Perfect Sets of Euler Tours of Complete Graphs

by

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Abstract

In this thesis we investigate perfect sets of Euler tours of complete graphs K_n and Hamilton decompositions of the line graphs of complete graphs $L(K_n)$. We also present some partial results in the area of pairwise compatible Hamilton path decompositions of the graph K_{2k} and pairwise compatible Hamilton decompositions of the graph K_{2k+1}.

Chapter 1 contains definitions and notation, and an introduction that outlines some of the work that has been done in the areas of pairwise compatible Euler tours of graphs, Hamilton decompositions of $L(K_n)$, and Dudeney sets. We also present the problems that will be considered in the thesis.

Kotzig conjectured in 1979 that K_{2k+1} has a perfect set of Euler tours for all positive integers k. In Chapter 2 we give a constructive proof of his conjecture. McKay conjectured that $L(K_n)$ has a Hamilton decomposition for all n. When n is odd, this conjecture is a corollary of Kotzig's conjecture.

In Chapter 3 we consider one way in which we could extend the definition of a perfect set of Euler tours to include K_{2k}, a graph that has no Euler tour. Since our goal is to have a Hamilton decomposition of $L(K_{2k})$ as a corollary, we define a perfect set of Euler tours of $K_{2k} + I$, where I is a 1-factor of K_{2k}, to be a set of Euler tours of $K_{2k} + I$ such that every 2-path of K_{2k} is in exactly one of the tours and such that for every edge $a\,b \in I$, each of the Euler tours either uses the digon $a\,b\,a$ or the digon $b\,a\,b$. We then give a constructive proof of a perfect set of Euler tours of $K_{2k} + I$, and thereby give a completion of the proof of McKay's conjecture.

The results in Chapter 4 were motivated by another question of Kotzig's: What is the smallest k for which there is a perfect set of Hamilton decompositions of K_{2k+1}?

We prove for all $k > 1$ that K_{2k} has at least $2k-2$ pairwise compatible Hamilton path decompositions. This is one less than the maximum possible of $2k-1$. In the case of K_4, it is straightforward to show it is best possible. We then construct a set of $4k-2$ Hamilton path decompositions of K_{2k} that between them contain every 2–path of the graph exactly twice. We also find a lower bound on the number of pairwise compatible Hamilton decompositions of K_{4m+1}.

We present our conclusions in Chapter 5.

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Chapter 1

Introduction

This chapter consists of two sections. Section 1.1 contains the definitions and notation that will be used in the thesis. Section 1.2 is background and a description of the problems that will be considered in the following chapters.

1.1 Definitions and notation

We will use K_n to denote the complete graph on n vertices. The line graph of K_n, denoted $L(K_n)$, is defined as follows: $V(L(K_n)) = E(K_n)$ and two vertices $e_1, e_2 \in V(L(K_n))$ are adjacent in $L(K_n)$ if and only if e_1 and e_2 are adjacent edges in K_n.

Let G be a finite graph on n vertices. A *trail* in G is a finite sequence

$$v_0,\ e_1,\ v_1,\ e_2, \ldots, v_{k-1},\ e_k,\ v_k$$

of vertices and edges in G such that for $1 \leq i \leq k$, $v_{i-1}\,v_i = e_i$, and for $1 \leq i < j \leq k$, $e_i \neq e_j$. We will write this trail as $v_0\,v_1 \cdots v_k$. A *tour* in G is a trail with the added condition that $v_0 = v_{k-1}$ and $v_1 = v_k$, (implying $e_1 = e_k$). Note that this definition allows a trail to begin and end on the same vertex and yet still not be a tour. (A tour is said to be a *closed* trail.) An *Euler tour* is a tour that contains every edge of the graph. If G has an Euler tour, G is said to be *Eulerian*. Similarly, if G does not have an Euler tour, G is *non-Eulerian.* A *walk* in G is a finite sequence

$$v_0,\ e_1,\ v_1,\ e_2, \ldots, v_{k-1},\ e_k,\ v_k$$

of vertices and edges in G such that for $1 \leq i \leq k$, $v_{i-1}\, v_i = e_i$, so the condition that the edges be all different is removed. Exactly as with a tour, a walk can be *closed.* A *path (cycle)* is a trail (tour) in which all the vertices are different, (except of course for the fact that in a tour $v_0 = v_{k-1}$ and $v_1 = v_k$). A Hamilton path (Hamilton cycle) is a path (cycle) containing all n vertices. We will call a decomposition of $E(G)$ into tours a *tour-decomposition.* An Euler tour of an Eulerian graph G is clearly a tour-decomposition of G into one tour. A k*-path* is a path on $k+1$ vertices. We will mostly be concerned with 2-paths, which we will write as $v_0\, v_1\, v_2$. We will call v_0 and v_2 the *end vertices* of the 2-path $v_0\, v_1\, v_2$, and v_1 its *centre* vertex, and we will say that $v_0\, v_1\, v_2$ is *centred at* v_1. Trails, tours and tour-decompositions of G can obviously be described by listing the set of 2-paths they contain. This idea will be used in all of the constructions in this thesis. A *digon* is a sequence of vertices and edges v_0, e_1, v_1, e_2, v_0, where $v_0 \neq v_1$, and $e_1 = e_2 = v_0\, v_1$; we will write this digon as $v_0\, v_1\, v_0$.

If ρ is an automorphism of G, and t is the trail $v_0\, v_1\, v_2\, \cdots\, v_{l-1}\, v_l$, then $\rho(t) = \rho(v_0\, v_1\, v_2\, \cdots\, v_{l-1}\, v_l)$ is the trail $\rho(v_0)\, \rho(v_1)\, \rho(v_2)\, \cdots\, \rho(v_{l-1})\, \rho(v_l)$. We will call two trails (and hence two tour-decompositions) t_1 and t_2 in G *similar* if there exists an automorphism ρ of G such that $t_2 = \rho(t_1)$. We are mostly concerned with complete graphs in this thesis so it will be enough for ρ to be a permutation of $V(G)$.

The constructions in Chapters 2 and 3 involve removing a 2-path from a trail. This does not mean that the edges in the 2-path are removed, only that the trail is broken. So, if t is the trail

$$v_0\, v_1\, v_2\, \cdots\, v_{l-1}\, v_l,$$

then $t - v_{i-1}\, v_i\, v_{i+1}$, where $1 \leq i \leq l-1$, is simply the following two trails:

$$v_0\, v_1\, \cdots\, v_{i-1}\, v_i \text{ and } v_i\, v_{i+1}\, \cdots\, v_{l-1}\, v_l.$$

Suppose n is even. Let $a\,b$ be an edge and v a vertex in G. By $v[a\,b]$ we mean the 2-path $a\,v\,b$. A 1*-factor* in G is a spanning subgraph of G in which every vertex has degree 1. In Chapter 2, we will be using 1–factors of K_{2k} to determine the end vertices of 2–paths in Euler tours of K_{2k+1}. Let F be a 1–factor of K_{2k}. If $(v_1 w_1, v_2 w_2, \ldots, v_k w_k)$ is an ordering of the edges in F, we will call $v_i w_i$ the i^{th} *edge* of F, for $i \in \{1,2,\ldots,k\}$, and denote it by $[F : i]$. Let $V(K_{2k+1}) = V(K_{2k}) \cup \{\infty\}$. For $u \in V(K_{2k})$, $u[F]$ will be the set of 2–paths

$$\{v_i\, u\, w_i : 1 \leq i \leq k \text{ and } v_i \neq u \neq w_i, v_i\, w_i \in E(F)\}$$

together with the 2–path

$$\infty\, u\, v_i, \text{ where } u = w_i, \text{ for some } i.$$

By $\infty[F]$ we mean the set of 2–paths $\{v_i \infty w_i : 1 \leq i \leq k\}$. We will often use the notation $v[F : i]$ intead of $v[u_i\, v_i]$, where $v \in V(K_{2k+1})$. We will sometimes use $v[F : i][F_j]$ for the two 2–paths $v[F : i]$ and $v[F : j]$, where $v \in V(K_{2k+1})$, and $1 \leq i, j \leq k$,

A 1–*factorization* of G is a partition of the edges of G into 1–factors. A 1-factorization $\mathcal{F}$ is said to be *perfect* if the union of any two of the 1–factors in $\mathcal{F}$ forms a Hamilton cycle in G. A partition of the edges of a graph G into Hamilton cycles or into Hamilton cycles and a 1–factor — depending on the parity of n — is called a *Hamilton decomposition* of G. In Chapter 3, we will be using Hamilton cycles of K_{2k-1} to determine the end vertices of 2–paths in Euler tours of the multigraph $K_{2k} + I$, where I is a 1–factor of K_{2k}. Let $V(K_{2k}) = V(K_{2k-1}) \cup \{\infty_1\}$. Let H be a Hamilton cycle of K_{2k-1}. If $(v_1 v_2, v_2 v_3, v_3 v_4, \ldots, v_{2k-1} v_1)$ is an ordering of the edges in H, we will call $v_i v_{i+1}$ the i^{th} *edge* of H, for $1 \leq i \leq 2k-1$, and denote it by $[H : i]$, where addition on the subscripts of the vertices is modulo $2k-1$ with residue classes $1, 2, \ldots, 2k-1$. If $v_i \neq v \neq v_{i+1}$, then by $v[H : i]$, we mean the 2–path $v_i\, v\, v_{i+1}$. If v_i equals v, then $v[H : i]$ is the 2–path $\infty_1\, v\, v_{i+1}$. Similarly, if $v = v_{i+1}$, $v[H : i] = v_i\, v\, \infty_1$. When it is obvious which Hamilton cycle the end vertices of the 2–paths centered at a vertex v are coming from, we will abbreviate $v[H : j]$ to $v[j]$.

The list of 2–paths and digons centred at vertex v in an Euler tour of K_{2k} in Chapter 3 will be specified in one of six ways. Let $1 \leq t \leq k-1$. By $v[H : 1, 3, 5, \ldots, 2t-1,\ 2t, 2t+2, \ldots, 2k-2]$, we mean the set of 2–paths $\{v[H : j] : j \in$

$\{1,3,5,\ldots,2t-1,2t,2t+2,\ldots,2k-2\}\}$. By $v[H:2,4,6,\ldots,2t,\ 2t+1,2t+3,\ldots,2k-1]$, we mean the set of 2–paths $\{v[H:j]:j\in\{2,4,6,\ldots,2t,2t+1,2t+3,\ldots,2k-1\}\}$. By $v[H:1,3,5,\ldots,2t-1,\ 2t+2,2t+4,\ldots,2k-2]$, we mean the set of 2–paths $\{v[H:j]:j\in\{1,3,5,\ldots,2t-1,2t+2,2t+4,\ldots,2k-2\}\}$ as well as the digon $u\,v\,u$, where $u=[H:2t]\cap[H:2t+1]$. By $v[H:2,4,6,\ldots,2t,\ 2t+3,2t+5,\ldots,2k-1]$, we mean the set of 2–paths $\{v[H:j]:j\in\{2,4,6,\ldots,2t,2t+3,2t+5,\ldots,2k-1\}\}$ as well as the digon $u\,v\,u$, where $u=[H:2t+1]\cap[H:2t+2]$. Towards the end of each of the two proofs in Chapter 3 we will also use $v[H:1,3,5,\ldots,2k-1]$ to mean the set of 2–paths $\{v[H:j]:j\in\{1,3,5,\ldots,2k-1\}\}$, and $v[H:2,4,6,\ldots,2k-2]$ to mean the set of 2–paths $\{v[H:j]:j\in\{2,4,6,\ldots,2k-2\}\}$ as well as the digon $u\,v\,u$, where $u=[H:1]\cap[H:2k-1]$. If in this last case $v=u$, then the digon will be $\infty_1\,v\,\infty_1$.

Since the degree of every vertex of K_{2k+1} is even, K_{2k+1} is Eulerian. An Euler tour of K_{2k+1} contains $k(2k+1)$ 2–paths. In total, K_{2k+1} contains $k(2k+1)(2k-1)$ 2–paths. It is natural to ask if a set of $2k-1$ Euler tours of K_{2k+1} can be found so that every 2–path of K_{2k+1} is in exactly one of the tours. Towards this end, we make the following two definitions: two tour–decompositions of an Eulerian graph are *compatible* if they have no 2–path in common; and a *perfect set of Euler tours* of K_{2k+1} is a set of $2k-1$ pairwise compatible Euler tours of K_{2k+1}. In other words, it is a set of $2k-1$ Euler tours that partition the set of 2–paths in K_{2k+1}. In Chapter 2, we construct a perfect set of Euler tours of K_{2k+1} for all k.

On the other hand, K_{2k} has no Euler tour because the degree of every vertex is odd. There are several ways we could modify the graph K_{2k} so that we could define for it something that approaches the idea of a perfect set of Euler tours of K_{2k+1}. We choose the following definition because it implies the existence of a Hamilton decomposition of $L(K_{2k})$. A *perfect set of Euler tours of* $K_{2k}+I$, where I is a 1–factor of K_{2k}, is a set of $2k-2$ Euler tours of $K_{2k}+I$ such that every 2–path of K_{2k} is in exactly one of the Euler tours, and for each of the edges $ab\in I$, each Euler tour either uses the digon aba or the digon bab, but not both. In Chapter 3, we construct a perfect set of Euler tours of $K_{2k}+I$ for all $k>1$.

Finally, for Chapter 4, we need the following definitions. A *Hamilton path decomposition* of K_{2k} is a decomposition of $E(K_{2k})$ into Hamilton paths. Since a Hamilton decomposition of K_{2k+1} is also a tour–decomposition, we have already defined two Hamilton decompositions of K_{2k+1} to be *compatible* if no 2–path in the graph is in more than one of the Hamilton cycles. We extend the definition of compatibility to a non–Eulerian graph by saying that two Hamilton path decompositions of K_{2k} are *compatible* if no 2–path in the graph is in more than one of the Hamilton path decompositions. We also define a *perfect set of Hamilton decompositions (Hamilton path decompositions) of* K_{2k+1} (K_{2k}) to be a set of $2k-1$ pairwise compatible Hamilton decompositions (Hamilton path decompositions) of the graph.

We will use the notation (a, b) for the greatest common factor of two integers a and b, and $\phi(n)$ for the Euler ϕ function. We will use $2^{-1}a \pmod{2k-1}$ to indicate either $\frac{a}{2} \pmod{2k-1}$, if a is even, or $\frac{a+2k-1}{2} \pmod{2k-1}$, if a is odd. This is multiplication by 2^{-1} in the ring Z_{2k-1}.

Finally, a *Dudeney set in* K_n is a set of $\frac{(n-1)(n-2)}{2}$ Hamilton cycles of K_n so that every 2–path of the graph is in exactly one of the Hamilton cycles.

1.2 Background and a Description of the Problems

In Chapter 2 we prove the following conjecture:

Conjecture 1.2.1 (Kotzig [12]) *The graph K_{2k+1} has a perfect set of Euler tours for all positive integers k.*

This is a special case of the following problem suggested by Hilton in 1985 at an Open University Combinatorics Workshop (see Jackson [7]).

Problem 1.2.2 (Hilton) *Determine the maximum number of pairwise compatible Euler tours in a given Eulerian graph G.*

In a related area, Bermond [2] has conjectured

Conjecture 1.2.3 (Bermond [2]) *If a graph G has a Hamilton decomposition then its line graph $L(G)$ can be decomposed into Hamilton cycles.*

More specifically, B. McKay (personal communication) conjectured

Conjecture 1.2.4 (McKay) *The line graph of the complete graph $L(K_n)$ can be decomposed into Hamilton cycles.*

The existence of a perfect set of Euler tours of K_{2k+1} immediately implies the existence of a Hamilton decomposition of $L(K_{2k+1})$: each Euler tour of K_{2k+1} induces a Hamilton cycle of $L(K_{2k+1})$, and since the Euler tours partition the 2-paths of K_{2k+1}, the induced Hamilton cycles partition the edges of $L(K_{2k+1})$. Therefore, when n is odd, a proof of McKay's conjecture is an immediate corollary of the validity of Kotzig's conjecture. The two conjectures are probably not equivalent: the two edges $ab\,bc$ and $bc\,bd$ in a line graph could certainly be adjacent in some cycle in the line graph, but, back in the original graph, the two 2-paths abc and cbd could not be adjacent in a tour.

It is not hard to construct a perfect set of Euler tours of K_3 or K_5, but, to my knowledge, no other perfect sets of Euler tours of complete graphs had been found until now.

In Chapter 3 we present results on one way of extending the idea of a perfect set of Euler tours to the graph K_{2k}, which itself has no Euler tour. We choose to define a perfect set of Euler tours of K_{2k} as we do because as a corollary we immediately have a Hamilton decomposition of $L(K_{2k})$. This seems to justify our definition as it parallels the odd case. Thus our construction of a perfect set of Euler tours of $K_{2k}+I$ completes the proof of McKay's conjecture since it implies that the graph $L(K_{2k})$ does have a Hamilton decomposition for all $k > 1$. Again there is no reason to suppose that the two results are equivalent.

There has been much work done trying to solve Problem 1.2.2. Jackson gives a review in [7]. We use $d(v)$ to indicate the degree of a vertex $v \in V(G)$ and $\delta(G)$ to indicate the minimum degree of G. A *block* in a graph is a maximal 2–connected subgraph. In giving an overview of the results in this area, we will assume for simplicity that the Eulerian graphs have no vertices of degree 2.

Suppose G is an Eulerian graph with $\delta(G) \geq 4$, and let v be a vertex of G of degree $\delta(G)$. If $u\,v \in E(G)$ then there are $\delta(G)-1$ 2–paths $u\,v\,w$, $w \in V(G)$. Therefore, there are at most $\delta(G)-1$ pairwise compatible Euler tours of G. Moreover, if there is a 2–path $u\,v\,x$ such that $G - u\,v - v\,x$ is disconnected, then no Euler tour of G could use the 2–path $u\,v\,x$, so there are at most $\delta(G)-2$ pairwise compatible Euler tours of G. Jackson conjectured that one of these bounds must hold:

Conjecture 1.2.5 (Jackson [6]) *The maximum number of pairwise compatible Euler tours of an Eulerian graph G is either $\delta(G)-1$ or $\delta(G)-2$.*

This conjecture is valid for $\delta(G) = 4$ [13] and for $\delta(G) = 6$ [8]. Although it has not been possible to prove this conjecture in general, Jackson and Wormald, by extending a result from [6], were able to prove:

Theorem 1.2.6 (Jackson and Wormald [9]) *An finite Eulerian graph G with $\delta(G) \geq 4$ has at least $\frac{1}{2}\delta(G)$ pairwise compatible Euler tours.*

Fleischner *et al.* [4] proved the following two theorems, using the first to prove the second.

Theorem 1.2.7 *Given a 1-factor L of K_{2k}, there is a 1-factorization $L_1, L_2, \ldots, L_{2k-2}$ of $K_{2k} - L$ such that $L \cup L_i$ is a Hamilton cycle of K_{2k} for $i \in \{1, 2, \ldots, 2k-2\}$.*

Theorem 1.2.8 *If G is a connected, finite, Eulerian graph with $\delta(G) \geq 4$ such that every cycle in G is a block of G, then G has $\delta(G) - 2$ pairwise compatible Euler tours.*

Note that in Theorem 1.2.8 the number $\delta(G) - 2$ is best possible.

Results about Hamilton decompositions of $L(K_n)$ tend to appear as corollaries to more general theorems.

Theorem 1.2.9 (Muthusumy and Paulraja [14]) *If G has a Hamilton decomposition into an even number of Hamilton cycles, then $L(G)$ has a Hamilton decomposition.*

Corollary 1.2.10 *The line graphs $L(K_{4m+1})$ and $L(K_{4m+2})$ each have a Hamilton decomposition for all m.*

Theorem 1.2.11 (Cox and Rodger [3]) *Let $l \equiv 0 \pmod 4$. If $n \equiv 1 \pmod{2l}$, or $n \equiv 0$ or $2 \pmod l$, then there exists a partition of the edges of $L(K_n)$ into cycles of length l.*

Corollary 1.2.12 *The line graph $L(K_{4m})$ has a Hamilton decomposition for all m.*

Theorem 1.2.13 (Muthusumy and Paulraja [14], Zhan [16]) *If G has a Hamilton decomposition into an odd number of Hamilton cycles, then the edges of $L(G)$ can be partitioned into Hamilton cycles and a 2-factor.*

Corollary 1.2.14 *The edges of the line graph $L(K_{4m+3})$ can be decomposed into Hamilton cycles and a 2-factor for all m.*

We also mention a result of Pike's that has implications for the existence of Hamilton decompositions of $L(K_{2k} - I)$.

Theorem 1.2.15 (Pike [15]) *If G is a $2k$–regular graph that has a perfect 1–factorization, then $L(G)$ has a Hamilton decomposition.*

Corollary 1.2.16 *The line graph of $K_{2k} - I$ has a Hamilton decomposition whenever K_{2k} has a perfect 1–factorization, where I is a 1–factor of K_{2k}.*

Pike provides a list of the values of k for which perfect 1–factorizations of K_{2k} exist. It includes k prime, $2k - 1$ prime, and 16 other values.

This is of interest here because the graph $K_{2k} - I$ is Eulerian, and asking for a perfect set of Euler tours of $K_{2k} - I$ would be another way of extending the idea behind Kotzig's Conjecture 1.2.1 to the graph K_{2k}. Corollary 1.2.16 would also be a corollary to such a result.

Chapter 4 is motivated by another question of Kotzig's [12]:

Problem 1.2.17 (Kotzig [12]) *What is the smallest $k > 1$ for which there is a perfect set of Hamilton decompositions of K_{2k+1}?*

It is possible that no such k exists. It is not hard to show that there cannot be two compatible Hamilton decompositions of K_5, let alone three, which is the number needed for a perfect set. Kotzig states in [12] that it is known that K_7 does not have a perfect set of Hamilton decompositions, but does not say how many pairwise compatible Hamilton decompositions are possible. The fact that perfect sets of Hamilton decompositions do not exist for these small cases leads us to ask instead:

Problem 1.2.18 *Given k, what is the maximum number of pairwise compatible Hamilton decompositions in K_{2k+1}?*

Since a set of l pairwise compatible Hamilton decompositions of K_{2k+1} implies the existence of a set of l pairwise compatible Hamilton path decompositions of K_{2k}, we can back up still further and ask:

Problem 1.2.19 *Given k, what is the maximum number of pairwise compatible Hamilton path decompositions in K_{2k}?*

Problems 1.2.17 and 1.2.18 are related to the existence of Dudeney sets in K_{2k+1} because a perfect set of Hamilton decompositions of K_{2k+1} is simply a resolvable Dudeney set. Also, since whenever there exists a Dudeney set of K_n, we immediately have a set of Hamilton paths of K_{n-1} that partition the 2–paths of K_{n-1}, results about Dudeney sets may have implications for Problem 1.2.19. Since Dudeney sets in K_n when n is odd have proven hard to find, we should perhaps assume that solving Problem 1.2.17 will be difficult. There is only one known infinite family of Dudeney sets of K_{2k+1}:

Theorem 1.2.20 (Heinrich, Kobayashi, Nakamura [5]) *There is a Dudeney set in K_{p+2} if p is prime and 2 is a generator of the multiplicative subgroup of $GF(p)$.*

There are also a few sporadic cases known: see [10].

However, when n is even, the existence of Dudeney sets has been solved completely.

Theorem 1.2.21 (Kobayashi, Kiyasu-Zen'iti, Nakamura [10]) *There exists a Dudeney set in K_n when n is even.*

Before proving Theorem 1.2.21, Kobayashi and Nakamura [11] gave an elegant construction of the following result.

Theorem 1.2.22 (Kobayashi, Nakamura [11]) *There exists a set of Hamilton cycles of K_n when n is even that between them contain every 2–path of K_n exactly twice.*

As a corollary, there is a set of Hamilton paths of K_{2k-1} that between them contain every 2–path of K_{2k-1} exactly twice. Similarly, if we change Problem 1.2.19 to ask for every 2–path twice instead of once, we are able to find a set of Hamilton path decompositions of K_{2k} so that every 2–path is in exactly two of the Hamilton paths.

We also give a construction for a set of $2k-2$ pairwise compatible Hamilton path decompositions of K_{2k} and thereby show that the solution to Problem 1.2.19 is either $2k-2$ or $2k-1$. (A perfect set of Hamilton path decompositions of K_{2k} would contain $2k-1$ Hamilton path decompositions.) In the case of $k=2$, two Hamilton path decompositions is best possible. These results are the first section of Chapter 4.

In the second section of Chapter 4, we give a lower bound to Problem 1.2.18 when $k > 2$ is even. The first and last vertices in a Hamilton path determine an edge, and the set of such edges determined by a Hamilton path decomposition is a 1–factor in K_{2k}. If we construct a set of l pairwise compatible Hamilton path decompositions of K_{2k} with the added condition that the 1–factors induced by each Hamilton path decomposition are pairwise disjoint, then we immediately have a set of l pairwise compatible Hamilton decompositions of K_{2k+1}.

When $k > 2$ is even we are able to show that there are at least

$$\max(\lceil\frac{2k}{3}\rceil - (k - 1 - \frac{\phi(2k-1)}{2}), 3)$$

pairwise compatible Hamilton decompositions of K_{2k+1}. When $k > 2$ is even and $2k - 1$ is prime, this means we have at least $\lceil\frac{2k}{3}\rceil$ pairwise compatible Hamilton decompositions of K_{2k+1}.

Chapter 2

A Perfect Set of Euler Tours of K_{2k+1}

2.1 Main Result

In this chapter we prove the following theorem and corollaries.

Theorem 2.1.1 *For all k, K_{2k+1} has a perfect set of Euler tours.*

Corollary 2.1.2 *For all k, $L(K_{2k+1})$ has a Hamilton decomposition.*

Corollary 2.1.3 *There exists a closed walk of K_{2k+1} in which every 2–path occurs exactly once.*

The proof of Corollary 2.1.2 is straightforward and we give it here. The proof of Corollary 2.1.3 requires details of the proof of Theorem 2.1.1, so we will present it in the last section of this chapter.

Proof of Corollary 2.1.2. Given a perfect set of Euler tours of K_{2k+1}, simply replace each 2–path $a\,b\,c$ in each of the tours by the edge $ab\,bc$ in $L(K_{2k+1})$. Since each tour covers each edge of K_{2k+1} exactly once, in the line graph the corresponding subgraph will cover each vertex exactly once, and hence be a Hamilton cycle. Since each 2–path of K_{2k+1} is used exactly once in exactly one of the tours, every edge of $L(K_{2k+1})$ is covered exactly once in the Hamilton cycles, giving a Hamilton decomposition. □

Since K_{2k+1} contains $k(2k+1)(2k-1)$ 2–paths, and an Euler tour of K_{2k+1} contains $k(2k+1)$ 2–paths, a perfect set of Euler tours of K_{2k+1} would have $2k-1$ Euler tours. The Euler tours in the perfect set of Euler tours of K_{2k+1} that we construct here are pairwise similar. In fact there exists a permutation σ of $V(K_{2k+1})$ such that if T is one of the Euler tours, then $\{\sigma^i(T) : 0 \leq i \leq 2k-2\}$ is the set of all the Euler tours. Thus, there exists a permutation τ of $V(L(K_{2k+1}))$, such that if H is one of the Hamilton cycles of $L(K_{2k+1})$, then $\{\tau^i(H) : 0 \leq i \leq 2k-2\}$ generates all of the Hamilton cycles in the Hamilton decomposition.

The proof of Theorem 2.1.1 is divided into two sections, the first for the case when k is even, and the second for the case when k is odd. The constructions are divided into a series of claims and proofs of the claims. In both sections, the key to the construction of the Euler tours is the choice of a particular 1–factorization $\mathcal{F}$ of K_{2k}. Let $V(K_{2k}) = \{1, 2, \ldots, 2k\}$. It is well known that the following generates a 1–factorization of K_{2k}. Let σ_1 be the permutation $(2\,3\,4\,\cdots\,2k-2\;2k-1\;2k)$ of the vertices of K_{2k} that fixes vertex 1 and cyclically rotates the others. Then $\mathcal{F} = \{F_0, F_1, \ldots, F_{2k-2}\}$, where F_0 is the 1–factor $\{1\,2, 3\,2k, 4\,2k-1, \ldots, k+1\;k+2\}$ and $F_i = \sigma_1^i(F_0)$, $1 \leq i \leq 2k-2$, is a 1–factorization of K_{2k}.

