MATHEMATICAL APPLICATIONS OF CONIC SECTIONS IN PROBLEM SOLVING IN ANCIENT GREECE AND MEDIEVAL ISLAM

by

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ABSTRACT

This thesis investigates mathematical applications of conic sections to problem solving from the time they were invented in 350 B.C. to the 13th century A.D. We begin with the classic problems of the ancient Greek geometrical tradition, then, we explore other problems, which arose through the course of the development of the theory of conic sections, that would also require the use of conic sections. In the second part, we present the methods of Islamic geometers in solving these same problems and compare their methods to those of the Greek geometers. Afterwards, we discuss some new problems solved by Islamic geometers, some of which they were able to translate into cubic equations. This will lead us to consider the important part conic sections played in the development of the theory of cubic equations.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my thesis supervisor Dr. J.L. Berggren. Through many reading courses and discussions, he helped me develop a strong background in the history of mathematics, and encouraged me to explore the many fascinating areas related to this subject. I thank him for the time and energy he has devoted to guiding me through my thesis, and indeed, through my education at Simon Fraser University. Lastly, I am grateful for the friendship that both he and his family has offered me over the past two years.

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The Ancient Greek Tradition

The evolution of Greek geometry was enormously influenced by efforts within the ancient tradition of problem solving. Researches in geometry from the time of Hippocrates of Chios in the 5th century B.C. spawned interest in problems of higher geometry, that is to say, the quadrature of the circle, the cube duplication and the construction of regular polygons. These problems, in turn, required the introduction of new geometrical methods, one of which, the use of conic sections, proved to be indispensable for solving the problems of the cube duplication and the angle trisection. While various geometers applied this technique, and discovered different solutions, the theory of conic sections slowly took shape. By the late 2nd century B.C., Apollonius had composed a thorough exposition of the theory of conics with the expressed purpose[1] of abetting the efforts of geometric problem solving. Indeed, many other problems arose, both in Hellenistic times and later in the Islamic world, whose solutions relied on the application of conic sections[2].

The ancient Greeks had a special classification scheme for geometrical problems. Pappus, who flourished at the beginning of the 4th century A.D., remarks in his *Collection* that the ancients divided problems into three classes: 'plane', 'solid', and 'curvilinear'. 'Plane' problems could be solved by means of ruler and compass; 'solid', by means of one or more sections of the cone but not by 'plane' methods; 'curvilinear', by means of special curves, but not by 'plane' or 'solid' methods[3]. He notes that both the cube duplication and the angle trisection fall within the 'solid' class, and that this posed problems for researchers, who were not able to construct conics in the plane. However, they did attack these problems with a host of different techniques[4]

[1]This intent is expressed in the prefaces to several books of the *Conics*. In particular, in the preface to book IV. See Ver Eecke, 1923, p. 282.

[2]Among them were applications to the problem of instruments that cause burning, such as parabolic mirrors, but these and similar 'applied' problems lie outside the scope of this thesis.

[3]This classification can be found in two passages preliminary to separate discussions, the one devoted to the cube duplication: *Collection* III, 1, the other to the angle trisection: *Collection* IV, 1.

[4]We shall provide examples of these methods in the sections dicussing the solutions of the cube duplication and the angle trisection.

including the intersection of solids, the construction of special curves, and the use of mechanical motions. Even after they succeeded in finding solutions by means of conics, 3rd century geometers continued to find ingenious mechanical procedures. Nevertheless, Pappus records that the use of conic sections came to be seen as the most appropriate approach to solving 'solid' problems[5].

[5]This division of problems and criterion for the choice of construction seems to have emerged in large part due to Pappus. Earlier commentators such as Proclus and Eutocius did not follow this scheme.

History of Conic Sections

Conic sections have been studied since the time of Menaechmus (mid fourth century B.C.) who first used them for the cube duplication, but our knowledge of the early stages of their study, including their discovery, is full of lacunae. In his *Collection*, Pappus refers to the work of Aristaeus (ca. 300 B.C.) and states that Euclid wrote four books of conics. There are several references by Archimedes to theorems proved 'in the elements of conics', though he says nothing of their authorship. Archimedes was probably born in 287 B.C. in Syracuse[6], and spent a considerable time in Alexandria where he may have worked with Euclid's successors. It is largely through his works[7] that we can determine the status of the theory of conics before Apollonius' treatise became the standard work on conics. This treatise, *Conics*, marked a big step in the development of the theory of conics for it initiated a conceptual change and presented a comprehensive rigorous treatment of conic sections. We are able, to some degree, to trace the evolution of the theory of conics by investigating the methods and constructions used by the ancient geometers to solve geometrical problems.

Before delving into a study of the solutions of these problems, we will present a brief overview of the theory of conics both before and after Apollonius. The pre-Apollonian stage is characterized first and foremost by the method of generation of the conic sections. The three sections are obtained by cutting a right circular cone by a plane at right angles to a generator. If the cone is right-angled this produces a parabola, if obtuse-angled a hyperbola, if acute angled an ellipse. The three sections were accordingly named "section of a right-angled cone", "section of an obtuse-angled cone", and "section of an acute-angled cone". Archimedes uses this terminology in his works, and Pappus attributes their naming to Apollonius's predecessors[8]. Diocles, a contemporary of Apollonius, uses this nomenclature in almost all his propositions

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[6]The fact that he was killed in the sack of Syracuse in 212 B.C. at the supposed age of 75 enables us to fix such a precise date.

[7]Most importantly *On Conoids and Spheroids*, and *The Quadrature of the Parabola*.

[8]*Collection* VII, Jones I p. 114.

(apart from proposition 8) in *On Burning Mirrors*.

With the above method of generation, each of the three curves can be characterized by what the Greeks called a "symptom"[9]. The right-angled cone with vertex A (Fig. 0), and axis (the straight line drawn from the vertex of the cone to the center of the base) AF cut by a plane perpendicular to a generator AV, produces a parabola with vertex V, axis VF. If KL be a line drawn from the curved section to the axis VF and at right angles to it, then VL lies on the axis, and one can prove that

(1) $KL^2 = 2AV.VL$

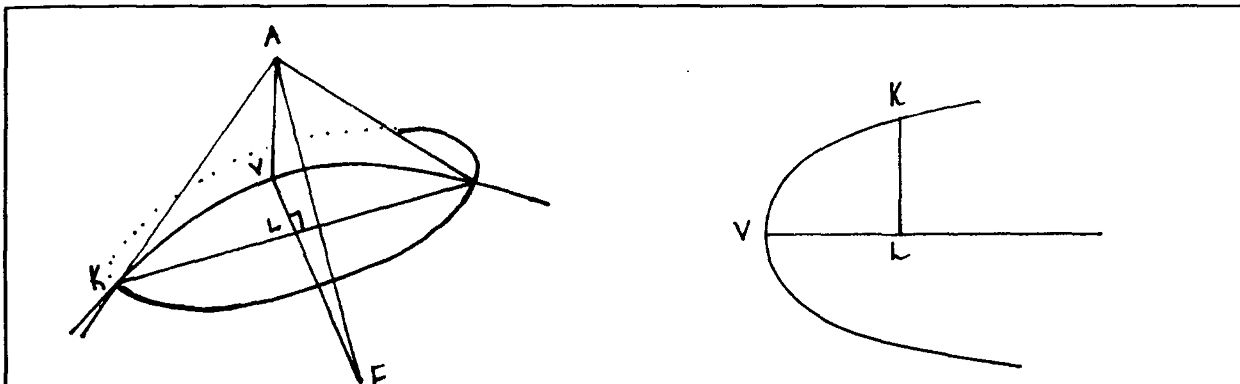

Figure 0

where the magnitude 2AV (the modern parameter) is called "the double of the distance from the vertex of the section to the axis". Similarly for the hyperbola and the ellipse (Figs 1, 2) one can prove that for a given section, KV, there is a constant length PV, such that

(2) $^{KL^2}/_{VL.PL} = {}^{2VF}/_{PV}$

where 2VF is a constant, twice the distance from the vertex of the section to the axis of the cone. In the case of the hyperbola, PV is called the transverse diameter, and in the case of the ellipse, PV is called the major axis. The most salient feature of this method of defining the curves is that they are in "orthogonal conjugation", that is the reference diameter VL always lies on the axis of symmetry of the curve and the ordinate KL is at right angles to the axis. Moreover, the fundamental property of these two conic sections is expressed as a proportion. From these

[9]Thus the Greeks referred to the fundamental property associated with the curve in question.

defining relations, Archimedes was able to deduce other properties of the conic sections which were instrumental in problem solving.

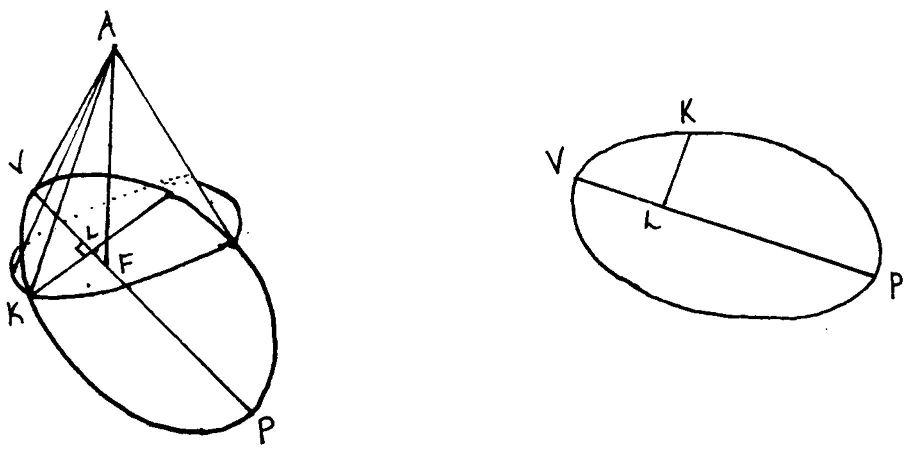

Figure 1

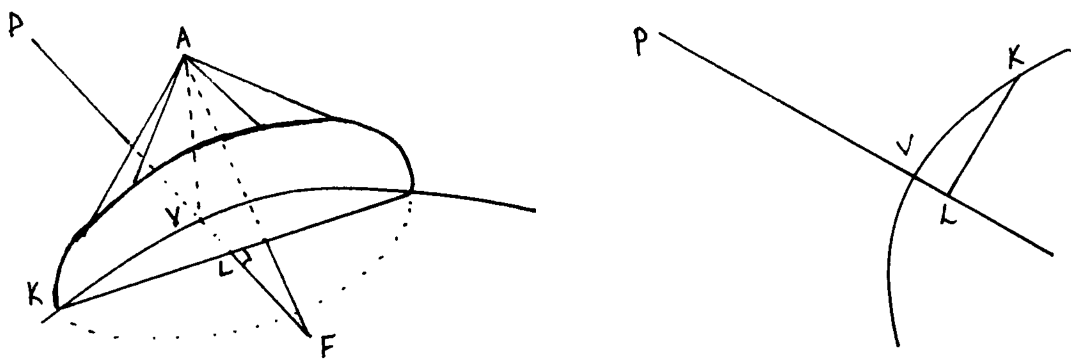

Figure 2

In his *Conics*, Apollonius introduced a new system for defining the three sections: he generates them by cutting the double oblique circular cone[10] by a plane. According to the different dispositions of the plane, the three curves can be generated from the same cone. Apollonius found symptoms for all three curves, and defined them by the method of application of areas[11] rather than as statements of proportion. In the case of the parabola, he represented the

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[10]A double cone whose axis is not necessarily perpendicular to its circular base.

[11]A standard procedure for formulating geometrically problems which are, algebraically, equations of the second degree.

symptom corresponding to (1a) $KL^2 = p.VL$, (Fig 3a)

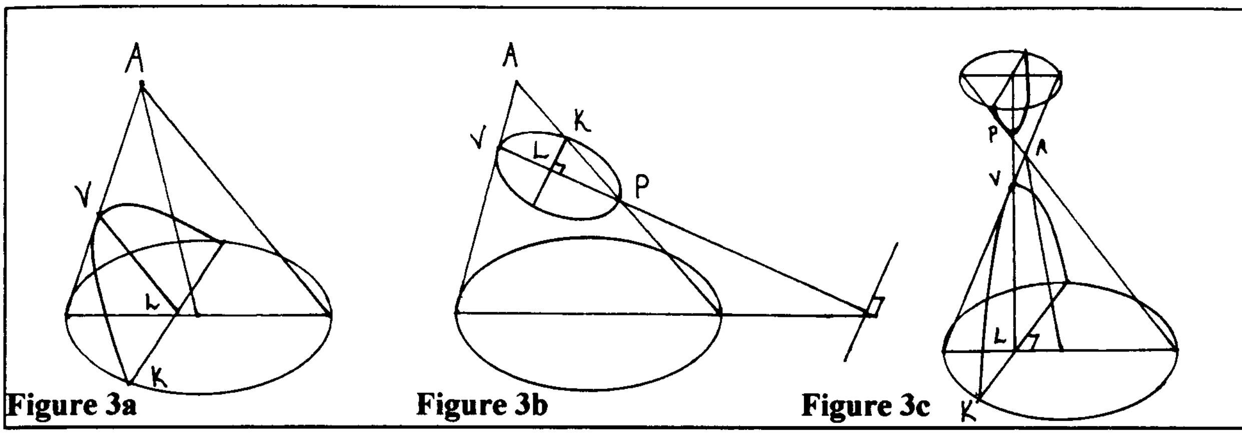

Figure 3a **Figure 3b** **Figure 3c**

by saying that the rectangle of side VL and area equal to the square on KL is applied to the line-length p (called the latus rectum). In the case of the hyperbola, since PV = PL - VL, equation (2) becomes (2a) $KL^2 = VL(p + {}^p/_a\, VL)$ (Fig. 3b)
where $a(=PV)$ is called the latus transversum. Apollonius expresses equation (2a) by saying that the rectangle with width VL and area KL^2 is applied to p so that it exceeds it by a rectangle similar and similarily situated[12] to the rectangle $p.a$. Similarly for the ellipse, since PV = PL + VL, equation (2) becomes

(3a) $KL^2 = VL(p - {}^p/_a\, VL)$. (Fig 3c)

Apollonius represents equation (3a) by saying that a rectangle of side VL is applied to p so that it falls short of it by a rectangle similar and similarly situated to the rectangle $p.a$.

This method of generation is far more general than the older approach for two reasons. Firstly, it uses a single type of cone for all sections; therefore, the conic sections are no longer associated with a type of cone, rather, they are determined by the position of the cutting plane relative to the double oblique circular cone. Secondly, it produces a symptom which applies not

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[12]This characterization is best understood from the diagram.

only to the axis[13] of the conic and orthogonal ordinates, but to any diameter[14] and the conjugate ordinates[15]. This method is known as "oblique conjugation" since the ordinates are not, in general, at right angles to the diameter. Furthermore, it immediately reveals the two branches of the hyperbola[16].

We will now briefly outline various solutions attributed to the ancient Greek geometers of the cube duplication and the angle trisection. Often, the original work is not available and we must rely on other sources. A compendia of known solutions is found in both Eutocius' commentary on Archimedes' *Sphere and Cylinder*[17] and Pappus' *Collection.* Since they both wrote after the time of Apollonius, they often recast the solutions in Apollonian terms. This makes it difficult to determine the actual content of the original solution. In each case, we will endeavour to situate the solution in its historical context and indicate significant changes that were made by the respective commentators.

Beforehand, we describe the particular way the ancients approached geometric problems. They used a method of analysis and synthesis for the discovery and construction of solutions to these problems. In the analysis, one assumes the problem to be solved and deduces properties of the constructed figure until an element of it emerges that is known from prior results to be constructible. Thus, geometers could refer to constructions in the *Elements* or the *Data* of Euclid, and to properties of the conics developed in the theory of the conic sections. The formal synthesis begins from these constructible terms and proceeds back through the steps of the analysis in a deductive sequence until one reaches the desired construction.

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[13]Archimedes used the term diameter to refer to the axis of a conic section.

[14]The diameter of the section is defined as a line which bisects all parallel chords in the section.

[15]The parallels to the tangent at that diameter.

[16]Apollonius calls them the "opposite sections".

[17]Solutions to the duplication of the cube follow *Sphere and Cylinder II:4*, where Archimedes assumes the construction of two mean proportionals.

The Cube Duplication

This ancient problem has a long and colourful history. One version of its origin is related by Eratosthenes, a younger contemporary of Archimedes, who relates that,

> "..when the god proclaimed to the Delians by the oracle that, if they would get rid of a plague, they could construct an altar double of the existing one, their craftsmen fell into a great perplexity in their efforts to discover how a solid could be made double of a (similar) solid..."[18]

(Both altars were supposed to be cubical.)

The story continues on to say that these geometers were sent to the Academy of Plato for the solution. In the 5th century, when Hippocrates of Chios reduced the problem to one of finding two mean proportionals[19] between two given lines, various solutions were found. A few examples include: the solution of Archytas using the intersection of solids, the use of the cissoid by Diocles, and the mechanical instrument known as the mesolabe attributed to Plato[20].

Eutocius attributes to Menaechmus[21] two solutions of the problem of finding two mean proportionals. Menaechmus was a pupil of Eudoxus and flourished in about the middle of the fourth century B.C. The problem is to find two segments X and Y given two segments A and B such that (1) A:X=X:Y=Y:B.

The first solution uses the intersection of a hyperbola with a parabola, and the second uses the intersection of two parabolas. It is difficult to imagine how Menaechmus conceived of constructing a solution to (1) using sections of a cone, and Eutocius' text is of little heuristic value in this regard for it is framed in conformity with a more developed theory of conics. There are two schools of thought on the issue of how conic sections and their fundamental properties

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[18]As quoted by Theon of Smyra, cf. Heath 1, vol. 1, p. 246.

[19]Given two straight line segments p and q, the problem is to construct two other straight segments ("mean proportionals") x and y in such a way that $p : x = x : y = y : q$. In modern algebraic notation the problem is equivalent to the cubic equation $x^3 = p^2q$.

[20]See Heath 1, v.1 for these solutions, and details of another mesolabe attributed to Eratosthenes.

[21]Actually, only one solution is explicitly attributed to Menaechmus. The second one follows the first one and Eutocius says nothing of its authorship.

were discovered, but all agree that Menaechmus is the first known source to use conic sections.

Some think that Menaechmus discovered them in plane sections of right angled cones, and that it was the properties of the ordinates in relation to the abscissae on the axes which he arrived at first[22]. In a summary of previous methods of extracting the mean proportionals, Eutocius quotes a reference made by Eratosthenes to cone-cutting "the Menaechmean triads[23]". This line has been important in supporting the view that Menaechmus was the discoverer of the conic sections, and that he used them for constructing the two mean proportionals. More evidence is given by the 1st century A.D. writer Geminus, who, in a discussion of the discovery and classification of curves, states that the conic sections were discovered by Menaechmus.

Knorr (1986, p. 63) argues a different point of view, and proposes that Menaechmus based his solution on curves defined with respect to the second-order relations among the mean proportional lines. From equation (1), three equations can be deduced;

(2) $X^2 = A.Y$,

(3) $Y^2 = B.X$ and,

(4) $X.Y = A.B$.

Menaechmus would then have drawn these solving curves on the basis of pointwise constructions[24], and defined them as special curves having the stated properties. Knorr argues that the shift from point constructions to sections of cones was born, around the time of Euclid, out of the Greek geometers' search for a more natural way of generation for these curves. Thus, Menaechmus' special curves would have been renamed as sections of a cone, either by himself, or another geometer. This point of view, though it seems plausible, especially as seen through a modern mathematical approach, contradicts the only historical information we have.

Nevertheless, the account given by Eutocius may serve as an accurate guide to the

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[22]Heath 1 v.I, p. 253

[23]The triads have long been understood as referring to the three conic sections. But, as Knorr has pointed out, Menaechmus uses two parabolas and a hyperbola.

[24]We will see an example of a pointwise construction of the parabola in Diocles' solution of this problem. Knorr (1986, pp 57 - 66) argues that a Menaechmean pointwise construction would be based on the pseudo-Platonic device known as the mesolabe.

essential line of thought Menaechmus followed; that is, to derive two curves that intersect in a point that determines the two mean proportionals. Of course, the point where they intersect solves equation (1). Although Eutocius provides both the analysis and the synthesis of the problem, Knorr (1989, p. 96) has suggested that the synthesis was Eutocius' own addition[25]. A summary of the analysis of the second solution as given by Eutocius is as follows:

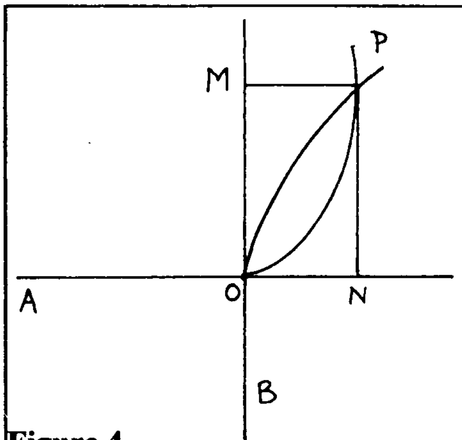

Figure 4

Analysis: (Fig 4) Let AO, OB equal the given magnitudes, and draw them at right angles to each other [26]. Suppose OM, ON have been found such that AO : OM = OM : ON = ON : OB. Measure OM along BO extended and ON along AO extended. Complete the rectangle OMPN.

Then, (5) $OB.OM = ON^2 = PM^2$

This defines a parabola $\mathcal{P}_1$ with vertex O, axis OM, and latus rectum OB.

Also, (6) $AO.ON = OM^2 = PN^2$

So P also lies on a parabola $\mathcal{P}_2$ with vertex O, axis ON, and latus rectum OA.

Thus, P is the point of intersection of the two curves, so

AO:PN = PN:PM = PM:OB,

and the problem is solved.

This solution involves the two curves satisfying equations (2), and (3). It can easily be seen that their intersection yields the desired mean proportional relationship.

The first solution as given by Eutocius is summarized as follows:

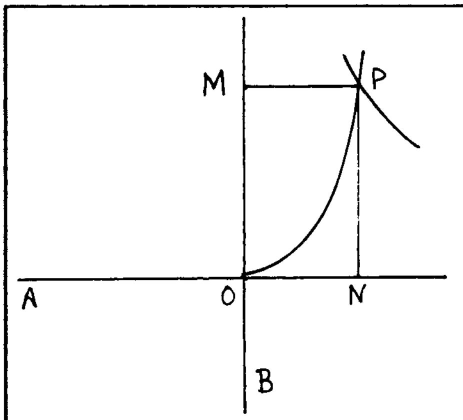

Figure 5

Analysis: (Fig 5) As above, complete the rectangle OMPN.

Then equation (5) holds.

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[25]He cites two reasons: (1) Pappus provides only the analysis of the Menaechmean solution; (2) Though the analyses of both commentators are characteristically pre-Apollonian in their terminology, the synthesis of Eutocius is decidedly Apollonian.

[26]We note the rectangle AO.OB represents the right hand side of equation (4).

Also, (7) AO.OB = OM.ON = PN.PM

So P lies on the section of a hyperbola $\mathcal{H}$ with asymptotes OM, ON

Thus, P is the point of intersection of $\mathcal{P}_1$ and $\mathcal{H}$, so

AO:PN = PN:PM = PM:OB,

And the problem is solved.

This hyperbola, in effect, is a locus satisfying a constant product relation as exemplified by equation (4). Although, the hyperbola was defined in terms of its symptom, this property was known to Archimedes, and can be stated as follows: if P be any point on the curve and PK, PL each be drawn parallel to one asymptote and meeting the other at K and L respectively, then PK.PL = (const.)[27].

Three related versions of the cube duplication have come down under Apollonius's name. Although these solutions, as preserved in our sources, all involve special constructions by means of sliding rulers such as the *neusis* [28] construction, Pappus (*Collection*, Book III) mentions the existence of yet another Apollonian solution by means of conic sections. He does not give the actual construction, but Knorr (1986, p.306) has suggested that it would probably be based on his neusis construction[29]. In fact, there are extant solutions within the Islamic tradition of classical geometry[30] that are based on the following Apollonian neusis construction. (Fig. 6)

Place the two given lines OA, OB at right angles. Complete rectangle OACB, and let D be the point where the diagonals bisect each other. Draw a circle about OACB. Extend sides OA, OB. Draw a line passing through C that meets the extended sides in Z and H and the circle in K such that ZK = CH. Then BZ and AH are the required means.

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[27]Archimedes knew the property for the rectangular hyperbola, that is, when the asymptotes are at right angles to each other.

[28]A neusis is the insertion of a segment of given length between two given straight or curved lines in such a way that the segment verges towards a given point.

[29]Apollonius solved many problems using neusis constructions in his work *On Neuses*, but he later produced other works investigating planar methods for solving some of these problems. This reveals a transition in thought concerning geometrical constructions that may account for the attempts to replace neusis constructions with solid methods.

[30]We will discuss these in the section on cube duplication in the Islamic tradition. See p. 28

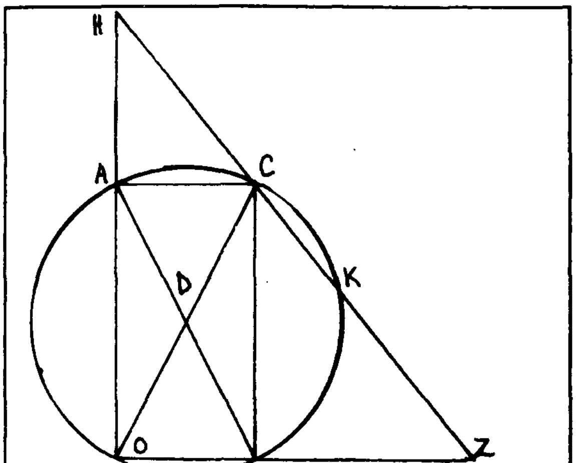

Figure 6

This neusis recasts the problem in a slightly different way than the Menaechmean analysis thereby revealing a new relationship between the mean proportionals and the given line segments. The relationship is that the straight line through H and Z cuts the circle in C and K, respectively, such that CH = ZK. We will see how the Islamic geometers used this relationship to describe a certain hyperbola, thereby eliminating the necessity of a sliding ruler.

A construction of the two mean proportionals given by Diocles introduces a different conception of the conic sections. In his work *On Burning Mirrors*, Prop. 10, Diocles provides the synthesis for the doubling of the cube using the intersection of two parabolas, the same method used by Menaechmus. The important feature of Diocles' solution is that he constructs these parabolas by means of a focus and directrix. This means of generating the parabola is not discussed by Apollonius in his *Conics*, and Toomer (1976, p. 17) has attributed its discovery to Diocles.

