Algebraic Semilattices of Groups

by

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Abstract

An *algebraic semigroup* $(S, \circ)$ is an affine variety S along with an associative product map $\circ : S \times S \to S$ which is also a morphism of varieties. Background material regarding algebraic semigroups is presented in Chapters 1, 2 and 3.

A *semilattice of groups* is a semilattice each of whose elements is a group, together with a set of group homomorphisms which is compatible with the semilattice structure. The union of these groups thus forms a semigroup where multiplication is determined by the group operations and the group homomorphisms. In Chapter 4 we characterize algebraic semilattices of groups. In particular we prove that a semilattice of groups is algebraic if and only if the semilattice is finite, the groups are algebraic groups and the connecting homomorphisms are morphisms of affine varieties. In order to show that the semilattice is finite we prove more generally that any semilattice of matrices is finite.

Let S be a semigroup and let $a, b \in S$. We say that $a <_{\mathcal{L}} b$ if and only if $S^1 a \subseteq S^1 b$ and $S^1 a \neq S^1 b$, where $S^1 = S \cup \{1\}$. Using S and the relation $<_{\mathcal{L}}$ we can form a semigroup $R(S)$ known as the *left Rhodes expansion* of S. In Chapter 5 we show that the left Rhodes expansion of an algebraic semilattice of groups is itself an algebraic semigroup.

Let S be a semigroup and let $a, b \in S$. We say $a\mathcal{J}b$ if there exist $x, y, x', y' \in S^1$ such that $xay = b$ and $x'by' = a$. The relation $\mathcal{J}$ is an equivalence relation. Further, a $\mathcal{J}$-class of S is *regular* if it contains an idempotent element. It is known that for any algebraic semigroup S the set $\mathcal{U}(S)$ of regular $\mathcal{J}$-classes is finite. Norman Reilly posed the more general question: *Is $\mathcal{U}(S)$ finite when S is a semigroup of $n \times n$ matrices over a field?* In Chapter 6 we show that the answer to this question is "no" by presenting such a semigroup having an infinite number of regular $\mathcal{J}$-classes.

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Chapter 1

Algebraic Geometry

The definitions and the statements of results in this chapter are taken from [3] and [6] where most results are stated without proof. For completeness and clarity I have provided proofs and additional details.

1.1 Introduction

Let k be an algebraically closed field and let $k[T_1, T_2, \ldots, T_n]$ be the algebra of polynomials in n indeterminates, $T_1, T_2, \ldots, T_n$, over k. We abbreviate $k[T_1, T_2, \ldots, T_n]$ to $k[T]$. An element $x \in k^n$ is a *zero of* f, $f \in k[T]$, if $f(x) = 0$. Moreover, x is a *zero of* S, $S \subseteq k[T]$, if $f(x) = 0$ for all $f \in S$. For $S \subseteq k[T]$, we denote by $\mathcal{V}(S)$ the set of zeros of S. An *algebraic set* is any subset of k^n of the form $\mathcal{V}(S)$ where $S \subseteq k[T]$. Also if $X \subseteq k^n$, we denote by $\mathcal{I}(X)$ the ideal formed by the $f \in k[T]$ vanishing on X. We will use the following lemma frequently and without comment.

Lemma 1.1.1 *Every algebraic set is of the form $\mathcal{V}(I)$ for some ideal $I \subseteq k[T]$.*

Proof. This follows from the easily verified fact that for all $S \subseteq k[T]$, $\mathcal{V}(S) = \mathcal{V}(< S >)$ where $< S >$ is the ideal of $k[T]$ generated by S. ■

We present two examples of algebraic sets.

Example 1.1.2 *Let $p \in k^n$, then the singleton set $\{p\}$ is an algebraic set.* If $(p_1, p_2, \dots p_n) \in k^n$, then $\{(p_1, p_2, \dots p_n)\} = \mathcal{V}(\{T_1 - p_1, T_2 - p_2, \dots, T_n - p_n\})$.

Example 1.1.3 *Let $S \subseteq k[T]$ and let $f_1, f_2, \dots, f_n \in k[T]$. Then*

$$P = \{p \in k^n : (f_1(p), f_2(p), \dots, f_n(p)) \in \mathcal{V}(S)\}$$

is an algebraic set.

For all $g \in S$ we create a polynomial $g^* \in k[T]$ as follows:

$$g^* = g(f_1(T_1, T_2, \dots, T_n), f_2(T_1, T_2, \dots, T_n), \dots, f_n(T_1, T_2, \dots, T_n)).$$

Further we let $S^* = \{g^* : g \in S\}$. It is straightforward to verify that $P = \mathcal{V}(S^*)$. Thus P is an algebraic set.

Proposition 1.1.4 *Let $X \subseteq k^n$, $Y \subseteq k^m$ be algebraic sets, then $X \times Y \subseteq k^{n+m}$ is an algebraic set.*

Proof. Let $S \subseteq k[T_1, T_2, \dots, T_r]$. In this proof we indicate that the elements of $\mathcal{V}(S)$ are r-tuples by writing $\mathcal{V}(S)$ as $\mathcal{V}_r(S)$.

Let $X = \mathcal{V}_n(I_X)$ and $Y = \mathcal{V}_m(I_Y)$ where I_X and I_Y are ideals of $k[T_1, T_2, \dots, T_n]$ and $k[U_1, U_2, \dots, U_m]$ respectively. Notice $I_X \cup I_Y$ can be viewed as a subset of

$$k[T_1, T_2, \dots, T_n, U_1, U_2, \dots, U_m].$$

Now for all (p, q) where $p \in X$, $q \in Y$ and for all $f \in I_X \cup I_Y$ we have $f(p, q) = 0$. Thus $X \times Y \subseteq \mathcal{V}_{n+m}(I_X \cup I_Y)$. Let $(r_1, r_2, \dots, r_n, s_1, s_2, \dots, s_m) = (r, s)$ be in $\mathcal{V}_{n+m}(I_X \cup I_Y)$, then for all $f \in I_X$, we have $f(r) = 0$. Thus $r \in X$. Similarly $s \in Y$. Hence $\mathcal{V}_{n+m}(I_X \cup I_Y) \subseteq X \times Y$. ∎

1.2 The Zariski Topology

It is convenient to define a topology on k^n. In this section we show that the so called Zariski topology is Noetherian.

Proposition 1.2.1

1. *$\emptyset$ and k^n are algebraic sets.*

2. *The union of two algebraic sets is an algebraic set.*

3. *The intersection of an arbitrary collection of algebraic sets is an algebraic set.*

Proof.

1. $\mathcal{V}(\{0\}) = k^n$, $\mathcal{V}(k[T]) = \emptyset$.

2. Let I, J be ideals of $k[T]$, then

$$IJ = \left\{ \sum_{i=1}^{r} a_i b_i \mid a_i \in I, b_i \in J, r \in Z^+ \right\}$$

is also an ideal of $k[T]$. Clearly $\mathcal{V}(I) \cup \mathcal{V}(J) \subseteq \mathcal{V}(IJ)$. Say $x \in \mathcal{V}(IJ)$ but $x \notin \mathcal{V}(I)$, then there exists $a \in I$ such that $a(x) \neq 0$. But for all $b \in J$ we have $ab(x) = a(x)b(x) = 0$. Thus for all $b \in J$, we have $b(x) = 0$ and $x \in \mathcal{V}(J)$. So $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(IJ)$.

3. Let $(I_\alpha)_{\alpha \in \Lambda}$ be a family of ideals of $k[T]$. Define $\sum_{\alpha \in \Lambda} I_\alpha$ as follows: $a \in \sum_{\alpha \in \Lambda} I_\alpha$ if and only if there exists a finite set $I_1, I_2, \ldots, I_h \in (I_\alpha)_{\alpha \in \Lambda}$ such that $a \in I_1 + I_2 + \ldots + I_h$. Then $\sum_{\alpha \in \Lambda} I_\alpha$ is an ideal of $k[T]$ and $\mathcal{V}\left(\sum_{\alpha \in \Lambda} I_\alpha\right) = \bigcap_{\alpha \in \Lambda} \mathcal{V}(I_\alpha)$.

■

We define the *Zariski topology* on k^n by taking the closed sets to be the algebraic sets. From Proposition 1.2.1 we see that the Zariski topology is indeed a topology. Let X be a topological space. We say that X satisfies the *descending chain condition on closed sets* if for any descending sequence of closed subsets, $X_1 \supseteq X_2 \supseteq \ldots$, of X there exists $h \in Z^+$ such that $X_i = X_h$ for all $i \geq h$. A topological space is *Noetherian* if it satisfies the descending chain condition on closed sets.

Lemma 1.2.2

1. *k^n with the Zariski topology is Noetherian.*

2. *A subset of a Noetherian space with the induced topology is Noetherian.*

3. *An algebraic set with the induced Zariski topology is Noetherian.*

Proof.

1. This follows from the fact that $k[T]$ is a *Noetherian ring*. For details we refer the reader to [5]

2. Let Y be a subset of a Noetherian space X, and let $Y_1 \supseteq Y_2 \supseteq \dots$ be a descending sequence of closed subsets of Y. Then there exist closed subsets $X_i \subseteq X$, $i = 1, 2, \dots$, such that $Y_i = X_i \cap Y$. Now $X_1 \supseteq X_1 \cap X_2 \supseteq X_1 \cap X_2 \cap X_3 \supseteq \dots$ is a descending sequence of closed subsets of X. Since X is Noetherian, there exists $h \in Z^+$ such that for all $h' \geq h$, $\bigcap_{i=1}^{h} X_i = \bigcap_{i=1}^{h'} X_i$. Rewriting $Y_1 \supseteq Y_2 \supseteq \dots$ we get $X_1 \cap Y \supseteq X_2 \cap Y \supseteq \dots$. Thus for all $i \in Z^+$ we have $X_i \cap Y = \bigcap_{j=1}^{i} X_j \cap Y$. We conclude that for $h' \geq h$, $Y_{h'} = \bigcap_{i=1}^{h'} X_i \cap Y = \bigcap_{i=1}^{h} X_i \cap Y = Y_h$.

3. This follows immediately from items 1 and 2.

■

1.3 Regular Functions and Ringed Spaces

To begin this section we define a k-algebra, $k[X]$, for each algebraic set X. An *F-algebra* consists of a vector space V over a field F, together with an operation of multiplication on V, such that for all $a \in F$ and $\alpha, \beta, \sigma \in V$, we have the following:

1. $(a\alpha)\beta = a(\alpha\beta) = \alpha(a\beta)$

2. $(\alpha + \beta)\sigma = \alpha\sigma + \beta\sigma$

3. $\alpha(\beta + \sigma) = \alpha\beta + \alpha\sigma$

4. $(\alpha\beta)\sigma = \alpha(\beta\sigma)$

Let $X \subseteq k^n$ be an algebraic set. We form a k-algebra, $k[X]$, by considering the restrictions to X of the polynomials of $k[T]$. That is for $f, g \in k[T]$ we say f is equivalent to g if and only if $f(x) = g(x)$, for all $x \in X$. We let the elements of our k-algebra be the equivalence classes of $k[T]$ with addition and multiplication defined as follows:

$$[f] + [g] = [f + g] \text{ and } [f][g] = [fg] \; (\forall [f], [g] \in k[X]).$$

By noting that $k[X]$ is isomorphic to $\frac{k[T]}{\mathcal{I}(X)}$ it is not hard to see that $k[X]$ is indeed a k-algebra. Strictly speaking the elements of $k[X]$ should be written as equivalence classes of polynomials; however where no confusion will occur we will write them simply as polynomials.

Let $X \subseteq k^n$ be an algebraic set and let x be a point in X. A k-valued function defined in a neighbourhood U of x is said to be *regular in* x if there exists both an open neighbourhood V of x and elements $g, h \in k[X]$ such that $V \subseteq U$, and for all $y \in V$, we have both $h(y) \neq 0$ and $f(y) = \frac{g(y)}{h(y)}$. A function f defined in a non-empty open subset U of X is *regular in* U if it is regular for all points of U. Let f and g be regular in U. We define $f + g$ and fg:

$$(f + g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(x)g(x) \;\; (\forall x \in U).$$

It is not hard to verify that the set of regular functions in U with the given addition and multiplication form a k-algebra which is denoted by $\mathcal{O}_X(U)$.

Definition 1.3.1 Let X be a topological space. Let $\mathcal{U}$ be the set of open subsets of X. Suppose that for each non-empty open subset U of X there is an associated k-algebra $\mathcal{O}(U)$ of k-valued functions of U such that, with $\mathcal{O}(\emptyset) = \{0\}$, we have

Sh1) If $\emptyset \neq U \subseteq V$ are open sets and $f \in \mathcal{O}(V)$, then $f \restriction U \in \mathcal{O}(U)$.

Sh2) Let U be a non-empty open set with an open covering $U_\alpha (\alpha \in \Lambda)$. Further let f, a k-valued function of U, be such that $f \restriction U_\alpha \in \mathcal{O}(U_\alpha) \; \forall \alpha \in \Lambda$, then $f \in \mathcal{O}(U)$.

Then $\mathcal{O} = \bigcup_{U \in \mathcal{U}} \mathcal{O}(U)$ is a *sheaf of functions* on X and the pair $(X, \mathcal{O})$ is a *ringed space*. We shall usually drop the $\mathcal{O}$ and speak of the ringed space X.

Definition 1.3.2 Let $(X, \mathcal{O})$ be a ringed space, and let $Y \subseteq X$. We form a ringed space, $(Y, \mathcal{O}')$, where Y is considered to have the induced topology and $\mathcal{O}'$ is defined as follows: $\mathcal{O}'(\emptyset) = \{0\}$ and, for $U \neq \emptyset$,

1. if U is open in Y but not in X, then $\mathcal{O}'(U)$ consists of all functions $f : U \to k$ such that there is a open covering $U \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$ by open sets of X such that for each $\alpha \in \Lambda, f \restriction U \cap U_\alpha = f_\alpha \restriction U \cap U_\alpha$ for some $f_\alpha \in \mathcal{O}(U_\alpha)$.
2. if U is open in Y and X, then $\mathcal{O}'(U) = \mathcal{O}(U)$.

We call $(Y, \mathcal{O}')$ the *ringed space induced by* X.

Verifying that $(Y, \mathcal{O}')$ is indeed a ringed space is straightforward.

Theorem 1.3.3 *Let X be an algebraic set with the induced Zariski topology. For each non-empty open subset U of X let $\mathcal{O}_X(U)$ be the associated k-algebra of regular functions and let $\mathcal{O}_X(\emptyset) = \{0\}$. Then $\mathcal{O} = \cup\{\mathcal{O}_X(U) : U \text{is open in} X\}$ is a sheaf of functions on X.*

Proof. Clearly the regular functions are k-valued. We verify that Sh1 is satisfied. Let U and V be open sets of X with $\emptyset \neq U \subseteq V$ and let $f \in \mathcal{O}_X(V)$. Since $f \in \mathcal{O}_X(V)$, for all $x \in V$ there exists A_x, an open neighbourhood of x, and $g_x, h_x \in k[X]$ such that

$$h_x(y) \neq 0 \text{ and } f(y) = \frac{g_x(y)}{h_x(y)} \ (\forall y \in A_x).$$

Hence for all $x \in U$, we have

$$h_x(y) \neq 0 \text{ and } f(y) = \frac{g_x(y)}{h_x(y)} \ (\forall y \in A_x \bigcap U).$$

Thus for all $x \in U$ there exits an open neighbourhood of x, namely $A_x \bigcap U$, and functions $g_x, h_x \in k[X]$ which meet the requirements needed to make $f \restriction U$ regular in x. We conclude that$f \restriction U \in \mathcal{O}_X(U)$.

