STUDYING STUDENTS' SENSE MAKING OF FRACTAL GEOMETRY

by

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Abstract

This study is a pilot attempt to introduce fractal geometry to a Survey Mathematics 12 class. The paper describes the methodology employed in implementation and attempts an analysis of the ways in which the students acquired the concepts incorporated in the study unit. Concluding remarks include suggestions for improvement of the unit and recommendations for subsequent implementations.

The method of inquiry is qualitative in nature. The data included eleven audio taped interviews and perusal of all assignments, quizzes, notebooks, and the unit test. The interview questions were intended to explore the ways in which students constructed their knowledge of fractals. The last two questions were designed to plumb for deeper understanding of the topic. They were framed to investigate students' conjectures regarding the appearance of structures that are between two and three dimensional; such structures were not specifically addressed in the study unit.

The taped interviews were transcribed and analyzed; the Piagetian theories of assimilation, accommodation, and reflective abstraction were applied to students' responses to the interview questions. Included in the analysis is an attempt to characterize students' cognition using the notions of operational and structural thinking.

The possibilities for cross curricular applications make fractal geometry an especially attractive topic of study. There are, however, aspects of the subject that appear to require special attention. The findings of this study indicate the accommodation of dimension and self-similarity dimension were particularly problematic for participants. Students also experienced difficulty in the effective characterization of a fractal. If fractal geometry is to be introduced, there are several conditions that must be met to ensure that the pedagogy employed in its presentation is appropriate to the subject matter.

First, allowance must be made for the novelty of the topic. Fractal geometry is not a subject that can be studied in two weeks; therefore, implementation should be spiral in nature to avoid overburdening students with concepts that will, for most, be completely foreign. Second, there must be suitable hardware, software, and print resources available. The vast potential for curricular innovation cannot be fully realized if students experience a "traditional" presentation of the material.

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Chapter 1

Introduction

Why fractal geometry?

Most high school mathematics curricula have traditionally been concerned with topics that have been a part of the mathematical body of knowledge for at least three hundred years. Such a situation would not be tolerated in either the social or empirical sciences. The advent of the pocket calculator has, in the eyes of many students, rendered topics such as the simplification of radicals and rationalization of irrational denominators largely irrelevant . The simplification of rational expressions is a skill that appears, to a great number of students, to be useful only in the solution of problems that appear in mathematics textbooks; as such students again question the relevance of the practice.

Why is such conservatism allowed to exist in the mathematics curriculum? Through our dated curriculum, and the drill and practice approach to mathematics teaching, students are given the impression that mathematics is the study of a static mass of esoteric facts, formulae, and algorithms that appear to have little application to the "real world." Without question the study of pure mathematics is critical; however, there should exist in the curriculum some balance between pure and applied mathematics at the secondary school level.

High school mathematics has, with the possible exception of Euclidean Geometry, inculcated students with the idea that mathematics problems always have one correct method of solution, and the teacher always knows the solution. A large number of students who enjoy mathematics say they do so for this very reason; they can always find that solution with no confusing ambiguities to cloud the issue. As a result, high school students tend to assume that the very essence of mathematics lies in finding "the solution." Many students experience a great deal of difficulty in attaining the "correct" solution, and as a result may become frustrated and bored with mathematics. Even students who are, or could be, proficient at solving traditional

mathematics problems may become bored with mathematics for completely different reasons; these students see mathematics as largely irrelevant. All of these students could benefit by seeing mathematics from a slightly different perspective.

It has been my experience that most students will, when confronted with a novel mathematical problem, become engaged in the problem and actively seek a solution. On many occasions, I have observed students' reactions to the Konigsberg Bridge problem or Fibonacci's rabbit population problem and noted that they are quite different from the "normal" situation wherein students are expected to complete seemingly countless mathematics "problems" in a sequential fashion; all mathematics teachers have observed the abundance of off task activity that occurs when students are given class time to do their homework assignments. The attitudes toward the discipline that manifest themselves in these two scenarios are markedly different and the change in the classroom atmosphere is tangible. As a teacher of mathematics one would like to cultivate that atmosphere of discovery more often. Perhaps the introduction of a new curricular topic could help to bring a renewed sense of discovery to the classroom.

I believe that students would benefit by doing a different type of mathematics: one that could lead students to a sense of mathematical discovery; one that could show students that there is a way to do some mathematical experimentation using current technology. It is my feeling that most high school mathematics students could benefit by doing a type of mathematics with which they are likely quite unfamiliar. An introduction to fractal geometry could fulfill all of these objectives.

Fractal geometry is a relatively new science; although its origins are a century old it has only recently been accepted into the mainstream of mathematical thought. It was with the advent of accessible computer technology that fractals and chaos theory were discovered to be two sides of the same coin. Had it not been for the "desktop" computer these two important topics would likely never have received any serious attention from mathematicians. Thus, the

nature of the study is inextricably tied to the use of technology. This provides further justification for the inclusion of the topic in high school curricula.

The latest curricular revisions from the Ministry appear to advocate the clustering of senior secondary mathematics students into two distinct streams: applied mathematics and principles of mathematics. The former is self-explanatory; the latter is oriented more toward pure mathematics. Fractals are well suited to both streams. The infinitely detailed nature and fractional dimensionality of these objects is within the realm of pure mathematics; the fact these structures closely model many natural phenomena is a connection to the physical world.

According to the National Council of Teachers of Mathematics (NCTM) curriculum standards, college bound students should investigate and compare different geometries.

> This component of the 9 - 12 geometry strand should provide experiences that deepen students' understanding of shapes and their properties with an emphasis on their wide applicability in human activity. The curriculum should be infused with examples of how geometry is used in recreations (as in billiards or sailing); in practical tasks (as in purchasing paint for a room); in the sciences (as in the description and analysis of mineral crystals); and in the arts (as in perspective drawing) (NCTM, 1991, p. 157).

If one places any credibility in NCTM recommendations, there is contained within the preceding statement ample justification for an introduction to fractal geometry.

Fractals are, without a doubt, foreign to a great many high school mathematics students. It is precisely because of the newness of the science and the unfamiliarity with the concepts that students should study fractal geometry. They could benefit from an introduction to an area of mathematical research that is, in some cases, barely a quarter century old. They could read about new discoveries in the field in current periodicals. They could see applications of the science in popular culture. They could see mathematics as a study of a dynamic system, rather than one that has remained static for centuries.

Although there is a desire to include different geometries in the curriculum, there is little knowledge of how students would learn the content of and react to alternate geometry curricula. There are many questions to be considered.

In his article regarding fractal geometry for secondary schools, Paul Goldenberg asks "What are the most promising topics and approaches for introducing this particular new mathematics into the curriculum? What parts [of fractal geometry] are appropriate for grades 7 to 12? How might the fractal content and the visual experimental approach broaden and deepen students' mathematical thinking and their interests in and perceptions of mathematics?" (Goldenberg, 1991, p. 39) There are many other questions that also need to be answered: Where will students' difficulties lie? What specific concepts will be particularly problematic? What would constitute the optimal scope and sequence of a study unit on fractals? Is it appropriate to expose high school students to such a complex topic? To date there has been no research done to determine how curricula might be designed in order to enhance student understanding of fractal geometry. My intent is to try to answer some of these questions.

This thesis is a pilot attempt to probe student understanding of the basics of and describe how students construct specific concepts related to fractal geometry. I hope to gain some insight into students' assimilation and internalization of the content that I have incorporated in an introductory unit on fractals. Specific concepts that will be discussed include the description of a fractal, self-similarity and self-similarity dimension. Chapter 2 briefly describes these and other concepts fundamental to fractal geometry.

Chapter 2

What is a Fractal?

A brief history of fractal geometry.

The word fractal, from the Latin word *frangere* which means to break, was coined by Benoit Mandelbröt in 1975. Mathematicians, however, have been aware of fractals for considerably longer.

> Many fractals and their descriptions go back to classical mathematics and mathematicians like Georg Cantor (1872), Giuseppe Peano (1890), David Hilbert (1891), Helge von Koch (1904), Waclaw Sierpinski (1916), and Gaston Julia (1918) to name just a few. However, these mathematicians did not think of their creations as conceptual steps toward a new geometry of nature. Rather, what we know as the Cantor set, the Koch curve, the Hilbert curve and the Sierpinski gasket were regarded as exceptional objects, as counter examples, as "mathematical monsters" (Peitgen, Jürgens, and Saupe, 1992, page 76).

For many decades fractals remained outside of mainstream mathematics because they were viewed merely as curiosities. Then, in the mid-1970s Mitchell Feigenbaum discovered that fractal geometry is the geometry of chaos theory. Since the discovery of that link, the mathematical community has taken great interest in fractals and serious research in the field has become legitimized.

> During the [early] 1970s, when both were in their infancy, chaos and fractals appeared to be unrelated. But they are mathematical cousins. Both grapple with the structure of irregularity. Fractals present us with a new language in which to describe the shape of chaos (Stewart, 1989, p. 216).

Properties of a fractal.

The question "what *exactly* is a fractal?" is often asked. According to Mandelbröt, a fractal is defined as a set whose Hausdorff (fractal) dimension is not an integer. Unfortunately,

this definition is not easily understood. In general, an object is considered to be a fractal if it exhibits certain properties. It must display some degree of self-similarity; that is, a small portion of the object when magnified will appear to be similar to the entire object. Mandelbröt has written that: "A fractal is a shape made of parts similar to the whole in some way (Mandelbröt, 1983)." Self-similarity is one of the central characteristics of a fractal; the self-similarity dimension is another of its distinguishing properties.

A fractal must have non-integral dimension. This means that it will be neither zero, one, two, nor three dimensional; its dimension will lie somewhere between these numbers. This means, for example, that "...a one dimensional line can, in some fashion, be bent so many times that the line begins to fill space [cover an area]. Thus, the wiggly line has dimension greater than one." (Kern & Mauk. 1990, p. 179) These characteristic properties are relatively easy to identify in simple geometric figures.

Self-similarity.

In *The Mathematical Tourist*, Ivars Peterson describes self-similarity as the property exhibited by focussing on progressively smaller and smaller segments of a figure. What one sees when zooming in on a fractal is that, unlike the structures studied in Euclidean geometry, the resulting magnification of the object does not tend to smooth the irregularities: "Instead, the objects tend to show the same degree of roughness at different levels of magnification." Peterson continues: "Fractal objects contain structures nested within one another. Each smaller structure is a miniature, though not necessarily identical, version of the larger form (Peterson, 1988, pp. 114 - 115)." In other words, one part of the object is a scaled down version of the entire object. The Koch curve and the Sierpinski gasket are classic, yet simple, examples of self-similar objects.

The Koch curve is one of the first fractal structures discovered; it was initially desribed by Helge von Koch approximately nine decades ago. The first three iterations of the Koch curve appear in *figure 2.1*.

figure 2.1

The first three iterations of the Koch curve.

stage 0

stage 1
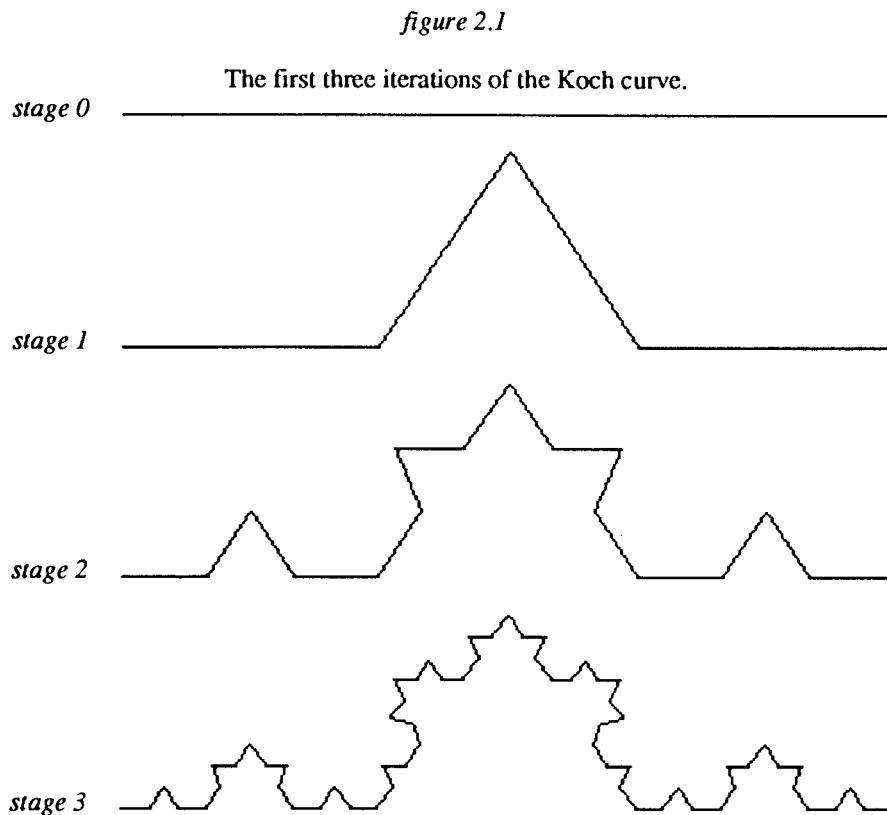
stage 2

stage 3

The Koch curve is generated by beginning with a straight line (*stage zero* of *figure 1*). The middle third is removed from the straight line and is replaced with two segments that are equal to one third of the length of the original segment; the resulting figure is a line segment interrupted by an equilateral triangle that has no base (see *stage 1* of *figure 1*). The Koch curve is a fractal in the mathematical limit of an infinite number of iterations.

Another well known fractal is called the Harter-Heightway curve. This structure, also known as the Dragon curve, was popularized in the book and motion picture *Jurassic Park*. The first few iterations of it are shown in *figure 2.2*.

figure 2.2
The Dragon curve

stage 1

stage 2

stage 3

stage 8
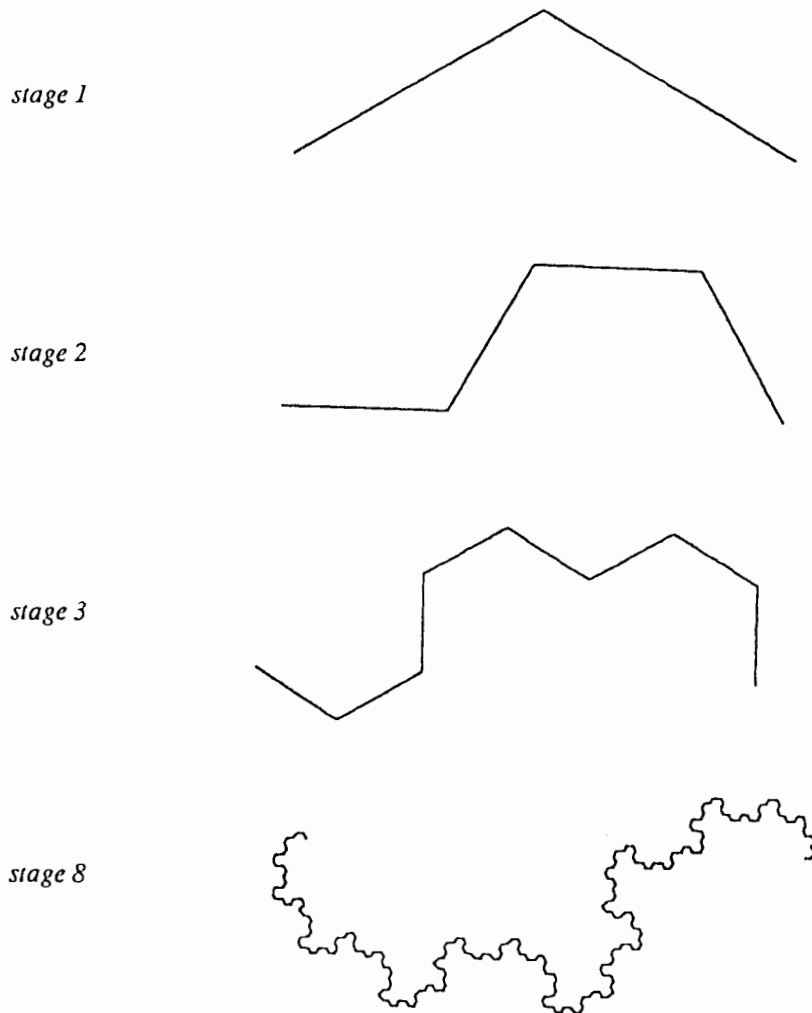
Self-similarity dimension.

Self-similarity dimension is a complex idea. In order to explain it at a level that the students could understand I needed a description that was simple and yet adequately conveyed the essence of the concept. To that end I employed an approach that I have seen in at least four different sources. It appears in Kern and Mauk (1990), and in Peitgen, Jürgens, and Saupe's *Fractals for the Classroom Part One: Introduction to Fractals and Chaos.* It is also contained in an unpublished manuscript by Jean Pederson and Peter Hilton under the working title *Mathematical Reflections in a Room with Many Mirrors.* A similar explanation can be found in

Peak and Frame's *Chaos Under Control.* In order to comprehend the approach, a little

vocabulary is required. (The information that follows is summarized for students in

Introduction to Fractal Geometry: Handout #2. This handout appears in Appendix B)

Most people are familiar with objects whose dimensions can be described in the

conventional manner: a point has zero dimension, a line is one dimensional, a square has two

dimensions, and a cube is considered to be three dimensional. Pederson and Hilton begin with

the assertion that any fractal can be broken up into self-similar objects. The objects into which

the fractal is broken are then said to be reduced by a certain numerical factor: this is referred to

as the *reduction factor* (or *magnification factor* depending upon perspective) and is assigned the

variable r. The number of copies, or "cells," of the original figure formed by this breaking

process is assigned the variable N. The *self-similarity dimension* is assigned the variable d

(Pederson and Hilton, 1995, p. 11).

figure 2.3
A line segment divided into 4 identical copies.
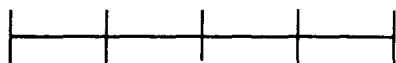
A square subdivided into 16 identical smaller squares.
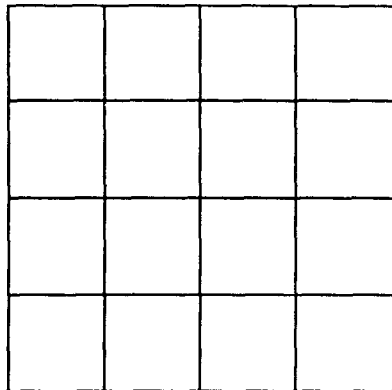
The straight line segment shown in *figure 2.3* has been subdivided into four equal parts. The original segment is linearly reduced by a factor of 4 and the number of identical self-similar copies formed is 4. The large square shown is subdivided into many smaller squares. Again, the linear reduction factor is 4, but this time the number of identical copies is 16. The cube diagrammed in *figure 2.4* has been subdivided into a number of smaller cubes. Yet again the reduction factor is 4, but now the number of identical copies is 64. These relationships can be described by the equation $r^d = N$.

Applying $r^d = N$ to the straight line we get $r = 4$ and $N = 4$; therefore, the value of d must be 1 in order to make the equation true. Progressing to the square, $r = 4$, $N = 16$, and so d must equal 2 for the equation to be true. Finally for the cube, $r = 4$, $N = 64$, and thus $d = 3$. These results illustrate the conventional facts; a line has a dimension of 1, a square is two dimensional, and a cube is three dimensional. The results become more interesting when the equation is applied to a figure that is considered to be a fractal.

figure 2.4

A cube subdivided in 64 identical smaller cubes.
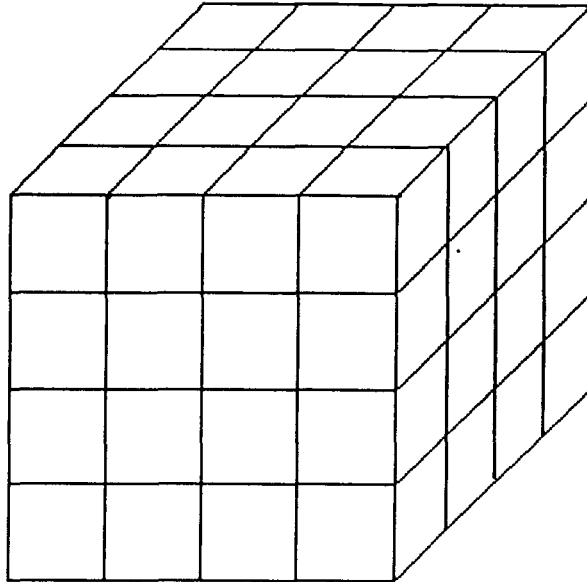
Using the Koch curve as an example (refer to *figure 2.1*), the first iteration yields four

identical copies of the original line segment, and each copy is one third as long as the original;

thus $N = 4$ and $r = 3$. Solving the equation for d we get

$$r^d = N; \therefore ,$$

$$d = \frac{\log N}{\log r}$$

$$d = \frac{\log 4}{\log 3}$$

$$d \approx 1.262$$

Therefore, the Koch curve would seem to have a dimension of approximately 1.262.

According to Pederson and Hilton, the equation $d = \dfrac{\log N}{\log r}$ is an acceptable definition for the

self-similarity dimension (Pederson and Hilton, 1995, p. 12). It should be noted that this is true

only in the context of an infinite number of iterations. Self-similarity dimension is actually

given by $d = \lim_{n \to \infty} \dfrac{\log N}{\log r}$ where n is the number of pieces into which the object is cut. Thus, not

only do such structures appear to have very unusual dimensions, but they also have other

interesting properties.

Perimeter of and area under the Koch Curve.

Referring to *figure 2.1*, one can see that with each subsequent iteration, the perimeter of the figure increases. If the original line segment is said to have unit length, then after the *stage 1* iteration the length of the curve has increased to 4/3. After the second iteration, the length has increased by another factor of 4/3; therefore, the length of the second iteration is $(4/3)^2$ or 16/9. After the third iteration the perimeter of the figure has increased to $(4/3)^3$ or 64/27. Thus, after the nth iteration, the perimeter is given by $(4/3)^n$. Since the base of this expression is greater than 1, as the exponent increases without bound the power will do the same; therefore, the perimeter is infinite. Calculation of the area under the curve yields a quite different result.

At *stage 0* the area under the curve is 0. At *stage 1* the area under the curve consists of one equilateral triangle whose area is given by

$$\frac{1}{2}bh = \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{6}\right)\text{units}^2$$

$$= \frac{\sqrt{3}}{36}\text{units}^2$$

At the *stage 2* iteration the area has increased slightly due to the addition of four smaller triangles that each have sides of length 1/9; therefore, the area under the curve is now given by

$$Area = \frac{\sqrt{3}}{36} + \frac{1}{2}\left(\frac{1}{9}\right)\left(\frac{\sqrt{3}}{18}\right)(4)$$

$$= \frac{\sqrt{3}}{36} + \frac{\sqrt{3}}{81}\text{units}^2$$

At *stage 3* sixteen smaller triangles that have sides of length 1/27 have been added; thus the area under the curve is now given by

$$Area = \frac{\sqrt{3}}{36} + \frac{\sqrt{3}}{81} + \frac{1}{2}\left(\frac{1}{27}\right)\left(\frac{\sqrt{3}}{54}\right)(16)$$

$$= \frac{\sqrt{3}}{36} + \frac{\sqrt{3}}{81} + \frac{4\sqrt{3}}{729} units^2$$

This process continues through successive iterations. The nth iteration adds another n^{th} power of 4 to the total number of triangles; each new set of triangles has sides of length $\left[\frac{1}{3}\right]^n$ units. The total area is thus given by

$$Area = \frac{\sqrt{3}}{36} + \frac{\sqrt{3}}{81} + \frac{4\sqrt{3}}{729} + ... + \frac{1}{2}\left[\frac{1}{3}\right]^n\left[\frac{\sqrt{3}}{2(3^n)}\right](4^n)$$

$$= \frac{\sqrt{3}}{4} \times \frac{1}{9} + \frac{\sqrt{3}}{4} \times \frac{4}{9^2} + \frac{\sqrt{3}}{4} \times \frac{4^2}{9^3} + ... + \frac{\sqrt{3}}{4} \times \frac{4^{n-1}}{9^n}$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{9} + \frac{4}{9^2} + \frac{4^2}{9^3} + ... + \frac{4^{n-1}}{9^n}\right]$$

$$= \frac{\sqrt{3}}{4} \times \frac{1}{9}\left[1 + \frac{4}{9} + \frac{4^2}{9^2} + ... + \frac{4^{n-1}}{9^{n-1}}\right]$$

The large brackets constitute an infinite geometric series with $a = 1$ and $r = \frac{4}{9}; \therefore,$

$$Area = \frac{\sqrt{3}}{36}\left[\frac{a}{1-r}\right]$$

$$= \frac{\sqrt{3}}{36}\left[\frac{1}{1-\frac{4}{9}}\right]$$

$$= \frac{\sqrt{3}}{20} units^2$$

It would seem that the greater the depth in which one studies these figures, the more complex they become. How is it that a seemingly simple collection of line segments could contain within it such paradoxical properties? My students were about to discover just exactly how mathematically unsettling these properties can be.

Chapter 3

Literature Review

This chapter consists of a review of the literature related to the implementation of fractal geometry in a secondary school classroom environment. It has been divided into three subsections: theory, practice, and resources.

The first section is a review of the literature that illustrates the potential for the curricular integration of fractal geometry. This is subdivided into two parts; the first summarizes the thoughts of researchers who advocate the introduction of fractals. The second part is an overview of the literature describing the experiences of teachers who have actually experimented with fractals in their classrooms. The chapter concludes with a summary of the resources that I found to be useful in planning and implementing my fractal geometry study unit.

Fractals in the Classroom.

