T-domains, Exotic Black Holes and Gravitational Collapse in a Tolman-Bondi Space-time

by

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T-Domains, Exotic Black Holes and Gravitational Collapse in a Tolman-Bondi Space-Time

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Abstract

This thesis presents investigations of selected solutions of the Einstein field equations of classical general relativity. This first chapter is a review of necessary material from tensor analysis on differentiable manifolds and general relativity. The second chapter studies the field equations in the case of spherical symmetry. The complete vacuum spherically symmetric solution is reviewed. The third chapter presents original solutions involving exotic matter derived within a spherically symmetric T-domain. Such solutions are exotic black holes because they resemble classical Schwarzschild black holes to external observers yet consist of exotic matter. The Tolman-Bondi solutions are studied in the fourth chapter. The first section reviews the integration of the field equations for spherically symmetric incoherent dust. Following that is an original detailed critical analysis the pressure-free collapse of an incoherent fluid body into a Schwarzschild black hole. This new analysis includes explicit transformation of the exterior Tolman-Bondi metric to the vacuum Schwarzschild metric and explicit verification of required junction conditions.

Acknowledgments

Ahem! . . . Hello, hello, is this thing on? . . . This is all so unexpected . . . gosh, I never thought I'd win . . . There are so many people to thank here tonight, so please forgive me if I leave you out!

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It's really been great working with all of you on this. Thank you and enjoy the rest of the evening!

Dedication

I dedicate this thesis to Johanna, of course! You didn't think I'd forgotten you just because I didn't mention you in the acknowledgments! Thank you, my love, because you certainly made a bigger contribution to this thesis than I can easily mention in a paragraph!

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Chapter 1

Tensor Analysis on Differentiable Manifolds

1.1 Notations and Conventions

To avoid confusion later on, a few convenient notations are defined here. While it is always desirable, consistency of notation is usually hard to achieve. In general relativity, use of the Greek and Latin alphabets alone makes it difficult to avoid repetition of the same symbol. The meaning of particular symbols is usually clear from the context and, where possible, there is a warning about recycled symbols.

The set $\mathbb{N}$ is taken to be the set of all positive integers: in particular, 0 is not included in $\mathbb{N}$. Any general n-tuple $(x^1, x^2, x^3, \ldots, x^n) \in \mathbb{R}^n$ ($n \in \mathbb{N}$) is denoted simply by the name x. The individual coordinates[1] x^i of $x \in \mathbb{R}^n$ are obtained by the *projection mappings* $\pi^i : \mathbb{R}^n \to \mathbb{R}$, defined by

$$\pi^i(x) = \pi^i(x^1, x^2, \ldots, x^n) := x^i,$$

where $i \in \{1, 2, \ldots, n\}$ and $x \in \mathbb{R}^n$.

The set $\mathcal{C}^r(A; B)$ ($r \geq 0$) is defined as the set of all functions $f : A \to B$ (where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are open sets in the usual topology) where all the functions

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[1] Do not confuse superscripts on indexed variables with exponents. The meaning is clear from the context.

$\pi^j \circ f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ $(j = 1 \ldots m)$ have continuous partial derivatives up to and including order r. A function $f : A \rightarrow B$ is said to be $C^r_p(A; B)$ if the domain A can be partitioned into a finite number of regular sub-domains, f is C^r on each sub-domain and f has finite jump discontinuities on the boundaries of the sub-domains. The image of a set A under some function f is denoted by $f[A]$, where

$$f[A] := \{y : y = f(x) \text{ for some } x \in A\}.$$

The right-hand and left-hand limits of a real-valued function f of a single real variable x as the x approaches $a \in \mathbb{R}$ is denoted by

$$f(a^+) := \lim_{x \downarrow a} f(x), \text{ and } f(a^-) := \lim_{x \uparrow a} f(x)$$

respectively. To analyze jump discontinuities, define

$$[\triangle f(a)] := f(a^+) - f(a^-).$$

A similar notation applies for a function of a single variable induced from a function of more variables by holding all but one of the arguments constant (ie. if x_0 is a constant and $f(y) := g(x_0, y)$ the expression $[\triangle g(x_0, y)]$ is defined to be $[\triangle f(y)]$).

A vital tool for the study of general relativity is the tensor calculus. This involves two significant conventions [25]:

Definition 1.1. *__The Range Convention__: In the event that an index variable (subscript or superscript) is unrepeated in a term, it is understood to vary over the range of values $\{1, 2, \ldots, n\}$, where n is the dimension of the space.*

Definition 1.2. *__The Summation Convention__: In the event that an index variable occurs once in a superscript and once in a subscript of a term, that variable is assumed to be a (dummy) summation variable to be summed over the range $1, 2 \ldots, n$.*

For example, the equation $a_i x^i T^{ab} = 0$ represents n^2 equations (one for each of the free indices a and b), each of which involves T^{ab} multiplied by the sum $\sum_{i=1}^n a_i x^i$. All exceptions to the range or summation conventions are stated explicitly to avoid ambiguity.

In the first chapter, the discussion of tensor analysis is set on a manifold of arbitrary dimension $n \in \mathbb{N}$. However, the discussion of general relativity in subsequent chapters assumes the setting is a specific pseudo-Riemannian differentiable manifold of dimension 4. Thus, throughout most of this thesis, the range and summation conventions hold with the additional assumption that $n = 4$, unless a different dimension is specified.

Various conventions are used in the literature on general relativity with regard to the signature of $\mathcal{M}$ and whether indices range from 0 to 3 or from 1 to 4. As is clear from the range convention, indices are assumed to range from 1 to 4 in this description of general relativity. The pseudo-Riemannian manifold that is the model for space-time is assumed to have signature $+2$. Hence, a local Minkowski coordinate frame admits a metric tensor represented by the line element

$$ds^2 = \eta_{ab}dx^a dx^b = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2.$$

The signs in the terms of the Riemann tensor and other geometrical constructions are clear from their definitions. Finally, the units for the field equations are geometrised units for mathematical simplicity in which the speed of light $c = 1$ and Newton's gravitational constant $G = 1$. The appropriate definitions of all the concepts mentioned above are given in this chapter.

1.2 Differentiable Manifolds

The model of space-time studied in classical general relativity is a *four-dimensional differentiable manifold*. To develop the idea of a differentiable manifold, a few preliminary ideas are necessary.

Definition 1.3. *Let $\mathcal{M}$ be a non-empty topologised set with Hausdorff topology; that is, for any distinct points $p, q \in \mathcal{M}$, there exist open sets $U_p \subset \mathcal{M}$ and $U_q \subset \mathcal{M}$ such that $p \in U_p$, $q \in U_q$ and $U_p \cap U_q = \emptyset$. A **chart** or **local coordinate system** (χ, U) is an ordered pair consisting of an open set $U \subseteq \mathcal{M}$ together with a continuous, one-to-one mapping $\chi : U \to D \subseteq \mathbb{R}^n$, where D is an open subset of $\mathbb{R}^n$ in the usual topology.*

*For each ordered n-tuple $x = (x^1, x^2, \ldots, x^n) \in D$, there is a point $p \in \mathcal{M}$ such that $x = \chi(p)$ (ie. χ maps U onto D). Under the chart (χ, U), the entries $x^1, x^2, \ldots, x^n$ of the n-tuple $x = \chi(p)$ are the **coordinates** of p.*

Part of the strength of this formalism for describing the set $\mathcal{M}$ is that a given open subset of $\mathcal{M}$ can be described by many different coordinate charts. This is often useful if certain properties of $\mathcal{M}$ or calculations or physical phenomena are more easily understood when describing $\mathcal{M}$ using the chart $(\widehat{\chi}, \widehat{U})$ rather than the chart (χ, U). (The language used here is particularly enlightening; the term "chart" alludes to a geographical map which is a useful metaphor. As some charts are more useful than others for navigational purposes, in a model $\mathcal{M}$ of the physical universe, some coordinate charts provide more insight than others.) If a neighbourhood of a point $p \in \mathcal{M}$ can be covered by two or more coordinate charts, it is possible to define transformations between the different local coordinates.

Definition 1.4. *Let (χ, U) and $(\widehat{\chi}, \widehat{U})$ be two charts such that the intersection $U \cap \widehat{U}$ is nonempty. Let $D := \chi(U \cap \widehat{U}) \subseteq \mathbb{R}^n$ and $\widehat{D} := \widehat{\chi}(U \cap \widehat{U}) \subseteq \mathbb{R}^n$. The two charts (χ, U) and $(\widehat{\chi}, \widehat{U})$ are said to be $\mathcal{C}^r$-related (where $r \geq 0$) if the intersection $U \cap \widehat{U} = \emptyset$ or the function $\widehat{X} : D \to \widehat{D}$ defined by $X := \chi \circ \widehat{\chi}^{-1}$ and its inverse $X : \widehat{D} \to D$ defined by $\widehat{X} := \widehat{\chi} \circ \chi^{-1}$ are $\mathcal{C}^r$ functions over their domains D and $\widehat{D}$ respectively. That is, the functions $X^i := \pi^i \circ X = \pi^i \circ \chi \circ \widehat{\chi}^{-1}$ and $\widehat{X}^j := \pi^j \circ \widehat{X} = \pi^j \circ \widehat{\chi} \circ \chi^{-1}$ (where $i, j \in \{1, 2, \ldots, n\}$) have continuous mixed partial derivatives up to and including order r over their respective domains.*

Using the notation above, the n-tuples x and $\widehat{x}$ associated with a point $p \in \mathcal{M}$ are

$$x = (x^1, x^2, \ldots, x^n) = \chi(p) = X(\widehat{x}) = (\chi \circ \widehat{\chi}^{-1})(\widehat{x}), \text{ and}$$

$$\widehat{x} = (\widehat{x}^1, \widehat{x}^2, \ldots, \widehat{x}^n) = \widehat{\chi}(p) = \widehat{X}(x) = (\widehat{\chi} \circ \chi^{-1})(x).$$

The kth components of these n-tuples are given by

$$\widehat{x}^k = [\pi^k \circ \widehat{\chi} \circ \chi^{-1}](x) \equiv \widehat{X}^k(x) = \widehat{X}^k(x^1, x^2, \ldots, x^n), \text{ and}$$

$$x^k = [\pi^k \circ \chi \circ \widehat{\gamma}^{-1}](\widehat{x}) \equiv X^k(\widehat{x}) = X^k(\widehat{x}^1, \widehat{x}^2, \dots, \widehat{x}^n),$$

where $x \in D$ and $\widehat{x} \in \widehat{D}$ with D and $\widehat{D}$ being the images of $U \cap \widehat{U}$ under the charts (χ, U) and $(\widehat{\gamma}, \widehat{U})$ respectively. The situation is clearly illustrated in the figure 1.1.

Definition 1.5. *Let Λ be some set of indices. A **sub-atlas of class $\mathcal{C}^r$** $(r \geq 0)$ is a collection of charts $\mathcal{A} = \{(\chi_\alpha, U_\alpha) : \alpha \in \Lambda\}$ such that $\mathcal{M} \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$ and all the charts (χ_α, U_α) are $\mathcal{C}^r$-related. If $\mathcal{A}$ is maximal (in the sense that any chart that is $\mathcal{C}^r$-related to every chart in $\mathcal{A}$ is also in $\mathcal{A}$), then $\mathcal{A}$ is an **atlas of class $\mathcal{C}^r$** .*

Finally, the necessary tools have been assembled to define what a differentiable manifold is.

Definition 1.6. *Let $(\mathcal{M}, \mathcal{A})$ consist of a set $\mathcal{M}$ with Hausdorff topology together with an atlas $\mathcal{A}$ of class $\mathcal{C}^r$ of coordinate charts that map open sets in $\mathcal{M}$ into open sets in $\mathbb{R}^n$. Then, the ordered pair $(\mathcal{M}, \mathcal{A})$ is an **n-dimensional differentiable manifold of class $\mathcal{C}^r$**.*

Again, the language is quite instructive. The goal is to create a model of space-time that approximates the physical universe. Just as one uses charts and atlases in geography to model the curved face of the earth, one uses mathematical charts and an atlas of space-time to better understand space-time.

1.3 Curves, Tangent Vectors, Tangent Spaces

Definition 1.7. *Let $[a, b]$ be a closed interval in $\mathbb{R}$. A **parametrised curve** γ is a mapping from $[a, b]$ into the manifold $\mathcal{M}$. (Note that the curve is the mapping γ and not the set of points $\gamma[[a, b]] \subset \mathcal{M}$.) Let the range of γ lie inside some neighborhood covered by a chart (χ, U). The coordinates associated with this curve are*

$$\mathcal{X}^k(t) := [\pi^k \circ \chi \circ \gamma](t) \equiv x^k,$$

*where $t \in [a, b]$. If the functions $\mathcal{X}^k$ have continuous ordinary derivatives with respect to the parameter t up to and including order r, then γ is said to be a **curve of class***
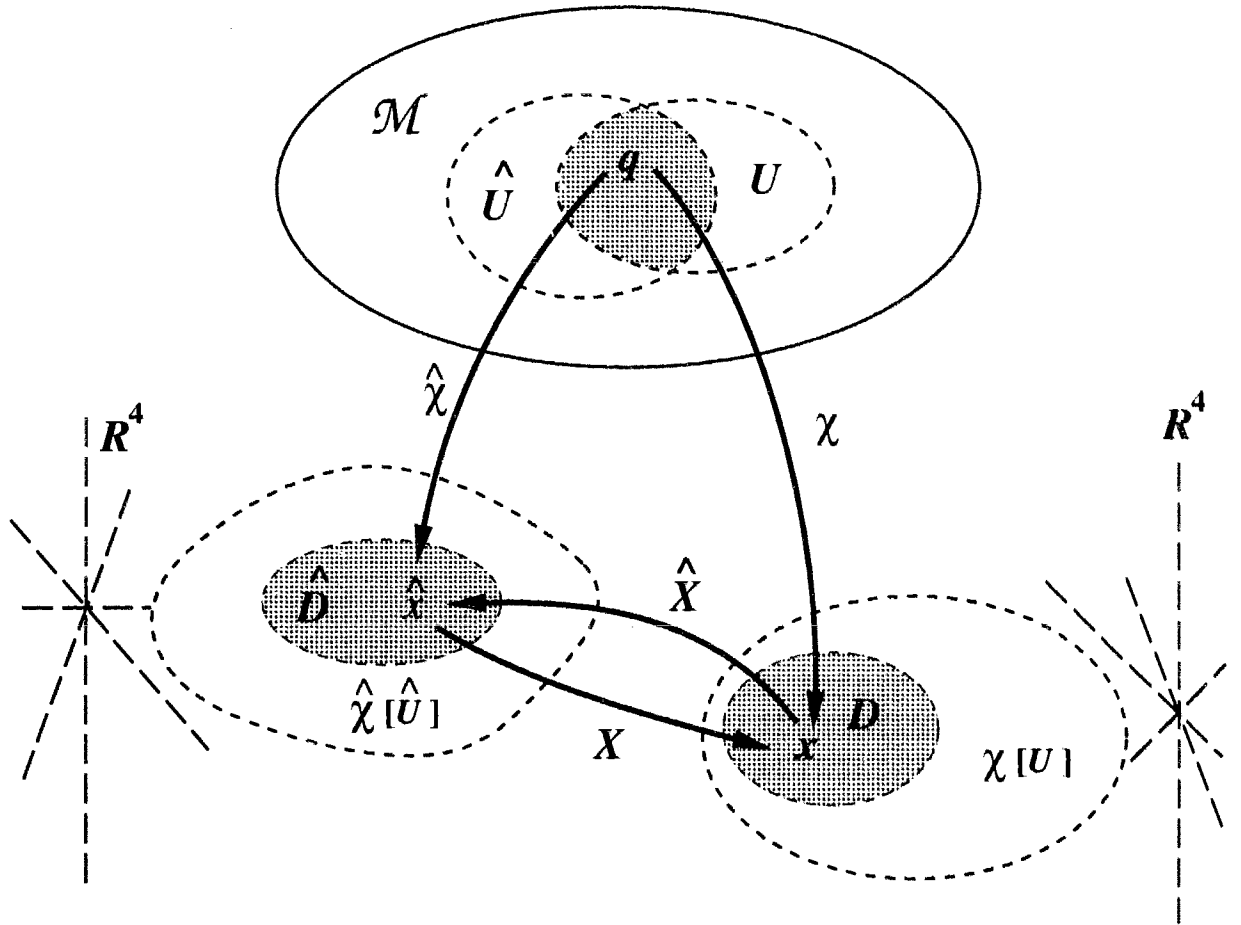
Figure 1.1: A differentiable manifold $\mathcal{M}$ with a graphical representation of two coordinate charts (χ, U) and $(\widehat{\chi}, \widehat{U})$ and the transformations between the two coordinate systems.

$\mathcal{C}^r$ *or a* $\mathcal{C}^r$-*curve. A non-degenerate curve is one such that*

$$\sum_{k=1}^{n} \left[\frac{d\mathcal{X}^k}{dt}(t)\right]^2 > 0,$$

at each point of differentiability.

Although a specific chart (χ, U) is used in the above definition, any general coordinate transformation to a $\mathcal{C}^r$-related chart $(\widehat{\chi}, \widehat{U})$ (where $r \geq 0$) suffices. That is, if the above definitions apply to the coordinates associated with a curve in a chart (χ, U), they apply also to the coordinates associated with the same curve in any other admissible chart$(\widehat{\chi}, \widehat{U})$.

Curves associated with a manifold allow specific differential operators – *tangent vectors* – to be defined. The totality of these operators constitutes a vector space which allows the construction of all the geometric objects that form the foundations of classical general relativity. This motivates the following definition.

Definition 1.8. *Let* $\gamma : [a, b] \to \mathcal{M}$ *be a curve whose image* $\gamma[a, b] \subset \mathcal{M}$ *is covered by a chart* (χ, U). *Let the point* $p = \gamma(t) \in \mathcal{M}$ *with* $t \in [a, b]$ *and let* $f \in \mathcal{C}^1(U; \mathbb{R})$ *be a differentiable real-valued function that is defined over all the points in the image of the curve* γ. *The* **tangent vector** *or* **contravariant vector** $\vec{t}_p$ *to the curve* γ *at the point* $p = \gamma(t)$ *is the map* $\vec{t}_p$ *defined by*

$$[\vec{t}_p(f)](t) := \frac{d}{dt}(f \circ \gamma)(t).$$

The components of the tangent vector $\vec{t}_p$ *to the curve* γ *(relative to the chart* (χ, U)) *are defined by*

$$\frac{d\mathcal{X}^k}{dt}(t) \equiv \frac{d}{dt}(\pi^k \circ \chi \circ \gamma)(t).$$

The set of all possible tangent vectors $\vec{t}_p$ *to all possible curves* γ *with ranges containing a point* $p \in \mathcal{M}$ *is denoted* $\mathcal{T}_p(\mathcal{M})$.

The tangent vector $\vec{t}_p$ maps real-valued functions defined in a neighbourhood of $p \in \mathcal{M}$ into the set of real-valued functions defined in some neighbourhood of $t \in [a, b]$. The

tangent vector $\mathbf{t}_p$ of γ at $p \in \mathcal{M}$ is visualised as a directed line segment emanating from p tangential to the image of $t \in [a,b]$ under γ. There is an intrinsic way of defining a tangent vector $\vec{\mathbf{t}}_p$ as the directional derivative

$$\vec{\mathbf{t}}_p = \vec{\mathbf{t}}_{\gamma(t)} \equiv \frac{d\mathfrak{X}^k}{dt}(t) \frac{\partial}{\partial x^k}\bigg|_{p=\gamma(t)}.$$

Consider the set $\mathcal{T}_p(\mathcal{M})$ of all possible tangent vectors to all possible differentiable curves whose ranges include a point $p \in \mathcal{M}$. It is possible to define addition of tangent vectors and multiplication of tangent vectors by scalars. Let $\vec{\mathbf{x}}_p, \vec{\mathbf{y}}_p \in \mathcal{T}_p(\mathcal{M})$ and let $\alpha \in \mathbb{R}$. Then, the vectors $\vec{\mathbf{x}}_p + \vec{\mathbf{y}}_p$ and $\alpha\vec{\mathbf{x}}_p$ are defined by the rules

$$[(\vec{\mathbf{x}}_p + \vec{\mathbf{y}}_p)(f)](t) := [\vec{\mathbf{x}}_p(f)](t) + [\vec{\mathbf{y}}_p(f)](t), \tag{1.1}$$

$$[(\alpha\vec{\mathbf{x}}_p)(f)](t) = \alpha[\vec{\mathbf{x}}_p(f)](t), \tag{1.2}$$

where $f \in \mathcal{C}^1(U;\mathbb{R})$ for some open set U containing p. It is clear that the set $\mathcal{T}_p(\mathcal{M})$ of tangent vectors at $p \in \mathcal{M}$ together with the rules for vector addition and scalar multiplication is a vector space over $\mathbb{R}$. Thus, $\mathcal{T}_p(\mathcal{M})$ is the *tangent vector space* at p. It can be shown [26] that if $\mathcal{M}$ is n-dimensional, then $\mathcal{T}_p(\mathcal{M})$ is n-dimensional also.

Each $p \in \mathcal{M}$ has an isomorphic copy of the tangent vector space $\mathcal{T}_p(\mathcal{M})$ associated with it. However, since distinct points have distinct tangent vector spaces, tangent vectors associated with *distinct* points *cannot* in general be added or subtracted. To simplify the notation $\vec{\mathbf{t}}_p$ for an element of $\mathcal{T}_p(\mathcal{M})$, drop the subscript p while remembering that $\vec{\mathbf{t}} \in \mathcal{T}_p(\mathcal{M})$ is a tangent vector strongly associated with $p \in \mathcal{M}$.

Interpreting a tangent vector as a directional derivative, a chart $(\chi, U) \in \mathcal{A}$ induces a natural coordinate basis $\{\vec{\mathbf{e}}_i\}_{i=1}^n$ for $\mathcal{T}_p(\mathcal{M})$, where $\vec{\mathbf{e}}_k := \frac{\partial}{\partial x^k}\big|_p$. That is, the basis for $\mathcal{T}_p(\mathcal{M})$ is the set of partial derivative operators with respect to the local coordinates x^k. Such a basis is called a *holonomic basis*. For convenience, when a particular chart is used, this holonomic basis is used for $\mathcal{T}_p(\mathcal{M})$. Then, every vector $\vec{\mathbf{t}} \in \mathcal{T}_p(\mathcal{M})$ can be expressed as $\vec{\mathbf{t}} = t^j \frac{\partial}{\partial x^j}$ for some suitable scalars $t^j \in \mathbb{R}$ which are the components of $\vec{\mathbf{t}}$ relative to the coordinate basis

$$\left\{ \frac{\partial}{\partial x^k}\bigg|_{x=\chi(p)} \right\}_{k=1}^n.$$

Given any vector space V, it is possible to define the dual space $\widetilde{V}$ of dual vectors. These dual vectors are linear functionals mapping V into $\mathbb{R}$. In this case, the dual vectors are referred to as *covariant* vectors to distinguish them from the contravariant vectors.

Definition 1.9. *A function* $\widetilde{u} : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}$ *is a* **covariant** *or* **cotangent vector** *if* $\widetilde{u}$ *is a linear function; that is, for every* $\vec{x}, \vec{y}$ *in* $\mathcal{T}_p(\mathcal{M})$ *and for every scalar* $\alpha \in \mathbb{R}$,

$$\widetilde{u}(\vec{x} + \vec{y}) = \widetilde{u}(\vec{x}) + \widetilde{u}(\vec{y}), \quad \text{and} \quad \widetilde{u}(\alpha\vec{x}) = \alpha\widetilde{u}(\vec{x}).$$

The set of all covariant vectors is denoted $\widetilde{\mathcal{T}}_p(\mathcal{M})$.

Addition and scalar multiplication of covariant vectors is defined by their action on contravariant vectors; given the covariant vectors $\widetilde{u}, \widetilde{v} \in \widetilde{\mathcal{T}}_p(\mathcal{M})$ and the scalar $\alpha \in \mathbb{R}$, the covariant vectors $[\widetilde{u} + \widetilde{v}]$ and $[\alpha\widetilde{u}]$ are defined by the rules

$$[\widetilde{u} + \widetilde{v}](\vec{t}) := \widetilde{u}(\vec{t}) + \widetilde{v}(\vec{t}), \tag{1.3a}$$

$$[\alpha\widetilde{u}](\vec{t}) := \alpha[\widetilde{u}(\vec{t})], \tag{1.3b}$$

for any vector $\vec{t}$ in $\mathcal{T}_p(\mathcal{M})$. The set $\widetilde{\mathcal{T}}_p(\mathcal{M})$ together with these rules for addition and scalar multiplication of covariant vectors is also a vector space over $\mathbb{R}$. As for any finite dimensional vector space, the dual space of $\widetilde{\mathcal{T}}_p(\mathcal{M})$ is isomorphic to $\mathcal{T}_p(\mathcal{M})$. Thus, $\mathcal{T}_p(\mathcal{M})$ is viewed as a space of linear functionals over $\widetilde{\mathcal{T}}_p(\mathcal{M})$; to show this, define $\vec{t}(\widetilde{u}) := \widetilde{u}(\vec{t})$, where $\vec{t} \in \mathcal{T}_p(\mathcal{M})$ and $\widetilde{u} \in \widetilde{\mathcal{T}}_p(\mathcal{M})$.

Given the basis $\{\vec{e}_i\}_{i=1}^n$ for $\mathcal{T}_p(\mathcal{M})$, there is a corresponding unique dual basis $\{\widetilde{e}^j\}_{j=1}^n$ for $\widetilde{\mathcal{T}}_p(\mathcal{M})$ defined by the relations $\widetilde{e}^j(\vec{e}_i) := \delta_i^j$. It follows that $\widetilde{\mathcal{T}}_p(\mathcal{M})$ has dimension n. Further, every covariant vector $\widetilde{u} \in \widetilde{\mathcal{T}}_p(\mathcal{M})$, then, can be written as $\widetilde{u} = u_k \widetilde{e}^k$ for suitable scalars $u_k \in \mathbb{R}$.

Just as the contravariant vectors in $\mathcal{T}_p(\mathcal{M})$ have an interpretation as directional derivative operators on scalar-valued functions on $\mathcal{M}$, the covariant vectors in $\widetilde{\mathcal{T}}_p(\mathcal{M})$ have an interpretation related to multivariate calculus. The *differential* of some function $f \in \mathcal{C}^1(\mathcal{M}; \mathbb{R})$ is defined as a covariant vector in $\widetilde{\mathcal{T}}_p(\mathcal{M})$.

Definition 1.10. *Let $f \in \mathcal{C}^1(\mathcal{M}; \mathbb{R})$. Then, the **differential of f** at $p \in \mathcal{M}$ is a covariant vector df that is defined by*

$$[df](\vec{\mathbf{t}}) := \vec{\mathbf{t}}f \qquad (\vec{\mathbf{t}} \in \mathcal{T}_p(\mathcal{M})).$$

If (χ, U) is a chart so that $p \in U$, let x^i denote the real-valued functions $x^i := \pi^i \circ \chi :$ $U \to \mathbb{R}$. The coordinate differentials $\{dx^i\}_{i=1}^n$ turn out to be the dual basis vectors for $\mathcal{T}_p(\mathcal{M})$ corresponding to the basis $\{(\partial/\partial x^j)_p\}_{j=1}^n$ [25], since

$$[dx^i](\partial/\partial x^j)_p = (\partial/\partial x^j)(x^i)\big|_p = \delta^i_j.$$

Hence, in terms of this coordinate basis, the differential of a function $f : U \to \mathbb{R}$ is $df = (\partial f/\partial x^k)dx^k$.

In practice, vectors at a specific point $p \in \mathcal{M}$ are not discussed as often as *vector fields* are. Let $\mathcal{T}(\mathcal{M}) := \cup_{p \in \mathcal{M}}\mathcal{T}_p(\mathcal{M})$ be the totality of all the tangent vector spaces of $\mathcal{M}$ (sometimes called the *tangent vector bundle* [13]). A contravariant vector field is a function mapping $\mathcal{M}$ into $\mathcal{T}(\mathcal{M})$; in other words, a vector field associates each point in the manifold $\mathcal{M}$ with a vector in the tangent space at that point. If the components $t^b(x)$ of some vector field $\vec{\mathbf{t}}$ relative to the basis $\{\partial/\partial x^b\}_{b=1}^n$ induced by some coordinate chart are smooth functions of the coordinates of the chart, the vector field $\vec{\mathbf{t}}$ is also smooth. This definition is made precise later, but it is sufficient to recognise that

$$\vec{\mathbf{t}} = t^b(x)\left.\frac{\partial}{\partial x^b}\right|_{x=\chi(p)}$$

expresses the vector field $\vec{\mathbf{t}}$ relative to the holonomic basis induced by the chart (χ, U). Obviously, a similar definition of a covariant vector field $\tilde{\mathbf{u}}$ is defined with components relative to some holonomic basis that vary smoothly.

For the most part, the bases used to describe $\mathcal{T}(\mathcal{M})$ and $\tilde{\mathcal{T}}(\mathcal{M})$ are be coordinate bases induced by some chart (χ, U). In that case, the vectors $\vec{\mathbf{t}} \in \mathcal{T}(\mathcal{M})$ and $\tilde{\mathbf{u}} \in \tilde{\mathcal{T}}(\mathcal{M})$ (that are images of some smooth vector fields) can be written as $\vec{\mathbf{t}} = t^a(x)\partial/\partial x^a$ and $\tilde{\mathbf{u}} = u_b(x)dx^b$, where

$$t^a := \vec{\mathbf{t}}(dx^a) \quad \text{and} \quad u_b := \tilde{\mathbf{u}}\left(\frac{\partial}{\partial x^b}\right)$$

are the components of $\vec{t}$ and $\tilde{u}$ relative to this coordinate induced basis. For a general coordinate transformation $\widehat{X}$ of class $\mathcal{C}^r$ ($r \geq 1$) relating the charts (χ, U) and $(\widehat{\chi}, \widehat{U})$, the contravariant basis vectors transform according to the multivariate chain rule. Hence, the basis induced by the chart $(\widehat{\chi}, \widehat{U})$ is related to the unhatted basis by the rule

$$\frac{\partial}{\partial \widehat{x}^a} = \frac{\partial X^k}{\partial \widehat{x}^a}(\widehat{x})\frac{\partial}{\partial x^k}. \tag{1.4}$$

It follows, then, that the components of a contravariant vector field $\vec{t}$ transform according to the rule

$$t^b(x) = \frac{\partial X^b}{\partial \widehat{x}^a}(\widehat{x})\widehat{t}^a(\widehat{x}), \tag{1.5}$$

since

$$\begin{aligned}
\vec{t} &= \widehat{t}^a \frac{\partial}{\partial \widehat{x}^a} = \widehat{t}^a \left(\frac{\partial X^b}{\partial \widehat{x}^a}(\widehat{x})\frac{\partial}{\partial x^b} \right) \\
&= \left(\frac{\partial X^b}{\partial \widehat{x}^a}(\widehat{x})\widehat{t}^a \right) \frac{\partial}{\partial x^b} \equiv t^b \frac{\partial}{\partial x^b}.
\end{aligned}$$

A similar argument gives the rule for the transformation of the covariant basis vectors as

$$d\widehat{x}^k = \frac{\partial \widehat{X}^k}{\partial x^l}(x)dx^l. \tag{1.6}$$

The corresponding transformation rule of components u_i of a covariant vector field $\tilde{u}$ as

$$\widehat{u}_j(\widehat{x}) = \frac{\partial X^i}{\partial \widehat{x}^j}(\widehat{x})u_i(x). \tag{1.7}$$

1.4 Tensors and Tensor Algebra

The existence of the spaces $\mathcal{T}_p(\mathcal{M})$ and $\widetilde{\mathcal{T}}_p(\mathcal{M})$ allow the construction of more complicated objects called *tensors*. Loosely speaking, tensors are multi-linear functionals that map Cartesian products of covariant and contravariant vectors into $\mathbb{R}$.

Definition 1.11. *Let* Π_r^s *denote the Cartesian product of* r *covariant and* s *contravariant vector spaces at* $p \in \mathcal{M}$ *defined by*

$$\Pi_r^s := \underbrace{\widetilde{\mathcal{T}}_p(\mathcal{M}) \times \widetilde{\mathcal{T}}_p(\mathcal{M}) \cdots \widetilde{\mathcal{T}}_p(\mathcal{M})}_{r \text{ times}} \times \underbrace{\mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \times \cdots \times \mathcal{T}_p(\mathcal{M})}_{s \text{ times}}.$$

*A **tensor of rank*** $(r+s)$ *is a mapping* $\mathbf{T} : \Pi_r^s \to \mathbb{R}$ *that is linear in each of the* $(r+s)$ *arguments. The set of all such linear functionals is denoted* ${}^r_s\mathcal{T}_p(\mathcal{M})$.

To elucidate, a tensor of rank $(0+s)$ is a *covariant* tensor of rank s, while a tensor of rank $(r+0)$ is a *contravariant* tensor of rank r. Thus, a contravariant tensor of rank 1 is a contravariant (tangent) vector while a covariant vector of rank 1 is a covariant (dual) vector. In terms of the notation just defined, ${}^1_0\mathcal{T}_p(\mathcal{M}) = \mathcal{T}_p(\mathcal{M})$ and ${}^0_1\mathcal{T}_p(\mathcal{M}) = \widetilde{\mathcal{T}}_p(\mathcal{M})$. Finally, a tensor of rank $(0+0)$ is a scalar $\alpha \in \mathbb{R}$.

Addition of two tensors $\mathbf{T}, \mathbf{S} \in {}^r_s\mathcal{T}_p(\mathcal{M})$ is defined by

$$[\mathbf{T}+\mathbf{S}](\widetilde{\mathbf{u}}_1, \cdots, \widetilde{\mathbf{u}}_r; \vec{\mathbf{t}}_1, \cdots, \vec{\mathbf{t}}_s) := \mathbf{T}(\widetilde{\mathbf{u}}_1, \cdots \widetilde{\mathbf{u}}_r; \vec{\mathbf{t}}_1, \cdots \vec{\mathbf{t}}_s) + \mathbf{S}(\widetilde{\mathbf{u}}_1, \cdots, \widetilde{\mathbf{u}}_r; \vec{\mathbf{t}}_1, \cdots, \vec{\mathbf{t}}_s),$$

where $\widetilde{\mathbf{u}}_1, \widetilde{\mathbf{u}}_2, \cdots, \widetilde{\mathbf{u}}_r \in \widetilde{\mathcal{T}}_p(\mathcal{M})$ and $\vec{\mathbf{t}}_1, \vec{\mathbf{t}}_2, \cdots \vec{\mathbf{t}}_s \in \mathcal{T}_p(\mathcal{M})$. Similarly, multiplication of a tensor $\mathbf{T} \in {}^r_s\mathcal{T}_p(\mathcal{M})$ by a scalar $\alpha \in \mathbb{R}$ is defined by

$$[\alpha\mathbf{T}](\widetilde{\mathbf{u}}_1, \cdots, \widetilde{\mathbf{u}}_r; \vec{\mathbf{t}}_1, \cdots \vec{\mathbf{t}}_s) := \alpha[\mathbf{T}(\widetilde{\mathbf{u}}_1, \cdots, \widetilde{\mathbf{u}}_r; \vec{\mathbf{t}}_1, \cdots \vec{\mathbf{t}}_s)],$$

where the arguments of the function are as above. With these rules, the set ${}^r_s\mathcal{T}_p(\mathcal{M})$ of all tensors of rank $(r+s)$ at an event $p \in \mathcal{M}$ turns out to be a vector space. Only tensors of the same rank can be added together.

Beyond the operations of addition and scalar multiplication, there is a *tensor product* for multiplying tensors.

Definition 1.12. *Let* $\mathbf{T} \in {}^{r_1}_{s_1}\mathcal{T}_p(\mathcal{M})$ *and* $\mathbf{S} \in {}^{r_2}_{s_2}\mathcal{T}_p(\mathcal{M})$. *The **tensor product*** $\mathbf{T} \otimes \mathbf{S} \in {}^{r_1+r_2}_{s_1+s_2}\mathcal{T}_p(\mathcal{M})$ *is the multi-linear functional defined by*

$$[\mathbf{T} \otimes \mathbf{S}](\widetilde{\mathbf{u}}_1, \ldots, \widetilde{\mathbf{u}}_{r_1+r_2}; \vec{\mathbf{t}}_1, \ldots, \vec{\mathbf{t}}_{s_1+s_2})$$
$$:= [\mathbf{T}(\widetilde{\mathbf{u}}_1, \ldots, \widetilde{\mathbf{u}}_{r_1}; \vec{\mathbf{t}}_1, \ldots, \vec{\mathbf{t}}_{s_1})] \cdot [\mathbf{S}(\widetilde{\mathbf{u}}_{r_1+1}, \ldots, \widetilde{\mathbf{u}}_{r_1+r_2}; \vec{\mathbf{t}}_{s_1+1}, \ldots, \vec{\mathbf{t}}_{s_1+s_2})]$$

for every $\vec{\mathbf{t}}_1, \ldots, \vec{\mathbf{t}}_{s_1+s_2} \in \mathcal{T}_p(\mathcal{M})$ *and every* $\widetilde{\mathbf{u}}_1, \ldots, \widetilde{\mathbf{u}}_{r_1+r_2} \in \widetilde{\mathcal{T}}_p(\mathcal{M})$.

Notice that the tensor product $\mathbf{T} \otimes \mathbf{S}$ is linear in every argument since the tensors $\mathbf{T}$ and $\mathbf{S}$ are linear in every argument. The tensor product is associative, so tensors like $\vec{t}_1 \otimes \cdots \vec{t}_r \otimes \widetilde{u}_1 \otimes \cdots \otimes \widetilde{u}_s \in {}^r_s\mathcal{T}_p(\mathcal{M})$ (where $\vec{t}_1, \ldots, \vec{t}_r \in \mathcal{T}_p(\mathcal{M})$ and $\widetilde{u}_1, \ldots, \widetilde{u}_s \in \widetilde{\mathcal{T}}_p(\mathcal{M})$) are well-defined. The tensor product of these contravariant and covariant vectors is the map $[\vec{t}_1 \otimes \cdots \vec{t}_r \otimes \widetilde{u}_1 \otimes \cdots \otimes \widetilde{u}_s] : \Pi^s_r \rightarrow \mathbb{R}$ defined by

$$[\vec{t}_1 \otimes \vec{t}_2 \otimes \cdots \otimes \vec{t}_r \otimes \widetilde{u}_1 \otimes \widetilde{u}_2 \otimes \cdots \otimes \widetilde{u}_s](\widetilde{v}_1, \widetilde{v}_2 \cdots, \widetilde{v}_r; \vec{y}_1, \vec{y}_2 \cdots, \vec{y}_s)$$
$$:= [\vec{t}_1(\widetilde{v}_1)] \cdot [\vec{t}_2(\widetilde{v}_2)] \cdots [\vec{t}_r(\widetilde{v}_r)] \cdot [\widetilde{u}_1(\vec{y}_1)] \cdot [\widetilde{u}_2(\vec{y}_2)] \cdots [\widetilde{u}_s(\vec{y}_s)],$$

for every $\widetilde{v}_1, \ldots, \widetilde{v}_r \in \widetilde{\mathcal{T}}_p(\mathcal{M})$ and $\vec{y}_1, \ldots, \vec{y}_s \in \mathcal{T}_p(\mathcal{M})$. Finally, the tensor product is also distributive over addition of tensors [13].