It is fundamental (though perhaps trivial) to understand how we will be joining together trails and 2–paths to form Euler tours. Given a trail in K_{2k+1} that ends at vertex v and another that starts at v, suppose we want to join them together at v to form a single trail. It is first necessary to know more about them. We need to know which 2–path centred at v this larger trail would use. In order to know that, we need to know the last edge of the first trail, say it is $u\,v$, and the first edge of the second, say $v\,w$. We can then take the two trails and the 2–path $u\,v\,w$ and form a single trail.

The main idea of the proof when k is even is to construct one Euler tour T_0 of K_{2k+1} and a permutation σ of the $V(K_{2k+1})$ so that $\{\sigma^i(T_0) : 0 \leq i \leq 2k-2\}$ is a perfect set of Euler tours. We describe T_0 by listing the 2–paths that are centred at each of the vertices in it. It should be clear that in an Euler tour, or indeed, in a tour decomposition of K_{2k+1}, that if we construct edges from the end vertices of each of the 2–paths centred at a given vertex v, then these edges form a 1–factor of $K_{2k} = K_{2k+1} - \{v\}$. Also, the union of the 1–factors formed by the end vertices of the 2–paths centred at v in each of the Euler tours in a perfect set of Euler tours forms a 1–factorization of K_{2k}. With this in mind, in listing the 2–paths centred at v in T_0, we start with a 1–factor F_0 of K_{2k}, such that $\{\sigma^i(F_0) : 0 \leq i \leq 2k-2\}$ is a 1–factorization of K_{2k}. We then say that the 2–paths centered at v in T_0 are $\{u\,v\,w : uw \in E(\sigma^j(F_0))\}$, where the cnoice of j depends on v. When we take $\sigma^i(T_0)$ for $0 \leq i \leq 2k+2$, we are effectively generating 2–paths centred at v with end vertices from each of the 1–factors $\sigma^{j+i}(F_0)$, $0 \leq i \leq 2k+2$. In other words, from a 1–factorization of K_{2k}. The difficulty lies in choosing which 1–factor $\sigma^j(F_0)$ will determine the end vertices of the 2–paths centred at a given vertex v. Having provided a list of the 2–paths in T_0, it is then necessary to prove that they do indeed form an Euler tour of K_{2k+1}, and not just a tour decomposition. (We necessarily have at least have constructed a tour decomposition.) To prove this, we consider T_0 minus the 2–paths centred at a fixed vertex ∞, and hence investigate and make use of the underlying structure of T_0.

The proof when k is odd, is similar to and relies heavily on the proof when k is even.

2.2 A Perfect Set of Euler Tours of K_{4m+1}

Let $k = 2m$. Denote the $4m-1$ Euler tours required in a perfect set of Euler tours of K_{4m+1} by $\{T_0, T_1, \ldots, T_{4m-2}\}$. We will construct T_0 by providing a list of the 2–paths that it contains, and construct T_i, $1 \leq i \leq 4m-2$, by defining a permutation σ of $V(K_{4m+1})$, and letting $T_i = \sigma^i(T_0)$. Thus the tours will be pairwise similar and it will only be necessary to prove that T_0 is an Euler tour and that the T_i, $0 \leq i \leq 4m-2$, partition the 2–paths of K_{4m+1}.

Construct the following 1–factorization of K_{4m} using the idea described in Section 2.1. Let $V(K_{4m}) = A \cup B \cup C$, where $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_m\}$, and $C = \{c_1, c_2, \ldots, c_{2m}\}$. Let $V(K_{4m+1}) = V(K_{4m}) \cup \{\infty\}$, and let σ be the permutation

$$(\infty)(c_1)(a_1\, b_1\, c_{2m}\, c_3\, a_m\, b_2\, c_{2m-2}\, c_5 \cdots a_{m-i+2}\, b_i\, c_{2m-2i+2}\, c_{2i+1} \cdots$$
$$\cdots a_3\, b_{m-1}\, c_4\, c_{2m-1}\, a_2\, b_m\, c_2).$$

of $V(K_{4m+1})$ that fixes ∞ and generates a 1–factorization of K_{4m} on the vertex set $A \cup B \cup C$, beginning with the initial 1–factor F_0, where F_0 is given by $\{a_i\, c_{2i-1} : 1 \leq i \leq m\} \cup \{b_i\, c_{2i} : 1 \leq i \leq m\}$. We now have the 1–factorization $\mathcal{F} = \{F_0, F_1, \ldots, F_{4m-2}\}$, where $F_i = \sigma^i(F_0)$, $0 \leq i \leq 4m-2$.

Since we want every edge of K_{4m+1} to be in T_0 exactly once, every vertex of T_0 will have $2m$ edge-disjoint 2–paths centred at it. The set of 2–paths used to specify T_0 will be based on the 1–factors F_{4m-2}, F_0, and F_1, and is listed below. From now on we will denote F_{4m-2} by F_{-1} in order to emphasize that $\sigma^{-1}(F_0) = F_{-1}$. The 2–paths in T_0 are:

$a_j[F_{-1}]$, for all $a_j \in A$,
$b_j[F_1]$, for all $b_j \in B$,
$c_j[F_0]$, for all $c_j \in C$, and
$\infty[F_0]$

where notation is as in Section 1.1.

Now let $T_i = \sigma^i(T_0)$, for $1 \leq i \leq 4m-2$. By definition the T_i are pairwise similar.

Claim 2.2.1 *The T_i, $0 \leq i \leq 4m-2$, partition the 2–paths in K_{4m+1}.*

Proof. Since σ fixes both ∞ and c_1, and $\mathcal{F}$ is a 1–factorization of K_{4m}, it is clear that the T_i partition all the 2–paths centered at either of these vertices.

Let $v \in V(K_{4m}) - \{c_1\}$. Let $t\,v\,s$ be a 2–path in K_{4m+1}, and assume $t \neq \infty$ and $s \neq \infty$. Then the edge $t\,s = [F_i : k]$ for a unique $i \in \{0, 1, 2, \ldots, 4m-2\}$ and a unique $k \in \{1, 2, \ldots, 2m\}$. There are three cases. If $v = \sigma^i(c_j)$ for some $c_j \in C$, then $t\,v\,s = \sigma^i(c_j[F_0 : k])$, and since $c_j[F_0 : k] \in T_0$, $t\,v\,s \in T_i$. If $v = \sigma^i(a_j)$ for some $a_j \in A$, then $v = \sigma^{i-1}(b_l)$ for some $b_l \in B$. So $t\,v\,s = \sigma^{i-1}(b_l[F_1 : k])$, and since $b_l[F_1 : k] \in T_0$, $t\,v\,s \in T_{i-1}$. If $v = \sigma^i(b_j)$ for some $b_j \in B$, then $v = \sigma^{i+1}(a_l)$ for some $a_l \in A$. So $t\,v\,s = \sigma^{i+1}(a_l[F_{-1} : k])$, and since $a_l[F_{-1} : k] \in T_0$, $t\,v\,s \in T_{i+1}$.

Now assume that $t = \infty$. Then there exist i and k such that $v\,s = [F_i : k]$. An argument similar to the above will show that $\infty\, v\, s \in T_j$ for some j. follows. □

Our goal now is to show that T_0 is an Euler tour. To accomplish this we give an exact description of the order in which the 2–paths occur in T_0. It is obvious that the removal of any one vertex divides an Euler tour of K_{4m+1} into exactly $2m$ trails. We consider the removal of vertex ∞ from T_0. Our first step is to partition all the 2–paths in T_0 except those centred at vertex ∞ into $2m$ parts, G_i, $1 \leq i \leq 2m$, and to prove that each part forms a single trail that begins and ends at vertex ∞; our second step is to prove that the 2–paths centred at vertex ∞, $\infty[F_0]$, join these trails together in such a way that they form an Euler tour.

We begin by ordering the edges in the three 1–factors used in the construction of T_0.

$$\begin{aligned} [F_0 : 2j-1] &= a_j\, c_{2j-1}, & 1 \leq j \leq m, \\ [F_0 : 2j] &= b_j\, c_{2j}, & 1 \leq j \leq m, \end{aligned}$$

$$\begin{aligned} [F_1 : 1] &= b_1\, c_1, & \\ [F_1 : 2j-1] &= a_j\, b_j, & 2 \leq j \leq m, \\ [F_1 : 2j] &= c_{2j}\, c_{2j+1}, & 1 \leq j \leq m-1, \\ [F_1 : 2m] &= a_1\, c_{2m}, & \end{aligned}$$

$$\begin{aligned} [F_{-1} : 2j-1] &= c_{2j-1}\, c_{2j}, & 1 \leq j \leq m, \\ [F_{-1} : 2j] &= b_j\, a_{j+1}, & 1 \leq j \leq m-1, \\ [F_{-1} : 2m] &= b_m\, a_1. & \end{aligned}$$

We now define a partition of all the 2–paths in T_0 except those centred at vertex ∞ and label the parts G_i, $1 \leq i \leq 2m$. To prove each G_i forms a single trail, we will show that for $i \leq m-1$, G_{i+1} contains a subtrail similar to G_i minus one 2–path, as well as nine other 2–paths, and for $i \geq m+2$, G_{i-1} contains a subtrail similar to G_i minus one 2–path, as well as nine other 2–paths. Before showing each G_i is a trail we will determine which 2–paths in G_i contain vertex ∞. This is necessary as we ultimately need to determine how the trails G_i will fit together when joined by the

2–paths centred at vertex ∞. In listing the 2–paths in G_i we will use the notation $u[F_i : j]$ described in the introduction in Chapter 1. If j should happen to be less than 1 or greater than $2m$, we assume that no 2–path results. To reiterate, we first simply assign certain 2–paths to G_i, and then, in a series of claims, verif, that they do indeed produce the trails as described.

The 2–paths in G_i, $i \in \{1, 2, \ldots, m\}$ are:

$$\begin{array}{ll} a_1[F_{-1} : 2i-1], & \\ a_k[F_{-1} : 2i-2k+1][F_{-1} : 2i-2k+2], & 2 \le k \le i, \\ b_k[F_1 : 2i-2k][F_1 : 2i-2k+1], & 1 \le k \le i, \\ c_k[F_0 : 2i-k][F_0 : 2i-k+1], & 1 \le k \le 2i. \end{array}$$

The 2–paths in G_i, $i \in \{m+1, m+2, \ldots, 2m\}$ are:

$$\begin{array}{ll} a_1[F_{-1} : 2i-2m], & \\ a_k[F_{-1} : 2i-2k+1][F_{-1} : 2i-2k+2], & i-m+1 \le k \le m, \\ b_k[F_1 : 2i-2k][F_1 : 2i-2k+1], & i-m \le k \le m, \\ c_k[F_0 : 2i-k][F_0 : 2i-k+1], & 2i-2m \le k \le 2m. \end{array}$$

Claim 2.2.2 *The G_i, $1 \le i \le 2m$, partition the 2–paths in $T_0 - \infty[F_0]$.*

Proof. It is straightforward to check that in the union of the G_i each vertex in $V(K_{4m})$ occurs as the centre vertex of $2m$ 2–paths with end vertices determined by the edges of the appropriate 1–factor of K_{4m}. □

Claim 2.2.3 *There are precisely two vertices in each G_i that are centres of 2–paths with vertex ∞ as an end vertex. These vertices are $b_{\frac{i+1}{2}}$ and c_i if i is odd, $a_{\frac{i}{2}+1}$ and c_i if i is even and $i \le 2m-2$, and a_1 and c_{2m} if $i = 2m$.*

Proof. It is easy to check that the 2–paths $c_1\, b_1\, \infty$ and $a_1\, c_1\, \infty$ are in G_1, that $a_{\frac{i+1}{2}}\, b_{\frac{i+1}{2}}\, \infty$ and $a_{\frac{i+1}{2}}\, c_i\, \infty$ are in G_i, $i > 1$ odd, that $b_{\frac{i}{2}}\, a_{\frac{i}{2}+1}\, \infty$ and $b_{\frac{i}{2}}\, c_i\, \infty$ are in G_i, $i < 2m$ even, and that $b_m\, a_1\, \infty$ and $b_m\, c_{2m}\, \infty$ are in G_{2m}.

By construction, there are exactly $4m$ 2–paths in T_0 that have ∞ as an end vertex. Since we have accounted for $4m$ such 2–paths, we are done. □

In order to prove that each G_i is a trail, we shall show that if $1 \leq i \leq m-1$, then G_{i+1} contains a subtrail similar to all of G_i minus one 2–path, and if $m+2 \leq i \leq 2m$, then G_{i-1} contains a subtrail similar to all of G_i minus one 2–path. Towards this end, let γ be the following permutation on the vertices of K_{4m+1}:

$$\gamma(c_i) = c_{i+1}, \quad 1 \leq i \leq 2m-1,$$
$$\gamma(c_{2m}) = c_1.$$

$$\gamma(a_i) = b_i, \quad 1 \leq i \leq m,$$
$$\gamma(b_i) = a_{i+1}, \quad 1 \leq i \leq m-1,$$
$$\gamma(b_m) = a_1.$$

$$\gamma(\infty) = \infty.$$

We next determine where γ maps the edges in the three 1–factors, F_0, F_1 and F_{-1}.

$$\gamma([F_0 : 2j-1]) = \gamma(a_j\, c_{2j-1}) = b_j\, c_{2j} = [F_0 : 2j], \quad 1 \leq j \leq m,$$
$$\gamma([F_0 : 2j]) = \gamma(b_j\, c_{2j}) = a_{j+1}\, c_{2j+1} = [F_0 : 2j+1], \quad 1 \leq j \leq m-1,$$
$$\gamma([F_0 : 2m]) = \gamma(b_m\, c_{2m}) = a_1\, c_1 = [F_0 : 1],$$

$$\gamma([F_1 : 1]) = \gamma(b_1\, c_1) = a_2\, c_2,$$
$$\gamma([F_1 : 2j-1]) = \gamma(a_j\, b_j) = b_j\, a_{j+1} = [F_{-1} : 2j], \quad 2 \leq j \leq m-1,$$
$$\gamma([F_1 : 2m-1]) = \gamma(a_m\, b_m) = b_m\, a_1 = [F_{-1} : 2m],$$
$$\gamma([F_1 : 2j]) = \gamma(c_{2j}\, c_{2j+1}) = c_{2j+1}\, c_{2j+2} = [F_{-1} : 2j+1], \quad 1 \leq j \leq m-1,$$
$$\gamma([F_1 : 2m]) = \gamma(a_1\, c_{2m}) = b_1\, c_1.$$

$$\gamma([F_{-1} : 2j-1]) = \gamma(c_{2j-1}\, c_{2j}) = c_{2j}\, c_{2j+1} = [F_1 : 2j], \quad 1 \leq j \leq m-1,$$
$$\gamma([F_{-1} : 2m-1]) = \gamma(c_{2m-1}\, c_{2m}) = c_{2m}\, c_1,$$
$$\gamma([F_{-1} : 2j]) = \gamma(b_j\, a_{j+1}) = a_{j+1}\, b_{j+1} = [F_1 : 2j+1], \quad 1 \leq j \leq m-1,$$
$$\gamma([F_{-1} : 2m]) = \gamma(b_m\, a_1) = a_1\, b_1.$$

So for $1 \leq k \leq 2m-1$, $\gamma([F_0 : k]) = [F_0 : k+1]$; for $2 \leq k \leq 2m-1$, $\gamma([F_1 : k]) = [F_{-1} : k+1]$; and for $1 \leq k \leq 2m-2$, $\gamma([F_{-1} : k]) = [F_1 : k+1]$.

Claim 2.2.4 *For $1 \le i \le m-1$, all the 2-paths in $\gamma(G_i)$ are in G_{i+1}, except for $\gamma(b_i[F_1 : 1])$.*

Proof. We know exactly which 2-paths are in G_i, $1 \le i \le m-1$, and how γ behaves on the vertices of K_{4m} and on the edges of F_0, F_1 and F_{-1}. We can therefore list the 2-paths in $\gamma(G_i)$, $1 \le i \le m-1$.

$$\gamma(a_1[F_{-1} : 2i-1]) = b_1[F_1 : 2i] = b_1[F_1 : 2(i+1)-2],$$

$$\begin{aligned}\gamma(a_k[F_{-1} : 2i-2k+1][F_{-1} : 2i-2k+2]),& \quad 2 \le k \le i\\ = b_k[F_1 : 2i-2k+2][F_1 : 2i-2k+3]&\\ = b_k[F_1 : 2(i+1)-2k][F_1 : 2(i+1)-2k+1],& \quad 2 \le k \le (i+1)-1\end{aligned}$$

$$\begin{aligned}\gamma(b_k[F_1 : 2i-2k][F_1 : 2i-2k+1]),& \quad 1 \le k \le i-1\\ = a_{k+1}[F_{-1} : 2i-2k+1][F_{-1} : 2i-2k+2]&\\ = a_{k+1}[F_{-1} : 2(i+1)-2(k+1)+1][F_{-1} : 2(i+1)-2(k+1)+2],&\\ 2 \le k+1 \le (i+1)-1&\end{aligned}$$

$$\gamma(b_i[F_1 : 1]) = a_{i+1}[a_2\, c_2], \qquad \notin G_{i+1}$$

$$\begin{aligned}\gamma(c_k[F_0 : 2i-k][F_0 : 2i-k+1]),& \quad 1 \le k \le 2i-1\\ = c_{k+1}[F_0 : 2i-k+1][F_0 : 2i-k+2]&\\ = c_{k+1}[F_0 : 2(i+1)-(k+1)][F_0 : 2(i+1)-(k+1)+1],&\\ 2 \le k+1 \le 2(i+1)-2&\end{aligned}$$

$$\gamma(c_{2i}[F_0 : 1]) = c_{2(i+1)-1}[F_0 : 2]$$

All of the resulting 2-paths above are in G_{i+1} except for $\gamma(b_i[F_1 : 1])$. □

We now prove by induction that each G_i, $1 \le i \le m$, is a single trail. The following two claims contain the basis of the induction and the induction step, respectively.

Claim 2.2.5 *The part G_1 is a single trail, and $G_1 - b_1[F_1 : 1]$ is the union of a trail with first edge $\infty\, c_1$ and last edge $c_1\, b_1$, and the single edge $b_1\, \infty$.*

Proof. It is easy to see from the list of the 2–paths in G_1 that G_1 is the trail $\infty\, c_1\, a_1\, c_2\, c_1\, b_1\, \infty$. The second point in the statement of the claim is obviously true. □

Claim 2.2.6 *Each of the parts, G_i, $1 \le i \le m$, is a trail, and $G_i - b_i[F_1 : 1])$ is one trail from $\infty\, c_i$ to $c_1\, b_i$, and a second from $b_i\, b_1$ to $b_{\frac{i+1}{2}}\, \infty$, if i is odd, and to $a_{\frac{i}{2}+1}\, \infty$, if i is even.*

Proof. Assume $i \le m-1$ is odd. By induction we can assume G_i is a trail and $G_i - b_i[F_1 : 1]$ is two trails: one from $\infty\, c_i$ to $c_1\, b_i$; and one from $b_i\, b_1$ to $b_{\frac{i+1}{2}}\, \infty$. By Claim 2.2.4, we know that $\gamma(G_i - b_i[F_1 : 1]) = \gamma(G_i - c_1\, b_i\, b_1)$ is a subset of G_{i+1}. So, G_{i+1} contains one trail from $\infty\, c_{i+1}$ to $c_2\, a_{i+1}$, and one trail from $a_{i+1}\, a_2$ to $a_{\frac{i+1}{2}+1}\, \infty$. Define these two subtrails of G_{i+1} to be t_1 and t_2, respectively. Note that t_2 is the single edge $a_2\, \infty$ if $i = 1$. From the list of the 2–paths in G_{i+1} and the proof of Claim 2.2.4, we see that the 2–paths in G_{i+1} that are not in $\gamma(G_i)$ are:

$$\begin{aligned}
b_1[F_1 : 2(i+1)-1] &= a_{i+1}\, b_1\, b_{i+1} \\
b_{i+1}[F_1 : 1] &= b_1\, b_{i+1}\, c_1 \\
a_1[F_{-1} : 2(i+1)-1] &= c_{2i+1}\, a_1\, c_{2i+2} \\
a_{i+1}[F_{-1} : 1][F_{-1} : 2] &= c_1\, a_{i+1}\, c_2 \text{ and } b_1\, a_{i+1}\, a_2 \\
c_1[F_0 : 2(i+1)-1][F_0 : 2(i+1)] &= a_{i+1}\, c_1\, c_{2i+1} \text{ and } b_{i+1}\, c_1\, c_{2i+2} \\
c_{2(i+1)-1}[F_0 : 1] &= a_1\, c_{2i+1}\, c_1 \\
c_{2(i+1)}[F_0 : 1] &= a_1\, c_{2i+2}\, c_1.
\end{aligned}$$

Then G_{i+1} is the trail

$$t_1\, c_1\, c_{2i+1}\, a_1\, c_{2i+2}\, c_1\, b_{i+1}\, b_1\, t_2.$$

When $i = 1$ note that $a_2[F_{-1} : 2] = b_1\, a_2\, \infty$.

Now suppose $i \le m-1$ is even.

The only difference in this case is that the final edge in G_i is $a_{\frac{i}{2}+1}\, \infty$. If we again let t_1 and t_2 be the two subtrails in $\gamma(G_i - c_1\, b_i\, b_1)$, then t_1 is still a trail from $\infty\, c_{i+1}$ to $c_2\, a_{i+1}$, but t_2 is now a trail from $a_{i+1}\, a_2$ to $b_{\frac{i}{2}+1}\, \infty$. The new 2–paths fit in exactly as in the case when i was odd.

The second result in the statement of the claim is easily seen by inspecting the above trail. □

We also know which 2–paths are in G_i when $m+2 \leq i \leq 2m$, and so can list the 2–paths in $\gamma^{-1}(G_i)$. Note that $b_{i-m}[F_1 : 2m]$ is a 2–path in G_i.

Claim 2.2.7 *For $m+2 \leq i \leq 2m$, all of the 2–paths in $\gamma^{-1}(G_i)$ are in G_{i-1}, except for $\gamma^{-1}(b_{i-m}[F_1 : 2m])$.*

Proof. Recall that for $2 \leq k \leq 2m$, $\gamma^{-1}([F_0 : k]) = [F_0 : k-1]$; and for $3 \leq k \leq 2m$, $\gamma^{-1}([F_{-1} : k]) = [F_1 : k-1]$; and for $2 \leq k \leq 2m-1$, $\gamma^{-1}([F_1 : k]) = [F_{-1} : k-1]$. We obtain the following:

$$\gamma^{-1}(a_1[F_{-1} : 2i - 2m]) = b_m[F_1 : 2i - 2m - 1] = b_m[F_1 : 2(i-1) - 2m + 1]$$

$$\begin{aligned}&\gamma^{-1}(a_k[F_{-1} : 2i - 2k + 1][F_{-1} : 2i - 2k + 2]), \qquad i - m + 1 \le k \le m \\ &\quad = b_{k-1}[F_1 : 2i - 2k][F_1 : 2i - 2k + 1] \\ &\quad = b_{k-1}[F_1 : 2(i-1) - 2(k-1)][F_1 : 2(i-1) - 2(k-1) + 1], \\ &\qquad\qquad (i-1) - m + 1 \le k - 1 \le m - 1\end{aligned}$$

$$\begin{aligned}&\gamma^{-1}(b_{i-m}[F_1 : 2m]) \\ &\quad = a_{i-m}[\gamma^{-1}(a_1)\,\gamma^{-1}(c_{2m})] = a_{i-m}[b_m\, c_{2m-1}] \notin G_{i-1}\end{aligned}$$

$$\begin{aligned}&\gamma^{-1}(b_k[F_1 : 2i - 2k][F_1 : 2i - 2k + 1]), \qquad i - m + 1 \le k \le m \\ &\quad = a_k[F_{-1} : 2i - 2k - 1][F_{-1} : 2i - 2k] \\ &\quad = a_k[F_{-1} : 2(i-1) - 2k + 1][F_{-1} : 2(i-1) - 2k + 2], \\ &\qquad\qquad (i-1) - m + 2 \le k \le m\end{aligned}$$

$$\begin{aligned}&\gamma^{-1}(c_{2i-2m}[F_0 : 2m]), \\ &\quad = c_{2(i-1)-2m+1}[F_0 : 2m - 1]. \\ &\gamma^{-1}(c_k[F_0 : 2i - k][F_0 : 2i - k + 1]), \qquad 2i - 2m + 1 \le k \le 2m \\ &\quad = c_{k-1}[F_0 : 2i - k - 1][F_0 : 2i - k] \\ &\quad = c_{k-1}[F_0 : 2(i-1) - (k-1)][F_0 : 2(i-1) - (k-1) + 1], \\ &\qquad\qquad 2(i-1) - 2m + 2 \le k - 1 \le 2m - 1\end{aligned}$$

The resulting 2–paths are all in G_{i-1} except $\gamma^{-1}(b_{i-m}[F_1 : 2m])$. □

We again use induction to prove in the following two claims that each of the parts G_i, $m + 1 \le i \le 2m$, is a trail.

Claim 2.2.8 *The part G_{2m} is a single trail. Also, $G_{2m} - b_m[F_1 : 2m]$ is made up of a trail from $\infty\, c_{2m}$ to $c_{2m}\, b_m$, and a trail from $b_m\, a_1$ to $a_1\, \infty$.*

Proof. The part G_{2m} consists of the trail $\infty\, c_{2m}\, b_m\, a_1\, \infty$. The second statement in the claim is obvious. □

Claim 2.2.9 *Each of the parts, G_i, $m+1 \leq i \leq 2m-1$, is a trail, and $G_i - b_{i-m}[F_1 : 2m]$ is one trail from $\infty\, c_i$ to $c_{2m}\, b_{i-m}$, and a second trail from $b_{i-m}\, a_1$ to $b_{\frac{i+1}{2}}\, \infty$, when i is odd. When i is even, the second trail starts on the edge $b_{i-m}\, a_1$ and goes to the edge $a_{\frac{i}{2}+1}\, \infty$.*

Proof. Assume by induction that G_i is a trail for some i, $m+2 \leq i \leq 2m$, and that $G_i - b_{i-m}[F_1 : 2m]$ is as described in the statement of the claim.

If i is even then G_i is a trail from $\infty\, c_i$ to $a_{\frac{i}{2}+1}\, \infty$, unless $i = 2m$, and then it is a trail to $a_1\, \infty$. By Claim 2.2.7, $\gamma^{-1}(G_i - b_{i-m}[a_1\, c_{2m}])$ is a subset of G_{i-1}, we know by induction that G_{i-1} must contain a trail from $\infty\, c_{i-1}$ to $c_{2m-1}\, a_{i-m}$, and one from $a_{i-m}\, b_m$ to $b_{\frac{i}{2}}\, \infty$. Define these two subtrails of G_{i-1} to be t_1 and t_2, respectively.

From the list of 2–paths in G_{i-1} and the proof of Claim 2.2.7, we know that the 2–paths that are in G_{i-1} that are not in $\gamma^{-1}(G_i)$ are:

$$\begin{aligned}
a_1[F_{-1} : 2(i-1) - 2m] &= a_{i-m}\, a_1\, b_{i-1-m} \\
a_{i-m}[F_{-1} : 2m-1][F_{-1} : 2m] &= c_{2m-1}\, a_{i-m}\, c_{2m} \text{ and} \\
& b_m\, a_{i-m}\, a_1 \\
b_m[F_1 : 2(i-1) - 2m] &= c_{2(i-1-m)}\, b_m\, c_{2(i-m)-1} \\
b_{i-1-m}[F_1 : 2m] &= a_1\, b_{i-1-m}\, c_{2m} \\
c_{2m}[F_0 : 2(i-1) - 2m][F_0 : 2(i-1) - 2m + 1] &= b_{i-1-m}\, c_{2m}\, c_{2(i-1-m)} \text{ and} \\
& a_{i-m}\, c_{2m}\, c_{2(i-m)-1} \\
c_{2(i-1)-2m}[F_0 : 2m] &= b_m\, c_{2(i-1-m)}\, c_{2m} \\
c_{2(i-m)-1}[F_0 : 2m] &= b_m\, c_{2(i-m)-1}\, c_{2m}
\end{aligned}$$

So G_{i-1} is:

$$t_1\, c_{2m}\, c_{2(i-m)-1}\, b_m\, c_{2(i-1-m)}\, c_{2m}\, b_{i-1-m}\, a_1\, t_2.$$

If i is odd then $G_i - b_{i-m}[a_1\, c_{2m}]$ is a trail from $\infty\, c_i$ to $c_{2m}\, b_{i-m}$, and a trail from $b_{i-m}\, a_1$ to $b_{\frac{i+1}{2}}\, \infty$. So G_{i-1} must contain a trail from $\infty\, c_{i-1}$ to $c_{2m-1}\, a_{i-m}$, and one from $a_{i-m}\, b_m$ to $a_{\frac{i+1}{2}}\, \infty$. The new 2–paths fit in exactly as they did when i was odd. Note that after removing the 2–path $b_{i-1-m}[F_1 : 2m]$ from G_{i-1}, we obtain the desired subtrails for the second part of the statement of the claim. □

Since each G_i is a trail by Claims 2.2.6 and 2.2.9, and since G_i starts and ends on the edges specified in Claim 2.2.3, we can now show that the union of the G_i, $1 \leq i \leq 2m$, with the 2–paths centred at vertex ∞ yields an Euler tour. We have seen that $\infty\, c_i$ is the first edge of G_i, for all i. Let f_i be the vertex such that $f_i\,\infty$ is the last edge in G_i. Then $\{f_i : 1 \leq i \leq 2m\} = A \cup B$. So $F^* = \{f_i\, c_i : 1 \leq i \leq 2m\}$ is a 1–factor of K_{4m}. By construction, the end vertices of the 2–paths centred at vertex ∞ are from F_0. The following claim proves that the union of these two 1–factors is a Hamilton cycle and thus that T_0 is an Euler tour.

