimate mathematical techniques. Cecil solved problem 7, the ski trip, by guess and check but then wanted to work backward to create an equation. "So we can, I know that's sort of cheating," he said. "We shouldn't be allowed calculators here. (Pause) Well, there's 20 club members going on the trip. (Laugh) So we got to work back to this somehow." [Cecil and Kevin]

It is more mathematical. Karla and Candy began problem 16, making change, by trying a few combinations but then they began to doubt that this approach was the right one. "Do we have to use any mathematical formula?" Candy asked. Karla answered, "Like, we're supposed to, but it doesn't matter how we solve it. (laughter) We have 15 minutes. That should give us enough time. Or, do you know a mathematical way to solve it?" This led them to spend some time attempting to create an equation.

I always do it that way. Carol hoped to find the area between the four circles by using triangles, because she always finds areas using triangles.

Unable to see how to implement the strategy. Often an approach was rejected because the student was unable to see how to implement it. Both Karla and Karen wanted to use a variable in their attempts to solve problem 5, the squares. But both rejected this strategy when they could not see how to define a variable in a reasonable manner.

Looks familiar. As Shelly and Carol worked on problem 18, factorial, Carol suggested that it might have to do with the fact that 100 is 10 squared. When Shelly questioned this, Carol replied that she had seen this before but she just could not remember.

You are incorrect. Occasionally one partner of a pair would reject the suggestion of the other partner, because he or she believed it to be incorrect. This was not always helpful, as, for example, when Cecil rejected a suggestion by Kevin to use simultaneous equations to find the intersection point of two lines, because he did not believe that that was the purpose of simultaneous equations.

Not clear enough. Diane twice suggested forming a right triangle with the radii of the two circles and tangent line segment in problem 11. The first time Sam rejected her idea since they did not know the length of the radii. The second time, however, Sam followed her suggestion. This time she presented it with a diagram and the suggestion to use Pythagoras' theorem. Her suggestion was much more complete.

Precision

Calculations were generally carried out correctly and precisely. The one notable exception to this was in Cecil's attempt to solve problem 10, the four circles. He chose to use 3.14 for π rather than using the key for π on the calculator. He then multiplied by 4, rounded the result to 12.5 and used this in further calculations. He recognized that his result would be inaccurate, referring to it as a rough estimate, but appeared to see this as of no importance. He willingly submitted his inaccurate result as his final answer to the problem. This attitude was an exception and most students aimed for accuracy in calculations, presenting exact answers such as 4 - π, or carrying as many decimal places as the calculator would allow.

Occasionally formulae were remembered incorrectly or not at all, but in all such cases the students asked for and received the correct formulae. They were then generally able to use them correctly. Algorithms were often not remembered correctly. When Sam attempted to use relative speed to solve problem 13, the tanks in the desert, he added the reciprocals of the times rather than the times themselves. On the other hand, when Simon attempted to solve problem 6, the automatic washer, he remembered the algorithm correctly, and constructed an appropriate equation. However, he made significant algebraic errors in attempting to solve the equation.

Inferences were often too imprecise to be of use in solving the problem at hand. Problem 5, squares, was puzzling to Carl and Randy. While Carl felt that the problem could be solved by fitting some sort of function to the situation, Randy attempted to engage him in a process of reasoning through the implications of an infinite process.

R If this process continues forever it would eventually reach all of it, wouldn't it?

C Ah, it's getting smaller, the amount that gets shaded each time is getting smaller. It's like the question, when does the, ah, it, it gets to, ah. Every time there's a smaller and smaller space but at some point it reaches nil.

R Yeah. But, it says, if the process continues forever. So if we did it like that, assuming that it, it would never end wouldn't the answer still come out.

[Randy and Carl] They were unfamiliar with infinite processes, and unable to develop sufficient precision in their thinking about these processes. A similar imprecision in inference was evident in most of the attempts to solve problem 2, division by nine. Although on the factorial problem, most subjects eventually came to consider the factors of 100!, they all omitted at least one of the required factors, due to their imprecise thinking about factorization. Here also, their lack of familiarity with the correct terminology made it difficult for them to express or discuss their ideas, further hindering them in their attempts to solve the problem. A lack of precision in the use of vocabulary was evident throughout. Equation, function and formula were used interchangeably, usually to refer to equations or algebraic expressions of any kind. The subjects found it especially difficult to express their ideas in problems where the concepts were less familiar, such as squares, division by nine and factorial; they appeared to lack the requisite vocabulary.

Diagrams were used extensively and were often crucial to finding a viable solution method. However, there was great variety in the precision of the diagrams constructed. In her attempt to solve problem 8, the triangle, Shelly created a clear and precise graph of the two lines. The precision of her graph allowed her to guess the intersection point of the two lines, thus saving her a calculation of which she was unsure. Sam's drawing of the four circles, problem 10, allowed him to see immediately that the radii could be drawn to create a square that would just enclose the shaded area. As noted, when working on the same problem, Karen began with a quite rough sketch, but as her solution progressed, she redrew and modified her diagram seven times, the important details becoming more precise with each redrawing. In contrast, Cecil's sketch for the same problem was so imprecise that he was unable to correctly divide it into smaller pieces. Thus, although he had developed a viable solution strategy his answer was, in the end, incorrect.

Cecil worked with Kevin on problem 8, the triangle, and together they produced a rough sketch of the two lines. When their later work produced results that contradicted their sketch they considered, but never actually carried out, redrawing the sketch to scale and using a ruler. Their resulting erroneous calculations led them to a wrong answer. In similar circumstances, Diane and Karen redrew their sketch for problem 9, the tangent circle. This helped them to discover their error. All those who attempted to solve problem 13, the tanks in the desert, began by drawing a sketch of the column of tanks and the messenger. However, none included on their drawings the movement of the column of tanks. This missing detail was crucial to their lack of success in solving this problem. It led two individuals to assume that the time the messenger spent travelling backward was equal to the time spent travelling forward.

Particularly in the area of algebra, subjects exhibited a lack of awareness of the usefulness of precision. A few students defined variables precisely, and in writing, but most made no attempt to do this, and some changed the meaning of a single variable several times during a single problem. Definitions, when given, were generally imprecise. Consider Tanya's attempt to solve problem 6, the automatic washer.

T Okay, so with the cold water valve open it takes 8 minutes, So, x equals cold water. Ah, okay, so it takes 8 minutes, for the cold valve. With both hot and cold water valves open, it takes only 5 minutes to fill the tub. So, x plus, we say y is the hot water, equals 5 minutes.

... So x, say x is the amount of cold water it takes and y is the amount of hot water it takes. Times the time so the amount of cold water over 8 minutes. x over 8, x, 8, 8 minutes.the hot and cold takes 5. That's 3 minutes less, with, 3 minutes less with the hot water as well.

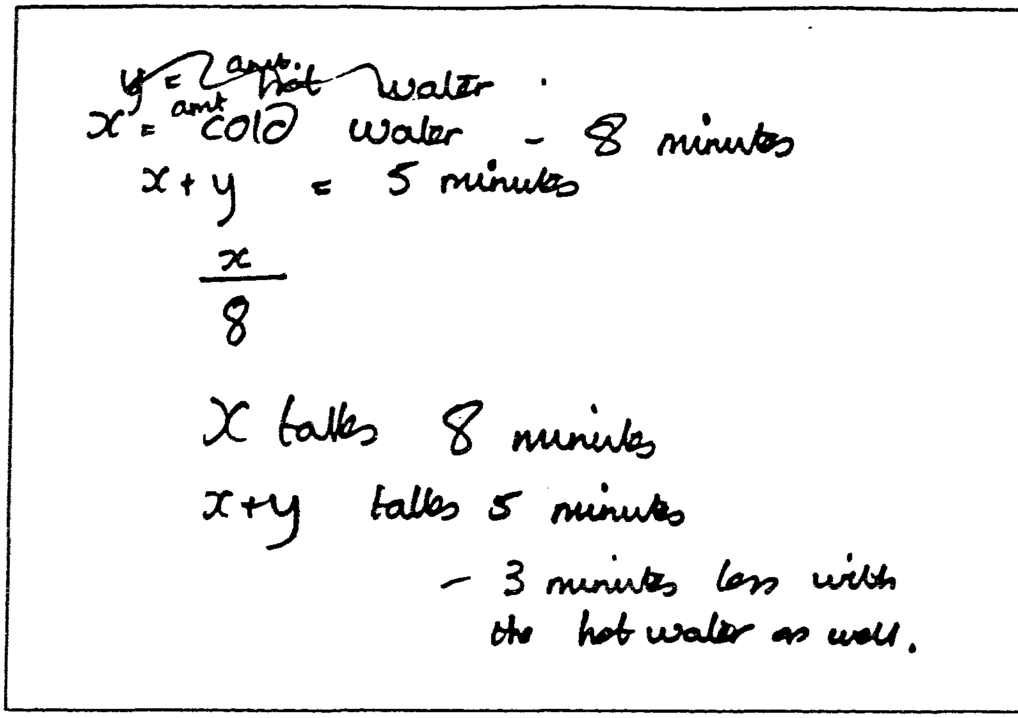

Figure 10 Tanya, problem 6.

[Tanya] We see that she began by simply using x and y as a kind of shorthand and then implied, by setting their sum equal to 5 minutes, that they represented times. However, she then redefined them to represent amounts. Further on in her solution she constructed and solved an equation for x, in which it appeared, once again, to represent time. This confusion in the use of variables has been well documented by Küchemann [1978]. The equals sign is also used imprecisely and with several apparent meanings. It is commonly used to designate definition in the naming of variables. It also appears to be used to mean "now calculate". A good example of this use is provided by Kevin and Janet in their trial and error solution to problem 3, the shopping trip. After trying $20 and $15 they decided to try $17.50:

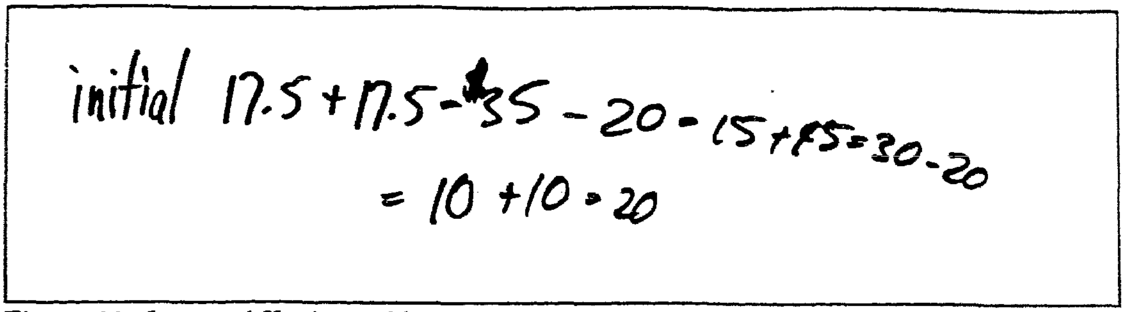

Figure 11 Janet and Kevin, problem 3

[Kevin and Janet]

Equations are also understood imprecisely, with quantities of different dimensions added together or set equal to each other. In the attempt by Diane and Sam to solve problem 15, the commuter, they had set x to be the time spent driving by the husband and y to be the distance which the commuter travels. The time saved by the husband is 1/6 of an hour. They put these quantities together to form the statement:

$$y + x - 1/6 = \text{distance.}$$

This problem elicited similar confusion of time and distance in most of those who attempted it. Confusion of time and quantity was common in solution attempts for problem 6, the automatic washer. Errors and misinterpretations in the use of algebraic notation has been analyzed in detail by several researchers. [See Küchemann, 1978 or Rosnick, 1981.]

Monitoring

Complete written records were not commonly kept. Most commonly, records consisted of diagrams and calculations only, sometimes with the answer given in a sentence, underlined or circled. Some went a step further and labelled the steps in a longer problem such as number 14, the cistern. Karen also did this in her attempt to solve problem 5, squares. Only a few subjects kept more extensive notes, including notes of their reasoning. One was Shelly, and it was particulary evident in her solution to problem 2, division by nine. Not only did she write down her examples, she wrote down her ideas using full sentences. She also, after asking me what exactly was required to "answer why," wrote down her understanding of my reply, "Explain why it works." Lack of written notes proved to be important in cases where assumptions were made. Karla decided to try problem 11, the two circles, using an assumption. She was quite explicit that this was simply an exploration, but did not write this down at all. What she did, instead, was to label the inner radius on her diagram as one unit long. Later, when she returned to the diagram again, she made use of this length, apparently forgetting that it was simply an assumption. Karen and Diane made a similar error in their attempt to solve problem 9, the tangent circle.

With complex problem statements, it was common to make a drawing and transfer as much information as possible from the problem statement to the drawing. This was common on problems 6, 13, 14, and 15, the automatic washer, the tanks, the cistern and the commuter. Janet even did this for the shopping trip. Some made no attempt at all to organize their solutions. This was true of Carol on problem 16, making change, and of most of those who attempted to solve problem 4, the spider and the fly. Most who tried this latter problem simply traced out several routes with no clear record kept. An exception was one pair, Karen and Karla, who traced out their routes in different colours. But they made no further attempt to organize the possible routes in some other way.

Most students did not rush through their problems, and did take some time for reflection. However, this was not true of Carol and Tanya, both of whom completed four problems in less time than most of the students took for three. Neither got any of their questions correct. Karla worked at a comfortable pace for about ten minutes on each problem and then, ceasing her analysis of the problem, submitted an answer which she called a guess and in which she expressed little confidence.

Generally, there was little monitoring of progress or of states of mind during individual problem sessions. However, Cecil and Carol both noted their states of mind often, even when working alone. Carol, for example, noted that she often chose an obvious answer, only to find that it was wrong, and later she reported that she had "worked herself up" and would now probably get the wrong answer. Janet often monitored her progress closely, asking herself if her work made sense and noting when she had gone as far as she thought she could. Shelly also occasionally noted her progress. When the subjects worked with partners there was more overt monitoring activity, especially of their states of mind. They often admitted to confusion when working with a partner. When Kevin and Cecil were faced with solving a complicated looking equation, they were both intimidated. "I don't even want to try it," Kevin said. Cecil replied, "I know, I was getting scared when I saw all this stuff up here." Karla and Candy also discussed their confusion which, during the solution to problem 6, led them to organize their count of the various ways to make change for a quarter. Janet was a notable exception to the trend towards more monitoring activity with partners than alone. She monitored her progress when working alone, but ceased doing this entirely with both her partners, Kevin and Carl.

Partners were no more likely than individuals to keep written notes. Individuals who kept more extensive notes when working alone, did not necessarily do so when working with a partner. Randy kept more notes than most when working alone, but kept almost none when working with Carl. Shelly, whose more extensive records are discussed above, continued to keep written notes during one pair interview, but not during the other. Partners did monitor each other's calculations, and occasionally, were able to spot errors. But, since calculation errors were not a major problem, this did not lead to any great advantage.

In general, monitoring, other than of state of mind, was not common and was often superficial. Written notes, adequate to the problems at hand, were not generally kept.

Beliefs

In this section, I will consider the subjects' specific beliefs about mathematics and problem solving.

There are rules to the game of mathematics. Many of the subjects appeared to believe that there are fixed rules to follow when doing mathematics, especially rules about how to present one's work and one's answer. Several students asked if I required written answers, and others asked how they should present their working steps. Candy was particularly concerned to know exactly what I required. When asked to read the question aloud, she asked if she should read the question number as well. Later, she wanted to know if she could write things down as well as say them aloud, and whether she could use a second sheet of paper. Felt pens and several sheets of paper had been placed on the table in front of her.

Several students appeared to believe that it was necessary to submit an answer whether one believed it correct or not. Carol, Karla and Janet all submitted answers in which they clearly had no confidence. Janet explained her decision to submit an answer that she thought was incorrect in this way, "So at this point, say, for example, I was writing a test, I would say, 'forget it.' This is my answer, 7.5 km long and I have no idea if that's right or wrong."

They may try to trick you. Several subjects believed that math problems were often constructed to trick the student. Carol saw the process of solving complex problems as one of removing all the superfluous information to find the single arithmetic calculation required to obtain an answer. Randy thought that problem 1, the sleepy passenger, was "just like those other questions, during the summer. The five minute mysteries." Diane, working with Karen, wondered if one of their problems was really a riddle, and whether another one might be "one of those weirdo calculus things."

Only some techniques are acceptable. At one point in their solution to problem 9, the tangent circle, Carl and Randy considered the use of Pythagoras' theorem but Carl rejected this, saying that it would be a form of cheating. Many students appeared to view trial and error solutions as unacceptable, and after finding an answer in this manner, would try to create an equation that would give them the same answer. Solutions involving equations were generally seen as preferable and as more mathematical. Karla and Candy solved problem 16, making change, by creating an organized list of all the possible combinations. However they viewed this as a "loser's way" since it did not involve an equation. They saw their solution as one in which they "didn't need to use math," clearly identifying mathematics with the use of equations. Karen and Diane were unable to solve this same problem, as they never attempted to count the combinations at all, but spent the whole of the allotted time trying to create a system of simultaneous equations that they hoped would give them the answer.

There is a formula for everything. Many students appeared to view mathematics as simply the application of the correct formula for the particular situation. Some students were so certain that a formula of some kind was required that they would find one even where one did not exist. While they saw the x's in the definition of 5! and 10! as multiplication symbols, Janet and Carol both misinterpreted the x's in the definition of n! as variables and wanted to use the definition as an equation to be solved. Similarly, Carl was so certain that there was some sort of logarithmic or exponential function which would give him the solution to problem 5, squares, that he could not be persuaded by his partner, Randy, to try any other approach. Cecil was certain that there must be some formula to give the required area in problem 10, four circles, and did not seem to see that, in essence, he was being asked to construct that formula himself.

There always is a solution. Many subjects clearly believed that all math questions have answers. If it is a math problem then the only question is, "How do you solve it?" not whether it has a solution. [Randy] And the answer cannot be too easy. "I always worry," Janet said, "If we get the answer right away." Furthermore, the answer should look right and this often means it will be a whole number, or end in a 5 or 0.

It is very difficult or impossible to do a problem you have not seen before. "Okay, so, I don't know how to approach this problem, as we haven't done anything on triangles or anything like that." Janet said to Carl as they began problem 8, the triangle. Cecil was similarly confused when asked to solve problem 10, the four circles, since he had never attempted a problem like that before. Both Cecil and Janet did go on to do substantial work on their problems, Janet with Carl's help, but both were initially intimidated by the unfamiliarity of the problems. Shelly and Carol, however, gave up and guessed on problem 18, factorial. Shelly said, "I think I was sick in school when we did this," and Carol added, "Shit! I've seen this before and I just can't remember." In her exit interview, Candy indicated that her favourite mathematics teacher had been her grade eleven teacher who explained everything and wrote extensive notes on overheads. As soon as she did not know what to do during any of her problem sessions, Candy immediately turned to me with questions or abandoned her attempt.

Some things are beyond the ordinary person's understanding. Several subjects seemed to believe that mathematics was somehow different from ordinary understanding. After trying a few examples of multiples of nine for problem 2, Cecil commented, "That's really strange. Hum, I'd probably have to ask a mathematician about this one." And, later in the same problem session, he indicated that he hoped I would give him the answer after the session and "I hope I can understand the answer for this one." Commenting on the same problem Janet said, "Because whoever invented math wanted it that way. How on earth could you explain that? It's a neat trick, though." Immediately after this she gave up her attempt at this problem.

One has to have a brain for math. Some subjects saw math as a special ability that you either have or do not have. Upon completing the solution to problem 10, four circles, Candy commented to her partner, Karla, "My brain kind of works that way." During his exit interview, Cecil stated that, "Unless you have a math brain, good study habits and a logical sense of reasoning would be of the greatest importance to success in math." Clearly he sees mathematical ability as something quite separate from ordinary logical reasoning, which is resorted to only in the absence of a special mathematical ability.

Sudden inspiration can be important in mathematics. A few subjects appeared to believe that sudden inspiration is important in solving problems. Karen and Diane felt that if they did not immediately follow up on such "brainstorms" they would forget them, clearly indicating that they do not see problem solving as being under their conscious control. "Yeah, if we both get a brainstorm at the same time we lose it," Karen said during a discussion with Karla about the difficulty of working with a partner. In her exit interview, she added, "I wish I did know that switch, you know, like when the light comes on suddenly."

When in doubt, use technology. During their session working on problem 18, factorial, Diane repeatedly turned to the calculator despite the fact that Sam told her it was of no help. She finally replied to Sam, "When in doubt, use technology." Others appeared to feel the same way, including Cecil who, when unable to solve problem 18, squares, wondered, "maybe it has something to do with this button [on the calculator] here."

Math can be intriguing. Several students found the problems intriguing even when they could not solve them. In particular, Janet and Cecil found some of their questions to be intriguing or "cool." And Karla reacted to problem 5, squares by noting, "Oh, this is a neat question." Carl and Shelly both said that they liked puzzles and problems, and found them fun. Several subjects, in exit interviews, noted that they liked problem solving, once they could "get a handle on it." [Diane, exit interview]

Reactions to Confusion and Frustration

Confusion was very common during all problem solving sessions in which the subjects did not have a solution method immediately available to them. Generally, confusion initially led to a rereading of the problem, or part of the problem, or a long silent pause. Occasionally, a student reacted by rereading or reassessing the steps that he or she had already taken. An example of such reassessment of work completed is when Karen, confused by the diagram, questioned Diane's assumption that, in problem 9, the tangent circle, the centre of the circle is above the x-intercept. Cecil became quite confused by what was wanted in question 2, division by nine, and after a length of time he also reviewed what he had discovered:

C Well, we have discovered that any numbers that add up to 9 are divisible by 9. We have also discovered that when we're multiplying the two. Um, if multiplication occurs, we subtract... If you subtract the number being multiplied by 9 from 9 you get the first digit of the number. And the number you're multiplying by becomes the last digit of the number. And they add up to nine.

[Cecil] More generally, confusion led to the abandonment of the particular line of reasoning being followed, without any attempt to see why this method might be flawed. In some problems, such as number 2, division by 9, confusion led most subjects to try more examples. If confusion continued for long, it generally led to some degree of frustration.

Reactions to frustration were varied. Many students simply gave up, especially when working alone. Others did not give up but submitted answers they clearly knew were wrong, apparently in an attempt to end the frustration and confusion. After spending a substantial amount of time trying to figure out a solution, Karla, on each of her problems, appeared to hit a point where she could tolerate the tension no longer, and she almost immediately submitted an answer in which she had no confidence at all.

When working with partners, there was no instance where a pair gave up before their time was up, and only one pair willingly submitted an answer which was clearly an attempt to end the problem session. Carol and Shelly were working on problem 18, factorial, when Carol asked Shelly, "Alright, are we ready to accede to this one?" Shelly replied, "I think we should guess 20 zeros." They immediately presented this answer, without any form of evaluation at all. A few students showed little sign of frustration. Sam was one of these. When his attempt to algorithmically solve the tanks in the desert problem failed, he immediately began a second solution attempt from scratch. His only frustration came when he was forced to leave the problem unsolved as time had run out.

In exit interviews, several students mentioned being frustrated with mathematics problems. Returning to school after a long absence, Diane found her first math course so frustrating at times that she almost cried in class. Karla and Karen both said that, if they were unable to get an answer very quickly, they became extremely frustrated and quit trying. Karen described her feelings, "I'm so impatient.... If it's not coming to me right away, forget it. I'm not going to work on it any more." Both Simon and Janet said that they found applied problems to be interesting, but that manipulative algebra was boring and very frustrating.

PAIR INTERACTIONS

Ideally, one might expect that working in dyads would require that the subjects attempt to construct an agreed upon representation of the problem and then decide upon the approach to be taken to solve the problem. However, the analysis shows that this is not what happened. Rather, they were generally so fixated upon finding an answer that little effort was put into analyzing the structure of the problem, or into generating and comparing various strategies. As well, concern about social interaction often worked against a rigorous analysis.

Skills and Strategies

Pairs tended to spend a little longer than individuals on the comprehension phase of the problem sessions. There was some more discussion of the particulars of a problem, and especially, of any assumptions made. Shelly and Carol, for example, made a conscious decision to solve problem 11, the two circles, under the assumption that the radius of the inner circle was equal to one. Even after they solved the problem in that manner, they noted that their answer was correct only under that assumption. Similarly, in two of three pair attempts to solve problem 15, the commuter, one partner had to be reminded of the convention that the 8 km/h from the second part of the question could not be used to solve the first part. However, the third pair broke this pattern, and both partners made use of the speed prematurely. Misunderstandings were also sometimes avoided in pair sessions. This is especially noticeable in two attempts by pairs to solve problem 18, factorial. In both cases, when one partner misinterpreted the definition as an equation the other partner corrected the misunderstanding.