We verify that Sh2 is satisfied. Let U be a non-empty open set with an open covering $U_\alpha(\alpha \in \Lambda)$. Let $f : U \to k$ be such that for all $\alpha \in \Lambda$, we have $f \restriction U_\alpha \in \mathcal{O}_X(U_\alpha)$. Since $U_\alpha(\alpha \in \Lambda)$ is an open covering of U we have that for all $x \in U$, there exists $\alpha \in \Lambda$ such that $x \in U_\alpha$. Now $f \upharpoonright U_\alpha \in \mathcal{O}_X(U_\alpha)$ so there exists an open neighbourhood, $A_x \subseteq U_\alpha$, of x and functions $g_x, h_x \in k[X]$ with

$$h_x(y) \neq 0 \text{ and } f(y) = \frac{g_x(y)}{h_x(y)} \; (\forall y \in A_x).$$

Since U_α is open, $A_x = A_x \bigcap U_\alpha$ is an open subset of U. Thus for all $x \in U$ there exist an open neighbourhood of x, namely $A_x = A_x \bigcap U_\alpha \subseteq U$, and functions $g_x, h_x \in k[X]$ which meet the requirements needed to make f regular in x. We conclude $f \in \mathcal{O}_X(U)$.

■

1.4 Affine Varieties

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces, and let $\phi : X \to Y$ be a continuous mapping. For each open set $V \subseteq Y$ we define a mapping ϕ_V^* from $\mathcal{O}_Y(V)$ into the set of k-valued functions on $\phi^{-1}(V)$ as follows: If $f \in \mathcal{O}_Y(V)$, then $\phi_V^*(f) = f \circ \phi$. We say that ϕ is a *morphism of ringed spaces* if, for each open $V \subseteq Y$ we have that ϕ_V^* maps $\mathcal{O}_Y(V)$ into $\mathcal{O}_X(\phi^{-1}(V))$. See Fig. 1.1. Let $\phi : X \to Y$ be a morphism of ringed spaces which is one-to-one and onto. If the mapping $\phi^{-1} : Y \to X$ is also a morphism of ringed spaces then ϕ is said to be an *isomorphism of ringed spaces*. An *affine variety* is a ringed space $(X, \mathcal{O})$ such that $(X, \mathcal{O})$ is isomorphic to a ringed space $(X', \mathcal{O}')$ where X' is an algebraic set with the induced Zariski topology and $\mathcal{O}'$ is the sheaf of regular functions. We shall usually drop the $\mathcal{O}$ and speak of the affine variety X. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are affine varieties then a morphism of ringed spaces $\phi : X \to Y$ is a *morphism of affine varieties*. An *isomorphism of affine varieties* is similarly defined.

Lemma 1.4.1 *Let $(X, \mathcal{O})$ and $(R, \mathcal{P})$ be ringed spaces, let $\phi : X \to R$ be an isomorphism of ringed spaces and let $Y \subseteq X$. Then $\phi \upharpoonright Y$ is an isomorphism of the induced ringed spaces $(Y, \mathcal{O}')$ and $(\phi(Y), \mathcal{P}')$.*

Proof. It follows from results in elementary topology that $\phi \upharpoonright Y$ is a homomorphism of the induced topologies. Thus to show that $\phi \upharpoonright Y$ is a morphism of the induced

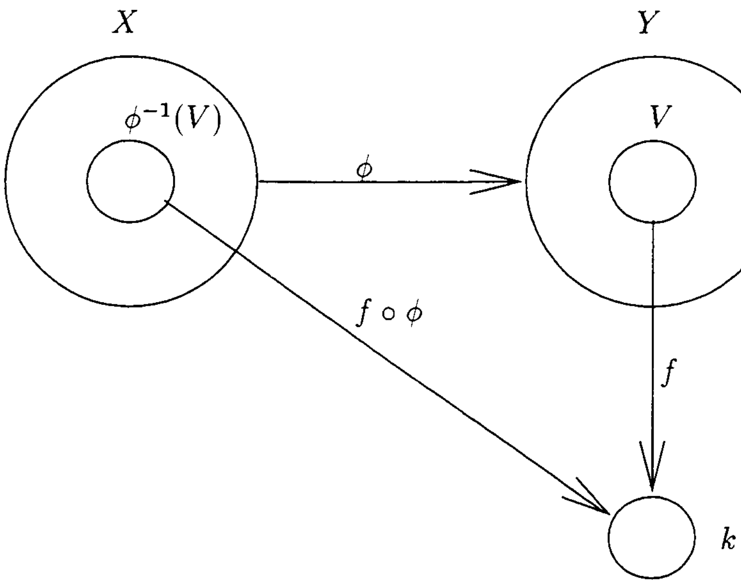

Figure 1.1: A morphism of the ringed spaces X and Y

ringed spaces it suffices to show that if $V \subseteq \phi(Y)$ is an open set and $f \in \mathcal{P}'(V)$ then $f \circ \phi \in \mathcal{O}'(\phi^{-1}(V))$. We examine two possibilities.

1. Suppose that V is open in $\phi(Y)$ and R. Since ϕ is an isomorphism we have that $\phi^{-1}(V)$ is open in X. Thus $\phi^{-1}(V)$ is open in Y and X, and, by Definition 1.3.2.2, we have that $\mathcal{O}'(\phi^{-1}(V)) = \mathcal{O}(\phi^{-1}(V))$. Also since ϕ is an isomorphism we have that $f \circ \phi \in \mathcal{O}(\phi^{-1}(V))$. Thus $f \circ \phi \in \mathcal{O}'(\phi^{-1}(V))$.

2. Suppose that V is open in $\phi(Y)$ but not in R. By Definition 1.3.2.1, f is such that there exists an open covering $V \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$ by open sets of R where for each $\alpha \in \Lambda$, $f \restriction V \cap V_\alpha = f_\alpha \restriction V \cap V_\alpha$ for some $f_\alpha \in \mathcal{O}(V_\alpha)$. Note that $\phi^{-1}(V \cap V_\alpha) = \phi^{-1}(V) \cap \phi^{-1}(V_\alpha)$. Thus $f \circ \phi : \phi^{-1}(V) \to k$ is such that there exists a covering $\phi^{-1}(V) \subseteq \bigcup_{\alpha \in \Lambda} \phi^{-1}(V_\alpha)$ by sets of X where for each $\alpha \in \Lambda$, $f \circ \phi \restriction \phi^{-1}(V) \cap \phi^{-1}(V_\alpha) = f_\alpha \circ \phi \restriction \phi^{-1}(V) \cap \phi^{-1}(V_\alpha)$ for some $f_\alpha \in \mathcal{O}(V_\alpha)$. Now, since ϕ is an isomorphism, each $\phi^{-1}(V_\alpha)$ is an open subset of X and each $f_\alpha \circ \phi \in \mathcal{O}(\phi^{-1}(V_\alpha))$. Thus by Definition 1.3.2.1 $f \circ \phi \in \mathcal{O}'(\phi^{-1}(V))$.

We conclude $\phi \restriction Y$ is a morphism of the induced ringed spaces. Since $\phi^{-1} : R \to Y$ is also an isomorphism an analogous proof will show that $\phi^{-1} \restriction \phi(Y)$ is a morphism of the induced ringed spaces.

■

The proof of the next lemma follows from the definitions of a regular function and an induced ringed space. The details are left to the reader.

Lemma 1.4.2 *Let X be an algebraic set, $\mathcal{O}_X$ be the sheaf of regular functions and Y be a closed subset of X. Then the induced sheaf of functions $(\mathcal{O}_X)'$ is equal to the sheaf of regular functions $\mathcal{O}_Y$.*

The following is a consequence of the definition of the induced Zariski topology, Lemma 1.4.1 and Lemma 1.4.2.

Corollary 1.4.3 *Let $(X, \mathcal{O}_X)$ be an affine variety and Y be a closed subset of X. Then $(Y, \mathcal{O}_Y)$ is an affine variety.*

Proposition 1.4.4 *Let $X \subseteq k^n$ be an affine variety and let f be in $k[X]$. Then $X_f = \{x \in X : f(x) \neq 0\}$ is an affine variety.*

Proof. Let $\mathcal{O}_X$ be the sheaf of regular functions on X and $\mathcal{O}'_X$ be the sheaf of functions defined on X_f in accordance with Definition 1.3.2. Then $(X_f, \mathcal{O}'_X)$ is a ringed space. The set $R = \{(x, \alpha) : x \in X,\ \ \alpha \in k,\ \ f(x)\alpha = 1\}$ is closed in k^{n+1}. Let $\phi : X_f \to R$ be the mapping which takes $x \in X_f$ to $(x, \frac{1}{f(x)})$. Clearly ϕ is one-to-one and onto. We show that ϕ is a morphism of ringed spaces. First we show that ϕ is continuous by showing that the pre-image of a closed subset of R is a closed subset of X_f. Let $V \subseteq R$ be closed. Then $V = R \cap \mathcal{V}(J)$ where J is an ideal of $k[T_1, T_2, \ldots, T_{n+1}]$. For all $g \in J$ we define a polynomial $g' \in k[T_1, T_2, \ldots, T_n]$ as follows. First expand g in terms of the variable T_{n+1} writing

$$g = g_0 + g_1 T_{n+1} + g_2 T_{n+1}^2 + \ldots + g_d T_{n+1}^d$$

where $d \geq 0$ and $g_0, g_1, \ldots, g_d \in k[T_1, T_2, \ldots, T_n]$. Then let

$$g' = g_0 f^d + g_1 f^{d-1} + g_2 f^{d-2} + \ldots + g_d.$$

Further let $J' = \{g' : g \in J\}$. We show $\phi^{-1}(V) = \mathcal{V}(J') \cap X_f$.

Suppose that $x \in \phi^{-1}(V)$. The polynomial g' is in J' only if there exists a $g \in J$ such that $g = g_0 + g_1 T_{n+1} + \ldots + g_d T_{n+1}$ and $g' = g_0 f^d + g_1 f^{d-1} + \ldots + g_d$. Now $\phi(x) = (x, \frac{1}{f(x)}) \in V$, thus

$$0 = g(x, \frac{1}{f(x)}) = g_0(x) + g_1(x)\frac{1}{f(x)} + g_2(x)(\frac{1}{f(x)})^2 + \ldots + g_d(\frac{1}{f(x)})^d.$$

Multiplying through by $f^d(x)$ we obtain

$$0 = g_0(x)f^d(x) + g_1(x)f^{d-1}(x) + \ldots + g_d(x) = g'(x).$$

We conclude $x \in \mathcal{V}(J') \cap X_f$.

Suppose that $x \in \mathcal{V}(J') \cap X_f$. Let $g \in J$, $g = g_0 + g_1 T_{n+1} + \ldots + g_d T_{n+1}^d$ and $g' = g_0 f^d + g_1 f^{d-1} + \ldots + g_d$. Then $g' \in \mathcal{V}(J')$. Since $x \in \mathcal{V}(J')$,we have that

$$0 = g'(x) = g_0(x)f^d(x) + g_1(x)f^{d-1}(x) + g_2(x)f^{d-2}(x) + \ldots + g_d(x).$$

So dividing through by $f^d(x) \neq 0$ we obtain

$$0 = g_0(x) + g_1(x)\frac{1}{f(x)} + \ldots + g_d(x)(\frac{1}{f(x)})^d = g(x, \frac{1}{f(x)}),$$

whence $(x, \frac{1}{f(x)}) \in \mathcal{V}(J) \cap R$. We conclude $x \in \phi^{-1}(V)$. Thus $\phi^{-1}(V)$ is a closed subset of X_f.

Let $U \subseteq R$ be an open set and let $\ell \in \mathcal{O}_R(U)$, the set of regular functions on U. We wish to show that $\ell \circ \phi$ is in $\mathcal{O}'_X(\phi^{-1}(U))$.

Suppose that $x \in \phi^{-1}(U)$, then $\phi(x) = (x, \frac{1}{f(x)}) \in U$. Since $\ell \in \mathcal{O}_R(U)$, there exists an open neighbourhood V of $(x, \frac{1}{f(x)})$ and polynomials $g, h \in k[R]$ such that

$$h(y) \neq 0 \text{ and } \ell(y) = \frac{g(y)}{h(y)} \quad (\forall y \in V).$$

Expanding g and h in terms of T_{n+1} we may write $g = g_0 + g_1 T_{n+1} + \ldots + g_d T_{n+1}^d$ and $h = h_0 + h_1 T_{n+1} + \ldots + h_d T_{n+1}^d$ where $d \geq 0$ and $g_o, g_1, \ldots, g_d, h_0, h_1, \ldots, h_d \in k[T_1, T_2, \ldots, T_n]$. Since ϕ is continuous, $\phi^{-1}(V)$ is an open neighbourhood of x. For all $z \in \phi^{-1}(V)$ we have $h_0(z)f^d(z) + h_1(z)f^{d-1}(z) + \ldots + h_d(z) \neq 0$. For if

$h_0(z)f^d(z) + h_1(z)f^{d-1}(z) + \ldots + h_d(z) = 0$ it would follow that $h(z, \frac{1}{f(z)}) = 0$ where $y = (z, \frac{1}{f(z)}) \in U$, a contradiction. Further for all $z \in \phi^{-1}(V)$, we have

$$\begin{aligned}
\ell \circ \phi(z) &= \ell(z, \frac{1}{f(z)}) = \frac{g(z, \frac{1}{f(z)})}{h(z, \frac{1}{f(z)})} \\
&= \frac{g_0(z) + g_1(z)\frac{1}{f(z)} + \ldots + g_d(z)(\frac{1}{f(z)})^d}{h_0(z) + h_1(z)\frac{1}{f(z)} + \ldots + h_d(z)(\frac{1}{f(z)})^d} \\
&= \frac{g_0(z)f^d(z) + g_1(z)f^{d-1}(z) + \ldots + g_d(z)}{h_0(z)f^d(z) + h_1(z)f^{d-1}(z) + \ldots + h_d(z)}.
\end{aligned}$$

Thus $\ell \circ \phi$ is regular in x. Our choice of x was arbitrary, so that $\ell \circ \phi \in \mathcal{O}'_X(\phi^{-1}(U))$ and $\phi^*_U(\mathcal{O}_R(U)) \subseteq \mathcal{O}_{X_f}(\phi^{-1}(U))$.

Next we show that ϕ^{-1} is also a morphism of ringed spaces. Let $V \subseteq X_f$ be closed. Then $V = X_f \cap V'$ where V' is a closed subset of X. Thus, by the definition of ϕ, we have that

$$\begin{aligned}
\phi(V) = \phi(X_f \cap V') &= \{(x, \alpha) : x \in V',\ \alpha \in k,\ f(x)\alpha = 1\} \\
&= R \cap (V' \times k)
\end{aligned}$$

which is a closed subset of R. Hence $\phi(V) \subseteq R$ is closed. We conclude that ϕ^{-1} is continuous.