Claim 2.2.10 *The union of the following two 1–factors of K_{4m} on the vertex set $A \cup B \cup C$ is a Hamilton cycle:*

$$F_0 = \{a_i\, c_{2i-1}, b_i\, c_{2i} : 1 \leq i \leq m\} \text{ and}$$

$$F^* = \{b_i\, c_{2i-1} : 1 \leq i \leq m-1\} \cup \{a_i\, c_{2i-2} : 2 \leq i \leq m\} \cup \{a_1\, c_{2m}\}.$$

Proof. The proof of this claim is easily seen. The Hamilton cycle is

$$(a_1\, c_1\, b_1\, c_2\, \cdots\, a_i\, c_{2i-1}\, b_i\, c_{2i}\, \cdots\, a_m\, c_{2m-1}\, b_m\, c_{2m}).$$

□

This completes the construction of a perfect set of Euler tours of K_{4m+1}.

2.3 A Perfect Set of Euler Tours of K_{4m+3}

Now let $k = 2m+1$. The construction of a perfect set of Euler tours of K_{4m+3} is very similar to our construction in Section 2.2. The proof requires the Euler tour T_0 that was constructed for K_{4m+1}, so, to avoid confusion, we will partition the 2–paths in K_{4m+3} into $\{S_0, S_1, \ldots, S_{4m}\}$, where each S_i is a tour–decomposition of K_{4m+3}. The S_i will be pairwise similar so that we need only check that S_0 is an Euler tour to be sure they all are.

We also want to let $\mathcal{F} = \{F_0, F_1, \ldots, F_{4m-2}\}$ be the same 1–factorization of K_{4m} as in Section 2.2, so we will let $\mathcal{E} = \{E_0, E_1, \ldots, E_{4m}\}$, as defined below, be the 1–factorization of K_{4m+2} on which we base the S_i.

Recall that $V(K_{4m}) = A \cup B \cup C$, where $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_m\}$, and $C = \{c_1, c_2, \ldots, c_{2m}\}$. Let $V(K_{4m+2}) = V(K_{4m}) \cup \{d_1, d_2\}$ and $V(K_{4m+3}) = V(K_{4m+2}) \cup \{\infty\}$. We use the same idea as before to construct the 1-factorization $\mathcal{E}$. Let τ be the following permutation of $V(K_{4m+3})$:

$$\begin{aligned}\tau = \quad & (\infty)(c_1)(a_1\, b_1\, c_{2m}\, c_3\, a_m\, b_2\, c_{2m-2}\, c_5\, a_{m-1} \\ & \cdots b_i\, c_{2m-2i+2}\, c_{2i+1}\, a_{m-i+1} \\ & \cdots b_{\frac{m}{2}}\, c_{m+2}\, c_{m+1}\, d_2\, d_1\, a_{\frac{m}{2}+1}\, b_{\frac{m}{2}+1}\, c_m\, c_{m+3}\, a_{\frac{m}{2}} \\ & \cdots b_i\, c_{2m-2i+2}\, c_{2i+1}\, a_{m-i+1} \\ & \cdots b_{m-1}\, c_4\, c_{2m-1}\, a_2\, b_m\, c_2), \text{ if } m \text{ is even, and}\end{aligned}$$

$$\begin{aligned}\tau = \quad & (\infty)(c_1)(a_1\, b_1\, c_{2m}\, c_3\, a_m\, b_2\, c_{2m-2}\, c_5\, a_{m-1} \\ & \cdots b_i\, c_{2m-2i+2}\, c_{2i+1}\, a_{m-i+1} \\ & \cdots b_{\frac{m+1}{2}}\, d_2\, d_1\, c_{m+1}\, c_{m+2}\, a_{\frac{m+1}{2}}\, b_{\frac{m+3}{2}}\, c_{m-1}\, c_{m+4}\, a_{\frac{m-1}{2}} \\ & \cdots b_i\, c_{2m-2i+2}\, c_{2i+1}\, a_{m-i+1} \\ & \cdots b_{m-1}\, c_4\, c_{2m-1}\, a_2\, b_m\, c_2), \text{ if } m \text{ is odd.}\end{aligned}$$

Let $\mathcal{E} = \{E_0, E_1, \ldots, E_{4m}\}$, where E_0 is the 1-factor $F_0 \cup \{d_1\, d_2\}$ and $E_i = \tau^i(E_0)$, $0 \leq i \leq 4m$.

The set of 2-paths for specifying S_0 will be based on the 1-factors E_{4m}, E_0, and E_1, as well as E_2 or E_3, and is listed below. We will refer to E_{4m} as E_{-1}.

$$\begin{aligned} & a_j[E_{-1}], \text{ for all } a_j \in A, \\ & b_j[E_1], \text{ for all } b_j \in B, \\ & c_j[E_0], \text{ for all } c_j \in C, \\ & d_1[E_0], \\ & d_2[E_0], \text{ and} \\ & \infty[E_2], \text{ if } m > 1, \text{ or } \infty[E_3], \text{ if } m = 1. \end{aligned}$$

Now let $S_i = \tau^i(S_0)$, for $0 \leq i \leq 4m$, so that the S_i are all pairwise similar. The proof of the following claim is very like that of Claim 2.2.1 and is not given.

Claim 2.3.1 *The S_i, $0 \leq i \leq 4m$, partition the 2-paths in K_{4m+3}.*

Before listing the 2–paths in S_0 and proving they form an Euler tour, we determine the edges in E_{-1}, E_0, E_1, and write them in terms of F_{-1}, F_0, and F_1.

When m is even, we have

$$\begin{aligned} E_0 &= \{[F_0 : j], 1 \le j \le 2m\} \\ &\cup \{d_1\, d_2\} \end{aligned}$$

$$\begin{aligned} E_1 &= \{[F_1 : j], 1 \le j \le 2m, j \ne m+1\} \\ &\cup \{a_{\frac{m}{2}+1}\, d_1\} \\ &\cup \{b_{\frac{m}{2}+1}\, d_2\} \end{aligned}$$

$$\begin{aligned} E_{-1} &= \{[F_{-1} : j], 1 \le j \le 2m, j \ne m+1\} \\ &\cup \{c_{m+1}\, d_2\} \\ &\cup \{c_{m+2}\, d_1\} \end{aligned}$$

Note that the edges $[F_1 : m+1] = a_{\frac{m}{2}+1}\, b_{\frac{m}{2}+1}$ and $[F_{-1} : m+1] = c_{m+1}\, c_{m+2}$ have each been removed and replaced by two new edges.

When m is odd, we have

$$\begin{aligned} E_0 &= \{[F_0 : j], 1 \le j \le 2m\} \\ &\cup \{d_1\, d_2\} \end{aligned}$$

$$\begin{aligned} E_1 &= \{[F_1 : j], 1 \le j \le 2m, j \ne m+1\} \\ &\cup \{c_{m+1}\, d_1\} \\ &\cup \{c_{m+2}\, d_2\} \end{aligned}$$

$$\begin{aligned} E_{-1} &= \{[F_{-1} : j], 1 \le j \le 2m, j \ne m+1\} \\ &\cup \{b_{\frac{m+1}{2}}\, d_2\} \\ &\cup \{a_{\frac{m+3}{2}}\, d_1\} \end{aligned}$$

Note that the edges $[F_1 : m+1] = c_{m+1}\, c_{m+2}$ and $[F_{-1} : m+1] = b_{\frac{m+1}{2}}\, a_{\frac{m+3}{2}}$ have again each been removed and replaced by two new edges.

Now partition all the 2–paths in S_0 except those centred at vertex ∞ into $2m+1$ parts, H_i, $0 \le i \le 2m$. For some of the $i \in \{1, 2, \ldots, 2m\}$, H_i will equal G_i. For the rest, H_i will be a copy of G_i with two of its 2–paths replaced by ten new 2–paths. The part H_0 is new. We will prove that each H_i is a trail starting and ending at vertex ∞, and then that the 2–paths centred at vertex ∞ do indeed join these trails into an Euler tour. To do this, we will show that the first and last edges of H_i, $1 \le i \le 2m$, are the first and last edges of G_i.

First consider m even. Since the H_i will be described in terms of the G_i, we need to determine which G_i will be affected by changing the 2–paths with end vertices from the edges $[F_1 : m+1]$ or $[F_{-1} : m+1]$. Since $m+1$ is odd, in both cases we would need $2i - 2k + 1 = m + 1$, implying $i = \frac{m}{2} + k$. Since $1 \le k \le m$, the only G_i that contain such 2–paths are those for which $\frac{m}{2} + 1 \le i \le \frac{3m}{2}$.

Construct the partition of the 2–paths of S_0 as follows:

The 2–paths in H_0 are simply $d_1[\infty\, d_2]$ and $d_2[\infty\, d_1]$.

For $1 \le i \le \frac{m}{2}$, and $\frac{3m}{2} + 1 \le i \le 2m$, $H_i = G_i$.

For $\frac{m}{2} + 1 \le i \le \frac{3m}{2}$,

$$\begin{aligned} H_i = G_i \;\; & \cup\, d_1[F_0 : 2i - m - 1][F_0 : 2i - m] \\ & \cup\, d_2[F_0 : 2i - m - 1][F_0 : 2i - m] \\ & \cup\, a_{i-\frac{m}{2}}[c_{m+1}\, d_2][c_{m+2}\, d_1] \\ & -\, a_{i-\frac{m}{2}}[c_{m+1}\, c_{m+2}] \\ & \cup\, b_{i-\frac{m}{2}}[a_{\frac{m}{2}+1}\, d_1][b_{\frac{m}{2}+1}\, d_2] \\ & -\, b_{i-\frac{m}{2}}[a_{\frac{m}{2}+1}\, b_{\frac{m}{2}+1}] \\ & \cup\, c_{2i-m-1}[d_1\, d_2] \\ & \cup\, c_{2i-m}[d_1\, d_2]. \end{aligned}$$

Claim 2.3.2 *When m is even, the H_i, $0 \le i \le 2m$, partition all the 2–paths of S_0 except those centred at vertex ∞.*

Proof. We show that for each vertex $v \in V(K_{4m+2})$, the H_i partition the 2–paths centred at v.

It is clear that the H_i partition the 2–paths centred at any vertex in $A \cup B \cup C$, given the way H_i is based on G_i.

We pick up the two 2–paths centred at vertex d_1, $d_1[F_0 : 2i - m - 1][F_0 : 2i - m]$ in the part H_i for $i \in \{\frac{m}{2}+1, \frac{m}{2}+2, \ldots, \frac{3m}{2}\}$. Similarly for d_2. The part H_0 contains the 2–paths $\infty\, d_1\, d_2$ and $\infty\, d_2\, d_1$. □

Claim 2.3.3 *When m is even, the part H_0 is a trail beginning on the edge $\infty\, d_1$ and ending on the edge $d_2\, \infty$. Each H_i, $1 \le i \le 2m$, is a trail beginning and ending on the same edges as G_i.*

Proof. Obviously H_0 is the trail $\infty\, d_1\, d_2\, \infty$.

When $1 \le i \le \frac{m}{2}$ and $\frac{3m}{2} + 1 \le i \le 2m$, $H_i = G_i$, and so H_i is a single trail beginning on the edge $\infty\, c_i$ for all i, and ending on the edge $b_{\frac{i+1}{2}}\, \infty$ if i is odd, on the edge $a_{\frac{i}{2}+1}\, \infty$ if i is even and $i \le 2m - 2$, and on the edge $a_1\, \infty$ if $i = 2m$.

When $\frac{m}{2} + 1 \le i \le \frac{3m}{2}$, we use the fact that G_i is a trail containing the 2–paths $c_{m+1}\, a_{i-\frac{m}{2}}\, c_{m+2}$ and $a_{\frac{m}{2}+1}\, b_{i-\frac{m}{2}}\, b_{\frac{m}{2}+1}$, not necessarily in this order. In H_i, the 2–path $c_{m+1}\, a_{i-\frac{m}{2}}\, c_{m+2}$ in G_i becomes the trail

$$c_{m+1}\, a_{i-\frac{m}{2}}\, d_2\, c_{2i-m-1}\, d_1\, a_{i-\frac{m}{2}}\, c_{m+2},$$

and the 2–path $a_{\frac{m}{2}+1}\, b_{i-\frac{m}{2}}\, b_{\frac{m}{2}+1}$ in G_i becomes

$$a_{\frac{m}{2}+1}\, b_{i-\frac{m}{2}}\, d_1\, c_{2i-m}\, d_2\, b_{i-\frac{m}{2}}\, b_{\frac{m}{2}+1}.$$

Since this includes all 2–paths in H_i, H_i is a trail. Since these trails do not contain vertex ∞, H_i begins and ends on the same edges as G_i. □

Now consider m odd. (Again we want to describe the H_i in terms of the G_i, show that each H_i is a trail, and prove that the first and last edges of H_i are the same as those of H_i, for $i \in \{1, 2 \ldots, 2m\}$.)

We again consider which 2–paths in G_i have end vertices from the edges $[F_1 : m+1]$ and $[F_{-1} : m+1]$. In this case, $m+1$ is even, so we have to consider the vertices in A and the vertices in B separately.

For any $b_k \in B$, the 2–path $b_k[F_1 : m+1]$ is in the part G_i when $2i - 2k = m+1$, or $i = \frac{m+1}{2} + k$. Since $1 \le k \le m$, the 2–paths in G_i that are centred at a vertex in B only have end vertices from $[F_1 : m+1]$ if $\frac{m+3}{2} \le i \le \frac{3m+1}{2}$.

The 2–path $a_1[F_{-1} : m+1]$ is in $G_{\frac{3m+1}{2}}$.

Now consider vertex $a_k \in A$, where $2 \le k \le m$. The 2–path $a_k[F_{-1} : m+1]$ is in the part G_i when $2i - 2k + 2 = m + 1$, or $i = \frac{m-1}{2} + k$. Since $2 \le k \le m$, the 2–paths in G_i that are centred at a vertex in $A - \{a_1\}$ have end vertices from $[F_{-1} : m+1]$ when $\frac{m+3}{2} \le i \le \frac{3m-1}{2}$.

The net result is that we define $H_i = G_i$ for $1 \le i \le \frac{m+1}{2}$ and for $\frac{3m+3}{2} \le i \le 2m$.

Now construct the remaining H_i. The 2–paths in H_0 are $d_1[\infty\, d_2]$ and $d_2[\infty\, d_1]$.

For $\frac{m+3}{2} \le i \le \frac{3m-1}{2}$,

$$\begin{aligned} H_i = G_i \;\; & \cup\, d_1[F_0 : 2i - m - 1][F_0 : 2i - m] \\ & \cup\, d_2[F_0 : 2i - m - 1][F_0 : 2i - m] \\ & \cup\, a_{i-\frac{m-1}{2}}[b_{\frac{m+1}{2}}\, d_2][a_{\frac{m+3}{2}}\, d_1] \\ & -\, a_{i-\frac{m-1}{2}}[b_{\frac{m+1}{2}}\, a_{\frac{m+3}{2}}] \\ & \cup\, b_{i-\frac{m+1}{2}}[c_{m+1}\, d_1][c_{m+2}\, d_2] \\ & -\, b_{i-\frac{m+1}{2}}[c_{m+1}\, c_{m+2}] \\ & \cup\, c_{2i-m-1}[d_1\, d_2]. \\ & \cup\, c_{2i-m}[d_1\, d_2] \end{aligned}$$

When $i = \frac{3m+1}{2}$,

$$\begin{aligned} H_{\frac{3m+1}{2}} = G_{\frac{3m+1}{2}} \;\; & \cup\, d_1[F_0 : 2m][F_0 : 1] \\ & \cup\, d_2[F_0 : 2m][F_0 : 1] \\ & \cup\, a_1[b_{\frac{m+1}{2}}\, d_2][a_{\frac{m+3}{2}}\, d_1] \\ & -\, a_1[b_{\frac{m+1}{2}}\, a_{\frac{m+3}{2}}] \\ & \cup\, b_m[c_{m+1}\, d_1][c_{m+2}\, d_2] \\ & -\, b_m[c_{m+1}\, c_{m+2}] \\ & \cup\, c_1[d_1\, d_2] \\ & \cup\, c_{2m}[d_1\, d_2]. \end{aligned}$$

The proof of the following claim is similar to that of Claim 2.3.2 and is not given.

Claim 2.3.4 *When m is odd, the H_i, $0 \le i \le 2m$, partition all of the 2–paths in S_0, except those centred at vertex ∞.*

Claim 2.3.5 *When m is odd, the part H_0 is a trail beginning on the edge $\infty\, d_1$ and ending on the edge $d_2\,\infty$. Each H_i, $1 \le i \le 2m$, is a trail beginning and ending on the same edges as G_i.*

Proof. H_0 is again the trail $\infty\, d_1\, d_2\, \infty$.

When $1 \le i \le \frac{m+1}{2}$ and $\frac{3m+3}{2} \le i \le 2m$, $H_i = G_i$.

When $\frac{m+3}{2} \le i \le \frac{3m-1}{2}$, G_i is a trail containing the 2–paths $b_{\frac{m+1}{2}}\, a_{i-\frac{m-1}{2}}\, a_{\frac{m+3}{2}}$ and $c_{m+1}\, b_{i-\frac{m+1}{2}}\, c_{m+2}$, not necessarily in this order. In H_i, $b_{\frac{m+1}{2}}\, a_{i-\frac{m-1}{2}}\, a_{\frac{m+3}{2}}$ becomes the trail

$$b_{\frac{m+1}{2}}\, a_{i-\frac{m-1}{2}}\, d_2\, c_{2i-m}\, d_1\, a_{i-\frac{m-1}{2}}\, a_{\frac{m+3}{2}},$$

and $c_{m+1}\, b_{i-\frac{m+1}{2}}\, c_{m+2}$ becomes the trail

$$c_{m+1}\, b_{i-\frac{m+1}{2}}\, d_1\, c_{2i-m-1}\, d_2\, b_{i-\frac{m+1}{2}}\, c_{m+2}.$$

When $i = \frac{3m+1}{2}$, $G_{\frac{3m+1}{2}}$ is a trail containing the 2–paths $b_{\frac{m+1}{2}}\, a_1\, a_{\frac{m+3}{2}}$ and $c_{m+1}\, b_m\, c_{m+2}$. In $H_{\frac{3m+1}{2}}$, the first of these 2–paths, $b_{\frac{m+1}{2}}\, a_1\, a_{\frac{m+3}{2}}$, becomes the trail

$$b_{\frac{m+1}{2}}\, a_1\, d_2\, c_1\, d_1\, a_1\, a_{\frac{m+3}{2}},$$

and the second, $c_{m+1}\, b_m\, c_{m+2}$, becomes

$$c_{m+1} b_m d_1\, c_{2m}\, d_2\, b_m\, c_{m+2}.$$

As before, we have not affected the first and last edges of G_i in constructing H_i. □

It remains to prove that the 2–paths centred at vertex ∞ join the H_i together to form an Euler tour. For $4m+3 \ge 11$, we consider the union of the 1–factor E_2 of K_{4m+2}, which dictates the end vertices of the 2–paths centred at vertex ∞ in S_0, with the 1–factor $F^* \cup \{d_1, d_2\}$. Recall that F^* is the 1–factor of K_{4m} whose i^{th} edge is $u\,v$ if $\infty\, u$ and $v\,\infty$ are the first and last edges of G_i. When $4m+3 = 7$, we use the 1–factor E_3 instead of E_2.

The proof of the following claim is easily seen.

Claim 2.3.6 *When $4m + 3 = 11$, the union of the two 1–factors of K_{10}, $E_2 = \{c_1\, c_4, b_1\, c_3, b_2\, a_2, d_1\, c_2, d_2\, a_1\}$ and $F^* \cup d_1\, d_2 = \{d_1\, d_2, c_1\, b_1, c_2\, a_2, c_3\, b_2, c_4\, a_1\}$ is a Hamilton cycle of K_{10}.*

Claim 2.3.7 *When m is even and $4m + 3 > 11$, the union of*

$$
\begin{aligned}
E_2 &= \{b_i\, c_{2i-2} : 2 \le i \le \frac{m}{2}, \frac{m}{2} + 2 \le i \le m\} \\
&\cup \{a_i\, c_{2i+1} : 2 \le i \le \frac{m}{2}, \frac{m}{2} + 2 \le i \le m-1\} \\
&\cup \{c_1\, c_{2m}, b_1\, c_3, a_1\, a_m, d_1\, c_m, d_2\, c_{m+3}, a_{\frac{m}{2}+1}\, b_{\frac{m}{2}+1}\}
\end{aligned}
$$

and

$$
\begin{aligned}
F^* \cup \{d_1, d_2\} &= \{a_i\, c_{2i-2}, 2 \le i \le m\} \\
&\cup \{b_i\, c_{2i-1}, 1 \le i \le m\} \\
&\cup \{a_1\, c_{2m}, d_1\, d_2\}
\end{aligned}
$$

is a Hamilton cycle of K_{4m+2}.

Proof. It is straightforward to check that the union of the two 1–factors is the following Hamilton cycle.

$$
(b_2\, c_2\, a_2\, c_5 \cdots b_{\frac{m}{2}}\, c_{m-2}\, a_{\frac{m}{2}}\, c_{m+1}
$$

$$
b_{\frac{m}{2}+1}\, a_{\frac{m}{2}+1}\, c_m\, d_1\, d_2\, c_{m+3}\, b_{\frac{m}{2}+2}\, c_{m+2}\, a_{\frac{m}{2}+2}\, c_{m+5} \cdots b_{m-1}\, c_{2m-4}\, a_{m-1}\, c_{2m-1}
$$

$$
b_m\, c_{2m-2}\, a_m\, a_1\, c_{2m}\, c_1\, b_1\, c_3).
$$

□

The proof of the next claim is again easily seen.

Claim 2.3.8 *When $4m + 3 = 7$, the union of the two 1–factors of K_6, $E_3 = \{c_1\, d_1, d_2\, c_2, a_1\, b_1\}$ and $F^* \cup \{d_1, d_2\} = \{d_1\, d_2, c_1\, b_1, c_2\, a_1\}$ is a Hamilton cycle.*

Claim 2.3.9 *When m is odd and $4m+3>7$, the union of*

$$\begin{aligned} E_2 &= \{b_i\, c_{2i-2} : 2 \le i \le \frac{m+1}{2}, \frac{m+5}{2} \le i \le m\} \\ &\cup \{a_i\, c_{2i+1} : 2 \le i \le \frac{m-1}{2}, \frac{m+3}{2} \le i \le m-1\} \\ &\cup \{c_1\, c_{2m}, b_1\, c_3, a_1\, a_m, d_1\, a_{\frac{m+1}{2}}, d_2\, b_{\frac{m+3}{2}}, c_{m+1}\, c_{m+2}\} \end{aligned}$$

and

$$\begin{aligned} F^* \cup \{d_1, d_2\} &= \{a_i\, c_{2i-2}, 2 \le i \le m\} \\ &\cup \{b_i\, c_{2i-1}, 1 \le i \le m\} \\ &\cup \{a_1\, c_{2m}, d_1\, d_2\} \end{aligned}$$

is a Hamilton cycle of K_{4m+3}.

Proof. The two 1–factors form the following Hamilton cycle.

$$(b_2\, c_2\, a_2\, c_5 \cdots b_{\frac{m-1}{2}}\, c_{m-3}\, a_{\frac{m-1}{2}}\, c_m\, b_{\frac{m+1}{2}}\, c_{m-1}\, a_{\frac{m+1}{2}}\, d_1\, d_2$$

$$b_{\frac{m+3}{2}}\, c_{m+2}\, c_{m+1}\, a_{\frac{m+3}{2}}\, c_{m+4}\, b_{\frac{m+5}{2}}\, c_{m+3}\, a_{\frac{m+5}{2}}\, c_{m+6} \cdots$$

$$b_{m-1}\, c_{2m-4}\, a_{m-1}\, c_{2m-1}\, b_m\, c_{2m-2}\, a_m\, a_1\, c_{2m}\, c_1\, b_1\, c_3).$$

□

By Claims 2.3.6 and 2.3.7, the 2–paths centred at vertex ∞ join the parts H_i, $0 \le i \le 2m$, together so that S_0 is an Euler tour when m is even. Similarly, by Claims 2.3.8 and 2.3.9, S_0 is an Euler tour when m is odd. We have shown that K_{4m+3} has a perfect set of Euler tours for all m.

This completes the proof of Theorem 2.1.1.

2.4 A Corollary of the Main Result

We now prove Corollary 2.1.3 which states that it is possible to traverse the edges of K_{2k+1} so that every 2–path occurs exactly once.

Proof of Corollary 2.1.3. Case 1: Suppose $k = 2m$ is even. The Euler tour T_i constructed in Section 2.2 contains the 2–path $\sigma^i(a_1)\infty c_1$ for all $i \in \{0, 1, 2, \ldots, 4m-2\}$. Let T_i^* be the trail $T_i \setminus \{\sigma^i(a_1)\infty c_1\}$ for all $i \in \{0, 1, 2, \ldots, 4m-2\}$ and assume that T_i^* goes from ∞c_1 to $\sigma^i(a_1)\infty$. Then the following union of trails and 2–paths is the required walk in K_{4m+1}:

$$T_0^* \, a_1 \infty c_1 \, T_1^* \, \sigma^1(a_1) \infty c_1 \, T_2^* \, \sigma^2(a_1) \infty c_1 \cdots T_{4m-2}^* \, \sigma^{4m-2}(a_1) \infty c_1.$$

Case 2: Now suppose $k = 2m + 1$ is odd. If $m > 1$, then the Euler tour S_i constructed in Section 2.3 contains the 2–path $\sigma^i(c_{2m})\infty c_1$ for all $i \in \{0, 1, 2, \ldots, 4m\}$. Let S_i^* be the trail $S_i \setminus \{\sigma^i(c_{2m})\infty c_1\}$ for all $i \in \{0, 1, 2, \ldots, 4m\}$ and assume that S_i^* goes from ∞c_1 to $\sigma^i(c_{2m})\infty$. Then the following union of trails and 2–paths is the required walk in K_{4m+3}:

$$S_0^* \, c_{2m} \infty c_1 \, S_1^* \, \sigma^1(c_{2m}) \infty c_1 \, S_2^* \, \sigma^2(c_{2m}) \infty c_1 \cdots S_{4m}^* \, \sigma^{4m}(c_{2m}) \infty c_1.$$

When $m = 1$, we had to use a different 1–factor to determine the 2–paths centred at ∞. The result follows exactly as before but now we have the 2–path $d_1 \infty c_1$ in S_0 instead of $c_{2m} \infty c_1$, so replace every occurrence c_{2m} in the above traversal by d_1. □

It is interesting to note that since we are merely tracing out the edges of one Euler trail after another, this ordering of the edges has the added property that each edge in K_{2k+1} is traversed exactly j times before any edge is traversed more than j times, for all $j \in \{1, 2, \ldots, 2k-2\}$.

