

From Weakly Reflexive Paraconsistent Inferences To Modal Logics

by

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Abstract

The study of paraconsistent inference is the study of inference that prohibits reasoning from inconsistent premises to arbitrary conclusions. We will first examine a pair of paraconsistent inference relations defined by Rescher. The chief interest in Rescher's work lies the fact that Rescher's inference relations sanction non-arbitrary reasoning in the presence of explicit self-inconsistencies. The technique is to refuse to assign any inferential role to any explicitly self-inconsistent premiss. From a practical standpoint, especially when greater tolerance for inconsistency is desired, Rescher's approach is an improvement over Jennings-Schotch's approach to paraconsistent inference which is inferentially explosive in the presence of self-inconsistent premiss.

After exploring the structural and 'preservational' properties of Rescher's inference relations, we will agglomerate Rescher's approach with Jennings-Schotch's approach to paraconsistent inference. A new inference relation will be defined and its connection with modal logics will be investigated.

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Dedication

For Fiona

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Chapter 1

Inference Relations

Let \mathcal{L} be a formal language and Φ be the set of well formed formulae of \mathcal{L} . It is well known that Tarski's consequence operation, $\mathbf{Cn}: 2^\Phi \rightarrow 2^\Phi$ (see [35]) satisfying

T1 Inclusion $\Gamma \subseteq \mathbf{Cn}(\Gamma)$

T2 Monotony $\Gamma \subseteq \Delta \Rightarrow \mathbf{Cn}(\Gamma) \subseteq \mathbf{Cn}(\Delta)$

T3 Idempotence $\mathbf{Cn}(\Gamma) = \mathbf{Cn}(\mathbf{Cn}(\Gamma))$

is in fact structurally isomorphic to Scott's relation defined over finite sets of sentences: $\vdash_s \subseteq 2^\Phi \times 2^\Phi$ satisfying

S1 Reflexivity $\Gamma \cap \Delta \neq \emptyset \Rightarrow \Gamma \vdash_s \Delta$

S2 Monotonicity¹ $\Gamma \vdash_s \Delta \Rightarrow \Gamma, \Gamma' \vdash_s \Delta, \Delta'$

S3 Transitivity $\Gamma, \alpha \vdash_s \Delta$ and $\Gamma \vdash_s \alpha, \Delta \Rightarrow \Gamma \vdash_s \Delta$

In [34], Scott not only shows that there is no loss in passing from \mathbf{Cn} theory to \vdash_s theory, in a more general setting, he shows that *any* formal system with deductive apparatus characterized by S1–S3 provides an adequate axiomatization of *conditional assertion*, where Γ conditionally asserts Δ iff every valuation (in some class of intended valuation) which makes every member of Γ true, makes some member of Δ true (i.e. every model of Γ is a model of δ , for some $\delta \in \Delta$)

The beauty of Scott's approach lies in the fact that it establishes *meta-theoretic* results with respect to a *class* of formal systems. Lindenbaum's lemma, for instance, can be easily established within the setting of \vdash_s (see [34] p.416), but in defining \vdash_s , we make no assumption about the closure conditions with respect to the sentential connectives in Φ . Φ is just

¹I will follow the standard convention to use Γ, Δ to denote $\Gamma \cup \Delta$ and Γ, α for $\Gamma \cup \{\alpha\}$.

the set of wffs of some formal language. So any formal system which has the structural features of S1–S3 will be complete with respect to the semantics of conditional assertion. The generality of Scott’s approach suggests an important strategy for studying formal systems quite independently of syntactic features of particular systems. In general, we can characterize the structural (or *connective-independent*) rules of a class of systems quite independent of its logical (or *connective-dependent*) rules. By tinkering with the structural rules we may obtain formal systems that are useful for different purposes. To mention some recent developments, computer scientists have studied formal systems that are deductively *non-monotonic* ([4], [7], [8], [13], [24], [26], and [30]). There, the motivation is to provide a formal representation of certain pattern of common sense reasoning. Since ordinary reasoners often revise their views in light of new information, the conclusion an agent draws at a given time need not always increase monotonically with that agent’s initially held assumptions. So some weakening of S2 seems to be the most natural route to go (see [11], [12], and [25]).

Our concern here, however, is not the issue of non-monotonicity but something closely related to it, namely, *paraconsistency*. Since newly acquired information is not always compatible with initial data, a reasoner must take measure to guard against drawing trivial conclusions. Revising one’s initial data to restore consistency may be an option available, but on occasion it is more important to maintain the integrity of the original data – perhaps the incompatible data is irrelevant to one’s overall project. On other occasions, it may even be ‘desirable’ to have inconsistencies in one’s database; for instance, inconsistencies may be deployed as directives to guide learning or inconsistencies in a taxpayer’s records can be used as reason to prompt further investigation. The important point is that many ordinary circumstances require us to reason in the presence of inconsistencies. Classical logics, when viewed as reasoning systems, do not offer a satisfactory representation of such reasoning patterns. To review the fundamental notions in classical logics, let Φ be the set of wffs of some formal language \mathcal{L} , and for ease of exposition let the classical provability relation \vdash be the single conclusion fragment of \vdash , satisfying S1–S3. In classical logics, a set is said to be inconsistent iff

$$(1) \Gamma \vdash \alpha \text{ and } \Gamma \vdash \neg\alpha, \text{ for some } \alpha \in \Phi$$

A set is said to be absolutely inconsistent iff

$$(2) \Gamma \vdash \alpha, \forall \alpha \in \Phi$$

Given the usual metatheoretic results of classical logics, inconsistent sets, absolute inconsistent sets and unsatisfiable sets (sets which have no model) are co-extensional collection of sets. On a very crude level, the classical \vdash allows us to discriminate between satisfiable sets and unsatisfiable sets. We can say that the classical \vdash *preserves* the satisfiability (or some metalinguistic property P) of a set Γ in the sense that

(Pres) If Γ is satisfiable (has P), then $\Gamma, \mathbf{Cn}(\Gamma)$ is also satisfiable (has P)²

Furthermore, we can say that the classical \vdash *distinguishes* a collection of sets under the property P in the sense that it preserves P for *that* collection of sets. On this preservation-theoretic approach, two related tasks are constitutive of the study of inference: the first is the identification of important metalinguistic properties of sets, and the second is the discovery of inference relations that preserve these properties. Provisionally, no restriction is imposed on the kind of properties or inference relations to be studied, except the properties in question must be non-monotonic, i.e. properties that are not closed under supersets, and the relations must be subsets of $2^\Phi \times \Phi$. Generalizing this methodological strategy, we can either begin by identifying a metalinguistic property of sets and ask what sort of inference relations will preserve such a property, or alternatively, begin by identifying an inference relation and ask what kind of properties are preserved under such an inference.

Deploying these conceptual resources to study classical inference is revealing. On the one hand, the classical \vdash preserves the satisfiability of a set Γ by ensuring that the set of models for $\Gamma, \mathbf{Cn}(\Gamma)$ is exactly the set of models for Γ , i.e.

$$\mathfrak{M}_{\Gamma, \mathbf{Cn}(\Gamma)} = \mathfrak{M}_\Gamma$$

In fact, this is a corollary of the soundness and completeness results of classical logics. On the other hand however, the classical \vdash does not distinguish different ways in which a set may fail to have a model. In particular, a set may be unsatisfiable because,

1. some wff in Γ is unsatisfiable
2. no wff in Γ is satisfiable
3. some satisfiable subsets, \mathbf{a}, \mathbf{b} of Γ are such that $\mathbf{a} \cup \mathbf{b}$ is unsatisfiable

²Since \mathbf{Cn} is a Tarski's consequence operation, $\Gamma, \mathbf{Cn}(\Gamma) = \mathbf{Cn}(\Gamma)$. But this is so because, the classical \vdash is reflexive. In general this does not hold for nonreflexive inference relations. I will continue to use $\Gamma, \mathbf{Cn}(\Gamma)$ for future comparison.

In our ordinary inferential life, encounter with 2 may be rare, but arguably 1 and 3 are not uncommon. An otherwise informative database, for instance, may be infected by a computer virus that generates arbitrary strings of incoherent sentences. In a court of trial or in a scientific research, incompatible evidence may be used to support competing claims. Since the classical \vdash does not distinguish between 1, 2, and 3, it trivializes the conclusion of all unsatisfiable (or inconsistent) sets indiscriminately. To use a colourful metaphor, all unsatisfiable sets are inferentially explosive under the classical \vdash – every sentence in the formal language can be inferred from an unsatisfiable set.³ Naturally, our initial effort must go towards finding inference relations which honour these distinctions. In particular, we want our inference relations to tolerate inconsistency at least in the sense of 3. Our starting point will be Rescher’s work in this area ([27], [28], and [29]), eventually we will venture into the theoretical terrain explored by Jennings and Schotch, especially in the context of the connection between paraconsistent inferences and modal logics ([19], [31], [33]). Rescher, like Jennings and Schotch, approaches the issue of paraconsistency by retaining substantial classical procedures and notions. The chief interest in Rescher’s work lies the fact that Rescher’s inference relations target 2 as the only inferentially deviant case and thus tolerate inconsistency in the sense of 1 and 3. In effect, Rescher’s inference relations permit an agent to reason in the presence of explicit self-inconsistencies.⁴ The technique, or rather the trick, is to refuse to assign any inferentially role to any explicitly self-inconsistent premiss. From a practical standpoint, especially when greater tolerance for inconsistency is desired, Rescher’s approach is an improvement over Jennings-Schotch’s approach to paraconsistent inference which targets 1 and 2 as the inferentially deviant case and as a result the Jennings-Schotch’s procedure is inferentially explosive in the presence of explicit self-inconsistencies. As we shall see however, Rescher’s method requires us to adopt weaken versions of reflexivity for our inference relations. This posts a very interesting problem when we agglomerate Rescher’s approach with Jennings-Schotch’s approach to modal paraconsistent logics. We defer discussion of the problem to subsequent chapters. We will first introduce the basic logical machinery and fill in the philosophical motivation along the way.

³I will continue to use ‘unsatisfiable sets’ and ‘inconsistent sets’ interchangeably as co-extensional terms. The traditional distinction between model theory and proof theory is maintained throughout.

⁴A wff, α , is self-inconsistent iff $\alpha \vdash \perp$ where \perp is the truth constant known as the falsum. There are two independent ways we may define \perp : we can treat \perp as a special symbol in the object language or we can regard \perp as a meta-linguistic variable for the negation of a theorem. In the current context, the usage of \perp is neutral to these two characterizations.

1.1 Maximal Consistent Subsets

In [27], [28], and [29], Rescher outlines a strategy to generate inference relations that are genuinely paraconsistent, i.e. from a classically inconsistent or unsatisfiable set of wffs, not every wff in the language is inferable. Rescher's strategy is to define a pair of relations in terms of the classical consequences of *maximal consistent subsets* (m.c.s) of an inconsistent set. The definition of m.c.s. is:

Definition 1.1 \mathbf{a} is a maximal consistent subset of Γ iff (1) \mathbf{a} is a non-empty subset of Γ , (2) \mathbf{a} is consistent, i.e. $\mathbf{a} \not\vdash \perp$, and (3) if $\alpha \in \Gamma - \mathbf{a}$, then $\mathbf{a}, \alpha \vdash \perp$

At bottom, the notion of maximal consistent subset is really a generalization of the notion of *maximal consistent set*. To see this, we simply observe that every maximal consistent set is a maximal consistent subset of Φ . To enrich our theoretical resources, we say that Γ is *m.c.s. undefinable* if no m.c.s. exist for Γ . Since the standard Henkin model for classical logics is usually defined in terms of the 'canonical universe' or the set of all maximal consistent sets, for an arbitrary set Γ we define the 'universe of Γ ' as $\mathcal{U}_\Gamma = \{\mathbf{a}: \mathbf{a} \text{ is m.c.s. of } \Gamma\}$, if Γ is m.c.s. definable, else $\mathcal{U}_\Gamma = \emptyset$. Observe that if $\Gamma = \Phi$, \mathcal{U}_Γ is simply the canonical universe of classical logics. A non-empty subset \mathbf{a} is said to be Γ -maximal if \mathbf{a} satisfies condition 3. In addition we offer the following classification of wffs:

Definition 1.2 Elements of a set Γ are classified as follows:

- (1) A wff α is an innocent bystander (i.b.) of Γ iff $\alpha \in \bigcap_{\mathbf{a} \in \mathcal{U}_\Gamma} \mathbf{a}$.
- (2) A wff α is a witness of Γ iff $\alpha \in \bigcup_{\mathbf{a} \in \mathcal{U}_\Gamma} \mathbf{a}$
- (3) A wff α is a culprit iff α is not a witness of Γ

Given definition 1.2,⁵ it is straightforward to verify that any member of Γ which is a theorem of *PL* is also an i.b. of Γ . Similarly, no member of Γ which is self-inconsistent is a witness of Γ . Now we can define two inference relations (\vdash_i for inevitable provability and \vdash_w for weak provability) and their corresponding consequence sets.

Definition 1.3 \vdash_i and \vdash_w are defined as follows:

- (1) $\Gamma \vdash_i \alpha$ iff $\forall \mathbf{a} \in \mathcal{U}_\Gamma, \mathbf{a} \vdash \alpha$

⁵'Innocent bystander' is a term used by Rescher. Rescher also uses the term 'culprit' for wffs which are co-extensive with wffs I call witnesses. I hope the terminological juggling is not so dazzling that we lap into incoherence. A self-consistent sentence should at least be considered to be a piece of potential evidence. The real villain should be the self-inconsistent sentence.

(2) $\Gamma \vdash_w \alpha$ iff $\exists \mathbf{a} \in \mathcal{U}_\Gamma, \mathbf{a} \vdash \alpha$

(3) the i -consequence set of Γ , $\mathbf{Cn}_i(\Gamma) = \{\alpha: \Gamma \vdash_i \alpha\}$

(4) the w -consequence set of Γ , $\mathbf{Cn}_w(\Gamma) = \{\alpha: \Gamma \vdash_w \alpha\}$

From an inferential perspective, \vdash_i and \vdash_w are both inferentially conservative with respect to classical consistent sets. By this we mean that they both retain a substantial fragment of the classical \vdash , in particular, where a set Γ is classically consistent, $\mathbf{Cn}(\Gamma) = \mathbf{Cn}_i(\Gamma) = \mathbf{Cn}_w(\Gamma)$. More interestingly however, where a set Γ is m.c.s. definable but classically inconsistent, it is straightforward to verify that $\mathbf{Cn}_i(\Gamma) = \bigcap_{\mathbf{a} \in \mathcal{U}_\Gamma} \mathbf{Cn}(\mathbf{a})$ and $\mathbf{Cn}_w(\Gamma) = \bigcup_{\mathbf{a} \in \mathcal{U}_\Gamma} \mathbf{Cn}(\mathbf{a})$. In effect, we can view \vdash_i and \vdash_w as reasoning strategies of two distinct types. Philosophically, we can characterize \vdash_i as a species of skeptical inference since it regards any conflicting claims as suspect and thus restricts a reasoner to draw conclusions only from the innocent bystanders of a set. If the innocent part of a set is empty, then \vdash_i licenses us to infer only tautologies from the set. \vdash_w , on the other hand, can be characterized as a species of liberal inference since it permits a reasoner to draw conclusions from any witness or consistent cluster of witnesses of a set. So in the presence of both α and $\neg\alpha$ in a set Γ , where α and $\neg\alpha$ are both witnesses of Γ , \vdash_w licenses us to infer both α and $\neg\alpha$ (but not $\alpha \wedge \neg\alpha$). Correspondingly, \vdash_w is also a species of non-adjunctive inference where arbitrary adjunction of $\alpha \wedge \beta$ from the set $\{\alpha, \beta\}$ is prohibited. Assuming that we have the usual sentential connectives and p, q, r are atomic sentences in our language, we can illustrate the difference between \vdash_i and \vdash_w with the following example:

Example 1. Let $\Gamma = \{p, p \rightarrow q, \neg q, r\}$

$\mathcal{U}_\Gamma = \{\{p, p \rightarrow q, r\}, \{p \rightarrow q, \neg q, r\}\}$

Clearly, r is an i -consequence of Γ since it is in every m.c.s. of Γ . But neither p nor $\neg p$ are i -consequences of Γ even though they are w -consequences of Γ .

From a preservation-theoretic standpoint, \vdash_i and \vdash_w are preservationally equivalent to \vdash for classically consistent sets. This follows immediately from the fact that $\mathbf{Cn}(\Gamma) = \mathbf{Cn}_i(\Gamma) = \mathbf{Cn}_w(\Gamma)$ for any classically consistent Γ . What we want to make sure is that \vdash_i and \vdash_w distinguish the right collection of inconsistent sets under the right properties. As a first step towards showing this, observe that a version of Lindenbaum's lemma holds for m.c.s.:

Theorem 1.1 *A set Γ is m.c.s. undefinable iff every finite subset of Γ is m.c.s. undefinable.*

Proof. The lemma holds trivially if Γ is finite. So assume that Γ is countably infinite.

(\Leftarrow) Suppose Γ is m.c.s. definable. Then $\exists \mathbf{a} \subseteq \Gamma$: \mathbf{a} is a m.c.s. of Γ . Hence $\mathbf{a} \not\vdash \perp$. If \mathbf{a} is finite, then clearly it is m.c.s. definable. If \mathbf{a} is infinite, then by compactness of \vdash , every finite subset of \mathbf{a} is consistent. So some finite subset of Γ is m.c.s. definable.