To my knowledge there have been no studies done on how secondary school students react to the introduction of fractal geometry. Although there exists ample literature describing the introduction of various aspects of the topic, these papers deal only with content; I discovered no detailed account of how the students responded to the content. Furthermore, I was unable to find any documentation of the implementation of an entire study unit on fractals. Nonetheless, there are many people who have spent a great deal of time and energy on bringing this topic to the classroom.

One need only leaf through any issue of *School Mathematics and Science*, or *The Mathematics Teacher*, or *Journal of Computers in Math and Science Teaching* to see that there are countless educators in the field that are experimenting with alternate curricula. Most of them are undoubtedly doing so in an attempt to find new ways of increasing student interest and

motivation. Many of them are trying to find stimulating new ways in which to refresh the delivery of a traditional curriculum. There exists a body of literature that suggests some teachers feel these objectives can be achieved through the study of fractals.

Fractal geometry combines mathematics with visual images that are fascinating in their intricacy. These images are often strikingly beautiful; there are few topics that combine mathematics and art in such spectacular fashion. Many students are attracted to the images, and this attraction offers teachers an opportunity to explore with students the mathematics behind the images.

Many educators involved in curricular experimentation are simultaneously attempting to tie mathematical activity more closely to current technology. There is ample justification for doing so, and fractal geometry is well suited to this objective. According to the NCTM's *Professional Standards for Teaching Mathematics*, "The teacher of mathematics, in order to enhance discourse, should encourage and accept the use of computers, calculators, and other technology." Elaborating on that statement, the *Standards* go on to say that "...Teachers must value and encourage the use of a variety of tools...[and]...should also help students learn to use calculators, computers, and other technological devices as tools for mathematical discourse." (NCTM Standards, 1991, p. 52) With the proliferation of personal computers and appropriate software applications, it has become incumbent upon the mathematics teacher to incorporate these innovations into his/her repertoire of teaching strategies.

Had it not been for the advent of more accessible computer technology it is unlikely that fractal geometry would yet be considered a legitimate field of mathematical study. It was in the early 1960s that Edward Lorenz discovered chaotic behavior in a system of differential equations while running a meteorological model on his "desk top" computer. His discovery remained largely unnoticed for almost a decade until 1971, when physicist Mitchell Feigenbaum discovered the link between fractal geomety and chaos theory. Thus, the study of fractals and

computer technology are inextricably linked, making it an ideal topic of study for secondary school students.

Suggestions for implementation

The theorists

Paul Goldenberg has spent some time exploring this idea. In his paper, *Seeing Beauty in Mathmatics: Using Fractal Geometry to Build a Spirit of Mathematical Enquiry*, Goldenberg with his colleagues sought "...to demonstrate that it is possible to make dramatic and fundamental changes in students' engagement in mathematics." This ambitious objective was motivated by several observations, including: (1) the perception that "...mathematics is the least creative of subjects: a dead, unchanging body of facts and techniques handed down from the ancients..."; (2) the resistance of the mathematics curriculum to change of any kind, and; (3) "Attracting students to mathematics, or, more precisely, maintaining and justifying their interest in mathematics." (Goldenberg, 1991, p. 40)

In my view, the latter provides a compelling argument for inclusion of fractals and technology as a unit of study. Goldenberg elaborates on this: "By tenth grade, many students have stopped taking the subject altogether leaving no future chance to discover parts of it that might appeal to them and cutting themselves off from opportunities to pursue studies that depend on higher mathematics (p. 40)." He mentions females as being particularly susceptible to the abandonment of mathematics as an optional course of study (p. 40). It is my feeling that this is one of the most pressing issues facing mathematics educators today. Goldenberg does not, however, limit his discussion to the realm of mathematics.

There is some evidence to suggest that fractals have many applications for modelling 'real life' phenomena in a wide variety of sciences. Here then, is a topic that could assist students to observe that mathematics does not exist in isolation: that it has the potential to be applied across curricula. Thus, Goldenberg sees "...the importance of fractal geometry as a tool

beyond the realm of academic mathematics and its potentially pivotal position in the curriculum as an organizing and unifying force for science and mathematics." (p. 50) William Egnatoff (1991) shares this view:

> Fractal geometry links mathematics, science and computer science in a web of complexity. The strands of that web - self-similarity, recursion, nonlinearity, randomness, chaos - are now visible to curious students. By constructing computational tools and asking leading questions in collaboration with peers and teachers, students can assemble fragments of personal experience and subject knowledge into a more coherent picture of how their world works and who they are within it (page 40).

A large portion of Goldenberg's paper consists of descriptions of various possible scenarios for studying fractals with computers. Much of what he suggests is "ideal"; there is little that I was able to incorporate into my students' experience due to logistical and technological constraints. Three suggestions of which I was able to make use, in slightly modified forms, included the drawing of fractal "trees," playing the "chaos game," and calculating the perimeter of, and area under, the Koch curve. Goldenberg then selects some topics from the traditional curriculum that can be applied to his proposed curricular innovations.

Goldenberg suggests that fractals could be studied as early as the seventh grade. There is, in fact, a case study of a precocious third grade student who, for a mathematics fair project, created a presentation on fractal geometry (Vacc, 1992). It should be noted that the student was considered to be "...one to two grade levels above his current grade placement (p. 280)." Nonetheless, this eight year old was able to grasp some of the fundamentals of fractals and communicate his knowledge to contest judges and classmates (p. 281). Ms. Vacc was so impressed that she decided incorporate the student and his project in a mathematics course she was teaching.

The course was a graduate level mathematics course for preservice and inservice teachers. At the beginning of the course she administered a questionnaire and determined that

none of her students considered fractal geometry to be an appropriate topic for elementary school students. She then introduced fractals to her students by having the eight year old present his project (p. 280). Not surprisingly, subsequent to the presentation her students decided "...that simple fractal concepts appear appropriate for inclusion in the elementary-school mathematics curriculum." However, she concedes that "...further exploration is needed in applying the subject's activities...[to]...other elementary school students (p. 285)."

Goldenberg acknowledges that, in order to introduce fractals to young children, there would necessarily be some early contact with more advanced mathematical ideas. He specifically mentions, among others, topics such as measure theory, trigonometry, calculus, and the concept of function. His argument in favour of early exposure to such ideas is that: "Initial contacts would be casual (even superficial), but the practical experience gained with these objects would make them quite homey and familiar by the time their properites are studied formally...." (p. 60)

Other threads Goldenberg has pulled from existing courses of study that can be tied to the study of fractals include limits, series, and "...such familiar notions as length, area, dimension, space, and randomness." (p. 51) Egnatoff (1991) augments this list with graphing, ratios and scaling, linear and quadratic expressions, equations of straight lines, logarithmic and exponential functions, and probability (p. 41). Several of these topics can be extended and generalized in a fractal context in ways which most students would not think possible. The idea that an object could have a fractional dimension is one that is completely counter-intuitive for many adolescents. A figure possessing an infinite perimeter enclosing a finite area might seem highly unlikely to anyone unfamiliar with fractal geometry.

These are but a few seemingly paradoxical situations that can arise from the study of fractals. Such conflicts may encourage students to reevaluate some of the ideas that they have considered sacred laws of mathematics. This reevaluation might possibly lead students to

"...wish to alter a definition to accommodate a new generalization of an old idea [so that] they may no longer see mathematics as rigid, unchanging, unforgiving and finished, but rather as a live and evolving study (page 51)." As revolutionary as this idea might seem to many secondary mathematics students and educators, it is not what Goldenberg perceives to be the key advantage to studying this new topic.

Perhaps the greatest impact that fractals and chaos have had on the mathematical community is the way in which they have revitalized the practice of mathematical experimentation. According to Mandelbröt, for a long period of time, experimental mathematics was not considered to be "real mathematics": an attitude that he claims dates to the time of Plato (Mandelbröt in Peitgen et al, 1992, p. 7). However, since the emergence of fractals and chaos as important fields of study, mathematical experimentation has enjoyed a renaissance. It is this that Goldenberg sees as perhaps the most important contribution that fractals can make to the mathematics classroom.

> The opportunity to perform mathematical experiments using interactive visual media can be as valuable to students as to mathematicians. Properly designed computer-supported environments can provide, through their concreteness, a scaffold for reasoning and a matrix for problem posing and...help students in grades 7 through 12 engage themselves in a visual, experimental mathematics, finding, posing, and attempting to solve problems much as a creative research mathematician would (page 41).

Goldenberg's assessment of the value of a mathematics student behaving as a scientist is iterated by Egnatoff: "Making sense of coastlines and population growth entails much of what scientists do - observing, classifying, describing, explaining, thinking with examples, modelling, and clarifying one's thoughts through reading, writing, and talking with colleagues." (p. 41) If fractal geometry were to be included in the school mathematics curriculum as Goldenberg suggests, it might truly be the most important curricular innovation in decades.

Unfortunately, the schools within my realm of experience possess neither the hardware nor the software required to explore the topic as Goldenberg has described; given the fiscal realities of public education in Canada today, it is unlikely that fractal geometry will occupy the role in secondary school mathematics curricula that he prescribes. Despite this, it is possible to provide students with some semblance of the mathematical experience he envisions as long as one has access to some relatively recent technology.

Jane Kern and Cherry Mauk (1990) advocate a teaching/learning exploration of fractal geometry that makes extensive use of computers and LOGO. LOGO is a programming language that was developed in the 1970s specifically for school mathematics students. It is easy to learn and is compatible with even the most obsolete hardware; although the article was written in 1990 the authors mention that, for their purposes, they employed a version of LOGO compatible with an Apple II (p. 181).

Even simple fractals are tedious and time consuming to draw by hand. Thus, computers make an excellent tool for exploration because they are designed to make repetitive calculations. It is this recursive aspect of computer programming and function that makes them ideally suited to drawing fractals.

Kern and Mauk list a number of natural phenomena that are worthwhile exploring in a fractal context. Their list includes the self-similar properties of twigs, leaves, trees, coastlines, faultlines, and runoff pattern (pp. 179 - 180).

> These interesting properties can be studied by students even at the secondary school level by considering some of the classic 'monster cuves,' such as the Koch snowflake...The generation of fractals using LOGO involves concepts from turtle geometry, Euclidean geometry, and fractal geometry (pp. 180 - 181).

A considerable portion of the remainder of their article deals with using LOGO to draw some of the classic fractals. The paper concludes with several justifications for the introduction

of fractal geometry to the secondary school student. An appendix contains several sample LOGO programmes that can be used to generate variations of the classic Koch curve.

Kern and Mauk are not without support in their advocacy of LOGO as a useful tool for the study of fractals. David Thomas has written a book entitled *Teenagers, Teachers, and Mathematics* (1992). Included within the book is a chapter on fractals and LOGO. There are suggestions for implementation and several sample programmes that will generate simple fractal structures. The January, 1985 issue of *Microquests* is devoted entirely to the exploration of fractal geometry using LOGO. Goldenberg also sees LOGO as a valuable implement in fractal exploration. While it is desirable to allow students to experiment and explore using LOGO and computers, the theorists fail to take into account that there is little "spare" time in a tightly packed curriculum to allow students to do so. In order to provide students with the sort of experience described above, there would appear to be two courses of action.

One option is to build a LOGO study unit into the existing curriculum. In this way students could be exposed to simple programming early on in their mathematical training. This study of LOGO could be ongoing throughout their school years, so that by the time they reach secondary school they are already proficient at the basics of the language. Hence, students would not of necessity be taught the entire language at once; they would simply modify their existing knowledge to include the creation of fractals in their repertoire of programming skills. There is insufficient time in the school year to provide students with their initial exposure to LOGO and then expect them to have the ability, within a relatively short period of time, to write programmes that will draw fractals.

The second option consists of sitting students in front of computers and giving them the programmes. A few would be capable of recognizing the algorithms and altering them to change the output of the programme. For these few it would be a worthwhile experience. The majority would not find it to be particularly stimulating; LOGO graphics are not exciting to

watch. In order to make a LOGO experience successful for a group of secondary school students, the classroom teacher must be prepared to devote a minimum of two weeks of instructional time to the topic. In and of itself this would be a worthwhile unit of study, but curricular pressure renders it virtually impossible. In order to teach and learn fractal geometry effectively, government mandated curricular reform is required.

Goldenberg posits that "...fractal geometry is a credible alternative to traditional precalculus courses. It ties together all prior mathematics..." (p. 63) Be that as it may, it has yet to be officially included in any mathematics curriculum of which I am aware. He advocates a "...fundamental restructuring of the precollege mathematics curriculum (p. 63)..." Ideally this would be a possibility. For the present it seems that fractals will necessarily be introduced in bits and pieces in assorted locations, rather than the large scale implementation that Goldenberg so desires. It will be the prerogative of individual teachers to provide students with experience in this "new" topic.

The practitioners

Ron Lewis of Sudbury, Ontario has developed and currently teaches an entire senior secondary course on fractal geometry and chaos theory (Lewis, 1990). Kari Oliver and Peter Penick of Port Orchard, Washington have compiled a set of activities that introduce some of the fundamentals of fractals and chaos. However, neither Lewis, nor Oliver and Penick have published any descriptions of either their experiences, or those of their students. What follows is a summary of the relevant literature that I was able to locate.

The "chaos game" is an activity that seems to be popular with teachers experimenting with fractals in the classroom. There are several versions of it; the most popular is that which I used with my class and is described in detail in the methodology section of this paper. Ray Barton is another teacher who had his students experiment with that particular variation of the chaos game. (In his article, *Chaos and Fractals*, he does not specify the age/grade level of the students in his class.)

Barton's students' initial exposure to the chaos game was to plot points using pencil, ruler, and paper. They were subsequently asked to predict the final outcome of the "game." Then, Barton encouraged anyone capable of programming to analyze the algorithm and write a programme that would "play" the game. His students reacted to the outcome of the programme (a Sierpinski gasket) "...with surprise and interest (Barton, 1990, p. 525)."

Barton goes on to describe a slightly different algorithm that has similar results. In this version he employs a matrix equation that transforms a point through rotations and translations:

The matrix equation for transforming a point (vector) is

$$\begin{bmatrix} X_{n+1} \\ Y_{n+1} \end{bmatrix} = \begin{bmatrix} R\cos(A) & -S\sin(A) \\ R\sin(A) & S\cos(A) \end{bmatrix} \bullet \begin{bmatrix} X_n \\ Y_n \end{bmatrix} + \begin{bmatrix} H \\ K \end{bmatrix}$$

where R is the horizontal scaling factor, S is the vertical scaling factor, A is the angle of rotation, H is the horizontal translation , K is the vertical translation , X_n and Y_n are the coordinates of the preimage, and X_{n+1} and Y_{n+1} are the coordinates of the image under the transformation. By using different values for the parameters R, S, A, H, and K, one can use this matrix equation to describe each of the transformations used in the chaos game (pp. 525 - 526).

Barton used this equation to facilitate the generalization of the chaos game programme . The equation also simplifies the programme modifications that are required to create different images. He used the generalized programme to encourage students "...to experiment with the chaos game by altering the parameters R, S, A, H, and K in the subroutines of the chaos game programme. This activity can afford an opportunity for students to investigate the question of whether the operations of rotating and translating are commutative." (p. 526 - 527) In this way he manages to tie a number of advanced mathematical concepts to a relatively simple activity.

Barton continues his article by describing image compression, one of the major commercial/industrial applications of fractal geometry. He states that by making minor alterations in the input parameters of the programme included in his article, many interesting, realistic looking images can be generated. He concludes by itemizing several concepts with

which students will gain familiarity by experimenting with the chaos game and a computer. His list includes "...randomness, transformations, computer use, problem analysis, and fractal self-similarity." (p. 529)

Barton's article is an excellent resource for anyone who is considering an extended, computer based investigation into the chaos game. The "game" is simple; although, superficially, the process appears to be completely random, the resulting structure is anything but. Hence, it provides an excellent bridge between chaos theory and fractal geometry. Another method by which chaos and fractals can be effectively combined is in a study of the Mandelbröt set (hereafter referred to as the M-set). If one has access to a computer lab containing some current hardware, this can be a fascinating exploration for high school students.

In order to understand the M-set students must first have some knowledge of complex numbers. This need not be an obstacle, as most senior secondary mathematics students are able to quickly grasp the fundamentals of complex numbers. Marny Frantz and Sylvia Lazarnick have published an article (1991) that details their methodology in the introduction of the M-set to a group of secondary school students. It is interesting to note that all of their objectives can be justified either through a traditional mathematics curriculum, or by following recommendations set out in the NCTM *Standards for Teaching Mathematics* (1991). They are listed as follows:

> Our students will-
> 1. become competent in complex number operations and graphing;
> 2. understand and use an iterative process;
> 3. understand how the M-set is generated;
> 4. gain experience and facility in using the calculator for complicated computations;
> 5. experience the power and utility of computers; and
> 6. experience a current and beautiful topic in mathematics (page 173).

Students spent three days mastering the requisite complex number concepts; these included the four basic operations, graphing, and calculating the distance from the origin to a

point in the complex plane. A fourth day was occupied by an introduction to fractals and iterative computations.

The students' task was to determine if a complex number was contained within the M-set. Initially, they were expected to do the iterative complex number computations using a calculator and the formula $z_{n+1} = z_n^2 + a + bi$, where $z_0 = 0$ and $a + bi$ is a given complex number. Frantz and Lazarnick report that students quickly discovered this becomes extremely tedious without the aid of a computer. The authors felt that this exercise provided students with "...a much more intimate understanding of the M-set iterative process than they might have had without [having done] any calculations (p. 177)."

The final three days involved computer based activities and discussion of results. Two programmes were used in the study unit; one of them was developed by a student at the school in which the experiment took place, and the other is commercially available. The class employed the former in deciding whether or not a particular point in the complex plane is included in the M-set. The latter was used to explore the M-set; apparently this programme has the capability of zooming in on any desired portion of the set. There is a plethora of available software that will do this; one of the best programmes of this nature that I saw is available as freeware on the Internet.

Although there is no detailed reference to students' reactions to the study unit, it was felt by the authors that "...almost all students finished the experience with some sense of the beauty, elegance, and power of mathematics (p. 177)." It is of interest to note that in both Barton's and Frantz and Lazarnick's classes, a few students were sufficiently motivated to write programmes to assist them in their studies. Obviously not all students have the expertise to perform this task; it might be appropriate for students such as these to pursue an independent exploration of fractals using LOGO.

Computer science and mathematics classes are obviously well suited for studying fractal geometry and chaos. It appears that many mathematics educators have considered providing students their initial contact with fractals; however, according to the literature, it would seem that it has largely been science teachers that have introduced fractals to their students. Tim Marks is one such teacher. He has written an account of his experience entitled *Focus on Fractals* (1992) that appears in *The Science Teacher*.

In his article, Marks describes how a class of his physics students performed an investigation in measuring the length, and then calculating the fractal dimension, of a coastline. He describes two approaches to the task. Although one method is more intuitive in terms of measuring distance than the other, the less intutive method yields a result that is easier to understand.

Marks describes how a map of a coastline can be measured using a pair of dividers. The students adjust the dividers so that they span, for example, 40 mm. They "walk" their dividers along the coastline, counting the number of "steps" required to cover the entire length of it. The scale on the map is then used to determine the length of the coastline. Once this has been accomplished they halve the length of the span and then measure the coastline again. This process is repeated until the span of the dividers is 5 mm. The measurements are tabulated and graphed on double logarithmic paper. The "best line" is then drawn through the data points. The fractal dimension of the coastline is given by $d = 1 + |m|$ where d is the dimension and m is the slope of the line of best fit (Lewis, 1990, p. 4). This same activity is included in the Survey Mathematics 12 materials that are available from the Ministry.

There is one advantage and one disadvantage that I see in using this method. Although the procedure for estimating the length of the coastline is easily understood, the calculation of the fractal dimension is not so straightforward. The formula $d = 1 + |m|$ would necessarily be given to students with no explanation; they would be expected to accept it at face value and

apply it to approximate the fractal dimension of the coastline. For this reason I chose to use the second method that Marks describes.

The box counting method is described in detail in the methodology section of this paper. It is not so easily perceived to be a measurement of the length of the coastline, but the resulting graph is much easier to interpret. The best line drawn through the data obtained in this fashion turns out to have a slope that approximates the fractal dimension directly. Not only is the graph more easily understood, but also the resource that I used includes activities leading up to the box counting technique that served to introduce students to it and assisted them in reaching the desired conclusions.

Physics educators appear to be particularly interested in the application of fractal geometry to their content area. In *The Physics Teacher* I learned of how fractals had been incorporated into four more physics classes. The Centre for Polymer Studies, Department of Physics at Boston University has collaborated with the Science and Mathematics Education Centre, School of Education, at the same University to develop some materials for linking science education and fractal geometry.

This consortium has published two articles; the first is entitled *Science Research in the Classroom*. (Buldyrev, Erickson, Garik, Shore, Stanley, Taylor, Trunfio, & Hickman, 1994) The classroom model employed is one whereby the "...high school teacher becomes a mentor and the student groups become the research community (p. 411)." The authors draw an analogy between high school teachers and their students, and research professors and graduate students. A mentoring process "...through *demonstrating*, *scaffolding*, and *fading*..." is described.

> During the initiation process the students confers frequently with the professor who *demonstrates* the research process...As students' strengths develop, they assume increasing responsibility for both accomplishing the research, and for defining the work to be done. The professor, however, continues to monitor the student's progress, and provides *scaffolding* by interjecting suggestions at appropriate times.

Finally, as the student achieves and demonstrates confidence in his/her own expertise, the professor's role as mentor *fades*, and the new role of colleague emerges (p. 412).

The authors elaborate: "...this process is the new role envisioned for secondary school science teachers who use our materials (p. 412)." This description reads very much like that which Goldenberg proposes for his ideal mathematics curriculum and classroom; it echoes the spirit of mathematical experimentation that Mandelbrot has repopularized. Such ideas also appear in the NCTM *Standards for Teaching Mathematics*. The difference is that while mathematics educators pontificate about such revolutionary ideas, the physics teachers "...are currently testing this and other projects with 32 teachers (p. 411)..."

The project entails students making use of computers and fractal geometry to model certain physical processes. In one scenario, students observe a pattern that develops from the plating of metals out of a solution. The pattern is then scanned into a computer and the fractal dimension of the pattern is calculated by interactive software; the software uses an algorithm identical to one of those described in Tim Marks' paper.

Several salient points are made in the article. "Doing research means moving ahead through a sea of ignorance, paying attention to a few essentials, and filling in one's background on the fly. We try to provide a similar experience to students." The authors ask and then answer the question: "In using our materials does the student become a science researcher? Not quite....We call this role student as investigator (p. 412)." The article closes with the following:

> Most current research is not just interdisciplinary: *the discipline is irrelevant*. That is also an important message of our materials. High school physics, chemistry, biology, and mathematics teachers also use our materials. Modern science research ranges indifferently across all science disciplines; some day our science *teaching* may also lose this disunion (pp. 415 - 416).

Overcoming Resistance with Fractals (Ching, Erickson, Garik, Hickman, Jordan, Schwarzer, & Shore, 1994) is the second paper published by this same group. This article

details how "...students construct a Sierpinski gasket from resistors and measure its resistance as a function of size." The Sierpinski gasket was chosen because "...it has proven a workhorse for testing physical theories on fractal geometry. Electrical conductivity, diffusive transport, and thermodynamics have all been studied extensively on the gasket (p. 546)..."

According to the authors, the understanding of "...materials properties of disordered solids is one of the central research themes in modern condensed matter and materials science." In this investigation, the Sierpinski gasket is chosen to model a "disordered solid." A fractal is chosen as the model because: "Such a study allows the building of physical intuition for the behavior of objects with complicated geometry (p. 546)."

Physics is not my area of expertise, and so much of the article has little meaning for me. However, the report concludes with a section entitled "*Student Response.*" In that section, the authors report that "...female students played an active role in the construction of the [resistor] network." Furthermore, one young lady became particularly engaged in the laboratory exercise, despite the fact it had been the authors' experience that "...young women students act primarily as recorders of data reported to them by male students (p. 550)." Here, then, is some anecdotal evidence supportive of Goldenberg's assertion that the study of fractal geometry could entice greater numbers of female students to consider mathematics to be a viable option in their senior high school years.

The Boston University project would appear to be a large, well organized, well thought out attempt at cross curricular studies incorporating fractal geometry. However, there are physics teachers who are doing the same on a much smaller scale. *A Simple Experiment that Demonstrates Fractal Behavior* (Ko and Bean, 1991) and *Fractal Bread* (Esbenshade, 1991) are descriptions of investigations into the dimension of objects that have dimension of between two and three.

Ko and Bean show that "...the crumpling of paper balls exhibits the concept of a nonintegral dimension in a way that is easily done in the classroom or laboratory (p. 79)." Their study was conducted during the summer of 1988 in a "Young Scholars Programme" sponsored by the National Science Foundation (p. 79). *Fractal Bread* describes an experiment performed by Esbenshade's physics class. His students compressed bread into wads and then calculated the dimension of the "fractal bread." He suggests that an application of this concept would be to "...compare different complex structures easily and quantitatively (p. 236)." For an an easily understood explanation of the mathematics behind these investigations, I refer the reader to David Peak and Michael Frame's book entitled *Chaos Under Control: The Art and Science of Complexity* (Peak & Frame, 1994, pp. 96 - 99).

It is of interest to note that, despite the fact fractal geometry is perceived to be a mathematical domain, it would seem to be the science teachers who are taking the lead in introducing this topic to students. It is possible that, because of curricular pressures, mathematics teachers are reluctant to introduce fractals; since there is insufficient time to do justice to the topic, it is ignored completely. Fractal geometry is not, in my opinion, a topic that lends itself well to an exploration that is one or two classes in duration. Such a treatment would tend to trivialize what is an extremely important area of mathematical research. However, despite the reluctance to make use of them, there are many high quality resource materials available that are appropriate to the secondary school level.