Diocles states the problem as follows: for a given line A, we seek another line such that the cube of A is twice that of the other line. This formulation of the problem is quite interesting; in the original problem we are to construct a cube which is twice a given cube, yet Diocles attacks this problem backwards. In fact, his procedure is like that of an analysis in that he assumes the cube doubled and seeks the cube which is half the given cube. Diocles sets the line GD(=½A) at right angles to a line EZ(=¼A)

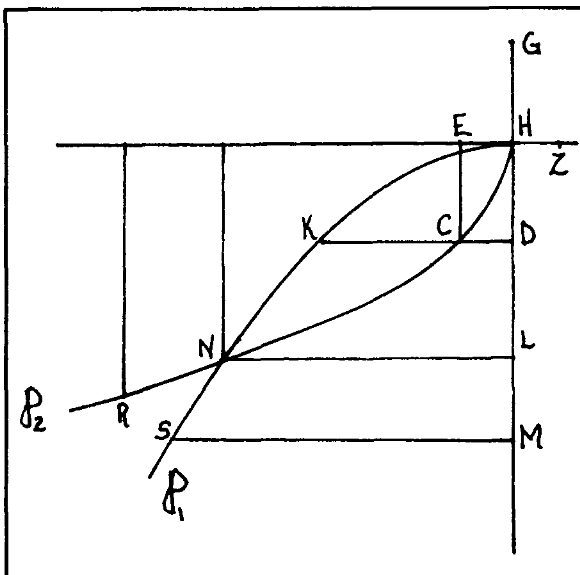

Figure 7

such that EZ and GD bisect each other at H, in other words, EH = HZ and GH = HD. (Fig. 7). In order to construct the first parabola $\mathcal{P}_1$, Diocles finds the points K, N, S such that their vertical distances from a line (the directrix) through G and parallel to EZ are equal to their respective distances from a fixed point D (the focus). He constructs the first parabola by joining the points H, K, N, and S, with a flexible ruler. Similarly, he constructs the second parabola $\mathcal{P}_2$ by finding the points C, N, R such that their vertical distances from a line through Z and parallel to GD, are equal to their respective distances from E. Then the curved line through C, N, R, and H cuts the curved line HKNS at the point N. He draws NL perpendicular to HM and proves that $A^3 = 2NL^3$.

Diocles proves that such a curved line is indeed a parabola in Prop. 5 of the same book by showing that it has the defining symptom of the parabola. Furthermore, Diocles proves that the parameter of this parabola is equal to four times the focal distance. An extension of the focus-directrix property to all three conic sections is found in Pappus' *Collection* VII. Pappus proves that, given a straight line AB and a fixed point G, the locus of a point D such that the ratio of its distance from G and its vertical distance from AB is constant will be a conic, and will be a parabola if the ratio is equal to 1, an ellipse if less than 1, and a hyperbola if greater than 1. The focus-directrix property of the parabola had its greatest use in the actual construction of the conic sections, especially in ancient Greek treatments of burning mirrors. However, it appears to have had minimal use in problem solving.

Diocles' work ***On Burning Mirrors*** is a treatment of the geometry of burning mirrors that investigates the properties of parabolic and spherical mirrors. He is not the only ancient Greek geometer to have studied burning mirrors, since both Archimedes and Anthemius (6th century A.D.) are also known to have investigated them. In fact, Anthemius even investigated the elliptical contour which reflects rays from one focus of the ellipse to the other. However, we will not include these applications of conic sections in our present research.

The Trisection of the Angle

In Book IV of the *Collection* Pappus presents a series of methods for trisecting the angle, followed by two methods for the general problem of dividing the angle in any given ratio. He uses these results to inscribe in a given circle a regular polygon having any specified number of sides. Heath (vol. I, p.235) has suggested that the problem of the trisection of any angle arose from attempts to continue the construction of regular polygons after that of the pentagon had been discovered. Indeed, the trisection of the angle is necessary to construct the nonagon, or any polygon the sides of which are a multiple of nine. Undoubtably though, since the ancient geometers were able to bisect the angle by means of ruler and compass, they sought also to trisect the angle. This problem proved far more difficult since it could not be solved by 'planar' methods. In fact, the first solutions involved neusis constructions.

In the latter part of the 3rd century B.C. Nicomedes invented the curve known as the conchoid for the specific purpose of solving a neusis required for the angle trisection. A method via neusis also underlies an Archimedean angle trisection preserved in an Arabic work attributed to him, the *Book of Lemmata.* A variant of this neusis is one of the three unattributed methods of trisection presented by Pappus in *Collection* IV, where Pappus shows how to effect it through the intersection of a given circle and a given hyperbola. The second method Pappus gives is a direct solution by means of conics. The third method, also without a neusis, uses the focus-directrix property to specify the hyperbola.

The trisection of the angle ABC can be reduced to a neusis as follows (Fig. 8) :

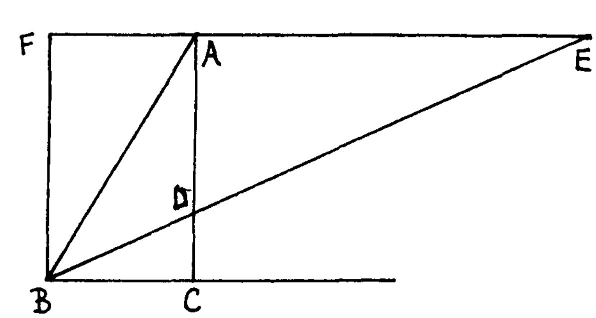

Figure 8

Consider the angle ABC. Draw AC perpendicular to BC. Complete rectangle ACBF. Produce the side FA to E. Now *insert a straight line ED of given length 2AB between AE and AC in such a way that ED verges towards B*. Then ∠DBC = ⅓∠ABC. Pappus shows how to solve this neusis problem in a more general way in the following analysis (Fig. 9):

Given a rectangle ADCB and a line segment M, construct a straight line FEA such that its intercept FE between CD and BC extended equals M.

Complete the parallelogram EDGF.

So M=EF=DG.

Therefore G lies on a circle with centre D and radius M.

Since BC.CD = BF.ED (*Elements* I.43)

= BF.FG,

G lies on a hyperbola $\mathcal{H}$ with BF, BA as asymptotes and passing through D.

So G lies on the intersection of the hyperbola and the circle.

In order to effect the construction we have only to draw the circle of centre D, radius M and the hyperbola specified by its asymptotes, passing through the point D. Pappus provides a separate solution of the construction of such a hyperbola. The intersection of these two curves gives the point G, which solves the problem. Pappus applies this neusis to the trisection of any acute angle and then extends the method to trisect both the right and obtuse angles.

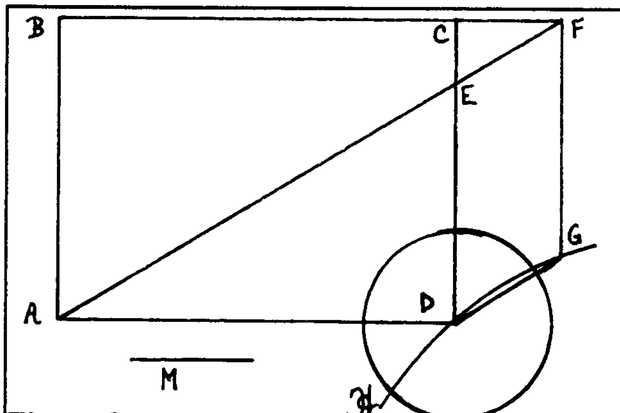

Figure 9

In his second solution, Pappus shows how to trisect a given arc of a circle by constructing a hyperbola determined by its Apollonian parameters. We shall present a brief summary of the analysis (Fig. 10):

Consider the triangle ABG of given base AG and whose angle at G is twice the angle at A. Construct H such that HG = ⅓GA. Draw BD perpendicular to AG.

Then, it can be shown that $3AD.DH = BD^2$.

so that $BD^2 : AD.DH = 3AH : AH$.

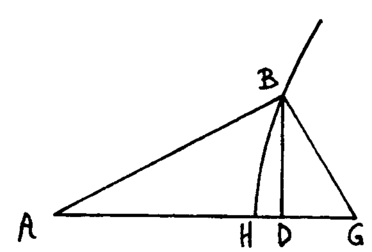

Figure 10

Thus B lies on a hyperbola $\mathcal{H}$ whose latus rectum is three times its latus transversum AH. Now, (Fig. 11)in order to trisect an arc GA of a circle of centre O, draw the chord GA and divide it at H so that AH = 2HG. Construct the hyperbola of vertex H, latus transversum AH, and latus rectum 3AH. Let the hyperbola meet the circular arc in P. Then by the above construction we have ∠PGA = 2 ∠BAG, so that ∠ AOP = 2∠OPG. Therefore OP trisects the arc APG. The construction of such a hyperbola relies on Apollonius' *Conics* I:54.

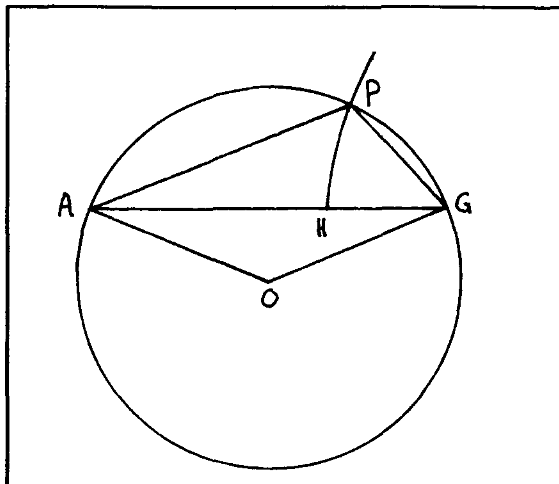

Figure 11

In the next solution to the problem, Pappus merely gives an alternate analysis of the problem. It is based on the same figure and leads to the same hyperbola, expressed as the locus relative to a given focus and directrix. He takes the same triangle ABG as before, (Fig. 12) but this time he bisects the angle G with the line GE, and draws EX and BD perpendicular to AG. Pappus thus obtains AE = EG and AX = XG. By similar triangles, AG : BG = AE : EB = AX : XD, and so BG = 2DX. It then follows that B lies on the locus of points such that the ratio of its distance from the focus G to its vertical distance to the directrix EX is equal to 2. Therefore, B lies on a hyperbola.

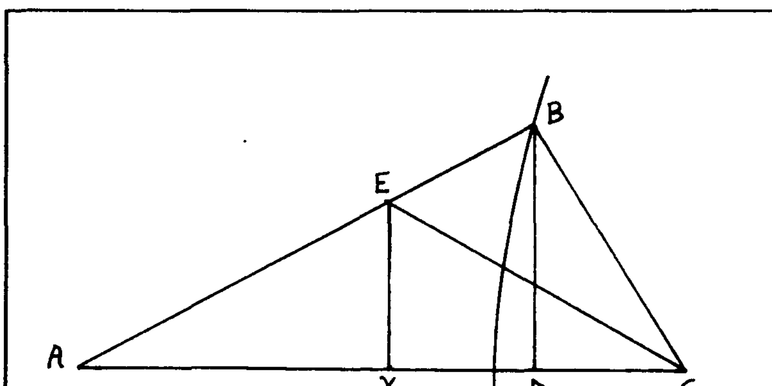

Figure 12

The assorted solutions given by Pappus in his *Collection* and especially by Eutocius in his commentary to Archimedes' *On the Sphere and Cylinder* attest to the interest of both in communicating different methods and their variations. Not only do the two collections of solutions expose different approaches to problem solving, but they reflect the ancient Greek geometers' skills in applying various aspects of the theory of conics.

These two problems were not the only ones that the Greek geometers solved by means of conic sections. In a passage preliminary to the discussion of the cube duplication and the angle trisection, Pappus says that geometers err when they solve a problem using an inappropriate method. He cites as an example the neusis assumed[31] by Archimedes in the book *On Spirals*, Prop. 5. According to Pappus, this problem is a 'solid' problem, and should be solved by means of conic sections.

At the end of Book IV of the *Collection*, Pappus provides a solution to this problem. Archimedes' neusis construction makes the following assumption:

(Fig. 13) Given a circle, a chord BC smaller than the diameter, and a point A on the circle such that ADE, perpendicular to BC, cuts BC at D such that BD>DC and meets the circle again in E, it is possible to place between the straight line BC and the circumference of the circle a straight line RP equal to DE and verging towards A.

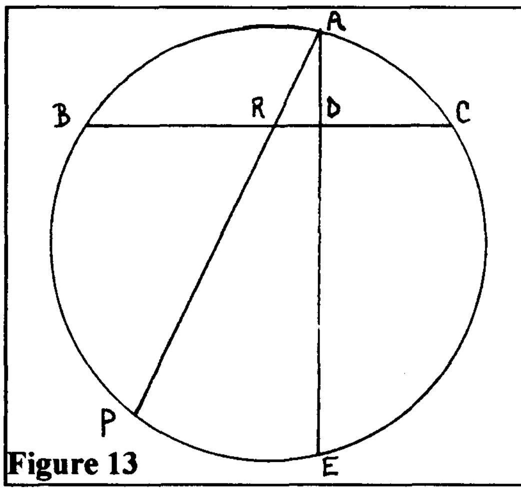

Figure 13

Pappus solves the more general problem of requiring RP to be equal to any given possible[32] length. He finds the solution by means of two lemmas (Fig. 14).

The first lemma states that, if from a given point A any straight line be drawn meeting a straight line BC given in position in R, and if RQ be drawn perpendicular to BC and of length bearing a given ratio, say ρ, to AR, the locus of Q is a hyperbola $\mathcal{H}$. He proves this

[31]The neusis is said to be assumed because there is no explanation as to how it is to be effected.

[32]After all, it cannot be larger than the diameter of the given circle.

by drawing ADA′ at right angles to BDC so that QR:RA = A′D:DA = the given ratio ρ. Then he takes DA″ along DA equal to DA′, QN perpendicular to AD, and obtains the relation

$$QN^2 : A''N.A'N = (\text{const.}).$$

The second lemma proves that, if BC is given in length, and Q is a point that, when QR is drawn perpendicular to BC, then BR.RC = k.QR, where k is a given length, then the locus of Q is a parabola $\mathcal{P}$. He achieves the required relation

$$QN'^2 = k.KN'$$

by taking O the middle point of BC, OK at right angles to BC and of length such that OC^2 = k. KO, and QN′ perpendicular to OK. To solve the problem Pappus constructs the parabola and the hyperbola in question, and their intersection gives Q, whence R, and therefore ARP, is determined.

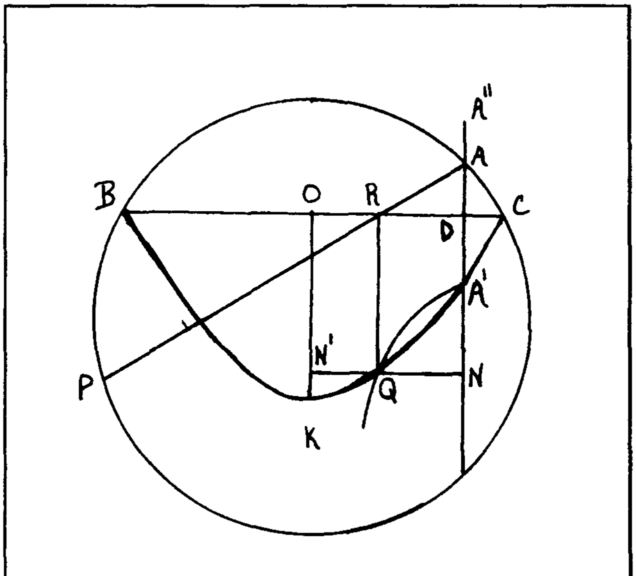

Figure 14

Archimedes' Problem

We are indebted to Eutocius for his commentary on Book II of Archimedes' *On the Sphere and Cylinder*, where a problem arises that requires yet another solution by means of conic sections. Although Archimedes promises a solution to this problem that arises as an auxiliary to Prop. 4, it is not included in the extant form of the treatise. Eutocius presents three solutions in his commentary: he credits the substance of the first one to Archimedes[33]; the second one, by Dionysodorus, takes a slightly different approach; the third solution is due to Diocles.

Proposition 4 poses the following problem: "to cut a given sphere by a plane in such a way that the volumes of the segments are to one another in a given ratio." Through a considerable manipulation of proportions and a skillful elimination of two unknowns, Archimedes reduces the problem to a subsidiary one, which he claims requires a *diorismos*[34] . He notes however, that due to the conditions subsisting in the present problem (Prop. 4), the subsidiary problem can be stated more specifically so that it does not require conditions for the existence of a solution.

Eutocius attacks the general subsidiary problem: (Fig. 15) Given two straight lines AB, AC and an area D, to divide AB at M so that AM : AC = D : MB^2. He gives both the analysis and the synthesis, and discusses the limits of possibility of the solution. We present in a slightly modern form the analysis and Eutocius' rendering of the conditions for a solution.

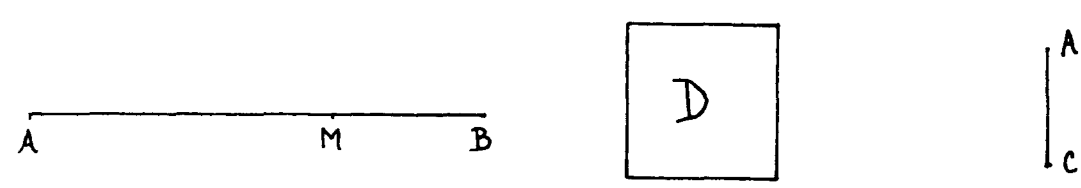

Figure 15

[33]Since Eutocius credits Archimedes with the solution, we henceforth refer to it as Archimedes' solution.

[34]An investigation into the limits of possibility, or the conditions in which a solution exists.

<u>Analysis</u> (Fig. 16) Assume M found. Draw AB, AC at right angles. Extend CM to N such that EBN is parallel to AC and CE is parallel to AB. Complete rectangle CENF, and draw PMH parallel to AC meeting FN in P and CE in H.

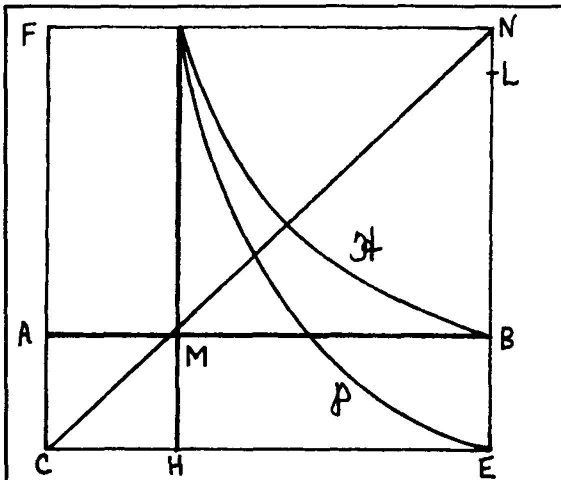

Figure 16

Let L be taken on EN so that CE.EL = D.

Then, AM : AC = CE.EL : MB^2 by hypothesis.

And by similar triangles AM : AC = CE : EN = CE.EL : EN.EL.

So $PN^2 = MB^2$ = EL.EN.

Hence P lies on a parabola 𝒫 with vertex E, axis EN, and parameter EL. Since EL is given[35], the parabola is given in position.

Next, since rectangles FH, AE are equal, FP.PH = AB.BE

Therefore P also lies on a hyperbola[36] ℋ with asymptotes CE, CF, passing through B. Thus P is determined as the intersection of the parabola and hyperbola[37]. Since P is given, M is also given[38].

Eutocius notes that since AM : AC = D : MB^2, it follows that AM. MB^2 = AC.D, but since AC.D is given and the maximum value of AM.MB^2 is attained when BM = 2AM a necessary and sufficient condition for the existence of a solution is that AC.D must not be greater than ⅓AB.(⅔AB)^2. The proof that the maximum value of AM.MB^2 is attained when BM = 2AM is given after the synthesis, but we shall investigate it immediately.

---

[35]Of course, what is given is EN and D, Diocles makes the interesting assumption that if an area D is given, along with a line segment EN, then the segment EL such that EL.EN = D is also given.

[36]This hyperbola is a rectangular hyperbola as its asymptotes are at right angles to each other.

[37]Since the two conic sections share a common axis and the vertex of the parabola lies within the hyperbola it is quite possible that they may not intersect.

[38]A 'given' is a term that is known by assumption or has been shown to be determined by such terms. This terminology is used in Euclid's *Data* and in works such as Aristotle's *Meteorologica*.

<u>Proof of *diorismos*</u> (Fig. 17) Let AB be divided at O so that BO = 2AO. Eutocius must show that $AO.OB^2$ is larger than $AM.MB^2$ where M is any point on AB.

Suppose that $AO : AC = CE.EL' : OB^2$,

so that $AO.OB^2 = CE.EL'.AC$.

Join CO and extend it to N; draw EBN parallel to AC, and complete the rectangle CENF. Draw POH parallel to AC meeting FN in P and CE in H. Draw the parabola $\mathcal{P}_1$ with vertex E, axis EN, and parameter EL′. It will pass through P, and beyond P will meet CF in some point. Next, draw a hyperbola $\mathcal{H}$ with asymptotes CE, CF passing through B. It will also pass through P. Extend NE to T so that NE = TE. Draw TP meeting CE in Y and CF extended in W. Thus TP is tangent to the parabola at P[39].

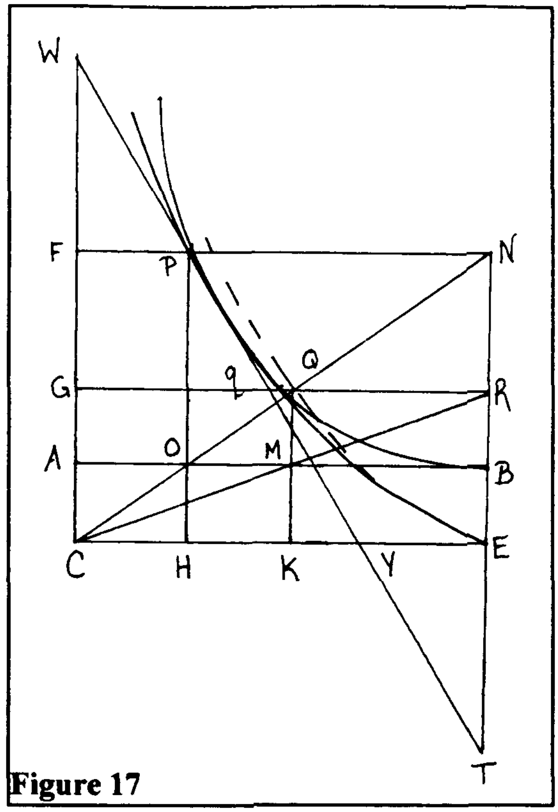

Figure 17

Then, PW = PY (since BO = 2AO and TP = 2PW = 2PY).

Since the segment WY, between the asymptotes, is bisected at P, and touches the hyperbola at P,

WY is tangent to the hyperbola. (*Conics* II.3)

Hence the hyperbola and the parabola have a common tangent at P.

Now suppose M is any point on AB, and through M draw QMK parallel to AC meeting the hyperbola in Q and CE in K. Draw GqQR through Q parallel to AB meeting CF in G, the parabola in q, and EN in R.

Since rectangles GK, AE are equal (property of the hyperbola), CMR is a straight line.

Now, $qR^2 = EL'.ER$ (by the property of the parabola)

[39]This is the converse of a proposition known to Archimedes; for a parabola with vertex V, diameter PV, and ordinate PQ, if the tangent at Q meets VP produced in T, then PV = VT.

So, $QR^2 < EL'.ER$.

Suppose $QR^2 = EL.ER$.

Then we have $AM : AC = CE : ER = CE.EL : QR^2 = CE.EL : MB^2$,

so $AM.MB^2 = CE.EL.AC$.

Therefore, $AM.MB^2 < CE.EL'.AC = AO.OB^2$, which is what we wanted to prove.

Eutocius goes on to discuss the number of solutions, which depends on the value of $AO.OB^2$ relative to AC.D:

(1) We have seen that, in order that the solution may be possible, $AC.D \ngtr AO.OB^2$.

(2) If $AC.D = AO.OB^2$, then the point O itself solves the problem.

(3) If $AC.D < AO.OB^2$, there are two solutions because there will be two points of intersection between the parabola and the hyperbola. This can be seen by drawing a parabola $\mathcal{P}_2$ with vertex E, axis EN, and parameter EL, it will pass through Q; and, since the parabola meets CF beyond Q, it must meet the hyperbola again (which has CF for its asymptote).

The original problem of dividing the sphere into segments having a given ratio is always possible for we can take our point of division to be anywhere along the diameter of the sphere. But the more general problem considered in the previous discussion by Eutocius, and suggested by Archimedes, will only exist for certain choices of the given line and the given area, subject to the restriction Eutocius provides.

Dionysodorus solves the less general form of the problem, as it is needed in Archimedes' problem. For AB the diameter of the sphere, it is required to find a plane cutting AB at right angles at a point M so that the segments into which the sphere is divided are in a given ratio, as CD:DE. (Fig.18)

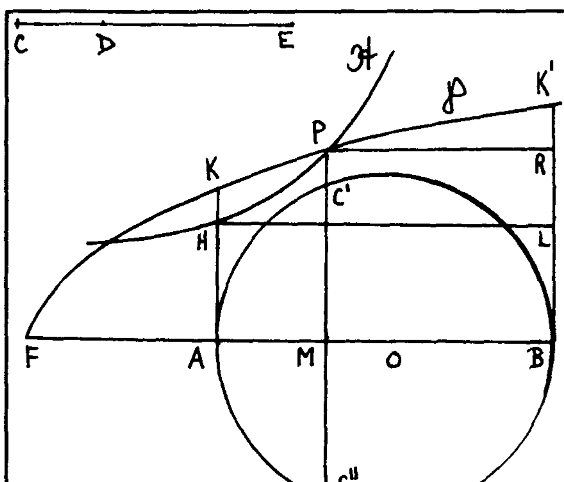

Figure 18

Produce BA to F so that AF = OA, where O is the centre of the sphere. Draw AH perpendicular to AB and of such length that FA : AH = CE : ED. Produce AH to K[40] so that AK^2 = FA.AH. Then K will lie on a parabola $\mathcal{P}$ with vertex F, axis FA, and parameter AH. Draw BK′ parallel to AK and meeting the parabola in K′; and with BF, BK′ as asymptotes describe a hyperbola $\mathcal{H}$ passing through H. This hyperbola will meet the parabola at some point P, between K and K′(since K lies above H and BK′ is an asymptote of the hyperbola). Draw PM perpendicular to AB meeting the great circle in C′, C″, and from H, P draw HL, PR both parallel to AB and meeting BK′ in L, R respectively. Then PM.PR= AH.HL by the property of the hyperbola,

so PM.MB = AH.AB.

Rearranging, $PM^2 : AH^2 = AB^2 : BM^2$.

Also, PM^2 = FM.AH by the property of the parabola,

or FM : AH = $PM^2 : AH^2 = AB^2 : BM^2$, from above.

From this point Dionysodorus uses Archimedes' propositions on the volume of a segment of a sphere and Prop. 2 to show that (segmt. AC′C″) : (segmt BC′C″) = CD : DE. He uses the same conic sections as Eutocius.

Diocles, proceeding in a different manner, reduces a slightly generalized form of Archimedes' problem to three simultaneous relations. Using the results of *On the Sphere and Cylinder* Prop. 2[41], he proceeds to state the problem in the following form: Given a straight line AB, a ratio C:D, and another straight line AK[42], to divide AB at M and to find two points H, H′ on BA and AB produced respectively so that the following relations may hold, (Fig. 19)

(1) C : D = HM : MH′

(2) HA : AM = AK : BM

---

[40]K lies on AH if AK>AH. Since we are dividing AB at H, we have AB>AH, and since AK^2 = AF.AH, it follows that AK>AH.

[41]If C′AC″ be a segment of a sphere, C′C″ a diameter of the base of the segment, and K the centre of the sphere, and if AB be the diameter of the sphere bisecting C′C″ in M, then the volume of the segment is equal to that of a cone whose base is the same as that of the segment and whose height is h, where h : AM = KB + BM : BM.

[42]By taking any given line AK, Diocles generalizes the problem, for which AK would be ½AB, the radius of the sphere.