Let $V \subseteq X_f$ be an open set and $\ell \in \mathcal{O}'_X(V)$. We show that $\ell \circ \phi^{-1} \in \mathcal{O}_R(\phi(V))$. Since V is open in X_f there exists V^* open in X such that $V = X_f \cap V^*$. Thus, since X_f is open in X, we may assume V is open in X. From this assumption it follows that $\ell \in \mathcal{O}_X(V)$. Hence for all $s \in V$ there exist $g_s, f_s \in k[X]$ and an open neighbourhood V_s of s such that

$$h_s(r) \neq 0 \text{ and } \ell(r) = \frac{g_s(r)}{h_s(r)}\ (\forall r \in V_s).$$

By letting $g_S(T_1, \ldots, T_n) = g_S(T_1, \ldots, T_n, T_{n+1})$ and $h_S(T_1, \ldots, T_n) = h_S(T_1, \ldots, T_n, T_{n+1})$ we can consider g_s and h_s to be elements of $k[R]$. Moreover, since ϕ^{-1} is continuous, $\phi(V_s)$ is an open neighbourhood of $(s, \frac{1}{f(s)})$. So for all $(s, \frac{1}{f(s)}) \in \phi(V)$ we have that

$$h_s\left(r, \frac{1}{f(r)}\right) \neq 0 \text{ and } \ell \circ \phi^{-1}\left(r, \frac{1}{f(r)}\right) = \frac{g_s(r, \frac{1}{f(r)})}{h_s(r, \frac{1}{f(r)})}\ \left(\forall \left(r, \frac{1}{f(r)}\right) \in \phi(V_s)\right).$$

We conclude that $\ell \circ \phi^{-1} \in \mathcal{O}_R(\phi(V))$. Thus ϕ^{-1} is a morphism of the ringed spaces, and $(X_f, \mathcal{O}'_X)$ and $(R, \mathcal{O}_R)$ are isomorphic ringed spaces. Since R is an algebraic set with the induced Zariski topology and $\mathcal{O}_R$ is the sheaf of regular functions, it follows that $(X_f, \mathcal{O}'_X)$ is an affine variety.

■

Let $X \subseteq k^n$, $Y \subseteq k^m$ be closed sets, then the map $\phi = (\phi_1, \phi_2, \ldots, \phi_m) : X \to Y$ where each $\phi_i \in k[X]$ is a *polynomial map.*

Proposition 1.4.5 *Let $X \subseteq k^n$, $Y \subseteq k^m$ be closed sets and $\mathcal{O}_X$, $\mathcal{O}_Y$ be the sheaves of functions formed by taking the regular functions. Further let $\phi = (\phi_1, \phi_2, \ldots, \phi_m) : X \to Y$ be a polynomial map, then ϕ is a morphism of affine varieties.*

Proof. We show ϕ is a continuous mapping by showing that the pre-image of a closed set of Y is closed in X. Let $V \subseteq Y$ be a closed set, then there exists $I \subseteq k[y_1, y_2, \ldots, y_m]$ such that $V = \mathcal{V}(I) \cap Y$. Let $f \in I$. We create a polynomial $f\phi \in k[x_1, x_2, \ldots, x_n]$ by substituting each occurrence of y_i in f with ϕ_i, $i = 1, 2, \ldots, m$. Let $(q_1, q_2 \ldots, q_n) \in \phi^{-1}(V)$, then since

$$(\phi_1(q_1, q_2 \ldots, q_n), \phi_2(q_1, q_2 \ldots, q_n), \ldots, \phi_m(q_1, q_2 \ldots, q_n)) \in V,$$

we have

$$f\phi(q_1, q_2 \ldots, q_n) = f(\phi_1(q_1, q_2 \ldots, q_n), \phi_2(q_1, q_2 \ldots, q_n), \ldots, \phi_m(q_1, q_2 \ldots, q_n)) = 0.$$

Therefore $(q_1, q_2, \ldots, q_n) \in \mathcal{V}(\{f\phi : f \in I\}) \cap X$ and $\phi^{-1}(V) \subseteq \mathcal{V}(\{f\phi : f \in I\}) \cap X$. Let$(q_1, q_2, \ldots, q_n) \in \mathcal{V}(\{f\phi : f \in I\}) \cap X$, then for all $f \in I$, we have $f\phi(q_1, q_2, \ldots, q_n) = 0$. So

$$f(\phi_1(q_1, q_2, \ldots, q_n), \phi_2(q_1, q_2, \ldots, q_n), \ldots, \phi_m(q_1, q_2, \ldots, q_n)) = 0 \ \ (\forall f \in I).$$

Hence $(\phi_1(q_1, q_2, \ldots, q_n), \phi_2(q_1, q_2, \ldots, q_n), \ldots, \phi_m(q_1, q_2, \ldots, q_n)) \in V$. We can conclude $(q_1, q_2, \ldots, q_n) \in \phi^{-1}(V)$ and $\mathcal{V}(f\phi : f \in I) \cap X \subseteq \phi^{-1}(V)$. Thus $\phi^{-1}(V) = \mathcal{V}(f\phi : f \in I) \cap X$ and ϕ is a continuous map.

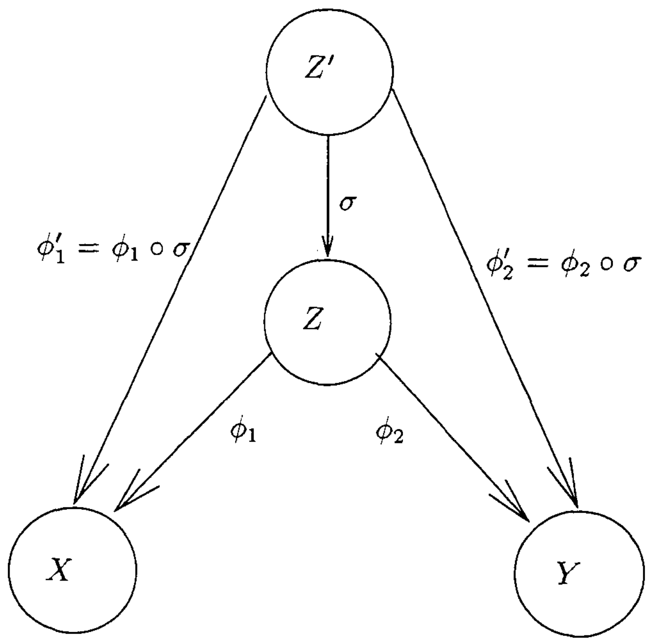

Figure 1.2: The product of the affine varieties X and Y

Let $V \subseteq Y$ be an open set and let $f \in \mathcal{O}_Y(V)$. We verify that $f \circ \phi \in \mathcal{O}_X(\phi^{-1}(V))$. Let $p \in \phi^{-1}(V)$, then $\phi(p) = q$ for some $q \in V$. Since $f \in \mathcal{O}_Y(V)$, there exist an open neighbourhood Q of q and elements $g, h \in k[Y]$ such that $h(y) \neq 0$ and $f(y) = \frac{g(y)}{h(y)}$ for all $y \in Q$. Since ϕ is continuous, $\phi^{-1}(Q)$ is an open neighbourhood of p. Further for all $a \in \phi^{-1}(Q)$, we have $f \circ \phi(a) = f(\phi(a)) = f(b)$ for some $b \in Q$. But $b \in Q$ implies that $h(b) \neq 0$ and $f(b) = \frac{g(b)}{h(b)}$ where $g(b) = g\phi(a)$ and $h(b) = h\phi(a)$. Moreover $h\phi, g\phi \in k[X]$. Therefore $f \circ \phi$ is regular in $\phi^{-1}(V)$. ∎

Definition 1.4.6 Let X and Y be two affine varieties. The *product* of X and Y is an affine variety Z together with morphisms $\phi_1 : Z \to X$ and $\phi_2 : Z \to Y$ such that the following holds: for any triple (Z', ϕ_1', ϕ_2') of an affine variety Z' together with morphisms $\phi_1' : Z' \to X$ and $\phi_2' : Z' \to Y$ there exists a unique morphism $\sigma : Z' \to Z$ such that $\phi_1' = \phi_1 \circ \sigma$ and $\phi_2' = \phi_2 \circ \sigma$. See Fig 1.2.

Note that this definition is in accordance with the general notion of a product in a category. A proof of the following theorem can be found in [6].

Theorem 1.4.7 *A product $X \times Y$ of two affine varieties X and Y exists and is unique up to isomorphism. Moreover the underlying set of $X \times Y$ can be identified with the Cartesian product of the sets X and Y.*

Chapter 2

Abstract Semigroups

The definitions and results in this chapter are taken from [2], [3] and [4].

2.1 Introduction

A *semigroup* $(S, \circ)$ is a set S with an associative operation $\circ$. We usually write S for $(S, \circ)$. An *idempotent* of S is an element $e \in S$ such that $e^2 = e$. We denote by $E(S)$ the set of idempotents of S. We can define a partial ordering on $E(S)$. If $e, f \in E(S)$, we say $e \geq f$ if and only if $ef = fe = f$. A commutative semigroup in which every element is idempotent is a *semilattice*. If S' is a semigroup, then the mapping $\phi : S \to S'$ is a *homomorphism* if and only if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in S$. A semigroup with an identity element is called a *monoid*. If S has no identity element we may adjoin an extra element 1 to the set S creating a monoid $S \cup \{1\}$ with the obvious multiplication. We will find it useful to define the semigroup S^1 as follows:

$$S^1 = \begin{cases} S & \text{if } S \text{ is a monoid} \\ S \cup \{1\} & \text{otherwise} \end{cases} .$$

Let $\mathcal{B}$ be an equivalence relation on a semigroup S. We will denote the fact that $a, b \in S$ are $\mathcal{B}$-related by $a\mathcal{B}b$. An equivalence relation $\mathcal{B}$ on S is called a *left congruence* if

$$(\forall s, t, a \in S) \;\; s\mathcal{B}t \;\; \Rightarrow \;\; as\mathcal{B}at.$$

Similarly $\mathcal{B}$ is called a *right congruence* if

$$(\forall s,t,a \in S) \;\; s\mathcal{B}t \;\; \Rightarrow \;\; sa\mathcal{B}ta.$$

Definition 2.1.1 Let I be a non-empty set and G be a group. We define a semigroup $(S, \circ)$ where $S = (I \times G \times I) \cup \{0\}$ and multiplication is as follows:

$$(i,g,j) \circ (m,h,n) = \begin{cases} (i,gh,n) & \text{if } j = m \\ 0 & \text{otherwise} \end{cases} \qquad (\forall\, (i,g,j),(m,h,n) \in S \backslash \{0\})$$

$$0 \circ (i,g,i) = (i,g,i) \circ 0 = 0 = 0 \circ 0 \quad (\forall\, (i,g,i) \in S \backslash \{0\}).$$

S is known as the $I \times I$ *Brandt semigroup over* G and is denoted by $\mu(I,G,I)$.

Definition 2.1.2 Let Y be a semilattice and let $\{G_\alpha : \alpha \in Y\}$ be a family of disjoint groups. For each pair $\alpha, \beta \in Y$ such that $\alpha \geq \beta$ let $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ be a group homomorphism such that

Sl1) $\phi_{\alpha,\alpha}$ is the identity mapping of G_α for each $\alpha \in Y$.

Sl2) for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$ we have $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ and let multiplication "$\circ$" on S be such that if $a \in G_\alpha$ and $b \in G_\beta$ then $a \circ b = \phi_{\alpha,\alpha\beta}(a)\phi_{\beta,\alpha\beta}(b)$ where the multiplication of $\phi_{\alpha,\alpha\beta}(a)$ and $\phi_{\beta,\alpha\beta}(b)$ takes place in $G_{\alpha\beta}$. Then S is a *semilattice of groups*. We will denote S by $[Y, G_\alpha, \phi_{\alpha,\beta}]$ in recognition of the fact that S is completely determined by $Y, \{G_\alpha : \alpha \in Y\}$ and $\{\phi_{\alpha,\beta} : \alpha, \beta \in Y \; \alpha \geq \beta\}$. Further we will denote by e_α the identity element of G_α.

It is straightforward to verify that multiplication is associative both on an $I \times I$ Brandt semigroup over G and on a semilattice of groups. Thus both are indeed semigroups. We will use the following lemma without comment.

Lemma 2.1.3 *Let $S = [Y, G_\alpha, \phi_{\alpha,\beta}]$ be a semilattice of groups and let $\alpha, \beta \in Y$ be such that $\alpha \geq \beta$. Then we have the following:*

1. $e_\alpha \geq e_\beta$.

2. $G_\alpha G_\beta \subseteq G_\beta$.

Proof.

1. Since $\phi_{\alpha,\beta}$ is a group homomorphism, we have that $\phi_{\alpha,\beta}(e_\alpha) = e_\beta$. So, by the definition of multiplication on S, we have that

$$e_\alpha \circ e_\beta = \phi_{\alpha,\beta}(e_\alpha)\phi_{\beta,\beta}(e_\beta) = e_\beta = \phi_{\beta,\beta}(e_\beta)\phi_{\alpha,\beta}(e_\alpha) = e_\beta \circ e_\alpha.$$

2. Let $a \in G_\alpha$ and $b \in G_\beta$. Then, by the definition of multiplication in S, we have that $a \circ b = \phi_{\alpha,\beta}(a)\phi_{\beta,\beta}(b) \in G_\beta$. ∎

2.2 Green's Equivalence Relations

Certain equivalence relations known as Green's equivalences are fundamental to the study of semigroups. In our investigations we will use four of these relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$.

Definition 2.2.1 Let a and b be elements of a semigroup S, then

1. we say a *divides* b if there exist $x, y \in S^1$ such that $xay = b$. We write $a \mid b$ to denote that a divides b.

2. $a\mathcal{J}b$ if and only if $a \mid b$ and $b \mid a$.

3. $a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$.

4. $a\mathcal{R}b$ if and only if there exist $x, y \in S^1$ such that $ax = b$ and $by = a$.

5. $a\mathcal{H}b$ if and only if $a\mathcal{R}b$ and $a\mathcal{L}b$.

It is straightforward to verify that $\mathcal{J}, \mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ are equivalence relations. Note the symmetric nature of the definitions of $\mathcal{L}$ and $\mathcal{R}$. While investigating $\mathcal{L}$ and $\mathcal{R}$ we will often state two results which have an obvious left-right symmetry. In these instances we will prove only one of the two results.

Let a be an element of a semigroup S. We denote the $\mathcal{J}$-class of a by J_a. The sets L_a, R_a and H_a are similarly defined. We will use the following lemma without comment.

Lemma 2.2.2 *Let S be a semigroup and $a, b \in S$. Then $a\mathcal{L}b$ if and only if $S^1a = S^1b$. Similarly $a\mathcal{R}b$ if and only if $aS^1 = bS^1$.*

Proof. Clearly $S^1a = S^1b$ implies that $a\mathcal{L}b$. If $a\mathcal{L}b$ then there exist $t, t' \in S^1$ such that $ta = b$ and $t'b = a$. Thus $S^1a = S^1t'b \subseteq S^1b$ and $S^1b = S^1ta \subseteq S^1a$. Hence we have $S^1a = S^1b$. In a similar fashion we can prove $a\mathcal{R}b$ if and only if $aS^1 = bS^1$. ■

Lemma 2.2.3 *The relation $\mathcal{L}$ is a right congruence and the relation $\mathcal{R}$ is a left congruence.*

Proof. Let S be a semigroup and let $s, t \in S$ be such that $s\mathcal{L}t$. Then by the definition of $\mathcal{L}$ there exists $u, v \in S^1$ such that $us = t$ and $vt = s$. So for all $a \in S$ we have that $usa = ta$ and $vta = sa$. Hence $sa\mathcal{L}ta$. We conclude that $\mathcal{L}$ is a right congruence. The proof that $\mathcal{R}$ is a left congruence is similar. ■

Let S be a semigroup. For all $x \in S^1$ we define mappings $\rho_x : S \to S$ and $\lambda_x : S \to S$ as follows:

$$\rho_x(s) = sx \text{ and } \lambda_x(s) = xs \quad (\forall s \in S).$$

The mappings ρ_x and λ_x are known as the *right translation by x* and the *left translation by x* respectively.