The phase of planning, assembling and implementing a solution did not differ much from that of individuals. Despite the expectation that the necessity to explain one's actions to one's partner would lead to more planning, little planning was apparent. Discussions were generally limited to one step or one idea at a time. The only exception was the case of Kevin and Janet's solution to problem 12, folded paper, quoted previously. Here Kevin described to Janet a complete plan to solve the problem using trigonometry, a topic with which Janet was unfamiliar. Attempts at analysis generally followed the pattern of the more able student, so that, for example, when Kevin and Janet solved problem 1, the sleepy passenger, it was essentially Kevin who solved the problem, explaining his reasoning to Janet. Similarly, attempts to construct equations usually followed the pattern used by the more able partner.

Pairs were just as likely, having chosen an inappropriate strategy, to stick with it even when it was not leading to a solution. This was especially evident when Karen and Diane attempted to solve problem 16, making change, by setting up a system of equations. Although neither of them appeared to have any idea how such a system would give them a count of different combinations, they stayed with it until their time was up. However, working with a partner did appear to help prevent some minor calculation or mechanical errors.

There was little difference in the use of verification between individuals and pairs. There were 14 verification attempts by individuals and 12 for pairs. It is interesting to recall that when Shelly attempted to check Carol's calculation in problem 17, handshakes, Carol objected very strongly and the check was not carried out. This calculation was Carol's one contribution to this problem.

The variety of strategies used to try to solve problems was as great for pairs as for individuals. However, strategy selection was more accessible to study. There was little evaluation of possible strategies, and decisions about the choice of strategy generally depended more on the personal interactions between the partners, than on any mathematical criteria. This will be discussed in more detail below.

Students showed no greater tendency for precision when working in pairs than when working alone. Variables were still not clearly defined and diagrams were often too messy for their purposes. Students often lacked the vocabulary to discuss the problems, and their own ideas, in detail. This is particulary true in problem 18, factorial, where no student used the term factor. This made discussions imprecise and awkward.

As has been discussed previously, subjects working with partners exhibited more monitoring of their states of mind than did subjects working alone. Unfortunately, the monitoring was generally superficial and did not lead to changes in behaviour. Partners made no more written notes of their progress than did individuals and in some cases, Randy and Shelly, for example, individuals kept fewer notes when working with a partner. Partners did however monitor each other's work and calculations. As a result there were no uncorrected calculation errors amongst partners and few algebraic errors. A notable exception was Shelly's error in not squaring the radii in problem 11, the two circles. Neither she nor Carol saw this mistake, which was crucial in preventing the simplification of the expression for the area.

Advantages of working in pairs

In discussing the advantages of working in pairs I limit myself, here, to discussing the advantages in problem solving efficiency and do not consider learning outcomes for individual students. Direct advantages of pair work appeared to come primarily from an increase in persistence, from the more able student leading the pair, and from monitoring of calculations.

There was a significant increase in persistence exhibited by pairs over individuals. When students worked alone there were a total of five problems in which the student simply gave up his or her attempt. There were no instances of quitting before the time was up amongst pairs. In only one case, the attempt by Shelly and Carol to solve the factorial problem, number 18, did a pair present a solution in which they had no confidence, clearly using this as a method to end the problem session without quitting. This behaviour was more common amongst individuals. In all three of her problems, Karla presented solutions in which she had little confidence. In each case she worked for at least ten minutes and then suddenly, as though she could endure the frustration no longer, she submitted a quick answer. "I guess. I guess that's the best I can do with that one," she said. Both Carol and Tanya, when working alone, completed four problems during a single problem session. It was planned that each student should attempt 3 problems per session, allowing approximately 15 minutes per problem. However, in these two cases they spent approximately five minutes on each problem and so there was time for each student to attempt a fourth problem. Seven of the eight problems were done incorrectly and in the eighth problem the student, Carol, quit. In both cases, but in Carol's work especially, the solution attempts were simplistic and no real attempt at understanding the structure of the problem was made. There was no case amongst the pairs where completing a problem, or a complete problem session, proceeded so quickly.

Candy was the only individual subject to attempt the cistern problem, number 14, and she quickly gave up. After spending some time drawing an elaborate picture and transferring the information from the problem statement to the picture, she quit, with no further steps taken towards solving the problem. While the three pairs which were given this problem were also initially overwhelmed by its apparent complexity, all of them were able to solve the problem correctly.

A major advantage for pairs arose from the pairing of a more able student with a less able student. The more able student often led the way to a solution that the less able student would not have seen. Subjects working in pairs were successful in 18 of the 34 solution attempts. However, 10 of these 18 problems were essentially solved by just one of the partners alone, with the second partner making no substantial contribution to the solution.

Shelly had a complete and correct strategy for solving problem 17, shaking hands. As Shelly explained it to Carol, Carol admitted that she would not have done it that way. "I just, I just want to automatically go, oh well, 20 people. Well, if each person shakes one person's hand that should mean there's, if you count the two that should be 10. [laughter] Half the class shakes the other half's hand, right? So yeah, 10." The situation was similar when Kevin and Cecil attempted to solve the ski trip problem, number 7.

Kevin was attempting to construct an equation when Cecil said, "The way I'd be doing it, I'd probably be just stumbling around and writing, drawing little pictures, pretty well, doing [laughter]." While Kevin was unsuccessful in his endeavour he greatly advanced their attempt toward an algebraic solution to the problem. When Carl and Janet attempted to solve problem 14, the cistern, it was Carl who broke the problem down into parts and knew to use the reciprocals of the times to obtain rates. In each of these cases, and in several others, a more able student knew, or was able to construct, an overall strategy to solve the problem.

In other cases a more able student, while not having a complete plan available, was able to correct the errors or misunderstandings of a less able student. During their attempt to solve the factorial problem, number 18, Carl corrected Janet's misunderstanding of the definition of n!. Similarly, in their attempt to solve the same problem, Sam prevented Diane from misinterpreting the calculator output and so from falling into the same error that Karen and Karla had made in their attempt to solve the problem.

In only a few cases did both students contribute parts of a complete solution that it is likely neither would have been able to find alone. Carl was able to see a general solution outline for problem 8, the triangle, but did not know how to find the point of intersection of the two lines. However, Janet was familiar with an algorithm to complete this step in the solution and together they were able to solve the problem. When Simon and Sam attempted to solve problem 7, the ski trip, it was Sam who was able to construct two equations in two unknowns while it was Simon who knew a technique to solve this system of equations. However, this balance of contributions leading to a successful solution was not common.

Pairs also benefitted by the monitoring, by one partner, of calculations performed by the other partner. As part of their solution to problem 7, the ski trip, Simon and Sam needed to solve a quadratic equation which they did using the quadratic formula. It was Simon who carried out the calculations, and he missed a negative sign which Sam noticed immediately:

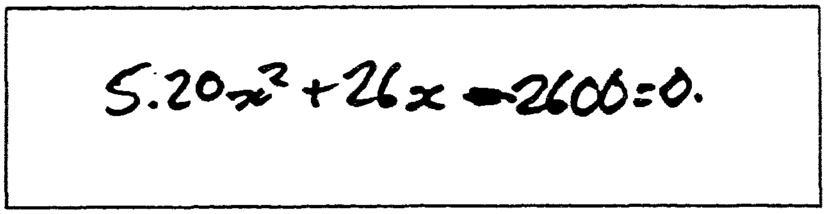

Figure 12 Sam and Simon, problem 7

Sa Sure, we can plug it into the quadratic formula.

Si Negative twenty six plus minus the square root of twenty six squared minus 4 times five point two zero times twenty six hundred

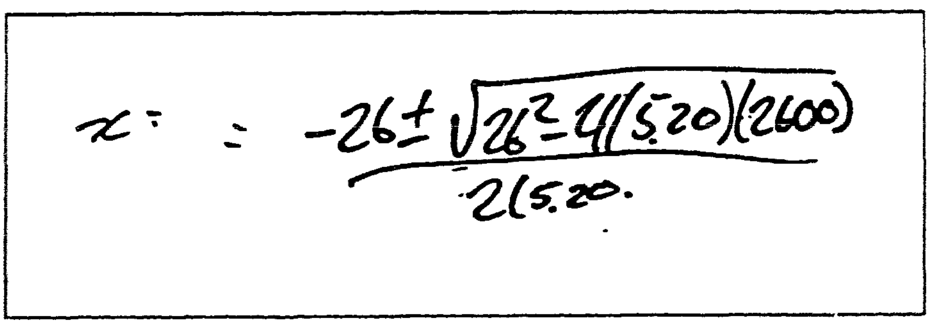

Figure 13 Sam and Simon, problem 7

Sa Minus twenty six hundred.

Si Minus twenty six hundred all over two times five point two zero.

[Simon and Sam]. Karen and Diane were on the last step of the cistern problem when Diane forgot to multiply by 3/4. Karen corrected her. There are several other examples. While one cannot know whether such errors would have been caught by an individual, it is clear that finding the error immediately increased the efficiency of the problem solving process.

In exit interviews, the students did not generally attribute many benefits to working with a partner. Most said that working with a partner brought a second perspective to the problem, but they were not specific as to how this actually helped them. One student, Simon, noted that with a partner there was more information available. It, Simon said, "is like a data base. The more data you can enter into it the more accurate that data is going to be."

Disadvantages of working in pairs

Having a partner was at times disadvantageous. When Karen and Diane attempted to solve problem 9, the tangent circle, Diane made the assumption that the centre must be on a vertical line with the point where the line intersects the x axis. Karen questioned this erroneous assumption but did not pursue her doubts in the face of Diane's certainty.

K Does that necess, that doesn't necessarily mean that that's the centre of the circle, now does it? See, I didn't really draw this to scale. ...

D You don't have to, really, draw this to scale, just, you just know that this length will be 5.

K Um hum

D Because, in any, the radius is always the, the same length, any, at any point on the circle, right?

K Um hum

D So any point I draw from here to the centre will be the radius, will always be 5.

K Okay.

[Karen and Diane] In this case it is not clear that Karen would have been able to solve the problem on her own. However, when Cecil and Kevin attempted to solve problem 8, the triangle, it appeared that Kevin would have been able to solve the problem if he had stuck with his own plan and not listened to Cecil. Kevin realized that all that was required to solve the problem was to find the intersection point of the two lines.

K I think we can do that with, ah, did you do that solving systems of equations? Where you have the two and then you, ah, like this, this, and then you subtract

C Um hum.

K one from the other and then you find the intersection?

C Subtraction and addition methods of

K Yeah. ... Use that to find the intersection point?

C Um ... I think it was just to find, to find the value of y, given two equations that relate to each other. You're supposed to find the value of y that works for both equations. I think that's what that was for. Um ... We just did that, ah, like three weeks ago (Inaudible). ... Ah. ... Well, it's a good question anyways. We know what the triangle looks like.

[Cecil and Kevin] They then abandoned the idea of solving the equations simultaneously and began an erroneous trigonometric analysis that led them far from any solution.

Another possible disadvantage of working in pairs is the lack of time to simply reflect. This is immediately apparent in viewing the video tapes. During almost all sessions with pairs, there was little silent time on the tapes, while during most sessions with individuals there were significant times when nothing was said.

In exit interviews several students noted that it was difficult to work with a partner because they became distracted from their trains of thought. Most of the students expressed a preference for working alone, and found working with a partner disadvantageous because of the necessity to focus on the social interaction to the detriment of mathematical reasoning. Diane expressed her feelings clearly, "There's politics involved in any kind of group situation. You know, you have to be more aware of what's going on, more sensitive to what the other person is feeling. Don't stomp all over their ego." Karen found working with a partner to be restraining, "Well, you want to be diplomatic and you want to let the other person have their say and things like that and if you both get an idea at the same time, you know, you both want to run with it and, I don't know, maybe it was me, I sort of want to take off on my own." [Karen, exit interview] Cecil found working with a stranger especially stressful as he was concerned with "not being on a par" mathematically with the other person. This, he said, "took away my ability to reason the way I wanted to reason." [Cecil, exit interview] Kevin found that he had to slow down in order to explain to his partner what he was doing, before moving on to the next step. Carol said that it was simply easier to concentrate when working alone.

Personal interactions

A two dimensional framework categorizing pair interactions emerged from the data collection and analysis. The first dimension is social interaction, beginning with very socially cooperative pairs and leading to those that were very uncooperative. The second dimension is mathematical interaction and is characterized by similarity or difference in mathematical background and ability. These two dimensions interact to create five categories. These categories may overlap somewhat and should not be conceived as rigid. While most pairs remained in a single category throughout their problem session, two moved between adjacent categories on different problems. As one might expect, such category switching depended upon the partners' relative mastery of the relevant mathematical material.

Table 4 Pair Categories

mathematical → **social ↓**	**similar**	**different**
cooperative	Socializers	
	Partners	Tutor/pupil
	Individuals	Individuals
uncooperative		Hostile pairs

Socializers. Socializers are generally of similar mathematical background and ability, and are very cooperative. For these pairs, social interaction is of central importance. They are very polite and non-assertive, seldom criticizing, and softening what criticisms they have by presenting them in the form of questions or by adding apologies. Dialogues contain a large number of supportive interjections. Strategic decisions are often made on the basis of social, rather than mathematical criteria. Pairs in this category were Karla and Karen, Karla and Candy, and Kevin and Cecil.

Karla and Karen's interactions are typical. They began their problem session with a show of politeness:

L Read the problem out loud and then start on it.

Kn Would you like to or shall I?

Ka Whatever you wish.

Kn I'll go ahead.

Ka Would you like to?

[Karen and Karla] They continue with this polite interaction throughout the session and, as well, they support each other with an almost constant stream of supportive interjections. They were both enthusiastic and talked very quickly, often both talking at once. These were not interruptions, but more in the character of completing the other's thoughts, or offering enthusiastic support. As a result the transcript of their problem session ran to 41 pages while the average length of pair transcripts was 20 pages. They generally suggested ideas in the form of questions, and did not challenge each other at all. Near the beginning of their attempt to solve problem 7, the ski trip, Karen wondered if they should look at the problem from the point of view of profit and loss. Although Karla appeared to have little enthusiasm for this line of thought, she did not prevent Karen from following it through. Neither did she ask for any justification.

Kn Is this a profit and loss thing? No, the profit is zero. Maybe it's a profit? You know the profit equals, er, profit equals revenue minus cost equation? Have you seen that before? It's just P equals R minus C.

Ka Yeah, I have seen that before.

Kn And if the profit is zero, the revenue would be y minus five twenty times x plus five.

Ka Well, do you want to go off on that tangent for a second?

Kn Yeah.

Ka	I wouldn't have thought of that but maybe we can look at this idea.

Kn	Let's see where we can go.

[Karen and Karla] This is entirely typical of each change in direction they made during their problem sessions. They never challenged each other and there was no attempt to evaluate ideas. When one wanted to change the direction of the work the other always acquiesced. Often they went off on a long tangent together. This happened in their attempt to solve problem 18, factorial. They decided to use the factorial button on the calculator to look for a pattern. Unfortunately, Karen misinterpreted the calculator display which gave the answer in scientific notation. Karla asked if the nondisplayed digits were all zero's, and when Karen replied that she thought so, Karla accepted this. They went forward under this assumption, leading them on a nonproductive, ten minute, tangent. While both Karen and Karla appeared enthusiastic and there was a great deal of comradery and laughter during their session they failed to get any of their questions correct. The other pairs in this category were more successful: Kevin and Cecil got one problem correct and Karla and Candy got two correct.

Partners. Partners have generally similar skill levels, but are not as concerned with social interaction as the Socializers. Their central focus is the problem. They may or may not be polite and supportive, but they are not hostile at all. They work together on the problem, both contributing ideas; these ideas are often evaluated and sometimes rejected. They question each other, asking for clarification or justification. One may do all the writing or all the drawing, but both are actively engaged in all steps of the problem solving process.

Representative of this category are Karen and Diane. Karen and Diane are polite and supportive, but direct. Decisions are made jointly, with discussion taking place. For example, during their attempt to solve problem 16, making change, they had constructed an equation in three variables when Diane decided that she would like to change that to a single variable.

D I was thinking. Up here, what if we made these all x's? No?

K Yeah, okay. Make them all one variable? So that, the, if we made dimes equal x the pennies would be x minus 9. And nickels would be x minus 5.

D Oh, I see what you're saying. You're, now, like

K So we have one variable for everything.

D Yeah, or, hum.

K If we let pennies equal x then dimes would be 10x and nickels would be 5x.

D That's better.

[Diane and Karen] A short while later Diane realized that what they were doing was not going to work and she had no hesitation in saying this.

D So, let's just take a number, stick a number in. It's not going to work. You know why?

K Why?

D 25 pennies.

K Um hum.

D So (inaudible). Um. It doesn't work. I think you're right.

[Diane and Karen] They worked together throughout, often monitoring and correcting each other's errors. While they were friendly, they were generally direct rather than overly polite.

Other pairs in this category were Shelly and Carol in the second session together and Simon and Sam on two of their three questions. Together the three pairs got four of eight questions correct.

Tutor/pupil. Tutor/pupil pairs have quite different mathematical skills, but are very, to moderately, cooperative. One student, the 'tutor', does most or all of the work on the problem, making most of the crucial decisions. This 'tutor' explains to his or her partner, the 'pupil', what is being done and may also explain why. The 'pupil' carefully follows the work of the 'tutor', sometimes asking for clarification. The 'tutor' may be very careful to include the partner and may even assign to him or her work that he or she is able to do.

Shelly and Carol, during their first interview together, were typical of the category, with Shelly playing the role of tutor and Carol that of pupil. Shelly took the lead immediately as she had an algorithm available for their first problem, number 17, shaking hands. She was however very careful to explain what she was doing and to ask Carol's opinion before proceeding. After describing the diagram that she would construct to solve the problem, Shelly asked, "Do you have any other suggestions? Shall we do it that way?" Carol watched what Shelly was doing carefully, added supportive interjections and did the calculations. Throughout all three problems Shelly led, while Carol followed closely, occasionally asked questions, and assisted with calculations. But Carol never took the initiative. When she did ask questions, they were requests for explanations rather than challenges to what Shelly had chosen to do. Shelly gave explanations without any apparent impatience. The following dialogue took place during the solving of problem 11, two circles. The radius of the inner circle had been named *a* and that of the outer circle *c* and Pythagoras' theorem had then been used to show that a equals the square root of c^2 minus 1.

C How do we know small *a* is that?

S Because it's this, *a* squared equals *c* squared minus 1. So you take the square off of that and you make that a square root. (Pause) So we started out with this, which is Pythagoran's. And we know that *b* squared is 1.

C Um hum.

S That one.

C Um hum, so that's 1.

S Therefore, that means that *c* squared equals *a* squared plus 1. Just, we just substituted that 1 in there.

C Yup. Okay.

[Shelly and Carol] Shelly was always careful to include Carol, suggesting that she complete certain tasks. At the beginning of the cistern problem, number 14, Shelly asked if Carol wanted to do the drawing, and then Shelly read the information to Carol who transferred it to the drawing she had created.

Other tutor/pupil pairs were Carl and Janet and Janet and Kevin. Together the three pairs got 8 of 10 questions correct and made substantial progress on the remaining two.

Individuals. Individuals may be of similar or of different mathematical background and skill but, in either case, they do not work cooperatively. While they may read the question together and begin working together, they spend most of the time pursuing separate lines of thought. They may take turns commenting on what they are doing, or one may provide a running commentary on his or her own work while the other works silently. They generally choose separate pens and may work on separate sheets of paper. At times, one may unsuccessfully attempt to get the other to cooperate on a solution.

Carl and Randy are representative of this category. The first problem they attempted was number 7, the ski trip. They began working together, attempting to construct algebraic expressions for the various quantities in the problem, and then to put these quantities into some kind of relationship to each other. At one point Randy believed that he had a correct equation, but Carl disagreed. "I don't think so," he said, but he offered no further critique. Randy decided to go ahead and solve the equation anyway, which he did with Carl looking on. When he obtained his (incorrect) answer, Carl simply did not acknowledge it and constructed and solved his own equation. Randy acknowledged that Carl was correct, and Carl replied that Randy was "awfully close, I think." In no way did they show hostility to each other. Rather, each simply worked on his own ideas separately from the other. This was particularly apparent in their third question, problem 5, squares. Once again, they began by discussing the problem together, and were able to construct the first three terms in an infinite series which, if extended would have represented the desired area. However, Carl immediately abandoned this line of thought and instead tried to fit some kind of exponential or logarithmic function to the problem. Randy attempted to try to further their original analysis of the situation. Throughout the rest of the session they took turns, each leading the discussion along his own particular line of thought, but listening to the other. In the end they ran out of time.

The only other pair in this category was Simon and Sam, and then only for the last of their three questions. Together the two pairs correctly solved two of four questions.

Hostile pair. Hostile pairs occupy the uncooperative end of the social dimension and are of different mathematical background and skill. One student, the more able, does most or all of the work on the problem, making most or all of the decisions. No attempt is made to include the less able student, and his or her participation may even be actively discouraged. As well, the dominant student makes no effort to explain strategies or techniques to his or her partner. There is no attempt to soften criticism, and they may openly show signs of frustration with each other. They may be rude or use a hostile tone of voice.

There was only one pair in this category, Sam and Diane. Sam was the more able and he dominated the problem session, becoming more impatient with Diane as the session went on. Sam took possession of the pens and paper for almost all of the session and he made all the decisions of strategy. Diane phrased most of her suggestions as questions, which Sam often ignored, or to which he often replied with a curt "No" and no further explanation. During their work on problem 15, the commuter, Diane hesitantly put forward an idea, summarily rejected by Sam:

D Arriving ten minutes early. On route. 'Cause he drove back again. 'Cause whatever time he's taken travelling is going to be divided by 2, because it's going to be 2 different directions.

S I don't think it's as simple as that. (takes a fresh sheet of paper.)

[Diane and Sam]

In their first problem, number 11, two circles, Diane almost immediately suggested drawing a triangle and using Pythagoras' theorem, but Sam rejected this since they only knew one side of the triangle. Diane made no attempt to pursue this idea for some time, during which the pair were involved in an extended exploration involving modifying diagrams. Finally, when the exploration led nowhere, Diane's idea was accepted by Sam when she explained it more completely. But it was Sam who made the decision to implement the idea, and Sam who carried it out. As the session went on, Sam became increasingly short with Diane, especially when she repeatedly tried to use the calculator for problem 18, factorial. "The calculator won't work," he told her, the impatience clear in his tone, but he never took the time to explain fully why the calculator was not useful in this case. As Sam became shorter and more impatient, Diane became visibly frustrated with him. She often put forward undeveloped ideas, while Sam appeared to see them as serious suggestions that Diane should be able to justify. "I'm just thinking out loud," Diane explained to Sam. Diane and Sam got one of their three questions correct.

Table 3 summarizes the results in each category. While one might expect that the partners would be the most successful category, since in this case two individuals are working together to solve a problem, this was not the case for this study. Rather the pupil/tutor pairs were far more successful than any other category, with 8 out of 10 correct. In this category, one partner led the solutions and essentially solved the problems alone. Partners were the next most successful category, with half of their questions correct. Eighteen problems were solved by pairs in all categories, but in only 8 of them did both partners contribute substantially to the solution. In the other 10, one partner solved the problem essentially alone.

Table 5 Problem results by category

CATEGORY	RESULTS
Socializers	5X, 3C, 1S
Partners	4X, 4C
Tutor/pupil	8C, 2S
Individuals	1X, 2C, 1S
Hostile	2X, 1C

X: incorrect or incomplete, S: incorrect but substantial progress made. C: correct

Interactions between pairs are highly varied and it is these interactions, rather than any mathematical criteria, that determine strategy selection. Socializers made no attempt to evaluate potential strategies based on mathematical validity but made choices, instead, on the basis of social interaction, often following any line of thought either partner brought to the fore. Sometimes one partner made most of the decisions. This is seen in the case of Kevin and Cecil. In all cases, Kevin went along with whatever Cecil wanted to do and did not question his decisions. As we have seen, at one point this led them to abandon the appropriate strategy of using simultaneous equations to find the intersection point of two lines, because Cecil did not believe this is what the technique would accomplish. Cecil is several years older than Kevin and this may have contributed to

Kevin's willingness to agree easily. The one hostile pair in this study also did not generally make strategy decisions based on a rational critique of the suggested strategy. Rather, most of what the less able partner suggested was dismissed out of hand.