(3) H′B : BM = AK : AM

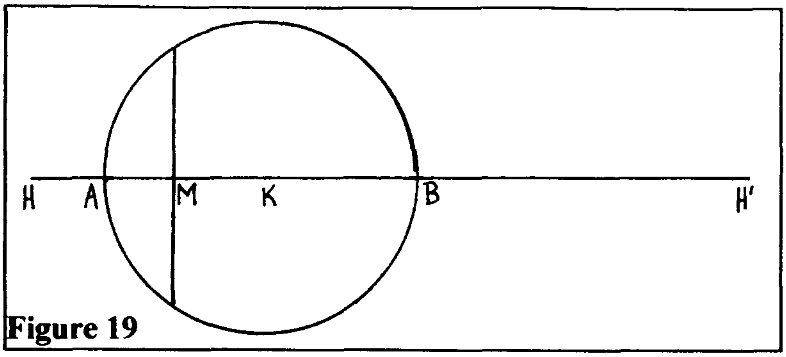

Figure 19

Analysis (Fig. 20) Suppose the problem solved, and that the points M, H, H′ are all found. Place AK at right angles to AB, and draw BK′ parallel and equal to AK. Join KM, K′M, and produce them to meet K′B, KA respectively in E, F. Draw KK′, draw EG through E parallel to BA meeting KF at G, and draw QMN parallel to AK meeting EG at Q and KK′ in N. From (2) and by similar triangles, HA : AM = FA : AM.

Therefore HA = FA and similarly H′B = BE,

so FA + AM = HM and EB + BM = H′M. Again, by similar triangles,

$$(FA + AM) : (BK' + BM) = AM : BM = (AK + AM) : (EB + BM).$$

Take AR along AH and BR′ along BH′ such that AR = BR′ + AK. Then we obtain

(4) HM.MH′ = RM.MR′.

For this reason, the position of R relative to H and A will determine the position of R′.

Now, (5) $C : D = RM.MR' : MH'^2$ by (1) and (4)

Measure MV along MN so that MV = BM. Join BV and extend it both ways. Draw RP, R′P′ perpendicular to RR′ meeting BV produced at P, P′ respectively. Then

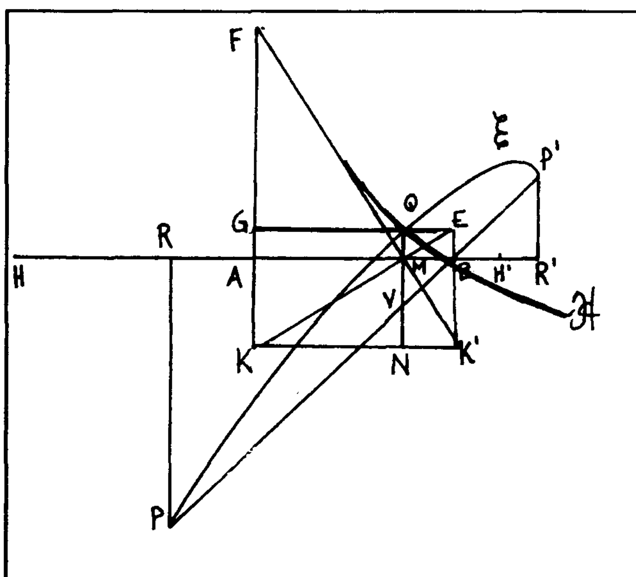

Figure 20

since angle MBV = 45°, PP′ is given in position, as are P, P′.

And, P′V : PV = R′M : MR by parallels.

Therefore PV. P′V = 2RM.MR′ (since $PV^2 = 2RM^2$)

Hence, PV. P′V : MH'^2 = 2C : D from (5), and so

QV^2 : PV.P′V = D : 2C, a given ratio (since QV = VM + MQ = MH′)

Suppose we take a line p such that D : 2C = p : PP″

Then Q lies on the ellipse $\mathscr{E}$ with diameter PP′, and parameter p[43], whose ordinates are inclined to PP′ at an angle of 45°[44].

Again, since EK is a diagonal of the parallelogram GK′,

GQ.QN = AB.BK′.

Therefore Q also lies on the hyperbola $\mathcal{H}$ with asymptotes KG, KK′, passing through B. Thus Q is determined as the intersection of a given ellipse and hyperbola, so M is given, and H, H′ can be found.

This analysis is indeed ingenious, and one wonders how Diocles arrived at it. In modern terms, he succeeded in reducing the original problem which contained four unknowns, to two equations in terms of two unknowns. The other remarkable feature of this solution is that Diocles uses the equation of an ellipse and applied in oblique conjugation. Not only is it rare to find the ellipse used in geometrical constructions, but Archimedes normally used the equation in orthogonal conjugation. Diocles continues on to give the synthesis, which we shall omit.

This concludes our treatment of the application of conic sections in problem solving in Greek geometry. We have seen how conic sections enabled the ancient Greeks to solve two of their classic problems. Moreover, the ancient Greek geometers saw how useful conic sections were both in replacing neusis constructions and in dividing certain line segments[45]. These two specific applications of conic sections would be called on by Islamic geometers to solve the same problems we have discussed above, as well as other related ones.

---

[43]In the notation of modern conics $p = DD'^2 / PP'$.

[44]The property of the ellipse as given in Apollonius I. 21 is QV^2 : PV.P'V = p : PP'.

[45]The trisection of the angle can be formulated as a neusis construction, as we have seen, and Archimedes' problem involves the construction of a certain line segment.

The Islamic Tradition

Arabic translations of Greek manuscripts began to be done in Baghdad on commission for the early 9th century caliphs Hārūn al-Rashīd and al-Ma'mūn. Euclid's *Elements* was studied and commented upon frequently, and became a basic textbook for the geometers of the Islamic world. The translation of Archimedes' works, such as *On the Sphere and Cylinder* and the (only partly genuine) *Heptagon in the Circle*, provided fertile ground for the revival of the Greek tradition of geometry and considerable stimulus for investigations in the conic sections. These investigations by Islamic mathematicians were greatly inspired by Apollonius' *Conics* , which formed a base for advanced research in geometry.

Although Islamic geometers did not contribute to the development of the theory of conic sections, they were adept at presenting new variants of constructions using known Greek methods. However, they often refined those methods, and occasionally proposed original solutions using conic sections to replace the Greek constructions. Islamic geometers worked on perfecting the inherited tool of conic sections for problem-solving in two ways: first, by seeking out problems, apart from those inherited from the ancient Greeks, that would require the use of conic sections, and second, by finding alternative constructions of the same problem.

Gradually, geometric problems were transformed into algebraic cubic equations, and Islamic geometers constructed roots to these equations. When they began to examine cubic equations with arbitrary coefficients, the emphasis shifted from constructing roots (which was now rather routine procedure) to determining when and why these cubic equations had roots. Through this work, they were able to display the relationships which revealed the unity underlying 'solid' problems, i.e. their geometrical researches led them to investigate the relationship of geometry to algebra and the role of conic sections in the theory of cubic equations.

We begin by surveying solutions to the classical Greek problems of the duplication of the cube and the angle trisection. Afterwards, we will present solutions to problems inspired by the works of Archimedes such as the construction of the heptagon and sectioning a sphere. Finally, we will examine constructions of a few specific cubic algebraic equations, and the subsequent generalization to a theory of cubic equations.

The Cube Duplication

Several constructions of two mean proportionals were transmitted from Greek into Arabic. This problem provided the Islamic geometers with ample opportunity to apply the theory of conic sections they had inherited from the Greeks. In most cases though, they rendered the solutions that had been found by their Greek predecessors. We will survey these solutions, as well as present a construction which is not found in Greek literature, and which is probably of Arabic origin.

The first method, by means of a circle and a hyperbola, appears in many slightly different forms in the Arabic tradition. This construction, probably of Greek origin (Knorr 1989, p. 259), effects Apollonius' neusis construction that was discussed in the section on Greek cube duplications (p. 12). The 9th century geometer Abū Bakr al-Harawī adopts this approach, and he expresses the problem as: " to construct, between two given lines, two lines so that the four straight lines follow proportionately in the same ratio."[46]

His construction can be summarized as follows (Fig. 21):

Draw the two give lines AB, BG at right angles to each other, complete rectangle ABGD, and draw its two diagonals which cut each other at the point E. Draw a hyperbolic section with asymptotes[47] AB, BG that passes through the point D. Draw a circle with centre E and radius EA, it will pass through the points A, B, G, D, and will cut the section at a point Z. Draw the line ZD and extend it to meet the asymptotes in H and T respectively. *

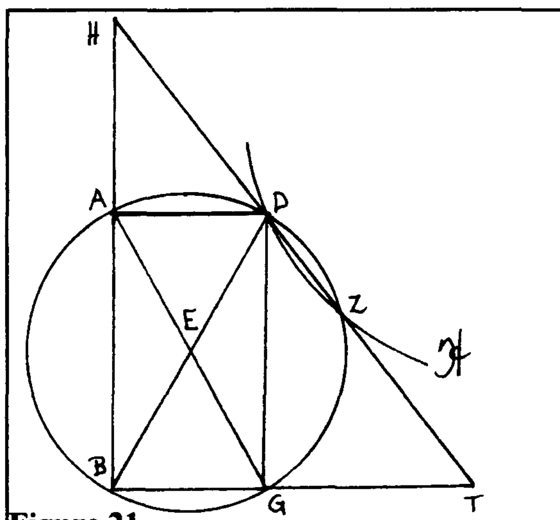

Figure 21

[46]Translation in Knorr 1, p. 251.

[47]Known as 'the two lines which do not fall on the section'.

Then HD = ZT by *Conics* II.8.

Now, ZH. DH = BH.AH since H is external to the circle and ZH, BH cut the circle. Similarly, DT.ZT = TB.TG and, since ZH = DT and ZT = DH, it follows that BH.AH = TB.TG.

Thus, TB : BH = AH : TG,

so TB : BH = AD : AH = AH : TG = TG : DG by similar triangles.

Hence, BG : AH = AH : TG = TG : AB,

and AH, TG are the required lines.

The neusis construction of Apollonius assumed drawing the line HDZT such that HD = ZT, and the proof continued similarly to above. The use of a hyperbola finds the point Z on the circle that will satisfy this property. The proof thus relies on *Conics* II.8 and the secant property of circles.

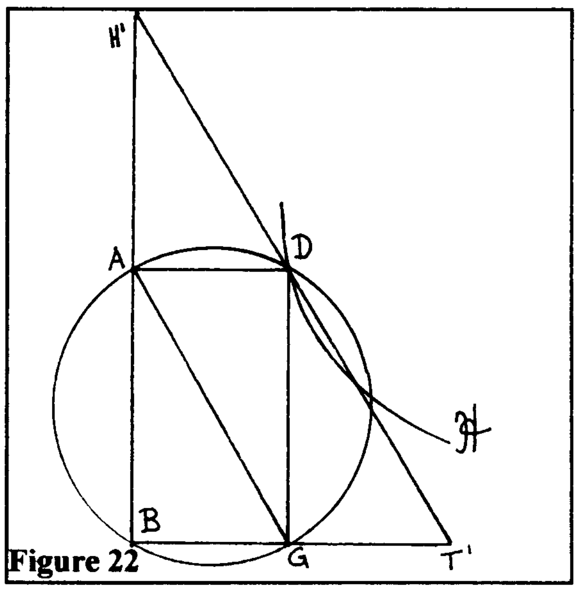

Figure 22

Abū Ja'far al-Khāzin, a 10th century geometer, gives a related method of cube duplication, and cites Abū Bakr as his source (Knorr 1989, p. 262). His construction is identical up to * except that D is specified as the vertex of the hyperbola. He then continues on to completely specify the hyperbola (Fig. 22). He does this by drawing through the point D a line parallel to AG which meets the asymptotes in H′, T′ respectively. Since AGT′D and AGDH′ are both parallelograms, H′D = DT′. If $H'T'^2 = p.2DB$, then 2DB is the diameter of the section and p is the latus rectum. Having specified the hyperbola needed for the solution, al-Khāzin resumes the proof as given by Abū Bakr.

Al-Khāzin ends his discussion with a method of cube duplication using a moving ruler; it is identical to the procedure used by Apollonius. This attests to his awareness of the essential identity of the two methods. Furthermore, in presenting them together, he makes his preference clear for the conic section method, which he call 'geometric'; the other 'instrumental' method is not a valid proof in his view. This echoes the attitude of Pappus and is a prominent feature in al-Khāzin's work. In fact, he expresses the same preference once again in another solution to the cube duplication using the method of Nicomedes. This method had been rendered into Arabic by Thābit ibn Qurra as part of his translation of Archimedes' *Sphere and Cylinder* and Eutocius' commentary on that work.

Thābit ibn Qurra was an important translator in the early period of Islamic science, working in the same circle of mathematicians as the Banū Mūsā in Baghdad at the time of the Caliph al-Ma'mūn. The Banū Mūsā travelled to the Byzantine provinces in order to acquire Greek scientific manuscripts and they were responsible for the translation and propagation of these works. One of their most important works is entitled ***The Book of the Knowledge of the Measurement of Plane and Spherical Figures by the Sons of Moses: Muḥammad, al-Ḥasan, and Aḥmad***. This work included tracts on the trisection of the angle, and the construction of two mean proportionals. Thābit worked on translating many of Archimedes' works, as well as Apollonius' *Conics*, and studied some problems inspired by these works.

In the following construction, al-Khāzin shows how a method via the hyperbola effects the neusis that Nicomedes works out via the conchoid.(Fig. 23)

Draw the given lines AB, BG at right angles, and complete rectangle ABGD.

Half AB, BG at the two points E, Z, and extend GB and DE so that they meet in H. Then BH = BG. Draw ZT perpendicular to BG, and GT(= AE = EB). Draw TH, and draw GK parallel to it. Extend BG to L. Then the required neusis is to draw a line from T to GL whose segment falling between GK and GL is equal to line GT.

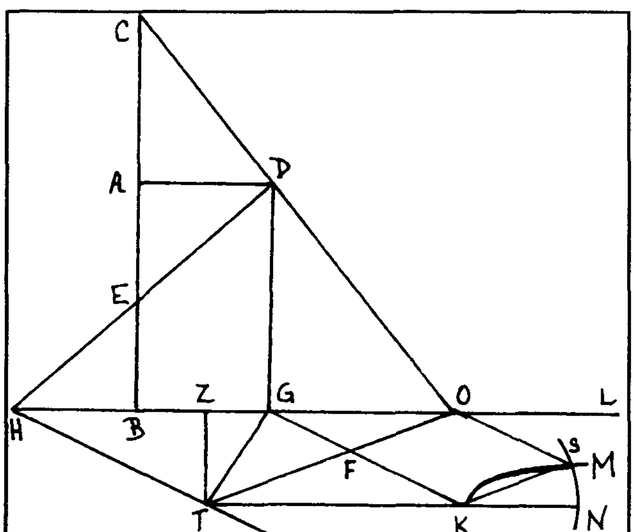

Figure 23

In order to do this, draw from T a line to GK parallel to HG, and let it be TK. Construct a hyperbola KM with asymptotes TH, HG passing through K. Extend TK to N such that KN = TG, and draw a circle with centre K and radius KN, which intersects the hyperbola at (say) S. Draw KS(=TG), and SO parallel to KG. Join TO cutting GK at F, so that

SO is equal and parallel to KF (since SO.OH = KG.GH via *Conics* II.12, and by similar triangles).

Then KS is equal and parallel to FO. Thus the neusis is completed.

Join OD and extend it to meet BA at C; then GO, CA are the required means between AB and BG.

The proof of this follows that of Nicomedes, which we will not include. An important feature of this construction is that the hyperbola is in oblique conjugation for the parallelogram GHTK. This method, which replaces a neusis construction, actually generalizes that of Pappus for the angle trisection[48]. Pappus used a hyperbola in orthogonal conjugation to effect the neusis in the case of a reference figure which is a rectangle rather than a parallelogram. Incidentally, Abū Ja'far presents a similar method for the angle trisection, which we will discuss in the following section.

Naṣīr al-Dīn al-Ṭūsī, who worked in the thirteenth century and is also well- known for his work in mathematical astronomy, adopts a method of the cube duplication similar to that of Abū Bakr (also in Knorr 1, p. 255). He inserts his method of finding two mean proportionals between two given lines as a marginal comment to both *Conics* V, 52 and ***Sphere and Cylinder*** II, prop. 1. Al-Ṭūsī presents a very thorough and rigorous construction; he carefully cites theorems from the *Conics* and shows his expertise in applying them to this problem.

He first considers the trivial case in which the two given lines AB, BG are equal and finds that the mean proportionals will be equal to them.

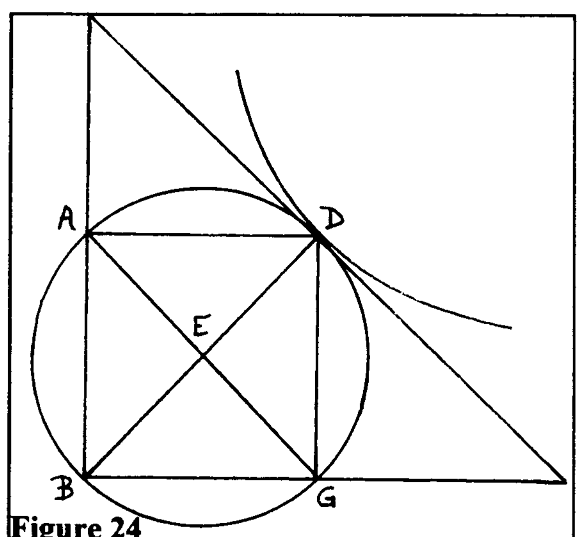

Figure 24

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[48]See p. 16 for Pappus' angle trisection. The required neusis is identical save for the reference figure ADCB.

This can be seen in (Fig. 24), since the circle and the hyperbola will not have a second point of intersection. Next, he takes AB > BG, places them at right angles, completes ABGD, draws the diagonals which intersect at E and constructs the circle with centre E, passing through A, B, G, D. He then draws the same line T′DD′H′ through D, parallel to AG and cutting the circle again at D′, as did Abū Ja'far, but explicitly states that it is tangent to the hyperbola with asymptotes AB, BG passing through D, at the point D (Fig. 25). This allows al-Ṭūsī to establish the existence and the position of the second point of intersection between the hyperbola and the circle. He first argues that the section and the circle cut each other between D and D′. Indeed, if not, the segment DD′ from the arc (of the circle) and the chords in this segment would fall between the section and the line tangent to it. But that is impossible, since, according to *Conics* I.32, no other line can fall between the section and the tangent. Al-Ṭūsī then argues that according to *Conics* IV. 35, since their convexities are opposed, they do not cut each other at more than two places. He proves in the same way as Abū Bakr that the second point of intersection Z determines the two mean proportionals.

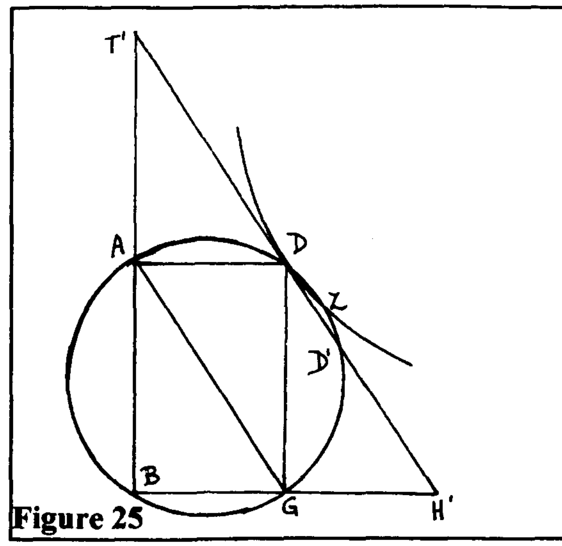

Figure 25

Together with the construction of two mean proportionals by means of a circle and a hyperbola, we find three other constructions of two mean proportionals in the book *Istikmāl* of the Andalusian geometer Yūsuf al-Mu'taman ibn Hūd. Al-Mu'taman, was a mathematician and astronomer, and was King of Saragossa from 1081 - 1085 A. D, when he was assassinated. He has recently become well-known in scholarly circles because of the discovery of his "Book of Perfection" (*Istikmāl*)[49] which unites an impressive amount of scientific literature which must have been available in Saragossa in the late 11th century A. D. The first construction, by means of two parabolas, is the same as in the commentary by Eutocius on Book II of Archimedes' *On*

[49]More information on the division of the *Istikmāl* into species and sections can be found in Hogendijk 3; published sections of the *Istikmāl* include Hogendijk 1, 2.

the Sphere and Cylinder[50]. This construction was known to Arabic geometers such as Abū Ja'far and al-Sijzī through the translation of Eutocius' collection of constructions. Abū 'Abdallāh al-Shannī, a 10th century geometer, gives the synthesis of this construction in his *Disclosure of the Fallacy of Abū'l - Jūd* (Kashf tamwīh Abī'l - Jūd)[51] and says that it was plagiarized by Abū'l-Jūd in his lost work "The book on Geometrical Subjects" *(Kitāb fī - l Handasiyyāt)*. The second construction al-Mu'taman gives, by means of a hyperbola and a parabola, is again the same as another one as found in Eutocius' commentary on Book II of Archimedes' *On the Sphere and Cylinder*[52]. Once again, through references to Eutocius' commentary, we know that this solution was known to a number of Arabic geometers. The fourth construction that al-Mu'taman gives, by means of a hyperbola and a circle, is precisely the one we have discussed above, and most resembles the variation given by Abū Bakr.

Al-Mu'taman's third construction, by means of a circle and a parabola, does not appear in any known geometrical work that was written in the entire Eastern Islamic world. Hogendijk (2, p. 19) argues that this construction was in fact unknown to Eastern Islamic geometers: since they were very fond of finding new solutions to old problems, it is unlikely that such a solution would have vanished without leaving a trace in the literature on conic sections that has survived from this period. Furthermore, since al-Mu'taman was a capable mathematician[53], we may assume that he was the author of this construction.

We now present a paraphrase of this construction: (Figure 26)

We are to find the two mean proportionals between two given segments $AB < BG$. Draw the two given lines at right angles, and through the points A, B, G draw a circle. AG will be the diameter of this circle since B is a right angle. Draw a parabola $\mathcal{P}$ with axis GB, vertex G and parameter GB, and let it meet the circle at the point D.

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[50]See p. 11 for this construction.

[51]See Hogendijk 7, p. 277, (M8).

[52]See also p. 11 for this construction.

[53]Hogendijk notes the mathematical abilities shown by al - Mu'taman in his remarkable simplification in the *Istikmāl* of the 'problem of Alhazen'.

Draw DE (an ordinate) perpendicular to the axis GB. Then ED and EG are the required mean proportionals, that is to say BG : ED = ED : EG = EG : AB.

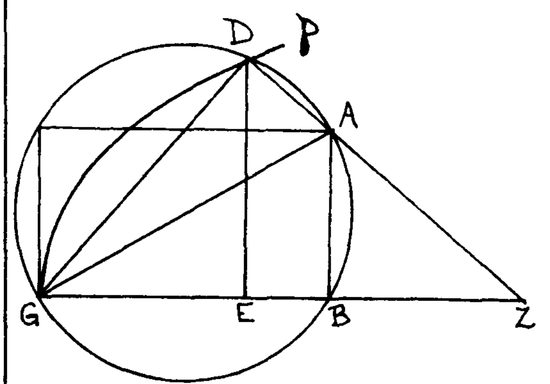

Figure 26

Proof: Draw DA and extend it to meet BG extended at Z. Draw DG.

Since ∠GDA is a right angle (AG diameter of the circle through D),

ZE : ED = ED : EG by similar triangles.

Also ED^2 = BG. EG since D is on the parabola. So,

(1) BG : ED = ED : EG, thus ZE = BG, and so

(2) ZB = EG. But,

(3) ZB : BA = ZE : ED by similar triangles,

thus by combining (1), (2), and (3), we obtain BG : ED = ED : EG = EG : AB.

Al-Mu'taman uses the same parabola in this construction as he does in the first two constructions. And the circle through A, B, G is the same as in his fourth construction. This shows how closely related the four constructions are, though their discoveries spanned more than 1300 years and scores of eminent mathematicians. We will return to this problem in the last section and examine how it was treated as a cubic equation. The read may want to refer to Chart I in the appendix which provides a brief comparison of Arabic and Greek solutions of the cube duplication.

The Angle Trisection

The trisection of the angle was a fundamental problem in Islamic geometry, as it was in classical Greek geometry, for two reasons: it was a great challenge to them since no 'successful' Greek solutions were transmitted to them, and, it became very popular as it drew the attention of many different geometers. We have seen how the Greeks trisected the angle using conic sections, and by means of the method known as neusis. Nevertheless, 10th century Islamic geometers such as al-Sijzī and Abū Ja'far al-Khāzin said that they did not know of successful trisections by the Ancients. They did know of the neusis construction which is used to trisect the angle in the pseudo-Archimedean *Book of Lemmata* through its translation into Arabic by Thābit ibn Qurra. However, at least in the 10th century, they did not consider the neusis as a legitimate geometrical construction, and attributed the first 'acceptable' angle trisection to Thābit ibn Qurra.

As we will see, Thābit's construction closely resembles the trisection found in Book IV of Pappus' *Collection*[54], as well as the one in the *Treatise by Aḥmad ibn Shākir on the Trisection of the Angle*[55]. This latter is not surprising, for we know that Thābit was a protegé of the three brothers known as the Banū Mūsā (one of whom was Ahmad ibn Mūsā ibn Shākir). What is surprising is that as far as we know, Books 1-7 of the *Collection* were not translated into Arabic. Hogendijk (8, p. 418) has suggested that Ahmad's construction was a translation of an unidentified Greek text closely related to Pappus' construction, and that Thābit wrote his solution to simplify the concise and difficult one by Aḥmad. As a preliminary, and for reference purposes, we present the neusis construction found in the *Book of Lemmata*[56]. Following that, we will describe both Ahmad and Thābit's solutions. Finally, we will investigate a series of other angle trisections found in Arabic literature using conic methods. Although we are only concerned with solutions restricting themselves to conic methods, they were not the only methods used by Islamic geometers to

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[54]This construction can be found on p. 16, above.

[55]See Hogendijk 9, p. 38 (25).

[56]Heath 3, pp. 301 - 318.

trisect the angle. The neusis construction was widely used; indeed, we still find this method among the many alternatives given by the geometer al-Bīrūnī in his "Canon" (*Al-qānūn al-Mas'ūdi*)[57], in the first half of the 11th century.

Proposition 8 of the *Book of Lemmata* contains a neusis construction which effects the trisection of the angle ADE (Fig. 27): Draw a circle with centre D, intersecting DA in A and DE in E. Extend ED and insert a line segment GB, equal to ED (the radius of the circle), between the circle and ED extended such that GB verges toward A. Then ∠BDG = ⅓∠ADE. This neusis is historically important since it arises in 'the problem of Alhazen', and is treated in its algebraic form by later mathematicians.