(\Rightarrow) Assume that $\exists \mathbf{b} \subseteq \Gamma$: \mathbf{b} is finite and m.c.s. definable. Then $\exists \mathbf{a} \subseteq \mathbf{b}$: $\mathbf{a} \in \mathcal{U}_{\mathbf{b}}$. We show that $\exists \mathbf{a}^+ \subseteq \Gamma$: $\mathbf{a} \subseteq \mathbf{a}^+$ and $\mathbf{a}^+ \in \mathcal{U}_{\Gamma}$. Let $\alpha_0, \dots, \alpha_n, \dots$ be enumeration of the wffs in $\Gamma - \mathbf{a}$. Define by recursion on the natural numbers the following sets:

$$\begin{aligned} \Sigma_0 &= \mathbf{a} \\ \Sigma_1 &= \begin{cases} \Sigma_0, \alpha_0 & \text{if } \Sigma_0, \alpha_0 \not\vdash \perp \\ \Sigma_0 & \text{otherwise} \end{cases} \\ &\quad \vdots \\ \Sigma_{n+1} &= \begin{cases} \Sigma_n, \alpha_n & \text{if } \Sigma_n, \alpha_n \not\vdash \perp \\ \Sigma_n & \text{otherwise} \end{cases} \\ &\quad \vdots \end{aligned}$$

Let $\Sigma = \bigcup_{0 \leq i < \infty} \Sigma_i$. Clearly, $\mathbf{a} \subseteq \Sigma \subseteq \Gamma$ and $\Sigma \in \mathcal{U}_{\Gamma}$. Hence, Γ is m.c.s. definable. \blacksquare

Using a construction similar to that of the preceding proof, the following result follows immediately.

Lemma 1.1 *Extension Lemma: every consistent subset of a set Γ has a Γ -maximal m.c.s. extension.*

Putting these two results together, it is not difficult to see the equivalence of the following claims:

- Γ is m.c.s. undefinable iff every finite subset of Γ is m.c.s. undefinable.
- Γ is m.c.s. undefinable iff every *singleton* subset of Γ is m.c.s. undefinable.
- Γ is m.c.s. undefinable iff every member of Γ is self-inconsistent.

In effect, these claims specify the defeasible condition for \vdash_i and \vdash_w as promised earlier; they imply that \vdash_i and \vdash_w inferentially malfunction just in case the premise set contains only self-inconsistent wffs, i.e. when a set is m.c.s. undefinable. In such a case, \vdash_i becomes inferentially explosive while \vdash_w becomes inferentially inert. Of course, this only tells us half of the story. The other half of the story requires us to spell out the preservational properties of \vdash_i and \vdash_w , if indeed they have any, for m.c.s. definable sets. Before we answer that query however, we shall detour to look at some of the structural and logical features of these relations, especially in the context of classical inference.

1.2 Some General Properties of \vdash_i and \vdash_w

Although \vdash_i and \vdash_w are defined in terms of the classical provability \vdash , strictly speaking, they are not consequence relations in Tarski-Scott's sense. It is easy to see that neither \vdash_i nor \vdash_w has the property of full reflexivity. A self-inconsistent member of Γ is neither i-provable nor w-provable from Γ . The failure of monotonicity for \vdash_i and the failure of transitivity for \vdash_w are equally transparent as the following examples will illustrate (we assume that $\{\neg, \wedge, \vee, \rightarrow\}$ is the set of connectives in the object language):

Example 2. Let $\Gamma = \{p \wedge q, \neg p \wedge r\}$ and $\Gamma' = \{p \wedge q, \neg p \wedge r, \neg q, \neg r\}$.

$$\mathcal{U}_\Gamma = \{\{p \wedge q\}, \{\neg p \wedge r\}\}$$

$$\mathcal{U}_{\Gamma'} = \{\{p \wedge q, \neg r\}, \{\neg p \wedge r, \neg q\}, \{\neg q, \neg r\}\}$$

Clearly, $\Gamma \subset \Gamma'$ and $\Gamma \vdash_i q \vee r$. But $\Gamma' \not\vdash_i q \vee r$.

Example 3. Let $\Gamma = \{p, \neg p\}$, $\alpha = p \rightarrow q$, and $\beta = q$

Clearly, $\Gamma \vdash_w \alpha$ and $\Gamma, \alpha \vdash_w \beta$, but $\Gamma \not\vdash_w \beta$

Nonetheless, \vdash_i and \vdash_w do have some of the structural properties of the Tarski-Scott relation. Of particular interest is the weakening of reflexivity.

Theorem 1.2 *Let Γ and Δ be sets of wffs, α and β be arbitrary wffs, then*

(1) \vdash_i satisfies the following properties:

(Ref*) α is i.b. of $\Gamma \Rightarrow \Gamma \vdash_i \alpha$

(Trans) $\Gamma, \alpha \vdash_i \beta$ and $\Gamma \vdash_i \alpha \Rightarrow \Gamma \vdash_i \beta$

(2) \vdash_w satisfies the following properties:

(Ref**) α is a witness of $\Gamma \Rightarrow \Gamma \vdash_w \alpha$

(Mon) $\Gamma \vdash_w \alpha \Rightarrow \Gamma, \Delta \vdash_w \alpha$ for any Δ

Proof.

(1) Obviously, \vdash_i satisfies Ref*. For Trans assume that $\Gamma, \alpha \vdash_i \beta$ and $\Gamma \vdash_i \alpha$. If Γ is m.c.s. undefinable, then $\Gamma \vdash_i \beta$ trivially. So assume that Γ is m.c.s. definable. Let $\mathbf{a} \in \mathcal{U}_\Gamma$ be arbitrary. So $\mathbf{a} \vdash \alpha$. But $\mathbf{a} \not\vdash \perp$, thus $\mathbf{a}, \alpha \not\vdash \perp$. So $\mathbf{a}, \alpha \in \mathcal{U}_{\Gamma, \alpha}$. By assumption $\Gamma, \alpha \vdash_i \beta$, so $\mathbf{a}, \alpha \vdash \beta$. By the Deduction Theorem of *PL*, $\mathbf{a} \vdash \alpha \rightarrow \beta$. But by assumption $\Gamma \vdash_i \alpha$ and \mathbf{a} was arbitrary, so $\forall \mathbf{a} \in \mathcal{U}_\Gamma, \mathbf{a} \vdash \beta$. Hence, $\Gamma \vdash_i \beta$.

(2) \vdash_w obviously satisfies Ref**. For Mon, assume that $\Gamma \vdash_w \alpha$, i.e. $\exists \mathbf{a} \in \mathcal{U}_\Gamma: \mathbf{a} \vdash \alpha$. Let \mathbf{a} be such a set. Let $\Delta \subseteq \Phi$. By assumption $\mathbf{a} \not\vdash \perp$ and $\mathbf{a} \subseteq \Gamma, \Delta$. So by the extension lemma $\exists \mathbf{a}^+: \mathbf{a} \subseteq \mathbf{a}^+$ and $\mathbf{a}^+ \in \mathcal{U}_{\Gamma, \Delta}$. So by monotonicity of \vdash , $\mathbf{a}^+ \vdash \alpha$. Hence $\Gamma, \Delta \vdash_w \alpha$. ■

What is the philosophical motivation for weakening reflexivity? In a tiny nutshell, it is an obvious strategy to neutralize the inferential impact of the ‘undesirable’ elements in a premiss set. Observe that in the case of \vdash_w , we take self-inconsistent data to be our villains and refuse to include them as conclusions. In the case of \vdash_i , we take both self-inconsistent *and* mutually conflicting data to be our villains and refuse to include them as conclusions. Generalizing this strategy, we can target any specific class of formulae in a premiss set and exclude them as conclusions. Our strategy complements an observation made by Jennings and Schotch in [33]. Their observation is that in some cases a theory (not necessary in the technical sense of a deductively closed set), while not inconsistent, may spew out counter-intuitive results or results which conflict with other well established theories. But in the absence of a better candidate for replacement, we simply have to retain our theory and minimize the impact of the ‘undesirable’ elements of the theory. Embracing full reflexivity as a correct pattern of reasoning has the unfortunate consequence that claims that verge on lunacy have to be accepted as conclusions when they are included as premises. If we are given a theory, or a set of claims, which contains outrageous or counter-intuitive assertions, one reasonable thing to do, exercising our good sense of the principle of charity, is to redirect our attention to things that are less odourous and more useful. Afterall, a theory may contain counter-intuitive claims and still have something important and even truthful to say. Weakening reflexivity in our inferential procedure is a tactic for avoiding being victimized by lunacy.

Returning to our theme however, other properties of \vdash_i and \vdash_w are similar to the classical

\vdash . In particular, they have the weak deduction property and are upwardly monotonic with respect to classical consequences of singular wffs.

Theorem 1.3 *Let Γ and Δ be sets of wffs, α and β be arbitrary wffs, then*

- (1) *If $\Gamma, \alpha \vdash_i \beta$, then $\Gamma \vdash_i \alpha \rightarrow \beta$.*
- (2) *If $\Gamma \vdash_i \alpha$ and $\alpha \vdash \beta$, then $\Gamma \vdash_i \beta$.*
- (3) *If $\Gamma, \alpha \vdash_w \beta$, then $\Gamma \vdash_w \alpha \rightarrow \beta$.*
- (4) *If $\Gamma \vdash_w \alpha$ and $\alpha \vdash \beta$, then $\Gamma \vdash_w \beta$.*

Proof.

(1) Assume that $\Gamma, \alpha \vdash_i \beta$. If Γ is m.c.s. undefined, then trivially $\Gamma \vdash_i \alpha \rightarrow \beta$. Assume that Γ is m.c.s. definable. Partition \mathcal{U}_Γ into 2 subsets, where

$$\Pi_\alpha = \{\mathbf{a} \in \mathcal{U}_\Gamma : \mathbf{a}, \alpha \not\vdash \perp\}$$

$$\mathcal{U}_\Gamma - \Pi_\alpha = \{\mathbf{b} \in \mathcal{U}_\Gamma : \mathbf{b}, \alpha \vdash \perp\}$$

Clearly, $\forall \mathbf{a} \in \Pi_\alpha, \mathbf{a}, \alpha \in \mathcal{U}_{\Gamma, \alpha}$ and $\forall \mathbf{b} \in \mathcal{U}_\Gamma - \Pi_\alpha, \mathbf{b} \in \mathcal{U}_{\Gamma, \alpha}$. So by the initial assumption, $\mathbf{a}, \alpha \vdash \beta$ and $\mathbf{b} \vdash \beta$. By the Deduction Theorem of *PL*, $\mathbf{a} \vdash \alpha \rightarrow \beta$, and by *PL*, $\mathbf{b} \vdash \alpha \rightarrow \beta$. But $\mathbf{a} \in \Pi_\alpha$ and $\mathbf{b} \in \mathcal{U}_\Gamma - \Pi_\alpha$ were arbitrary. Hence either way, $\Gamma \vdash_i \alpha \rightarrow \beta$.

(2) The claim clearly holds if Γ is m.c.s. undefinable, so assume that Γ is m.c.s. definable. Assume that $\Gamma \vdash_i \alpha$ and $\alpha \vdash \beta$. Let $\mathbf{a} \in \mathcal{U}_\Gamma$ be arbitrary. So by initial assumptions $\mathbf{a} \vdash \alpha$ and $\alpha \vdash \beta$. By monotonicity of \vdash , $\mathbf{a}, \alpha \vdash \beta$. By transitivity of \vdash , $\mathbf{a} \vdash \beta$. But \mathbf{a} was arbitrary, so $\forall \mathbf{a} \in \mathcal{U}_\Gamma, \mathbf{a} \vdash \beta$. Hence, $\Gamma \vdash_i \beta$.

(3) Assume that $\Gamma, \alpha \vdash_w \beta$. So $\exists \mathbf{a} \in \mathcal{U}_\Gamma : \mathbf{a} \vdash \beta$. But either $\alpha \in \mathbf{a}$ or $\alpha \notin \mathbf{a}$. If $\alpha \notin \mathbf{a}$, then $\mathbf{a} \in \mathcal{U}_\Gamma$. So $\Gamma \vdash_w \beta$ by assumption. So $\Gamma \vdash_w \alpha \rightarrow \beta$. If $\alpha \in \mathbf{a}$, then $\mathbf{a} - \{\alpha\} \vdash \alpha \rightarrow \beta$ by the Deduction Theorem of *PL*. But then $\mathbf{a} - \{\alpha\} \in \mathcal{U}_\Gamma$. Hence, $\Gamma \vdash_w \alpha \rightarrow \beta$. So either way $\Gamma \vdash_w \alpha \rightarrow \beta$.

(4) Assume that $\Gamma \vdash_w \alpha$ and $\alpha \vdash \beta$. So, $\exists \mathbf{a} \in \mathcal{U}_\Gamma : \mathbf{a} \vdash \alpha$. By monotonicity of \vdash , $\mathbf{a}, \alpha \vdash \beta$. So by transitivity of \vdash , $\mathbf{a} \vdash \beta$. Hence, $\Gamma \vdash_w \beta$. ■

Now we can state more explicitly what logical rules are retained by \vdash_i and \vdash_w .

Theorem 1.4 *Let Γ and Δ be sets of wffs, α and β be arbitrary wffs, then*

- (1) *the following rules hold for \vdash_i*

- $(\neg\neg)$ If $\Gamma \vdash_i \neg\neg\alpha$, then $\Gamma \vdash_i \alpha$
 $(\wedge E)$ If $\Gamma \vdash_i \alpha \wedge \beta$, then $\Gamma \vdash_i \alpha$ and $\Gamma \vdash_i \beta$
 $(\wedge I)$ If $\Gamma \vdash_i \alpha$ and $\Gamma \vdash_i \beta$, then $\Gamma \vdash_i \alpha \wedge \beta$
 $(\vee E)$ If $\Gamma \vdash_i \alpha \vee \beta$ and $\Gamma \vdash_i \neg\beta$, then $\Gamma \vdash_i \alpha$
 $(\vee I)$ If $\Gamma \vdash_i \alpha$, then $\Gamma \vdash_i \alpha \vee \beta$
 $(\rightarrow E)$ If $\Gamma \vdash_i \alpha \rightarrow \beta$ and $\Gamma \vdash_i \alpha$, then $\Gamma \vdash_i \beta$
 $(\rightarrow I)$ If $\Gamma \vdash_i \beta$, then $\Gamma \vdash_i \alpha \rightarrow \beta$

(2) the following rules hold for \vdash_w

- $(\neg\neg)$ If $\Gamma \vdash_w \neg\neg\alpha$, then $\Gamma \vdash_w \alpha$
 $(\wedge E)$ If $\Gamma \vdash_w \alpha \wedge \beta$, then $\Gamma \vdash_w \alpha$ and $\Gamma \vdash_w \beta$
 $(\vee I)$ If $\Gamma \vdash_w \alpha$, then $\Gamma \vdash_w \alpha \vee \beta$
 $(\rightarrow I)$ If $\Gamma \vdash_w \beta$, then $\Gamma \vdash_w \alpha \rightarrow \beta$

Proof.

(1) $(\neg\neg)$, $(\wedge E)$, $(\vee I)$, and $(\rightarrow I)$ follow immediately from theorem 1.3 (2). For $(\wedge I)$, observe that the corresponding rule holds with respect to \vdash . $(\vee E)$ and $(\rightarrow I)$ are immediate consequences of $(\wedge I)$ and theorem 1.3 (2).

(2) They follow immediately from theorem 1.3 (4). ■

Although theorem 1.3 and theorem 1.4 give us a partial proof-theoretic characterization of \vdash_i and \vdash_w , without a full account of the semantics for these relations however, it is too early to declare victory. In particular we need to know whether the structural and logical rules we have given are indeed adequate for some semantical theory. So far, no semantical account has been discovered with respect to these relations. Nonetheless, as a step towards that discovery, we can give a more detailed description of the preservational properties of \vdash_i and \vdash_w .

Chapter 2

Inference And Preservation

The study of logic is usually defined in the standard first year textbook as the study of correct reasoning. When pressured, the standard text usually spells out the ideology of correctness in terms of rules which allow us to draw true conclusions on the basis of true premises, in other words, rules which prevent reasoners from drawing false conclusions from true premises. On the standard view, correct reasoning patterns are just inferences that preserve truth.¹ While not incorrect, this story is of no help when we are confronted with inconsistent data. Since all inconsistent sets are unsatisfiable in classical semantics, the issue of truth is irrelevant for inconsistent sets. Nonetheless, our dissatisfaction with classical logics should not lead to a complete rejection of the standard view. We can still salvage the basic insight of the standard view by observing that truth or a particular distribution of truth values are not the only properties worth preserving in a set. Good probabilistic reasoning, for instance, should prohibit us to draw highly improbable conclusions given highly probable premises. On the philosophically more general and intriguing conception of logic then, good reasoning patterns are inferences that allow us to preserve important metalinguistic properties of premiss sets. Barwise echoes the same philosophical sentiment when he says:

... the study of valid inference as a situated activity shifts attention from *truth preservation to information extraction and information processing*. Valid inference is seen not as a relation between sentences that simply preserves truth, but

¹Speaking more precisely and less metaphysically, it is the truth-value, 1, that is preserved under classical rules.

rather as a situated, purposeful activity whose aim is the extraction of information from a situation, information relevant to the agent. ([3], p.xiv)

Information and situation semanticists are not the sole advocates of reform here. In [17], [19], [31], [33] Jennings and Schotch have argued that many metalinguistic properties other than truth are worth preserving, one in particular, gives us a measurement of the instability of sets. The intuitive appeal is that it allows us to distinguish ‘the amount of work required to winnow stable subsets from an unstable set depends upon the set in question’ ([33], p.311). To see the thrust of their claim, we first observe the following definition:

Definition 2.1 *A n -covering family of a set Γ is a collection, $\mathfrak{d} = \{\mathfrak{g}_1, \dots, \mathfrak{g}_n\}$, of subsets of Γ such that $\Gamma = \bigcup_{1 \leq k \leq n} \mathfrak{g}_k$ (where $n \leq \omega$).*

Elements of a n -covering family are called *constituents*. A n -covering family, \mathfrak{d} , of Γ is a n -partition of Γ just in case elements of \mathfrak{d} are pairwise disjoint. Now if we define a function which maps a set to its smallest n -covering family containing all classically consistent constituents, then a measurement theory is in the offing. More specifically, we define an onto mapping from 2^Φ to $\mathbb{N} \cup \{\infty\}$. Following Jennings and Schotch, we call this a *level* function.