Members of pairs in the individual category each followed his own line of thought. Strategy decisions were not discussed or critiqued. In their parallel work on problem 5, squares, both Carl and Randy monitored each other's work, but they never even acknowledged that they were following different strategies, and certainly made no attempt to evaluate the two approaches and decide which would be more useful. In the tutor/pupil pairs, the more able student explained his or her choices to the less able student, thus having to provide a justification for his or her strategy choice. In each case, the less able student asked questions that the more able student had to answer before going on. These were certainly the most successful pairs. Pairs in the partners category had to come to mutual decisions, so that justifications for strategy choice were often provided. This was the second most successful category.

Gender issues.

In Table 4 we see that there are twelve women to six men in the top, more cooperative, half of the table and one woman to five men in the bottom, less cooperative half of the table. It is also notable that all of the mixed sex pairs are either in tutor/pupil pairs or are hostile, and in each of the mixed pupil/tutor pairs the man fills the role of tutor. In general, the men were less likely to be cooperative than the women. The one pair of men in the partners category overlapped the individuals category on one of their problems. The women appeared more cooperative, but were often unable to openly challenge each other or to evaluate ideas freely. While the sample is small, the results are suggestive.

Table 6 Categories by sex

mathematical → **social ↓**	**similar**	**different**
cooperative	FF FF MM	
	FF FF MM	FF FM FM
	MM	MM
uncooperative		MF

Diane, the woman who was in the hostile pair, had strong opinions on the differences between her experience when working with Karen and when working with Sam.

D One thing I did notice was working with [Karen] as another woman was much easier. There was a lot more flow and a lot of give and take than working with [Sam], because he was a male, um, this is my own perception, from my experience with relationships in life, is with men it's a little more different. Because, immediately [Sam] wanted to be in control of the situation which men normally want to and that's okay, like, I don't have a problem with that. But it makes, it made it a little more difficult, for like I felt to, like, get it across, some of what I was saying.... 'Cause I was feeling we were going around in a big circle and it was getting too complicated on that one problem we were doing. And, finally, I got him to do the Pythagorean theorem and we solved it.

[Diane, exit interview]

Subjects' reactions.

All but one subject expressed reservations about working with a partner. They all usually worked alone when studying for their mathematics courses, and this was generally by choice, but also because getting a group together could be difficult. Too many distractions was cited as the biggest disadvantage of working on homework or studying with others. It was simply easier to concentrate on the work when working alone. The biggest advantage to working with a partner was seen as the provision of a second point of view.

With regard to their experiences in working with a partner during the study most students said it was more difficult in most ways. Kevin thought that working alone was much faster, since you did not have to take time to explain what you were doing before moving on. With a partner there was a greater need for communication skills. Cecil found working with a stranger to be especially stressful. He was concerned with how he would compare with his partner and this, he said, interfered with his ability to think rationally. Janet felt inadequate when someone had to tell her what was going on. Shelly also preferred to work alone since, when she is with a partner, she is reluctant to argue, holds back so as not to take over, defers to more assertive people and gets annoyed by others inability to see what she sees. Karla felt that the problem with the pairs was that they were both talking and neither was really listening. She found that the necessity to persuade another person held her back from pursuing her own ideas. Diane noted that sometimes, when working with a partner, one becomes more concerned with what that partner is thinking and feeling, than with the mathematics that is being done. Carl was concerned that, when working with a partner, one might not realize where one is having difficulty. Karen held a similar opinion:

> If you get truly stuck it is really, or not even truly stuck but just a little bit stuck, it is really easy to ask the person who is sitting across from you. Oh, this way. Okay. And it's back to the same thing. I've had that one thing explained to me. Whereas if you're alone and really don't have anything to fall back on other than your own resources, yourself, that's the only way to really learn. Because you have reasoned it through yourself and you have, I like to think of it as I've made a new pathway in my brain.

[Karen, exit interview] Karen also found that working with a partner was restraining as she felt that she had to be diplomatic. Sam found that working with a partner was interesting and more problems were solved that way, but he was certain that he could have solved them all on his own, given enough time. Simon was the one person who had no negative feelings about working with a partner. He felt that the feedback that one gets from a partner, whether negative or positive, makes one really think about the problem.

It is worth noting that, generally, those in the most cooperative pairs mentioned most often that they felt restrained by the necessity to take the social interactions into consideration. Sam, Simon and Carl made no mention of the social dimensions of their experiences in dyads.

SOME SUBJECTS' REACTIONS TO THE STUDY

Three subjects commented that their participation in the study was beneficial to them. Sitting with a mathematical problem in front of a video camera for 15 minutes forced them to concentrate and not to give up. For all three, this was the first time they had been able to work with such persistence and they discovered that it was possible to persevere and therefore succeed in solving at least some of the problems. Janet said, "I had fun and it made me realize that I can do it. Maybe I should set up a video camera and tape recorder when I study." Karen went further:

> It has helped me with my Physics and Chemistry and all of my problem solving things so much. And I mean it's only been, like I said, three or four sessions, fifty minutes time, and it's amazing how much more tolerance I have, how much more patience I have to sit down and figure this stuff out. I feel better about even getting a part of something down and understanding it and knowing that at least that part is right. Whereas, before if I couldn't get the whole concept on the page immediately, I was, forget it.

[Karen, exit interview] Cecil also enjoyed the sessions and expressed his preference for now learning in the experimental situation rather than the usual classroom setting. "Working in a closed room with a time limit gave me the ability to focus much more. Thank you for the chance to do this."

CHAPTER V DISCUSSION AND CONCLUSION

Mathematics education researchers have been interested in problem solving for a number of years. Central to the investigation of problem solving is the question of what people actually do when they solve problems. In this study I have looked at what fourteen average college algebra students did when they attempted to solve problems alone and in dyads. I have focused on the skills, strategies, beliefs and attitudes that these college students displayed as they tried to solve nonroutine problems, and on what differences there were in the process when the students worked in pairs rather than alone.

This chapter gives a conclusion to the study, including addressing the research questions, looking at the limitations of the study and implications for further research.

PAIR INTERACTIONS

In exit interviews all the students in the study stated that the main benefit of working with another person, or persons, was the provision of a different point of view and the pooling of information. The literature mentions several factors which could account for the efficacy of group problem solving:

1. An increased focus on the task at hand. [Bossert, 1988/89, Dees, 1985, Rosenthal, 1995]
2. Increased opportunities to rehearse information orally leading to greater integration of the information. [Bossert, 1988/89, p.234, Dees, 1985, 1991, Stacey, 1992, Webb, 1991]

3. Constructive controversy, in which students encounter challenge and disbelief, in which they challenge others and then use discussion to examine beliefs and strategies more closely. [Bossert, 1988/89, Noddings, 1985,]

4. The pooling of ideas and strategies and background information. [Noddings, 1985, Stacey, 1992]

5. Reduction of anxiety and corresponding increase in confidence. [Dees, 1985, Stacey, 1992]

6. Encouragement from peers, a warmer, welcoming and supportive atmosphere. [Bossert, 1988/89, Rosenthal, 1995]

Ideally, then, what we would expect from small groups is a discussion of the problem, possibly including background information, leading to an agreed upon understanding of the problem. This would then lead to suggestions of possible strategies to follow, challenges leading to more constructive discussion and a rational decision about what strategy to follow. As the solution attempt proceeded partners would encourage and assist each other, ask for explanations and give explanations, and challenge and evaluate each other's ideas. This would, ideally, lead to a clear solution to the problem.

However this is not at all what I saw. There was generally little discussion of individual interpretations of the problem. Only in a very few cases (the meaning of factorial and the use of speed in the first part of the commuter problem) did one partner supply necessary background information that the other lacked. The students did not generally spend any time advancing ideas for solution methods and then discussing the relative merits of the various methods. That is, strategies and ideas were not pooled and then evaluated. Instead, the first strategy that appeared to lead towards a solution was generally followed immediately.

Constructive controversy was almost entirely absent. In most cases there was no controversy at all. In those cases where there was controversy it was generally not resolved constructively. What happened during each session was determined in great part by the social interactions of the pairs. Pairs in the socializer category exhibited no controversy at all. The students were so focused on maintaining a smooth social interaction that they never challenged each other. Rather, when either student suggested a direction his or her partner would acquiesce immediately and then supply support and encouragement. Each student in the individuals category simply followed his or her own strategy alone rather than trying to convince his or her partner that it was a viable solution method.

In the case of tutor/pupil pairs, the strategy followed was that of the tutor. While the pupil might ask for explanations, she or he never challenged the tutor's decision and seldom made suggestions of her or his own. There was controversy between the partners in the hostile pair, but it was not resolved constructively. When Sam did not agree with Diane he simply ignored her or overruled her, but without any discussion or explanation. It is amongst the pairs in the partners category that we would expect to find constructive controversy playing a part, and here it is more evident, although still in a minor role. There was some discussion of ideas and proposed strategies, but few direct challenges that led to a defence or a real controversy. Diane and Karen are typical here. When they worked on the tangent circle problem, Karen was not comfortable with Diane's placement of the centre of the circle. While she expressed her doubts and they were discussed, she did not challenge Diane to defend her assumption. Thus, the discussion was not as helpful as it might have been.

Real encouragement from peers was not generally present. While partners often made supportive interjections ("yes", "umm", "go for it"), it was much rarer to hear positive evaluative statements from one partner to another. When it was seen, it was usually within socializer pairs or tutor/pupil pairs. Karla, for example, praised Candy after she was able to solve the four circle problem. This, however, must be seen as contributing more to a positive social exchange than to the solution of the problem, since the praise was given after the problem was complete.

For some students, working with a partner reduced anxiety and increased confidence, but for others working with a partner increased their anxiety levels. For example, Cecil stated that when working with someone else he was distracted by worry about how he would compare with his partner.

In this study, I did not see an increase in focus on the task at hand. However, the experimental situation may account for this. Even when they worked as individuals, the students were aware that their work was being recorded, and this awareness kept them focused on the task. Several students commented, in exit interviews, that they found that participation in the study had helped them to focus. In a more natural situation, it is possible that the students would have been more focused during pair problems sessions than they would be alone. However, I have no evidence for this.

Only for some of the students in this study did the pair sessions provide increased opportunities to rehearse information orally. This was so for the students playing the part of tutor in the tutor/pupil pairs. These students described what they were attempting to do and explained to their partners why they wanted to do this. However, students in other categories did not spend much time explaining what they understood or why they wanted to follow a certain strategy. Ideas and strategies were often introduced with no explanation at all. Since partners seldom challenged each other, there was little call to explain choices.

Thus, the study does not support the hypothesis that it is the six factors listed above that provide substantial benefits to pairs attempting to solve non-routine mathematical problems. Nevertheless, this study shows that the pairs did significantly better at solving the problems than did the individuals. Individuals correctly solved 9 of 44 problems, or about 20%, while pairs solved 18 of 34 problems, or about 53%. In order to determine what characteristics of pair work contributed to this increase in success, we need to examine the pair interactions in some detail. I will begin by discussing some factors that seem not to have contributed significantly to the increased success of pairs in problem solving.

There was only a slight increase over individuals in the time spent on the comprehension phase of the process. Assumptions and conventions were more explicitly noted. In some cases one partner used his or her understanding to explain an aspect of the question to the other partner. For example, Carl explained the concept of factorial to Janet, while Karla ensured that Candy did not use the 8 km/h from the second part of the commuter problem during the first part.

The planning phase differed little from that seen with individuals. There was very little planning at all. The notable exception to this was Kevin's detailed plan for a solution to the folded paper problem. Since he had seen that he could use trigonometry to solve the problem, and since Janet had not studied trigonometry, he gave a very brief description of what trigonometry could tell him and then explained how he planned to solve the problem. It is possible that Kevin or other subjects do prepare such plans when working alone, but simply do not verbalize them. However, there is no evidence of this extensive planning and the solution attempts implemented by both pairs and individuals do not generally reflect such planning.

There was also little difference in verification. Pairs were no more likely to attempt to verify any aspect of their solutions than were individuals. In fact, in one case, Shelly and Carol, one partner vetoed the other's attempt to check a calculation. Stacey [1992] had noticed this lack of checking behaviour with groups of younger students.

Pairs generally displayed very similar strategies to those displayed by individuals, with a few important exceptions. The less useful strategies of guessing, of guessing a single operation, and of eliminating most of the information in order to simplify the problem, disappeared almost entirely for partners. These strategies had generally been displayed by the weaker students and when they were paired with more capable students it appeared that the more capable partner had other strategies available. Strategy choice was more open to observation with pairs than with individuals. However, there was generally little more analysis. Just as with individuals, the first idea to come to mind was usually the idea that was pursued. Stacey [1992] noted that, with small groups of seventh, eighth and ninth grade students, chosen strategies were often those that were easier to understand or easier to carry out. This was often the case with the college students in the present study. Another important criterion was whether the method was "mathematical," that is, generally, whether it involved an equation.

Pairs made more comments on their states of mind. They were more likely to verbalize their confusion and frustration. However, they still did not generally use confusion as a signal for the need to reconsider their approach. Rather, the verbalization of frustration was more likely to be part of a social interaction than of a mathematical one.

There were also, in certain cases, some clear disadvantages to working with a partner. In three different pairs, there were disagreements in which a useful strategy was not followed because a partner rejected it. In two of these cases, the partner rejecting the useful strategy was the more socially dominant partner. Some subjects noted, in their exit interviews, that they felt they were held back by their concern not to hurt their partners feelings. Others mentioned a lack of time for reflection as an impediment to solving a problem while working with a partner.

Nevertheless, the pairs were significantly more successful. I contend that this study points to four factors which contributed to the increased success of the pairs. These are:

1. an increase in persistence,
2. the more able partner leading the pair,
3. oral rehearsal of ideas,

4. the correction of minor errors.

There was a significant increase in persistence: Most pairs either solved the problem correctly or used the full 15 minutes in the attempt. In only one case did a pair submit a solution which had been guessed and in which they had little confidence. There were 10 cases of quitting or guessing amongst the individuals. This increase in persistence appears to come about by a combination of three factors; the pair working at the persistence level of the more persistent partner, the sharing of frustration and confusion lowering the overall frustration, and the need for both partners to agree in order to quit. This increase in persistence can be credited, I believe, with much of the increase in success rate.

Another benefit came from a more able partner leading the solution process. Of 18 problems solved correctly by pairs, 10 were solved essentially by one partner alone. This can be most clearly seen in the pairs in the tutor/pupil category. For example Shelly solved the handshake problem almost entirely on her own. Carol only assisted by doing the calculations suggested to her by Shelly. Also, Kevin solved the folded paper problem essentially alone since he did it using trigonometry and Janet knew no trigonometry. In other cases, while a more able partner led the solution attempt, the less able partner did contribute to parts of the solution.

While oral rehearsal of ideas and strategies was limited almost entirely to those in the tutor/pupil pairs, it is also these pairs who were the most successful. In each case, the tutor explained to his or her partner what he or she was doing. It is likely that the necessity of explaining caused the student to think more precisely about what he or she wished to do or that the act of explaining brought about clarification of the process. Also, in the pupil/tutor pairs the pupil often asked questions which the partner then had to answer, leading to another opportunity for clarification and perception of possible errors. Pairs in other categories did not go through this rehearsal process and so did not have this opportunity to confirm or extend their understanding of the situation. The greater success of the tutor/pupil pairs adds strength to my contention that oral rehearsal is an important contributing factor to the increased success of the pairs.

Monitoring of calculations also contributed to correct solution attempts. Often one partner was able to see simple arithmetic or algebraic errors made by the other partner. In some cases conceptual errors were also prevented. This was true for two of the three pairs who attempted the factorial problem. In each case, one partner understood the concept of factorial and corrected the other's misunderstanding.

Researchers have speculated that group work provides social and affective benefits to students. The following list is a summary of such benefits:

1. Enhanced enjoyment of mathematics and mathematics classes. Students sense a warmer, more welcoming and more caring atmosphere. [Good, Mulryan and McCaslin, 1992, Rosenthal, 1995]
2. Enhancement of self esteem and self confidence. [Dees, 1985]
3. Increased practice in learning to work and communicate with others. [Good, Mulryan and McCaslin, 1992, Rosenthal, 1995]

In exit interviews, all students stated clearly that they did not usually work with someone else and all but one stated that they preferred to work alone. Many said that, when working with another person, they had to spend too much time and energy focussing on the social interaction to the detriment of pursuing the mathematical goal. Some cited worry over hurting the other person's feelings, while others worried about how they would compare mathematically with their partners. One student felt that, when working with a partner, she would not learn as much as she would working on her own, since she would not have to work through all the details herself. Being shown by a partner was not, she asserted, as effective or satisfying as figuring it out oneself. This almost universal negative attitude to group work certainly does not indicate an enhanced enjoyment of solving mathematical problems or an enhancement of self esteem and self confidence. While, in some of the pair sessions, there was a friendly and positive atmosphere, in the case of the pairs in the individuals category and the hostile category this was not the case. Certainly, it is clear that simply pairing students will not ensure a positive experience for all, or even for most.

Finally, these students clearly did not know how to work together constructively at mathematical problem solving. It would seem that any experience they have had in the past with cooperative work has not led them to develop the skills needed to benefit from the experience of working with another person. It is also clear that simply assigning them to pairs to work on a problem did not lead to practice of positive communication skills. The socializers were too fixated on maintaining social harmony to make any attempts at developing skills in communicating mathematical ideas. Most of the others were too fixated on finding the right answers to expend any effort in talking with a partner about problem solving strategies. Communication was most effective in the tutor/pupil pairs, but there the communication was mostly one way.

In summary, pair problem solving sessions did not work at all as one might expect and hope they would. Benefits did not arise from the factors generally cited in the literature, but came from an increase in persistence, the pair working at the level of the more able partner, from oral rehearsal of ideas, and, to a lesser extent, from the correction of minor errors. The expected social benefits of the small group process are also not present. In fact most of the subjects of this study reported negative reactions to working with others.

ANALYSIS OF PROBLEM SOLVING SESSIONS

The students in this study generally had available to them a wide variety of specific strategies for problem solving. Twenty four distinct strategies were identified in chapter four. These strategies are, as one would expect, far more diverse and potentially more useful than the seven strategies identified by Sowder [1988] in use by sixth and eighth grade students (see chapter 2). Most of the strategies seen in the present study are potentially useful; however a few were detrimental. These inappropriate strategies are very similar to the inappropriate strategies identified in Sowder's study of younger students, where they generally involved some kind of guessing or looking for clues, as opposed to analyzing the meaning of the problem. Similar strategies exhibited by the college students included; calculate everything you can, guess, guess which operation to use, and eliminate most of the information in order to do a single step calculation. Positive strategies included such things as: draw and modify a diagram, try a similar problem, make an estimate, look for a pattern, and use a physical model. These are all useful heuristics that might be taught in a problem solving course. While not all students exhibited all, or even most, of these positive strategies, most exhibited a good variety of strategies. Exceptions to this were Carol and Tanya, both of whom predominantly displayed the negative strategies such as guessing or simplifying the problem down to a single step.

One important strategy was noticeably rare. This was to analyze the mathematical structure of the problem; especially to analyze relationships amongst various quantities in the problem. The subjects would often write expressions for various quantities in the problem, but were seldom able to focus on the relationships amongst these quantities. They did not appear to see an equation as an algebraic model of a relationship. Schoenfeld [1985b] has identified four cognitive and metacognitive factors important in problem solving. Two of these are resources and heuristics. It seems that while these students knew the important heuristic of writing an equation, they may have lacked the mathematical resources to carry it out, that is, they did not know how to construct the equation.

With the variety of strategies available to them, one might have expected the subjects to have been able to make substantial progress towards solving most of the problems. This was not the case. They were held back by several factors.

Polya [1973] put particular emphasis on the first step, understanding the problem, of his four step problem solving plan. However, the students generally spent very little time attempting to understand the problem. While they often reread the problem several times, they did not appear to have many skills in analysis. The comprehension phase of the problem solving process usually consisted of reading and rereading the problem, noting the knowns and unknowns, and often drawing a picture or a diagram. However, if that did not lead to an apparent solution method, they did not have many tools for further analysis. They occasionally tried some examples in an attempt to understand what was going on and, in geometric problems, they often modified diagrams. What many seemed unable to do, was to analyze the structure of the problem. Especially in algebraic problems, this often left them without any way to construct an appropriate equation. Most students focused entirely on the goal of finding a solution to the exclusion of trying to fully understand the problem.

Most students spent little, if any, time planning a solution. There were notable exceptions to this. Carl planned and estimated before going on to calculate in his solution to the spider and the fly. Kevin had planned his solution to the folded paper problem in detail and explained it to his partner before implementing it. But these occasions stand out for their rarity. Most of the students, after a very short comprehension phase jumped immediately into calculations of some kind.

The students generally saw little need to verify their answers, but wanted to be told by me whether they were right or wrong. I had not originally planned to do this, but the students in the pilot study were clear that they wanted to know how well they had done, as well as the correct answers. The validation of their solutions had to come from an external source. I see this as part of an overall attitude of not really being responsible for the solution process; of seeing mathematics as being something that someone else must teach you how to do.

The students generally lacked good strategy selection criteria. Often they did not know when a particular strategy might prove useful, and they seldom evaluated strategies before implementing them. Strategy selection often came down to following the first idea that came to mind. This confirms what Schoenfeld had noted in his research [1985a, p.372]. While some strategy choices were based on the mathematical structure of the problem, for example, the use of diagrams in geometric problems, this was often not the case. The students' beliefs about mathematics were very important in their choice of a strategy. Some methods, such as trial and error, were considered illegitimate, while others, such as writing an equation or system of equations, were seen as more mathematical and hence to be preferred. Familiarity and ease of use were important criteria in the choice of what method to follow.

Even when strategies were evaluated this did not necessarily lead to an appropriate choice. This was clearly seen when Kevin and Cecil worked on the triangle problem and Kevin suggested that they solve the pair of equations simultaneously to find the intersection point. Cecil rejected this, believing that the purpose of solving simultaneous equations was not to find an intersection point. Similarly, Sam originally rejected Diane's suggestion of using Pythagoras' theorem to solve the two circle problem. In both these cases, the rejection of an appropriate strategy was essentially for interpersonal reasons rather than mathematical ones. The more assertive student rejected the suggestion of the less assertive one.

Once a strategy was chosen its use was seldom reviewed. Rather, it was generally followed until it led to a clear dead end. Then the next strategy to come to mind was similarly followed. One notable exception to this was Candy and Karla's solution to the making change problem. They began by counting combinations, then switched to trying to write an equation since that was seen as more mathematical, and then they re-evaluated that and returned to counting, devising a system to organize the count. This behaviour, however, was the exception rather than the rule. There was generally very little monitoring behaviour at all. Few written notes were kept, meaning that it would have been difficult to review their progress on a problem had they decided to do so.

A lack of precision, including a lack of appreciation for the value of precision, was clearly detrimental to the problem solving process. Inferences were often very general and not well considered. This was clearly seen in the simplifying misinterpretation of the sleepy traveller problem, which led to an incorrect solution of one quarter, and also in the vague interpretations of "continues forever" in the squares in squares problem. Diagrams were also often very imprecise and this sometimes led to incorrect solutions. Cecil's messy drawing in the four circles problem led him to miscount the number of regions to be subtracted, and Karen and Diane's rough graph in the tangent circle problem led them to assume, incorrectly, that the centre of the circle was in a particular location.

Lack of precision in the use of variables was an important contributor to incorrect solutions. Variables were seldom formally defined, and sometimes their meanings were changed part way through a solution attempt. Often letters were simply used as a kind of shorthand for recording the information from the problem statement. Although this might be a legitimate use of letters in some situations, it leads to confusion when this shorthand is then converted to variables. Sometimes it was clear that the student had not even considered whether the variable represented time or distance. This was particularly evident in Candy and Karla's attempt to solve the commuter problem. Most of the students appeared to feel no need to be more precise. Cecil, for example, was quite aware that his answer to the four circle problem was inaccurate, but seemed to see no problem with that.

Schoenfeld [1983a] has noted that the cognitive behaviours of problem solvers are embedded in and shaped by metacognitive and social factors. In the present study, we can see that the students' attitudes and beliefs contributed to their inability to solve the problems given them. This is especially notable since all these students were volunteers and so would not be expected to have particularly negative attitudes toward mathematics. Often the attitudes that held them back were not necessarily negative but were simply not useful.