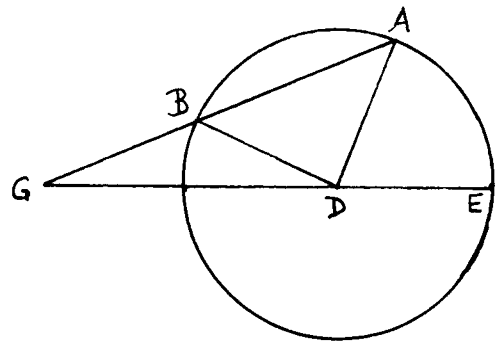

Figure 27

The solution of Aḥmad ibn Mūsā is divided into the same three parts as Pappus' solution: 1) a lemma on the construction of the hyperbola; 2) a lemma effecting a required neusis[58]; 3) the trisection via the neusis. We point out the noteworthy differences by referring to Figure 28, which is a mirror image of Pappus' Figure 9. Firstly, Aḥmad does not give the analysis of the problem, but effects the same general neusis as Pappus (that is, to draw a straight line

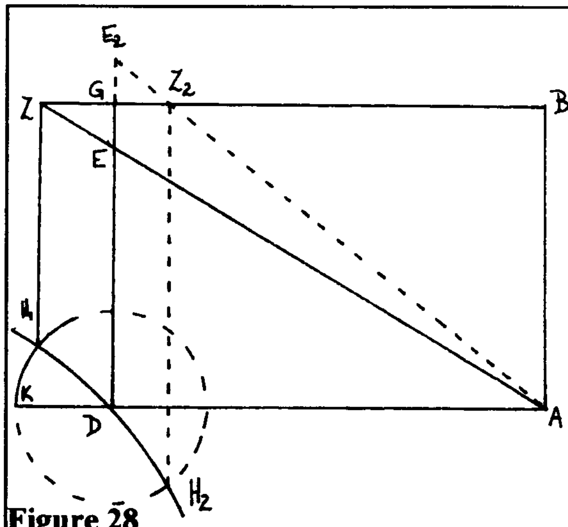

Figure 28

[57]See for the Arabic text Al-Bīrūnī, *Al-Qānūn al-Mas'ūdī*, edited with introduction by M. Nizam ud-Din, H.J.J. Winter, and Hasan Barani, Hyderabad 1954 - 1956.

[58]It is not the same neusis as in the *Book of Lemmata*; it has been described on p. 15, above, in conjunction with Pappus' construction.

from A to BG extended, such that the part of it which falls between GD and BG extended is equal to a given line M). Secondly, the point K is completely determined by Aḥmad, as lying on AD extended. Pappus says only: let DK = M. More importantly, Ahmad draws a segment of a circle KH_1 instead of the circumference of a circle. This is more correct, for a complete circumference would intersect the hyperbola in two points H_1 and H_2, and this would lead to the construction of two straight lines, as indicated by the dotted lines in Figure 28.

The next difference is in the proof of the neusis construction, and is a very subtle one. Pappus proceeds by establishing the equality of the areas of the rectangles BZ, ED and BG, GD to show that ZB : BG = GD : DE. On the other hand, Aḥmad uses an argument of similarity based on: (1) AB is parallel to EG, hence ZB : BG = ZA : AE, and (2) ZG is parallel to AD, hence ZA : AE = GD : DE, to show that ZB : BG = GD : DE.

Both geometers give the trisection of the acute angle using the above neusis construction[59]. Aḥmad remarks that the trisection of the right angle is easy; he then shows how to trisect the obtuse angle ABG (Fig. 29) by drawing BD at right angles to BG, and trisecting both the acute and right angles to obtain angle EBZ.

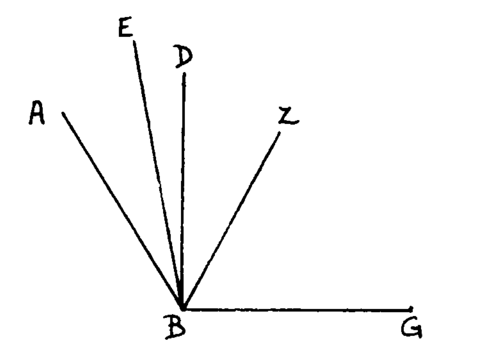

Figure 29

The following trisection, found in a text called *The Trisection of the Rectilineal Angle, Composed by Thābit ibn Qurra*[60], follows the same three part structure. Thābit first provides the construction of hyperbola in oblique conjugation as in *Conics* II.4. Next, he formulates the neusis construction in a slightly different manner (Fig. 30):

"The surface ABGD parallel of sides[61], and the side BG has been extended in its straight line in the direction of G and we do not make for it an extremity.

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[59]This is shown on p. 36, above. The method of Aḥmad is identical.

[60]See Hogendijk 9, p. 38 (24).

[61]That is, the parallelogram ABGD, however; in his diagram, Thābit actually draws a rectangle.

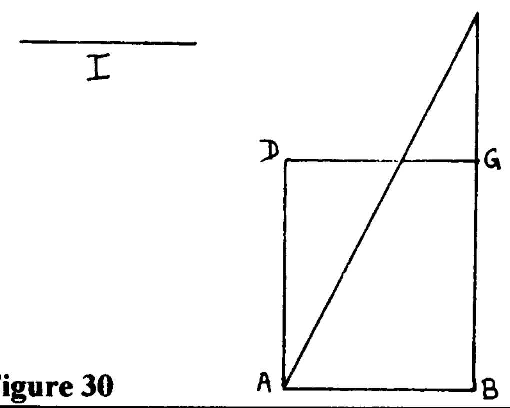

Figure 30

And we wish to extend from point A a straight line which is inclined toward it so that what subtends the angle G from it is equal to the given line I". (Knorr 1, p. 278)

Thābit views the given line segment as subtending the given angle, whereas we saw that Aḥmad (like Pappus) conceived it as drawn across two different lines. Also, Thābit's statement of the neusis is more general as it applies to a given parallelogram. Of course, the context of this particular problem of the angle trisection can be specialized to the rectangle, as we have seen in the previous solutions. The actual construction and proof, by means of a circle and a hyperbola, follows exactly the same lines as above, however; Thābit quotes *Conics* II.12 in full, the theorem he appeals to in his proof[62], before establishing the required equality of areas.

Two other solutions of problems, leading to the trisection of the angle, are present in the *Optics*[63] of Ibn al-Haytham. He solves both these problems as lemmas to a more general one, that is: to construct the points of reflection on a (convex or concave) circular mirror, given the positions of the eye and the observer. Just as Diocles' researches into the burning mirrors provided him with fertile ground for studying and using conic sections, Ibn al-Haytham used conic sections in order to solve problems arising from his studies in the field of optics. His experience in solving other geometric problems by means of conic sections, such as the construction of the heptagon, must have been extremely useful in his solution to the famous "problem of Alhazen".

In Book V, Prop. 33 of the *Optics*, Ibn al-Haytham effects the following neusis construction, by means of a circle and a hyperbola: (Fig. 31)

From a given point A on a circle ABG, draw a line that cuts the diameter extended in E

[62]Again, refer to p. 16 to see when *Conics* II, 12 is invoked by Pappus.

[63]See Sabra (1982) for a translation of the relevant parts.

and the circumference of the circle in G in such a way that ED equals a given line. This implies a construction of Proposition 8 of the *Lemmata,* for the give line equal to the radius of the circle, and Ibn al-Haytham uses the same construction as in Pappus' *Collection.* However, by using a different property of the hyperbola (from *Conics* II.8), Ibn al-Haytham arrives at a simpler proof. The solutions of Pappus, Thābit, and Aḥmad, as we have discussed above, are all based on *Conics* II.12, and require a longer proof.

In Prop. 34 of the same book, Ibn al-Haytham effects a slightly different neusis, also by means of a circle and a hyperbola: (Fig. 32) From a given point A on a circle ABG, draw a line that cuts the diameter BG in E and the circumference in D in such a way that ED equals the given line. This problem also leads to the trisection of the angle for the given line equal in length to the radius of the circle. In order to make this clear, suppose ED = radius of the circle. Let M be the center of the circle, and let MF be a perpendicular to GB, which cuts ED produced in F. If A is not on FM produced, the problem has two solutions, and we consider the solution in which $E \neq M$ (otherwise we are just joining A to the center M). Since angle EMF = 90° and ED = DM, we have DF = DM. Then it can easily be shown that the angle between AM and FM produced is three

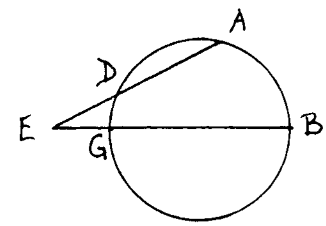

Figure 31

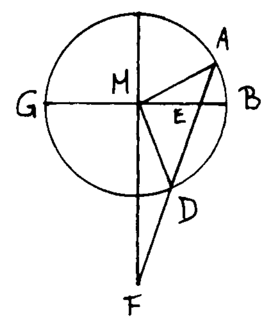

Figure 32

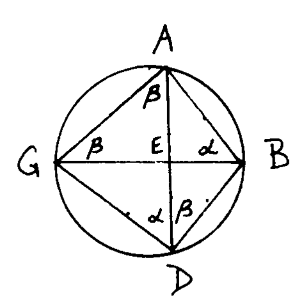

Figure 33

times the angle F.

Proposition 34 is therefore more general than the problem of the angle trisection; however, since the construction of Ibn al-Haytham introduces some new concepts, we present his solution. He uses Figure 33 in order to construct Figure 34, in which he constructs a certain angle that he will use, in turn, in Figure 33 to effect the neusis construction.

Let HZ (in Fig. 34) be the given line. On either side of HZ, construct angles α and β equal to $\angle$ABE and $\angle$AGE respectively (from Fig. 33). Complete parallelogram HKZT(back to Fig. 34), and draw through T the branch of the hyperbola $\mathcal{H}_1$ with asymptotes KH, KZ. Then, with T as center and a radius equal to BG, draw a circle that may or may not cut the opposite branch of the hyperbola $\mathcal{H}_2$. Suppose that the circle and $\mathcal{H}_2$ do meet, say at the point S. Join TS, cutting the asymptotes at F and Q; and, through Z, draw LZM parallel to TS, and, like TS, cutting both asymptotes. LZM will cut the extension of HT, say in M. Finally, (in Fig. 33) draw GD at an angle to BG equal to $\angle$MLH, and join BD. Considerations of similar triangles entail the equality of DE and the given line HZ.

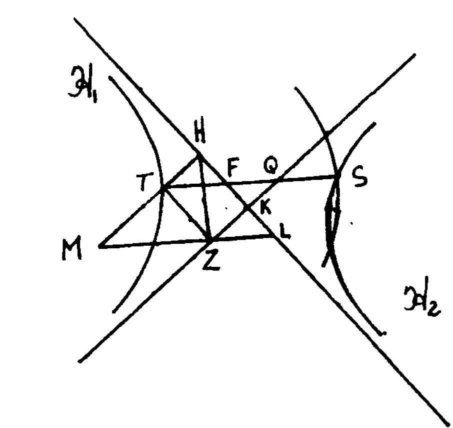

Figure 34

Ibn al-Haytham states that from T on $\mathcal{H}_1$, it may not be possible to draw more than one line that reaches $\mathcal{H}_2$. This, of course, would be the case when the circle touches $\mathcal{H}_2$ at a point. He also notes that in some cases two such lines may be drawn, and, further, for the construction of the required line to be at all possible, it is necessary that BG must not be shorter than the shortest line that can be drawn from T to $\mathcal{H}_2$. He refers the reader to Propositions 34 and 61 of Bk. V of the *Conics* on how this shortest line should be determined.

In this construction, Ibn al-Haytham uses three interesting techniques. The first is the construction of one angle equal to a given one, this can be found in the *Elements* I, 23. The second is his explicit use of both branches of the hyperbola; although geometers were aware of the two branches of a hyperbola, they rarely used them in their constructions. The third remarkable feature of Ibn al-Haytham's construction is his appeal to Book V of the *Conics* , which deals with *maxima* and *minima* problems. He displays his geometric rigour by considering the necessary conditions for a solution, and, in doing so, appeals to propositions that no other geometer had previously used.

It is noteworthy that al-Mu'taman, in his *Book of Perfection*, also treats the "problem of Alhazen". He presents a simplification and generalization of propositions 33 and 34 of Ibn al-Haytham by combining the two cases of the line equal to the given line being either outside (Prop. 33, Fig. 31) or inside (Prop. 34, Fig. 32) of the circle. His general solution is obtained by intersecting a circle with two branches of a hyperbola.

Al-Sijzī worked almost a century after Thābit, and is responsible for transmitting a series of related methods of the angle trisection. They appear in a tract *Treatise on the division of the angle into three equal parts*[64] where he reviews lemmas required to trisect the angle devised by himself and other geometers. In the preface to this tract he affirms that the problem was first solved by Thābit ibn Qurra and after him by Abū Sahl al-Kūhī. Since the lemma he attributes to al-Kūhī is very similar to his own, al-Kūhī probably invented this method and passed it on to al-Sijzī (Knorr, Hogendijk). In fact, these two geometers worked together at an observatory in Shirāz in the same time period that al-Sijzī first copied the method. We begin by presenting al-Kūhī's method as found in his *On the determination of two means between the two lines and the division of the angle into three parts by Abū Sahl Wayjan b. Rustam al-Kūhī*[65], and then by discussing similar lemmas which al-Sijzī attributes to himself. The lemma of al-Kūhī is as follows: (Fig. 35)

Let GD, DA be two given lines containing the angle D; draw lines GB, BA such that

BA = BG and GD : AD = AD : DB.

Although it is not evident from the above lemma, we will see that AB is in fact arbitrary. Al-Kūhī

[64]See Hogendijk 9, p. 36 (12).

[65]English translation by A. Sayili "The trisection of the angle by Abū Sahl Wayjan ibn al-Kūhī". *Belleten* **26** (1962), p. 693 - 700.

will be trisecting the complement of angle GDA, the length of AB is arbitrarily set by the choices of lengths for the given lines GD and DA. The solution begins with the synthesis, and al-Kūhī trisects the angle E directly, using the lemma. (Fig. 35)

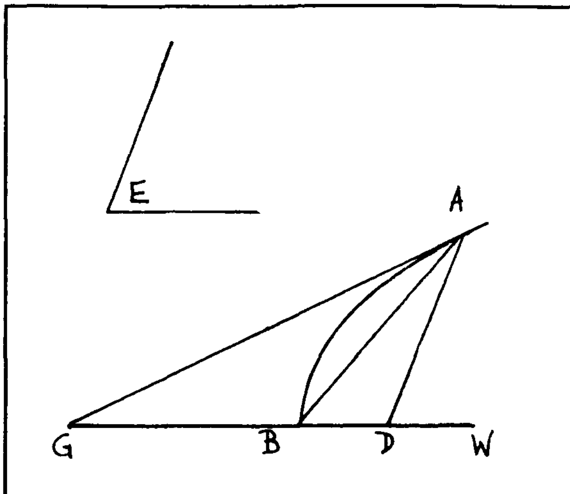

Figure 35

Draw a hyperbolic section, AB, such that its latus rectum is equal to its latus transversum BG, and such that both are equal to the chord AB, and let the angle of arrangement[66] ADW be equal to the angle of E. (This construction is demonstrated in *Conics* I.55)

Since GD.DB : AD^2 = AB : BG[67] via *Conics* I.21 (this is the ratio property for the hyperbola), and AB = BG, it follows that GD.DB = AD^2.

This is the construction required in the lemma, now al-Kūhī proves angle E has been trisected.

Then ∠ABD = ∠GAD (triangles GDA, ADB are similar since GD : AD = AD : DB).

And ∠DBA = 2 ∠AGD, because AB = BG, so that ∠GAD = 2 ∠AGD.

Therefore ∠GAD + ∠AGD = 3 ∠AGD.

But ∠ADW = ∠GAD + ∠AGD, so ∠ADW = 3 ∠G, therefore ∠G = ⅓∠ADW which is the angle of ordinates of the section.

Hence ∠AGB = ⅓∠E.

---

[66]The angle between the diameter and the corresponding ordinate. This angle does not depend on the choice of the ordinate, since all ordinates corresponding to a certain diameter are parallel.

[67]Al-Kūhī actually gives the incorrect ratio by inverting the right hand side. (Knorr 1, p. 308)

This version of the angle trisection is comparable to Pappus' second method of trisection[68]. Pappus gives the analysis of constructing the triangle ABG (Fig. 36), of given base AG, whose angles at A and G are in the ratio 1 : 2; he claims that B lies on a hyperbola whose latus rectum is three times its latus transversum. Al-Kūhī's construction is based on a similar triangle, but with the side AB given. It is possible he derived his solution from a Greek prototype related to Pappus, although Hogendijk (9, p. 13) believes it was an original invention.

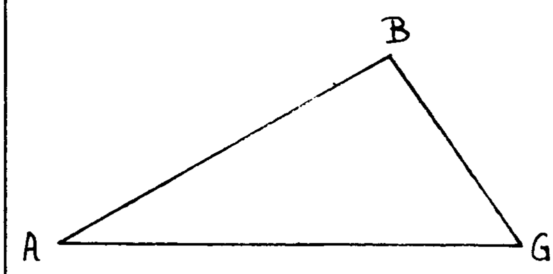

Figure 36

Al-Sijzī , in his *Treatise on the division of the angle into three equal parts*[69], presents all the problems to which Islamic geometers reduced the trisection of the angle. He shows, one by one, that they are all equivalent to the following problem: (Fig. 37) given the angle KCD, extend KC to A such that CA = CD and from D draw a line segment DE to AC such that $DE.EC + EC^2 = CD^2$. Then angle CDE is one third of angle KCD. His construction of this problem is virtually identical to the one of al-Kūhī we have just seen, using exactly the same hyperbola.

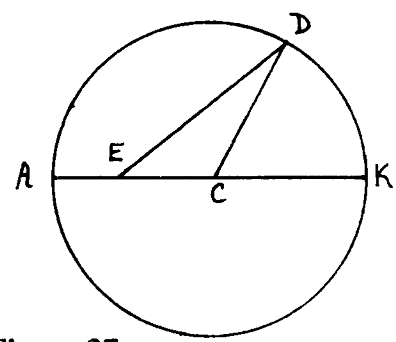

Figure 37

Al-Sijzī also gives another construction by means of conic sections of a problem, equivalent to the trisection of the angle that was invented by al-Bīrūnī. Al-Bīrūnī had proved that: (Fig. 38)given an isoleces triangle ABC, and a point D on the base BC

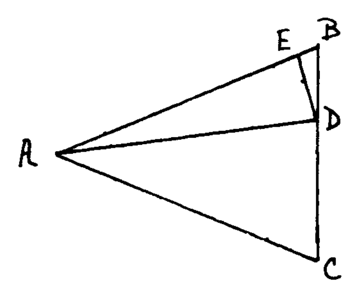

Figure 38

[68]See p. 16, above.

[69]See Woepcke (1851) pp. 117 - 127 for a partial translation of this treatise.

such that AB : BD = AD : DE, (so we have E on AB and AD = AE), then ∠BAD = ⅓∠BAC. It is interesting that Al-Sijzī uses here the same circle and the opposite branch of the hyperbola used by al-Kūhī; the method therefore is very similar.

Finally, there is solution by Abū'l-Jūd of a problem of al-Bīrūnī, which leads to the trisection of the angle[70]. It is very similar to one of the problems discussed by al-Sijzī, but the construction involves the intersection of a parabola and a hyperbola. Al-Bīrūnī proposes this problem, but is unable to solve it: (Fig. 39) given a line BC and a point A not on the line, to draw a line AD from A to BC such that $AD.BC + BD^2 = BC^2$. We present only Abū'l-Jūd's constuction, and omit the details of the proof.

Draw AL perpendicular to BC, and WB also perpendicular to BC such that WB = BC. Draw the parabola $\mathcal{P}$ with vertex W, axis WB and parameter BC, and the hyperbola $\mathcal{H}$ through A with transverse axis AL and parameter 2.AL. Let the two conics intersect at a point Z. Draw ZD perpendicular to BC, then D is the required point.

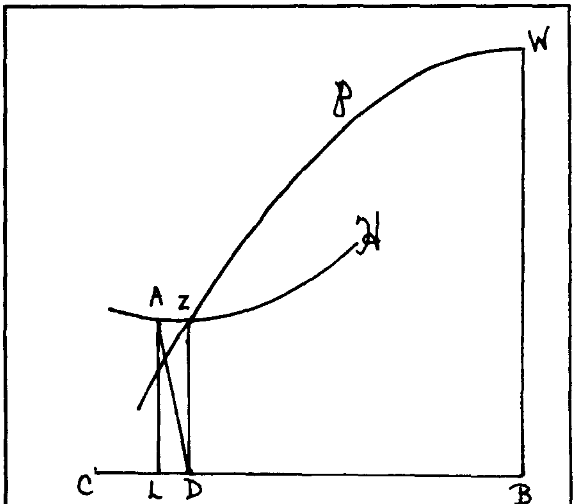

Figure 39

Abū'l-Jūd notes that if the conics intersect twice, and perpendiculars are drawn to BC, then both points on BC will satisfy the given conditions. Furthermore, it is possible that there be no solution if for every point D on BC we have AD > BC because then $AD.BC + BD^2 > BC^2$.

This concludes the section on the trisection of the angle. The plethora of solutions and related problems attests to the significance of this famous problem in Islamic mathematics. In addition to its usefulness in the 'problem of Alhazen', the trisection of the angle also enabled geometers to construct regular n-gons with n = 3k if they were able to construct regular k-gons.

[70]This solution can be found in Woepcke (1851), pp. 114 - 116.

The Construction of Regular Polygons

The ancient Greeks showed some interest in the construction of regular polygons, for instance: Euclid showed how, by means of ruler and compass, to construct both a regular pentagon and a regular 15-gon in a circle (*Elements* IV.11), and Archimedes worked on the construction of the heptagon. We will present a problem related to Euclid's on the construction of a pentagon that Islamic geometers formulated, that is, the construction of an equilateral pentagon in a given square, a problem which may be solved by conic sections. Of course, the construction of the hexagon in a circle is trivial and is found in Euclid's *Elements* Book IV. However, the construction of the heptagon in a circle proved to be more complicated. Although Archimedes had solved this problem by means of a neusis construction, it was not until 10th century, when Islamic geometers attacked this problem, that a solution was found by means of conic sections. With the twelve different solutions they constructed, this problem offers us many interesting applications of conic sections. We will also look at solutions by Islamic geometers to the problem of constructing the nonagon. This problem follows immediately from the construction of an equilateral triangle and the trisection of the angle, but other methods were found by the ever-diligent geometers of the Islamic world.

The Pentagon in a Square

The problem of inscribing an equilateral pentagon in a given square was not, to our knowledge, formulated by the ancient Greeks. A combination of problems found in the *Elements* leads to the construction of an equilateral and equiangular pentagon in a given square (Hogendijk 7, p.102). However, this can only be done with four of the five angular points on the square. By dropping the requirement of equiangularity, Abū Kāmil was able to solve the problem in the early tenth century, and his solution to the problem *to find the side of an equilateral pentagon constructed within a square of side 10* is found in the second part of his famous work on algebra *On the Pentagon and Decagon*[71]. He solves this case (Fig. 40) by algebraic reasoning. The more difficult case, involving a quartic equation, was solved by al-Kūhī in the late tenth century (Fig. 41). Al-Kūhī, considered the best geometer of his time, also supervised astronomical observations in an observatory in Baghdad, and worked as we have said with geometers such as al-Sizjī and al-Bīrūnī.

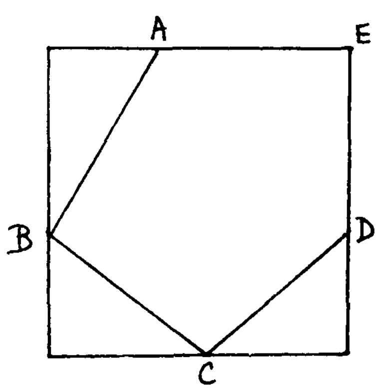

Figure 40

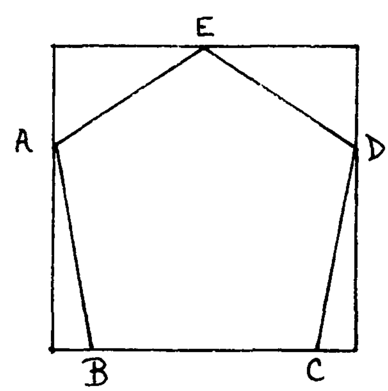

Figure 41

Al-Kūhī's construction is remarkable because it contains the proof of the focus-directrix property of the hyperbola with eccentricity $e = 2$. The focus-directrix property of the parabola was known to Diocles, and Pappus used the focus-directrix property of the hyperbola in his

[71]This is the second part of Abū Kāmil's *Algebra;* several modern translations of this work have been made, see Lorch 1, p. 215 for a complete list.

solution to the trisection of the angle. However, Apollonius does not mention this property in his *Conics*, and there are no other occurrences of it in any text by a medieval author.

Al-Kūhī divides his solution into an analysis and a synthesis, each of which contain three propositions. The first two propositions reduce the initial problem to one of constructing in a given rectangle ABGD (which is half the original square) three lines AE, EK, KD such that AE = 2EK = KD, and EK is parallel to AB (Fig. 42). This problem is solved in proposition 3, where K is found by means of intersecting two hyperbolas. Al-Kūhī shows that the point K lies on two different hyperbolas by deriving their fundamental properties, and then he identifies the elements needed to construct them. The details of al-Kūhī's analysis are as follows, and are divided into two parts:

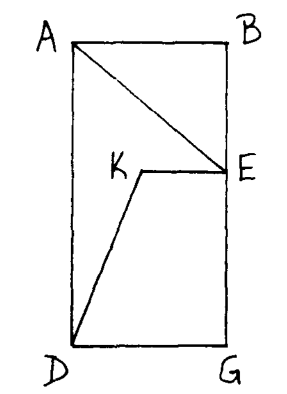

Figure 42

(1) (Fig. 43) Assume the construction done and drop KL perpendicular to AB.

Then $AE^2 = BE^2 + AB^2 = LK^2 + AB^2$. But AE = 2EK by assumption, and EK = BL, therefore, $KL^2 + AB^2 = 4BL^2$ (α).

Bisect AB in N, take S on AB extended such that BS = ½AB, then

$4BL^2 - AB^2 = 4(BL^2 - BN^2) = 4(BL + BN)(BL - BN) = 4LS.LN$. (β)

Combining (α) and (β) we obtain

$KL^2 = 4LS.LN$, i.e.

$KL^2 : (LN.LS) = 4SN : SN$.

This is precisely the fundamental property of a hyperbola $\mathcal{H}$ with vertex N, transverse axis SN and latus rectum equal to 4SN. Since the rectangle ABGD is given, so are the points S and N, and therefore $\mathcal{H}$ can

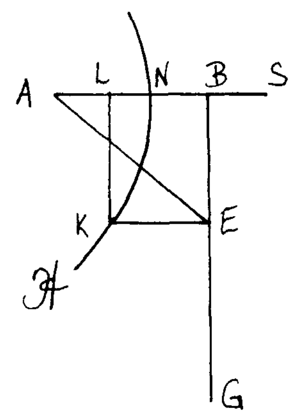

Figure 43

be found by *Conics* I.54.

(2) (Fig. 44) Assume the construction done and

drop KM perpendicular onto GD, extend it to meet AG in O.

Since AD = 2DG, then by similar triangles MO = 2MG, and since DK = 2KE, then MO = DK.

Therefore $OM^2 = DK^2 = DM^2 + MK^2$. (γ)

Extend KM to W such that MW = MK, then

$OM^2 - MK^2 = (OM - MK)(OM + MK) = OK.OW$ (δ)

Combining (γ) and (δ) we obtain $DM^2 = OK.OW$ and,

$OK.OW : OA^2 = DM^2 : OA^2$ (ζ)

Let C be on AG such that CF is perpendicular to GD and CF = FD.

Then CF = 2FG since AD = 2DG and AD is parallel to CF, therefore FD = 2FG or GD = 3FG.

Hence F is known.