Definition 2.2 *The level of a set Γ , $\ell: 2^\Phi \rightarrow \mathbb{N} \cup \{\infty\}$ is defined as follows:*

$$\ell(\Gamma) = \begin{cases} 0 & \text{if } \Gamma = \emptyset \text{ or } \Gamma \subseteq \{\alpha: \vdash \alpha\} \\ \text{the cardinality of the} \\ \text{least covering family of} \\ \Gamma \text{ into consistent subsets} \\ \text{up to and including } \omega & \text{if such covering family exists} \\ \infty & \text{otherwise} \end{cases}$$

The sentence ‘ $\ell(\Gamma) = \infty$ ’ does not say that Γ has infinite level; rather it says that Γ has no level at all. So we must distinguish between ‘ $\ell(\Gamma) = \infty$ ’ and ‘ $\ell(\Gamma) = \omega$ ’. Having said that, it is time to demonstrate the virtue of ℓ . For example, it allows us to distinguish between

$$\Gamma = \{p \wedge q, \neg p \wedge q, \neg q\}$$

$$\Delta = \{p, \neg p, q\}$$

Clearly, there is a sense in which Γ is less stable than Δ since there are more conflicting sentences in Γ than in Δ . To no one surprise, inferentially more unstable sets have higher ℓ -value than inferentially less unstable sets. Using the definition, it is straightforward to verify that $\ell(\Gamma) = 3$ while $\ell(\Delta) = 2$. The notion of level in fact offers us a very natural

way to taxonomize sets. Classically consistent sets are sets with level less than or equal to 1 while classically inconsistent sets are sets with level 2 or greater. Sets containing self-inconsistent sentences are simply sets without a level. Having said that, we can now introduce an inference relation called forcing.

Definition 2.3 A set Γ forces α , $\Gamma \Vdash \alpha$ iff $\forall \mathfrak{d} \in \mathfrak{D}_{\ell(\Gamma)}(\Gamma)$, $\exists \mathfrak{g} \in \mathfrak{d}$: $\mathfrak{g} \vdash \alpha$ where $\mathfrak{D}_{\ell(\Gamma)}(\Gamma)$ is the set of $\ell(\Gamma)$ -fold covering families of Γ .

By inspecting the definition, it is apparent that \Vdash is reflexive, transitive, and more importantly, ℓ -preserving for any Γ , i.e. $\ell(\mathbf{Cn}_f(\Gamma), \Gamma) = \ell(\Gamma)$, where $\mathbf{Cn}_f(\Gamma) = \{\alpha: \Gamma \Vdash \alpha\}$. Since adding sentences that are forced by a set will not result in a set with greater inferential instability, from the preservation-theoretic standpoint, forcing is a plausible strategy for reasoning in the presence of classically inconsistent premiss set. To see that \Vdash is indeed ℓ -preserving we only need to observe that if $\ell(\Gamma) = n \neq \infty$, then $\exists \mathfrak{d} \in \mathfrak{D}_n(\Gamma): \mathfrak{d} = \{\mathfrak{g}_1, \dots, \mathfrak{g}_n\}$ where $\forall k_{(1 \leq k \leq n)}$, $\mathfrak{g}_k \not\vdash \perp$, and $\Gamma = \bigcup_{1 \leq k \leq n} \mathfrak{g}_k$, and elements in \mathfrak{d} are pairwise distinct. So any α that is forced by Γ is in fact classical consequence of some consistent constituent of \mathfrak{d} . Details of the proof and other results, especially in connection with weak aggregative modal logics, are well documented in [19], [31], and [33]. For now, we only need to observe that \Vdash cuts down the number of inferentially explosive sets by rendering all inconsistent sets with a defined level to be nonexplosive.

Given the general methodological lesson we learn from Jennings and Schotch, it is time to make good some of our earlier promise. We shall introduce some notations to facilitate discussion. We use ' $\kappa(\Gamma)$ ' to denote the set of culprits of Γ and for $\Gamma - \kappa(\Gamma)$ we write ' Γ^* '. We can now introduce a measurement function similar to ℓ . In honour of Rescher, we call this the Rescher function denoted by ' λ ':

Definition 2.4 The Rescher value of a set Γ , $\lambda: 2^\Phi \rightarrow \mathbb{N} \cup \{\infty\}$, is defined as follows:

$$\lambda(\Gamma) = \begin{cases} 0 & \text{if } \Gamma = \emptyset \text{ or } \Gamma \subseteq \{\alpha: \vdash \alpha\} \\ \text{Card}(\mathcal{U}_{\Gamma^*}) \leq \omega & \text{if } \Gamma \text{ is m.c.s. definable} \\ \infty & \text{otherwise} \end{cases}$$

A few remarks are in order. Firstly, $\lambda(\Gamma) = \lambda(\Gamma^*)$ if Γ is m.c.s. definable, so we could have defined λ in terms of Γ in the relevant clause instead. Secondly, it is equally possible to define λ in terms of covering families of a set.² But we forego that in favor of a more Rescher-like

²In particular, λ maps a set to the cardinality (up to and including ω) of the collection of subsets of Γ^* , $\mathfrak{d} = \{\mathfrak{g}_1, \dots, \mathfrak{g}_n\}$ such that $\forall k_{(1 \leq k \leq n)}$ \mathfrak{g}_k is Γ^* -maximally consistent and $\Gamma^* = \bigcup \mathfrak{g}_k$.

definition. The precise philosophical significance of λ measurement theory, however, has not been well understood. Nonetheless, in a recent article [22], Lozinskii has developed a measurement theory of *information* that is related to the λ -value of a set. More specifically, Lozinskii defines the quantity of information of an unsatisfiable set Γ as a decreasing function of the cardinality of the collection of models for each m.c.s. of Γ . A detailed discussion of Lozinskii's work is quite beyond the scope of this study.³

The ℓ -value and λ -value of a set are obviously related in an interesting way. In particular, by the extension lemma of chapter one it is not difficult to prove that if $\ell(\Gamma) \neq \infty$, then the ℓ -value is the lower bound of the λ -value, i.e., every set that has a defined level, n , has at least n number of m.c.s. This gives us reason to think that there may be a measurement function whose value is the lower bound of the λ -value for any set. As it turns out, there is such a function. Moreover, we can define a corresponding inference relation which preserves its value. We call this the level* function and its inference relation forcing*.

Definition 2.5 *The level* of a set Γ , $\ell^*: 2^\Phi \rightarrow \mathbb{N} \cup \{\infty\}$, is defined as follows:*

$$\ell^*(\Gamma) = \begin{cases} 0 & \text{if } \Gamma = \emptyset \text{ or } \Gamma \subseteq \{\alpha: \vdash \alpha\} \\ \ell(\Gamma^*) & \text{if } \Gamma \text{ is m.c.s. definable} \\ \infty & \text{otherwise} \end{cases}$$

Definition 2.6 *A set Γ forces* α , $\Gamma \Vdash^* \alpha$ iff $\forall \mathfrak{d} \in \mathfrak{D}_{\ell^*(\Gamma)}(\Gamma^*)$, $\exists \mathfrak{g} \in \mathfrak{d}: \mathfrak{g} \vdash \alpha$ where $\mathfrak{D}_{\ell^*(\Gamma)}(\Gamma^*)$ is the set of $\ell(\Gamma^*)$ -fold covering families of Γ^* .*

The generalization from ℓ to ℓ^* obviously gives us a non-trivial measurement value for sets whose culprit sector is non-empty. Indeed, this is the main difference between ℓ function and ℓ^* function. ℓ^* is really just ℓ applied to Γ^* . Correspondingly, we can interpret ℓ^* as a measurement function for the instability of the inferentially useful part of a premise set.

³The issue is complicated by the fact that Lozinskii is primarily concerned with first order systems restricted to the Herbrand universe. The general idea is to define the set of all quasi-Herbrand-models of a first order set Γ as the union of all Herbrand-models of each m.c.s. of Γ , i.e.

$$\Omega_\Gamma = \bigcup_{\alpha \in \mathcal{U}_\Gamma} \mathfrak{M}_\alpha$$

Then define the *base* of Γ , $Base(\Gamma)$, to be the set of all ground atomic formulae of Γ . The informational quantity of Γ is now defined as:

$$\mathcal{I}_\Gamma = \log_2 \frac{2^{Card(Base(\Gamma))}}{Card(\Omega_\Gamma)}$$

Within this new measure theory, a set which has no defined level need not be inferentially useless, for there may be a subset of that set which has a defined level. With respect to \vdash^* , it is obviously a modification of \vdash by weakening the property of reflexivity. In so doing, \vdash^* and \vdash_w clearly share the same property of weak reflexivity. Like \vdash_w , \vdash^* targets all self-inconsistent premises as villains and refuses to assign any inferential role to them. We can now show that ℓ^* -value is the lower bound of λ -value for any set. This claim holds trivially for cases in which $\ell^*(\Gamma) = \infty$ or $\ell^*(\Gamma) = 0$, we consider cases in which $1 \leq \ell^*(\Gamma) \leq \omega$.

Theorem 2.1 *For any Γ , if $\ell^*(\Gamma) = n$, then $\lambda(\Gamma) \geq n$ where $1 \leq n \leq \omega$.*

Proof. The proof proceeds by induction on n for an arbitrary Γ . For the basis, $n = 1$ so $\ell^*(\Gamma) = \ell(\Gamma^*)$ and $\{\Gamma^*\} = \mathcal{U}_\Gamma$. So $\ell^*(\Gamma) = \lambda(\Gamma)$. Assume that the claim holds when $n = k$, we prove that it also holds when $n = k + 1$. So let $\ell^*(\Gamma) = k + 1$, then by definition, $\exists \mathfrak{d} \in \mathfrak{D}_{k+1}(\Gamma^*): \mathfrak{d} = \{\mathfrak{g}_1, \dots, \mathfrak{g}_{k+1}\}$ where $\forall j_{(1 \leq j \leq k+1)} \mathfrak{g}_j \not\vdash \perp$ and $\Gamma^* = \bigcup \mathfrak{g}_j$. By the extension lemma, each constituent, \mathfrak{g}_j can be extended to a consistent Γ -maximal extension. Since there are $k + 1$ distinct constituents in \mathfrak{d} , there are $k + 1$ distinct corresponding extensions. Hence $\lambda(\Gamma) \not\prec k + 1$, i.e. $\lambda(\Gamma) \geq k + 1$. ■

Clearly, the ℓ^* -value of a set, Γ , coincides with its λ -value when one of the $\ell^*(\Gamma)$ -fold covering families contains constituents that are m.c.s. of Γ . For example:

Example 4. $\Gamma = \{p \wedge q, \neg p \wedge \neg q\}$
 $\ell(\Gamma) = \ell^*(\Gamma) = \lambda(\Gamma) = 2$

However, a set may have completely different ℓ , ℓ^* , and λ values. For example:

Example 5. $\Gamma^+ = \{p \wedge q, \neg p \wedge \neg q, p, \neg q, p \wedge \neg p\}$
 $\ell(\Gamma^+) = \infty$
 $\ell^*(\Gamma^+) = 2$
 $\lambda(\Gamma^+) = 3$

We can summarize the relation between various measurement functions with table 1.

$\ell^*(\Gamma) = \infty$	$\ell(\Gamma) = \infty$	$\lambda(\Gamma) = \infty$
$\ell^*(\Gamma) = 0$	$\ell(\Gamma) = 0$	$\lambda(\Gamma) = 0$
$\ell^*(\Gamma) = 1$	$\ell(\Gamma) = \infty$	$\lambda(\Gamma) = 1$
	$\ell(\Gamma) = 1$	
$\ell^*(\Gamma) = n \geq 2$	$\ell(\Gamma) = \infty$	$\lambda(\Gamma) \geq n$
	$\ell(\Gamma) = n$	

table 1

Now we deliver the preservational result for \vdash_i .

Theorem 2.2 *The following claims hold with respect to \vdash_i :*

- (1) $\ell^*(\mathbf{Cn}_i(\Gamma), \Gamma) = \ell^*(\Gamma)$ for any m.c.s. definable Γ
- (2) $\lambda(\mathbf{Cn}_i(\Gamma), \Gamma) = \lambda(\Gamma)$ for any m.c.s. definable Γ
- (3) $\ell(\mathbf{Cn}_i(\Gamma), \Gamma) = \ell(\Gamma)$ for any Γ

Proof.

(1) Let Γ be m.c.s. definable set. The proof proceeds by induction on the ℓ^* -value. If $\ell^*(\Gamma) = 1$, $\ell(\Gamma^*) = 1$. But $\{\Gamma^*\} = \mathcal{U}_\Gamma$ so $\mathbf{Cn}_i(\Gamma) = \mathbf{Cn}(\Gamma^*)$. So $\ell^*(\mathbf{Cn}_i(\Gamma), \Gamma) = 1$. Assume that the claim holds for $\ell^*(\Gamma) = n$. We show that the claim holds for $\ell^*(\Gamma) = n + 1$. Let $\ell^*(\Gamma) = n + 1$. Then $\exists \mathfrak{d} \in \mathcal{D}_{n+1}(\Gamma^*) : \mathfrak{d} = \{\mathbf{g}_1, \dots, \mathbf{g}_{n+1}\}$ where $\forall k_{(1 \leq k \leq n+1)} \mathbf{g}_k \not\vdash \perp$ and $\Gamma^* = \bigcup \mathbf{g}_k$. But $\mathbf{Cn}_i(\Gamma) = \bigcap_{\mathbf{a} \in \mathcal{U}_\Gamma} \mathbf{Cn}(\mathbf{a})$ and $\forall \mathbf{a} \in \mathcal{U}_\Gamma$, $\ell^*(\mathbf{Cn}(\mathbf{a})) = 1$. So $\ell^*(\bigcap_{\mathbf{a} \in \mathcal{U}_\Gamma} \mathbf{Cn}(\mathbf{a})) = 1$. Hence $\ell^*(\mathbf{Cn}_i(\Gamma)) = 1$. But $\forall \alpha \in \Gamma^*$, $\mathbf{Cn}_i(\Gamma), \alpha \not\vdash \perp$. So, $\forall k_{(1 \leq k \leq n+1)} \mathbf{Cn}_i(\Gamma), \mathbf{g}_k \not\vdash \perp$. But there are $n+1$ distinct constituents in \mathfrak{d} , hence, $\ell^*(\mathbf{Cn}_i(\Gamma), \Gamma) = n + 1$.

(2) To prove claims (2) we first establish the following lemma:

Lemma 2.1 *For any Γ and Δ , if $\forall \mathbf{a} \in \mathcal{U}_\Gamma$, $\mathbf{a}, \Delta^* \not\vdash \perp$, then $\lambda(\Gamma) = \lambda(\Gamma, \Delta)$ (Thinning Lemma).*

Proof of Lemma. Let Γ and Δ be arbitrary sets of wffs. Let Γ be m.c.s. definable. Assume that $\forall \mathbf{a} \in \mathcal{U}_\Gamma$, $\mathbf{a}, \Delta^* \not\vdash \perp$. So, $\forall \mathbf{a} \in \mathcal{U}_\Gamma$, $\mathbf{a}, \Delta^* \in \mathcal{U}_{\Gamma, \Delta}$. So clearly $\lambda(\Gamma) \leq \lambda(\Gamma, \Delta)$. Since, every maximal consistent extension of Δ^* in Γ, Δ includes \mathbf{a}, Δ^* , for each $\mathbf{a} \in \mathcal{U}_\Gamma$, it follows that $\lambda(\Gamma) \not\leq \lambda(\Gamma, \Delta)$. Hence, $\lambda(\Gamma) = \lambda(\Gamma, \Delta)$.

To complete the proof, let Γ be any m.c.s. definable set. By reasoning similar to the argument in part (1), $\ell(\mathbf{Cn}_i(\Gamma)) = 1$. So $\forall \mathbf{a} \in \mathcal{U}_\Gamma$, $\mathbf{Cn}_i(\Gamma), \mathbf{a} \not\vdash \perp$. So by the thinning lemma $\lambda(\mathbf{Cn}_i(\Gamma), \Gamma) = \lambda(\Gamma)$.