If $\{\vec{e}_i\}_{i=1}^n$ and $\{\widetilde{e}^j\}_{j=1}^n$ are dual bases for $\mathcal{T}_p(\mathcal{M})$ and $\widetilde{\mathcal{T}}_p(\mathcal{M})$ respectively, the set of n^{r+s} tensor products

$$\{\vec{e}_{i_1} \otimes \vec{e}_{i_2} \otimes \cdots \otimes \vec{e}_{i_r} \otimes \widetilde{e}^{j_1} \otimes \widetilde{e}^{j_2} \otimes \cdots \otimes \widetilde{e}^{j_s} : i_1, \ldots i_r, j_1, \ldots j_s \in \{1, \ldots, n\}\}$$

is a basis set for the vector space ${}^r_s\mathcal{T}_p(\mathcal{M})$ [25]. In terms of this basis, any tensor ${}^r_s\mathbf{T} \in {}^r_s\mathcal{T}_p(\mathcal{M})$ can be written

$$ {}^r_s\mathbf{T} = T^{i_1 i_2 \cdots i_r}{}_{j_1 j_2 \cdots j_s} \vec{e}_{i_1} \otimes \cdots \otimes \vec{e}_{i_r} \otimes \widetilde{e}^{j_1} \otimes \cdots \otimes \widetilde{e}^{j_s}, \tag{1.8}$$

where $T^{i_1 i_2 \cdots i_r}{}_{j_1 j_2 \cdots j_s} \in \mathbb{R}$ are the components of ${}^r_s\mathbf{T}$ relative to the dual bases $\{\vec{e}_i\}_{i=1}^n$ and $\{\widetilde{e}^j\}_{j=1}^n$. Explicitly, the components can be found by the equation

$$T^{i_1 i_2 \cdots i_r}{}_{j_1 j_2 \cdots j_s} = {}^r_s\mathbf{T}(\widetilde{e}^{i_1}, \ldots, \widetilde{e}^{i_r}; \vec{e}_{j_1}, \ldots, \vec{e}_{j_s}). \tag{1.9}$$

Knowing how to define components of tensors, it is possible to consider symmetries of tensors. A tensor $T^{i_1 i_2 \cdots i_r}$ of rank $(r + 0)$ is *symmetric* in the kth and lth indices $(1 \leq k \leq l \leq r)$ if $T^{i_1 \cdots i_k \cdots i_l \cdots i_r} = T^{i_1 \cdots i_l \cdots i_k \cdots i_r}$; it is *antisymmetric* in the kth and lth indices $(1 \leq k \leq l \leq r)$ if $T^{i_1 \cdots i_k \cdots i_l \cdots i_r} = -T^{i_1 \cdots i_l \cdots i_k \cdots i_r}$.

Another important operation on tensors is *contraction*. The contraction of a tensor of rank $(r + s)$ results in a tensor of rank $((r - 1) + (s - 1))$. This operation is called contraction because it involves summing the components of a tensor (relative to some basis set) over one of the covariant and one of the contravariant indices, contracting it to a tensor of lower rank.

Definition 1.13. *Let $_s^r\mathbf{T} \in {_s^r}\mathcal{T}_p(\mathcal{M})$ be a tensor of order $(r+s)$ that has components $T^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_s}$ relative to the dual bases $\{\vec{e}_i\}_{i=1}^n$ and $\{\tilde{e}^j\}_{j=1}^n$. The* **contraction** *of the tensor $_s^r\mathbf{T}$ in the pth contravariant index and the qth covariant index $(1 \le p \le r, 1 \le q \le s)$ is the tensor $C_q^p({_s^r}\mathbf{T}) \in {_{s-1}^{r-1}}\mathcal{T}_p(\mathcal{M})$ defined by*

$$C_q^p({_s^r}\mathbf{T}) := [T^{i_1 \ldots i_{p-1}\, m\, i_{p+1} \ldots i_r}{}_{j_1 \ldots j_{q-1}\, m\, j_{q+1} \ldots j_s}] \cdot$$
$$[\vec{e}_{i_1} \otimes \cdots \otimes \vec{e}_{i_{p-1}} \otimes \vec{e}_{i_{p+1}} \otimes \cdots \otimes \vec{e}_{i_r} \otimes \tilde{e}^{j_1} \otimes \cdots \otimes \tilde{e}^{j_{q-1}} \otimes \tilde{e}^{j_{q+1}} \otimes \cdots \otimes \tilde{e}^{j_s}].$$

That is, the components of the contracted tensor are obtained by replacing the pth contravariant and the qth covariant index in the components of the tensor with a dummy variable and summing.

It can be proved that $C_q^p({_s^r}\mathbf{T})$ is in fact a tensor. Also, in spite of the appearance of basis dependence in this definition, the contraction of a tensor is in fact basis-independent [25].

To make the earlier notion of a vector field precise, define a *tensor field* [13]. Since a vector is simply a tensor of rank (0+1) or (1+0), this definition includes both contravariant and covariant vector fields.

Definition 1.14. *Let* $\mathbf{T}$ *be a mapping*

$$\mathbf{T} : \mathcal{M} \to {_s^r}\mathcal{T}(\mathcal{M}) \text{ where } {_s^r}\mathcal{T}(\mathcal{M}) := \cup_{p \in \mathcal{M}} {_s^r}\mathcal{T}_p(\mathcal{M}).$$

($_s^r\mathcal{T}(\mathcal{M})$ is the totality of all the tangent tensor spaces for every $p \in \mathcal{M}$). $\mathbf{T}$ *associates each point $p \in \mathcal{M}$ with a tensor of rank $(r+s)$ in the associated tangent tensor space. Given any chart (χ, U) in the atlas for space-time, let $T^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_s}(x)$ be the components of these tensors at each point $p \in \mathcal{M}$ with respect to the coordinate bases $\{\partial/\partial x^i\}_{i=1}^n\}$ and $\{dx_j\}_{j=1}^n$ induced by (χ, U) at each point. If these components are $\mathcal{C}^m$ functions $(m \ge 1)$ of the coordinates $x = (x^1, x^2, \ldots, x^n)$, then $\mathbf{T}$ is a $\mathcal{C}^m$-tensor field of rank $(r+s)$ on $\mathcal{M}$. More simply, a tensor field is a tensor-valued function on $\mathcal{M}$ whose components vary smoothly over $\mathcal{M}$.*

Although this definition appears to depend on a particular chart, it is not difficult to show that it holds for all charts in the atlas if it holds for one. For computational purposes, the definition of a tensor field as a map associating points in $\mathcal{M}$ with multi-linear

functionals of ordered $(r + s)$-tuples of vectors in $\mathcal{T}_p(\mathcal{M})$ and $\widetilde{\mathcal{T}}_p(\mathcal{M})$ is impractical. Following the classical literature, a tensor field $^r_s\mathbf{T}$ is referred to by its components $T^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s}(\ )$ with respect to some local coordinates on $\mathcal{M}$. The choice of chart induces a natural basis for $\mathcal{T}_p(\mathcal{M})$, namely the holonomic basis associated with the local coordinates. That basis induces a dual basis for $\widetilde{\mathcal{T}}_p(\mathcal{M})$ and in turn for $^r_s\mathcal{T}_p(\mathcal{M})$. Hence, the choice of local coordinate systems fixes the representation $T^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s}$ of a tensor field $^r_s\mathbf{T}$ in some open set U containing $p \in \mathcal{M}$. The complicated nature of the Einstein field equations (1.35) necessitate using coordinate representations of tensors throughout this thesis. Further, unless necessary, the argument x of such components are left out; it is implicitly understood that such components actually do depend on the coordinates of points in $\mathcal{M}$.

The contravariant and covariant indices of the components of a tensor field transform just as the components of contravariant and covariant vector fields. Explicitly, if the functions X and $\widehat{X} = X^{-1}$ describe a $\mathcal{C}^r$ $(r \geq 1)$ general coordinate transformation relating the charts (χ, U) and $(\widehat{\chi}, \widehat{U})$, the components of a tensor field $^r_s\mathbf{T}$ of rank $(r + s)$ transform according to the rule

$$\widehat{T}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}(\widehat{x}) = \left[\frac{\partial \widehat{X}^{i_1}}{\partial x^{k_1}}\right] \cdots \left[\frac{\partial \widehat{X}^{i_r}}{\partial x^{k_r}}\right] \cdot \left[\frac{\partial X^{l_1}}{\partial \widehat{x}^{j_1}}\right] \cdots \left[\frac{\partial X^{l_s}}{\partial \widehat{x}^{j_s}}\right] T^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s}(x). \qquad (1.10)$$

Very often, since tensors are referred to by their components, the tensor character of an object is proved by verifying this transformation law. In fact, this transformation rule is sometimes given as the definition of a tensor.

1.5 The Metric Tensor

This discussion to this point has been based on a differentiable manifold $(\mathcal{M}, \mathcal{A})$ and the vector spaces $\mathcal{T}_p(\mathcal{M})$ included in the tangent vector bundle. No inner product or vector norm has yet been assigned to these spaces. A *metric tensor field* g_{oo} defines an inner product on each tangent space which determines the geometric properties of $\mathcal{M}$.

Definition 1.15. *Let g_{oo} be a tensor field of rank (0+2) on $\mathcal{M}$ with components $g_{ab}(x)$ relative to some chart (χ, U). The tensor field g_{oo} is a **metric tensor field** if the*

mapping g_{oo} satisfies the following criteria at each point $p = \chi^{-1}(x) \in \mathcal{M}$ for any vectors $\vec{t}_1, \vec{t}_2, \vec{s}_1 \in \mathcal{T}_p(\mathcal{M})$ and any scalars $\alpha, \beta \in \mathbb{R}$:

1. $[g_{oo}(x)](\vec{t}_1, \vec{t}_2) \in \mathbb{R}$.

2. $[g_{oo}(x)](\alpha\vec{t}_1 + \beta\vec{s}_1, \vec{t}_2) = \alpha[g_{oo}(x)](\vec{t}_1, \vec{t}_2) + \beta[g_{oo}(x)](\vec{s}_1, \vec{t}_2)$.

3. $[g_{oo}](x)(\vec{t}_1, \vec{t}_2) = [g_{oo}(x)](\vec{t}_2, \vec{t}_1)$.

4. $[\forall \vec{t} \in \mathcal{T}_p(\mathcal{M}), \ [g_{oo}(x)](\vec{t}, \vec{a}) = \vec{0}] \Leftrightarrow \vec{a} = \vec{0}$.

(These requirements imply that g_{sssoo} defines an inner product on each tangent vector space $\mathcal{T}_p(\mathcal{M})$.)

The first two criteria simply assert that $g_{oo}(x)$ is a tensor, ie. that $g_{oo}(x)$ is scalar-valued and bilinear. The next criterion asserts that $g_{oo}(x)$ is symmetric, so the components $g_{ab}(x)$ satisfy $g_{ab}(x) \equiv g_{ba}(x)$. The last requirement is known as the *axiom of non-degeneracy* (see [8]); it follows from this that $\det[g_{ab}(x)] \neq 0$ over the domain of g_{oo}, where $[g_{ab}(x)]$ is the $n \times n$ matrix formed by all the components of g_{oo} [11].

The non-degeneracy of the metric tensor has two important consequences. Firstly, the matrix inverse $[g_{ab}(x)]^{-1}$ exists. This allows the definition of a contravariant from of the metric tensor field.

Definition 1.16. *The **contravariant metric tensor field** g^{oo} is the tensor field of rank $(2+0)$ that has components $g^{ab}(x)$ defined by*

$$g^{ab}(x)g_{bc}(x) \equiv \delta^a_c.$$

That is, the components of the contravariant metric tensor are given by the elements of the matrix inverse $[g^{ab}(x)] := [g_{ab}(x)]^{-1}$ of the matrix $[g_{ab}(x)]$.

A second consequence of the axiom of non-degeneracy is that none of the eigenvalues of $[g_{ab}(x)]$ are zero. This makes it possible to enumerate the eigenvalues of $[g_{ab}(x)]$ in the form

$$\epsilon_1 g_1(x), \epsilon_2 g_2(x), \ldots, \epsilon_n g_n(x), \tag{1.11}$$

where $g_j(x) > 0$ and the ϵ_j are indicators of the signs of the eigenvalues, ie. $\epsilon_j = +1$ or $\epsilon_j = -1$. Thus, the metric tensor can be characterised by its eigenvalues [11].

Definition 1.17. *Let* $\mathcal{M}$ *be endowed with a metric tensor field whose components are* $g_{ab}(x)$ *relative to some chart* (χ, U). *The* **signature** *of the metric tensor is the sum*

$$\sum_{j=1}^{n} \epsilon_j,$$

where ϵ_j *are the indicators of the eigenvalues as given in* (1.11). *If the signature of the metric is* n *(the dimension of the manifold), the metric is said to be* **Riemannian**; *otherwise, it is* **pseudo-Riemannian**.

The signature of the metric tensor field is given by the sum of the number of positive eigenvalues of $[g_{ab}(x)]$ minus the number of negative eigenvalues of $[g_{ab}(x)]$. Although the definition of signature refers to the signature of a metric tensor at some specific point $p \in \mathcal{M}$, it is common to refer to the signature of the manifold $\mathcal{M}$ with the implicit assumption that $\mathcal{M}$ is endowed with a metric tensor field of fixed signature.

The true importance of the metric tensor field lies in its characterisation of (contravariant) vectors.

Definition 1.18. *Let* g_{ab} *be the components of a metric tensor relative to some fixed basis* $\{\vec{e}_k\}_{k=1}^{n}$ *at some point* $p \in \mathcal{M}$. *Let* $\vec{v} = v^a \vec{e}_a \in \mathcal{T}_p(\mathcal{M})$ *be a nonzero contravariant vector. Consider the quantity*

$$\Phi := g_{oo}(\vec{v}, \vec{v}) = g_{ab} v^a v^b.$$

The vector $\vec{v}$ *is said to be* **space-like** *if* $\Phi > 0$, **time-like** *if* $\Phi < 0$ *or* **null** *if* $\Phi = 0$. *A vector field is space-like, time-like or null if it is space-like, time-like or null respectively at each point of its domain. Similarly, a curve is space-like, time-like or null according to the space-like, time-like or null chararcterisation of its tangent vectors.*

If the metric is Riemannian at a point, the inner product is positive definite; that is, every nonzero vector is space-like. For a metric of pseudo-Riemannian signature, nonzero time-like and null vectors exist. This makes the name "metric" a bit misleading. It seems plausible to think of $\mathcal{M}$ as a metric space with a distance function related in some way to the metric tensor field. While this is possible with Riemannian

metrics, the notion of distance between points in $\mathcal{M}$ is not as useful as the notion of *separation* between points in $\mathcal{M}$.

Definition 1.19. *Let* $\gamma : [a, b] \subset \mathbb{R} \to \mathcal{M}$ *be a non-degenerate, smooth curve with endpoints* $p = \gamma(a)$ *and* $q = \gamma(b)$ *in* $\mathcal{M}$. *Let* (χ, U) *be a chart covering the image of* γ *and let the mappings* $\mathcal{X} := \chi \circ \gamma$ *and* $\mathcal{X}^a := \pi^a \circ \chi \circ \gamma$ *be the coordinate mappings induced by the curve* γ *and the chart. Then, the **separation** of* p *and* q *along the curve* γ *is defined to be*

$$\sigma(\gamma; p, q) := \int_a^b \left| g_{ab}(\mathcal{X}(t)) \frac{d\mathcal{X}^a}{dt}(t) \frac{d\mathcal{X}^b}{dt}(t) \right|^{\frac{1}{2}} dt \geq 0.$$

To make $\mathcal{M}$ a metric space, it is necessary to have a Riemannian metric (see [1]). Define

$$d(p, q) := \inf\{\sigma(\gamma; p, q) : \gamma \text{ is any smooth curve with endpoints } p, q\}.$$

With a pseudo-Riemannian metric, the metric tensor has negative eigenvalues and so the inner product determined is indefinite. As such, the separation between distinct points along a null curve is zero which obviously means that $(\mathcal{M}, d)$ cannot be a metric space.

The inner product given by the metric tensor induces an isomorphism between the tangent space $\mathcal{T}_p(\mathcal{M})$ and its associated dual space $\widetilde{\mathcal{T}}_p(\mathcal{M})$. Associated with a covariant (dual) vector field $\widetilde{\mathbf{v}} = v_k(x)\widetilde{\mathbf{e}}^k(x)$, there is a contravariant vector field $\vec{\mathbf{v}} = v^j(x)\vec{\mathbf{e}}_j(x)$ where

$$v^j(x) = g^{jk}(x)v_k(x) \text{ and } v_k(x) = g_{kl}(x)v^l(x). \tag{1.12}$$

The contravariant vector field v^j results from *raising* the covariant index in the covariant vector field v_k and the covariant vector field v_k results from *lowering* the contravariant index in v^l. Similarly, the covariant metric tensor and its contravariant counterpart are used to raise and lower indices in arbitrary tensor fields. For lowering the pth contravariant index ($1 \leq p \leq r$) or raising the qth covariant index ($1 \leq q \leq s$) of the components of a tensor field $T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}$ of rank $(r + s)$ (which yields a tensor

of rank $((r - 1) + (s + 1))$ or $((r + 1) + (s - 1))$ respectively), the rules are:

$$T^{i_1 \cdots i_{p-1}}{}_k{}^{i_{p+1} \cdots i_r}{}_{j_1 \cdots j_s}(x) = g_{k i_p}(x) T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}(x), \tag{1.13a}$$

$$T^{i_1 \cdots i_r}{}_{j_1 \cdots j_{q-1}}{}^k{}_{j_{q+1} \cdots j_s}(x) = g^{k j_q}(x) T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}(x). \tag{1.13b}$$

Thus, for a manifold endowed with a metric, the covariant or contravariant rank of a tensor can be changed at will through successive raising and lowering operations.

Due to the connection of the metric tensor to distance in Riemannian spaces, it is common to see the components of g_{oo} relative to some holonomic basis given as a *line element*:

$$ds^2 = g_{ab}(x) dx^a dx^b.$$

The quantity ds^2 is meant to denote the square of the infinitesimal displacement between two points with coordinates x and $x + dx$ respectively in a Riemannian manifold. It would be more accurate to write $g_{oo} = g_{ab}(x) dx^a \otimes dx^b$ to express all the components of the metric tensor. However, to save space, it is more convenient to use a line element.

1.6   The Covariant Derivative

Before proceeding further, two definitions are useful.

Definition 1.20. *The **Christoffel symbols of the first kind** associated with a given metric tensor g_{ab} are defined by*

$$[ij, k] := \frac{1}{2}(\partial_i g_{jk}(x) + \partial_j g_{ki}(x) - \partial_k g_{ij}(x)). \tag{1.14}$$

*The **Christoffel symbols of the second kind** (or **connection coefficients**) are defined by*

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} := g^{im}(x)[jk, m]. \tag{1.15}$$

From the definitions, it is clear that

$$[ij,k] = [ji,k] \text{ and } \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \left\{ \begin{matrix} i \\ kj \end{matrix} \right\}. \tag{1.16}$$

The Christoffel symbols of the second kind are related to the symbols of the first kind by raising one of the indices. However, these are not components of a tensor as can be verified by looking at the relation between the Christoffel symbols of the second kind in different coordinates [25]:

$$\widehat{\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}} = \frac{\partial \widehat{X}^i}{\partial x^a} \frac{\partial X^b}{\partial \widehat{x}^j} \frac{\partial X^c}{\partial \widehat{x}^k} \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} + \frac{\partial \widehat{X}^i}{\partial x^m} \frac{\partial^2 X^m}{\partial \widehat{x}^j \widehat{x}^k}. \tag{1.17}$$

There is an additional term involving the second order derivatives of the transformation between coordinates.

Given a scalar field $\Phi : \mathcal{M} \to \mathbb{R}$ and two coordinate charts (χ, U) and $(\widehat{\chi}, \widehat{U})$ that overlap, in the region of intersection $U \cap \widehat{U}$, define $\widehat{\phi} := \Phi \circ \chi^{-1}$ and $\phi := \Phi \circ \widehat{\chi}^{-1}$. Then, $\Phi(p) \equiv \phi(x) \equiv \widehat{\phi}(\widehat{x})$ where $p \in U \cap \widehat{U}$, $x \in \chi[U \cap \widehat{U}]$ and $\widehat{x} \in \widehat{\chi}[U \cap \widehat{U}]$. Define $v_a(x) := (\partial_a phi)(x)$ as the partial derivatives of the scalar field in the chart (χ, U). Then, the v_a are the components of a covariant vector field, since they transform by the rule

$$\widehat{v}_b(\widehat{x}) = \frac{\partial \widehat{\phi}}{\partial \widehat{x}^b}(\widehat{x}) = \frac{\partial X^a}{\partial \widehat{x}^b}(\widehat{x}) \frac{\partial \phi}{\partial x^a}(x) = \frac{\partial X^a}{\partial \widehat{x}^b}(\widehat{x}) v_a(x).$$

Define $v_{ab}(x) := (\partial_a \partial_b \phi)(x)$. With the second order partial derivatives of ϕ, there is a different transformation under a change of charts:

$$\begin{aligned} \widehat{v}_{mn}(\widehat{x}) &= \frac{\partial^2 \widehat{\phi}}{\partial \widehat{x}^m \widehat{x}^n}(\widehat{x}) = \frac{\partial}{\partial \widehat{x}^m} \left(\frac{\partial X^b}{\partial \widehat{x}^n}(\widehat{x}) \frac{\partial \phi}{\partial x^b}(x) \right) \\ &= \frac{\partial X^a}{\partial \widehat{x}^m} \frac{\partial X^b}{\partial \widehat{x}^n} v_{ab}(x) + \frac{\partial^2 X^a}{\partial \widehat{x}^m \partial \widehat{x}^n}(\widehat{x}) v_a(x). \end{aligned}$$

Repeating this argument with tensor fields of higher rank, it becomes clear that differentiating the components of tensors relative to some coordinate chart generally yields a set of quantities that are not the components of a tensor (consider the Christoffel

symbols, for example). As the methods of multivariate calculus are desired to formulate differential equations and provide other reasonable tools for the description of space-time, some kind of derivative operator is desired that operates on tensors and produces tensors. Since ordinary partial derivatives of components vector fields relative to some coordinate basis do not transform as tensors do, a covariant form of the derivative operator is desired.

Definition 1.21. *Let* $T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}$ *be a tensor field of rank* $(r + s)$ *over some region in* $U \subset \mathcal{M}$ *covered by the chart* (χ, U). *The* **covariant derivative** *of* $T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}$ *is the tensor field* $\nabla_a T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}$ *with components defined by*

$$\nabla_a T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}(x) := \partial_a T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}(x)$$

$$+ \begin{Bmatrix} i_1 \\ ak \end{Bmatrix} T^{k i_2 \cdots i_r}{}_{j_1 \ldots j_s}(x) + \begin{Bmatrix} i_2 \\ ak \end{Bmatrix} T^{i_1 k \cdots i_r}{}_{j_1 \ldots j_s}(x) +$$

$$\cdots + \begin{Bmatrix} i_r \\ ak \end{Bmatrix} T^{i_1 i_2 \cdots k}{}_{j_1 \ldots j_s}(x) - \begin{Bmatrix} m \\ aj_1 \end{Bmatrix} T^{i_1 \cdots i_r}{}_{m j_2 \ldots j_s}(x)$$

$$- \begin{Bmatrix} m \\ aj_2 \end{Bmatrix} T^{i_1 \cdots i_r}{}_{j_1 m \ldots j_s}(x) - \cdots - \begin{Bmatrix} m \\ aj_s \end{Bmatrix} T^{i_1 \cdots i_r}{}_{j_1 j_2 \ldots m}(x).$$

Let $\gamma : I \subset \mathbb{R} \to \mathcal{M}$ *be a curve whose range is covered by the chart* (χ, U). *The* **absolute derivative** *of the tensor field* $T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}(x)$ *along* γ *is the tensor field of rank* $(r + (s + 1))$ *defined along* $\gamma[I]$ *by*

$$\frac{D}{Du} T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}(x) \Big|_{x = \chi(u)} := [\nabla_a T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}(x)]\Big|_{x = \chi(u)} \frac{dX^a}{du}(u).$$

The occurrence of second order partial derivatives in the transformation (1.17) of the Christoffel symbols makes the covariant derivative transform as a tensor. Although the above definitions seem complex, it is easiest to remember that each term in the covariant derivative is added if it differs from $T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}$ in a contravariant index and subtracted if it differs from $T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}$ in a covariant index. In such terms, the displaced index is put into a Christoffel symbol of the second kind and replaced by a dummy index of summation.

For a scalar field ϕ, the *gradient* of ϕ is the contravariant vector $\nabla^a \phi := g^{ab} \nabla_b \phi \equiv g^{ab} \partial_b \phi$; in covariant form, the gradient is $\nabla_a \phi = \partial_b \phi$. The covariant derivative also

satisfies the Leibnitz product rule of ordinary calculus:

$$\nabla_a(T^{i_1\ldots i_{r_1}}{}_{j_1\ldots j_{s_1}} S^{k_1\ldots k_{r_2}}{}_{l_1\ldots l_{s_2}}) = S^{k_1\ldots k_{r_2}}{}_{l_1\ldots l_{s_2}} \nabla_a(T^{i_1\ldots i_{r_1}}{}_{j_1\ldots j_{s_1}})$$
$$+T^{i_1\ldots i_{r_1}}{}_{j_1\ldots j_{s_1}} \nabla_a(S^{k_1\ldots k_{r_2}}{}_{l_1\ldots l_{s_2}}).$$

As a consequence, the absolute derivative also has this Leibnitzian property.

It is possible to define a structure called a linear connection on $\mathcal{M}$ that produces the Christoffel symbols and the covariant derivative (see [26, 13]). In this thesis, a metric-induced connection is considered, so the Christoffel symbols are defined from the metric tensor. As a result (see [25]), the covariant derivative of the metric tensor vanishes:

$$\nabla_a g_{bc} \equiv 0. \tag{1.18}$$

Combining this property with the Leibnitzian property and the rules for raising and lowering indices, the metric tensor components can be treated as constants when taking covariant derivatives. Thus, for instance,

$$\nabla_a T^{ij} = \nabla_a(g^{ik}g^{jl}T_{jl}) = g^{ik}g^{jl}\nabla_a T_{kl}$$

for any tensor field T^{ij}.

1.7 Geodesics

Geodesics in Riemannian spaces are curves of stationary length. They represent the paths of shortest arc length connecting distinct points. For a curved space embedded in a space of higher dimension, they can be fundamentally different than the geodesics of the larger space. For instance, in $\mathbb{E}^3$, the geodesics are straight lines. However, on the manifold S^2 which is the surface of a sphere embedded in $\mathbb{E}^3$, the geodesics are great circles. For a pseudo-Riemannian space, curves can be time-like or null, so an interpretation related to separation rather than arc length is needed.

Towards finding a suitable analogy, let $p, q \in \mathcal{M}$ be two distinct fixed points. Consider the class of curves

$$\mathcal{C}_0^1(I \subset \mathbb{R}; \mathcal{M}) := \{\gamma : \gamma \in \mathcal{C}^1(I; \mathcal{M}) \text{ for some interval } I = [a, b] \subset \mathbb{R}$$
$$\text{and } \gamma(a) = p, \ \gamma(b) = q\}$$

consisting of every parametrised curve γ joining p and q. For each $\gamma \in \mathcal{C}_0^1(I \subset \mathbb{R}; \mathcal{M})$, the functional

$$\mathcal{L}[\gamma] := \int_a^b \left| g_{ij}(\mathcal{X}(u)) \frac{d\mathcal{X}^i}{du}(u) \frac{d\mathcal{X}^j}{du}(u) \right|^{\frac{1}{2}} du,$$

has some definite value. This functional in some sense describes the separation of the two points; in fact, for a Riemannian metric, it gives the arc length along the curve γ that extends between p and q. Suppose γ is a curve for which $\mathcal{L}[\gamma]$ is a stationary value of the functional $\mathcal{L}$, then, as a consequence of the Euler-Lagrange equations (see [11]), it is possible to select a special parameter u so that the components $d\mathcal{X}^a/du$ of the tangent vectors along the curve necessarily obey the following differential equations:

$$\frac{d}{du}\left(g_{ab}(\mathcal{X}(u)) \frac{d\mathcal{X}^b}{du}(u) \right) = \frac{1}{2}\partial_a g_{ij}(\mathcal{X}(u)) \frac{d\mathcal{X}^i}{du}(u) \frac{d\mathcal{X}^j}{du}(u).$$

These equations can be more concisely written using the Christoffel symbols of the first kind:

$$g_{ac}(x)\Big|_{x=\mathcal{X}(u)} \frac{d^2\mathcal{X}^a}{du^2} + [ab, c]\Big|_{x=\mathcal{X}(u)} \frac{d\mathcal{X}^a}{du}\frac{d\mathcal{X}^b}{du} = 0.$$

Finally, multiplying by g^{ca} and contracting, a second order coupled system of non-linear ordinary equations results:

$$\frac{d^2\mathcal{X}^m}{du^2}(u) + \left\{ \begin{matrix} m \\ ab \end{matrix} \right\}\Bigg|_{x=\mathcal{X}(u)} \frac{d\mathcal{X}^a}{du}(u)\frac{d\mathcal{X}^b}{du}(u) = 0. \tag{1.19}$$

Definition 1.22. *Let γ be a curve whose image is covered by some chart (χ, U). Let $x^a = \mathcal{X}^a(u)$ be the components of $\gamma(u)$ under this chart. Then, the curve γ is a* **geodesic curve** *if the components $\mathcal{X}^a$ satisfy the equations (1.19). The equations (1.19) are the* **geodesic equations** *associated with the metric tensor.*

The equations (1.19) are tensorial since they can be expressed in terms of the absolute derivative:

$$\left(\frac{D}{Du}\frac{d\mathcal{X}^a}{du} \right)\Bigg|_{x=\mathcal{X}(u)} = 0. \tag{1.20}$$

Changing the parameter u generally changes the system (1.19); the right hand side of (1.19) will be non-homogeneous if the parameter u is substituted for some general parameter $s = \mathcal{S}(u)$. However, if $\mathcal{S}(u) = au + b$ for some real constants a, b, then the equations (1.19) remains homogeneous. Thus, the geodesic equations define a family of curves and a class of *affine parameters* related by linear transformations that are associated with each curve.

Along a geodesic curve, $g_{mn}(d\mathcal{X}^m/du)(d\mathcal{X}^n/du)$ is a constant (see [25]). By a permissible change of the affine parameter along the curve, this becomes

$$g_{mn}(x)\Big|_{x=\mathcal{X}(u)} \frac{d\mathcal{X}^m}{du}(u) \frac{d\mathcal{X}^n}{du}(u) = \epsilon, \qquad (1.21)$$

where $\epsilon = 0, \pm 1$ depending on the space-like, time-like or null character of the geodesic curve. In general, when attempting to solve the system (1.19), the additional constraint (1.21) is helpful. Results from the theory of o.d.e's guarantee the existence of a unique solution to the geodesic equations given suitable initial or boundary conditions provided the Christoffel symbols satisfy suitable continuity requirements.

1.8   The Tensors of Riemann, Ricci and Einstein

For a scalar field $\phi : D \subset \mathbb{R}^n \to \mathbb{R}$ that has continuous partial derivatives up to and including order two, the second order mixed partial derivatives $\partial_i \partial_j \phi$ and $\partial_j \partial_i \phi$ are identical. Having constructed the covariant derivative to extend the partial derivative operator in a tensorial manner, it is worthwhile to consider the commutator $\nabla_k \nabla_s - \nabla_s \nabla_k$ of the covariant derivative operator. A quick calculation shows that, for a scalar field $f \in \mathcal{C}^2(\mathcal{M}; \mathbb{R})$, $(\nabla_k \nabla_s - \nabla_s \nabla_k)f \equiv 0$ due to the symmetries of the Christoffel symbols and the fact $(\partial_k \partial_s - \partial_s \partial_k)f \equiv 0$. For a covariant vector field V_r,

$$(\nabla_k \nabla_s - \nabla_s \nabla_k)V_r(x) = \left[\partial_s \begin{Bmatrix} i \\ kr \end{Bmatrix} - \partial_k \begin{Bmatrix} i \\ sr \end{Bmatrix} + \begin{Bmatrix} j \\ kr \end{Bmatrix} \begin{Bmatrix} i \\ sj \end{Bmatrix} - \begin{Bmatrix} j \\ sr \end{Bmatrix} \begin{Bmatrix} i \\ kj \end{Bmatrix} \right] V_i(x).$$

$$(1.22)$$

The coefficient of V_i on the right-hand side is a tensor of rank (1+3). This motivates the following definition.

Definition 1.23. *The **Riemann curvature tensor** is a tensor field $R^i{}_{rsk}$ of rank $(1 + 3)$ defined by*

$$R^i{}_{rsk} := \partial_s \left\{ {i \atop kr} \right\} - \partial_k \left\{ {i \atop sr} \right\} + \left\{ {j \atop kr} \right\} \left\{ {i \atop sj} \right\} - \left\{ {j \atop sr} \right\} \left\{ {i \atop kj} \right\}.$$

Using the definition of the Riemann tensor to evaluate the commutator of the covariant derivative of a covariant vector field in (1.22), $(\nabla_k \nabla_s - \nabla_s \nabla_k)V_r \equiv R^i{}_{rsk}V_i$. Thus, the commutator of the covariant derivative operator vanishes only where $R^i{}_{rsk} \equiv 0$ (such a region is said to be *flat*).

As the name indicates, the Riemann curvature tensor includes information about the curvature of the manifold under consideration. From the definition, it depends on the first and second order derivatives of the components of the metric tensor and hence is determined by intrinsic properties of the space. The covariant Riemann tensor R_{ijkl} provides insight into anti-symmetries, symmetries and other properties of the Riemann tensor.

Proposition 1.1. *Let $R_{ijkl}(x) = g_{ip}(x)R^p{}_{jkl}(x)$. Then the following identities hold:*

1. $R_{ijkl} = \frac{1}{2}(\partial_j \partial_k g_{il} + \partial_i \partial_l g_{jk} - \partial_j \partial_l g_{ik} + \partial_i \partial_k g_{jl})$

   $+ g^{mn}([il, m][jk, n] - [ik, m][jl, n]).$

   (1.23a)

2. $R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$ (1.23b)

3. $R_{ijkl} + R_{iklj} + R_{iljk} \equiv 0.$ (1.23c)

4. $\nabla_m R_{ijkl} + \nabla_l R_{ijmk} + \nabla_k R_{ijlm} \equiv 0.$ (1.23d)

The proofs of the above are straightforward computations (see [25]). The equations (1.23d) are known as the *Bianchi identities*. For $n \geq 4$, the symmetries and anti-symmetries of the Riemann tensor reduce the number of independent components of the Riemann tensor from n^4 to $n^2(n^2 - 1)/12$.

Various contractions of the Riemann tensor yield tensors essential for relativity.

Definition 1.24. *The **Ricci tensor** R_{ij} , the **curvature invariant** R and the **Einstein tensor** G_{ij} are defined as follows:*

$$1. R_{ij} := R^k{}_{ijk} = g^{kl} R_{lijk}, \tag{1.24}$$

$$2. R := R^i{}_i = g^{ij} R_{ij}, \tag{1.25}$$

$$3. G_{ij} := R_{ij} - \frac{1}{2} R g_{ij}. \tag{1.26}$$

From the above definitions, the Ricci and Einstein tensors are both symmetric covariant tensor fields of rank (0+2). Further, the Bianchi identities (1.23d) imply the first and second contracted Bianchi identities:

$$\nabla_k R_{ij} - \nabla_j R_{ik} + \nabla_l R^l{}_{ikj} \equiv 0, \tag{1.27a}$$

$$\nabla_j G^{ij} = \nabla_j (R^{ij} - \frac{1}{2} g^{ij}) \equiv 0. \tag{1.27b}$$

1.9 The Tetrad Formalism

Most of the tools for developing the theory of relativity are in place. With the ideas from differential geometry and the tensors needed to describe curvature, the tetrad formalism provides a useful way of representing tensors. Thorough treatments can be found in [15, 23, 5]. The underlying manifold that models space-time is assumed to be four-dimensional with a metric of signature +2.

Definition 1.25. *Let $\mathcal{M}$ be a differentiable manifold of dimension 4 with a metric of signature +2. A set of four linearly independent contravariant vector fields $\{e_{(a)}\}_{a=1}^4$ on $\mathcal{M}$ is an **orthonormal tetrad** if, relative to any coordinate chart, the components $e_{(a)}{}^i$ of the vectors $e_{(a)}$ satisfy the equations*

$$g_{ij} e_{(a)}{}^i e_{(b)}{}^j = \eta_{(a)(b)}$$

*where $\eta_{(a)(b)} \equiv \eta^{(a)(b)} := \mathrm{diag}(1,1,1-1)$. The bracketed indices are **tetrad indices** which are raised and lowered as follows:*

$$e^{(a)i} = \eta^{(a)(b)} e_{(b)}{}^i; \quad e_{(a)j} = \eta_{(a)(b)} e^{(b)}{}_j.$$

This definition is not the most general; sometimes, complex null tetrads with a different choice of $\eta_{(a)(b)}$ is useful (see [15, 23]), but for present purposes, $\eta_{(a)(b)} = \text{diag}(1, 1, 1 - 1)$ is used.

Notice that $e_{(a)j} = e_{(a)}{}^i g_{ij}$ are the components of four covariant vector fields obtained from the contravariant vectors $e_{(a)}{}^i$. A bit of manipulation reveals

$$e^{(a)}{}_r e_{(a)_s} = g_{rs}, \quad e_{(a)}{}^r e^{(a)s} = g^{rs}, \quad e_{(a)}{}^r e^{(a)}{}_s = \delta^r{}_s, \quad \text{and} \quad e^{(a)}{}_r e_{(b)}{}^r = \delta^{(a)}{}_{(b)}.$$

For an orthonormal tetrad consisting of three space-like vectors $e^{(1)}{}_i$, $e^{(2)}{}_j$, $e^{(3)}{}_k$ and one time-like vector $e^{(4)}{}_l$, the definitions above imply that

$$g_{rs} = \eta_{(a)(b)} e^{(a)}{}_r e^{(b)}{}_s = e^{(1)}{}_r e^{(1)}{}_s + e^{(2)}{}_r e^{(2)}{}_s + e^{(3)}{}_r e^{(3)}{}_s - e^{(4)}{}_r e^{(4)}{}_s. \tag{1.28}$$

This provides a decomposition of g_{rs} in terms of four linearly independent vector fields.