Since AD is parallel to OM is parallel to CF, then $DM^2 : OA^2 = DF^2 : CA^2 = FC^2 : CA^2$, and (ζ) becomes

$(OK.OW) : OA^2 = FC^2 : CA^2$ (λ)

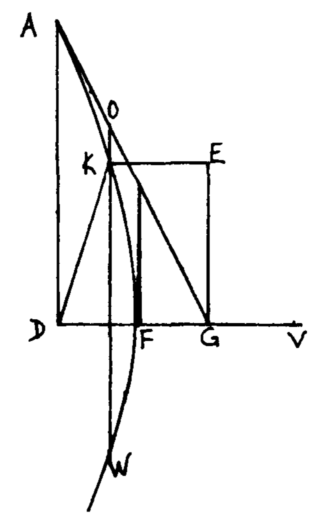

Figure 44

Al-Kūhī uses this relationship to show that K, W lie on a conic section by using the converse of *Conics* III. 16 which states: if AC, CF are tangent to a conic section at A, F and if OKW is parallel to CF intersects AC in O and the conic in K, W, then (λ) holds.

Therefore K, W lie on a conic $\mathcal{J}$ which is tangent to FC at F and to AC at A.

There are two more things that al-Kūhī must do: he must specify the conic section; and he must find the information necessary to construct the conic section. Firstly, he uses *Conics* II.7 to establish that MF is the axis of the conic section.

Since KW is perpendicular to FM and is bisected by FM, and since KW is parallel to the tangent FC at F, it follows that MF is the axis.

This allows him to specify the conic section:

By *Conics* I.35, if the conic section is a parabola then GF = FD, but GF = $\frac{1}{2}$FD, therefore $\mathfrak{J}$ is either an ellipse or a hyperbola.

Then, for transverse axis FV, we have GF : FD = GV : VD by *Conics* I.36.

But GF : FD = 1 : 2 = GV : VD.

This is only possible if G lies between F and V,

hence $\mathfrak{J}$ is a hyperbola with transverse axis FV.

Lastly, al-Kūhī determines the value of the parameter *p*, which enables him to construct the hyperbola $\mathfrak{J}$ using *Conics* I.54.

By the fundamental property of the hyperbola, AD^2 : (DF.DV) = p : VF.

Since AD =DV and DF = $\frac{2}{3}$DG = $\frac{1}{3}$AD, then p = 3VF.

And since both T, F are known, we have sufficient information to construct $\mathfrak{J}$.

In this proposition, al-Kūhī has proved that the curve satisfying the focus-directrix property with eccentricity 2 also satisfies the fundamental property of the hyperbola. In proposition 4 of the synthesis, al-Kūhī shows the reverse, thereby proving the equivalence of the two properties. In addition, al-Kūhī has shown considerable creativity in solving this geometrical problem, and skill in applying the theorems of the *Conics*.

The Regular Heptagon

The only extant Greek exact construction of the regular heptagon has been preserved in a 9th century Arabic translation of the *Book of the Construction of the Circle, Divided into Seven Equal Parts*, attributed to Archimedes[72]. This construction is effected by means of a moving ruler using a method similar to a neusis, in order to construct a certain line segment which is used to find a side of the heptagon. Many of the Islamic geometers from the 10th and 11th century considered such a construction of the regular heptagon inadmissible, and began to seek constructions by means of ruler, compass and conics only. Of the twelve constructions by Islamic geometers which have come down to us, we will discuss four constructions that represent the different methods used.

These methods were greatly influenced by the Archimedean construction found in Propositions 17 and 18[73] of the *Book of the Construction of the Circle, Divided into Seven Equal Parts.* Proposition 17(Figure 45) is a construction of point K on a given segment AB and point Z on AB extended such that

(1) $AB.KB = ZA^2$, and $ZK.AK = KB^2$.

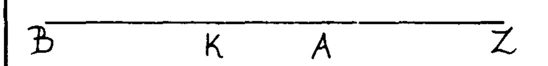

Figure 45

The first step is to draw the square ABDG, its diagonal BG, and to extend BA to E (Figure 46). Next, one places a ruler with one end on D such that it intersects BG, AG and BA extended, at the points T, H and Z respectively, and then one somehow determines a position of the ruler such that the area of $\triangle ZAH$ equals the area $\triangle GTD$ (henceforth written as $\triangle ZAH$ =

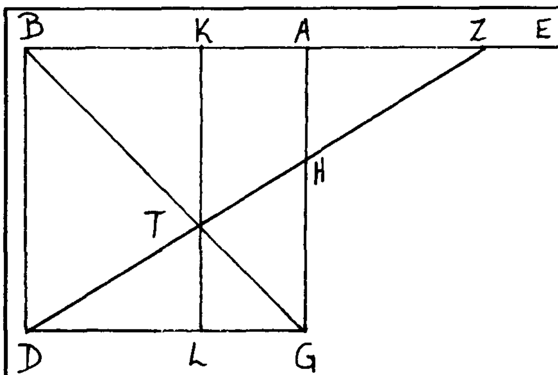

Figure 46

[72]The book as it stands is not considered to be entirely Archimedes' work.

[73]The question of Archimedes' authorship of these two propositions is unsettled. See Hogendijk 7, p. 212, 213.

△GTD). The construction of the straight line DZ resembles the neusis constructions we have seen; however, none of the three segments ZH, ZT and HT has a given length. Once DZ has been found as required, one draws KTL perpendicular to AB, and Z, K are the desired points. The proof of this is easily deduced by considering the similar triangles ZAH, DLT and ZKT, DLT.

Proposition 18 is the construction of a circle divided into seven equal parts using the line segment ZAKB as constructed above.

One constructs the triangle AKE (Figure 47) such that AE = AZ, and KE = KB. The text does not make it clear why this construction is possible, however, according to *Elements* I.20-22, a triangle AKE can be constructed from three segments ZA, AK, KB if and only if each of the segments is smaller than the sum of the two remaining ones. These conditions can be proved from equation (1) above. Now one joins ZE and EB, and circumscribes a circle about triangle ZEB and extends EA and EK to meet the circle in G and D. Then arcs ZG, GD and BE are each one-seventh of a complete circumference, and arcs DB and ZE are two-sevenths of it. By bisecting arcs DB and ZE in P and Q, the circle has been divided into seven equal parts. The proof consists of two parts: first, one shows that arc ZG = arc EB = arc GD; secondly, one shows that arc ZE = arc DB = 2arc EB. Thus the complete circumference of the circle is seven times arc EB.

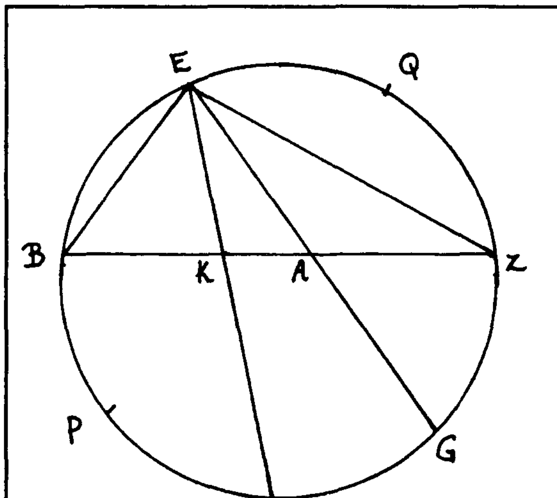

Figure 47

One approach used by Islamic geometers to find an admissible construction of the regular heptagon was to discover an admissible construction of a line DTHZ in a square ABDG such that △GTD = △ZAH. The geometer Abū Ḥāmid Aḥmad ibn Muḥammed ibn al-Ḥusayn al-Ṣaghānī succeeded in doing this by means of the intersection of two hyperbolas. He claims to have discovered this solution in 970 A.D. which he encloses in a letter to the prince 'Aḍud al-Dawla[74]. (Figure 48)

Extend DG towards C such that GC = DG. Draw two "opposite hyperbolas" $\mathcal{H}_1$ and $\mathcal{H}_2$ through D and C respectively, with asymptotes GB and GA. Draw a hyperbola $\mathcal{H}_3$ through G with asymptotes BD and BA. The branch of the hyperbola $\mathcal{H}_2$ intersects the hyperbola $\mathcal{H}_3$ in a point F between DC and BA since it intersects the asymptote BA of the hyperbola $\mathcal{H}_3$ through G. Draw FZ perpendicular to A extended. Draw DZ cutting BG at T and AG at H. Then $\triangle$GTD = $\triangle$ZAH.

This construction is interesting for two reasons: both branches of the hyperbola, $\mathcal{H}_1$ and $\mathcal{H}_2$, are drawn and their asymptotes are not perpendicular to each other. The term hyperbola, in the ancient sense, meant one single branch of what is nowadays the hyperbola. For example, Abū'l-Jūd refers to al-Ṣaghānī's solution as a construction by means of "three hyperbolas"[75]. Although geometers were well acquainted with the properties of the two branches of the hyperbola, they rarely used them both in their constructions.

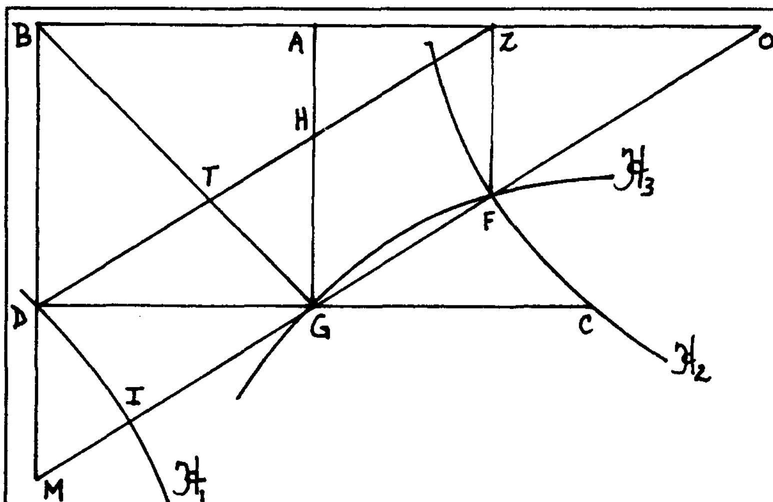

Figure 48

<u>Proof</u> Draw MGFO such that M lies on BD extended and O on BA extended.

Then MG intersects $\mathcal{H}_1$ in I. Since BO and BM are asymptotes of $\mathcal{H}_3$,

FO = GM (*Conics* II.8).

But $\triangle$MDG is similar to $\triangle$FZO, since MD is parallel to FZ and DG is parallel to ZO. Hence DG = ZO, whence DGOZ is a parallelogram. Since GH is parallel to ZF, GHZF

[74]See Hogendijk 7, p. 277 (M9).

[75]Hogendijk 7, p.223

is also a parallelogram, and therefore ZH = FG.

Also, since F and I are points on the opposite hyperbolas $\mathcal{H}_1$ and $\mathcal{H}_2$ and the straight line FI passes through their centre[76] G,

FG = IG (*Conics* I.47).

And since DH intersects the asymptotes GB and GA of $\mathcal{H}_1$ (passing through I and D, with centre G) in T and H, and DH is parallel to IG,

$IG^2 = DT.DH$ (*Conics* II.11).

It follows that $ZH^2 = DT.TH$.

Since $\triangle DGH$ is similar to $\triangle HAZ$, we have ZH/DT = DG/ZA and $\angle GDT = \angle HZA$, so

$\triangle GTD = \triangle ZAH$ (*Elements* VI.15).

The construction attributed to Archimedes together with this admissible construction of the line DTHZ constitute an admissible construction of the heptagon. Three other Islamic geometers succeeded in finding admissible constructions of the line DTHZ by means of conic sections, and we briefly summarize them below. In the late 10th century, Abū'l-Jūd constructed the line in the square ABGD by means of intersecting a parabola and an equilateral hyperbola; this construction is in the *Book of the Construction of the Heptagon in the Circle*[77]. The geometer Kamāl al-Dīn ibn Yūnus, in the late 12th or early 13th century, constructed it by means of intersecting two equilateral hyperbolas, and uses an inside/outside[78] argument to determine the position of their point of intersection. Ibn al-Haytham, who actually believed that Archimedes possessed an admissible construction of DZ by means of conic sections which wasn't included in the treatise, used the intersection of two parabolas.

The salient feature of Ibn al-Haytham's construction, as found in his *Chapter on the Lemma for the Side of the Heptagon*[79], can be seen in his formulation of the problem: (Figure 49) to

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[76]The point which divides a diameter of a hyperbola into two equal parts.

[77]See Hogendijk 7, p. 277 (M10).

[78]Of course, the terms inside and outside refer to the curvature of the hyperbola. We will see this kind of argument in the work of Sharaf al-Dīn al-Ṭūsī.

[79]See Hogendijk 7, p. 278 (M16).

construct a square ABGD together with a straight line through B that intersects the diagonal AG in Z, GD in H and AD extended in E such that $\triangle BZG = \triangle DHE$. Ibn al-Haytham does not assume that the square is a given one, realizing that any square ABGD together with a line BZHE is sufficient for constructing a regular heptagon by means of Proposition 18 of the treatise attributed to Archimedes. That is, he determines the point of intersection of the two parabolas without assuming that the square is a given square. In contrast, in al-Ṣaghānī's construction discussed above, it was assumed that the square is a given square.

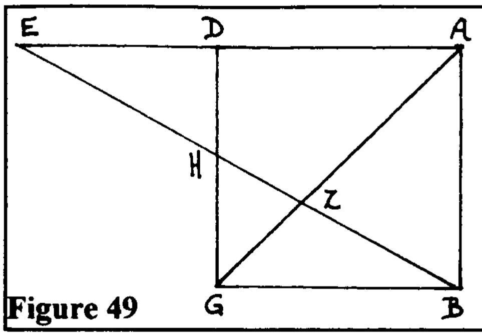

Figure 49

A second approach that Islamic geometers took was to construct a line segment AGDB in Proposition 18 such that $AD.GD = DB^2$ and $GB.DB = AG^2$ (Figure 50), without referring to the square used by Archimedes in Proposition 17 of his treatise. Three of these constructions have come down to us, and seem to be easier than the previous ones discussed.

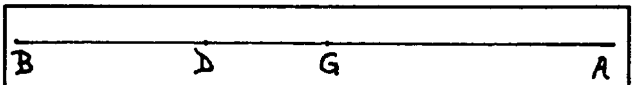

Figure 50

The first construction is due to Ibn al-Haytham, who found five constructions of the regular heptagon during the first half of the 11th century, which he gives in his *Chapter* and which he does according to the method of analysis. (Figure 51)

Assume that AGDB is such that (1) $AD.GD = DB^2$ and (2) $GD.DB = AG^2$.

Draw GX perpendicular to AB such that $GX = AG$.

Ibn al-Haytham now shows that X lies on a

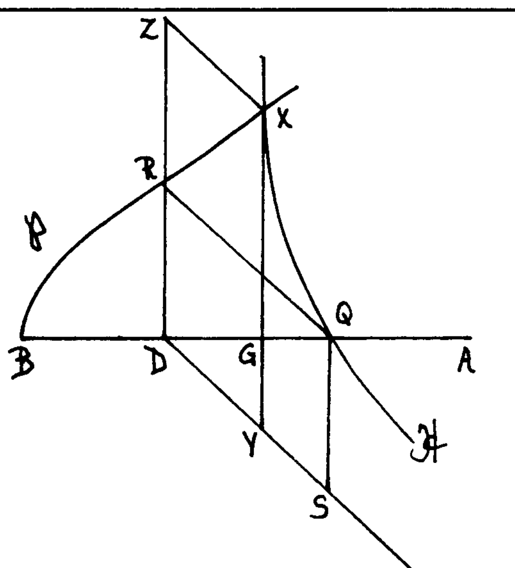

Figure 51

parabola $\mathcal{P}$ and a hyperbola $\mathcal{H}$.

Since (2) holds, X is on a parabola $\mathcal{P}$ with vertex B, axis DB and parameter DB. Extend BD to Q such that DQ = DB. Draw DR and QS perpendicular to AB such that

(3) DR = QS = DB.

Draw QR and DS. Extend XG to meet DS in Y. Draw XZ parallel to YD such that Z lies on DR extended. Since GY = GD, and GX = GA,

(4) XY = AD.

From (1), (3), and (4) it follows that XY.GD = QS.QD. Since GD/XZ = QD/QR, we obtain XY.XZ = QS.QR.

Hence X is on a hyperbola $\mathcal{H}$ through Q with asymptotes DS and DR (*Conics* II.12)[80]. The point X determines the point G, and since XG = AG, also the point A. If we assume that B and D are known, then the required line AGDB is found. Ibn al-Haytham uses this line to construct the regular heptagon in a slightly different manner than that of the Archimedean Proposition 18[81].

In the late 10th century, al-Kūhī found two solutions to the problem of constructing the line AGDB. In his *Letter on the Derivation of the Side of the Equilateral Heptagon in the Circle*[82], he intersects an equilateral hyperbola and a parabola to construct the line, and in the *Letter on the Construction of the Side of the Equilateral Heptagon in the Circle*[83], he intersects two hyperbolas. Abū'l-Jūd also uses an equilateral hyperbola and a parabola in his above-mentioned solution; however, he constructs the Archimedean square whereas al-Kūhī constructs a certain line segment. In both cases, al-Kūhī uses a method similar to the Archimedean Proposition 18 in

[80]The asymptotes of this parabola form an angle of 135° and ∠QDR = 90°. Ibn al-Haytham has shown that in this case, XY.DG = QD^2.

[81]See Rashed, "La Construction de l'heptagon régulier par Ibn al-Haytham", *Journal for the History of Arabic Science* **3** (1979), 309 - 387, for this construction.

[82]Published by Samplonius in German translation; see Hogendijk 7, p. 277 (M4).

[83]Discussed by Anbouba 3, p. 319; see Hogendijk 7, p. 277 (M5).

order to construct the regular heptagon[84].

The third approach taken by Islamic geometers resembles the previous one; they also construct a certain line segment by means of conic sections. By means of this line segment, they construct a certain triangle, and by inscribing a similar triangle in a given circle, they are able to construct the side of a regular heptagon. There are four kinds of triangles, whose angles are integral multiples of $\alpha = 180°/7$, that can be related to the construction of the regular heptagon:

(1) with angles α, 3α, 3α,

(2) with angles 2α, 3α, 2α,

(3) with angles α, 5α, α,

(4) with angles α, 2α, 4α.

Naṣr ibn ʿAbdallāh[85] shows how the problem of constructing a regular heptagon leads to the problem of constructing the triangle (1). He then analyses the construction of such a triangle as follows (Figure 52):

Assume ABG is such a triangle, and that $\angle A = \alpha$. Extend AB to D such that $\angle DGB = \alpha$. Extend AD to E such that DE = DG. Then $\angle GDA = 2\alpha$, and $\angle DEG = \angle DGE = \alpha$. Thus EG = GA = AB. Since $\triangle AGD$ is similar to $\triangle GBD$[86], we have AD/GD = GD/BD; therefore

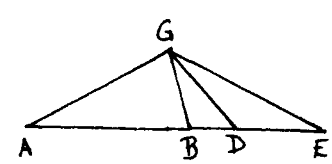

Figure 52

(a) $AD.BD = GD^2 = DE^2$.

Since $\triangle EGA \sim \triangle ADG$[87],

we have AE/EG = EG/DE; therefore

(b) $AE.DE = EG^2 = AB^2$.

The problem has now become to construct a line segment ABDE such that (a) and (b) hold.

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[84]See Hogendijk 7, p. 209 for details.

[85]The solution is found in a manuscript written in 1277A.D., but the date of the actual construction is unknown.

[86]These triangles both have angles α, 4α, 2α.

[87]These triangles both have angles α, 5α, α.

(Figure 53)

Assume that ABDE is like this, and draw DX perpendicular to ABDE such that DX = DE. Then X is on an equilateral hyperbola $\mathcal{H}_1$ with vertex B and transverse axis AB since

$$DX^2 = DE^2 = AD.BD.$$

Draw AN parallel to DX such that AN = AB. Draw NB, draw MAK parallel to NB and extend XD to meet these parallels at L and K. So KX = AE since KL = AB, LD = BD, and DX = DE.

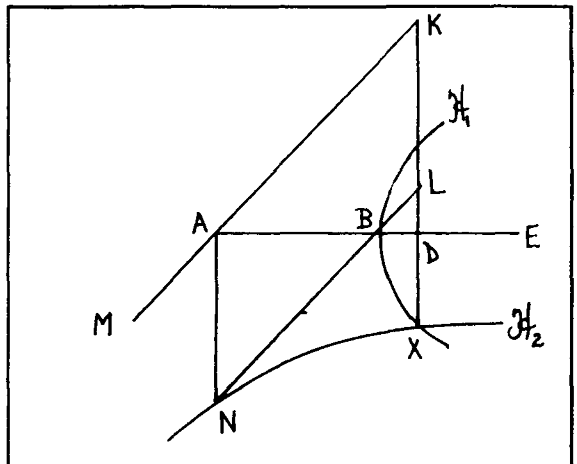

Figure 53

Hence X is on a hyperbola $\mathcal{H}_2$ through N with asymptotes AM and AB since

$$AN^2 = AB^2 = AE.DE = KX.DX \text{ (\textit{Conics} II.11).}$$

The point X determines D, and since DX = DE, it also determines E, and the analysis is complete.

This construction of the line segment ABDE such that AD.BD = DE^2 and AE.DE = AB^2 is very similar to Ibn al-Haytham's construction of the line AGDB such that AD.GD = DB^2 and GB.DB = AG^2. In fact, the latter construction corresponds to the construction of triangle (4), whereas the former, as we have seen, corresponds to the construction of triangle (1). Triangles (2) and (3) can be constructed with similar line segments. Ibn al-Haytham, in his *Treatise*, presents constructions all four triangles by means of conic sections and shows how they relate to their respective line segments. The four problems of constructing triangles are essentially rephrasings of one problem, namely to construct an angle $\alpha = 180°/7$.

The last construction we will discuss is probably the earliest Arabic construction of the regular heptagon (Hogendijk 7, p. 238), and is given in al-Sijzī's *On the Construction of the Heptagon*[88]. This construction is a joint effort between Abū'l-Jūd, who first attempted to solve

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[88]See Hogendijk 7, p. 277 (M7).

the problem using a neusis construction, and Al-ʿAlā' ibn Sahl, who, at the request of al-Sijzī, devised a certain construction by means of conic sections. Little is known of Ibn Sahl, though he is thought to have been one generation older than Abū'l-Jūd and al-Sijzī. The history of the development of this construction is very colourful, and includes slanderous remarks made by competing geometers, and several accusations of plagiarism. The Islamic geometers showed tremendous eagerness to solve the problem of constructing the regular heptagon, and although they made many erroneous attempts, their combined efforts finally paid off.

Abū'l-Jūd's task is to construct a triangle ABD (Figure 54) such that $\angle A = \angle D = 3\angle B$. Once he has done this, he inscribes a triangle similar to ADB in a given circle, and the side corresponding to AD is the side of the inscribed regular heptagon. In order to construct this triangle, Abū'l-Jūd uses a line segment AGB such that $AB.AG = \lambda^2$ for a segment λ satisfying $\lambda/BG = AB/(AB + BG)$.

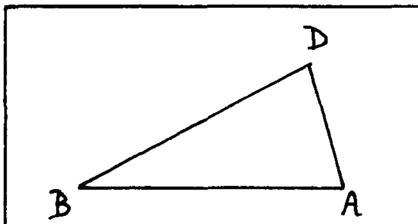

Figure 54

In a letter to another geometer[89], Abū'l-Jūd claims he arrived at this line segment by considering an analysis of a triangle ABD such that $\angle A = \angle D = 3\angle B$. By drawing DX such that $\angle ADX = \angle ABD$, and XW such that $\angle BXW = \angle XBW$, he proves that $AD^2 = AX.AB$ and $AD/BX = AB/(AB + BX)$ (Figure 55). He says that he was brought to consider the triangle ABD by following the method used by Euclid for the construction of the pentagon in the circle (*Elements* IV. 10-11). Euclid treats as a preliminary (*Elements* IV.10) an isosceles triangle such that each of the angles at the base is two times the angle (β) at the vertex; then the sum of its three angles is five times β, or $\beta = 180°/5$. Similarly, Abū'L-Jūd treats the isosceles triangle such that each of the angles at the base is three times the remaining angle α, so that $\alpha = 180°/7$.

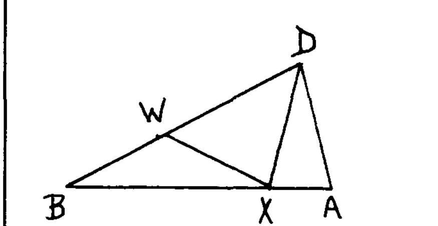

Figure 55

However, Abū'l-Jūd was unable to construct the required line segment; the construction was the work of Ibn Sahl (Figure 56):

[89]See Hogendijk 7, p. 277 (M9).

For a given line segment AB, choose D on BA extended such that AD = AB. Complete the square ADEZ.

Draw a hyperbola $\mathcal{H}$ through A with asymptotes EZ, ED.

Draw a parabola $\mathcal{P}$ with vertex B, axis BD and parameter AB.

Let the two conics intersect in H. Draw HG perpendicular to AB, then G is the required point.

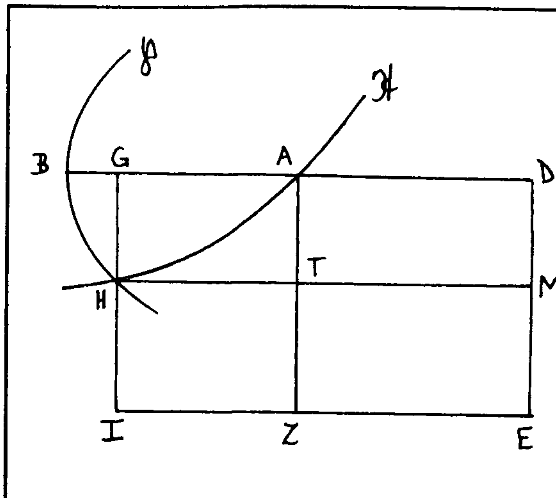

Figure 56

<u>Proof</u> Complete rectangle GDEI, draw HTM parallel to IE and ATZ parallel to GI.

HMEI = ADEZ since H and A are on the hyperbola (*Conics* II.12).

Then GAZI = GDMH, by subtracting TMEZ and adding GHTA.

Thus AZ.AG = GH.GD and hence

(1) GH/AG = AB/GD (since AZ = AB). But

(2) GH^2 = AB.BG, since H is on the parabola.

Since GD = GA + AB, and for λ = GH, we have found the required line segment ABG such that from (2) AB.GH = λ^2 where from (1) λ/AG = AB/(AB + AG).

The essential part of this construction of the regular heptagon, as with the previous ones, is the construction by means of conic sections of a straight line segment divided in points such that certain conditions are satisfied. As in the third method, this line segment enables one to draw a certain triangle which is inscribed in a given circle in order to determine the side of the heptagon.

In finding all these constructions of the regular heptagon the Islamic geometers were stimulated to study the theory of conic sections and its applications. Ibn al-Haytham, in his work on this problem, realized how seemingly different constructions all led to the construction of certain triangles which could be used to construct the heptagon. Islamic mathematicians would need this kind of realization of an underlying unity between solutions to 'solid' problems in order to translate them into specific cubic equations. Moreover, their experience with conic sections would help them in their studies of geometrical solutions to more general cubic equations.