The resources

There are several resources upon which I drew heavily when planning my study unit. *Fractals for the Classroom Part One: Introduction to Fractals and Chaos* by Peitgen, Jürgens, and Saupe was invaluable. Another source that I found extremely useful was Bernt Wahl's book entitled *Exploring Fractals on the Macintosh*. A volume that I found to be effective in communicating the basics of the field was Peak and Frame's *Chaos Under Control: The Art and*

Science of Complexity. All three of these books include many suggestions for implementation that are suitable for a variety of secondary school levels.

Fractals for the Classroom is a thorough examination of fractals and chaos. It consists of two textbooks, Volumes *1* and *2*; volume 1 focusses on fractal geometry, and volume 2 deals more with chaos theory. Each volume is accompanied by a soft cover book entitled *Strategic Activities*. Photocopying of all investigations in the *Strategic Activities* workbooks is permitted. These books include more than enough content and experimentation to occupy an entire course on fractals and chaos.

Although the textbooks are aimed at the senior secondary school student, my feeling is that they are somewhat too advanced for most high school aged adolescents. Nevertheless, I found the textbooks to be extremely useful for the enhancement of my grounding in the topic. In contrast, the *Strategic Activities* workbooks include worksheets that are suitable for a broad spectrum of mathematical sophistication; there are materials contained within these texts that are appropriate for any secondary school mathematics class.

The textbooks are written so that no chapter is necessarily dependent upon the preceding one: each can stand on its own and be read independently from the others. This enables the classroom teacher to select one or more chapters to be studied without feeling the need to teach both books in their entirety. At the conclusion of each chapter there is a "Programme of the Chapter." These programmes are written in BASIC and serve to "...highlight one of the most prominent experiments of the respective chapter (p. ix)."

The *Strategic Activities* workbooks contain investigations that parallel the textbook chapters, but again, the teacher is free to pick and choose the investigations that best suit his/her needs and time constraints. This flexibility is a very attractive feature given the nature of the secondary mathematics curriculum. For the purposes of my study unit I selected several

worksheets. These included the authors' version of the chaos game, an investigation based on Pascal's triangle, and a series of four activities which enable students to calculate the fractal dimension of a coastline.

The original concept for *Fractals for the Classroom* emerged from a series of lectures given by one of the authors, Heinz-Otto Peitgen. Through these lectures it was discovered that there existed within the mathematics education community a great deal of interest in both fractal geometry and chaos. This interest culminated in an agreement between Springer-Verlag and the NCTM to cooperate in the production of materials suitable for secondary schools.

The endorsement and involvement of the NCTM generally implies a progressive approach to a topic and *Fractals for the Classroom* is no exception. There is ample potential for exploration incorporating contemporary technologies, including video tape, computers, and graphics calculators. Alternatively, if access to such facilites is limited, there are many classroom activities intended primarily for student based explorations. With these materials any secondary mathematics class can experience as much or as little fractal geometry as the teacher sees fit. If one is fortunate enough to have discretionary use of current hardware, there exist countless software applications for the study of fractals.

Exploring Fractals on the Macintosh is an excellent resource for any teacher who has access to a Mac lab. This book is more "readable" for secondary school students than is *Fractals for the Classroom* but, as the title implies, computer hardware is prerequisite. The most important advantage of this publication is that included with the book is a programme entitled *FractaSketch*, which is described in more detail in the methodology section of this paper. Like the *Fractals for the Classroom* series, there is enough information and investigation contained within *Exploring Fractals on the Macintosh* to occupy an entire course on fractal geometry. Since focus of my study unit was a general introduction to fractals, I used only a

small portion of *Exploring Fractals on the Macintosh*; chapters one, two, and three were of particular use to me.

Chapters one and two of the book provide an overview of fractal geometry. Chapter one is primarily concerned with the many "real life" phenomena that would appear to be closely modelled by fractal geometry. Many of these are discussed in chapter one; Wahl's list includes phenomena as diverse as river systems, architecture, and music. Chapter two begins with a brief history of the field followed by a description of two different types of fractals. Although he does not use the terms affine and deterministic, he essentially distinguishes between these two categories of fractals. There is a brief discussion of fractal dimension and chaos theory, after which the chapter concludes with applications of fractals and chaos in science and technology. These chapters comprise an excellent introduction to the topic for anyone who is unfamiliar with fractals, making it ideal for most secondary school students.

Chapter three was, by far, the most useful to me in the development of my study unit. This chapter contains detailed instructions on the use of *FractaSketch* and countless suggestions for student (and teacher) exploration using the programme; it would be possible to engage a group of students for several days with *FractaSketch* and this chapter alone. One can easily draw simple fractals after a few minutes of experimenting with the programme. However, as one gains familiarity with the various features of *FractaSketch*, it becomes relatively simple to draw figures of astonishing complexity.

Exploring Fractals on the Macintosh and the accompanying software was, perhaps, the most valuable resource that I discovered. Based on observations of my students I can say with some confidence that, from their perspective, the *FractaSketch* explorations were the most enjoyable part of the study unit. Even the "underachievers" were genuinely engaged in exploration of the programme. One such student remarked that a drawing he created "...looks just like an island." Any teacher who is planning an extended exploration of fractal geometry,

and has access to a Macintosh lab, should obtain a copy of *Exploring Fractals on the Macintosh*. This publication not only greatly improved the quality of the study unit, but it also enriched my knowledge of fractals and enhanced my teaching of the topic.

There are other resources that I employed to a lesser extent. One of these is *Fractals in Your Future* by Ron Lewis. Chapters one and two provide a good introduction to fractals. It is obvious that Lewis has expended countless hours developing investigations to mitigate the discovery of fractal properties. This is another resource that was undoubtedly of more use to me in augmenting my personal knowledge than in furnishing student activities. The fractal geometry materials contained in the Ministry resource book for Survey Mathematics 12 provided me with a starting point for some of the ideas in the handouts that I prepared for my students. It was this material, and the November 1991 issue of the NCTM's *Student Math Notes* entitled *Fracturing our Ideas about Dimension*, that initially piqued my curiosity in fractal geometry.

Without question, there exist many excellent resources that I have yet to discover; thus, the foregoing is not an exhaustive bibliography. Nonetheless, for educators who are interested in teaching students the fundamentals of fractal geometry, I feel that I have provided a good foundation upon which they can build even better study units.

Chapter 4

Cognitive Theories of Knowledge Acquisition

This chapter deals with some of the cognitive psychological aspects of knowledge

acquisition. It begins with a brief overview of Piagetian reflective abstraction as it applies to the

learning of novel mathematical concepts. Following that is a summary of what Ed Dubinsky has

labelled the alpha, beta, and gamma behaviors exhibited by learners. The chapter concludes

with a description of how these ideas relate to structural versus operational cognition in a

mathematical context.

The way in which students acquire new information is a phenomenon that has received a

great deal of attention in the education research community. However, despite the attention this

cognitive activity remains enigmatic. Jean Piaget is "...probably the first and most important

contributor to the epistemology of mathematics from the point of view of psychology

(Vergnaud, 1990, p. 15)." Piaget spent a great deal of time and effort defining the process of

knowledge acquisition and describing the ways in which it can be enhanced.

Equilibration and reflective abstraction.

Piaget postulated that a learner's cognitive condition is in a state of dynamic equilibrium.

Periodically, this equilibrium may be disturbed; such disequilibrating experiences are known as

perturbations. The learner responds to perturbations by attempting to restore equilibrium

through a process known as equilibration. The method by which equilibration occurs is a topic

that has received much attention from educational researchers over the years.

Equilibration "...is the process by which a knower attempts to understand a given item of

information by situating that item in the knower's overall cognitive system...[and]...refers to a

series of cognitive actions performed by a knower seeking to understand cognitive aliments

(Dubinsky & Lewin, pp. 59 - 60)." The new information is "...integrated by making the

appropriate modifications in one or more [pre-existing] cognitive structures (p. 58)." According to Piaget, these cognitive structures are known as schemae, or schemes.

Piaget theorized that all knowledge is organized according to schemes. Perturbations are caused by new information that conflicts with, or is not satisfactorily internalized by, pre-existing schemes. They are said to consist of three parts:

1. recognition of a certain situation;
2. association of a specific activity with that kind of situation, and the;
3. expectation of a certain result (von Glasersfeld, 1989, page 127).

Mathematical knowledge is thought to consist of an interconnected collection of schemes corresponding to distinct mathematical concepts (Dubinsky, 1989, p. 286). If a learner encounters information for which he/she does not have a preconceived scheme, a perturbation in his/her equilibrium is said to have occurred. This perturbation sets the stage for equilibration; the learner will recognize and respond to the perturbation by constructing a new scheme, or modifying a pre-existing one.

While this process of construction, or reconstruction, is occurring a learner progresses from one mental state to another. In Piagetian terms, this progression is known as a transitional state. A transitional state is characterized by "...the experience of previous ideas conflicting with new elements (Tall, 1991, p. 9)." Piaget posited that such transitions are facilitated by two processes: assimilation and accommodation. Tall describes these processes as follows:

> Piaget uses the term *assimilation* to describe the process by which an individual takes in new data, and *accommodation* the process by which the individuals' cognitive structure must be modified. He sees assimilation and accommodation as complimentary (Tall, 1991, p. 9).

Acording to Dubinsky and Lewin (1986), a learner will apply the concept that is to be learned to his/her preexisting schemes. If these structures prove inadequate for assimilation of the new concept, then either a particular scheme will be modified, or a completely new scheme

will be constructed. Once the concept in question is perceived as having been "understood," the learner's cognitive system has equilibrated itself and the novel concept is said to have been accommodated (Dubinsky & Lewin, 1986, p. 60). Both assimilation and accommodation are intended to be employed within the framework of Piagetian scheme theory (von Glasersfeld, 1989, p. 127). von Glasersfeld has postulated the existence of three main causes of perturbations.

The first source of perturbation is when a subject has recognized a situation as one that requires a specific preexisting scheme. However, when that scheme fails to produce the expected result, accommodation of the recognition process may occur. The second cause is when an existing scheme produces, instead of the usual result, another result that is satisfactory. In this case, a new scheme may be constructed that will then be expected to produce the new result. Finally, a perturbation may occur when an activity that is associated with a particular situation leads to a result that is recognized as the expected outcome of another scheme (von Glasersfeld, 1991, p. 101). However, in order for any accommodation to occur a prerequisite process known as reflective abstraction must take place.

Reflective abstraction is "...the situating of information in a cognitive system [that] occurs as the knower builds an understanding of the item [of information] (Dubinsky & Lewin, 1986, p. 55)..." It is the process whereby old schemes are modified, or new schemes are constructed, in order to "understand" novel information. This facet of knowledge acquisition is of particular importance for the purposes of this paper because Piagetian theory states that the construction aspects of reflective abstraction "...are the most important for the development of mathematical thought during adolescence and beyond (Dubinsky, 1991, pp. 95 - 96)."

Piaget has written that "...the development of cognitive structures is due to reflective abstraction (Piaget, 1985, p. 145)..." and that "...new mathematical constructions proceed by reflective abstraction (Beth & Piaget, 1965, p. 205)..." He identified four different types of

construction of which reflective abstraction is composed: interiorization, coordination, encapsulation, generalization. A fifth aspect, known as reversal, was apparently considered by Piaget, but he ultimately deemed it irrelevant to reflective abstraction. Dubinsky believes otherwise (Dubinsky, 1991, p. 103).

Interiorization has occurred when a student has mentally constructed a representation of a concept. According to Dubinsky, interiorization is "...the formation of an internal process corresponding to some mathematical transformation (Dubinsky, 1989, p. 286)." He cites the commutativity of addition as an example of interiorization. A child might discover that addition is commutative by counting, reordering and counting again, and repeating this process over and over again; the child has interiorized the concept of commutativity when he/she realizes that the same result is always obtained (Dubinsky, 1991, p. 100).

Coordination is a process of composition of two or more schemes to construct a new one (Dubinsky, 1989, p. 286). He uses the concept of number to illustrate this process: this concept "...is constructed by coordinating the two schemes of classification (construction of a set in which the elements are units, indistinguishable from each other) and seriation (which is itself a coordination of the various actions of pairing, tripling, etc.) (Dubinsky, 1991, p. 100)"

Encapsulation is the aspect of reflective abstraction that enables an individual to correlate several cognitive structures and project them into a single entity that describes a complex concept. Dubinsky and Lewin posit that encapsulation is likely the most important and powerful aspect of reflective abstraction (Dubinsky & Lewin, 1986, p. 62). In explaining this process they use the Klein-4 group as an example.

Initially, a student utilizes each of the four operations (identity, negation, reciprocity, and correlation) separately. Eventually, he/she may perceive that these four operations compose a single structure: the Klein-4 group. It is at this point that encapsulation is said to have occurred

(Dubinsky and Lewin, 1986, pp. 62-63). In describing this process, Dubinsky paraphrases Piaget: encapsulation involves "...building new forms that bear on previous forms and include them as contents." and "...reflective abstractions that draw from more elementary forms the elements used to construct new forms (Piaget in Dubinsky, 1991, p. 101)."

Generalization is said to have occurred when a child becomes aware of the wider applicability of a given scheme (Dubinsky, 1991, p. 101). To illustrate this idea Dubinsky and Lewin cite the example of a child extending the property of commutativity of addition to include the same property of multiplication (Dubinsky & Lewin, 1986, p. 62).

The fifth, and final, facet of reflective abstraction to be considered is known as reversal. It is thought to be a method by which a learner constructs a new process which consists of reversing the original process; this means that if the learner has successfully assimilated the new concept, he/she should be able to make sense of it in the reverse order (Dubinsky, 1991, p. 102). An simple example of reversal is the translation of an exponential function to a logarithmic function, and vice versa.

Many high school students experience difficulty with the equality of exponential and logarithmic expressions such as $y = 2^x \Leftrightarrow \log_2 y = x$; however, once this obstacle is overcome the student's understanding of both logarithms and logarithmic functions is greatly enhanced. Reversal, then, is the final of the five cognitive construction activities associated with a transitional state. While the process of reflective abstraction is ongoing Dubinsky has identified several different behaviors that may or may not be manifested by a learner.

Dubinsky believes that as a learner who is attempting to "understand" a new concept he/she may exhibit alpha, beta, and gamma behaviors . In alpha behavior, the subject may believe that he/she has understood the information but, in reality, no reflective abstraction has

taken place. Alpha behavior is characterized by a conception of the information that is unstable and hence shifting from moment to moment (Dubinsky and Lewin, 1989, p. 63).

Dubinsky characterizes beta behavior as that which is exhibited when the learner has successfully integrated the concept into his/her repertoire of mathematical schemes: reflective abstraction has taken place. He writes that beta behavior is the "...paradigm case of successful learning (Dubinsky & Lewin, 1986, p. 64)." Gamma behavior is exhibited by a subject that possesses a cognitive system that is "...sufficiently rich to integrate the novel [information] without constructing new cognitive structures. Nonetheless, reflective abstraction occurs as the new aliment induces the existing system of concepts to accommodate it through extending itself (Dubinsky & Lewin, 1986, p. 65)."

Based on the above descriptions of alpha, beta, and gamma behavior it has been my experience that most students display a great deal of alpha behavior. It seems that there are always a few that assimilate and accommodate very quickly; however, it is rarely that one encounters a student that possesses sufficient mathematical sophistication to manifest gamma behavior. According to cognitive psychologists, these few are able to quickly make the transition from an operative mode of knowing to a figurative one.

Operational versus structural cognition.

It is thought by most influential researchers that mathematics students must construct their knowledge. Much of their argument is based on Piagetian theory; to many educators, Piaget was the pioneer constructivist. Among his many contributions to cognitive psychology includes the distinction between two different modes of mathematical thinking: figurative and operative. These two types of thinking are very closely tied to his notion of reflective abstraction.

> The essential character of mental life is its close connection with our actions, and intelligence itself must be conceived as a system of operations, that is to

say, interiorized actions, made reversible and coordinated in the form of
'operational structures'...if we call this aspect of consciousness which relates
to actions and operations 'operational' there also exists a 'figural' aspect, that
is to say relative to the perceptible configurations (for example, perception
and the mental image) (Beth and Piaget, 1966, page 156).

As is the case with many Piagetian ideas these have, over the years, been analyzed,
modified, and refined by countless researchers. The duality of operative and figurative thinking
have thus been redefined in different but analagous ways by these researchers. One way in
which this has been done is to rethink figurative cognition as "structural," and operative
cognition as "operational."

Similar dualities exists under many different names. Skemp (1976) describes relational
versus instrumental understanding, and Hiebert and Lefevre (1986) distinguish between
conceptual and procedural knowledge The terms operational and structural appear to have been
coined by Sfard (1991). In the case of mathematics, the difference between the two modes of
thinking is epistemological. Operational thinking encompasses the algorithmic, process
oriented, aspect of mathematics. Students who perceive math to be nothing more than a litany
of rules to be memorized and invoked when appropriate would be said to have a strictly
operational understanding of the discipline.

Although, historically, operational thinking has been considered inferior to structural, it
need not necessarily be considered as such; many students who are extremely proficient at
mathematics work solely on an operational basis. Indeed, Sfard suggests that there are
comparatively few that can operate mathematically on a structural level. She writes: "...the
structural conception is very difficult to attain (that is probably why some people feel, intuitively
of course, that the special ability to develop a structural conception is what distinguishes
mathematicians from 'mere mortals')." (1991, pp. 9 - 10)

Structural thought implies a deeper understanding of mathematical ideas; a student who is able to see a concept as an object, and who can mentally manipulate that object, would be considered to possess a structural understanding of that concept. While many see operational versus structural knowledge as a dichotomy, Sfard perceives the two to be complimentary. She argues that "...we have good reasons to expect that in the process of concept formation, operational conceptions would precede the structural." and continues "...the operational and structural elements cannot be separated from each other (Sfard, 1991, p. 11)." In other words, "...the same representation, the same mathematical concepts may sometimes be interpreted as processes and at other times as objects (Sfard, 1994, p. 193)...."

Sfard believes that all mathematical thought has, historically, developed in an operational way, followed by a deeper structural understanding when a given concept has received sufficient study. She cautions, however, that: "...one should be careful not to make automatic projections from history to psychology. After all, the deliberately guided process of reconstruction may not follow the meandering path of those who were the first travellers through untrodden area (1994, p. 195)."

Despite this proviso Sfard argues that since this is the way in which all mathematics has evolved, it is likely that mathematical development in the individual would follow a similar pattern. She suggests the existence of three stages in concept development: interiorization, condensation, and reification (Sfard, 1991, p. 18).

Sfard's connotation of interiorization is similar to that of Piaget's; during this phase "...the learner gets acquainted with the process which will eventually give rise to a new concept...[and]...a process has been interiorized if it can be carried out through mental representations, and in order to be considered, analyzed, and compared it needs no longer be actually performed (Sfard, 1991, p. 18)." For example, junior secondary school students are taught a specific algorithm that is used to solve linear equations in one variable. For many

students, by the end of grade 9 mathematics they need not actually perform the steps of transposing terms from one side of the equality sign to the other; at this point the process is said to have been interiorized. Once interiorization has been achieved the learner is ready to proceed to the condensation phase.

Sfard writes: "Condensation is a period of squeezing lengthy sequences of operations into more manageable units. At this stage a person becomes more and more capable of thinking about a given process as a whole...This is the point at which a new concept is born." She elaborates: "Thanks to condensation, combining the process with other processes, making comparisons, and generalizing become much easier (Sfard, 1991, p. 19)." This idea appears to me to be a synthesis of the Piagetian ideas of coordination and generalization.

The condensation phase of concept development may be quite lengthy; it seems likely that with much of the curriculum, many senior secondary mathematics students get no further than condensation. "The condensation phase lasts as long as a new entity remains tightly connected to a certain process. Only when a person becomes capable of conceiving the notion as a full fledged object, we shall say that the object has been reified (Sfard, 1991, p. 19)." Once a student has reached this stage of concept development he/she is capable of shifting the focus of his/her problem solving activity as the situation dictates. "The problem solver oscillates between the operational and structural approach, and between one structural interpretation and another. "It is this flexibility of perspective that distinguishes a student who is learning a concept from one who truly 'understands' (Sfard, 1994, p. 202)."

It seems to be generally accepted that reflective abstraction is a crucial aspect of knowledge acquisition. Although Sfard's analysis has a somewhat different perspective it is still based on Piagetian theory and, as such, is not in strict disagreement with Dubinsky's position. Based on my cursory analysis of these complex processes, it appears to me that their perceptions are complimentary; reflective abstraction is a requisite step in the progression from operative to

structural thinking. It is of interest to note that both researchers suggest the use of computers to assist students in their construction of mathematical concepts (Sfard, 1994, p. 224 and Dubinsky, 1991, p. 123). This observation bodes well for my Survey Mathematics students and their study of fractal geometry.

Chapter 5

Methodology

What follows is a description of the methodology employed in the study. The first section is intended to provide an overall context for the study. The school setting, the students involved, the course in which the study took place, and the researcher are described. The second section is a detailed account of the pedagogy employed in the instructional unit. This is followed by a summary of the difficulties encountered in the implementation of the study unit. The chapter concludes with a section that addresses data collection.

The Context.

The School.

The study was carried out at a school that I wil call Burnaby West Secondary School. Like most Burnaby secondary schools, Burnaby West is on a linear timetable; a teacher will meet a given class for 77 minutes on alternate days. The student demographics are typical of those found in many suburban British Columbia high schools. The school population is approximately 40% Caucasian and 60% Asian. Many students are learning disabled to some degree, and we have the requisite number of underachieving and/or under motivated students.

The researcher and the course.

The students involved in this study were enrolled in Survey Mathematics 12. This course is particularly appropriate for the following reasons: (1) The intent of the course is to broaden students' mathematical horizons through the inclusion of non-traditional topics. For example, the curriculum contains matrices, graph theory, and the "mathematics of finance" to name a few. (2) There is no government examination, and thus there is little or no curricular pressure. (3) The majority of students enrolled in the course intend to pursue their education at some post-secondary institution. As I am the sole teacher of the course in our school, my role in the study was that of teacher/researcher.

My experience as a mathematics teacher spans fourteen years. Prior to undertaking the master's programme at Simon Fraser University, I had a B.Sc. from the University of Alberta and a teaching certificate obtained via the Professional Development Programme at Simon Fraser. Combining the roles of full time teacher and part time researcher was challenging; each one is extremely time consuming. The greatest challenge was making the time to be effective at both.

This paper describes a portion of my fifth experience in teaching Survey Mathematics 12. I have enjoyed teaching this particular course a great deal and for several reasons. Because there is no curricular pressure, both students and teacher are allowed the freedom to pursue topics in as little or as much detail as they choose. Another aspect of the course that I find to be of value is that it does more to enhance divergent thinking in students than any other high school mathematics course offered in British Columbia. Furthermore, several ex-students who have completed the course and subsequently returned to visit have informed me that Survey math has been of more use to them in their university studies than has Mathematics 12. Finally, as it attracts both academically and non-academically inclined students, there invariably exists an interesting classroom dynamic. It is unfortunate that the Ministry has decreed that the course will be offered for the final time in the 1996-97 school year.

In other provinces Survey Mathematics 12 is, more appropriately, called Finite Mathematics. The misnomer, and the fact the course is not provincially examinable, tends to attract a wide variety of students. Many enroll in the course with an eye to obtaining an easy University entrance credit; these young adults are generally extremely disappointed with the level of mathematics studied and the amount of work required. As a rule they are not interested in mathematics and are not particularly good students. I mention this because I think that the feedback obtained from these students could be valuable in my research; it should be useful to analyze the perceptions and conceptualizations of students who are somewhat apathetic toward mathematics.

The participants.

Survey Mathematics thus provides me with students possessing a wide spectrum of mathematical aptitudes and attitudes. Another aspect that makes the course ideal for my purposes is that fractal geometry is included in the curriculum as an optional topic and so a teacher is fully justified in exploring the subject in greater depth.

Fourteen students were enrolled in the course at the time the study began. Three of the fourteen were Caucasian and the remainder were Oriental. Of the fourteen, three were seriously underachieving students. Although these three were scheduled for interviews none of them appeared for their scheduled appointments; indeed, none of them has appeared in class since prior to the conclusion of the study unit. One additional student joined the class about half way through the study unit but because of his late arrival, he was not interviewed.

As a result I have interview data for eleven students. Eight of the eleven interviewees were also enrolled in Mathematics 12, and two of the eleven were taking Advanced Placement (A.P.) Calculus. Of these eleven, several were English as a second language (E.S.L.) students. The interviews for these proved very difficult for both the students and for me. Subsequent transcription of the data was equally as difficult.

Since I was new to the school in September, 1 had no previous experience with any of the students in the class. Because I was new, my rapport with the class was not as open and familiar as I would have liked it to be. Over the years I have developed a style that, with most classes, works quite well for me; I like to keep the classroom relaxed and informal. It is my feeling that it is more comfortable for all concerned if I am able to maintain such an atmosphere. However, if pushed by uncooperative students, I can also play the authoritarian game. Unfortunately, two students in the class did not respond well to my teaching style. As a result, the tone in the

classroom was strained too frequently for my taste. When we began the fractal geometry study unit, I had been teaching the class for approximately two months.

My initial question was "How many of you have ever heard of fractal geometry?" None of the students answered in the affirmative. In retrospect I have no doubt that the students were feeling some apprehension. Most of the participants are expecting to attend some post secondary institution and so they are justifiably concerned with achievement in their classes. Now they were not only being exposed to a new teacher, but this person also expected them to learn a topic with which they were completely unfamiliar.

The study unit spanned a period of approximately six weeks. During that time I became much more familiar with my students, and by the end of the unit I was on good terms with everyone who attended the class regularly.

Software.