Some students displayed a belief that there must be a formula or algorithm for any problem. If they did not know the formula, or had not seen a similar problem before, then they did not believe that they would be able to solve the problem they were given. They did not see mathematics as something that they were able to generate, but only as something that they must learn from someone else. Cecil demonstrated this attitude clearly during his attempt to solve the division by nine problem. He simply hoped he would be able to understand the solution when I showed it to him later. This attitude hampered many of the students, leaving them at a loss for what to do when faced with a unfamiliar problem.

Other students seemed to have a strong belief in inspiration, and when inspiration failed to appear, they had no idea what to do next. Others were hampered by their belief that only certain techniques were acceptable in solving a mathematics problem. The problem not only had to be solved but it also had to be solved "mathematically." This led Randy to decide not to present his correct solution to one problem.

The students in this study generally knew all of the mathematical techniques necessary to solve the problems they were given. They also had a wide variety of specific strategies available to them. Based on content knowledge alone, they ought to have been able to solve most of these problems. However, they lacked the general skills and attitudes necessary to use the knowledge that they had. They often did not see comprehension as an essential part of the problem solving process, they did not monitor their progress and assess their strategies, they believed that mathematics comes from outside of them and that there are acceptable and unacceptable solution methods, and they did not understand the need for precision in inference, calculations, and diagrams. All of this hampered them in trying to solve unfamiliar problems.

IMPLICATIONS FROM THE OPERATION/STRUCTURAL ANALYSIS

I believe that the analysis of the solution attempts in the light of Sfard's theory of the dual operation/structural nature of mathematical understanding deepens our understanding of the students' actions. It is a crucial factor in making sense of their attempts.

The students often did not spend time trying to understand the mathematical structure of the problems because they were still working at an operational level with regard to algebraic problems. They lacked the crucial skill of seeing the relationships in the problems as algebraic relationships. We have seen that more substantially correct solutions were produced for problems that could be approached operationally than for those that required a describe first, calculate later, structural approach. A clear example of this was the difference in success between the two different circle problems. In the four circle problem, they needed only to see the geometric relationship and then calculate an answer. In the two circle problem, they had first to see the geometric relationship and then model it algebraically. This they found much more difficult. We have seen that for problems, like the folded paper problem, which could be done operationally or structurally, the operational approach was chosen.

Problems that required the construction of equations were particulary difficult, something that should be surprising when we consider that these are all algebra students. However, the students did not see the structures of the problems that were presented to them because, I believe, they did not yet possess the necessary mathematical structures. The structure they lacked was algebraic. They did not yet see, for example, that distance divided by time can be conceived of as a single entity, speed, which they can relate to other speeds in a problem. They had not fully made the transition from arithmetic to algebra and still saw expressions such as 520/(x+5) as directions to do certain arithmetic operations, rather than as a single quantity that itself can be operated upon. That is, they could not operate on or with the unknown. Many were still at the level of seeing algebra only as operation or as processes. As Gray and Tall have said, "The less able child who is fixed in process can only solve problems at the next level up by coordinating sequential processes. This is, for them, an extremely difficult process" [1994, p.135]

In this study, the subjects were often faced with problems requiring structural approaches while they were only prepared to attempt the problems in an operational manner. We have seen that this may lead a student to attempt to solve a problem without the use of algebra when it is needed. Several other consequences which became apparent during the study, were lack of direction, reliance on (often incorrectly memorized) algorithms, the separation of solving the problem from understanding the problem, and the belief in mathematics as a kind of magic.

When faced with a problem whose structure they were unable to see, the students appeared to lack any strategy that would give them direction. This often led them to "go around in circles" or to become distracted by extraneous details. Karla and Candy were in this situation in trying to solve problem 15, the commuter. The first part of this problem can be solved in a single step once the structure of the problem is understood. However, they were unable to see this structure and could find no strategy to follow. They drew an elaborate picture of a train station and train and they repeatedly got sidetracked by unimportant details such as the hair colour of the commuter.

C A nice spectacular question ... It's a trick question. Um ...

K She took a taxi.

(Laughter)

C What if she walks really fast?

K 8 km, ah, that's pretty fast. Yeah, that's a pretty brisk clip.

C Yeah.

K But we don't know that for the first part.

C Chopping along.

K A redhead too. Fiery red hair. (pointing back to their drawing)

C Pretty good, eh?

K Okay.

[Karla and Candy] They came back to the problem between episodes of this kind, but were unable to make any headway at all.

When the students could not understand the structure of a problem, they often resorted to the application of algorithms, sometimes incorrectly memorized or inappropriate to the problem. Working on problem 5, squares in the square, with Randy, Carl began by analyzing the structure of the problem and was able to construct the first three terms of the series. However, he did not recognize a series as an acceptable solution type, or even as an initial step towards a solution, and so he dropped this strategy. The series appeared to remind him of his recent study of interest rates and the development of exponential functions. Abandoning any attempt to understand the problem, he spent the rest of the session trying to fit an exponential equation to the situation. It appears that, in the absence of any recognized strategy to understand the structure of the problem, Carl's recognition of the iterative nature of the problem had led him to the only other iterative function he knew, the compound interest function.

Janet correctly decided that the way to solve problem 11, the two circles, was to find the areas of the two circles and then subtract the area of the inner one from the area of the outer one. But, since neither radius was given directly, she was unable to discern how to carry out this plan and she concluded, "that must be wrong 'cause it's, you can't do it that way." In fact, she needed to assign variables to the radii and then construct an algebraic expression for the difference of the two areas. Janet was acting at an operational level on this problem and did not see this structural solution. At an impasse, she resorted to using the area and length formulae that she had available. She attempted to measure both radii but then did not use these measurements in the formulae for the areas of the circles. She appeared to have forgotten her original strategy when she was unable to carry it out immediately. Instead she calculated both circumferences and then subtracted one from the other, using the result as a radius in the formula for the area of a circle. She appeared to be applying a heuristic that suggests that one can use length formulae for lengths and area formulae for areas, without any reference to the geometric structure of the problem.

Students were sometimes quite willing to attempt to solve problems which they knew they did not understand. This is clearly seen in Karla's attempt to solve problem 5, the squares in the square. She read and reread the problem, trying to understand what it meant for the process to continue forever. "If the process continues forever. See, I don't, don't really see how that process can continue forever. What fraction of the original square? ... Hum. What I have to do is come up with some formula that's going to tell me." As though it was common to attempt to solve problems she did not understand, Karla then attempted to find a solution, initially guessing the one operation answer of one ninth. She rejected this and continued to try to find a formula. She had no doubt that she did not understand, stating, "I think the key is, if the process continues forever." Eventually Karla admitted defeat, clearly aware that she was missing the central idea.

Several students misinterpreted part of the problem statement of problem 18, factorial. They read the x's used to signify multiplication in the definition of n factorial as a variable x rather than as the symbol ×. This happened with Janet when she worked on the problem with Carl. Carl read the problem and was clearly familiar with factorials. Nevertheless he read the definition of n! as "n factorial is x to the n minus 1 by n" and this may have contributed to Janet's confusion.

J Okay, I understand the first two. [referring to the definitions of 5! and 10!] I'm just trying to get through the n one, yeah. 'Cause that would be the key, wouldn't it? To figure out that one?

C Yeah, for n number

J Yeah ... So ...

C Well, it's going to have, ah,

J Would this be the equation then? Like 100, bracket, ... no, that would [inaudible]. Could you just solve for x? Plug the 100 where all the n's are and then solve for x and that would be the answer. Would that work? You think?

[Janet and Carl] It is clear that Janet had not understood the definition and that she was not attempting to do so. Rather, she was looking for a solution method without having first understood the problem, as though the two processes, understanding and solving, were quite unrelated.

Good, Mulryan and McCaslin [1992, p.173] see problem solving as adaptive learning in a social setting. Thus, to fully understand these actions and attitudes, we need to consider how the students may have adapted to the situations in which they have learned mathematics. Students in algebra class are often presented with problems which they are unable to solve. This is then generally followed by a demonstration of the "correct" solution by the instructor. If the students are repeatedly presented with structural methods and solutions for material that they view only procedurally, they may be unable to see where the solutions have come from. Rather, it will be as though the teacher had some magical formula that allowed her to conjure a solution out of emptiness. Repeated exposure to such a situation might easily lead to a belief in mathematics as magic and not as rational problem solving. Thus, when I presented these students with problems for which they had no ready made solutions, their first reaction was often not to analyze the problem but to reach for their inventory of "magic tricks" and hope that they would find one that works. The skill, then, is to know what particular trick to grab in any situation. So much time is spent on this that they often did not even look at the meaning and structure of the problem. Mathematics, then, appears to come from outside and to be validated from outside. It is not something that one can expect to generate oneself or to really own.

Sfard's theory of the operational/structural duality, the realization that the students have not fully completed the transition to algebra, and an understanding of the attitudes developed in the mathematics classroom all help to clarify much of what was seen in the student's attempts to solve problems. Although these students were studying algebra, and had studied algebra in the past, they had not yet arrived at the point where they were able to use algebra to analyze the structure of these unfamiliar problems. In giving them problems which required a structural approach when they were only able to proceed operationally, I had presented them with an extremely difficult task. This not only led to their inability to solve the problems, but also led to a lack of direction, inappropriate use of algorithms, and detachment from meaning.

LIMITATIONS OF THE STUDY

A qualitative study is, by its very nature, not generalizable. The intent of the study, however, was not to produce general results, but to explore the problem solving process in detail, and, thus, to point the direction for further research. The results of this study have come from a small group of students, at one particular community college, at one particular time. The sample size was small, only fourteen subjects, and these were all volunteers. The students were generally average, to somewhat above average, in their mathematical achievement. This was expected, as below average students would likely be more reluctant to volunteer for a study of this nature. Self selection would also be expected to produce a bias in favour of those who had more positive experiences with, and attitudes toward, mathematics. While in many characteristics such as age, educational background, family and employment status, this group is representative of the diversity of algebra students at Kwantlen College, in other characteristics, such as cultural background and English fluency they are not representative. With a different group of students, from the same or a different institute, the results might have been different. This means that results must be seen as tentative, until confirmed by further study. This is especially true of the classification of the pairs developed in chapter four. While I believe it is a useful tool for understanding the problem solving process amongst the pairs, with the small sample size it must remain tentative.

Limitations of scope relate to the problem statement. The focus of this study was to examine what actually happened during problem solving sessions. There was no attempt at intervention, or at teaching problem solving. What was examined was the strategies, skills and attitudes of the subjects and their interactions when working in pairs. There was no attempt to look at learning outcomes. That is, I made no attempt to examine whether the students learned any new skills, or developed new attitudes, due to taking part in the study. Specifically, while I looked at how the sessions with pairs differed from those with individuals I did not attempt to look at whether students learned more or learned something different when working in pairs, as compared to when working as individuals. Thus, while this study allows for tentative conclusions about how the process and results differ between individuals and pairs, it does not allow for any conclusions about whether working in pairs may produce beneficial learning outcomes.

FURTHER RESEARCH

This study suggests two main avenues for further research: Research with larger groups needed to confirm tentative results, and research which could extend the scope of the study.

With a group of only fourteen students, results must be considered suggestive and tentative. There is a need to repeat the research with a larger group of students, to see whether the same set of strategies appears, the same general skills are displayed, and the same attitudes and emotional reactions are apparent. It has been seen, in this study, that poor strategy selection, lack of precision, and lack of perseverance were important factors contributing to the inability to solve many of the problems. Further studies focusing on each of these areas could be useful. The examination of the differences between the problem solving processes of individuals and those of pairs needs to be extended by a larger study. It is necessary to confirm whether the two way classification system for dyads developed in chapter four is valid in general, and to determine what percentage of pairs would fit into each category. If it is confirmed, this classification system can help to explain the different experiences that different subjects have working in pairs. It would also be very useful to extend the analysis of this study to groups of three or more. Clearly, even pairs present a very complex phenomenon to study. Larger groups involve many more interactions, both mathematical and social. It is necessary to understand these interactions in order to determine how group work can most profitably fit into the mathematics classroom.

The research questions that guided this study, while in some ways quite broad, were, in other ways, quite focused. There is a need for further research to extend the scope of the questions and to extend results. The students in this study focused almost exclusively on the goal of obtaining an answer to the given problem. While this was the goal they were given, they were often so focused on obtaining an answer, any answer, that they put little effort into really understanding the problem. It is possible that this was an effect of the experimental situation, but I believe it is more common than that, and that students often do not see understanding the problem as part of the solution process. Research needs to be designed to focus on this aspect of students' problem solving.

Another factor contributing to the failure to solve many of the problems was a lack of understanding of the importance of precision. It would be valuable to understand this attitude in greater detail. In light of the comments by several students that participation in this study helped them in their general problem solving by helping them to focus and persevere, it would be especially important to investigate on this aspect.

An important theme to emerge from this study is the importance of the operational/structural duality. The idea of action or process becoming understood as mental objects or structures has appeared often in the literature. The process is discussed by Freudenthal [1991], by Harel and Kaput [1991] where it is called entification, by Dubinsky [1991] where it is discussed as encapsulation, and by Sfard [1994] where it is called reification. Yet, strangely, discussion of this dimension is often missing from the investigation of problem solving. More commonly, it appears in the discussion of the acquisition of new concepts, or of transitions, such as that from arithmetic to algebra. I feel it would be especially useful to extend problem solving research in this direction, to focus on students' actions as they try to solve problems that have been specifically designed to focus on this theme. This is especially important for algebra students who need to be able to see equations as algebraic models for relationships that arise in various situations.

Group work is becoming a regular part of many mathematics classrooms. However, the students in this study were almost unanimous in their dislike of group work. The immediate questions are: Why do they dislike it? What negative experiences have they had? Why have these experiences been negative? One possible explanation comes from the students' focus on finding the answer rather than on improving learning outcomes. Group work at college level often consists of group projects on which the students will all receive the same grade. Since they are being graded, they may be more focused on obtaining acceptable solutions than on increasing learning. In this case, working with others can be seen as requiring an extra effort to maintain social relations, an effort that is diverted from the process of creating an acceptable project for the instructor to grade. If group work is to continue to be a part of the college mathematics classroom, it is necessary to design studies that address the question of why students are not more positive about it. If students are expected to work in pairs despite the fact that they do not like doing so, then there needs to be sufficient evidence that working in pairs produces positive learning outcomes. We need more studies which explore learning outcomes from group work in mathematics at the college level. This study has indicated that social interactions may be just as important as mathematical interactions in both the success of group problem solving and in the type of experiences the students have. This suggests the need for studies that would focus on both the social and mathematical interactions amongst students working in groups.

PERSONAL BENEFITS

As a college instructor, I generally see only the results of a problem solving session as presented to me on assignments and examinations. What I do not see is the process that the students go through to produce the work they submit to me. This research has increased my understanding of the complexity of this process. It has led me to realize that while my students often have a reasonably good grasp of the mathematical techniques needed to solve a particular problem, there are many conditions that may make it difficult or impossible for them to apply the techniques they know. In my algebra and precalculus classes, I now spend a significantly larger portion of the time concentrating on solving problems. Specifically, I spend a lot more time on the part of an algebraic problem that comes before the step of "write an equation." I have begun to include, in my lesson plans, more problems that do not call for a solution, but rather require an algebraic description of a situation.

I am much more aware of the importance of perseverance in learning to solve problems. I am moving towards assignments with fewer, but more complex, problems so that I send the message to my students that they can expect to spend a reasonable length of time on any one problem. I am continuing to increase the amount of classtime that I allow for the students to work on problems while I have a chance to circulate around the room, keeping the students focussed on the task at hand.

I have also become much more aware of the complexities of group work. I am aware that many of my students strongly resent being asked to work in groups and being marked as a group, and I am aware that they can have very negative experiences if the situation is not well planned. In assigning students to groups, much more than just academic considerations come into play. I generally do not have access to enough information about the particular social interactions in my classroom to make group selections in which I am confident the experience will be positive for all concerned. Thus, I now allow the students to chose their own groups, and they appear somewhat happier with this. Group work in mathematics class is much more that just assigning students to groups and handing out assignments. For it to work well, the instructor has to be prepared to act as facilitator, monitor, coach and role model.

CONCLUSION

For this study I have drawn heavily on the seminal work of Schoenfeld, both for background and for methodology. While my data analysis methods were different than his, my think aloud problem sessions were modelled on his similar sessions with more advanced and more able university students. Schoenfeld concentrated his studies on the executive behaviour of his subjects and I found that my less advanced and less able subjects exhibited many of the same behaviours as his more advanced students. My students failed to evaluate strategies before implementing them, they spent a lot of time on "wild goose chases" and they did not monitor their progress or ask where a particular calculation might lead them. Schoenfeld has generally chosen to have his subjects work in pairs in order to lessen their reactivity to the experimental situation. It is simply more natural to talk to a partner than to speak aloud when working alone. He has acknowledged that there is a risk that the behaviours seen will not necessarily be the same as those that might have been seen if the students worked alone. [Schoenfeld 1985b] The present study has specifically looked at this issue. I have found that there is little difference in the strategies brought to bear on the problems by pairs or by individuals, and the decision points in a session are more open to study with pairs. However, I have seen that the character of the problem session when students work with a partner, is greatly determined by the social interactions of the pair. Useful strategies, for example may be rejected for essentially social reasons. Schoenfeld [1985b] cited the reduction anxiety as another reason to choose pairs. However, I found that this was not always the case for pairs, and for many students anxiety actually increased. Some worried about how they would appear to their partners while other felt they had to hold back in order to appear considerate of their partners self esteem. Thus, while speak aloud problem sessions with pairs may be used to infer general student behaviour, it must be done with caution when generalizing to the behaviour to individuals.

The aim of this study was to investigate the actions of average college algebra students working alone and in pairs. While all but one of the subjects stated that they preferred to work alone they did cite the provision of a second point of view as being of major benefit when working with a partner. The research literature would also lead one to expect that working in pairs would require that the students would attempt to construct an agreed upon representation of the problem and then decide upon the approach to be taken to solve it. However, this is not what happened. There was little discussion of the structure of the problem and almost no analysis of proposed strategies. Constructive controversy was almost entirely absent. Nevertheless, the pairs were substantially more successful in solving the problems. This increased success arose from four factors: an increase in persistence, the more able partner leading the pair, an increased opportunity for oral rehearsal, and the correction of minor errors. The particular character of any problem session depended on both the academic and social interactions of the partners. Five categories of pairs emerged from the study: socializers, tutor/pupil pairs, partners, individuals and hostile pairs. The students, whether working alone or in pairs, exhibited a wide variety of mathematical skills and strategies in their attempts to solve the problems given them. Despite this, they were not successful in solving many of the problems. Several factors contributed to their lack of success. They were generally so fixated upon finding an answer that little effort was put into analyzing the structure of the problem or generating and comparing various strategies. A major factor in their lack of success was that while the problems given them often required a structural approach, the students were generally working at an operational level for this material.

In conclusion, I see problem solving as central to the college algebra curriculum, but there is still a great deal to be learned about what students actually do when solving problems and what they learn by solving problems.

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APPENDIX A
REQUEST FOR VOLUNTEERS

Research Project Description
Problem Solving in College Algebra Students
by
Lin Hammill

I am doing a study on problem solving amongst college algebra students. This study is being conducted as part of my dissertation for the doctoral degree in mathematics, through the Department of Mathematics and Statistics of Simon Fraser University.

I am currently looking for students who would like to take part in this study. There will be a small remuneration for each volunteer. I need 12-16 students currently enrolled in Math 112. The volunteers need not be especially good students. I am as interested in C students as I am in A or B students. Could you please announce this to your Math 112 classes. If any students are interested they can find me in office 3335 or call 599-2556 (V.M. 9606). If you like, I could visit your classes to explain the project to them in person. This would take no more than 5 minutes. Thanks for your help. Below is a more detailed description of the study.

* * * * * * *

The study consists of two parts. One part consists in subjects working, over a period of about two months, to attempt to solve a complex mathematical problem. The record of this work is to be kept in a notebook to be given to the researcher at the end of the time period. The second part consists of a series of task oriented interviews, each of which will be videotaped. Interviews will be conducted first with a single subject and then with small groups of subjects working together. Subjects will be asked to attempt to solve a series of mathematical problems during the interviews. It is expected that each subject will be asked to participate in at least three interviews, held at approximately two week intervals.

If you agree to take part in this study you will be asked to take part in the entire series of interviews and to submit the problem-solving record described above. You may, however, decide to withdraw your participation at any time. Participation or non-participation in this study is voluntary and will not affect your marks in any mathematics course in which you are enrolled at Kwantlen College.

If you agree to participate in this study you will be paid a stipend of $10.00 for each interview in which you take part and a stipend of $20.00 for completion of the problem solving notebook.

The data obtained in this study will be kept strictly confidential. Videotapes and written material will be kept only until analysis of them is complete They will then be destroyed by erasure (for videotapes) or shredding (for documents). Pseudonyms will be used in any report of the study.

APPENDIX B
CONSENT LETTER FROM PARTICIPANTS

Simon Fraser University and those conducting this study subscribe to the ethical conduct of research and to the protection at all times of the interests, comfort, and safety of subjects. This form and the information it contains are given to you for your protection and full understanding of the procedures involved. Your signature on this form will signify that you have received the document described below regarding this study, that you have received an adequate opportunity to consider the information in the document, and that you voluntarily agree to participate in this study.

Having been asked by Lin Hammill to participate in the research study described in the document entitled, "Research Project Description, Problem Solving in College Algebra Students," I understand the procedures to be used in this study.

I understand that I may withdraw my participation from this study at any time.

I also understand that if I have any concerns or complaints I may register them with Ms. Hammill or with Dr. K. Heinrich, Chair, Department of Mathematics and Statistics, Simon Fraser University.

I agree to participate in this study by giving permission for my written work to be used as data for this study and by taking part in the task oriented interviews as described in the above named document. I understand that I will be videotaped during these interviews.

Name____

Address______________________________________

Signature____________________

Witness_____________________

Date________________________

Once signed, a copy of this consent form and a subject feedback form will be provided to you.

APPENDIX C
PROBLEMS AND SOLUTIONS

1. A passenger who had travelled half of his journey fell asleep. When he awoke, he still had to travel half the distance that he had travelled while sleeping. For what part of the entire journey had he been sleeping?

Solution:

Let x be the fraction of the total distance during which he slept. Then when he awoke he still had a distance of 1/2 x left to travel. Thus the distance from when he fell asleep to the end of the journey was 3/2 x. This is one half of the total distance. Thus we have:

$$\frac{3}{2}x=\frac{1}{2}$$
$$x=\frac{2}{3}\times\frac{1}{2}$$
$$x=\frac{1}{3}$$

Thus he slept for one third of the journey.

2. There is a rule regarding division by nine that you may know. It says that a number is divisible by nine if the sum of its digits is divisible by nine. Can you show why this rule works?

Solution:

We will demonstrate for a three digit number. Let abc be any three digit number. Then:

$$
\begin{aligned}
\text{"}abc\text{"} &= 100a + 10b + c \\
&= (99+1)a + (9+1)b + c \\
&= (99a + 9b) + a + b + c \\
&= 9(11a + b) + (a + b + c)
\end{aligned}
$$

The first term is divisible by 9. Thus the integer abc is divisible by 9 if and only if the second term, a + b + c, is also divisible by 9.

3. A boy went shopping with his father. He found a hat he wanted for $20. He said to his father, "If you will lend me as much money as I have in my wallet, I will buy the hat." His father agreed. They then did it again with a $20 shirt and with a $20 belt. The boy was finally out of money. How much had he started with?