Chapter 3

A Perfect Set of Euler Tours of $K_{2k} + I$

3.1 Main Result

In this chapter we prove the following theorem and corollary.

Theorem 3.1.1 *For all $k > 1$, $K_{2k} + I$ has a perfect set of Euler tours.*

Recall that we defined a perfect set of Euler tours of $K_{2k} + I$, where I is a 1-factor of K_{2k} to be a set of $2k - 2$ Euler tours of $K_{2k} + I$ such that every 2-path of K_{2k} is in exactly one of the Euler tours, and for each of the edges $a\,b \in I$, each Euler tour either uses the digon $a\,b\,a$ or the digon $b\,a\,b$.

Corollary 3.1.2 *For all $k > 1$, $L(K_{2k})$ has a Hamilton decomposition.*

Proof. Given a perfect set of Euler tours of $K_{2k} + I$, replace each 2-path $a\,b\,c$ with the edge $ab\,bc$ in $L(K_{2k})$, and ignore the digons. (A sequence of 2-paths and digons such as $a\,b\,c, b\,c\,b, c\,b\,d$ will become the two adjacent edges $ab\,bc$ and $cb\,bd$ in $L(K_{2k})$.) The proof now follows in exactly the same way as that of Corollary 2.1.2. □

This chapter is divided into two sections as the two cases of k even and k odd are again considered separately. In both sections we use the following well-known construction of a Hamilton decomposition $\mathcal{H}$ of K_{2k-1}: Let $V(K_{2k-1}) = \{1, 2, \ldots, 2k-1\}$. Let σ_1 be the permutation

$$(1)(2\ 3\ 4 \cdots 2k{-}3\ 2k{-}2\ 2k{-}1)$$

of the vertices of K_{2k-1} that fixes vertex 1 and cyclically rotates the others. Then $\mathcal{H} = \{H_0, H_1, \ldots, H_{k-2}\}$, where H_0 is the Hamilton cycle

$$(1\ 2\ 2k{-}1\ 3\ 2k{-}2\ 4 \cdots k\ k{+}1),$$

and $H_i = \sigma_1^i(H_0)$, $0 \le i \le k-2$, is a Hamilton decomposition of K_{2k-1}. We actually want to construct $H_i = \sigma_1^i(H_0)$, for $0 \le i \le 2k-3$, so that we generate each Hamilton cycle twice. Thus for all $i \in \{0, 1, 2, \ldots, 2k-3\}$, $H_{i+k-1} = H_i$, where subscript addition is modulo $2k-2$ on the residue classes $0, 1, \ldots, 2k-3$.

3.2 A Perfect Set of Euler Tours of $K_{4m}+I$

Let $k = 2m$. Let $V(K_{4m-1}) = \{\infty_2\} \cup A \cup B \cup C \cup D$, where $A = \{a_1, a_2, \ldots, a_{m-1}\}$, $B = \{b_1, b_2, \ldots, b_{m-1}\}$, $C = \{c_0, c_1, c_2, \ldots, c_{m-1}\}$ and $D = \{d_1, d_2, \ldots, d_{m-1}, d_m\}$. Let $V(K_{4m}) = V(K_{4m-1}) \cup \{\infty_1\}$.

We use the above construction of a Hamilton decomposition of K_{4m-1} but with the new labeling on the vertex set. Let σ be the following permutation of $V(K_{4m})$

$$(\infty_1)(\infty_2)(a_1\, b_1\, a_2\, b_2 \cdots a_{m-1}\, b_{m-1}\, d_m\, c_{m-1}\, d_{m-1}\, c_{m-2}\, d_{m-2} \cdots c_1\, d_1\, c_0)$$

that fixes ∞_1 and generates a Hamilton decomposition of K_{4m-1} on the vertex set $\{\infty_2\} \cup A \cup B \cup C \cup D$. The Hamilton cycle H_0 (shown in Figure 3.1) is

$$(\infty_2\, c_0\, d_1\, a_1\, c_1\, b_1\, d_2\, a_2\, c_2\, b_2 \cdots d_i\, a_i\, c_i\, b_i \cdots d_{m-1}\, a_{m-1}\, c_{m-1}\, b_{m-1}\, d_m),$$

and we now have the Hamilton decomposition $\mathcal{H} = \{H_0, H_1, \ldots, H_{2m-2}\}$, where $H_i = \sigma^i(H_0)$, $0 \le i \le 2m-2$. As mentioned in Section 3.1, we actually want to consider $H_i = \sigma^i(H_0)$, $0 \le i \le 4m-3$.

We order the edges of H_i, $i \in \{0, 1, 2, \ldots, 4m-3\}$, so that $\infty_2\, \sigma^i(c_0)$ is its first edge, $\sigma^i(c_0)\, \sigma^i(d_1)$ is its second edge, and so on, counting off the edges around the

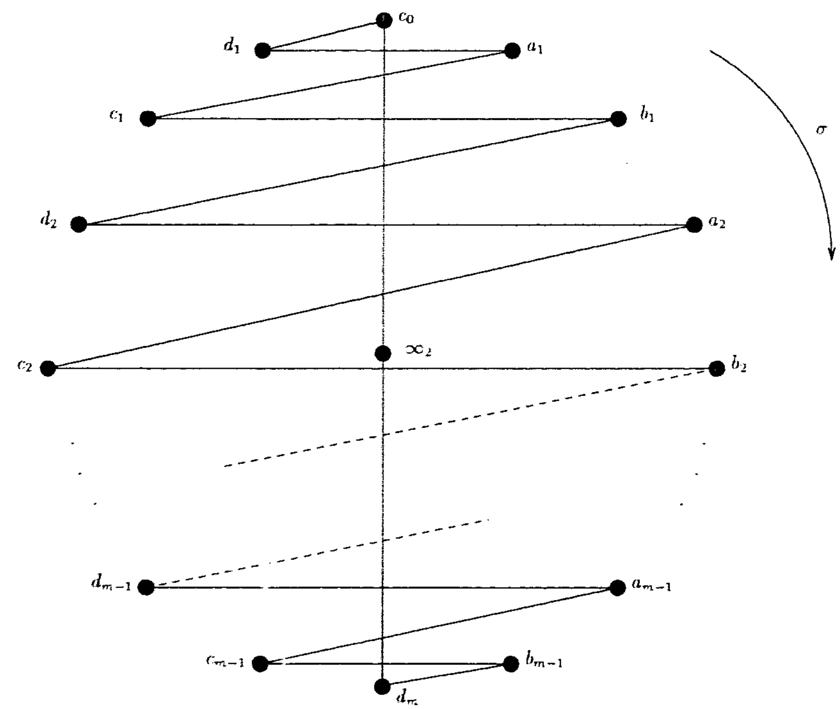


Figure 3.1: H_0 and σ

cycle, so that $\sigma^i(b_{m-1})\,\sigma^i(d_m)$ and $\sigma^i(d_m)\,\infty_2$ are its $(4m-2)^{th}$ and $(4m-1)^{th}$ edges, respectively. (We will be using that fact that $[H_i : j] = [H_{i+2m-1} : 4m-j]$.)

Our aim is to find a perfect set of Euler tours of $K_{4m}+I$. We choose the 1–factor I to be those edges of K_{4m} that are fixed by σ^{2m-1}, so that I is itself fixed (setwise) by σ. Thus,

$$I = \{\infty_1\,\infty_2, c_0\,d_m\} \cup \{a_i\,c_{m-i} : 1 \le i \le m-1\} \cup \{b_i\,d_{m-i} : 1 \le i \le m-1\}.$$

We call the $4m-2$ Euler tours of $K_{4m}+I$ that we will construct T_j, $0 \le j \le 4m-3$. It is sufficient to specify each T_j by providing a list of the 2–paths and digons it contains. It is then necessary to prove that the T_j do indeed partition the set of 2–paths of K_{4m} and to prove that each T_j is really an Euler tour of $K_{4m}+I$ that satisfies the condition on the digons.

In order to construct T_j, $0 \le j \le 4m-3$, we first construct T'_j, $0 \le j \le 4m-3$, where each T'_j contains only the 2–paths in T_j that are centred at vertices in $A \cup B \cup C \cup D$, and not those that are centred at ∞_1 or ∞_2. We do this because the T'_j will all be pairwise similar, and we can establish their basic structure merely by considering T_0'. Once we have proved that T_0' is a set of $2m$ trails that start at vertex ∞_1 and end at vertex ∞_2, we will know that the same is true of each T_j'. We can then find 2–paths centred at vertices ∞_1 and ∞_2 that will join the trails in each T_j' into the Euler tour, T_j.

The set of 2–paths for specifying T_0' will be based on the edges in the Hamilton cycles H_{4m-3}, H_0, and H_1. As usual, we denote H_{4m-3} by H_{-1} in order to emphasize that $\sigma^{-1}(H_0) = H_{4m-3}$. The edges of these three Hamilton cycles are ordered as described above and listed below.

$$\begin{aligned}
[H_0 : 1] &= \infty_2\, c_0, \\
[H_0 : 2] &= c_0\, d_1, \\
[H_0 : 4k-1] &= d_k\, a_k, 1 \le k \le m-1, \\
[H_0 : 4k] &= a_k\, c_k, 1 \le k \le m-1, \\
[H_0 : 4k+1] &= c_k\, b_k, 1 \le k \le m-1, \\
[H_0 : 4k+2] &= b_k\, d_{k+1}, 1 \le k \le m-1, \\
[H_0 : 4m-1] &= d_m\, \infty_2.
\end{aligned}$$

$$\begin{aligned}
[H_1 : 1] &= \infty_2\, a_1, \\
[H_1 : 4k-2] &= a_k\, c_{k-1}, 1 \le k \le m-1, \\
[H_1 : 4k-1] &= c_{k-1}\, b_k, 1 \le k \le m-1, \\
[H_1 : 4k] &= b_k\, d_k, 1 \le k \le m-1, \\
[H_1 : 4k+1] &= d_k\, a_{k+1}, 1 \le k \le m-2, \\
[H_1 : 4m-3] &= d_{m-1}\, d_m, \\
[H_1 : 4m-2] &= d_m\, c_{m-1}, \\
[H_1 : 4m-1] &= c_{m-1}\, \infty_2.
\end{aligned}$$

$$
\begin{aligned}
[H_{-1} : 1] &= \infty_2\, d_1, \\
[H_{-1} : 2] &= d_1\, c_1, \\
[H_{-1} : 3] &= c_1\, c_0, \\
[H_{-1} : 4] &= c_0\, d_2, \\
[H_{-1} : 4k+1] &= d_{k+1}\, a_k, 1 \le k \le m-1, \\
[H_{-1} : 4k+2] &= a_k\, c_{k+1}, 1 \le k \le m-2, \\
[H_{-1} : 4k+3] &= c_{k+1}\, b_k, 1 \le k \le m-2, \\
[H_{-1} : 4k+4] &= b_k\, d_{k+2}, 1 \le k \le m-2, \\
[H_{-1} : 4m-2] &= a_{m-1}\, b_{m-1}, \\
[H_{-1} : 4m-1] &= b_{m-1}\, \infty_2.
\end{aligned}
$$

We use the notation described in Chapter 1 to list the 2–paths in T_0':

$$a_i[H_{-1} : 1, 3, 5, \ldots, 4(m-i)-3,\ 4(m-i), 4(m-i)+2, 4(m-i)+4, \ldots, 4m-2],$$

$$b_i[H_1 : 1, 3, 5, \ldots, 4(m-i)-1,\ 4(m-i)+2, 4(m-i)+4, 4(m-i)+6, \ldots, 4m-2],$$

$$c_i[H_0 : 1, 3, 5, \ldots, 4(m-i)-1,\ 4(m-i), 4(m-i)+2, 4(m-i)+4, \ldots, 4m-2],$$

$$d_i[H_0 : 1, 3, 5, \ldots, 4(m-i)+1,\ 4(m-i)+2, 4(m-i)+4, 4(m-i)+6, \ldots, 4m-2],$$

all for $1 \le i \le m-1$, as well as

$$c_0[H_0 : 1, 3, 5, \ldots, 4m-3] \text{ and } d_m[H_0 : 1, 2, 4, \ldots, 4m-2].$$

For each centre vertex v, the two adjacent edges at which the change from an odd numbered edge to an even numbered edge is made are the two edges that contain v', where $v\,v' \in I$. This ensures that the correct digons are included in the tour.

Now let $T_j' = \sigma^j(T_0')$, for $1 \le j \le 4m-3$. Since σ fixes I, σ is an automorphism of $K_{4m} + I$. Therefore, the T_j', $0 \le j \le 4m-3$, are pairwise similar. We need to prove they partition the set of 2–paths of K_{4m} that are not centred at ∞_1 or ∞_2.

Claim 3.2.1 *The T'_j, $0 \le j \le 4m-3$, partition the set of 2–paths in K_{4m} that are centred at a vertex in $A \cup B \cup C \cup D$.*

Proof.

We show that for each vertex v in $A \cup B \cup C \cup D$, for each $r \in \{1,2,\dots,4m-1\}$, and each $j \in \{0,1,2,\dots,2m-2\}$, we have each 2–path $\sigma^j(v[H_0 : r])$ in one of the T'_l. These 2–paths are all different and we have $(4m-2)(2m-1)(4m-1)$ of them. As this is the number of 2–paths in K_{4m} that are not centred at ∞_1 or ∞_2, we must have every such 2–path exactly once. Addition on the subscripts of the T'_j will be modulo $4m-2$.

Case 1: Consider $c_i \in C$, $1 \le i \le m-1$. By construction, the 2–paths in $c_i[H_0 : 1,3,\dots,4(m-i)-1,\ 4(m-i), 4(m-i)+2,\dots,4m-2]$ and in $b_{m-i}[H_1 : 1,3,\dots,4i-1,\ 4i+2, 4i+4,\dots,4m-2]$ are in T'_0. Now $\sigma^{2m-2}(b_{m-i}) = c_i$ for all $i \in \{1,2,\dots,m-1\}$, and $\sigma^{2m-2}(H_1) = H_{2m-1}$. Therefore, $T'_{2m-2} = \sigma^{2m-2}(T'_0)$ will contain the 2–paths in:

$$\begin{aligned}
&\sigma^{2m-2}(b_{m-i}[H_1 : 1,3,\dots,4i-1,\ 4i+2,4i+4,\dots,4m-2]) \\
=\ & c_i[H_{2m-1} : 1,3,\dots,4i-1,\ 4i+2,4i+4,\dots,4m-2] \\
=\ & c_i[H_0 : 2,4,\dots,4(m-i)-2,\ 4(m-i)+1,4(m-i)+3,\dots,4m-1],
\end{aligned}$$

using the fact that $[H_{j+2m-1} : k] = [H_j : 4m-k]$.

Therefore, we have each of the 2–paths $c_i[H_0 : r]$, $r \in \{1,2,\dots,4m-1\}$, at least once in one of T'_0 or T'_{2m-2}, and hence each of the 2–paths $\sigma^j(c_i[H_0 : r])$, $r \in \{1,2,\dots,4m-1\}$, in one of $T'_j = \sigma^j(T'_0)$, or $T'_{2m-2+j} = \sigma^j(T'_{2m-2})$, for each $j \in \{0,1,2,\dots,4m-3\}$. This is equivalent to having each of the 2–paths $\sigma^j(c_i[H_0 : r])$, $r \in \{1,2,\dots,4m-1\}$, $0 \le j \le 2m-2$, and $\sigma^j(a_{m-i}[H_0 : r])$, $r \in \{1,2,\dots,4m-1\}$, $0 \le j \le 2m-2$, in some T'_l, since $\sigma^{2m-1}(c_i) = a_{m-i}$, for all $i \in \{1,2,\dots,m-1\}$, and since $H_0 = H_{2m-1}$.

Case 2: Consider $d_i \in D$, $1 \le i \le m-1$. The 2–paths in $d_i[H_0 : 1,3,\dots,4m-4i+1,\ 4m-4i+2,4m-4i+4,\dots,4m-2]$ and in $a_{m-i}[H_{-1} : 1,3,\dots,4i-3,\ 4i,4i+2,\dots,4m-2]$ are in T'_0, and $\sigma^{2m}(a_{m-i}) = d_i$, for all $i \in \{1,2,\dots,m-1\}$, and

$\sigma^{2m}(H_{-1}) = H_{2m-1}$. Thus T'_{2m} contains the 2–paths in

$$\begin{aligned} & \sigma^{2m}(a_{m-i}[H_{-1}:1,3,\ldots,4i-3,\ 4i,4i+2,\ldots,4m-2]) \\ = \ & d_i[H_{2m-1}:1,3,\ldots,4i-3,\ 4i,4i+2,\ldots,4m-2] \\ = \ & d_i[H_0:2,4,\ldots,4m-4i,\ 4m-4i+3,4m+4i+5,\ldots,4m-1]. \end{aligned}$$

Therefore, we have each of the 2–paths $d_i[H_0:r]$, $r\in\{1,2,\ldots,4m-1\}$, at least once in one of T'_0 or T'_{2m}, and hence each of $\sigma^j(d_i[H_0:r])$, $r\in\{1,2,\ldots,4m-1\}$, in at least one of T'_j or T'_{2m+j}, for each $j\in\{0,1,2,\ldots,4m-3\}$. This is equivalent to having each of the 2–paths $\sigma^j(d_i[H_0:r])$, $r\in\{1,2,\ldots,4m-1\}$, $0\le j\le 2m-2$, and $\sigma^j(b_{m-i}[H_0:r])$, $\{1,2,\ldots,4m-1\}$, $0\le j\le 2m-2$, at least once, since $\sigma^{2m-1}(d_i)=b_{m-i}$, for all $i\in\{1,2,\ldots,m-1\}$,

Case 3: Finally, we consider the vertex c_0. The 2–paths in $c_0[H_0:1,3,5,\ldots,4m-3]$ and in $d_m[H_0:1,2,4,\ldots,4m-2]$ are in T'_0. Also, $\sigma^{2m-1}(d_m)=c_0$, and $\sigma^{2m-1}(H_0)=H_{2m-1}$. Therefore, T'_{2m-1} contains the 2–paths in

$$\begin{aligned} & \sigma^{2m-1}(d_m[H_0:1,2,4,\ldots,4m-2]) \\ = \ & c_0[H_{2m-1}:1,2,4,\ldots,4m-2] \\ = \ & c_0[H_0:2,4,\ldots,4m-2,4m-1], \end{aligned}$$

giving $c_0[H_0:r]$, $r\in\{1,2,\ldots,4m-1\}$, at least once in one of T'_0 or T'_{2m-1}. Therefore, we have the 2–paths $\sigma^j(c_0[H_0:r])$, $r\in\{1,2,\ldots,4m-1\}$ in either T'_j or T'_{2m-1+j}, for each $j\in\{0,1,2,\ldots,4m-3\}$. This is equivalent to having $\sigma^j(c_0[H_0:r])$, $r\in\{1,2,\ldots,4m-1\}$, $0\le j\le 2m-2$, and $\sigma^j(d_m[H_0:r])$, $r\in\{1,2,\ldots,4m-1\}$, $0\le j\le 2m-2$, each at least once.

In total, for $v\in A\cup B\cup C\cup D$, we have $\sigma^j(v[H_0:1,2,\ldots,4m-1])$ for all $j\in\{0,1,\ldots,2m-2\}$. □

The following claim establishes the structure of T'_0. It will be important to know the first and last edges of each trail in T'_0 when we come to put in the 2–paths centred at ∞_1 and ∞_2.

Claim 3.2.2 *The list of 2–paths given for T'_0 forms a set of $2m-1$ trails, each of which starts on ∞_1 and ends on ∞_2. We will call the trail that starts on the edge*

$\infty_1 v$, P_v, for all $v \in A \cup B \cup C \cup D$. The trails, with their first and last edges, are as follows:

P_{a_i}, $1 \leq i \leq \lceil \frac{m}{2} \rceil - 1$, *a trail from* $\infty_1 a_i$ *to* $a_{2i} \infty_2$,
P_{b_i}, $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$, *a trail from* $\infty_1 b_i$ *to* $c_{2i-1} \infty_2$,
P_{c_i}, $0 \leq i \leq \lceil \frac{m}{2} \rceil - 1$, *a trail from* $\infty_1 c_i$ *to* $c_{2i} \infty_2$,
P_{d_i}, $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$, *a trail from* $\infty_1 d_i$ *to* $a_{2i-1} \infty_2$,
$P_{c_{\frac{m}{2}}}$ *when* m *is even, a trail from* $\infty_1 c_{\frac{m}{2}}$ *to* $d_m \infty_2$,
$P_{d_{\frac{m+1}{2}}}$, *when* m *is odd, a trail from* $\infty_1 d_{\frac{m+1}{2}}$ *to* $d_m \infty_2$,
P_{a_i}, $\lceil \frac{m}{2} \rceil \leq i \leq m - 1$, *a trail from* $\infty_1 a_i$ *to* $b_{2(m-i)-1} \infty_2$,
P_{b_i}, $\lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m - 1$, *a trail from* $\infty_1 b_i$ *to* $d_{2(m-i)} \infty_2$,
P_{c_i}, $\lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m - 1$, *a trail from* $\infty_1 c_i$ *to* $b_{2(m-i)} \infty_2$, *and*
P_{d_i}, $\lceil \frac{m}{2} \rceil + 1 \leq i \leq m$, *a trail from* $\infty_1 d_i$ *to* $d_{2(m-i)+1} \infty_2$.

Proof.

To prove this claim, we list the order in which the 2–paths occur in the trails. We will not mention which Hamilton cycle the edges that determine the end vertices of a 2–path come from, as it should be clear that 2–paths centred at a vertex in A, B or $C \cup D$, have their end vertices from H_{-1}, H_1, or H_0, respectively. Some of the 2–paths have a superscript. These superscripts are relevant only in Section 3.3.2.

In order to check that we have covered every edge exactly once, it is enough to verify that every 2–path listed below is actually in T_0', that every two adjacent 2–paths overlap in an edge, and to count the number of edges covered by the trails. Since any edge containing ∞_1 or ∞_2 is in only one 2–path in T_0', these edges must determine the ends of trails. If these trails cover $8m^2 - 2$ edges, then we have covered all the edges in $K_{4m} - \{\infty_1 \infty_2\} \cup (I - \{\infty_1 \infty_2\})$ exactly once. It is not necessary to check that the edges covered are all distinct: once you start a trail at an edge, say, $\infty_1 v$, then the rest of the trail is completely determined because every edge (except those containing ∞_1 or ∞_2) is in exactly two 2–paths. It is obvious that the following trails all start on different edges.

Counting the edges in the P_v does yield $8m^2 - 2$ edges, as required.

The trails are as follows and the verification of the above although dreadfully tedious is not difficult. Within each trail there are several patterns on sets of four 2–paths. We will show a pattern and specify that it occurs for $q_1 \leq j \leq q_2$, for some q_1 and q_2. We will also show the pattern for $j = q_1$ and $j = q_2$ because this helps when verifying that every pair of adjacent 2–paths do overlap in an edge. A 2–path will be underlined if it happens to be the end of a pattern for some j.

$P_{a_i}, 1 \le i \le \lceil \frac{m}{2} \rceil - 1:$

$a_i[4i+1]$	$d_{i+1}[4i-1]$	$d_i[4i+3]$	$a_{i+1}[4i-3]$	$(j=0)$
$a_{i-j}[4i+4j+1]$	$d_{i+j+1}[4i-4j-1]$	$d_{i-j}[4i+4j+3]$	$a_{i+j+1}[4i-4j-3]$	$0 \le j \le i-1$
$a_1[8i-3]$	$d_{2i}[3]$	$d_1[8i-1]$	$a_{2i}[1]$	$(j=i-1)$

$P_{a_i}, \lceil \frac{m}{2} \rceil \le i \le m-1:$

$a_i[4i+2]$	$c_{i+1}[4i]$	$c_i[4i+4]$	$a_{i+1}[4i-2]$	$(j=0)$
$a_{i-j}[4i+4j+2]$	$c_{i+j+1}[4i-4j]$	$c_{i-j}[4i+4j+4]$	$a_{i+j+1}[4i-4j-2]$	$0 \le j \le m-i-2$
$a_{2i-m+2}[4m-6]$	$c_{m-1}[8i-4m+8]$	$c_{2i-m+2}[4m-4]$	$a_{m-1}[8i-4m+6]$	$(j=m-i-2)$
$\underline{a_{2i-m+1}[4m-2]}^*$				
$b_{m-1}[8i-4m+2]$	$c_{2i-m}[4m-2]$	$d_m[8i-4m]$	$a_{2i-m}[4m-4]$	$(j=0)$
$b_{m-j-1}[8i-4m-4j+2]$	$c_{2i-m-j}[4m-4j-2]$	$d_{m-j}[8i-4m-4j]$	$a_{2i-m-j}[4m-4j-4]$	$0 \le j \le i-\lfloor \frac{m}{2} \rfloor -1$
$b_{\lfloor \frac{3m}{2} \rfloor -i}[4(i-\lceil \frac{m}{2} \rceil)+6]^{*(o)}$	$c_{i-\lceil \frac{m}{2} \rceil+1}[4(\lfloor \frac{3m}{2} \rfloor -i)+2]^{\dagger(o)}$	$d_{\lfloor \frac{3m}{2} \rfloor -i+1}[4(i-\lceil \frac{m}{2} \rceil)+4]^{\dagger(o)}$	$a_{i-\lceil \frac{m}{2} \rceil+1}[4(\lfloor \frac{3m}{2} \rfloor -i)]^{*(o)}$	$(j=i-\lfloor \frac{m}{2} \rfloor -1)$
$\underline{b_{\lfloor \frac{3m}{2} \rfloor -i-1}[4(i-\lceil \frac{m}{2} \rceil)+1]}$	$d_{i-\lceil \frac{m}{2} \rceil}[4(\lfloor \frac{3m}{2} \rfloor -i)-3]$	$c_{\lfloor \frac{3m}{2} \rfloor -i-1}[4(i-\lceil \frac{m}{2} \rceil)-1]$	$a_{i-\lceil \frac{m}{2} \rceil}[4(\lfloor \frac{3m}{2} \rfloor -i)-5]$	$(j=0)$
$b_{\lfloor \frac{3m}{2} \rfloor -i-j-1}[4(i-\lceil \frac{m}{2} \rceil -j)+1]$	$d_{i-\lceil \frac{m}{2} \rceil -j}[4(\lfloor \frac{3m}{2} \rfloor -i-j)-3]$	$c_{\lfloor \frac{3m}{2} \rfloor -i-j-1}[4(i-\lceil \frac{m}{2} \rceil -j)-1]$	$a_{i-\lceil \frac{m}{2} \rceil -j}[4(\lfloor \frac{3m}{2} \rfloor -i-j)-5]$	$0 \le j \le i-\lceil \frac{m}{2} \rceil -1$
$b_{2(m-i)}[5]$	$d_1[8(m-i)+1]$	$c_{2(m-i)}[3]$	$a_1[8(m-i)-1]$	$(j=i-\lceil \frac{m}{2} \rceil -1)$
$b_{2(m-i)-1}[1]$				

$P_{b_i}, 1 \le i \le \lfloor \frac{m}{2} \rfloor$:

$b_i[4i-1]$	$c_{i-1}[4i+1]$	$c_i[4i-3]$	$b_{i-1}[4i+3]$	$(j=0)$
$b_{i+j}[4i-4j-1]$	$c_{i-j-1}[4i+4j+1]$	$c_{i+j}[4i-4j-3]$	$b_{i-j-1}[4i+4j+3]$	$0 \le j \le i-2$
$b_{2i-2}[7]$	$c_1[8i-7]$	$c_{2i-2}[5]$	$b_1[8i-5]$	$(j=i-2)$
$b_{2i-1}[3]$	$c_0[8i-3]$	$c_{2i-1}[1]$		

$P_{b_{\frac{m+1}{2}}}, m$ odd :

$b_{\frac{m+1}{2}}[2m+2]$	$d_{\frac{m+1}{2}}[2m+4]$	$d_{\frac{m+3}{2}}[2m]$	$b_{\frac{m-1}{2}}[2m+6]$	$(j=0)$
$b_{\frac{m+1}{2}+j}[2m-4j+2]$	$d_{\frac{m+1}{2}-j}[2m+4j+4]$	$d_{\frac{m+3}{2}+j}[2m-4j]$	$b_{\frac{m-1}{2}-j}[2m+4j+6]$	$0 \le j \le \frac{m-5}{2}$
$b_{m-2}[12]$	$d_3[4m-6]$	$d_{m-1}[10]$	$b_2[4m-4]$	$(j=\frac{m-5}{2})$
$b_{m-1}[8]$	$d_2[4m-2]$	$d_m[6]$	$b_1[4m-2]^{*(o)}$	
$c_{m-1}[6]^{\dagger(o)}$	$d_2[4m-4]^{\dagger(o)}$	$a_{m-1}[4]^{*(o)}$	$c_0[4m-5]$	
$d_{m-1}[1]$				