In order for students to study fractals effectively, they must have access to appropriate hardware and software. I was able to locate two programmes that were useful in facilitating students' understanding of some of the basic concepts. These programmes have several functions. They expedite the drawing of the figures and provide a visual display of the recursive process. One of the two also assists in the understanding of the self-similarity dimension.

The first programme that students used is called *Fract-O-Graph*. It is included with the Survey Mathematics 12 resource package that is available from the Ministry.
In addition to *Fract-O-Graph*, I discovered a programme called *FractaSketch*. It is an excellent programme that is included with Bernt Wahl's book entitled *Exploring Fractals with the Macintosh*. This programme was perfect for the purposes of my research; it is easy to use and has the capability to generate simple and complex fractal images. The publisher Addison Wesley was kind enough to give me permission to use the programme in my research.

With the assistance of these two programmes, students were able to draw simple fractals very quickly. I had the students begin with *Fract-O-Graph* because it is easier to use. Furthermore, *FractaSketch* has some more advanced features that require slightly greater knowledge on the part of the user.

There are at least three distinct advantages to using *FractaSketch*. Although it is more difficult to use, it yields more a accurate drawing of the desired fractal and allows a greater freedom in the types of fractals that one is able to draw. This is because with *Fract-O-Graph*, the angles that one is allowed to use are predetermined; with *FractaSketch*, the user specifies the angles contained in the drawing.

Once the user has completed his/her sketch of the generator, he/she double-clicks the mouse and *FractaSketch* automatically switches to a screen in which the generator may be redrawn to as many iterations as the user desires. The programme automatically calculates and displays the fractal dimension of the diagram in the upper right corner of this screen. This feature was extremely useful to students in completing one of the required assignments.

A third advantage to using *FractaSketch* is that included with the programme is a folder containing many predrawn fractals that very closely imitate naturally occurring structures. Users are able to see how a very simple line drawing that initially consists of perhaps three or four line segments can very quickly become a fern leaf or a tree. Many students used the programme to draw realistic looking trees of their own creation.

A third source of software was provided by an Ontario high school teacher named Ron Lewis who has written a textbook that he uses in teaching an entire course on fractals and chaos. He included with his textbook a disk that contains several programmes, some of which Mr. Lewis obtained from the Internet; all are available as freeware.

These programmes are written for IBM compatible machines; however, the IBM computers in the lab at our school have insufficient memory to run the software. Fortunately we have, in the mathematics department, an IBM 486 that is equipped with a liquid crystal overhead projector display. Access to this equipment made the programmes very useful for demonstration purposes.

The disk contains, among other things, generators for the Koch and Dragon curves. The software allows one to begin generating a fractal with a straight line and progress to as many iterations of the curve as is desired. It is very fast, even at relatively high levels of iteration. The programme and the computer were extremely effective tools in demonstrating the concept of self-similarity.

The Study Unit.

The objectives.

The study unit is of my own design. I attempted to focus on five effective behavioral objectives. It was hoped that the students would acquire the ability to: (1) describe the characteristics of a fractal; (2) sketch simple fractal structures; (3) recognize and define self-similarity; (4) manipulate the self-similarity dimension and apply it to various fractal structures under differing conditions, and; (5) compare and contrast the properties of fractal structures with those of some naturally occurring phenomena.

Two additional objectives that I hoped to achieve were affective in nature. The primary one was that students be provided with a different lens with which to view mathematics. The second was to take this new lens, and use it in combination with objective (5) to decide for themselves if they believe that fractal geometry is indeed the geometry of nature.

Contained within these broad parameters are other, more specific, behavioral objectives that varied from lesson to lesson. To aid the students in achieving these objectives it was my task to gather the appropriate resources.

In order to assist students in using the software that I had found, and to facilitate their study of the required concepts, I prepared a series of six handouts. I will be referring to these handouts from time to time throughout my description of the methodology. They are included in their entirety in appendix B.

The slide show.

We began with a slide show that I had prepared. Some of these slides are included with *Fractals for the Classroom Part One: An Introduction to Fractals and Chaos*. The remainder were photographed from colour plates contained within the same book.

There were eighteen slides that consisted of photographs of various naturally occurring phenomena juxtaposed with fractal "forgeries." Some of the "forgeries" are quite realistic and most of the students were convinced that they were looking at pictures of real landscapes. When it was revealed that many of the slides were photographs of fractals, students seemed to warm up to the topic. Following the slide show, I briefly mentioned some of the pioneers of the field and began an informal description of self-similarity.

I introduced the concept by using some of the slides that we had viewed. I attempted to point out that several of the images they had seen appeared to have bits and pieces that recurred again and again at differing levels of magnification. Of course this was difficult given the limited resolution of the slides and the projector; however, I was confident that I had available to me the tools required to give the students a clear idea of what is meant by self-similarity.

The Chaos Game.

To conclude the first class the students engaged in the "Chaos Game." My rationale for inclusion of this activity was that I hoped it would help them begin to establish a link between chaos theory and fractal geometry.

This activity is not so much a game as it is a mathematical exercise, albeit a tedious one. The students were given a diagram of an equilateral triangle consisting of only three points: one at each vertex, labelled T, L, and R. They chose an arbitrary point within the triangle, and then rolled a die that I had provided. After each roll of the die they plotted a new point. If the die showed either a 1 or a 2, they "moved" their point half the distance to vertex L. If a 3 or a 4 was rolled, they "moved" their point half the distance to vertex T. Finally, if the die came up either 5 or 6, they translated the point half the distance to vertex R.

The students "played" the game for no longer than twenty minutes at a time so that they did not become bored with the task. They were given six opportunities to "play" the game. In order to maintain their interest, I informed them that their output of the game would be graded. Furthermore, students were told that I would be able to tell if they had played the "game" properly because I knew how the result should appear and they did not. This seemed to be enough incentive to keep most of them on task.

Simple Fractal Structures.

Next the students were exposed to some simple fractal structures. (See the appendix for *Introduction to Fractal Geometry: Handout #1*. This handout includes a list of structures studied.) We began with constructions like the Koch curve.

Initially the students were expected to draw the first few iterations using pencil, ruler, and squared paper. My rationale for having them perform this exercise was to give them some sense of the complexity of these seemingly simple figures; what begins as a small number of

line segments very quickly becomes extremely difficult and time consuming to draw. It is my feeling that students cannot appreciate this complexity by watching a computer draw the structure.

After having allowed the students some time to draw their sketches, I made use of an IBM 486 that belongs to the math department. The IBM is equipped with a liquid crystal display for an overhead projector. This allowed me to project images of several different fractals and point out exactly what is meant by self-similarity.

The self-similarity dimension.

Next we began a more formal discussion of self-similarity . In order to apply the self-similarity dimension, students required a basic knowledge of logarithms. Many of my Survey Math 12 students who were also enrolled in Math 12 were comfortable with the required concepts. For those that were not I kept the explanation simple. The requisite knowledge was that they have the ability to solve simple exponential equations such as $3^n = 5$. We discussed how this may be solved by taking the logarithm of both sides of the equation:

$$3^n = 5$$
$$\log 3^n = \log 5$$
$$n \log 3 = \log 5$$
$$n = \frac{\log 5}{\log 3}$$
$$n \approx 1.465$$

Since this was the only knowledge of logarithms that was required I did not go into any greater depth. The next activity in which the students were engaged involved the calculation of the self-similarity dimension for the simple fractal structures that we had studied up to that point.

Technologically based activities.

In total, about twenty hours of class time were spent studying fractal geometry. Of these twenty, computer lab time comprised approximately six hours. This total includes a two hour

session in which the class was engaged in using software in the Workshop for Computer Aided Tutoring in Mathematics (WCAT) lab at Simon Fraser University (SFU). Ideally, we could have made use of more lab time, but logistical problems made it almost impossible for us to gain more access to the computer facilities at Burnaby West.

After the students had become more familiar with some of the classic curves, we went to the Mac lab and the students made use of *Fract-O-Graph*. As they progressed through the study unit, the primary use of lab time evolved into an opportunity for them to explore the software and to sketch the fractals they had been studying. In this application, the age of the hardware worked in the students' favour. Because the machines are so slow, they could watch as each subsequent iteration of a fractal was drawn, thus enabling them to see how the structures grow and become more complex.

On our first trip to the lab, the class was expected to use both the first and second handouts that I had prepared. On these handouts were diagrams of simple fractals that they could "draw" using the computers. I had intended to give students a brief explanation on how to use *Fract-O-Graph*; as it turned out they required no instruction whatsoever on the use of the programme. After some preliminary exploration, they quite quickly became proficient at drawing the figures.

On another trip to the lab I provided students with *Introduction to Fractal Geometry: Handout #4* and a new programme: *FractaSketch*. The handout contained several suggestions for implementation of the programme, but I noticed that most students went about it in their own way. Many of them were struck by the realistic looking "trees" that could be generated and spent a great deal of time trying to make their diagrams more natural in appearance. *Figure 5.1* shows several iterations of a fractal "tree" as drawn by *FractaSketch*. Once again, more computer time would have been useful. In order to fully explore the potential of the *FractaSketch* one requires considerably longer than two hours exposure to the programme.
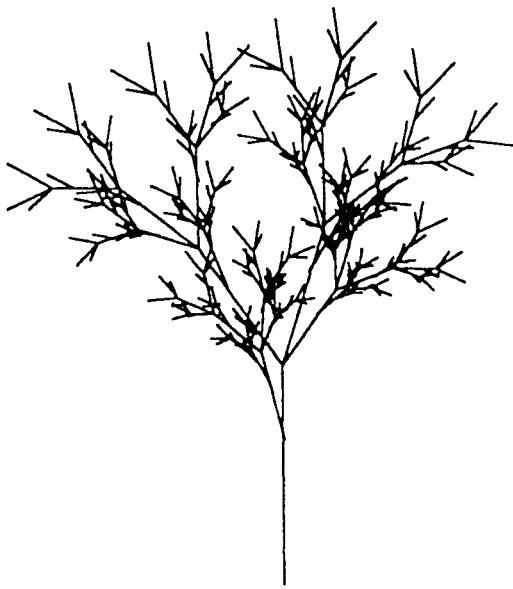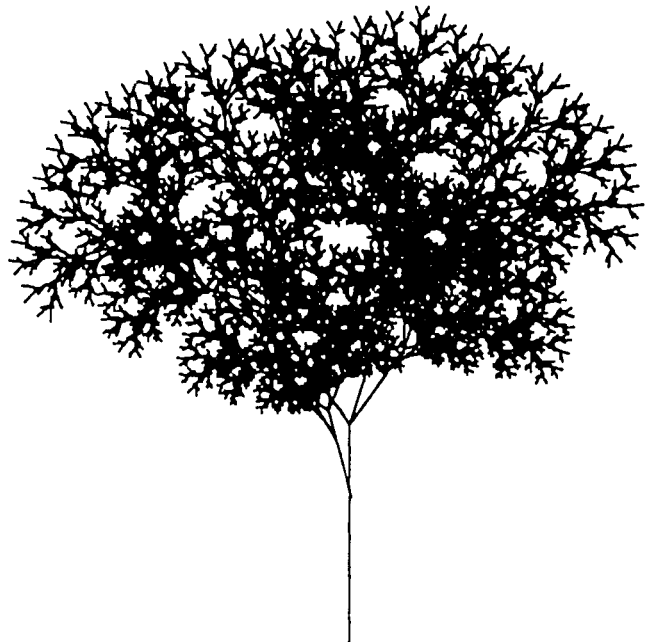
figure 5.1

A fractal "tree."
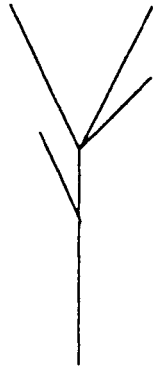
stage 1
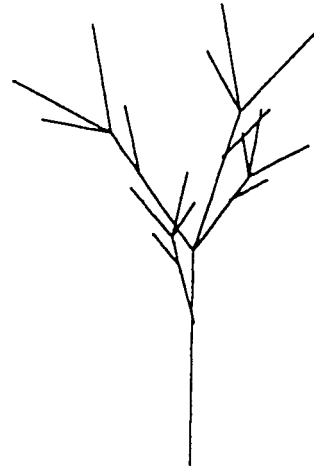
stage 2

stage 4

stage 7

<u>Classroom activities.</u>

Since unlimited access to the computer lab was not feasible, I required a number of student activities that were not computer based. Fortunately, we were able to employ several relevant exercises that I had located.

After having allowed considerable time for students to explore some of the basic structures, my instruction returned to a more mathematical side of the discipline. In order that the students gain more familiarity with the self-similarity dimension, I provided them with a list of fractal generators and the dimensions of the resulting fractals. They were then expected to use trigonometry to calculate the angles required to sketch the curves (see *Introduction to Fractal Geometry: Handout #3*).

The level of trigonometry required included application of simple trigonometric ratios and, for the more complex structures, use of the sine and cosine laws. Students had some limited access to the computer hardware and software while working on the assignment; several of the them made use of *FractaSketch* to check their solutions to the problems. They did so by sketching a generator using their calculated angles and then making note of the fractal dimension of the resulting figure. This was done with no prompting on my part.

<u>More fractal structures.</u>

Pascal's triangle contains within it some surprising patterns. We spent a few minutes at the beginning of one class generating the numbers in the triangle. The students were then split into groups and each group was given a large scale copy of the triangle completed up to the twenty fifth row. Different groups were then asked to shade different parts of the triangle. One group was to shade the even numbers, another was asked to shade all multiples of three, a third group was to shade the multiples of four, and a fourth group shaded the multiples of five. The resulting diagrams were then compared and contrasted. Many of the students were rather

surprised at the outcome of the exercise. Although each diagram turns out slightly differently, they are eerily alike, and are definitely self-similar.

I used this activity as a springboard to introduce one of the more widely recognized fractals: the Sierpinsky gasket (*figure 5.1*). We calculated the self-similarity dimension of it and some similar structures. At the conclusion of this particular class students again "played" the chaos game. It was at that point that several of the students concluded that the "game" would result in a Sierpinsky gasket.

figure 5.2

The first two iterations of the Sierpinski gasket

stage 0   stage 1   stage 2
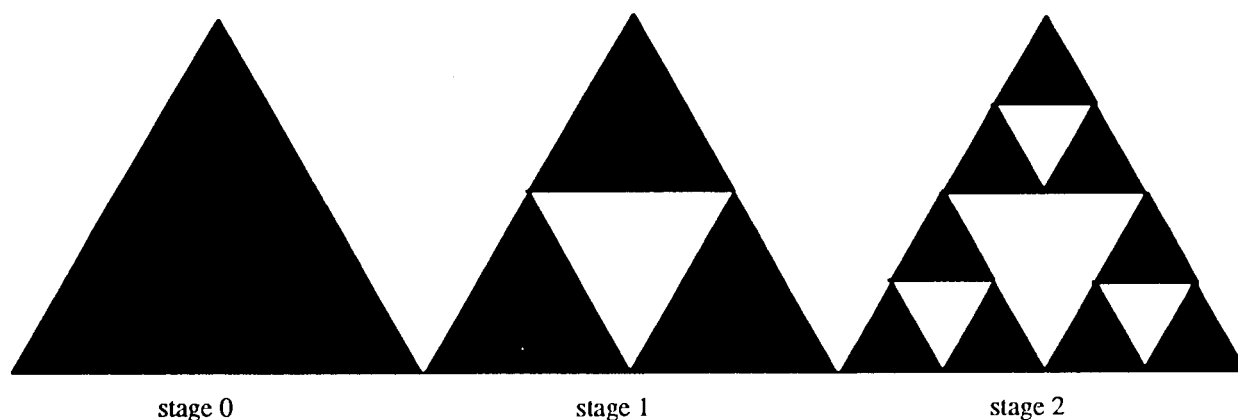
A second look at the Koch curve.

Other concepts investigated without the use of a computer included the perimeter of the Koch curve and the area contained under the curve. These were revelations of sorts for many students. It seemed counterintuitive that a figure that spans a finite distance could have an infinite perimeter. The explanation of the area contained under the curve provided even more confusion. Although many of the students were confounded by the explanation, there were equally as many that were able to comprehend it. Those that did were confronted with yet another paradox; here was a figure that contained a finite area, but had infinite perimeter. If one was so inclined, one might choose to believe that this paradox could be applied to model the geometry of an island.

The investigations.

In *The Fractal Geometry of Nature* Benoit Mandelbröt asks "How long is the coastline of England?" In answering the question he posits that the length of the coastline is dependent upon the level of accuracy with which it is measured. Mandelbröt believes that the smaller the unit of measurement, the longer the coastline becomes. If students are adequately prepared, they can see how a coastline might be considered a fractal; it contains a finite area, and the perimeter could well be infinite. The final set of investigations that I used are designed to guide students to just such a conclusion.

These investigations are contained in the workbook entitled *Strategic Activities for the Classroom Volume One* that accompanies *Fractals for the Classroom Part 1.* (The four investigations that I used are contained in the appendix.) Investigations 3.3, 3.4, and 3.6 were photocopied and used exactly as they are published. I modified investigation 3.7 to make it seem less contrived and to give it a more "local" flavour.

Investigation 3.3: Curve fitting.

The first activity in the set requires that students graph three tables of values in three different ways: on standard graph paper, on semi-logarithmic paper, and on double logarithmic paper. The first set of data represents the distance travelled by an automobile that is moving at a fixed velocity. The second relation depicts how world population has changed in the past 350 years. The final data set displays the relationship between the elapsed time and distance travelled by a free falling skydiver. The objective of this exercise is to show students that different types of functions have characteristically shaped graphs and that the shapes of those graphs can be altered by changing the scale used in the plot.

The relationship between elapsed time and distance travelled by the automobile is linear and its graph demonstrates that fact on standard graph paper. When world population is plotted

versus time, the exponential nature of the curve is shown clearly on standard graph paper; when plotted on semi-logarithmic paper, the curve is "straightened out." When a distance versus time plot is carried out on the skydiver data the curve does not "straighten out" unless it is graphed on double logarithmic paper, indicating that there is a power relationship between the two variables. There are questions at the conclusion of the activity that are intended to guide the student to these conclusions.

Investigation 3.4: Curve fitting using logs.

This investigation is designed to illustrate exactly why exponential and power functions are "straightened out" by semi-log and double log paper. Students again graph linear, exponential, and power functions, but this time the relations are given by equations rather than data tables, and the students are required to supply a table of values for each function. Once again, there are questions at the end of the investigation that are designed to lead the students to the required conclusions.

Investigation 3.6: Box Dimension.

The third activity is, conceptually, the crucial one. In this activity there are three shapes that are superimposed onto square grids: a sine wave, a darkened circle (the authors call it a "black hole"), and a wildly irregular graph of a "function." The shapes are printed on grids of six scales varying from relatively large to relatively small. Students were expected to count the number of boxes that contain any portion of each shape. The results of these counts were tabulated and graphed on the three different types of graph paper.

Students were to deduce from the graphs the relationship that exists between the grid size and the boxcount: linear, exponential, or power. The graphs of the data tables were curves on both standard and semi-logarithmic graph paper. Most of the class was readily able to determine that it was a power function, as the graphical representation of the data tables "straightened out" only when plotted on double logarithmic paper.

Next, they were asked to calculate the slope of the resulting linear graphs. Most students obtained slope values of slightly less than 1 for the graph of the data obtained from the sine wave, just under 2 for the "black hole" data, and approximately 1.8 for the "function." From this it was hoped they would infer that the slope of the line is equal to the dimension of the figure.

It should be pointed out that through the duration of the graphing activities I provided very little input. In fact, I was asked several times how these investigations were related to fractal geometry. My reticence was intentional; I did not want to prejudice the students' thinking by making any suggestions regarding the direction the investigations were taking. The students were working in pairs and, generally, were able to assist one another when required. At this point in the study, I felt that it was more appropriate to restrict my role to that of a guide or "coach" in the hope that they would reach the desired conclusions with minimal assistance from me.

Investigation 3.7: Box Dimension and coastlines.

The final investigation involved the same sort of exercise as the preceding the one, except that I substituted a map of the Queen Charlotte Islands for the three figures that were analyzed in Investigation 3.6.

Using a computer I generated grids that were approximately the same scale as those employed in the preceding exercises. I then photocopied them onto overhead projector sheets so that they could be placed over the maps. Students were to count the number of boxes that contained any portion of the coastlines of the Graham and Moresby Islands. Averaging class data and graphing it on double logarithmic paper rendered a linear graph with a slope of approximately 1.3. From this it was hoped that the students would conclude that the coastlines of the islands have a fractal dimension.

It was with these last two activities that students encountered the most difficulty intellectually. Most of them were unable to make the connection between the slope of the line and the dimension of the object. The nature of the investigation makes it virtually impossible to calculate whole number dimensions for the sine wave and the "black hole"; when I (very carefully!) performed the investigations myself the values at which I arrived were close to the actual dimensions. However, they were not close enough to allow students to make the desired leap of insight. The observed discrepancy is due to the fact a limiting process is invoked; the box counting process becomes accurate only when an infinitely small grid is used in the procedure.

A field trip.

One of our final fractal excursions consisted of a field trip to the WCAT computer lab at SFU. Dr. John Hebron kindly consented to be our guide in a journey through the considerable amount of fractal software that is on file in the lab. He did an outstanding job of leading the students through many applications that truly demonstrated the beauty and wonder of fractal geometry. It was an excellent opportunity for the students to work with some state of the art hardware and some sophisticated software.

We were in the lab for approximately two hours, and John spent no longer than fifteen minutes on any one programme. As a result, the students had no opportunity to get bored with a given application. For many of them this trip crystallized the concept of self-similarity and provided them with further evidence that, just perhaps, fractal geometry is the geometry of nature.

Tying things together.

Subsequent to the field trip there remained only two classes in which to tie up loose ends and test the students on what they had retained. During one of the final classes I returned to the IBM 486 and the software that I had obtained from Ron Lewis.

The disk contains a file which will sketch several fractals, including a Koch Island. The Koch Island is very similar to the Koch curve; instead of beginning with a straight line, it begins with an equilateral triangle and a Koch curve is constructed on each side of the triangle. As one reaches higher stages of the "island" the perimeter is obviously increasing, while the area contained within the structure remains, for all intents and purposes, constant. My intent was to use this as a model for the geometry of a real island.

Another fractal that the programme will draw results in a familiar figure: the Sierpinsky gasket. Early stages of the fractal bear no resemblance whatsoever to the gasket. However, as one attains higher stages the structure becomes strikingly obvious. It was following this demonstration that I employed the students' output of the Chaos game.

The students played the game in pairs on clear acetate overhead projector sheets. A total of six pairs of students played the game, and I included a transparency that I had prepared myself. When the sheets were stacked on top of one another and projected onto the screen, the results were hardly perfect, but they resembled a Sierpinsky gasket closely enough that many students were mystified and intrigued. I then projected a computer generated Chaos "game" that plots the points so quickly they could witness the gasket materialize in front of their eyes.

The final concept discussed in class was the difference between deterministic and affine self-similarity. A fractal that is exactly identical at any level of magnification is said to exhibit deterministic self-similarity. The Dragon and Koch curves are examples of such structures. On the other hand, a fractal that displays affine self-similarity does not consist of selfsame copies of itself at increasing levels of magnification. Rather, when one "zooms in" on an affine fractal, the component fragments of the "curve" appear to be *approximately* the same. Examples of affine fractals include coastlines, silhouettes of mountain ranges, and riverbeds. Like deterministic fractals, however, affine curves are said to have infinite length.

<u>Implementation difficulties.</u>

The greatest difficulties in implementation of the study unit were logistic and technological in nature. My primary difficulty was gaining access to the lab. As is the case at most schools, computer lab time at Burnaby West is in demand. The business education, humanities, and computer studies courses have priority access to the labs; the mathematics department is forced to take whatever time is left over. I had to be very persistent in my efforts in order to secure any lab time at all. Furthermore, gaining access to the lab was no guarantee of a successful experience once we were there.

It would be an understatement to say that the computer facilities at our school are somewhat dated. There are three computer labs: one containing IBMs , another with Commodore machines, and a third lab that has Macintosh computers. Each facility has an ample number of workstations, but none of the three labs is capable of running much of the current software; the hardware is simply too old. The IBM lab is the most technologically "advanced," but I could not find software suitable for my purposes that was IBM compatible. To complicate matters further, any relevant IBM software that I was able to locate would not run on our machines because there is insufficient memory available. As a result, Iwas forced to use our Macintosh lab.

Unfortunately, our Mac lab is a working museum. It consists of twenty seven Macintosh Plus computers that are a minimum of eight years old. They so obsolete that one day I discovered, much to my dismay, that they will not read high density disks. Even the age of the hardware was not the greatest challenge. The network is so tenuous that even the slightest disturbance can cause it to crash. As a result of this, and other unknown bugs in the system, at any given moment one or more computers may freeze, and/or students may not be able to print their work.

A further difficulty we experienced was that not only was our computer time limited, but it was also fragmented. We were allowed access for one class per week if scheduling was fortuitous. If it happened that the network would not cooperate on that particular day we were out of luck until the next time that I was able to secure lab time. There was some improvement in this situation toward the conclusion of the study unit; we were able to access the lab at my discretion because the computer studies teacher had grown frustrated and abandoned it altogether.

Despite these problems, I am confident that most students had a worthwhile experience with the software that I was able to locate. I believe that their computer time assisted them in their acquisition and assimilation of the concepts presented. Another objective that was met by the lab time was that we were able to take the mathematics class out of the mathematics classroom and introduce to the discipline a measure of investigation and discovery.

Data Collection.

The data to be analyzed in this study come from two primary sources: student work and student interviews. The participants were informed prior to the beginning of the study that all assigned work would be collected and analyzed. Such work included quizzes, the unit test, written reports of all investigations, output from the chaos game, homework assignments, and computer printouts of fractals generated during lab time. The quizzes, tests, and written reports were immediately graded and returned to the students. These, along with the other information, were to be included in a fractal notebook that would be collected subsequent to the completion of the study unit. I also encouraged students to keep a written record of any thoughts regarding and/or impressions of the content, and scope and sequence of the study unit.

