On Numerical Modeling of Corporate Strategic Debt Service

By

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PROJECT SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF FINANCIAL RISK MANAGEMENT

In the
Faculty
Of
Business Administration

Financial Risk Management Program

Summer 2009

APPROVAL

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Degree: **Master of Financial Risk Management**

Title of Project: **On Numerical Modeling of Corporate Strategic Debt Services**

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Abstract

The point(s) which divide the state variables space into different phases and give rise to different debt service flows based on game theory can be determined either analytically for perpetual debt, or numerically for debt of finite maturity. The fact that debt service flow can vary through its life span and that such a change can be modeled is of importance to credit risk management. Numerical implementation of the strategic debt service theory poses many challenging aspects. The aim of this thesis is to explore feasible options that can be implemented in dealing with such challenges.

Keywords: debt service flow, state variable space, critical value

To my family

Lisa, Tony and Justin

for

their understanding and support in my pursuit

of

further education

ACKNOWLEDGEMENTS

I wish to thank Professor Theunissen for his intuition, guidance, discussion and support during this project. I thank Professor Jones for his support of this work.

I thank Professors Aidan Vining, Robert Gauer; Drs. Andrey Pavlov, Evan Gatev, Peter Tingling, Phil Goddard for their academic insight and being patient with me.

I thank the staffs of Segal Graduate School of Business, particularly Mr. Brian Chand, who have extended their continuous and timely assistance and support throughout my period of study.

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1. Review & Introduction

Firms borrow money to fund business activities and by doing so, take on obligations. The obligation in question can be a contracted debt which must be serviced during the term of the contract and its principal value is expected to return to the lender at the end of the contract. A firm can service the debt service fees from the cash flows it generates. In this thesis we define Corporate Debt, or Debt, as the money a firm borrows to fund its business activities and the firm is termed the Borrower. The entity that provides the funds is termed the Lender. The Lender provides funds to a firm with the expectation of financial return as represented by the sum of regular debt service fees throughout the contract and the principal value at the end of the contract. To facilitate the deal, the firm's assets are often held as collateral together with terms and conditions on default and liquidation. Default is defined as the Borrower failing to pay, or paying less debt service fees than is stipulated in the loan agreement. Liquidation means the Lender can take control and sell the firm's assets when the Borrower defaults in order to recover its loan value from the proceeds of such a sale. At the time contract is signed, a firm's asset (book) value is known; however, asset values can change during the lifetime of the debt, and so can the cash flows generated by the firm. The debt service flow comes from the cash the firm generates and therefore it can also vary. In reality, neither the Borrower nor the Lender knows the current value of the assets until it is sold. However, the Borrower may hold more information about the value of the assets on a continuous basis and that the Lender may audit at a regular time intervals to ascertain the value of the assets. In this thesis, it is assumed that both the Borrower and the Lender know the asset value with certainty on a continuous basis. The change in asset value may trigger the Borrower's urge to alter the debt service flow accordingly to reflect its financial reality; we define this event as default by the Borrower. The Lender has the option to accept a default situation, and hence receive a lower rate of debt service fees with the hope that the firm's value will eventually increase; or to reject and liquidate the firm's assets according to the contract in order to recover its loan value. In a liquidation state, if a constant liquidation cost is assumed, then the Lender would receive the minimum of either: the difference in the firm's asset value and liquidation cost, or the principal value of the debt. The Borrowers receive the maximum of zero or the difference between the firm's asset value and the sum of principal value of the debt and the liquidation costs.

The state in between a firm servicing its debt according to the contract and liquidation is the ***Strategic Debt Service*** state. In this thesis, the firm's business activities still continue in the strategic debt service state; the firm still generates cash; there is no change of ownership; and the debt servicing is at a lower level than that outlined in the loan agreement. The fact that the strategic debt service state exists is due to the cost associated with liquidation. Often, the debt service flow is reduced but the Borrower will propose a rate that makes the Lender indifferent between receiving the lower rate of debt service or liquidating the firm.

There are many variables to consider when it comes to corporate debt, and much of the literature discusses the link between capital structure and corporate debt valuation. These variables include maturity, coupon rate, convertible and callable options, credit spreads, changing interest rates and volatility of the underlying asset's return, to name just a few. In terms of debt structure, some of the research focuses on the heterogeneity of bank and non-bank sources of corporate debt funding; and the structured claims by priority classes. For example, Rauh and Sufi (2008) assess the empirical relationship between credit quality and debt structure.

The level of corporate debt, or leverage, is also a dynamic indicator of the business cycle a firm is operating in. Firms issue corporate debt of various maturities accordingly. Myers (1977) argues that firms can mitigate the underinvestment problem by issuing short-term debt. Under this argument, using short-term debt that matures before a firm exercises its growth options allows stockholders to capture a larger proportion of the value created by positive net present value projects. Short-term debt effectively allows stockholders to "buy back" debt at prices that do not reflect the value of new profitable investment opportunities. Jensen (1986) argues that debt can reduce the agency costs of free cash flow, which are most severe for firms with low growth opportunities. Goyal et al. (2001) establishes the relationship between growth opportunities and the level and structure of debt during the Reagan period using data from 61 US military defense firms and 61 manufacturing firms. Lewellen (1971) argues the merger of two or more firms whose earnings streams are correlated reduces the risk of default of the merged firms and increases the borrowing ability of the combined enterprise. Lewellen attributes the increased total borrowing capacity of the resulting firm to the well-known effect of tax-deductible interest payments, which provides an economic incentive for firms seeking to maximize shareholder wealth to engage in mergers.

The behaviour of Borrowers and Lenders in times of financial stress is also an interesting subject that has attracted the attention of many researchers. An example is Weiss (2002) who studies a sample of 37 New York and American Stock Exchange firms that filed for bankruptcy between November 1979 and December 1986, and finds violation in 29 cases. Weiss concludes that the breakdown in priority of claims occurs primarily among the unsecured creditors and between the unsecured creditors and equity holders. Secured creditors' contracts are generally upheld. In other parts of the world, Borrowers seem to factor in renegotiation. If the rights of creditors are strengthened by regulators, the total amount of debt the Borrowers are willing to commit would reduce, like the Indian case in Vig (2006).

In modern finance, valuation of corporate debt and quantification of credit risk exposure is important but not straightforward for two reasons. First, data is limited. For example, most corporate debt is partly or completely illiquid, making mark-to-market values unavailable and forcing one to derive the credit exposure from other related known variables. Secondly, effectively modeling covariation in credit risk across different credit exposures is very difficult. Notably, a class of models that have been used to value corporate debt in the presence of credit risk is the contingent claims analysis. The simplest form of contingent claims analysis was initiated by Merton.

Using the Black-Scholes formula, with certain assumptions of the firm's underlying asset and boundary conditions, Merton (1974) pioneered a structured pricing model to value corporate debt based on a simplified corporate financing structure: (i) a single, homogenous class of zero-coupon debt and (ii) the residual claim, equity. The Borrower promises to pay the Lender a total amount at maturity only, and if the Borrower fails to do so, the Lender can immediately seize the firm's assets. Seizure is assumed to be a simplified, seamless and costless process. The value of equity is a call option on the firm's assets with a strike price equal to the face value of the debt. The bond's value at maturity T is equal to the minimum of the debt principal and the value of the firm, $B(V,T) = \min(P,V)$. Many researchers, such as Black and Cox (1976), Ingersoll (1977) and Jones et al. (1984), have contributed to the comprehensiveness of structured models with their work. Despite the elegant closed-form analytical solutions for the valuation of corporate debt, there appear to be gaps in the application of structured model theory to the observed market data. The theories systematically overestimate the corporate debt value. A key feature of this group of contingent claims analysis is that the lower boundary, at which the firm is re-organised (bankruptcy occurs) and creditors take possession of the firm or receive their claims through a court settlement, is given exogenously. Longstaff and Schwatz (1995) use a stochastic lower boundary to model the critical condition although it is still given exogenously.

The main issue with an exogenously defined lower boundary is that it does not reflect the fact that the Borrower and the Lender negotiate all debt obligations during a period of financial stress. Given the behaviour of Borrowers and Lenders throughout the life of a debt contract, another type of contingent claims analysis has attracted considerable research efforts: Strategic Contingent Claims Analysis. In this class of models, the Borrower uses bankruptcy costs to her advantage and hijacks the Lender's claim on interest payments and on the firm's assets, and persuades the Lender to accept a compromised level of debt service. This strategic play can happen when the firm is in financial stress, as well as when firm is doing well and the underlying asset is much greater than its value at the time the debt is issued. Compared to simple contingent claims analysis, the strategic one differs mainly in three areas: (i) permits default to occur before the maturity date of the debt; (ii) includes liquidation costs in the model; (iii) recognizes the negotiating power of the Borrower, using liquidation costs to persuade the debt holder to compromise on the debt contract terms. The key is that the Borrower will offer debt service fees that make the Lender indifferent between liquidation or extending the credit.

Mella-Barral and Perraudin (1997) report that by incorporating the strategic contingent claims model and assuming perpetual debt, they are able to interpret risk premiums as reflected in market data. With proper assumptions, they claim that strategic debt service may explain 30% to 40% of the default risk premium observed in market.

Anderson et al. (1996) (AST) develop a strategic debt service model based on game theory that incorporates the mentioned feature. The paper accounts for three phases of state variable – asset value for c<r case (i.e. the coupon rate is lower than the risk-free rate, leading to a negative risk premium). These phases are: normal debt service phase; strategic debt service phase and liquidation phase. The focal points are the value of V*, which is the switching point (or trigger point, the highest asset value point in the strategic debt service phase), and service flow rate $S^*(V_t)$ inside the strategic debt service region.

AST does not clearly define the Lender's 'indifference' or that of V*. AST implies that there is a V* and its position is located through value matching and smooth pasting. AST provide a closed-form solution to the perpetual debt case, and state that there is a jump in debt service flow when compared the flows in two different states. Their paper mentions some of the results of strategic debt service modeling for debt with a finite maturity, but provides no details of their modeling parameters. Additionally, the paper does not mention if the service flow continues to show a jump at the switching point for debt of finite maturity. More details of discussion can be found in the following chapters.