Lemma 2.2.4 *(Green's Lemma) Let S be a semigroup and let $a, b \in S$ be such that $a\mathcal{R}b$. Further let s, s' be the elements of S^1 such that $as = b$ and $bs' = a$. Then the* *right translations $\rho_s \restriction L_a$, $\rho_{s'} \restriction L_b$ are mutually inverse $\mathcal{R}$-class preserving bijections from L_a onto L_b and L_b onto L_a respectively.*

Proof. Since, by Lemma 2.2.3, $\mathcal{L}$ is a right congruence, it is clear that $\rho_s \restriction L_a$ maps L_a into L_b and $\rho_{s'} \restriction L_b$ maps L_b into L_a. Let $\ell \in L_a$ then there exists $u \in S^1$ such that $ua = \ell$. Thus we have that

$$\rho_{s'}\rho_s(\ell) = \ell ss' = uass' = ubs' = ua = \ell,$$

whence $\rho_{s'}\rho_s : L_a \to L_a$ is the identity mapping. Similarly we may show that $\rho_s\rho_{s'} \restriction L_b$ is the identity mapping on L_b. We conclude that $\rho_s \restriction L_a$ and $\rho_{s'} \restriction L_b$ are mutually inverse bijections. If $\ell \in L_a$, then $\ell s \in L_b$ has the property that $(\ell s)s' = \ell$. Thus $\ell s \mathcal{R} \ell$ and, so, the mapping $\rho_s \restriction L_a$ is $\mathcal{R}$-class preserving. Similarly $\rho_{s'} \restriction L_b$ is $\mathcal{R}$-class preserving. ■

The left-right dual which follows is proved in an analogous fashion.

Lemma 2.2.5 *(Green's Lemma) Let S be a semigroup and let $a, b \in S$ be such that $a\mathcal{L}b$. Further let t, t' be the elements of S^1 such that $ta = b$ and $t'b = a$. Then the left translations $\lambda_t \restriction R_a$, $\lambda_{t'} \restriction R_b$ are mutually inverse $\mathcal{L}$-class preserving bijections from R_a onto R_b and R_b onto R_a respectively.*

Proposition 2.2.6 *Let S be a semigroup, $e \in E(S)$ and $a \in S$. If $a\mathcal{H}e$, then H_a is a subgroup of S.*

Proof. We show H_a is a subgroup of S by verifying that the condition

$$H_a t = tH_a = H_a \quad (\forall t \in H_a)$$

is satisfied. It is not hard to see that this condition holds if and only if H_a is a group.

Let $t \in H_a$. By Lemma 2.2.4 we have that $\rho_t \restriction H_a$ is a bijection of H_a onto itself. Thus $H_a t = H_a$. We may similarly show that $tH_a = H_a$.

■

2.3 The Rhodes Expansion

Let S be a semigroup. We define the relations $\leq_{\mathcal{L}}$ and $<_{\mathcal{L}}$ on S as follows:

- $a \leq_{\mathcal{L}} b$ if and only if $S^1 a \subseteq S^1 b$ $(\forall a, b \in S)$
- $a <_{\mathcal{L}} b$ if and only if $S^1 a \subseteq S^1 b$ and $S^1 a \neq S^1 b$ $(\forall a, b \in S)$

Let $\overline{a} = (a_1, a_2, \ldots, a_n)$ where each a_i is an element of S. Then $\overline{a}$ is an *$\mathcal{L}$-chain* provided that $a_1 \leq_{\mathcal{L}} a_2 \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} a_n$. Further $\overline{a}$ is a *reduced $\mathcal{L}$-chain* provided that $a_1 <_{\mathcal{L}} a_2 <_{\mathcal{L}} \ldots <_{\mathcal{L}} a_n$. If $\overline{a}$ is an $\mathcal{L}$-chain we define $red(\overline{a})$ to be the reduced $\mathcal{L}$-chain formed by removing all but the left most element from any string of $\mathcal{L}$-related elements in $\overline{a}$. For example if S is the integers with the usual multiplication then $(0, 4, -4, -2, 2)$ is an $\mathcal{L}$-chain and $red((0, 4, -4, -2, 2)) = (0, 4, -2)$. Clearly $red(\overline{a})$ is a unique reduced $\mathcal{L}$-chain. We define a semigroup, $(R(S), \circ)$, known as the *left Rhodes expansion*: Let $R(S)$ be the set of all reduced $\mathcal{L}$-chains and let multiplication be defined via

$$(a_1, a_2, \ldots, a_n) \circ (b_1, b_2, \ldots, b_m) = red(a_1 b_1, a_2 b_1, \ldots, a_n b_1, b_1, b_2, \ldots, b_m).$$

It is straightforward to verify that multiplication on $R(S)$ is associative. The *right Rhodes expansion* is defined analogously.

Chapter 3

Algebraic Semigroups

With the exception of Theorem 3.2.3 the results and proofs in this chapter have been adapted from [3]. Theorem 3.2.3 is my own work.

3.1 Introduction

A semigroup $(S, \circ)$ is a *(linear) algebraic semigroup* provided that $(S, \circ)$ is isomorphic to $(S', \circ')$ where S' is an affine variety and $\circ' : S' \times S' \to S'$ is an associative product map which is also a morphism of varieties. Recall that an affine variety is a ringed space which is isomorphic to a ringed space $(T, \mathcal{O})$ where T is an algebraic set with the induced Zariski topology and $\mathcal{O}$ is the sheaf of regular functions. Note that Theorem 1.4.7 assures us that if S' is an affine variety then $S' \times S'$ is an affine variety and, thus, that $\circ'$ may be a morphism of varieties. To illustrate this definition we present two examples of algebraic semigroups.

Example 3.1.1 *Any finite semigroup is an algebraic semigroup.* Clearly any finite set S can be represented by the zeros of a polynomial in $k[T_1]$ and, thus is an affine variety. It is a straightforward exercise to construct a polynomial map which gives the desired multiplication.

We let $M_n(k)$ denote the semigroup of $n \times n$ matrices over k with the usual matrix multiplication.

Example 3.1.2 *Let $S \subseteq M_n(k)$ be both a semigroup and an algebraic subset of k^{n^2}, then S is an algebraic semigroup.* Let $A, B \in S$. Since each entry of the product AB is a polynomial expression of the elements of A and B, we have that the multiplication is given by a polynomial map. Therefore S is an algebraic semigroup. In particular, $M_n(k)$ is an algebraic semigroup.

The following lemma is a consequence of Corollary 1.4.3. We will use it without comment.

Lemma 3.1.3 *Let S be an algebraic semigroup. Any closed subsemigroup of S is an algebraic semigroup.*

A *homomorphism* between two algebraic semigroups is a semigroup homomorphism which is also a morphism of affine varieties. An *isomorphism* is similarly defined.

3.2 The Matrix Structure of Algebraic Semigroups

The following well-known theorem and its corollary are fundamental to our investigation of algebraic semigroups. The reader is referred to [3] for a proof of the theorem.

Theorem 3.2.1 *Let M be an algebraic monoid. Then M is isomorphic to a closed submonoid of some $M_n(k)$.*

Corollary 3.2.2 *Let S be an algebraic semigroup, then S is isomorphic to a closed subsemigroup of some $M_n(k)$.*

Proof. We may assume that S is a closed subset of some k^d. Let $u \in S$ and let $M = \{S \times \{0\}\} \bigcup \{(u, 1)\} \subseteq k^{d+1}$. On M define multiplication as follows:

- $(a, 0) \circ (b, 0) = (ab, 0) \ \ (\forall a, b \in S)$
- $(u, 1) \circ (u, 1) = (u, 1)$
- $(u, 1) \circ (a, 0) = (a, 0) \circ (u, 1) = (a, 0) \ \ (\forall a \in S)$

Clearly $(M, \circ)$ is a monoid with identity $(u, 1)$. Note that we may write

$$(a,x)\circ(b,y) = ab(1-x)(1-y) + axy + bx(1-y) + ay(1-x) \quad (\forall (a,x)(b,y) \in M)$$

Thus $\circ$ is a polynomial expression of multiplication in S. It is not hard to show that such a polynomial expression is a morphism of varieties and , thus, that M is an algebraic monoid. Consequently, by Theorem 3.2.1, M is isomorphic to a closed submonoid of some $M_n(k)$. But S is isomorphic to $S \times \{0\}$, which, by Proposition 1.1.4 and the definition of M, is a closed subsemigroup of M. Thus S is isomorphic to a closed subsemigroup of some $M_n(k)$. ∎

The following is a typical application of Theorem 3.2.1.

Theorem 3.2.3 *Let G be an algebraic group and I be a finite set. Then $\mu(I, G, I)$, the $I \times I$ Brandt semigroup over G, is an algebraic semigroup and $\mu(I, G, I)^1$ is an algebraic monoid.*

Proof. Let $|I| = m$. Given Theorem 3.2.1, we can assume G is an algebraic subgroup of $M_n(k)$. In the following we consider an $mn \times mn$ matrix to be partitioned into m^2 $n \times n$ blocks. Let M be the set of $mn \times mn$ matrices whose entries are zero except for exactly one of its $n \times n$ blocks; this remaining block belongs to G. We show that $M' = M \cup \{0\}$, where 0 is the zero matrix, is an algebraic semigroup. First we show that M' is an algebraic subset of $k^{m^2n^2}$. Since G is an algebraic subset of k^{n^2}, we have that $G = \mathcal{V}(I)$ for some ideal

$$I \subseteq k[T_{11}, T_{12}, \ldots, T_{1n}, T_{21}, T_{22}, \ldots, T_{2n}, \ldots, T_{n1}, T_{n2}, \ldots, T_{nn}].$$

Let $X = \{x_{pqij} : p, q \in \{1, 2, \ldots, n\}\ i, j \in \{1, 2, \ldots, m\}\}$ be a set of $(mn)^2$ indeterminates. For all $f \in I$ and $i, j \in \{1, 2, \ldots, m\}$ we form the polynomial $f^*_{ij} \in k[X]$ by replacing each instance of T_{pq} in f by x_{pqij}. For all $i, j \in \{1, 2, \ldots, m\}$ let I_{ij} be the ideal of $k[X]$ generated by

$$\{f^*_{ij} : f \in I\} \cup \{x_{pqi'j'} : p, q \in \{1, 2, \ldots, n\}\ (i', j') \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\} \backslash \{(i, j)\}\}.$$

Then

$$M' = M \cup \{0\} = \bigcup_{i,j \in \{1,2,\ldots,m\}} \mathcal{V}(I_{ij}) \cup \{0\}.$$

We conclude that M' is a closed set.

Given $a \in G$ and $i,j \in \{1,2,\ldots,m\}$, let M_{aij} be the element of M which has the matrix a in its ij'th block. Then for all $a,b \in G$ and $i,j,i',j' \in \{1,2,\ldots,m\}$

- $M_{aij} \cdot 0 = 0 \cdot M_{aij} = 0 \cdot 0 = 0$
- $M_{aij} \cdot M_{bi'j'} = \begin{cases} 0 & \text{if } j \neq i' \\ M_{abij'} & \text{if } j = i' \end{cases}$

Thus M' is closed under matrix multiplication and we conclude that M' is an algebraic semigroup. Clearly the mapping which takes $(i,a,j) \in \mu(I,G,I)$ to $M_{a,i,j}$ is an isomorphism between $\mu(I,G,I)$ and M'. We conclude that $\mu(I,G,I)$ is an algebraic semigroup.

Let 1 be the identity matrix. Since any finite subset of $k^{m^2n^2}$ is an algebraic set, $\{1\}$ is an algebraic set. Thus $M' \cup \{1\}$ is an algebraic set. Moreover $M' \cup \{1\}$ is closed under multiplication. We conclude $M' \cup \{1\}$ is an algebraic semigroup. Clearly $M' \cup \{1\}$ is isomorphic to $\mu(I,G,I)^1$. ■

3.3 Preparatory Results

The results in this section are used in Chapters 4 through 6.

Lemma 3.3.1 *If S is an algebraic semigroup and $e \in E(S)$, then $eS = \{ex : x \in S\}$, Se and eSe are algebraic semigroups.*

Proof. We may assume $S \subseteq M_n(k)$. Let T be a matrix of n^2 indeterminates. Then, since $S \subseteq M_n(k)$, the matrix $T - eT$ may be viewed as n^2 polynomials in $k[T]$. Thus the set $\{p \in k^{n^2} : p - ep = 0\}$ is an algebraic set. But $x \in S$ is in eS if and only if $x = ex$. Thus $eS = S \cap \{p \in k^{n^2} : p - ep = 0\}$ is an algebraic set. Clearly eS is a subsemigroup of S. We conclude that eS is a closed subsemigroup of S and hence an algebraic semigroup. An analogous proof will serve to show that Se and eSe are algebraic semigroups. ■

In [3] the following result was proved in the process of proving a theorem. Here it is presented in isolation.

Lemma 3.3.2 *Let S be an algebraic subsemigroup of $M_n(k)$ and let $b \in S$. If there exists an idempotent $e \in M_n(k)$ such that $b\mathcal{H}e$ in $M_n(k)$, then $e \in S$ and $b\mathcal{H}e$ in S.*

Proof. Let $e = e^2 \in M_n(k)$ be such that $e\mathcal{H}b$ in $M_n(k)$ and let $S_1 = \{x \in S : ex = xe = x\}$. Then $b \in S_1$, and, since

$$S_1 = S \cap \{p \in k^{n^2} : ep - p = 0,\ pe - p = 0\},$$

S_1 is an algebraic subset of S. Since $b\mathcal{H}e$, we have $eb = be = b$ and by Proposition 2.2.6, we have that there exists $c \in M_n(k)$ such that $ec = ce = c$, and $bc = cb = e$. Note that we do not know whether $c \in S$. Let $i \in Z^+$. Using the fact that for all $x \in S_1$ we have $c^i b^i x = b^i c^i x = x$, it is easy to show that $b^i S_1 = \{x \in S_1 : c^i x \in S_1\}$. So, by Example 1.1.3, $b^i S_1$ is closed. Further we have that $bS_1 \supseteq b^2 S_1 \supseteq \ldots$, so, since S is a Noetherian space, by Lemma 1.2.2 we have that there exists $i \in Z^+$ such that $b^i S_1 = b^{i+1} S_1$. Thus $S_1 = eS_1 = c^i b^i S_1 = c^i b^{i+1} S_1 = ebS_1 = bS_1$. Similarly $S_1 = S_1 b$. Therefore there exists $x \in S_1$ such that $b = bx$. So $x = ex = cbx = cb = e$. Hence $e \in S_1$. Further $e \in S_1$ and $S_1 = bS_1 = S_1 b$ imply that there exist $y, z \in S_1$ such that $by = e = zb$. It follows that $b\mathcal{H}e$ in S. ■

Definition 3.3.3 *If $e \in M_n(k)$ is an idempotent and $a \in M_n(k)$, then $det_e(a) = det(eae + 1 - e)$.*

Lemma 3.3.4 *Let S be an algebraic subsemigroup of $M_n(k)$, let $e \in E(S)$ and let $a \in S$. Then $det_e(a) \neq 0$ if and only if $eae\mathcal{H}e$ in S.*

Proof. Suppose $det_e(a) \neq 0$. Then there exists $x \in M_n(k)$ such that $(eae + 1 - e)x = 1$. Thus $eaex = e(eae + 1 - e)x = e$ and by duality we have that $eae\mathcal{H}e$ in $M_n(k)$. Thus, by Lemma 3.3.2, we have that $eae\mathcal{H}e$ in S.