Formal instruction was completed on a Thursday afternoon. Class time on the following Monday was occupied by tying up some loose ends, and review of the study unit. On

Wednesday afternoon we travelled to SFU for our field trip, and on Friday students wrote the unit test.

Clinical, semi-structured interviews were administered to the students in the class. The interviews were, on average, approximately 45 minutes in duration and were conducted during the week of the review, field trip, and unit test. The students were scheduled to appear for their interviews in one of three daily time slots: before school, at lunch, or after school. With the exception of three who failed to show, my students were extremely cooperative; I was impressed with the willingess demonstrated in giving up their free time to participate in the process.

Several students were interviewed prior to both the field trip and the unit test. These interviews would undoubtedly have been more successful had the students involved been exposed to both experiences. There was, without question, some crucial review and illustration of concepts during the field trip; hopefully, the same occurred during test preparation.

My primary objective for the interview was to determine how (and if) the concepts discussed in class had been internalized and assimilated. The audio tapes of the interviews have been compared and contrasted with the written work that the students submitted so that it may be possible to determine how the students' understanding of the topic progressed.

The interviews were structured in accordance with the following list of questions:
1. What does the word "dimension" mean to you?
2. What does the term "self-similarity" mean to you?
3. What does the concept of self-similarity dimension mean to you?
4. How would you characterize the properties of a fractal?
5. For a given fractal at the third stage of iteration make a quick sketch of second stage of the iterations (see *figure 6.1*).
6. Consider two coastlines. One has a fractal dimension of 1.5 and the other has dimension 1.9. How do they look alike? How are they different?

7. From what you have seen and learned in this study do you think that it is possible for some structure to have a dimension that lies somewhere between 2 and 3? What would such an object look like?

8. Which would have a higher self-similarity dimension: a slightly crumpled wad of paper, or one that is tightly wadded into a ball? Why?

figure 6.1

Stage 3 drawing of a fractal from which students were expected to draw stage 2.
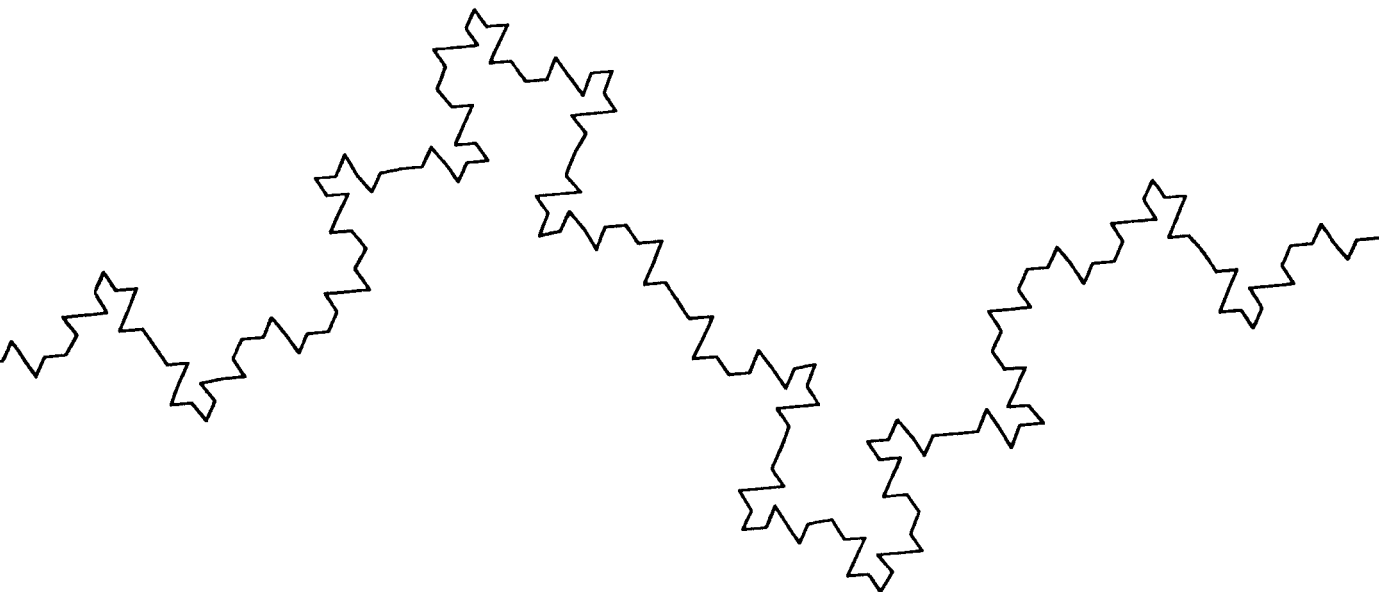
Not all questions asked in the interviews were necessarily based on information covered in the teaching unit; some questions were designed to probe for deeper understanding of the topic. In the chapter that follows, transcriptions of student responses to the eight questions listed will be analyzed according to the theoretical framework outlined in Chapter 4.

Chapter 6

Results and Analysis

In this chapter I will attempt to organize and interpret the data I have collected from my students. The first section consists of my rationale for each of the interview questions. This is followed by an analysis of the data collected. In the data analysis section I will employ the principles of reflective abstraction, and operational versus structural cognition, in an effort to interpret students' responses to the interview questions. The chapter concludes with a discussion of potential improvements to the study unit and the implications of the improvements for student learning.

The interview questions.

Early in the school term, when I first informed my Survey Math 12 students that they would be participating in this study, none of my students was familiar with any aspect of fractal geometry. Everything that followed was based on this naiveté with the topic; behavioral objectives and, hence, the study unit were designed with this fact in mind. However, prior even to the development of the objectives and the study unit, I had a tentative list of interview questions in mind (see page 66 for the interview questions).

My intent was to frame the questions so that they would require some independent thought on the part of the respondent; as such, the questions would ensure that a student who was proficient at memorizing facts and procedures might not necessarily have a "successful" interview. Yet, because the students had never heard of fractal geometry, the questions were out of necessity based on the fundamentals of the discipline. Dimension is one of those fundamentals.

Dimension is critical to the understanding of fractals; thus, two of the first three interview questions dealt directly with different aspects of this notion. If students were to have

their conceptions of dimension altered, they must possess a solid grounding in the conventional sense of the word; thus, my first question was designed to gauge their prior knowledge of one, two, and three dimensional structures. My third question regarding self-similarity dimension was intended to reveal not only the how, but also if students' perceptions of dimension had evolved over the course of the study unit.

Self-similarity is another idea that is critical to the understanding of fractals. It was my hope that the second interview question would induce students to articulate a coherent perception of self-similarity. English is problematic for many of them; because of this they were encouraged to draw sketches to illustrate their explanations. The rationale, I felt, for exploring their understanding of dimension, self-similarity, and self-similarity dimension respectively was that perhaps by vocalizing their perceptions of these concepts they would be better prepared to respond to the fourth question regarding the characteristics of a fractal.

Because the true definition of what constitutes a fractal is not easily understood, it was hoped that students would infer the characteristics of these structures. This ambiguity proved difficult for the students who rely heavily upon memory in doing mathematics. If, however, they were able to demonstrate sound concepts of self-similarity, and self-similarity dimension, then they should have possessed the ability to effectively characterize the concept of fractal. Once again, they were encouraged to draw diagrams, if necessary, to clarify their explanations. These first four questions were crucial in determining whether a student had acquired the fundamentals of fractal geometry. The remaining questions were designed to probe for a deeper understanding of the concepts covered in class.

The study unit contained a considerable amount of drawing: both computer generated and pencil and paper drawings were important components of student activities. The fifth interview question required that students sketch the second stage of a simple fractal having been given the third stage. Although most students had considerable experience and expertise in

drawing figures of progressively increasing complexity, they had yet to reverse the process. As discussed earlier, reversal is considered to be an indication that reflective abstraction has occurred. A successful attempt would indicate that he/she was capable of reversing the process that had been practised during the unit. This could be construed as an indication that at least one aspect of reflective abstraction had occurred. Generalization is another aspect of reflective abstraction; question 6 was an attempt to determine whether students could generalize and apply the self-similarity dimension to a specific phenomenon.

Because we had spent a considerable amount of time studying coastline geometry, it seemed appropriate to include a question pertaining to that topic. The response that I sought was closely related to that of question 3; I was hoping students would deduce that self-similarity dimension can be interpreted as a measure of complexity. Perhaps, in turn, this could be generalized and applied to describe the relative "roughness" or "smoothness" of a coastline. Responses to questions 7 and 8 required further generalization of this notion of complexity.

In class, our discussion of fractals was restricted to structures that had a dimension of between one and two; we did not consider the possibility that such ideas could apply to objects possessing a dimension greater than two, but not quite three. It is thought by some that there are many naturally occurring structures that have fractal dimensions. Question 7 was an attempt to introduce the possibility that such objects exist, and question 8 was intended to tease out specifics regarding the appearance of these objects. The interviews concluded with an attempt to have students name a "real life" structure that has a dimension of between two and three.

The interview questions can hardly be considered an exhaustive investigation of students' perceptions and conceptions of fractal geometry; nonetheless, it has been my experience that a student's understanding is often more successfully communicated in a conversational context than it is when I try to decipher something he/she has written. Therefore, when the responses to the questions are compared to and contrasted with their written work, a more complete image of

their cognition should emerge. In the next section I will organize the data around the behavioral objectives outlined earlier. Furthermore, I will endeavor to apply the principles of reflective abstraction, and structural and procedural knowledge, in an effort to determine the depth of my students' understanding.

Data analysis.

All student/participants were in the twelfth grade; I have used aliases to protect their identities. Because all had completed Mathematics 11 I assumed a certain amount of mathematical sophistication. Generally speaking, this turned out to be a safe assumption. It, therefore, came as a surprise to me that so few of them understood the concept of dimension in its conventional context. Very few of them were able to correctly identify structures of zero, one, two, or three dimensions.

Alison was among the "better" students in the class: a very quiet and conscientious young lady. She was extremely proficient at applying algorithms and duplicating the procedures necessary for the acquisition of "correct" answers. To be fair, her response to question 1 may have been a result of her unfamiliarity with the language. That being said, Alison's answer to the question was not atypical of many that I received.

I: Before we started all of this you must have had some idea about what the word dimension means. Can you explain that word to me?

Alison: I don't know.

I: (Indicating a point, a line, and a square that I have drawn on a page.) How many dimensions are each of these?

Alison: (The point) One. (The line) No, is zero.

I: Which is zero? Both are zero?

Alison: Yes.

I: How about this (indicating a square that I have drawn.) Or, how about the whole piece of paper?

Alison: Two. Is it two?

I: Why do you say this (the paper) has two ? Or why does this (the line) have two?

Alison: Is flat.

I: Anything else?

Alison: No.

I: How about length and width? Would those be two dimensions?

Alison: I think so.

I: What about that (the line) then. What does that have?

Alison: No dimension.

I: Does it have length? Does it have width?

Alison: Yes.

I: It has both of those?

Alison: Just length.

I: So how many dimensions?

Alison: Zero.

William is another extremely capable student whose concept of dimension is lacking. He was, at times, very insightful and quickly acquired new concepts. This was despite the fact English is his second language. William's perspicacity is, I think, illustrated by the following anecdote.

When I was introducing the concept of self-similarity dimension I developed the notion according to the outline that appears in the methodology section of this paper. I was using the formula $r^d=N$ to describe one, two, and three dimensional structures. Although I had specified the meaning of the parameters r and N, I had deliberately not done so for d. I had neither completed the explanation nor initiated any discussion of it before William deduced that d represents dimension. Yet, as is evident from his reply to question 1, his concept of Euclidean dimension is incomplete.

I: Before we started this did you have any idea that there was a possibility that something could have a dimension of between one and two?

William: No.

I: So now that you have had some exposure to this topic how has your idea of dimension changed?

William: My idea of dimension is still the same. Because I cannot even touch what is between dimension one and two: no, I mean two and three. I don't know what it is supposed to be like except that it is a bunch of lines.

I: So you still have the same basic idea of dimension; but do you think that those sorts of things exist, eh?

William: I guess so. But one dimension is a dot, and two dimension is a line and three dimension is a box: a 3-D box. Of that I am very sure. But 1 point something dimension I don't know what it's looking like.

Here again, we see an imperfect concept of dimension: this from a student who was concurrently enrolled in Mathematics 12 and A.P. Calculus. One student of the eleven interviewed was able to give me a correct answer to question 1. Valerie was another E.S.L. student. She was, perhaps, not as insightful as some of the students in the class, but she was of above average ability, an enthusiastic learner, and a contributor to the positive tone in the classroom.

I: Before we started all of this you must have had some idea of the meaning of the word dimension.

Valerie: Yes. The shape or...I don't know.

I: Well, what is the dimension of a point? A line? A square? A cube?

Valerie: Zero, one, two, and three. But I don't know how to explain it.

Although she asserted that she could not explain it, she correctly summarized the idea of dimension in its conventional sense. Her feeling of inadequacy in her explanation might have been a result of her unfamiliarity with the language; I have no doubt that she could have explained it very well in Mandarin.

Rosemary joined the class late after having discovered that Mathematics 12 required more study than she had anticipated. The Math 12 curriculum is not, in my opinion, beyond her

abilities; she is quite perceptive and has the potential to be a good student. Rather, she was somewhat undermotivated and was not willing to spend the time needed to achieve the kind of grade she felt she required to maintain her GPA. She joined my class because she saw Survey Mathematics as a means by which she could obtain a decent grade with a minimum of effort and as an easy university entrance credit. The following is an excerpt of her response to question 1.

I: Before we started all of this you must have had some conception of
          what dimension means. Has that changed?

Rosemary: A little. I only knew about 3-D. I didn't even realize that a line was one
          dimensional.

I: What does it mean to you now?

Rosemary: I don't know.

I: Confusing?

Rosemary: A bit, yeah. Because I don't really know when something is two
          dimensional. I was trying to figure out if it goes on both sides of the
          line.

During a class discussion of self-similarity dimension, I recall Rosemary saying she had no idea that dimension was so confusing. It is apparent from the above exchange that, at the time of the interview, she still found it to be perplexing. Her last statement is of interest to me; it appears, based on this statement, that self-similarity dimension is also problematic for her.

It seems that Rosemary was attempting to apply integral dimension to a structure that has a fractal dimension. The source of her confusion could lie in her attempt to contrast a fractal such as the Koch curve with a fractal exemplified by the diagrams shown in *figure 6.1* In Rosemary's way of thinking the Koch curve remains on "one side of the line," while the other structure "...goes on both sides of the line." Thus, according to Rosemary, the Koch curve might be uni-dimensional, while the figure on the right might be two dimensional. If this is the case it would seem that she is not only confused about dimension, but the entire notion of a fractal is also unclear.

figure 6.2

generator for the Koch curve   generator for a variation on the Koch curve
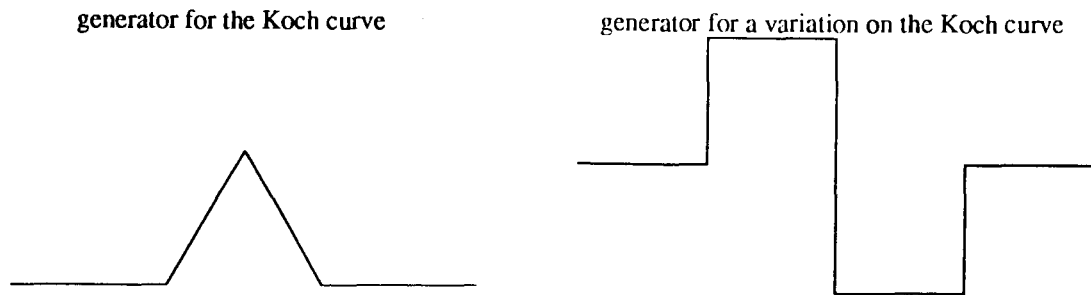
This is an extremely small sample; nonetheless, it is disconcerting to discover that in a group that has progressed as far as the high school mathematics curriculum will allow, fewer than ten percent can effectively communicate a geometric concept as fundamental as dimension. My suspicion is the blame does not lie with the students. Most mathematics teachers likely make the same assumption that I did: students already know what dimension means and for that reason it does not need to be spelled out for them. Despite these answers to the initial question, the responses to question 2 were more encouraging.

Stephen was likely the least motivated of all my Survey Mathematics students. In mathematics class, his work ethic was nonexistent. He completed none of required assignments, and it is unlikely that he did any homework whatsoever. Although Stephen is probably of average to above average intelligence he, like many high school students, was not particularly interested in academics. Despite this it was he who provided me with the most succinct description of self-similarity.

I: What about self-similarity? What does that mean to you?

Stephen: Self similiarity. Like from a fractal point of view?

I: Yes.

Stephen: You know how in the shapes you can see certain characteristics of the shape, right? Like say, you see a triangle, and in that triangle there are smaller ones and in that smaller one you see smaller ones and so it is the exact same shape, only in different sizes.

I: Did what you saw yesterday up in the [WCAT] computer lab help you to get a better idea of what self-similarity is?

Stephen: Yeah. When you could, you know, really zoom in on the points, and you know it kept going and going and going, I thought that was wicked.

I: Can you give me a little bit more information on the "zooming in" that you were talking about?

Stephen: Like you can see smaller...Like was at first there was a big shape, right? And then in the big shape you can see the smaller shapes that are exactly the same. And then if you zoom in on those you can see smaller shapes in those ones. And then you can bring those up, and then you can see the shape again.

None of the students possessed the capacity to articulate a succinct definition of self-similarity, although most were capable of placing it in a context that made sense to me. From a constructivist perspective this could be interpreted as a positive result. The students were not merely parroting a definition that they had obtained from me or from a textbook: most had constructed their own definition. There were only two who admitted they had no concept of self-similarity whatsoever.

Some of the responses to question 3 were quite interesting. Tara was one of several students who seemed to have similar ideas regarding self-similarity dimension. Tara had a very strong work ethic; she was one of the most conscientious students in the class. Tara was always overly concerned about her grades as she was hoping to attend university after finishing grade 12. A segment of our conversation about self-similarity dimension follows.

I: What does self-similarity dimension mean to you?

Tara: Like if you choose a part of the fractal like the dimension is the same as the big one. Can I draw it?

I: Sure.

At this juncture Tara drew the first and second stages of the Koch curve. While she was making the statement that follows, she circled several line segments on her diagram. She was trying to indicate that the self-similarity dimension of the entire structure is equal to that of any constituent part of the curve.

Tara: Like if you chose this one it would be the same as the whole rest of them. How you calculated the dimension you find r and you find N and it's like r and N are the same. Like for that little section and for the whole thing. Therefore this dimension is the same as the whole thing. That's what I think it means.

I: O.K. When you were asked to find the self-similarity dimension you made that calculation $\left[d = \dfrac{\log N}{\log r} \right]$. So then you found a number that gave you the value of d. Besides the word dimension, can you use one or two words to describe what that value d means?

Tara: Like slope?

I: Uh, that's part of it. See, what I was trying to get the other people to do was to think of something that has a dimension of a little bit more than one, and something else that has a dimension of a little bit less than two. Now how are these two figures different?

Tara: It's like got a bigger slope. Like the rise is higher. Like a dimension of one would be a straight line, right? And then a dimension of two would be sort of just like that (indicating on her diagram). And then dimension of three would be like this (indicating on the diagram again). It's like the slope keeps like, rising. Like it becomes like, sharper. More clear, like...(here she loses the thought altogether.)

It was while Tara was explaining her ideas regarding slope that I began to see that she was experiencing some difficulty with all aspects of dimension. In order to explicate her statement regarding "bigger slope" she drew some simple diagrams of a curve that, in class, we knew as the Dragon curve. I have reproduced her drawings in *figure 6.2* below. It appears, based on her diagrams, that she was in the process of accommodating the idea of self-similarity dimension.

figure 6.3

Tara's perception of the relationship between slope and dimension.
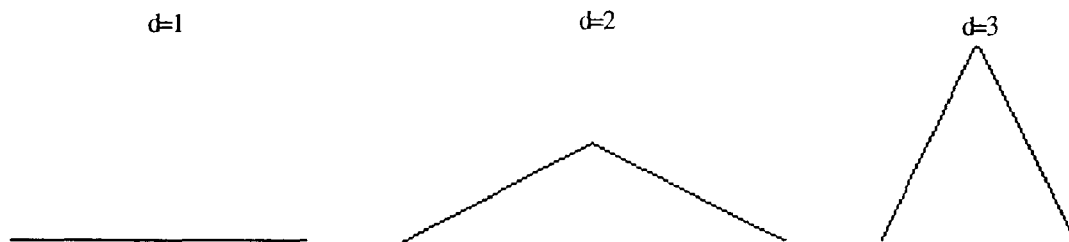
Tara's thoughts relating slope and dimension are, when one considers her diagrams, correct except for one large misinterpretation. She has correctly perceived that as the angle between the horizontal and the leading segment of the curve increases, so does the dimension of the resulting fractal. However, she has incorrectly labelled the dimensions as one, two, and three rather than dimensions that are increasingly close to two. This is, perhaps, evidence that she was in the process of constructing a new scheme to accommodate self-similarity dimension while invoking the scheme for Euclidean dimension. This is very much in character for Tara. If she encounters a concept that proves resilient to assimilation she will persevere until she reaches the point where she feels that she "understands."

According to Tall (1991), students in transition experience new ideas in conflict with previously acquired knowledge. While Tara was in transition from her prior cognitive state to the new one that she was trying to construct, she may have been invoking schemes involving slope and dimension (in its conventional sense) in an attempt to make sense of her new concept of dimension. These schemes proved inadequate to the task, and so she was experiencing considerable confusion while the transitional process was ongoing.

This idea of slope was one that several students mentioned in their efforts to define self-similarity dimension. Rosemary made a similar point in her explanation.

I: O.K. So now what does the phrase self-similarity dimension mean to you? There [indicating the formula that was used] is the calculation that you had to make to calculate it. So aside from it equalling dimension, when you came up with a number to apply to some figure that you were supposed to find the dimension of, what are one or two words that could be used to describe what this number represents.

Rosemary: Slope?

I: Slope of what?

Rosemary: Slope of the object, or the fractal, or whatever.

I: If you have something that has a dimension of really close to one compared to something that has a dimension of really close to two, how are those things different?

> Rosemary: The one that is really close to one is really close to a straight line, and the other one would be closer to a two dimensional object, or a triangle.

This last statement is precisely correct; it seems to signify that she has sorted out some of her difficulties with dimension and is perhaps indicative of some beta behavior. Her response continues:

> I: So how would they look different?
>
> Rosemary: Like, something between one and two would be like a triangle without a bottom, or a square without the, like, a half a square without the other half.
>
> I: O.K. So you are talking about something really close to a line as opposed to something that...So it would be a measure of the....
>
> Rosemary: Box dimension? I don't know. One has shape and one doesn't.

Tara and Rosemary were able, given a diagram of a fractal, to determine correct values of r and N and then calculate the self-similarity dimension for the fractal. However, based on the excerpts from our conversations, it seems that neither of them understood exactly what it was they were calculating. Based on their interview responses, these two students would appear to have a purely operational understanding of self-similarity dimension. As such, when the interviews took place they were seemingly in the process of constructing an understanding of the concept.

Mary and William were, without a doubt, the top two students in the class. Like William, Mary was ESL and was concomitantly enrolled in Mathematics 12 and A.P. Calculus. These two students completed Survey Mathematics 12 with averages in the 90's. When I asked Mary about self-similarity dimension, this was her reply.

> I: You remember making the calculation for self-similarity dimension, right?
>
> Mary: Yeah.
>
> I: So then you found this number, and besides it being a measure of dimension you could think of it as being a measure of ......?
>
> Mary: I don't know.

I: Suppose you see something that has a dimension of very close to 1, and you compare it to something that has a dimension very close to 2. What is the difference between the two figures?

Mary: This one is more...no, it's not more complicated.

I: Why? Why did you say "more complicated?"

Mary: I don't know.

I: Which one is more complicated?

Mary: Two.

I: How do you mean, more complicated.

Mary: The shape, no, you can see more, no. I don't know. The angle is bigger for this one sir.

I: What angle?

Mary: I don't know.

I: Why? Why did you say "more complicated?"

Mary: This one is more realistic I think.

I: By complicated what do you mean?

Mary: I don't know. I just wanted to say that word.

Here is another student whose concept of self-similarity dimension includes some aspect of angle. So although Mary's perception is somewhat more advanced than either Tara's or Rosemary's, she too maintains that "angle" is central to the definition. William's answer was, in some respects, quite similar to Mary's.

I: Say you were looking at something that has a dimension of very close to one, and something else that has a dimension of very close to two. How are those things different?

William: Is that the slope?

I: No, it's not necessarily the slope. I am just looking for one word that you could use that would tell me what the difference is between something that is very close to one dimensional and something that is very close to two dimensional.

William: The shape is different.

I: What about the shape?

William: It's more detailed.

I: So you could say, then, that this number is a measure of...

William: Details.

I: You used another word a minute ago.

William: Complex.

William and Mary are essentially correct in their perceptions of self-similarity dimension. Once again, language could be a barrier to the effective communication of understanding. William was the only interviewee that used "complex" to describe the difference between the two structures to which I alluded in the question; Mary used "complicated", which could be construed as having much the same meaning. However, despite a seemingly good grasp of self-similarity dimension, later in the interview William admitted to me that "...I don't know what 1.3 dimension means to me, like for the last three weeks. All I know how to do is calculate it."