(3) If a set Γ is m.c.s. undefinable, then $\ell(\Gamma) = \infty$ and so any superset of Γ has ℓ -value ∞ . If Γ is m.c.s. definable, then its ℓ^* -value is preserved under \vdash_i and so its ℓ -value is also preserved under \vdash_i . ■

In effect, lemma 2.1 specifies a sufficient condition for λ -preservation: any inference relation whose output consequence set is consistent with every m.c.s. of the premise set will preserve the λ -value. However, an inference relation whose output consequence set is consistent with *some* m.c.s. of the premise set may also preserve the λ -value. So for instance,

Example 6. $\Gamma = \{p \wedge q, \neg p \wedge \neg q\}$, $\Delta = \{p\}$
 $\lambda(\Gamma, \Delta) = \lambda(\Gamma)$ even though $\{\neg p \wedge \neg q\}, \Delta \vdash \perp$

Strictly speaking, there is no smallest inference relation which is λ -preserving for m.c.s. definable sets. By this we mean the intersection of all λ -preserving inference relation for m.c.s. definable sets is empty. So for instance:

Example 7. Define an inference relation $\vdash_{\perp} \subseteq 2^{\Phi} \times \Phi$ such that
 $\langle \Gamma, \alpha \rangle \in \vdash_{\perp}$ iff $\alpha \in \{\beta : \beta \vdash \perp\}$ and Γ is m.c.s. definable.

Trivially, \vdash_{\perp} preserves λ -value for m.c.s. definable sets.⁴ But $\vdash_{\perp} \cap \vdash_i = \emptyset$.

As it turns out, none of \vdash , \vdash^* , and \vdash_w preserve λ -value. More surprisingly however, \vdash_w does not even preserve ℓ^* -value. So from the standpoint of the relevant measurement theory, \vdash_w yields a more unstable consequence set for a given inconsistent premise set. We can observe the following counterexample:

Example 8. $\Gamma = \{p \wedge q, \neg p \wedge \neg q\}$ $\Delta = \{p, \neg q\}$

Clearly, $p \in \mathbf{Cn}_{\xi}(\Gamma)$ and $\neg q \in \mathbf{Cn}_{\xi}(\Gamma)$ (where $\xi \in \{f, f^*, w\}$) and $\lambda(\Gamma) = 2$. But $\lambda(\Gamma, \Delta) = 3$. Moreover, $p \wedge \neg q \in \mathbf{Cn}_w(\Gamma, \Delta)$, but $\ell^*(\Gamma, \Delta) = 2$ and $\ell^*(\Gamma, \Delta, \{p \wedge \neg q\}) = 3$.

We can give an overall picture of the inferential behavior of \vdash_i , \vdash_w and \vdash^* with the following table:

⁴ \vdash_{\perp} also preserves ℓ^* -value for m.c.s. definable sets. However, it is not known whether all λ -preserving inference relations are ℓ^* -preserving relation.

		$\mathbf{Cn}_i(\Gamma)$	$\mathbf{Cn}_{f^*}(\Gamma)$	$\mathbf{Cn}_w(\Gamma)$
$\ell^*(\Gamma) = n \geq 2$	$\lambda(\Gamma) \geq n$	$\bigcap_{a \in \mathcal{U}_\Gamma} \mathbf{Cn}(a)$	$\subseteq \mathbf{Cn}_{f^*}(\Gamma) \subseteq$	$\bigcup_{a \in \mathcal{U}_\Gamma} \mathbf{Cn}(a)$
$\ell^*(\Gamma) = 1$	$\lambda(\Gamma) = 1$	$\mathbf{Cn}(\Gamma^*)$	$\mathbf{Cn}(\Gamma^*)$	$\mathbf{Cn}(\Gamma^*)$
$\ell^*(\Gamma) = 0$	$\lambda(\Gamma) = 0$	PL	PL	PL
$\ell^*(\Gamma) = \infty$	$\lambda(\Gamma) = \infty$	Φ	Φ	\emptyset

table 2

Recall that in chapter one we have offered a philosophical interpretation of \vdash_w and \vdash_i as species of liberal and skeptical reasoning strategy respectively. Obviously, \vdash^* is a plausible alternative to both \vdash_i and \vdash_w . From the preservation-theoretic perspective, it provides sufficient restriction to preserve the ℓ^* -value of the conclusion set, but not so restrictive to simply restore consistency in the conclusion set. From the paraconsistent theoretic perspective, it allows us to reason in the presence of self-inconsistencies in a premise set.

Chapter 3

Inference And Modality

It is generally known that the smallest modal logic determined (i.e. sound and complete) by the class of kripkean binary relational frame is the logic K , most economically axiomatized by adding to PL the single rule called Scott's Rule:

$$[SR] \frac{\Gamma \vdash \alpha}{\Box[\Gamma] \vdash \Box\alpha}$$

where $\Box[\Gamma] = \{\Box\beta : \beta \in \Gamma\}$. Alternatively, K can also be axiomatized by adding to theorems of PL

$$[K_{\rightarrow}] \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$$

and the rule of necessitation

$$[RN] \frac{\vdash \alpha}{\vdash \Box\alpha}$$

One of the most interesting feature of $[SR]$ is that it allows us modalize sentences in the context of classical inference ($[SR]$ is also known as the rule of modalization [9]). Thus if α is classically inferred from a set Γ , $[SR]$ licenses us to infer $\Box\alpha$, in classical style, from the fully modalized set $\Box[\Gamma]$. In [19] and [31], Jennings and Schotch have bridged the gap between paraconsistent inference and modal logics by deploying the fact that modalization of classically inconsistent sets have the effect of restoring full consistency of a set.¹ In measurement theory, this amounts to reducing ℓ , ℓ^* and λ value of a set to 1 or less. Apparently, this allows us to represent each inconsistent set with a consistent modalized set

¹The historical root of this approach to inference and modality traces back to the work of C. I. Lewis and C. H. Langford in [21]

and to use modal logics to study the inferential properties of inconsistent sets. They have discovered that the modal logic K_n axiomatized by adding to PL , the single rule:

$$[JS] \frac{\Gamma \vdash \alpha}{\Box[\Gamma] \vdash \Box \alpha} \ell(\Gamma) = n$$

is determined by the class of $n+1$ -ary relational frame or alternatively called Jennings-Schotch frame. A JS-frame is a pair $\mathfrak{F} = \langle \mathcal{U}, \mathcal{R} \rangle$ where $\mathcal{U} \neq \emptyset$ and $\mathcal{R} \subseteq \mathcal{U}^{n+1}$. A JS-model is a JS-frame with a valuation, $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{V} \rangle$ where $\mathcal{V}: At \rightarrow 2^{\mathcal{U}}$. The truth condition for non-modal wffs are defined inductively in the usual way. For modal wffs, the truth condition is given by:

$$\models_x^{\mathfrak{M}} \Box \alpha \Leftrightarrow \forall y_1, \dots, y_n \in \mathcal{U}, \mathcal{R}xy_1, \dots, y_n \Rightarrow \exists i(1 \leq i \leq n): \models_{y_i}^{\mathfrak{M}} \alpha$$

Obviously, JS-semantics is a natural generalization of the kripkean semantics.² Observe that when $n = 1$, $[JS]$ is equivalent to $[SR]$ and the truth condition for modal formulae in JS-model is equivalent to the kripkean truth condition, so the logic K and the kripkean binary relational frame are uniquely recovered when $n = 1$.³ More importantly from the inferential point of view, JS-semantics is a clever way to project the inferential properties of forcing into the modal fragment of K_n . To illustrate, for a given set Γ whose ℓ -value is n or less, we may have $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg \alpha$ but not $\Gamma \vdash \alpha \wedge \neg \alpha$. To no one surprise, given the truth condition for modal formulae in JS-model, we have $\Box[\Gamma] \vdash \Box \alpha$ and $\Box[\Gamma] \vdash \Box \neg \alpha$ but $\Box[\Gamma] \not\vdash \Box(\alpha \wedge \neg \alpha)$. This is so precisely because the modal semantics has been arranged to invalidate the strong aggregation principle,

$$[K_{1\wedge}] \Box \alpha \wedge \Box \beta \rightarrow \Box(\alpha \wedge \beta)$$

Instead, K_n accepts the weaker aggregation principle:

$$[K_{n\wedge}] \Box \alpha_1 \wedge \dots \wedge \Box \alpha_{n+1} \rightarrow \Box \frac{2}{n+1}(\alpha_1, \dots, \alpha_{n+1})$$

where $\frac{2}{n+1}$ is the operation of forming disjunction of all distinct pairwise conjunctions from $n+1$ -tuple. More generally, we use $\frac{j}{k}$ where $j, k \in \mathbb{N}$ as the operation of forming disjunction of all distinct j -conjunctions from k -tuples.

²Other attempts to generalize the kripkean semantics can be founded in Dov Gabbay [10], Robert Goldblatt [14] and [15], David K. Johnston [20], Kang (Kenneth) Lu [23], and Jennings and Schotch [32].

³This is not surprising since the algebraic foundations of kripkean relational semantics was articulated by Tarski and Jónsson in their seminal paper in [36]

It is not difficult to see that for each collection of sets with a given ℓ -value, the modal logic generated by $[JS]$ for *that* collection admits a distinct aggregation principle. So for instance, for sets with level 2, we need the $\frac{2}{3}$ aggregation principle

$$[K_{2\wedge}] \Box\alpha_1 \wedge \Box\alpha_2 \wedge \Box\alpha_3 \rightarrow \Box\frac{2}{3}(\alpha_1, \alpha_2, \alpha_3)$$

The generalization to level greater than 2 is obvious. In fact, we have a countably infinite descending sequence of logics,

$$K_1 \supseteq K_2 \supseteq \dots \supseteq K_{i-1} \supseteq K_i \dots$$

endorsing a progressively weaker forms of the aggregation principle. The intersection of these logics, $\bigcap_{i \in \mathbb{N}} K_i$, is the logic N axiomatized by adding to PL , the rule $[RN]$ and the rule of regularity (see [18]):

$$[RR] \frac{\vdash \alpha \rightarrow \beta}{\vdash \Box\alpha \rightarrow \Box\beta}$$

Now in stating the modalization rule corresponding to each of these K_i logics, where $i \in \mathbb{N}$, we have fixed the ℓ -value (i -value) according to the arity of the relational frame. This marks out two different but related notions of forcing known as general and fixed level forcing. Unlike general forcing where the ℓ -value varies from sets to sets, the notion of forcing deployed in the study of modal logics is fixed at a given level. In a binary relation frame, the ℓ -value is 1, in ternary relational frame, the ℓ -value is 2. Again the generalization is obvious.

There is more, however. By projecting the properties of \vdash (at a given level) into the intensional operator, ' \Box ', $[JS]$ also restores classical inference completely. As far as proof theory is concerned, the deductive apparatus of K_n is completely classical. \vdash_{K_n} has the usual structural and logical properties. Again, this is not surprising since in extending PL to K_n , we ensure that no unintended properties are added to the logic.

In light of these discoveries, it is natural to ask whether we can project the properties of \vdash_i , \vdash_w and \vdash^* into the object language by recovering the corresponding modal logic. In each case we can recover the modal logic by stating the corresponding modalization rules:

$$[RI] \frac{\Gamma \vdash_i \alpha}{\Box[\Gamma] \vdash \Box\alpha} \lambda(\Gamma) = n$$

$$[JS^*] \frac{\Gamma \vdash^* \alpha}{\Box[\Gamma] \vdash \Box\alpha} \ell^*(\Gamma) = n$$

$$[RW] \frac{\Gamma \vdash_w \alpha}{\Box[\Gamma] \vdash \Box\alpha} \lambda(\Gamma) = n$$

One may suspect that given that the logic K_n generated by $[JS]$ is alternatively axiomatizable by $[K_{n\wedge}]$, $[RR]$ and $[RN]$ (and thus all K_n theorems are effectively enumerable), the axiomatization of the logic generated by $[JS^*]$ should be straightforward. But the problem turns out to be more complicated than expected.⁴ The complication involves the failure of reflexivity and thus the failure of the rule of extensionality to project the properties of these relations in general:

$$[RE] \frac{\vdash \alpha \leftrightarrow \beta}{\vdash \Box\alpha \leftrightarrow \Box\beta}$$

To see how $[RE]$ may fail, we observe that if $\Box(p \wedge \neg p) \in \Box[\Gamma]$, then by the reflexivity of the \vdash , $\Box[\Gamma] \vdash \Box(p \wedge \neg p)$. However, on the assumption that our logic is an extension of PL , we have $\vdash (p \wedge \neg p) \leftrightarrow (q \wedge \neg q)$. So if $[RE]$ were acceptable, then we would have $\Box[\Gamma] \vdash \Box(q \wedge \neg q)$. But clearly, none of \vdash_i , \vdash_w , and \vdash^* license us to infer $q \wedge \neg q$ by the mere presence of $p \wedge \neg p$ in Γ (unless $\ell^*(\Gamma) = \infty$). Similar remarks should also go to the rule $[RR]$. Since we have $\vdash \perp \rightarrow \alpha$ for any $\alpha \in \Phi$, in the presence of $[RR]$, we have $\vdash \Box\perp \rightarrow \Box\alpha$. So if $\Box\perp \in \Box[\Gamma]$, then $\Box[\Gamma] \vdash \Box\alpha$ for any $\alpha \in \Phi$ and we are back in the soup. The failure of $[RR]$ in fact points to a larger problem: since \vdash_i , \vdash_w , and \vdash^* treat m.c.s. undefinable sets as inferentially deviant, we want our modal semantics to respect the distinction between a modalized set containing only $\Box\perp$ as members and a modalized set containing some $\Box\perp$ as member. Thus we need to prevent some modalized set containing $\Box\perp$ from proving every $\Box\alpha \in \Phi$, in spite of the fact that $\{\perp\} \vdash_i \alpha$ and $\{\perp\} \vdash^* \alpha$ for any $\alpha \in \Phi$. But given the deduction theorem of PL , it is difficult to see how we can avoid $\Box\perp \rightarrow \Box\alpha$ as a theorem in the logic generated by $[RI]$ or $[JS^*]$.

Our solution to these two problems will, in part, be a compromise. The solution to the problem with $[RE]$, or rather a concession, is to doctor our inference relations in such a way that they sanction replication of self-inconsistent wffs, while keeping most of their preservational properties in tact. By this we mean that we allow self-inconsistent wffs to be reproduced in the conclusion set only from self-inconsistent wffs appearing in the premise set. Since we are more experienced with \vdash and its corresponding modal logics, we will focus

⁴The issue of finite axiomatizability of K_n is discussed by Peter Apostoli and Bryson Brown in [2]. As it turns out it is a non-trivial matter to show that in the presence of $[N]$ and $[RR]$, $[K_{n\wedge}]$ yields closure under $[JS]$. Both Apostoli [1] and Brown [5] have independently shown that K_n is finitely axiomatizable under $[JS]$. Apostoli's proof in particular, brings out some deep connections between hypergraph colouring problem and weak aggregative modal logics.

on \vdash^* . First of all, define the relation \vdash_c as follows:

Definition 3.1 $\Gamma \vdash_c \alpha$ iff $\exists \beta \in \Gamma: \beta \vdash \alpha$ and $\alpha \vdash \perp$

We now define a new relation from \vdash^* and \vdash_c .

Definition 3.2 For any Γ and α , we define \triangleright , (general) yielding, as follows:

- (1) if $\ell^*(\Gamma) = \infty$, then $\Gamma \triangleright \alpha$ iff $\Gamma \vdash_c \alpha$
- (2) if $\ell^*(\Gamma) \neq \infty$, then $\Gamma \triangleright \alpha$ iff either $\Gamma \vdash^* \alpha$ or $\Gamma \vdash_c \alpha$

Provisionally, we can defend \triangleright by pointing out that in switching from \vdash^* to \triangleright we have not lost any preservational power. \triangleright clearly preserves the ℓ^* -value of any set as before. The only difference is that \triangleright permits replication of self-inconsistent wffs for any set and thus no set is inferentially explosive under \triangleright . From a syntactic point of view, perhaps we have lost some distinction between different types of self-inconsistency. But from a semantical point of view we can justify our move by qualifying that our treatment regards self-inconsistent wffs as inferentially uninformative and thus there is no loss or gain in permitting replication of self-inconsistency. Having said that, we can now state the corresponding modalization rule for \triangleright ; again, the ℓ^* -value is fixed at a given level n :

$$[RY_n] \frac{\Gamma \triangleright \alpha}{\Box[\Gamma] \vdash \Box \alpha} \ell^*(\Gamma) = n$$

In stating $[RY_n]$ as a rule, we in fact bite the bullet and concede that $\Box[\Gamma] \vdash \Box(q \wedge \neg q)$ when $\Box(p \wedge \neg p) \in \Box[\Gamma]$. But this is acceptable provided that $[RR]$ is not a rule of inference. With this in mind, we solve the second part of the problem by defining our modal semantics specifically to suppress $[RR]$. Since we also need to distinguish self-inconsistent wffs from other wffs, we need to impart in our object language sufficient expressive power to make such a distinction. To accomplish the task, we will exploit the resources of $S5$, and introduce bi-modal formulae in our formal language. This means that our base logic is $S5$ instead of PL . Semantically, the $S5$ necessity, \blacksquare , and the $S5$ possibility, \blacklozenge , will bear their usual Leibnizian truth conditions. In the subsequent presentation, we will also need to redefine the notion of level, level*, and yielding according to the underlying logic $S5$.