Having introduced the rules for using the matrices $\eta_{(a)(b)}$ and $\eta^{(a)(b)}$ to obtain the fields $e^{(a)i}$ from $e_{(c)}{}^i$, extend the use of tetrads to describe general tensors other than the metric tensor.

Definition 1.26. *Let $T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s}$ be the components of some tensor field. The **tetrad components** $T^{(a_1) \ldots (a_r)}{}_{(b_1) \ldots (b_s)}$ are given by*

$$T^{(a_1) \ldots (a_r)}{}_{(b_1) \ldots (b_s)} := T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s} e^{(a_1)}{}_{i_1} \cdots e^{(a_r)}{}_{i_r} e_{(b_1)}{}^{j_1} \cdots e_{(b_s)}{}^{j_s}.$$

It follows from above that

$$T^{i_1 \cdots i_r}{}_{j_1 \ldots j_s} = T^{(a_1) \ldots (a_r)}{}_{(b_1) \ldots (b_s)} e_{(a_1)}{}^{i_1} \cdots e_{(a_r)}{}^{i_r} e^{(b_1)}{}_{j_1} \cdots e^{(b_s)}{}_{j_s}.$$

The tetrad indices of a tensor are raised and lowered with $\eta^{(a)(b)}$ and $\eta_{(a)(b)}$ in the same way tensor indices are raised and lowered with g^{ab} and g_{ab}.

The main advantage of using tetrads is that the tetrad components of tensors transform as scalars. That is, an admissible change of coordinates does not change the numerical value of the tetrad components of a tensor. From a physical perspective, this makes the components of tensors *measurable* which is of utmost importance for experiments. Tetrad components of tensors also help identify and rule out possible

singular regions of a manifold because scalar invariants may reveal more about a region than the individual components of a tensor relative to some fixed chart. The invariance of tetrad components also provides a useful way to classify tensors according to their algebraic properties (see [5, 15]). Finally, the use of tetrads simplifies the computation of the components of tensors in a coordinate-independent manner. (In particular, the *Kretchmann scalar* $R^{ijkl}R_{ijkl} \equiv R^{(i)(j)(k)(l)}R_{(i)(j)(k)(l)}$ is often easier to compute using the tetrad components of the Riemann tensor.)

1.10 The Stress-Energy-Momentum Tensor

With the tools from differential geometry assembled so far, consider some ideas from physics. Since, in Newtonian gravitation, gravitational potential fields are related to the mass-density in space, some version of this must be included in the new theory. In keeping with the ideas developed in this chapter, the new "mass-density" should be a tensor field. The tensor analogue of mass-density is the *stress-energy-momentum tensor* T_{ab}, a relativistic version of the tensor of a similar name from continuum mechanics. It embodies the mechanical properties of matter and acts as the source of gravitational fields. The properties of T_{ab} can be found using statistical arguments, integrating the flux of 4-momentum[2] of all the particles of matter through a 3-dimensional hypersurface (space-like or time-like) in space-time. The exact physical arguments are in [24], but it is sufficient to acknowledge the primary algebraic and differential identities that the tensor T_{ab} must satisfy. These are

$$T_{ab} \equiv T_{ba} \qquad \text{(symmetry of } T_{ab}\text{)}, \tag{1.29a}$$

$$\nabla_i T^{ij} \equiv 0 \qquad \text{(conservation equations)}. \tag{1.29b}$$

The identities (1.29b) are called conservation equations due to their relation to equations of conservation of energy, momentum, etc. that arise in classical mechanics.

To interpret T_{ab} physically, it is necessary to use coordinate-independent invariants. This is most easily achieved with the help of an orthonormal tetrad. A natural choice

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[2]The 4-momentum p^a of a particle of mass m moving along a world-line with unit 4-velocity u^a is given by $p^a := mu^a$.

of tetrads comes from solving the covariant eigenvalue problem

$$T_{ab}V_{(i)}{}^b = \lambda_{(i)}g_{ab}V_{(i)}{}^b, \tag{1.30}$$

where there is no sum over i in the right-hand side of the equation (1.30). The 4 contravariant vectors $V_{(i)}{}^b$ are eigenvectors of this problem and the corresponding real scalars $\lambda_{(i)}$ are the eigenvalues[3]. The eigenvectors are orthogonal, with three space-like eigenvectors $V_{(1)}{}^b$, $V_{(2)}{}^b$, $V_{(3)}{}^b$ and a time-like eigenvector $V_{(4)}{}^b$ which can be defined to be future-directed[4].

Armed with this orthonormal tetrad, it is possible to find the invariant tetrad components of T_{ab}. These are given by

$$T_{(a)(b)} := T_{ij}V_{(a)}{}^i V_{(b)}{}^j \tag{1.31}$$

and simplify matters considerably. These non-vanishing invariants are

$$\lambda_{(1)} := T_{(1)(1)}, \quad \lambda_{(2)} := T_{(2)(2)}, \quad \lambda_{(3)} := T_{(3)(3)}, \quad \lambda_{(4)} := T_{(4)(4)}. \tag{1.32}$$

The eigenvalues $\lambda_{(1)}$, $\lambda_{(2)}$ and $\lambda_{(3)}$ are the *principal stresses* which are referred to as pressures if they are positive, tensions if they are negative. Corresponding to the principal stresses are the space-like eigenvectors $V_{(1)}{}^a$, $V_{(2)}{}^a$ and $V_{(3)}{}^a$ which are the *principal directions of stress*. The eigenvalue $\lambda_{(4)}$ of the time-like eigenvector is called the *density* of the medium (this can be density of mass or energy, which are equivalent in the general theory of relativity). The time-like eigenvector $V_{(4)}{}^a$ is often denoted u^a and referred to as the 4-velocity of the medium.

If $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)} =: p > 0$, the medium is a *perfect fluid*. Defining $\mu := \lambda_{(4)} > 0$, for a perfect fluid, by (1.28), T_{ab} can be written

$$T_{ab} = (p + \mu)u_a u_b + pg_{ab}, \tag{1.33}$$

where p is the pressure inside the fluid and μ is the density. The equality of all three space-like eigenvalues indicates that the pressure is isotropic. In the Segré notation [15], the tensor T_{ab} is of type $[(1\,1\,1),1]$ for a perfect fluid.

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[3] The symmetry of T_{ab} guarantees the eigenvalues are real.

[4] This is possible provided T_{ab} is of Segré class A1 [15]; there are situations with null eigenvectors and complex eigenvalues (which are appropriate to study radiation or electromagnetic fields) but those are not be considered here.

Since T_{ab} describes the distribution of matter in space-time, choosing the components of T_{ab} arbitrarily do not generally result in physically acceptable space-times. To further constrain the possible choices of T_{ab}, the invariant characterisation of T_{ab} in terms of its eigenvalues is very useful. Hawking and Ellis [13] give two conditions and Wald [30] gives another that restrict choices for the stress-energy-momentum tensor:

1. *The weak energy condition:* $T_{ab}V^a V^b \geq 0$ for every time-like vector field V^a. (This is equivalent to $-\lambda_{(4)} \geq 0$ and $\lambda_{(\alpha)} - \lambda_{(4)} \geq 0$, where $\alpha = 1, 2, 3$ provided T_{ab} is of Segré class A1.)

2. *The strong energy condition:* $T_{ab}V^a V^b \geq -\frac{1}{2}g^{ab}T_{ab}$ for every time-like vector field V^a. (This is equivalent to $-\lambda_{(4)} \geq 0$ and $\lambda_{(1)} + \lambda_{(2)} + \lambda_{(3)} - \lambda_{(4)} \geq 0$ and $\lambda_{(\alpha)} - \lambda_{(4)} \geq 0$, where $\alpha = 1, 2, 3$ provided T_{ab} is of Segré class A1.)

3. *The dominant energy condition:* $T^{ab}V_a V_b \geq 0$ and $T^{ab}V_a$ is time-like or null for every time-like vector field V_a. (This is equivalent to $-\lambda_{(4)} \geq 0$ and $|\lambda_{(\alpha)}| \leq -\lambda_{(4)}$, where $\alpha = 1, 2, 3$ provided provided T_{ab} is of Segré class A1.)

These conditions can be simplified depending on the algebraic classification of T_{ab}. The weak energy condition insists that the energy density is positive everywhere while the dominant energy condition insists that the magnitude of the principal stresses does not exceed the energy density. These conditions are physically reasonable but there are examples of interesting space-times (see [27, 10]) involving matter that violates some of the energy conditions.

1.11 The Einstein Field Equations

The underlying model for the space-time of general relativity is a four-dimensional differentiable manifold $(\mathcal{M}, \mathcal{A})$ of class $\mathcal{C}_p^3$ with a metric of signature $+2$. This differentiable manifold is referred to as a *space-time* and points in it are *events*. This differs from the Newtonian model in which space is Riemannian and flat. Having established a manifold structure for the model of space-time, the tensor fields on $\mathcal{M}$ represent the physical quantities of interest in space-time.

Another necessary property is *time-orientability*.

Definition 1.27. *A differentiable manifold* $(\mathcal{M}, \mathcal{A})$ *with an indefinite metric* g_{oo} *is* **time-orientable** *if there exists a continuous vector field* $\vec{\mathbf{T}}$ *that is everywhere time-like.*

Once a manifold has such a vector field $\vec{\mathbf{T}}$, a classification of all time-like vectors at each point in $\mathcal{M}$ is established: given a time-like vector $\vec{\mathbf{V}} \in \mathcal{T}_p(\mathcal{M})$. Then, $\vec{\mathbf{V}}$ is *future-directed* if $[g_{oo}(p)](\vec{\mathbf{V}}, \vec{\mathbf{T}}(p)) < 0$ and is *past-directed* otherwise. It is this property that fixes the arrow of time and ensures material particles traveling along time-like world-lines have a fixed definition of the future as opposed to the past. This is obviously in accord with intuitive notions about time.

What is desired is a way to map the properties of "gravitational force" onto the properties of a pseudo-Riemannian differentiable manifold. In Newtonian gravitation, space is associated with the Euclidean manifold $\mathbb{E}^3$ which can be covered globally by a Cartesian chart $\chi : \mathbb{E}^3 \to \mathbb{R}^3$. Let ρ be a scalar field defined on $\chi[\mathbb{E}^3] \subset \mathbb{R}^3$ that is nonzero only in some open set $D \subset \chi[\mathbb{E}^3]$. If ρ is taken to be some mass density distribution, then there exists a gravitational potential V that satisfies Poisson's equation inside matter and Laplace's equation outside:

$$\nabla^2 V(x) = \begin{cases} -4\pi\rho(x) & x \in D, \\ 0 & x \notin D. \end{cases} \tag{1.34}$$

The solution to this problem is found using the appropriate Green's function and is given by the integral

$$V(x) = -\int_{\mathbb{R}^3} \frac{\rho(y) d^3 y}{\|x - y\|}.$$

One of the postulates of the post-Newtonian way of looking at gravity is the *geodesic hypothesis*: the world-line of a free particle in space-time is a geodesic curve. Thus, the perceived attraction between massive bodies is actually due to the curvature of the underlying manifold. Time-like geodesics correspond to the world-lines of material particles while null geodesics correspond to the world-lines of photons or

particles of light. (Space-like geodesics do not have obvious interpretations; theoretical particles known as *tachyons* follow space-like curves which, by necessity, travel faster than light.)

However, given a manifold with metric, the geodesic hypothesis alone does not capture all the aspects of post-Newtonian gravitation. In particular, some field equations similar to the Poisson equation (1.34) relating gravitational potential to mass density are needed. Clearly, these field equations should be tensorial. This reflects the fact that underlying laws of nature maintain the same basic form regardless of the coordinate system imposed to express them. The field equations (like others) should be partial differential equations in the functions to be determined. In the limit of low gravity, the field equations should reduce to the Poisson equation (1.34) above. The role of a gravitational potential in (1.34) should be assumed by tensors related to curvature, as the geodesic hypothesis asserts that the perceived gravitational potential is a consequence of curvature of space-time. Since the stress-energy-momentum tensor is the analogue of mass density, T_{ab} should act as the source of gravitation in this theory. Finally, in the case of a flat space with a Minkowskian metric, there is presumed to be no mass, so T^{ab} should vanish.

The best candidates (see [24, 23, 15]) for these field equations are the *Einstein field equations*[5]:

$$\mathcal{E}_{ab} := G_{ab} + \kappa T_{ab} = R_{ab} - \frac{1}{2} R g_{ab} + \kappa T_{ab} = 0. \tag{1.35}$$

The equations (1.35) relate the distribution of matter and energy in space-time to the curvature and geometric properties of space-time. Assuming that the stress-energy-momentum tensor is known explicitly, the fields equations are a coupled, semi-linear system of second order p.d.e's in the unknown metric tensor components. The symmetry of the Einstein, Ricci and metric tensors reduce the number of unknown functions g_{ab} from 16 to 10. The number of independent equations is not 10, however, due to the additional 4 differential identities $\nabla_a G^a{}_b \equiv 0$. Hence, an additional four conditions (known as *coordinate conditions*) are also permitted.

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[5] Geometrised units are chosen so that $\kappa = 8\pi$ and $c = G = 1$, where G is Newton's gravitational constant and c is the speed of light.

There is a certain ambiguity in the interpretation of the field equations; if approached using a specific coordinate system, the tensor T_{ab} must be specified before the metric tensor can be found. Thus, the dependence of T_{ab} on the coordinates is known before the specific geometric interpretations of the coordinates is determined. Conversely, the tensor g_{ab} can be specified which generates a specific T_{ab}. This may not result in a physically meaningful distribution of matter and energy, but it is a useful way of discovering solutions. Another common approach is a mixed method (see [24]) in which some constraints on various components of g_{ab} and T_{ab} are specified to assist in finding a solution. Most often, it is best to think of the field equations as a set of restrictions on the choices of the 20 unknown quantities g_{ab} and T_{ab}.

Finding solutions of the field equations is a nontrivial task. This stems in part due to the large number of unknowns and also to the nonlinear nature of the differential equations. Solutions are generally found by making assumptions of symmetry to reduce the number of unknown functions and the number of dependent variables.

An final consideration in solving the field equations is the question of *junction conditions*. Suppose there is some vacuum domain of space-time in which the T_{ab} vanishes and some reason where $T_{ab} \neq 0$. Let $S \subset \mathcal{M}$ be a hypersurface representing the boundary between matter and vacuum. In some coordinate chart, the set S can be described by $\Sigma = \chi[S]$ where $\Sigma = \{x \in \mathbb{R}^4 : f(x) = 0\}$. (Notice Σ divides the region into two disjoint connected parts, one where $f(x) > 0$ and another where $f(x) < 0$.) Some or all of the components T_{ab} are discontinuous across Σ. As a consequence of the Einstein field equations, the derivatives of 2nd or 3rd order of the metric tensor components may also be discontinuous across Σ. As a first junction condition, it is required that, relative to an admissible coordinate chart in a domain of space-time, the quantities g_{ab} and $\partial_c g_{ab}$ should be continuous functions across Σ in some admissible coordinate chart. The second junction condition relies on the covariant normal[6] $n_j(x) := (\nabla_j f)(x) = \partial_j f(x)$ to the hypersurface Σ. Explicitly, the

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[6]Usually, n_j is scaled so that it has unit "length" which is always possible provided $\nabla_j f$ is not a null vector.

second junction condition requires that

$$T^i{}_j n_i\big|_\Sigma = 0; \tag{1.36}$$

physically, this requires the continuity of the flux of 4-momentum across Σ. Other junction conditions do exist; for instance, the requirement of Israel [14] that the second fundamental form is continuous across Σ is another possible junction condition. The junction conditions presented above are due to Synge [24] and are adopted in this thesis.

Chapter 2

Spherical Symmetry in General Relativity

Generally speaking, the Einstein field equations (1.35) are an under-determined system since the number of independent equations exceeds the number of unknown functions. The process of finding exact solutions is simplified by making additional assumptions about the form of the metric tensor and symmetries of the underlying space-time. A common assumption is that of *spherical symmetry*. Vacuum solutions, in addition to the convenience of having a vanishing stress-energy-momentum tensor, are useful for astronomical calculations. This chapter discusses spherical symmetry in general terms and provides a complete description of the maximal spherically symmetric vacuum space-time in general relativity.

2.1 Spherical Symmetry

A detailed discussion of Lie groups, Lie algebras, infinitesimal translations and groups of motions is necessary to formulate a precise definition of a spherically symmetric pseudo-Riemannian manifold [11, 15]. For the present purposes, it is sufficient to consider the following definition:

Definition 2.1. *A differentiable manifold* $(\mathcal{M}, \mathcal{A})$ *is **spherically symmetric** if it can be covered by charts of the form* (χ, U) *in which the metric tensor components are given by the line element*

$$ds^2 = e^{\alpha(x^1, x^4)}(dx^1)^2 + e^{\beta(x^1, x^4)}((dx^2)^2 + \sin^2(x^2)(dx^3)^2) - e^{\gamma(x^1, x^4)}(dx^4)^2. \quad (2.1)$$

(The functions α, β *and* γ *are assumed to be* $\mathcal{C}^3_p$ *functions over their domains).*

The Axiom of Lorentzian signature [8] has been used to determine the signs of the metric tensor components in (2.1). In general, the coordinates x^2 and x^3 are restricted to lie in the ranges $2k\pi < x^2 < (2k+1)\pi$ and $(n-1)\pi < x^3 < (n+1)\pi$ for some $k, n \in \mathbb{Z}$. From the countably infinite possible domains of charts included in the atlas for $\mathcal{M}$, choose $k = n = 0$, so $0 < x^2 < \pi$ and $-\pi < x^3 < \pi$. Hence, in the chart (χ, U) that has the line element (2.1), x^1 is a space-like radial coordinate, x^2 and x^3 are space-like angular coordinates on a sphere and x^4 is a time-like coordinate. The metric tensor components g_{12}, g_{23}, g_{13}, g_{24} and g_{34} are identically zero indicating there is no preferred spatial direction nor any angular motion. (In general, g_{14} is not identically zero, but a transformation of coordinates can be found to put the metric in diagonal form.)

The components of the mixed Einstein tensor are

$$G^1{}_1 = e^{-\alpha} \left(-\frac{1}{4}(\partial_1\beta)^2 - \frac{1}{2}(\partial_1\beta)(\partial_1\gamma) \right) + e^{-\beta} \tag{2.2a}$$

$$+ e^{-\gamma} \left(\partial_4^2\beta + \frac{3}{4}(\partial_4\beta)^2 - \frac{1}{2}(\partial_4\beta)(\partial_4\gamma) \right),$$

$$G^2{}_2 = e^{-\alpha} \left(-\frac{1}{2}\partial_1^2\beta - \frac{1}{4}(\partial_1\beta)^2 - \frac{1}{2}\partial_1\partial_1\gamma - \frac{1}{4}(\partial_1\gamma)^2 - \frac{1}{4}(\partial_1\beta)(\partial_1\gamma) \right. \tag{2.2b}$$

$$+ \frac{1}{4}(\partial_1\alpha)(\partial_1\beta) + \frac{1}{4}(\partial_1\alpha)(\partial_1\gamma) \Big)$$

$$+ e^{-\gamma} \left(\frac{1}{2}\partial_4^2\beta + \frac{1}{4}(\partial_4\beta)^2 + \frac{1}{2}\partial_4^2\alpha + \frac{1}{4}(\partial_4\alpha)^2 + \frac{1}{4}(\partial_4\alpha)(\partial_4\beta) \right.$$

$$- \frac{1}{4}(\partial_4\beta)(\partial_4\gamma) - \frac{1}{4}(\partial_4\alpha)(\partial_4\gamma) \Big),$$

$$G^3{}_3 = G^2{}_2, \tag{2.2c}$$

$$G^1{}_4 = e^{-\alpha} \left(-\partial_1^2\beta - \frac{3}{4}(\partial_1\beta)^2 + \frac{1}{2}(\partial_1\alpha)(\partial_1\beta) \right) + e^{-\beta}$$

$$+ e^{-\gamma} \left(\frac{1}{4}(\partial_4\beta)^2 + \frac{1}{2}(\partial_4\alpha)(\partial_4\beta) \right), \tag{2.2d}$$

$$e^\alpha G^1{}_4 = -e^\gamma G^1{}_1 = \partial_1\partial_4\beta + \frac{1}{2}(\partial_1\beta)(\partial_4\beta) - \frac{1}{2}(\partial_4\alpha)(\partial_1\beta) - \frac{1}{2}(\partial_4\beta)(\partial_1\gamma). \tag{2.2e}$$

These are useful for formulating the Einstein field equations in specific instances. Notice that the mixed Einstein tensor components depend only on x^1 and x^4. Then the field equations imply that the mixed components of the stress-energy momentum tensor depend only on x^1 and x^4. Furthermore, the equivalence of the Einstein tensor components $G^2{}_2 = G^3{}_3$ implies $T^2{}_2 = T^3{}_3$. The equivalence $T^2{}_2 = T^3{}_3$ implies two of the principal stresses of the stress-energy momentum tensor are equal, so the stresses within the matter do not single out any angular direction.

Any hypersurface described by $\mathcal{D}_2 := \{x \in \mathcal{N}[U] : x^1 = c_1 = \text{constant}, \ x^4 = c_4 = \text{constant}\}$ induces a two-dimensional sub-manifold $\mathcal{M}_2$ metrically and topologically equivalent to S^2, the surface of a sphere in Euclidean space. The induced metric tensor on $\mathcal{M}_2$ is

$$ds^2_{|\mathcal{D}_2} = e^{\beta(c_1,c_4)}[(dx^2)^2 + \sin^2(x^2)(dx^3)^2].$$

Consider the quantity $A = 4\pi \exp(\beta(x^1, x^1))$ that is the surface area of such a 2-sphere. The radius of curvature is $\exp(\beta(x^1, x^1))$ and the covariant components of the gradient of A are $4\pi e^\beta \partial_a \beta$. The space-like or time-like character of $\partial_a(A)$ is given by

$$g^{ab}(\partial_a(A)\partial_b(A)) = 16\pi^2 e^{2\beta}(e^{-\alpha}(\partial_1 \beta)^2 - e^{-\gamma}(\partial_4 \beta)^2).$$

Then, the domain of space-time has a different characterisation depending on whether the surfaces $A = $ constant have space-like or time-like normal vectors [33].

Definition 2.2. *Let* $\mathcal{D} \equiv \chi[U] \subset \mathbb{R}^4$ *be the image of a spherically symmetric domain under a suitable chart with a metric tensor of the form (2.1). The domain* $\mathcal{D}$ *is an* **R-domain** *if* β *is space-like throughout* $\mathcal{D}$ *(ie.* $g^{ab}\nabla_a\beta\nabla_b\beta > 0$*). If* β *is time-like throughout* $\mathcal{D}$, *(ie.* $g^{ab}\nabla_a\beta\nabla_b\beta < 0$*) then* $\mathcal{D}$ *is a* **T-domain**. *This is equivalent to the requirement* $f(x^1, x^4) > 0$ *for an R-domain and* $f(x^1, x^4) < 0$ *for a T-domain, where* $f : \mathcal{D} \to \mathbb{R}$ *is defined by*

$$f(x^1, x^4) := [\partial_1\beta(x^1, x^4)]^2 e^{-\alpha(x^1, x^4)} - [\partial_4\beta(x^1, x^4)]^2 e^{-\gamma(x^1, x^4)}. \tag{2.3}$$

For a hypersurface along which the gradient of the surface area of the 2-spheres is a null vector, the points are not in an R-domain nor a T-domain. Far from matter concentrations where the metric is asymptotically flat, events lie in an R-domain. T-domains usually exist where concentrations of mass are very high. Within an R-domain, it is usually possible to find a coordinate transformation to *curvature coordinates* [24] which takes the line element into the form[1]

$$ds^2 = e^{\hat{\alpha}(r,t)}dr^2 + r^2 d\Omega^2 - e^{\hat{\gamma}(r,t)}dt^2.$$

For a T-domain, a similar transformation can be found to *T-coordinates* that gives the line element[2]

$$ds^2 = e^{\tilde{\alpha}(R,T)}dR^2 + T^2 d\Omega^2 - e^{\tilde{\gamma}(R,T)}dT^2.$$

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[1]Hats are used to indicate that the functions $\hat{\alpha}$ and $\hat{\gamma}$ are different from the functions α and γ.

[2]Tildes are used to indicate that the functions $\tilde{\alpha}$ and $\tilde{\gamma}$ are different from the functions α and γ.

2.2   The Schwarzschild Solution

The Schwarzschild solution is the most well known exact solution of the Einstein field equations (1.35). It describes the gravitational field in the vacuum outside an isolated spherically symmetric static body. Provided a star is close to spherical, this metric is used to model space-time in the vacuum outside. The original major experiments to verify Einstein's theory of gravitation—measuring the perihelion precession of Mercury and the bending of light due to the Sun's gravity among them—rely on the assumption that gravity in the solar system is reasonably modeled by Schwarzschild's vacuum solution. It is also the first known example of a black hole coming out of Einstein's equations. Detailed descriptions of these experimental results and details of the the geodesics in Schwarzschild space-time can be found in [5, 23, 19].

The line element for the Schwarzschild metric is

$$ds^2 = \left(1 - \frac{2m}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta\phi^2) - \left(1 - \frac{2m}{r}\right)^{-1} dt^2, \qquad (2.4)$$

where m is a constant. This is obviously the same form as (2.1) with the identification $r \equiv x^1$, $\theta \equiv x^2$, $\phi \equiv x^3$ and $t \equiv x^4$ in (2.1), and defining $\alpha(r,t) := -\ln(1 - 2m/r)$, $\beta(r,t) := 2\ln r$ and $\gamma(r,t) := \ln(1 - 2m/r)$. The constant m is the *Schwarzschild mass* of the gravitating body because, in the Newtonian limit, this constant is proportional to the mass of the gravitating body (see [23]).

This is a vacuum solution derived from the assumption of spherical symmetry and the field equations $G_{ab} = 0$. The solution is also static in that the metric tensor components and any derived quantities are all independent of the time-like coordinate t. A result known as Birkhoff's theorem (see the appendix of [13]) guarantees that any $\mathcal{C}^2$ vacuum spherically symmetric solution of the Einstein field equations is part of the maximally extended Schwarzschild solution. In studies of static solutions of the Einstein field equations, the domains of matter are assumed to have a boundary $r = r_b > 2m > 0$ in order to be joined successfully with the Schwarzschild metric outside matter. The metric is also *asymptotically flat* which (in this context) means, in the limit as $r \to \infty$, the metric tensor components go over that of a Minkowski chart, namely $\eta_{ab} = \text{diag}(1, 1, 1, -1)$.

The gravitational radius $r = 2m > 0$ lies at the edge of the domain that the Schwarzschild chart can cover. The metric tensor appears to have a singularity on the hypersurface $r = 2m$ since $g_{11}(r) \to +\infty$ as $r \downarrow 2m$. If the source of the Schwarzschild field is a static, spherically symmetric body whose outer boundary has radius $r_b > 2m$, this causes no problem in describing the entirety of space-time in a single chart. However, for a dense body (such as a neutron star) or a vacuum spherically symmetric space-time, the nature of this apparent singularity must be examined further.

2.3 Extending the Schwarzschild Solution

In classical physics, a singularity of a field is identified as somewhere that the field diverges. For a static electric or gravitational field with potential of the form $V = k/r$, the singularity at $r = 0$ is due to the "infinite charge or mass density" of the idealised "point" charge/mass. Non-linear theories do not necessarily have such easy interpretations of singularities.

In precise terms, a space-time $\mathcal{M}$ is *time-like geodesically incomplete* if there exists a time-like geodesic curve in $\mathcal{M}$ that cannot be extended to arbitrary values of its affine parameter. In a sense, this means that a freely falling observer could reach the boundary of space-time or "run out of manifold" in a finite amount of time. Since time-like geodesic incompleteness is a bit too restrictive, a similar definition applies for null geodesic incompleteness. However, particles are not restricted to moving along geodesics; there are other time-like curves in space-time that are not geodesics. To allow for other kinds of incomplete curves, *b-completeness* and *generalised affine parameters* are introduced (for details, see [6, 13]). These constructions are necessary to define a space-time $\mathcal{M}$ as *singular* if it contains time-like or null curves that are incomplete. It turns out that the definition of a singularity is much more involved than defining singular space-times and the identification is difficult.

A singularity in a local coordinate system is particularly misleading. For example, the flat differentiable manifold $\mathbb{E}^3$ is covered almost globally by a spherical coordinate

system with coordinates (r, θ, ϕ). In such a case, the line element takes the form

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $0 < r < +\infty, 0 < \theta < \pi$ and $-\pi < \phi < \pi$. The determinant of the metric tensor is $\det(g) = r^4 \sin^2 \theta$ which tends to zero as $r \downarrow 0$ or $\theta \downarrow 0$. There is, however, nothing irregular about those subsets of the manifold. The apparent singular behaviour is an artifice of the choice of coordinate charts; simply converting to Cartesian coordinates does away with the problem. The components of the metric tensor, when considered as functions of the coordinates, are not scalar fields (so they do not transform as tensor fields of rank (0+0)). As such, no real geometrical meaning is prescribed to singularities in the individual components of the metric (or any other) tensor.

One indication that the singular behaviour on the hypersurface $r = 2m$ of the Schwarzschild solution is not indicative of a singularity is given by the scalar invariants of the metric. For instance, the Kretchmann scalar is completely regular except as $r \downarrow 0$:

$$\left(R^{ijkl} R_{ijkl} \right)\big|_{r=2m} = \left(48m^2/r^6 \right)\big|_{r=2m} \equiv \frac{3}{4m^4} > 0.$$

This scalar has the same value in any coordinate system, unlike the component g_{11} of the metric tensor. It seems, then, that something is lacking in the choice of Schwarzschild (r, t)-coordinates to describe this space-time.

Other coordinate charts are used to circumvent this difficulty. For simplicity, the transformations discussed here will be on the two-dimensional sub-manifold $\mathcal{M}_2$ where $\theta =$constant, $\phi =$constant. It is the transformations of the coordinates r and t for the sub-manifold that are of interest and the angular coordinates play passive roles in the transformations. Lemaître found a way to eliminate the discontinuity in the metric

tensor components. This transformation is as follows:

$$\widehat{r} \equiv \widehat{\mathcal{R}}(r,t) := t + 2\sqrt{2mr} + 2m \ln \left| \frac{\sqrt{r} - \sqrt{2m}}{\sqrt{r} + \sqrt{2m}} \right| + \frac{2}{3}\sqrt{\frac{r^3}{2m}}, \tag{2.5a}$$

$$\widehat{t} \equiv \widehat{\mathcal{T}}(r,t) := t - 2\sqrt{2mr} + 2m \ln \left| \frac{\sqrt{r} - \sqrt{2m}}{\sqrt{r} + \sqrt{2m}} \right|, \tag{2.5b}$$

$$\frac{\partial(\widehat{r},\widehat{t})}{\partial(r,t)} = \sqrt{\frac{r}{2m}} = \left(\frac{3(\widehat{r} - \widehat{t})}{4m} \right)^{\frac{1}{3}} > 0; \tag{2.5c}$$

The resulting metric from this change of chart is the Lemaître metric [23, 33] that is described by the line element

$$ds^2 = \frac{2m}{\mathcal{R}(\widehat{r},\widehat{t})}d\widehat{r}^2 + [\mathcal{R}(\widehat{r},\widehat{t})]^2 d\widehat{\Omega}^2 - d\widehat{t}^2. \tag{2.6}$$

The function $\mathcal{R}$ in the line element (2.6) is defined in the inverse transformation from Lemaître $(\widehat{r},\widehat{t})$-coordinates to Schwarzschild (r,t)-coordinates:

$$r \equiv \mathcal{R}(\widehat{r},\widehat{t}) := \left(\frac{3}{2}\sqrt{2m}(\widehat{r} - \widehat{t}) \right)^{\frac{2}{3}}, \tag{2.7a}$$

$$t \equiv \mathcal{T}(\widehat{r},\widehat{t}) := \widehat{t} - 2\sqrt{2m}(3\sqrt{2m}(\widehat{r} - \widehat{t})/2)^{\frac{1}{3}} \tag{2.7b}$$

$$2m \ln \left(\frac{(3\sqrt{2m}(\widehat{r} - \widehat{t})/2)^{\frac{1}{3}} + \sqrt{2m}}{(3\sqrt{2m}(\widehat{r} - \widehat{t})/2)^{\frac{1}{3}} - \sqrt{2m}} \right).$$

This choice of charts is physically interpreted as the coordinates co-moving with an observer falling radially from infinity.

Looking at the line element (2.6), the components of the metric tensor are clearly regular as $r \downarrow 2m$. The hypersurface $r = 2m$ at the boundary of the Schwarzschild domain corresponds to the hypersurface $\widehat{r} - \widehat{t} = (4/3)m$ in the Lemaître domain. Defining

$$D_R := \{(r,t) : 2m < r < +\infty, t \in \mathbb{R}\}, \tag{2.8a}$$

$$\widehat{D}_R := \left\{ (\widehat{r},\widehat{t}) : \widehat{r} - \widehat{t} > \frac{4}{3}m \right\}, \tag{2.8b}$$

$$\widehat{D}_T := \left\{ (\widehat{r},\widehat{t}) : 0 < \widehat{r} - \widehat{t} < \frac{4}{3}m \right\}, \tag{2.8c}$$

it becomes clear that the Schwarzschild domain D_R is homeomorphic to the domain $\widehat{D}_R$ covered by the Lemaître chart, so the domain $\widehat{D}_R$ is an R-domain. However, the Lemaître coordinates are also valid in the T-domain $\widehat{D}_T$ that extends beyond the domain where Schwarzschild (r, t)-coordinates are valid. The metric (2.6) is defined and regular in $\widehat{D}_R \cup \widehat{D}_T$ and on the boundary hypersurface where $\widehat{r} - \widehat{t} = 4m/3$ (although there is a singularity along the hypersurface $\widehat{r} - \widehat{t} = 0$ along which $\mathcal{R}(\widehat{r}, \widehat{t}) = 0$ at the boundary of the domain $\widehat{D}_T$; see figure 2.1).

It is possible to cover the T-domain beyond $r = 2m$ with Schwarzschild-like coordinates. The domain in the new coordinate system is thus called the *vacuum Schwarzschild T-domain*, denoted D_T. The transformation from the Lemaître $(\widehat{r}, \widehat{t})$-coordinates in $\widehat{D}_T$ to the Schwarzschild (R, T)-coordinates in the domain D_T is defined as follows:

$$R \equiv \widetilde{\mathcal{R}}(\widehat{r}, \widehat{t}) := \widehat{t} - 2\sqrt{2m}(3\sqrt{2m}(\widehat{r} - \widehat{t})/2)^{\frac{1}{3}} \tag{2.9a}$$
$$+ 2m \ln \left(\frac{\sqrt{2m} + (3\sqrt{2m}(\widehat{r} - \widehat{t})/2)^{\frac{1}{3}}}{\sqrt{2m} - (3\sqrt{2m}(\widehat{r} - \widehat{t})/2)^{\frac{1}{3}}} \right),$$

$$T \equiv \widetilde{T}(\widehat{r}, \widehat{t}) := \left(\frac{3}{2}\sqrt{2m}(\widehat{r} - \widehat{t}) \right)^{\frac{2}{3}}, \tag{2.9b}$$

$$\frac{\partial(R, T)}{\partial(\widehat{r}, \widehat{t})} = \left(\frac{4m}{3(\widehat{r} - \widehat{t})} \right)^{\frac{1}{3}} = \sqrt{\frac{2m}{\widetilde{T}(\widehat{r}, \widehat{t})}} > 0 \ ((\widehat{r}, \widehat{t}) \in \widehat{D}_T). \tag{2.9c}$$

The range of the transformation (2.9) is

$$D_T := \{(R, T) \in \mathbb{R}^2 : R \in \mathbb{R}, 0 < T < 2m\}.$$

Using the transformation (2.9), the new line element is

$$ds^2 = \left(\frac{2m}{T} - 1 \right) dR^2 + T^2 d\Omega^2 - \left(\frac{2m}{T} - 1 \right)^{-1} dT^2 \tag{2.10}$$

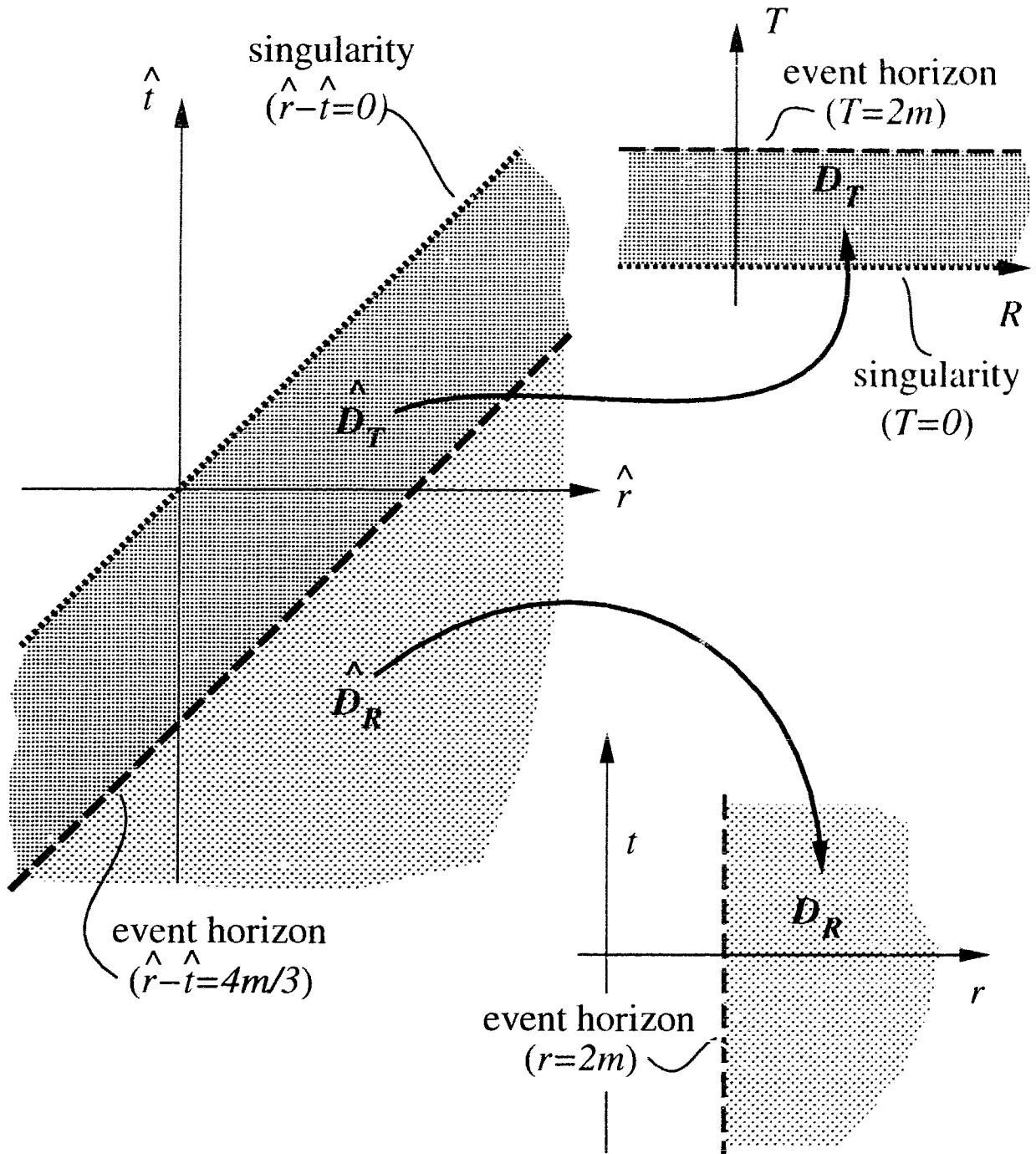where $(R, T) \in D_T$. This metric obviously resembles the Schwarzschild metric (2.4)

Figure 2.1: Depiction of the transformation between Lemaitre $(\hat{r}, \hat{t})$-coordinates and Schwarzschild (r, t) and (R, T)-coordinates in the respective R and T-domains.

where the radial variable r is replaced by the time-like variable T in the metric tensor components. The Lemaître $(\hat{r}, \hat{t})$-coordinates can be recovered from the (R, T)-coordinates using the same functions $\mathcal{R}, \mathcal{T}$ as in (2.7a) and (2.7b):

$$\hat{r} \equiv \mathcal{R}(T, R) := R + 2\sqrt{2mT} + 2m \ln \left| \frac{\sqrt{2m} - \sqrt{T}}{\sqrt{2m} + \sqrt{T}} \right| + \frac{2}{3}\sqrt{\frac{T^3}{2m}}, \tag{2.11a}$$

$$\hat{t} \equiv \mathcal{T}(T, R) := R - 2\sqrt{2mT} + 2m \ln \left| \frac{\sqrt{2m} - \sqrt{T}}{\sqrt{2m} + \sqrt{T}} \right|. \tag{2.11b}$$

The line element in the T-domain $\widehat{D}_T$ is exactly as in (2.6).