It is interesting to note that the idea of slope was so frequently employed in an attempt to convey the notion of self-similarity. Stephen and Mary, in my opinion, used an analagous idea when they mentioned "angle" in their descriptions. In the context of the deterministic fractals to which the students had been exposed, these ideas are correct. However, what they have said regarding slope and angle does not apply to affine fractals. It seems that, at least for this part of the interview, students are restricting their definition of fractal to include only those that display deterministic self-similarity.

Tara and Rosemary were not the only students to experience difficulty with self-similarity dimension. Of the eleven participants, five admitted that they had no idea of what was meant by the phrase. Only two were able to provide a satisfactory description of it, yet all but one student performed the calculation flawlessly on the unit test. Based on these results, it would appear that the participants had a purely operational understanding of self-similarity dimension. If this was indeed the case, then little or no reflective abstraction had taken place;

thus students required more time to deal with the concept at the operational level in order to achieve structural understanding. It follows that accommodation of self-similarity dimension had not occurred.

When the interviews took place Tara, Rosemary, William, and Mary all seemed to be in transition regarding self-similarity dimension; they may have been in the process of adapting preexisting schemes regarding dimension in its traditional sense, and/or constructing new ones in an attempt to accommodate self-similarity dimension. This could explain the manifestation of alpha behavior amongst the participants. Likewise, it seems they were still struggling with the integration of self-similarity into their cognitve schemes.

Upon discovering that few of my students were able to provide concise answers to the first three questions, I did not expect that many would be able to effectively characterize the properties of a fractal. Renata's explanation of her connotation of fractal was not unlike many that I documented. She is another student for whom English is problematic. She is like Alison in several other respects as well; Renata is quiet, conscientious, is proficient at performing algorithms, and achieved a final grade of close to 90% in Survey Mathematics. And, like Alison, she appears to be performing at a purely operational level.

I: Can you tell me what fractal means to you?

Renata: Some drawing.

I: Just some drawing, eh? Anything else?

Renata: No.

I: Is there anything about a fractal that you could tell me about: just by looking at something you could say "yes. that is a fractal."

Renata: Yes.

I: How could you tell?

Renata: It has a pattern to form the fractal. We can calculate it.

According to Renata's response fractal, in its totality, lies in the calculation of self-similarity dimension. Here is a student capable of flawless computation of that property of a fractal while seemingly having no concept of what constitutes a fractal. It is possible, however, that while Renata is unable to express her thoughts in English she may be capable of correctly answering the questions in her first language.

In general, none of my students was able to provide a concise characterization of a fractal, although most were capable of articulating bits and pieces of the concept. This is not surprising since there was no strict definition discussed in class to be memorized and regurgitated. It is obvious from the transcript that follows that at the time of the interview Rosemary was struggling to identify some properties of a fractal To her credit, she tried valiantly to answer the question.

I: How would you characterize the properties of a fractal?

Rosemary: Many dimensions. No. Below two dimensional. Between one and two dimensional. Repetitive. Like one specific...Like say you draw half of a star it keeps on repeating on each line. Each line is the same each time you draw it.

I: How often would it repeat?

Rosemary: However many dimensions you want to go up to. The more stages you have the more times it will repeat.

I: Think about what you were talking about a minute ago about self-similarity; relate those two.

Rosemary: Each stage of a certain part of the picture you are drawing repeats itself. Like the main part, like if you are drawing a triangle with two lines it's always going to have a triangle with two lines. Like if you just take a piece of it, it will be the one that you started with.

Here again we see Rosemary's difficulty with the idea of dimension. It could be said that her response is a reflection of more alpha behavior; her conceptions of fractals and dimension are unstable and seem to be shifting from moment to moment.

Clearly Rosemary, among others, experienced considerable difficulty with the fundamentals of fractal geometry; questions 1 to 4 were intended to probe students'

understanding of these fundamentals. The remaining questions designed to gauge students' depth of understanding and ability to generalize. If the preceding is any indication of the degree to which participants accommodated the basics, then the responses to questions 5 to 8 should expose further weaknesses in the subjects' conceptions of fractals.

Because participants were expected to have drawn a considerable number of fractals, I anticipated that the responses to question 5 would be largely correct. Yet there were few sketches that accurately recreated the stage two drawing of the stage three fractal I provided. Of the eleven student sketches that I received only one could be considered "correct." Three others were reasonably close to correct, and seven students could not draw stage two having been given stage three. Most drew the generator correctly; only two respondents were unable to do this.

Three of the four "successful" attempts were initiated in the same manner; William, Gerald, and Rosemary began by drawing stage one, or the generator , of the fractal. These students then "fractified" the generator to obtain the stage two drawing. Valerie used a slightly different approach. She initially traced the stage two drawing superimposed onto the diagram she was given, and then copied the tracing below the original drawing. Interestingly, hers was the most accurate representation that I received.

It is difficult to speculate why so few students could successfully draw the stage two fractal. It is possible that, because of deficiencies observed with conceptions of self-similarity, they were incapable of recognizing the key aspect of the diagram (the generator) that would allow them to correctly draw stage two. It is worth mentioning that students had never seen the fractal they were expected to reproduce. I deliberately selected one with which they were unfamiliar in order to eliminate the possibility that students would recall the image from prior experiences. I have little doubt that if it had been a Koch curve, or one the others that we had studied in class, the success rate would have been much higher.

That fractals can be used to model coastlines is an idea many students are able to intuit. James was one of the underachievers in the class; he eventually dropped the course without having participated in an interview. One day we were in the Mac lab using an *Fract-O-Graph* to draw simple fractals. James observed that one of the generators he had drawn, after many iterations, resembled an island. This event occurred before we had had any discussion regarding the fractal dimension of coastlines. Despite the apparentness of the phenomenon, and the time we had spent in class on the concept, most participants were unable to explain the difference between two coastlines of differing dimension.

Renata and Alison were capable of neither articulating nor drawing the visible contrast between two coastlines of slightly differing dimension. Neither student perceived the series of activities she performed leading up to the calculation of coastline dimension to be anything more than another assignment that was part of the course requirements. In their interviews both Renata and Alison admitted that the investigations were meaningless to them. Hopefully, for others in the class these activities were slightly more enlightening.

Rosemary, Gerald, William and Armond sketched diagrams that effectively communicated the way in which the two coastlines would differ in appearance. The remainder of the students experienced varying degrees of success in responding to this question. Valerie and Stephen were two students with similar, yet incorrect, thoughts regarding this concept.

I: If I have a map that has a dimension of 1.2 and another map of a different coastline that has a dimension of 1.7 could you tell me how those two things would look different or how they would look the same?

Valerie: They would look different. The one with the 1.2 ....The 1.7 one would be wider...

I: Want to draw a picture?

[Valerie draws two simple diagrams, neither of which is correct. Her diagrams show that the island having a coastline of smaller dimension has a smaller area than the island with the larger dimension coastline.]

I: You are actually talking about the area more than you are about the coastline. I was thinking about the coastline itself.

Valerie: I don't know.

Here are Stephen's thoughts regarding the same question.

I: O.K. Say you have two coastlines. One of them has a dimension of 1.3 and the other one has a dimension of 1.8. Can you tell me the difference between the two, besides the fact they have different dimensions?

Stephen: Yeah. The angle that the first part of it goes out is different.

I: Can you draw it? Can you draw me a sketch showing the difference between the two coastlines?

Stephen: Yeah. How do you want me to show it? Like a triangle or something?

I: Just draw some little islands; one that has a coastline of dimension 1.3 and another that has dimension of 1.8.

Stephen: Yeah, I think I could do that. Like, one shape would be like, smaller than the other. Like, you're saying that it would be exactly the same, right? Exactly the same shape?

I: Well, yeah. Say some imaginary island. They have different dimensions so they couldn't really be the same shape.

Stephen: Well yeah, that's true. I don't know if I can draw it.

I: Think about that piece of paper that I gave you a long time ago that had those really simple fractals in different dimensions.

Stephen: Yeah. I remember those. That is exactly what I am....Like, I know what it is, I just can't say it. I think the island with the bigger dimension, the smaller one could fit inside. I don't know how to explain it.

Self-similarity dimension and area are linked, although not the way in which Stephen and Valerie have indicated. Both students believed an island having a coastline of smaller dimension should fit inside the island with the larger dimension coastline. These remarks expose further misconceptions of the basic concept of dimension. The question asked required that students elucidate the difference between two structures: one whose dimension is just greater than 1, and the other having a dimension closer to 2. Stephen and Valerie were under the impression that the smaller the area, the smaller the dimension.

Based on their responses that it seems that Stephen's and Valerie's ideas regarding dimension in its conventional context are in a state of flux. The reader will recall that Valerie was the only student to provide a strictly correct answer to question 1. It is possible that both students' thoughts regarding dimension in its traditional context have been thrown into disarray by exposure to dimension in its new context. The preceding excerpts could be indicative of Dubinsky's alpha behavior.

When Tara was asked the same question she once again mentioned slope. However, her use of slope in this context is different from that which she employed in answering question 3 regarding self-similarity dimension; in her response to question 6 she is referring to the slope of a linear graph.

I: So if I told you that I had a map that had a coastline of dimension 1.2 and another map that had a coastline of dimension 1.7 could you tell me basically how those two things would look different?

Tara: The 1.7 would have like steeper lines. Like the slopes cause like it's the slope that....

I: Can you draw it for me? Just a quick sketch.

Tara: Which curve?

I: It doesn't matter. We're not talking about the curves that you guys did on the computer any more now. We are talking about an actual coastline, O.K.? So if you saw a picture of a coastline...[Tara begins to draw graphs now.] So this one has 1.7 and this one has 1.2?

Tara: Yeah.

I: Can you tell me in words how this one and this one (indicating her sketches) look different?

Tara: The 1.7 is steeper.

I: What do you mean by steeper?

Tara: It's like because you get the slope by measuring the rise over the run, right? So if the rise is a bigger number than the run then therefore you are going to get that number and let's say rise is this and the run is this, then you get that (indicating on the paper again).

I: So you are talking about the graphs, right? Not the coastlines themselves.

Tara: Yeah.

This time Tara was attempting to apply the methods she had used in analyzing the box dimension of a coastline. First she drew two diagrams of linear graphs: one which she perceived to have a "dimension" of 1.2, and the other a "dimension" of 1.7. Initially the graphs were straight lines. When she realized that this response was incorrect she superimposed wavy lines on to the straight lines. The wavy lines were intended to represent curves of different dimension although, visibly, there is little difference between the two sketches.

Approximately thirty six per cent of the participants were able to correctly respond to question 6. This, despite the results from question 3 which indicated that only two of the eleven students seemed to have grasped the essence of self-similarity dimension. On the surface these results appear contradictory. Most students were unable to identify self-similarity dimension as a measure of complexity, yet they were capable of sketching a diagram depicting the difference between the two coastlines. This may be explained by the considerable time spent in class drawing, analyzing, and calculating the dimension of fractal structures, whereas no literal definition of self-similarity dimension was discussed.

The final two interview questions dealt with concepts that were not mentioned in class. It is likely because of the unfamiliarity of the ideas that the responses to these questions often occupied the greater portion of a student's interview time. Responses to question 7, being basically a matter of opinion, were generally correct. When subjects were asked to justify that opinion in question 8, the results were somewhat more interesting.

Most participants thought that, given what they had learned about fractals, it was likely that there exist structures that have a fractal dimension of between 2.0 and 3.0: three said no, one said maybe, and seven replied in the affirmative. The challenge was describing the appearance of such structures. In order to guide students toward some perception of the

appearance, I adapted an investigation that I discovered in Peak and Frame's book *Chaos Under Control: The Art and Science of Complexity.*

In *Chaos Under Control*, Peak and Frame describe an activity in which students calculate the self-similarity dimension of wadded up sheets of paper. The authors suggest that most three dimensional objects that are uniformly solid throughout contain within them spaces which are predominantly the same size and shape. Peak and Frame use the example of a styrofoam ball: many air spaces, but mostly of the same size and shape. They maintain that because the spaces contained within a wadded up paper ball are of irregular size and shape, that a paper ball has a fractal dimension (page 98). It should be noted, however, that crumpling a piece of paper into a ball does not "magically" change its dimension; the resulting paper wad is intended merely to approximately model a structure that has a dimension of between two and three.

There was insufficient time for inclusion of this investigation in the study unit, so I had students try to imagine the results of it. My goal was to lead them to the conclusion that, because of the irregularity of the spaces contained within the paper wads, it is possible that the wads have a fractal dimension. Rather than using the styrofoam ball analogy, I had students imagine that they had filled an empty cube with marbles. I then had them mentally compare and contrast the interior appearance of the cube full of marbles with a solid cube, and then with the interior of a paper wad. Finally, I asked students to think of a "real life" object that might have a fractal dimension. Typically, this part of the interview was lengthy, generally running to several pages of transcribed text. Several students provided interesting responses to my questions; however, for the sake of brevity I will include only one sample transcript.

Janet is a very personable young lady and an above average student. She was a willing and able participant in classroom discussions. Janet and I had an engaging conversation regarding this particular line of questioning. She had a tendency to answer a question with a

question; although her reply was phrased as an answer, her intonation suggested a question. There is much to a conversation that a transcript does not effectively communicate.

I: Do you think that anything could exist that has a dimension of somewhere between two and three?

Janet: Something in nature. Like an object?

I: What would it look like?

Janet: It would look weird.

I: Can you describe it?

Janet: No.

I: You said "...something in nature." Why did you say that?

Janet: Because nature has all these wierd looking things that can't be explained. I don't know. It's really like...

It is likely that Janet suggested a naturally occurring structure because of the work that we had previously done with fractals that had a dimension of between one and two. Students had seen a great many images that began simply and, after many iterations, closely resembled the outline of a leaf, a tree, or a fern. When she said "...nature has all these wierd looking things that can't be explained..." she may have been alluding to the fact these wierd looking things cannot be explained by traditional mathematics.

I: O.K. When you think of something three dimensional what do you think of?

Janet: A cube.

I: A solid cube or a hollow cube?

Janet: Solid.

I: We can say that this piece of paper is basically how many dimensional?

Janet: Two.

I: [Crumpling the paper.] How many dimensions does it have now?

Janet: Three.

I: How about if I crumple it so it isn't so tightly wadded together?

Janet: A number of dimensions.

I: Why?

Janet: Because it is not totally solid.

Here, Janet makes an interesting statement. In her view, a tightly crumpled ball is three dimensional, while one that is not so tightly crumpled has "a number of dimensions...because it is not totally solid." Her indecision regarding the two objects is undoubtedly due to my question regarding the existence of objects that have dimension of between two and three. In retrospect I know that I should have explored what she meant by the more loosely crumpled ball having "a number of dimensions."

I: If you take that and think about what you said earlier about something in nature...

Janet: Some things are not totally solid.

I: Let's back up a little bit. You said earlier that a solid cube comes to mind when you think of something three dimensional. How about a hollow cube; is that three dimensional?

Janet: Yes.

I: What do you think that you would have to do to the inside of the cube to make it somewhere between two and three dimensional?

Janet: Decrease the sides? No. Increase the sides.

I: Increase the dimensions [read "length"] of the sides?

Janet: Yes.

I: You said that if I crumple the paper really tightly it is three dimensional.

Janet: Yes.

I: But if it is not crumpled so tightly it is not three dimensional.

Janet: Right.

I: Why?

Janet: I don't know. Wait. Because it doesn't have a definite shape.

I: Anything else? Think about the things that you looked at that were somewhere between one and two dimensional and see if you can take that and extend it.

Janet: I don't know. It's in between dimensions. It's not two and it's not three.

I: You think that the more tightly its compacted the more three dimensional it is, and the more loosely it's packed it is closer to two dimensional.

Janet: Yes.

I: What is the difference between one that is really tightly packed and one that is less tightly packed?

Janet: Their dimension is different.

At this juncture it is difficult to determine which of the ideas are her own and which are a result of my prompting. It is obvious that despite a conscious effort to remain noncommittal I was not; the phrasing and tone of the questions undoubtedly coloured Janet's responses. I am quite sure that Janet was trying to give me the answer she thought I wanted to hear when she replied, "Their dimension is different."

I: Why? Think about what it actually looks like; a densely compressed paper ball compared to one that is not so compacted. What is the difference in appearance?

Janet: Appearance; one doesn't have a different shape, but the other one does. And then...

I: What about inside?

Janet: It's more hollow and the other one is more solid and compact.

I: What about an empty box then; that's completely hollow.

Janet: Outside it has a definite shape, though.

I: It is still three dimensional, though eh?

Janet: Yes.

I: When it's densely packed what about the empty space within the wadded up piece of paper.

Janet: As in inside the space?

I: Yes.

Janet: In the solid one?

I: Even when you really squeeze this together. There is still some empty space in there, right?

Janet: Lesser though.

I: What do those spaces look like?

Janet: Smaller.

I: Are they all regular shaped or are they all different shaped?

Janet: They are all different shapes.

I: What about when you do this [uncrumpling the ball somewhat]?

Janet: It has greater spacings.

I: So what can you think of in nature that might have this sort of structure, but unlike an empty box where you can see this space that is well defined and it's a definite shape.

Janet: In nature? Animals, as in sea animals.

I: What kind of sea animals?

Janet: Those sea urchins. Or a jellyfish, it doesn't have a definite shape.

Despite the hints provided in the questions, Janet is of the opinion that the dimension of an object is dictated by its outward appearance. In what follows she seems to vacillate to some extent, but toward the end of the interview she returns to her initial position.

I: Say you have a sponge in each hand, and one of them has a dimension of 2.4 and the other one has dimension 2.8. What would be the difference?

Janet: The shape.

I: Can you be a little more specific?

Janet: Their insides?

I: How do you think they would look different inside?

Janet: The spaces. The one with the less dimension has greater, has more space.

I: What kind of space?

Janet: More undefinite shape.

I: Relate it to this (holding up the crumpled paper.)

Janet: It has lots of spaces inside but it doesn't really have a shape.

I: And what about the one that is closer to three dimensional?

Janet: It has less space inside but it still has...it is more compacted.

Janet's answer is, according to Peak and Frame, essentially correct. She concluded that a less densely packed object has a smaller dimension than a more dense one. However, she did not specifically cite the irregularity of the spaces as the reason for fractal dimension. In the following excerpt the questions are an attempt to direct students' attention to the shapes and sizes of the spaces within a structure that is not completely solid. Here are Mary's thoughts regarding this line of questioning

I: Give me an example of something simple that is 3-dimensional.

Mary: A cube.

I: Are you thinking of a solid cube or a hollow cube?

Mary: Solid cube.

I: Does it matter? Are they both three dimensional?

Mary: Yeah.

I: Say we have two of them: a solid one and a hollow one. Say we take the hollow one and fill it with marbles that are all the same size and shape. Now you take these two cubes and cut them right down the middle. The solid one is obvious. It's just solid. What about the other one? What does it look like?

Mary: There will be marbles, like half a circle shape.

I: What else is in there besides marbles?

Mary: Space.

I: Where?

Mary: Between the marbles.

I: If the marbles are all the same size and shape are the spaces all the same size and shape?

Mary: Yes.

I: So when you cut the cube down the middle...

Mary: There's going to be a pattern.

I: It all looks pretty much the same, right. If you cut this thing [the paper wad] down the middle, does it all look pretty much the same?

Mary: Yes. No.

I: Why

Mary: Because it just is. I don't think it will make a pattern.

I: So what will the difference be between this thing [the paper wad]...Say I was able to compress this into a cube shape but keep the same basic structure and then I cut it down the middle. How would the inside of this cube look compared with the insides of the other cubes?

Mary: There is no pattern there.

I: Can you think of anything in nature like that?

Mary: (Long pause) I don't know. Everything is sort of the same. Give me an example and I will think of one for you.

For my students these were completely new ideas; it is to be expected that they would display some confusion regarding them. Many likely found the line of questioning to be confusing; when I read the transcripts I see that my input is not always helpful. This could be a contributing factor in Janet's alpha behavior regarding the dimension of the objects. At one point she seems to understand the essence of the fractal dimension, but by the end of the interview she has abandoned that idea. Mary was able to follow the line of questioning, but could not quite put it all together at the end to synthesize a coherent picture of a "real life" object having a fractal dimension. However, there were other students who offered perceptive suggestions regarding naturally occurring structures with fractal dimensions.

Tara gave two examples of such structures. Her first thought was that the surface of the earth would have a fractal dimension. If this thought occurred to her spontaneously then it is extremely insightful indeed. I know this was not discussed in class, but it is possible she read it somewhere. If, for example, one considers a mountain range it could be said that the *surface* of the range has a fractal dimension. There is no fixed pattern to the surface contours of a mountain range. Thus, a rectangular prism superimposed atop the range would be only partially filled and would have spaces if irregular size and shape contained within it. Tara's other suggestion was a tree.

I: Can you think of anything in nature that is like that?

Tara: (Long pause) A tree. It isn't exactly the same throughout.

I: What part of the tree are we talking about here?

Tara: Well, like the bushy part at the top. It's like the computer drawings that we did. Those were really neat because it is self-similar all the way throughout. And it does look realistic, but in a real tree there are deformities all over the place because of whatever.

Here again Tara has taken what appears to be a three dimensional object and "zoomed in" on it. A cedar tree, for example, appears from a distance to be a conical solid. However, when one "zooms in" one sees that the tree is anything but solid. As with the mountain range there are countless spaces of irregular size and shape contained within the foliage of the tree.

Upon concluding the discussion of naturally occurring fractal structures we reached the end of each interview. Overall, I was very impressed with my students' deportment in the interviews. Without exception they demonstrated a serious approach to the process and genuinely tried their best to answer the questions asked. For this, and their many other contributions to this study, I owe them a tremendous debt of gratitude.

Chapter 7

Discussion

Anytime one tries something new and different one learns from the experience. Hopefully, the lessons learned are applied to future attempts at similar endeavors. If I was to teach fractal geometry again, there are lessons I have learned from my initial "teaching experiment" that will guide me in my next experience. What follows is a summary of improvements and changes I would make to ameliorate the teaching/learning experience for my students and myself. The behavioral objectives listed in chapter 3 are revisited and the relative success of their achievement is assessed in terms of student performance. The chapter concludes with some suggestions for others who might be interested in the implementation of a study unit on fractal geometry.

Revisiting the behavioral objectives.

For the sake of convenience the behavioral objectives outlined earlier are listed again. Effective objectives specified that students have the ability to:

(1) describe the characteristics of a fractal;
(2) sketch simple fractal structures;
(3) recognize and define self-similarity;
(4) manipulate the self-similarity dimension and apply it to various fractal structures under differing conditions, and;
(5) compare and contrast the properties of fractal structures with those of some naturally occurring phenomena.

The affective objectives were that:
(6) students be provided with a different lens with which to view mathematics, and;
(7) take this new lens, and use it in combination with objective (5) to decide for themselves if they believe that fractal geometry is indeed the geometry of nature.

The initial stages of the study unit were successful in terms of student engagement and assimilation. Reactions to the slide show were very positive and, although the chaos game was tedious, it ultimately justified my rationale for its inclusion. The second behavioral objective was successfully achieved as almost all students were able to draw successive iterations of simple, deterministic fractals. As reported earlier, the success rate for reversal of the process was not as good as I might have hoped, but that was not among the objectives that I had set for my students.

This portion of the unit was somewhat less intellectually demanding than what followed, but it was a critical starting point. Any modifications that I would make to this introduction to the topic would be minor. There are two improvements that I would make given more favourable circumstances: (1) significantly more computer lab time; and (2) a more modern network. However, there are several key modifications that could make the remainder of the study unit much more effective.

As previously documented, there appeared to be only one student in my class who had a solid concept of dimension in its conventional sense. It is possible that prior to the study unit many participants were capable of discerning the dimension of structures explained by Euclidean geometry, and that as a result of their experience with fractals their prior conceptions of dimension became disordered. Nonetheless, if one is to alter students' perceptions of dimension, then students must clearly understand the traditional connotation of the notion from the outset. I made the mistake of assuming that my grade 12 students understood what was meant by zero, one, two, and three dimensional structures.

It is my feeling that this assumption impeded my students' assimilation of the fundamentals of fractal geometry. Thus, a minor adjustment to the study unit could have major implications for students' acquisition of the requisite concepts. As parts of the preface and conclusion to the study unit I would be sure to include a short lesson on dimension as it is

connoted in the traditional sense. That way, regardless of whether their initial conceptions of dimension were faulty, or if those conceptions were altered due to contact with fractals, students would at least be assured of what is meant by dimension in a Euclidean context. They would then be much better prepared to deal with the more complex notion of self-similarity dimension.

Generally speaking, I was satisfied with students' conception of self-similarity. Although no literal definition of the idea was discussed, almost all participants were able to articulate a functional understanding of it. I would, however, most certainly spend more time on a concise description of self-similarity dimension. The reader may recall that participants struggled with the characterization of a fractal. Subsequent to analysis of the data, it is my feeling that if students were able to demonstrate a sound concept of self-similarity dimension, the achievement of my first behavioral objective would be greatly enhanced.

Given a simple fractal generator, it is a trivial matter to identify values for r and N and apply the formula $r^d=N$ in order to calculate d. Perusal of students' tests and quizzes reveals that for most simple deterministic fractals, calculation of self-similarity dimension was not problematic. There was a slightly lower success rate when students were given the value of d and asked to calculate the angle between the horizontal and the leading edge of a given generator. A problem such as this required application of the self-similarity dimension and right triangle trigonometry (see the unit test in appendix B). In general I feel that students acquired an operational understanding of self-similarity dimension and thus, my fourth behavioral objective was achieved. It should be noted, however, that the correct application of an algorithm does not necessarily imply a structural understanding of the concept represented by that formula.