Most of the work published on the topic of strategic debt service suggests that the possibility of bargaining by the equity holder pushes the debt risk premium up, and lowers the debt value. Intuitively, this should be the case. One exception is Archarya et al. (2006) who argue that, with optimal cash management, defaults generated by deliberate underperformance (strategic defaults) and those forced by inadequate cash (liquidity defaults) work as substitutes: allowing for strategic debt service leads to a decline in the equilibrium likelihood of liquidity defaults. In some cases, this decline is sufficiently sharp that equilibrium debt values actually increase and yield spreads decline.

The bargaining power of each player has also been researched. In a more recent study, Fan and Sundaresan (2000) present a more balanced view on the bargaining power of the equity holder and bond holder while including the tax shield effect on equity and debt valuation. By including tax benefits, Fan et al. distinguish between the value of the firm's assets and the firm's value as an ongoing entity. By doing so, they are able to insert the corporate tax term into the standard partial differential equation. They find that the negotiation between the Borrower and the Lender is over the value of the firm when tax is at presence due to tax shield benefits. Closed-form solutions are provided for the perpetual debt case only. A formula for equity valuation in the case of finite length of debt is given and that numerical solution is recommended. Their work supports the partition of three phases of the debt and that service flow is discontinuous from one phase to the other, although their results show that trigger points occur later (at a lower value of the firm's assets) in their model as compared to Anderson et al. (1996).

Most of the aforementioned work provides analytical solutions to strategic debt service for the perpetual debt scenario. For debt with a finite maturity, the calculation of debt service flow becomes challenging. In particular, a numerical scheme's ability to approximate the 'trigger point' -- the highest value of the firm's assets where strategic debt service takes place, is of practical importance as it is closely linked to credit risk management. The ability to model the trigger point and the service flow accurately would enable the lender to: i) price the debt more accurately; ii) plan the income stream in advance; iii) better manage financial risk. Therefore, the focus of this thesis is on the numerical implementation of strategic contingent claims analysis in a game theory setting. In the following sections, I shall discuss the environment within which the numerical modeling takes place; the methodology employed; modeling and results interpretation, followed by a conclusion.

2. The world of this thesis

The following assumptions are made in order to simplify the issues of debt in order to focus on the point of switching from normal debt service to strategic debt service. These assumptions are applicable throughout all sections in this thesis.

A. The firm's capital structure consists of both equity and a single type of perpetual debt or finite maturity debt with a coupon rate of c
B. Asset value of the firm generates cash at the constant rate of β irrespective of the value of the assets; the value of the assets follow a geometric Brownian motion with volatility of return σ being constant throughout the life of the debt
C. Information regarding the value of the firm's assets are readily available to both the Lender and the Borrower on a continuous basis
D. The debt is a private debt, i.e., provided by institutions and cannot be traded; the firm cannot issue new debt or issue new shares
E. The risk free rate is assumed to be constant throughout the life of the debt and that $c<r$, i.e. a negative risk premium
F. The debt service state is divided into 3 phases: normal service flow phase (the firm honours its contractual obligation and the contracted coupon rate is paid), discounted service flow phase (the firm is unable or unwilling to honour its contractual obligation and is in default by offering a lower debt service flow rate), and liquidation phase (the Lender liquidates the firm in an attempt to recover the value of the debt from the sales proceeds)
G. The Borrower decides whether or not to honour the debt renegotiation and offers a reduced service flow at the rate that makes the Lender indifferent between accepting or rejecting the offer
H. The Lender can choose to accept or reject by comparing only the payoff if liquidate or payoff if continue which equals to 'adjusted' service flow plus expected future debt value. If accept, Lender receives a lower rate of debt service in the hope that the firm's asset value increases (and with it the cash flow it generates) in the future and that the debt service flow returns to normal
I. The tax shield benefits that the firm enjoys as long as it honors the terms and conditions of the debt contract is not considered in this thesis
J. Liquidation costs are fixed

Of all the assumptions, C, F, G and H are critical assumptions. But for the purpose of this thesis, they are assumed to be true.

3. Methodology

3.1 AST framework

The AST's framework can be summarized as during periods of financial stress, the Borrower is to default on his debt service obligation, pay less service fees than specified in the contract. With a small time-step, the level of payment is determined by

$$S(V_t) = Min(cP, Max(0, Min(P, Max(V_t - K, 0)) - \frac{\pi B_u + (1-\pi)B_d}{R}) \quad \ldots\ldots. \qquad \text{AST-1}$$

where V_t is the value of the firm's assets at time t; $S(V_t)$ is the debt service flow which is a function of V_t; c is debt's coupon rate and P is the principal value of the debt; K is the constant liquidation cost. π is the Martingale probability of the asset value increasing (in scale of u) in the next time period; $1-\pi$ is the Martingale probability of asset value decreasing (in scale of d) in the next time period. R is one plus the interest rate per time-step; $B_u = B(uV_t)$ is the value of the debt when the value of the firm's assets increase to uV_t ; $B_d = B(uV_d)$ is the value of the debt when the value of the firm's assets move down to dV_t. Finally, following the Cox-Ross-Rubinstein (CRR) convention, $u = 1/d$.

AST-1 describes that in the event of default by the Borrower, the Lender will select the best action based on two values: $\max(V_t - K, 0)$, the liquidation payoff; and the payoff of $S(V_t) + \frac{\pi B_u + (1-\pi)B_d}{R}$ if the Lender accepts the Borrower's offer. The payoff when the offer is accepted is the sum of the service flow and the continuation of value of debt that follows Martingale probabilities. AST claim that the service flow is chosen so that the Lender sees no difference between liquidating the firm and accepting the reduced service flow, therefore allowing the contract to continue.
Therefore, at each time, the value of the debt is

$$B(V_t) = S(V_t) + \frac{\pi B_u + (1-\pi)B_d}{R} \qquad \text{AST-2}$$

3.2 The Service Flow in Continuous Time

Equation AST-2 can also be written in continuous time. As the form of strategic debt service in continuous time can help with the discussion below, I derive the continuous time representation of AST-2 in this thesis. Following the CRR convention for the binomial scheme, and let h be the time step, the probability of an upward move can be expressed as:

$$\pi = \frac{e^{(r-\beta)h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \ ; \ 1-\pi = \frac{e^{\sigma\sqrt{h}} - e^{(r-\beta)h}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \qquad (1)$$

Assuming the h is very small and using Taylor expansion, we have

$$e^{(r-\beta)h} = 1 + (r-\beta)h + O(h^2)$$

$$u = e^{\sigma\sqrt{h}} = 1 + \sigma\sqrt{h} + \frac{1}{2}\sigma^2 h + O(h^2) \text{ and}$$

$$d = e^{-\sigma\sqrt{h}} = 1 - \sigma\sqrt{h} + \frac{1}{2}\sigma^2 h + O(h^2) \qquad (2)$$

Let $B_u = B(uV_t, t+h)$ and $B_d = B(dV_t, t+h)$ be the debt value when asset increases and decreases respectively, and use Taylor expansion to expand the $B_u and B_d$ terms, we have:

$$B_u = B_t + (e^{\sigma\sqrt{h}} - 1)V_t \frac{\partial B}{\partial V_t} + \frac{1}{2}(e^{\sigma\sqrt{h}} - 1)^2 V_t^2 \frac{\partial^2 B}{\partial V_t^2} + h\frac{\partial B}{\partial t} + O(h^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$B_d = B_t + (e^{-\sigma\sqrt{h}} - 1)V_t \frac{\partial B}{\partial V_t} + \frac{1}{2}(e^{-\sigma\sqrt{h}} - 1)^2 V_t^2 \frac{\partial^2 B}{\partial V_t^2} + h\frac{\partial B}{\partial t} + O(h^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Moving $S(V_t)$ to the right-hand side and multiplying both sides of the equation **AST-2** by R, and bringing all expanded expressions above into the (AST-2) equation, we have:

$$(B_t - S(V_t))R = \frac{(r-\beta)h - (-\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h)}{2\sigma\sqrt{h}} B_t + \frac{(\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h) - (r-\beta)h}{2\sigma\sqrt{h}} B_t$$

$$+ \frac{(r-\beta)h - (-\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h)}{2\sigma\sqrt{h}} (\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h) V_t \frac{\partial B}{\partial V_t} + \frac{(\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h) - (r-\beta)h}{2\sigma\sqrt{h}} (-\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h) V_t \frac{\partial B}{\partial V_t}$$

$$+ (\frac{(r-\beta)h - (-\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h)}{2\sigma\sqrt{h}} + \frac{(\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h) - (r-\beta)h}{2\sigma\sqrt{h}}) * \frac{1}{2}\sigma^2 h V_t^2 \frac{\partial^2 B}{\partial V_t^2}$$

$$+ (\frac{(r-\beta)h - (-\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h)}{2\sigma\sqrt{h}} + \frac{(\sigma\sqrt{h} + \frac{1}{2}\sigma^2 h) - (r-\beta)h}{2\sigma\sqrt{h}}) * h\frac{\partial B}{\partial t}$$

$$+ O(h^{\geq 1.5})$$

……………………………………………………………………………………….. (5)

In equation (5), $O(h^{\geq 1.5})$ means all the terms of 1.5 power or higher of h. We can simplify the above expression and this gives:

$$(B_t - S(V_t))R = B_t + (r-\beta)hV_t \frac{\partial B}{\partial V_t} + \frac{1}{2}\sigma^2 h V_t^{\,2} \frac{\partial^2 B}{\partial V_t^{\,2}} + h\frac{\partial B}{\partial t} + O(h^{\geq 1.5}) \quad ....(6)$$

Please note subscript t does not mean partial differentiation over time; it means the value at time t. Since R is dollar capitalized over one period at the risk-free rate, it can be written as $(1+r)^h \approx 1 + rh$; we can get a more simplified equation of the form:

$$rhB_t - S(V_t)(1+rh) = (r-\beta)hV_t \frac{\partial B}{\partial V_t} + \frac{1}{2}\sigma^2 h V_t^{\,2} \frac{\partial^2 B}{\partial V_t^{\,2}} + h\frac{\partial B}{\partial t} + O(h^{\geq 1.5}) \;\ldots\ldots\; (7)$$