In fact, Islamic geometers did work on the relation between the regular heptagon and cubic equations. In an untitled work on algebra, Al-Khayyām mentions the work of Abū Naṣr ibn 'Irāq :

> "He used the terminology of the algebraists, and the analysis led him to (the equation): a cube and squares are equal to numbers. He solved it by means of conics". (Hogendijk 9, p. 240)

Unfortunately, this work by Abū Naṣr is not extant, and we have no further references of it. We investigated the conic sections used by the other geometers who solved the Archimedean lemma, and found that al-Ṣaghānī's construction, for instance, is equivalent to the construction of an equation of the form: a cube is equal to numbers and squares. However, it is difficult to suggest a plausible reconstruction of Abū Naṣr's reasoning.

The Nonagon

The construction of the nonagon may be considered as a special case of the trisection of the angle. If one places an angle of 40° in the centre of the circle, its chord is the side of the inscribed regular nonagon in the circle. The angle of 40° can be constructed by taking two angles of 20° together, and one constructs an angle of 20° merely by trisecting an angle of an equilateral triangle[90]. Of course, one could also construct a nonagon by inscribing an equilateral triangle in a circle and trisecting its angles. This latter procedure is similar to the one Euclid uses in *Elements* IV, 11 in order to inscribe a regular pentagon in the circle. He first constructs an isoceles triangle with angles 36°, 36°, 18°, then inscribes this triangle in a circle, and finally bisects both 36° angles to obtain the result. The former procedure is based on the central angle, that is, constructing the angle at the center of the circle, whose chord corresponds to one side of the nonagon.

As such, this problem might have been of little interest, but in fact, it stimulated the discovery of a new problem in Islamic geometry. We quote the introduction of Abu'l-Jūd's answer to a question put forth by al-Bīrūnī:

> "Why have we stated in the seventh proposition of the seventh section of the fourth part of our *Book on Geometrical Subjects*[91] that the construction of the nonagon is by this proposition possible by means of algebra?" (Hogendijk, 1979)

In his answer, Abū'l-Jūd shows how the construction of the regular nonagon can be reduced to another problem: solving the equation $x^3 + 1 = 3x$. This reduction to a cubic equation is as follows. (Fig. 57)

Abū'l-Jūd considers the isosceles triangle ABG, in which AG = BG and AGB = 20°[92]. He takes D on BG such that AD = AB. By completing the triangle as in figure 57, where EZ = ED = AD, and ZK is perpendicular to AG, he arrives at the following relationship:

(1) $AG^2 = AB(GD + 2BG) = (3BG - DB)$.

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[90]The construction of an equilateral triangle is shown in the *Elements* Book I, prop. 1

[91]This lost work is the collection of tracts mentioned by ʿUmar al-Khayyām in the appendix to his *Algebra*.

[92]Of course, the construction of this angle is achieved by trisecting the 60° angle.

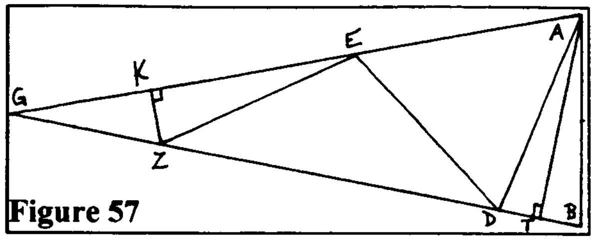

Figure 57

Now Abū'l-Jūd assumes that $AG = BG = 1$, and $AB = x$ (*jidhr* - the unknown quantity), so x is the side of a regular 18 sided polygon in the circle with radius $AG = 1$.

Then $AB^2 = DB.AG$, (since triangles ABD and GAB are similar) or $DB = x^2$ (*māl* - the square of the unknown).

From (1) it follows that $1 = x(3 - x^2)$, so $x^3 + 1 = 3x$.

From this text we learn that in the 7th proposition of the seventh section of the fourth part of the *Book on Geometrical Subjects*, Abu'l-Jūd solves this equation by conic sections. Unfortunately, this book has disappeared.

In his *Canon*, al-Bīrūnī derives the equation $x^3 + 1 = 3x$ in exactly the same way as described above. In addition, he reduces the construction of the nonagon to yet another cubic equation: $x^3 = 1 + 3x$; x is the chord of two-ninths of the circumference of the circle, in which the side of the inscribed regular nonagon has length 1. Contrary to the above method, this problem is formulated in a way that bears little relation to the trisection of an angle (Hogendijk 9, p. 26).

The construction of the nonagon, as we have said, is a special case of the angle trisection, that is, of the 60° angle. Although 10th century Islamic geometers were well aware of the equivalence of the construction of the nonagon to a cubic equation, they had not yet established a similar equivalence for the trisection of an arbitrary angle. However, in the 15th century, the Islamic mathematician Jamshīd Ghiyāṭ al-Dīn al-Kāshi reduced the trisection of an arbitrary angle α to an equivalent of a cubic equation involving the chord of the angle α, placed in the centre of a circle with radius 1[93].

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[93]See A.P. Youschkewitch, *Geschichte der Mathematik im Mittelalter*, Moscow, 1961 (Russian), Leipzig, 1964 (German translation), p. 321.

Archimedes' Problem

In the fourth proposition of *On the Sphere and the Cylinder* II, Archimedes formulates a subsidiary problem which the ancient Greeks were able to solve by means of conic sections. Eutocius gives three solutions to this problem in his *Commentary,* each of which we have discussed in the first section. The subsidiary problem can be stated as follows (Fig. 58): to divide a line DZ at a point H such that the ratio of HZ to ZT is equal to the ratio of BD^2 to DH^2 where DB, BZ are given lengths and the ratio of BZ to BT is known[94].

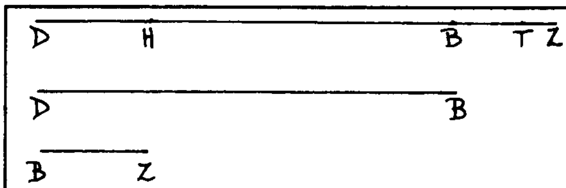

Figure 58

According to al-Khayyām, the first Islamic mathematician to attempt to solve this problem was al-Māhānī, who translated it into its algebraic equivalent, ***a cube plus numbers equals squares.*** In its algebraic equivalent, the "coefficient of the squares" is the length of the line DZ and the "numbers" depends on the given lengths DB, BZ and the ratio of BZ to BT[95]. He failed to solve it but his efforts merited the problem being named after him. Al-Khayyām goes on to say that Abū Jaʿfar al-Khāzin successfully constructed a root of this equation in a certain treatise, by means of conic sections[96]. Almost at the same time, the problem was solved by means of a parabola and a hyperbola by Ibn al-Haytham[97], whose construction is as follows. (Fig. 59)

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[94]In Archimedes' problem DB is the diameter of the sphere, DB = 2BZ, and BZ : BT is known in terms of the given ratio of the problem. See p. 20 for the problem statement.

[95]For a=DZ, b=BD, and c=BZ/BT, we have $x^3 + b(1/c - 1) = ax^2$.

[96]Hogendijk 7 argues that Abū Jaʿfar knew of Eutocius' construction of the problem (as found in his commentary on *On the Sphere and Cylinder* II), and thus remarked that al-Māhānī's cubic equation could be solved geometrically by means of the same construction as Eutocius, which was not available to al-Māhānī.

[97]For a translation of this tract, see Woepcke (1851), p. 91 - 94.

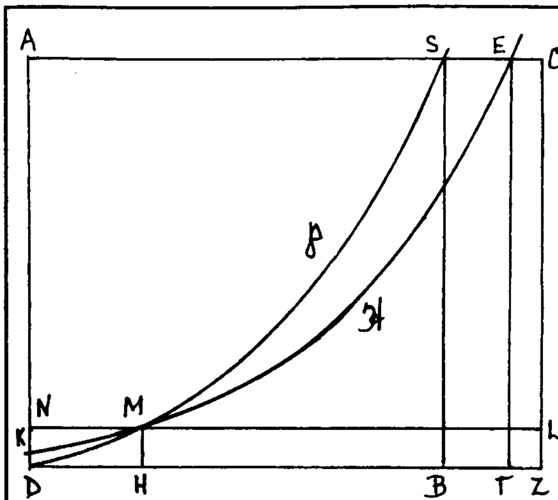

Figure 59

Draw AD, ET, CZ perpendicular to the line DBTZ and all equal to DB. Draw a straight line through their endpoints A, E, C, and through the point E, draw a hyperbola $\mathcal{H}$ with asymptotes CZ, ZD. Then $\mathcal{H}$ will intersect AD in a point K between A and D. Draw a parabola $\mathcal{P}$ with vertex D, axis DA and parameter DB. Then $\mathcal{P}$ will intersect AC in a point S such that $AS^2 = AD.BD = BD^2$. Therefore, AS = BD, and since AE = DT > DB[98], we have AE > AS. Therefore E is outside $\mathcal{P}$. And since K is inside $\mathcal{P}$, the two conics $\mathcal{P}$ and $\mathcal{H}$ will necessarily intersect, say, at the point M. Draw from M a perpendicular to DZ meeting DZ in H. Then H is the required point.

Proof: Draw NML parallel to DZ. Since M lies on the parabola, we have $MN^2 = BD.DN$ and also $DH^2 = BD.MH$. Thus

(1) $BD^2 : DH^2 = BD : MH$.

Since M also lies on $\mathcal{H}$, we have ML : EC = ET : MH, and also

(2) HZ : ZT = BD : MH.

Combining (1) and (2), we obtain the required result $BD^2 : DH^2 = HZ : ZT$.

Ibn al-Haytham's construction corresponds to the auxiliary problem specific to Archimedes' problem in Prop. 4 of *On the Sphere and Cylinder* and, as such, does not require an investigation into the conditions for a solution. The construction Eutocius gives of Archimedes' solution, that we presented earlier, solves a more general problem for which it is necessary to investigate the limits of possibility. This general problem is also solved in a Leyden manuscript by an Islamic geometer, who essentially discovers the same limits of possibility as Archimedes[99]. F. Woepcke

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[98]This follows from the condition in Archimedes' original problem that BZ > ZT.(Heath 3, p. 64)

[99]Woepcke has argued that there is a fundamental difference in their conditions - we discuss this on p. 66.

(1851) has attributed the construction to al-Kūhī (Woepcke, p. 102)[100] however, there is no conclusive evidence.

The problem of Archimedes is generalized in the Leyden manuscript to the following one: (Fig. 60) given two lines AB and C, to divide AB at D such that (*) AD : C = $C^2 : BD^2$. This is equivalent to the construction of the equation (AB - x) : C = $C^2 : x^2$[101]. The geometer who solves this problem notes that if $C^3 > \frac{1}{3}AB.(\frac{2}{3}AB)^2 = \frac{4}{27}AB^3$, the problem as it stands is impossible. That is, it would be impossible to (1; Fig. 60) divide AB at D such that (*) holds; however, he adds that for $C^3 > \frac{4}{27}AB^3$, one can extend AB to D such that (*) holds (2; Fig. 61).

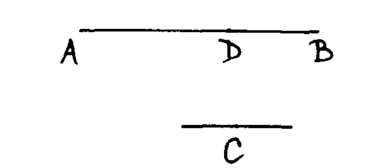

Figure 60

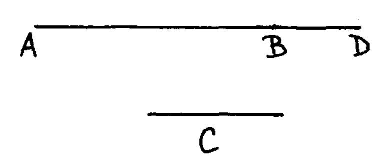

Figure 61

Now, the constructions of both (1) and (2) can be effected using the following lemma: to find AD such that $AD.BD^2 = C^3$, where A, B, and D are collinear. The construction of this lemma is as follows. (Fig. 62, 63)

Take BE = C, and construct the square BHZE. Draw a parabola $\mathcal{P}$ with vertex A, axis AB and parameter BE, and a hyperbola $\mathcal{H}$ through Z with asymptotes EB, BH. Suppose $\mathcal{P}$ and $\mathcal{H}$ intersect at a point T.

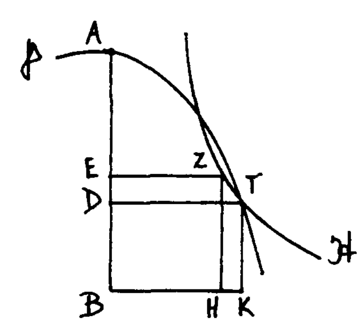

Figure 62

[100]There are two reasons for this opinion: (1) the construction ends on the last line of a page, and the following page begins by attributing a problem to al-Kūhī and continues with the resolution of another problem. It is therefore unclear whether the previous or following problem of the manuscript is being attributed to al-Kūhī, and (2) al-Kūhī is the author of another construction related to an Archimedean problem which investigates the limits of possibility.

[101]Or, $ABx^2 = x^3 + C^3$.

Draw perpendiculars TK, TD to BH, EB respectively. Then

(I) AD : TD = TD : BE since T lies on $\mathcal{P}$, and

(ii) BK : BE = EZ : KT i.e. TD : BE = BE : BD since Z lies on $\mathcal{H}$.

Since C = BE, combine (i) and (ii) to find that AD : C = C^2 : BD^2 or

(iii) AD.BD^2 = C^3.

Suppose (1; Fig. 62) that AD = AB - BD, then (iii) becomes AB.BD^2 - BD^3 = C^3, which corresponds to the cubic equation $x^3 + c = ax^2$. Otherwise, we have (2; Fig. 63) that AD = AB + BD in which case (iii) becomes AB.BD^2 + BD^3 = C^3, and this corresponds to the cubic equation $x^3 + ax^2 = c$. The geometer states that case (1) is limited since it has a solution only when

(*) $C^3 \leq {}^4/_{27}.AB^3$,

and that case (2) always has a solution.

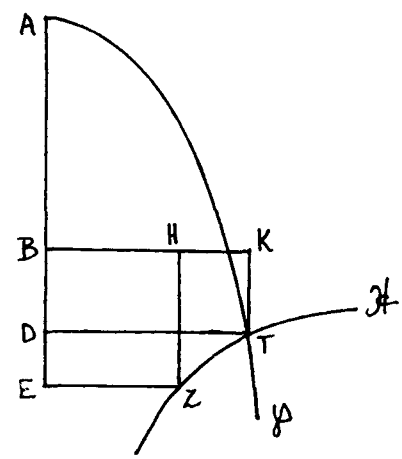

Figure 63

The author does not indicate how he arrived at the limit of possibility for a solution in case (1). However, we have already seen Archimedes' derivation of an equivalent condition, that is, that the product of the given surface and the given line segment cannot be greater than the product of AE.EB^2 when BE = 2EA. Woepcke notes that in substituting the product of the two givens, linear and planar, by the cube of a single given line segment, the Islamic geometer succeeded in finding the modern expression (*) of this limit. Therefore, this geometer changed Archimedes' geometric condition for solving a particular problem to a general algebraic condition for solving a class of problems. This general condition will also be obtained in a clear and systematic way by the geometer Sharaf al-Dīn al-Ṭūsī in his discussion of the corresponding general cubic equation[102]. It is very interesting that in generalizing the problem of Archimedes, the geometer actually constructed two different forms of a cubic equation.

[102]See p. 81 for a discussion of this cubic equation and its limits of possibility.

Al-Kūhī's Problem

Al-Kūhī was inspired by Archimedes to solve another problem by means of conic sections. The problem was never actually stated in Archimedes' *On the Sphere and Cylinder*, yet al-Kūhī claims it arises naturally from propositions 5 and 6[103] of this work. The problem is to construct a spherical segment whose spherical surface is equal to that of one segment and whose volume is equal to that of another. However, arbitrary choices of these segments will not necessarily lead to a problem with a solution. Not only does al-Kūhī solve this very difficult problem, but he establishes the conditions in which a solution exists. His analysis is summarized as follows[104]:

(Fig. 64) Al-Kūhī takes a point B known in position[105] on a line BE, also known in position, and assumes that the problem is solved by the segment ABG of height BZ in a sphere of diameter BD. The area of this segment is known, and by *On the Sphere and Cylinder* I. 42 & 43, it is equal to the area of a circle whose radius is the chord BA.

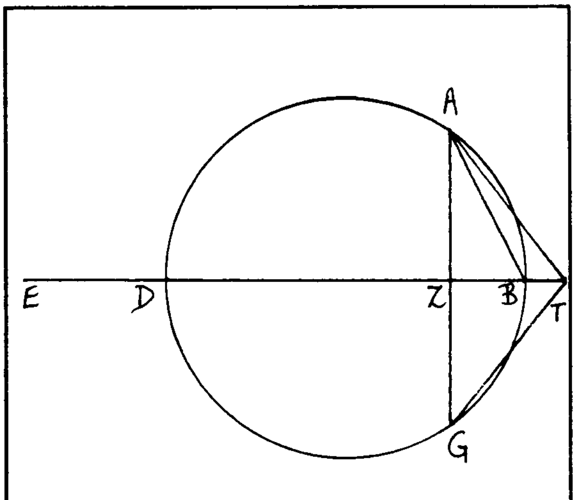

Figure 64

Therefore, (1) Area = πBA^2.

Now, the volume of this segment is also known, and by *On the Sphere and Cylinder* II.2 it is equal to the volume of a right cone having the same base as the segment and height TZ defined by TZ/ZB = EZ/ ZD, where EZ = ZD + ½BD.

Therefore, (2) Volume = $\frac{1}{3}\pi AZ^2.TZ$

[103]Proposition 5: To construct a segment of a sphere similar to one segment and equal in volume to another. Proposition 6: Given two segments of spheres, to find a third segment of a sphere similar to one of the segments and having its surface equal to that of the other. See Heath 3, pp 79 - 84.

[104]The following account is based on that of in Berggren 1996.

[105]We find definitions of the senses in which things are *given* in Euclid's *Data*. A straight line or an area is said to be given *in magnitude* when we can make others equal to them. A point or a line or an angle is said to be given in position when they occupy the same place.

Now, Al-Kūhī shows how to recast expression (1) into the form of the symptom of the hyperbola.

From (1), since the area is known, so is the magnitude of the segment AB. Moreover, since $\pi BA^2 = \pi BD.BZ$, we have an expression which describes a hyperbola with abscissa BD and ordinate BZ.

The case of relation (2) is not so straightforward; however, al-Kūhī will recast it to define a parabola with known vertex and known parameter. In fact he derives the required expression by using both the 'givens' (the area and the volume): he transforms the known volume of the spherical segment into the cone AGT, and, he defines another cone whose base has the same area as the given surface. Finally, the point of intersection of the two conic sections that al-Kūhī will have described determines the position of D on the diameter BD.

Al-Kūhī defines a 'cone of surface' , C, as the right isosceles cone whose base has radius AB. Since BA is known, so are C and the ratio of C to the cone AGT (call this ratio R). Now, $R = {}^{AB.AB^2}/_{TZ.AZ^2}$, and by similarity of right triangles and the definition of TZ,

(3) $R = {}^{AB.DB}/_{BZ.ZE}$.

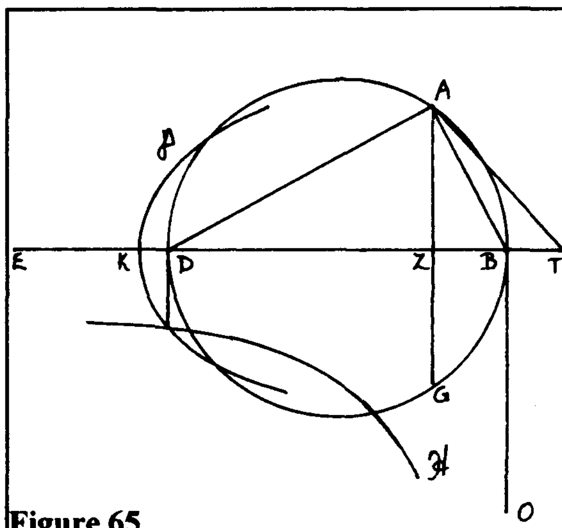

Figure 65

At this point, Al-Kūhī introduces DK, and shows that the point K, lying on BD extended, is known in position. This point K will be the vertex of the parabola. The trick is to show that BK is known in magnitude. (Fig. 65)

Define DK by the following proportion

(4) $R = {}^{AB.DK}/_{BZ^2}$, in other words,

DK/BZ = DB/ZE, which assumes the existence of a fourth proportional, DK.

Combining, (3) and (4),

$R = {}^{AB.BD + AB.DK}/_{BZ.ZE + BZ^2}$ (by *Elements* V.12[106]), and since ZE = 3.½BD - BZ,

(5) $R = {}^{AB(BD + DK)}/_{3.\frac{1}{2}BD.ZT}$

Since AB is the mean proportional between BD and BZ,

(6) $R = {}^{AB(BD + DK)}/_{3.\frac{1}{2}AB^2}$.

[106]That is, if X/Y = a/b = c/d then X/Y = (a+c)/(b+d).

But BD + DK = BK, so it follows from (6) that BK is known in magnitude on the basis of the givens in the problem.

It thus remains to determine the parameter of the parabola.

Define a known line segment *s* by the proportion

(7) $^{AB}/_{s} = R$.

Then by (4) and (7), we see that $s.DK = BZ^2$, so if we draw DM perpendicular to BD such that DM = BZ, then DM is the ordinate of a parabola with vertex K and abscissa KD.

Al-Kūhī now proceeds with the construction of these two conic sections which will intersect under certain conditions. We discuss these conditions later; let us first assume a solution exists.

Draw the parabola $\mathcal{P}$ with vertex K, axis KB and parameter *s*. Draw the rectangular hyperbola $\mathcal{H}$ whose asymptotes are BE, BO, where BO is perpendicular to BE.

Since $s.DK = DM^2$, the parabola passes through M, and as a consequence of (1), $BD.DM = BD.BZ\ (=AB^2)$, the hyperbola also passes through M.

Then the point of intersection of these two conic sections M, is known in position, as is the point D. Thus the diameter BD is now determined and therefore the sphere is determined. Also BZ = DM is determined, so the segment of the sphere is determined.

To establish the conditions under which the conic sections intersect, Al-Kūhī investigates the ratio R. Of course, the conic sections themselves are defined in terms of R and AB, which are in turn defined in terms of the givens of the problem: area and volume. Al-Kūhī observes that for a fixed AB (so the numerator of the ratio R is fixed) $R^2 = {}^{DB^3}/_{BZ.ZE^2}$ [107] ≥ 2. Therefore, for $R^2 > 2$, the conic sections intersect twice, giving rise to two spherical segments: one smaller than a hemisphere and one larger. When $R^2 = 2$, the two sections are tangent to each other at the point M, and in this case, BZ = ½ZE, and the segment is a hemisphere. Of course, if $R^2 < 2$, there is no solution. Al-Kūhī concludes with the remark that when R = 2, the solution determined by the perpendicular nearer B corresponds to a segment equal to the whole sphere, while that determined by the perpendicular further away corresponds to a segment smaller than

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[107] From (3) we have $R = {}^{AB.DB}/_{BZ.ZE} = {}^{DB^3}/_{AB.ZE.DB}$. Also from (3), $R = {}^{AB.DB.ZE}/_{BZ.ZE^2}$, hence it follows that $R^2 = {}^{DB^3}/_{BZ.ZE^2}$.

a hemisphere. Thus, R has an upper bound at 2 when the segment is larger than a hemisphere because the segment cannot exceed the whole sphere.

We do not render the details of his proof here, as his demonstration is independent of conic sections. However, we note that Eutocius had shown in his *Commentary on the Sphere and Cylinder* II.4 an equivalent relationship, that is, that $BZ.ZE^2$ (the cone which is equivalent to the given volume of the segment) is a maximum when $BZ = \frac{1}{2}ZE$[108]. We will discuss this result and compare two proofs of it in the very last section[109].

Al-Kūhī was able to formulate a problem that no one had yet worked on, not even the ancient Greeks, and his solution shows remarkable ingenuity. By introducing the ratio R, he was able to recast his initial expressions into the form of symptoms for a hyperbola and a parabola. For some comments on how al-Kūhī may have hit upon the utility of R, see Berggren 1996.

[108]See p. 21 for the details of this discussion.

[109]See section entitled **A Comparison**.

Some Constructions of Algebraic Equations

In this section, we discuss examples of specific problems which were transformed into cubic equations, and then solved by means of conic sections. The geometrical solution of a cubic equation is the construction of a line segment, by means of two intersecting conic sections, which satisfies the cubic equation. That is, the line segment enables one to construct two equal solids, each corresponding to a side of the cubic equation. A discussion on the development of algebra in the Islamic world falls outside the scope of this paper; however, we will attempt to trace the emergence of algebraic cubic equations used in problem solving.

Al-Māhānī's work on the problem arising from Prop. 4 of Archimedes' *On the Sphere and Cylinder II* marked the beginning of the study of cubic equations. Although he was unsuccessful in constructing a solution, we have seen how other geometers attacked and eventually solved the problem. We have already seen how Abū'l-Jūd transformed the construction of the regular nonagon into a certain cubic equation, and how al-Bīrūnī discovered another, equivalent form of cubic equation.

We also have evidence of the work done by Abū Naṣr on the solution of a cubic equation associated with the construction of the regular heptagon. Furthermore, there is an extant geometrical construction of a solution to the cubic equation $x^3 + p = qx^3 + rx$ in a Manisa manuscript[110] by Kamāl al-Dīn ibn Yūnus, which resembles the earliest construction of the heptagon, though Kamāl al-Dīn makes no mention of the relation between this construction and the regular heptagon. However, it is possible that he found his solution by generalizing the heptagon construction of Ibn Sahl that we discussed previously, which he knew of from al-Sijzī's *On the Construction of the Heptagon*[111]. For, if we give p, q, and r specific values, the construction of Kamāl al-Dīn is precisely the one of Ibn Sahl. The construction is as follows (Fig. 66)

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[110]Published by Hogendijk 7, pp. 240 - 241.

[111]In the *Letter on the proof of the assumption of the lemma* (Hogendijk 7, p. 277 (M17)) Kamāl al-Dīn mentions *On the Construction of the Heptagon*, and even discusses the construction of G on AB as required in the earliest construction of the heptagon mentioned by al-Sijzī.

Let AB = q, $s = p/q^2$, and AG = r/s[112]. Draw GE perpendicular to AG with GE = AB. Complete rectangle AE and draw BD parallel to GE and ED parallel to GB. Draw a parabola $\mathcal{P}$ with vertex A, axis AB and parameter s. Draw a hyperbola $\mathcal{H}$ through B with asymptotes EG and ED. Let $\mathcal{H}$ and $\mathcal{P}$ intersect at H. Draw THK perpendicular to AG and HM perpendicular to EG. Then HK = x.

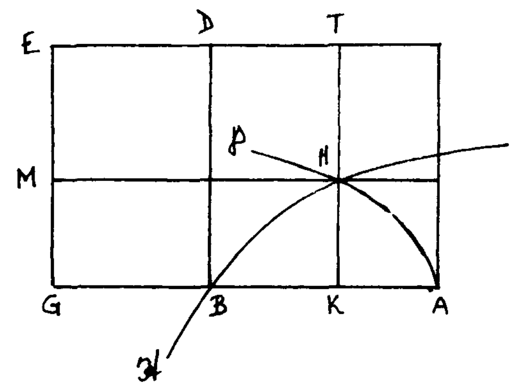

Figure 66

The proof follows immediately. Since H and B are on $\mathcal{H}$, then

rectangle HTEM = rectangle BDEG, this translates into $(q - x)(r/s - z) = q(r/s - q)$ or

$rx + qzs = q^2.s + zxs$, where AK = z.

Now, $HK^2 = AK.s$ since H is on the parabola, and $q^2.s = p$ by construction.

Hence $rx + qx^2 = p + x^3$.