For further analysis of the vacuum T-domain D_T, it is convenient to use an alternate chart using null or light-cone coordinates. The term "light-cone coordinates" refers to the fact that the coordinate curves are null curves and hence represent the worldlines of particles of light. A doubly null coordinate system describes the the maximal extension of the Schwarzschild solution. The angular coordinates θ and ϕ play passive roles once again and so this discussion focuses on a two-dimensional submanifold. Consider the two-dimensional domain D in the (U, V)-plane where $UV < 1$. This domain D can be divided up into open sub-domains in the separate quadrants:

$$D_{\text{I}} := \{(U, V) : U < 0, V > 0\}, \tag{2.12a}$$

$$D_{\text{II}} := \{(U, V) : U > 0, V > 0, UV < 1\}, \tag{2.12b}$$

$$D_{\text{III}} := \{(U, V) : U > 0, V < 0\}, \tag{2.12c}$$

$$D_{\text{IV}} := \{(U, V) : U < 0, V < 0, UV < 1\}, \tag{2.12d}$$

$$D_{\text{II}} := \{(U, V) : UV = 0\} \tag{2.12e}$$

Define the Kruskal-Szekeres metric [16] in D by the following line element:

$$ds^2 = -\frac{32m^2}{\mathcal{Y}(U, V)} \exp \left(\frac{-\mathcal{Y}(U, V)}{2m} \right) dU dV + [\mathcal{Y}(U, V)]^2 d\Omega^2, \text{ where} \tag{2.13a}$$

$$\mathcal{Y}(U, V) := 2m(1 + \mathcal{W}(-UV/e)) \text{ and} \tag{2.13b}$$

$$\mathcal{W}(z)e^{\mathcal{W}(z)} \equiv z, \text{ for } z \in (-1/e, +\infty). \tag{2.13c}$$

This is the maximal extension of the spherically symmetric vacuum Schwarzschild solution.

The function $\mathcal{Y} : D \to (-1, +\infty)$ in the line element (2.13a) includes the Lambert-$\mathcal{W}$ function[3] [7] defined implicitly by the equation (2.13c) so that it satisfies the transcendental identity

$$UV + \exp\left(\frac{\mathcal{Y}(U,V)}{2m}\right)\left(\frac{cm\mathcal{Y}(U,V)}{2m} - 1\right) \equiv 0. \tag{2.14}$$

The Lambert-$\mathcal{W}$ function is the inverse of the function f defined by $f(x) := xe^x$. On the semi-infinite interval $(-1, +\infty)$, the function f is monotone increasing and infinitely differentiable, so the inverse $\mathcal{W}$ is well-defined and infinitely differentiable on the semi-infinite interval $(-1/e, +\infty)$. The function $\mathcal{W}$ as it occurs in $\mathcal{Y}$ has as its argument $-e^{-1}UV$. By the definition of the domain D under consideration in the (U,V)-plane, $-\infty < UV < 1$, so $-1/e < -UV/e < +\infty$. As such, the function $\mathcal{Y}$ is smooth over the domain D considered. (Notice that there are other branches of the function $\mathcal{W}$; the point $-1/e$ is a branch point and, in the function $\mathcal{Y}$ defined in (2.13b), corresponds to the hyperbolae $UV = 1$ in D which are the genuine singularities of this space-time.)

Let the functions $\mathcal{Y}_{\mathrm{I}}$, $\mathcal{Y}_{\mathrm{II}}$, $\mathcal{Y}_{\mathrm{III}}$ and $\mathcal{Y}_{\mathrm{IV}}$ denote the restrictions of $\mathcal{Y}$ to the domains D_{I}, D_{II}, D_{III} and D_{IV} respectively. These particular functions occur in the transformations from Kruskal-Szekeres coordinates to Schwarzschild-like coordinates. On the hypersurface D_{II}, $UV = 0$; as such, for $(U, V) \in D_{\mathrm{II}}$, $Y(U,V) = 2m(1 + \mathcal{W}(0)) = 2m$. Thus, even though parts of the null hypersurface corresponding to D_{II} cannot be described by Schwarzschild-like coordinates in D_R or D_T, the metric is smooth and regular there.

The curves $U = $ constant and $V = $ constant have tangent vectors that are everywhere null. The coordinate lines, then, are the the worldlines of photons. Material observers must have worldlines lying strictly within the coordinate lines. Geometrically, in the domain D_{I}, the tangents drawn to the world-line of any material observer have to have strictly positive and finite slopes. That is, at an event with Kruskal-Szekeres coordinates $(U_0, V_0) \in D$, any time-like curve passing through (U_0, V_0) lies

---

[3]This transformation is always defined implicitly in the literature. I introduce the Lambert-W function in recognition of the fact that an implicitly defined function is still a function in its own right.

strictly in the set $I^+ \cup I^-$, where

$$I^+ := \{(U,V) \in D : U_0 < U, V_0 < V\} \text{ and } I^- := \{(U,V) \in D : U_0 > U, V_0 > V\}.$$

These sets are, respectively, the *chronological future* and the *chronological past* of (U_0, V_0) [13]. They include all possible events in the past that could affect (U_0, V_0) through a time-like curve terminating at (U_0, V_0) and all the possible events in the future that can be affected by a time-like curve originating from (U_0, V_0) (the causal future and past include null trajectories also). For events in D_I, the causal and chronological pasts lie strictly in D_IV and D_I while the chronological and causal futures lie strictly within D_I and D_II. Thus, events in D_II can be affected by events in D_I but the converse is not true. The null hypersurface $UV = 0$ is often referred to as an *event horizon* due to this asymmetry; observers within D_I can never receive signals from observers in D_II and thus have no knowledge of what exists beyond (see figure 2.2).

The relevance of this particular space-time is that its various quadrants can be covered with Schwarzschild-type coordinates; thus, it satisfies the vacuum field equations and hence is an extension of the spherically symmetric vacuum solution of the Einstein field equations (1.35). The Kruskal-Szekeres (U, V)-coordinates in D_I are obtained from the Schwarzschild (r, t)-coordinates in the domain D_R by the following relations:

$$U \equiv \mathcal{U}_\mathrm{R}(r,t) := -\exp\left(\frac{r-t}{4m}\right)\sqrt{\frac{r}{2m} - 1}, \tag{2.15a}$$

$$V \equiv \mathcal{V}_\mathrm{R}(r,t) := \exp\left(\frac{r+t}{4m}\right)\sqrt{\frac{r}{2m} - 1}, \tag{2.15b}$$

$$\frac{\partial(U,V)}{\partial(r,t)} = -\frac{r e^{\frac{r}{2m}}}{16m^3} < 0, \ (r,t) \in D_R, \tag{2.15c}$$

$$D_R := \{(r,t) : 2m < r < +\infty, t \in \mathbb{R}\}. \tag{2.15d}$$

The inverse transformation from D_I to D_R is as follows:

$$r \equiv \mathcal{Y}_\mathrm{I}(UV) := 2m(1 + \mathcal{W}(-e^{-1}UV)), \tag{2.15e}$$

$$t \equiv \mathcal{T}_\mathrm{I}(UV) := 2\ln(-V/U). \tag{2.15f}$$

Consider the line $r = r_c =$ constant $> 2m$ in the Schwarzschild domain. The image of such vertical lines in the Kruskal-Szekeres domain D_1 is a branch of a hyperbola $UV = -(r_c/2m - 1)e^{r_c/2m} < 0$. As r_c decreases towards $2m$, the images of the hyperbolae approach the coordinate axes. Thus, the line $r = 2m$ in the Schwarzschild domain is mapped to the coordinate axes $U = 0, V > 0$ and $V = 0, U < 0$ bounding the domain D_I.

The horizontal lines $t = t_c =$ constant $\in \mathbb{R}$ in the Schwarzschild domain are mapped into half-lines $V = -(e^{t_c/2m})U$ incident on the origin in the Kruskal-Szekeres domain D_I. As $t \to -\infty$, the half-lines are nearly horizontal and hence are nearer to the U-axis. The line $t = 0$ maps to the half-line of slope -1 that cuts D_1 in half. Finally, as $t \to \infty$, the slopes of the half-lines become almost vertical and near the V-axis.

A similar transformation exists from D_{II} to the T-domain described before in (2.9). The null surface $UV = 0$ which bounds D_I is the image of the event horizon $r - 2m$ in the transformation (2.15) and it divides the domains D_{II} and D_1 in the Kruskal-Szekeres domain. The transformation between D_T and D_{II} is as follows:

$$U \equiv \mathcal{U}_{\mathrm{T}}(R,T) := \exp\left(\frac{T-R}{4m}\right)\sqrt{\frac{T}{2m} - 1}, \tag{2.16a}$$

$$V \equiv \mathcal{V}_{\mathrm{T}}(R,T) := \exp\left(\frac{T+R}{4m}\right)\sqrt{\frac{T}{2m} - 1}, \tag{2.16b}$$

$$\frac{\partial(U,V)}{\partial(R,T)} = \frac{Te^{\frac{T}{2m}}}{16m^3} > 0, \quad (R,T) \in D_T, \tag{2.16c}$$

$$D_T := \{(R,T) : R \in \mathbb{R}, 0 < T < 2m\}; \tag{2.16d}$$

$$R \equiv \mathcal{Y}_{\mathrm{II}}(UV) := 2m(1 + \mathcal{W}(-e^{-1}UV)), \tag{2.16e}$$

$$T \equiv \mathcal{T}_{\mathrm{II}}(V/U) := 2\ln(V/U). \tag{2.16f}$$

The radial half-lines and hyperbolae in D_{II} are similar to those in D_I. For instance, the images of the lines $R =$ constant in D_{II} are radial half-lines from the origin while the images of the lines $T =$ constant are branches of hyperbolae $UV = c$ with $0 < c < 1$. However, in D_{II}, there is a genuine singularity, the hyperbola $UV = 1$. In this domain, any future-directed curve, null or time-like, starting from an event

$(U_0, V_0) \in D_{\mathrm{II}}$ necessarily intersects this singularity. This singularity is called a *black hole* for this reason; not even light can escape from the domain D_{II} (see figure 2.2).

Finally, there are the domains D_{III} and D_{IV}. The line element is invariant upon reflection through the origin; that is, the mapping $(U, V) \mapsto (-U, -V)$ does not change the metric tensor components at all. The R-domain D_{III} can be covered by coordinates identical to the (r, t)-coordinates covering D_{I} while the T-domain D_{IV} can be covered by coordinates identical to the (R, T)-coordinates covering D_{II}. The transformation from D_{III} to D_R is

$$U \equiv \mathcal{U}_{\mathrm{R'}}(r', t') := \exp\left(\frac{r' - t'}{4m}\right) \sqrt{\frac{r'}{2m} - 1}, \tag{2.17a}$$

$$V \equiv \mathcal{V}_{\mathrm{R'}}(r', t') := -\exp\left(\frac{r' + t'}{4m}\right) \sqrt{\frac{r'}{2m} - 1}, \tag{2.17b}$$

$$\frac{\partial(U, V)}{\partial(r', t')} = -\frac{r' e^{\frac{r'}{2m}}}{16m^3} < 0, \ (r', t') \in D_R, \tag{2.17c}$$

$$r' \equiv \mathcal{Y}_{\mathrm{III}}(UV) := 2m(1 + \mathcal{W}(-e^{-1}UV)), \tag{2.17d}$$

$$t' \equiv \mathcal{T}_{\mathrm{III}}(V/U) := 2\ln(-V/U), \tag{2.17e}$$

and the transformation from D_{IV} to D_T is

$$U \equiv \mathcal{U}_{\mathrm{T'}}(R', T') := -\exp\left(\frac{T' - R'}{4m}\right) \sqrt{\frac{T'}{2m} - 1}, \tag{2.18a}$$

$$V \equiv \mathcal{V}_{\mathrm{T'}}(R', T') := -\exp\left(\frac{T' + R'}{4m}\right) \sqrt{\frac{T'}{2m} - 1}, \tag{2.18b}$$

$$\frac{\partial(U, V)}{\partial(R', T')} = \frac{T' e^{\frac{T'}{2m}}}{16m^3} > 0, \ (R', T') \in D_T \tag{2.18c}$$

$$R' \equiv \mathcal{Y}_{\mathrm{IV}}(UV) := 2m(1 + \mathcal{W}(-e^{-1}UV)), \tag{2.18d}$$

$$T' \equiv \mathcal{T}_{\mathrm{IV}}(V/U) := 2\ln(V/U). \tag{2.18e}$$

Primed coordinates are used in the Schwarzschild-like coordinates for these domains is because the orientation of time is different in these domains. In particular, the domain D_{III} is almost identical to D_{I} in that future-directed time-like or null curves can fall into the future singularity in D_{II} . However, the singularity in D_{IV} is a *past*
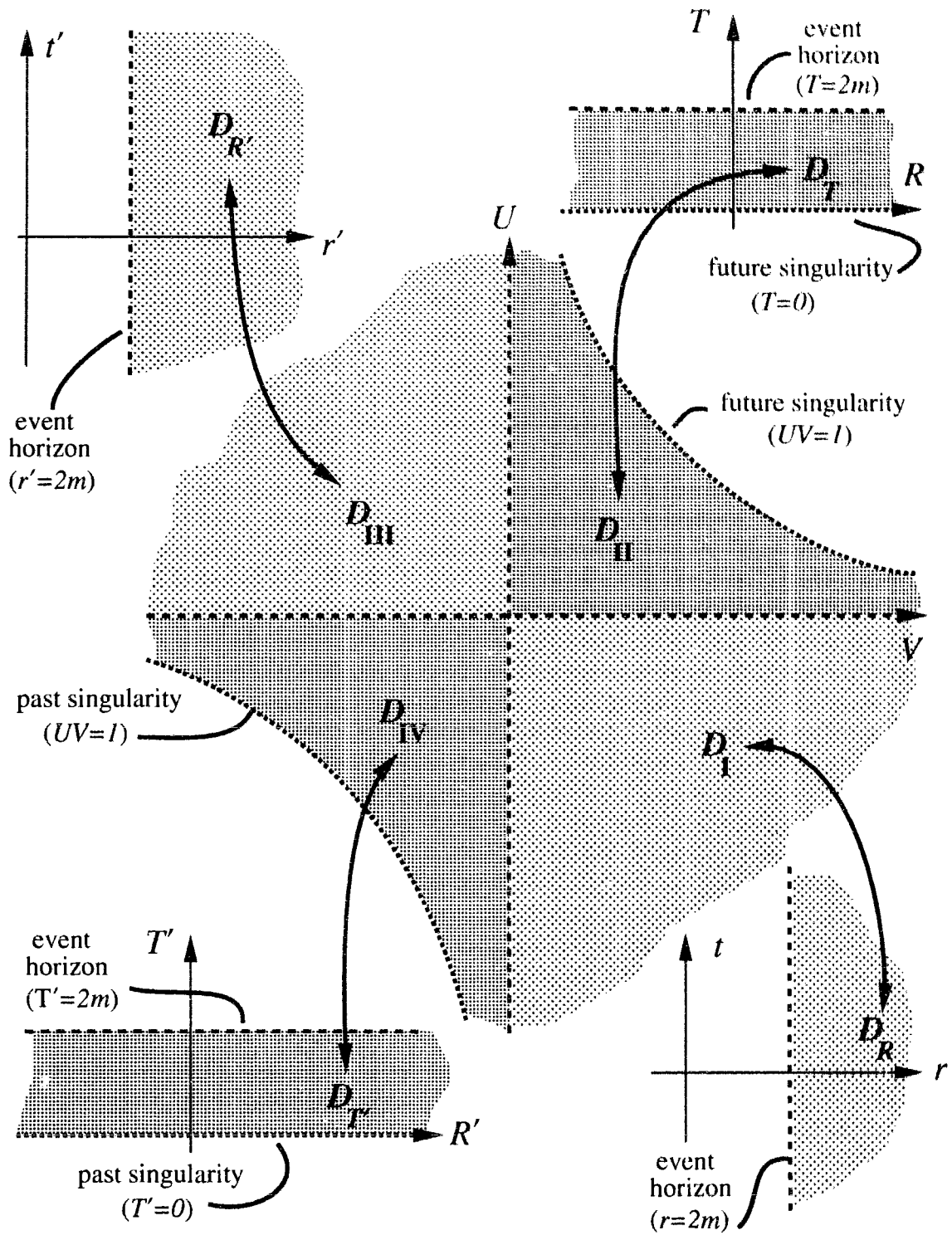
Figure 2.2: Depiction of the transformation between Kruskal-Szekeres (U,V)-coordinates and Schwarzschild-type coordinates in each of the sub-domains.

singularity or a *white hole* unlike that in D_{II}. Future-directed time-like or null curves starting in D_{IV} can travel into D_{I} or D_{III} but no observer or photon starting in any of the other three domains can travel back there. Further, the domains D_{I} and D_{III} can be viewed as distinct universes separated by the black hole; however, no material particle or light-ray can ever traverse from one to the other due to the singularity separating them. (The situation is clearly illustrated in figure 2.2).

Chapter 3

T-domains and exotic black holes

Static spherically symmetric stars have been studied extensively in the literature on general relativity. Many classes of static solutions for bodies composed of perfect isotropic fluids or anisotropic fluids are documented in [15]. Some of the models are more realistic in the choice of an equation of state (see [4, 2]) than others derived from mathematical considerations alone (see [29, 32, 17, 18]), although it should be noted that most physically realistic models cannot be solved exactly and require numerical methods. These exact solutions all arise from the assumption of time-independence of the solutions of the field equations (1.35) in a spherically symmetric R-domain that is joined to a vacuum Schwarzschild metric using suitable junction conditions.

Ruban [21, 22] did examine T-models of a sphere. Later, questions about time machines and the possible existence of closed time-like curves in space-time prompted Morris and Thorne [27] to study wormholes. These wormhole solutions are built from *exotic* matter that violates the energy conditions that restrict the stress-energy-momentum tensor. More recent studies of gravitational collapse have revealed models of spherically symmetric stars consisting of anisotropic fluid matter undergoing a transition into exotic matter after collapsing past the event horizon of a black hole (see [10, 9]). This chapter presents a survey of general solutions within a spherically symmetric T-domain that involve similar exotic matter. The solutions obtained differ from other spherically symmetric solutions in R-domains in that the gravitational effects are due to radial tension within the matter rather than mass-energy density.

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For external observers, the space-times resemble Schwarzschild black holes, and hence are *exotic black holes*.

3.1 Analysis of a General T-domain

In the most general case, the Einstein field equations together with all other constraints are written as follows:

$$\mathcal{E}_{ij} := G_{ij} + \kappa T_{ij} = 0, \tag{3.1a}$$

$$\mathcal{T}^i := \nabla_j T^{ij} = 0, \tag{3.1b}$$

$$\mathcal{C}^a(g_{ij}, \partial_k g_{ij}) = 0. \tag{3.1c}$$

The $\mathcal{C}^a$ are 4 possible coordinate conditions that can be chosen. There are 20 unknown functions (g_{ij} and T_{ij}) to be determined and 18 equations in total in (3.1). The equations (3.1) are not independent, however, due to the 4 differential identities $\nabla_j \mathcal{E}^{ij} + \kappa \mathcal{T}^i \equiv 0$. Thus, there are 14 independent equations to solve and 20 unknown functions so the system (3.1) is under-determined.

For a general spherically symmetric T-domain, the spherically symmetric ansatz (2.1) with the identification[1] $x^1 \equiv R$, $x^2 \equiv \theta$, $x^3 \equiv \phi$, $x^4 \equiv T$ and $\exp(\beta(R,T)) := T^2$ is

$$ds^2 = e^{\alpha(R,T)}dR^2 + T^2 d\Omega^2 - e^{\gamma(R,T)}dT^2, \tag{3.2a}$$

$$d\Omega^2 := d\theta^2 + \sin^2\theta d\phi^2. \tag{3.2b}$$

The unknown functions α and γ are assumed to be of the differentiability class $\mathcal{C}^3_p$ over their domains. These coordinates are valid in some domain $\tilde{D}_I \subset \mathbb{R}^4$. The angular coordinates play passive roles so the analysis is restricted to a sub-manifold $\mathcal{M}_2$ on which $\theta = $ constant and $\phi = $ constant. In this analysis, some kind of matter ($T^a{}_b \neq 0$) is embedded within the T-domain

$$D_T := \{(R,T) : R \in \mathbb{R}, 0 < T < 2m\}$$

---

[1]The coordinate names R and T should not be confused with the tensors represented by similar letters. The meaning is generally clear from the context.

of the chart covering a domain diffeomorphic to the sub-domain $D_{\rm II}$ of the Kruskal space-time (2.16). The matter exists in some open set $D_I \subset D_\Gamma$ in the (R,T)-plane. The boundary between matter and the spherically symmetric vacuum domain outside is denoted ∂D_I and the vacuum domain is denoted D_E. The boundary can be described by some curve $F(R,T) = 0$ for some suitable function F; with an additional assumption $\partial_r F(R,T) \neq 0$ on the boundary ∂D_I, the implicit function theorem guarantees that the function F can be inverted locally on ∂D_I. Thus, the domains under consideration can be given by

$$D_I := \{(R,T) \in \mathbb{R}^2 : R \in \mathbb{R}, 0 < T_1 < T < B(R)\}, \qquad (3.3\text{a})$$

$$\partial D_I := \{(R,T) \in \mathbb{R}^2 : R \in \mathbb{R}, T = B(R)\}, \qquad (3.3\text{b})$$

$$D_E := \{(R,T) \in \mathbb{R}^2 : R \in \mathbb{R}, B(R) < T < 2m\}. \qquad (3.3\text{c})$$

The line element (3.2) is valid in the domain D_I while the line element in the vacuum domain D_E is

$$ds_E^2 = \left(\frac{2m}{T} - 1\right) dR^2 + T^2 d\Omega^2 - \left(\frac{2m}{T} - 1\right)^{-1} dT^2. \qquad (3.4)$$

In the definition of D_E and in this line element, the parameter m is the invariant Schwarzschild mass m of the space-time. (This will be related to the non-vanishing stress-energy tensor components in equation (3.12)).

The components of the mixed Einstein tensor computed from (2.2) give the non-trivial field equations (for simplicity, denote $\alpha = \alpha(R,T)$ and $\gamma = \gamma(R,T)$).

$$\mathcal{E}^1_{\ 1} = G^1_{\ 1} + \kappa T^1_{\ 1} = \frac{1}{T^2}\left(1 + e^{-\gamma}(1 - T\partial_T\gamma)\right) + \kappa T^1_{\ 1} = 0, \qquad (3.5\text{a})$$

$$\mathcal{E}^2_{\ 2} = G^2_{\ 2} + \kappa T^2_{\ 2}$$
$$= e^{-\gamma}\left(\frac{1}{2}(\partial_T^2\alpha) - \frac{1}{4}(\partial_T\alpha)(\partial_T\gamma) + \frac{1}{4}(\partial_T\alpha)^2 + \frac{1}{2T}\partial_T(\alpha - \gamma)\right)$$
$$+ e^{-\alpha}\left(\frac{1}{4}(\partial_R\alpha)(\partial_R\gamma) - \frac{1}{2}\partial_R^2\gamma - \frac{1}{4}(\partial_R\gamma)^2\right) + \kappa T^2_{\ 2} = 0, \qquad (3.5\text{b})$$

$$\mathcal{E}^3_{\ 3} = \mathcal{E}^2_{\ 2}, \qquad (3.5\text{c})$$

$$\mathcal{E}^4_{\ 4} = G^4_{\ 4} + \kappa T^4_{\ 4} = \frac{1}{T^2}(1 + e^{-\gamma}(1 + T\partial_T\alpha)) + \kappa T^4_{\ 4} = 0, \qquad (3.5\text{d})$$

$$\mathcal{E}^1_{\ 4} = G^1_{\ 4} + \kappa T^1_{\ 4} = -e^{-\alpha}\frac{\partial_R\gamma}{T} + \kappa T^1_{\ 4} = 0, \qquad (3.5\text{e})$$

$$\mathcal{E}^4_{\ 1} = -e^{\alpha-\gamma}\mathcal{E}^1_{\ 4}. \qquad (3.5\text{f})$$

The conservation equations $\mathcal{T}_a = \nabla_k T^k{}_a = 0$ reduce to two nontrivial equations:

$$\mathcal{T}_1 = \partial_R T^1{}_1 + \partial_r T^4{}_1 + \frac{1}{2}(\partial_R\gamma)(T^1{}_1 - T^4{}_4) + \left(\frac{1}{2}\partial_T(\alpha+\gamma) + \frac{2}{T}\right)T^4{}_1 = 0, \quad (3.6a)$$

$$\mathcal{T}_4 = \partial_R T^1{}_4 + \partial_T T^4{}_4 - \frac{1}{2}(\partial_T\alpha)T^1{}_1 - \frac{2}{T}T^2{}_2 + \frac{1}{2}\partial_T(\alpha+\gamma)T^1{}_4 \qquad (3.6b)$$

$$+ \left(\frac{1}{2}\partial_T\alpha + \frac{2}{T}\right)T^4{}_4 = 0.$$

There are 4 independent equations and 6 unknown functions: α, γ, $T^1{}_1$, $T^1{}_4$, $T^2{}_2$ and $T^4{}_4$ which depend only on (R,T). The system is under-determined and two functions can be prescribed. Thus, assuming $T^1{}_1$ and $T^4{}_4$ are known functions of (R,T), the remaining four functions can be determined.

The field equation $\mathcal{E}^1{}_1 = 0$ (3.5a) can be integrated, since it becomes

$$\partial_T(Te^{-\gamma}) = -\kappa(T^2)T^1{}_1 - 1.$$

Hence, the metric tensor component g_{44} can be determined directly from the pre-scribed function $T^1{}_1$.

$$e^{-\gamma(R,T)} = \frac{2M(R,T)}{T} - 1, \quad \text{where} \qquad (3.7a)$$

$$M(R,T) := \frac{1}{2}\left(f(R) - \kappa\int_{T_0}^{T} \tau^2 T^1{}_1(R,\tau)d\tau\right) \qquad (3.7b)$$

for some arbitrary differentiable function $f \in \mathcal{C}^3_p(D_I;\mathbb{R})$. The function $T^1{}_1$ must be negative in order to make a positive contribution to the effective mass function $M(R,T) > 0$. Integrating the equation (3.5d) gives the function α needed for g_{11}:

$$\alpha(R,T) = -\int_{T_0}^{T}\left(\frac{e^{\gamma(R,\tau)}+1}{\tau} + \kappa\tau e^{\gamma(R,\tau)}T^4{}_4(R,\tau)\right)d\tau. \qquad (3.8)$$

The unknown function $T^4{}_1$ is obtained by differentiating $e^{-\gamma(R,T)}$ with respect to R (given in the equation (3.7a)) to find $\partial_R\gamma$ and substituting the result into the field equation $\mathcal{E}^4{}_1 = 0$ (3.5e). Thus,

$$T^4{}_1 = \frac{1}{T^2}\left(\frac{f'(R)}{\kappa} - \int_{T_0}^{T}\tau^2\partial_R T^1{}_1(R,\tau)d\tau\right). \qquad (3.9)$$

Finally, the last unknown $T^2{}_2$ is given using the differential identity $\nabla_a T^a{}_4 = 0$ (3.6b).

$$
\begin{aligned}
T^2{}_2(R,T) = {} & \frac{T}{2}\left(\partial_R T^1{}_4 + \partial_T T^4{}_1\right) - \frac{T}{4}(\partial_T \alpha) + \frac{T}{4}\partial_R(\alpha + \gamma)T^1{}_4 \\
& + \left(\frac{T}{4}(\partial_T \alpha) + 1\right) T^4{}_4.
\end{aligned}
\tag{3.10}
$$

The junction conditions (1.36) must be satisfied at the interface ∂D_I between matter and vacuum. The boundary as defined in (3.3) can be viewed as a level curve of the function $F(R,T) := B(R) - T = 0$. As such, the non-vanishing components of the gradient $\nabla_a F$ are $\nabla_1 F = B'(R)$ and $\nabla_4 F = -1$. Using $\nabla_a F$ as a vector normal to the boundary, the junction conditions $[T^a{}_b n_a]\big|_{T=B(R)} = 0$ become

$$
[T^1{}_1 B'(R) - T^4{}_1]\big|_{T=B(R)} = 0, \tag{3.11a}
$$

$$
[T^1{}_4 B'(R) - T^4{}_4]\big|_{T=B(R)} = 0. \tag{3.11b}
$$

Specific choices of the boundary $B(R)$, the stress-energy tensor components $T^1{}_1$, $T^1{}_4$, and $T^4{}_4$ are necessary for explicit verification of these junction conditions. Notice that some solutions of the field equations (3.1) are not able to meet these junction conditions. Such solutions are still useful as local solutions that can be matched to other solutions that do satisfy the required junction conditions at the interface between matter and vacuum [17, 18].

In the domain D_E outside the matter, $T^a{}_b = 0$. As such, for $T > B(R)$, the solution derived for g_{44} gives

$$
g_{44} = e^{\gamma(R,T)} = \left(\frac{2M(R, B(R))}{T} - 1\right)^{-1}.
$$

This appears to depend upon the time-like coordinate T and the radial coordinate R. In fact, for $T > B(R)$, the quantity $M(R,T)$ is shown to be constant using the equation (3.9) for $T^4{}_1$ and the junction condition (3.11a):

$$
\begin{aligned}
\left[\frac{d}{dR} 2M(R,T)\right]\Big|_{T=B(R)} &= \left[\frac{d}{dR}\left(f(R) - \kappa \int_{T_0}^{T} \tau^2 T^1{}_1(R,\tau)d\tau\right)\right]\Big|_{T=B(R)} \\
&= \left[-\kappa B'(R)(T^2 T^1{}_1) + \left(f'(R) - \kappa \int_{T_0}^{T} \tau^2 \partial_R T^1{}_1(R,\tau)d\tau\right)\right]\Big|_{T=B(R)} \\
&= -\kappa [B(R)]^2 [B'(R)T^1{}_1 - T^4{}_1]\big|_{T=B(R)} = 0.
\end{aligned}
\tag{3.12}
$$

Thus, for $T > B(R)$, $M(R,T) = m$ where m is a constant and g_{44} depends only on the coordinate T. When matched with the vacuum Schwarzschild T-domain metric, it is clear that this parameter m is identical to the Schwarzschild mass as observed by an external observer. In this instance, the Schwarzschild mass is different from that in the static spherically symmetric case in an R-domain because the mass in the T-domain is *tension generated mass*. That is, the effects of gravity observed from the vacuum domain coincide with the usual Schwarzschild picture for a Schwarzschild black hole or a star of mass m. Inside, however, the effective mass is determined by the radial tension $T^1{}_1 < 0$ rather than the energy density $T^4{}_4$.

3.2   Particular Solutions

In practice, the preceding analysis is extremely difficult to apply towards obtaining exact solutions. Rather than assuming the stress-energy-momentum tensor components $T^1{}_1$ and $T^4{}_4$ are prescribed, it is usually more practical to use a mixed method (see [24]) that prescribes some constraints on the functions g_{ab} and $T^a{}_b$ to determine the rest. To derive some solutions to the Einstein field equations in the T-domain, restrict the analysis to the case where the metric functions are functions of the time-like coordinate T only. This resembles assuming the static case for an R-domain in which the metric functions are independent of r (special cases of this include the Schwarzschild vacuum solution and many other static solutions such as in [15, 17, 18, 32, 4, 31]). The line element (3.2) goes over into

$$ds^2 \;=\; e^{\alpha(T)}dR^2 + T^2 d\Omega^2 - e^{\gamma(T)}dT^2, \tag{3.13a}$$

$$d\Omega^2 \;=\; d\theta^2 + \sin^2\theta d\phi^2, \tag{3.13b}$$

and the field equations (3.5) become

$$\mathcal{E}^1{}_1 = G^1{}_1 + \kappa T^1{}_1 = \frac{1}{T^2}\left(1 + \frac{d}{dT}(Te^{-\gamma})\right) + \kappa T^1{}_1 = 0, \tag{3.14a}$$

$$\begin{aligned}\mathcal{E}^2{}_2 &= G^2{}_2 + \kappa T^2{}_2 \\ &= e^{-\gamma}\left(\frac{1}{2}\alpha'' - \frac{1}{4}\alpha'\gamma' + \frac{1}{4}(\alpha')^2 + \frac{1}{2T}(\alpha' - \gamma')\right) + \kappa T^2{}_2 = 0.\end{aligned} \tag{3.14b}$$

$$\mathcal{E}^3{}_3 = \mathcal{E}^2{}_2, \tag{3.14c}$$

$$\mathcal{E}^4{}_4 = G^4{}_4 + \kappa T^4{}_4 = \frac{1}{T^2}(1 + e^{-\gamma}(1 + T\alpha')) + \kappa T^4{}_4 = 0, \tag{3.14d}$$

where the prime denotes the total derivative with respect to the T coordinate. In addition, the conservation equations $\nabla_a T^{ab} = 0$ reduce to a single nontrivial equation:

$$\nabla_a T^a{}_4 = (T^4{}_4)' - \frac{\alpha'}{2}T^1{}_1 + \frac{2}{T}(T^4{}_4 - T^2{}_2) + \frac{\alpha'}{2}T^4{}_4 = 0. \tag{3.15}$$

The system at this point is under-determined. It consists of 3 nontrivial field equations and 1 conservation equation; however, there are 5 unknown functions: α, γ, $T^1{}_1$, $T^2{}_2$ and $T^4{}_4$. For the static spherically symmetric case in an R-domain, the requirement $T^1{}_1 = T^2{}_2(= T^3{}_3)$ is often used; this amounts to requiring isotropy of pressure within a perfect fluid. By analogy, to make the system of equations above more determinate, introduce the requirement

$$T^4{}_4 = T^2{}_2. \tag{3.16}$$

Then, the above system can be written as follows (see [29]):

$$\frac{d}{dT}\left(\frac{e^{-\gamma} + 1}{T}\right) + \frac{d}{dT}\left(\frac{\alpha' e^{-\gamma}}{2T}\right) + e^{-\alpha - \gamma}\frac{d}{dT}\left(\frac{\alpha' e^{-\alpha}}{2T}\right) = 0, \tag{3.17a}$$

$$\frac{1}{T^2}(1 + e^{-\gamma}(1 - T\gamma')) + \kappa T^1{}_1 = 0, \tag{3.17b}$$

$$\frac{1}{T^2}(1 + e^{-\gamma}(1 + T\alpha')) + \kappa T^4{}_4 = 0, \tag{3.17c}$$

$$(T^4{}_4)' + \frac{\alpha'}{2}(T^4{}_4 - T^1{}_1) = 0. \tag{3.17d}$$

The equation (3.17d) is identically satisfied as a consequence of (3.17a), (3.17b), (3.17c). In that case, the system to be solved is (3.17a), (3.17b) and (3.17c) for the unknowns α, γ, $T^1{}_1$ and $T^4{}_4$.

The strategy adopted is as follows.

1. Prescribe some constraint on α, γ or both to make the equation (3.17a) integrable.

2. Integrate the equation (3.17a) and use with the additional constraint to find the solutions for α and γ.

3. Substitute the expressions for α and γ into (3.17b) and (3.17c) to give $T^1{}_1$ and $T^4{}_4$.

At this point the system is solved.

Some solutions obtained in this manner are tabulated in Table 3.1. (Admittedly, the solution VII is derived using a different procedure closer to that in [31] which uses an equation of state as an additional assumption.) These solutions are mostly local solutions that satisfy the field equations involving a number of arbitrary constants that arise out of the integration and the initial assumptions about the functions. Some of the solutions cannot satisfy the necessary junctions conditions for any values of these constants. In such a case, the matter described by the solution can be joined to some other solution with matter of a different kind that can meet the junction conditions (see [17]). Such an analysis is done for the solution I in the next section.

3.3 Analysis of Solution I

The solution I closely resembles the interior Schwarzschild model of a static star (see [15]) with constant energy density. It can be derived by assuming that $g_{44} = -(QT^2-1)^{-1}$ for some constant $Q > 0$ and following the approach outlined previously. Alternately, assume that $T^1{}_1 = -3Q/\kappa = $ constant and integrate (3.7a) to obtain g_{44}.