Please note that in AST paper, they missed the multiplier of R on $S(V_t)$ when introducing their equation (3) from the binomial game setting in their paper. Additionally, observing equation (7), we can notice that all terms contain an 'h' except $S(V_t)$. However, it is a small error and when $h \to 0$, the error is small. Dropping the higher order h-term, and dividing both sides of the equation by h, and using AST's definition of $S^*(V_t)$ we have our final alternative form of AST2 written in continuous time:

$$rB_t - S^*(V_t)(1+rh) = (r-\beta)V_t \frac{\partial B}{\partial V_t} + \frac{1}{2}\sigma^2 V_t^{\,2} \frac{\partial^2 B}{\partial V_t^{\,2}} + \frac{\partial B}{\partial t} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (8)$$

Equation (8) is equation **AST2** in continuous time. Perhaps we can re-write equation (6) to make our discussion easier:

$$B_t = S(V_t) + \frac{B_t + (r-\beta)hV_t \frac{\partial B}{\partial V_t} + \frac{1}{2}\sigma^2 h V_t^{\,2} \frac{\partial^2 B}{\partial V_t^{\,2}} + h\frac{\partial B}{\partial t}}{R} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (9)$$

Equation (9) is written for comparison with AST2. The second term on the right side of the equation is equivalent to the binomial expression of $\frac{\pi B_u + (1-\pi)B_d}{R}$. It is important to recall the assumptions when deriving binomial scheme: 1) very small h (time-step); 2) constant risk-free rate; 3) constant (lower is better for approximation) volatility of return on assets; 4) each recombining node has the same probability. At a very small time-step, the expected value of the debt equals its current value and weighted sum of first and second derivatives of debt over asset value and time.

In AST, an instantaneous service flow rate function is defined as $S(V_t) = S^*(V_t)h$. Using their definition, in strategic debt service region of $B(V,t) = V - K$, place this condition into equation (9), it is not difficult to prove that $S^*(V_t) = \beta V - rK$ which is what AST derive in their paper.

A conflict exists here. If we use AST convention AST-1, $S(V_t)$ is of the unit of value. However, $S^*(V_t)$, although defined as rate of value, appears to be of the unit of value by its solution. One way to solve this conflict is to define $S(V_t)$ in AST-1 with an 'h' term to the right side of the equation, i.e. make $S(V_t)$ a flow rate from the start and that AST-1 is a definition of $S^*(V_t)$. For the purposes of this thesis, I continue to use the AST definition.

$S^*(V_t)$ can be substantially lower than c*P which is the contractual debt service obligation. Following the AST logic, there is a jump at the scale of $cP-(\beta V - rK)$ in service flow at the switching point. If one uses a numerical scheme, such as the explicit finite difference scheme, to calculate the service flow based on the AST framework, and provided the scheme is set up properly to handle the continuous time scenario; the service flow curves should be straight lines as the theory predicts and there should be a service flow jump at the switching point denoted as V*.

4. Numerical Modeling

4.1 Confirming the AST Results for Perpetual Debt

It is important to confirm the AST result for a perpetual bond. Perpetual can be interrupted as a debt with infinite maturity: $T \to \infty$. Analytical solutions to the bond values B(V, t) were provided by AST using the modified PDE in two state spaces separated by V*:

$$\frac{1}{2}\sigma^2 V^2 B_{vv} + (r-\beta)VB_v - rB + cP = 0; \quad \text{for } V > V^*, \text{ where } B_t = 0$$

and

$\frac{1}{2}\sigma^2 V^2 B_{vv} + (r-\beta)VB_v - rB + S^*(V) = 0;$ for $V \le V^*$ where $S^*(V)$ is the service flow

In solving the problem, AST followed this logic:

1) Provide a general solution to the PDE $B(V,t) = A_1 V^{\gamma_1} + A_2 V^{\gamma_2} + U$
2) Focus on when $V \to \infty$ ends and assume risk-free bond valuation to derive $B(V,t) = A_2 V^{\gamma_2} + \frac{cP}{r}$ where γ_2 <0 and $\gamma_2 = .5 - \frac{r-\beta}{\sigma^2} - \sqrt{((r-\beta)/\sigma^2 - .5)^2 + 2r/\sigma^2}$
3) Assume that in the strategic debt service region $B(V,t) = V - K$ where K is the fixed liquidation costs;
4) Solve V^*, A_2, and B(V, t) using value matching and smooth pasting at V*

Analytical solution for V^*, A_2 are obtained as:

$A_2 = [V^* - (\frac{cP}{r} + K)]/V^{*\gamma_2}$ and $V^* = (\frac{cP}{r} + K)/(1 - \frac{1}{\gamma_2})$

Numerical modeling confirms the dotted line of the perpetual debt case in Fig 2. of AST, with $V^* \approx 0.51$ for P=1.0, c=0.05, r=0.1, β =0.08, K=0.2, $\sigma^2 = 0.03$. V* is the point which separates the straight line of V-K and the concave curve approaching 0.5 bond value asymptotically.

It can be proved analytically that the AST solution in step (2) is a solution for large asset values with a normal debt service flow (positive part) of cP. This is done by taking the positive portion of the general solution into the PDE, we have:

$$A_2 V^{\gamma_2}(\frac{1}{2}\sigma^2\gamma_2(\gamma_2 - 1) + (r-\beta)\gamma_2 - r) + S(V_t) - Ur = 0$$

Since the first term in the bracket is zero, $S(V_t) - Ur = 0, i.e., S(V_t) = cP$

There is another way to verify the AST result of V* obtained by 'connecting' the positive half of the general solution of the PDE with strategic debt service state through value matching condition only. It follows that at the trigger point V*, the value of the two states must be the same, hence:

$$V^* - K = A_2 V^{*\gamma_2} + \frac{cP}{r}$$

Defining the value matching function $F(V, A_2) = A_2 V^{\gamma_2} + \frac{cP}{r} - (V - K)$ as a function of V and A_2, we can plot out the value matching function on a two-dimensional grid of V and A_2. According to the value matching condition, the location of the trigger point should be where F=0 (ideally, it should be a flat top). The exercise is conducted using these parameters.

Table 1. Modeling Parameters		
P=1.0	A2	V
K=0.2	Min = -0.1	min = 0.2
c=0.05	Max = 0.0	max = 2.5
r=0.1	dA2 = 0.0001	dV = 0.01
β=0.08	N = 1000	M = 230
σ^2=0.03		

The value of $F(V, A_2)$ is represented in the three-dimensional surface in figure 1a, 1b and 1c.

Figure 1a. Value matching function $F(V, A_2)$, as defined in the text, plotted on a 2D grid of V and A_2. The point where the value of the function equals to zero verify asset value location, V*

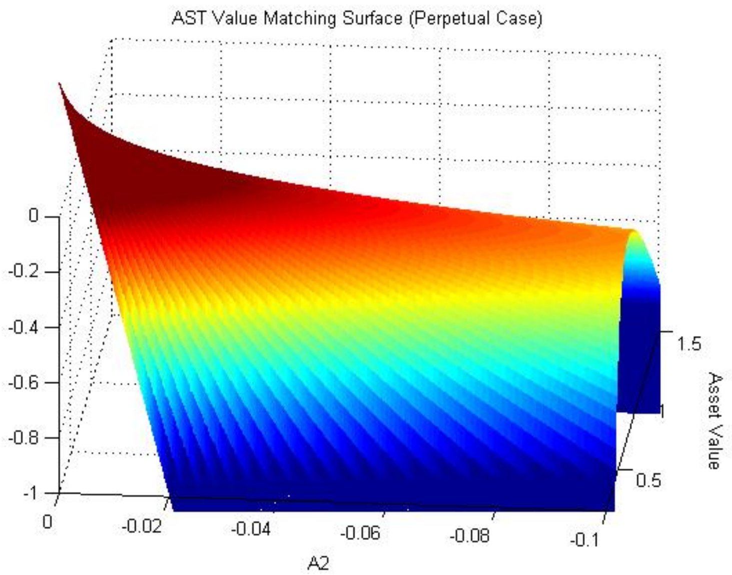


Figure 1b. Value matching function $F(V, A_2)$ plotted on a 2D grid of V and A_2; view from a different angle

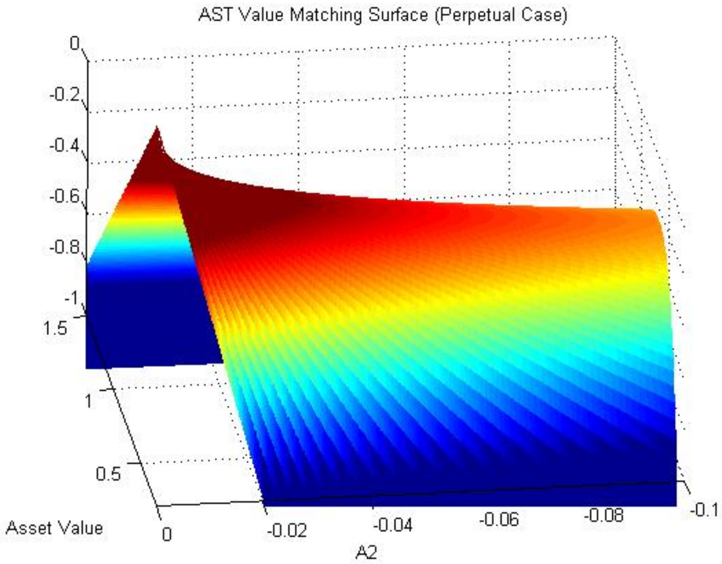
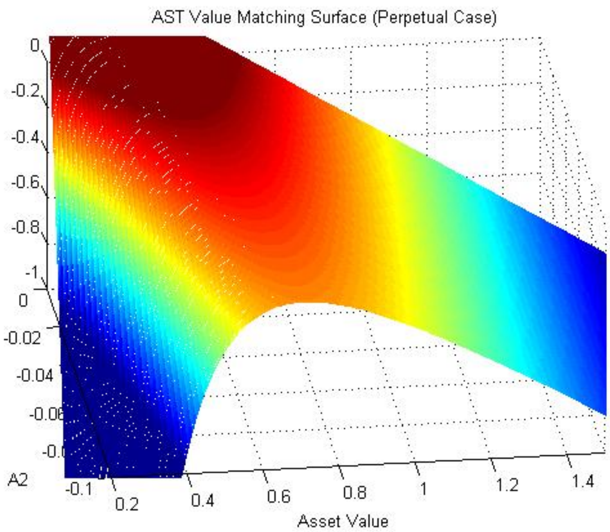