$P_{b_i}, \lceil \frac{m}{2} \rceil + 1 \le i \le m-1$:

$b_i[4i]$	$d_i[4i+2]$	$d_{i+1}[4i-2]$	$b_{i-1}[4i+4]$	$(j=0)$
$b_{i+j}[4i-4j]$	$d_{i-j}[4i+4j+2]$	$d_{i+j+1}[4i-4j-2]$	$b_{i-j-1}[4i+4j+4]$	$0 \le j \le m-i-2$
$b_{m-2}[8i-4m+8]$	$d_{2i-m+2}[4m-6]$	$d_{m-1}[8i-4m+6]$	$b_{2i-m+1}[4m-4]$	$(j=m-i-2)$
$\underline{b_{m-1}[8i-4m+4]}$	$\underline{d_{2i-m+1}[4m-2]}$	$\underline{d_m[8i-4m+2]}$		
$\underline{b_{2i-m}[4m-2]}^*$	$c_{m-1}[8i-4m+2]$	$d_{2i-m+1}[4m-4]$	$a_{m-1}[8i-4m]$	$(j=0)$
$b_{2i-m-j}[4m-4j-2]$	$c_{m-j-1}[8i-4m-4j+2]$	$d_{2i-m-j+1}[4m-4j-4]$	$a_{m-j-1}[8i-4m-4j]$	$0 \le j \le i-\lfloor \frac{m}{2} \rfloor -1$
$b_{i-\lceil \frac{m}{2} \rceil+1}[4(\lfloor \frac{3m}{2} \rfloor -i)+2]^{*(o)}$	$c_{\lfloor \frac{3m}{2} \rfloor -i}[4(i-\lceil \frac{m}{2} \rceil)+6]^{\dagger(o)}$	$d_{i-\lceil \frac{m}{2} \rceil+2}[4(\lfloor \frac{3m}{2} \rfloor -i)]^{\dagger(o)}$	$a_{\lfloor \frac{3m}{2} \rfloor -i}[4(i-\lceil \frac{m}{2} \rceil+1)]^{*(o)}$	$(j=i-\lfloor \frac{m}{2} \rfloor -1)$
$b_{i-\lceil \frac{m}{2} \rceil}[4(\lfloor \frac{3m}{2} \rfloor -i)-3]$	$d_{\lfloor \frac{3m}{2} \rfloor -i-1}[4(i-\lceil \frac{m}{2} \rceil)+1]$	$c_{i-\lceil \frac{m}{2} \rceil}[4(\lfloor \frac{3m}{2} \rfloor -i)-5]$	$a_{\lfloor \frac{3m}{2} \rfloor -i-1}[4(i-\lceil \frac{m}{2} \rceil)-1]$	$(j=0)$
$b_{i-\lceil \frac{m}{2} \rceil -j}[4(\lfloor \frac{3m}{2} \rfloor -i-j)-3]$	$d_{\lfloor \frac{3m}{2} \rfloor -i-j-1}[4(i-j-\lceil \frac{m}{2} \rceil)+1]$	$c_{i-\lceil \frac{m}{2} \rceil -j}[4(\lfloor \frac{3m}{2} \rfloor -i-j)-5]$	$a_{\lfloor \frac{3m}{2} \rfloor -i-j-1}[4(i-\lceil \frac{m}{2} \rceil -j)-1]$	$0 \le j \le i-\lceil \frac{m}{2} \rceil -2$
$b_2[8(m-i)+5]$	$d_{2(m-i)+1}[9]$	$c_2[8(m-i)+3]$	$a_{2(m-i)+1}[7]$	$(j=i-\lceil \frac{m}{2} \rceil -2)$
$b_1[8(m-i)+1]$	$d_{2(m-i)}[5]$	$c_1[8(m-i)-1]$	$a_{2(m-i)}[3]$	
$c_0[8(m-i)-1]$	$\underline{d_{2(m-i)}[1]}$			

$P_{c_i}, 0 \le i \le \lceil \frac{m}{2} \rceil - 1:$

$c_i[4i+1]$	$b_i[4i+3]$	$b_{i+1}[4i-1]$	$c_{i-1}[4i+5]$	$(j=0)$
$c_{i+j}[4i-4j+1]$	$b_{i-j}[4i+4j+3]$	$b_{i+j+1}[4i-4j-1]$	$c_{i-j-1}[4i+4j+5]$	$0 \le j \le i-1$
$c_{2i-1}[5]$	$b_1[8i-1]$	$b_{2i}[3]$	$c_0[8i+1]$	$(j=i-1)$
$c_{2i}[1]$				

$P_{c_i}, \lfloor \frac{m}{2} \rfloor + 1 \le i \le m-1:$

$c_i[4i]$	$a_i[4i-2]$	$a_{i-1}[4i+2]$	$c_{i+1}[4i-4]$	$(j=0)$
$c_{i-j}[4i+4j]$	$a_{i+j}[4i-4j-2]$	$a_{i-j-1}[4i+4j+2]$	$c_{i+j+1}[4i-4j-4]$	$0 \le j \le m-i-2$
$c_{2i-m+2}[4m-8]$	$a_{m-2}[8i-4m+6]$	$a_{2i-m+1}[4m-6]$	$c_{m-1}[8i-4m+4]$	$(j=m-i-2)$
$\underline{c_{2i-m+1}[4m-4]}$	$\underline{a_{m-1}[8i-4m+2]}$	$\underline{a_{2i-m}[4m-2]}^*$		
$b_{m-1}[8i-4m-2]$	$c_{2i-m-1}[4m-2]$	$d_m[8i-4m-4]$	$a_{2i-m-1}[4m-4]$	$(j=0)$
$b_{m-j-1}[8i-4m-4j-2]$	$c_{2i-m-j-1}[4m-4j-2]$	$d_{m-j}[8i-4m-4j-4]$	$a_{2i-m-j-1}[4m-4j-4]$	$0 \le j \le i-\lceil \frac{m}{2} \rceil-1$
$b_{\lceil \frac{3m}{2} \rceil -i}[4(i-\lfloor \frac{m}{2} \rfloor)+2]^{*(e)}$	$c_{i-\lfloor \frac{m}{2} \rfloor}[4(\lceil \frac{3m}{2} \rceil -i)+2]^{\dagger(e)}$	$d_{\lceil \frac{3m}{2} \rceil -i+1}[4(i-\lfloor \frac{m}{2} \rfloor)]^{\dagger(e)}$	$a_{i-\lfloor \frac{m}{2} \rfloor}[4(\lceil \frac{3m}{2} \rceil -i)]^{*(e)}$	$(j=i-\lceil \frac{m}{2} \rceil-1)$
$\underline{b_{\lceil \frac{3m}{2} \rceil -i-1}[4(i-\lfloor \frac{m}{2} \rfloor)-3]}$				
$d_{i-\lfloor \frac{m}{2} \rfloor -1}[4(\lceil \frac{3m}{2} \rceil -i)-3]$	$c_{\lceil \frac{3m}{2} \rceil -i-1}[4(i-\lfloor \frac{m}{2} \rfloor)-5]$	$a_{i-\lfloor \frac{m}{2} \rfloor -1}[4(\lceil \frac{3m}{2} \rceil -i)-5]$	$b_{\lceil \frac{3m}{2} \rceil -i-2}[4(i-\lfloor \frac{m}{2} \rfloor)-7]$	$(j=0)$
$d_{i-\lfloor \frac{m}{2} \rfloor -j-1}[4(\lceil \frac{3m}{2} \rceil -i-j)-3]$	$c_{\lceil \frac{3m}{2} \rceil -i-j-1}[4(i-\lfloor \frac{m}{2} \rfloor -j)-5]$	$a_{i-\lfloor \frac{m}{2} \rfloor -j-1}[4(\lceil \frac{3m}{2} \rceil -i-j)-5]$	$b_{\lceil \frac{3m}{2} \rceil -i-j-2}[4(i-\lfloor \frac{m}{2} \rfloor -j)-7]$	$0 \le j \le i-\lfloor \frac{m}{2} \rfloor -2$
$d_1[8(m-i)+5]$	$c_{2(m-i)+1}[3]$	$a_1[8(m-i)+3]$	$b_{2(m-i)}[1]$	$(j=i-\lfloor \frac{m}{2} \rfloor -2)$

$P_{d_i}, 1 \le i \le \lfloor \frac{m}{2} \rfloor$:

$d_i[4i-1]$	$a_i[4i-3]$	$a_{i-1}[4i+1]$	$d_{i+1}[4i-5]$	$(j=0)$
$d_{i-j}[4i+4j-1]$	$a_{i+j}[4i-4j-3]$	$a_{i-j-1}[4i+4j+1]$	$d_{i+j+1}[4i-4j-5]$	$0 \le j \le i-2$
$d_2[8i-9]$	$a_{2i-2}[5]$	$a_1[8i-7]$	$d_{2i-1}[3]$	$(j=i-2)$
$d_1[8i-5]$	$a_{2i-1}[1]$			

$P_{d_{\frac{m}{2}+1}}, m$ even

$d_{\frac{m}{2}+1}[2m+2]$	$b_{\frac{m}{2}}[2m+4]$	$b_{\frac{m}{2}+1}[2m]$	$d_{\frac{m}{2}}[2m+6]$	$(j=0)$
$d_{\frac{m}{2}+j+1}[2m-4j+2]$	$b_{\frac{m}{2}-j}[2m+4j+4]$	$b_{\frac{m}{2}+j+1}[2m-4j]$	$d_{\frac{m}{2}-j}[2m+4j+6]$	$0 \le j \le \frac{m}{2}-2$
$d_{m-1}[10]$	$b_2[4m-4]$	$b_{m-1}[8]$	$d_2[4m-2]$	$(j=\frac{m}{2}-2)$
$d_m[6]$	$b_1[4m-2]^{*(e)}$	$c_{m-1}[6]^{\dagger(e)}$	$d_2[4m-4]^{\dagger(e)}$	
$a_{m-1}[4]^{*(e)}$	$c_0[4m-5]$	$d_{m-1}[1]$		

$P_{d_i}, \lfloor \frac{m}{2} \rfloor + 2 \le i \le m$:

$d_i[4i-2]$	$b_{i-1}[4i]$	$b_i[4i-4]$	$d_{i-1}[4i+2]$	$(j=0)$
$d_{i+j}[4i-4j-2]$	$b_{i-j-1}[4i+4j]$	$b_{i+j}[4i-4j-4]$	$d_{i-j-1}[4i+4j+2]$	$0 \le j \le m-i-1$
$d_{m-1}[8i-4m+2]$	$b_{2i-m}[4m-4]$	$b_{m-1}[8i-4m]$	$d_{2i-m}[4m-2]$	$(j=m-i-1)$
$\underline{d_m[8i-4m-2]}$				
$b_{2i-m-1}[4m-2]^*$	$c_{m-1}[8i-4m-2]$	$d_{2i-m}[4m-4]$	$a_{m-1}[8i-4m-4]$	$(j=0)$
$b_{2i-m-j-1}[4m-4j-2]$	$c_{m-j-1}[8i-4m-4j-2]$	$d_{2i-m-j}[4m-4j-4]$	$a_{m-j-1}[8i-4m-4j-4]$	$0 \le j \le i-\lceil \frac{m}{2} \rceil -1$
$b_{i-\lfloor \frac{m}{2} \rfloor}[4(\lceil \frac{3m}{2} \rceil - i)+2]^{*(e)}$	$c_{\lceil \frac{3m}{2} \rceil - i}[4(i-\lfloor \frac{m}{2} \rfloor)+2]^{\dagger(e)}$	$d_{i-\lfloor \frac{m}{2} \rfloor+1}[4(\lceil \frac{3m}{2} \rceil - i)]^{\dagger(e)}$	$a_{\lceil \frac{3m}{2} \rceil - i}[4(i-\lfloor \frac{m}{2} \rfloor)]^{*(e)}$	$(j=i-\lceil \frac{m}{2} \rceil -1)$
$b_{i-\lfloor \frac{m}{2} \rfloor-1}[4(\lceil \frac{3m}{2} \rceil - i)-3]$	$d_{\lceil \frac{3m}{2} \rceil - i-1}[4(i-\lfloor \frac{m}{2} \rfloor)-3]$	$c_{i-\lfloor \frac{m}{2} \rfloor-1}[4(\lceil \frac{3m}{2} \rceil - i)-5]$	$a_{\lceil \frac{3m}{2} \rceil - i-1}[4(i-\lfloor \frac{m}{2} \rfloor)-5]$	$(j=0)$
$b_{i-\lfloor \frac{m}{2} \rfloor-j-1}[4(\lceil \frac{3m}{2} \rceil - i-j)-3]$	$d_{\lceil \frac{3m}{2} \rceil - i-j-1}[4(i-\lfloor \frac{m}{2} \rfloor-j)-3]$	$c_{i-\lfloor \frac{m}{2} \rfloor-j-1}[4(\lceil \frac{3m}{2} \rceil - i-j)-5]$	$a_{\lceil \frac{3m}{2} \rceil - i-j-1}[4(i-\lfloor \frac{m}{2} \rfloor-j)-5]$	$0 \le j \le i-\lfloor \frac{m}{2} \rfloor -2$
$b_1[8(m-i)+5]$	$d_{2(m-i)+1}[5]$	$c_1[8(m-i)+3]$	$a_{2(m-i)+1}[3]$	$(j=i-\lfloor \frac{m}{2} \rfloor -2)$
$\underline{c_0[8(m-i)+3]}$	$\underline{d_{2(m-i)+1}[1]}$			

$P_{c_{\frac{m}{2}}}$, when m is even :

$$\begin{array}{rcll}
c_{\frac{m}{2}}[2m]^{*(e)} & a_{\frac{m}{2}} & c_{\frac{m}{2}}[2m-1]^{\ddagger(e)} & \\
d_{\frac{m}{2}}[2m+1]^{\ddagger(e)} & b_{\frac{m}{2}} & d_{\frac{m}{2}}[2m+2]^{*(e)} & \\
d_{\frac{m}{2}+1}[2m-2]^{*(e)} & b_{\frac{m}{2}-1} & d_{\frac{m}{2}+1}[2m-3]^{\ddagger(e)} & \\
c_{\frac{m}{2}-1}[2m+3]^{\ddagger(e)} & a_{\frac{m}{2}+1} & c_{\frac{m}{2}-1}[2m+4]^{*(e)} & (j=0)
\end{array}$$

$$\begin{array}{rcll}
c_{\frac{m}{2}+j}[2m-4j]^{*(e)} & a_{\frac{m}{2}-j} & c_{\frac{m}{2}+j}[2m-4j-1]^{\ddagger(e)} & \\
d_{\frac{m}{2}-j}[2m+4j+1]^{\ddagger(e)} & b_{\frac{m}{2}+j} & d_{\frac{m}{2}-j}[2m+4j+2]^{*(e)} & \\
d_{\frac{m}{2}+j+1}[2m-4j-2]^{*(e)} & b_{\frac{m}{2}-j-1} & d_{\frac{m}{2}+j+1}[2m-4j-3]^{\ddagger(e)} & \\
c_{\frac{m}{2}-j-1}[2m+4j+3]^{\ddagger(e)} & a_{\frac{m}{2}+j+1} & c_{\frac{m}{2}-j-1}[2m+4j+4]^{*(e)} & 0 \le j \le \frac{m}{2}-2
\end{array}$$

$$\begin{array}{rcll}
c_{m-2}[8]^{*(e)} & a_2 & c_{m-2}[7]^{\ddagger(e)} & \\
d_2[4m-7]^{\ddagger(e)} & b_{m-2} & d_2[4m-6]^{*(e)} & \\
d_{m-1}[6]^{*(e)} & b_1 & d_{m-1}[5]^{\ddagger(e)} & \\
c_1[4m-5]^{\ddagger(e)} & a_{m-1} & c_1[4m-4]^{*(e)} & (j=\frac{m}{2}-2)
\end{array}$$

$$\begin{array}{rcl}
c_{m-1}[4]^{*(e)} & a_1 & c_{m-1}[3]^{\ddagger(e)} \\
d_1[4m-3]^{\ddagger(e)} & b_{m-1} & d_1[4m-2]^{*(e)} \\
d_m[2]^{*(e)} & c_0 & d_m[1]^{\ddagger(e)}
\end{array}$$

$P_{d_{\frac{m+1}{2}}}$, m odd :

$$\begin{array}{rcll}
d_{\frac{m+1}{2}}[2m]^{*(o)} & b_{\frac{m-1}{2}} & d_{\frac{m+1}{2}}[2m-1]^{\ddagger(o)} & \\
c_{\frac{m-1}{2}}[2m+1]^{\ddagger(o)} & a_{\frac{m+1}{2}} & c_{\frac{m-1}{2}}[2m+2]^{*(o)} & \\
c_{\frac{m+1}{2}}[2m-2]^{*(o)} & a_{\frac{m-1}{2}} & c_{\frac{m+1}{2}}[2m-3]^{\ddagger(o)} & \\
d_{\frac{m-1}{2}}[2m+3]^{\ddagger(o)} & b_{\frac{m+1}{2}} & d_{\frac{m-1}{2}}[2m+4]^{*(o)} & (j=0)
\end{array}$$

$$\begin{array}{rcll}
d_{\frac{m+1}{2}+j}[2m-4j]^{*(o)} & b_{\frac{m-1}{2}-j} & d_{\frac{m+1}{2}+j}[2m-4j-1]^{\ddagger(o)} & \\
c_{\frac{m-1}{2}-j}[2m+4j+1]^{\ddagger(o)} & a_{\frac{m+1}{2}+j} & c_{\frac{m-1}{2}-j}[2m+4j+2]^{*(o)} & \\
c_{\frac{m+1}{2}+j}[2m-4j-2]^{*(o)} & a_{\frac{m-1}{2}-j} & c_{\frac{m+1}{2}+j}[2m-4j-3]^{\ddagger(o)} & \\
d_{\frac{m-1}{2}-j}[2m+4j+3]^{\ddagger(o)} & b_{\frac{m+1}{2}+j} & d_{\frac{m-1}{2}-j}[2m+4j+4]^{*(o)} & 0 \le j \le \frac{m-3}{2}
\end{array}$$

$$\begin{array}{rcll}
d_{m-1}[6]^{*(o)} & b_1 & d_{m-1}[5]^{\ddagger(o)} & \\
c_1[4m-5]^{\ddagger(o)} & a_{m-1} & c_1[4m-4]^{*(o)} & \\
c_{m-1}[4]^{*(o)} & a_1 & c_{m-1}[3]^{\ddagger(o)} & \\
d_1[4m-3]^{\ddagger(o)} & b_{m-1} & d_1[4m-2]^{*(o)} & (j=\frac{m-3}{2})
\end{array}$$

$$d_m[2]^{*(o)} \quad c_0 \quad d_m[1]^{\ddagger(o)}$$

□

We are now at the point of joining the trails in each of the T_i' together with 2–paths centred at ∞_1 and ∞_2 to form the Euler tours T_i. Currently, the T_i' are all pairwise similar. However, in order to use every 2–path centred at ∞_1 or ∞_2 exactly once, we have to have some Euler tours that contain the digon $\infty_1\,\infty_2\,\infty_1$, and some that contain the digon $\infty_2\,\infty_1\,\infty_2$. The following two claims, covering the two cases of m odd and m even, show how to join the trails in T_0' together with 2–paths centred at ∞_1 and ∞_2 to construct one Euler tour of $K_{4m}+I$ that contains the digon $\infty_1\,\infty_2\,\infty_1$ and another that contains the digon $\infty_2\,\infty_1\,\infty_2$. We then use these two Euler tours to generate a perfect set of Euler tours.

Claim 3.2.3 *Assume m is odd. Let*

$$T_a = T_0' \cup \infty_1[H_0 : 2,4,6,\ldots,4m-2] \cup \infty_2[H_0 : 1,3,5,\ldots,4m-1] \text{ and}$$

$$T_b = T_0' \cup \infty_1[H_0 : 1,3,5,\ldots,4m-1] \cup \infty_2[H_m : 2,4,6,\ldots,4m-2].$$

Then

1. *T_a and T_b are Euler tours of $K_{4m}+I$, and*

2. *The set of Euler tours, $\{T_i : 0 \leq i \leq 4m-3\}$, where $T_i = \sigma^i(T_a)$ if $0 \leq i \leq 2m-2$ and $T_i = \sigma^i(T_b)$ if $2m-1 \leq i \leq 4m-3$, is a perfect set of Euler tours of $K_{4m}+I$.*

Proof.

Assume $m > 1$.

Proof of 1): To construct T_a we need to know that the 2–paths and digon that are in $\infty_1[H_0 : 2,4,\ldots,4m-2]$ are $c_0\,\infty_1\,d_1$, $a_j\,\infty_1\,c_j$, $1 \leq j \leq m-1$, $b_j\,\infty_1\,d_{j+1}$, $1 \leq j \leq m-1$, and $\infty_2\,\infty_1\,\infty_2$. The 2–paths in $\infty_2[H_0 : 1,3,\ldots,4m-1]$ are $\infty_1\,\infty_2\,c_0$, $d_j\,\infty_2\,a_j$, $1 \leq j \leq m-1$, $c_j\,\infty_2\,b_j$, $1 \leq j \leq m-1$, and $d_m\,\infty_2\,\infty_1$.

To construct T_b we need to know that the 2–paths in $\infty_1[H_0 : 1,3,\ldots,4m-1]$ are $\infty_2\,\infty_1\,c_0$, $d_j\,\infty_1\,a_j$, $1 \leq j \leq m-1$, $c_j\,\infty_1\,b_j$, $1 \leq j \leq m-1$, and $d_m\,\infty_1\,\infty_2$. The 2–paths in $\infty_2[H_m : 2,4,\ldots,4m-2]$ are $a_j\,\infty_2\,b_{m-j}$, $1 \leq j \leq m-1$, and $c_j\,\infty_2\,d_{m-j}$, $0 \leq j \leq m-1$, together with the digon $\infty_1\,\infty_2\,\infty_1$.

The left-hand diagram in Figure 3.2 shows how the 2–paths centred at ∞_1 and ∞_2 join the trails in T_0' together to form the Euler tour T_a. As well as ∞_1 and ∞_2, there are two columns of vertices in the diagram, each containing $V(K_{4m-2})$. A dashed line between vertex l in the left-hand column and vertex r in the right-hand column indicates the trail in T_0' that starts on the edge $\infty_1\,l$ and ends on the edge $r\,\infty_2$. This is the trail labeled P_l in Claim 3.2.2. A solid line between two vertices l_1 and l_2 in the left-hand column indicates the 2–path $l_1\,\infty_1\,l_2$. A solid line between two vertices r_1 and r_2 in the right-hand column indicates the 2–path $r_1\,\infty_2\,r_2$. Finally, the dotted lines represent actual edges in the Euler tour T_0.

In exactly the same manner, the right-hand diagram in Figure 3.2 shows how the 2–paths centred at ∞_1 and at ∞_2 join the trails in T_0' together to form the Euler tour T_b.

Proof of 2): It is not hard to see that the set of Euler tours $\{T_i : 0 \leq i \leq 4m-3\}$ contains every 2–path centred at ∞_1 or ∞_2 exactly once, and hence that we have constructed a perfect set of Euler tours of $K_{4m}+I$.

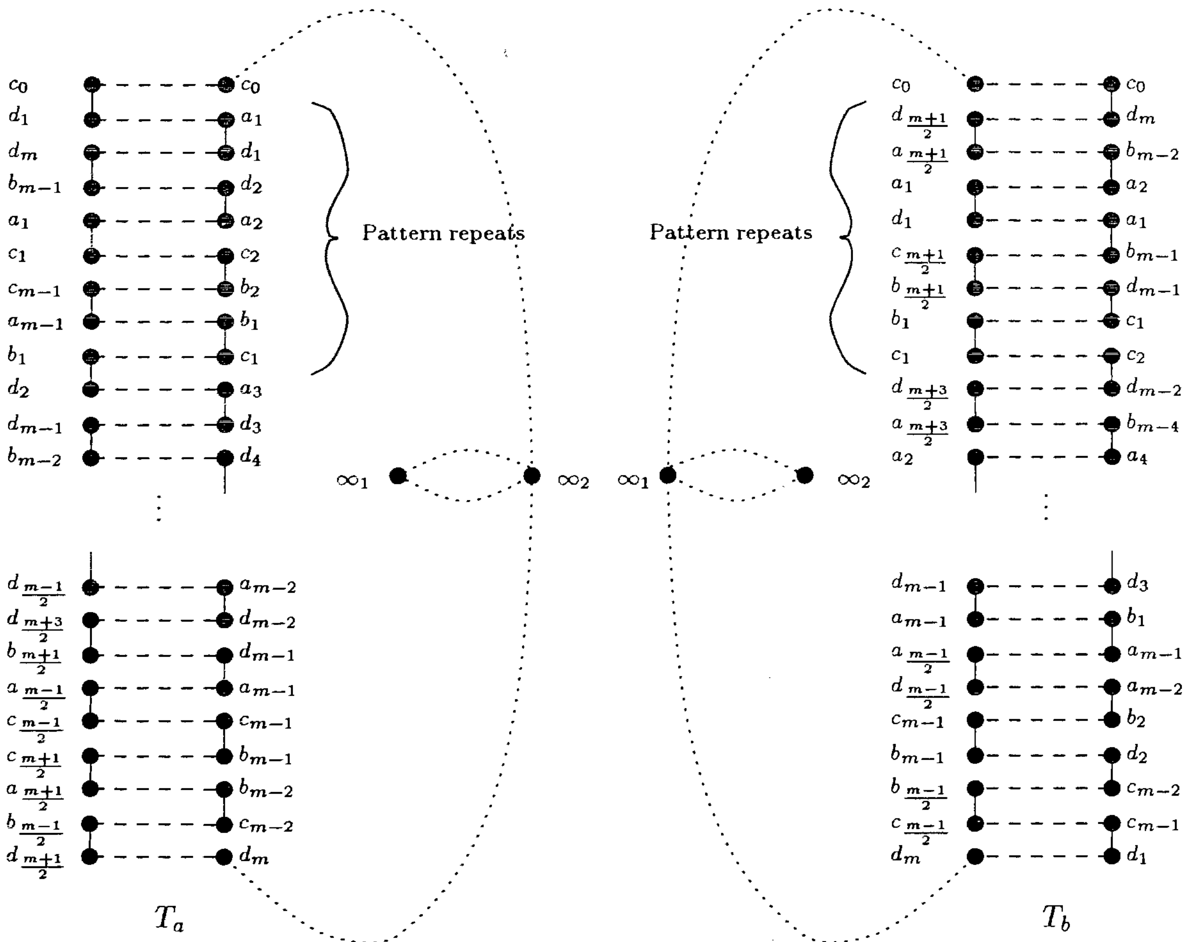


Figure 3.2: T_a and T_b when m is odd and $m \geq 3$.

When $m = 1$ the diagrams in Figure 3.2 do not apply. It is however easy to check this case separately. □

Claim 3.2.4 *Assume m is even. Let*

$$T_a = T_0' \cup \infty_1[H_0 : 2, 4, 6, \ldots, 4m-2] \cup \infty_2[H_0 : 1, 3, 5, \ldots, 4m-1] \textit{ and}$$

$$T_b = T_0' \cup \infty_1[H_0 : 1, 3, 5, \ldots, 4m-1] \cup \infty_2[H_{m+1} : 2, 4, 6, \ldots, 4m-2] \textit{ if } m > 2 \textit{ and}$$

$$T_b = T_0' \cup \infty_1[H_1 : 1, 3, 5, \ldots, 4m-1] \cup \infty_2[H_1 : 2, 4, 6, \ldots, 4m-2] \textit{ if } m = 2.$$

Then T_a and T_b are Euler tours of $K_{4m}+I$, and the set of Euler tours, $\{T_i : 0 \leq i \leq 4m-3\}$, where $T_i = \sigma^i(T_a)$ if $0 \leq i \leq 2m-2$ and $T_i = \sigma^i(T_b)$ if $2m-1 \leq i \leq 4m-3$, is a perfect set of Euler tours of $K_{4m}+I$.