Suppose $ea e\mathcal{H}e$ in S. Then there exists $x \in S$ such that $eaex = e$. Thus $(eae + 1 - e)(exe - e + 1) = 1$ and we have that $det_e(a) \neq 0$. ■

Lemma 3.3.5 *Let $S \subseteq M_n(k)$ be an algebraic semigroup and let $e \in E(S)$. Then the set $I = \{a \in S : \ a \not| e\}$ is closed in S.*

Proof. Let H denote H_e, and let $X = eSe \cap \{x \in k^{n^2} : \ det_e(x) = 0\}$. Then, by Lemma 3.3.1, X is closed. Further by Lemma 3.3.4 we have that $X = eSe \setminus H$. Let $x, y \in X$ and $a \in I$. Suppose $exaye$ is not in X. Then $exaye \in H$, a subgroup of S, whence $a \mid e$, a contradiction. Therefore $exaye \in X$. Now let $a \in S$ be such that $exaye \in X$ for all $x, y \in S$. We show that $a \in I$. Suppose $a \notin I$. Then $xay = e$ for some $x, y \in S$. So $exaye = e \in H$, a contradiction. We conclude that $I = \{a \in S : exaye \in X \ \ \forall x, y \in S\}$. Thus I is closed. ■

For the purposes of this thesis we call a $\mathcal{J}$-class *regular* if and only if it contains an idempotent element. This is not the standard definition of a regular $\mathcal{J}$-class. The reader is referred to [3] pages 3 and 4 and [2] for further information. We denote the set of regular $\mathcal{J}$-classes of S by $\mathcal{U}(S)$. In [3] there was an error in the proof of the following theorem. Norman Reilly provided the correction which is presented here.

Theorem 3.3.6 *Let S be an algebraic semigroup, then $\mathcal{U}(S)$ is a finite set.*

Proof. Suppose the theorem is false. Then there exists an infinite set $X \subseteq E(S)$ such that for all $e, f \in X$ we have $e\mathcal{J}f$ if and only if $e = f$. For $e \in X$, let $I(e) = \{a : a \in S \ a \not| e\}$ which is closed by Lemma 3.3.5. We claim that there exists an infinite subset Y of X such that for all $e \in Y$, $I(e) \cap Y$ is finite. Suppose not, then X itself is not such a set. Therefore there exists $f_1 \in X$ such that $X_1 = I(f_1) \cap X$ is infinite. Similarly there exists $f_2 \in X_1$ such that $X_2 = I(f_2) \cap X_1$ is infinite. Continuing we find a sequence $f_1, f_2, \ldots$ in X such that for all $i \in Z^+$ we have $f_{i+1} \in X_i = I(f_1) \cap I(f_2) \cap \ldots \cap I(f_i) \cap X$. By the definition of $I(f_{i+1})$ we have that $f_{i+1} \notin I(f_{i+1})$. So we have a strictly descending chain of closed sets

$$I(f_1) \supset I(f_1) \cap I(f_2) \supset I(f_1) \cap I(f_2) \cap I(f_3) \supset \ldots$$

This contradicts the fact that S is Noetherian. Therefore there exists an infinite set $Y \subseteq X$ such that for all $e \in Y$, $I(e) \cap Y$ is finite. Choose $e_1 \in Y$. Since $Y \cap I(e_1)$ is finite, $Y \setminus I(e_1)$ is infinite. Thus there exists $e_2 \in Y \setminus I(e_1)$ such that $e_2 \neq e_1$. Similarly there exists $e_3 \in Y \setminus (I(e_1) \cup I(e_2))$ such that $e_3 \neq e_1$ and $e_3 \neq e_2$. Hence we find distinct idempotents $e_1, e_2, \ldots$ in X such that for $i > j$ we have $e_i \mid e_j$. Let $m \in Z^+$. Consider the chain $e_m \mid e_{m-1} \mid \ldots \mid e_2 \mid e_1$. Since $e_m \mid e_{m-1}$ there exist $x, y \in S$ such that $xe_my = e_{m-1}$. Let $e'_{m-1} = e_m y e_{m-1} x e_m \in E(S)$. Then $e_{m-1} \mathcal{J} e'_{m-1}$ and $e'_{m-1} \leq e_m$. By the choice of X this implies $e'_{m-1} < e_m$. For if $e'_{m-1} = e_m$, then $e_m \mathcal{J} e_{m-1}$, a contradiction. Thus we have $e_m > e'_{m-1} \mid e_{m-2} \mid \ldots \mid e_2 \mid e_1$. Since $e'_{m-1} \mid e_{m-2}$ there exist $x', y' \in S$ such that $x'e'_{m-1}y' = e_{m-2}$. Let $e'_{m-2} = e'_{m-1}y'e_{m-2}x'e'_{m-1}$, then $e_{m-2} \mathcal{J} e'_{m-2}$ and $e'_{m-2} < e'_{m-1}$. Thus we have $e_m > e'_{m-1} > e'_{m-2} \mid e_{m-3} \mid \ldots \mid e_2 \mid e_1$. Continuing we find a sequence of idempotents $e_m > e'_{m-1} > e'_{m-2} > \ldots > e'_2 > e_1$. Since m was chosen arbitrarily this means that we can find a descending sequence of idempotents of arbitrary length. Since S is a matrix semigroup, this is a contradiction. ■

Chapter 4

Algebraic Semilattices of Groups

4.1 Introduction

In this chapter we characterize algebraic semilattices of groups. Although Theorem 4.2.4 was almost certainly previously known, we present here an origional proof based on a suggestion by Jan van der Heuvel. The remaining results and proofs are new.

4.2 Algebraic Semilattices

The goal of this section is to prove that any algebraic semilattice is finite. In order to do this we will prove a stronger result: Any semilattice $S \subseteq M_n(k)$ is finite. While this result is accessible using the following theorem from linear algebra we give a complete proof.

Theorem 4.2.1 *Let F be a commutative family of diagonalizable $n \times n$ matrices over an algebraically closed field k. There exists an invertible matrix $P \in M_n(k)$ such that $P^{-1}AP$ is diagonal for every $A \in F$.*

For further information the reader is refered to [1].

Throughout this section we will view a matrix $A \subseteq M_n(k)$ as a linear operator on the set of $n \times 1$ matrices over k, $A : x \mapsto Ax$. We will denote the range and kernel of A by $Rng(A)$ and $Ker(A)$ respectively.

Lemma 4.2.2 *Let R be the set of $n \times 1$ matrices over k. If $A \in M_n(k)$ is such that $A^2 = A$, then $R = Ker(A) \oplus Rng(A)$.*

Proof. Since the rank of $Rng(A)$ plus the rank of $Ker(A)$ equals n, it suffices to show that $Ker(A) \cap Rng(A) = \{\overline{0}\}$. Suppose that $\overline{x} \in Rng(A) \cap Ker(A)$. Then, since $\overline{x} \in Rng(A)$, there exists $\overline{y} \in R$ such that $\overline{x} = A\overline{y}$. So we have that $A\overline{x} = A(A\overline{x}) = A\overline{y} = \overline{x}$. But $\overline{x} \in Ker(A)$ implies that $A\overline{x} = \overline{0}$. Hence $\overline{x} = A\overline{x} = \overline{0}$. ■

Lemma 4.2.3 *Let S be a semilattice such that $S \subseteq M_n(k)$, and let $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_p\}$ and $\{C\}$ be disjoint subsets of S. Further let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for*

$$R = \bigcap_{i=1}^{m} Rng(A_i) \cap \bigcap_{i=1}^{p} Ker(B_i) \cap Rng(C)$$

and

$$T = \bigcap_{i=1}^{m} Rng(A_i) \cap \bigcap_{i=1}^{p} Ker(B_i) \cap Ker(C)$$

respectively. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $\bigcap_{i=1}^{m} Rng(A_i) \cap \bigcap_{i=1}^{p} Ker(B_i)$.

Proof. Let $\overline{x} \in \bigcap_{i=1}^{m} Rng(A_i) \cap \bigcap_{i=1}^{p} Ker(B_i)$. By Lemma 4.2.2, $\overline{x} = \overline{x_1} + \overline{x_2}$ for some $\overline{x_1} \in Rng(C)$ and $\overline{x_2} \in Ker(C)$. As in the proof of the previous lemma we observe that if $E \in M_n(k)$ is an idempotent and $\overline{y} \in Rng(E)$, then we have that $E\overline{y} = \overline{y}$. Now $\overline{x_1} \in Rng(C)$ and $\overline{x_2} \in Ker(C)$, thus $C\overline{x} = C\overline{x_1} + C\overline{x_2} = \overline{x_1}$. For all $i \in \{1, 2, \ldots, m\}$ we have $\overline{x} \in Rng(A_i)$, implying $\overline{x_1} = C\overline{x} = CA_i\overline{x} = A_iC\overline{x} = A_i\overline{x_1}$ and, hence, $\overline{x_1} \in Rng(A_i)$. For all $i \in \{1, 2, \ldots, p\}$ we have $\overline{x} \in Ker(B_i)$, implying $B_i\overline{x_1} = B_iC\overline{x} = CB_i\overline{x} = C\overline{0} = \overline{0}$ and, hence, that $\overline{x_1} \in Ker(B_i)$. We conclude that $\overline{x_1} \in \bigcap_{i=1}^{m} Rng(A_i) \cap \bigcap_{i=1}^{p} Ker(B_i)$. Also since $\overline{x_2} = \overline{x} - \overline{x_1}$, we have that $\overline{x_2} \in \bigcap_{i=1}^{m} Rng(A_i) \cap \bigcap_{i=1}^{p} Ker(B_i)$. Thus $\overline{x_1}$ is in the space spanned by $\mathcal{B}_1$ and $\overline{x_2}$ is in the space spanned by $\mathcal{B}_2$. We conclude $\mathcal{B}_1 \cup \mathcal{B}_2$ spans $\bigcap_{i=1}^{m} Rng(A_i) \cap \bigcap_{i=1}^{p} Ker(B_i)$.

Now say that $\overline{0} = c_1\overline{v_1} + c_2\overline{v_2} + \ldots + c_r\overline{v_r} + d_1\overline{w_1} + d_2\overline{w_2} + \ldots + d_s\overline{w_s}$ where for $i \in \{1, 2, \ldots, r\}$, $c_i \in k$ and $\overline{v_i} \in \mathcal{B}_1$, and for $j \in \{1, 2, \ldots, s\}$, $d_j \in k$ and $\overline{w_j} \in \mathcal{B}_2$.

Then

$$C\overline{0} = c_1C\overline{v_1} + c_2C\overline{v_2} + \ldots + c_rC\overline{v_r} + d_1C\overline{w_1} + d_2C\overline{w_2} + \ldots + d_sC\overline{w_s}.$$

So $\overline{0} = c_1\overline{v_1} + c_2\overline{v_2} + \ldots + c_r\overline{v_r}$, and, since $\mathcal{B}_1$ is a basis for R, we have that $c_1 = c_2 = \ldots = c_r = 0$. Thus $\overline{0} = d_1\overline{w_1} + d_2\overline{w_2} + \ldots + d_s\overline{w_s}$, and, since $\mathcal{B}_2$ is a basis for T, we have that $d_1 = d_2 = \ldots = d_s = 0$. Therefore we have that $\mathcal{B}_1 \cup \mathcal{B}_2$ is an independent set of vectors.

We conclude that $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $\bigcap_{i=1}^{m} Rng(A_i) \cap \bigcap_{i=1}^{p} Ker(B_i)$. ■

Theorem 4.2.4 *If S is a semilattice and $S \subseteq M_n(k)$, then S is finite.*

Proof. Let $\{A_1, A_2, \ldots, A_p\}$ be a subset of S. To prove our claim it is enough to show that the matrices $A_1, A_2, \ldots, A_p$ are simultaneously diagonalizable. That is, to show that there exists an $n \times n$ matrix T such that for all $A_i \in \{A_1, A_2, \ldots, A_p\}$ there exists D_i, a $(0,1)$ diagonal matrix with $A_i = TD_iT^{-1}$. For if such a T exists, then $\{A_1, A_2, \ldots, A_p\} = \{TD_1T^{-1}, TD_2T^{-1}, \ldots, TD_pT^{-1}\}$. Since there are 2^n distinct $(0,1)$ diagonal matrices, we have that $p \leq 2^n$. We conclude that S has size less than or equal to 2^n and, so, is finite.

We proceed to show that $A_1, A_2, \ldots, A_p$ are simultaneously diagonalizable. For all $\alpha \in \{1, 2, \ldots, p\}$ we define $Y_\alpha = \{y_1y_2\ldots y_\alpha : \forall i \in \{1, 2, \ldots, \alpha\} \quad y_i \in \{0, 1\}\}$. That is Y_α is the set of all $(0,1)$ strings of length α. Further, for all $y_1y_2\ldots y_\alpha \in Y_\alpha$ we define $S_{y_1y_2\ldots y_\alpha} = f(y_1)(A_1) \cap f(y_2)(A_2) \cap \ldots \cap f(y_\alpha)(A_\alpha)$ where $f(1) = Rng$ and $f(0) = Ker$. For example $S_{0010} = Ker(A_1) \cap Ker(A_2) \cap Rng(A_3) \cap Ker(A_4)$. Also we define $\mathcal{B}_{y_1y_2\ldots y_\alpha}$ to be the set of all bases for $S_{y_1y_2\ldots y_\alpha}$. We further define $\mathcal{B}$ to be the set of bases of all $n \times 1$ vectors over k.

Since A_1 is idempotent we have that for all $\beta_0 \in \mathcal{B}_0$ and $\beta_1 \in \mathcal{B}_1$, $\beta_0 \cup \beta_1 \in \mathcal{B}$. Lemma 4.2.3 implies that for all $y_1y_2 \in Y_2$ and $\beta_{y_1y_2} \in \mathcal{B}_{y_1y_2}$ we have that $\beta_{00} \cup \beta_{01} \in \mathcal{B}_0$ and $\beta_{10} \cup \beta_{11} \in \mathcal{B}_1$, whence

$$(\beta_{00} \cup \beta_{01}) \cup (\beta_{10} \cup \beta_{11}) \in \mathcal{B}.$$

Using Lemma 4.2.3 again gives that for all $y_1y_2y_3 \in Y_3$ and $\beta_{y_1y_2y_3} \in \mathcal{B}_{y_1y_2y_3}$ we have

$$\beta_{000} \cup \beta_{001} \in \mathcal{B}_{00}, \quad \beta_{010} \cup \beta_{011} \in \mathcal{B}_{01}, \quad \beta_{100} \cup \beta_{101} \in \mathcal{B}_{10}, \quad \beta_{100} \cup \beta_{111} \in \mathcal{B}_{11}.$$

Hence

$$(\beta_{000} \cup \beta_{001}) \cup (\beta_{010} \cup \beta_{011}) \cup (\beta_{100} \cup \beta_{101}) \cup (\beta_{100} \cup \beta_{111}) \in \mathcal{B}.$$

It is clear that we can use Lemma 4.2.3 in this manner $p-1$ times. Thus we can eventually show that, given an arbitrary $\beta_{\overline{y}} \in \mathcal{B}_{\overline{y}}$ for each $\overline{y} \in Y_p$, we have $\bigcup_{\overline{y} \in Y_p} \beta_{\overline{y}} \in \mathcal{B}$.