	$g_{11}(T) = e^{\alpha(T)}$	$g_{44}(T) = e^{\gamma(T)}$
I	$k^2 \left(3C - (QT^2 - 1)^{\frac{1}{2}} \right)^2$	$(QT^2 - 1)^{-1}$
II	A^2	$(kT^2 - 1)^{-1}$
III	$B^2 (1 + T^2/A^2)$	$(1 + 2T^2/A^2)(1 + T^2/A^2)^{-1}(T^2/C^2 - 1)^{-1}$
IV	$T^2(AT^n + BT^{-n})$	$(n^2 - 1)^{-1}$
V	$A^2 T^{2n}$	$(n^2 - 2n - 1)/(1 + kT^{-(m+1)})$ where $m := (2n+1)(n-3)/(n+1)$
VI	$AT^2 e^{-BT}$	$[(B^2T^2/16)e^{BT-4}\mathrm{Ei}(BT-4)$ $-(5B^2T^2/16)e^{BT}\mathrm{Ei}(BT)$ $+(BT)/2 - 2 + CT^2 e^{BT}]^{-1}$
VII	$n(B-1)$ $+(1 + A^2T^2)^{\frac{1}{2}}[(C/T)$ $-((nB)/(AT))\mathrm{arcsinh}(AT)]$	$(1/n)(1 + A^2T^2)[n(B-1)$ $+(1 + A^2T^2)^{\frac{1}{2}}((C/T)$ $-((nB)/(AT))\mathrm{arcsinh}(AT))]^{-1}$

Table 3.1: Particular solutions for g_{ab} in the T-domain

In either case, for the solution I in Table 3.1, the line element in the domain D_I is

$$ds^2 = k^2 \left(3C - (QT^2 - 1)^{\frac{1}{2}} \right)^2 dR^2 + T^2 d\Omega^2 - \frac{dT^2}{QT^2 - 1}, \qquad (3.18a)$$

$$D_I = \{(R, T) \in \mathbb{R}^2 : R \in \mathbb{R}, T_0 < T < T_b\} \qquad (3.18b)$$

for some suitable constants $T_0 > 0$ and T_b that determine the bounds on T. The outer boundary of the matter ∂U_I is given by the curve $T = B(R) := T_b = $ constant, so

$$\partial D_I := \{(R, T) \in \mathbb{R}^2 : R \in \mathbb{R}, T = T_b\}. \qquad (3.19)$$

Using (3.17b) and (3.17c), the nonzero components of the tensor $T^a{}_b$ are

$$T^1{}_1 = -\frac{3Q}{\kappa} < 0; \quad T^4{}_4 = T^2{}_2 = -\frac{3Q(C - (QT^2 - 1)^{\frac{1}{2}})}{\kappa(3C - (QT^2 - 1)^{\frac{1}{2}})}. \qquad (3.20)$$

While it is immediately obvious that $T^1{}_1 < 0$, more information about the parameter C is needed to determine whether $T^4{}_4$ is positive or negative.

The arbitrary constants C, k in the metric tensor components arise out of the integration of the system (3.17). These can be related to the boundary parameters T_0, T_b by considering the boundary of D_I. Clearly, the metric tensor component

$g_{44} = (QT^2 - 1)^{-1}$ becomes infinite as $T \downarrow Q^{-\frac{1}{2}}$. Thus, it is reasonable to assume $T_0 > Q^{-\frac{1}{2}} > 0$, so $Q^{-\frac{1}{2}}$ is a lower bound on the parameter T_0. The other constants can be related to the boundary parameter T_b through use of the junction conditions.

The junction conditions (3.11) take a simple form due to the form of the boundary curve $T = B(R) = T_b$ and the fact that $T^1{}_4 = T^4{}_1 = 0$. As such, the equation (3.11a) is identically satisfied:

$$[(T^1{}_1) \cdot (B'(R)) - T^4{}_1]\big|_{T=T_b} = [(T^1{}_1) \cdot (0) - T^4{}_1]\big|_{T=T_b} = 0.$$

The equation (3.11b) implies

$$[(T^4{}_1) \cdot (B'(R))T^4{}_4]\big|_{T=T_b} = [(T^4{}_1) \cdot (0) - T^4{}_4]\big|_{T=T_b} = -T^4{}_4(T_b) = 0.$$

Looking at the earlier equation (3.20) for $T^4{}_4$, this condition simplifies to $C^2 = QT_b^2 - 1$ or $C = (QT_b^2 - 1)^{\frac{1}{2}}$.

To find the parameter k that occurs in $g_{11} = e^\alpha$, use the fact that the metric must match the vacuum Schwarzschild T-domain metric at the boundary. This means

$$g_{11}(T_b) = k^2 \left(3C - \left(QT_b^2 - 1\right)^{\frac{1}{2}}\right)^2 = \frac{2m}{T_b} - 1$$

and

$$g_{44}(T_b) = \left(QT_b^2 - 1\right)^{-1} = \left(\frac{2m}{T_b} - 1\right)^{-1}.$$

In terms of the line element (3.18a) with the parameter C as determined previously, this reduces to

$$k^2 \left(3 \left(QT_b^2 - 1\right)^{\frac{1}{2}} - \left(QT_b^2 - 1\right)^{\frac{1}{2}}\right)^2 \left(\frac{1}{QT_b^2 - 1}\right) = 1 \Rightarrow k^2 = \frac{1}{4}. \qquad (3.21)$$

Thus, with the constants C and k determined, the line element (3.18a) is

$$\frac{1}{4} \left(3 \left(QT_b^2 - 1\right)^{\frac{1}{2}} - \left(QT^2 - 1\right)^{\frac{1}{2}}\right)^2 dR^2 + T^2 d\Omega^2 - \frac{dT^2}{QT^2 - 1}. \qquad (3.22)$$

Having determined the constants C and k, the question of the sign of the component $T^4{}_1$ of the stress-energy-momentum tensor is resolved:

$$T^4{}_1 = -\frac{3Q}{\kappa} \left(\frac{(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}}}{3(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}}}\right) < 0. \qquad (3.23)$$

For $0 < T_0 < T < T_b$, the component $T^1{}_4$ is obviously negative.

Consider the energy conditions. Solving the eigenvalue problem $T^a{}_b v^b = \lambda v^a = 0$ turns out to be trivial because the tensor $T^a{}_b$ is diagonal. Thus, the solution to this eigenvalue problem defines an orthonormal tetrad:

$$v_{(1)}{}^a := 2\left(3\left(QT_b^2 - 1\right)^{\frac{1}{2}} - \left(QT^2 - 1\right)^{\frac{1}{2}}\right)^{-1} \delta_{(1)}{}^a, \tag{3.24}$$

$$v_{(2)}{}^a := T^{-1} \delta_{(2)}{}^a, \tag{3.25}$$

$$v_{(3)}{}^a := (T \sin\theta)^{-1} \delta_{(3)}{}^a, \tag{3.26}$$

$$v_{(4)}{}^a := \left(QT^2 - 1\right)^{\frac{1}{2}} \delta_{(4)}{}^a. \tag{3.27}$$

The eigenvalues corresponding to these eigenvectors are also the non-vanishing tetrad components $T^{(a)}{}_{(b)}$ of the stress-energy-momentum tensor relative to the orthonormal tetrad:

$$\lambda_{(1)} := T^{(1)}{}_{(1)} = T^1{}_1 = -\frac{3Q}{\kappa} < 0, \tag{3.28}$$

$$\lambda_{(2)} := T^{(2)}{}_{(2)} = T^2{}_2 = -\frac{3Q}{\kappa}\left(\frac{(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}}}{3(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}}}\right) < 0, \tag{3.29}$$

$$\lambda_{(3)} = \lambda_{(2)}, \tag{3.30}$$

$$\lambda_{(4)} := T^{(4)}{}_{(4)} = T^1{}_4 = -\frac{3Q}{\kappa}\left(\frac{(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}}}{3(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}}}\right) < 0. \tag{3.31}$$

Recall that the metric tensor can be decomposed expressed in terms of an orthonormal tetrad as in (1.28); writing this in mixed form gives

$$\delta^a{}_b = v_{(1)}{}^a v^{(1)}{}_b + v_{(2)}{}^a v^{(2)}{}_b + v_{(3)}{}^a v^{(3)}{}_b + v_{(4)}{}^a v^{(4)}{}_b \tag{3.32}$$

Thus, the mixed stress-energy-momentum tensor admits a similar decomposition in terms of its eigenvalues and eigenvectors. Applying the fact that $\lambda_{(2)} = \lambda_{(3)}$ and using the equation (3.32), this decomposition is

$$\begin{aligned}
T^a{}_b &= T^{(k)}{}_{(l)} v_{(k)}{}^a v^{(l)}{}_b \\
&= \lambda_{(1)} v_{(1)}{}^a v^{(1)}{}_b + \lambda_{(2)}[v_{(2)}{}^a v^{(2)}{}_b + v_{(3)}{}^a v^{(3)}{}_b] + \lambda_{(4)} v_{(4)}{}^a v^{(4)}{}_b \\
&= \lambda_{(1)} v_{(1)}{}^a v^{(1)}{}_b + \lambda_{(2)}[\delta^a{}_b - v_{(1)}{}^a v^{(1)}{}_b - v_{(4)}{}^a v^{(4)}{}_b] + \lambda_{(4)} v_{(4)}{}^a v^{(4)}{}_b \\
&= [\lambda_{(1)} - \lambda_{(2)}] v_{(1)}{}^a v^{(1)}{}_b + \lambda_{(2)} \delta^a{}_b + [\lambda_{(4)} - \lambda_{(2)}] v_{(4)}{}^a v^{(4)}{}_b.
\end{aligned}$$

Defining the velocity of the medium as

$$u^a := v_{(1)}{}^a,$$

the energy density of the medium as

$$\mu := -\lambda_{(4)} = \frac{3Q}{\kappa} \left(\frac{(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}}}{3(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}}} \right) > 0$$

and the radial tension (because $\tau < 0$) as

$$\tau := \lambda_{(1)} = -\frac{3Q}{\kappa} < 0,$$

the stress-energy momentum tensor takes the form

$$T^a{}_b = (\tau + \mu)u^a u_b + \mu \delta^a{}_b. \qquad (3.33)$$

This is similar to the case for a perfect fluid (see (1.33)) except the isotropic pressure is replaced by a radial tension and the velocity of the fluid is *space-like*. This makes the fluid *tachyonic* in nature rather than a perfect fluid. The fluid is anisotropic because the angular stresses are tensions equal in magnitude to μ which is different from the radial tension τ. Although the energy density $\mu = -\lambda_{(4)}$ is positive, the weak, strong and dominant energy conditions are not satisfied since

$$\lambda_{(2)} - \lambda_{(4)} = 0 \text{ and } \lambda_{(1)} - \lambda_{(4)} = \frac{-6Q(QT_b^2 - 1)^{\frac{1}{2}}}{\kappa \left(3(QT_b^2 - 1)^{\frac{1}{2}} - (QT^2 - 1)^{\frac{1}{2}} \right)}.$$

Thus, the fluid matter is *exotic* matter[2]. This solution constitutes an exotic black hole because the exotic matter lies entirely within the T-domain; as such, observers in domains D_I (see (2.15)) of the Kruskal-Szekeres space-time see a black hole of Schwarzschild mass $m = QT_b^3/2$ even though the T-domain is substantially different.

---

[2]In the literature, exotic matter usually violates energy conditions because the energy density $\mu < 0$ which is not the case here. However, another common feature in studies of exotic matter is that the principal stresses are tensions rather than pressures. For this reason, the matter is still called exotic.

To complete this analysis, consider the tension-generated Schwarzschild mass m as a constant and the parameter Q as a variable. In terms of m and Q, set the boundary parameters as

$$T_0 := Q^{-\frac{1}{2}} \text{ and } T_b := \left(\frac{2m}{Q}\right)^{\frac{1}{3}}. \tag{3.34}$$

Hence, D_I is as wide as it possibly can be since T_0 is as small as it can be for a prescribed value of Q. Holding m constant and letting Q increase without bound, both boundary parameters T_0 and T_b decrease towards zero. Thus, as Q increases, the tachyonic fluid domain shrinks down to a singularity and the entire Schwarzschild T-domain is recovered. The situation is illustrated in figure 3.1.
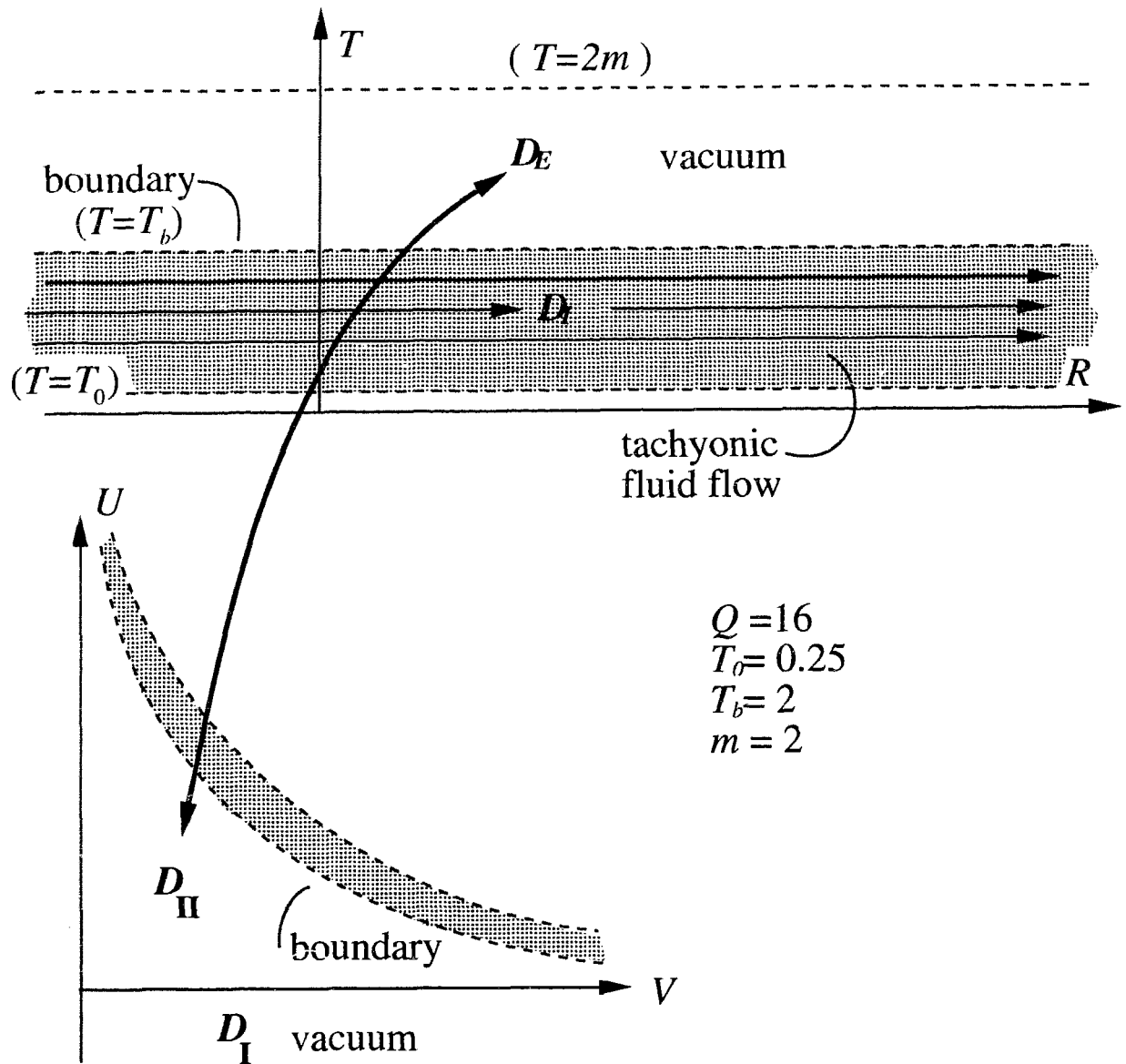
Figure 3.1: Qualitative representation of the tachyonic fluid within the T-domain and the image in the Kruskal space-time.

Chapter 4

The Tolman-Bondi solutions

The Tolman-Bondi solutions consist of spherically symmetric space-times containing the simplest kind of perfect fluid. It is assumed that the fluid matter consists of a fine dust; as a result, there is no pressure. The stress-energy tensor is greatly simplified by virtue of this assumption and it is the resulting mass density alone that is responsible for gravity. This model was first applied by Tolman [28] to model a star by a spherically symmetric, inhomogeneous, pressure-free fluid body. It was further analysed numerically by Oppenheimer and Snyder [20] to study gravitational contraction. This class of space-times bears Bondi's name also due to his later research (see [3]). The Tolman-Bondi solutions have been applied primarily to cosmological models. Tolman-Bondi space-times have also been used recently to construct counterexamples to the Cosmic Censorship Hypothesis[1]. However, more than fifty years after their initial discovery, it is difficult to find a mathematically thorough description of gravitational collapse in a Tolman-Bondi space-time.

This chapter provides a complete global analysis describing the gravitational collapse of an inhomogeneous, pressure-free fluid body into a black hole. This includes three possible cases (parabolic, elliptic and hyperbolic) complete with necessary junction conditions applied at the boundary of the body. The detailed analysis includes the transformation of the exterior metric to the exterior Schwarzschild form and the

---

[1]This conjecture is due to Penrose and roughly states that no acceptable solution of the Einstein field equations will result in a singularity that does not lie behind some kind of event horizon.

exterior form in the T-domain of the vacuum spherically symmetric solution.

4.1 Integrating the Field Equations

For a perfect fluid, three of the eigenvalues of the tensor $T^a{}_b$ are equivalent, so $T^a{}_b$ can be expressed in covariant form as $T_{ab} = (p + \mu)u_a u_b + p g_{ab}$, where μ is the energy density, u^a is the 4-velocity field of the fluid matter and p is the isotropic pressure within the fluid. In the case of an incoherent fluid or dust, the pressure is assumed to be zero. Thus, Einstein's field equations are given by the system

$$\mathcal{E}_{ij} := G_{ij} + \kappa T_{ij} = 0, \tag{4.1a}$$

$$T^{ab} := \mu u^a u^b, \tag{4.1b}$$

$$\mathcal{T}^a := \nabla_b T^{ab} = 0, \tag{4.1c}$$

$$\mathcal{U} := u^a u_a + 1 = 0, \tag{4.1d}$$

The mixed tensor $T^a{}_b$ has some interesting algebraic properties. Multiplying T^{ab} by $g_{bk} u^k$ and contracting gives

$$T^a{}_k u^k = -\mu u^a,$$

since $u_k u^k = -1$. Therefore, $-\mu$ is an eigenvalue of the 4×4 matrix $[T^a{}_b]$; the corresponding eigenvector components are given by the time-like vector u^a. The other three eigenvalues are exactly zero (which is the pressure in the perfect fluid).

Using the equation (4.1b) for the stress-energy-momentum tensor, the conservation equations (4.1c) imply that

$$\mathcal{T}^a := u^a \nabla_b(\mu u^b) + \mu u^b \nabla_b(u^a) = 0. \tag{4.2}$$

Multiplying through by $(-u_a)$ and substituting (4.1d) again, the above equation yields

$$-u_a \mathcal{T}^a = \nabla_b(\mu u^b) = 0, \tag{4.3}$$

since $u^a \nabla_b u_a = 0$. The equation (4.3) is the continuity equation for the four-dimensional fluid flow: together with (4.2) and an additional requirement $\mu \neq 0$, it follows that the stream-lines are geodesics, since $u^b \nabla_b u^a = 0$.

Looking at the tensor T_{ab} in (4.1b), the three energy conditions are satisfied (since the pressure $p = 0$) provided

$$\mu \geq 0. \tag{4.4}$$

In a spherically symmetric comoving coordinate system, the line element can be written as [15]

$$ds^2 = e^\lambda d\rho^2 + r^2 d\Omega^2 - d\tau^2, \tag{4.5a}$$

$$\lambda = \Lambda(\rho, \tau), \tag{4.5b}$$

$$r = \mathfrak{R}(\rho, \tau) > 0, \tag{4.5c}$$

$$d\Omega^2 := d\theta^2 + \sin^2\theta \, d\phi^2. \tag{4.5d}$$

(This is equivalent to identifying $\rho \equiv x^1$, $\theta \equiv x^2$, $\phi \equiv x^3$, $\tau \equiv x^4$, $\alpha(x^1, x^4) \equiv \Lambda(\rho, \tau)$, $\beta(x^1, x^4) \equiv 2 \ln \mathfrak{R}(\rho, \tau)$ and $\gamma(x^1, x^4) \equiv 0$ in the general ansatz (2.1) for spherical symmetry.) In a coordinate system in which $r > 0$ and $\theta \in (0, \pi)$, typical domains of validity are

$$\widetilde{D}_I := \{(\rho, \theta, \phi, \tau) : \rho_c < \rho < \rho_b, \ 0 < \theta < \pi, \ -\pi < \phi < \pi, \ \mathfrak{T}_0(\rho) < \tau < \mathfrak{T}_1(\rho)\},$$

$$\widetilde{D}_E := \{(\rho, \theta, \phi, \tau) : \rho_b < \rho, \ 0 < \theta < \pi, \ -\pi < \phi < \pi, \ \mathfrak{T}_0(\rho) < \tau < \mathfrak{T}_1(\rho)\},$$

$$\partial\widetilde{D}_I := \{(\rho, \theta, \phi, \tau) : \rho = \rho_b, \ 0 < \theta < \pi, -\pi < \phi < \pi, \ \mathfrak{T}_0(\rho_b) < \tau < \mathfrak{T}_1(\rho_b)\}.$$

A special class of exact solutions of the geodesic equations is given by

$$\dot{\rho} \equiv 0, \ \dot{\theta} \equiv 0, \ \dot{\phi} \equiv 0, \ \dot{\tau} \equiv 1,$$

$$\rho(s) = \text{constant}, \ \theta(s) = \text{constant}, \ \phi(s) = \text{constant}, \ \tau(s) = s,$$

where dots refer to differentiation with respect to the parameter s. This parameterised curve is a time-like radial geodesic curve. For a collapsing dust cloud, choose the fluid

velocities along such geodesics. Therefore, the components u^a are expressed in the comoving coordinate system as

$$u^1 = u^2 = u^3 \equiv 0, \quad u^4 \equiv 1 \equiv -u_4. \tag{4.6}$$

Hence, each surface $\rho = \rho_0 = \text{constant}$ is associated with a collapsing spherical shell of dust particles at rest in this frame.

The choice of the 4-velocity u^a in (4.6) simplifies the stress-energy-momentum tensor; using (4.5a) and (4.5d), all the components of $T^a{}_b$ vanish except $T^4{}_4 = -\mu$. The Einstein field equations (1.35) reduce to the following four non-trivial equations:

$$
\begin{aligned}
\mathcal{E}^1{}_1 \ &:= \ G^1{}_1 + \kappa T^1{}_1 \\
&= \ \left(\frac{2}{r}\right)(\partial_\tau^2 r) + (\partial_\tau \ln r)^2 - (\partial_\rho \ln r)^2 e^{-\lambda} + r^{-2} = 0,
\end{aligned}
\tag{4.7a}
$$

$$
\begin{aligned}
\mathcal{E}^2{}_2 \ &\equiv \ \mathcal{E}^3{}_3 := G^3{}_3 + \kappa T^3{}_3 \\
&= \ \frac{1}{r}(\partial_\tau^2 r) + \frac{1}{2}(\partial_\tau^2 \lambda) + \left(\frac{1}{2}\partial_\tau \lambda\right)^2 + \left(\frac{1}{2}(\partial_\rho \ln r)(\partial_\rho \lambda) - r^{-1}(\partial_\rho^2 r)\right) e^{-\lambda} \\
&\quad + \frac{1}{2r}(\partial_\tau r)(\partial_\tau \lambda) = 0,
\end{aligned}
\tag{4.7b}
$$

$$
\begin{aligned}
\mathcal{E}^4{}_4 \ &:= \ G^4{}_4 + \kappa T^4{}_4 \\
&= \ \left((\partial_\rho \ln r)(\partial_\rho \lambda) - \frac{2}{r}(\partial_\rho^2 r) - (\partial_\rho \ln r)^2\right) e^{-\lambda} \\
&\quad + (\partial_\tau \ln r)^2 + (\partial_\tau \ln r)(\partial_\tau \lambda) + r^{-2} - \kappa\mu = 0,
\end{aligned}
\tag{4.7c}
$$

$$
\begin{aligned}
\mathcal{E}^1{}_4 \ &:= \ G^1{}_4 + \kappa T^1{}_4 \\
&= \ r^{-1}\left(2(\partial_\tau \partial_\rho r) - (\partial_\tau \lambda)(\partial_\rho r)\right) e^{-\lambda} = 0.
\end{aligned}
\tag{4.7d}
$$

The conservation equations (4.1c) go over into one non-trivial equation:

$$
\begin{aligned}
\mathcal{T}_4 \ &= \ \partial_\rho T^1{}_4 + \left(\frac{1}{2}(\partial_\rho \lambda) + 2\partial_\rho(\ln r)\right) T^1{}_4 \\
&\quad - \left(\frac{1}{2}(\partial_\tau \lambda)T^1{}_1 + 2(\partial_\tau \ln r)T^2{}_2\right) + \partial_\tau T^4{}_4 + \left(\frac{1}{2}(\partial_\tau \lambda) + 2\partial_\tau \ln r\right) T^4{}_4 \\
&= \ -\left\{\partial_\tau \mu + \mu \partial_\tau \left(\frac{\lambda}{2} + 2\ln r\right)\right\} = 0.
\end{aligned}
\tag{4.8}
$$

The differential identities $\nabla_a \mathcal{E}^a{}_b + \kappa \mathcal{T}_b \equiv 0$ reduce to two non-trivial identities:

$$\partial_\rho T^1{}_1 + 2\,(\partial_\rho \ln r)\; T^1{}_1 + \partial_\tau T^4{}_1 + \left(\partial_\tau \left(\frac{\lambda}{2} + 2\ln r\right)\right) T^4{}_1$$
$$- 2(\partial_\rho \ln r) T^2{}_2 \equiv 0, \tag{4.9a}$$

$$\partial_\rho T^1{}_4 + \left(\partial_\rho \left(\frac{\lambda}{2} + 2\ln r\right)\right) T^1{}_4 - \left(\frac{1}{2}(\partial_\tau \lambda) T^1{}_1 + 2(\partial_\tau \ln r)\; T^2{}_2\right) + \partial_\tau T^4{}_4$$
$$+ \left(\partial_\tau \left(\frac{\lambda}{2} + 2\ln r\right)\right) T^1{}_4 - \kappa \mathcal{T}_4 \equiv 0. \tag{4.9b}$$

There are three unknown functions Λ, $\mathcal{R}$ and μ and five partial differential equations (4.7) and (4.8). Moreover, there are two differential identities (4.9). Thus, this is a determinate system of partial differential equations in a two-dimensional domain.

The strategy for solving this system is given in the following steps.

1. Solve the two equations $\mathcal{E}^1{}_4 = 0$ and $\mathcal{E}^1{}_1 = 0$.

2. At this stage, by the differential identity (4.9a) and an additional assumption $\partial_\rho r \neq 0$, the equation $\mathcal{E}^2{}_2 = 0$ must hold. (For the case $\partial_\rho r = 0$, see [15].)

3. Solve the equation $\mathcal{E}^4{}_4 = 0$ by defining $\mu(\rho, \tau)$.

4. It follows from the identity (4.9b) that the equation $\mathcal{T}_4 = 0$ must be satisfied.

Thus, the whole system of equations is solved. Notice that by solving the first order partial differential equation $\mathcal{E}^1{}_4 = 0$ for Λ and the second order partial differential equation $\mathcal{E}^1{}_1 = 0$ for $\mathcal{R}$, three arbitrary functions of a single variable appear in the general solution. (These arbitrary functions are denoted f, F and $\mathcal{T}_0$.)

Following this strategy for finding the solution, the equation (4.7d) yields

$$-re^\lambda (\partial_\rho r)^{-1} \mathcal{E}^1{}_4 = \partial_\tau (\lambda - 2\ln|\partial_\rho r|) = 0.$$

Integrating the above equation with respect to τ in a convex domain of the (ρ, τ)-plane gives

$$\lambda = \Lambda(\rho, \tau) = 2\ln\left|\frac{\partial \mathcal{R}}{\partial \rho}(\rho, \tau)\right| + h(\rho).$$

Here, h is an arbitrary function of integration and it belongs to the class $\mathcal{C}_p^3$. It turns out that the solutions can be conveniently classified into three distinct cases according to whether $\exp[-h(\rho)]$ is less than, greater than or equal to one. A convenient way to describe these three cases is to set

$$\exp[-h(\rho)] = 1 - \epsilon[f(\rho)]^2 > 0, \quad (\epsilon = 0, \pm 1).$$

Here, f is an arbitrary function of class $\mathcal{C}_p^3$. Using the above results, the metric tensor component g_{11} is

$$g_{11}(\rho, \tau) = \exp[\Lambda(\rho, \tau)] = \frac{\left[\frac{\partial \mathcal{R}}{\partial \rho}(\rho, \tau)\right]^2}{1 - \epsilon[f(\rho)]^2} > 0, \tag{4.10}$$

with the restrictions $\partial_\rho \mathcal{R} \neq 0$ and $\epsilon[f(\rho)]^2 < 1$.

Substituting the equation (4.10) for g_{11} into the field equation $\mathcal{E}^1{}_1 = 0$ (4.7a) gives $\mathcal{E}^1{}_1 = r^{-2}(2r\partial_\tau^2 r + (\partial_\tau r)^2 + 1) - r^{-2}(1 - \epsilon[f(\rho)]^2) = 0$. The result of canceling r^{-2} and multiplying with $\partial_\tau r$ is

$$\partial_\tau(r(\partial_\tau r)^2) = -\epsilon[f(\rho)]^2 \partial_\tau r.$$

Integrating with respect to τ gives

$$[\partial_\tau r]^2 = \left[\frac{\partial \mathcal{R}}{\partial \tau}(\rho, \tau)\right]^2 = \frac{F(\rho)}{\mathcal{R}(\rho, \tau)} - \epsilon[f(\rho)]^2. \tag{4.11}$$

The function F is an arbitrary function of integration of class $\mathcal{C}_p^3$, subject to the constraint $F(\rho) \geq \epsilon\mathcal{R}(\rho, \tau)[f(\rho)]^2$. The p.d.e. (4.11) ultimately determines the unknown function $\mathcal{R}$ in the general solution. It is studied extensively in the separate cases $\epsilon = 0, \pm 1$ in the following sections.

Using the p.d.e (4.11) and the expression for g_{11} in (4.10), it follows that

$$-\epsilon[f(\rho)]^2 = e^{-\lambda}(\partial_\rho r)^2 - 1 = (\partial_\tau r)^2 - F(\rho)/r.$$

Therefore, it is possible to express F in terms of $\mathcal{R}$ and Λ and to find the total

derivative F'. It turns out, upon differentiating, that F' is proportional to $G^4{}_4$:

$$F(\rho) = r + r(\partial_\tau r)^2 - re^{-\lambda}(\partial_\rho r)^2, \text{ so,}$$

$$F'(\rho) = r^2\partial_\rho r[e^{-\lambda}((\partial_\rho \ln r)(\partial_\rho\lambda) - 2r^{-1}\partial_\rho^2 r - (\partial_\rho \ln r)^2)$$

$$+(\partial_\tau \ln r)^2 + (\partial_\tau \ln r)(\partial_\tau\lambda) + r^{-2}]$$

$$= (r^2\partial_\rho r)G^4{}_4.$$

Thus, to satisfy the field equation $\mathcal{E}^4{}_4 = 0$ in (4.7c), define

$$\mu(\rho,\tau) := \kappa^{-1}G^4{}_4 = \frac{F'(\rho)}{8\pi[\mathfrak{R}(\rho,\tau)]^2\left[\frac{\partial\mathfrak{R}}{\partial\rho}(\rho,\tau)\right]}. \tag{4.12}$$

This definition implies $\mu(\rho,\tau) = 0 \Leftrightarrow F(\rho) = \text{constant}$.

The energy conditions (4.4) require $\mu \geq 0$. From the definition of μ in (4.12) and the restriction $\partial_\rho\mathfrak{R} \neq 0$, it follows that $F'(\rho)[\partial_\rho\mathfrak{R}]^{-1} \geq 0$. To satisfy this requirement, select $F'(\rho) \geq 0$ and $\partial_\rho\mathfrak{R} > 0$. In the interior of the collapsing star (corresponding to the interval $\rho_c < \rho < \rho_b$), assume that the proper mass density μ is strictly positive. Thus, assume that $F'(\rho) > 0$ inside the fluid body and

$$F(\rho_{c+}) = \lim_{\rho\downarrow\rho_c} F(\rho) := 0. \tag{4.13}$$

Returning to the problem of solving the field equations (4.7), all the equations have been reduced to the remaining non-linear, first order, second degree equation (4.11). It implies two distinct first order partial differential equations

$$\frac{\partial\mathfrak{R}}{\partial\tau}(\rho,\tau) = \pm\sqrt{\frac{F(\rho)}{\mathfrak{R}(\rho,\tau)} - \epsilon[f(\rho)]^2}. \tag{4.14}$$

For the gravitational collapse of a fluid body, the negative sign is physically reasonable. (For the expanding phase of the cosmological model, the positive sign would be the wiser choice.) The differential equation (4.14) is considered in the two-dimensional domains corresponding to the interior and exterior of the star as well as the intermediate boundary:

$$D_I := \{(\rho,\tau) : \rho_c < \rho < \rho_b,\ \mathfrak{T}_0(\rho) < \tau < \mathfrak{T}_1(\rho)\}, \tag{4.15a}$$

$$D_E := \{(\rho,\tau) : \rho_b < \rho,\ \mathfrak{T}_0(\rho) < \tau < \mathfrak{T}_1(\rho)\}, \tag{4.15b}$$

$$\partial D_I := \{(\rho,\tau) : \rho = \rho_b,\ \mathfrak{T}_0(\rho_b) < \tau < \mathfrak{T}_1(\rho_b)\}. \tag{4.15c}$$

If $B : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a scalar-valued function defined by $B(\rho, \tau) := \rho - \rho_b$, then the boundary ∂D_I can also be described as a level curve of B:

$$\partial D_I := \{(\rho, \tau) : B(\rho, \tau) := \rho - \rho_b = 0, \ \mathfrak{T}_0(\rho_b) < \tau < \mathfrak{T}_1(\rho_b)\},$$

$$\partial_\rho B \equiv 1, \ \partial_\tau B \equiv 0.$$

It is still necessary to verify the junction conditions. Since the boundary between matter and vacuum is given by the curve $B(\rho, \tau) \equiv 0$, the normal direction to this curve is given by $\nabla_a B$, the gradient of B. Thus, the junction conditions reduce to $[T^a{}_b \partial_a B]_{\partial D_I} = 0$. The explicit normal components can be used to reduce the junction conditions to the following two nontrivial equations:

$$[T^1{}_1 \partial_\rho B + T^1{}_1 \partial_\tau B]_{|\partial D_I} = [T^1{}_1]_{|\partial D_I} = 0,$$

$$[T^1{}_4 \partial_\rho B + T^1{}_4 \partial_\tau B]_{|\partial D_I} = [T^1{}_4]_{|\partial D_I} = 0.$$

Using the choice of stress-energy-momentum tensor in (4.1b) with the choice of fluid velocities in (4.6), the above junction conditions are identically satisfied.

Returning to the arbitrary function F that came out of the integration of the field equations, a physical interpretation is useful. The equation (4.12) relates F' to the mass density μ. It is reasonable, then , to look for a relationship between $F(\rho)$ and the total mass included in the spherical domain $(\rho_c, \rho]$. Define the "total effective mass function" $M^\#$ by

$$M^\#(\rho, \tau) := \int_{\rho_c}^{\rho} \int_0^\pi \int_{-\pi}^{\pi} \mu(x, \tau)[\mathfrak{R}(x, \tau)]^2 \sin\theta \frac{\partial \mathfrak{R}}{\partial x}(x, \tau) \, d\phi \, d\theta \, dx$$

$$= \int_0^{\mathfrak{R}(\rho, \tau)} \int_0^\pi \int_{-\pi}^{\pi} \mu r^2 \sin\theta \, d\phi \, d\theta \, dr$$

Using the expression (4.12) for μ and assumption (4.13), the integral in this definition can be evaluated:

$$M^\#(\rho, \tau) = \frac{1}{2}[F(\rho) - F(\rho_{c+})] = \frac{1}{2}F(\rho) > 0,$$

so $\partial_\tau M^\#(\rho, \tau) \equiv 0$. Therefore, the total effective mass of a spherical core corresponding to the interval (ρ_c, ρ) is conserved. Since the total effective mass function does not depend explicitly on τ, redefine the total effective mass function M by $M(\rho) :=$

$M^{\#}(\rho, \tau_c)$, where τ_c is a constant [3]. Replacing $M^{\#}$ by M in the above equations and recalling the p.d.e. (4.11), the function F is related to the total effective mass function M by the relations

$$F(\rho) = 2M(\rho) > 0, \quad \text{and} \tag{4.16a}$$

$$\lim_{\rho \downarrow \rho_c} M(\rho) = \frac{1}{2} \lim_{\rho \downarrow \rho_c} F(\rho) = 0, \tag{4.16b}$$

$$\frac{\partial^2 R}{\partial \tau^2}(\rho, \tau) = -\frac{M(\rho)}{[\mathcal{R}(\rho, \tau)]^2} < 0. \tag{4.16c}$$

The points corresponding to $\rho = \rho_c$ represent the world line of the centre of the star. The condition (4.16b) prevents a singularity from appearing at the centre before the final collapse. Also, notice that the equation (4.16c) resembles the classical inverse square law of Newtonian gravitation, even though this "equation of motion" emerges from Einstein's field equations.