3.1 The Syntax And Semantics Of JSR_n

The syntax of our formal language \mathcal{L} is defined as follows:

Definition 3.3 \mathcal{L} is a triple $\langle At, k, \Phi \rangle$ where

- At is a denumerable set atomic sentences, $\{p_i: i \in \mathbb{N}\}$
- $k = \{\neg, \rightarrow, \Box, \blacksquare, (\dots)\}$ where \neg, \Box, \blacksquare are unary connectives, \rightarrow a binary connective, and (\dots) are left and right brackets

We define the set of wffs, Φ , to be the least inductive set closed under propositional connectives, $\neg, \rightarrow, \Box, \blacksquare$ as follows:

1. $At \subset \Phi$
2. if $\alpha \in \Phi$ and $\beta \in \Phi$, then $\neg\alpha \in \Phi$, $(\alpha \rightarrow \beta) \in \Phi$, $\Box\alpha \in \Phi$, and $\blacksquare\alpha \in \Phi$
3. nothing is in Φ except by step 1 or 2

For readability, we adopt the convention of omitting outer brackets. In addition, the falsum \perp , the truth \top , the connectives, \vee, \wedge, \diamond , and \blacklozenge are introduced as abbreviations: \perp for any constant false sentence, \top for $\neg\perp$, $(\alpha \vee \beta)$ for $(\neg\alpha \rightarrow \beta)$, $(\alpha \wedge \beta)$ for $\neg(\alpha \rightarrow \neg\beta)$, $\diamond\alpha$ for $\neg\Box\neg\alpha$, and $\blacklozenge\alpha$ for $\neg\blacksquare\neg\alpha$. We now introduce the modal logic JSR_n . To simplify matters, we shall deploy our abbreviations in the subsequent presentation and use ' \vdash_{Λ_n} ' instead of ' \vdash_{JSR_n} ' to denote theoremhood or provability in JSR_n .

Definition 3.4 The logic $JSR_n (= \Lambda_n)$ is a subset of Φ satisfying

- $PL \subset \Lambda_n$
- Λ_n is closed under Modus Ponens

In addition, Λ_n contains the following axiom schemata and rules:

- $[T_{\blacksquare}] \vdash_{\Lambda_n} \blacksquare\alpha \rightarrow \alpha$
- $[5_{\blacksquare}] \vdash_{\Lambda_n} \neg\blacksquare\alpha \rightarrow \blacksquare\neg\blacksquare\alpha$
- $[RR_{\blacksquare}]$

$$\frac{\vdash_{\Lambda_n} \alpha \rightarrow \beta}{\vdash_{\Lambda_n} \blacksquare\alpha \rightarrow \blacksquare\beta}$$

- $[N] \vdash_{\Lambda_n} \Box\top$
- $[Br] \vdash_{\Lambda_n} \blacksquare\alpha \rightarrow \Box\alpha$

- $[K_n^{\Box\Diamond}]$

$$\vdash_{\Lambda_n} \bigwedge_{1 \leq k \leq n+1} (\Box \alpha_k \wedge \Diamond \alpha_k) \rightarrow (\Box \frac{2}{n+1} (\alpha_1, \dots, \alpha_{n+1}) \wedge \Diamond \frac{2}{n+1} (\alpha_1, \dots, \alpha_{n+1}))$$

- $[RR_{\Box}]$

$$\frac{\vdash_{\Lambda_n} \alpha \rightarrow \beta}{\vdash_{\Lambda_n} \Box \alpha \rightarrow (\Diamond \alpha \rightarrow \Box \beta)}$$

- $[RE_{\Box}]$

$$\frac{\vdash_{\Lambda_n} \alpha \leftrightarrow \beta}{\vdash_{\Lambda_n} \Box \alpha \leftrightarrow \Box \beta}$$

- $[RY_n]$

$$\frac{\Gamma \triangleright \alpha}{\Box[\Gamma] \vdash_{\Lambda_n} \Box \alpha} \ell^*(\Gamma) = n$$

Given the logic $JSR_n (= \Lambda_n)$, a wff α is a Λ_n -theorem iff $\alpha \in \Lambda_n$ and α is Λ_n -derivable from Γ , $\Gamma \vdash_{\Lambda_n} \alpha$ iff there exist $\alpha_1, \dots, \alpha_n \in \Gamma$ such that $(\alpha_1 \wedge \dots \wedge \alpha_n \rightarrow \alpha)$ is a Λ_n -theorem. A set Γ is Λ_n -consistent iff $\Gamma \not\vdash_{\Lambda_n} \perp$. Γ is maximal Λ_n -consistent iff Γ is Λ_n -consistent and for any $\alpha \in \Phi$, either $\alpha \in \Gamma$ or $\neg \alpha \in \Gamma$. Given these definitions, it is straightforward to verify that every Λ_n -consistent set has a maximal Λ_n -consistent extension. Also, observe that ℓ , ℓ^* , and \triangleright are redefined in terms of \vdash_{Λ_n} .

Definition 3.5 A NR_n Frame is a triple $\mathfrak{F} = \langle \mathcal{U}, \mathcal{N}, \mathcal{R} \rangle$ where

$$\mathcal{U} \neq \emptyset$$

$$\mathcal{N}: \mathcal{U} \rightarrow \{\{\mathcal{U}\}, \{\emptyset\}\}$$

$$\mathcal{R} \subseteq \mathcal{U}^{n+1}$$

A NR_n Model is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{V} \rangle$ where

$$\mathcal{V}: At \rightarrow 2^{\mathcal{U}}$$

\mathcal{V} is extended to $\|\cdot\|^{\mathfrak{M}}$ inductively by:

$$1. \|p_i\|^{\mathfrak{M}} = \mathcal{V}(p_i) \quad p_i \in At$$

$$2. \|\top\|^{\mathfrak{M}} = \mathcal{U}$$

$$3. \|\perp\|^{\mathfrak{M}} = \emptyset$$

$$4. \|\neg \alpha\|^{\mathfrak{M}} = \mathcal{U} - \|\alpha\|^{\mathfrak{M}}$$

$$5. \|\alpha \rightarrow \beta\|^{\mathfrak{M}} = (\mathcal{U} - \|\alpha\|^{\mathfrak{M}}) \cup \|\beta\|^{\mathfrak{M}}$$

$$6. \|\blacksquare\alpha\|^{\mathfrak{M}} = \{x \in \mathcal{U} : \|\alpha\|^{\mathfrak{M}} = \mathcal{U}\}$$

$$7. \|\Box\alpha\|^{\mathfrak{M}} = \{x \in \mathcal{U} : \forall y_1, \dots, y_n \in \mathcal{U}, \mathcal{R}xy_1, \dots, y_n \Rightarrow \exists i(1 \leq i \leq n) : y_i \in \|\alpha\|^{\mathfrak{M}} \text{ or } \|\alpha\|^{\mathfrak{M}} \in \mathcal{N}(x)\}$$

Definition 3.6 For any $\alpha, \beta \in \Phi$, any $\Gamma \subseteq \Phi$, we say α is satisfiable in a NR_n -model, \mathfrak{M} , iff $\|\alpha\|^{\mathfrak{M}} \neq \emptyset$. α is valid in a NR_n -model, \mathfrak{M} , iff $\|\alpha\|^{\mathfrak{M}} = \mathcal{U}$. α is valid in a NR_n -frame, \mathfrak{F} , iff α is valid in every model on \mathfrak{F} . We say that Γ entails α in a NR_n -model, \mathfrak{M} , iff $\|\Gamma\|^{\mathfrak{M}} \subseteq \|\alpha\|^{\mathfrak{M}}$, where $\|\Gamma\|^{\mathfrak{M}} = \{x \in \mathcal{U} : x \in \bigcap_{\gamma \in \Gamma} \|\gamma\|^{\mathfrak{M}}\}$. Γ entails α in a NR_n -frame, \mathfrak{F} , iff for Γ entails α in every model on \mathfrak{F} . Lastly, we say that α and β are semantically equivalent iff for every model, \mathfrak{M} , on every NR_n -frame, $\|\alpha\|^{\mathfrak{M}} = \|\beta\|^{\mathfrak{M}}$.

Now to see that $[RR\Box]$ does not preserve validity on a NR_n -frame \mathfrak{F} in general simply observe that even though $\|\perp\|^{\mathfrak{M}} \subseteq \|\beta\|^{\mathfrak{M}}$ for some model \mathfrak{M} in \mathfrak{F} , $\|\Box\perp\|^{\mathfrak{M}} \not\subseteq \|\Box\beta\|^{\mathfrak{M}}$ when $\|\beta\|^{\mathfrak{M}} \neq \emptyset$.

3.2 Determination Of JSR_n

Theorem 3.1 The logic $JSR_n (= \Lambda_n)$ is sound with respect to the class, \mathfrak{C}_{NR_n} , of all NR_n -frames, i.e. $\forall \alpha \in \Phi$, if $\vdash_{\Lambda_n} \alpha$, then $\forall \mathfrak{F} \in \mathfrak{C}_{NR_n}$, $\mathfrak{F} \models \alpha$

Proof.

Let $\mathfrak{F} = \langle \mathcal{U}, \mathcal{N}, \mathcal{R} \rangle$, be an arbitrary NR_n -frame in \mathfrak{C}_{NR_n} ; \mathfrak{M} an arbitrary model on \mathfrak{F} . The cases for PL , $[RR\blacksquare]$, $[T\blacksquare]$, $[5\blacksquare]$, $[N]$, and $[Br]$ are trivial. We will consider the rest of the cases:

(1) $[RE\Box]$:

Let α and β be arbitrary wffs such that $\|\alpha\|^{\mathfrak{M}} = \|\beta\|^{\mathfrak{M}}$. We need to show that $\|\Box\alpha\|^{\mathfrak{M}} = \|\Box\beta\|^{\mathfrak{M}}$.

(i) $\|\Box\alpha\|^{\mathfrak{M}} \subseteq \|\Box\beta\|^{\mathfrak{M}}$:

let $x \in \|\Box\alpha\|^{\mathfrak{M}}$ be arbitrary. Then either $\forall y_1, \dots, y_n \in \mathcal{U}, \mathcal{R}xy_1, \dots, y_n \Rightarrow \exists i(1 \leq i \leq n) : y_i \in \|\alpha\|^{\mathfrak{M}}$ or $\|\alpha\|^{\mathfrak{M}} \in \mathcal{N}(x)$ or But $\|\alpha\|^{\mathfrak{M}} = \|\beta\|^{\mathfrak{M}}$. So either ways $x \in \|\Box\beta\|^{\mathfrak{M}}$. But x was arbitrary, so $\|\Box\alpha\|^{\mathfrak{M}} \subseteq \|\Box\beta\|^{\mathfrak{M}}$.

(ii) $\|\Box\beta\|^{\mathfrak{M}} \subseteq \|\Box\alpha\|^{\mathfrak{M}}$: similar

(2) $[RR^-]$:

Assume that $\|\alpha\|^{\mathfrak{M}} \subseteq \|\beta\|^{\mathfrak{M}}$ for some arbitrary wffs α and β . We need to show that $(\|\Box\alpha\|^{\mathfrak{M}} \cap \|\Diamond\alpha\|^{\mathfrak{M}}) \subseteq \|\Box\beta\|^{\mathfrak{M}}$. Let $x \in \|\Box\alpha\|^{\mathfrak{M}} \cap \|\Diamond\alpha\|^{\mathfrak{M}}$ be arbitrary. So clearly, $x \in \{y \in \mathcal{U}: \|\alpha\|^{\mathfrak{M}} \neq \emptyset\}$. But $x \in \|\Box\alpha\|^{\mathfrak{M}}$, so either $\|\alpha\|^{\mathfrak{M}} \in \mathcal{N}(x)$ or $\forall y_1, \dots, y_n \in \mathcal{U}, \mathcal{R}xy_1, \dots, y_n \Rightarrow \exists i_{(1 \leq i \leq n)}: y_i \in \|\alpha\|^{\mathfrak{M}}$. Suppose, $\|\alpha\|^{\mathfrak{M}} \in \mathcal{N}(x)$. Since $\|\alpha\|^{\mathfrak{M}} \neq \emptyset$, $\mathcal{N}(x) = \{\mathcal{U}\}$. But $\|\alpha\|^{\mathfrak{M}} \subseteq \|\beta\|^{\mathfrak{M}}$, so $\|\beta\|^{\mathfrak{M}} \in \mathcal{N}(x)$. So $x \in \|\Box\beta\|^{\mathfrak{M}}$. Assume that $\|\alpha\|^{\mathfrak{M}} \notin \mathcal{N}(x)$. Let $y_1, \dots, y_n \in \mathcal{U}$ be arbitrary and such that $\mathcal{R}xy_1, \dots, y_n$. Then $\exists i_{(1 \leq i \leq n)}: y_i \in \|\alpha\|^{\mathfrak{M}}$. But $\|\alpha\|^{\mathfrak{M}} \subseteq \|\beta\|^{\mathfrak{M}}$. So $\exists i_{(1 \leq i \leq n)}: y_i \in \|\beta\|^{\mathfrak{M}}$. But y_1, \dots, y_n were arbitrary, so $x \in \|\Box\beta\|^{\mathfrak{M}}$. Since x was arbitrary, $(\|\Box\alpha\|^{\mathfrak{M}} \cap \|\Diamond\alpha\|^{\mathfrak{M}}) \subseteq \|\Box\beta\|^{\mathfrak{M}}$.

(3) $[K_n^{\Box\Diamond}]$:

Let $\alpha_1, \dots, \alpha_{n+1} \in \Phi$ be arbitrary. Let $x \in \bigcap_{1 \leq i \leq n+1} \|\Box\alpha_i \wedge \Diamond\alpha_i\|^{\mathfrak{M}}$ be arbitrary. It suffices to show that $x \in \|\Box_{\frac{2}{n+1}}(\alpha_1, \dots, \alpha_{n+1})\|^{\mathfrak{M}} \cap \|\Diamond_{\frac{2}{n+1}}(\alpha_1, \dots, \alpha_{n+1})\|^{\mathfrak{M}}$. Let $y_1, \dots, y_n \in \mathcal{U}$ be such that $\mathcal{R}xy_1, \dots, y_n$. We consider two cases:

(i) $\exists j_{(1 \leq j \leq n+1)}: \|\alpha_j\|^{\mathfrak{M}} = \mathcal{U}$:

Since $x \in \bigcap_{1 \leq i \leq n+1} \|\Diamond\alpha_i\|^{\mathfrak{M}}$, $\forall i_{(1 \leq i \leq n+1)}$, if $\|\alpha_i\|^{\mathfrak{M}} \neq \|\alpha_j\|^{\mathfrak{M}}$, then $\|\alpha_i\|^{\mathfrak{M}} \notin \mathcal{N}(x)$. Let $\|\alpha_k\|^{\mathfrak{M}} \neq \|\alpha_j\|^{\mathfrak{M}}$, where $1 \leq k \leq n+1$. Then $\|\alpha_k\|^{\mathfrak{M}} \notin \mathcal{N}(x)$. But $x \in \|\Box\alpha_k\|^{\mathfrak{M}}$. So, $\exists i_{(1 \leq i \leq n)}: y_i \in \|\alpha_k\|^{\mathfrak{M}}$. But $\|\alpha_k\|^{\mathfrak{M}} \cap \|\alpha_j\|^{\mathfrak{M}} = \|\alpha_k\|^{\mathfrak{M}}$. So $\exists i_{(1 \leq i \leq n)}: y_i \in \|\alpha_k \wedge \alpha_j\|^{\mathfrak{M}}$. So $\exists i_{(1 \leq i \leq n)}: y_i \in \|\frac{2}{n+1}(\alpha_1, \dots, \alpha_{n+1})\|^{\mathfrak{M}}$. Since y_1, \dots, y_n were arbitrary, it follows that $x \in \|\Box_{\frac{2}{n+1}}(\alpha_1, \dots, \alpha_{n+1})\|^{\mathfrak{M}} \cap \|\Diamond_{\frac{2}{n+1}}(\alpha_1, \dots, \alpha_{n+1})\|^{\mathfrak{M}}$.

(ii) $\forall i_{(1 \leq i \leq n+1)}, \|\alpha_i\|^{\mathfrak{M}} \neq \mathcal{U}$:

Then $\forall i_{(1 \leq i \leq n+1)}, \|\alpha_i\|^{\mathfrak{M}} \neq \mathcal{N}(x)$ since $x \in \bigcap_{1 \leq i \leq n+1} \|\Diamond\alpha_i\|^{\mathfrak{M}}$. But by our the initial assumption $x \in \bigcap_{1 \leq i \leq n+1} \|\Box\alpha_i\|^{\mathfrak{M}}$, so $\forall i_{(1 \leq i \leq n+1)}, \exists j_{(1 \leq j \leq n)}: y_j \in \|\alpha_i\|^{\mathfrak{M}}$. It follows that $\exists j_{(1 \leq j \leq n)}: y_j \in \|\frac{2}{n+1}(\alpha_1, \dots, \alpha_{n+1})\|^{\mathfrak{M}}$. Given that y_1, \dots, y_n were arbitrary this suffices to show that $x \in \|\Box_{\frac{2}{n+1}}(\alpha_1, \dots, \alpha_{n+1})\|^{\mathfrak{M}} \cap \|\Diamond_{\frac{2}{n+1}}(\alpha_1, \dots, \alpha_{n+1})\|^{\mathfrak{M}}$.