Proof.

When $m>2$ is even the only difference in 2–paths centred at ∞_1 or ∞_2 from the odd case is the set of 2–paths centred at ∞_2 in T_b. These 2–paths are $a_j\,\infty_2\,b_{m+1-j}$, $2\le j\le m-1$, $c_j\,\infty_2\,d_{m-1-j}$, $0\le j\le m-2$, $a_1\,\infty_2\,c_{m-1}$, $b_1\,\infty_2\,d_m$ and $\infty_1\,\infty_2\,\infty_1$.

Figure 3.3 proves that T_a and T_b are Euler tours. The result follows exactly as in the odd case.

The case $m=2$ is readily verified. □

This completes the construction of a perfect set of Euler tours of $K_{4m}+I$.

3.3 A Perfect Set of Euler Tours of $K_{4m+2}+J$

We have constructed a perfect set of Euler tours of $K_{2k}+I$ when k is even. Now assume k is odd and let $k=2m+1$.

3.3.1 A Perfect Set of Euler Tours of K_6+J

The general construction that follows in Section 3.3.2 for a perfect set of Euler tours of $K_{4m+2}+J$ does not work when $m=1$, so we do this case separately by giving four Euler tours T_0, T_1, T_2, T_3, that form a perfect set of Euler tours. Let $V(K_6)=\{1,2,3,4,5,6\}$ and let $J=\{1\,2,3\,4,5\,6\}$.

T_0:	4	1	2	1	3	2	5	1	6	2	4	3	4	5	6	5	3	6	4	1
T_1:	4	2	1	2	3	4	3	6	2	5	6	5	1	3	5	4	6	1	4	2
T_2:	5	1	2	1	6	3	1	4	3	4	6	5	6	2	3	5	2	4	5	1
T_3:	5	2	1	2	6	4	2	3	6	5	6	1	3	4	3	5	1	4	5	2

3.3.2 A Perfect Set of Euler Tours of $K_{4m+2}+J$, $m>1$

Let $V(K_{4m+1})=\{\infty_2\}\cup A\cup B\cup C'\cup D'$, where $C'=C\cup\{c_m\}$ and $D'=D\cup\{d_{m+1}\}$ and A, B, C and D are as in Section 3.2. Let $V(K_{4m+2})=V(K_{4m+1})\cup\{\infty_1\}$. We construct a perfect set of Euler tours of $K_{4m+2}+J$, where J is a 1–factor of K_{4m+2},

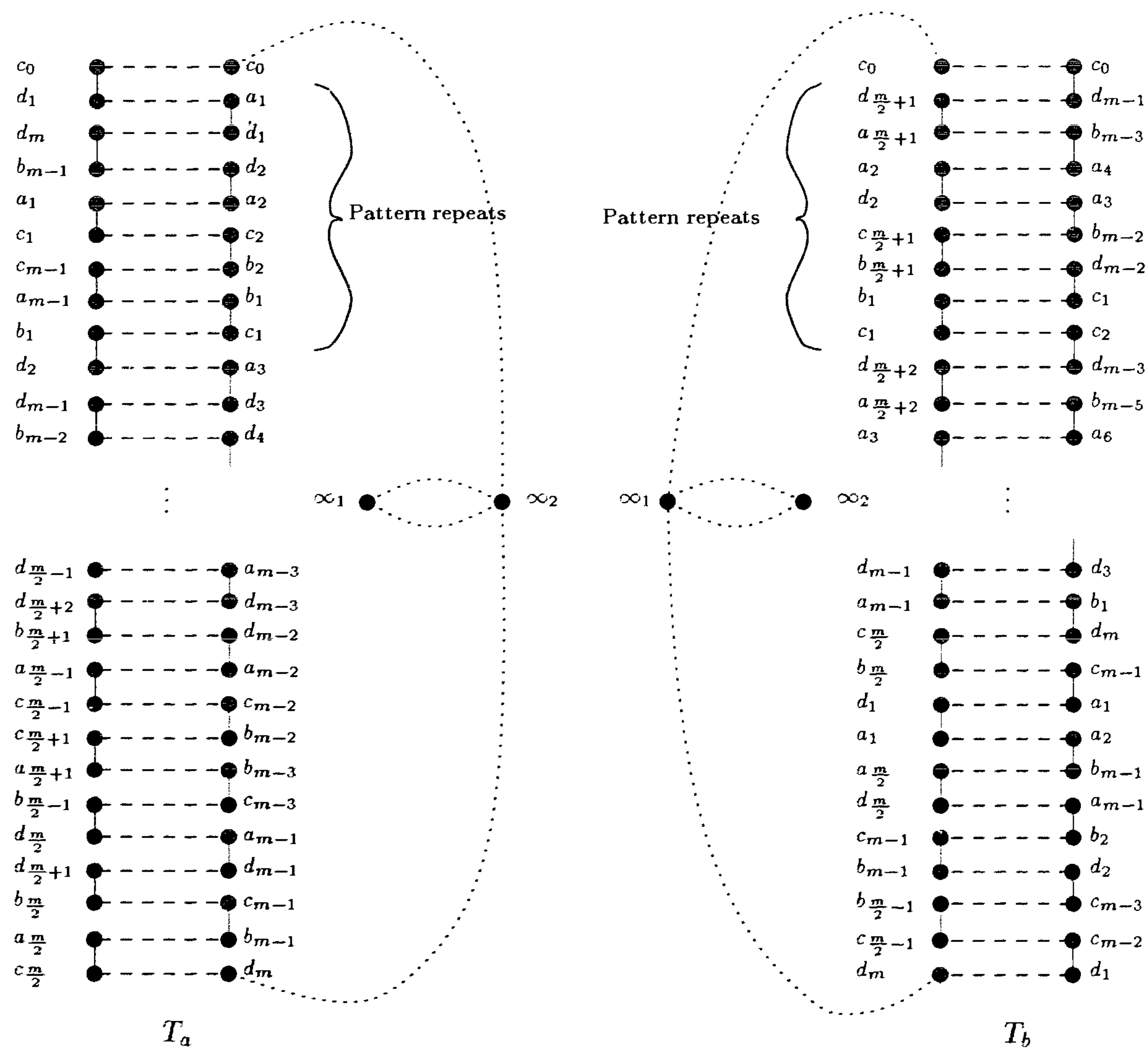


Figure 3.3: T_a and T_b when m is even.

by partitioning the 2–paths of K_{4m+2} into $4m$ parts, and then showing that the 2–paths in each part do indeed form an Euler tour of $K_{4m+2}+J$. We will be using the trails in T'_0 that were constructed for $K_{4m}+I$ to accomplish the latter half of this, so in this section we will partition the 2–paths in K_{4m+2} into $\{S_0, S_1, \ldots, S_{4m-1}\}$. For $i \in \{0, 1, 2, \ldots, 4m-1\}$, let S'_i be only those 2–paths in S_i that are centred at a vertex in $A \cup B \cup C' \cup D'$.

We use the construction mentioned in Section 3.1 to obtain a Hamilton decomposition of K_{4m+1}. We label the vertices so that as many of the trails as possible in S'_0 will be the same as, or similar to, a trail in T'_0. Let τ be the following permutation

$$(\infty_1)(\infty_2)(a_1\, b_1\, a_2\, b_2\, \ldots\, a_{m-1}\, b_{m-1}\, d_{m+1}\, c_m\, d_m\, c_{m-1}\, d_{m-1}\, \ldots\, c_1\, d_1\, c_0)$$

of $V(K_{4m+2})$ that fixes ∞_1 and generates a Hamilton decomposition

$$\mathcal{C} = \{C_0, C_1, \ldots, C_{2m-1}\}$$

of K_{4m+1} on the vertex set $\{\infty_2\} \cup A \cup B \cup C' \cup D'$. The Hamilton cycle C_0 (shown in Figure 3.4) is given by

$$(\infty_2\, c_0\, d_1\, a_1\, c_1\, b_1\, d_2\, a_2\, c_2\, b_2 \cdots d_i\, a_i\, c_i\, b_i \cdots d_{m-1}\, a_{m-1}\, c_{m-1}\, b_{m-1}\, d_m\, d_{m+1}\, c_m),$$

and we now have the Hamilton decomposition $\mathcal{C}$, where $C_i = \tau^i(C_0)$, $0 \le i \le 2m-1$. We can obtain a set of $4m$ Hamilton cycles by letting $C_i = \tau^i(C_0)$ for $i \in \{0, 1, 2, \ldots, 4m-1\}$. Note that $C_i = C_{i+2m}$, for all i, where addition on the subscripts is modulo $4m$. It is easy to see from Figure 3.4 that when we choose edges that are fixed by τ^{2m} to be the 1-factor J of K_{4m+2}, we have

$$\begin{aligned} J &= \{\infty_1\, \infty_2, c_0\, c_m, d_1\, d_{m+1}\} \\ &\cup \{a_i\, d_{m-i+1} : 1 \le i \le m-1\} \\ &\cup \{b_i\, c_{m-i} : 1 \le i \le m-1\}, \end{aligned}$$

which is itself fixed (setwise) by τ.

We will use the Hamilton cycles C_0, C_1 and $C_{4m-1} = C_{-1}$ to list the 2–paths in S'_0. Order the edges in these three cycles as follows:

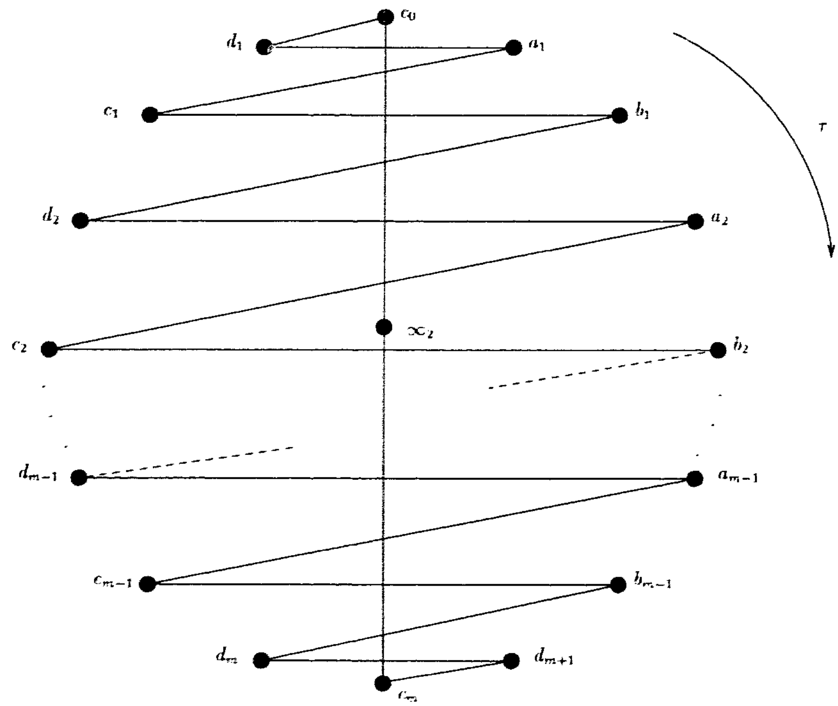


Figure 3.4: C_0 and τ

$$
\begin{aligned}
[C_0 : 1] &= \infty_2\, c_0, \\
[C_0 : 2] &= c_0\, d_1, \\
[C_0 : 4k-1] &= d_k\, a_k, 1 \le k \le m-1, \\
[C_0 : 4k] &= a_k\, c_k, 1 \le k \le m-1, \\
[C_0 : 4k+1] &= c_k\, b_k, 1 \le k \le m-1, \\
[C_0 : 4k+2] &= b_k\, d_{k+1}, 1 \le k \le m-1, \\
[C_0 : 4m-1] &= d_m\, d_{m+1}, \\
[C_0 : 4m] &= d_{m+1}\, c_m, \\
[C_0 : 4m+1] &= c_m\, \infty_2.
\end{aligned}
$$

$$
\begin{aligned}
[C_1 : 1] &= \infty_2\, a_1, \\
[C_1 : 4k-2] &= a_k\, c_{k-1}, 1 \le k \le m-1,
\end{aligned}
$$

$$
\begin{aligned}
[C_1 : 4k-1] &= c_{k-1}\, b_k, 1 \le k \le m-1,\\
[C_1 : 4k] &= b_k\, d_k, 1 \le k \le m-1,\\
[C_1 : 4k+1] &= d_k\, a_{k+1}, 1 \le k \le m-2,\\
[C_1 : 4m-3] &= d_{m-1}\, d_{m+1},\\
[C_1 : 4m-2] &= d_{m+1}\, c_{m-1},\\
[C_1 : 4m-1] &= c_{m-1}\, c_m,\\
[C_1 : 4m] &= c_m\, d_m,\\
[C_1 : 4m+1] &= d_m\, \infty_2,
\end{aligned}
$$

$$
\begin{aligned}
[C_{-1} : 1] &= \infty_2\, d_1,\\
[C_{-1} : 2] &= d_1\, c_1,\\
[C_{-1} : 3] &= c_1\, c_0,\\
[C_{-1} : 4] &= c_0\, d_2,\\
[C_{-1} : 4k+1] &= d_{k+1}\, a_k, 1 \le k \le m-1,\\
[C_{-1} : 4k+2] &= a_k\, c_{k+1}, 1 \le k \le m-1,\\
[C_{-1} : 4k+3] &= c_{k+1}\, b_k, 1 \le k \le m-1,\\
[C_{-1} : 4k+4] &= b_k\, d_{k+2}, 1 \le k \le m-1,\\
[C_{-1} : 4m+1] &= d_{m+1}\, \infty_2.
\end{aligned}
$$

The 2–paths in S_0' are:

$a_i[C_{-1} : 1, 3, 5, \dots, 4(m-i)-1,\ 4(m-i)+2, 4(m-i)+4, 4(m-i)+6, \dots, 4m]$ and

$b_i[C_1 : 1, 3, 5, \dots, 4(m-i)+1,\ 4(m-i)+4, 4(m-i)+6, 4(m-i)+8, \dots, 4m]$,

for $i \in \{1, 2, \dots, m-1\}$,

$c_i[C_0 : 1, 3, 5, \dots, 4(m-i)+1,\ 4(m-i)+2, 4(m-i)+4, 4(m-i)+6, \dots, 4m]$ and

$d_i[C_0 : 1, 3, 5, \dots, 4(m-i)+3,\ 4(m-i)+4, 4(m-i)+6, 4(m-i)+8, \dots, 4m]$, for $i \in \{1, 2, \ldots, m\}$, and

$$c_0[C_0 : 1, 3, 5, \ldots, 4m-1] \text{ and } d_{m+1}[C_0 : 1, 4, 6, 8, \ldots, 4m].$$

Define $S'_i = \tau^j(S'_0)$, $0 \le j \le 4m-1$. Since τ is an automorphism of $K_{4m+2}+I$, the S'_i are all pairwise similar.

Claim 3.3.1 *The S'_j, $0 \le j \le 4m-1$, partition the set of 2–paths in K_{4m+2} that are centred at a vertex in $A \cup B \cup C' \cup D'$.*

Proof. The proof is very similar to the proof of Claim 3.2.1.

Case 1: Let $c_i \in C' \setminus \{c_0, c_m\}$. Then the 2–paths

$$c_i[C_0 : 1, 3, \ldots, 4(m-i)+1, 4(m-i)+2, 4(m-i)+4, \ldots, 4m]$$

are in S'_0 and the 2–paths

$$\tau^{2m+1}(a_{m-i}[C_{-1} : 1, 3, \ldots, 4i-1, 4i+2, 4i+4, \ldots, 4m])$$

are in S'_{2m+1}. This second set of 2–paths is equal to

$$c_i[C_0 : 2, 4, \ldots, 4(m-i), 4(m-i)+3, 4(m-i)+5, \ldots, 4m+1].$$

Combining these two sets, we have each 2–path in $c_i[C_0 : r]$, $r \in \{1, 2, \ldots, 4m+1\}$, at least once in S'_0 or S'_{2m+1}. Therefore, we have each 2–path in $\tau^j(c_i[C_0 : r])$, $r \in \{1, 2, \ldots, 4m+1\}$, $0 \le j \le 4m-1$, at least once somewhere in the S'_l. This is equivalent to having each 2–path in $\tau^j(c_i[C_0 : r])$, $r \in \{1, 2, \ldots, 4m+1\}$, $0 \le j \le 2m-1$, and in $\tau^j(b_{m-i}[C_0 : r])$, $r \in \{1, 2, \ldots, 4m+1\}$, $0 \le j \le 2m-1$, at least once, since $\tau^{2m}(c_i) = b_{m-i}$ for all $i \in \{1, 2, \ldots, m-1\}$.

Case 2: Let $d_i \in D' \setminus \{d_1, d_{m+1}\}$. Then the 2–paths

$$d_i[C_0 : 1, 3, \ldots, 4(m-i)+3, 4(m-i)+4, 4(m-i)+6, \ldots, 4m]$$

are in S'_0 and the 2–paths

$$\tau^{2m-1}(b_{m-i+1}[C_1 : 1, 3, \ldots, 4i-3, 4i, 4i+2, \ldots, 4m])$$

are in S'_{2m-1}. This second set of 2–paths is equivalent to

$$d_i[C_0:2,4,\dots,4(m-i)+2,4(m-i)+5,4(m-i)+7,\dots,4m+1].$$

Combining these two sets, we have each 2–path in $d_i[C_0:r]$, $r\in\{1,2,\dots,4m+1\}$, at least once in S'_0 or S'_{2m-1}. So we get each 2–path in $\tau^j(d_i[C_0:r])$, $r\in\{1,2,\dots,4m+1\}$, $0\le j\le 2m-1$, and in $\tau^j(a_{m-i+1}[C_0:r])$, $r\in\{1,2,\dots,4m+1\}$, $0\le j\le 2m-1$, at least once, since $\tau^{2m}(d_i)=a_{m-i+1}$ for all $i\in\{2,3,\dots,m\}$.

Case 3: The 2–paths $c_0[C_0:1,3,\dots,4m-1]$ in S'_0 and the 2–paths

$$\tau^{2m}(c_m[C_0:1,2,4,\dots,4m])=c_0[C_0:2,4,\dots,4m,4m+1]$$

in S'_{2m} together give $c_0[C_0:r]$, $r\in\{1,2,\dots,4m+1\}$, at least once in S'_0 or S'_{2m}. Therefore, we have each 2–path in $\tau^j(c_0[C_0:r])$, $r\in\{1,2,\dots,4m+1\}$, $0\le j\le 2m-1$, and in $\tau^j(c_m[C_0:r])$, $r\in\{1,2,\dots,4m+1\}$, $0\le j\le 2m-1$, at least once somewhere in the S'_l.

Case 4: The 2–paths $d_1[C_0:1,3,\dots,4m-1,4m]$ in S'_0 and the 2–paths

$$\tau^{2m}(d_{m+1}[C_0:1,4,6,\dots,4m])=d_1[C_0:2,4,\dots,4m-2,4m+1]$$

in S'_{2m} together give $d_1[C_0:r]$, $r\in\{1,2,\dots,4m+1\}$, at least once in S'_0 or S'_{2m}. Therefore, we have each 2–path in $\tau^j(d_1[C_0:r])$, $r\in\{1,2,\dots,4m+1\}$, $0\le j\le 2m-1$, and in $\tau^j(d_{m+1}[C_0:r])$, $r\in\{1,2,\dots,4m+1\}$, $0\le j\le 2m-1$, at least once.

Altogether, for each $v\in A\cup B\cup C'\cup D'$ and each $j\in\{0,1,\dots,2m-1\}$ we have $\tau^j(v[C_0:r])$, $r\in\{1,2,\dots,4m+1\}$, at least once. This means we have every 2–path at least once, and hence, exactly once. □

Claim 3.3.2 *The 2–paths in S'_0 fit together to form $2m$ trails. Of these, $2m-2$ start on an edge containing ∞_1 and end on an edge containing ∞_2. Label such a trail P'_v, where $\infty_1 v$ is the first edge of the trail. The trails in S'_0 with their first and last edges are as follows:*

P'_{a_i}, $1\le i\le\lceil\frac{m}{2}\rceil-1$, from $\infty_1 a_i$ to $a_{2i}\infty_2$,

P'_{b_i}, $1\le i\le\lfloor\frac{m}{2}\rfloor$, from $\infty_1 b_i$ to $c_{2i-1}\infty_2$,

P'_{c_i}, $0 \leq i \leq \lceil \frac{m}{2} \rceil - 1$, *from* $\infty_1\, c_i$ *to* $c_{2i}\, \infty_2$,
P'_{d_i}, $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$, *from* $\infty_1\, d_i$ *to* $a_{2i-1}\, \infty_2$,
$P'_{c_{\frac{m}{2}}}$, *when* m *is even, from* $\infty_1\, c_{\frac{m}{2}}$ *to* $c_m\, \infty_2$,
$P'_{d_{\frac{m+1}{2}}}$, *when* m *is odd, from* $\infty_1\, d_{\frac{m+1}{2}}$ *to* $c_m\, \infty_2$,
P'_{a_i}, $\lceil \frac{m}{2} \rceil \leq i \leq m-1$, *from* $\infty_1\, a_i$ *to* $b_{2(m-i)-1}\, \infty_2$,
P'_{b_i}, $\lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m-1$, *from* $\infty_1\, b_i$ *to* $d_{2(m-i)}\, \infty_2$,
P'_{c_i}, $\lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m-1$, *from* $\infty_1\, c_i$ *to* $b_{2(m-i)}\, \infty_2$,
P'_{d_i}, $\lceil \frac{m}{2} \rceil + 1 \leq i \leq m$, *from* $\infty_1\, d_i$ *to* $d_{2(m-i)+1}\, \infty_2$.

In addition we have the two trails

$$c_m[4m]\, d_{m+1}[4m] = \infty_1\, c_m\, d_{m+1}\, \infty_1 \text{ and}$$

$$d_{m+1}[1]\, c_0[4m-1]\, d_m[1] = \infty_2\, d_{m+1}\, c_0\, d_m\, \infty_2.$$

Proof.

There are relatively few 2–paths on which T'_0 and S'_0 differ and we can use the trails in T'_0 to determine the structure of the trails in S'_0. We will do this by considering where the 2–paths in S'_0 differ from those in T'_0. For most pairs α and k, $\alpha \in \{-1, 0, 1\}$, $k \in \{1, 2, \ldots, 4m-2\}$, the edges $[H_\alpha : k]$ and $[C_\alpha : k]$ are the same. The edges that differ that will affect 2–paths in S'_0 are $C_1[4m-2] \neq H_1[4m-2]$ and $C_{-1}[4m-2] \neq H_{-1}[4m-2]$. Also, any 2–path in S'_0 that is centred at c_m or d_{m+1}, or has end vertices from the $4m^{th}$ or $(4m-1)^{th}$ edge of one of the Hamilton cycles, C_0, C_{-1} or C_1, must be new.

From now on we will no longer mention which Hamilton cycle C_0, C_1, or C_{-1} the end vertices of the 2–paths in S'_0 come from, since 2–paths centred at a vertex in A always have end–vertices from C_{-1}, 2–paths centred at a vertex in B always have end–vertices from C_1, and 2–paths centred at a vertex in $C' \cup D'$ always have end–vertices from C_0. We will however mention which Hamilton cycle, H_{-1}, H_0, or H_1, the 2–paths in T'_0 come from, mostly to stress that we are considering a 2–path in T'_0 and not one in S'_0.

The 2–paths that are in T'_0 but not in S'_0 are those that are marked in the trails of T'_0 with a superscript $*$, $*(o)$ (only applies when m is odd), or $*(e)$ (only applies when

m is even). Whether m is odd or even, the 2–paths that are marked in this way are:

$$a_i[H_{-1}:4(m-i)],\ 1\le i\le m-1,$$

$$b_i[H_1:4(m-i)+2],\ 1\le i\le m-1,$$

$$c_i[H_0:4(m-i)],\ 1\le i\le m-1,$$

$$d_i[H_0:4(m-i)+2],\ 1\le i\le m,$$

$$a_j[H_{-1}:4m-2],\ \text{for all } j\in\{1,2,\ldots,m-1\},\ \text{and}$$

$$b_j[H_1:4m-2],\ \text{for all } j\in\{2,3,\ldots,m-1\}.$$

The 2–paths that are in S_0' but not in T_0' are:

$$v[4m],\ \text{for all } v\in A\cup B\cup C'\cup D'\setminus\{c_0\}, c_0[4m-1],\ \text{and } d_1[4m-1],$$

$$c_m[1,2,4,6,\ldots,4m-2] \text{ and } d_{m+1}[1,4,6,8,\ldots,4m-2],$$

$$a_i[4(m-i)-1],\ 1\le i\le m-1,$$

$$b_i[4(m-i)+1],\ 1\le i\le m-1,$$

$$c_i[4(m-i)+1],\ 1\le i\le m-1,$$

$$d_i[4(m-i)+3],\ 2\le i\le m,$$

$$a_j[4m-2],\ \text{for all } j\in\{1,2,\ldots,m-1\},\ (\text{because } C_{-1}[4m-2]\ne H_{-1}[4m-2])$$

and

$$b_j[4m-2],\ \text{for all } j\in\{2,3,\ldots,m-1\}\ (\text{because } C_1[4m-2]\ne H_1[4m-2]).$$

First of all, $P'_{a_i}=P_{a_i}$, $1\le i\le\lceil\frac{m}{2}\rceil-1$, $P'_{b_i}=P_{b_i}$, $1\le i\le\lfloor\frac{m}{2}\rfloor$, $P'_{c_i}=P_{c_i}$, $1\le i\le\lceil\frac{m}{2}\rceil-1$, and $P'_{d_i}=P_{d_i}$, $1\le i\le\lfloor\frac{m}{2}\rfloor$, because none of the 2–paths in these trails is one that was either removed or changed by using the Hamilton cycles in $\mathcal{C}$ instead of the Hamilton cycles in $\mathcal{H}$.

The trail $P'_{c_{\frac{m}{2}}}$ when m is even and the trail $P'_{d_{\frac{m+1}{2}}}$ when m is odd are completely different from $P_{c_{\frac{m}{2}}}$ and $P_{d_{\frac{m+1}{2}}}$, respectively. This is not surprising given that we need a different set of digons in S_0' than in T_0'. They use the 2–paths $c_i[4(m-i)+2]$,

$1 \le i \le m-1$, and $d_i[4(m-i)+4]$, $2 \le i \le m$. These are marked with a superscript $\dagger(e)$ or $\dagger(o)$ in the trails of T_0'. (Again, (e) stands for the case when m is even and (o) stand for the case when m is odd.) They also use the new 2–paths $c_m[1,2]$, $d_1[4m]$, $c_i[4(m-i)+1]$, $1 \le i \le m-1$, and $d_i[4(m-i)+3]$, $1 \le i \le m$. Note that they do not use any of the 2–paths that were in $P_{c_{\frac{m}{2}}}$ or $P_{d_{\frac{m+1}{2}}}$.