The invariant volume element of the spatial hypersurface inherent in the metric (4.5a) is given by $e^{\lambda/2} r^2 \sin\theta \, d\rho \, d\phi \, d\theta$. Define the "total proper mass" function M_p [3] by

$$M_p(\rho) = \int_{\rho_c}^{\rho}\int_0^{\pi}\int_{-\pi}^{\pi} \mu(x, \tau) \exp[\Lambda(x, \tau)/2][\mathcal{R}(x, \tau)]^2 \sin\vartheta \, d\phi \, d\theta \, dx$$

$$= \frac{1}{2} \int_{\rho_c}^{\rho} \frac{F'(x)}{\sqrt{1 - \epsilon[f(x)]^2}} \, dx.$$

The "gravitational binding energy"[30, 3] is the difference of the effective and proper masses:

$$M_p(\rho) - M(\rho) = \frac{1}{2} \int_{\rho_c}^{\rho} \left[\frac{1}{\sqrt{1 - \epsilon[f(x)]^2}} - 1 \right] F'(x) \, dx. \tag{4.17}$$

The physical meaning of the arbitrary function f appearing in g_{11} and the p.d.e. 4.11 can be understood by studying a time-like radial geodesic in the Schwarzschild metric. It is given by the ordinary differential equation [5]

$$\frac{1}{2}\left[\frac{dR}{ds}(s)\right]^2 - \frac{m}{\mathcal{R}(s)} = \frac{1}{2}[E^2 - 1].$$

In the above o.d.e., s is the proper time parameter, m is the Schwarzschild mass of the spherically symmetric central body and E is the conserved total energy (including

the rest energy) of a unit mass test particle freely falling along the geodesic. The left-hand side of this o.d.e. resembles the "kinetic energy" plus the "potential energy" of the unit mass particle according to Newtonian physics. The p.d.e. (4.11), expressed with $2M$ rather than F at a particular value $\rho = \rho_0$, becomes

$$\frac{1}{2}\left[\frac{\partial \mathcal{R}}{\partial \tau}(\rho_0, \tau)\right]^2 - \frac{M(\rho_0)}{\mathcal{R}(\rho_0, \tau)} = -\left(\frac{\epsilon}{2}\right)[f(\rho_0)]^2 =: \frac{1}{2}\{[E(\rho_0)]^2 - 1\}.$$

The preceding p.d.e. and o.d.e. are remarkably similar. Therefore, physically speaking, it is reasonable to conclude that $\sqrt{1 - \epsilon[f(\rho_0)]^2}$ represents the total energy $E(\rho_0)$ (including the rest energy) of a unit mass dust particle following the radial geodesics characterised by $\rho = \rho_0$. (The negative root $-\sqrt{1 - \epsilon[f(\rho_0)]^2}$ is ignored.) In the p.d.e. above, the "potential energy" stems from the mass $M(\rho_0) > 0$ contained within the spherical core corresponding to the interval (ρ_c, ρ_0); as in Newtonian gravitation, the total mass of the external spherical shell (outside the interval (ρ_0, ρ_b)) does not affect the motion of the particle at $\rho = \rho_0$.

The curvature of the metric in (4.5a) also merits investigation, particularly for the identification of possible singularities. The corresponding orthonormal tetrad can be defined by

$$e^k_{(1)} = e^{-\lambda/2}\delta^k_{(1)}, \quad e^k_{(2)} = r^{-1}\delta^k_{(2)}, \quad e^k_{(3)} = r^{-1}(\sin\theta)^{-1}\delta^k_{(3)}, \quad e^k_{(4)} = \delta^k_{(4)}. \tag{4.18}$$

The non-zero Riemann invariants of the metric (4.5a) relative to the tetrad in (4.18) are the following:

$$R_{(2)(3)(2)(3)} = r^{-2}[1 - e^{-\lambda}(\partial_\rho r)^2 + (\partial_\tau r)^2] = \frac{F(\rho)}{r^3} = \frac{2M(\rho)}{r^3}, \tag{4.19a}$$

$$\begin{aligned} R_{(1)(2)(1)(2)} &= \frac{e^{-\lambda}}{r}\left(\frac{1}{2}(\partial_\rho r)(\partial_\rho \lambda) - (\partial_\rho^2 r)\right) + \frac{1}{2}(\partial_\tau \ln r)(\partial_\tau \lambda) \\ &= \frac{\kappa}{2}\mu - \frac{M(\rho)}{r^3} = R_{(1)(3)(1)(3)}, \end{aligned} \tag{4.19b}$$

$$R_{(1)(2)(2)(4)} = \frac{e^{-(\lambda/2)}}{r}\left(\partial_\rho \partial_\tau r - \frac{1}{2}(\partial_\rho r)(\partial_\tau \lambda)\right) \equiv 0 = R_{(1)(3)(3)(4)}, \tag{4.19c}$$

$$R_{(1)(4)(1)(4)} = -e^{-(\lambda/2)}\partial_\tau^2(e^{(\lambda/2)}) = \frac{\kappa}{2}\mu - \frac{2M(\rho)}{r^3},\qquad\qquad(4.19\text{d})$$

$$R_{(2)(4)(2)(4)} = -r^{-1}\partial_\tau^2 r = \frac{M(\rho)}{r^3} = R_{(3)(4)(3)(4)},\qquad\qquad(4.19\text{e})$$

$$\begin{aligned}R^{abcd}R_{abcd} &= R^{(a)(b)(c)(d)}R_{(a)(b)(c)(d)}\\ &= 3[\kappa\mu]^2 - 16\kappa\mu\frac{M(\rho)}{[\Re(\rho,\tau)]^3} + \frac{48[M(\rho)]^2}{[\Re(\rho,\tau)]^6}.\end{aligned}\qquad\qquad(4.19\text{f})$$

The Riemann invariants have been simplified with the aid of the field equations (4.7) as well as the equations (4.10) for e^λ, (4.12) for μ, (4.16) for M and the p.d.e. (4.11). It is interesting to note that the Riemann invariants reduce to algebraic functions of μ, M, and r and not on their derivatives.

The metric can be written as

$$ds^2 = \frac{\left[\frac{\partial\Re}{\partial\rho}(\rho,\tau)\right]^2}{1-\epsilon[f(\rho)]^2}d\rho^2 + [\Re(\rho,\tau)]^2 d\Omega^2 - d\tau^2.\qquad\qquad(4.20)$$

Alternately, the local solution of the p.d.e's (4.14) can be written in a unified fashion [23, 15] as parametric equations

$$r = \frac{F(\rho)}{2[f(\rho)]^2}h_\epsilon'(\eta),$$

$$\mathfrak{T}_0(\rho) - \tau = \pm\frac{F(\rho)}{2[f(\rho)]^3}h_\epsilon(\eta),$$

$$h_\epsilon(\eta) = \begin{cases} \eta^3/6 & \text{for } \epsilon = 0,\\ \eta\pm\sin\eta & \text{for } \epsilon = +1,\\ \sinh\eta - \eta & \text{for } \epsilon = -1,\end{cases}$$

This unified solution requires $f(\rho)\neq 0$. Further, $r = 0 \Rightarrow F(\rho) = 0$ or $h_\epsilon'(\eta) = 0$. However, expressing the metric tensor in terms of the parameters ρ and η proves to be very complicated. For global or semi-global analysis, the unified solutions are not convenient.

In the following sections, the p.d.e. (4.14) is solved for the distinct cases $\epsilon = 0, +1, -1$ respectively to give $\Re$, the last unknown function remaining. Borrowing

terminology from the theory of p.d.e's, these separate cases are called parabolic, elliptic and hyperbolic. The resulting metric tensor components is be analysed in detail. Matching conditions are found at the boundary ∂D_I and the external metric is transformed into the vacuum Schwarzschild metric (2.4) to complete the analysis in each case.

4.2 The Parabolic Case: $\epsilon = 0$

The partial differential equation (4.11) particular to this case is

$$\frac{1}{2}\left(\frac{\partial \mathcal{R}}{\partial \tau}(\rho_0, \tau)\right)^2 - \frac{M(\rho_0)}{\mathcal{R}(\rho_0, \tau)} \equiv 0.$$

Recalling the analogy with classical physics, this implies that the "kinetic energy" and the "potential energy" of a dust particle are exactly balanced. In a sense, the dust particles are "coasting."

The collapsing dust particles obey the p.d.e. (4.14):

$$\frac{\partial \mathcal{R}}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{F(\rho)}{\mathcal{R}(\rho, \tau)}}. \tag{4.21}$$

The earlier restriction of $\mathcal{R}$ as a positive real-valued function forces F to be positive also; thus, the right hand side of (4.21) is well-defined. The solution of this p.d.e. can be found by integrating with respect to τ:

$$r = \mathcal{R}(\rho, \tau) = \left(\frac{3}{2}\right)^{\frac{2}{3}} [F(\rho)]^{\frac{1}{3}} (\mathcal{T}_1(\rho) - \tau)^{\frac{2}{3}}, \tag{4.22a}$$

$$\mathcal{T}_1'(\rho) > 0, \tag{4.22b}$$

where $\mathcal{T}_1$ is the arbitrary function of integration and must be of class $\mathcal{C}_p^3$. Note that this solution implies

$$\lim_{\tau \to -\infty} \mathcal{R}(\rho, \tau) \to \infty, \quad \text{and} \quad \lim_{\tau \to -\infty} \frac{\partial \mathcal{R}}{\partial \tau}(\rho, \tau) = 0.$$

Moreover, for this solution,

$$\mathcal{R}(\rho, \tau) = 0 \Rightarrow F(\rho) = 0, \text{ or } \tau = \mathcal{T}_1(\rho).$$

Since F is nonnegative and monotone increasing, $\rho = \rho_c$ is the only root of the equation $F(\rho) = 0$ in $[\rho_c, \rho_b)$.

The two-dimensional domains corresponding to the interior, the exterior and the boundary of the star are

$$D_I := \{(\rho, \tau) : \rho_c < \rho < \rho_b, \ -\infty < \tau < \mathcal{T}_1(\rho)\}, \tag{4.23a}$$

$$D_E := \{(\rho, \tau) : \rho_b < \rho, \ -\infty < \tau < \mathcal{T}_1(\rho)\}, \tag{4.23b}$$

$$\partial D_I := \{(\rho, \tau) : \rho = \rho_b, \ -\infty < \tau < \mathcal{T}_1(\rho_b)\} \tag{4.23c}$$

(compare these to (4.15)). This metric tensor components for this solution include the the functions F and $\mathcal{T}_1$. These arbitrary functions are of class $\mathcal{C}_p^3$ in the unbounded domain $D_I \cup D_E \cup \partial D_I$. Since jump discontinuities are permissible on ∂D_I, introduce the following notation to denote the very smooth pieces of these functions by the following equations:

$$F(\rho) = \begin{cases} F_1(\rho) & \text{for } \rho_c < \rho < \rho_b, \\ F_E(\rho) & \text{for } \rho_b < \rho, \end{cases} \tag{4.24a}$$

$$\mathcal{T}_1(\rho) = \begin{cases} \mathcal{T}_1(\rho) & \text{for } \rho_c < \rho < \rho_b, \\ \mathcal{T}_E(\rho) & \text{for } \rho_b < \rho, \end{cases} \tag{4.24b}$$

$$\mathcal{R}(\rho, \tau) = \begin{cases} \mathcal{R}_1(\rho, \tau) & \text{for } (\rho, \tau) \in D_I, \\ \mathcal{R}_E(\rho, \tau) & \text{for } (\rho, \tau) \in D_E. \end{cases} \tag{4.24c}$$

In the exterior domain, the mass density $\mu(\rho, \tau) \equiv 0$. Recalling the relationship (4.12) between μ and F in the exterior domain, it follows that

$$F_E(\rho) = 2m > 0. \tag{4.25}$$

Here, m is a positive constant. Using the definition of the total effective mass function M together with the previous equations (4.24a) and (4.25), M is given by

$$M(\rho) = \frac{F(\rho)}{2} = \begin{cases} \frac{1}{2}F_1(\rho) & \text{for } \rho_c < \rho < \rho_b, \\ m & \text{for } \rho_b < \rho. \end{cases} \tag{4.26}$$

Therefore, $m > 0$ stands for the total (Schwarzschild) mass of the star.

With the notation just introduced, the interior and exterior line elements for the case $\epsilon = 0$ can be written using the general line element (4.20):

$$ds_1^2 = \left[\frac{\partial \mathcal{R}_1}{\partial \rho}(\rho, \tau)\right]^2 d\rho^2 + [\mathcal{R}_1(\rho, \tau)]^2 d\Omega^2 - d\tau^2, \tag{4.27a}$$

$$ds_E^2 = \left[\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau)\right]^2 d\rho^2 + [\mathcal{R}_E(\rho, \tau)]^2 d\Omega^2 - d\tau^2, \tag{4.27b}$$

where $0 < \mathcal{R}_1(\rho, \tau) < \mathcal{R}_E(\rho, \tau)$, $\partial_\rho \mathcal{R}_1 > 0$ and $\partial_\rho \mathcal{R}_E > 0$.

To continuously match the metric tensor components and their first partial derivatives across the boundary ∂D_I, introduce another convenient notation. Consider a function H belonging to the class $\mathcal{C}_p^3$ in $D_I \cup D_E$. The jump across the the boundary ∂D_I is defined in the usual manner as

$$\begin{aligned} [\Delta H(\rho_b, \tau)] &:= \lim_{\rho \downarrow \rho_b} H(\rho, \tau) - \lim_{\rho \uparrow \rho_b} H(\rho, \tau) \\ &\equiv H(\rho_{b+}, \tau) - H(\rho_{b-}, \tau) \end{aligned}$$

for all $-\infty < \tau < \mathcal{T}_1(\rho_b)$. In the case

$$[\Delta H(\rho_b, \tau)] \equiv 0,$$

the function H is said to have a removable discontinuity on ∂D_I. Henceforth, a removable discontinuity is always eliminated by definitions like

$$H(\rho_b, \tau) := \lim_{\rho \downarrow \rho_b} H(\rho, \tau) = \lim_{\rho \uparrow \rho_b} H(\rho, \tau). \tag{4.28}$$

Hence, the function H is now continuous across the boundary.

Proposition 4.1. *A necessary and sufficient condition for the continuity of the metric tensor components of* (4.27) *and their first order partial derivatives across the boundary* ∂D_I *is that the functions* F, F', F'', $\mathcal{T}_1$, $\mathcal{T}_1'$ *and* $\mathcal{T}_1''$ *(where primes denote total derivatives) are all continuous across* ∂D_I.

Proof. Using the solution (4.22a) and the definitions (4.24) of the interior and exterior branches, it is possible to summarise all the partial derivatives of $\mathcal{R}$ up to and including 2nd order:

$$\mathcal{R}_{\mathrm{I}}(\rho,\tau) = \left(\frac{3}{2}\right)^{\frac{2}{3}} [F_{\mathrm{I}}(\rho)]^{\frac{1}{3}} (\mathcal{T}_{\mathrm{I}}(\rho) - \tau)^{\frac{2}{3}} > 0, \tag{4.29a}$$

$$\mathcal{R}_{\mathrm{E}}(\rho,\tau) = \left(\frac{3}{2}\right)^{\frac{2}{3}} [2m]^{\frac{1}{3}} (\mathcal{T}_{\mathrm{E}}(\rho) - \tau)^{\frac{2}{3}} > 0, \tag{4.29b}$$

$$\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \rho}(\rho,\tau) = \frac{1}{3}\mathcal{R}_{\mathrm{I}}(\rho,\tau) \left([\ln F_{\mathrm{I}}(\rho)]' + \frac{2\mathcal{T}_{\mathrm{I}}'(\rho)}{\mathcal{T}_{\mathrm{I}}(\rho) - \tau}\right) > 0, \tag{4.29c}$$

$$\frac{\partial \mathcal{R}_{\mathrm{E}}}{\partial \rho}(\rho,\tau) = \frac{2}{3}\mathcal{R}_{\mathrm{E}}(\rho,\tau) \left(\frac{\mathcal{T}_{\mathrm{E}}'(\rho)}{\mathcal{T}_{\mathrm{E}}(\rho) - \tau}\right) > 0, \tag{4.29d}$$

$$\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \tau}(\rho,\tau) = -\frac{2}{3}\frac{\mathcal{R}_{\mathrm{I}}(\rho,\tau)}{(\mathcal{T}_{\mathrm{I}}(\rho) - \tau)} < 0, \tag{4.29e}$$

$$\frac{\partial \mathcal{R}_{\mathrm{E}}}{\partial \tau}(\rho,\tau) = -\frac{2}{3}\left(\frac{\mathcal{R}_{\mathrm{E}}(\rho,\tau)}{\mathcal{T}_{\mathrm{E}}(\rho) - \tau}\right) < 0, \tag{4.29f}$$

$$\frac{\partial^2 \mathcal{R}_{\mathrm{I}}}{\partial \rho^2}(\rho,\tau) = \frac{1}{3}\mathcal{R}_{\mathrm{I}}(\rho,\tau) \left([\ln F_{\mathrm{I}}(\rho)]'' - 2\left[\frac{\mathcal{T}_{\mathrm{I}}'(\rho)}{\mathcal{T}_{\mathrm{I}}(\rho) - \tau}\right]^2 + \frac{2\mathcal{T}_{\mathrm{I}}''(\rho)}{\mathcal{T}_{\mathrm{I}}(\rho) - \tau}\right)$$
$$+ \frac{1}{3}\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \rho}(\rho,\tau) \left([\ln F_{\mathrm{I}}(\rho)]' + \frac{2\mathcal{T}_{\mathrm{I}}'(\rho)}{\mathcal{T}_{\mathrm{I}}(\rho) - \tau}\right), \tag{4.29g}$$

$$\frac{\partial^2 \mathcal{R}_{\mathrm{E}}}{\partial \rho^2}(\rho,\tau) = \frac{2}{3}\mathcal{R}_{\mathrm{E}}(\rho,\tau) \left(\frac{\mathcal{T}_{\mathrm{E}}''(\rho)}{\mathcal{T}_{\mathrm{E}}(\rho) - \tau} - \left[\frac{\mathcal{T}_{\mathrm{E}}'(\rho)}{\mathcal{T}_{\mathrm{E}}(\rho) - \tau}\right]^2\right)$$
$$+ \frac{2}{3}\left(\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \rho}(\rho,\tau)\frac{\mathcal{T}_{\mathrm{E}}'(\rho)}{\mathcal{T}_{\mathrm{I}}(\rho) - \tau}\right), \tag{4.29h}$$

$$\frac{\partial^2 \mathcal{R}_{\mathrm{I}}}{\partial \tau \partial \rho}(\rho,\tau) = \frac{1}{3}\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \tau}(\rho,\tau) \left([\ln F_{\mathrm{I}}(\rho)]' + \frac{2\mathcal{T}_{\mathrm{I}}'(\rho)}{\mathcal{T}_{\mathrm{I}}(\rho) - \tau}\right) + \frac{2}{3}\frac{\mathcal{R}_{\mathrm{I}}(\rho,\tau)\mathcal{T}_{\mathrm{I}}'(\rho)}{(\mathcal{T}_{\mathrm{I}}(\rho) - \tau)^2}, \tag{4.29i}$$

$$\frac{\partial^2 \mathcal{R}_{\mathrm{E}}}{\partial \tau \partial \rho}(\rho,\tau) = \frac{2}{3}\frac{\partial \mathcal{R}_{\mathrm{E}}}{\partial \tau}(\rho,\tau) \left(\frac{\mathcal{T}_{\mathrm{E}}'(\rho)}{\mathcal{T}_{\mathrm{E}}(\rho) - \tau}\right) + \frac{2}{3}\frac{\mathcal{R}_{\mathrm{E}}(\rho,\tau)\mathcal{T}_{\mathrm{E}}'(\rho)}{(\mathcal{T}_{\mathrm{E}}(\rho) - \tau)^2}, \tag{4.29j}$$

Looking at the line element (4.27), the continuity of the metric tensor component g_{22} across the boundary ∂D_I implies

$$[\triangle\mathcal{R}(\rho_b, \tau)] := \lim_{\rho\downarrow\rho_b}[\mathcal{R}_I(\rho, \tau)] - \lim_{\rho\downarrow\rho_b}[\mathcal{R}_E(\rho, \tau)] \equiv 0. \tag{4.30}$$

Using the expressions (4.29a,b) for $\mathcal{R}_I$ and $\mathcal{R}_E$, the condition (4.30) implies the identity

$$\sqrt{F_I(\rho_{b-})}[\mathcal{T}_I(\rho_{b-}) - \tau] \equiv \sqrt{2m}[\mathcal{T}_E(\rho_{b+}) - \tau]. \tag{4.31}$$

Differentiating this identity with respect to τ, and recalling F_E, F_I as described in (4.26) gives

$$F_I(\rho_{b-}) = 2m = F_E(\rho_{b+}). \tag{4.32}$$

Since F is nonnegative and monotone increasing, the condition (4.32) implies that the total mass function M attains its only extremum in the interval $(\rho_c, \rho_b]$ at the boundary point $\rho = \rho_b$ and its maximum value is m. This conclusion is physically reasonable.

Now, substitute the previous condition (4.32) into (4.31):

$$\mathcal{T}_I(\rho_{b-}) = \mathcal{T}_E(\rho_{b+}) =: \mathcal{T}_I(\rho_b). \tag{4.33}$$

At this stage, the functions F and $\mathcal{T}_I$ are continuous across the boundary.

Consider now the continuities of the first partial derivatives of g_{22}. Using the line element given in (4.27) and the continuity of $\mathcal{R}$ in (4.30), the continuity of $\partial_\tau g_{22}$ demands that

$$\lim_{\rho\downarrow\rho_b}\left[\frac{\partial\mathcal{R}_I}{\partial\tau}(\rho, \tau)\right] - \lim_{\rho\downarrow\rho_b}\left[\frac{\partial\mathcal{R}_E}{\partial\tau}(\rho, \tau)\right] \equiv 0. \tag{4.34}$$

The explicit forms of $\partial_\tau\mathcal{R}_I$ and $\partial_\tau\mathcal{R}_E$ are given in (4.29e,f); using the continuity of $\mathcal{R}$ given in (4.30), together with the continuity of F and $\mathcal{T}_I$ given in (4.32) and (4.33), the above conditions are automatically satisfied. (The continuity of $\partial_\tau\mathcal{R}$ given by (4.34) implies that the radial "velocities" of dust particles moving along radial geodesics across the boundary of the star are continuous).

To consider the implications of the continuity of $\partial_\rho g_{22}$, explore the condition

$$\lim_{\rho \uparrow \rho_b} \left[\frac{\partial \mathcal{R}_I}{\partial \rho}(\rho, \tau) \right] - \lim_{\rho \downarrow \rho_b} \left[\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau) \right] \equiv 0. \tag{4.35}$$

Using the explicit expressions for $\partial_\rho \mathcal{R}_I$ and $\partial_\rho \mathcal{R}_E$ given in (4.29c,d), together with the continuities of $\mathcal{R}$, F and $\mathcal{T}_I$, the continuity of $\partial_\rho \mathcal{R}$ in equation (4.35) implies

$$[\mathcal{T}_I(\rho_b) - \tau] \lim_{\rho \uparrow \rho_b} [\ln F_I(\rho)]' + 2 \left(\lim_{\rho \uparrow \rho_b} [\mathcal{T}_I'(\rho)] - \lim_{\rho \uparrow \rho_b} [\mathcal{T}_E'(\rho)] \right) \equiv 0. \tag{4.36}$$

Differentiating with respect to τ and using the relation between F and μ given in (4.11) gives

$$\lim_{\rho \uparrow \rho_b} F_I'(\rho) = 0, \tag{4.37a}$$

$$\lim_{\rho \uparrow \rho_b} \mu(\rho, \tau) \equiv 0, \tag{4.37b}$$

$$\mu(\rho, \tau) \equiv 0 \text{ for } \rho \geq \rho_b. \tag{4.37c}$$

The above conditions (4.37) and the definition of F_E in (4.25) make the functions F' and μ continuous across the boundary. The continuity of F' at $\rho = \rho_b$ in (4.37a), when used in (4.36), yields

$$\lim_{\rho \uparrow \rho_b} [\mathcal{T}_I'(\rho)] - \lim_{\rho \downarrow \rho_b} [\mathcal{T}_E'(\rho)] \equiv 0. \tag{4.38}$$

Therefore, the function $\mathcal{T}_I'$ is also continuous across the boundary.

Returning again to the line element (4.27), consider the continuity of the derivatives of g_{11}. The continuity of g_{11} implies the continuity of $\partial_\rho \mathcal{R}$; this condition has already been investigated (see (4.35)), so no additional information emerges. The continuity of $\partial_\rho g_{11}$ implies

$$\lim_{\rho \uparrow \rho_b} \left[\frac{\partial^2 \mathcal{R}_I}{\partial \rho^2}(\rho, \tau) \right] - \lim_{\rho \downarrow \rho_b} \left[\frac{\partial^2 \mathcal{R}_E}{\partial \rho^2}(\rho, \tau) \right] \equiv 0. \tag{4.39}$$

The explicit expressions (4.29g,h) for $\partial_\rho^2 \mathcal{R}_I$ and $\partial_\rho^2 \mathcal{R}_E$ can be used with the continuities of $\mathcal{R}$, $\partial_\rho \mathcal{R}$, F, F', $\mathcal{T}_I$ and $\mathcal{T}_I'$ to derive the identity

$$(\mathcal{T}_I(\rho_b) - \tau) \left(\lim_{\rho \uparrow \rho_b} [\ln F_I(\rho)]'' \right) + 2 \left(\lim_{\rho \uparrow \rho_b} [\mathcal{T}_I''(\rho)] - \lim_{\rho \uparrow \rho_b} [\mathcal{T}_E''(\rho)] \right) \equiv 0. \tag{4.40}$$

Differentiating the above with respect to τ,

$$\lim_{\rho \downarrow \rho_b}[\ln F_{\rm I}(\rho)]'' = 0. \tag{4.41}$$

Substituting (4.41) back into (4.40) yields

$$\lim_{\rho \downarrow \rho_b}[\mathcal{T}_{\rm I}''(\rho)] - \lim_{\rho \downarrow \rho_b}[\mathcal{T}_{\rm E}''(\rho)] \equiv 0. \tag{4.42}$$

Thus, it follows that the functions F'' and $\mathcal{T}_{\rm I}''$ are both continuous across the boundary. The continuity of $\partial_\tau g_{11}$ does not yield new conditions upon looking at the explicit expression for $\partial_\tau \partial_\rho \mathcal{R}$ and observing the continuities already established. The continuities of g_{33} are equivalent to those of g_{22} and the continuities of $g_{44} = -1$ are trivial. Therefore, it has been proved that the continuities of the metric tensor components and of their first partial derivatives across the boundary ∂D_I imply the continuities of F, F', F'', $\mathcal{T}_{\rm I}$, $\mathcal{T}_{\rm I}'$ and $\mathcal{T}_{\rm I}''$ across the boundary ∂D_I. The converse statement follows from the explicit expressions (4.29) for all the derivatives of $\mathcal{R}_{\rm I}$ and $\mathcal{R}_{\rm E}$ and observing the continuities of F, F', F'', $\mathcal{T}_{\rm I}$, $\mathcal{T}_{\rm I}'$ and $\mathcal{T}_{\rm I}''$ across the boundary ∂D_I.   □

To complete this analysis, a general coordinate transformation relating this local coordinate system to the Schwarzschild coordinate system (2.4) is needed. The metric in equations (4.27) can be expressed with the help of the explicit expressions for the solution $\mathcal{R}$ and its derivatives given in

$$ds_{\rm I}^2 = [\mathcal{R}_{\rm I}(\rho, \tau)]^2 \left((1/9) \left[\frac{F_{\rm I}'(\rho)}{F_{\rm I}(\rho)} + \frac{2\mathcal{T}_{\rm I}'(\rho)}{\mathcal{T}_{\rm I}(\rho) - \tau} \right]^2 d\rho^2 + d\Omega^2 \right) - d\tau^2,$$

$$ds_{\rm E}^2 = \left(\frac{4m}{3(\mathcal{T}_{\rm E}(\rho) - \tau)} \right)^{\frac{2}{3}} [\mathcal{T}_{\rm E}'(\rho)]^2 d\rho^2 + \left(\frac{3}{2}\sqrt{2m}(\mathcal{T}_{\rm E}(\rho) - \tau) \right)^{\frac{4}{3}} d\Omega^2 - d\tau^2.$$

From the equation (4.22b), $\mathcal{T}_{\rm I}'(\rho) > 0$ and $\mathcal{T}_{\rm E}'(\rho) > 0$ so both are monotone increasing.

This allows the introduction of another coordinate system:

$$\widehat{r} = \mathcal{T}_1(\rho),$$ (4.43a)

$$\widehat{t} = \tau,$$ (4.43b)

$$\frac{\partial(\widehat{r},\widehat{t})}{\partial(\rho,\tau)} = \mathcal{T}_1'(\rho) > 0, \quad (\rho,\tau) \in D_I \cup D_E \cup \partial D_I,$$ (4.43c)

$$\widehat{D} := \widehat{D}_I \cup \widehat{D}_E \cup \partial \widehat{D}_I$$ (4.43d)

$$\widehat{D}_I := \{(\widehat{r},\widehat{t}) : \widehat{r}_c < \widehat{r} < \widehat{r}_b, \ -\infty < \widehat{t} < \widehat{r}\},$$ (4.43e)

$$\widehat{D}_E := \{(\widehat{r},\widehat{t}) : \widehat{r}_b < \widehat{r}, \ -\infty < \widehat{t} < \widehat{r}\},$$ (4.43f)

$$\partial\widehat{D}_I := \{(\widehat{r},\widehat{t}) : \widehat{r} = \widehat{r}_b, \ -\infty < \widehat{t} < \widehat{r}_b\},$$ (4.43g)

$$\widehat{r}_c := \mathcal{T}_1(\rho_c), \quad \widehat{r}_b := \mathcal{T}_1(\rho_b).$$ (4.43h)

In this transformation, $\widehat{\theta} = \theta$ and $\widehat{\phi} = \phi$ and thus play passive roles. The metric tensor components in the new $(\widehat{r},\widehat{t})$-coordinates are found from under the usual transformation of coordinates:

$$ds_I^2 = \frac{2\widehat{M}(\widehat{r})\left(1 + \left[\ln\sqrt{2\widehat{M}(\widehat{r})}\right]'(\widehat{r}-\widehat{t})\right)^2 (d\widehat{r})^2}{\left[\frac{3}{2}\sqrt{2\widehat{M}(\widehat{r})}(\widehat{r}-\widehat{t})\right]^{\frac{2}{3}}}$$
$$+ \left[\frac{3}{2}\sqrt{2\widehat{M}(\widehat{r})}(\widehat{r}-\widehat{t})\right]^{\frac{4}{3}} (d\widehat{\Omega})^2 - (d\widehat{t})^2,$$

$$ds_E^2 = \frac{2m(d\widehat{r})^2}{[(3/2)\sqrt{2m}(\widehat{r}-\widehat{t})]^{\frac{2}{3}}} + \left[\frac{3}{2}\sqrt{2m}(\widehat{r}-\widehat{t})\right]^{\frac{4}{3}} (d\widehat{\Omega})^2 - (d\widehat{t})^2,$$

where $2\widehat{M}(\widehat{r}) = \widehat{F}(\widehat{r}) := (F \circ \mathcal{T}_1^{-1})(\widehat{r})$. The line element in the exterior domain is the Lemaître line element (2.5) for the spherically symmetric vacuum solution. By analogy, the metric in the interior domain is called the interior Lemaître metric.

Having obtained the Lemaître metric in the exterior domain where $\widehat{r} > \widehat{r}_b$, recall that the event horizon in these coordinates is the curve $\widehat{r}-\widehat{t} = 4m/3$ in the $(\widehat{r},\widehat{t})$-plane. Hence, in the exterior domain, the Lemaître $(\widehat{r},\widehat{t})$-coordinates can be transformed into Schwarzschild coordinates (2.4) in the R-domain $\widehat{r} - \widehat{t} > 4m/3$ (see (2.5)). In the T-domain where $\widehat{r} > \widehat{r}_b$ and $\widehat{r} - \widehat{t} < 4m/3$, the $(\widehat{r},\widehat{t})$-coordinates can be transformed
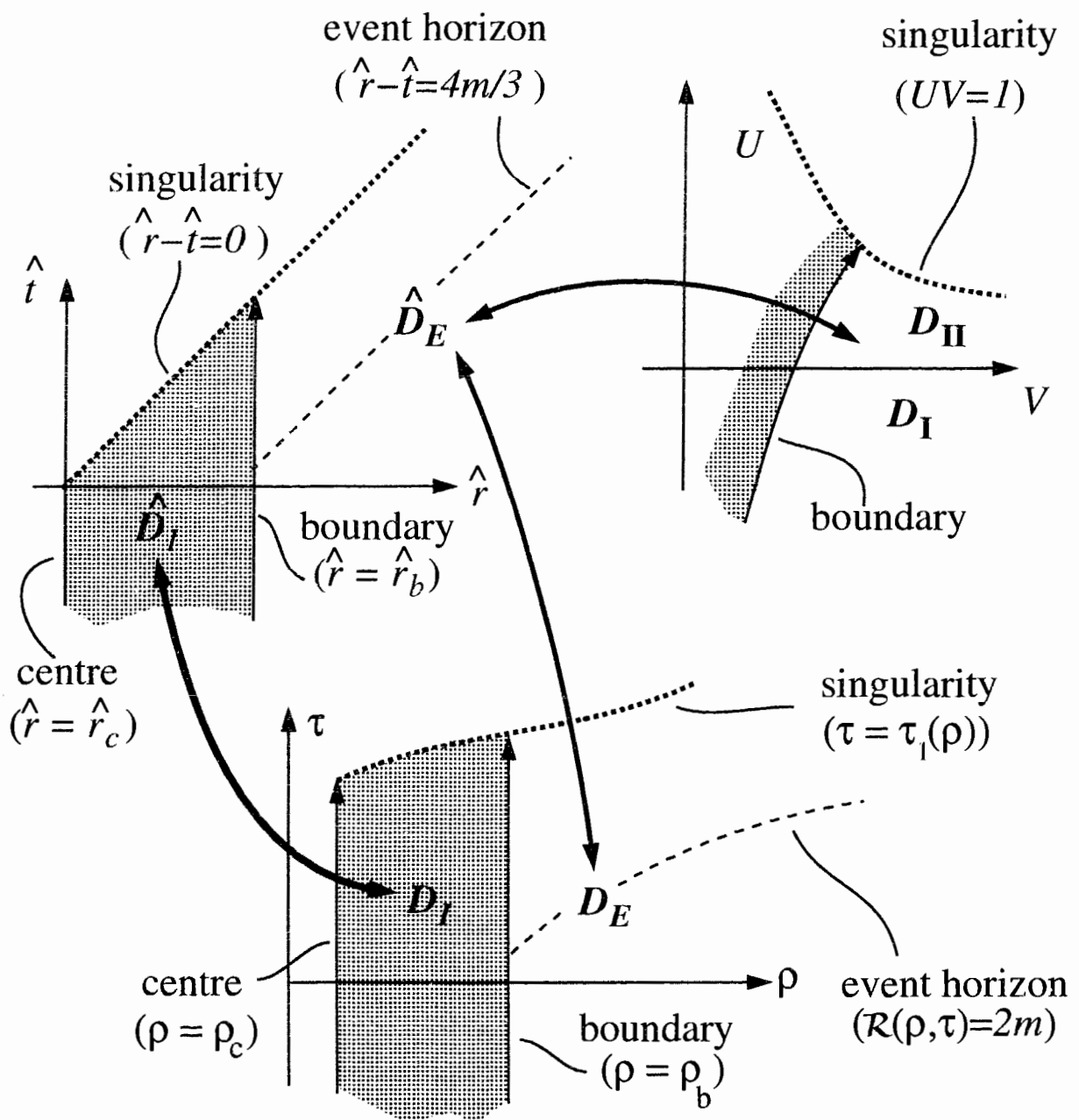
Figure 4.1: Gravitational collapse of a dust sphere of parabolic ($\epsilon = 0$) type.

into vacuum Schwarzschild (R, T)-coordinates (see (2.9)). Both of these domains can then be mapped using (2.15) and (2.16) into the Kruskal sub-domain $D_i \cup D_{ii} \cup D_{ii}$. This is illustrated in figure 4.1 which shows the gravitational collapse of a spherical dust ball of parabolic type into a black hole.

4.3 The Elliptic Case: $\epsilon = +1$

The equation (4.11) yields

$$\frac{1}{2}\left[\frac{\partial \mathcal{R}}{\partial \tau}(\rho_0, \tau)\right]^2 - \frac{M(\rho_0)}{\mathcal{R}(\rho_0, \tau)} = -\frac{1}{2}[f(\rho_0)]^2 < 0,$$

where $f(\rho_0) \neq 0$. In this case, the absolute value of the "potential energy" exceeds the "kinetic energy." The p.d.e. (4.14) for the gravitational collapse becomes

$$\frac{\partial \mathcal{R}}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{F(\rho)}{\mathcal{R}(\rho, \tau)} - [f(\rho)]^2} < 0, \tag{4.44}$$

where $0 < [f(\rho)]^2 < F(\rho)/\mathcal{R}(\rho, \tau)$. Conclude from this inequality that

$$\lim_{\rho \downarrow \rho_c} F(\rho) = 0 \Rightarrow \lim_{\rho \downarrow \rho_c} \mathcal{R}(\rho, \tau) \equiv 0. \tag{4.45}$$

Every solution of (4.44) can define a curve in the (ρ, τ)-plane by the equation

$$\frac{F(\rho)}{[f(\rho)]^2 \mathcal{R}(\rho, \tau)} - 1 = 0.$$

On this curve, $\partial_\tau \mathcal{R} = 0$. If this curve is taken as an initial boundary of the domain of consideration $\tau = \mathcal{T}_0(\rho)$ then all collapsing dust particles start from rest at finite proper times. Therefore, out of the three cases considered for this dust model, this case is the most pertinent in describing the collapse of a spherically symmetric star into a black hole. If a particular $\rho_0 \in (\rho_c, \rho_b]$ is chosen, then by the equality (4.44) and the first mean value theorem,

$$\begin{aligned} \mathcal{R}(\rho_0, \tau) &= \mathcal{R}(\rho_0, \mathcal{T}_0(\rho_0)) + (\tau - \mathcal{T}_0(\rho_0))\left[\frac{\partial \mathcal{R}}{\partial \tau}(\rho_0, (1-\theta)\mathcal{T}_0(\rho_0) + \theta\tau)\right] \\ &< \mathcal{R}(\rho_0, \mathcal{T}_0(\rho_0)), \end{aligned}$$

where $(0 < \theta < 1)$. Therefore, $\mathcal{R}(\rho_0, \mathcal{T}_0(\rho_0))$ is the maximum value of $\mathcal{R}(\rho_0, \tau)$ for $\tau \in (\mathcal{T}_0(\rho_0), \mathcal{T}_1(\rho_0)]$.

The general component g_{11} in (4.10) in the case $\epsilon = +1$, subject to the constraints $\partial_\rho \mathcal{R} > 0$ and $f(\rho) \neq 0$, is

$$\exp[\Lambda(\rho, \tau)] = \frac{\left[\frac{\partial \mathcal{R}}{\partial \rho}(\rho, \tau)\right]^2}{1 - [f(\rho)]^2} > 0,$$

where $0 < |f(\rho)| < 1$.