Figure 1c. Value matching function $F(V, A_2)$ plotted on a 2D grid of V and A_2 from yet another angle

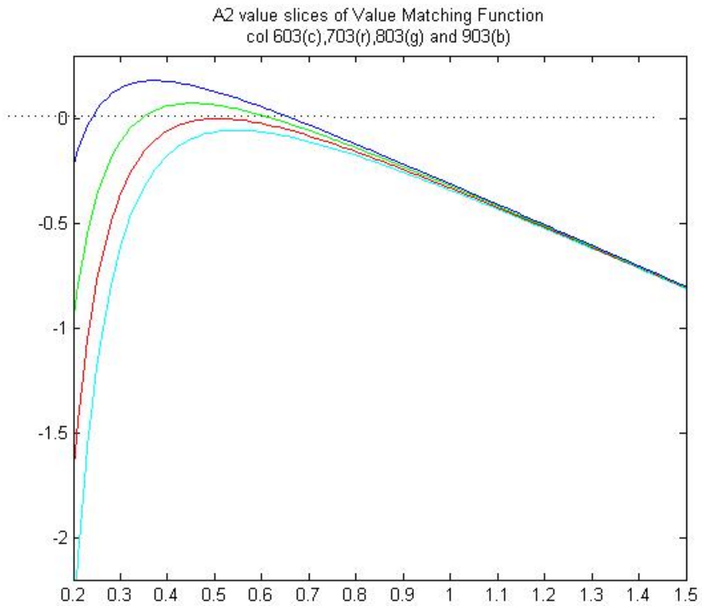


The shape of the value matching surface has been restricted such that any value larger than 1.0 is set to 1.0. Not surprisingly, as the asset value is small and the A_2 term is small, we get a peak in the value matching function. This is due to the fact that we only evaluate the 'positive' term of the general solution of the PDE. To dampen the peak, we need the other half of the general solution to the PDE. It is not very convenient to evaluate if the solution obtained by AST of A*=-0.0297, V*=0.51 is the optimal solution of the whole region. To evaluate, we need to slice the value matching surface along the A_2 direction. Four slices are taken at A_2 (column) position of 603, 703, 803 and 903. These positions correspond to A_2 values of -0.0397, -0.0297, -0.0197 and -0.0097 respectively. The value match curves at these A_2 positions are plotted in figure 2.

As shown, only the red line (AST solution of A2=-0.0297 & V*=0.51) gives tangent zero $F(V,t)$ value at A2=-0.0297 in the neighborhood of V* = 0.5. All other curves do not. All curves converge to a point on the right side of the plot due to the debt value asymptotically approaching 0.5 using AST parameters, and because that the value matching function is debt value subtract V* - K; giving the match value surface a straight line (surface) of slope -1 to the high V end. The curves rapidly descend towards the left as the asset value become small. This is because A2 is negative in AST setting. Nevertheless, in the AST model, after the V* point, the straight line of debt value at V(t)-K takes over so there is no need to worry about the matching value function anymore. Below K, the debt value becomes zero. It is worth mentioning that at V*, the service flow $S^*(V_t) = \beta V - rK$ is approximately 0.02, a drop from the next (V*+dv) asset value which gives service flow of c*P = 0.05.

Thus, I confirm AST results for the perpetual bonds case without using the smooth pasting condition.

Figure 2. 'Optimal' value V* can be found by checking the value of matching function $F(V, A_2)$ at various slices of A2. The 'optimal' value of the matching function is where it is equal to zero. The position of zero on the value matching surface gives the value of A_2 and V*



The general solution to the PDE is 2nd order continuous. The strategic debt service region is characterized by $B(V,t) = V - K$, which is a straight line. The 'positive portion' of the general solution (step 2) for the PDE confirms that at high V, the service flow is cP. In the strategic debt service region, with $B(V,t) = V - K$, the service flow has to jump at the trigger point since the service flow is $S^*(V_t) = \beta V - rK$, no longer cP. The only way to avoid a jump in service flow is to modify the expression of the bond value in the region of strategic debt service and add higher power V terms.

4.2 Scheme for Numerical Modeling

Modeling strategic debt service involves solving for three key unknowns at each step backwards in time: i) the debt value for each possible asset value; ii) the 'trigger point', the highest asset value where strategic debt service applies; iii) the service flow. To numerically implement the scheme, recognize that the debt value, service flow and V* are all functions of V and t, then a discrete grid of M nodes on the asset axis and N nodes on the time axis is set up. In the grid, asset value 0 and maximum asset value correspond to nodes 1 and M respectively; time zero (today) and time of maturity of the debt correspond to nodes 1 and N. Debt and service flow can only have values on the nodes; in between the nodes, there is nothing.

According to AST-2, a scheme has been set up:

i) Start at maturity and move backwards in time. At each time step, calculate the service flow function for M nodes according to AST-1 using the debt value from the last step (the step closer to maturity)

ii) At each time step, apply finite difference scheme operator to obtain the debt value for M nodes at the present time step

iii) Add i) and ii) to obtain debt value at the present time step for M nodes

iv) Determine V* value using the present value of debt using value matching $B(V^*{}_t) = V - K$ and smooth pasting $\frac{\partial B}{\partial V} = 1$

4.3 Numerical Modeling I – Finite Maturity Debt with Explicit Finite Difference

An explicit finite difference scheme is used to calculate the debt value backwards at each node according to the PDE for the provided boundary condition. Since explicit finite difference schemes can be unstable, a small time step is selected. Matlab is used for the calculation. Parameters used in numerical modeling of the service flow function using AST-1 are shown in table 2.

Table 2. Modeling Parameters - Finite Maturity Debt

P=1.0 (Debt Principal)	T (maturity)	V (asset value)
K=0.2 (Liquidation costs)	Min = 0	min = 0
c=0.05 (coupon rate)	Max = 5.0	max = 2.5
r=0.1 (risk free rate)	N = 60001	M = 151
β=0.08 (cash flow generated)		
σ^2=0.03 (std deviation of the	return on firm	Asset)

The boundary conditions are:

1) At maturity (T=5), the debt value equals the service flow function, which follows the payoff function of $\min(V_T - K, P)$;
2) At 'large' asset values (the nodes along the V=2.5 line in the mesh grid), the debt becomes a risk-free bond and its value follows a function of $B = (1 - e^{-r(T-t)})\dfrac{cP}{r} + Pe^{-r(T-t)}$; the service flow equals c*P
3) At the zero asset value line, the debt value is also zero and the service flow is also zero

At each time step, the service flow has been calculated using AST-1 based on the previous step's debt value; the service flow is then fed into AST-2 in the finite difference backwardation to calculate the debt value for the current time. The backwardation process continues until time zero is reached.

Figure 3. Modeling of service flow rate for finite maturity debt using explicit finite difference scheme following the example in AST using AST-1; the size of the grid is 151 (asset axis) by 60000 (time axis)

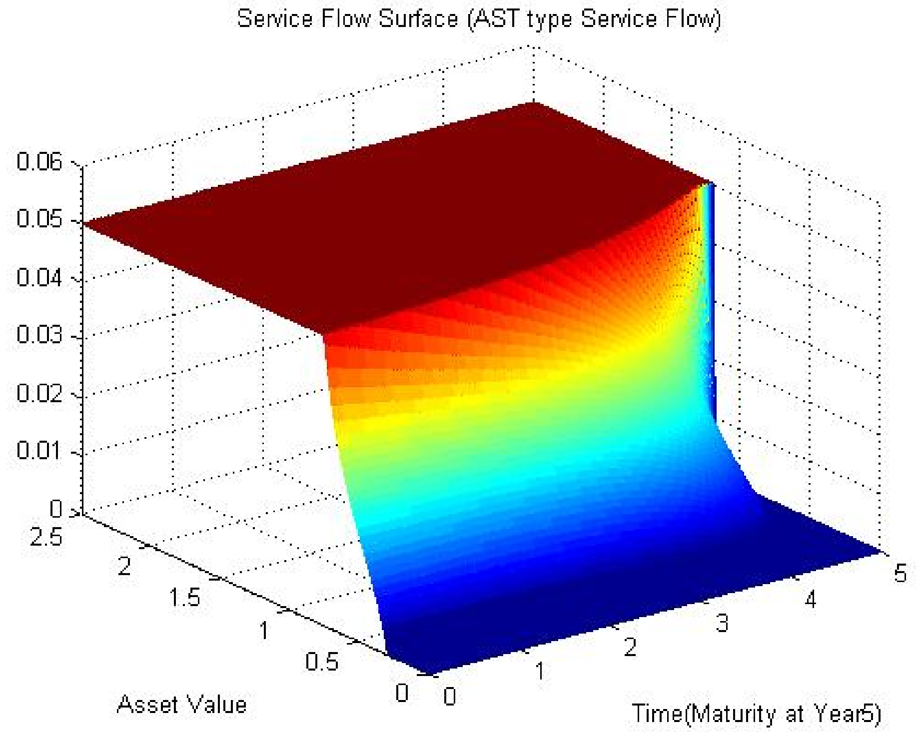
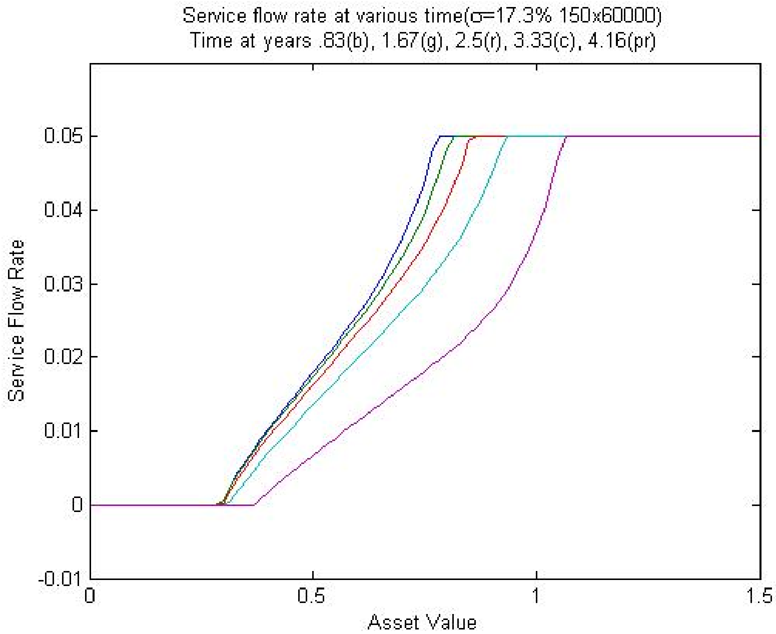