Solution:

Let x be the amount of money (in dollars) that the boy started with. Then his father lent him x more dollars so that he had a total of 2x dollars. Of this he spent $20 on the hat leaving him with:

$$2x-20$$

His father then lent him as much as he already had so that he then had:

$$2(2x-20)$$

He then bought the $20 shirt, leaving him with:

$$2(2x-20)-20$$

His father doubled that giving him:

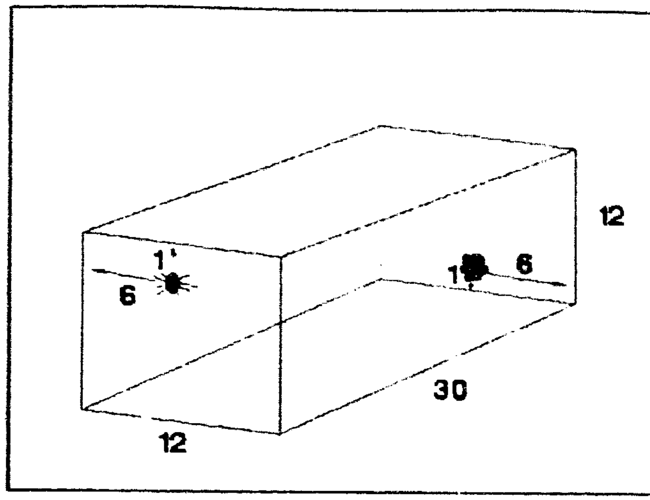
$$2(2(2x-20)-20)$$

He spent $20 more and had no money left. Thus we have:

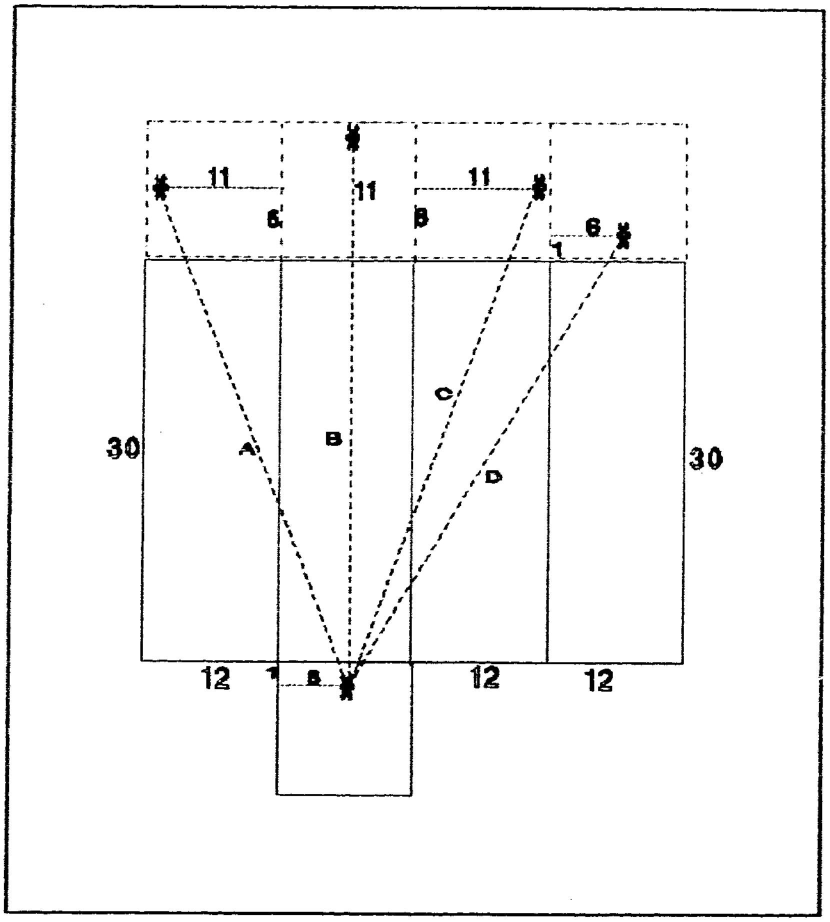
$$\begin{aligned} 2(2(2x-20)-20)-20&=0 \\ 8x-140&=0 \\ 8x&=140 \\ x&=17.5 \end{aligned}$$

Thus the boy started out with $17.50.

4. Suppose a spider and a fly are on opposite walls of a rectangular room, as shown in the drawing. The spider wants to get to the fly and must do so by travelling on the surfaces of the room. What is the shortest path the spider can take?



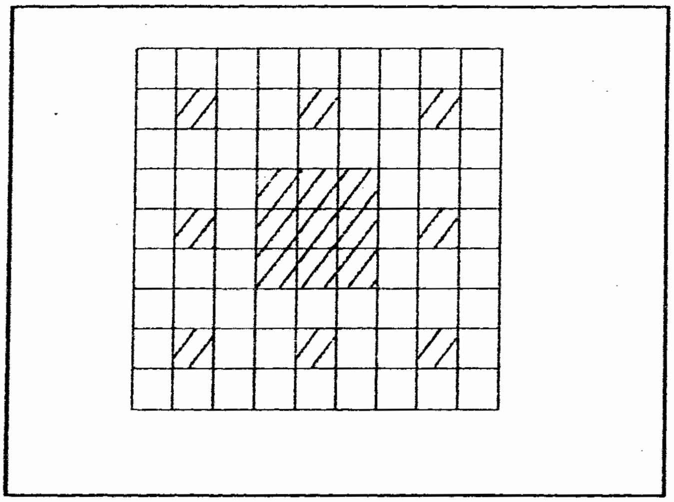
Solution:



The shortest distance between two points on a plane surface is a straight line. Thus we open the room out to form a plane surface as shown in the diagram below. Note that there are four ways in which the back side of the room may be attached to the rest, as shown by the dotted lines. Then there are 4 possible straight line routes from the spider to the fly, as indicated by the dashed lines. We need only calculate the lengths of these lines. Routes A and C are the same length, which is $\sqrt{17^2+37^2} \approx 40.7$. Route B is $1+30+11=42$.

Route D is $\sqrt{24^2+32^2}=40$. Thus route D is the shortest route.

5. A square is divided into nine smaller squares and the centre square is shaded, as in the drawing. Each of the eight unshaded squares is then divided into nine smaller squares and the centre of each is shaded. If the process continues forever, what fraction of the original square will be shaded?



Solution:

Let x be the fraction of the whole square which is shaded. The large shaded square at the centre in 1/9 of the whole square. Each of the eight sqares of the same size surrounding it is a 1/9 size copy of the whole square. Thus:

$$x = \frac{1}{9} + 8\left(\frac{1}{9}x\right)$$
$$x - \frac{8}{9}x = \frac{1}{9}$$
$$\frac{1}{9}x = \frac{1}{9}$$
$$x = 1$$

Thus all of the square is shaded.

6. With only the cold water valve open, it takes eight minutes to fill the tub of an automatic washer. With both hot and cold water valves open, it takes only 5 minutes to fill the tub. How long will it take to fill the tub with only the hot water valve open?

Solution:

It is clear that with both valves open:

$$\frac{\text{water from}}{\text{hot water valve}} + \frac{\text{water from}}{\text{cold water valve}} = \text{one tub full}$$

Then:

$$\frac{\text{rate of}}{\text{hot water}} \times \text{time} + \frac{\text{rate of}}{\text{cold water}} \times \text{time} = \text{one tub full}$$

The time to fill when both valves are open is 5 minutes. We need to find expressions for the rates. But the rate of flow of water is the volumn divided by the time. We will measure in tubs per minute.

Then the rate of the cold water is 1/8 tubs per minute.

Let x be the time (in minutes) it would take the hot water valve alone to fill the tub. The the rate for the hot water is 1/x tubs per minutes.

Thus we have:

$$\frac{1}{x}5 + \frac{1}{8}5 = 1$$

$$5 + \frac{5}{8}x = x$$

$$5 = \frac{3}{8}x$$

$$x = \frac{40}{3}$$

It takes the hot water valve thirteen and one third minutes to fill the tub.

7. A ski club chartered a bus for a ski trip at a cost of $520. In an attempt to lower the bus fare per skier, the club invited nonmembers to go along. After five nonmembers joined the trip, the fare per skier decreased by $5.20. How many club members are going on the bus?

Solution:

We see that:

Cost per skier after	=	Cost per skier before	-	$5.20

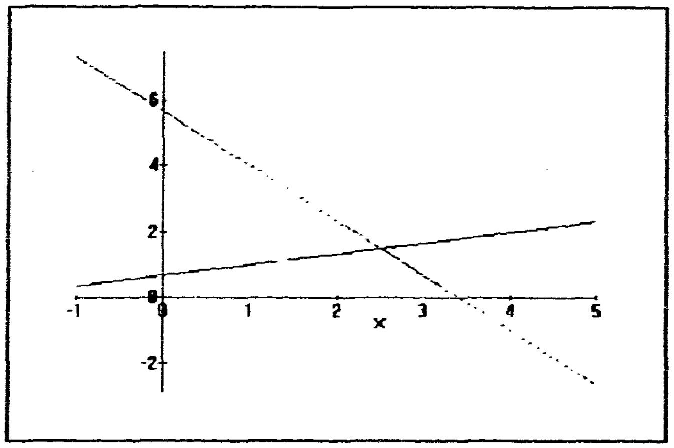
Cost per skier is in each case equal to $520 divided by the number of skiers. Let x be the number of club members. Then x + 5 is the number of skiers who actually went on the trip. Thus:

$$\frac{520}{x+5}=\frac{520}{x}-5.2$$
$$520x=520(x+5)-5.2x(x+5)$$
$$0=5.2x^2+26x-2600$$
$$0=x^2+5x-500$$
$$0=(x-20)(x+25)$$
$$x=20,\ x=-25$$

X = -25 is rejected. There were 20 club members.

8 . Find the area of the triangle bounded by the y-axis and the two lines given by:
$x - 3y = -2$ and
$5x + 3y = 17$.

Solution:



We begin by graphing the lines.

To find the area of the triangle we let the base be along the y-axis. Thus to find the length of the base we must find the y intercepts and to find the height we must find the x coordinate of the intersection point of the lines.

To find the y intercept of the line with equation $x - 3y = -2$ we set $x = 0$ and find $y = 2/3$.

To find the y intercept of the line with equation $5x + 3y = 17$ we set $x = 0$ and find $y = 17/3$.

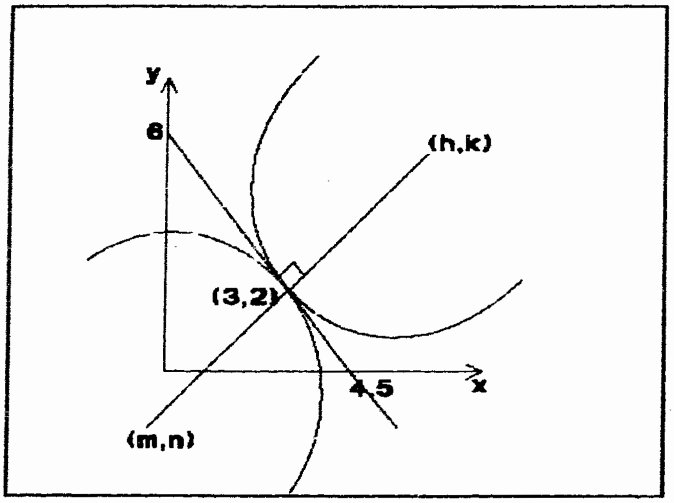
Thus the length of the base of the triangle is $17/3 - 2/3 = 15/3 = 5$.

To find the x coordinate of the intersection point of the two lines we add the two equations together to obtain $6x = 15$ or $x = 5/2$.

Thus the area of the triangle is $\frac{1}{2} \times 5 \times \frac{5}{2} = \frac{25}{4}$. The area is 25/4 square units.

9. A circle of radius 5 is tangent at the point (3,2) to the line given by $y = -\frac{4}{3}x+6$. Find the centre of the circle. (Note: a line is tangent to a circle if it touches the circle in only one point and if it forms a right angle with a radius of the circle.)

Solution:



We begin by graphing the line and sketching the circle.

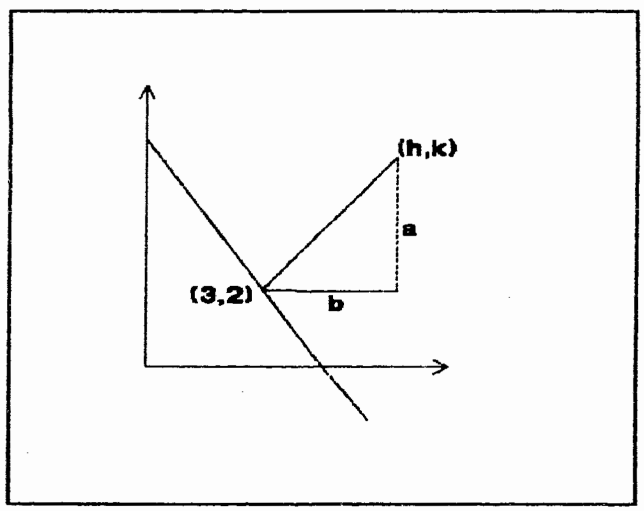
We note that there are two circles which could satisfy the conditions of the problem. We will find the centre of the upper one.

Since the circle is tangent to the line the radius of the circle joining its centre to the point of intersection, (3,2), will be perpendicular to the given line. The slope of the given line is -4/3, so the slope of the line formed by the radius is 3/4.

We now draw and label another diagram.

Then we see that since the slope of the radius is 3/4, a = 3/4 b. Also $a^2 + b^2 = 25$, by Pythagoras' theorem. Thus:

$$(\frac{3}{4}b)^2+b^2=25$$
$$\frac{9}{16}b^2+b^2=25$$
$$\frac{25}{16}b^2=25$$
$$b^2=16$$
$$b=4$$



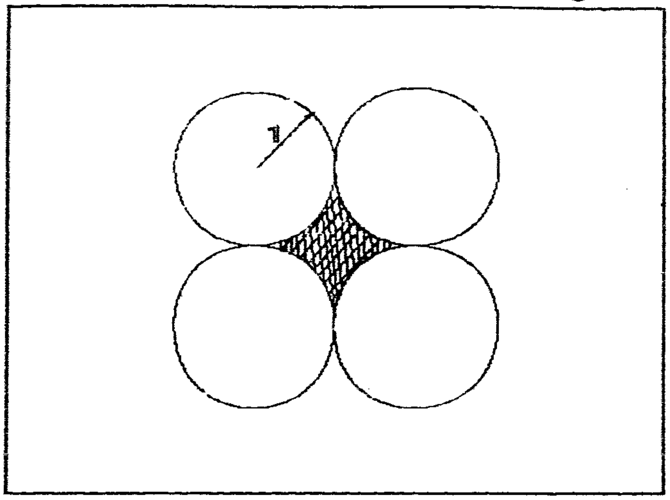
So a = 3. Thus the x coordinate of the centre is 3 + 4 = 7 and the y coordinate of the centre is 2 + 3 = 5.

The centre is at (7,5).

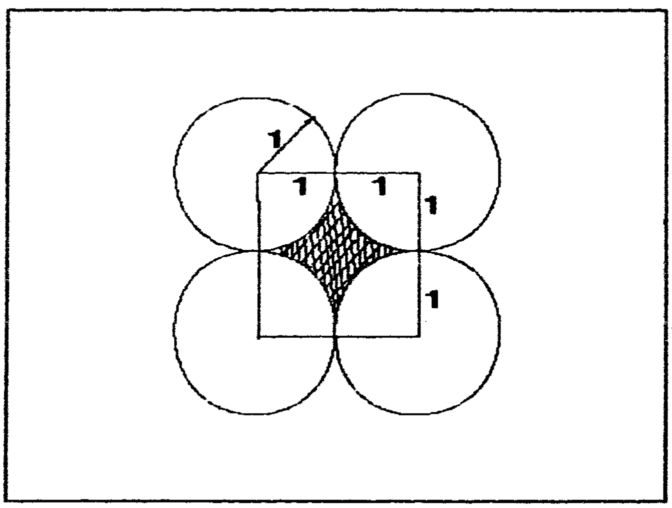
Using the same method, it can be shown that the centre of the other circle is at

(-1,-1).

10. Find the area of the shaded region.



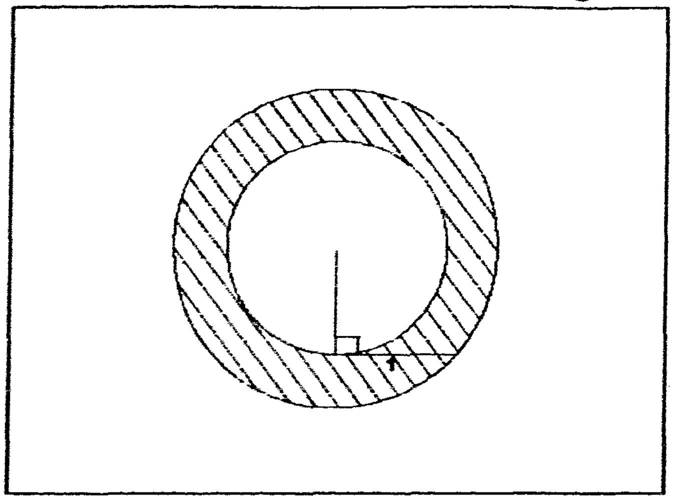
Solution:



If we draw the radii from the centres of the circles to the points of tangency of the circles a square is formed as in the diagram. Then the area of the shaded region is equal to the area of the square minus the combined areas of the sectors of the circles that are within the square. Since each sector is a quarter circle we have

Area $= 2^2 - \pi(1^2) = 4 - \pi$ square units.

11. Find the area of the shaded region.



Solution:

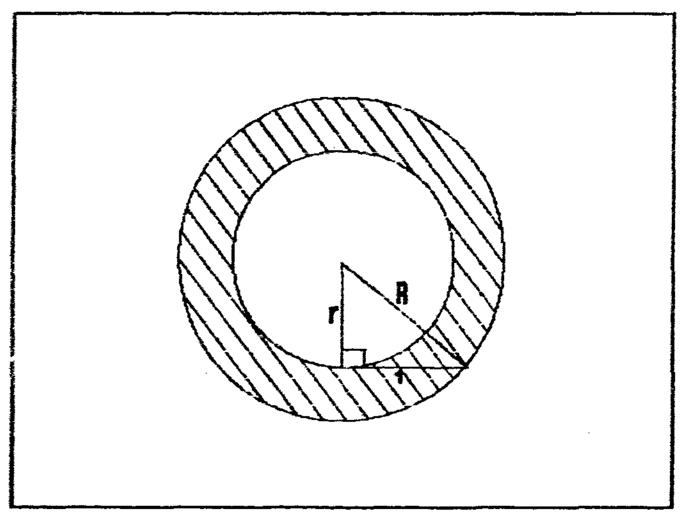
To find the area of the shaded region we need to subtract the area of the smaller circle from the area of the larger circle. Complete the triangle by drawing in the line as shown. Then the raddii of the two circles are r and R.

Then the required area is:

$$\pi R^2 - \pi r^2 = \pi (R^2 - r^2)$$

Now we consider the relationship between the two radii by applying Pythagoras' theorem to the triangle:

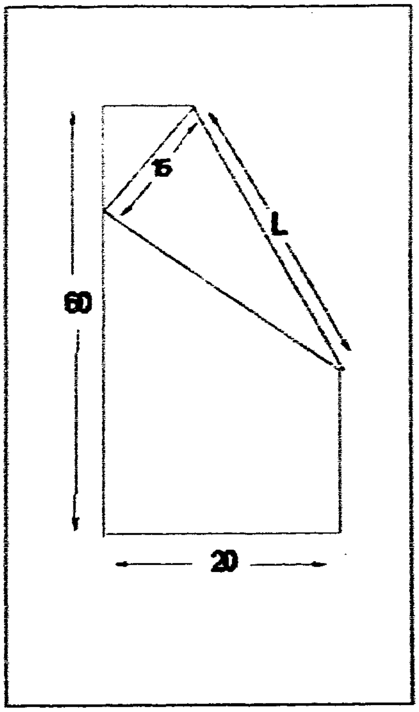
$$r^2 + 1^2 = R^2$$
$$1 = R^2 - r^2$$



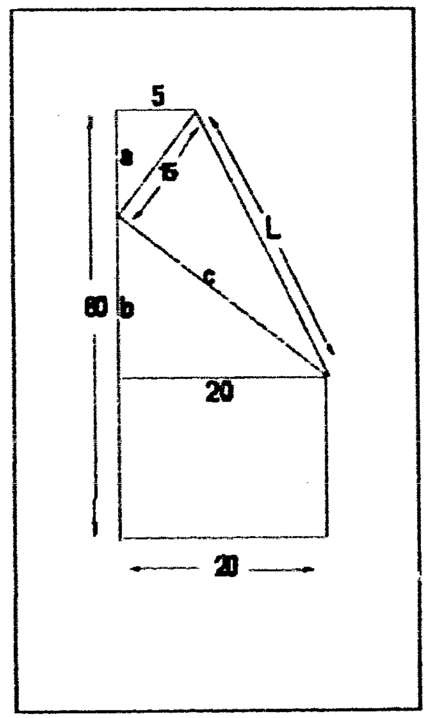
Thus the area is:

$$\pi (R^2 - r^2) = \pi (1) = \pi$$

12. A 60cm by 20cm rectangular piece of paper is folded as shown. Find L.



Solution:



Draw a horizontal line and label the diagram as shown. Note that the horizontal part of the top of the paper is of length 5 cm and the added horizontal line is of length 20 cm. Note also that each of the corners of the paper is a right angle. Thus each of the triangles formed is a right triangle.

Apply Pythagoras' theorem to the small tiangle in the upper left hand corner to obtain:

$$a^2+5^2=15^2$$
$$a^2=15^2-5^2$$
$$a^2=200$$
$$a=\sqrt{200}$$

Now notice that a+b = c so that b = c–a. Thus $b=c-\sqrt{200}$. Now apply Pythagoras' theorem to the lower triangle to obtain:

$$
\begin{aligned}
b^2+20^2 &= c^2 \\
(c-\sqrt{200})^2+400 &= c^2 \\
c^2-2\sqrt{200}c+200+400 &= c^2 \\
-2\sqrt{200}c+600 &= 0 \\
c &= \frac{300}{\sqrt{200}}
\end{aligned}
$$

Finally, applying Pythagoras' theorem to the last triangle, we obtain:

$$
\begin{aligned}
L^2 &= 15^2+c^2 \\
L &= \sqrt{15^2+\left(\frac{300}{\sqrt{200}}\right)^2} \\
L &= \sqrt{675}
\end{aligned}
$$

Thus the folded side is approximately 25.98 cm long.

13. A column of tanks is moving across the desert at a steady speed of 50 km per hour. A messenger travels from the front of the column to the rear of the column and then immediately returns to the front. If the messenger travels at a constant speed of 75 km per hour and the round trip takes 18 minutes, how long is the column?

Solution:

We see that:

$$\text{time to the back of column} + \text{time to the front of column} = 18/60 \text{ h.}$$

The speed of the messenger relative to the column of tanks is 75 + 50 = 125 km/h when moving to the back of the column and 75 - 50 = 25 km/h when moving from the back to the front. The time for each part of the round trip is the length of the column divided by the relative speed. Let d be the length of the column. Then:

$$\frac{d}{125}+\frac{d}{25}=\frac{18}{60}$$

$$\frac{6d}{125}=\frac{18}{60}$$

$$6d=125\cdot\frac{18}{60}$$

$$d=6.25$$

Thus, the column is 6.25 km long.

14. A cistern used to collect rainwater has 3 inlets that channel water into it and two outlet drains. If the cistern is full, the smaller outlet can empty it in 24 hours by itself, while the larger outlet can empty it in 12 hours working by itself. In a severe rainstorm with both outlets closed, either the right or the left inlet, working alone, can fill the cistern in 8 hours, while it would take the centre inlet only 6 hours to fill the cistern it the other two were closed. A long hard rainstorm hits at noon when the cistern is empty, both outlets are closed, and all the inlets are open. At 1:00 p.m. both outlets are opened. At 2:00 p.m. the right inlet becomes clogged with leaves and fails to work. At what time will the cistern be full?

Solution:

We will consider the level of water in the cistern hour by hour. First, we will find the flow rates of each inlet and outlet.

smaller outlet:	−1/24 cistern/h
larger outlet:	−1/12 cistern/h
right inlet:	1/8 cistern/h
left inlet:	1/8 cistern/h
centre inlet:	1/6 cistern/h.

Time	Rate	Accumulation
noon − 1 pm	1/8 + 1/8 + 1/6 = 10/24	10/24
1 pm − 2 pm	10/24 − 3/24 = 7/24	17/24
2 pm − ?	7/24 − 1/8 = 4/24	24/24

From 2 pm on the rate of filling is 4/24 cistern/h and there is 7/24 cistern left to fill. If t is the time (in hours) after 2 pm until the cistern is full, then we have

$$\frac{4}{24}t=\frac{7}{24}$$
$$t=\frac{7}{4}$$

The cistern will be full at 3:45 pm.