(4) $[RY_n]$:

Assume that for arbitrary Γ and α , $\Gamma \triangleright \alpha$ and $\ell^*(\Gamma) = n$. We show that $\Box[\Gamma]$ entails $\Box\alpha$, i.e. $\|\Box[\Gamma]\|^{\mathfrak{M}} \subseteq \|\Box\alpha\|^{\mathfrak{M}}$. There are two cases:

(i) $\exists \gamma_\alpha \in \Gamma: \gamma_\alpha \vdash_{\Lambda_n} \alpha$ and $\alpha \vdash_{\Lambda_n} \perp$. So $\|\Gamma\|^{\mathfrak{M}} \subseteq \|\gamma_\alpha\|^{\mathfrak{M}} \subseteq \|\alpha\|^{\mathfrak{M}} \subseteq \|\perp\|^{\mathfrak{M}}$. So $\|\gamma_\alpha\|^{\mathfrak{M}} = \|\alpha\|^{\mathfrak{M}} = \|\perp\|^{\mathfrak{M}}$. Let $x \in \|\Box[\Gamma]\|^{\mathfrak{M}}$ be arbitrary. Then $x \in \bigcap_{\gamma \in \Gamma} \|\Box\gamma\|^{\mathfrak{M}}$. But $\gamma_\alpha \in \Gamma$ by assumption. So $x \in \|\Box\gamma_\alpha\|^{\mathfrak{M}}$. So either $\|\gamma_\alpha\|^{\mathfrak{M}} \in \mathcal{N}(x)$ or $\forall y_1, \dots, y_n \in \mathcal{U}, \mathcal{R}xy_1, \dots, y_n \Rightarrow$

$\exists i_{(1 \leq i \leq n)}: y_i \in \|\gamma_\alpha\|^{\mathfrak{M}}$. But $\|\gamma_\alpha\|^{\mathfrak{M}} = \|\alpha\|^{\mathfrak{M}}$. Thus, $x \in \|\Box\alpha\|^{\mathfrak{M}}$.

(ii) $\Gamma \vdash^* \alpha$. Assume that $\exists \mathfrak{d} \in \mathfrak{D}_n: \mathfrak{d} = \{\mathfrak{g}_1, \dots, \mathfrak{g}_n\}$ and $\forall i_{(1 \leq i \leq n)} \|\mathfrak{g}_i\|^{\mathfrak{M}} \neq \emptyset$ and $\bigcup \mathfrak{g}_i = \Gamma^*$ and $\exists \mathfrak{g}_\alpha \in \mathfrak{d}: \|\mathfrak{g}_\alpha\|^{\mathfrak{M}} \subseteq \|\alpha\|^{\mathfrak{M}}$. Since $\mathfrak{g}_\alpha \subseteq \Gamma$, $\|\Box[\Gamma]\|^{\mathfrak{M}} \subseteq \|\Box[\mathfrak{g}_\alpha]\|^{\mathfrak{M}}$. Let $x \in \|\Box[\Gamma]\|^{\mathfrak{M}}$ be arbitrary. Then $x \in \|\Box[\mathfrak{g}_\alpha]\|^{\mathfrak{M}}$. So either $\|\mathfrak{g}_\alpha\|^{\mathfrak{M}} \in \mathcal{N}(x)$ or $\forall y_1, \dots, y_n \in \mathcal{U}, \mathcal{R}xy_1, \dots, y_n \Rightarrow \exists i_{(1 \leq i \leq n)}: y_i \in \|\mathfrak{g}_\alpha\|^{\mathfrak{M}}$. If $\|\mathfrak{g}_\alpha\|^{\mathfrak{M}} = \mathcal{U}$, then $\|\alpha\|^{\mathfrak{M}} = \mathcal{U}$. So $x \in \|\Box\alpha\|^{\mathfrak{M}}$ holds trivially. Assume that $\|\mathfrak{g}_\alpha\|^{\mathfrak{M}} \neq \mathcal{U}$. Then $\|\mathfrak{g}_\alpha\|^{\mathfrak{M}} \notin \mathcal{N}(x)$. So $\forall y_1, \dots, y_n \in \mathcal{U}, \mathcal{R}xy_1, \dots, y_n \Rightarrow \exists i_{(1 \leq i \leq n)}: y_i \in \|\mathfrak{g}_\alpha\|^{\mathfrak{M}}$. But $\|\mathfrak{g}_\alpha\|^{\mathfrak{M}} \subseteq \|\alpha\|^{\mathfrak{M}}$, thus $\forall y_1, \dots, y_n \in \mathcal{U}, \mathcal{R}xy_1, \dots, y_n \Rightarrow \exists i_{(1 \leq i \leq n)}: y_i \in \|\alpha\|^{\mathfrak{M}}$. Hence, $x \in \|\Box\alpha\|^{\mathfrak{M}}$ as required. \blacksquare

Before we present the completeness result for JSR_n . We need an additional definition and a lemma.

Definition 3.7 Let Γ be a set of wffs and Σ a collection of finite subsets of Γ . Σ is a m -cluster over Γ iff for every m -covering family, \mathfrak{d} , of Γ there is a constituent $\mathfrak{g} \in \mathfrak{d}$ such that some element of Σ is a subset of \mathfrak{g} .

Lemma 3.1 (Cluster Compactness) Let Γ be a set of wffs and Σ be a m -cluster over Γ . Then there is a $\Sigma^- \subseteq_{fin} \Sigma$ such that Σ^- is a m -cluster of Γ (where ' \subseteq_{fin} ' denotes the finite subset relation).

Proof. Our strategy is to prove the contrapositive, i.e. if every finite subset of Σ is not a m -cluster of Γ , then Σ is not a m -cluster of Γ . We proceed to construct a first order theory, \mathcal{T} , such that every finite subset of Σ is not a m -cluster of Γ iff every finite subset of \mathcal{T} has a model. So by the compactness theorem of first order logic, if every finite subset of Σ is not a m -cluster of Γ , then \mathcal{T} has a model and hence by the construction of \mathcal{T} , Σ is not a m -cluster of Γ . Let $\Gamma \subseteq \Phi$. For simplicity let Σ be a countable set of finite subsets of Γ :

$$\Sigma = \{\mathfrak{a}_1, \dots, \mathfrak{a}_i, \dots\}$$

For each $i \in \mathbb{N}$, let

$$\mathfrak{a}_i = \{\alpha_{i1}, \dots, \alpha_{ik_i}\}$$

Assume that every finite subset of Σ is not a m -cluster of Γ , i.e. for each finite subset Σ^- of Σ , there is a m -covering family, \mathfrak{d} , of Γ such that no element of \mathfrak{d} is a superset of elements of Σ^- . Now extend first order language \mathcal{L}_f with

- m unary predicate symbols, P_1, \dots, P_m , each symbol represents a constituent in a m -covering family
- denumerably many constant symbols, $c_{11}, \dots, c_{1k_1}, \dots, c_{i1}, \dots, c_{ik_i}, \dots$ each indexed to the corresponding element of Σ .

Let

$$\Theta = \{\forall x (\bigvee_{1 \leq h \leq m} P_h x)\}$$

Consider the following infinite sequence of sets:

$$\begin{aligned} \Omega_1 &= \{ \bigwedge_{1 \leq h \leq m} (\bigvee_{11 \leq i_j \leq 1k_1} \neg P_h c_{1j}) \} \\ &\vdots \\ \Omega_i &= \{ \bigwedge_{1 \leq h \leq m} (\bigvee_{i1 \leq i_j \leq ik_i} \neg P_h c_{ij}) \} \\ &\vdots \end{aligned}$$

Let $\Omega_\omega = \bigcup_{1 \leq n \leq \omega} \Omega_n$. By our assumption every finite subset of Σ is not a m -cluster of Γ so every finite subset of $\Omega_\omega \cup \Theta$ has a model. By first order compactness, $\Omega_\omega \cup \Theta$ has a model. Hence there is a m -covering family, \mathfrak{d} , of Γ such that no element of \mathfrak{d} is a superset of elements of Σ . Hence, Σ is not a m -cluster of Γ . ■

We are now in the position to prove the completeness result and the corresponding fundamental theorem for JSR_n .

Theorem 3.2 *Let Δ be any Λ_n -consistent set of wffs. Then Δ is satisfiable in a NR_n -model, $\mathfrak{M} = \langle \mathcal{U}, \mathcal{N}, \mathcal{R}, \mathcal{V} \rangle$, on a NR_n -frame.*

Proof.

Let Δ be Λ_n -consistent set of wffs. Let Δ^+ be a maximal Λ_n -consistent extension of Δ . Define the canonical Δ^+ model, $\mathfrak{M}_{\Delta^+} = \langle \mathcal{U}_{\Delta^+}, \mathcal{N}_{\Delta^+}, \mathcal{R}_{\Delta^+}, \mathcal{V}_{\Delta^+} \rangle$, as follows:

$\mathcal{U}_{\Delta^+} = \{x: x \text{ is a maximal } \Lambda_n\text{-consistent set and } \blacksquare(\Delta^+) \subseteq x\}$,
 where $\blacksquare(\Delta^+) = \{\alpha: \blacksquare\alpha \in \Delta^+\}$

$\mathcal{N}_{\Delta^+}: \mathcal{U}_{\Delta^+} \longrightarrow \{\{\mathcal{U}_{\Delta^+}\}, \{\emptyset\}\}$ is defined by:

$\forall x \in \mathcal{U}_{\Delta^+}$,

$$\mathcal{N}_{\Delta^+}(x) = \begin{cases} \{\mathcal{U}_{\Delta^+}\} & \text{if } \Box \perp \notin x \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

$\mathcal{R}_{\Delta^+} \subseteq \mathcal{U}_{\Delta^+}^{n+1}$ is defined by:

$$\forall xy_1, \dots, y_n \in \mathcal{U}_{\Delta^+}, \mathcal{R}_{\Delta^+}xy_1, \dots, y_n \Leftrightarrow \Box(x)^* \subseteq \bigcup_{1 \leq i \leq n} y_i$$

where $\Box(x)^* = \{\alpha: \Box\alpha \in x \text{ and } \alpha \not\vdash_{\Lambda_n} \perp\}$

$\mathcal{V}_{\Delta^+}: At \rightarrow 2^{\mathcal{U}_{\Delta^+}}$ defined as:

$$\forall x \in \mathcal{U}_{\Delta^+}, \forall p_i \in At, x \in \mathcal{V}_{\Delta^+}(p_i) \Leftrightarrow p_i \in x$$

We prove that $\forall \alpha \in \Phi, \forall x \in \mathcal{U}_{\Delta^+}, \alpha \in x$ iff $x \in \|\alpha\|^{\mathfrak{M}_{\Delta^+}}$ (i.e. the fundamental theorem). The proof is by induction on the complexity of α . If $\alpha = p_i$, for some $p_i \in At$, then the claim follows immediately from the definition of \mathcal{V}_{Δ^+} . We prove that the claim holds for α of the form $\neg\beta, \beta \rightarrow \gamma, \blacksquare\beta$, and $\Box\beta$. The cases for $\alpha = \neg\beta$, and $\alpha = \beta \rightarrow \gamma$ are trivial. We will only consider (A) $\alpha = \blacksquare\beta$, and (B) $\alpha = \Box\beta$. Notice that the induction hypothesis states that $\forall x \in \mathcal{U}_{\Delta^+}, \beta \in x$ iff $x \in \|\beta\|^{\mathfrak{M}_{\Delta^+}}$.

(A1) Assume that $x \in \mathcal{U}_{\Delta^+}$ and that $x \notin \|\blacksquare\beta\|^{\mathfrak{M}_{\Delta^+}}$. Then $\exists y \in \mathcal{U}_{\Delta^+}: y \in \|\neg\beta\|^{\mathfrak{M}_{\Delta^+}}$. By the induction hypothesis, $\beta \notin y$. By the definition of \mathcal{U}_{Δ^+} , $\neg\blacksquare\beta \in \Delta^+$. By [5 \blacksquare], $\blacksquare\neg\blacksquare\beta \in \Delta^+$. By definition of \mathcal{U}_{Δ^+} , $\neg\blacksquare\beta \in x$. But x is Λ_n -consistent, so $\blacksquare\beta \notin x$.

(A2) Assume that $\blacksquare\beta \notin x$. We show the consistency of $\Sigma = \blacksquare(\Delta^+) \cup \{\neg\beta\}$. Assume not. Then $\exists i: \gamma_1, \dots, \gamma_i \in \blacksquare(\Delta^+)$ and $\vdash_{\Lambda_n} \gamma_1 \wedge \dots \wedge \gamma_i \rightarrow \beta$.

By [RR \blacksquare], $\vdash_{\Lambda_n} \blacksquare(\gamma_1 \wedge \dots \wedge \gamma_i) \rightarrow \blacksquare\beta$.

But $\vdash_{\Lambda_n} \blacksquare\gamma_1 \wedge \dots \wedge \blacksquare\gamma_i \rightarrow \blacksquare(\gamma_1 \wedge \dots \wedge \gamma_i)$.

So, $\vdash_{\Lambda_n} \blacksquare\gamma_1 \wedge \dots \wedge \blacksquare\gamma_i \rightarrow \blacksquare\beta$.

By definition of $\blacksquare(\Delta^+)$, $\blacksquare\gamma_1, \dots, \blacksquare\gamma_i \in \Delta^+$.

Thus, $\blacksquare\beta \in \Delta^+$. But $\vdash_{\Lambda_n} \blacksquare\beta \rightarrow \blacksquare\blacksquare\beta$. So $\blacksquare\blacksquare\beta \in \Delta^+$. By the definition of \mathcal{U}_{Δ^+} , $\forall y \in \mathcal{U}_{\Delta^+}, \blacksquare\beta \in y$. In particular, $\blacksquare\beta \in x$. But this contradicts our assumption. So Σ is Λ_n -consistent. Let Σ^+ be maximal Λ_n -consistent extension of Σ . So $\Sigma^+ \in \mathcal{U}_{\Delta^+}$, and $\beta \notin \Sigma^+$. Hence, by the induction hypothesis, $\Sigma^+ \notin \|\beta\|^{\mathfrak{M}_{\Delta^+}}$, i.e. $\|\beta\|^{\mathfrak{M}_{\Delta^+}} \neq \mathcal{U}_{\Delta^+}$. Therefore, $x \notin \|\blacksquare\beta\|^{\mathfrak{M}_{\Delta^+}}$.

(B1) Assume that $x \in \mathcal{U}_{\Delta+}$, and that $\Box\beta \in x$. Then there are two subcases:

(a) $\beta = \perp$: Then $\vdash_{\Lambda_n} \beta \leftrightarrow \perp$. So by $[RE_{\Box}]$, $\vdash_{\Lambda_n} \Box\beta \leftrightarrow \Box\perp$. So $\Box\perp \in x$. By the definition of $\mathcal{N}_{\Delta+}$, $\mathcal{N}(x) = \{\emptyset\}$. By the soundness of JSR_n , $\|\beta\|^{\mathfrak{M}_{\Delta+}} = \|\perp\|^{\mathfrak{M}_{\Delta+}} = \emptyset$. Hence, $\|\beta\|^{\mathfrak{M}_{\Delta+}} \in \mathcal{N}_{\Delta+}(x)$. Therefore, $x \in \|\Box\beta\|^{\mathfrak{M}_{\Delta+}}$.

(b) $\beta \neq \perp$: Let $y_1, \dots, y_n \in \mathcal{U}_{\Delta+}$ be arbitrary and such that $\mathcal{R}_{\Delta+}xy_1, \dots, y_n$. Then $\Box(x)^* \subseteq \bigcup_{1 \leq i \leq n} y_i$. But $\beta \in \Box(x)^*$, so $\beta \in \bigcup_{1 \leq i \leq n} y_i$. Hence, $\exists i_{(1 \leq i \leq n)}: \beta \in y_i$. By the induction hypothesis, $\exists i_{(1 \leq i \leq n)}: y_i \in \|\beta\|^{\mathfrak{M}_{\Delta+}}$. But y_1, \dots, y_n were arbitrary, thus $\forall y_1, \dots, y_n \in \mathcal{U}_{\Delta+}$, $\mathcal{R}_{\Delta+}xy_1, \dots, y_n \Rightarrow \exists i_{(1 \leq i \leq n)}: y_i \in \|\beta\|^{\mathfrak{M}_{\Delta+}}$. Therefore, $x \in \|\Box\beta\|^{\mathfrak{M}_{\Delta+}}$.