Figure 4. Service flow, $S^*(V_t)$, curves at selected time from time zero with maturity at year 5. The slices were taken at five different times giving a good over view of how service flow progress as the time to maturity changes (f.g. 4 year time is closer to maturity than 1 year time). The curves are colored and initial of different colour is written in the bracket.



To compare the calculated service flow curves with the theoretical value of $S^*(V_t) = \beta V - rK$, these service flows are plotted together. Figures 4a and 4b show similar curves except that volatility is different.

Figure 4a. Calculated service flow curves using AST-1 during debt value calculation using finite difference scheme (σ =17.3%). Service flow curves at various time (from time 0, 1.67 yr means 1.67yr from time 0 and 3.33yr to maturity which is 5yr) are compared with theoretical flow rate $\beta V - rK$

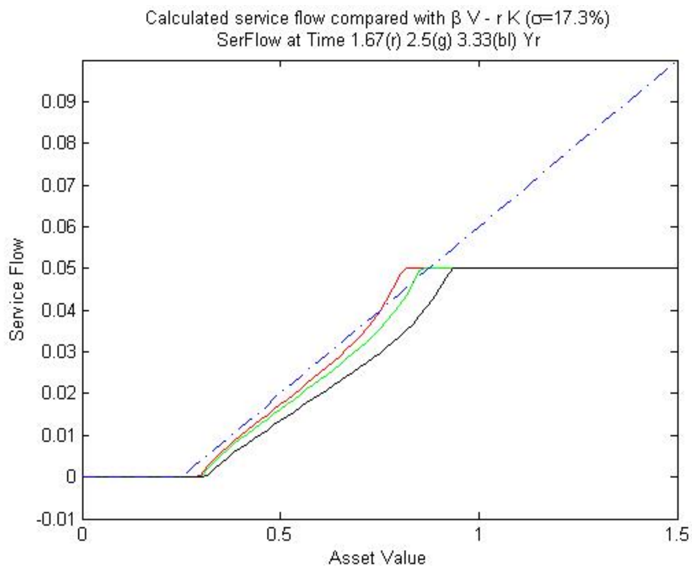
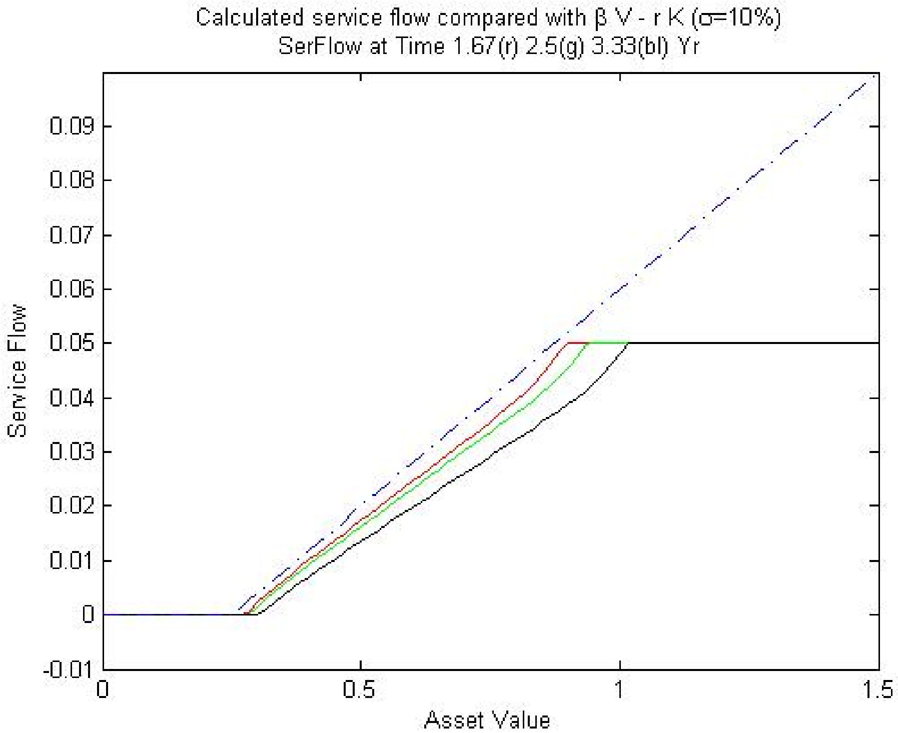


Figure 4b. Calculated service flow curves using AST-1 during debt value calculation using finite difference scheme. σ =10%



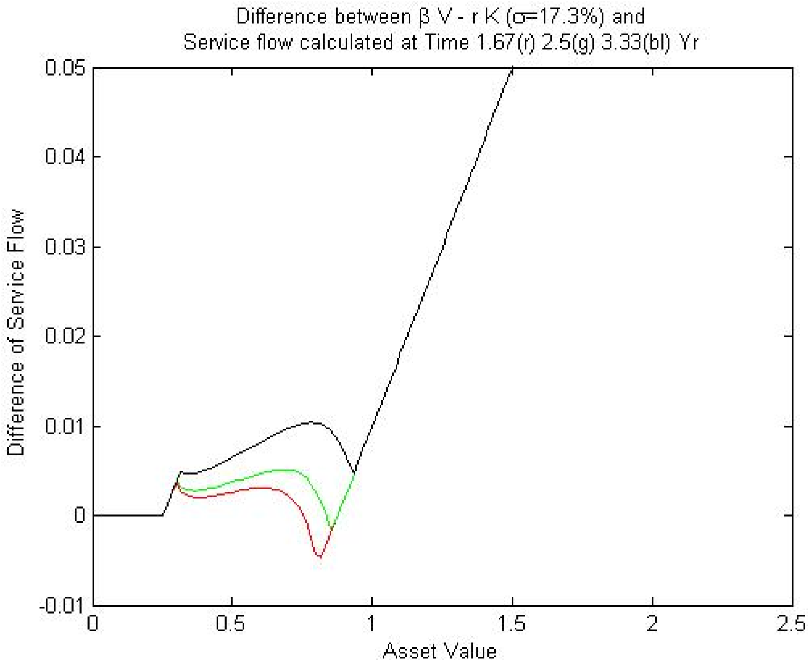
Lower volatility gives 'straighter' service flow curves; however, in both cases, the calculated service flow gives lower service flow levels than that predicted by the theory. Notably, the shorter time to maturity, the larger the difference.

For shorter times to maturity, the debt value still closely resembles the payoff at maturity, hence the difference between V-K and the debt value is small around the region of strategic debt service, hence the difference is smaller and thus pushing the service flow curve to be flatter.

The fact that the service flow curves are not straight and that they deviate from the theoretical linear service flow (density) line significantly (about 0.01 at maximum, and mainly in the strategic debt service region, refer to the figure below) is disturbing, yet understandable. It has already been shown that the binomial term in AST-1 is equivalent to PDE for small time steps. Therefore, when applying AST-1 at a fine time step (the modeling dt=8.3333e-005 year), should provide a good approximation of PDE continuous time and therefore support the analytical form of solution of debt service flow: $S^*(V_T) = \beta V - rK$. The service flow curves become 'straighter' after the volatility is reduced from 17% to 10% causing the approximation of binomial to PDE to improve for the same small time step.

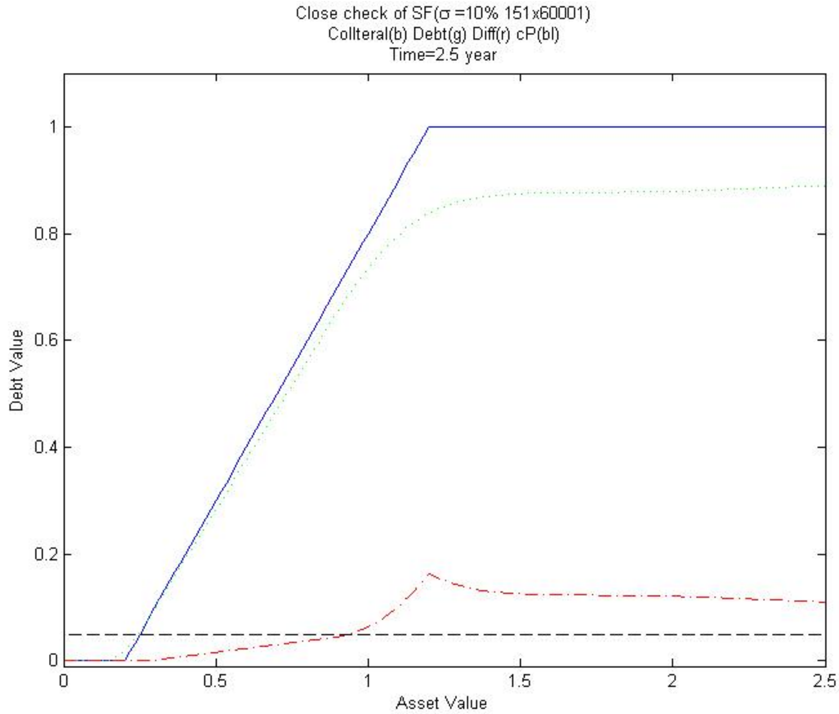
The level of service flow is lower than $S^*(V_T) = \beta V - rK$, meaning that the debt value modeled is higher than expected. One possibility is that it may be due to the finite difference scheme employed, which is modeling sensitive parameters. To test this hypothesis, I calculate the service flow function using a Crank-Nicolson finite difference scheme using the same set of parameters in the next section. Other possibilities explanations could be the negative risk premium, or AST-1 is not suitable for direct numerical implementation. Another way to solve the service flow is then presented.