15. A commuter is picked up by her husband at the train station every afternoon. Her husband leaves the house at the same time every day, always drives at the same speed, and regularly arrives at the station just as his wife's train pulls in. One day she takes a different train and arrives at the station one hour earlier than usual. She starts immediately to walk home at a constant speed. Her husband meets her along the way, picks her up and drives back home. They arrive there 10 minutes earlier that usual. How long did she spend walking? If she walks at 8 kms per hour, how fast does he drive?

Solution:

Since the husband's trip is 10 minutes less than usual, it is 5 minutes less each way and he picks her up 5 minutes earlier than usual. Since she started walking 60 minutes before he was to pick her up, she must have walked for 55 minutes.

The distance she walked is the distance that he could have driven in 5 minutes. She walked for 55 minutes at 8 km/h, so she walked

$$\frac{55}{60} \times 8 = \frac{22}{3} \text{ km.}$$

Thus he could drive 22/3 km in 5 minutes, so his speed was

$$\frac{\frac{22}{3}}{\frac{5}{60}} = 88 \text{ km/h.}$$

16. How many ways are there to make change for a quarter, using dimes, nickles and pennies?

Solution:

We will set up a table to list all the possible ways to make change for a quarter using dimes, nickles, and pennies.

dimes	nickles	pennies
0	5	0
	4	5
	3	10
	2	15
	1	20
	0	25
1	3	0
	2	5
	1	10
	0	15
2	1	0
	0	5

Thus we see that there are 12 possible ways to make change.

17. On the first day of math class 20 people are present in the room. To become acquainted with one another, each person shakes hands just once with every one else. How many handshakes take place?

Solution:

The first person in the class shakes hands with everyone else, that is, with 19 people.

The second person in the class has already shaken hands with the first person and must shake hands with everyone else, that is, with 18 people.

The third person in the class has already shaken hands with the first two and must shake hands with everyone else, that is, with 17 people.

This continues in this manner until the second to last person shakes hands with the last person.

Thus the number of handshakes is:

$19 + 18 + 17 + \ldots + 2 + 1 = 190.$

18. $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

$10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 3628800$

$n! = 1 \times 2 \times \ldots \times (n-1) \times n$

How many zeros are at the end of the number

$100! = 1 \times 2 \times 3 \times \ldots \times 98 \times 99 \times 100$?

Solution:

The zeros at the end of the number are produced by factors which are equal to multiples of ten.

100! has factors 10, 20, 30, 40 50, 60, 70, 80, and 90, each of which contributes one 0 to 100!. That makes 9 zeros.

Each even number contributes at least one 2 as a factor. Whenever these 2's are matched with 5's we have another 10. Thus we get zeros for 5, 15, 25, 35, 45, 55, 65, 75, 85, and 95. There are three additional 5's contributed by 25, 50 and 75. That makes 13 more zeros.

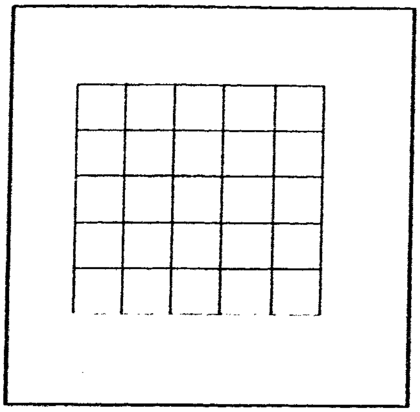
100 is a factor of 100!. 100 contributes 2 more zeros.

None of the other factors of 100! contributes a zero.

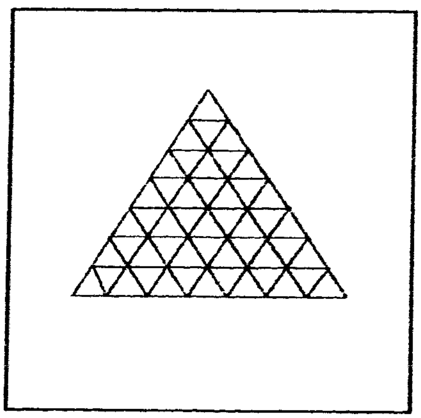
Thus there are 24 zeros at the end of 100!

APPENDIX D
WORKBOOK PROBLEMS

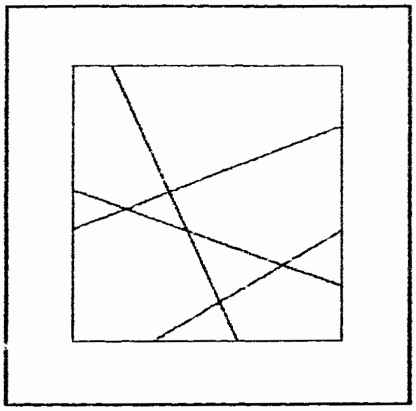
1. How many squares are there on a rectangular grid? Consider different sizes of grids.



How many equilateral triangles are there on an eightfold triangular grid?



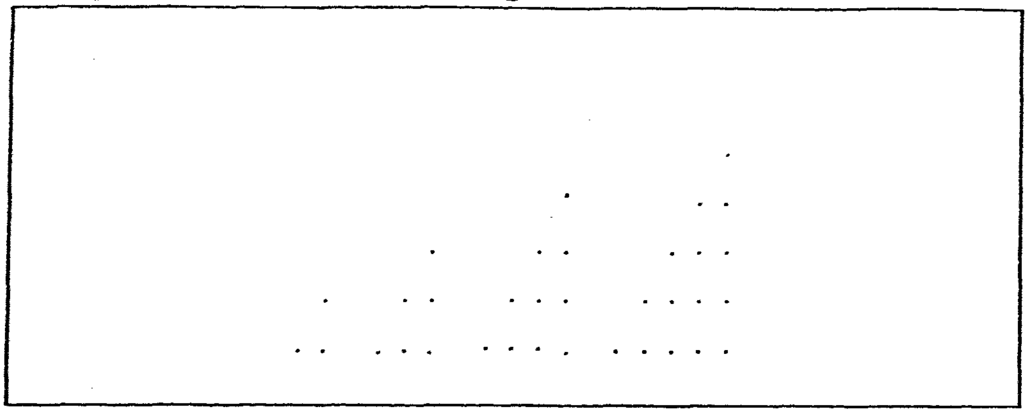
2. Take a square and draw a straight line across it. Draw several more lines in any arrangement you like, so that all the lines cross the square, dividing it into several regions. The task is to colour the regions in such a way that adjacent regions are never coloured the same. (Regions having only one point in common are not considered adjacent.) How few colours are needed to colour any such arrangement?



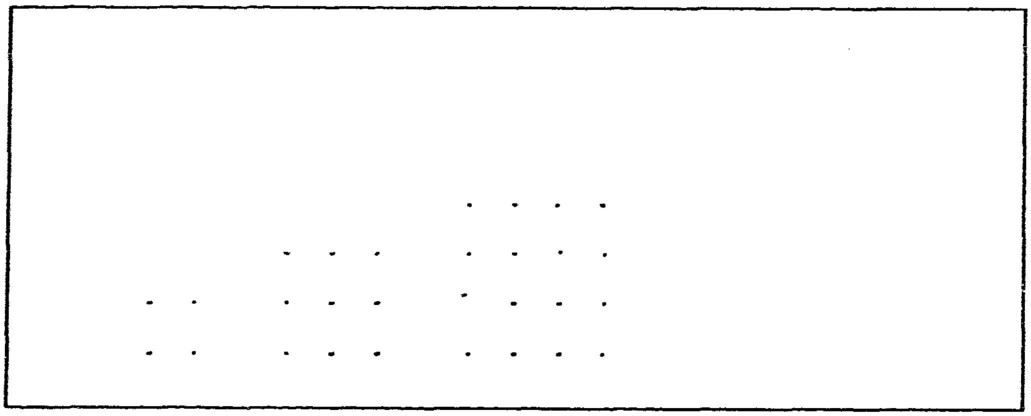
Now allow the lines to be curves and loops and remove the restriction that they go right across the square. Now how many colours will it take?

Can you change the square into the surface of a three dimensional object?

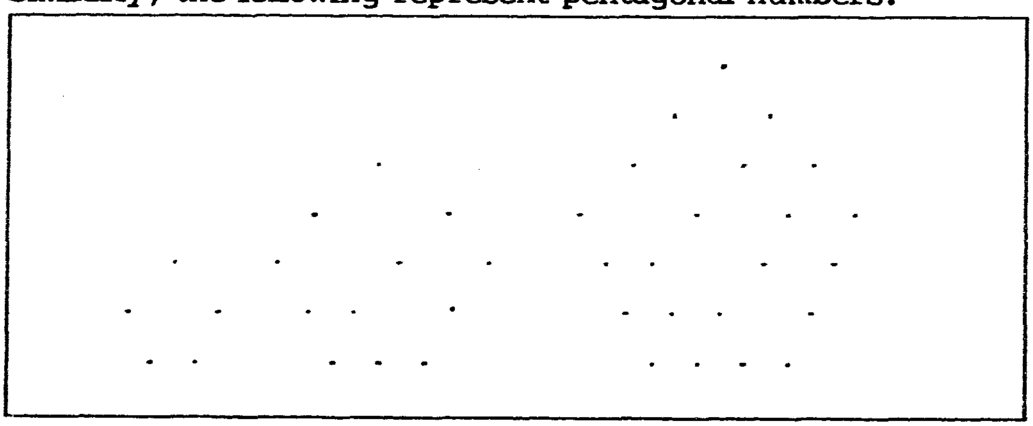
3. A number which can be represented as the number of dots in a triangular array is called triangular.



A number which can be represented as the number of dots in a square array is called square.

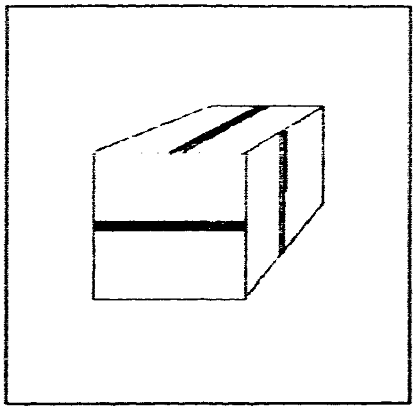


Similarly, the following represent pentagonal numbers.

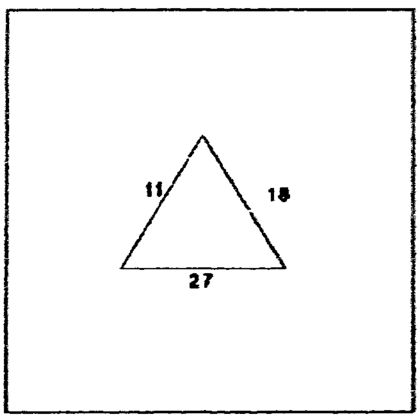


Which numbers are triangular, which square, which pentagonal? Generally which are P-polygonal?

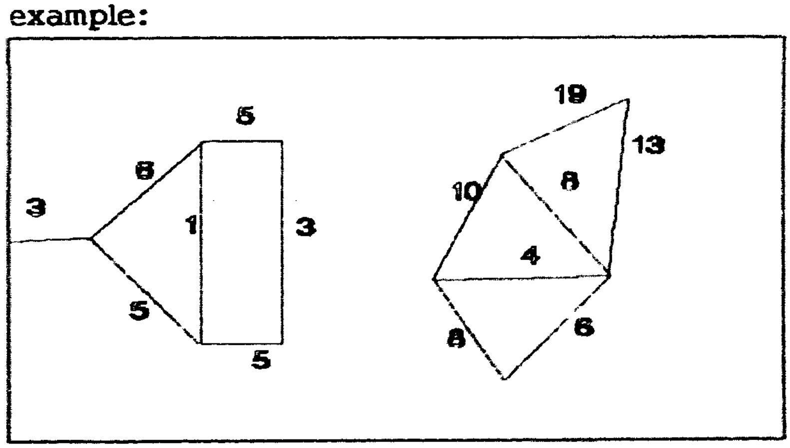
4. How many different cubes can be made such that each face has a single line joining the midpoints of a pair of opposite edges? Same question for a diagonal stripe. Try a tetrahedron?



5. A secret number is assigned to each vertex of a triangle. On each side of the triangle is written the sum of the secret numbers at its ends. Find a simple rule for revealing the secret numbers. For example secret numbers 1, 10, 17 produce:



Generalize to other polygons. Consider more general arrangements of vertices and edges. For example:



Consider operations other than adding.

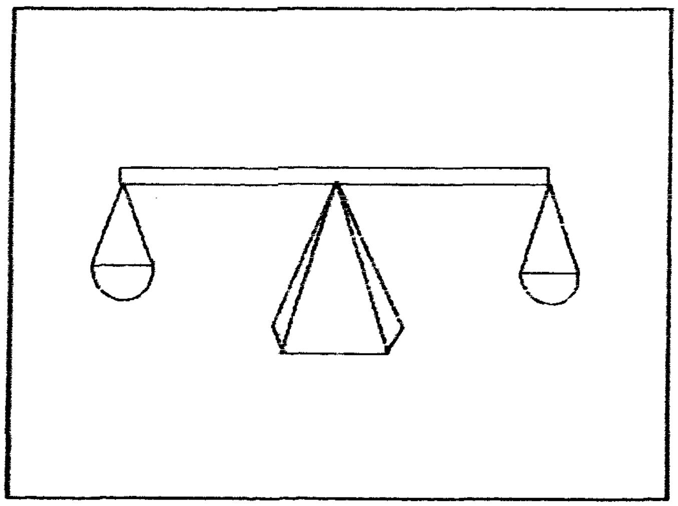
6. Amongst nine apparently identical tennis balls, one is lighter than the others which all have the same weight. How quickly can you guarantee to find the light ball using only a simple balance?

What if there are more than nine balls?

What if you know only that one ball has a different weight?

What if there are two kinds of balls, heavy and light, but unknown numbers of each?

What if the balls are all of different weights, and you wish to line them up in order of weight?



APPENDIX E
TRANSCRIPTS OF TWO PROBLEM SESSIONS

... indicates a pause of up to four seconds.
PAUSE indicates a pause of more than four seconds.

Interview 10 Tape 2
Cecil alone, Problems 3 10 2

Problem 3 Father and son

L So here's your first question.

C Okay. A boy went shopping with his father. He found a hat he wanted for $20. He said to his father, "If you will lend me as much money as I have in my wallet, I will buy the hat." His father agreed. They then did it again with a $20 shirt and a $20 belt. The boy was finally out of money. How much had he started with?

Let's see here ... hum ... It's going to be a tricky question ... Well, I think I could probably figure it out in my head but I don't know if I know any formulas to figure it out....

Okay, so the hat, the hat was 20 dollars ... and ... he borrowed as much money as he had. He borrowed as much money off his dad as he had in his wallet, to buy the hat. So, ... he spent the 20 dollars ... spent, and, ah, ... and was left with 40. ... And he did it again. And they they did it again with a 20 dollar shirt and a twenty dollar belt. They did the same thing over and over.

PAUSE

Hum, I am stumped on this question ... Let's see, ... I am not too sure exactly what I'm doing in this.

PAUSE

Okay. Total money spent, 60 dollars. And, hum.

PAUSE

Well, I'm not too sure about the formula for it but if he started out with ... 40 bucks and his father lent him 40, that would put him over the limit. So, he started out with less than 40 dollars. ...

If he started, if he started with 20, with 20 dollars he would have had 40 before he bought the hat. 20 borrowed and we've got the 20 dollars again. ... And if he borrowed the 20 again, so he, if he borrowed the 20 dollars again he would have 40 all over again. After the shirt was bought he'd be down to 20. And then if he borrowed 20 dollars again he would be left with 20 dollars left over at the end of it. But he is broke so 20 dollars is wrong.

PAUSE

So if we let x equal initial amount, initial amount of money ... Hum

PAUSE

I'm thinking of some sort of exponential idea here, but, uh, I could be wrong. ... So ... well, what if 15 dollars for the initial amount. His dad lent him 15. After buying the hat for 20 he has 10 dollars left over. If he borrowed 10 dollars off his dad to buy the shirt, he would be broke. ... hum ...

I'm thinking it has to be between 15 and 20 dollars. We know that, for sure. ...

So let's try 18. Okay, 18, 18, 36. (unclear) buying the hat. Left with 16 dollars left over. ... Borrows 16 off his dad. After buying, that's 32 ... So we know that this is not going to work out. ... 'Cause 20 off, from the shirt. Let's see, the shirt, that's 12. If he borrowed 12, that would be 24, which would leave him 4 dollars left over after buying the belt. ...

Oh, boy, well, I'm doing this the trial method, the trial and error method, but what's it (unclear). I'm sure there's got to be a formula in here somewhere but I really don't know it. Um, okay, well, it's not 18. ...

It's probably 17.50 something, yup. ...Okay, we'll go with this here. 17 ... Okay, ... buying the hat for 20. ... Is 14 , and borrowing 14. Which leaves us with 28. ... That's the hat. We took the shirt out of there. Leaves us with 8. ... Oh, borrows 8. He's 4 dollars shy.

So, the answer is, the answer is 17.50. We aren't sure about that so we're just going to check it. Go back over here again. 17.50, 50, that's 35. And after the hat for 20, it's 15. And after the shirt of 20, ... That leaves us with 10 dollars after he buys the shirt. So he borrows 10 dollars from his dad. Okay, good, we know it is 17.50.

Answer. ... Good thing it wasn't complicated, otherwise I would have never

HAT $20.00

20.00 spent ans. $17.50

Total $60.00 $

~~40.00~~

Iwe let x = Amount $

Started w $20.00

Borrow 40.00

20.00

shirt 40.00

20.00

20.00

20.00

int 15.00

15.00

30.00

Hat 20.00

10.00

shirt 10.00

17.50

17.50

35.00

20.00

15.

Shirt 10.00

10.

17.00

17.00

34.00

Hat 20.00

14.00

14.00

28.00

20.00

8.00

8.00

8.

18.00

18.00

36.00

Hat 16.00

16.00

32.00

shirt 20.00

12.00

12.00

4.00

4.

15 17.50.

figured it out. (laugh) Now look at all my writing here, my goodness. Um, so I have solved this one, but I didn't do it by formula. I just did it by a bunch of scribbling and trial and error.

L Well, we are just looking for the solution, not necessarily with a formula.

C Okay, well, we found the formula. Or we didn't find the formula, we just found the answer.

L Okay.

Problem 10 Four circles

L Okay, so, here's the next problem.

C Okay. Find the area of the shaded region. Woo hoo. Okay.

PAUSE

So the area of each circle is, it's right here, pi, where's that hiding? (looking for the button the calculator)

L Here it is.

C Oh, I'm sure I can find it here somewhere. ...

L Um, the pi's the second, no, third, and there.

C Oh, here, okay.

PAUSE

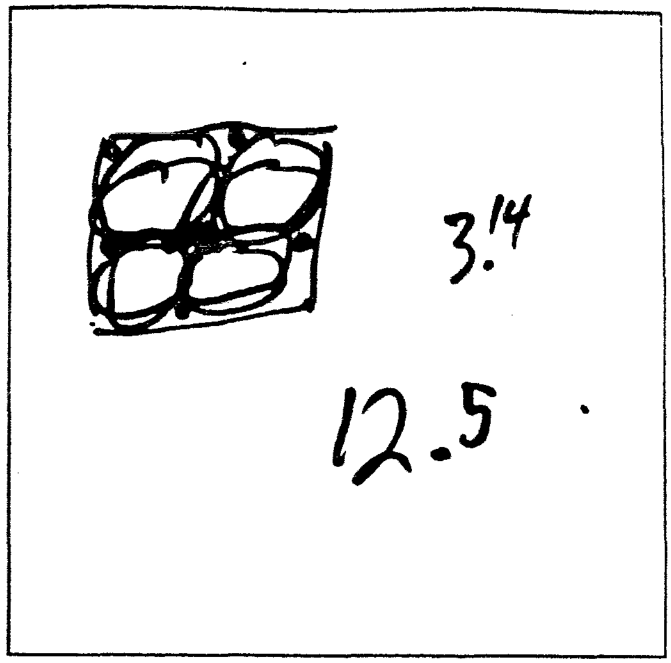
hum ... Okay ... So when you hit this, ah, pi. Does it, it multiply it automatically?

L No, you'll have to put the multiply in.

C Okay ... times ... clear ... times ... pi r squared. Equals. Decimal points, how far you want me to go to?

L Oh, you can give the answer in any form you like.

C Okay, well, I'll say the area of each of these is 3.14, whatever units that may be. Um, I should know that anyways. ... Here we go. 1 2 3 4, shaded region, um ... hum ... This is much trickier that the last one, definitely



PAUSE

I'm thinking that, ah, this is a trick question, um, ... If you know the area of each circle there must be a formula for the space in between. ... And it doesn't matter how big the area is, that space in between 4 circles will always be the same percentage per area. ... So if the area is 3.14 ... That area has to be something like half of one circle ... or if you know the area of the circle you should be able to figure out the circumference of a circle. ...And the circumference of the shaded is exactly, or should be exactly one quarter, one quarter of the circumference of the circle. ... C of circle

PAUSE

Hum ... This one has got me really good. ... Okay, the area of the 4 circles is ... 12.5.

PAUSE

The shaded region is still going to fit inside one of the circles completely ... and touch, touch the edge on each 4 points. ...

I've never done a question like this one before.

PAUSE

(unclear) ... circumference is 2 pi r ... circumference is equal to 2 ... hum, ... alright ... 2 times ... (unclear) circumference is 6.3

PAUSE

Now, I don't think I'm heading in the right direction here. Ooh.

PAUSE

Must be some sort of inverted formula or something. Um.

PAUSE

If the circumference is 6.3 ... the length of each of these, sides of the shaded area would be 2 divided by 4 which is 1.6. 1.6, that's length of each side ... which, I'm sure has nothing to do with it, but we could pursue it, 1.6.

PAUSE

Or, ... aha, okay, so, if the radius is 1, we could make a square out of this and it would be exactly 4 units by 4, 4 units by 4, which is equal to 16. And the shaded area, the area of the shaded area would be exactly the same as the area of each of these points here. My diagram is not very good. So, area of this shaded area is the same as point here, point here, point there, point there, point there. The area of the square is 16 and the area of each circle is 3.14, which we rounded to be 12.5 from the 16, which leaves 3.5. Divided that by half because there's two areas which we are going for here. Which is 1.75.

1.75 and that's going to give us sort of a rough estimate because I was, wasn't too accurate with my decimal points. I did round the up at the 100ths.

So area of shaded area is 1.75 give or take a couple hundredths of units. Hah. Pretty proud of myself on that one.

Okay, next, huh.

L Okay.

Problem 2 division by 9

L There you go.

C Thanks. There's a rule regarding division by nine that you may know. Rings bells, but I can't remember it. It says that a number is divisible by nine if the sum

90 ~~99~~

~~144~~

81

72 63 54 45 36 27 18 9 14 3 1 =159

18 14 8 9

When x

4x9 36

3x9 27

5x9=45 45

72

7x9 72

63 6x9 63

9

4

5x9 45

45 2x9 18

of the digits is divisible. Okay, a number is divisible by nine if the sum of the digits is divisible by 9. Can you show why this rule works? (laugh)

I remember this one. I have no idea. Oh, ... Okay, if the sum of it's digits is divisible by 9. So if I have 27, that's divisible by 9 and it says ... hum ... some examples.

Hum, 18 and of course 9. Let's go backwards here for a change. Ah, 27, ... is, ah, 9 times 4 is what? ... Oh, my goodness, yup 36, knew it was there somewhere. Um, so why does it work? ...

That's really strange. Hum, I'd probably have to ask a mathematician about this one. Um, okay, ... um,

PAUSE

It wouldn't have anything to do with the thing we're multiplying. It has noting to do with, like 3 times 6 which would be 18, 2 times 7 which is 14 there. 8 and just 9. Hum. Why is it? ... 3 and 6 ... so that's 4 nines. 3 nines, nines. So, if I have 5 nines ... 45, 54, 63, 72, 9, 8, 7, 6, 5, 4, 3, 2. I need 1. That's really wierd. 81, hum, and then back to 90. ... Got a definite pattern here. Um

PAUSE

But if you have 99, adds up to 18, hah.

Anyways, I guess you have to stay within, ah, under 10, under 10 times 9. ... Unless, say, if you had 100 and , 144, is not divisible by 9, so I guess it doesn't work with anything over the hundreds bracket, Say you have 9 and, well, let's see. 144 divided by 9, 16. It does work. Okay, then, so. We have 1, 4, 3, 1. 1, 4, 3, 1, 159, 159, hum 159 times 9. We got 14 there. ... Still looks strange, hum. ... You had to pick this question for me didn't you? (laugh) ...