Kamāl al-Dīn does not say why the parabola and the hyperbola intersect, nor does he state the position of their intersection H relative to A, B, and G. It is possible, for certain choices of coefficients, that G be situated between A and B; this would have an effect on the position of H. He does not discuss the number of intersections either. In fact, it seems that he has specific values of *p*, *q*, and *r* in mind since he does not consider the different cases that could arise from this general cubic equation.

We now turn to a series of new problems which Islamic geometers formulated as cubic equations and solved by means of conic sections. The first is in a tract by Ibn al-Haytham, where he solves the following problem: to divide a given number k into two parts in such a way that the one part is the cube of the other, in other words, to solve the cubic equation $x^3 + x = k$. The method he uses to resolve this problem is first, to find four quantities a > b > c > d such that

(i) a : b = b : c = c : d, (they are in continued proportion)and

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[112]Of course, in the words of Kamāl al-Dīn, we construct the solid equal to *p* with base the square on *q* and height *s*, and we construct the rectangle equal to *r* with length *s*.

(ii) (a-b) : d = k^3 : k. (This ratio is known since k, and thus k^3 are given)

We present the synthesis of the problem based on the translation by Sesiano 1976. (Fig. 67)

Suppose the segment AB is given. Draw BG perpendicular to AB. Let D be on BG such that AB/BD equals the given ratio k^3 : k, and draw DE parallel to AB. Complete rectangle AEDB. Through E, draw a hyperbola $\mathcal{H}$ with asymptotes AB, BG. Extend GB to N such that BN = BD. Draw NQ parallel to AB meeting EA extended in Q. Draw a parabola $\mathcal{P}$ with vertex N, axis NQ, and parameter NB. As $\mathcal{P}$ moves towards O from N, it moves away from its axis NQ as well as from the segment AB arbitrarily closely, and as $\mathcal{H}$ moves towards E, it approaches the segment AB. Hence, $\mathcal{P}$ will intersect $\mathcal{H}$ at a point Z. Draw ZT parallel to AB and ZKL parallel to GBN.

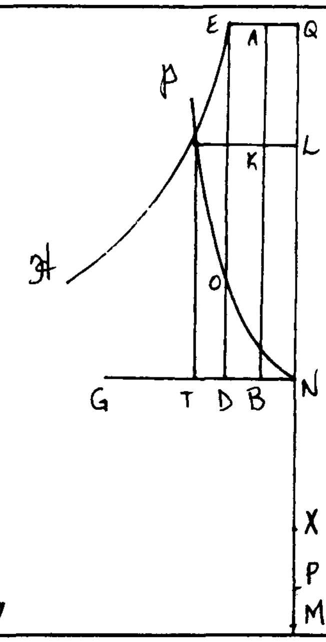

Figure 67

Then TZ.ZK = DE.EA *Conics* II.12, so AB/ZT = ZK/BD, and

(1) AB/ZT = ZK/KL. Extend QN to M such that NM = ZT. Then

(2) AB/ZT = QN/NM. From (1) and (2), it follows that

(3) QN/NM = ZK/KL. Now, by the property of the parabola, LZ^2 = NL.LK. Hence

(4) NL/LZ = ZL/LK. From (3) and (4), it follows that QM/MN = NL/LZ = ZL/LK.

Let X on NM be such that MX = LZ, and P on MX such that MP = LK. Then the quantities QM, MN, MX, MP are in continued proportion, so (I) holds.

Moreover, since QN = BA and MP = KL = BN = BD, then

QN/PM = AB/BD, which is the given ratio.

But QN = QM - MN, so (ii) also holds.

Ibn al-Haytham states that once we have found the four quantities QM, MN, MX, MP satisfying (i) and (ii), then we can solve the initial problem: to divide a given number into parts such that one is the cube of the other. Using elementary manipulations of proportions, he proves that if we divide our given number, say XY, at θ such that $X\theta/\theta Y = QM \cdot MN/MN$, (*Elements* VI, 10) then $X\theta = Y\theta^3$. Hence, $Y\theta + Y\theta^3 = XY$.

The work of Ibn al-Haytham is extremely precise and thorough; he quotes every theorem from the *Elements* and the *Conics* as they are used and carefully proves that the two conic sections do indeed intersect.

The next construction of a cubic equation we consider is found in a treatise written approximately two hundred years later by al-Khayyām entitled *On the Division of the Quarter Circle*[113]. The problem is to divide the quarter circle AB of a circle ABCD (Fig. 68) with center E in two parts at the point G, and to draw GH perpendicular to the diameter BD such that the ratio of AE to GH is equal to the ratio of EH to HB.

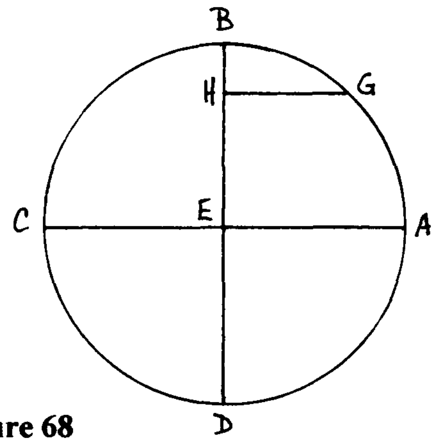

Figure 68

In this treatise, al-Khayyām seems to have more in mind than just solving the problem. He begins by showing how a certain analysis does not result in a figure which we know how to construct. He states that he did not pursue this analysis any further due to its difficulty and the knowledge about conic sections it would require. Then, he gives another analysis of the problem, which eventually leads to a cubic equation, and which he solves by means of conic sections. Strangely enough, the following problem in the manuscript containing *On the Division of the Quarter Circle* is an unattributed construction of the identical problem, by a method similar to the previously unsuccessful one of al-Khayyām.

We summarize the two similar methods first, and then give a more detailed description of the construction of the cubic equation in the second of al-Khayyām's attempts.

First attempt: (Fig. 69)Al-Khayyām draws a tangent line IBM through B, and GK perpendicular to EA such that they intersect in I. Then he draws the hyperbola $\mathcal{H}$ through E with asymptotes IBM, IGK. One seeks to determine the position of a point L on the hyperbola

[113]Published by Rashed and Djebbar 1981, as part of the *Algebra*.

so that once the line LHG is drawn parallel to IBM, G would be known in position. However, L is not known in position through this analysis.

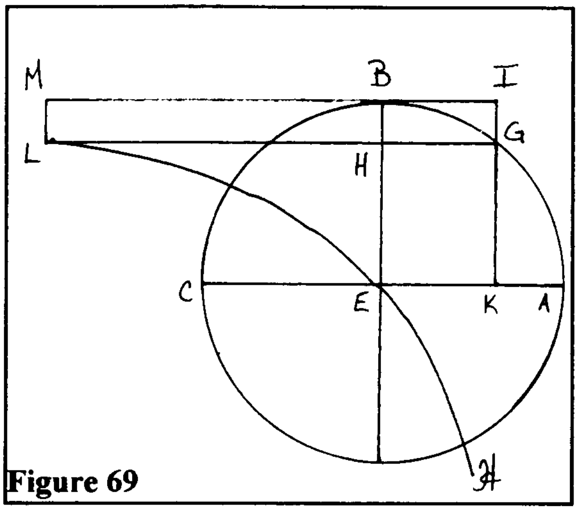

Figure 69

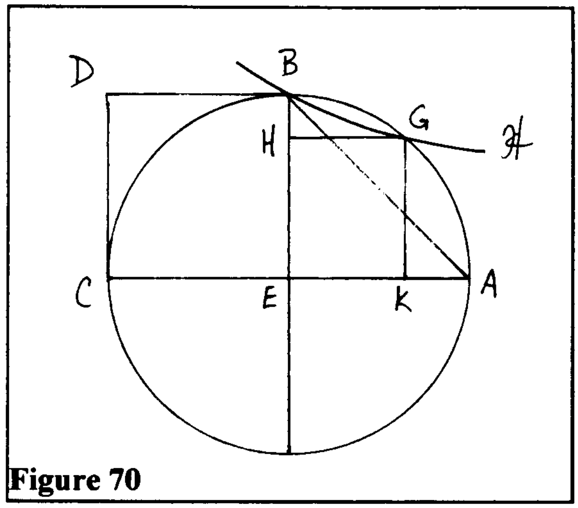

Figure 70

<u>Similar successful method</u>: (Fig. 70)

Complete the square CDBE. Draw a hyperbola ℋ through B with asymptotes CE, CD. Join AB, then AB is tangent to the hyperbola and lies inside the circle. Consequently, ℋ cuts the circle at another point, say G. Draw GH perpendicular to AB and GK perpendicular to BC. Then CK.GK = CE.EB. Subtracting rectangle HC, we obtain GH.EH = BD.HB. But BD = AE so AE/GH = EH/HB.

By constructing the hyperbola through B instead of through E, we can find its point of intersection with the circle and thus determine the required point G.

<u>Second attempt</u>: (Fig. 71)

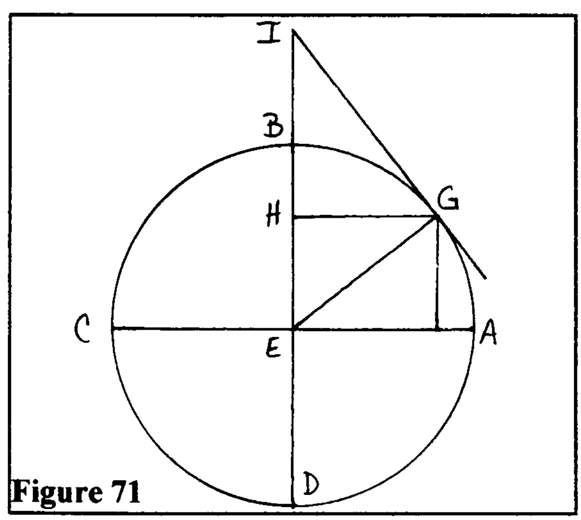

Figure 71

Draw GI tangent to the circle. Extend EB to I, and join GE. Al-Khayyām shows, through a series of manipulations of proportions, that

GH = BI, GE = EB, and EG + GH = EI.

His analysis has thus led him to the construction of a right-angled triangle (since ∠EGI = 90°) whose hypotenuse is equal to the sum of one of its sides and the line drawn from the right angle perpendicular to the hypotenuse. At this point, al-Khayyām states a few other properties of the triangle: (1) the triangle is not isosceles; (2) EG < GI; (3) EG + EH = GI. Having established and proved these properties, al-Khayyām turns to the construction of the triangle (Fig. 72) ABC, with right angle at B, and BD drawn perpendicular to AC, satisfying the required properties. To begin, he proposes to use to the language of the algebraists, and, he assigns a specific length of twenty to the segment AD. For BD the 'thing'[114], he uses the relations obtained above, and some algebra, to obtain the following cubic equation: $x^3 + 200x = 20x^2 + 2000$.

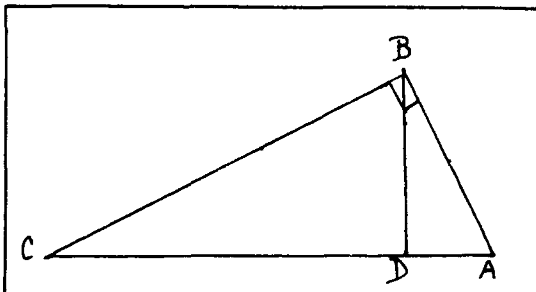

Figure 72

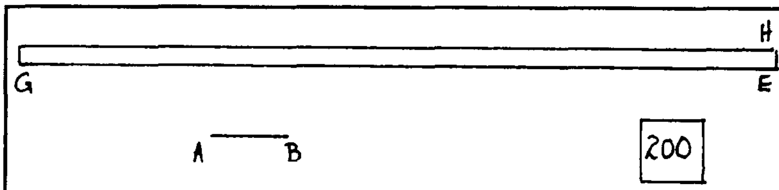

Figure 73

Solving for 'x': (Fig. 73) Let AB = 20, EG = 200, and EH = 1. Then the rectangle HG = 200. Construct a square equal to the rectangle HG (*Elements* II.14), let the side of this square equal AH. Draw AH perpendicular to AB (Fig. 74). Take D on AB such that AD = 10. Draw a semi-circle with diameter BD, and draw ED parallel to AH. Complete the rectangle ADEH. Draw a hyperbola $\mathcal{H}$ through D with asymptotes

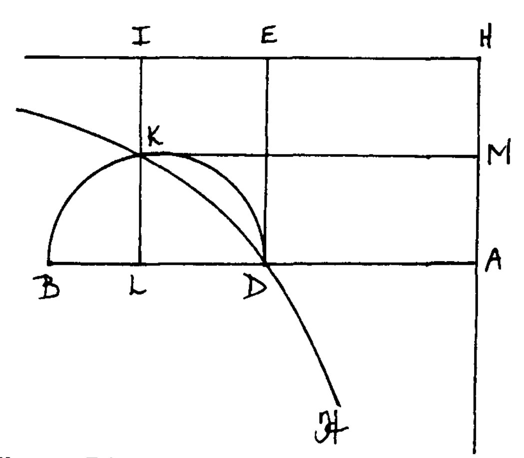

Figure 74

[114]The 'thing' is the unknown quantity, which we call x.

AH, HE. Then $\mathcal{H}$ cuts the circle again at a point K. Draw KL perpendicular to AB. Then AL is the required segment 'x'.

Proof: Extend LK to meet HE extended at I. Draw KM parallel to AL.

Then AD.DE = KM.MH *(Conics* II.12), since K, D are on $\mathcal{H}$.

Subtract EH.MH, so that AD.AM = KI.IE.

Add LK.DL, so that AL.LK = DL.LI, or

(1) AL/LI = DL/LK. Also,

(2) DL/LK = LK/LB, so combining (1) and (2), we have

$AL^2/LI^2 = DL^2/LK^2 = DL^2 / LB.DL = DL/LB$. Therefore, $AL^2.LB = LI^2.DL$.

Add $LI^2.AD$, so that $LI^2.AL = LI^2.AD + AL^2.LB$. But $LI^2 = 200$, and AL = x.

Hence, $200x = LI^2.AD + AL^2.LB$. But $LI^2.AD = 2000$, therefore,

$2000 + AL^2.LB = 200x$. Add AL^3 which is equal to $AL^2.AL$, then

$AL^3 + 200.AL = 2000 + AL^2.AL + AL.LB = 2000 + AL^2.AB$.

Since we have assumed that AB = 20, we have shown that

$AL^3 + 200.AL = 2000 + 20AL^2$.

Al-Khayyām shows how to construct the triangle (Fig. 72) ABC, with BD perpendicular to AC. He takes AD = 10, DB equal to the segment AL which has been shown to be of known length. Join AB, draw a perpendicular to AB from B such that it meets AD extended in C, then ABC is the required triangle. Next, al-Khayyām shows how to construct the original problem with the use of this triangle. He also mentions that if one wanted to find a solution using 'arithmetic', one could use approximations by refering to the tables of chords in Ptolemy's *Almagest*, or the tables of sines of a trustworthy *Zij*[115].

In the course of solving this problem, al-Khayyām informs the reader that the geometer Abu'l-Jūd succeeded in solving a certain cubic equation which had stumped eminent geometers such as al-Kūhī and al-Ṣaghāni. The problem is as follows: If you divide 10 into two parts, the

[115]Persian word for chord; also, the name for astronomical handbooks containing extensive astronomical and mathematical tables, among the latter one containing values of the sine function.

sum of their squares plus the quotient of the greater over the smaller is 72[116]; al-Khayyām says that the analysis carried Abu'l-Jūd to the cubic equation 'squares are equal to a cube plus roots plus numbers'[117]. In his *Algebra*, al-Khayyām also informs the reader that Abū'l-Jūd was unsuccessful in his attempt to solve the cubic equation: a cube plus numbers are equal to squares. He reproduces and comments upon Abu'l-Jūd's solutions of both cubic equations.

An algebraic quartic equation arises in an anonymous treatise[118], and is solved by means of conic sections. The problem is to construct the trapezoid ABCD such that each of its sides AB, BC, AD equal 10, and that the area equals 90 (Fig. 75). One first draws AK perpendicular to CD, then, taking DK = x, we obtain (AB - x).AK = 90. And, since AB = 10, $AK^2 = 10^2 - x^2$, and consequently $(10 - x)^2 . (100 - x^2) = 8100$, or $x^4 + 2000x = 20x^3 + 1900$. The construction is summarized as follows:

Let AB = 10, and draw EB perpendicular to AB such that EB = $^9/_{10}$ AB[119]. Complete the rectangle BZ and construct a hyperbola $\mathcal{H}$ which passes through E and has AB, AZ for asymptotes. Draw a circle with center B and radius AB. It will intersect $\mathcal{H}$ since AB > BE. Let C be the point of intersection. Draw BC (= AB), and construct the angle BAD = ABC such that AD = BC. Thus, ABCD will be the required trapeze.

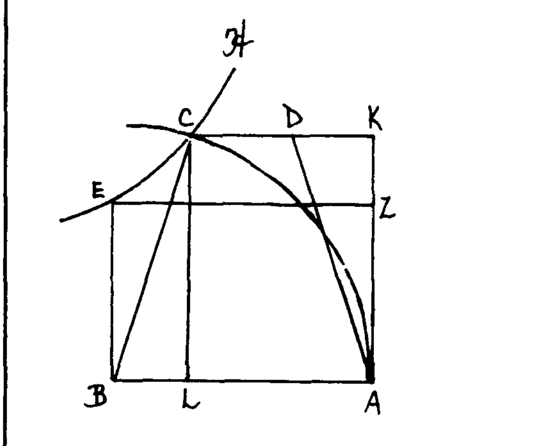

Figure 75

The proof follows easily: draw CL perpendicular to AB, then triangle CBL is similar to triangle

[116]The rational root 2 of this equation is obvious, however, the other positive one is not so evident.

[117]Indeed, the equation for $5<x<10$ is as follows: $x^2 + (10 - x)^2 + x/(10-x) = 72$.

[118]Published by Woepcke 1851, pp 115 - 116.

[119]That is, EB is equal to the ratio of the given area 90 to the given line segment 10.

ADK, hence rectangle ABCD = rectangle ALCK; but since C and E both lie on $\mathcal{H}$,
we also have rectangle ALCK = rectangle ABEZ. Therefore,
rectangle ABCD = rectangle ABEZ = AB.EB = 90.

Naturally, the number and importance of problems reduced to different forms of cubic equations suggested the necessity of a more general theory. Abū'l-Jūd seems to have been the first one to have attempted, on the basis of ancient Greek geometrical procedures, to develop a general theory of cubic equations. But his work ***Book on Geometrical Subjects***, mentioned by al-Khayyām was lost. Included in this book were constructions of many kinds of cubic equations, notably $x^3 + c = ax^2$, $x^3 + bx + c = ax^2$, and probably $x^3 + c = bx$ and $x^3 = c$[120]. Fortunately, the algebraic treatise of al-Khayyām has survived, for it counts as one of the greatest works of Islamic mathematics.

[120]See Hogendijk 9, p 33 on why this is probable.

'Umar Al-Khayyām and Cubic Equations

'Umar Al-Khayyām was born in the middle of the 11th century and lived for approximately 80 years. Outside the Islamic world, he is mostly admired as a poet, especially for the famous verses ascribed to him under the name of *The Rub'āyāt*. However, al-Khayyām made significant contributions in both the sciences of astronomy and mathematics: at a research observatory in Isfahan, he conducted a programme of astronomical investigations which enabled him to prepare a reform of the calender, and in 1070, he completed his great mathematical work found in his book *Algebra*.

In the *Algebra*, which is dedicated to the chief judge of the city of Samarqand, Abū Ṭāhir, al-Khayyām embarks upon the project of treating all cubic equations in their general forms; a feat no other geometer had yet accomplished. This work was foreshadowed by his *Opuscule*[121] in which he presents an exposition of the history of cubic equations and which serves as an introduction to his great work. The *Algebra* contains a classification of cubic equations and geometrical constructions of the roots of these equations as line segments obtained from the intersection of conic sections. Its realization draws on many different sources: the works of Thābit ibn Qurra and al-Khwārizmī on the theory of equations of degree ≤ 2, the methods and results, as we have seen, of previous geometers such as al-Kūhī and Ibn al-Haytham, and the algebraic translation by Abū'l-Jūd and al-Bīrūnī of certain solid problems.

In the first part of the *Algebra*, al-Khayyām lists all types of equations with degree ≤ 3. There are 25 species altogether, since he considers only positive coefficients, 14 of which require the use of conic sections for their construction. The classification of these equations depends not only on the degree, but also on the number of terms. Although they are expressed in words rather than symbols, we refer to them by their modern equivalent. Hence, the 14 species requiring the use of conic sections are listed as follows:

<u>Binomials</u>: (1) $x^3 = c$.

<u>Trinomials</u>: (1) $x^3 + bx = c$, (2) $x^3 + c = bx$, (3) $c + bx = x^3$,

(4) $x^3 + ax^2 = c$, (5) $x^3 + c = ax^2$, (6) $c + ax^2 = x^3$.

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[121] Published by Rashed and Djebbar (1981); it includes the problem "On the division of the quarter circle" previously discussed.

Tetranomials: (1) $x^3 + ax^2 + bx = c$, (2) $x^3 + ax^2 + c = bx$, (3) $x^3 + bx + c = ax^2$,

(4) $x^3 = ax^2 + bx + c$,

And (5) $x^3 + ax^2 = bx + c$, (6) $x^3 + bx = ax^2 + c$, (7) $x^3 + c = ax^2 + bx$.

In the second part, al-Khayyām shows how, for each case, conics can be used to produce a line segment from which solids that satisfy the required relation can be constructed. Before commencing the geometrical resolution of an equation, al-Khayyām writes each equation in homogeneous form. That is, an equation such as $x^3 + bx = c$ is written as $x^3 + p^2x = p^2q$. This transformation depends on a few theorems of elementary geometry. Throughout the discussion, he warns the reader that a particular case may have no solutions, one solution, or two solutions, depending on whether the conics intersect in 0, 1, or 2 points.

First, we present al-Khayyām's construction of the equation $x^3 + c = ax^2$, which appears in Archimedes' problem, also known as al-Māhānī's problem. We know that Abū'l-Jūd attempted to solve this equation in its algebraic form, but, according to al-Khayyām, was unsuccessful. In our presentation of al-Khayyām's construction, we include his corrections of the mistakes he found in Abū'l-Jūd's construction. (Fig. 76)

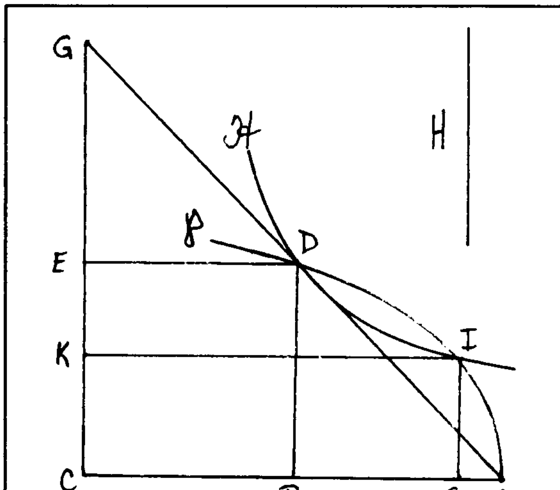

Figure 76

Let AC be equal to the number of squares, AC = a, and construct a cube equal to the given number c, with side H, so that $H^3 = c$. Then there are three cases; either H is equal to AC, is greater than AC, or is less than AC. The first two cases are impossible, since we can only have positive coefficients, therefore, H < AC. Take B on AC such that BC = H. Once again, we consider three cases: (1) BC = AB, (2) BC > AB, and (3) BC < AB. Complete the square BDEC. Draw a hyperbola $\mathcal{H}$ through D with asymptotes AC, CE, and a parabola $\mathcal{P}$ with vertex A, axis AC, and parameter BC.

Case 1: Since $DB^2 = AB.BC$, $\mathcal{P}$ also passes through D. There is another intersection point, say I, between A and B. (Fig. 76)

•Al-Khayyām says here that the eminent geometer Abū'l-Jūd commited an error in claiming that the two sections are tangent at the point D. This is incorrect. Draw AD, extend it to meet CE in G, then AD = DG, so ADG is tangent to $\mathcal{H}$. Now, if the two sections were tangent then the segment from D to an arbitrary point on the section AD of $\mathcal{P}$ would fall between the section $\mathcal{H}$ and its tangent. But this is impossible.

Case 2: D is outside $\mathcal{P}$ since $DB^2 > AB.BC$. If the two sections intersect, then they do so once or twice between B and A. (Fig. 77)

• Again, al-Khayyām says here that the eminent geometer Abū'l-Jūd commited an error in claiming that this case is impossible, that is, that the two conic section do not intersect. Al-Khayyām gives a numerical example, corresponding to the equation $x^3 + 144 = 10x^2$, which disproves Abū'l-Jūd's claim.

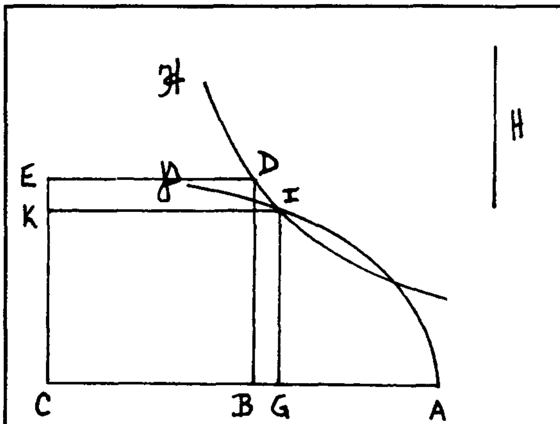

Figure 77

Case 3: D is inside $\mathcal{P}$ since $DB^2 < AB.BC$. There are two points of intersection, one between C and B, say I, and the other between B and A. (Fig. 78)

(All figures) Draw IG perpendicular to AB and IK perpendicular to CE.

Then IG.IK = DB.DE. Therefore, GC/BC = BC/IG.

But IG is an ordinate to $\mathcal{P}$, so $IG^2 = AG.BC$, and BC/IG = IG/GA.

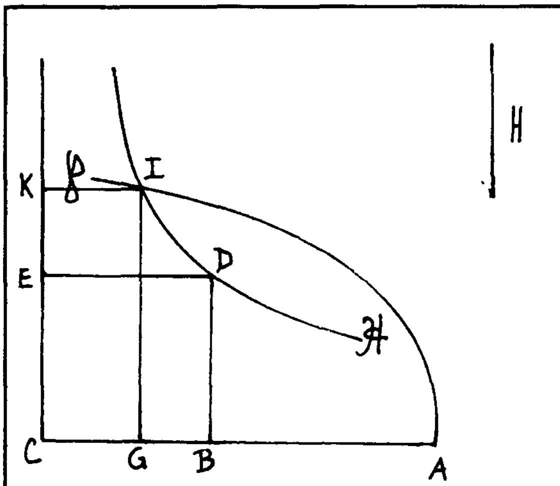

Figure 78

Therefore $GC/CB = CB/IG = IG/GA$, and $GC^2/BC^2 = BC/GA$.

Hence $BC^3 = GC^2.GA$, and, we obtain

$BC^3 + GC^3 = GC^2.AC$ by adding GC^3 to both sides of the equation.

Therefore, GC is the required length.

Although he considers different cases and determines conditions under which positive solutions will exist, al-Khayyām does not exhaust the problem. Already, Eutocius[122] and, later on, al-Kūhī had noticed that the limit of positive roots is determined by the conditions

(I) $c = {}^{4a^3}/_{27}$[123], there is one solution, and for

$c < {}^{4a^3}/_{27}$, there are two positive solutions.