Figure 4c. Difference between theoretical service flow and the calculated service flow curves using AST-1 during debt value calculation using finite difference scheme. σ =17. 3%



In perpetual case, we observe a jump in service flow from 0.05 to 0.02. For debt with finite maturity, numerical modelling should give different result, but there should be a jump in the modeling. The fact that no jump has been observed in this modeling may suggest that the debt service should be determined differently (rather than use AST-1 directly). To illustrate the valuation process when using AST-1, a plot is provided below in which each key component of AST-1, the $\min(\max(V_t - K, 0), P)$ term, the binomial term, the difference between the afore mentioned to terms, and the c*P term, is modeled. It can be seen that AST-1 is equivalent of truncating the red difference curve at and above c*P level giving the service flow curve the shape that we observe in the modeling.

Figure 4d. Visualize the valuation process of using AST-1to determine service flow rate at time= 2.5year. The blue line is the $\min(\max(V_t - K, 0), P)$ which is lender's payoff when liquidate; the green dotted line is the binomial term which is equivalent to that obtained by finite difference scheme; the difference between aforementioned two terms is plotted in red, and the c*P line is plotted in black. Using AST-1 directly to determine the service flow rate function is equivalent of truncating the difference line (red) at c*P level, leaving the service flow curve that we observe as results by applying AST-1 in numerical modelling.



The calculation of V* is also sensitive to the ' ε ' used and other factors.

4.4 Numerical Modeling I – Finite Maturity Debt with Crank-Nicolson

All other factors are equal; the above experiment is repeated using Crank-Nicolson finite difference (CNFD) scheme. CNFD is more stable than explicit scheme and converges at the rates of $O(\delta t^2) and O(\delta s^2)$, while the explicit scheme converges at $O(\delta t) and O(\delta s^2)$.

Similar to the explicit case, volatility factor of σ =17.3% and 10% is modeled. For σ =17.3%, the debt value at the middle of the grid at time=0 by CNFD is 0.6316, a little higher than the EFD scheme of 0.6306. For σ =10%, the debt value calculated a0.6771 compared to that of EFD scheme of 0.6768 the middle of the grid at time=0 by CNFD

Figure 5a. Calculated service flow curves using AST-1 equation during debt value calculation using Crank Nicolson finite difference scheme. σ =17%

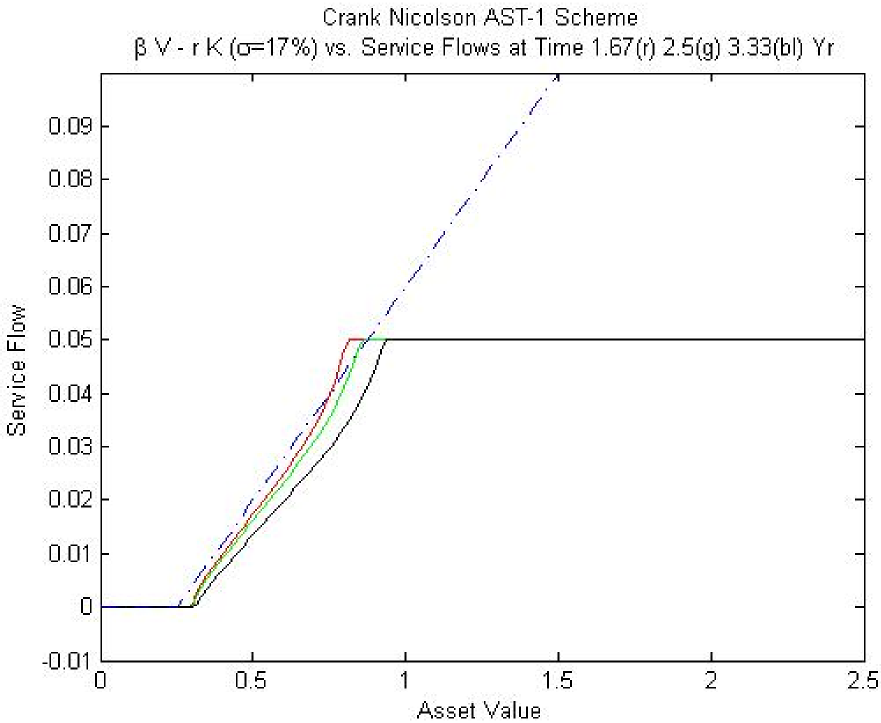
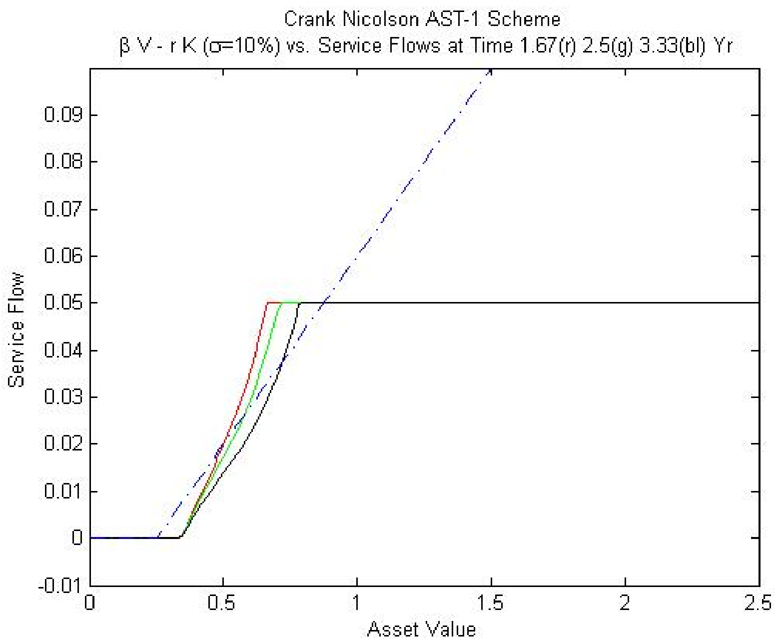


Figure 5b. Calculated service flow curves using AST-1 equation during debt value calculation using Crank-Nicolson finite difference scheme. σ =17%

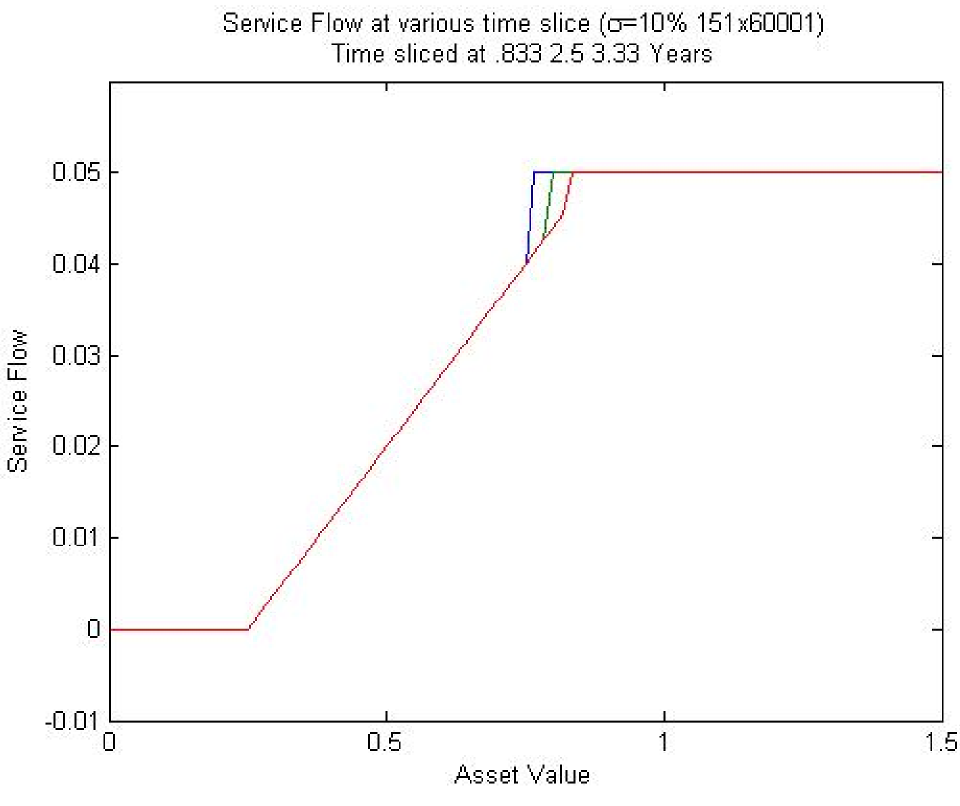


4.5 Numerical Modeling II – Finite Maturity Debt

Using difference approach to calculate the same service flow according to the following steps:

a) Set $V^* = P + K$ for time at maturity
b) At each following time step, use finite difference method to calculate debt value until $V^* + dV$ (dv is the grid size along asset value axis). In this region $S^*(V_t) = cP$, thus the service flow rate is added to the calculated debt value to obtain the present time debt value
c) Check if $B(V^*{}_t) = V - K$ and $\frac{\partial B}{\partial V} = 1$ at the lowest asset value. If yes, V* is found and proceed to the next step; if not let $V^* = V^* - dV$, i.e. move down the grid, and calculate one more debt value down one node, using the finite difference method, test if value matching and smooth pasting conditions are met. If yes, go to step d); if not move V* down by another node and the repeat the process until the V* position is found. Here the monotonic nature of V* decreases as the time to maturity increases is used
d) In the strategic debt service region, $B(V^*{}_t) = V - K$; $S^*(V_t) = \beta V - rK$ and in the region below K, $B = 0, S^*(V_t) = 0$
e) Complete all the time steps until reaching time=0

Figure 6. Calculate debt service flow using an approach which set service flow to $\beta V - rK$ in the strategic debt service phase. The disadvantage of this approach is lacking "strategic debt service element'.