(B2) Assume that $\Box\beta \notin x$. It suffices to show that $\exists y_1, \dots, y_n \in \mathcal{U}_{\Delta+}: \Box(x)^* \subseteq \bigcup_{1 \leq i \leq n} y_i$ and $\forall i_{(1 \leq i \leq n)}, \neg\beta \in y_i$ and $\|\beta\|^{\mathfrak{M}_{\Delta+}} \notin \mathcal{N}_{\Delta+}(x)$. We consider two cases:

(a) $\beta = \perp$: Then $\vdash_{\Lambda_n} \beta \leftrightarrow \perp$. By $[RE_{\Box}]$, $\vdash_{\Lambda_n} \Box\beta \leftrightarrow \Box\perp$. So $\Box\perp \notin x$. By the definition of $\mathcal{N}_{\Delta+}$, $\mathcal{N}_{\Delta+}(x) = \{\mathcal{U}_{\Delta+}\}$. But by soundness of JSR_n , $\|\beta\|^{\mathfrak{M}_{\Delta+}} = \|\perp\|^{\mathfrak{M}_{\Delta+}} = \emptyset$. Hence, $\|\beta\|^{\mathfrak{M}_{\Delta+}} \notin \mathcal{N}_{\Delta+}(x)$. Since, by assumption $\beta = \perp$, $\neg\beta = \top$. So, $\forall y \in \mathcal{U}_{\Delta+}$, $\neg\beta \in y$. It remains to be proven that $\exists y_1, \dots, y_n \in \mathcal{U}_{\Delta+}: \Box(x) \subseteq \bigcup_{1 \leq i \leq n} y_i$ (since $\Box\perp \notin x$, $\Box(x)^* = \Box(x)$). We first show that there is a covering family of $\Box(x)$ into n Λ_n -consistent constituents. Suppose for reductio that such n -covering family does not exist, i.e. $\forall \mathfrak{d} \in \mathfrak{D}_n(\Box(x)^*), \exists \mathfrak{g} \in \mathfrak{d}: \mathfrak{g} \vdash_{\Lambda_n} \perp$. Then $\Box(x) \triangleright \perp$. By the rule $[RY_n]$, $x \vdash_{\Lambda_n} \Box\perp$. So $\Box\perp \in x$. This contradicts our hypothesis. So $\exists \mathfrak{d} \in \mathfrak{D}_n(\Box(x))$: $\mathfrak{d} = \{\mathfrak{g}_1, \dots, \mathfrak{g}_n\}$ and $\forall i_{(1 \leq i \leq n)}, \mathfrak{g}_i \not\vdash_{\Lambda_n} \perp$. By Lindenbaum's Lemma, each constituent can be extended to maximality. Hence, $\exists y_1, \dots, y_n \in \mathcal{U}_{\Delta+}: \Box(x) \subseteq \bigcup_{1 \leq i \leq n} y_i$

(b) $\beta \neq \perp$: Then $\|\beta\|^{\mathfrak{M}_{\Delta+}} \neq \emptyset$, and $\|\beta\|^{\mathfrak{M}_{\Delta+}} \neq \mathcal{U}_{\Delta+}$. So $\forall y \in \mathcal{U}_{\Delta+}$, $\|\beta\|^{\mathfrak{M}_{\Delta+}} \notin \mathcal{N}_{\Delta+}(y)$. We show that $\exists \mathfrak{d} \in \mathfrak{D}_n(\Box(x)^*)$: $\mathfrak{d} = \{\mathfrak{g}_1, \dots, \mathfrak{g}_n\}$ and $\forall i_{(1 \leq i \leq n)}, \mathfrak{g}_i, \neg\beta \not\vdash_{\Lambda_n} \perp$. Assume for reductio that such \mathfrak{d} does not exist. There are two subcases:

(i) $\Box(x)^*$ is not decomposable into n Λ_n -consistent constituents: Then by argument similar to (a), $\Box(x)^* \triangleright \beta$. So by the rule $[RY_n]$, $x \vdash_{\Lambda_n} \Box\beta$. So $\Box\beta \in x$ which contradicts the initial assumption.

(ii) $\Box(x)^*$ is decomposable into n Λ_n -consistent constituents. Assume that $\forall \mathfrak{d} \in \mathfrak{D}_n(\Box(x)^*), \mathfrak{d} = \{\mathfrak{g}_1, \dots, \mathfrak{g}_n\}$ and $\forall i_{(1 \leq i \leq n)}, \mathfrak{g}_i \not\vdash_{\Lambda_n} \perp, \exists j_{(1 \leq j \leq n)}: \mathfrak{g}_j \vdash_{\Lambda_n} \beta$. Let

$$B = \{\mathfrak{b}: \mathfrak{b} \subseteq_{fin} \mathfrak{g}_j \text{ and } \mathfrak{b} \vdash_{\Lambda_n} \beta\}$$

Clearly, B is an n -cluster over $\Box(x)^*$. By lemma 3.1, $\exists B_0 = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$: $B_0 \subseteq_{fin} B$ and B_0 is a n -cluster over $\Box(x)^*$. Let

$$\alpha = \bigvee_{1 \leq i \leq k, \mathbf{b}_i \in B_0} \bigwedge \mathbf{b}_i$$

where $\bigwedge \mathbf{b}_i$ is the conjunction of all members of \mathbf{b}_i . Then clearly, $\Box(x)^* \triangleright \alpha$. By $[RY_n]$, $x \vdash_{\Lambda_n} \Box \alpha$. But by assumption, $\Box(x)^*$ is decomposable into n Λ_n -consistent constituents, and B_0 is an n -cluster over $\Box(x)^*$, thus $\alpha \not\vdash_{\Lambda_n} \perp$. Hence, $\exists z \in \mathcal{U}_{\Delta^+}$: $\alpha \in z$. So, $z \in \|\alpha\|^{\mathfrak{M}_{\Delta^+}}$. Hence, $\|\alpha\|^{\mathfrak{M}_{\Delta^+}} \neq \emptyset$. So, $\forall y \in \mathcal{U}_{\Delta^+}$, $y \in \|\diamond \alpha\|^{\mathfrak{M}_{\Delta^+}}$. Hence, $\|\diamond \alpha\|^{\mathfrak{M}_{\Delta^+}} = \mathcal{U}_{\Delta^+}$. So, $\forall y \in \mathcal{U}_{\Delta^+}$, $\diamond \alpha \in y$. So in particular, $\diamond \alpha \in x$. But by assumption, $\forall i_{(1 \leq i \leq k)}$, $\mathbf{b}_i, \neg \beta \vdash_{\Lambda_n} \perp$. So, $\forall i_{(1 \leq i \leq k)}$, $\vdash_{\Lambda_n} \bigwedge \mathbf{b}_i \rightarrow \beta$. Hence, $\vdash_{\Lambda_n} \alpha \rightarrow \beta$. By $[RR_{\Box}^-]$, $\vdash_{\Lambda_n} \Box \alpha \rightarrow (\diamond \alpha \rightarrow \Box \beta)$. By the deductive closure of x , $\Box \alpha \in x$ and $\Box \alpha \rightarrow (\diamond \alpha \rightarrow \Box \beta) \in x$. But $\diamond \alpha \in x$, so $\Box \beta \in x$. This contradicts our hypothesis. Hence, $\Box(x)^*$ is decomposable into n Λ_n -consistent constituents such that each constituent is Λ_n -consistent with $\neg \beta$. By Lindenbaum's Lemma, each constituent can be extended to a maximal Λ_n -consistent set which includes $\neg \beta$. Let y_1, \dots, y_n be such extensions. Hence by the induction hypothesis, the desired result obtains. ■

Chapter 4

Residual Issues And Problems

In this final chapter, we will consider some residual issues and open problems that are related to our study. Some general methodological lesson will also be brought to bear on possible research in the future.

4.1 A Descending Sequence Of Logics

We have seen that the logic $JSR_n (= \Lambda_n)$ is determined by the class of NR_n -frames. An obvious implication of our result is that, similar to K_i logics where $i \in \mathbb{N}$, the sequence of countably many logics:

$$\Lambda_1 \supseteq \Lambda_2 \supseteq \dots \supseteq \Lambda_{i-1} \supseteq \Lambda_i \supseteq \dots$$

are determined by their corresponding unrestricted class of structures. Since each of these logics endorses a progressively weaker form of the aggregation principle and the modalization rule, the corresponding sequence of classes of structures will have a progressively higher arity. Naturally, it is reasonable to assume that there is a limit to the sequence of Λ_i logics. In particular, we hypothesize that analogous to K_i logics where $\bigcap_{i \in \mathbb{N}} K_i = N$, the intersection of Λ_i logics, $\bigcap_{i \in \mathbb{N}} \Lambda_i$, is finitely axiomatizable by adding to the the base logic $S5$, the principle $[N]$, the rule $\lceil RR_{\square}^- \rceil$, and the rule $\lceil RE_{\square} \rceil$. We name this logic N^- .

To show that our hypothesis is indeed correct, it is instructive to recall the argument deployed by Jennings and Schotch in their proof that $\bigcap_{i \in \mathbb{N}} K_i = N$. Their argument essentially shows that:

1. N is determined by the class of locale frames, and

2. Any formula fails in a locale frame also fails in the class of relational frame of rank i , where i is the arity of the relation.

Since for each $i \in \mathbb{N}$, K_i is determined by the class of frame of rank i , it follows immediately, from 1. and 2., that every non-theorem of N is a non-theorem of K_i , and hence $\bigcap_{i \in \mathbb{N}} K_i \subseteq N$. We will give a similar argument to show that $\bigcap_{i \in \mathbb{N}} \Lambda_i = N^-$.

Definition 4.1 A *NL-frame* is a triple $\mathfrak{F} = \langle \mathcal{U}, \mathcal{N}, \mathcal{L} \rangle$ where

$$\mathcal{U} \neq \emptyset$$

$$\mathcal{N}: \mathcal{U} \longrightarrow \{\{\mathcal{U}\}, \{\emptyset\}\}$$

$$\mathcal{L}: \mathcal{U} \longrightarrow 2^{2^{\mathcal{U}}} \text{ satisfying}$$

Minimality $\forall x \in \mathcal{U}, \forall \mathbf{a} \subseteq \mathcal{U}, \mathbf{a} \in \mathcal{L}(x) \Rightarrow (\forall \mathbf{b}, \mathbf{b} \subset \mathbf{a} \Rightarrow \mathbf{b} \notin \mathcal{L}(x))$

Normality $\forall x \in \mathcal{U}, \mathcal{L}(x) \neq \emptyset$

Exclusivity $\forall x \in \mathcal{U}, \emptyset \notin \mathcal{L}(x)$

A *NL-model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{V} \rangle$ where

$$\mathcal{V}: At \longrightarrow 2^{\mathcal{U}}$$

\mathcal{V} is extended to $\|\cdot\|^{\mathfrak{M}}$ in the usual way for truth functional connectives.

For the modal cases, we have:

- $\|\blacksquare\alpha\|^{\mathfrak{M}} = \{x \in \mathcal{U}: \|\alpha\|^{\mathfrak{M}} = \mathcal{U}\}$
- $\|\Box\alpha\|^{\mathfrak{M}} = \{x \in \mathcal{U}: (\exists \mathbf{a} \in \mathcal{L}(x): \mathbf{a} \subseteq \|\alpha\|^{\mathfrak{M}}) \text{ or } \|\alpha\|^{\mathfrak{M}} \in \mathcal{N}(x)\}$

Theorem 4.1 N^- is determined by the class of *NL frames*.

Proof. The soundness of N^- is routine calculation, we will prove the completeness result instead.

The proof of completeness is by Henkin construction. In particular, we show that any N^- -consistent set, Δ , is satisfiable in a *NL model* on a *NL frame*. Let Δ^+ be a maximal N^- -extension of Δ , then define the canonical Δ^+ model $\mathfrak{M}_{\Delta^+} = \langle \mathcal{U}_{\Delta^+}, \mathcal{N}_{\Delta^+}, \mathcal{L}_{\Delta^+}, \mathcal{V}_{\Delta^+} \rangle$, as follows:

$$\mathcal{U}_{\Delta^+} = \{x: x \text{ is a maximal } \Lambda_n\text{-consistent set and } \blacksquare(\Delta^+) \subseteq x\},$$

where $\blacksquare(\Delta^+) = \{\alpha: \blacksquare\alpha \in \Delta^+\}$

$\mathcal{N}_{\Delta^+}: \mathcal{U}_{\Delta^+} \longrightarrow \{\{\mathcal{U}_{\Delta^+}\}, \{\emptyset\}\}$ is defined by:

$\forall x \in \mathcal{U}_{\Delta^+},$

$$\mathcal{N}_{\Delta^+}(x) = \begin{cases} \{\mathcal{U}_{\Delta^+}\} & \text{if } \Box\perp \notin x \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

To define \mathcal{L}_{Δ^+} we need some preliminary definitions:

- $\forall \alpha \in \Phi, |\alpha|_{N^-} = \{x \in \mathcal{U}_{\Delta^+}: \alpha \in x\}$
- $\forall x \in \mathcal{U}_{\Delta^+}, \Psi(x) = \{|\alpha|_{N^-}: \Box\alpha \in x \text{ and } \alpha \neq \perp\}$
- A maximal chain, C , of $\Psi(x)$ is a subset of $\Psi(x)$ such that
 1. $\forall \mathbf{a}, \mathbf{b} \in C, \mathbf{a} \subseteq \mathbf{b} \text{ or } \mathbf{b} \subseteq \mathbf{a}$ (i.e. C is a chain of $\Psi(x)$)
 2. $\forall \mathbf{c} \in \Psi(x) - C, \exists \mathbf{a} \in C: \mathbf{a} \not\subseteq \mathbf{c} \text{ and } \mathbf{c} \not\subseteq \mathbf{a}$ (i.e. C is maximal)

Clearly,

$$\bigcup \{C: C \text{ is a maximal chain of } \Psi(x)\} = \Psi(x)$$

$\mathcal{L}_{\Delta^+}: \mathcal{U}_{\Delta^+} \longrightarrow 2^{2^{\mathcal{U}_{\Delta^+}}}$ is defined by:

$$\forall x \in \mathcal{U}_{\Delta^+}, \mathcal{L}_{\Delta^+}(x) = \{\cap C: \emptyset \neq C \text{ is a maximal chain of } \Psi(x)\}$$

$\mathcal{V}_{\Delta^+}: At \longrightarrow 2^{\mathcal{U}_{\Delta^+}}$ is defined by:

$$\forall x \in \mathcal{U}_{\Delta^+}, \forall p_i \in At, x \in \mathcal{V}_{\Delta^+}(p_i) \Leftrightarrow p_i \in x$$

We now prove that $\forall \alpha \in \Phi, |\alpha|_{N^-} = \|\alpha\|^{\mathfrak{M}_{\Delta^+}}$. The proof is by induction on the complexity of α . If $\alpha = p_i$, for some $p_i \in At$, then the claim follows immediately from the definition of \mathcal{V}_{Δ^+} . The case for $\alpha = \neg\beta$, $\alpha = \beta \rightarrow \gamma$, are trivial. We consider cases in which (A) $\alpha = \blacksquare\beta$ and (B) $\alpha = \Box\beta$. Notice that the induction hypothesis states that $|\beta|_{N^-} = \|\beta\|^{\mathfrak{M}_{\Delta^+}}$.

(A) see the completeness proof of JSR_n , theorem 3.2, for the relevant case.

(B1) $|\Box\beta|_{N^-} \subseteq \|\Box\beta\|^{\mathfrak{M}_{\Delta^+}}$:

Assume that $\Box\beta \in x$ for some arbitrary $x \in \mathcal{U}_{\Delta^+}$, i.e. assume that $x \in |\Box\beta|_{N^-}$. Then there are two cases:

(i) $\beta = \perp$: Then by the definition of $\mathcal{N}_{\Delta+}$, $\mathcal{N}_{\Delta+}(x) = \{\emptyset\}$. So $|\beta|_{N-} \in \mathcal{N}_{\Delta+}(x)$. By the induction hypothesis, $\|\beta\|^{\mathfrak{M}_{\Delta+}} \in \mathcal{N}_{\Delta+}(x)$. Hence, $x \in \|\square\beta\|^{\mathfrak{M}_{\Delta+}}$.

(ii) $\beta \neq \perp$: Then $|\beta|_{N-} \in \Psi(x)$. So there is a maximal chain C_β of $\Psi(x)$ such that $|\beta|_{N-} \in C_\beta$. But $\cap C_\beta \subseteq |\beta|_{N-}$ and $\cap C_\beta \in \mathcal{L}_{\Delta+}(x)$. Clearly, $\cap C_\beta \neq \emptyset$, since $\emptyset \notin \Psi(x)$. So by the induction hypothesis, $\exists \mathbf{a} \in \mathcal{L}_{\Delta+}(x)$: $\mathbf{a} \subseteq \|\beta\|^{\mathfrak{M}_{\Delta+}}$. Hence, $x \in \|\square\beta\|^{\mathfrak{M}_{\Delta+}}$.

(B2) $\|\square\beta\|^{\mathfrak{M}_{\Delta+}} \subseteq |\square\beta|_{N-}$:

Assume that for an arbitrary $x \in \mathcal{U}_{\Delta+}$, $x \in \|\square\beta\|^{\mathfrak{M}_{\Delta+}}$. There are two cases:

(i) $\beta = \perp$: Then $\|\beta\|^{\mathfrak{M}_{\Delta+}} \in \mathcal{N}_{\Delta+}(x)$. By the induction hypothesis, $|\beta|_{N-} \in \mathcal{N}_{\Delta+}(x)$. So $\emptyset \in \mathcal{N}_{\Delta+}(x)$, i.e. $\mathcal{N}_{\Delta+}(x) = \{\emptyset\}$. By the definition of $\mathcal{N}_{\Delta+}$, $\square\beta \in x$. Hence, $x \in |\square\beta|_{N-}$.