For all $\overline{y} \in Y_p$ we choose $\beta_{\overline{y}} \in \mathcal{B}_{\overline{y}}$. Since $\beta = \bigcup_{\overline{y} \in Y_p} \beta_{\overline{y}} \in \mathcal{B}$, β is a set of n linearly independent $n \times 1$ vectors. We form our matrix T from these n column vectors. Since these vectors are linearly independent, T is an $n \times n$ invertible matrix. We now show that for $1 \leq i \leq p$, $T^{-1}A_iT$ is a diagonal matrix. Let the columns of T be $\overline{t_1}, \overline{t_2}, \ldots, \overline{t_n}$ and let $\overline{v_i}$ be the $(0,1)$ column vector with a 1 in the i'th position and 0's elsewhere. If $\overline{x} = [x_1, x_2, \ldots, x_n]^T$ is any $n \times 1$ matrix over k, then $\overline{x} = x_1\overline{v_1} + x_2\overline{v_2} + \ldots + x_n\overline{v_n}$. So

$$T\overline{x} = x_1T\overline{v_1} + x_2T\overline{v_2} + \ldots + x_nT\overline{v_n} = x_1\overline{t_1} + x_2\overline{t_2} + \ldots + x_n\overline{t_n}.$$

Now for all $\overline{t_i}$, $i \in \{1, 2, \ldots, n\}$, there exists $\overline{y} \in Y_p$ such that $\overline{t_i} \in \beta_{\overline{y}}$. Further for $j \in \{1, 2, \ldots, p\}$, $y_j = 1$ implies $\overline{t_i} \in Rng(A_j)$ and $y_j = 0$ implies $\overline{t_i} \in Ker(A_j)$. So that

$$A_j\overline{t_i} = \begin{cases} \overline{t_i} & \text{if } y_j = 1 \\ \overline{0} & \text{if } y_j = 0. \end{cases}$$

Hence $(A_jT)\overline{x} = \sum_{\overline{t_i} \in Rng(A_j)} x_i\overline{t_i}$. Therefore

$$\begin{aligned} (T^{-1}A_jT)\overline{x} &= T^{-1}(A_jT)\overline{x} = T^{-1} \sum_{\overline{t_i} \in Rng(A_j)} x_i\overline{t_i} \\ &= \sum_{\overline{t_i} \in Rng(A_j)} x_iT^{-1}\overline{t_i} = \sum_{\overline{t_i} \in Rng(A_j)} x_i\overline{v_i}. \end{aligned}$$

Since this holds for all $n \times 1$ matrices $\overline{x}$ over k, $T^{-1}A_jT = D_j$ where D_j is the $(0,1)$ diagonal matrix which has a 1 in the position (i,i) if $\overline{t_i} \in Rng(A_j)$ and 0 elsewhere. We conclude that $\{A_1, A_2 \ldots, A_p\} = \{T^{-1}D_1T, T^{-1}D_2T, \ldots, T^{-1}D_pT\}$.

■

The following is a direct consequence of Theorem 4.2.4 and Corollary 3.2.2.

Corollary 4.2.5 *Any algebraic semilattice is finite.*

Notice that if $S = [Y, G_\alpha, \phi_{\alpha,\beta}]$ is a semilattice of groups, then Y is isomorphic to $\{e \in S : e = e^2\}$. Thus if S is algebraic, then Y is also algebraic. Further, given Lemma 2.1.3, we have that for all $\alpha \in Y$, $J_{e_\alpha} = \{e_\alpha\}$. We can conclude that Corollary 4.2.5 also follows from Theorem 3.3.6.

4.3 The Characterization of Algebraic Semilattices of Groups

In this section we give necessary and sufficient conditions for a semilattice of groups to be isomorphic to an algebraic semigroup.

Lemma 4.3.1 *Let $S = [Y, G_\alpha, \phi_{\alpha,\beta}]$ be a subsemigroup of $M_n(k)$. Then we have the following:*

1. *If $\alpha \geq \delta$ and $g \in G_\alpha$, then $det_{e_\delta}(g) \neq 0$.*
2. *If $\alpha \geq \delta$, $g_\alpha \in G_\alpha$ and $g_\beta \in G_\beta$, then $det_{e_\delta}(g_\alpha)det_{e_\delta}(g_\beta) = det_{e_\delta}(g_\alpha g_\beta)$.*

Proof.

1. For all $g \in G_\alpha$ there exists $g^{-1} \in G_\alpha$ such that $gg^{-1} = g^{-1}g = e_\alpha$. Now since e_δ is an idempotent,

$$(e_\delta g e_\delta + 1 - e_\delta)(e_\delta g^{-1} e_\delta + 1 - e_\delta) = e_\delta g e_\delta g^{-1} e_\delta + 1 - e_\delta.$$

We have that $e_\delta g \in G_\delta$, thus $e_\delta g e_\delta = e_\delta g$. Hence $e_\delta g e_\delta g^{-1} e_\delta = e_\delta g g^{-1} e_\delta = e_\delta e_\alpha e_\delta$. However $e_\alpha \geq e_\delta$, so $e_\delta e_\alpha = e_\delta$, and we have that $(e_\delta g e_\delta + 1 - e_\delta)(e_\delta g^{-1} e_\delta + 1 - e_\delta) = 1$. We conclude that $det_{e_\delta}(g) \neq 0$.

2. Notice that

$$\begin{aligned} det_{e_\delta}(g_\alpha)det_{e_\delta}(g_\beta) &= det(e_\delta g_\alpha e_\delta + 1 - e_\delta)det(e_\delta g_\beta e_\delta + 1 - e_\delta) \\ &= det((e_\delta g_\alpha e_\delta + 1 - e_\delta)(e_\delta g_\beta e_\delta + 1 - e_\delta)). \end{aligned}$$

Since $\alpha > \delta$, we have that $e_\delta g_\alpha \in G_\delta$, whence $e_\delta g_\alpha e_\delta = e_\delta g_\alpha$. This, along with the fact that e_δ is idempotent, implies

$$det((e_\delta g_\alpha e_\delta + 1 - e_\delta)(e_\delta g_\beta e_\delta + 1 - e_\delta)) = det(e_\delta g_\alpha g_\beta e_\delta + 1 - e_\delta) = det_{e_\delta}(g_\alpha g_\beta).$$

■

Theorem 4.3.2 *Let $S = [Y, G_\alpha, \phi_{\alpha,\beta}]$ be an algebraic semilattice of groups and let $\alpha \in Y$. Then G_α is an algebraic group. Further, if $S \subseteq M_n(k)$, then $G_\alpha = \{x \in e_\alpha S \ : \ det_{e_\alpha}(x) \neq 0\}$.*

Proof. We can assume that S is a subsemigroup of $M_n(k)$. Pick any $\alpha \in Y$. Let $e = e_\alpha$, $f = det_e$, and $eS_f = \{x \in eS : f(x) \neq 0\}$. We show that $G_\alpha = eS_f$. Let $g \in G_\alpha$, then by Lemma 4.3.1.1 we have that $f(g) \neq 0$. Clearly $g \in eS$. We conclude $G_\alpha \subseteq eS_f$.

Let $b \in G_\beta \cap eS_f$. Since $f(b) \neq 0$, there exists $c \in M_n(k)$ such that $(ebe + 1 - e)c = 1$. Hence $e(ebe+1-e)c = e$. Rewriting we get $(ebe)c = e$. Similarly, since $c(ebe + 1 - e) = 1$, we have $c(ebe) = e$. Therefore $ebe\mathcal{H}e$ in $M_n(k)$. Thus, by Lemma 3.3.2, $ebe\mathcal{H}e$ in S. So there exists $d \in G_\delta$ such that $ebed = e$. Given the definition of S, $ebed = e$ implies that $\alpha\beta\delta = \alpha$. Thus $ee_\beta e_\delta = e$ and we have that $ee_\beta = (ee_\beta e_\delta)e_\beta = ee_\beta e_\delta = e$. But $b \in eS_f$ implies that $ee_\beta = e_\beta$. Therefore $e_\beta = ee_\beta = e$. We conclude $eS_f \subseteq G_\alpha$.

By Proposition 1.4.4 and Lemma 3.3.1 we have that eS_f is an affine variety. Thus $G_\alpha = eS_f$ is an algebraic group. ■

With this theorem and Corollary 4.2.5 we have proved the two main conditions necessary for a semilattice of groups to be algebraic i.e. $S = [Y, G_\alpha, \phi_{\alpha,\beta}]$ is algebraic only if Y is finite and each G_α is an algebraic group. Further, since we can assume that S is a subsemigroup of $M_n(k)$, it is straightforward to show that the connecting homomorphisms are morphisms of affine varieties. In addition to stating this formally, the next lemma gives further necessary conditions concerning the G_α that allow us to prove Theorem 4.3.4, characterizing algebraic semilattices of groups.

Lemma 4.3.3 *If $S = [Y, G_\alpha, \phi_{\alpha,\beta}]$ is an algebraic semilattice of groups, then S is isomorphic to a semilattice of groups $S' = [Y, G'_\alpha, \phi'_{\alpha,\beta}]$ where*

1. *Y is finite,*
2. *each G'_α is an algebraic group. Furthermore each G'_α is an algebraic set of k^m, for some suitable m, with multiplication given by a polynomial map.*
3. *each $\phi'_{\alpha,\beta}$ is a polynomial map.*

Proof. We can assume $S \subsetneq M_n(k)$. We begin our proof by defining a G'_α for each $\alpha \in Y$ and by showing that these G'_α's meet condition 2 above. By Theorem 4.2.4 we have that Y is finite. Let $Y = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, and for all $i \in \{1, 2, \ldots, r\}$, let $e_i = e_{\alpha_i}$ and $det_i = det_{e_i}$. For all $j \in \{1, 2, \ldots, r\}$ we define a function $h_j : S \to k$ as follows:

$$h_j(x) = \begin{cases} 0 & \text{if } x \in G_{\alpha_i} \text{ and } \alpha_i \not\geq \alpha_j \\ \frac{1}{det_j(x)} & \text{if } x \in G_{\alpha_i} \text{ and } \alpha_i \geq \alpha_j \end{cases}$$

By Lemma 4.3.1.1 we have $det_j(x) \neq 0$ in the second case. For all $\alpha_i \in Y$ let

$$G'_{\alpha_i} = \{(a, h_1(a), h_2(a), \ldots, h_r(a)) : a \in G_{\alpha_i}\}.$$

We define multiplication " $\circ$ " on G'_{α_i} via

$$(a, h_1(a), h_2(a), \ldots, h_r(a)) \circ (b, h_1(b), h_2(b), \ldots, h_r(b)) = (ab, h_1(ab), h_2(ab), \ldots, h_r(ab)).$$

It is clear that G'_{α_i} is a group. Let $L = \{\ell : \alpha_\ell \geq \alpha_i\}$ and $\overline{L} = \{1, 2, \ldots, r\} \setminus L$. Then by the proof of Proposition 1.4.4 and Theorem 4.3.2 we have that

$$G'_{\alpha_i} = \left\{(a, v_1, v_2, \ldots, v_r) : a \in e_i S,\ \forall \ell \in L\ v_\ell = \frac{1}{det_\ell(a)},\ \ \forall \ell \in \overline{L}\ \ v_\ell = 0\right\},$$

is an algebraic set. Let $a, a' \in G_{\alpha_i}$. If $\alpha_j \leq \alpha_i$, then by Lemma 4.3.1.2, $det_j(a) det_j(a') = det_j(aa')$. If $\alpha_j \not\leq \alpha_i$ then $h_j(a) h_j(a') = 0 = h_j(aa')$. Thus for all $j \in \{1, 2, \ldots, r\}$, we have that $h_j(a) h_j(a') = h_j(aa')$. We conclude that multiplication on G'_{α_i} is given by a polynomial map, whence G'_{α_i} is an algebraic group.

We will now define the homomorphisms $\phi'_{\alpha,\beta}$. Further we will verify that they are homomorphisms which meet conditions Sl1 and Sl2 and that they are polynomial maps. For all $\alpha_i \geq \alpha_j$ define $\phi'_{ij} = \phi'_{\alpha_i,\alpha_j} : G'_{\alpha_i} \to G'_{\alpha_j}$ as follows. For all $(a, h_1(a), h_2(a), \ldots, h_r(a)) \in G'_{\alpha_i}$, we let

$$\phi'_{ij}\left((a, h_1(a), h_2(a), \ldots, h_r(a))\right) = (ae_j, h_1(a)h_1(e_j), h_2(a)h_2(e_j), \ldots, h_r(a)h_r(e_j)).$$

Notice that each ϕ'_{ij} is a polynomial map. We verify that each ϕ'_{ij} is a group homomorphism. Let $a \in G_{\alpha_i}$ and $\ell \in \{1, 2, \ldots, r\}$. Suppose $\alpha_\ell \leq \alpha_j$. Then, since $ae_j \in G_{\alpha_j}$, $h_\ell(ae_j) = \frac{1}{det_j(ae_j)}$. But $\alpha_i \geq \alpha_j \geq \alpha_\ell$, so $h_\ell(a) = \frac{1}{det_j(a)}$, $h_\ell(e_j) = \frac{1}{det_j(e_j)}$. Thus by Lemma 4.3.1.2, $h_\ell(a)h_\ell(e_j) = h_\ell(ae_j)$. Suppose $\alpha_\ell \not\leq \alpha_j$. Then, since $ae_j \in G_{\alpha_j}$, $h_\ell(ae_j) = 0$. But if $\alpha_\ell \not\leq \alpha_j$, then $h_\ell(e_j) = 0$, whence $h_\ell(a)h_\ell(e_j) = h_\ell(ae_j)$. We have shown that

$$(ae_j, h_1(a)h_1(e_j), \ldots, h_r(a)h_r(e_j)) = (ae_j, h_1(ae_j), \ldots, h_r(ae_j)). \tag{4.1}$$

Further for all $a, b \in G_{\alpha_i}$, $\ell \in \{1, 2, \ldots, r\}$, we have $h_\ell(ae_j)h_\ell(be_j) = h_\ell(ae_jbe_j) = h_\ell(abe_j)$. Using the two above facts it is straightforward to show that ϕ'_{ij} is a group homomorphism. By (4.1), it follows that for all $\alpha_i \in Y$ and $a \in G_{\alpha_i}$

$$\begin{aligned} \phi'_{ii}(a, h_1(a), \ldots, h_r(a)) &= (ae_i, h_1(ae_i), \ldots, h_r(ae_i)) \\ &= (a, h_1(a), \ldots, h_r(a)). \end{aligned}$$

Thus Sl1 holds. Let $\alpha_i, \alpha_j, \alpha_\ell \in Y$ be such that $\alpha_i \geq \alpha_j \geq \alpha_\ell$ and let $a \in G_{al_i}$ then, by (4.1), we have the following:

$$\begin{aligned} \phi_{j,\ell}\phi_{i,j}(a, h_1(a), \ldots, h_r(a)) &= \phi_{j,\ell}(ae_j, h_1(ae_j), \ldots, h_r(ae_j)) \\ &= (ae_je_\ell, h_1(ae_je_\ell), \ldots, h_r(ae_je_\ell)) \\ &= (ae_\ell, h_1(ae_\ell), \ldots, h_r(ae_\ell)) \\ &= \phi_{i,\ell}(a, h_1(a), \ldots, h_r(a)). \end{aligned}$$

Hence Sl2 holds. We conclude that $S' = [Y, G'_\alpha, \phi'_{\alpha,\beta}]$ is a semilattice of groups satisfying the given conditions.

We complete our proof by showing that S is isomorphic to S'. Let λ be the mapping from S to S' defined as follows: For all $a \in S$ we let $\lambda(a) = (a, h_1(a), h_2(a), \ldots, h_r(a))$. Clearly λ is one-to-one and onto. We verify that λ is a homomorphism. Let $a \in G_{\alpha_i}$, $b \in G_{\alpha_j}$ where $\alpha_i\alpha_j = \alpha_k$, then

$$\begin{aligned}
\lambda(a)\lambda(b) &= (a, h_1(a), h_2(a), \ldots, h_r(a)) \circ (b, h_1(b), h_2(b), \ldots, h_r(b)) \\
&= \phi'_{ik}(a, h_1(a), h_2(a), \ldots, h_r(a))\phi'_{jk}(b, h_1(b), h_2(b), \ldots, h_r(b)) \\
&= (ae_k, h_1(ae_k), h_2(ae_k), \ldots, h_r(ae_k))(be_k, h_1(be_k), h_2(be_k), \ldots, h_r(be_k)) \\
&= (abe_k, h_1(abe_k), h_2(abe_k), \ldots, h_r(abe_k)) \\
&= \lambda(ab).
\end{aligned}$$

Thus S is isomorphic to S', and our result is proved.