The partial differential equation (4.44) can be regarded as an ordinary differential equation in τ; the result of integrating with respect to τ is

$$\tau - \mathcal{T}_0(\rho) = \frac{\mathcal{R}(\rho, \tau)}{|f(\rho)|} \sqrt{\frac{F(\rho)}{[f(\rho)]^2 \mathcal{R}(\rho, \tau)} - 1}$$

$$\pm \frac{F(\rho)}{|f(\rho)|^3} \arctan \sqrt{\frac{F(\rho)}{[f(\rho)]^2 \mathcal{R}(\rho, \tau)} - 1}. \tag{4.46}$$

This equation defines the function $\mathcal{R}$ implicitly. The arbitrary function $\mathcal{T}_0$ of class $\mathcal{C}_\rho^3$ arises out of the integration. (The expression under the root sign is always positive by virtue of the inequality (4.44).) The principal branch of the arctan function is chosen from now on without loss of generality.

The interior, exterior and the boundary of the star are assumed to be corresponding to the following domains (compare to equations (4.23ab,c)) :

$$D_I := \{(\rho, \tau) : \rho_c < \rho < \rho_b, \mathcal{T}_0(\rho) < \tau < \mathcal{T}_1(\rho)\}, \tag{4.47a}$$

$$D_E := \{(\rho, \tau) : \rho_b < \rho, \mathcal{T}_0(\rho) < \tau < \mathcal{T}_1(\rho)\}, \tag{4.47b}$$

$$\partial D_I := \{(\rho, \tau) : \rho = \rho_b, \mathcal{T}_0(\rho_b) < \tau < \mathcal{T}_1(\rho_b)\}, \tag{4.47c}$$

$$\mathcal{T}_1(\rho) := \mathcal{T}_0(\rho) + \frac{\pi F(\rho)}{2|f(\rho)|^3}. \tag{4.47d}$$

The choice of $\mathcal{T}_1$ is essentially explained in the equation (4.71f). Note also that D_I is a bounded domain.

Since differentiation of the absolute value function is essentially complicated, a simplifying assumption is made:

$$0 < f(\rho) < 1. \tag{4.48}$$

The smooth branches of the functions F, $\mathfrak{T}_0$ and $\mathfrak{R}$ are denoted exactly as in (4.24). A similar notation is used for the function f. Moreover, the function F is again nonnegative, monotone increasing and constant for $\rho > \rho_b$. The interior and exterior line elements are

$$ds_{\mathrm{I}}^2 = \frac{\left[\frac{\partial \mathfrak{R}_{\mathrm{I}}}{\partial \rho}(\rho, \tau) \right]^2}{1 - [f_{\mathrm{I}}(\rho)]^2} d\rho^2 + [\mathfrak{R}_{\mathrm{I}}(\rho, \tau)]^2 d\Omega^2 - d\tau^2, \qquad (4.49a)$$

$$ds_{\mathrm{E}}^2 = \frac{\left[\frac{\partial \mathfrak{R}_{\mathrm{E}}}{\partial \rho}(\rho, \tau) \right]^2}{1 - [f_{\mathrm{E}}(\rho)]^2} d\rho^2 + [\mathfrak{R}_{\mathrm{E}}(\rho, \tau)]^2 d\Omega^2 - d\tau^2. \qquad (4.49b)$$

Once again, the conditions for the matching of the metric tensor components and their first order partial derivatives must be found.

Proposition 4.2. *A necessary and sufficient condition for the continuity of the metric tensor components of* (4.49) *and their first order partial derivatives across the boundary* ∂D_I *is that the functions* F, F', F'', $\mathfrak{T}_1$, $\mathfrak{T}_1'$ *and* $\mathfrak{T}_1''$ *(where primes denote total derivatives) are all continuous across* ∂D_I.

Proof. From the implicit solution (4.46) for $\mathfrak{R}$, the interior and exterior branches of the function $\mathfrak{R}$ satisfy

$$\tau - \mathfrak{T}_{\mathrm{I}}(\rho) = \frac{\mathfrak{R}_{\mathrm{I}}(\rho, \tau)}{f_{\mathrm{I}}(\rho)} W_{\mathrm{I}}(\rho, \tau) + \frac{F_{\mathrm{I}}(\rho)}{[f_{\mathrm{I}}(\rho)]^3} \arctan[W_{\mathrm{I}}(\rho, \tau)], \qquad (4.50a)$$

$$\tau - \mathfrak{T}_{\mathrm{E}}(\rho) = \frac{\mathfrak{R}_{\mathrm{E}}(\rho, \tau)}{f_{\mathrm{E}}(\rho)} W_{\mathrm{E}}(\rho, \tau) + \frac{2m}{[f_{\mathrm{E}}(\rho)]^3} \arctan[W_{\mathrm{E}}(\rho, \tau)], \qquad (4.50b)$$

$$
\begin{aligned}
\frac{\partial \mathfrak{R}_{\mathrm{I}}}{\partial \rho}(\rho, \tau) &= \left(\frac{F_{\mathrm{I}}'(\rho)}{F_{\mathrm{I}}(\rho)} - \frac{2 f_{\mathrm{I}}'(\rho)}{f_{\mathrm{I}}(\rho)} \right) \mathfrak{R}_{\mathrm{I}}(\rho, \tau) \\
&\quad + W_{\mathrm{I}}(\rho, \tau) \left\{ \mathfrak{T}_{\mathrm{I}}'(\rho) + \left(\frac{F_{\mathrm{I}}'(\rho)}{F_{\mathrm{I}}(\rho)} - \frac{3 f_{\mathrm{I}}'(\rho)}{f_{\mathrm{I}}(\rho)} \right) (\tau - \mathfrak{T}_{\mathrm{I}}(\rho)) \right\} \\
&= \frac{f_{\mathrm{I}}'(\rho)}{f_{\mathrm{I}}(\rho)} \mathfrak{R}_{\mathrm{I}}(\rho, \tau) + f_{\mathrm{I}}(\rho) \mathfrak{T}_{\mathrm{I}}'(\rho) W_{\mathrm{I}}(\rho, \tau) \\
&\quad + \left(\frac{F_{\mathrm{I}}'(\rho)}{[f_{\mathrm{I}}(\rho)]^2} - \frac{3 F_{\mathrm{I}}(\rho) f_{\mathrm{I}}'(\rho)}{[f_{\mathrm{I}}(\rho)]^3} \right) \{ 1 + W_{\mathrm{I}}(\rho, \tau) \arctan[W_{\mathrm{I}}(\rho, \tau)] \}, \qquad (4.50c)
\end{aligned}
$$

$$\frac{\partial \mathcal{R}_{\mathrm{E}}}{\partial \rho}(\rho, \tau) = \frac{f_{\mathrm{E}}'(\rho)}{f_{\mathrm{E}}(\rho)} \mathcal{R}_{\mathrm{E}}(\rho, \tau) + f_{\mathrm{E}}(\rho) \mathcal{T}_{\mathrm{E}}'(\rho) W_{\mathrm{E}}(\rho, \tau)$$
$$- \frac{6m f_{\mathrm{E}}'(\rho)}{[f_{\mathrm{E}}(\rho)]^3} \{ 1 + W_{\mathrm{E}}(\rho, \tau) \arctan[W_{\mathrm{E}}(\rho, \tau)] \}. \tag{4.50d}$$

$$\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{F_{\mathrm{I}}(\rho)}{\mathcal{R}_{\mathrm{I}}(\rho, \tau)} - [f_{\mathrm{I}}(\rho)]^2} < 0, \tag{4.50e}$$

$$\frac{\partial \mathcal{R}_{\mathrm{E}}}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{2m}{\mathcal{R}_{\mathrm{E}}(\rho, \tau)} - [f_{\mathrm{E}}(\rho)]^2} < 0, \tag{4.50f}$$

$$\frac{\partial^2 \mathcal{R}_{\mathrm{I}}}{\partial \tau \partial \rho}(\rho, \tau) = \frac{1}{2} \left(\frac{F_{\mathrm{I}}'(\rho)}{\mathcal{R}_{\mathrm{I}}(\rho, \tau)} - \frac{F_{\mathrm{I}}(\rho) \frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \rho}(\rho, \tau)}{[\mathcal{R}_{\mathrm{I}}(\rho, \tau)]^2} - 2 f_{\mathrm{I}}(\rho) f_{\mathrm{I}}'(\rho) \right) \left[\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \tau}(\rho, \tau) \right]^{-1}, \tag{4.50g}$$

$$\frac{\partial^2 \mathcal{R}_{\mathrm{E}}}{\partial \tau \partial \rho}(\rho, \tau) = - \left(\frac{m}{[\mathcal{R}_{\mathrm{E}}(\rho, \tau)]^2} \frac{\partial \mathcal{R}_{\mathrm{E}}}{\partial \rho}(\rho, \tau) + f_{\mathrm{E}}(\rho) f_{\mathrm{E}}'(\rho) \right) \left[\frac{\partial \mathcal{R}_{\mathrm{E}}}{\partial \tau}(\rho, \tau) \right]^{-1}, \tag{4.50h}$$

$$\frac{\partial^2 \mathcal{R}_{\mathrm{I}}}{\partial \rho^2}(\rho, \tau) = [\ln f_{\mathrm{I}}(\rho)]' \frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \rho}(\rho, \tau) + \mathcal{R}_{\mathrm{I}}(\rho, \tau) [\ln f_{\mathrm{I}}(\rho)]''$$
$$+ \frac{1}{2} \left(\frac{F_{\mathrm{I}}'(\rho)}{[f_{\mathrm{I}}(\rho)]^2} - \frac{3 F_{\mathrm{I}}(\rho) f_{\mathrm{I}}'(\rho)}{[f_{\mathrm{I}}(\rho)]^3} \right) \left(\left[\ln \frac{F_{\mathrm{I}}(\rho)}{[f_{\mathrm{I}}(\rho)]^2} \right]' - \frac{\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \rho}(\rho, \tau)}{\mathcal{R}_{\mathrm{I}}(\rho, \tau)} \right)$$
$$+ \frac{1}{2} \frac{F_{\mathrm{I}}(\rho)}{[f_{\mathrm{I}}(\rho)]^2 \mathcal{R}_{\mathrm{I}}(\rho, \tau) W_{\mathrm{I}}(\rho, \tau)} \left(\left[\ln \frac{F_{\mathrm{I}}(\rho)}{[f_{\mathrm{I}}(\rho)]^2} \right]' - \frac{\frac{\partial \mathcal{R}_{\mathrm{I}}}{\partial \rho}(\rho, \tau)}{\mathcal{R}_{\mathrm{I}}(\rho, \tau)} \right) \times$$
$$\left(f_{\mathrm{I}}(\rho) \mathcal{T}_{\mathrm{I}}'(\rho) + \left(\frac{F_{\mathrm{I}}'(\rho)}{[f_{\mathrm{I}}(\rho)]^2} - \frac{3 F_{\mathrm{I}}(\rho) f_{\mathrm{I}}'(\rho)}{[f_{\mathrm{I}}(\rho)]^3} \right) \arctan[W_{\mathrm{I}}(\rho, \tau)] \right)$$
$$+ W_{\mathrm{I}}(\rho, \tau) [f_{\mathrm{I}}(\rho) \mathcal{T}_{\mathrm{I}}'(\rho)]' + (1 + W_{\mathrm{I}}(\rho, \tau) \arctan[W_{\mathrm{I}}(\rho, \tau)]) \times$$
$$\left(\frac{F_{\mathrm{I}}''(\rho)}{[f_{\mathrm{I}}(\rho)]^2} - 5 \frac{F_{\mathrm{I}}'(\rho) f_{\mathrm{I}}'(\rho)}{[f_{\mathrm{I}}(\rho)]^3} - 3 \frac{F_{\mathrm{I}}(\rho) f_{\mathrm{I}}''(\rho)}{[f_{\mathrm{I}}(\rho)]^3} + 9 \frac{F_{\mathrm{I}}(\rho) [f_{\mathrm{I}}'(\rho)]^2}{[f_{\mathrm{I}}(\rho)]^4} \right), \tag{4.50i}$$

$$\frac{\partial^2 \mathcal{R}_E}{\partial \rho^2}(\rho, \tau) = [\ln f_E(\rho)]' \frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau) + \mathcal{R}_E(\rho, \tau)[\ln f_E(\rho)]''$$

$$+ \frac{3m f_E'(\rho)}{[f_E(\rho)]^3} \left(2[\ln f_E(\rho)]' + \frac{\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau)}{\mathcal{R}_E(\rho, \tau)} \right) + W_E(\rho, \tau)[f_E(\rho)\mathcal{T}_E'(\rho)]'$$

$$- \frac{m}{[f_E(\rho)]^2 \mathcal{R}_E(\rho, \tau) W_E(\rho, \tau)} \left(2[\ln f_E(\rho)]' + \frac{\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau)}{\mathcal{R}_E(\rho, \tau)} \right) \times$$

$$\left(f_E(\rho)\mathcal{T}_E'(\rho) - \frac{6m f_E'(\rho)}{[f_E(\rho)]^3} \arctan[W_E(\rho, \tau)] \right)$$

$$- 6m \left(1 + W_E(\rho, \tau) \arctan[W_E(\rho, \tau)] \right) \left(\frac{f_E''(\rho)}{[f_E(\rho)]^3} - 3\frac{[f_E'(\rho)]^2}{[f_L(\rho)]^4} \right), (4.50j)$$

$$W_I(\rho, \tau) := \sqrt{\frac{F_I(\rho)}{[f_I(\rho)]^2 \mathcal{R}_I(\rho, \tau)} - 1}, \tag{4.50k}$$

$$W_E(\rho, \tau) := \sqrt{\frac{2m}{[f_E(\rho)]^2 \mathcal{R}_E(\rho, \tau)} - 1}. \tag{4.50l}$$

Consider the continuities of the metric tensor components (4.49) and of their first order partial derivatives. Recall that the continuity of g_{22} implies (compare with the equation (4.30))

$$\mathcal{R}_E(\rho_{b+}, \tau) - \mathcal{R}_I(\rho_{b-}, \tau) \equiv 0. \tag{4.51}$$

The continuity of $\partial_\tau g_{22}$ implies $[\Delta(\partial_\tau \mathcal{R})^2(\rho_b, \tau] \equiv 0$, so the explicit equations (4.50e,f) and the previous condition (4.51) imply that

$$\frac{[F_I(\rho_{b-}) - 2m]}{\mathcal{R}(\rho_b, \tau)} \equiv [\Delta(f(\rho_b))^2]. \tag{4.52}$$

Differentiate (4.52) with respect to τ. Recalling that $\partial_\tau \mathcal{R}(\rho, \tau) < 0$ and using (4.52) again, the result is

$$F_I(\rho_{b-}) - 2m = 0, \tag{4.53a}$$

$$[\Delta f(\rho_b)] = 0. \tag{4.53b}$$

Therefore, the functions F and f are continuous across the boundary. Continuities of $\mathcal{R}$, F and f used in the expressions (4.50a,b) for $\partial_\tau \mathcal{R}$ imply that

$$[\Delta \tau_o(\rho_b)] = 0. \tag{4.54}$$

(Notice here that the continuities of F, f and $\mathfrak{T}_0$ also imply the continuity of $\mathfrak{R}$.)

The continuity of $\partial_\rho g_{22}$ implies that $[\Delta\partial_\rho\mathfrak{R}(\rho_b,\tau)] \equiv 0$. Using the continuity of $\mathfrak{R}$ in (4.51) and the explicit expressions for $\partial_\rho\mathfrak{R}$ in (4.50c,d),

$$(f(\rho_b)[\Delta\mathfrak{T}_0'(\rho_b)])w(\tau) + \left(\frac{2m[\Delta f'(\rho_b)]}{[f(\rho_b)]^3}\right)(1 + [w(\tau)]^2)^{-1}$$
$$+ \left(\frac{[\Delta F'(\rho_b)]}{[f(\rho_b)]^2} - \frac{6m[\Delta f'(\rho_b)]}{[f(\rho_b)]^3}\right)(1 + w(\tau)\arctan(w(\tau))) \equiv 0, \tag{4.55}$$

where

$$*w(\tau) := \sqrt{\frac{2m}{[f(\rho_b)]^2\mathfrak{R}(\rho_b,\tau)} - 1}.$$

Now, the Wronskian of the set of functions $\{\theta,(1+\theta^2)^{-1},1+\theta\arctan\theta\}$ in an interval is given by

$$\begin{vmatrix} \theta & (1+\theta^2)^{-1} & 1+\theta\arctan\theta \\ 1 & -2\theta(1+\theta^2)^{-2} & \arctan\theta+\theta(1+\theta^2)^{-1} \\ 0 & -2(1-3\theta^2)(1+\theta^2)^{-3} & 2(1+\theta^2)^{-2} \end{vmatrix} = -4(1+\theta^2)^{-4} < 0.$$

Therefore, the set of functions $\{w(\tau),(1+[w(\tau)]^2)^{-1},1+w(\tau)\arctan(w(\tau))\}$ is linearly independent in the interval $\mathfrak{T}_0(\rho) < \tau < \mathfrak{T}_1(\rho)$. It follows from the relation (4.55) that the coefficients of the independent functions must all be zero:

$$f(\rho_b)[\Delta\mathfrak{T}_0'(\rho_b)] = 0,$$

$$\frac{2m[\Delta f'(\rho_b)]}{[f(\rho_b)]^3} = 0,$$

$$\frac{[\Delta F'(\rho_b)]}{[f(\rho_b)]^2} - \frac{6m[\Delta f'(\rho_b)]}{[f(\rho_b)]^3} = 0.$$

The above equations imply the three independent continuities:

$$[\Delta\mathfrak{T}_0'(\rho_b)] = 0, \tag{4.56a}$$

$$[\triangle f'(\rho_b)] = 0,$$ (4.56b)

$$[\triangle F'(\rho_b)] = 0.$$ (4.56c)

Hence, the functions f, F and $\mathfrak{T}_0$ and their first derivatives are all continuous across the boundary $\rho = \rho_b$.

Now, the continuities of g_{11} and f imply the continuity of $\partial_\rho \mathfrak{R}$. This condition has already been examined, so no other additional information emerges. The continuity of $\partial_\tau g_{11}$ implies the continuity of $\partial_\tau \partial_\rho \mathfrak{R}$. Looking at the explicit equations (4.50g,h) for $\partial_\tau \partial_\rho \mathfrak{R}$, no new equation arises. The continuity of $\partial_\rho g_{11}$ implies that

$$\left[\triangle \frac{\partial^2 \mathfrak{R}}{\partial \rho^2}(\rho_b, \tau) \right] \equiv 0.$$ (4.57)

The continuities already established reduce the above identity to

$$2m \frac{[\triangle f''(\rho_b)]}{[f(\rho_b)]^3} \{ 1 + [w(\tau)]^2 \}^{-1} + f(\rho_b)[\triangle \mathfrak{T}_0''(\rho_b)]w(\tau)$$

$$+ \frac{1}{[f(\rho_b)]^2} \left(F''(\rho_{b-}) - 6m \frac{[\triangle f''(\rho_b)]}{f(\rho_b)} \right) \{ 1 + w(\tau)\arctan[w(\tau)] \} \equiv 0.$$ (4.58)

Recalling that the set of functions $\{ w(\tau), (1 + [w(\tau)]^2)^{-1}, 1 + w(\tau)\arctan[w(\tau)] \}$ is linearly independent in any interval, the identity (4.58) reduces to three equalities:

$$F''(\rho_{b-}) = 0,$$ (4.59a)

$$[\triangle f''(\rho_b)] = 0,$$ (4.59b)

$$[\triangle \mathfrak{T}_0''(\rho_b)] = 0.$$ (4.59c)

Therefore, it has been established that the continuities of the metric tensor components in (4.49a,b) and of their first partial derivatives imply that the functions $F, f, \mathfrak{T}_0, F', f', \mathfrak{T}_0', F'', f''$ and $\mathfrak{T}_0''$ are continuous across the boundary. $\square$

As before, a transformation from the exterior metric (4.49b) into the vacuum Schwarzschild metric (2.4) is desired to complete this description. Since the solution R of the p.d.e. (4.44) is given implicitly, this transformation is not as straightforward as it is for the case $\epsilon = 0$. The transformation of the coordinate chart is

$$\hat{r} = \widehat{\mathcal{R}}(\rho, \tau) := \mathcal{R}_E(\rho, \tau),$$
$$\hat{t} = \widehat{T}(\rho, \tau),$$

where the coordinates $\hat{\theta} \equiv \theta$ and $\hat{\phi} \equiv \phi$ play passive roles in this transformation. As such, this can be regarded as a transformation from one two-dimensional domain to another. Since $\mathcal{R}$ is known implicitly, the only unknown function is $\widehat{T}$. To derive $\widehat{T}$, use the transformation rule for the transformation of the contravariant metric tensor components to derive at a pair of p.d.e.'s for $\widehat{T}$ [20]:

$$\hat{g}^{ab}(\hat{x}) = \frac{\partial \widehat{X}^a}{\partial x^c}(x) \frac{\partial \widehat{X}^b}{\partial x^d}(x) g^{cd}(x),$$

where $(\hat{x}^1, \hat{x}^4) := (\hat{r}, \hat{t})$ and $(x^1, x^4) := (\rho, \tau)$. The hatted metric tensor components should be those of the Schwarzschild metric (see (2.4)); recalling the exterior line element (4.49b), the following equations result:

$$0 = \hat{g}^{14} = \left[\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau) \right] \left[\frac{\partial \widehat{T}}{\partial \rho}(\rho, \tau) \right] \left(\frac{1 - [f_E(\rho)]^2}{\left[\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau) \right]^2} \right) - \left[\frac{\partial \mathcal{R}_E}{\partial \tau}(\rho, \tau) \right] \left[\frac{\partial \widehat{T}}{\partial \tau}(\rho, \tau) \right],$$

$$\tag{4.60a}$$

$$-\left(1 - \frac{2m}{\hat{r}} \right)^{-1} = \hat{g}^{44} = \left[\frac{\partial \widehat{T}}{\partial \rho}(\rho, \tau) \right]^2 \left(\frac{1 - [f_E(\rho)]^2}{\left[\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau) \right]^2} \right) - \left[\frac{\partial \widehat{T}}{\partial \tau}(\rho, \tau) \right]^2, \tag{4.60b}$$

$$1 - \frac{2m}{\hat{r}} = \hat{g}^{11} = \left[\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau) \right]^2 \left(\frac{1 - [f_E(\rho)]^2}{\left[\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau) \right]^2} \right) - \left[\frac{\partial \mathcal{R}_E}{\partial \tau}(\rho, \tau) \right]^2. \tag{4.60c}$$

Recalling the p.d.e (4.50f) for the case $\epsilon = +1$, the third equation (4.60c) above is identically satisfied. Simplifying the first equation (4.60a) above gives a linear,

homogeneous first order p.d.e. in the unknown function $\widehat{T}$:

$$\frac{\partial \widehat{T}}{\partial \rho}(\rho, \tau) = \frac{\partial \mathfrak{R}_{E}}{\partial \rho}(\rho, \tau)\frac{\partial \mathfrak{R}_{E}}{\partial \tau}(\rho, \tau)[1 - [f_{E}(\rho)]^{2}]^{-1}\frac{\partial \widehat{T}}{\partial \tau}(\rho, \tau). \tag{4.61}$$

Substituting the above into (4.60b) and using (4.50f), the result is the first order, second degree equation

$$\left[\frac{\partial \widehat{T}}{\partial \tau}(\rho, \tau)\right]^{2} = \left(1 - \left[\frac{\partial \mathfrak{R}_{E}}{\partial \tau}(\rho, \tau)\right]^{2}[1 - [f_{E}(\rho)]^{2}]^{-1}\right)^{-1}\left(1 - \frac{2m}{\widehat{r}}\right)^{-1}$$

$$= [1 - [f_{E}(\rho)]^{2}]\left(1 - \frac{2m}{\widehat{r}}\right)^{-2} > 0. \tag{4.62}$$

Out of two possibilities in (4.62), choose $\partial_{\tau}\widehat{T} > 0$ to preserve the orientation of the time-like variable. Therefore, the equation (4.62) yields

$$\frac{\partial \widehat{T}}{\partial \tau} = \sqrt{1 - [f_{E}(\rho)]^{2}}\left(1 - \frac{2m}{\widehat{r}}\right)^{-1} > 0. \tag{4.63}$$

Substituting (4.63) into (4.61) gives

$$\frac{\partial \widehat{T}}{\partial \rho}(\rho, \tau) = \frac{\partial \mathfrak{R}_{E}}{\partial \rho}(\rho, \tau)\frac{\partial \mathfrak{R}_{E}}{\partial \tau}(\rho, \tau)[1 - [f_{E}(\rho)]^{2}]^{-\frac{1}{2}}\left(1 - \frac{2m}{\widehat{r}}\right)^{-1} < 0. \tag{4.64}$$

The pair of first order partial differential equations (4.63) and (4.64) is soluble provided the right hand sides satisfy an integrability condition. Checking the integrability condition in these (ρ, τ)-coordinates is quite difficult. It is more convenient to introduce an intermediate coordinate chart by the following equations:

$$\xi = \Xi(\rho, \tau) := \rho, \tag{4.65a}$$

$$\eta = \mathcal{H}(\rho, \tau) := 2\arctan\sqrt{\frac{2m}{[f_{E}(\rho)]^{2}\mathfrak{R}_{E}(\rho, \tau)} - 1}, \tag{4.65b}$$

$$\tan^{2}\left(\frac{\eta}{2}\right) = \frac{2m}{[f_{E}(\rho)]^{2}\mathfrak{R}_{E}(\rho, \tau)} - 1 > 0, \tag{4.65c}$$

$$\frac{\partial(\xi, \eta)}{\partial(\rho, \tau)} = \frac{\partial \mathcal{H}}{\partial \tau}(\rho, \tau) = \frac{f_{E}(\rho)}{\mathfrak{R}_{E}(\rho, \tau)} > 0, \tag{4.65d}$$

$$D_{E} = \{(\rho, \tau) : \rho_{b} < \rho, \mathcal{T}_{E}(\rho) < \tau < \mathcal{T}_{1}(\rho)\}. \tag{4.65e}$$

Furthermore, use the notation $f_E^{\#}(\xi) \equiv f_E(\rho) = f_E(\xi)$ and $\mathcal{T}_E^{\#}(\xi) \equiv \mathcal{T}_E(\rho) = \mathcal{T}_E(\xi)$. The inverse transformation to (4.65) can be derived with the assistance of the implicit equation (4.50b) for $\mathcal{R}_E$:

$$\rho = \mathcal{R}(\xi) := \xi, \tag{4.66a}$$

$$\tau = \mathcal{T}(\xi, \eta) := \mathcal{T}_E(\xi) + \frac{m}{[f_E(\xi)]^3}(\eta + \sin \eta), \tag{4.66b}$$

$$\frac{\partial(\rho, \tau)}{\partial(\xi, \eta)} = 2m \frac{\cos^2(\eta/2)}{[f_E(\xi)]^3} > 0, \tag{4.66c}$$

$$D_E^{\#} := \{(\xi, \eta) : \rho_b < \xi, 0 < \eta < \pi\}. \tag{4.66d}$$

(It is necessary to restrict $\eta \in (0, \pi)$ to preserve the inequality (4.44)). The transformation of (ξ, η)-coordinates into the Schwarzschild coordinates can be written explicitly by the transformation

$$\hat{r} = \mathcal{R}_E(\rho, \tau) = 2m \frac{\cos^2(\eta/2)}{[f_E(\xi)]^2} = m \frac{(1 + \cos \eta)}{[f_E(\xi)]^2} =: \mathcal{R}^{\#}(\xi, \eta), \tag{4.67a}$$

$$\hat{t} = \widehat{T}(\rho, \tau) =: T^{\#}(\xi, \eta), \tag{4.67b}$$

$$D_S^{\#} := \{(\xi, \eta) : \rho_b < \xi, 0 < \eta < \eta_h(\xi)\} \subset D_E^{\#}, \tag{4.67c}$$

$$\eta_h(\xi) := 2 \arccos[f_E(\xi)]. \tag{4.67d}$$

The above choice of $\eta_h(\xi)$ implies by (4.67a) that $\hat{r} = \mathcal{R}^{\#}(\xi, \eta_h(\xi)) \equiv 2m$. The function $T^{\#}$ in (4.67b) is still unknown and has to be determined. The pair of p.d.e's (4.63) and (4.64) to be solved can now be expressed in terms of (ξ, η)-coordinates with the assistance of the chain rule and the transformations (4.66) and (4.67). The

resulting p.d.e's are used to verify the integrability condition:

$$P(\xi,\eta) := \frac{\partial T^{\#}}{\partial \eta}(\xi,\eta) \tag{4.68a}$$

$$= \frac{\partial \widehat{T}}{\partial \rho}(\rho,\tau)\frac{\partial \mathcal{R}}{\partial \eta}(\xi,\eta) + \frac{\partial \widehat{T}}{\partial \tau}(\rho,\tau)\frac{\partial \mathcal{T}}{\partial \eta}(\xi,\eta)$$

$$= 0 + \sqrt{1 - [f_E(\xi)]^2}\left(1 - \frac{2m}{\mathcal{R}^{\#}(\xi,\eta)}\right)^{-1}\left(\frac{m(1+\cos\eta)}{[f_E(\xi)]^3}\right)$$

$$= \frac{2m\Sigma(\xi)}{[f_E(\xi)]^3}\left(\frac{\cos^2(\eta/2)}{1 - [f_E(\xi)]^2/\cos^2(\eta/2)}\right)$$

$$= \frac{2m\Sigma(\xi)}{[f_E(\xi)]^3}\left(\cos^2(\eta/2) + [f_E(\xi)]^2 + \frac{[f_E(\xi)]^4}{\cos^2(\eta/2) - [f_E(\xi)]^2}\right),$$

$$Q(\xi,\eta) := \frac{\partial T^{\#}}{\partial \xi}(\xi,\eta) \tag{4.68b}$$

$$= \frac{\partial \widehat{T}}{\partial \rho}(\rho,\tau)\frac{\partial \mathcal{R}}{\partial \xi}(\xi,\eta) + \frac{\partial \widehat{T}}{\partial \tau}(\rho,\tau)\frac{\partial \mathcal{T}}{\partial \xi}(\xi,\eta)$$

$$= \frac{\mathcal{T}_E'(\xi)}{\Sigma(\xi)} + \frac{mf_E'(\xi)}{\Sigma(\xi)[f_E(\xi)]^2}\left(-3\frac{\eta+\sin\eta}{[f_E(\xi)]^2} + 4\frac{\sin(\eta/2)\cos^3(\eta/2)}{\cos^2(\eta/2) - [f_E(\xi)]^2}\right),$$

$$\Sigma(\xi) := \sqrt{1 - [f_E(\xi)]^2}. \tag{4.68c}$$

The integrability condition for the pair of first order p.d.e's (4.68a,b) is

$$\frac{\partial P}{\partial \xi}(\xi,\eta) = \frac{-2mf_E'(\xi)\cos^4(\eta/2)}{[f_E(\xi)]^4\Sigma(\xi)} \times$$

$$\left(\frac{4[f_E(\xi)]^4 - 5[f_E(\xi)]^2 - 2[f_E(\xi)]^2\cos^2(\eta/2) + 3\cos^2(\eta/2)}{(\cos^2(\eta/2) - [f_E(\xi)]^2)^2}\right)$$

$$\equiv \frac{\partial Q}{\partial \eta}(\xi,\eta). \tag{4.69}$$

This integrability condition is identically satisfied. Therefore, a solution $T^{\#}(\xi,\eta)$ exists. In this coordinate system, the integration of this pair of p.d.e's is manageable. The solution is given explicitly by a line integral (along any continuous and piecewise differentiable curve Γ in $D_S^{\#}$):

$$\widehat{t} = T^{\#}(\xi,\eta) = \int_{(\xi,0)}^{(\xi,\eta)}\left[\frac{\partial T^{\#}}{\partial x}(x,y)dx + \frac{\partial T^{\#}}{\partial y}(x,y)dy\right]$$

$$= \int_{\xi_0}^{\xi}\frac{\mathcal{T}_E'(x)}{\sqrt{1 - [f_E(x)]^2}}dx + m\sqrt{1 - [f_E(\xi)]^2}\left(\frac{(\eta+\sin\eta)}{[f_E(\xi)]^3} + \frac{2\eta}{f_E(\xi)}\right)$$

$$+ 2m\ln\left[\frac{\tan(\eta_h(\xi)/2) + \tan(\eta/2)}{\tan(\eta_h(\xi)/2) - \tan(\eta/2)}\right], \tag{4.70a}$$

$$\tan(\eta_{\mathrm{H}}(\xi)/2) = [f_{\mathrm{E}}(\xi)]^{-1}\sqrt{1 - [f_{\mathrm{E}}(\xi)]^2}. \tag{4.70b}$$

Now, by (4.50b), (4.67b), (4.70) and (4.65b,c), the transformation from the (ρ, τ)-coordinates into the Schwarzschild $(\hat{r}, \hat{t})$-coordinates is

$$\hat{r} = \mathcal{R}_{\mathrm{E}}(\rho, \tau), \tag{4.71a}$$

$$
\begin{aligned}
\hat{t} = \widehat{T}(\rho, \tau) &= \int_{\rho_0}^{\rho} \frac{\mathcal{T}_{\mathrm{E}}'(x)}{\sqrt{1 - [f_{\mathrm{E}}(x)]^2}} dx \\
&+ \sqrt{1 - [f_{\mathrm{E}}(\rho)]^2}\left(\tau - \mathcal{T}_{\mathrm{E}}(\rho) + \frac{4m}{f_{\mathrm{E}}(\rho)}\arctan\sqrt{\frac{2m}{[f_{\mathrm{E}}(\rho)]^2 \mathcal{R}_{\mathrm{E}}(\rho, \tau)} - 1}\right) \\
&+ 2m\ln\left[\frac{\tan(\eta_{\mathrm{H}}(\rho)/2) + \sqrt{(2m/[f_{\mathrm{E}}(\rho)]^2 \mathcal{R}_{\mathrm{E}}(\rho, \tau)) - 1}}{\tan(\eta_{\mathrm{H}}(\rho)/2) - \sqrt{(2m/[f_{\mathrm{E}}(\rho)]^2 \mathcal{R}_{\mathrm{E}}(\rho, \tau)) - 1}}\right],
\end{aligned} \tag{4.71b}
$$

$$\tan(\eta_{\mathrm{H}}(\rho)/2) = [f_{\mathrm{E}}(\rho)]^{-1}\sqrt{1 - [f_{\mathrm{E}}(\rho)]^2}, \tag{4.71c}$$

$$\mathcal{T}_{\mathrm{H}}(\rho) := \mathcal{T}_{\mathrm{E}}(\rho) + \frac{m}{[f_{\mathrm{E}}(\rho)]^3}(\eta_{\mathrm{H}}(\rho) + \sin[\eta_{\mathrm{H}}(\rho)]), \tag{4.71d}$$

$$D_S := \{(\rho, \tau) : \rho_b < \rho, \mathcal{T}_{\mathrm{E}}(\rho) < \tau < \mathcal{T}_{\mathrm{H}}(\rho)\}. \tag{4.71e}$$

$$\mathcal{T}_{\mathrm{H}}(\rho) < \mathcal{T}_{\mathrm{I}}(\rho); \ \mathcal{T}_{\mathrm{I}}(\rho) := \mathcal{T}_{\mathrm{E}}(\rho) + \frac{m\pi}{[f_{\mathrm{E}}(\rho)]^3}, \tag{4.71f}$$

This domain of the above transformation from the comoving (ρ, τ)-coordinates to Schwarzschild $(\hat{r}, \hat{t})$-coordinates is given by D_S in (4.71e). The corresponding range $\widehat{D}_S$ of the transformation (4.71) is found by considering the boundary curve $\hat{r} = \widehat{B}(\hat{t})$ defined by the parametric equations

$$\hat{r} = \mathcal{R}_{\mathrm{E}}(\rho_b, \tau), \ \hat{t} = \widehat{T}(\rho_b, \tau),$$

where $\mathcal{T}_{\mathrm{E}}(\rho_b) < \tau < \mathcal{T}_{\mathrm{H}}(\rho_b)$. Thus, Schwarzschild coordinates are valid for

$$
\begin{aligned}
\widehat{D}_S &:= \{(\hat{r}, \hat{t}) : \widehat{B}(\hat{t}) < \hat{r}, \hat{t}_0 < \hat{t} < \infty\}, \text{ where} \\
\hat{t}_0 &:= \int_{\rho_0}^{\rho_b} \frac{\mathcal{T}_{\mathrm{E}}'(x)}{\sqrt{1 - [f_{\mathrm{E}}(x)]^2}} dx.
\end{aligned}
$$

There is an analogue of the above transformation that transforms (ρ, τ)-coordinates into (R, T)-coordinates in the Schwarzschild T-domain. To find this mapping, consider
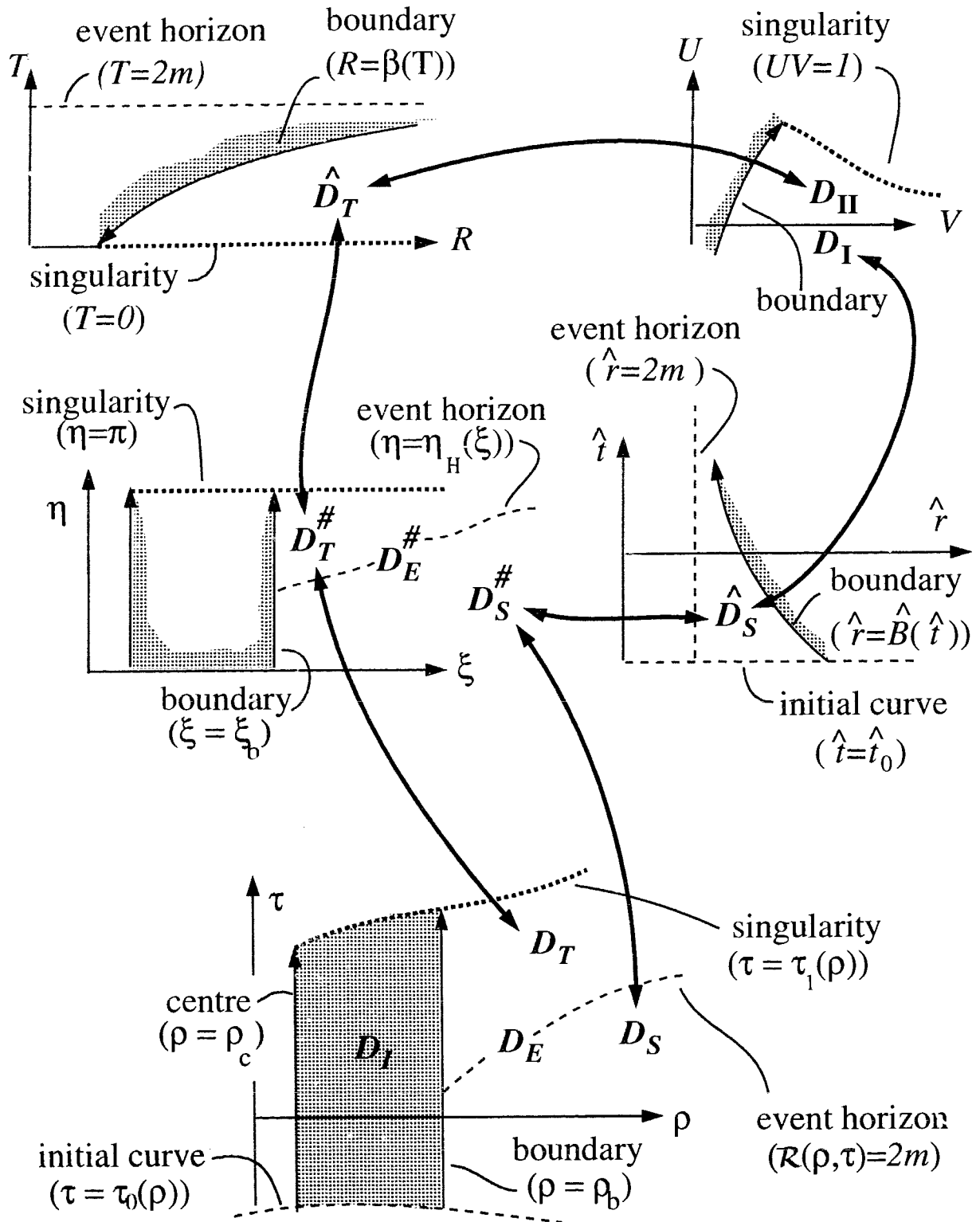
Figure 4.2: Gravitational collapse of a dust sphere of elliptic ($\epsilon = +1$) type.

the (ξ, η)-coordinates in the complementary domain $D_T^{\#}$ (within the event horizon) given by

$$D_T^{\#} := \{(\xi, \eta) : \rho_b < \xi, \ \eta_{\text{hi}}(\xi) < \eta < \pi\} \subset D_E^{\#}.$$

In this domain, $\mathcal{R}^{\#}(\xi, \eta) < 2m$, so the boundary of the star has already collapsed past its event horizon and a black hole is forming. The transformation from $D_T^{\#}$ into $\mathbb{R}^2$ is

$$
\begin{aligned}
R &= R^{\#}(\xi, \eta) \\
&:= \int_{\xi_0}^{\xi} \frac{\mathcal{T}_E'(x)}{\sqrt{1 - [f_E(x)]^2}} dx + m\sqrt{1 - [f_E(\xi)]^2} \left(\frac{(\eta + \sin \eta)}{[f_E(\xi)]^3} + \frac{2\eta}{f_E(\xi)} \right) \\
&\quad + 2m \ln \left[\frac{\tan(\eta/2) + \tan(\eta_{\text{hi}}(\xi)/2)}{\tan(\eta/2) - \tan(\eta_{\text{hi}}(\xi)/2)} \right], \\
T &= \Theta^{\#}(\xi, \eta) := \frac{2m}{[f_E(\xi)]^2} \cos^2(\eta/2).
\end{aligned}
$$

Looking at the boundary curve $R = \beta(T)$ given parametrically by the equations

$$R = R^{\#}(\rho_b, \eta), \quad T = \Theta^{\#}(\rho_b, \eta),$$

where $\eta \in (\eta_{\text{hi}}(\rho_b), \pi)$, the range $\widehat{D}_T$ of the transformation to Schwarzschild T-coordinates is

$$\widehat{D}_T := \{(R, T) : \beta(T) < R < \infty, \ 0 < T < 2m\}.$$

As $\eta \uparrow \pi$, $T = \Theta^{\#}(\rho, \eta) \downarrow 0$ and the entire spherical body collapses into the ultimate singularity.