The disadvantage of this scheme is there is little strategic game play in it.

4.6 Calculate the Controlled Service Flow

Previously, results from two different approaches in determining the service flow function were presented. We see that when using the 'relaxed' (i.e. allowing it to apply to all phases of asset values) AST Eq.1, the calculated service flow appears smooth and there is no jump in the service flow. However, using an 'extrapolate-and-test' approach to locate V* and assign an analytical debt service flow rate in the strategic debt service region thereafter gives no room to incorporate the strategic play element in the process; a process where at each time step the lender compares the payoff under liquidation with the payoff where the firm continues operations.

An alternative solution is to apply AST Eq.1 only in the strategic debt service region between K and V*, that is, $V \in [K, V^*]$. For asset values greater than V*, the service flow is c*P. For asset values less than K, the service flow is zero. In between K and V*, the strategic debt service flow is obtained by applying AST Eq.1. This approach is denoted Controlled AST Eq.1 method.

Figure 7. Calculated debt service flow surface using Crank-Nicolson finite difference scheme and Controlled AST Eq.1 method (the vertical blue at the far right end is due to V* initial value is set to P+K. V* quickly drops down to 1.0 level in the next time step and then lower). The red colour 'wall' represents vertical line, i.e., jump from lower level (yellow ring). In the yellow to dark blue transition zone, the service flow is continuous.

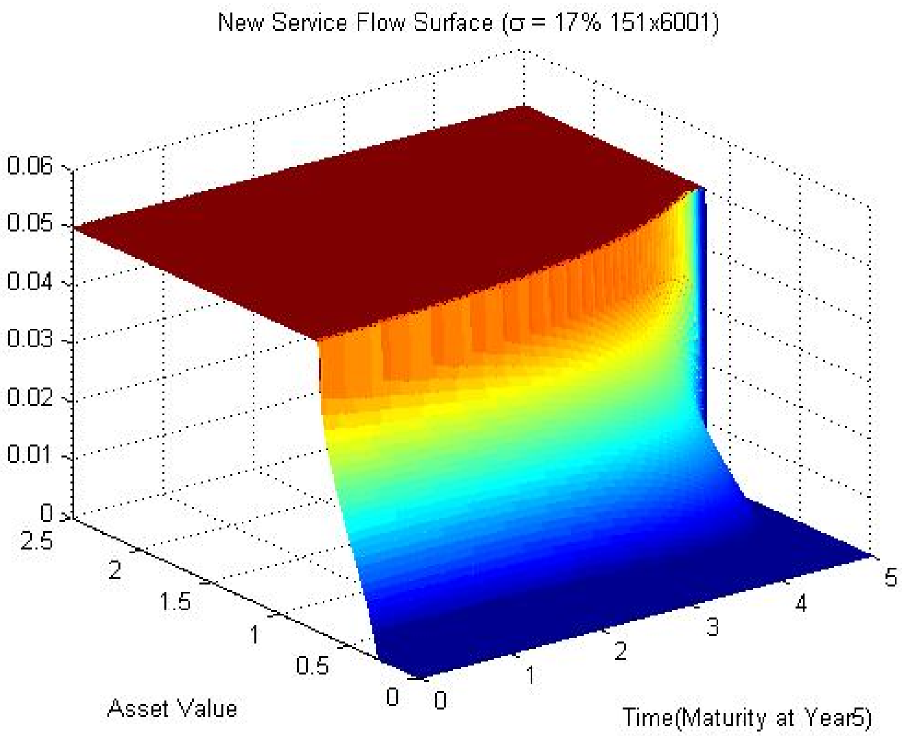


Figure 8. Calculated debt service flow displayed at 3 different times from time 0 (time slices). The linear dashed line represents the theoretical service flow line $S^*(V_t) = \beta V - rK$. Red line is of time 1.67 years from time zero or 3.33 year to maturity, the green line is 2.5 years to maturity and the black line is 1.67 years to maturity. The V* calculation is a very delicate process and many parameters can influence the outcome, these will be discussed in the text.

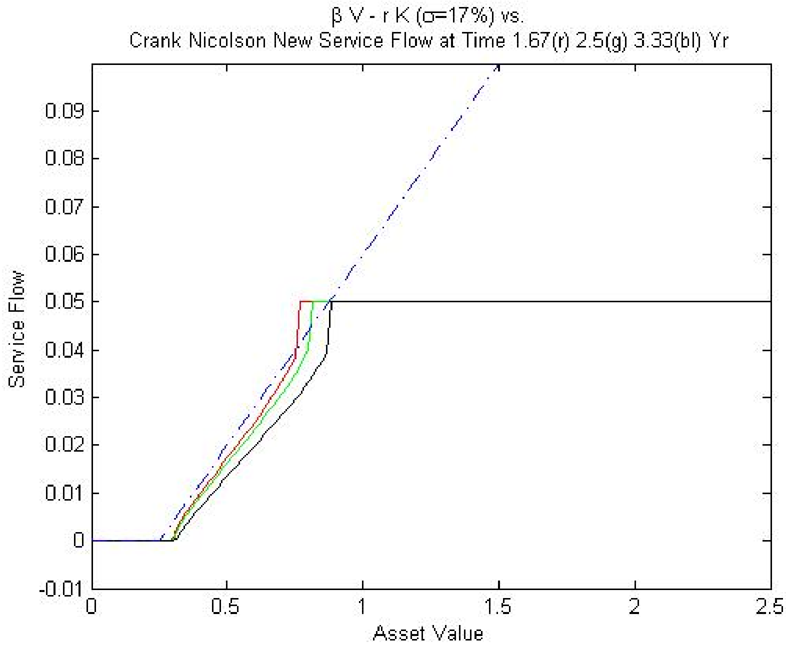
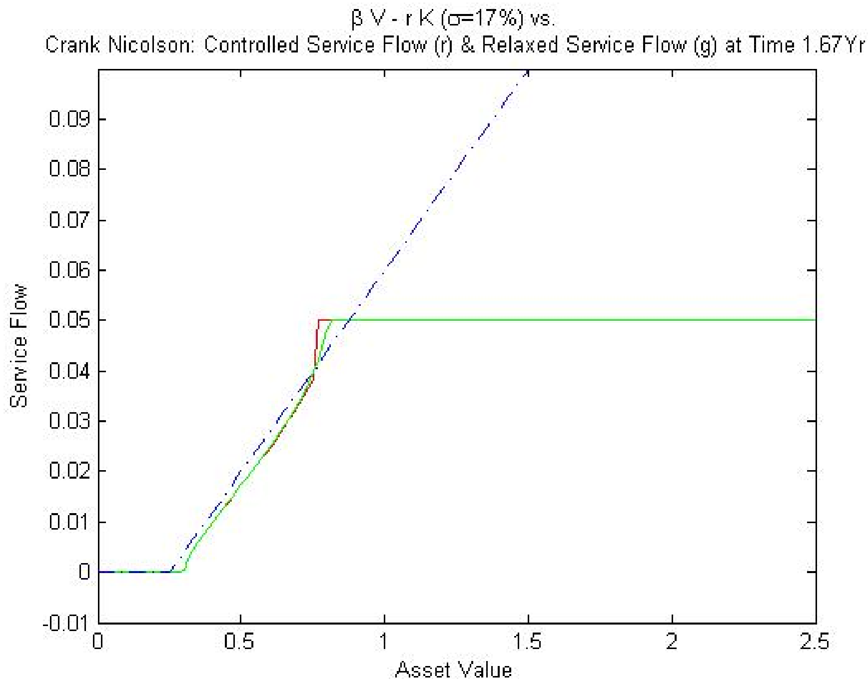


Figure 9. Comparing two different means of determining the debt service flow: one applies AST Eq.1 in all asset value regions (relaxed) which is in green colour and shows no jump of service flow at V*. The alternative is apply AST Eq.1 strictly within [K,V*] region, i.e. ‘controlled’ application.



5. Calculate the V*

The calculation of V* is a highly delicate process. The following parameters may impact the value of V* wildly and consequently impact the service flow function, even though the service flow is a small fraction of the debt value.

1) Numerical scheme for calculating the first derivative
2) Size of grid on the asset axis
3) ε which is the small number in validating that a numerical scheme is close to the true first derivatives
4) Volatility of the return on firm's asset

The impact of these factors is examined individually with modeling conducted.

5.1 Numerical Scheme for Calculating the First Derivative

As the calculation of V* involves smooth pasting, we need to evaluate the way we calculate the first derivative of debt value with respect to firm's asset value. Three different approaches of calculating the first derivative are tested. The first one is a standard finite difference approach to approximate the first derivative:

$$f'(x_i) = \lim_{h\to 0} \frac{f(x_i + h) - f(x_i)}{h}$$

Instead of letting $h \to 0$, finite difference uses first order forward difference

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

as an approximation for the first derivative; where h is the size of the grid and $f(x_{i+1}), f(x_i)$ are functions at two adjacent grids. The error of the first order forward difference is of the order of h.