■

Theorem 4.3.4 *Let S be a semilattice of groups. S is an algebraic semigroup if and only if there exists and algebraically closed field k such that S is isomorphic to a semilattice of groups $S' = [Y, G_\alpha, \phi_{\alpha,\beta}]$ where*

1. *Y is finite,*
2. *each G_α is an algebraic group. Furthermore each G_α is an algebraic set of k^m, for some suitable integer m, with multiplication given by a polynomial map.*
3. *each $\phi_{\alpha,\beta}$ is a polynomial map.*

Proof. Let $S = [Y, H_\alpha, \delta_{\alpha,\beta}]$ be an algebraic semilattice of groups, then by Lemma 4.3.3 S is isomorphic to a semilattice of groups $[Y, H'_\alpha, \delta'_{\alpha,\beta}]$ which satisfies conditions 1, 2, and 3.

Now say S is isomorphic to $S' = [Y, G_\alpha, \phi_{\alpha,\beta}]$ where conditions 1, 2 and 3 are satisfied. Since $S' = \bigcup_{\alpha \in Y} G_\alpha$, we have that S' is a finite union of algebraic sets and so is itself an algebraic set. Thus in order to complete our proof all we need verify is that multiplication on S' is a morphism of varieties. Let $* : S' \times S' \to S'$ be multiplication on S', let $Y = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and let ϕ_{ij} denote ϕ_{α_i,α_j}. Further let $\circ_i$ denote multiplication on G_{α_i}. We begin our demonstration that $*$ is a morphism of varieties by showing that $*$ is continuous. More specifically we show that the preimage of a closed set is a closed set. Say V is a closed set in S'. Let P_ℓ be the set of pairs (i,j) such that $\alpha_i\alpha_j = \alpha_\ell$. From the definition of $S' \times S'$ and $*$ we have that

$$*^{-1}(V) = \bigcup_{\ell\in\{1,2,\ldots,r\}} \bigcup_{(i,j)\in P_\ell} \{(a,b) \in G_{\alpha_i} \times G_{\alpha_j} : \phi_{i\ell}(a) \circ_\ell \phi_{j,\ell}(b) \in V\}.$$

Now for any $(i,j) \in P_\ell$ we have that $\phi_{i\ell}$, $\phi_{j\ell}$ and $\circ_\ell$ are polynomial maps. Thus their "composition", $\phi_{i\ell} \circ_\ell \phi_{j\ell}$, is a polynomial map from $G_{\alpha_i} \times G_{\alpha_j}$ into G_{α_ℓ}. Thus by Proposition 1.4.5 we have that $\{(a,b) \in G_{\alpha_i} \times G_{\alpha_j} : \phi_{i\ell}(a) \circ_\ell \phi_{j\ell}(b) \in V\}$ is a closed subset of $G_{\alpha_i} \times G_{\alpha_j}$. But, by Proposition 1.1.4, $G_{\alpha_i} \times G_{\alpha_j}$ is a closed subset of $S' \times S'$. Thus $\{(a,b) \in G_{\alpha_i} \times G_{\alpha_j} : \phi_{i\ell}(a) \circ_\ell \phi_{j\ell}(b) \in V\}$ is a closed subset of $S' \times S'$. Hence $*^{-1}(V)$ is a finite union of closed sets and is itself a closed set. We conclude that $*$ is continuous.

Let $V \subseteq S'$ be an open set and let $f \in \mathcal{O}_{S'}(V)$. We complete our demonstration that $*$ is a morphism of varieties by showing that $f \circ * \in \mathcal{O}_{S'\times S'}(*^{-1}(V))$. As noted before

$$*^{-1}(V) = \bigcup_{\ell\in\{1,2,\ldots,r\}} \bigcup_{(i,j)\in P_\ell} \{(a,b) \in G_{\alpha_i} \times G_{\alpha_j} : \phi_{i\ell}(a) \circ_\ell \phi_{j\ell}(b) \in V\}$$

where for any $(i,j) \in P_\ell$ the composition $\phi_{i\ell} \circ_\ell \phi_{j,\ell} : G_{\alpha_i} \times G_{\alpha_j} \to G_{\alpha_\ell}$, is a polynomial map. Further for any pair (i,j) there exists a unique $\ell \in \{1,2,\ldots,r\}$ such that $(i,j) \in P_\ell$. Thus $* \restriction G_{\alpha_i} \times G_{\alpha_j} = \phi_{i\ell} \circ_\ell \phi_{j,\ell}$. So by Proposition 1.4.5 we have that $f \circ * \restriction G_{\alpha_i} \times G_{\alpha_j} \in \mathcal{O}_{G_{\alpha_i}\times G_{\alpha_j}}(*^{-1}(V))$. Now since Y is finite each $G_{\alpha_i} \times G_{\alpha_j}$ is the complement of a closed set namely

$$\bigcup_{(g,h)\in\{1,2,\ldots,r\}\times\{1,2,\ldots,r\}\setminus\{(i,j)\}} G_{\alpha_g} \times G_{\alpha_h}.$$

Thus each $G_{\alpha_i} \times G_{\alpha_j}$ is an open set and $G_{\alpha_i} \times G_{\alpha_j}$ $((i,j) \in \{1,2,\ldots,r\}\times\{1,2,\ldots,r\})$ is an open covering of $S' \times S'$. So by Definition 1.3.1.2 we have that $f \circ * \in \mathcal{O}_{S'\times S'}(*^{-1}(V))$.

We conclude that $*$ is a morphism of varieties, whence S' is an algebraic semigroup. ■

Chapter 5

A Rhodes Expansion

In this chapter we show that the Rhodes expansion of an algebraic semilattice of groups is an algebraic semigroup. All results in this chapter are new.

Lemma 5.1.5 *Let f be a map from a semigroup S onto a set A with a binary operation $*$. Further let f be such that for all $a, b \in A$, $f(ab) = f(a) * f(b)$. Then $*$ is an associative operation.*

Proof. Let $x_1, x_2, x_3 \in A$. Since f is onto, there exist $a_1, a_2, a_3 \in S$ such that $f(a_1) = x_1$, $f(a_2) = x_2$ and $f(a_3) = x_3$. Then

$$(x_1 * x_2) * x_3 = (f(a_1) * f(a_2)) * f(a_3) = f(a_1a_2) * f(a_3) = f((a_1a_2)a_3),$$

and

$$x_1 * (x_2 * x_3) = f(a_1) * (f(a_2) * f(a_3)) = f(a_1) * f(a_2a_3) = f(a_1(a_2a_3)).$$

Now since S is a semigroup, $(a_1a_2)a_3 = a_1(a_2a_3)$, whence $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$. We conclude $*$ is an associative operation. ■

Lemma 5.1.6 *Let $S = [Y, G_\alpha, \phi_{\alpha,\beta}]$ be a semilattice of groups and let $a \in G_\alpha$, $b \in G_\beta$, then we have the following:*

1. *$a\mathcal{L}b$ if and only if $\alpha = \beta$.*
2. *$a <_{\mathcal{L}} b$ if and only if $\alpha < \beta$.*

Proof.

1. Suppose $a \in G_\alpha$, $b \in G_\beta$ are such that $a\mathcal{L}b$. Then there exists $x \in S^1$ such that $a = xb$. Thus $ae_\alpha = xbe_\beta$, and we have

$$e_\alpha = e_\alpha e_\alpha = (a^{-1}a)e_\alpha = a^{-1}(ae_\alpha) = a^{-1}(xbe_\beta) = (a^{-1}(xb))e_\beta = a^{-1}ae_\beta = e_\alpha e_\beta.$$

Similarly $e_\beta = e_\beta e_\alpha$. Now $e_\alpha e_\beta = e_{\alpha\beta} = e_\beta e_\alpha$. Thus $e_\alpha = e_\beta$ and we can conclude $\alpha = \beta$. Conversely suppose $a, b \in G_\alpha$. Then $(ba^{-1})a = be_\alpha = b$ and $(ab^{-1})b = ae_\alpha = a$. Thus $a\mathcal{L}b$.

2. Suppose $a \in G_\alpha$, $b \in G_\beta$ are such that $a <_\mathcal{L} b$. Then there exists $x \in S^1$ such that $a = xb$. So, as shown in the previous paragraph, $e_\alpha = e_\alpha e_\beta$. From the definition of S, $e_\alpha = e_\alpha e_\beta$ implies that $\alpha \leq \beta$. From 1. we have that $\alpha \neq \beta$. Hence $\alpha < \beta$. Conversely suppose $\alpha < \beta$. Then $a = ae_\alpha = ae_\alpha e_\beta = (ae_\alpha b^{-1})b$, so $a \leq_\mathcal{L} b$. From 1. we have that a is not $\mathcal{L}$-related to b. Hence $a <_\mathcal{L} b$.

■

Theorem 5.1.7 *If S is an algebraic semilattice of groups, the Rhodes expansion $R(S)$ of S is an algebraic semigroup.*

Proof. By Theorem 4.3.4 we can assume that $S = [Y, G_\alpha, \phi_{\alpha,\beta}]$, where Y is finite, each G_α is an algebraic set, and multiplication on S is given by a polynomial map.

This proof has three parts. In *part 2* we construct an algebraic semigroup $(A, *)$ and in *part 3* we show that A is isomorphic to $R(S)$. To construct A we use a finite semigroup M isomorphic to $R(Y)$. Describing M is our task in *part 1.*

Part 1. Let $Y = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. With each $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_\ell}) \in R(Y)$ we associate a $(0,1)$ r-tuple, $\overline{x}$, which is defined as follows:

- in the i_1'th, i_2'th, $\ldots$, i_ℓ'th position of $\overline{x}$ we place a 1.
- in all other positions of $\overline{x}$ we place a 0.

Let

$$M = \{\overline{x} \in k^r : \exists s \in R(Y) \text{ such that } \overline{x} \text{ is the } r\text{-tuple associated with } s\}$$

We now define multiplication on M. Notice that for any set $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_t}\} \subseteq Y$ there can be at most one ordering of this set in $R(Y)$. Therefore the function $f : R(Y) \to M$ which takes $s \in R(Y)$ to the r-tuple associated with s above is a bijection. Let $\overline{x_1}, \overline{x_2} \in M$. We define $\overline{x_1} \circ \overline{x_2} = f(f^{-1}(\overline{x_1})f^{-1}(\overline{x_2}))$. Since f is a bijection, this multiplication is well defined. By Lemma 5.1.5, all we need in order to verify that $\circ$ is associative (and thus M is a semigroup), is that for all $a, b \in R(Y)$, $f(ab) = f(a) \circ f(b)$. This will also confirm that f is a homomorphism and, so, we will have that $R(Y) \cong (M, \circ)$. For all $a, b \in R(Y)$ we have that

$$f(ab) = f(f^{-1}(f(a))f^{-1}(f(b))) = f(a) \circ f(b).$$

Observe that since Y is finite, M is finite. Thus M is an algebraic semigroup.

Part 2. Let n be such that $S \subseteq k^n$ and let $\overline{0}$ be the n-tuple of zeros. For all $\overline{x} \in M$ we let

$$A_{\overline{x}} = \left\{ (\overline{x}, \overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}) \;\middle|\; \begin{array}{l} \text{if } x_i = 1, \text{ then } \overline{a_i} \in G_{\alpha_i} \\ \text{if } x_i = 0, \text{ then } \overline{a_i} = \overline{0} \end{array} \right\}.$$

Further we let $A = \bigcup_{\overline{x} \in M} A_{\overline{x}}$. Each $A_{\overline{x}}$ is a finite direct product of algebraic sets and so, by Proposition 1.1.4, is an algebraic set. Therefore A, being a finite union of algebraic sets, is an algebraic set.

We now prepare to define multiplication on A. Let $\overline{x}, \overline{y} \in M$. Let

$$f^{-1}(\overline{x}) = (\pi_1, \pi_2, \ldots, \pi_d), \text{and} f^{-1}(\overline{y}) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_e).$$

For each $\overline{x}, \overline{y} \in M$, $\alpha_h \in Y$ we wish to record whether or not α_h is contained in the sequence $(\pi_1\epsilon_1, \pi_2\epsilon_1, \ldots, \pi_d\epsilon_1, \epsilon_1, \epsilon_2, \ldots, \epsilon_e)$ and if α_h does occur what is the "nature" of its left most occurrence. To do this we form functions $P_{h,i,j}$ as follows:

- For $1 \leq h \leq r$, $1 \leq i \leq r$ and $1 \leq j \leq r$

$$P_{h,i,j}(\overline{x}, \overline{y}) = \begin{cases} 1 & \text{if the left most occurrence of } \alpha_h \text{ in the sequence is } \alpha_i\alpha_j \\ 0 & \text{otherwise} \end{cases} \qquad (\forall \overline{x}, \overline{y} \in M)$$

- For $1 \leq h \leq r$, $i = r+1$, and $1 \leq j \leq r$

$$P_{h,i,j}(\overline{x},\overline{y}) = \begin{cases} 1 & \text{if the left most occurrence of } \alpha_h \text{ in} \\ & \text{the sequence is } \alpha_j \\ 0 & \text{otherwise} \end{cases} \qquad (\forall \overline{x},\overline{y} \in M)$$

Notice that if $P_{h,r+1,j}(\overline{x},\overline{y}) = 1$, then $h = j$.