(ii) $\beta \neq \perp$: Then $\exists \mathbf{a} \in \mathcal{L}_{\Delta+}(x)$: $\mathbf{a} \subseteq \|\beta\|^{\mathfrak{M}_{\Delta+}}$ or $\|\beta\|^{\mathfrak{M}_{\Delta+}} \in \mathcal{N}_{\Delta+}(x)$. If $\beta = \top$, then trivially $\square\beta \in x$, for all $x \in \mathcal{U}_{\Delta+}$. So assume that $\beta \neq \top$. Then, $\|\beta\|^{\mathfrak{M}_{\Delta+}} \neq \mathcal{U}_{\Delta+}$. So, $\|\beta\|^{\mathfrak{M}_{\Delta+}} \notin \mathcal{N}_{\Delta+}(x)$. Thus by the induction hypothesis, $\exists \mathbf{a} \in \mathcal{L}_{\Delta+}(x)$: $\mathbf{a} \subseteq |\beta|_{N-}$. But $\mathbf{a} = \cap C_\beta$ for some maximal chain C_β of $\Psi(x)$. So we have

$$\Gamma = \{\gamma: \square\gamma \in x \ \& \ |\gamma|_{N-} \in C_\beta\} \vdash_{N-} \beta$$

So for some finite subset, $\{\gamma_1, \dots, \gamma_n\}$, of Γ ,

$$\vdash_{N-} \gamma_1 \wedge \dots \wedge \gamma_n \rightarrow \beta$$

But $\{|\gamma_1|_{N-}, \dots, |\gamma_n|_{N-}\}$ is a chain. So there must be an ordering on this set such that $\forall j(1 \leq j \leq n), \vdash_{N-} \gamma_j \rightarrow \gamma_{j+1}$. So $\exists k(1 \leq k \leq n): \vdash_{N-} \gamma_k \rightarrow \gamma_1 \wedge \dots \wedge \gamma_n$. Hence, $\vdash_{N-} \gamma_k \rightarrow \beta$. Clearly, $\gamma_k \neq \perp$, for otherwise $|\gamma_k|_{N-} \notin \Psi(x)$ and so $|\gamma_k|_{N-} \notin C_\beta$. By $[RR_{\square}^-]$, $\vdash_{N-} \square\gamma_k \rightarrow (\blacklozenge\gamma_k \rightarrow \square\beta)$. But $\square\gamma_k \in x$ and $\gamma_k \neq \perp$, so $\blacklozenge\gamma_k \in x$. Hence, by the deductive closure of x , $\square\beta \in x$, i.e. $x \in |\square\beta|_{N-}$. \blacksquare

We are now in a position to prove our initial hypothesis.

Theorem 4.2 $\bigcap_{i \in \mathbb{N}} \Lambda_i = N^-$

Proof. $N^- \subseteq \bigcap_{i \in \mathbb{N}} \Lambda_i$ is trivial. We show that $\bigcap_{i \in \mathbb{N}} \Lambda_i \subseteq N^-$. Since N^- is determined by the class of NL frames and Λ_i is determined by the NR_i frames of rank i , it suffices to show that $\forall \alpha \in \Phi$, if α fail in the class of NL frame, then α fails in the class of NR_i frames of rank i . Let $\mathfrak{M} = \langle \mathcal{U}, \mathcal{N}, \mathcal{L}, \mathcal{V} \rangle$ be a NL model on a arbitrary NL frame such that α fails

in \mathfrak{M} , i.e. $\exists x \in \mathcal{U}: x \notin \|\alpha\|^{\mathfrak{M}}$. We prove that α fail in a model $\mathfrak{M}^* = \langle \mathcal{U}^*, \mathcal{N}^*, \mathcal{R}^*, \mathcal{V}^* \rangle$ on a NR_n frame of rank n . The proof is exactly analogous to the one given by Jennings and Schotch in [18].

Define \mathfrak{M}^* as follows:

$$\mathcal{U}^* = \mathcal{U}$$

$$\mathcal{N}^* = \mathcal{N}$$

$$\mathcal{V}^* = \mathcal{V}$$

$\forall x y_1, \dots, y_n \in \mathcal{U}$, define the function, ρ , such that $\rho(x) =$

$$\{\langle y_1, \dots, y_n \rangle: \langle y_1, \dots, y_n \rangle \in \prod_n (\mathbf{a}_1, \dots, \mathbf{a}_n), \text{ for some } (\mathbf{a}_1, \dots, \mathbf{a}_n) \in (\mathcal{L}(x))^n\}$$

$\forall x \in \mathcal{U}^*$, define $\mathcal{R}^* \subseteq \mathcal{U}^{*n+1}$ as follows:

$$\mathcal{R}^* x y_1, \dots, y_n \Leftrightarrow \langle y_1, \dots, y_n \rangle \in \rho(x)$$

We show by induction on the complexity of α that if $x \notin \|\alpha\|^{\mathfrak{M}}$, then $x \notin \|\alpha\|^{\mathfrak{M}^*}$. The basis is given trivially by \mathcal{V}^* . We assume the induction hypothesis and only consider the case where $\alpha = \Box\beta$.

Assume that $x \notin \|\Box\beta\|^{\mathfrak{M}}$. Then $\forall \mathbf{a} \in \mathcal{L}(x)$, $\exists y \in \mathbf{a}: y \notin \|\beta\|^{\mathfrak{M}}$ and $\|\beta\|^{\mathfrak{M}} \notin \mathcal{N}(x)$. So $\exists y_1, \dots, y_n \in \mathcal{U}^*: \mathcal{R}^* x y_1, \dots, y_n$ and $\forall i (1 \leq i \leq n), y_i \notin \|\beta\|^{\mathfrak{M}^*}$ and $\|\beta\|^{\mathfrak{M}^*} \notin \mathcal{N}^*(x)$. Hence $x \notin \|\Box\beta\|^{\mathfrak{M}^*}$. ■

4.2 The Decidability of Λ_i Logics

Recall that the standard method for establishing the decidability of a logic, L , is to show that the set of L -theorems *and* its complement are both effectively enumerable. In modal logics, the usual way to proceed is show that

1. L is finitely axiomatizable, and
2. L has the finite model property

For 1., it suffices to show that there is a decidable set of L -axioms whose closure under a finite set of ‘reasonable’ rules is exactly the set of L -theorems.¹ And for 2., it suffices to show that all non- L -theorems fail in a finite universe of a filtration of the L -canonical model. Normally, 2. is the non-trivial part of the proof of decidability; 1. is usually guaranteed by the definition of L —we explicitly define L under the appropriate closure conditions on a finite set of L -axioms. In our case however, the finite model property of Λ_i logics can easily be proven by modifying the standard procedure.² Proving the finite axiomatizability of Λ_i logics turns out to be a non-trivial problem.

The problem of the finite axiomatizability of Λ_i logics can be restated as follows. In characterizing Λ_i logics in terms of their corresponding modalization rule $[RY_i]$, we have not shown that $[RY_i]$ yields closure under the axioms of Λ_i . In fact, we have not even shown that $[\triangleright]$ is a compact inference relation. Apparently, the method developed by Apostoli and Brown ([2]) for proving the finite axiomatizability of K_n will not do in our case. Their proof relies on the fact that \vdash has a certain structural property. But as it turns out such a property does not hold with respect to $[\triangleright]$. The finite axiomatizability of Λ_i logics, and hence the decidability of these logics, remains an open problem.

4.3 Modal Logics Without $[RE]$

As we have seen, for a weakly reflexive relation such as \vdash^* we can recover a fully reflexive inference relation, $[\triangleright]$; the technique is to permit reproduction of self-inconsistencies. The chief interest in recovering such a relation is that it allows us to study modal logics which admit the rule $[RE]$. Arguably however, there are occasions in which even $[RE]$ may seem inappropriate. Consider for instance, the variety of ‘doxastic’ or ‘epistemic’ logics which accept the principle $[K_{\rightarrow}]$ and the rule $[RR]$. One common complaint against such logics is that only ‘perfect logicians’ can reason in a manner depicted by these logics; more specifically, the objection is that in the presence of $[K_{\rightarrow}]$ and $[RR]$, any knowledge or belief set has to be closed under logical consequence. Surely, there is room for disagreement here. On occasion, even the brightest amongst us may have difficulty seeing the logical implications of their beliefs, especially when these beliefs are sufficiently complex. Now a similar complaint can also be made against $[RE]$: it is equally problematic to assume that our knowledge or

¹A rule is said to be ‘reasonable’ iff there is an effective method for determining the correct application of the rule (see Chellas [6]).

²See appendix A.

belief sets are closed under logical equivalence.

In light of these considerations, we have no more reason to insist on reviving reflexivity in an inference relation in order to study modal logic, then to insist on salvaging $[RE]$ as a correct rule of inference at all cost. Logics generated by $[RI]$, $[RW]$, and $[JS^*]$ are perfectly respectable modal logics which merit attention. The difficulty and the challenge is to offer a semantics for these logics that will contribute to a more general understanding of modal logics without $[RE]$.

4.4 A General Approach To Modalization And Inference

Let Φ be the set of wffs of some formal language \mathcal{L} . Consider any inference relation $I \subseteq 2^\Phi \times \Phi$ which satisfy some structural properties, P_1, \dots, P_n . With respect to such a class of inference relations, what sort of modal logics are generated under the corresponding modalization rule?

$$[IR] \frac{\langle \Gamma, \alpha \rangle \in I}{\Box[\Gamma] \vdash \Box\alpha}$$

Appendix: A

The Finite Model Property of JSR_n

In this appendix, we will show that the logic JSR_n ($= \Lambda_n$) has the finite model property and thus all non-theorems of JSR_n are effectively enumerable. It will become apparent that in the proof of the filtration theorem for NR_n -model, we need to both generalize the proof from binary relation to $n + 1$ -ary relation and modify the notion of suitability as known in the standard literature (see [6] [9], [14], and [16]). The filtration neighbourhood however can be defined in the standard way (see [6]). It is clear that the strategy employed here is applicable to all Λ_i logics, where $i \in \mathbb{N}$.

Definition 4.2 *Let $\Sigma \subseteq \Phi$ be closed under subffs. Let $\mathfrak{M} = \langle \mathcal{U}, \mathcal{N}, \mathcal{R}, \mathcal{V} \rangle$ be a NR_n -model. The Σ equivalent relation, $\simeq_\Sigma \subseteq \mathcal{U}^2$ is defined as:*

$$\forall x, y \in \mathcal{U}, \forall \sigma \in \Sigma, x \simeq_\Sigma y \text{ iff } x \in \|\sigma\|^{\mathfrak{M}} \Leftrightarrow y \in \|\sigma\|^{\mathfrak{M}}$$

A x - Σ equivalent class is defined as:

$$\forall x \in \mathcal{U}, [x]_\Sigma = \{y \in \mathcal{U} : x \simeq_\Sigma y\}$$

A class of x - Σ equivalent classes is defined as:

$$\forall X \subseteq \mathcal{U}, \forall x \in \mathcal{U}, [X]_\Sigma = \{[x]_\Sigma : x \in X\}$$

The filtration of \mathfrak{M} through Σ is a NR_n model, $\mathfrak{M}_\Sigma = \langle \mathcal{U}_\Sigma, \mathcal{N}_\Sigma, \mathcal{R}_\Sigma, \mathcal{V}_\Sigma \rangle$ in which:

- $\mathcal{U}_\Sigma = [\mathcal{U}]_\Sigma$
- $\forall x \in \mathcal{U}, \forall \square \sigma \in \Sigma, \|\sigma\|^{\mathfrak{M}} \in \mathcal{N}(x) \Leftrightarrow \|\sigma\|^{\mathfrak{M}}_\Sigma \in \mathcal{N}_\Sigma([x]_\Sigma)$
- $\mathcal{R}_\Sigma \subseteq \mathcal{U}_\Sigma^{n+1}$ is suitable, i.e. $\forall x y_1, \dots, y_n \in \mathcal{U}, \forall \square \sigma \in \Sigma,$

1. $\mathcal{R}xy_1, \dots, y_n \Rightarrow \mathcal{R}_\Sigma[x]_\Sigma[y_1]_\Sigma, \dots, [y_n]_\Sigma$
2. $\mathcal{R}_\Sigma[x]_\Sigma[y_1]_\Sigma, \dots, [y_n]_\Sigma \Rightarrow (\sim \mathcal{R}xy_1, \dots, y_n \Rightarrow \exists i_{(1 \leq i \leq n)}: y_i \in \|\sigma\|^{\mathfrak{M}})$

- $\forall p_i \in At,$

$$\mathcal{V}_\Sigma(p_i) = \begin{cases} \{[x]_\Sigma: p_i \in x\} & \text{if } p_i \in \Sigma \\ \text{any arbitrary non-empty} & \\ \text{proper subset of } \mathcal{U}_\Sigma & \text{otherwise} \end{cases}$$

Theorem 4.3 *Let $\Sigma \subseteq \Phi$ be closed under subwffs. Let $\mathfrak{M}_\Sigma = \langle \mathcal{U}_\Sigma, \mathcal{N}_\Sigma, \mathcal{R}_\Sigma, \mathcal{V}_\Sigma \rangle$ be a Σ -filtration of the NR_n model $\mathfrak{M} = \langle \mathcal{U}, \mathcal{N}, \mathcal{R}, \mathcal{V} \rangle$. Then for every $\sigma \in \Sigma$, for every $x \in \mathcal{U}$, $\|[\sigma]\|_{\mathfrak{M}_\Sigma} = \|\sigma\|^{\mathfrak{M}_\Sigma}$.*

Proof. The proof proceeds by induction on the complexity of σ . The basis of the induction is given by the definition of \mathcal{V}_Σ . We only consider the case in which $\sigma = \Box\beta$. Note that the induction hypothesis states that $\|[\beta]\|_{\mathfrak{M}_\Sigma} = \|\beta\|^{\mathfrak{M}_\Sigma}$. We need to prove that $\|[\Box\beta]\|_{\mathfrak{M}_\Sigma} = \|\Box\beta\|^{\mathfrak{M}_\Sigma}$.

(i) $\|\Box\beta\|^{\mathfrak{M}_\Sigma} \subseteq \|[\Box\beta]\|_{\mathfrak{M}_\Sigma}$:

Let $[x]_\Sigma \notin \|[\Box\beta]\|_{\mathfrak{M}_\Sigma}$ be arbitrary. So $x \notin \|\Box\beta\|^{\mathfrak{M}}$. Then, $\exists y_1, \dots, y_n \in \mathcal{U}: \mathcal{R}xy_1, \dots, y_n$ and $\forall i_{(1 \leq i \leq n)}, y_i \notin \|\beta\|^{\mathfrak{M}}$, and $\|\beta\|^{\mathfrak{M}} \notin \mathcal{N}(x)$. By the definition of \mathcal{N}_Σ , $\|[\beta]\|_{\mathfrak{M}_\Sigma} \notin \mathcal{N}_\Sigma([x]_\Sigma)$. So by the induction hypothesis, $\|\beta\|^{\mathfrak{M}_\Sigma} \notin \mathcal{N}_\Sigma([x]_\Sigma)$. By the suitability of \mathcal{R}_Σ (1), $\mathcal{R}_\Sigma[x]_\Sigma[y_1]_\Sigma, \dots, [y_n]_\Sigma$. But $\forall i_{(1 \leq i \leq n)}, y_i \notin \|\beta\|^{\mathfrak{M}}$, hence $\forall i_{(1 \leq i \leq n)}, [y_i]_\Sigma \notin \|[\beta]\|_{\mathfrak{M}_\Sigma}$. So by the induction hypothesis, $\forall i_{(1 \leq i \leq n)}, [y_i]_\Sigma \notin \|\beta\|^{\mathfrak{M}_\Sigma}$. Hence, $[x]_\Sigma \notin \|\Box\beta\|^{\mathfrak{M}_\Sigma}$. But $[x]_\Sigma$ was arbitrary, so $\|\Box\beta\|^{\mathfrak{M}_\Sigma} \subseteq \|[\Box\beta]\|_{\mathfrak{M}_\Sigma}$.

(ii) $\|[\Box\beta]\|_{\mathfrak{M}_\Sigma} \subseteq \|\Box\beta\|^{\mathfrak{M}_\Sigma}$:

Let $[x]_\Sigma \notin \|\Box\beta\|^{\mathfrak{M}_\Sigma}$ be arbitrary. So $\|\beta\|^{\mathfrak{M}_\Sigma} \notin \mathcal{N}_\Sigma([x]_\Sigma)$, and $\exists [y_1]_\Sigma, \dots, [y_n]_\Sigma \in \mathcal{U}_\Sigma: \mathcal{R}[x]_\Sigma[y_1]_\Sigma, \dots, [y_n]_\Sigma$ and $\forall i_{(1 \leq i \leq n)}, [y_i]_\Sigma \notin \|\beta\|^{\mathfrak{M}_\Sigma}$. By the definition of \mathcal{N}_Σ , clearly, $\|\beta\|^{\mathfrak{M}_\Sigma} \notin \mathcal{N}(x)$. By the induction hypothesis, $\forall i_{(1 \leq i \leq n)}, [y_i]_\Sigma \notin \|[\beta]\|_{\mathfrak{M}_\Sigma}$, so $\forall i_{(1 \leq i \leq n)}, y_i \notin \|\beta\|^{\mathfrak{M}}$. By the suitability of \mathcal{R}_Σ (2), $\mathcal{R}xy_1, \dots, y_n$. So, $x \notin \|\Box\beta\|^{\mathfrak{M}}$. Hence, $[x]_\Sigma \notin \|[\Box\beta]\|_{\mathfrak{M}_\Sigma}$. ■

Theorem 4.4 *JSR_n has the finite model property, i.e. every non-theorem of JSR_n fails in a finite NR_n -model.*

Proof. By the fundamental theorem for JSR_n logics, every non-theorem fails at a point in the universe of the canonical model. Simply construct a filtration on the canonical model

and by the filtration theorem, every non-theorem fails in the filtration, which has at most finitely many points in the filtration universe. ■

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