Not too sure where to start on this. Except, um, ... well, there has to be some sort of formula to show this. ... But what is it? Actually, there doesn't have to be. But knowing math there probably is. Um.

PAUSE

So 4 nines ... times 9. You can think of that as ... (unclear) there and 4 off there ... No ... 1 off there (unclear) 1 off there and 3 off there. Let's see what happens if we take (unclear) 36 perhaps and 27 which is 3 times, 3 times 9. If we take 1 off here and 2 off. Okay, we take 1 off here, leaves this with a 2. So that means we want to take 2 off this 9, which leaves us with a 7. Okay, try 63 here. 63 is

... 6 times 9, so we don't take any off there, hum. It was a neat theory but it didn't work.

63, 7 times 9 is 72. 63 ... 6 times 9, 63. So we're not taking any off here. So we're taking 6 off there. Which is why it works. Okay....

So, we've, we're working on a formula for why it works. We don't know why it works. Okay. ...So any number, so, we have 5 times 9 ... That's 45 ... Why? Why does it work that way? ... Subtract 1 up here (unclear) that's 4. That leaves us with 4 there. So if we take the 4 off the 9 that leaves us with 5. ...

And why does it only work with nines? ... (unclear) me. I don't know that, so, uh, ... Okay ... So 45 ... We could say that it works because it does and some things you just can't change. Um, 45, it's got to be something to do with this.

Oh, maybe this is just it in reverse. ... 5 times 9 is 45. So I take the 1 off here ... and ... any number times 9 ... (unclear) the difference is 4 ... between these two. 4, 5 ... So that's reversed and 4 minus (unclear) ...Nope, uh. Let's try the ... 18 which is 9 times 2. Okay, so we know that the first digit is 1. ... (unclear) subtract 1 from there which leaves us 1. 1 nine here is 1, oh, boy.

Okay, why does that work? ... Probably got something to do with square roots or something. I don't know. Um ...

Well, I really wanted to go 3 for 3 today if possible but it doesn't look like it's going to happen here. Um, hum. Okay, I hope I get the answer, answer for this one, once, if I don't figure it out myself. ... I hope I can understand the answer for this, if I don't figure it out. ...

Okay, 5 times 9 is, the difference between these two is 4. ... So we take the difference of 4 (unclear) ... Is that coincidence? No, no ... (unclear) 45 ...

Well, we have discovered that any numbers that add up to 9 are divisible by 9. We have also discovered that when we're multiplying the two ... Um, if multiplication occurs ... we subtract ... (unclear) if you subtract ...the number being multiplied by 9 from 9 you get the first digit of the number. ...And the number you're multiplying by becomes the last digit of the number. ... and they add up to 9.

Hum, hu, hum. And why does it work? I don't know. ...Hum ... When we did 72 which is 7 times ... 9, subtract the 2. 2 (unclear) 2 there and 7 comes (unclear) so that's wrong. Forget all that stuff I just said. I think it is wrong.

Um, I think I'm stuck on this one.

L Okay.

C Yup. That's a hard one, for me.

Interview 18 Tape 7
Karla and Candy Problems 16 15 10

Problem 16 Making change

L Here's your question.

C (gestures to K to read)

K How many ways are there to make change for a quarter, using dimes, nickels and pennies?

How many ways? Yeah, okay. So we've got our quarter.

C 25 cents.

K Right, and of course the first one is what? We could actually do this by thinking all the different ways.

C Um

K If you wanted to.

C (drawing)

K Okay, good, good, ah, 25 pennies. Which I wouldn't draw all of them, just

C Oh, we've got pennies, here?

K Yeah, yeah, tricky, yeah.

Yeah, okay, so 25 pennies.

C (laugh)

K How about 2 dimes and 5 pennies?

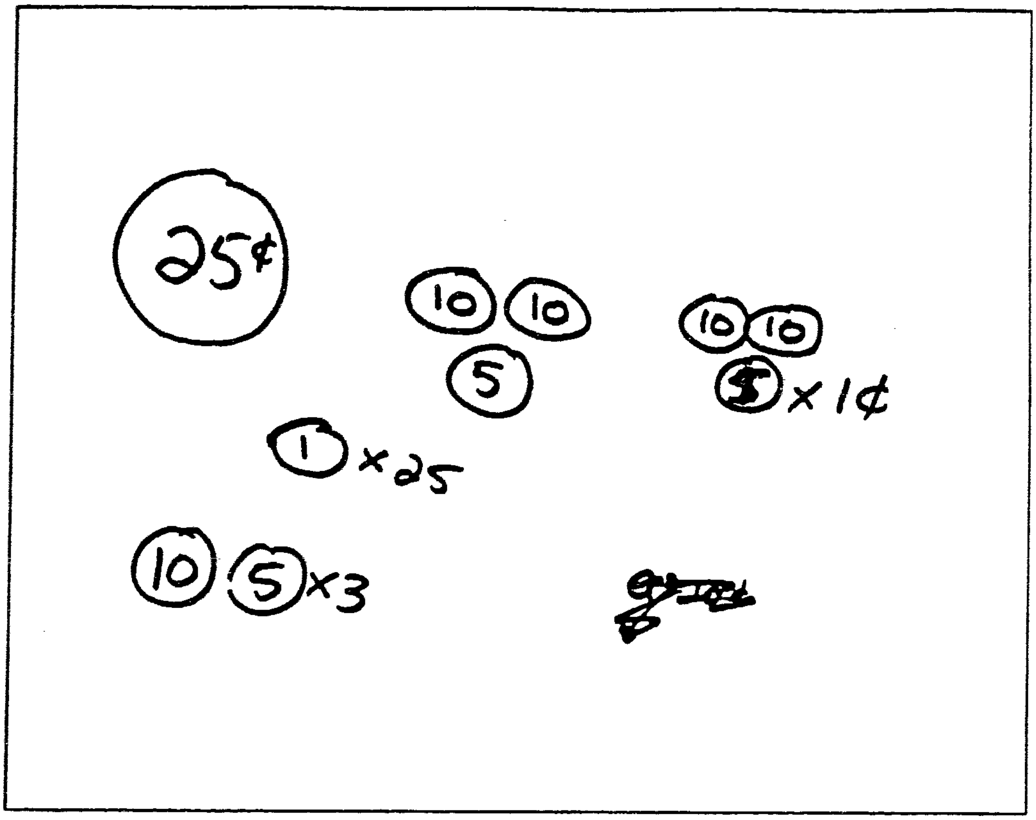
C 2 dimes (unclear) er, nickels and dimes and pennies (unclear)

K Yeah, it's going to get confused, though.

C Opps, yeah.

K 1 (unclear)

K&C (unclear)



K It's just that once we get like 10 of them written down or whatever, or 105, however many different ways.

Oh, how about a dime and three nickels?

C A dime and

K Or 5 nickels, too. 3 nickels, 5 nickels.

C Do we have to like use any mathematical formula?

K Like, we're supposed to, but it doesn't matter how we solve it.

(laughter)

We have 15 minutes. That should give us enough time. Or, do you know a mathematical way to solve it?

C No.

K Okay, good, so (laugh) what else?

C Well, maybe if got technical. a equals

K Yeah

C b

K Do you really want to do that?

(laughter)

That's true, that's true too. If we could think of an equation.

C We could try.

K We're taking the loser's way out, eh?

C Let's try it. You want to try?

K Okay

C Okay

(laughter)

K Um. Look how many variables we have.

a = 10¢
b = 5¢
c = 1¢

C 3.

K Yeah, that those are

C But, if you can go

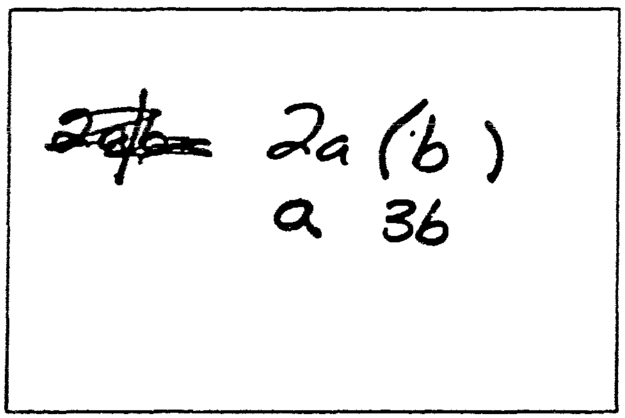
K So

C 2 a b (unclear)

K That would be 2 of these and 2 of these which would be 30 cents. Ah, no it wouldn't.

C No.

K It would be 2 a and b



C Yeah

K Sorry about that.

C Geeze, you confuse me.

K Yeah, sorry. ... 'Cause you didn't bracket it. Okay.

C That's it. Put brackets in.

K Good, good.

C How's that?

K Okay.

C Um, a, b, just keeping my own brain, um, ah

K How many ways are there? How many ways, sounds like a

C Lots.

K combinations and permutations question. How many ways are there? I liked our old way.

C Yeah.

K 'Cause I think, I think we were almost there.

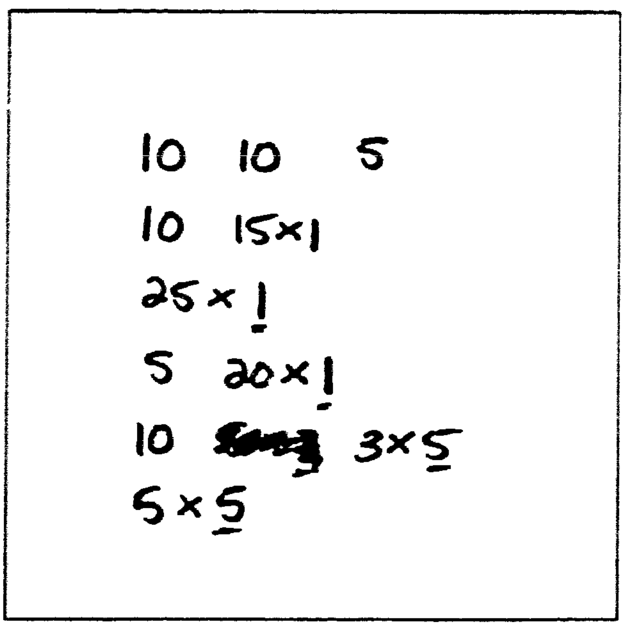
C I think that was easier.

K Yeah, yeah, yeah. Okay, what about a dime and 15 pennies?

C A dime (C begins to record their ideas)

K Okay and how about a nickel and twenty pennies?

Okay, let's see. Oh, 5 nickels... 5 nickels ... 2 nickels and 15 pennies.



C Just a sec.

K Oh, sorry.

C I'll just be (unclear)

K Yeah. 2 nickels and 15 pennies.

C 2 nickels

K A nickel, a dime and 10 pennies. Or do we have

C No

K 10 pennies

C Not any 10 times 1.

K Oh, okay, so in that case we can also have 3 nickels and 10 pennies. Any time you can do a dime you can do a couple of nickels, anyhow.

C Maybe, if we started making this smaller (unclear)

K Yeah, yeah.

C Just

K I think we're almost out of possibilities. ... 5 nickels.

C What else is there?

K Do you think we have 2 nickels and 15 pennies? We have a dime and 15 pennies.

C Oh, that's 3 nickels.

K Dime and 15 pennies.

C Yeah

K 25 pennies, um ... Oh, 2 dimes and 5 pennies. Do we have that? ...

C 2 dimes and 5 pennies. No.

K I thought we'd know that.

C We've got 2 dimes and a nickel.

K Okay. 2 dimes and 5 pennies. ... A dime, 2 nickels and 5 pennies. Did we do that?

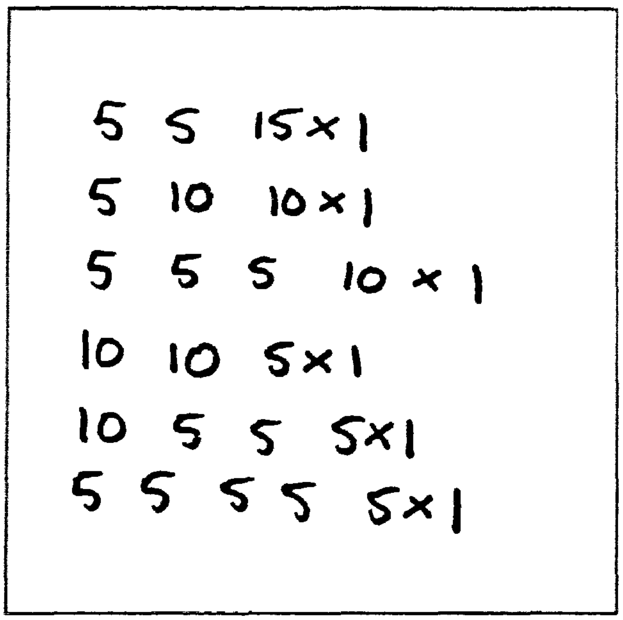
(laughter)

4 nickels and 5 pennies (laugh)

C	I'm glad I'm writing. (laugh)

K	Um, that's a lot of ways. But I'm sure we're missing some. With pennies it's like endless.

C	Um hum



K	You know? Ah, this is hard to keep track of, isn't it?

C	Um hum.

K	Very hard.

C	It would help if we have pennies and nickels and dimes (laugh).

K	Yeah, where are the props? (laugh) Hum.

C	(laugh) I'll just get some sissors. (C is lifting paper to look under) (laugh)

K	(laugh) How about, did we do, 3 nickels and 10 pennies? Yes, right there.

C	Yeah.

K	3 nickels and 10 pennies, Okay. Did we do, we did a dime, 2 nickels and 10 pennies, eh?

C Ah

K A dime, 2 nickels and 10 pennies? I mean, no, it would be a dime, 2 nickels and, ah, 5 pennies.

C Right there.

K Oh, yeah, okay.

C Elimination kind of thing.

K Yeah ... So, maybe if we think of it from the penny angle.

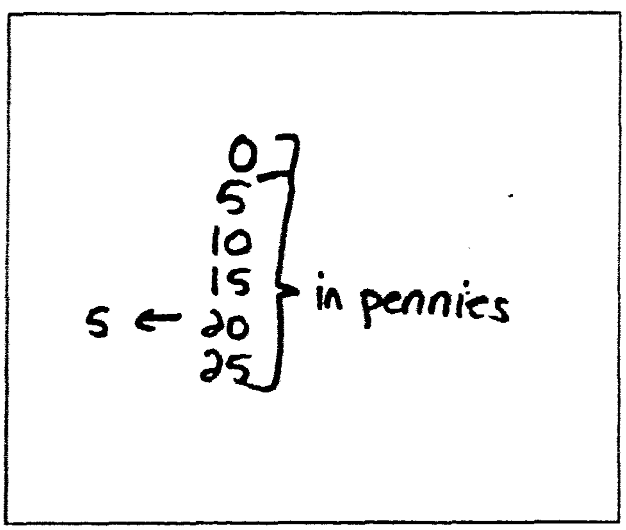
C The penny angle?

K Using dimes and nickels. If we're going to do, it has to be in, in groups of 5. It always has to be, so we can dot of thinking of pennies. Like we either have 25 pennies or we have 20 pennies.

C Yeah

K And a nickel. 'Cause we can only have a, it'll always be groups of 5. Otherwise it won't work.

C Um (writing)



K Or zero. We don't even have to have any pennies.

Ah, so there we go. And we can branch out because within each of these there's a few possibilities too.

C (unclear)

K So there's only one way we can have 25 pennies. We can have 25 pennies. That's it. So that's it. There's just 1.

C (unclear) 1 nickel, 5

K I thought of something. Maybe if we wrote how many ways, like there's only 1 way, on this side you can write 1. 'Cause we know there's only 1 way.

(C writes 0,5, ... 25 down the side of paper)

On, on this side, on this side we can write how to do them and on this side how many ways.

If you want?

C 1 way.

K Just 1.

C Okay, that's 0 on this side.

K Yup. And then, and, ah, to have 20 pennies there's only one way. It has to be a nickel and 20 pennies.

C We could have 5 more pennies. No, that would be that. We've done that.

K Yeah, yeah, so there's a, now it gets interesting. With 15 pennies, we can have 2, 2 nickels and 15 pennies or a dime and 15 pennies. 2 ways.

C (unclear)

K Can you think of another one? No.

C Not immediately.

K No. And then this one we can have a couple more ways.

C Yeah, 10 and 15 ... 5, 5, 5.

K Well, that's to have, oh, right. 'Cause we need 15 cents. I see what you mean.

C Just got to be 25.

K Yup

C (unclear)

K 10, is that it. I thought there was more ways we could.

C 10 cents, 5, 25, 25

K So just 2 ways, I guess. But then we're going to have more for this one.

C Yeah.

K For this one we can have 2 dimes.

C 2 dimes

K Or we can have 4, opps

C 2 dimes

K nickels ... or 1 dime and 2 nickels.

C That is a 5, hah!

K Oh, yeah, okay. So just 3 ways?

C Um

K We need to make 20 cents. How can we do that?

(C writes)...

It seems like doing this we're getting less ways than we did here, That's ... so, just, so ...

C I don't know.

K So, 3 ways.

C Is that it?

K Yeah ... 'Cause we're only working with nickels and dimes 3 ways. And then to have absolutely no pennies we could have 5 nickels ...

C Yeah

K And we can have 2 dimes and a nickel or we could have 2 nickels. No.

C 3

K 3 nickels and a dime.

0 - 3
5 - 3
10 - 2
15 - 2
20 - 1
25 - 1

C Um

PAUSE

K That's

C 1, 1, 2, 2, 3, 3, a pattern.

K Um, yeah. So so, it's 12 ways.

C Um hum. 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

K So we thought of 12 and we proved it. 12 ways.

C Ta da!

K Ta da! 12 ways.

(laughter)

C Okay.

K That was a neat question.

C Yeah

K 'Cause we didn't have to use math.

(laughter)

Problem 10 four circles

C Number 10

K Okay

C I'll read.

K Go ahead, I'm listening.

C Find the area of the shaded region.

K Geometry, oh.

C I'm afraid so.

K Oh.

C Try to figure it out. Can I draw?

K Oh, by all means.

C Thank you.

K Geeze, I really wanted to, but if you have your heart set on it (said jokingly).

C (unclear)

K No (unclear) ... Okay, very nice!

C I'm trying.

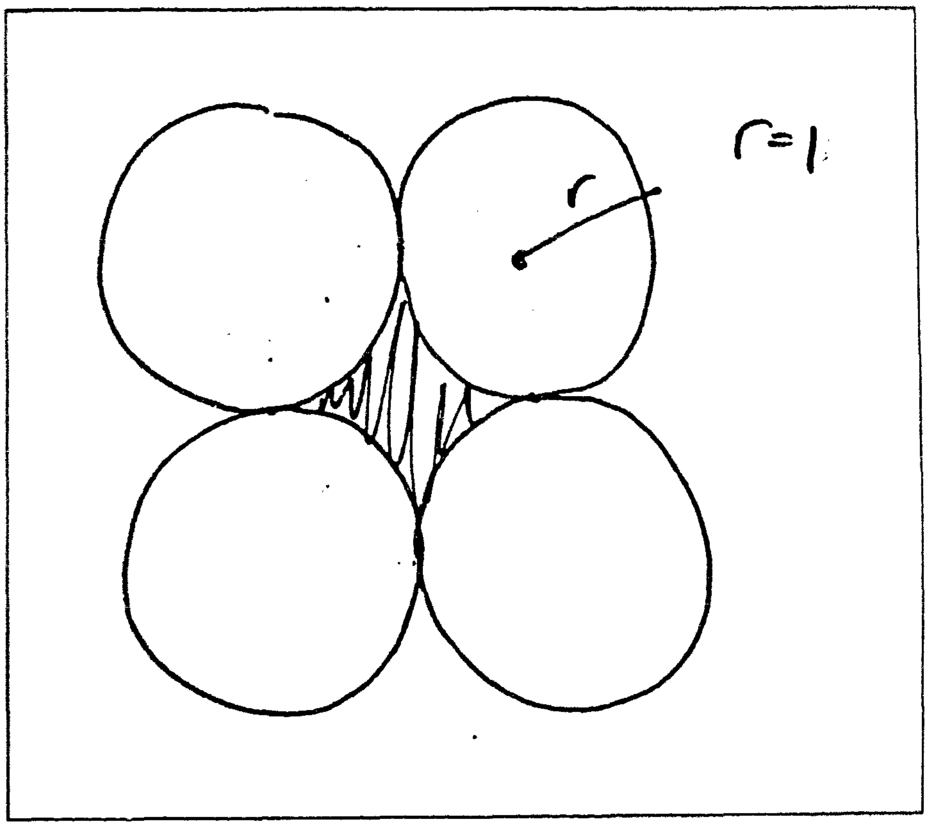
K And this is the area we want.

C Radius of 1, oop.

K Okay, radius is 1.

C What's the area of a circle?

K pi r squared. ... That's my contribution.



(laughter)

C That's the area of one?

K Yeah. ... So basically all these areas will be, pi?

C Um, 1. How are you supposed to get that?

K Yeah, I know. Good question.

What's this (pointing to the shaded region) in relation to all of these (pointing to the circles)? Is there, is there any relation, like that ...

C Maybe if we had an outside

K Yeah that's what I was thinking, eh? If we found the square. Yeah, you thought of that too, eh?

C We could do a big circle. I don't know.

K That would confuse us.

C Yeah, it would.

K But, it's a good thought. Um, well, wait a second. If we did a big circle all these areas put together would equal this, wouldn't it (indicating areas outside the circles)?

C Well, if you minus the area of the big circle from the area of the four small circles you would have all of that.

K Yeah, yeah.

C I don't know how you would. But maybe.

K But, then wouldn't 4 of these equal this thing.

C I don't know.

K Because it's all, it's all sort of

C I don't know how you'd prove it.

K I can't (laugh)

C (laugh)

K It's just a thought.

Area of. We have 4 pi for all the areas. And then what's this thing?

C Maybe if we played with some radius in here. (begins to draw the inner square)

K Ah, what about this circle?

C What circle?

K We could do a circle around the diamond. Or a square.

C That worked. (looking at the drawing) 1, 1, 1, 1, 1, 1, ...

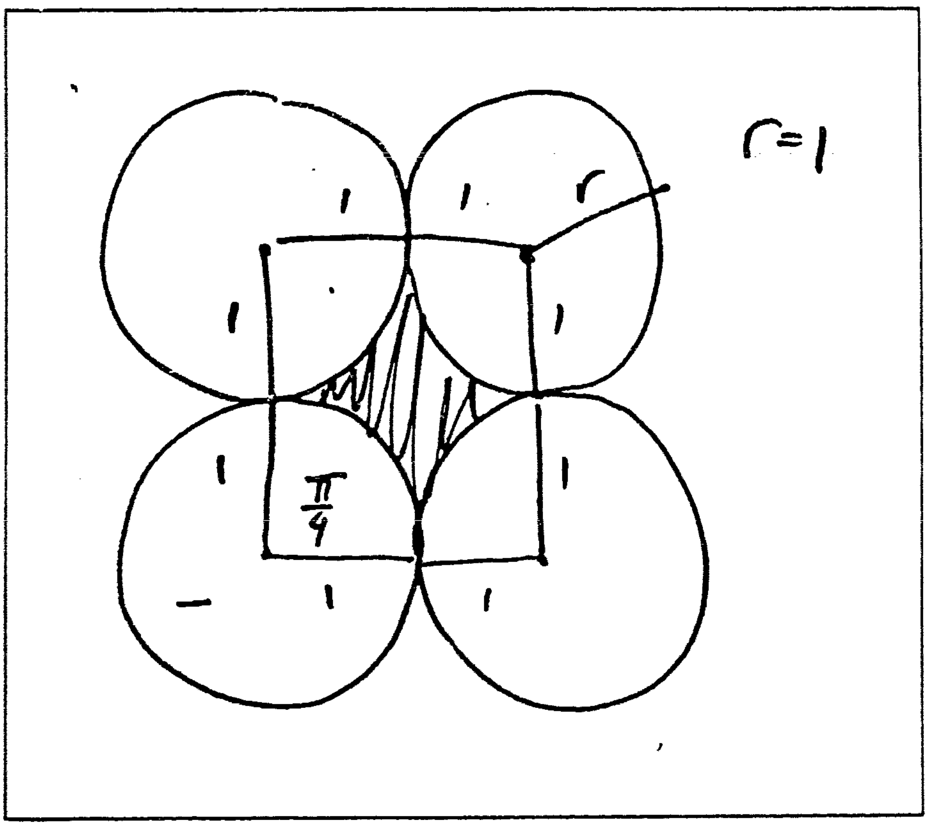
K Ah.