Let $(\overline{x},\overline{a}) = (\overline{x},\overline{a_1},\overline{a_2},\ldots,\overline{a_r})$, $(\overline{y},\overline{b}) = (\overline{y},\overline{b_1},\overline{b_2},\ldots,\overline{b_r})$ and $(\overline{z},\overline{c}) = (\overline{z},\overline{c_1},\overline{c_2},\ldots,\overline{c_r})$ where

$$\overline{x} = \{x_1,x_2,\ldots,x_r\}, \overline{y} = \{y_1,y_2,\ldots,y_r\}, \overline{z} = \{z_1,z_2,\ldots,z_r\} \in M,$$

and

$$\overline{a} = \{\overline{a_1},\overline{a_2},\ldots,\overline{a_r}\}, \overline{b} = \{\overline{b_1},\overline{b_2},\ldots,\overline{b_r}\}, \overline{c} = \{\overline{c_1},\overline{c_2},\ldots,\overline{c_r}\}$$

where $\overline{a_i},\overline{b_i},\overline{c_i} \in S \subseteq k^n$. Further for all $1 \leq d \leq r$ and $(\overline{x},\overline{a}),(\overline{y},\overline{b}) \in A$ let

$$\begin{aligned} g_d((\overline{x},\overline{a})(\overline{y},\overline{b})) \;=\; & \overline{a_1}\,\overline{b_1}P_{d,1,1}(\overline{x},\overline{y}) + \overline{a_1}\,\overline{b_2}P_{d,1,2}(\overline{x},\overline{y}) + \ldots + \overline{a_1}\,\overline{b_r}P_{d,1,r}(\overline{x},\overline{y}) \\ & + \overline{a_2}\,\overline{b_1}P_{d,2,1}(\overline{x},\overline{y}) + \overline{a_2}\,\overline{b_2}P_{d,2,2}(\overline{x},\overline{y}) + \ldots + \overline{a_2}\,\overline{b_r}P_{d,2,r}(\overline{x},\overline{y}) \\ & \vdots \\ & + \overline{a_r}\,\overline{b_1}P_{d,r,1}(\overline{x},\overline{y}) + \overline{a_r}\,\overline{b_2}P_{d,r,2}(\overline{x},\overline{y}) + \ldots + \overline{a_r}\,\overline{b_r}P_{d,r,r}(\overline{x},\overline{y}) \\ & + \overline{b_1}P_{d,r+1,1}(\overline{x},\overline{y}) + \overline{b_2}P_{d,r+1,2}(\overline{x},\overline{y}) + \ldots + \overline{b_r}P_{d,r+1,r}(\overline{x},\overline{y}). \end{aligned}$$

We are now ready to define multiplication on A. For all $(\overline{x},\overline{a})(\overline{y},\overline{b}) \in A$ we let

$$(\overline{x},\overline{a}) * (\overline{y},\overline{b}) = (\overline{x} \circ \overline{y}, g_1((\overline{x},\overline{a})(\overline{y},\overline{b})), g_2((\overline{x},\overline{a})(\overline{y},\overline{b})),\ldots, g_r((\overline{x},\overline{a})(\overline{y},\overline{b}))).$$

Since the domain of each $P_{h,i,j}$ is finite we may assume that each $P_{h,i,j}$ is a polynomial map. Further multiplication in S is a polynomial map. Thus multiplication in A is given by a polynomial map. We now verify that A is closed under this multiplication; that is, we verify that for all $(\overline{x},\overline{a}),(\overline{y},\overline{b}) \in A$, $(\overline{x},\overline{a}) * (\overline{y},\overline{b}) \in A$. Let $(\overline{x},\overline{a}) * (\overline{y},\overline{b}) = (\overline{z},\overline{c})$. For all $1 \leq h \leq r$ either $z_h = 1$ or $z_h = 0$. If $z_h = 0$, then α_h does not occur in $f^{-1}(x)f^{-1}(y)$. Thus for all $1 \leq h \leq r$, $1 \leq i \leq r+1$ and $1 \leq j \leq r$ we have that $P_{h,i,j}(\overline{x},\overline{y}) = 0$. Hence $\overline{c_h} = \overline{0}$ as required for $(\overline{x},\overline{a})*(\overline{y},\overline{b})$ to be in A. If $z_h = 1$, then α_h

occurs in $f^{-1}(\overline{x})f^{-1}(\overline{y})$ and there exist a unique pair i,j where $1 \leq i \leq r+1$, $1 \leq j \leq r$ such that $P_{h,i,j}(\overline{x},\overline{y}) \neq 0$. Suppose $1 \leq i \leq r$. Then $\overline{c_h} = g_h((\overline{x},\overline{a})(\overline{y},\overline{b})) = \overline{a_i}\overline{b_j} \in G_{\alpha_h}$ as required for $(\overline{x},\overline{a}) * (\overline{y},\overline{b})$ to be in A. Alternatively suppose $i = r+1$. Then $\overline{c_h} = g_h((\overline{x},\overline{a})(\overline{y},\overline{b})) = \overline{b_h} \in G_{\alpha_h}$ as required for $(\overline{x},\overline{a}) * (\overline{y},\overline{b})$ to be in A.

Part 3. We define our isomorphism $\phi : R(S) \rightarrow A$ as follows: For $s = (\overline{s_1},\overline{s_2},\ldots,\overline{s_b}) \in R(S)$, $\overline{s_i} \in G_{\beta_i}$ we let $\phi(s) = (f((\beta_1,\beta_2,\ldots,\beta_b)),\overline{\sigma_1},\overline{\sigma_2},\ldots,\overline{\sigma_r})$ where

$$\overline{\sigma_i} = \begin{cases} \overline{s_j} & \text{if } \alpha_i = \beta_j \text{ for some } j = 1,2,\ldots,b \\ \overline{0} & \text{if there does not exist } \beta_j \ (j = 1,2,\ldots,b) \text{ such that } \alpha_i = \beta_j. \end{cases}$$

Note that $\overline{s_1} <_{\mathcal{L}} \overline{s_2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} \overline{s_b}$, so, by Lemma 5.1.6, $\beta_1 < \beta_2 < \ldots < \beta_b$ and $f((\beta_1,\beta_2,\ldots,\beta_b)) \in M$, whence $(f((\beta_1,\beta_2,\ldots,\beta_b)),\overline{\sigma_1},\overline{\sigma_2},\ldots,\overline{\sigma_r}) \in A$. We show that ϕ is one-to-one. Say $s_1, s_2 \in R(S)$ are such that $\phi(s_1) = \phi(s_2)$. Let $s_1 = (\overline{a_1},\overline{a_2},\ldots,\overline{a_\ell})$ where $\overline{a_i} \in G_{\lambda_i}$ and let $s_2 = (\overline{b_1},\overline{b_2},\ldots,\overline{b_t})$ where $\overline{b_i} \in G_{\tau_i}$. Then $\phi(s_1) = \phi(s_2)$ implies that $f((\lambda_1,\lambda_2,\ldots,\lambda_\ell)) = f((\tau_1,\tau_2,\ldots,\tau_t))$. So, since f is a bijection, we have that $(\lambda_1,\lambda_2,\ldots,\lambda_\ell) = (\tau_1,\tau_2,\ldots,\tau_t)$. Thus $\ell = t$ and for $i = 1,2,\ldots,t$, $\lambda_i = \tau_i$. Moreover for all $i = 1,2,\ldots,t$ there exists $j = 1,2,\ldots,r$ such that $\alpha_j = \lambda_i = \tau_i$. Thus by the definition of ϕ we have that $\overline{a_i} = \overline{b_i}$. We conclude $s_1 = (\overline{a_1},\overline{a_2},\ldots,\overline{a_\ell}) = (\overline{b_1},\overline{b_2},\ldots,\overline{b_t}) = s_2$. Next we show that ϕ is onto. Let $(\overline{x},\overline{a_1},\overline{a_2},\ldots,\overline{a_r}) \in A$ and let $f^{-1} = (\alpha_{w_1},\alpha_{w_2},\ldots,\alpha_{w_\xi})$. Then, by Lemma 5.1.6, $s = (\overline{a_{w_1}},\overline{a_{w_2}},\ldots,\overline{a_{w_\xi}}) \in R(S)$, so, by the definition of ϕ, $\phi(s) = (\overline{x},\overline{a_1},\overline{a_2},\ldots,\overline{a_r})$. We now show ϕ is a homomorphism. That is we show that for all $s_1, s_2 \in R(S)$, $\phi(s_1) * \phi(s_2) = \phi(s_1 s_2)$. Let $s_1 = (\overline{a_1},\overline{a_2},\ldots,\overline{a_\ell})$ where $\overline{a_i} \in G_{\lambda_i}$ and let $s_2 = (\overline{b_1},\overline{b_2},\ldots,\overline{b_t})$ where $\overline{b_i} \in G_{\tau_i}$. Further let $\phi(s_1) = (\overline{x},\overline{A_1},\overline{A_2},\ldots,\overline{A_r})$, $\phi(s_2) = (\overline{y},\overline{B_1},\overline{B_2},\ldots,\overline{B_r})$, and $\phi(s_1 s_2) = (\overline{z},\overline{D_1},\overline{D_2},\ldots,\overline{D_r})$. We have that $\phi(s_1) * \phi(s_2) = (\overline{x} \circ \overline{y},\overline{C_1},\overline{C_2},\ldots,\overline{C_r})$ where for $1 \leq d \leq r$, $\overline{C_d} = g_d((\overline{x},\overline{A_1},\overline{A_2},\ldots,\overline{A_r})(\overline{y},\overline{B_1},\overline{B_2},\ldots,\overline{B_r}))$. It is not hard to see that $\overline{z} = \overline{x} \circ \overline{y}$, for $\phi(s_1 s_2) = \phi(red(\overline{a_1}\overline{b_1},\overline{a_2}\overline{b_1},\ldots,\overline{a_\ell}\overline{b_1},\overline{b_1},\overline{b_2},\ldots,\overline{b_t}))$. So, by Lemma 5.1.6 and the definition of ϕ, we have that

$$\begin{aligned} z &= f(red(\lambda_1\tau_1,\lambda_2\tau_1,\ldots,\lambda_\ell\tau_1,\tau_1,\tau_2,\ldots,\tau_t)) \\ & f((\lambda_1,\lambda_2,\ldots,\lambda_\ell)(\tau_1,\tau_2,\ldots,\tau_t)) \\ & f(\lambda_1,\lambda_2,\ldots,\lambda_\ell) \circ f(\tau_1,\tau_2,\ldots,\tau_t) = \overline{x} \circ \overline{y}. \end{aligned}$$

To see that for all $h = 1, 2, \ldots, r$, we have $\overline{C_h} = \overline{D_h}$ we examine three possibilities.

1. The element $\alpha_h \in Y$ does not occur in the sequence $\lambda_1\tau_1, \lambda_2\tau_1, \ldots, \lambda_\ell\tau_1, \tau_1, \tau_2, \ldots, \tau_t$. In this case the h'th position of both $\overline{z}$ and $\overline{x} \circ \overline{y}$ will be 0. Thus $\overline{D_h} = \overline{0} = \overline{C_h}$.

2. The element $\alpha_h \in Y$ occurs in the sequence $\lambda_1\tau_1, \lambda_2\tau_1, \ldots, \lambda_\ell\tau_1, \tau_1, \tau_2, \ldots, \tau_t$ and the left most occurrence of α_h in the sequence is $\lambda_m\tau_1$ for some $m \in \{1, 2, \ldots, \ell\}$. In this case there exists a unique pair i', j' where $1 \leq i' \leq r$, $1 \leq j' \leq r$ such that $\alpha_{i'} = \lambda_m$ and $\alpha_{j'} = \tau_1$. From the definition of the functions $P_{h,i,j}$ we see that i' and j' are the only values of i and j for which $P_{h,i,j}(\overline{x}, \overline{y}) \neq 0$. Thus $\overline{C_h} = g_h((\overline{x}, \overline{A_1}, \overline{A_2}, \ldots, \overline{A_r})(\overline{y}, \overline{B_1}, \overline{B_2}, \ldots, \overline{B_r})) = \overline{A_{i'}}\,\overline{B_{j'}} = \overline{a_m}\overline{b_1}$. To find $\overline{D_h}$ we notice that by Lemma 5.1.6 we have that if the left most occurrence of α_h in the sequence $\lambda_1\tau_1, \lambda_2\tau_1, \ldots, \lambda_\ell\tau_1, \tau_1, \tau_2, \ldots, \tau_t$ is $\lambda_m\tau_1 = \alpha_{i'}\alpha_{j'}$, then the left most element of G_{α_h} in the sequence $\overline{a_1}\overline{b_1}, \overline{a_2}\overline{b_1}, \ldots, \overline{a_\ell}\overline{b_1}, \overline{b_1}, \overline{b_2}, \ldots, \overline{b_t}$ is $\overline{a_m}\overline{b_1}$. Thus $\overline{a_m}\overline{b_1}$ is the only element of G_{α_h} in $s_1 * s_2 = red(\overline{a_1}\overline{b_1}, \overline{a_2}\overline{b_1}, \ldots, \overline{a_\ell}\overline{b_1}, \overline{b_1}, \overline{b_2}, \ldots, \overline{b_t})$. Therefore $\overline{D_h} = \overline{a_m}\overline{b_1} = \overline{C_h}$.

3. The element $\alpha_h \in Y$ occurs in the sequence $\lambda_1\tau_1, \lambda_2\tau_1, \ldots, \lambda_\ell\tau_1, \tau_1, \tau_2, \ldots, \tau_t$ and the left most occurrence of α_h in the sequence is τ_m for some $m \in \{1, 2, \ldots, t\}$. This case is similar to 2. The details are left to the reader.

We conclude that

$$\phi(s_1s_2) = (\overline{z}, \overline{D_1}, \overline{D_2}, \ldots, \overline{D_r}) = (\overline{x} \circ \overline{y}, \overline{C_1}, \overline{C_2}, \ldots, \overline{C_r}) = \phi(s_1) * \phi(s_2)$$

and, thus, that ϕ is a morphism.

Showing that ϕ is a morphism completes the verification that ϕ is an isomorphism. Further it shows, via Lemma 5.1.5, that $(A, *)$ is associative and thus is a semigroup. We have already shown that A is an algebraic set and that $*$ is a polynomial map. Therefore we can now conclude that $R(S)$ is isomorphic to $(A, *)$, an algebraic semigroup.

■

Chapter 6

A Counterexample

In light of Theorem 3.3.6 and Theorem 4.2.4 Norman Reilly posed the following question: If S is a subsemigroup of $M_n(k)$, then is $\mathcal{U}(S)$ finite? In this chapter we show that the answer to this question is "no". To do this we present a semigroup S such that $\mathcal{U}(S)$ is infinite. This counterexample is new.

We let C denote the field of complex numbers.

Example 6.1.8 *There exists a semigroup $S \subseteq M_2(C)$ where $\mathcal{U}(S)$ is infinite.* Let X be the set

$$\left\{ \begin{bmatrix} a & 1 \\ a(1-a) & 1-a \end{bmatrix} : a \in Z \right\}$$

and let S be the semigroup generated by X. Clearly $X \subseteq E(S)$ and X is infinite. Thus to verify that $\mathcal{U}(S)$ is infinite all we need show is that for all $e, f \in X$ we have $e\mathcal{J}f$ implies $e = f$. For $a \in Z$ we define $M(a)$ to be

$$\begin{bmatrix} a & 1 \\ a(1-a) & 1-a \end{bmatrix}.$$

Let $a_1, a_2, \ldots, a_n \in Z$. Then

$$M(a_1)M(a_2) = (a_1 + 1 - a_2) \begin{bmatrix} a_2 & 1 \\ a_2(1-a_1) & 1-a_1 \end{bmatrix}$$

and

$$M(a_1)M(a_2)M(a_3) = (a_1+1-a_2)(a_2+1-a_3)\begin{bmatrix} a_3 & 1 \\ a_3(1-a_1) & 1-a_1 \end{bmatrix}.$$

In general

$$M(a_1)M(a_2)\dots M(a_n) = \prod_{i=1}^{n-1}(a_i+1-a_{i+1})\begin{bmatrix} a_n & 1 \\ a_n(1-a_1) & 1-a_1 \end{bmatrix}.$$

Let $M(x), M(y) \in X$ be such that $M(x)\mathcal{J}M(y)$. Then, since S is generated by X, there exists $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \in Z$ such that

$$M(a_1)M(a_2)\dots M(a_n)M(x)M(b_1)M(b_2)\dots M(b_m) = M(y).$$

Thus

$$(a_1+1-a_2)\dots(a_n+1-x)(x+1-b_1)\dots(b_{m-1}+1-b_m)\begin{bmatrix} b_m & 1 \\ b_m(1-a_1) & 1-a_1 \end{bmatrix} = M(y).$$

Let

$$\ell = (a_1+1-a_2)(a_2+1-a_3)\dots(a_n+1-x)(x+1-b_1)\dots(b_{m-1}+1-b_m).$$

Then we have that

$$\begin{bmatrix} \ell b_m & \ell \\ \ell b_m(1-a_1) & \ell(1-a_1) \end{bmatrix} = M(y).$$

So $\ell = 1$, $b_m = y$ and $a_1 = y$. Suppose that we have integers $x_1, x_2, \dots, x_r$ such that

$$(x_1+1-x_2)(x_2+1-x_3)\dots(x_{r-1}+1-x_r) = 1.$$

Clearly each term, x_i+1-x_{i+1}, must equal 1 or -1. So either $x_{i+1} = x_i$ or $x_{i+1} = x_i+2$. Thus we see that either $x_1 = x_2 = \dots = x_r$ or $x_1 < x_r$. Now $\ell = 1$ and $a_1 = y = b_m$, so we can conclude that

$$y = a_1 = a_2 = \dots = a_n = x = b_1 = b_2 = \dots = b_m = y,$$

whence $M(x) = M(y)$.

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