Thus, outside the dust ball, the (ρ, τ)-coordinates can be transformed into Schwarzschild-type coordinates in both the vacuum R-domain and the vacuum T-domain (see (4.71)). The Schwarzschild-type coordinates can then be transformed into Kruskal coordinates as in (2.15) and (2.16). The gravitational collapse of a spherical dust body into a black hole is depicted in the figure 4.2.

4.4   The Hyperbolic Case: $\epsilon = -1$

The p.d.e. (4.11) describing the collapsing radial velocity yields

$$\frac{\partial \mathcal{R}}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{F(\rho)}{\mathcal{R}(\rho, \tau)} + [f(\rho)]^2} < 0, \tag{4.72}$$

where $F(\rho)/\mathcal{R}(\rho, \tau) + [f(\rho)]^2 > 0$, and $f(\rho) \neq 0$. This choice of $\epsilon = -1$ gives the metric tensor component g_{11} in (4.10) as

$$\exp[\Lambda(\rho, \tau)] = \frac{\left[\frac{\partial \mathcal{R}}{\partial \rho}(\rho, \tau)\right]^2}{1 + [f(\rho)]^2} > 0,$$

where $\partial_\rho \mathcal{R}(\rho, \tau) > 0$.

Integrating the partial differential equation (4.72) with respect to τ gives

$$
\begin{aligned}
\tau - \mathcal{T}_1(\rho) \;=\; & -\frac{\mathcal{R}(\rho, \tau)}{|f(\rho)|}\sqrt{\frac{F(\rho)}{[f(\rho)]^2 \mathcal{R}(\rho, \tau)} + 1} \\
& + \frac{F(\rho)}{|f(\rho)|^3}\,\mathrm{arctanh}\sqrt{\frac{F(\rho)}{[f(\rho)]^2 \mathcal{R}(\rho, \tau)} + 1}\,,
\end{aligned}
$$

for $\sqrt{F(\rho)/(\mathcal{R}(\rho, \tau)) + [f(\rho)]^2} < |f(\rho)|$, and

$$
\begin{aligned}
\tau - \mathcal{T}_1(\rho) \;=\; & -\frac{\mathcal{R}(\rho, \tau)}{|f(\rho)|}\sqrt{\frac{F(\rho)}{[f(\rho)]^2 \mathcal{R}(\rho, \tau)} + 1} \\
& + \frac{F(\rho)}{|f(\rho)|^3}\,\mathrm{arccoth}\sqrt{\frac{F(\rho)}{[f(\rho)]^2 \mathcal{R}(\rho, \tau)} + 1}\,,
\end{aligned}
$$

for $|f(\rho)| < \sqrt{F(\rho)(\mathcal{R}(\rho, \tau)) + [f(\rho)]^2}$. The function $\mathcal{T}_1$ is an arbitrary $\mathcal{C}_p^3$ function of integration. The first case requires that $F(\rho)/\mathcal{R}(\rho, \tau) < 0$, and which is not considered because both $\mathcal{R}$ and F are positive. Therefore, consider the second equation only. Moreover, for the sake of simplicity, choose $f(\rho) > 0$. Thus, the case involving the function arccoth above together with the assumption $f > 0$ yields an implicit equation for the function $\mathcal{R}$:

$$\mathcal{T}_1(\rho) - \tau =$$

$$\frac{\mathfrak{R}(\rho,\tau)}{f(\rho)}\sqrt{\frac{F(\rho)}{[f(\rho)]^2\mathfrak{R}(\rho,\tau)}+1}-\frac{F(\rho)}{[f(\rho)]^3}\operatorname{arccoth}\sqrt{\frac{F(\rho)}{[f(\rho)]^2\mathfrak{R}(\rho,\tau)}+1}\,. \qquad (4.73)$$

Once, the function $\mathfrak{R}$ is defined implicitly.

To consider the matching of metric tensor components and their derivatives at the boundary of the star, adopt a notation as in (4.24) to distinguish branches of the functions $\mathfrak{R}$, F, f and $\mathfrak{T}_1$ within the interior domain and within the exterior domain. Notice also that $F_E(\rho) \equiv 2m$ as before. At the boundary of the star at $\rho = \rho_b$, the line elements in the interior and exterior domains are given by

$$ds_I^2 = \frac{\left[\frac{\partial\mathfrak{R}_I}{\partial\rho}(\rho,\tau)\right]^2}{1+[f_I(\rho)]^2}d\rho^2 + [\mathfrak{R}_I(\rho,\tau)]^2 d\Omega^2 - d\tau^2, \qquad (4.74a)$$

$$ds_E^2 = \frac{\left[\frac{\partial\mathfrak{R}_E}{\partial\rho}(\rho,\tau)\right]^2}{1+[f_E(\rho)]^2}d\rho^2 + [\mathfrak{R}_E(\rho,\tau)]^2 d\Omega^2 - d\tau^2. \qquad (4.74b)$$

Using the solution (4.73), the interior and the exterior smooth branches of the function $\mathfrak{R}$ satisfy

$$\mathfrak{T}_I(\rho) - \tau = \frac{\mathfrak{R}_I(\rho,\tau)}{f_I(\rho)}V_I(\rho,\tau) - \frac{F_I(\rho)}{[f_I(\rho)]^3}\operatorname{arccoth}[V_I(\rho,\tau)], \qquad (4.75a)$$

$$\mathfrak{T}_E(\rho) - \tau = \frac{\mathfrak{R}_E(\rho,\tau)}{f_E(\rho)}V_E(\rho,\tau) - \frac{2m}{[f_I(\rho)]^3}\operatorname{arccoth}[V_E(\rho,\tau)], \qquad (4.75b)$$

$$D_I \quad := \quad \{(\rho,\tau) : \rho_c < \rho < \rho_b, -\infty < \tau < \mathfrak{T}_I(\rho)\}, \qquad (4.75c)$$

$$D_E \quad := \quad \{(\rho,\tau) : \rho_b < \rho, -\infty < \tau < \mathfrak{T}_E(\rho)\}, \qquad (4.75d)$$

$$\partial D_I \quad := \quad \{(\rho,\tau) : \rho = \rho_b, -\infty < \tau < \mathfrak{T}_I(\rho_b)\}, \qquad (4.75e)$$

$$V_I(\rho,\tau) := \sqrt{\frac{F_I(\rho)}{[f_I(\rho)]^2\mathfrak{R}_I(\rho,\tau)}+1}, \qquad (4.75f)$$

$$V_E(\rho,\tau) := \sqrt{\frac{2m}{[f_E(\rho)]^2\mathfrak{R}_E(\rho,\tau)}+1}. \qquad (4.75g)$$

Having established the line element inside in D_I and D_E, it is now possible to check the continuity requirements.

Proposition 4.3. *A necessary and sufficient condition for the continuity of the metric tensor components of* (4.74) *and their first order partial derivatives across the boundary* ∂D_I *is that the functions* F, F', F'', $\mathfrak{T}_1$, $\mathfrak{T}_1'$ *and* $\mathfrak{T}_1''$ *(where primes denote a total derivative) are all continuous across* ∂D_I.

Proof. The results from differentiating equations (4.75) are as follows:

$$
\begin{aligned}
\frac{\partial \mathfrak{R}_I}{\partial \rho}(\rho, \tau) &= \left[\ln \frac{F_I(\rho)}{[f_I(\rho)]^2} \right]' \mathfrak{R}_I(\rho, \tau) + V_I(\rho, \tau) \left(\mathfrak{T}_I'(\rho) + \left[\ln \frac{F_I(\rho)}{[f_I(\rho)]^3} \right]' (\tau - \mathfrak{T}_I(\rho)) \right) \\
&= \left(\frac{F_I'(\rho)}{[f_I(\rho)]^2} - \frac{3 F_I(\rho) f_I'(\rho)}{[f_I(\rho)]^3} \right) (V_I(\rho, \tau) \, \text{arccoth}[V_I(\rho, \tau)] - 1) \\
&\quad + \frac{f_I'(\rho)}{f_I(\rho)} \mathfrak{R}_I(\rho, \tau) + f_I(\rho) \mathfrak{T}_I'(\rho) V_I(\rho, \tau),
\end{aligned}
\tag{4.76a}
$$

$$
\begin{aligned}
\frac{\partial \mathfrak{R}_E}{\partial \rho}(\rho, \tau) &= \frac{6 m f_E'(\rho)}{[f_E(\rho)]^3} (1 - V_E(\rho, \tau) \, \text{arccoth}[V_E(\rho, \tau)]) \\
&\quad + \frac{f_E'(\rho)}{f_E(\rho)} \mathfrak{R}_E(\rho, \tau) + f_E(\rho) \mathfrak{T}_E'(\rho) V_E(\rho, \tau),
\end{aligned}
\tag{4.76b}
$$

$$
\frac{\partial \mathfrak{R}_I}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{F_I(\rho)}{\mathfrak{R}_I(\rho, \tau)} + [f_I(\rho)]^2} < 0,
\tag{4.76c}
$$

$$
\frac{\partial \mathfrak{R}_E}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{2m}{\mathfrak{R}_E(\rho, \tau)} + [f_E(\rho)]^2} < 0,
\tag{4.76d}
$$

$$
\frac{\partial^2 \mathfrak{R}_I}{\partial \tau \partial \rho}(\rho, \tau) = \frac{1}{2} \left(\frac{F_I'(\rho)}{\mathfrak{R}_I(\rho, \tau)} - \frac{F_I(\rho) \frac{\partial \mathfrak{R}_I}{\partial \rho}(\rho, \tau)}{[\mathfrak{R}_I(\rho, \tau)]^2} + 2 f_I(\rho) f_I'(\rho) \right) \left(\frac{\partial \mathfrak{R}_I}{\partial \tau}(\rho, \tau) \right)^{-1},
\tag{4.76e}
$$

$$
\frac{\partial^2 \mathfrak{R}_E}{\partial \tau \partial \rho}(\rho, \tau) = -\left(\frac{m}{[\mathfrak{R}_E(\rho, \tau)]^2} \frac{\partial \mathfrak{R}_E}{\partial \rho}(\rho, \tau) - f_E(\rho) f_E'(\rho) \right) \left(\frac{\partial \mathfrak{R}_E}{\partial \tau}(\rho, \tau) \right)^{-1},
\tag{4.76f}
$$

$$\frac{\partial^2 \mathcal{R}_I}{\partial \rho^2}(\rho, \tau) = [\ln f_I(\rho)]' \frac{\partial \mathcal{R}_I}{\partial \rho}(\rho, \tau) + \mathcal{R}_I(\rho, \tau)[\ln f_I(\rho)]''$$

$$- \frac{1}{2}\left(\frac{F_I'(\rho)}{[f_I(\rho)]^2} - \frac{3F_I(\rho)f_I'(\rho)}{[f_I(\rho)]^3}\right)\left(\left[\ln \frac{F_I(\rho)}{[f_I(\rho)]^2}\right]' - \frac{\frac{\partial \mathcal{R}_I}{\partial \rho}(\rho, \tau)}{\mathcal{R}_I(\rho, \tau)}\right)$$

$$+ \frac{1}{2}\frac{F_I(\rho)}{[f_I(\rho)]^2 \mathcal{R}_I(\rho, \tau) V_I(\rho, \tau)}\left(\left[\ln \frac{F_I'(\rho)}{[f_I(\rho)]^2}\right]' - \frac{\frac{\partial \mathcal{R}_I}{\partial \rho}(\rho, \tau)}{\mathcal{R}_I(\rho, \tau)}\right) \times$$

$$\left\{f_I(\rho)\mathcal{T}_I'(\rho) + \left(\frac{F_I'(\rho)}{[f_I(\rho)]^2} - \frac{3F_I(\rho)f_I'(\rho)}{[f_I(\rho)]^3}\right) \text{arccoth}[V_I(\rho, \tau)]\right\}$$

$$+ V_I(\rho, \tau)[f_I(\rho)\mathcal{T}_I'(\rho)]' + (V_I(\rho, \tau)\,\text{arccoth}[V_I(\rho, \tau)] - 1) \times$$

$$\left(\frac{F_I''(\rho)}{[f_I(\rho)]^2} - 5\frac{F_I'(\rho)f_I'(\rho)}{[f_I(\rho)]^3} - 3\frac{F_I(\rho)f_I''(\rho)}{[f_I(\rho)]^3} + 9\frac{F_I(\rho)[f_I'(\rho)]^2}{[f_I(\rho)]^4}\right), \quad (4.76\text{g})$$

$$\frac{\partial^2 \mathcal{R}_E}{\partial \rho^2}(\rho, \tau) = [\ln f_E(\rho)]' \frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau) + \mathcal{R}_E(\rho, \tau)[\ln f_E(\rho)]''$$

$$- \frac{3mf_E'(\rho)}{[f_E(\rho)]^3}\left(2[\ln f_E(\rho)]' + \frac{\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau)}{\mathcal{R}_E(\rho, \tau)}\right) + V_E(\rho, \tau)[f_E(\rho)\mathcal{T}_E'(\rho)]'$$

$$- \frac{m}{[f_E(\rho)]^2 \mathcal{R}_E(\rho, \tau) V_E(\rho, \tau)}\left(2[\ln f_E(\rho)]' + \frac{\frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau)}{\mathcal{R}_E(\rho, \tau)}\right) \times$$

$$\left\{f_E(\rho)\mathcal{T}_E'(\rho) - \frac{6mf_E'(\rho)}{[f_E(\rho)]^3} \text{arccoth}[V_E(\rho, \tau)]\right\}$$

$$- 6m\,(V_E(\rho, \tau)\,\text{arccoth}[V_E(\rho, \tau)] - 1)\left(\frac{f_E''(\rho)}{[f_E(\rho)]^3} - \frac{3[f_E'(\rho)]^2}{[f_E(\rho)]^4}\right). \quad (4.76\text{h})$$

The necessity is established first. Suppose that the functions F, f, $\mathcal{T}_I$, F', f', $\mathcal{T}_I'$, F'', f'' and $\mathcal{T}_I''$ are continuous across the boundary ∂D_I. Then, by equations (4.75a,b) that implicitly define $\mathcal{R}_I$ and $\mathcal{R}_E$,

$$\mathcal{R}_I(\rho_{b-}, \tau)\sqrt{\frac{2m}{[f(\rho_b)]^2 \mathcal{R}_I(\rho_{b-}, \tau)} + 1} - \frac{2m}{[f(\rho_b)]^2}\text{arccoth}\sqrt{\frac{2m}{[f(\rho_b)]^2 \mathcal{R}_I(\rho_{b-}, \tau)} + 1}$$

$$\equiv \mathcal{R}_E(\rho_{b+}, \tau)\sqrt{\frac{2m}{[f(\rho_b)]^2 \mathcal{R}_E(\rho_{b+}, \tau)} + 1} - \frac{2m}{[f(\rho_b)]^2}\text{arccoth}\sqrt{\frac{2m}{[f(\rho_b)]^2 \mathcal{R}_E(\rho_{b+}, \tau)} + 1}.$$

Define the functions

$$w_I = \mathcal{W}_I(\tau) := \sqrt{\frac{2m}{[f(\rho_b)]^2 \mathcal{R}_{\mathrm{I}}(\rho_{b-},\tau)} + 1} > 1,$$

$$w_E = \mathcal{W}_E(\tau) := \sqrt{\frac{2m}{[f(\rho_b)]^2 \mathcal{R}_{\mathrm{E}}(\rho_{b+},\tau)} + 1} > 1 \text{ and}$$

$$J(w) := w(w^2 - 1)^{-1} - \operatorname{arccoth} w, w > 1.$$

With the definitions of w_I, w_E and J, the previous identity can be expressed as

$$J(w_I) \equiv J(w_E).$$

The derivative of J is

$$J'(w) = -\frac{2}{(w^2 - 1)^2} < 0.$$

This strict inequality shows that the function J is one-to-one. Since J is one-to-one and $J(w_I) \equiv J(w_E)$, conclude that

$$\mathcal{W}_I(\tau) = w_I \equiv w_E = \mathcal{W}_E(\tau);$$

$$\Rightarrow [\mathcal{W}_I(\tau)]^2 - 1 \equiv [\mathcal{W}_E(\tau)]^2 - 1;$$

$$\Rightarrow [\triangle\mathcal{R}(\rho_b,\tau)] \equiv 0.$$

Therefore, the continuity of $\mathcal{R}$ across the boundary ∂D_I is established. So, the metric tensor component g_{22} is continuous. The continuities of F, f and R together with the equations (4.76c,d) give the continuity of $\partial_\tau \mathcal{R}$ at $\rho = \rho_b$. Thus, $\partial_\tau g_{22}$ is continuous. By the continuities of F, f, $\mathcal{R}$, F' and f' and the equations (4.76a,b), the continuities of $\partial_\rho\mathcal{R}$, $\partial_\rho g_{22}$ and g_{11} are proved. It follows that the continuities of $F, f, R, F', f', \partial_\rho\mathcal{R}$ and $\partial_\tau\mathcal{R}$ and the equations (4.76e,f) yield the continuity of $\partial_\tau g_{11}$. Finally, from the equations (4.76g,h) and the continuities already established, $\partial_\rho g_{11}$ is continuous. Therefore, it has been proved that the continuities of $F, f, \mathcal{T}_1, F', f', \mathcal{T}_1', F'', f''$ and $\mathcal{T}_1''$ imply the continuities of the metric tensor components and of their first partial derivatives across the boundary ∂D_I. The converse result follows in a manner similar to that used in previous sections. $\qquad\square$

Once again, a transformation of the exterior line element (4.74b) into the Schwarzschild line element (2.4) is sought. Such a transformation takes the form

$$\widehat{r} = \widehat{\mathcal{R}}(\rho, \tau) := \mathcal{R}_E(\rho, \tau), \quad \widehat{t} = \widehat{T}(\rho, \tau),$$

where $\mathcal{R}_E$ is given implicitly by (4.75b) and $\widehat{T}$ has yet to be determined. The angular coordinates $\widehat{\theta} = \theta$ and $\widehat{\phi} = \phi$ play passive roles and are left out. The function $\widehat{T}$ has to be determined. The procedure is as before; obtain a pair of first order, linear p.d.e's for $\widehat{T}$, use a suitable intermediate coordinate chart to verify the integrability condition and perform the integration, express the transformation explicitly in terms of (ρ, τ)-coordinates again and obtain the range of the transformation from the boundary curve.

Following the argument leading up to (4.63) and (4.64), the pair of first order, linear p.d.e's for $\widehat{T}$ are as follows:

$$\frac{\partial \widehat{T}}{\partial \tau}(\rho, \tau) = \sqrt{1 + [f_E(\rho)]^2} \left(1 - \frac{2m}{\mathcal{R}_E(\rho, \tau)}\right)^{-1} > 0, \qquad (4.77a)$$

$$\frac{\partial \widehat{T}}{\partial \rho}(\rho, \tau) = \frac{\partial \mathcal{R}_E}{\partial \rho}(\rho, \tau)\frac{\partial \mathcal{R}_E}{\partial \tau}(\rho, \tau)[1 + [f_E(\rho)]^2]^{-\frac{1}{2}} \left(1 - \frac{2m}{\mathcal{R}_E(\rho, \tau)}\right)^{-1} < 0 \quad (4.77b)$$

Again, checking the integrability condition for (4.77) is more convenient in another coordinate system. The transformation to the intermediate coordinate chart is given by the following equations:

$$\xi = \Xi(\rho, \tau) := \rho, \qquad (4.78a)$$

$$\zeta = \mathcal{Z}(\rho, \tau) := 2 \operatorname{arccoth} \sqrt{\frac{2m}{[f_E(\rho)]^2 \mathcal{R}_E(\rho, \tau)} + 1} > 0, \qquad (4.78b)$$

$$\operatorname{coth}^2\left(\frac{\zeta}{2}\right) = \frac{2m}{[f_E(\rho)]^2 \mathcal{R}_E(\rho, \tau)} + 1, \qquad (4.78c)$$

$$\frac{\partial(\xi, \zeta)}{\partial(\rho, \tau)} = \frac{\partial \mathcal{Z}}{\partial \tau}(\rho, \tau) = -\frac{f_E(\rho)}{\mathcal{R}_E(\rho, \tau)} < 0, \qquad (4.78d)$$

$$D_E = \{(\rho, \tau) : \rho_b < \rho, \ -\infty < \tau < \mathcal{T}_E(\rho)\}. \qquad (4.78e)$$

To be precise in the notation, denote $f_E^{\#}(\xi) \equiv f_E(\rho) = f_E(\xi)$ and $\mathcal{T}_E^{\#}(\xi) \equiv \mathcal{T}_E(\rho) = \mathcal{T}_E(\xi)$.

The transformation (4.78) can be inverted through use of the p.d.e. (4.75b); the inverse transformation is

$$\rho = \mathcal{R}^{\#}(\xi) := \xi, \tag{4.79a}$$

$$\tau = T^{\#}(\xi,\zeta) := \mathcal{T}_{\mathrm{E}}(\xi) + \frac{m}{[f_{\mathrm{E}}(\xi)]^3}(\zeta - \sinh\zeta), \tag{4.79b}$$

$$\frac{\partial(\rho,\tau)}{\partial(\xi,\zeta)} = -2m\frac{\sinh^2(\zeta/2)}{[f_{\mathrm{E}}(\xi)]^3} < 0, \tag{4.79c}$$

$$D_E^{\#} := \{(\xi,\zeta) : \rho_b < \xi, \, 0 < \zeta < \infty\}. \tag{4.79d}$$

The transformation of (ξ,ζ)-coordinates into Schwarzschild coordinates can found with the the transformation (4.78):

$$\widehat{r} = \mathcal{R}_{\mathrm{E}}(\rho,\tau) = 2m\frac{\sinh^2(\zeta/2)}{[f_{\mathrm{E}}(\xi)]^2} = \frac{m}{[f_{\mathrm{E}}(\xi)]^2}(\cosh\zeta - 1) =: \mathcal{R}^{\#}(\xi,\zeta), \tag{4.80a}$$

$$\widehat{t} = \widehat{T}(\rho,\tau) = T^{\#}(\xi,\zeta), \tag{4.80b}$$

$$D_S^{\#} := \{(\xi,\zeta) : \rho_b < \xi, \, \zeta_{\mathrm{II}}(\xi) < \zeta\}, \tag{4.80c}$$

$$\zeta_{\mathrm{II}}(\xi) := 2\,\mathrm{arcsinh}[f_{\mathrm{E}}(\xi)]. \tag{4.80d}$$

The function $T^{\#}$ is unknown. Use the chain rule and (4.78b) to convert the system (4.77) into the following p.d.e's for $T^{\#}$:

$$P^{\#}(\xi,\zeta) := \frac{\partial T^{\#}}{\partial\zeta}(\xi,\zeta) \tag{4.81a}$$

$$= \frac{-2m\Sigma^{\#}(\xi)}{[f_{\mathrm{E}}(\xi)]^3}\frac{\sinh^4(\zeta/2)}{\sinh^2(\zeta/2) - [f_{\mathrm{E}}(\xi)]^2},$$

$$Q^{\#}(\xi,\zeta) := \frac{\partial T^{\#}}{\partial\xi}(\xi,\zeta) \tag{4.81b}$$

$$= \frac{\partial\widehat{T}}{\partial\rho}(\rho,\tau)\frac{\partial\mathcal{R}}{\partial\xi}(\xi,\zeta) + \frac{\partial\widehat{T}}{\partial\tau}(\rho,\tau)\frac{\partial T}{\partial\xi}(\xi,\zeta)$$

$$= \frac{\mathcal{T}_{\mathrm{E}}'(\xi)}{\Sigma^{\#}(\xi)} + \frac{mf_{\mathrm{E}}'(\xi)}{\Sigma^{\#}(\xi)[f_{\mathrm{E}}(\xi)]^2}\left(3\frac{\sinh\zeta - \zeta}{[f_{\mathrm{E}}(\xi)]^2} + 4\frac{\cosh(\zeta/2)\sinh^3(\zeta/2)}{\sinh^2(\zeta/2) - [f_{\mathrm{E}}(\xi)]^2}\right),$$

$$\Sigma^{\#}(\xi) := \sqrt{1 + [f_{\mathrm{E}}(\xi)]^2}. \tag{4.81c}$$

The integrability condition for the equations (4.81) is

$$
\begin{aligned}
\frac{\partial P^{\#}}{\partial \xi}(\xi,\zeta) &= \frac{-2m f_E'(\xi)\sinh^4(\zeta/2)}{\Sigma^{\#}(\xi)[f_E(\xi)]^4} \times \\
&\quad \left(\frac{4[f_E(\xi)]^4 + 5[f_E(\xi)]^2 - 2[f_E(\xi)]^2 \sinh^2(\zeta/2) - 3\sinh^2(\zeta/2)}{(\sinh^2(\zeta/2) - [f_E(\xi)]^2)^2} \right) \\
&\equiv \frac{\partial Q^{\#}}{\partial \zeta}(\xi,\zeta).
\end{aligned}
\tag{4.82}
$$

This condition is identically satisfied. Therefore, the solution $T^{\#}(\xi,\zeta)$ is obtained by a line integral (and the equation (4.80d)):

$$
\begin{aligned}
\widehat{t} &= T^{\#}(\xi,\zeta) = \int_{(\xi_0,0)}^{(\xi,\zeta)} \left[\frac{\partial T^{\#}}{\partial x}(x,y)\,dx + \frac{\partial T^{\#}}{\partial y}(x,y)\,dy \right] \\
&= \int_{\xi_0}^{\xi} \frac{T_E'(x)}{\sqrt{1+[f_E(x)]^2}}\,dx - m\sqrt{1+[f_E(\xi)]^2}\left(\frac{(\sinh\zeta - \zeta)}{[f_E(\xi)]^3} + \frac{2\zeta}{f_E(\xi)} \right) \\
&\quad + 2m\ln\left[\frac{\coth[\zeta_H(\xi)/2] + \coth[\zeta/2]}{\coth[\zeta_H(\xi)/2] - \coth[\zeta/2]} \right],
\end{aligned}
\tag{4.83a}
$$

$$
\coth[\zeta_H(\xi)/2] = \sqrt{1+[f_E(\xi)]^{-2}}.
\tag{4.83b}
$$

Using the above solution (4.83) in (ξ,η)-coordinates, the transformation from the (ρ,τ)-coordinates into the Schwarzschild $(\widehat{r},\widehat{t})$−coordinates is

$$
\widehat{r} = \mathcal{R}_E(\rho,\tau),
\tag{4.84a}
$$

$$
\begin{aligned}
\widehat{t} = \widehat{T}(\rho,\tau) &= \int_{\rho_0}^{\rho} \frac{T_E'(x)}{\sqrt{1+[f_E(x)]^2}}\,dx \\
&\quad + \sqrt{1+[f_E(\rho)]^2}\left(\tau - T_E(\rho) - \frac{4m}{f_E(\rho)}\operatorname{arccoth}\sqrt{\frac{2m}{[f_E(\rho)]^2\mathcal{R}_E(\rho,\tau)} + 1} \right) \\
&\quad + 2m\ln\left[\frac{\coth[\zeta_H(\rho)/2] + \sqrt{(2m/[f_E(\rho)]^2\mathcal{R}_E(\rho,\tau)) + 1}}{\coth[\zeta_H(\rho)/2] - \sqrt{(2m/[f_E(\rho)]^2\mathcal{R}_E(\rho,\tau)) + 1}} \right],
\end{aligned}
\tag{4.84b}
$$

$$
\coth[\zeta_H(\rho)/2] = \sqrt{1+[f_E(\rho)]^{-2}},
\tag{4.84c}
$$

$$
T_H(\rho) := T_E(\rho) - \frac{m}{[f_E(\rho)]^3}\{\sinh[\zeta_H(\rho)] - \zeta_H(\rho)\},
\tag{4.84d}
$$

$$
D_S := \{(\rho,\tau) : \rho_b < \rho,\ -\infty < \tau < T_H(\rho)\}.
\tag{4.84e}
$$

The function ζ_{II} is defined so that $\mathcal{R}_{\text{E}}(\rho, \mathcal{T}_{\text{II}}(\rho)) \equiv 2m$, ie. the image of $\mathcal{R}^{\#}$ evaluated along the curve $\zeta = \zeta_{\text{II}}(\xi)$ in the (ξ, η)-plane is a portion of the event horizon in the Schwarzschild space-time.

The range $\widehat{D}_S$ of the transformation from (ρ, τ)-coordinates into the Schwarzschild R-domain is given by the boundary curve $\widehat{r} = \widehat{B}(\widehat{t})$ defined by the parametric equations

$$\widehat{r} = \mathcal{R}_{\text{E}}(\rho_b, \tau), \quad \widehat{t} = \widehat{T}(\rho_b, \tau),$$

where $-\infty < \tau < \mathcal{T}_{\text{II}}(\rho_b)$. Thus, the Schwarzschild coordinate chart is valid for

$$\widehat{D}_S := \{(\widehat{r}, \widehat{t}) : \widehat{B}(\widehat{t}) < \widehat{r}, \widehat{t} \in \mathbb{R}\}.$$

For the domain where $\rho > \rho_b$ and $\tau > \mathcal{T}_{\text{II}}(\rho)$ in (ρ, τ)-coordinates, the coordinates transform into (R, T)-coordinates in the Schwarzschild T-domain. To find the range of this transformation, consider the (ξ, ζ)-coordinates in the alternate domain $D_T^{\#}$ given by

$$D_T^{\#} := \{(\xi, \zeta) : \rho_b < \xi, \, 0 < \zeta < \zeta_{\text{II}}(\xi)\} \subset D_E^{\#}.$$

The transformation itself is

$$
\begin{aligned}
R &= R^{\#}(\xi, \zeta) \\
&:= \int_{\xi_0}^{\xi} \frac{\mathcal{T}_{\text{E}}'(x)}{\sqrt{1 + [f_{\text{E}}(x)]^2}} dx - m\sqrt{1 + [f_{\text{E}}(\xi)]^2} \left(\frac{(\sinh \zeta - \zeta)}{[f_{\text{E}}(\xi)]^3} + \frac{2\zeta}{f_{\text{E}}(\xi)} \right) \\
&\quad + 2m \ln \left[\frac{\coth(\zeta/2) + \coth(\zeta_{\text{II}}(\xi)/2)}{\coth(\zeta/2) - \coth(\zeta_{\text{II}}(\xi)/2)} \right], \\
T &= \Theta^{\#}(\xi, \zeta) := \frac{2m}{[f_{\text{E}}(\xi)]^2} \sinh^2(\zeta/2).
\end{aligned}
$$

The domain $\widehat{D}_T$ of validity for the (R, T) coordinate chart is given by

$$\widehat{D}_T := \{(R, T) : \widehat{\beta}(T) < R < \infty, \, 0 < T < 2m\},$$

where, the boundary curve $R = \widehat{\beta}(T)$ is provided by the parametric equations

$$R = R^{\#}(\rho_b, \zeta), \quad T = \Theta^{\#}(\rho_b, \zeta),$$
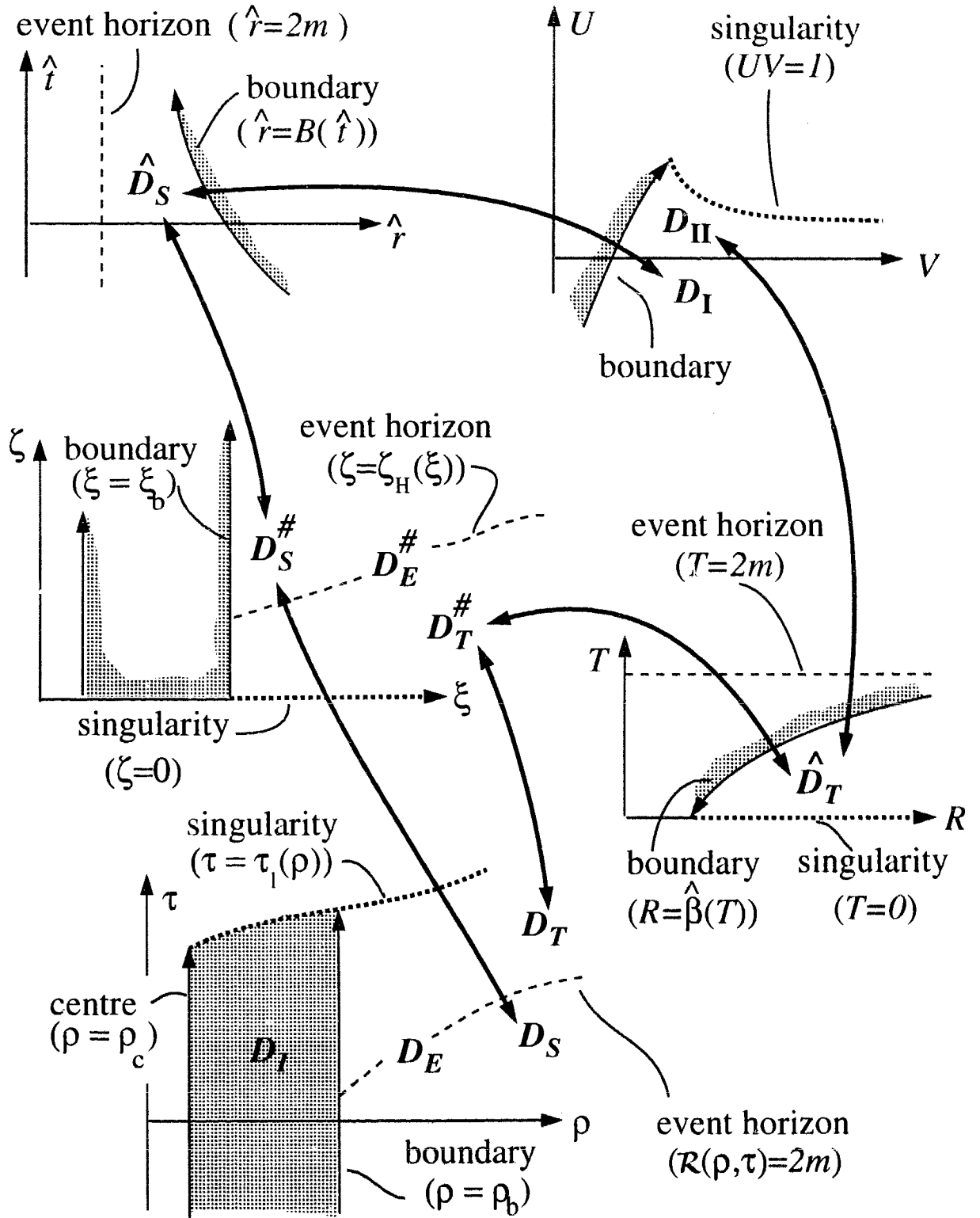
Figure 4.3: Gravitational collapse of a dust sphere of hyperbolic ($\epsilon = -1$) type.

for $0 < \zeta < \zeta_{\mathrm{II}}(\rho_b)$. The Schwarzschild metric in the R-domain and T-domain can both be transformed into Kruskal coordinates by the transformations (2.15) and (2.16) respectively. The gravitational collapse of a stellar dust ball of hyperbolic type is depicted in figure 4.3.

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