$P'_{c_{\frac{m}{2}}}$, m even :

$$\begin{array}{rcll}
c_{\frac{m}{2}}[2m+1] & b_{\frac{m}{2}} & c_{\frac{m}{2}}[2m+2] & \\
d_{\frac{m}{2}+1}[2m] & a_{\frac{m}{2}} & d_{\frac{m}{2}+1}[2m-1] & \\
d_{\frac{m}{2}}[2m+3] & a_{\frac{m}{2}+1} & d_{\frac{m}{2}}[2m+4] & \\
c_{\frac{m}{2}+1}[2m-2] & b_{\frac{m}{2}-1} & c_{\frac{m}{2}+1}[2m-3] & (j=0)
\end{array}$$

$$\begin{array}{rcll}
c_{\frac{m}{2}-j}[2m+4j+1] & b_{\frac{m}{2}+j} & c_{\frac{m}{2}-j}[2m+4j+2] & \\
d_{\frac{m}{2}+j+1}[2m-4j] & a_{\frac{m}{2}-j} & d_{\frac{m}{2}+j+1}[2m-4j-1] & \\
d_{\frac{m}{2}-j}[2m+4j+3] & a_{\frac{m}{2}+j+1} & d_{\frac{m}{2}-j}[2m+4j+4] & \\
c_{\frac{m}{2}+j+1}[2m-4j-2] & b_{\frac{m}{2}-j-1} & c_{\frac{m}{2}+j+1}[2m-4j-3] & 0 \le j \le \frac{m}{2}-2
\end{array}$$

$$\begin{array}{rcll}
c_2[4m-7] & b_{m-2} & c_2[4m-6] & \\
d_{m-1}[8] & a_2 & d_{m-1}[7] & \\
d_2[4m-5] & a_{m-1} & d_2[4m-4] & \\
c_{m-1}[6] & b_1 & c_{m-1}[5] & (j=\frac{m}{2}-2)
\end{array}$$

$$\begin{array}{rcl}
c_1[4m-3] & b_{m-1} & c_1[4m-2] \\
d_m[4] & a_1 & d_m[3] \\
d_1[4m-1] & d_{m+1} & d_1[4m] \\
c_m[2] & c_0 & c_m[1]
\end{array}$$

and

$P'_{d_{\frac{m+1}{2}}}, m$ odd :

$$\begin{array}{rcll}
d_{\frac{m+1}{2}}[2m+1] & a_{\frac{m+1}{2}} & d_{\frac{m+1}{2}}[2m+2] & \\
c_{\frac{m+1}{2}}[2m] & b_{\frac{m-1}{2}} & c_{\frac{m+1}{2}}[2m-1] & \\
c_{\frac{m-1}{2}}[2m+3] & b_{\frac{m+1}{2}} & c_{\frac{m-1}{2}}[2m+4] & \\
d_{\frac{m+3}{2}}[2m-2] & a_{\frac{m-1}{2}} & d_{\frac{m+3}{2}}[2m-3] & (j=0)
\end{array}$$

$$\begin{array}{rcll}
d_{\frac{m+1}{2}-j}[2m+4j+1] & a_{\frac{m+1}{2}+j} & d_{\frac{m+1}{2}-j}[2m+4j+2] & \\
c_{\frac{m+1}{2}+j}[2m-4j] & b_{\frac{m-1}{2}-j} & c_{\frac{m+1}{2}+j}[2m-4j-1] & \\
c_{\frac{m-1}{2}-j}[2m+4j+3] & b_{\frac{m+1}{2}+j} & c_{\frac{m-1}{2}-j}[2m+4j+4] & \\
d_{\frac{m+1}{2}+j+1}[2m-4j-2] & a_{\frac{m-1}{2}-j} & d_{\frac{m+1}{2}+j+1}[2m-4j-3] & 0 \le j \le \frac{m-3}{2}
\end{array}$$

$$\begin{array}{rcll}
d_2[4m-5] & a_{m-1} & d_2[4m-4] & \\
c_{m-1}[6] & b_1 & c_{m-1}[5] & \\
c_1[4m-3] & b_{m-1} & c_1[4m-2] & \\
d_m[4] & a_1 & d_m[3] & (j=\frac{m-3}{2})
\end{array}$$

$$\begin{array}{rcl}
d_1[4m-1] & d_{m+1} & d_1[4m] \\
c_m[2] & c_0 & c_m[1]
\end{array}$$

The remaining trails of T'_0 are all modified at least once to make them the trails in S'_0. We remove 1 or 2 subtrails from each and then use the new 2-paths that are still available and the 2-paths marked with a superscript $\ddagger(e)$ or $\ddagger(o)$ in $P_{c_{\frac{m}{2}}}$ or $P_{d_{\frac{m+1}{2}}}$, respectively, to join the trails together again and to create two new trails.

First consider the 2-paths that are marked with a superscript $*$ in the trails on T'_0. These are $a_{2i-m+1}[H_{-1}:4m-2]$ in P_{a_i}, $\lceil\frac{m}{2}\rceil \le i \le m-1$, and $a_{2i-m}[H_{-1}:4m-2]$ in P_{c_i}, $\lfloor\frac{m}{2}\rfloor+1 \le i \le m-1$, giving $a_j[H_{-1}:4m-2]$ exactly once for each $j \in \{1,2,\ldots,m-1\}$. Replace the 2-path $a_j[H_{-1}:4m-2] = a_{m-1}\,a_j\,b_{m-1}$ with the following trail:

$$a_j[4m-2]\, c_m[4j]\, c_j[4m]\, d_{m+1}[4j]\, a_j[4m] = a_{m-1}\, a_j\, c_m\, c_j\, d_{m+1}\, a_j\, b_{m-1}.$$

The 2–paths $b_{2i-m}[H_1 : 4m-2]$ in P_{b_i}, $\lceil \frac{m}{2} \rceil + 1 \leq i \leq m-1$, and the 2–paths $b_{2i-m-1}[H_1 : 4m-2]$ in P_{d_i}, $\lfloor \frac{m}{2} \rfloor + 2 \leq i \leq m$, are also marked with a superscript $*$, giving $b_j[H_1 : 4m-2]$ exactly once for each $j \in \{2, 3, \ldots, m-1\}$. For each $j \in \{2, 3, \ldots, m-1\}$, replace the 2–path $b_j[H_1 : 4m-2] = d_m\, b_j\, c_{m-1}$ with the trail

$$b_j[4m]\, c_m[4j+2]\, d_{j+1}[4m]\, d_{m+1}[4j+2]\, b_j[4m-2] = d_m\, b_j\, c_m\, d_{j+1}\, d_{m+1}\, b_j\, c_{m-1}.$$

The following subtrail is marked with superscripts $*(o)$ or $\dagger(o)$ in $P_{b_{\frac{m+1}{2}}}$ when m is odd, and with superscripts $*(e)$ or $\dagger(e)$ in $P_{d_{\frac{m}{2}+1}}$ when m is even.

$$b_1[H_1 : 4m-2]\, c_{m-1}[H_0 : 6]\, d_2[H_0 : 4m-4]\, a_{m-1}[H_{-1} : 4]$$

$$= d_m\, b_1\, c_{m-1}\, d_2\, a_{m-1}\, c_0.$$

Replace it with

$$b_1[4m]\, c_m[6]\, d_2[4m]\, d_{m+1}[6]\, b_1[4m-3]\, d_{m-1}[5]\, c_1[4m-5]\, a_{m-1}[3]$$

$$= d_m\, b_1\, c_m\, d_2\, d_{m+1}\, b_1\, d_{m-1}\, c_1\, a_{m-1}\, c_0.$$

For the remaining changes to the trails in T_0', we have to consider m even and odd separately.

Case 1: If m is odd, then P_{c_i}, $\lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m-1$, and P_{d_i}, $\lceil \frac{m}{2} \rceil + 1 \leq i \leq m$, require no more changes to become P'_{c_i}, $\lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m-1$, and P'_{d_i}, $\lceil \frac{m}{2} \rceil + 1 \leq i \leq m$, respectively. The subtrail in P_{a_i} comprising the four 2–paths that are marked with superscripts $*(o)$ or $\dagger(o)$,

$$b_{\frac{3m-1}{2}-i}[H_1 : 4(i - \tfrac{m-1}{2}) + 2]\, c_{i-\frac{m-1}{2}}[H_0 : 4(\tfrac{3m-1}{2} - i) + 2]$$
$$d_{\frac{3m-1}{2}-i+1}[H_0 : 4(i - \tfrac{m-1}{2})]\, a_{i-\frac{m-1}{2}}[H_{-1} : 4(\tfrac{3m-1}{2} - i)]$$

$$= a_{i-\frac{m-1}{2}+1}\, b_{\frac{3m-1}{2}-i}\, c_{i-\frac{m-1}{2}}\, d_{\frac{3m-1}{2}-i+1}\, a_{i-\frac{m-1}{2}}\, b_{\frac{3m-1}{2}-i-1}$$

becomes

$$b_{\frac{3m-1}{2}-i}[4(i-\tfrac{m-1}{2})+1]\, d_{i-\frac{m-1}{2}}[4(\tfrac{3m-1}{2}-i)+1]$$
$$c_{\frac{3m-1}{2}-i}[4(i-\tfrac{m-1}{2})-1]\, a_{i-\frac{m-1}{2}}[4(\tfrac{3m-1}{2}-i)-1]$$

$$= a_{i-\frac{m-1}{2}+1}\, b_{\frac{3m-1}{2}-i}\, d_{i-\frac{m-1}{2}}\, c_{\frac{3m-1}{2}-i}\, a_{i-\frac{m-1}{2}}\, b_{\frac{3m-1}{2}-i-1},$$

completing the trail P'_{a_i}, for $\frac{m+1}{2} \le i \le m-1$.

The subtrail in P_{b_i}, $\frac{m+3}{2} \le i \le m-1$, comprising the four 2–paths that are marked with superscript $*(o)$ or $\dagger(o)$,

$$b_{i-\frac{m-1}{2}}[H_1 : 4(\tfrac{3m-1}{2}-i)+2]\, c_{\frac{3m-1}{2}-i}[H_0 : 4(i-\tfrac{m-1}{2})+2]$$
$$d_{i-\frac{m-1}{2}+1}[H_0 : 4(\tfrac{3m-1}{2}-i)]\, a_{\frac{3m-1}{2}-i}[H_{-1} : 4(i-\tfrac{m-1}{2})]$$

$$= a_{\frac{3m-1}{2}-i+1}\, b_{i-\frac{m-1}{2}}\, c_{\frac{3m-1}{2}-i}\, d_{i-\frac{m-1}{2}+1}\, a_{\frac{3m-1}{2}-i}\, b_{i-\frac{m-1}{2}-1},$$

becomes

$$b_{i-\frac{m-1}{2}}[4(\tfrac{3m-1}{2}-i)+1]\, d_{\frac{3m-1}{2}-i}[4(i-\tfrac{m-1}{2})+1]$$
$$c_{i-\frac{m-1}{2}}[4(\tfrac{3m-1}{2}-i)-1]\, a_{\frac{3m-1}{2}-i}[4(i-\tfrac{m-1}{2})-1]$$

$$= a_{\frac{3m-1}{2}-i+1}\, b_{i-\frac{m-1}{2}}\, d_{\frac{3m-1}{2}-i}\, c_{i-\frac{m-1}{2}}\, a_{\frac{3m-1}{2}-i}\, b_{i-\frac{m-1}{2}-1},$$

completing the trail P'_{b_i}, for $\frac{m+3}{2} \le i \le m-1$.

Case 2: If m is even, then P_{a_i}, $\lceil \frac{m}{2} \rceil \le i \le m-1$ and P_{b_i}, $\lfloor \frac{m}{2} \rfloor + 1 \le i \le m-1$, require no more changes to become P'_{a_i}, $\lceil \frac{m}{2} \rceil \le i \le m-1$ and P'_{b_i}, $\lfloor \frac{m}{2} \rfloor + 1 \le i \le m-1$, respectively. The subtrail in P_{c_i}, $\frac{m}{2}+1 \le i \le m-1$, comprising the four 2–paths that are marked with superscript $*(e)$ or $\dagger(e)$,

$$b_{\frac{3m}{2}-i}[H_1 : 4(i-\tfrac{m}{2})+2]\, c_{i-\frac{m}{2}}[H_0 : 4(\tfrac{3m}{2}-i)+2]$$
$$d_{\frac{3m}{2}-i+1}[H_0 : 4(i-\tfrac{m}{2})]\, a_{i-\frac{m}{2}}[H_{-1} : 4(\tfrac{3m}{2}-i)]$$

$$= a_{i-\frac{m}{2}+1}\, b_{\frac{3m}{2}-i}\, c_{i-\frac{m}{2}}\, d_{\frac{3m}{2}-i+1}\, a_{i-\frac{m}{2}}\, b_{\frac{3m}{2}-i-1}$$

becomes

$$b_{\frac{3m}{2}-i}[4(i-\tfrac{m}{2})+1]\, d_{i-\frac{m}{2}}[4(\tfrac{3m}{2}-i)+1]$$
$$c_{\frac{3m}{2}-i}[4(i-\tfrac{m}{2})-1]\, a_{i-\frac{m}{2}}[4(\tfrac{3m}{2}-i)-1]$$

$$= a_{i-\frac{m}{2}+1}\, b_{\frac{3m}{2}-i}\, d_{i-\frac{m}{2}}\, c_{\frac{3m}{2}-i}\, a_{i-\frac{m}{2}}\, b_{\frac{3m}{2}-i-1},$$

completing the trail P'_{c_i}, for $\frac{m}{2}+1 \leq i \leq m-1$.

The subtrail in P_{d_i}, $\frac{m}{2}+2 \leq i \leq m$, comprising the four 2-paths that are marked with superscript $*(e)$ or $\dagger(e)$,

$$b_{i-\frac{m}{2}}[H_1 : 4(\tfrac{3m}{2}-i)+2]\, c_{\frac{3m}{2}-i}[H_0 : 4(i-\tfrac{m}{2})+2]$$
$$d_{i-\frac{m}{2}+1}[H_0 : 4(\tfrac{3m}{2}-i)]\, a_{\frac{3m}{2}-i}[H_{-1} : 4(i-\tfrac{m}{2})]$$
$$= a_{\frac{3m}{2}-i+1}\, b_{i-\frac{m}{2}}\, c_{\frac{3m}{2}-i}\, d_{i-\frac{m}{2}+1}\, a_{\frac{3m}{2}-i}\, b_{i-\frac{m}{2}-1}$$

becomes

$$b_{i-\frac{m}{2}}[4(\tfrac{3m}{2}-i)+1]\, d_{\frac{3m}{2}-i}[4(i-\tfrac{m}{2})+1]$$
$$c_{i-\frac{m}{2}}[4(\tfrac{3m}{2}-i)-1]\, a_{\frac{3m}{2}-i}[4(i-\tfrac{m}{2})-1]$$
$$= a_{\frac{3m}{2}-i+1}\, b_{i-\frac{m}{2}}\, d_{\frac{3m}{2}-i}\, c_{i-\frac{m}{2}}\, a_{\frac{3m}{2}-i}\, b_{i-\frac{m}{2}-1},$$

completing the trail P'_{d_i} for $\frac{m}{2}+2 \leq i \leq m$.

The remaining two trails that use the four edges $\infty_1 c_m$, $\infty_1 d_{m+1}$, $\infty_2 d_m$, and $\infty_2 d_{m+1}$, do not follow the pattern of starting on an edge containing ∞_1 and ending on an edge containing ∞_2. Instead, they are

$$c_m[4m]\, d_{m+1}[4m] = \infty_1\, c_m\, d_{m+1}\, \infty_1 \text{ and}$$

$$d_{m+1}[1]\, c_0[4m-1]\, d_m[1] = \infty_2\, d_{m+1}\, c_0\, d_m\, \infty_2.$$

We have now used all of the new 2-paths as well as those that were marked with a superscript $\ddagger(e)$ in $P_{c_{\frac{m}{2}}}$ or with a superscript $\ddagger(o)$ in $P_{d_{\frac{m+1}{2}}}$. □

In the following claims, we show how to use the 2-paths centred at ∞_1 and ∞_2 to complete the S'_i into Euler tours.

Claim 3.3.3 *Assume m is odd. Let*

$$S_a = S'_0 \cup \infty_1[C_1 : 2,4,6,\ldots,4m] \cup \infty_2[C_1 : 1,3,5,\ldots,4m+1] \text{ and}$$

$$S_b = S'_0 \cup \infty_1[C_m : 1,3,5,\ldots,4m+1] \cup \infty_2[C_1 : 2,4,6,\ldots,4m].$$

Then S_a and S_b are Euler tours of $K_{4m+2}+J$, and the set of Euler tours, $\{S_i : 0 \leq i \leq 4m-1\}$, where $S_i = \tau^i(S_a)$ if $0 \leq i \leq 2m-1$, and $S_i = \tau^i(S_b)$ if $2m \leq i \leq 4m-1$, is a perfect set of Euler tours of $K_{4m+2}+J$.

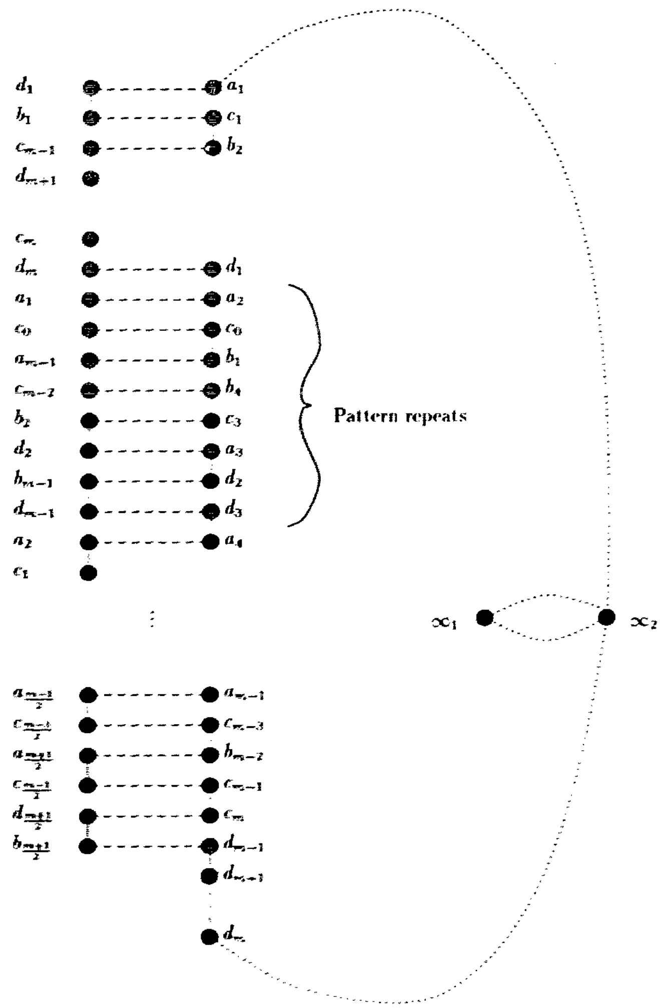


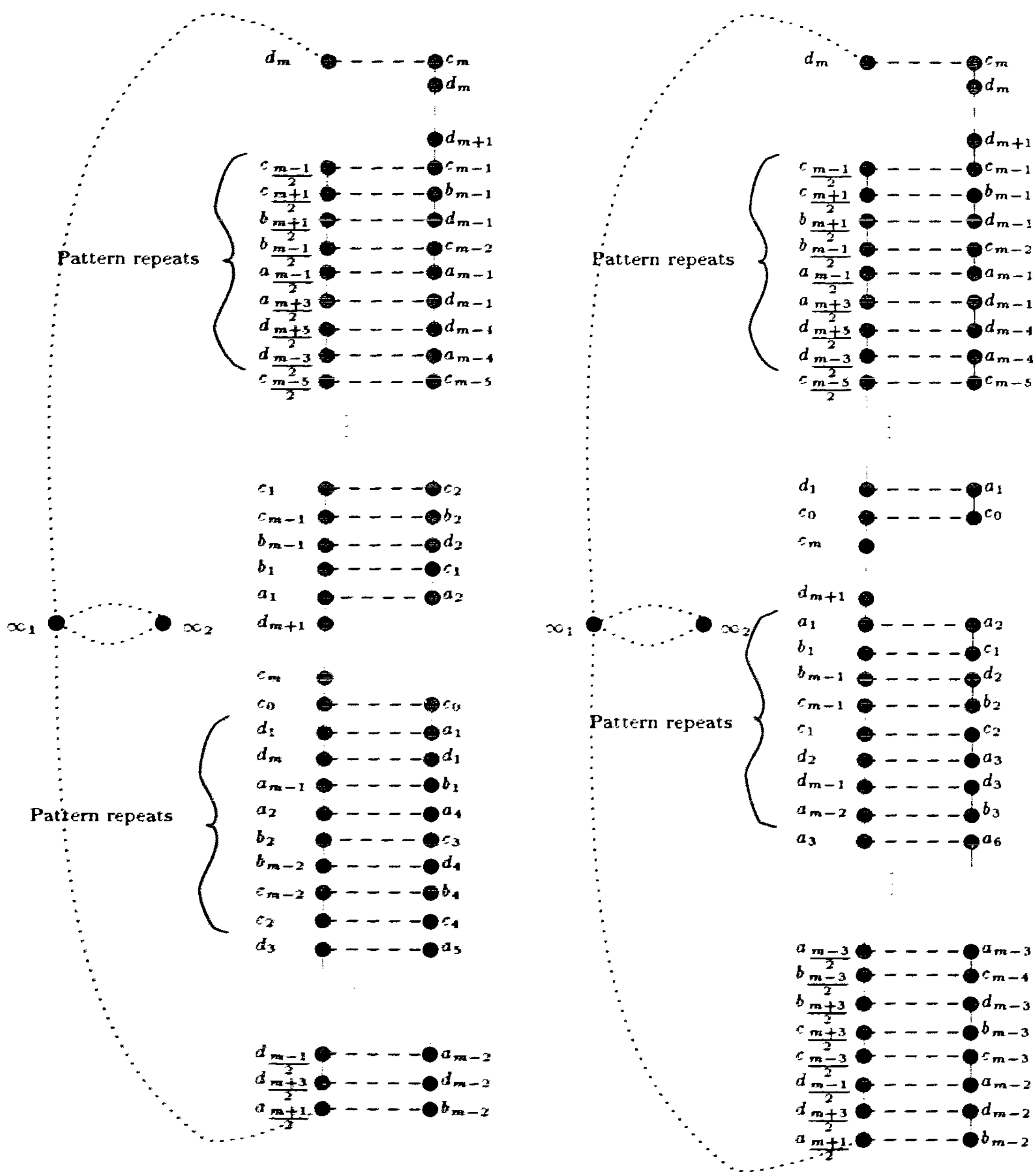
Figure 3.5: S_a when m is odd and $m \geq 3$.

Proof.

Figures 3.5 and 3.6 show, respectively, that S_a and S_b are Euler tours. The different edges in the graph are defined the same way as those for T_a and T_b with the additional vertical dashed edge from c_m to d_{m+1} in the left-hand column of vertices representing the trail $\infty_1\, c_m\, d_{m+1}\, \infty_1$, and the vertical dashed edge from d_{m+1} to d_m in the right-hand column representing the trail $\infty_2\, d_{m+1}\, c_0\, d_m\, \infty_2$.

We should probably note for the sake of the proof of Claim 3.3.1, that for all $i \in \{0, 1, 2, \ldots, 4m-1\}$, S_i' is indeed a subset of the S_i defined in this claim.

□



$m \equiv 3 \pmod 4$ $m \equiv 1 \pmod 4$

Figure 3.6: S_b when m is odd and $m \geq 3$.

The following claim for the case of $m = 2$ is given without proof.

Claim 3.3.4 *Let*

$$S_a = S_0' \cup \infty_1[C_2 : 2,4,6,8] \cup \infty_2[C_1 : 1,3,5,7,9] \text{ and}$$

$$S_b = S_0' \cup \infty_1[C_1 : 1,3,5,7,9] \cup \infty_2[C_0 : 2,4,6,8].$$

Then S_a and S_b are Euler tours of $K_{10}+J$, and the set of Euler tours, $\{S_i : 0 \le i \le 7\}$, where $S_i = \tau^i(S_a)$ if $0 \le i \le 3$, and $S_i = \tau^i(S_b)$ if $4 \le i \le 7$, is a perfect set of Euler tours of $K_{10}+J$.

Claim 3.3.5 *Assume $m > 2$ is even. Let*

$$S_a = S_0' \cup \infty_1[C_m : 2,4,6,\ldots,4m] \cup \infty_2[C_1 : 1,3,5,\ldots,4m+1] \text{ and}$$

$$S_b = S_0' \cup \infty_1[C_{m+1} : 1,3,5,\ldots,4m+1] \cup \infty_2[C_1 : 2,4,6,\ldots,4m].$$

Then S_a and S_b are Euler tours of $K_{4m+2}+J$, and the set of Euler tours, $\{S_i : 0 \le i \le 4m-1\}$, where $S_i = \tau^i(S_a)$ if $0 \le i \le 2m-1$, and $S_i = \tau^i(S_b)$ if $2m \le i \le 4m-1$, is a perfect set of Euler tours of $K_{4m+2}+J$.

Proof.

Figures 3.7 and 3.8 show S_a and S_b are Euler tours.

□

This completes the construction of a perfect set of Euler tours of $K_{4m+2}+J$ and the proof of Theorem 3.1.1.

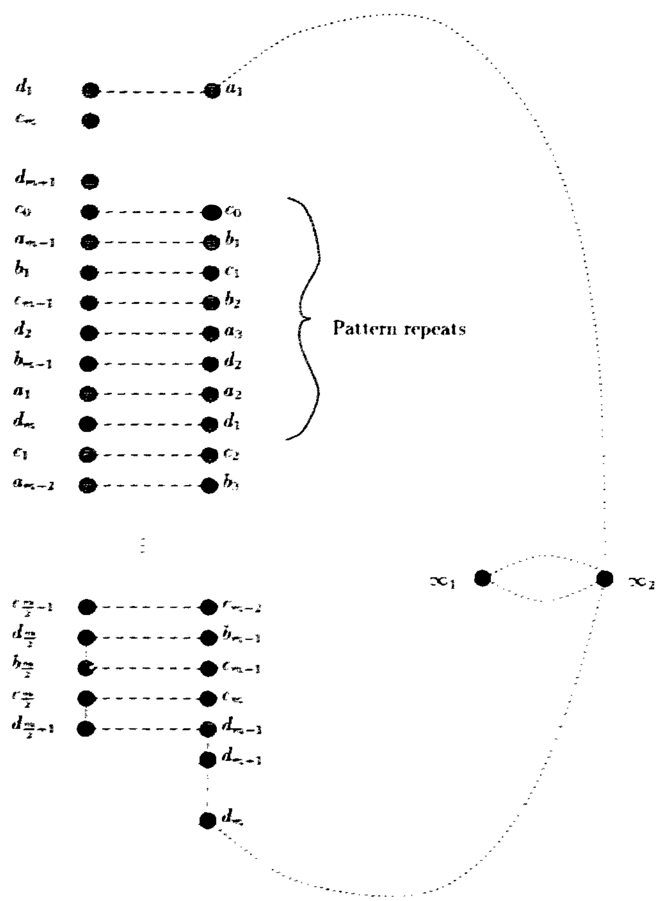


Figure 3.7: S_a when m is even and $m \geq 4$.

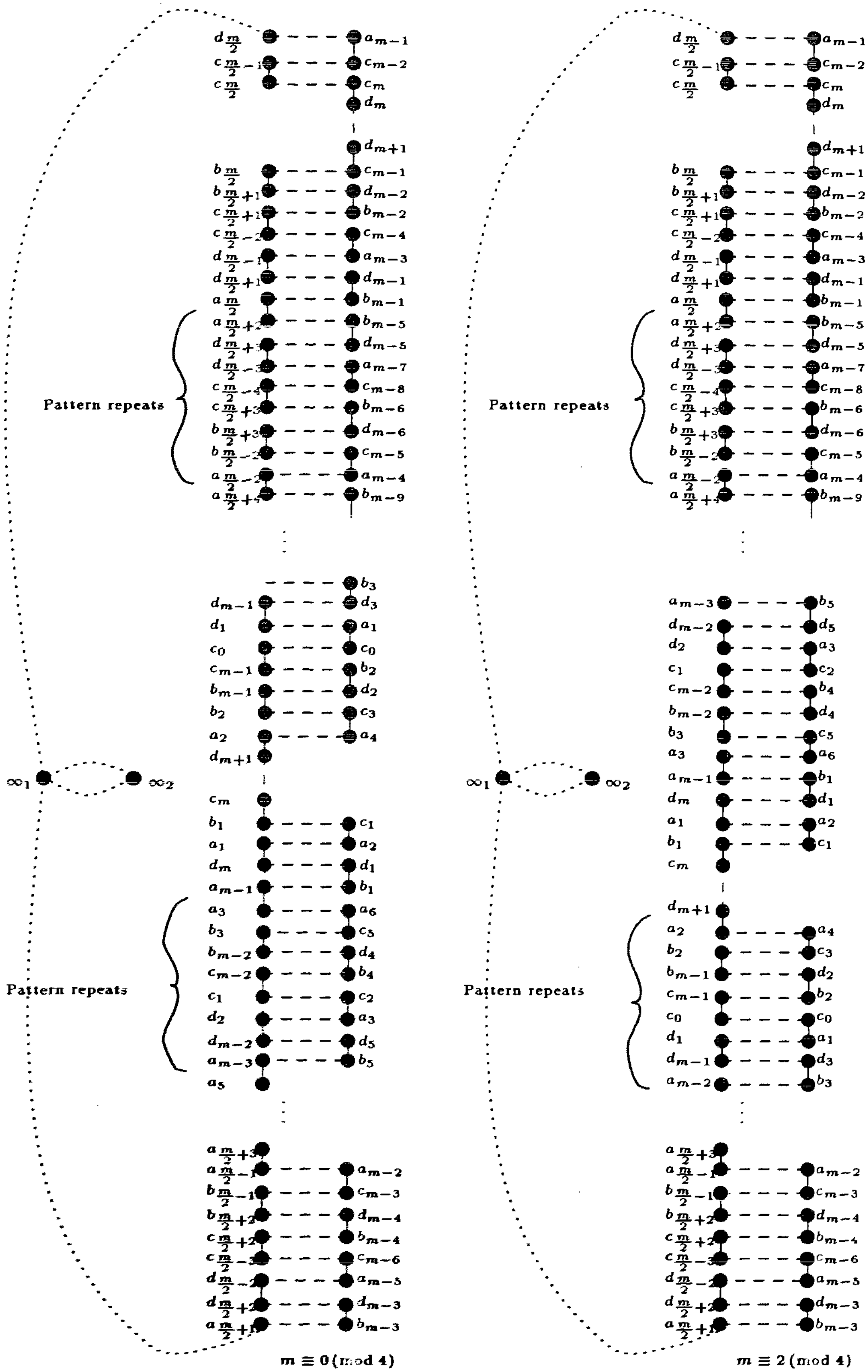


Figure 3.8: S_b when m is even and $m \geq 4$.