The second approach is to use the central difference scheme to approximate the first derivative:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

The error of the central difference scheme is of the order of h^2

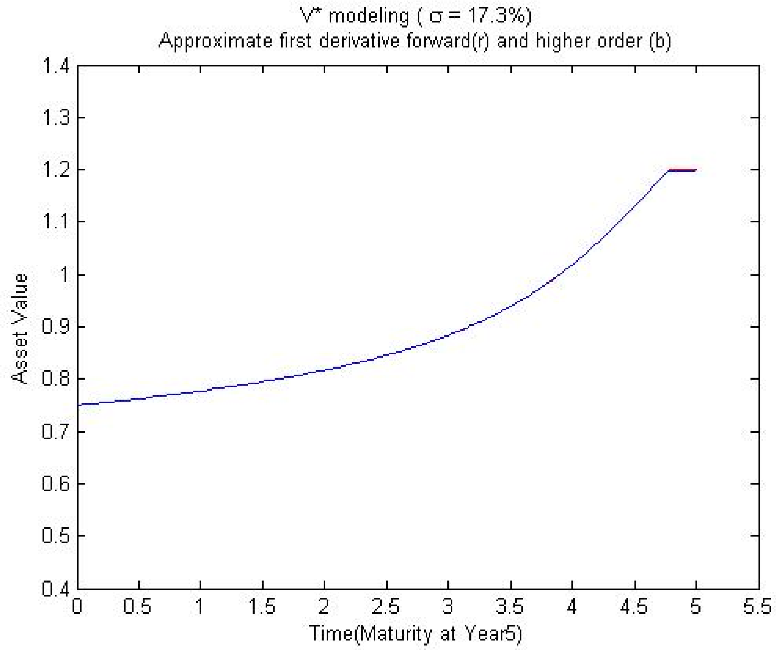
The third approach is to use more grids in finite difference in order to better approximate the first derivative. One of these being:

$$f'(x_i) \approx \frac{-f(x_{i-3}) + 6f(x_{i-2}) - 18f(x_{i-1}) + 10f(x_i) + 3f(x_{i+1})}{12h}$$

This error of this finite difference scheme is of the order of $O(\frac{1}{20}h^4)$

Using these different schemes to approximate the first derivatives generate more or less the same V* results. The finite difference scheme used seems to be of less importance in determining V* .

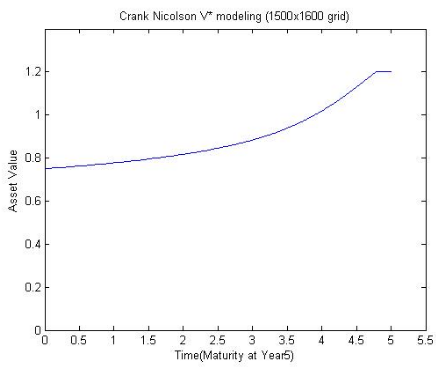
Figure 10. Comparing different methods to approximate first order derivative in determining V*. Here the forward finite difference scheme is compared to that of a higher order of approximation. There is almost no difference between the two algorithms.

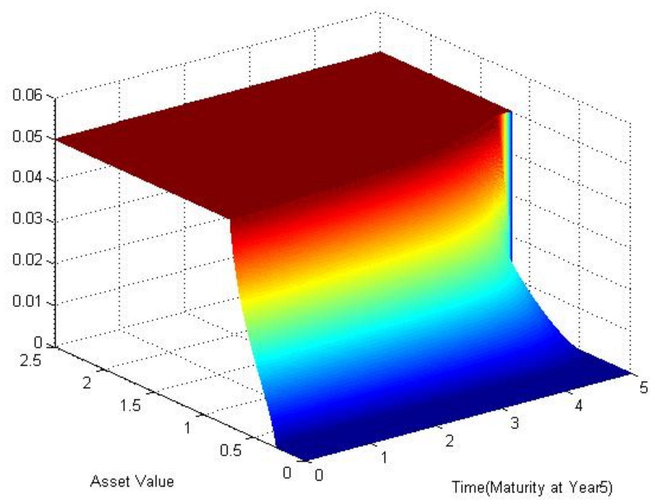


5.2 Size of grid

The size of the grid is of less importance as well, as shown in the results using 150 points along the asset direction (0.0 – 2.5) or 1500 points in the asset direction. The results confirm the grid size is of less importance to the calculation of V*.

Figure 11. Reducing the grid interval (or increasing the number of nodes) appears to enable a smoother V* curve, but does not change the value of it. The service flow rate surface and the calculated V* curves are shown in 2 plots.





5.3 The Small Number

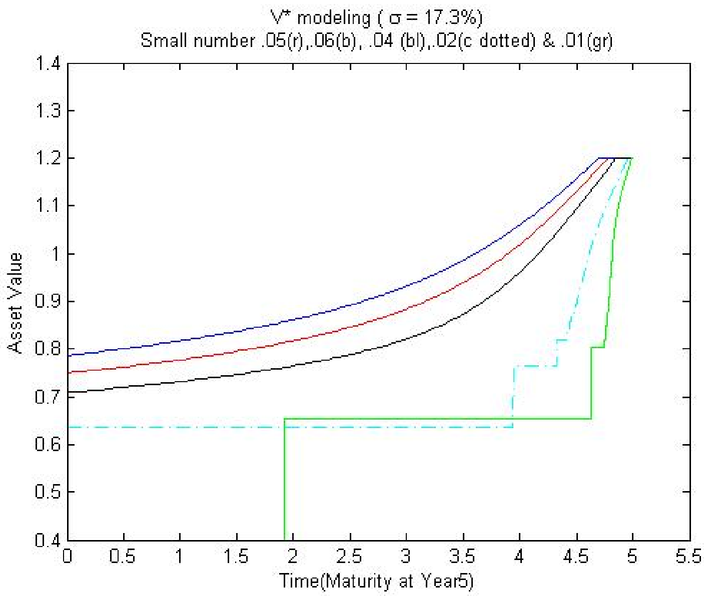
The small number used is single most important factor in calculating of V*. The small number (ε) is used as a threshold for numerical approximation of value matching and smooth pasting conditions.

$B(V_i) - (V_i + K) \le \varepsilon$ for value matching and

$B(V_{i+1}) - B(V_i) - h \le \varepsilon$ for smooth pasting

Using small numbers ε =0.01, 0.04, 0.05 and 0.06 in numerical modeling under the Crank-Nicolson scheme to model V* curve, four different V* curves are obtained:

Figure 13. Modeling of V* curves using different small numbers in numerical approximation of value matching and smooth pasting conditions. The small numbers are .06 (blue line), .05(red line), .04(black line) .02 (c and dotted line) and .01(green line)

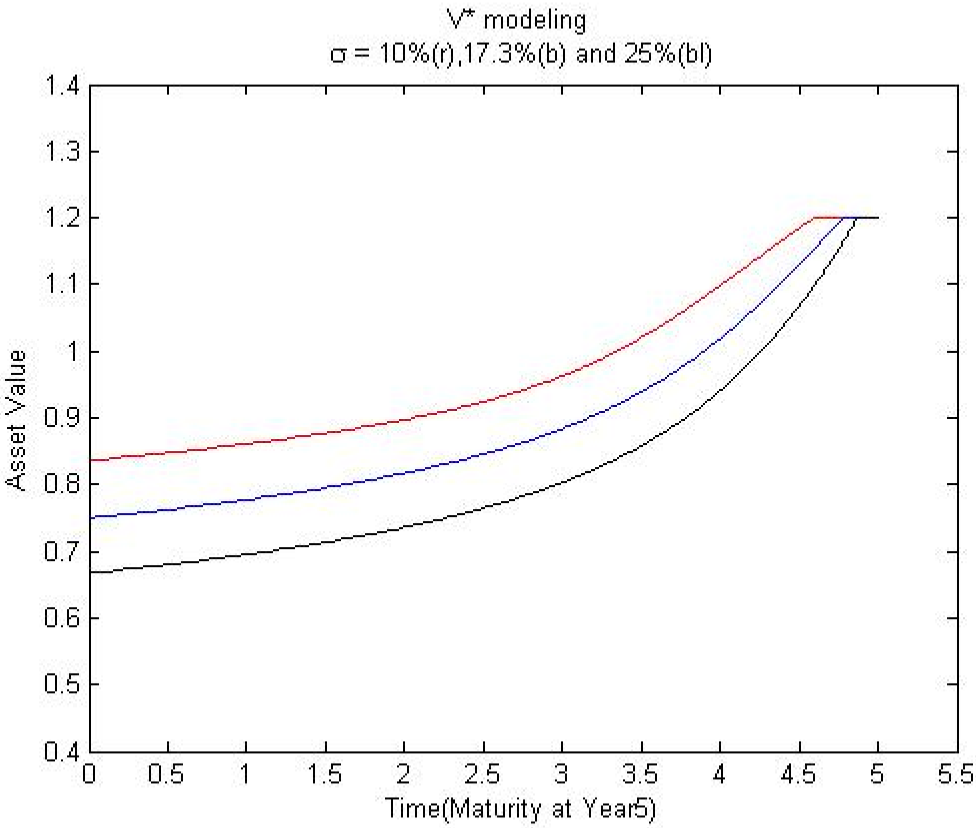


It is a concern that V* calculation is highly sensitive to the choice of ε . In the experiment, there are 1500 grids along the asset value axis with asset value ranges of 0 – 2.5, giving a grid spacing of 0.017. It appears that the small number needs to be set higher than the size of the grid; however, there seems to be no requirement on the upper limit of the small number.

5.4 Volatility

Volatility impacts V* value, where greater volatility reduces the V* value. This is shown in the figure below. For perpetual debt, the analytical solution indicates that there exists a monotonic relationship between V* and volatility; as volatility increases, the value of V* reduces. It seems for debt of finite maturity, such a relationship still holds.

Figure 14. Monotonic relationship between volatility and V*



6. Conclusion

I would like to conclude on these points:

1) Alternative approach confirms the location of V* and amplitude A2 for perpetual debt of AST model;
2) Debt service flow calculation follows Eq.1 of AST provided that it is applicable strictly within the region of $V \in [K, V^*]$. When this condition is relaxed, the debt service flow no longer displays a 'jump' at the location of trigger point (V*)
3) Calculation of V* is a delicate process which depends on a few modeling parameters. Of these modeling parameters, the selected small number for validating approximation of value matching and smooth pasting is most important.
4) There exist a monotonic relationship between volatility and V* value

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