The final three objectives were closely related. It is with these three that I feel students achieved the greatest success. In their work with computers, students were constantly reminded of the theoretical applications to "real world" phenomena. In the Mac lab at Burnaby West students watched as the computers drew fractal "trees," "shrubs," and "islands" based on

generators they had sketched themselves. In the WCAT lab at SFU they were able to generate fractal mountains and watch as the they "flew" over computer generated fractal landscapes below. Students witnessed combinations of simple Euclidean geometrical figures become complex leaves and flowers through iterations and reiterations of the original patterns. It is my opinion, based on informal, incidental conversations that took place during the interviews, that by the end of the study unit most students felt that fractal geometry effectively describes the geometry of nature.

In assessing the overall effectiveness of the study unit there are several factors to be considered. Foremost among these is students' lack of prior knowledge of fractal geometry. One must bear in mind that participants progressed from having absolutely no concept of fractal to a stage where they could identify some key characteristics and calculate self-similarity dimension. Furthermore, most students became aware of the potential for wide ranging applications of fractal geometry. In this context the teaching experiment was, in my opinion, quite successful. A second equally important factor is the inexperience of the teacher with regard to the subject matter. There is always room for improvement in one's practice; there are several changes I would make to enhance achievement of the behavioral objectives.

Suggestions for improvement of pedagogy.

When I was planning the study unit, I made a conscious decision to provide considerably less structure than that to which students had become accustomed in a traditional mathematics class. My hope was that in doing so it might foster an atmosphere of "discovery" in my classroom. According to the responses that I received, the class seemed equally divided in their opinions regarding the style that I adopted for the study unit: some students liked the flexible structure and some did not. In retrospect, I feel that all students would have benefited from a somewhat more conventional approach. Should I have the opportunity to repeat the experience, my students could expect slightly more traditional pedagogy.

Periodically, I would try to summarize main ideas in a more structured way. Examination of participants' notebooks revealed that there were very few notes taken. In the absence of a textbook it is incumbent upon me to provide students with qualified reference material to effectively review for quizzes and a unit test. Subsequent to an initial exposure in which students would be permitted to explore a concept independently, elucidation of key aspects in the prescriptive manner might enhance student performance. It is not necessary to "regress" to a "chalk and talk" style of teaching; however, I feel that some type of formal presentation is required for effective communication of concepts.

To cite an example, at one point I recognized that much of the class was experiencing difficulty with the concept of box dimension. At that point I should have intervened and led a discussion of my objectives for the activity and the objectives implicit in the activity itself. If students have some record of the discussion to facilitate their learning, so much the better. Calculation of self-similarity dimension was one area in which I adopted a more traditional approach and, generally speaking, students were able to apply the formula and algorithm correctly.

Most mathematics teachers are extremely proficient at communicating algorithms and procedures; for many, the effective objectives are the embodiment of mathematics. When one studies the British Columbia mathematics curriculum one finds few, if any affective objectives. As a result, most math teachers are unfamiliar with the methods used in the evaluation of these objectives. Fractal geometry offers an opportunity to explore alternatives to the traditional assessment techniques employed by mathematics teachers.

In order to evaluate student achievement of the final three objectives, students could be assigned a short (500 word) essay describing their thoughts regarding the existence of a fractal geometry of nature. To make the assignment worthwhile, one of the criteria would stipulate that

the essay include a list of references; in that way it would be incumbent upon students to do some research to support their opinions. Unfortunately, there is a shortage of appropriate reference materials; much of the existing literature is too advanced for secondary school students. There are several books and articles listed in the bibliography that are suitable for a high school audience, but gaining access to a class set would be problematic.

Recommendations for implementation.

In hindsight, it occurs to me that my behavioral objectives were likely too ambitious. Given that my students had no previous exposure to the topic, my expectations were unrealistic. In Canada the traditional mathematics curriculum is spiral in nature; that is, topics are visited and revisited from year to year and with each encounter, concepts are covered in slightly greater depth. In this way many students are able to acquire a structural understanding of, for example, solving single variable equations or right triangle trigonometry. My students were given no such opportunity with fractal geometry.

We studied the topic for approximately six weeks; this represents a threefold increase in the time that is typically allotted for a unit in the traditional curriculum. We did not address three times the material, but the concepts were completely foreign. Thus, only a gifted student would have the ability to accommodate all of the ideas disseminated during those six weeks. In a sense, fractals are like any mathematical topic. Students need time to think about the ideas, to make sense of them, and to construct their own meanings. They need time to ponder notions such as infinity and dimensions that are not natural numbers. Historically, British Columbia mathematics curricula do not allow sufficient time for students to reflect on the information they are expected to acquire.

It is rumoured that the Ministry is considering the introduction of fractal geometry to the secondary curriculum. As a result of my experience, it is my feeling that there are two ways in which this can be accomplished. The first is that fractal geometry be introduced at the grade 10

level, and the topic is visited and revisited like any other in grades 11 and 12. If the government opts to include fractals in only the grade 12 curriculum, then a significant period of time must be devoted to the study of the fundamentals. The former option would be ideal but the latter is what is rumoured to be under consideration.

Should the Ministry decide to implement the second option, it would require the deletion of more than one topic from the existing curriculum in order to allow sufficient time for an effective study of fractals. It is, however, unlikely that this will occur. If it does not, then it is unlikely the curricular experiment will succeed; it will go the way of the Math 12 calculus unit. Like calculus, it is impossible to study fractal geometry for two or three weeks and expect students to acquire a significant amount of relevant information.

Other considerations include available hardware, software, and print resources. In terms of technology, most schools are ill equipped to deal effectively with the content of a unit on fractal geometry. Because of the fiscal realities of public education this situation is unlikely to change in the foreseeable future. Hence, most students will be required to study fractals using pencil, paper, and perhaps a textbook. It is possible that this teaching/learning situation could work; however, in my opinion it is counter to much of the philosophical rationale for introduction of the topic.

Concluding remarks.

As outlined in the introduction and in the literature review, there exists ample justification for teaching fractal geometry in high schools. Its cross curricular relevance makes it ideal for inclusion in the arts and the sciences alike. When my students were informed that I would be collecting their notebooks, I asked that they keep a journal of sorts. Tara was the only one to do so. The following is a quote from one of her entries:

These are some major discoveries which have changed the way that I view things. I feel that I have been ripped off; I should have been introduced to this when I was five years old. This is not a difficult concept. This was an amazing section of math. It wasn't the usual exercises from a textbook. This made me think and even incorporate philosophy into my thoughts and discoveries. This should be taught or at least introduced at a much earlier age.

Tara's thoughts reflect Benoit Mandelbröt's feelings. In his words: "...if fractals' use in teaching is confirmed and proves lasting, it is likely to dwarf all their other uses (Mandelbröt, 1992, p. 78)."

Appendices

Appendix A: Letters of authorization

The following is a letter requesting parental permission for student participation in the study.

Dear Parent/Guardian: March 25th, 1996

Your son/daughter is being asked to participate in a study that is to take place in his/her Survey Mathematics 12 class. Participants in the study will be taught an experimental unit on fractal geometry, a topic that is included in the Survey Mathematics 12 curriculum. The study unit and students' responses to it are to be documented as part of my graduate studies at Simon Fraser University.

The content of the study unit is non-traditional, but the methods used in the teaching will largely be those with which students are familiar. Participants will be responsible for all information taught and all assignments and/or evaluative measures prescribed by the teacher.

One of the major differences between this unit and traditional ones is that upon conclusion the students will be asked to participate in an interview. The purpose of the interview is to determine if and how the students learned the information presented to them. The interviews will be approximately 45 minutes in length. Your son/daughter will be asked to appear either before school, at lunch, or after school. Each interview will be audio taped for the purposes of analysis. The tapes will be destroyed upon completion of the study; confidentiality is guaranteed. If you would be interested in obtaining a copy of the results of the study, please feel free to request one from me and I will be happy to oblige.

Participation is completely voluntary; should you decide that your child will not participate, or should your son/daughter choose not to participate in the study, that decision will be respected and there will be no penalty to the student for not taking part in the project. Furthermore, your son/daughter will be free to opt out of the study at any time without incurring any penalty to his/her class grade. Finally, should he/she become dissatisfied with, or have any complaints regarding, any part of the study he/she will have the opportunity to express his/her concerns. If your child wishes to express his/her dissatisfaction to a third party, complaints should be addressed to Dr. Robin Barrow, Dean of Education, Simon Fraser University, V5A 1S6. Dr. Barrow can also be contacted by telephone at (604)291-3148.

If you are willing to give your consent for your son/daughter to participate, but only if he/she wishes to participate, please fill in the appropriate blank below, sign this letter and return it with your son/daughter . Should you require further information regarding any aspect of the study please contact me at school (664-8535) or at home (421-2326).

Thank you very much for your attention.

Yours truly,

Michael Langille

I, _____, give my son/daughter _____ my permission to participate in the study described above.

I, _____, do not wish my child to to participate.

Signed: _____

Date: _____

To whom it may concern: March 25th, 1996

This letter is to acknowledge that I am aware of research on Survey Mathematics 12 students' assimilation and acquisition of fractal geometry that is to be carried out at Burnaby West Secondary School by Michael Langille, a teacher at Alpha and a student in the Master's Degree Programme in mathematics education at Simon Fraser University.

I understand that the research methodology is to consist of a six week study unit on fractal geometry to be administered by Mr. Langille. His students will then be subjected to an interview. The intent of the interview is to determine if, and how, the students have acquired the concepts presented through the course of the study unit. The interviews will be audio taped and these tapes will provide most of the data to be studied. Other data will consist of students' work that will be collected and analyzed. At the outset the students will be informed of their right to withdraw from the study, or any part of the study, at any time.

In my opinion, these activities are well within the educational context of the Survey Mathematics curriculum and the mathematics department at Burnaby West.

Principal
Burnaby West Secondary School

Appendix B

<u>Worksheet #1</u>

Introduction to Fractal Geometry

The figure that appears below is one of the earliest fractals studied. It was discovered by the Swedish mathematician Helge von Koch in 1904 and thus, is known as the Koch curve.

The first three iterations of the Koch Curve.

stage 0 _____

stage 1

stage 2

stage 3
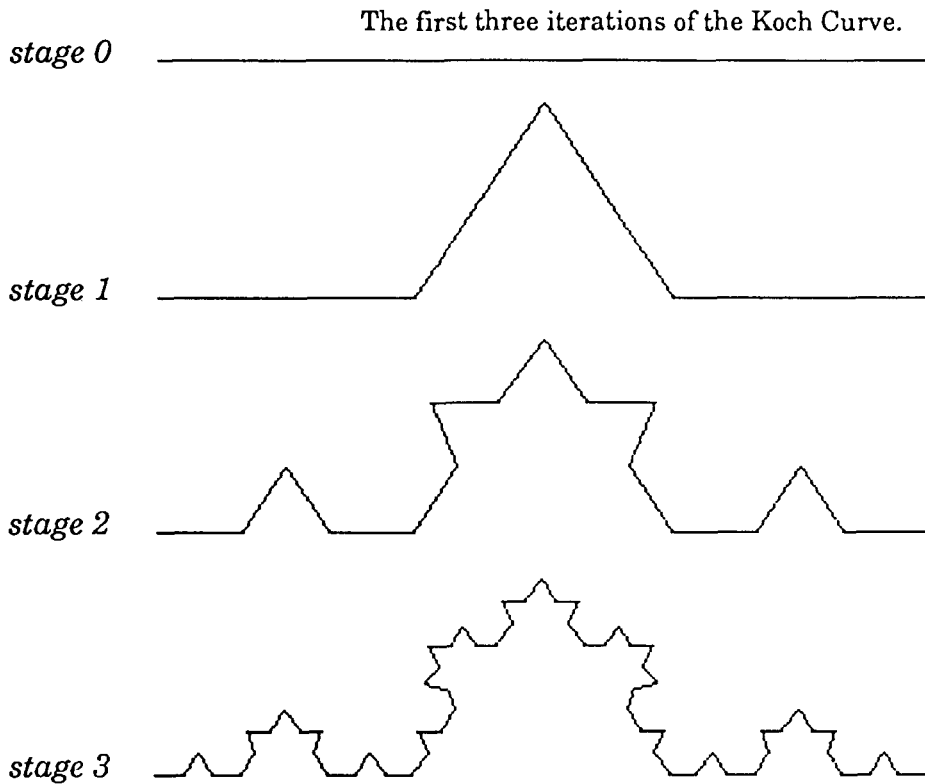
As is the case with all of the simple fractals that we will be studying initially, the Koch curve begins with a straight line and becomes more and more complex with each iteration. Each line segment in a given iteration is replaced with the generator.

For each of the generators shown below, use pencil, ruler, and squared paper to sketch the first three iterations of the fractal.

1.
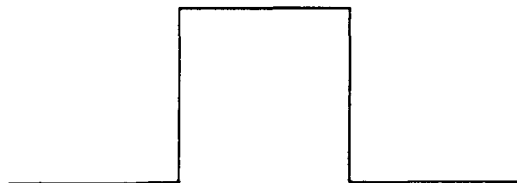
2.
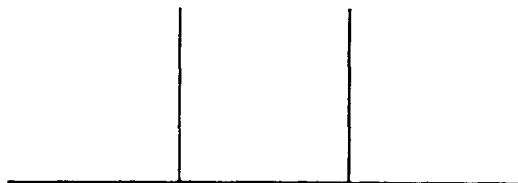
3.
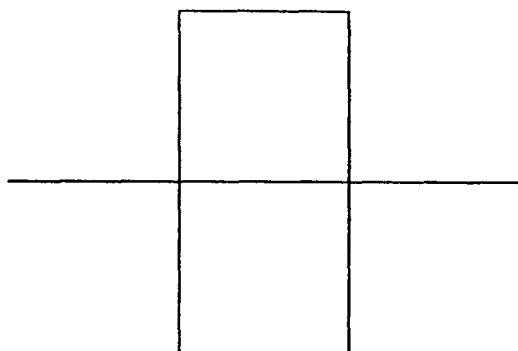
4.
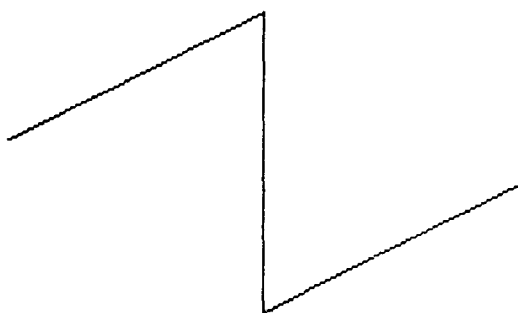
The Chaos Game

The Chaos Game is not so much of a game as it is an activity that will (hopefully) eventually help you to understand the relationship between chaos and fractals. Chaos is a relatively new science that was discovered by meteorologist Edward Lorenz in 1961. It has since been discovered that Fractal geometry is the geometry of chaos. Because of the complexity of chaos theory, and the equally complex relationship between it and fractal geometry, we will not be studying chaos in any detail.

Here is how the Chaos "game" is played. Mark a point anywhere inside the vertices of Δ LTR.

Step 1. Roll the die and move according to these rules:

for a 1 or a 2, move halfway to vertex L;
for a 3 or a 4, move halfway to vertex T;
for a 5 or a 6 move halfway to vertex R.

Step 2. For each point that you locate in this manner, plot a small point on the acetate overhead sheet provided.

For the first day, plot as many points as you can in the time provided. By the completion of the unit you should have plotted approximately 200 points (that

works out to about 20 per day). This activity will be graded; I know what the result is supposed to look like, so take care in making your measurements and plotting your points. This is an activity that you will be given class time to do every day that we study fractals. You will not be allowed to take this activity to work on at home.

Introduction to Fractal Geometry: Handout #2

Begin by sketching the next two iterations of the figures shown below.

1.

2.
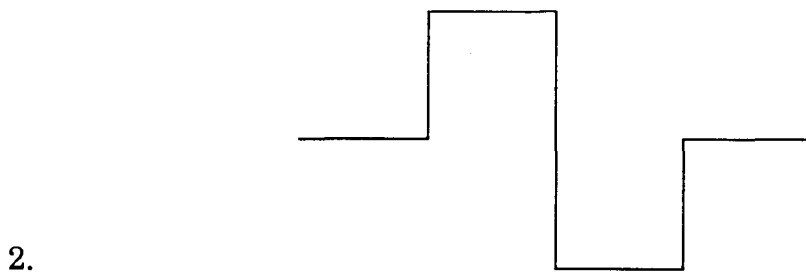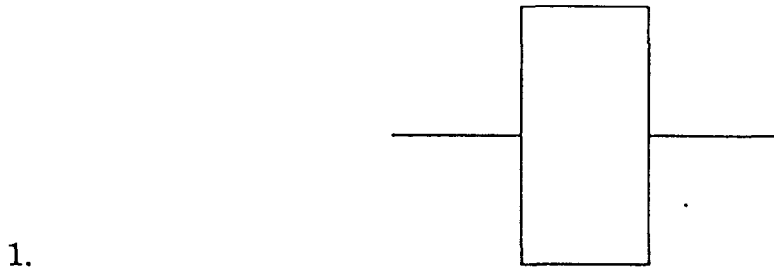
Consider the following diagrams:

figure 1

A line segment divided into 4 identical copies.

A square subdivided into 16 identical smaller squares.
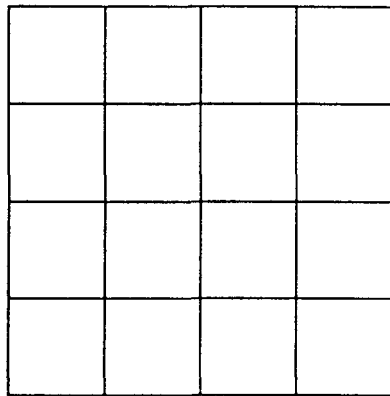
The straight line segment shown in *figure 1* has been subdivided into four equal parts. The original segment is reduced by a factor of 4 and the number of identical copies formed is 4. The large square shown is subdivided into many smaller squares. Again, the reduction factor is 4 but this time the number of identical copies is 16. The cube diagrammed in *figure 2* (overleaf) has been subdivided into a number of smaller cubes. Yet again the reduction factor is 4, but now the number

of identical copies is 64. These relationships can be described by the equation $r^d = N$.

In this equation, r represents the factor by which each segment has been reduced in the transition from the original figure to the one that has been subdivided. The parameter N represents the number of copies of the original figure that are contained in the new figure.

figure 2

A cube subdivided in 64 identical smaller cubes.
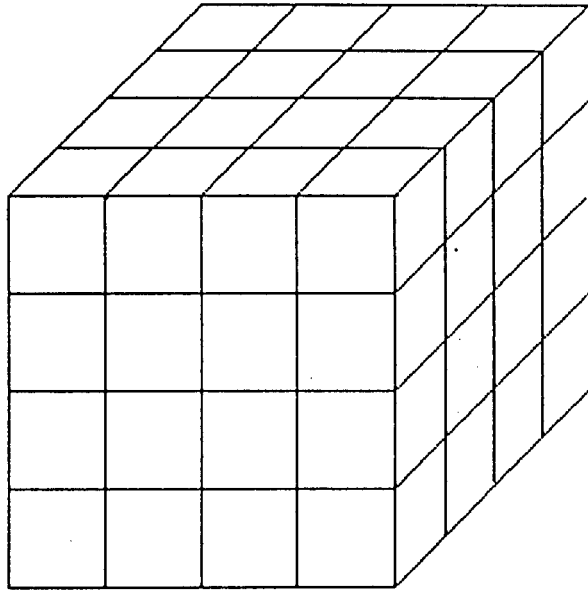
1. Below are some of the figures that you were to draw for homework. Calculate the dimension of these figures.

a. *stage 1*

b.
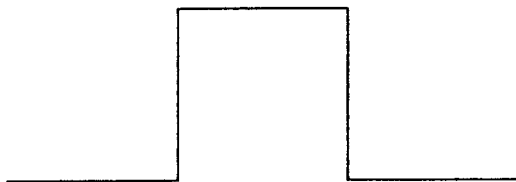
c.
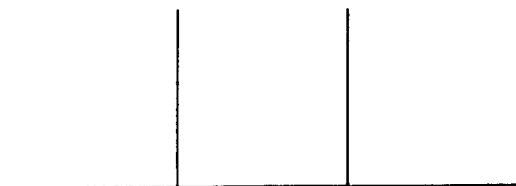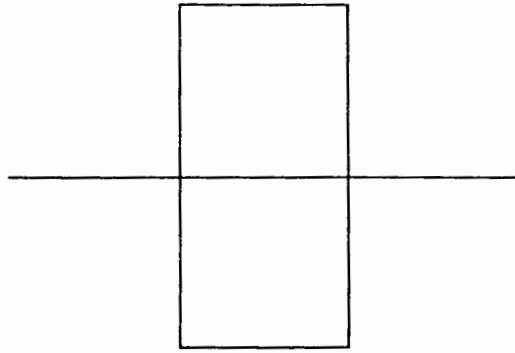
d.
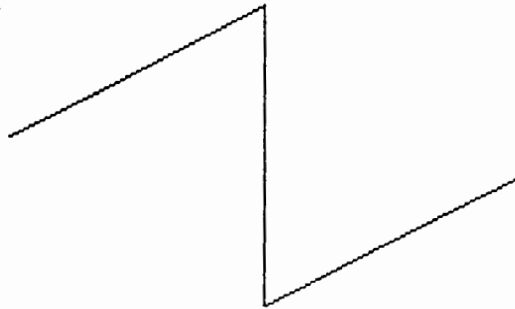
e.

2. Calculate the self-similarity dimension of the two figures that you were asked to draw at the beginning of the class.

Introduction to Fractal Geometry: Handout #3

Use the programme called *Fract-O-graph* to draw some of the fractals that you have been asked to sketch in the two previous classes. The programme will sketch as many iterations as you want; all you have to do is push the button labelled *fractify*.

The sketches that the computer draws may be different than the ones that you sketched yourself. This does not necessarily mean that your drawings are incorrect. There are a few problems with the programme that prohibit it from sketching certain fractals. One in particular that it does not like is the Gosper curve. If you need some clarification just ask. Print a couple of them and include them in your notebook.
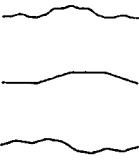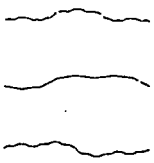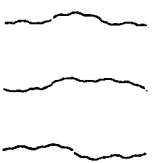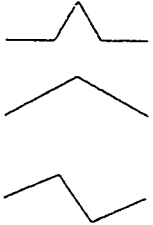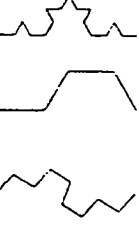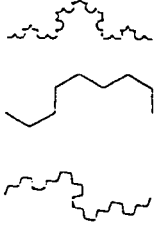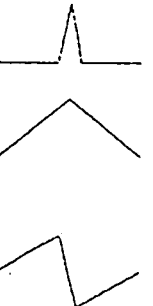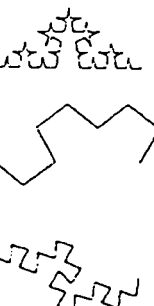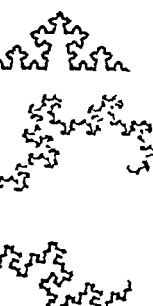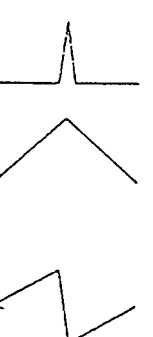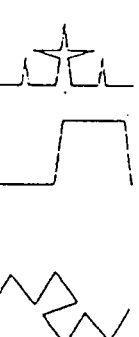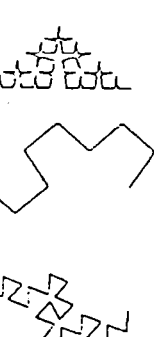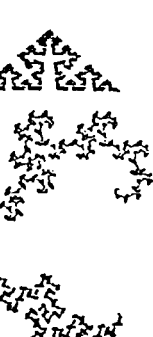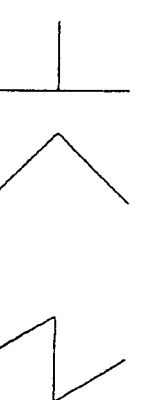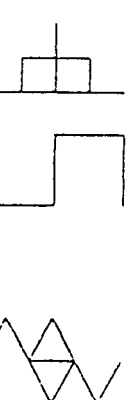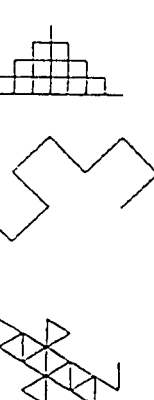
After you have explored the programme using some of the generators discussed in class try making up some of your own. I would like you to print a copy of any of these that you do and include them in your notebook also.

On page 2 of this handout there are several different versions of curves that we have studied so far in class. Complete the table below for each of the curves shown using the self-similarity dimension $r^d = N$ and some trigonometry. We will discuss a few of them in class. The subsequent iterations of each generator are there for the purposes of illustration; you need only analyze the generators themselves. You should show some work that I can see when these handouts are collected with your notebooks.

curve	*d*	*N*	*r*	*θ*
Koch H.H. Gosper	1.02			
Koch H.H. Gosper	1.26			

curve	d	N	r	θ
Koch H.H. Gosper	1.58			
Koch H.H. Gosper	1.70			
Koch H.H. Gosper	2.00			

This page photocopied from *Fractals in Your Future*, by Ron Lewis: page 39.

Dimension	Generators (Stage 1)	Stage 2	Stage 3	Later Stage
1.02				
1.26				
1.58				
1.70				
2.00				

Introduction to Fractal Geometry: Handout #4

Open the programme called *FractaSketch*. Use the programme to draw some of the fractals that you studied in *Handout #1*. As with the other programme that we used, (*Fract-O-Graph*) when you have completed sketching your generator double click the mouse and the screen will automatically be changed to the "fractify" mode. On the menu bar is a menu called *Grid*. You may want to pull down this menu and change the grid to *None*. This will make it easier to specify the angle that you want to use to draw your generator; otherwise you are pretty much restricted to a certain range of angle measures. Remember that for the fractals that we have looked at to date all segments are the same length. Clicking on the numbers across the bottom of this screen enables you to see your generator at different stages.