Chapter 4

Another Question of Kotzig's

The results in this chapter were motivated by Kotzig's question [12]: What is the smallest k for which there is a perfect set of Hamilton decompositions of K_{2k+1}? The difficulty of this question led us to consider two related problems. In Section 4.1 we show that for any k there are at least $2k-2$ pairwise compatible Hamilton path decompositions of K_{2k}. A simple corollary of the proof of this theorem is that there exists a set of $4k-2$ Hamilton path decompositions of K_{2k} such that every 2-path is in exactly two of the Hamilton paths. In Section 4.2 we add a new vertex ∞ to Hamilton path decompositions similar to those constructed in Section 4.1 to get a lower bound on the number of pairwise compatible Hamilton decompositions of K_{2k+1}, when k is even.

4.1 Pairwise Compatible Hamilton Path Decompositions

The graph K_{2k} has $k(2k-1)(2k-2)$ 2-paths. A Hamilton path decomposition of K_{2k} contains $k(2k-2)$ 2-paths. We would like to construct a set of $2k-1$ pairwise compatible Hamilton path decompositions of K_{2k}: a perfect set of Hamilton path decompositions of K_{2k}. However, when $k=2$, it is possible to find at most two compatible Hamilton path decompositions. In Theorem 4.1.1 we extend this result by constructing $2k-2$ pairwise compatible Hamilton path decompositions of K_{2k} for all values of k. There is however no reason to suppose for $k > 2$ that it is not possible to find $2k-1$ pairwise compatible Hamilton path decompositions.

Theorem 4.1.1 *The complete graph K_{2k} has a set of $2k-2$ pairwise compatible Hamilton path decompositions for all $k > 1$.*

We first prove three lemmas. The second lemma and part 2 of the first are only used in Section 4.2, but it is convenient to prove the results all at once.

We assume that all addition is modulo $2k-1$ with residue classes $0, 1, \ldots, 2k-2$, unless otherwise stated. Let $V(K_{2k}) = \{\infty_1\} \cup \{0, 1, \ldots, 2k-2\}$ and $V(K_{2k+1}) = V(K_{2k}) \cup \{\infty\}$. For $0 \leq i \leq 2k-2$ and $x, y \in \{0, 1, 2, \ldots, 2k-2\}$, let $F_i = \{\infty_1\, i\} \cup \{x\, y : x \neq y \text{ and } x + y \equiv 2i \pmod{2k-1}\}$.

We define a "length" function on the edges in K_{2k} that do not contain vertex ∞_1 as follows. Let $\ell(x\, y) = \min(x - y \pmod{2k-1}, y - x \pmod{2k-1})$. We say two edges $v_1 v_2$ and $u_1 u_2$ in K_{2k} are *parallel* if none of the vertices is ∞_1 and if $u_1 + u_2 \equiv v_1 + v_2 \pmod{2k-1}$. For example, for each $i \in \{0, 1, 2, \ldots, 2k-2\}$, the edges in F_i that do not contain ∞_1 are pairwise parallel.

Suppose for some $a, b \in \{0, 1, \ldots, 2k-2\}$ that $F_a \cup F_b$ is a Hamilton cycle H of K_{2k}. We can assume that $H = (w_1\, w_2 \cdots w_{2k})$, that the edge $w_1 w_2$ is in F_a, and that $w_1 = \infty_1$. We want to consider the 2–paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b]\colon 1 \leq j \leq k\}$. This set contains 2–paths of the form $\infty\, u\, v$ and so the union of the 2–paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b]\colon 1 \leq j \leq k\}$ will contain trails that start and end at vertex ∞. For the moment we want to consider trails in K_{2k} and not in K_{2k+1}, so we will omit 2–paths containing ∞. This is equivalent to constructing the trails in K_{2k+1} and then removing ∞. We don't want to forget about the 2–paths that contain ∞ altogether, because in the next section, we will use these 2–paths to join the Hamilton paths in K_{2k} into Hamilton cycles in K_{2k+1}.

Lemma 4.1.2 *Given that $F_a \cup F_b$ is a Hamilton cycle $H = (w_1\, w_2 \cdots w_{2k})$ of K_{2k}, where $w_1 = \infty_1$ and $w_1 w_2 \in F_a$, the trails formed by the set of 2–paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b] : 1 \leq j \leq k\}$ have the following two properties:*

1. *They form a Hamilton path decomposition of K_{2k}, and*

2. *The Hamilton path that begins on vertex $w_1 = \infty_1$ ends on vertex $w_{k+1} = 2^{-1}(a+b)(mod\ 2k-1)$.*

Proof.

The outer cycle in Figure 4 1 is the Hamilton cycle $H = F_a \cup F_b$ when k is even. When k is odd, a similar figure is obtained.

Proof of 1): The subtrail of $\{w_{2j-1}[F_a] \cup w_{2j}[F_b] : 1 \leq j \leq k\}$ in K_{2k} that starts on w_1 is the Hamilton path P given by the boldface edges. It is not hard to see that the trails that start on the other vertices form Hamilton paths in exactly the same way. In fact, if we let ρ be the following permutation of $V(K_{2k})$,

$$\rho = (w_1\, w_2\, \cdots\, w_{2k}),$$

then the other trails formed by the set of 2–paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b] : 1 \leq j \leq k\}$ are $\rho^j(P)$, for $1 \leq j \leq k-1$.

Proof of 2): By the definitions of F_a and F_b, we can describe vertices w_i, $2 \leq i \leq 2k$, in terms of a and b. The Hamilton path P shown in this figure obviously starts at $w_1 = \infty_1$ and ends at $w_{k+1} \equiv ka-(k-1)b \equiv kb-(k-1)a \equiv 2^{-1}(a+b)(\text{mod } 2k-1)$. □

Lemma 4.1.3 *When k is even, the Hamilton paths formed by the set of 2–paths in $\{w_{2j-1}[F_a] \cup w_{2j}[F_b]$: $1 \leq j \leq k\}$, have the following property:*

The length of the edges in K_{2k} determined by the first and last vertices of each of the Hamilton paths, except P, is a constant. That constant is

$$\min(2^{-1}(a-b)(mod\ 2k-1), 2^{-1}(b-a)(mod\ 2k-1)).$$

Proof.

Assume k is even. From the action of ρ on P in Figure 4.1, we see that if we start a trail at vertex w_i, $2 \leq i \leq k$, that it will finish at w_{i+k}, where addition on the subscripts is modulo $2k$, with residue classes $1, 2, \ldots, 2k$. By definition of

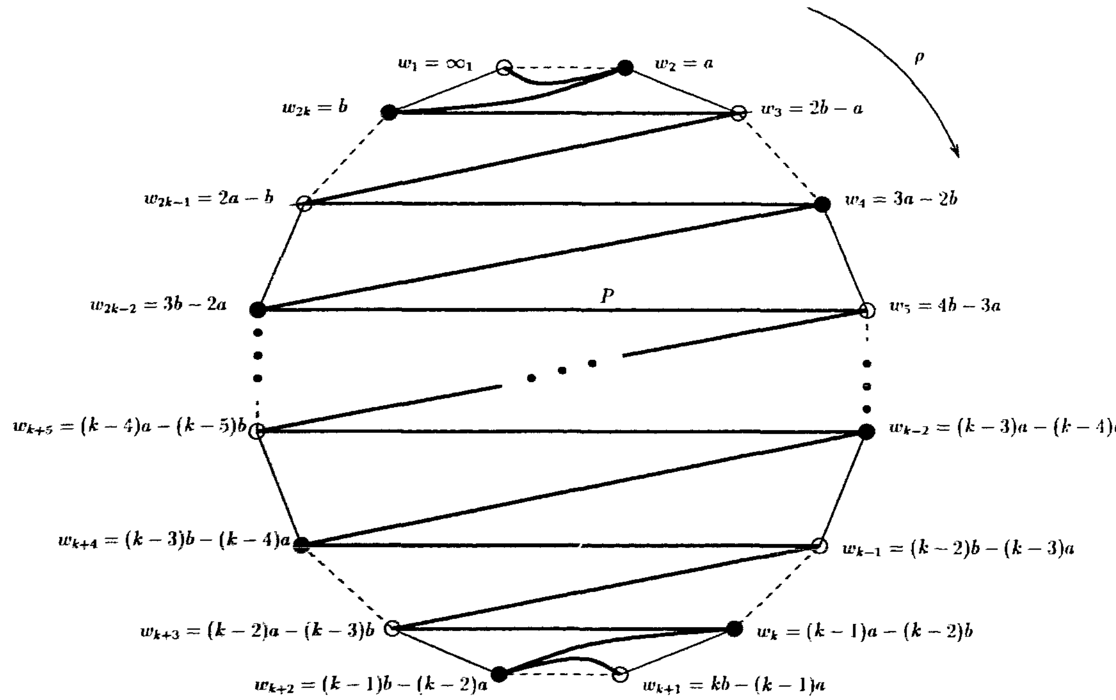


Figure 4.1: P and ρ

F_a and F_b, if i is even, $w_{i+k} \equiv k(a-b) + w_i \equiv 2^{-1}(a-b) + w_i (\text{mod } 2k-1)$. If i is odd, then $w_{i+k} \equiv k(b-a) + w_i \equiv 2^{-1}(b-a) + w_i (\text{mod } 2k-1)$. In either case, $\ell(w_i\, w_{i+k}) \equiv \min(2^{-1}(a-b) (\text{mod } 2k-1), 2^{-1}(b-a) (\text{mod } 2k-1))$. □

The proof of the third lemma is heavily based on the proof of Theorem 1 in [1]. Note that k can again be odd as well as even.

Lemma 4.1.4 *Assume that $c > d$, where $c, d \in \{0, 1, 2, \ldots, 2k-2\}$. If $c - d$ and $2k-1$ are relatively prime, then $F_c \cup F_d$ is a Hamilton cycle, where $F_i = \{\infty_1\, i\} \cup \{x\, y : x \neq y \text{ and } x + y \equiv 2i (\text{mod } 2k-1)\}$, for $i \in \{c, d\}$.*

Proof.

Let F_c and F_d be two such 1–factors of K_{2k} so that $c - d$ and $2k-1$ are relatively prime. Consider an l–subset of those edges in F_c that do not contain ∞_1. The sum of the vertices in these edges will be congruent to $2lc(\text{mod } 2k-1)$, since an edge $x\,y$ in F_c, $x \neq \infty_1 \neq y$, satisfies $x + y \equiv 2c(\text{mod } 2k-1)$. Similarly for F_d. Suppose $F_c \cup F_d$ is not a Hamilton cycle of K_{2k}. Then there is an even length $2m$–cycle in $F_c \cup F_d$ that does not contain ∞_1, where $2 \leq m \leq k-1$. We can sum the vertices in this cycle as edges of F_c or as edges of F_d to get that $2mc \equiv 2md \pmod{2k-1}$. This contradicts the fact that $c-d$ and $2k-1$ are relatively prime. □

Define σ and τ to be the following permutations of $V(K_{2k})$:

$$\sigma = (\infty_1)(0\,1\,2\,\cdots\,2k{-}2) \text{ and}$$

$$\tau = (\infty_1)(k)(0\,1)(2\,2k{-}2)(3\,2k{-}3)\cdots(k{-}1\;k{+}1).$$

Note that $\tau(F_0) = F_1$ and $\tau(F_1) = F_0$.

Each of $H_0, H_1, \ldots, H_{k-2}$ and $H'_0, H'_1, \ldots, H'_{k-2}$ will be a set of 2–paths, and our objective is to show that each of these sets of 2–paths is a Hamilton path decomposition of K_{2k}. We will list the 2–paths in H_0, show how to determine the H_j and H'_j so they are similar to H_0, show that no two of $\{H_0, H_1, \ldots, H_{k-2}\} \cup \{H'_0, H'_1, \ldots, H'_{k-2}\}$ have a 2–path in common, and prove that H_0 is a Hamilton path decomposition of K_{2k}.

Define the 2–paths in H_0 to be

$\infty_1[F_0]$
$0[F_1]$
$2i[F_0]$ for $i \in \{1, 2, \ldots, k-1\}$, and
$(2i-1)[F_1]$ for $i \in \{1, 2, \ldots, k-1\}$.

Let $H'_0 = \tau(H_0)$, $H_j = \sigma^{2j}(H_0)$, for $1 \leq j \leq k-2$, and $H'_j = \sigma^{2j}(H'_0)$, for $1 \leq j \leq k-2$. By definition, the H_j and H'_j are all similar to H_0.

Claim 4.1.5 *The 2–paths in H'_0 are $\infty_1[F_1]$, $0[F_0]$, and $2i[F_1]$ and $(2i-1)[F_0]$ for $i \in \{1, 2, \ldots, k-1\}$.*

Proof.

This follows immediately since $\tau(F_0) = F_1$ and $\tau(F_1) = F_0$. □

Claim 4.1.6 *For any $j \in \{0, 1, \ldots, k-2\}$, the set of 2–paths in H_j and H'_j contains every 2–path in K_{2k} with end vertices from an edge in F_{2j} or F_{2j+1} exactly once.*

Proof.

By definition and by Claim 4.1.5, we know that H_0 and H_0' between them contain every 2–path with end vertices from F_0 or F_1, exactly once. Let $j \in \{0, 1, \ldots, k-2\}$. Since $H_j = \sigma^{2j}(H_0)$ and $H_j' = \sigma^{2j}(H_0')$, and $F_{2j} = \sigma^{2j}(F_0)$ and $F_{2j+1} = \sigma^{2j}(F_1)$, we know that H_j and H_j' between them contain every 2–path in K_{2k} with end vertices from an edge in F_{2j} or F_{2j+1} exactly once. □

It follows that no two of $\{H_0, H_1, \ldots, H_{k-2}\} \cup \{H_0', H_1', \ldots, H_{k-2}'\}$ have a 2–path in common. In fact we have all possible 2–paths exactly once except those with end vertices an edge in F_{2k-2}.

Claim 4.1.7 *The 2–paths in H_0 form a Hamilton path decomposition of K_{2k}.*

Proof.

By Lemma 4.1.4, $F_0 \cup F_1$ is a Hamilton cycle of K_{2k}. We can therefore use part 1 of Lemma 4.1.2 to prove that the 2–paths in H_0 form a Hamilton path decomposition. □

This completes the proof of Theorem 4.1.1.

It would seem to be difficult to find a perfect set of Hamilton path decompositions of K_{2k}. However, we can find a set of Hamilton path decompositions of K_{2k} that contain every 2–path exactly twice as a simple corollary to the proof of Theorem 4.1.1.

Corollary 4.1.8 *The complete graph K_{2k} has a set of $4k-2$ Hamilton path decompositions so that every 2–path in K_{2k} is in exactly two of them.*

Proof.

Let $H_0, H_1, \ldots, H_{2k-2}$ and $H_0', H_1', \ldots, H_{2k-2}'$ be the Hamilton path decompositions we want to construct. Define H_0 and H_0' as in the proof of Theorem 4.1.1. Let $H_j = \sigma^{2j}(H_0)$, $0 \leq j \leq 2k-2$, and $H_j' = \sigma^{2j}(H_0')$, $0 \leq j \leq 2k-2$. Exactly as before, we can show that for all $j \in \{0, 1, \ldots, 2k-2\}$, H_j and H_j' between them contain every 2–path in K_{2k} with end vertices from an edge in F_{2j} or F_{2j+1}, where addition on the subscripts of the 1–factors is modulo $2k-1$, with residue classes $0, 1, \ldots, 2k-2$. □

It seems appropriate to mention the next two results as they tie in with the result in Theorem 1.2.22. The first is an obvious corollary of Corollary 4.1.8; the second is a corollary of Theorem 1.2.22 [11].

Corollary 4.1.9 *There exists a set of Hamilton paths of* K_{2k} *that between them contain every* 2*-path of* K_{2k} *exactly twice.*

Corollary 4.1.10 *There exists a set of Hamilton paths of* K_{2k+1} *that between them contain every* 2*-path of* K_{2k+1} *exactly twice.*

4.2 Pairwise Compatible Hamilton Cycle Decompositions

In Section 4.1 we found a set of $2k-2$ pairwise compatible Hamilton path decompositions of K_{2k}. If the edges determined by the end vertices of each of the Hamilton paths were distinct, we could add a new vertex ∞ to each Hamilton path decomposition and join the ends of each Hamilton path through ∞ to construct $2k-2$ pairwise compatible Hamilton decompositions of K_{2k+1}. Sadly this doesn't happen. We now attempt to get a lower bound on the number of pairwise compatible Hamilton decompositions of K_{2k+1}, when k is even, by constructing a different (smaller) set of pairwise compatible Hamilton path decompositions of K_{2k}, and making sure that we will be able to join the ends of all the Hamilton paths together with distinct 2-paths centred at a new vertex ∞. (The restriction to even k arises because the result in Lemma 4.1.3 does not hold for odd k.)

The following lemmas are needed to find pairs of 1-factors of K_{2k}, $F_a \cup F_b$, on which the Hamilton decompositions will be based. The 1-factors, $V(K_{2k})$, and $V(K_{2k+1})$ are still defined as in Section 4.1.

Lemma 4.2.1 *Let* $u\,v$ *and* $x\,y$ *be two edges in* K_{2k} *such that none of the vertices is* ∞_1. *If* $u\,v$ *and* $x\,y$ *are not parallel, then* $2^{-1}(u+v) \not\equiv 2^{-1}(x+y) (mod\ 2k-1)$.

Proof. Assume $2^{-1}(u+v) \equiv 2^{-1}(x+y) \pmod{2k-1}$. Then $u+v \equiv x+y \pmod{2k-1}$, and $u\,v$ and $x\,y$ are parallel. □

Lemma 4.2.2 *If $k > 2$ and even, then there exists a set S of $\lceil \frac{2k}{3} \rceil$ disjoint edges in K_{2k} such that:*

1. *No two of the edges are parallel,*
2. *No two of the edges have the same length, and*
3. *None of the edges contains the vertex ∞_1.*

Moreover, we can always find a subset S^* of S with at least three edges that have lengths relatively prime to $2k-1$.

If $k = 2$ there is only one such edge.

Proof. The proof is divided into the three cases of $k \equiv 0 \pmod 6$, $k \equiv 2 \pmod 6$, and $k \equiv 4 \pmod 6$.

If $k \equiv 0 \pmod 6$:

$$\begin{aligned} S &= \{0\,k{-}1,\ 1\,k{-}3,\ 2\,k{-}5, \ldots, \tfrac{k}{3}{-}1\ \tfrac{k}{3}{+}1\} \\ &\cup \{2k{-}2\,k{+}1,\ 2k{-}3\,k{+}3,\ 2k{-}4\,k{+}5, \ldots, \tfrac{5k}{3}\ \tfrac{5k}{3}{-}3\} \\ &\cup \{k{-}2\,k{+}2\}. \end{aligned}$$

The set S has $\frac{2k}{3}$ edges. Let $S^* = \{0\,k{-}1,\ \frac{k}{3}{-}1\ \frac{k}{3}{+}1,\ k{-}2\,k{+}2\}$.

If $k \equiv 2 \pmod 6$:

$$\begin{aligned} S &= \{0\,k{-}1,\ 1\,k{-}3,\ 2\,k{-}5, \ldots, \tfrac{k-2}{3}\ \tfrac{k+1}{3}\} \\ &\cup \{2k{-}2\,k{+}1,\ 2k{-}3\,k{+}3,\ 2k{-}4\,k{+}5, \ldots, \tfrac{5k-1}{3}\ \tfrac{5k-7}{3}\} \\ &\cup \{k{-}4\,k{+}2\} \end{aligned}$$

In this case, S has $\frac{2k+2}{3}$ edges if $k > 2$. (It has only one edge if $k = 2$.) Let $S^* = \{0\,k{-}1,\ \frac{k-2}{3}\ \frac{k+1}{3},\ \frac{5k-1}{3}\ \frac{5k-7}{3}\}$ when $k > 2$.

If $k \equiv 4 \pmod 6$:

$$\begin{aligned} S &= \{0\,k{-}1,\ 1\,k{-}3,\ 2\,k{-}5,\ldots,\tfrac{k-4}{3}\,\tfrac{k+5}{3}\} \\ &\cup \{2k{-}2\,k{+}1,\ 2k{-}3\,k{+}3,\ 2k{-}4\,k{+}5,\ldots,\tfrac{5k-2}{3}\,\tfrac{5k-5}{3}\} \\ &\cup \{k{-}2\,k\}. \end{aligned}$$

In this case S has $\frac{2k+1}{3}$ edges. Let $S^* = \{0\,k{-}1,\ \frac{5k-2}{3}\,\frac{5k-5}{3},\ k{-}2\,k\}$.

□

Theorem 4.2.3 *Suppose $k > 2$ is even. There are at least* $\max(\lceil\frac{2k}{3}\rceil - (k-1-\frac{\phi(2k-1)}{2}), 3)$ *pairwise compatible Hamilton decompositions of K_{2k+1}.*

Proof. By Lemma 4.2.2 we can find a set S of $\lceil\frac{2k}{3}\rceil$ disjoint edges in K_{2k} so that no two of the edges are parallel, no two of the edges have the same length, and so that none of the edges contains ∞_1. There are at least $\lceil\frac{2k}{3}\rceil - (k-1-\frac{\phi(2k-1)}{2})$ disjoint edges $a\,b \in S$ such that $(a-b, 2k-1) = 1$. If $\lceil\frac{2k}{3}\rceil - (k-1-\frac{\phi(2k-1)}{2}) \geq 3$, choose S' to be this subset of S. If $\lceil\frac{2k}{3}\rceil - (k-1-\frac{\phi(2k-1)}{2}) < 3$, choose S' to be the set S^* defined in Lemma 4.2.2, so that $|S'|$ is always at least 3. Consider an edge $a\,b \in S'$. Since $\infty_1 \notin \{a, b\}$, both F_a and F_b are defined and, by Lemma 4.1.4, we know that $F_a \cup F_b$ is a Hamilton cycle. By Lemma 4.1.2 and (since k is even) Lemma 4.1.3 we can construct a Hamilton path decomposition of K_{2k} with the property that the Hamilton path that starts on vertex ∞_1 ends on vertex $2^{-1}(a+b) \pmod{2k-1}$, and the length of each the edges, $\{w_i\,w_{i+k} : 2 \leq i \leq k\}$, determined by the first and last vertices of each of the other Hamilton paths is a constant, $\min(2^{-1}(a-b) \pmod{2k-1}, 2^{-1}(b-a) \pmod{2k-1})$, dependent on the length of the edge $a\,b$. We can extend these Hamilton paths to Hamilton cycles of K_{2k+1} by adding the 2-paths $\infty_1\,\infty\,2^{-1}(a+b)$ and $\{w_i\,\infty\,w_{i+k}\}$. These Hamilton cycles together comprise a Hamilton decomposition of K_{2k+1}. Doing this for each such edge $a\,b \in S'$ gives $\lceil\frac{2k}{3}\rceil - (k-1-\frac{\phi(2k-1)}{2})$ Hamilton decompositions of K_{2k+1}. Since the edges in S' are disjoint, the end vertices of 2-paths centred at any vertex $v \in V(K_{2k})$ come from different 1-factors in each of the Hamilton path decompositions. Since no two edges in S have the same length, all the 2-paths centered at ∞ that do not contain ∞_1 will be distinct. And since none of the edges in S are parallel, we know by Lemma 4.2.1 that all the 2–paths centered at ∞ that do contain ∞_1 will be distinct. □

Given k, we can possibly do better than Theorem 4.2.3 by actually counting the number of edges in the set S that have lengths relatively prime to $2k-1$. Also, given k, we could deliberately construct a set $S^{\dagger}$, as in the following corollary, so as to improve the number of pairwise compatible Hamilton decompositions.

Corollary 4.2.4 *Suppose k is even. Let $S^{\dagger}$ be any set of disjoint edges in K_{2k} such that ∞ is not in any of the edges, no two of the edges are parallel, no two of the edges have the same length, and such that $(a-b, 2k-1) = 1$ for all edges $a\,b \in S^{\dagger}$. There are at least $|S^{\dagger}|$ pairwise compatible Hamilton decompositions of K_{2k+1}.*

More specifically, if $2k-1$ is prime, then the union of any two of the 1–factors of K_{2k} is a Hamilton cycle.

Corollary 4.2.5 *Suppose k is even and $2k-1$ is prime. Then there at least $\lceil \frac{2k}{3} \rceil$ pairwise compatible Hamilton decompositions of K_{2k+1}.*

Chapter 5

Conclusions

In Chapters 2 and 3 we verify Kotzig's and McKay's conjectures by constructing perfect sets of Euler tours of K_{2k+1} and of $K_{2k}+I$, and by showing that they lead to Hamilton decompositions of the line graph of the complete graph.

Chapter 3 was motivated by a desire to extend the idea of Conjecture 1.2.1 to K_{2k}. We chose to define a perfect set of Euler tours of $K_{2k}+I$ as we did because we wanted to complete the verification of McKay's conjecture. For completeness, we mention here a couple of other suggestions for extending Kotzig's conjecture to complete graphs on an even number of vertices.

Problem 5.1.6 *Let I be a 1-factor of K_{2k}. Does there exist a set of Euler tours of $K_{2k}-I$, such that every 2-path of $K_{2k}-I$ is in exactly one of the tours?*

Necessarily, this would require $2k-3$ Euler tours. (This is trivial to do when $k=2$ and not hard when $k=3$.) It would however be more satisfying to have a definition that contains every 2-path of K_{2k}.

Problem 5.1.7 *Suppose that $\mathcal{I}=\{I_1, I_2, \ldots, I_{2k-1}\}$ is a given 1-factorization of K_{2k}. Does there exist an Euler tour of each $K_{2k}-I_i$, $1\le i\le 2k-1$, so that every 2-path of K_{2k} is in exactly one of the tours?*

In this case we would need $2k-1$ Euler tours. This again is trivial when $k=2$ and not hard when $k=3$. A solution would imply the existence of a decomposition of $L(K_{2k})$ into cycles of length $k(2k-2)$, so that each vertex of the graph is missed by exactly one of the cycles. Certainly a desirable result. However, the choice of $\mathcal{I}$ might radically affect the problem.

The problems that were posed at the end of Chapter 1 about pairwise compatible Hamilton decompositions and pairwise compatible Hamilton path decompositions are still open. We have shown that K_{2k} has at least $2k-2$ pairwise compatible Hamilton path decompositions for all $k \geq 2$, and have mentioned that this is best possible when $k = 2$. It remains to discover for which k it is possible to find $2k-1$ pairwise compatible Hamilton path decompositions.

It is interesting that it is so much harder to find pairwise compatible Hamilton decompositions of K_{2k+1} than it is to find pairwise compatible Euler tours, and that perfect sets of Hamilton decompositions of K_{2k+1} do not even exist for small k. Perhaps another way of tackling this problem would be to look for properties of K_{2k+1} that might put an upper bound on the maximum number of pairwise compatible Hamilton decompositions. Finally, when k is odd, there is nothing known about the maximum number of pairwise compatible Hamilton decompositions of K_{2k+1}, beyond the fatuous statement that there must be at least one. Is it even possible to show that there must be at least three, as we have shown when k is even?

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