C I think you. Geeze, I'm confusing myself. Okay, a quarter of the area.

K Hay, that's, that's good.

C 1/4, this would be

K pi over 4



C pi over 4

K And this is, this is, ah, the area's

C The rest of it.

K Yeah. So it's this minus this.

C I think you want (unclear) the circle and the square. I know what we're talking about, see.

K Yeah.

C It would be minus those 4. That would equal 1 circle.

K How, how do you get that?

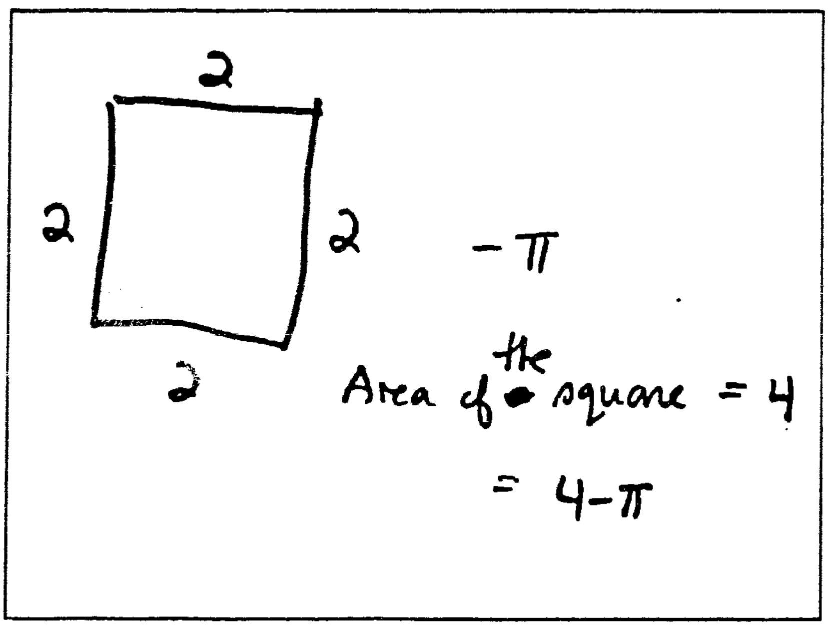
C A quarter of 4 circles is 1.

K Oh, oh, you mean these would equal 1, see.

C Yeah.

K Yeah, yeah.

C So you minus the pi, huh! From the area of that one.



K Yeah.

C That would be the square.

K Oh, yeah, because 4 of these is pi anyways. That's right.

C The area of a square, or the area of the square equals 4.

K Yup, yeah, 4.

C It would be 4 minus pi.

K Yeah, 4 minus pi. Hay that's great. I don't know if it's right but it looks really good.

C My brain kind of works that way.

K Yeah, that's good. 4 minus pi.

C pi over 4, pi over 4, that's pi. (labelling the drawing) pi over 4 times 4 equals pi equals one circle.

K I think that's right.

C Okay, prove me wrong.

K I think it's right.

C I think we're right.

K Yeah.

C 4 minus pi. (said to I)

K Hay, that was good.

C Yeah.

K Except I wanted to draw a circle around that but you had the right idea.

C Well I didn't see how.

K No, I didn't.

C I mean, you still can if you want.

K No, that's okay.

Problem 15 the commuter

K A commuter is picked up by her husband at the train station every afternoon. Her husband leaves the house at the same time every day, always drives at the same speed, and regularly arrives at the train station just as his wife's train pulls in.

Take it.

C	One day she takes a different train and arrives at the station one hour earlier than usual. She starts immediately to walk home at a constant speed. Her husband meets her along the way, picks her up and drives back home. They arrive 10 minutes earlier than usual. How long did she spend walking? If she walks at 8 km per hour, how fast does he drive?

K	This is a double question. Okay, so why don't we do one of those, like, distance, rate

C	Can I draw a train station?

K	Yeah

C	Cool. Wow. Are we supposed to use all of these or what? (gesturing to the different coloured felt pens)

I	Use as many as you like.

C&K	(unclear) (laughter)

C	Okay

K	A commuter is picked up by her husband at the train station every afternoon.

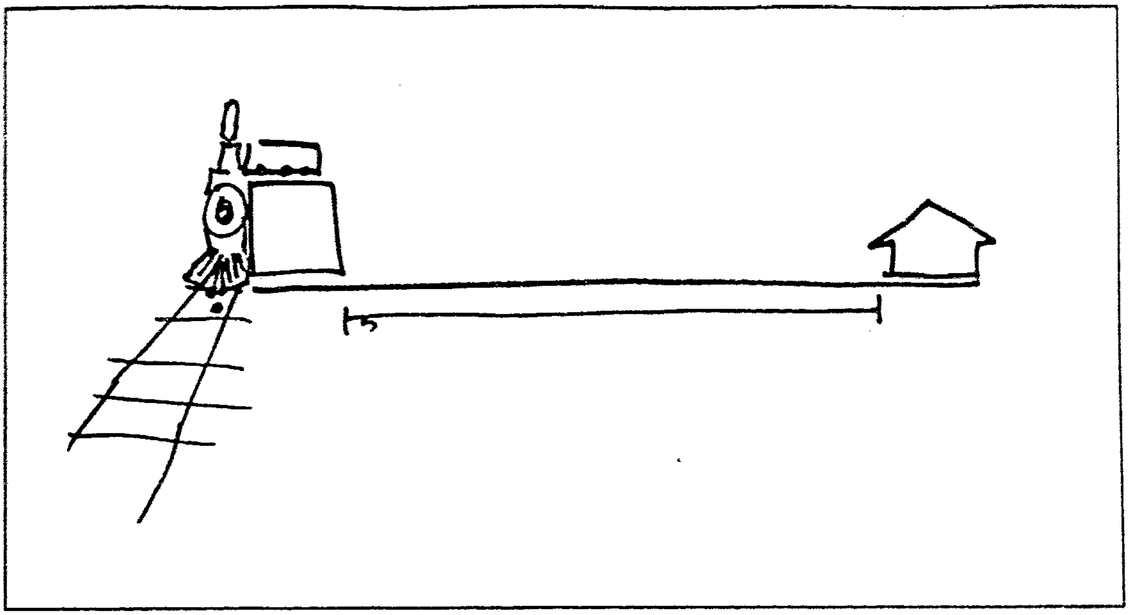
C	Train station, from where? His house?

K	Her husband leaves the house at the same time. Okay. We, we'll put the house. At the same time every day, always drives at the same speed, regularly arrives at the station just as his wife's train pulls in. So we can have, like, the train coming in here or something. (indicates a direction perpendicular to the line of car travel drawn) So we know that it happens at the same time. Yeah, that's good. ... Gee, I like that. That's good.

Okay, so then, so then, as this is happening, this happens. (using hand gestures to indicate perpendicular motions meeting at the station) It all, this is the same, like. Um, it takes the same time.

C	We need to know, the train?

K	Her husband leaves the house at the same time, always drives at the same speed, regularly arrives at the station just as his wife's train pulls in. One day she takes a different train and arrives at the station one hour earlier than usual. She starts immediately to walk home at a constant speed.



C At a constant speed. So he'll drive over here from here at the same time the train comes.

K Yeah, but they'll

C Does it say how long it'll take him? No.

K No.

C We need to know.

K That'd be too easy. (laugh) One day she takes a different train and arrives at the station an hour earlier than usual.

C So can we, like, do, now how?

K (moan)

C Okay, so this is

K Distance, time, yeah.

C Okay, he got in the car.

K And we don't even know that these distances are the same, 'cause we don't know how fast the train's going. It may not be the same distance.

C Arrival time, driving time

K They arrive 10 minutes earlier than usual. 10 minutes earlier than usual. How long did she spend walking?

See, the thing I don't get is, she takes a different train and arrives at the station an hour earlier. But

C How much?

K is that different train going the same speed as the other train was? Does that matter?

C Does he have to go over there then? Um.

K (unclear)

C If we did, like, he drives there at that same time this train comes in.

K Yup.

C If we just do it something,

K Yeah

C to confuse ourselves more,

K Yeah

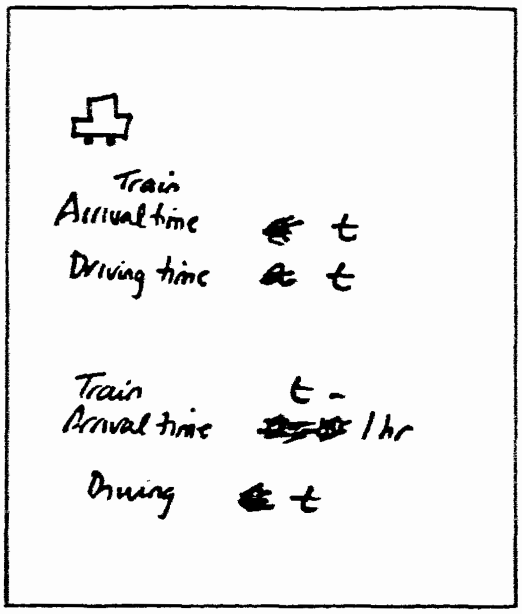
C we just put his arrival time, we take a hours, whatever, and the train arrival time would take a, a, if you know what I mean

K Yeah, yeah, I do, okay, yeah.

C So, like, if the train arrival time, the other train, ... a minus 10

K a minus 10?

C I don't know. 10 minutes off of the time 'cause she got there early, right?



K	No, no. What happens is, she was there an hour earlier than usual and then she starts walking home. Her husband picks, intercepts her.

C	No, but, this is before, this is the other train.

K	Yeah, it's an hour.

C	The time when it's early.

K	An hour earlier, arrives at the station an hour earlier.

C	Okay, an hour.

K	So she takes an hour.

C	An hour, yeah, opps.

K	Except the arrival times isn't a minus an hour because she could have taken a train that, that was just an hour earlier. The time could be the same. You know what I mean? Instead of the 10 o'clock train she took a 9 o'clock train. It's the time, the actual arrival time could change.

C	Yeah, that's what, that's what they mean. Yeah.

K	Yeah. ... 'Cause it isn't. It's not the, ah

C	Well, she arrives ah hour earlier.

K Yeah, yeah.

C 'Cause a was the time that train came. Now, if this train comes in earlier, it'd be a minus 1 hour, right? Because

K How does this relate to the distance?

C I don't know that. I'm just trying to (laugh) trying to make sense ...

K So then ... how long did she spend walking? So this time plus the time she spends walking is going to equal a time that's 10 minutes earlier than the time they usually get home. So whatever this time is plus whatever time she walks ...

Meets her along the way. Right where does he meet her? Maybe we should draw, what, what happened, like, where, the wife's walking home and the husband intercepts her, so we can put a label on the distances.

C She's had an hour to walk. (unclear) (drawing)

K Just on the way. We don't know where is that.

PAUSE

So, this is home. She starts walking and he meets her. She starts walking this way. ...

C 8 km per hour

K That's only after, though.

C After what?

K That's the second question. Here we don't know that.

C Oh, yeah.

K Yeah. We don't know what speed she's walking at here.

C How are you supposed to do this is there's no numbers? (laugh)

K Yeah, um, okay, so then, so then the train's here. It's an hour early, like you say, t, C	Um hum

K	let's say times minus an hour, t minus an hour, right? Husband starts driving and he meets her part way. So they continue the rest of the drive and get there 10 minutes earlier than usual. We don't have a clue where he picked her up. We don't know how long she was walking.

C	What are we looking for?

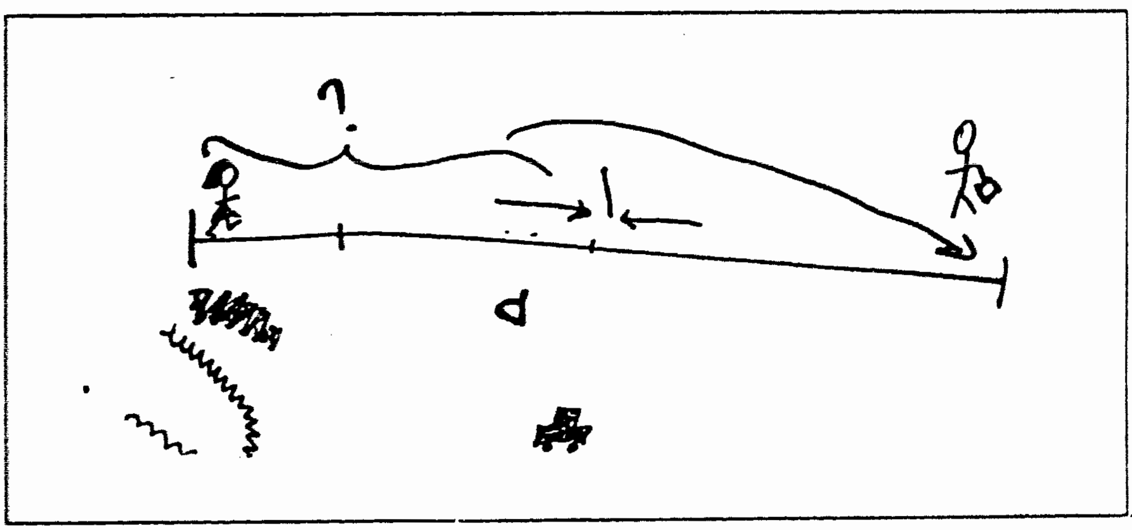
K	We're looking for how long she was walking. Ah, we're looking for this distance. Let's say that this is the point, that, opps, well, okay, it's convenient, half way, but let's just say

C	Okay, sure

K	that this is where the husband's car and she meet. Right there. Okay, so what we want to know is this time.

C	Um hum

K	How long was she walking before the husband picked her up? He was driving this way, picks her up, drives back and they arrive home together 10 minutes earlier.



Well, we don't want to say t though, again do we?

C	You didn't make anything from it.

K	Yeah. Well this, the distance, no the distance. It's not the distance. It's not the distance.

C Maybe we should (unclear)

K Yeah, we should put distance, put distance in there.

C What if we made this into distance?

K From the home to the train station.

C Like (unclear)

K Right, 'cause that's constant. It's always going to be the same.

C Yeah. I guess we can't give it a number.

K d, distance.

C Yeah ... d. We should give how long she went, like, another variable, just to really confuse us.

K Right, right, right, and all we know is they were back 10 minutes earlier that usual. But we also know she was an hour earlier and she walked for part of the time.

C Um hum. She's got to be walking for awhile.

K Yeah ... He drives. It doesn't say at all how long it takes?

C No.

K The husband leaves the house the same time every day and always drives and regularly arrives at the station just as his wife's train pulls in. Just as his wife's train pulls in. So, okay. This guy here, Mr Husband.

C Um hum

K Okay, with his, ah, with his briefcase (draws), that's the husband. And, ah, ... (unclear)

C She looks pretty naked (draws a dress on wife figure).

K That's true. So, he doesn't know. He just left the same time as usual. He starts his drive. She's already walking. ...

C She would be somewhere on this line then, isn't she?

K Yup.

C So, while he's driving she's still walking.

K Yeah, she's already been walking for at least an hour. Because didn't he just leave at the same time. Does it say, unknown to her husband?

C Oh, you sneaky one! Yeah, she would be walking for an hour.

K One day she catches an earlier train.

C (unclear) he isn't leaving. Oh, maybe, (unclear)

K It doesn't say if he know or not. It doesn't say if he knows she left an hour earlier. Because he'd, if he knew that, he'd have driven and maybe met her just as she started. If he doesn't know he's going to leave at the same time.

C How long does he usually drive?

K Yeah.

(laughter)

C 'Cause he, he leaves

K I say she had to have been walking at least an hour plus whatever.

C (unclear)

K It has to be.

C She was in an hour early, right?

K Yeah.

C And he would pick her up at that end of the hour so he, whatever long he drives takes, it would be minus that if she'd been walking. Do you understand? And a little less. 'Cause he would have left before that hour was up.

K Yeah, yeah, that's what we don't know. Ah, okay, so one hour minus whatever time if takes him to pick her up, is where they meet, is what you're saying.

C Yeah, I think, no, I don't know. An hour, K Well, why don't we put a time on it just to make it clear what our. Let's say, he usually leaves at 6 p.m. to pick her up. Let's make this so we can, in our minds, get it clear.

C Right, ah.

K Yeah. So she usually takes. She, let's say, she's usually on her train that gets her there for 7 p.m. and let's just say that it takes an hour for him to drive and pick her up, so

C Okay

K bang, 7 p.m. Everything's, she's happy, there he is, 7 p.m. This time, let's say he still leaves at 6 'cause he doesn't know

C Um hum

K that this time though her train got there at 5 instead of 6.

C 5

K Yeah, I mean 6 instead of 7.

C Yeah

K So they leave at, ah, so the train's getting there as he's leaving.

C So she walks for an hour and he drives for an hour. If she's going 8 km per hour she'll have gone 8 km and he would pick her up at whatever 8 is. If it takes an hour.

K Yeah, we're just, we're just kind of saying that.

C (laugh)

K Okay, so, so, if she got there at 6 and she started walking home and he left at 6 as usual, then it takes an hour to, Ah ah ah eeee (throws hands up)

C (laughs)

K There's at least 3 numbers that aren't written here, eh?

C (laugh) Ah, you kill me.

K Ah, ah.

C A nice spectacular question ... It's a trick question. Um ...

K She took the taxi.

(laughter)

C What if she walks really (unclear)

K 8 km, ah, that's pretty fast. Yeah, that's a pretty brisk clip.

C Yeah.

K But we don't know that for the first part.

C Chopping along.

K A redhead too, fiery red hair.(referring back to their drawing)

C Pretty good, eh?

K Okay.

C Um. Do we get any hints from the corner (indicating I)

(laughter)

K Um. Well, we got to, okay, let's try and make an equation out of this mess and try. I know that we should make an equation out of this.

C Sure

K Okay, so what are they asking? How long was she walking? Variable, how long, it's time, we want to know the time it takes her to walk. That's what, that's what we're

C How long did she walk? She's walking at 8 km per hour. She was walking under an hour.

K But, we don't know the 8 km yet. You, you can't say that yet. We can't.

C I keep it in there.

K Yeah, I know. It's just so nice to have it in there.

C It's another number, you know.

K Yeah. Okay, so we'll let t equal the time it takes her, time to walk.

C t?

K Okay? Whatever time it takes her to walk before the husband picks her up. (takes a fresh sheet of paper) Keep this (the old sheet) here so we know what we've done.

C t equals time.

K How much time do we have left?

I Lots of time.

K We do?

I Um hum.

C We do? No kidding? (laugh)

K We have lots of time. Okay.

C Time to walk. What?

K The time it takes her to walk between the train station and where the husband intercepts.

C To walk (writing)

K From train station to where husband picks up.

C Station to pickup.

K 'Cause that's what we want to know, so time to walk from the station to pickup. ... Well, she had under an hour to do it.

C 'Cause we use that thing that she gets here at, she's supposed to get there at 7, but she gets there at 6.

K Yeah, yeah.

C And he leaves at 6.

K You know that's confusing us, 'cause it's an hour. Why don't we say it only takes him half an hour. Then we can get it better in our mind. 'Cause that hour's confusing us.

C What does he leave later?

K Yeah, let's say he leaves at 6:30, ah, 5:50. If we say he leaves at 5:30 and it takes him half an hour to pick her up and her train comes in at 6.

C Yeah, that'll work.

K Yeah.

C And, then she'd be an hour early.

K Right. So, let's say

C She'll come at 5.

K Let's say she'd come at 5 and, ah, he's only, see, he's only leaving at 5:30.

C So, she's got half an hour to walk plus the time it would take him to drive the distance that she hasn't walked. And that she's walking while he's driving. (laugh)

K Yup.

C Ah, let's try it in another language. It'll be easier, I think, ah ...

K Um, so it takes him half an hour to get there and he leaves at 5:30. Well if he's leaving at 5:30 she's walked at least a half an hour already...

C Yup.

K At least, plus, like you say, plus wherever, however more time it takes him to drive to wherever she is.

C Yeah.

K Yeah.

C I don't know.

K So that's what we want to know. That's what we want to know.

C Well, we want to know that, too. We want to know this and we want to know

K We want to know, yeah. Ah, so, why don't we call this distance d.

C Distance? Where?

K This will be distance d. (indicates distance from station to home) Yeah,

C Yeah.

K So this is distance d. Ah, therefore, wait a second, this is d minus t.

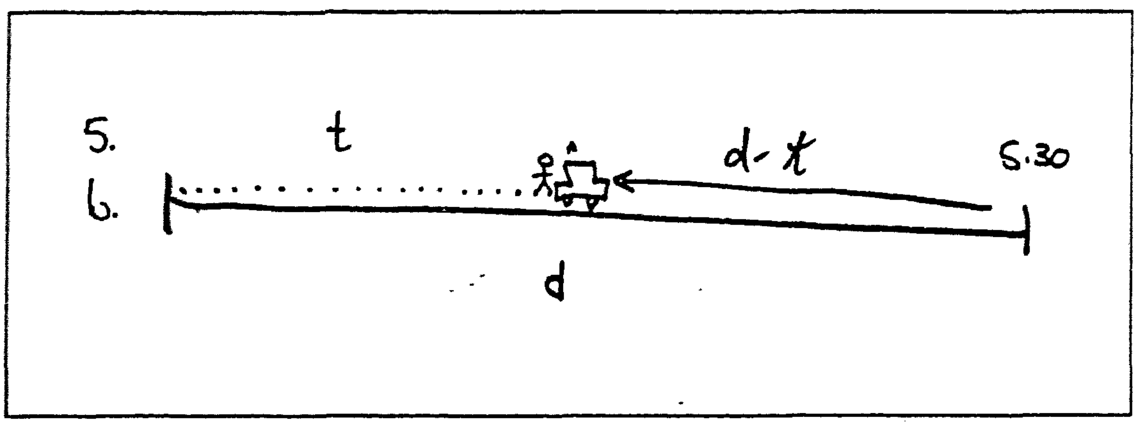
C d minus t

K Right? d minus t and that's t. Now what does all of that equal? What is all that going to equal?

C And how are we going to relate it to rates?

K Yeah, that's the thing, yeah, yeah.

C So, we want t and



K So rate, rate and distance and time, how about?

C d and d minus t.

K Rate. Isn't rate distance over time? Is that what it is?

C Um hum, rate, km per hour.

K Right, right, so that's what we have to do really. 'Cause we've got a lot of distance and time and if we know that rate at which she was walking and the rate at which he was driving.

C How do we say what distance she moved? d minus d minus t?

K Well, okay. We know that the husband drove d minus t. We know that, okay, so that's the actual distance he drove. We know it's d minus t. The distance over the time will give that rate at which he's driving How can we use that? ... Oh, ah! ... oh, it vanished.

C (laugh)

K Well, we should use that one hour that we were given because she's been walking for part of that one hour, for. Actually, hasn't she been walking for the, the whole hour?

C Are we doing the half hour or the hour?

K She, as soon as the train got in and it was an hour early.

C Train came at 5.

K Yeah, well.

C Not really, we were just

K That we just, that we just

C Yeah

K sort of arbitrarily assigned, yeah.

C And he, you said he left at 5:30.

K Yeah. But basically the time she was walking plus the distance minus time, right, because whatever time she was walking plus this distance will give us like the meeting point. That's what that would give us. What?

C I'm trying to think but my brain is going blueee ueeee.

K Yeah, I know. That's not right.

C So if we used half an hour he'd drive half an hour. She's an hour early. She's got half an hour to walk before he leaves. Half an hour. To garage.

K Total distance would be distance minus time plus t. Which we know because that's going to cancel out and give us just the d.

$$d = (d - t) + (t)$$

Right? So whatever the entire distance is is going to be her walking time plus his driving time, ...

C huh

K Her walking time is going to be his driving time plus an hour, right?

Since she got there an hour earlier. Wouldn't, would that make sense? ...

No, but it depends on when he leaves. It depends when he leaves. So actually her walking time might be an hour minus whatever time it takes him to, to pick her up.

$$d - t = 1\,hr - t$$

C How long would it take her to walk home?

K That's the thing we don't know, yeah. ... Like, does he leave before the hour is up or not? That's what we don't know. ...

But if we substitute this for her walking time we're saying that the d minus t is an hour minus. It doesn't make sense.

C (laugh)

I Well, your time is up.

K Okay, good.

(laughter)