Across the bottom of the screen are the types of segments that are available for use in drawing your generators. Experiment with these to discover what difference they make in the fractals that you draw. One of the more useful ones is on the right hand side of the screen; this enables you to draw "invisible" segments. When you use this tool the segment that is drawn will not be included in the subsequent drawing of the fractal.

At the upper right of the screen you will see the equation "*D =*." This will tell you the fractal dimension of the generator that you have drawn. Use *FractaSketch* to attempt to draw some of the fractals that appear on the second page of *Handout #3*.

Try drawing the variations of Peano's curve that are shown below. I have tried to draw the arrowheads so that they are visible. Note that in the fourth drawing the segments are not all the same length. The fifth drawing should give you the basic idea of what to draw. Play around with the angles for awhile; you will know when you have drawn the generator using the correct angles. The sixth drawing is another that you should play around with for awhile. Once again, you will recognize the fractal when you have found the correct angles.

For more ideas see the reverse of this page.

5.

6.
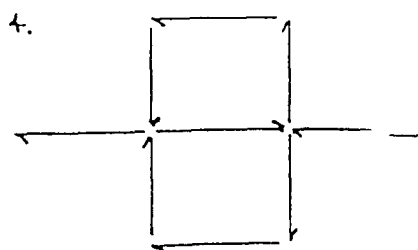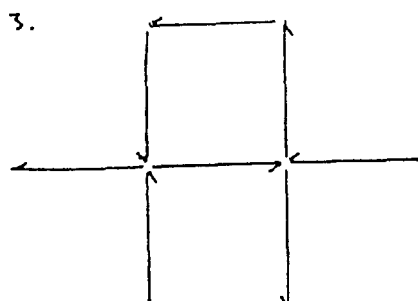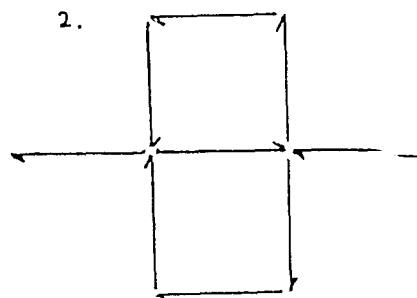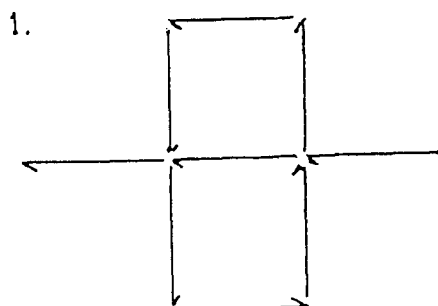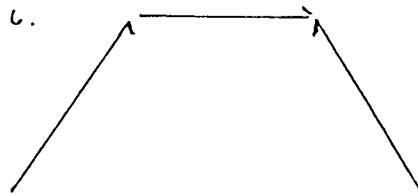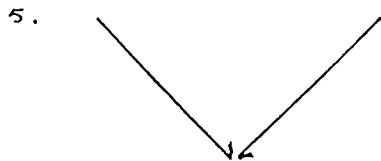
The diagram below and to the left shows a number of generators that will yield variations of the dragon curve. See if you can match the generator to the fractal that is shown several stages later (below right).

The diagrams below are photocopied from *Exploring Fractals on the Macintosh*, by Bernt Wahl: pages 61 and 62.

a

b

c

d

e

f

g

h

i

j

k

l

m

n

o

p

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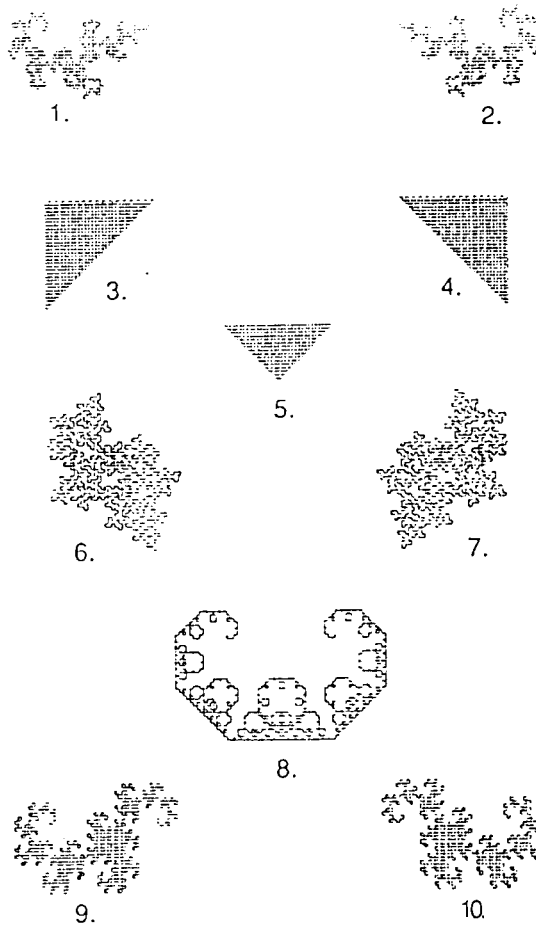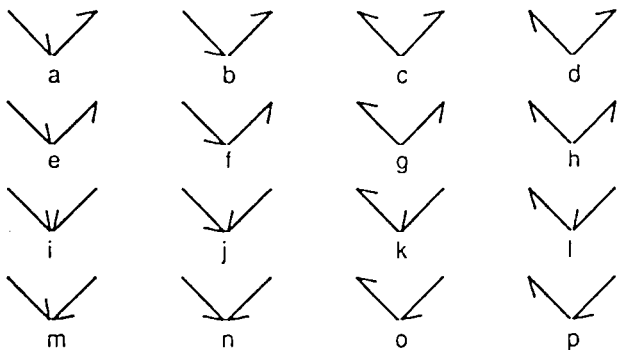
Try drawing some trees using the instructions that follow.

The following is photocopied from Exploring Fractals on the Macintosh, by Bernt Wahl: page 67.

Basic Tree Construction with FractaSketch

Let's draw a tree, beginning with a seed created in FractaSketch. See Figure 3.57.
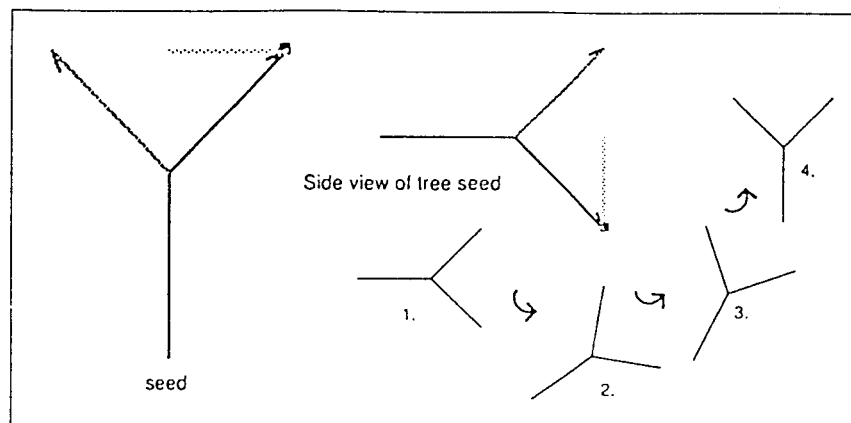

Figure 3.57
Y seed—the most basic construction for a fractal tree—shown with the tree's side drawing and its corresponding upright rotation

Hint: It is easiest to construct a fallen tree, which is a tree lying 90 degrees on its side. When the construction is completed, you can use the Rotate feature from the Scale menu to put it in an upright position.
Let's begin construction of our fallen tree with three segments.

Step 1 Choose box 9 (the nonreplaced line segment), draw a horizontal line roughly halfway across the screen, and click. This is the tree's trunk.

Step 2 Choose box 1 and draw the tree's first branch. Do this by drawing a line from the trunk with an ascending 45-degree angle toward the upper right.

Step 3 Choose box 0, and use it to backtrack invisibly to the top of the tree trunk (the right-hand side). These light-gray lines are used to let you draw figures that consist of noncontinuous or disconnected sections. In this case, use this feature to draw a symmetric branch on the opposite side.

Step 4 Choose box 2 and draw a line equal in length to the other branch, only this time it should descend by 45 degrees toward the lower left. This branch should be symmetrical about the tree trunk.

Step 5 Choose box 0, finish the drawing by double-clicking, and rotate the tree into an upright position. When you're finished, if the Edit menu is set to Left to Right, an invisible line will automatically be drawn to the tree's middle top, which will define the seed's length. Otherwise, if the Edit menu is set to First to Last, this line will be ignored.
The image should look like a giant Y. You can proceed to higher levels and see how your tree takes shape.

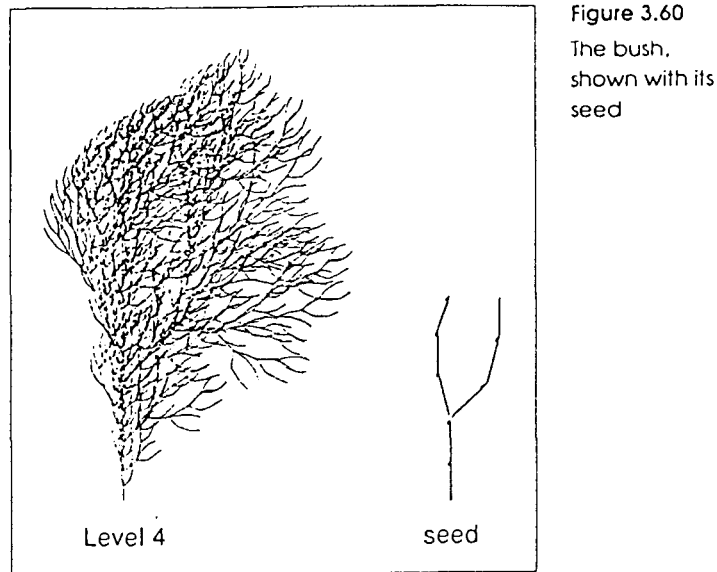The diagrams that appear below were photocopied from Exploring Fractals on the Macintosh, by Bernt Wahl: page 69.

Figure 3.60
The bush, shown with its seed

Level 4   seed

Figure 3.61 shows some templates that produce familiar trees. You can create your own fractal seeds and see what kinds of trees you come up with.
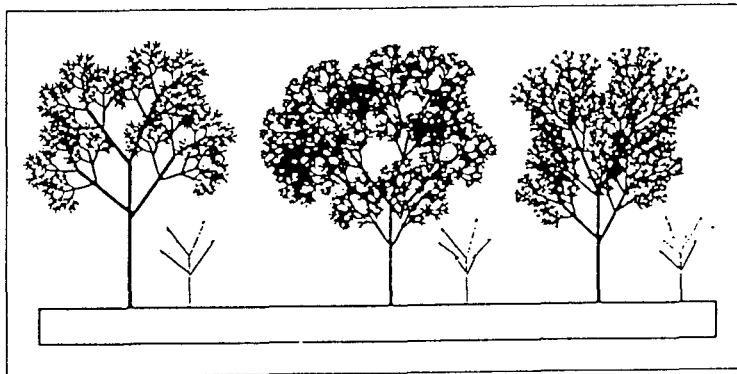

Figure 3.61
Recognizable structures of familiar trees

Included on the disc with *FractaSketch* are some ready made fractals that you can view. Open the file labelled *EF FractaSketch file folder*. Contained within that folder are several other folders. Open, say, *EF Nature folder* and click on *EF Fern II*. It may be interesting for you to click back to stage one of the fractal and watch how the fern leaf is generated as you click through various stages of the fractal. Take some time to explore some of the other folders contained within the *EF FractaSketch file folder*.

<u>Worksheet #5</u>

Introduction to Fractal Geometry: Handout #5

Use the box counting technique practiced in the previous activity to measure the complexity of the coastline of the Queen Charlotte Islands. There are many islands in the archipelago. To speed things up count only the boxes that contain parts of the coastlines of the two main islands; Graham Island is the northernmost island, and Moresby is the large southern island. The coastlines of these two islands have been highlighted to make them easier to distinguish.

Make sure that you include in your count every box in the grid that contains any part of the coastline within the interior of the box. Enter the results in the table below. The "scale" that has been used for a particular grid can be found in the lower left corner of the grid. The count for the grid of the smallest scale has been done for you.

Group Data

1/(scale *x*)	4	8	12	16	24	32
boxcount *y*						277

Class Data (average)

1/(scale *x*)	4	8	12	16	24	32
boxcount *y*						277

Make two double logarithmic plots of the boxcount *y* versus the reciprocal of the scale *x*; use your group data and the average of the class data. Draw your graphs on the same type of double logarithmic graph paper used in the previous activity. Use two separate pieces of graph paper.

1. Calculate the slopes of the "best fit" lines that result from the plots that you drew.

2. What dimension does the coastline of the Queen Charlottes appear to have?

Introduction to Fractal Geometry: Handout #6

One of the programmes with which you will be interacting today provides a visual display of the iteration of a simple function. This display is called the Mandelbröt set (it will be referred to as the M-set from now on),and is named after its discoverer Benoit Mandelbröt. At the beginning of this unit we viewed a slide show in which there were several slides showing a part of the M-set. At that time several of you asked what it was. Without getting too technical I will now try to explain in a little more detail.

If you recall the unit we studied on finance and growth you should remember using recursive formulas. In studying those formulas you would begin with a number, plug it into the formula, get a second number, and then plug that one in to generate a third number, and then plug that one in to generate a fourth number, and so on and so on. This is precisely how the M-set is generated.

When we studied recursive formulas you may recall having observed one of two results. The results of the iterations either became progressively larger and larger, or they got smaller and smaller until they were very close to some limiting value. The successive iterations of the formula used by Mandelbröt behave in much the same way.

Mandelbröt began with the formula $z_{n+1} = \left(z_n\right)^2 + c$. You should recognize this as a simple recursive formula; however, Mandelbröt used complex numbers (sometimes known as imaginary numbers) which make the calculations more complicated. Start with the initial value $z_0 = 0$, and choose some value $c = x + yi$, and begin the recursion (the value i is defined as $\sqrt{-1}$). The values resulting from the recursion are graphed on the complex coordinate plane where the x-axis is as usual and the y axis is the complex axis (see the graph below).
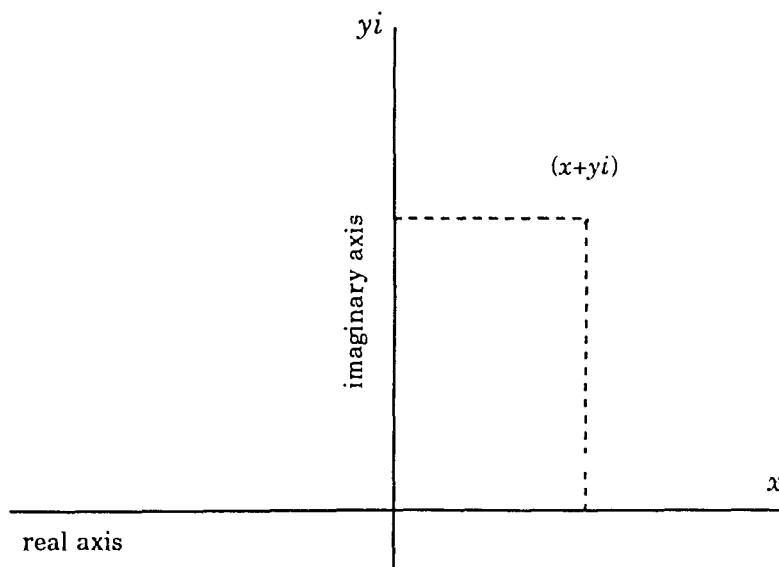
The M-set consists of all the points for which the iterated function $z_{n+1} = (z_n)^2 + c$ with the initial value $z_0 = 0$ does not produce values which grow to infinity. In order to be contained within the M-set the iterated values of the function must lie within the circular region of the coordinate plane defined by $-2 \leq x \leq 2$ and $-2i \leq yi \leq 2i$.

An initial value of c is chosen and the recursion begins. If the chosen initial value of c is contained within the M-set one of three things will happen within the region mentioned above:

1. The resulting sequence of values may converge to a single point, or;
2. the resulting sequence of values may oscillate between two or more points, or;
3. the resulting sequence of values may forever bounce around randomly.

On your computer screen the values that are contained within the M-set are coloured black. It is the values that are not contained within the set that give the pictures their colours. The colours are determined by the speed with which the iterated values of the function escape to infinity. For example, the areas that are coloured blue are the values of c that escape to infinity very quickly. The points contained within the areas that are coloured yellow escape to infinity, but not nearly as quickly as the ones coloured blue.

A simple diagram of the M-set appears below. Using the computer, explore the M-set and try to find the following areas. Mark them on the diagram. Try to find:

• an area where a miniature version of the M-set appears;
• an example of self-similarity (has the same structure at any magnification);
• the area known as "Seahorse Valley."

The diagram below is photocopied from the Survey Mathematics 12 resource binder
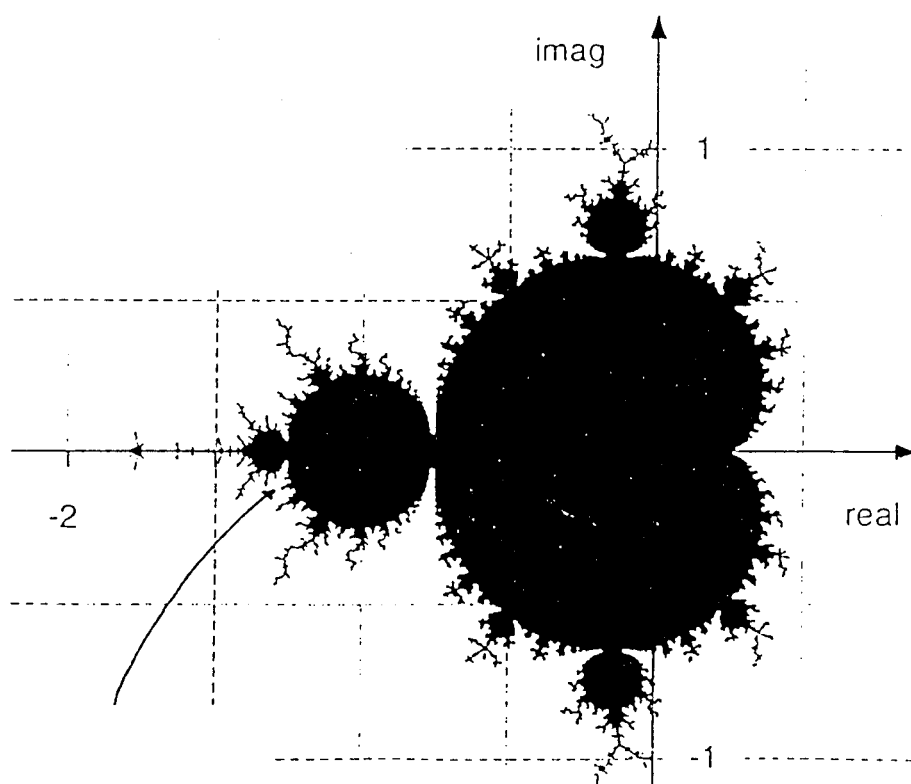
Aside from these exercises play around with the programme and just admire some of the images generated. The computer must make extremely complicated calculations to produce these images. This is one of the reasons that fractal geometry did not become an important area of study until very recently.

Another programme that you will use today generates realistic looking fractal landscapes. John will explain how to change some of the controls within the programme so that you can customize your image. I have no idea how this programme works so I will not even try to explain. Play around with it and think about how real these pictures can be made to look. Think about some of the philosophical questions that arise from these pictures.

Last, but not least, I want you to use *FractaSketch* again. If you have not already had a chance, look at some of the images contained within the file called *EF FractaSketch Nature folder*. When you look at these pictures think about the landscape images you produced today. Also, if you have not had the chance to print some fractals from the computer this would be a good opportunity to do so. I will be expecting to see some computer generated fractals in your notebooks along with the ones that you were supposed to draw using pencil and paper. Remember: I want your notebooks by Friday, December 22nd at the **latest**! Remember also that I asked you to include in your notebook some of your thoughts regarding and impressions of fractal geometry: particularly with regard to what you have seen and learned today.

In Mandelbröt's book *The Fractal Geometry of Nature* he writes:

> "Why is geometry often described as 'cold' or 'dry'? One reason lies in its inability to describe the shape of a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line... Nature exhibits not simply a higher degree but an altogether different level of complexity."
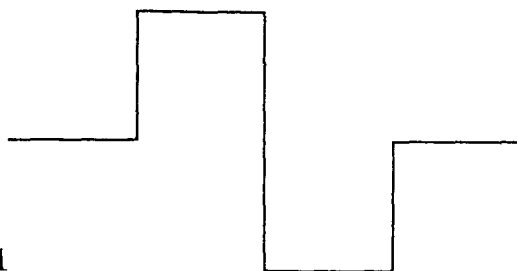
This is likely his most widely quoted statement. He goes on to propose that fractal geometry can be used to explain many natural phenomena. Drawing on what you have seen and learned in this unit and on the assigned readings write a short response to Mandelbröt's theory. Your essay should be a minimum of two double spaced type written pages (about 500 words).
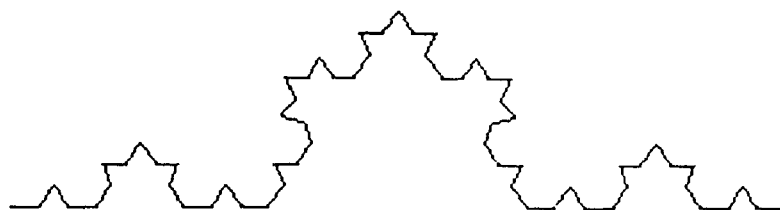
Fractal Geometry Quiz #1

1. Given the following stage 0 and stage 1 diagrams;

    a. Draw the stage 2 diagram

    b. calculate the fractal dimension of the generator.

    Stage 0

    Stage 1

2. Given the stage three diagram shown below, sketch the stage two diagram.

Quiz #2

S.Ma. 12 Fractal Quiz #2

1. a. Stage 0 and stage 1 of a Sierpinski carpet are shown above. Calculate its fractal dimension. Show all work.

   b. Briefly describe the appearance of stage 2 of the fractal. Draw a diagram if you wish.

2. Given a Koch curve that has dimension 1.3, calculate the angle required in order for stage 1 to be drawn. Show all work and draw a sketch. Use the reverse side of this page.

Introduction to Fractal Geometry: Unit Test

Name:_____

1. Calculate the fractal dimension of the figures resulting from many iterations of the following generators. Show all work.

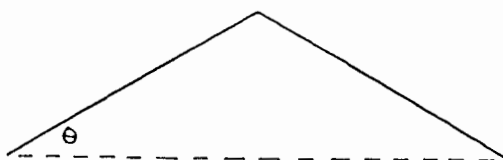Hint: in a given generator all segments are the same length.
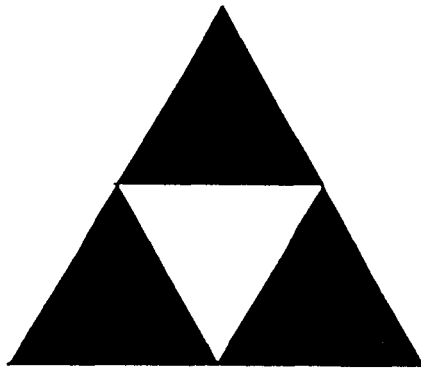
a.

(2)

b.

(2)

c. $\theta = 30°$

(3)

1. (continued...)

d.

(2)

2. The figure shown below could be used to generate a Koch curve of   dimension $d \approx 1.5606$. Calculate the angle between the horizontal and the   second segment of the generator. Show all work. Round your answer to   the nearest tenth of a degree.
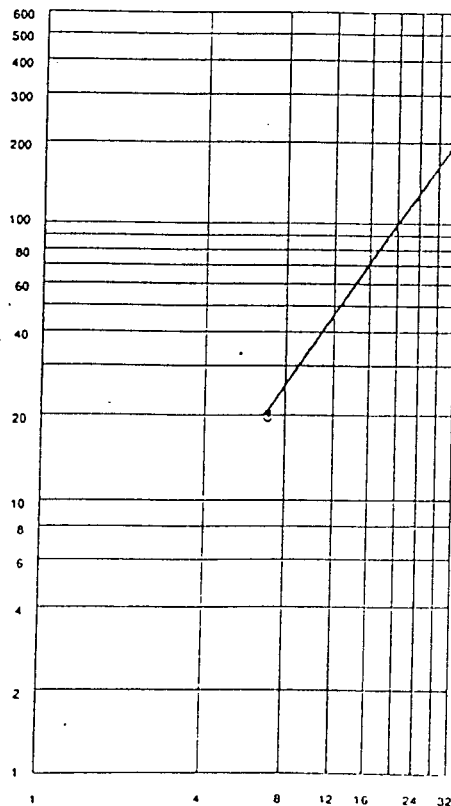

(3)

3. Make a sketch of the third iteration of the figure shown in question 1(a). **Make it neat** and be as accurate as you can.

(3)

4. The graph shown below represents a double logarithmic plot of a boxcount that was done on the coastline of Dogbreath Island. Calculate the fractal dimension of the coastline. Show work.

(1)

5. Two "curves" have fractal dimension 1.2 and 1.8 respectively. Write a paragraph comparing and contrasting the two curves. Please **note: a** paragraph consists of **several** sentences. A sentence begins **with a capital** letter and ends with a period; what is in between must make **some sense.** You may support your description with diagrams if you wish.

(2)

$\overline{20}$

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