On the other hand, al-Khayyām shows that for

(II) $c \leq (\frac{1}{2}a)^3$ (cases 1 and 3), there can be two roots, for

$c > (\frac{1}{2}a)^3$ (case 2), there can be either one, two, or no solution, and for

$c \geq a^3$, there are no solutions.

By writing $(\frac{1}{2}a)^3 = {}^{3\frac{3}{8}a^3}/_{27}$, we can more easily compare conditions (I) and (II).

We now consider a different species of cubic equation, the tetranomial. Al-Khayyām writes, in his *Algebra,* that Abū'l-Jūd correctly constructed the solution to a certain tetranomial cubic equation but that since it had specified coefficients, he was not led to investigate the different cases that arise from the general cubic equation. The general form of the tetranomial in question is $x^3 + bx + c = ax^2$. Al-Khayyām's construction is as follows (Fig. 79):

Let BE be equal to the number of squares, that is, $BE = a$, and BC be such that BC^2 is equal to the number of roots, that is, $BC^2 = b$. Draw BC perpendicular to BE.

Construct a solid whose base is the square on BC, and let this solid be equal to the given number c[124]. Let the height of this solid be AB, (then $BC^2.AB = c$) drawn on BE extended. Draw the semi-circle AGE on AE. Then either C is inside the circle, on the

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[122]In an equivalent form, see p. 21, above.

[123]See p. 23 for the discussion of Eutocius' solution, and p. 65 for a discussion of al-Kūhī's solution.

[124]This construction is possible due to a lemma that al-Khayyām establishes before treating the equations requiring the use of conic sections.

circumference of the circle, or outside of it.

<u>Case 1</u>: (in which C is inside the circle) Extend BC until it cuts the circle in G, and complete the rectangle AC. Construct on GC a rectangle equal to the rectangle AC, say rectangle CH. Then H is known in position. Now, H is either inside the circle, on the circumference of the circle, or outside of it.

<u>Case 1a</u>: (in which H is inside the circle) Construct a hyperbola $\mathcal{H}$ through the point H with asymptotes GC, CM. Then $\mathcal{H}$ necessarily cuts the circle in two points, say L and N; they are thus known in position. Draw LK and NP perpendicular to AE, and LI perpendicular to BG.

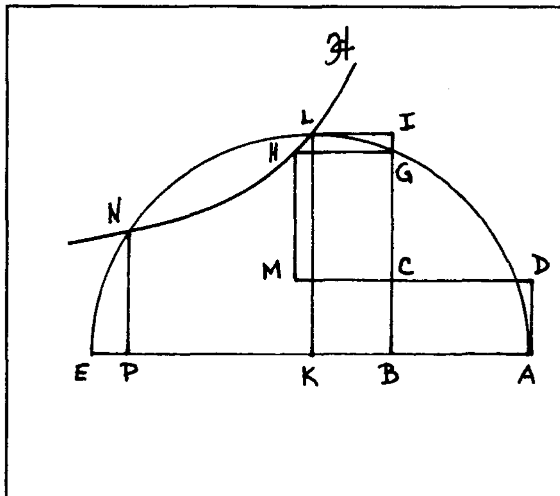

Figure 79

Then rectangle LC = rectangle CH = rectangle CA (property of hyperbola). Adding rectangle CK to both, we obtain rectangle DK = rectangle IK, so

$LK/KA = DA/LI$ and $LK^2/KA^2 = DA^2/LI^2$.

But $LK^2/KA^2 = EK/KA$, since, by a property of circle, $LK/KA = EK/LK$; therefore $BC^2/BK^2 = EK/KA$, and so $BC^2.KA = BK^2.KE$.

But $BC^2.KA = b.BK + c$. Now add BK^3 to both sides, so that

$BK^3 + b.BK + c = BK^2(KE + BK) = BE.BK^2 = a.BK^2$.

Hence BK is a solution to the equation, and similarly BP is a solution.

<u>Case 1b</u>: (in which H is outside the circle) This is the case Abū'l-Jūd mentioned for the determination of his problem. If $\mathcal{H}$ meets $\mathcal{C}$ in a point of tangency or in two points of intersection, then the problem is as in Case 1a. If $\mathcal{H}$ cannot be made to meet $\mathcal{C}$, then the problem is impossible[125].

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[125] Al - Khayyām does not prove this; presumably, one is meant to draw the conic sections for a given choice of coefficients and determine whether they can be made to intersect.

Case 2: (in which C is on the circumference or is outside the circle). Extend GC and construct a rectangle on it equal to the rectangle AC with one angle on C, and such that, if we construct $\mathcal{H}$ with the above-mentioned properties passing through the angle opposite to C, it will intersect the circle either by a point of tangency or by intersection. We apply the same reasoning as in the first case. This can be determined emperically by drawing the conic sections.

Al-Khayyām finishes by noting that for a solution to exist, it must be less than EB, which is the number of squares (EB = a). This is certainly true, although it does not, once again, exhaust the problem. In fact, we see at the end of Case 1a and Case 2, that he does not give a precise condition for the existence of a solution, that is, for the two curves to intersect. He merely states that if they can be made to intersect then there will be one or two solutions, and if they cannot, then there will be no solution. For a given particular equation, on must attempt the construction first to see whether a solution is possible.

Al-Khayyām, makes his equation homogeneous at the outset: he constructs the side BC of a square such that $BC^2 = b$, and then a solid of height AB such that $BC^2.AB = c$. These two constructions are essential to his geometric method, and are steps that were not previously taken in the construction of solid problems. The construction of the point H is also interesting, and provides a good example of a method that was fundamental in ancient Greek geometry: the application of areas[126]. We are essentially trying to find H such that HG.GC = AB.BC. Thus we are solving for HG, and since AB.BC/GC is a given magnitude, HG is known. But HG is to be drawn perpendicular to GC, thus the point H is known in position.

These are two examples of al-Khayyām's treatment of cubic equations, and his geometric constructions by means of conic sections of their roots. There is a question which arises naturally from al-Khayyām's work: how to determine the precise conditions under which there will be one, two, or no solutions to cubic equations. The answer to this question, as we will see, would be given a century later by another mathematician Sharaf al-Dīn al-Ṭūsī. There is another question,

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[126]In this case, we are to *apply* to a given straight line GC an *area* equal to a given rectangle AB.BC. This *application*, or construction of a rectangle equal to a given rectangle can be found in the *Elements* I, 45.

which al-Khayyām asked himself but was unable to answer, that is: how to find numerical solutions to these cubic equations. Numerical solutions to quadratic equations were known to exist, and could be found in terms of the coefficients. However, neither al-Khayyām nor al-Ṭūsī was able to determine the roots of cubic equations by means of an algebraic formula in terms of the coefficients, this would have to wait until the 16th century.

Finally, there are two other facts that were not recognized by al-Khayyām: (1) there are cases where a cubic equation can have not only two but three positive roots for suitable coefficients; (2) similarly, there are cases when cubic equations can have one, two, or three negative roots; it is not surprising that al-Khayyām did not realize this since he dealt only with positive coefficients[127].

[127]This does not mean to say that only equations with negative coefficients can have negative roots; however, al-Khayyām seemed only interested in positive quantities since he was solving these cubic equations by constructing geometrical figures.

Al-Ṭūsī and the Theory of Cubic Equations

Sharaf al-Dīn al-Ṭūsī worked in the late 12th century, and taught geometers such as Kamāl al-Dīn ibn Yūnus (author of a construction of the heptagon). In his *Algebra*, the next step in the development of the theory of cubic equations after that of al-Khayyām, al-Ṭūsī had essentially two goals: first, to give a complete discussion of the conditions under which each cubic equation can be solved by means of intersecting conics; secondly, to devise an algorithm for solving the possible cases numerically[128]. In this paper, as we are mainly interested in the use of conic sections for solving problems, we will concentrate on the first of al-Ṭūsī's contributions.

The first part of al-Ṭūsī's work is very similar to that of al-Khayyām, however, there is one notable difference. Al-Ṭūsī makes consistent efforts to prove the existence of points of intersection of the conics, whereas al-Khayyām does not. These proofs are based on interior/exterior arguments that call upon the convexity and continuity of conic sections. The following example illustrates the method of al-Ṭūsī; it is the construction of the cubic equation $x^3 + ax^2 = c$, which always has a solution. He uses the same conic sections as al-Khayyām, a parabola and a hyperbola. (Fig. 80)

Take a cube of side K and equal to c, so $K^3 = c$, and let $AB = a$. Let C be on AB extended such that $BC = K$, and complete BCED, a square of side BC. Draw a hyperbola $\mathcal{H}$ with vertex E and asymptotes BC, BD and a parabola $\mathcal{P}$ with vertex A and parameter BC. Draw MC perpendicular to AC such that it meets $\mathcal{P}$ at M, then $BC.AC = MC^2$, therefore $MC^2 > BC^2$, since $AC > BC$. Consequently M is above E,

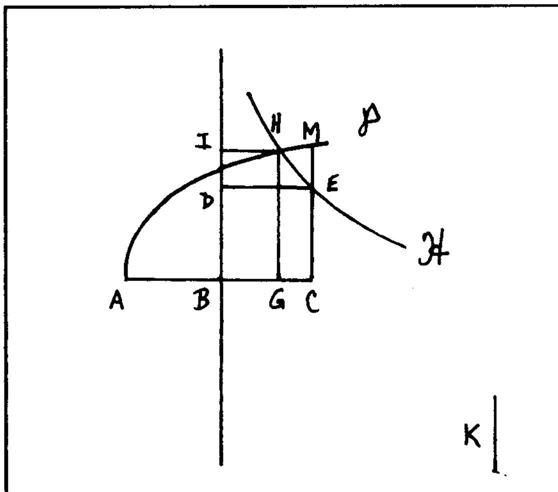

Figure 80

[128]In fact, he gave what is essentially the Ruffini-Horner algorithm for cubic equations. This method was used for the computation of cube roots before the middle of the 3rd century A.D. in China, and in the 10th century in the Islamic world. The generalization to arbitrary cubic equations may have been used in the 11th century by al-Bīrūnī (Hogendijk 4, p. 79).

which is on $\mathcal{H}$; M is then interior to $\mathcal{H}$, therefore $\mathcal{P}$ and $\mathcal{H}$ necessarily intersect. Let H be their point of intersection, draw IH perpendicular to BD and GH perpendicular to AC, then $BC.AG = HG^2$ (equation of $\mathcal{P}$), hence

(1) $AG/HG = HG/BC$, but we also have

$BI.IH = BD^2 = BC^2$ (equation of $\mathcal{H}$), therefore $BI/BC = BC/IH$, that is

(2) $HG/BC = BC/BG$, hence, from (1) and (2) we obtain

$AG/HG = HG/BC = BC/BG$, therefore $BG^2.AG = BC^3$, whence

$BC^3 = BG^2.BG + BG^2.AB = BG^3 + AB.BG^2$, and consequently

$c = BG^3 + b.BG^2$; therefore BG is the required solution.

Al-Khayyām executes the same construction as above, however, he does not prove that the conic sections intersect. Of course, it is possible that it seemed obvious to him since hyperbolas get arbitrarily close to their asymptotes. Nevertheless, al-Ṭūsī's arguments show his determination in rigourously establishing the existence of points of intersection. In order to show that the conic sections necessarily intersect, al-Ṭūsī first takes a suitable point M on $\mathcal{P}$ and shows that M is in the interior of $\mathcal{H}$ since MC > BC, that is, M is above E. Since the point A, which is the vertex of $\mathcal{P}$, is on the exterior of $\mathcal{H}$ and the continuous[129] curve $\mathcal{P}$ goes through the points A and M; then $\mathcal{P}$ intersects $\mathcal{H}$. This proof also suggests where the point of intersection will occur relative to the coefficients. It is obvious from the argument that the point G must fall between A and C, hence, the solution must be strictly smaller than K. Al-Khayyām finds the same upper limit of the solution.

The second part of al-Ṭūsī's *Algebra* is dedicated to five equations that, according to him, give rise to impossible cases, that is, cases where no positive solution exists. This part diverges significantly from the work of al-Khayyām, who was content to simply note when impossible cases could arise. Al-Ṭūsī, apart from being interested in the existence of points of intersection, and thus of the existence of roots, also wanted to seek out the reasons for the impossible cases. This would enable him to characterize them, that is, to identify them through the coefficients of the equation at hand. The five equations are

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[129]Al-Ṭūsī does not explicitly speak of continuity, but it is assumed in his argument.

(1) $x^3 + c = ax^2$, (2) $x^3 + c = bx$, (3) $x^3 + ax^2 + c = bx$,

(4) $x^3 + bx + c = ax^2$, (5) $x^3 + c = ax^2 + bx$.

Al-Ṭūsī actually shows that each of the above equations can be reduced to a species found in the first part of the *Algebra*, thus showing that al-Khayyām's separate geometrical constructions for these equations are superfluous. The species to which each equation can be reduced, provided a solution exists, depends on the coefficients of the equation. Although there are no conic sections in the second part of the *Algebra*, we give a brief summary of al-Ṭūsī's procedure.

For the sake of completeness, we investigate equation (1), which is the one found in Archimedes' problem.

Step 1: Al-Ṭūsī first remarks that $a > x$ since $a.x^2 = x^2.x + c$, then $a.x^2 > x^2.x$.

Step 2: Determination of 'maximum'. Write equation (1) as $c = x^2(a - x)$. Al-Tūsī determines[130] a quantity $m = (⅔)a$ and shows that $x^2(a - x) < (⅔a)^2(a - ⅔a)$ for all positive $x \neq m$. That is, he shows that the expression $x^2(a - x)$ attains its maximum for $x = ⅔a$. This immediately indicates that if $c > (⅔a)^2(a - ⅔a) = {}^4/_{27}a^3$, there is no solution and if $c = {}^4/_{27}a^3$ there is exactly one solution $x = m = ⅔a$.

Step 3: Reduction of equation. Al-Ṭūsī supposes that $c < {}^4/_{27}a^3$, and for $d = {}^4/_{27}a^3 - c$, he considers the equation

(6) $y^3 + ay^2 = d$.

Step 4: Computation of a first root. The unique positive root of (6) has already been constructed geometrically in the first part of the *Algebra* by means of a parabola and a hyperbola, and an algorithm for the computation of this root (say y_1) has also been described. Al-Ṭūsī then proves that $x_1 = m + y_1$ is a root of the original equation (1). The existence of this root is guaranteed by the geometrical construction of y_1, and we are assured of at least one solution x_1 such that $m < x_1 < a$. Now (1) may be rewritten as $c = ax^2 - x^3$, an expression whose right hand side is 0 at $x = 0$ and $x = a$ and attains its

[130]Al-Ṭūsī does not explicitly state how he determines this quantity. There are two opinions on the matter: Rashed (1985) claims that al-Ṭūsī determined local minima and maxima essentially by means of 17th century methods, and Hogendijk 4 argues that he probably found his results by means of manipulations of squares and rectangles on the basis of Book II of Euclid's *Elements*.

maximum $^4/_{27}a^3$, (> c) at x = ⅔a. Hence (1) has two roots on (0, a). Al-Tūsī recognized this and, so, went on to:

Step 5: Computation of second root. Al-Ṭūsī geometrically constructs a segment of length p such that

$p^2 + p(a - x_1) = x_1(a - x_1)$. (As done in the first part of the *Algebra*)

He shows that $x_2 = a - x_1 + p$ is another root of (1) with $x_2 < m$.

In summary, for equation (1), al-Tūsī distinguishes three cases:

(I) $c > {^4/_{27}}a^3$, the problem is impossible;

(II) $c = {^4/_{27}}a^3$, the solution is $m = ⅔a$;

(III) $c < {^4/_{27}}a^3$, there are two positive solutions x_1 and x_2 with $0 < x_1 < ⅔a < x_2 < a$.

Al-Ṭūsī finds that the existence of a solution, in this case, depends on the maximum value attained by the cubic curve. This discovery enables him to give an exact relationship between the number of roots and the coefficients of the equation. Although he describes a numerical procedure for approximating the roots, he was not able to determine them in terms of the coefficients; there is no solid evidence that the algebraic solution of the cubic equation was known before the Italian Renaissance.

A Comparison (al-Kūhī's Problem)

In this section, we compare the solution of al-Kūhī's problem[131] with the construction of the root of the associated cubic equation. We are especially interested in comparing the limits of solvability al-Kūhī obtains with those obtained by al-Khayyām and al-Ṭūsī, respectively, for the corresponding cubic equation. We first recall the results of each geometer, then recast their solutions in a way that will permit us to compare them directly.

The problem: to construct a spherical segment whose spherical surface is equal to that of one segment and whose volume is equal to that of another.

<u>Al-Kūhī's analysis</u>

The parameters: (Fig. 81) Let the spherical segment ABG solve the problem, AZ perpendicular to BD. We have shown that the problem is to determine BD, the diameter of the sphere, and BZ, the height of the segment. We remind the reader that

ρ = segment AB, k = BK, x = BD, and y = DM,

R = ratio of 'cone of surface' C (the right isosceles cone whose base has radius AB) to the cone AGT[132], and

s = ρ/R.

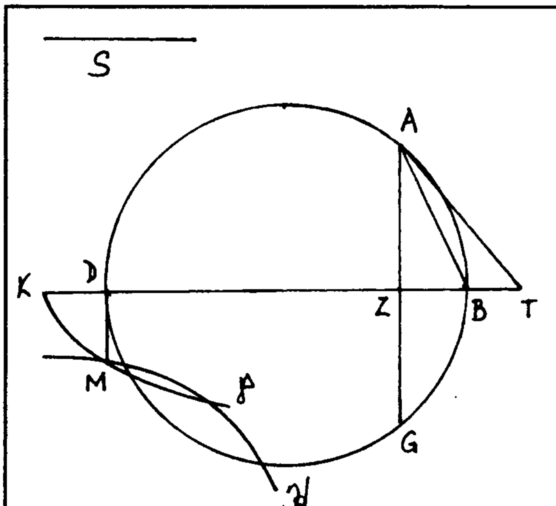

Figure 81

The solution: To determine the point M, if it exists, al-Kūhī uses the intersection of two conics. If M is known in position, then so is D, hence the magnitude BD. The hyperbola is defined by

$$\mathcal{H}_1: \quad xy = \rho^2,$$

and the parabola defined by $\quad \mathcal{P}_1: \quad y^2 = s(k - x),$

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[131]See p. 67 for our discussion of this problem.

[132]The point T is known since the volume of the segment is known: by *On the Sphere and Cylinder* II.2 the volume is equal to the volume of a right cone having the same base as the segment and height TZ defined by TZ/ZB = EZ/ ZD, where EZ = ZD + ½BD.

Eliminating y yields the cubic equation

$$x^3 + \rho^3 R = kx^2.$$

The conditions: Al-Kūhī identifies the following five cases:

(1) if $R < \sqrt{2}$, no solution.

(2) if $R = \sqrt{2}$, one solution since $\mathcal{H}_1$ and $\mathcal{P}_1$ are tangent, and $x = 2k/3$.

(3) if $\sqrt{2} < R < 2$, two solutions; $x_1 > \frac{2}{3}k$, $x_2 < \frac{2}{3}k$.

(4) if $R = 2$, two solutions, $x_1 = \frac{1}{3}k$, $x_2 > \frac{2}{3}k$[133].

(5) if $R > 2$, such a ratio can only correspond to a solution $x > \frac{2}{3}k$.

For $x = 2k/3$, the spherical segment corresponds to a hemisphere (BZ = ZD), and for $x = \frac{1}{3}k$ (that is, ρ = BD) the spherical segment corresponds to the whole sphere. Hence values of $x < \frac{2}{3}k$ correspond to spherical segments larger than a hemisphere (BZ > ZD), and values of $x > \frac{2}{3}k$ correspond to spherical segments smaller than a hemisphere (BZ < ZD).

Now we take a look at the corresponding cubic equation, and al-Khayyām's construction of the root of the cubic equation $x'^3 + c = ax'^2$.

<u>Al-Khayyām's analysis</u>

In order to construct this cubic equation, al-Khayyām uses the intersection of a hyperbola and a parabola:

$$\mathcal{H}_2: \quad x'y' = c^{2/3},$$

and the parabola defined by $\mathcal{P}_2$: $\quad y'^2 = c^{1/3}(a - x')$.

The conditions: Al-Khayyām identifies four cases:

(6) if $c > a^3$, no root.

(7) if $(\frac{1}{2}a)^3 < c < a^3$, there can be one, two, or no roots. There is one root when $x' = \frac{2}{3}c$, and if there are two roots then $\frac{1}{2}c < x_1', x_2' < c$.

(8) if $c = (\frac{1}{2}a)^3$, two roots, and $x_1' = \frac{1}{2}c$.

(9) if $c < (\frac{1}{2}a)^3$, there are necessarily two roots.

<u>Al-Tūsī's analysis</u>

He also uses $\mathcal{H}_2$ and $\mathcal{P}_2$, however, he obtains the correct limits of solvability. Al-Khayyām's

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[133]Actually, al-Kūhī approximates the second solution; he claims it will yield a segment whose height is nearly on-eighth of the diameter of the sphere, or rather larger than this by a small quantity.

conditions (6) and (7) above are inaccurate, in fact, there are no roots for $^4/_{27}a^3 < c < a^3$, as we will see below in al-Ṭūsī's conditions.

The conditions: Al-Ṭūsī identifies three cases:

(10) if $c > {^4/_{27}}a^3$, no roots.

(11) if $c = {^4/_{27}}a^3$, one root, $x' = {^4/_{27}}a^3$

(12) if $c < {^4/_{27}}a^3$, two roots such that $0 < x_1' < ⅔c < x_2' < c$.

We now examine both al-Khayyām's and al-Ṭūsī's conditions in terms of the ratio R. Therefore, from the cubic equation we take $a = \rho^3 R$, and $c = k$. Now, since $R = ⅔k/\rho$ [134], al-Khayyām's four conditions become:

(6′) if $R < \sqrt{^8/_{27}}$, no root.

(7′) if $\sqrt{^8/_{27}} < R < \sqrt{2\ ^{10}/_{27}}$, there can be one, two, or no roots, There is one root when $R = \sqrt{2}$.

(8′) if $R = \sqrt{2\ ^{10}/_{27}}$, there are two roots, and $x_1' = ½k$.

(9′) if $R > \sqrt{2\ ^{10}/_{27}}$, there are necessarily two roots.

Similarly, al-Ṭūsī's three conditions become:

(10′) if $R < \sqrt{2}$, no root.

(11′) if $R = \sqrt{2}$, one root, $x' = ⅔k$.

(12′) if $R > \sqrt{2}$, there are necessarily two roots such that $0 < x_1' < ⅔k < x_2' < k$.

The conic sections in both the above solutions become

$\mathcal{H}_2 : x'y' = k^{2/3}$, and

$\mathcal{P}_2 : y'^2 = k^{1/3}(\rho^3 R - x')$.

For $x' = x$, and $y' = R^{2/3}y$, we have

$\mathcal{H}_2 :\ x R^{2/3} y = k^{2/3}$, hence $xy = k^{2/3}/R^{2/3} = \rho^2$, or $\mathcal{H}_1 = \mathcal{H}_2$, and

$\mathcal{P}_2 :\ (R^{2/3}y)^2 = k^{1/3}(\rho^3 R - x)$, hence $y^2 = \rho/R(k - x) = s(k - x)$, or $\mathcal{P}_1 = \mathcal{P}_2$.

Therefore, $\mathcal{H}_1$ intersects $\mathcal{P}_1$ if and only if $\mathcal{H}_2$ intersects $\mathcal{P}_2$.

The following Table 1 makes the comparison much clearer, as all conditions are plotted

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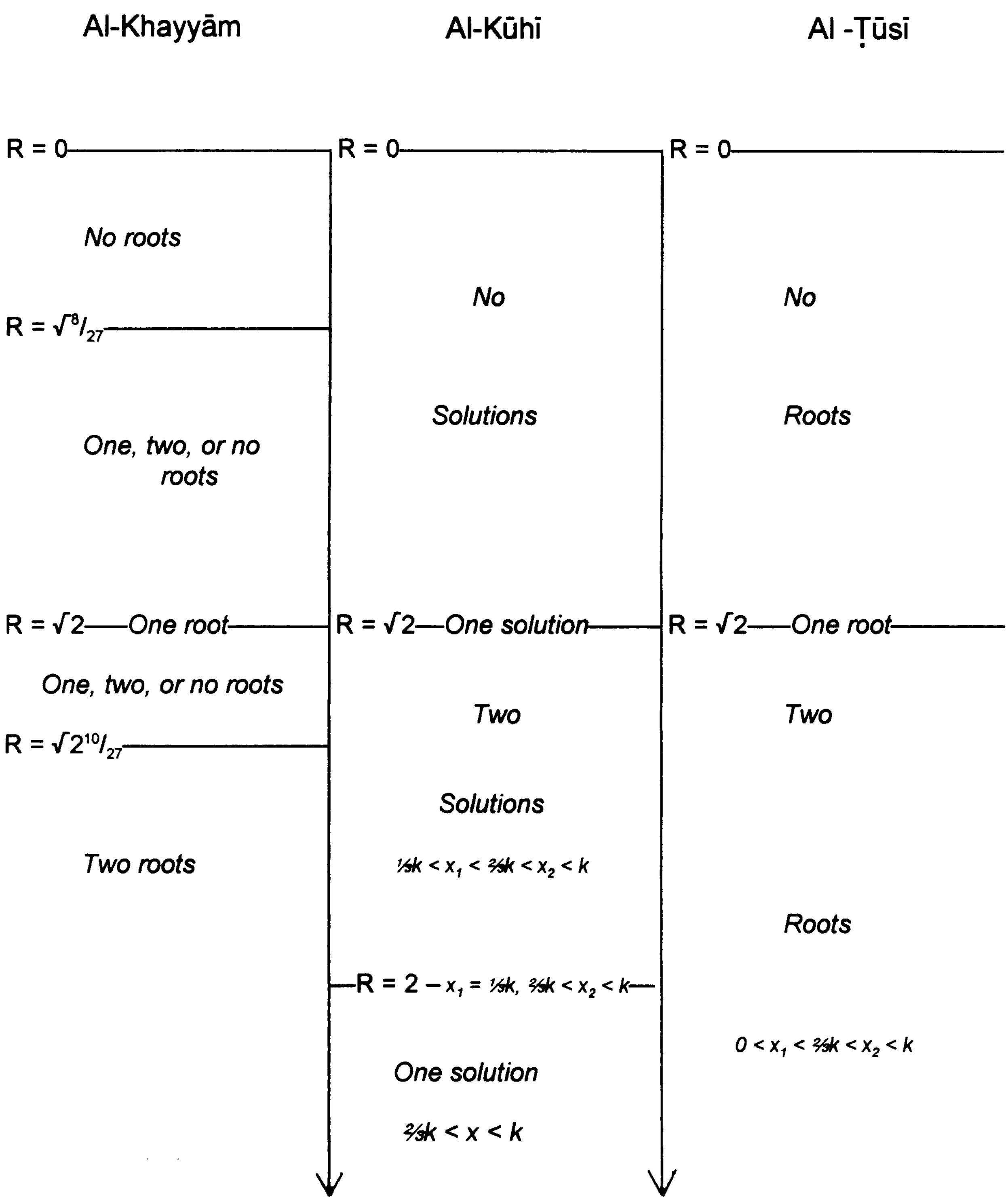
[134] This follows directly from the definition of R and BK.

in terms of the ratio R. There are two R-axes that run vertically, increasing from R = 0.

TABLE 1



The three geometers guarantee two roots for values of $R > \sqrt{2\ {}^{10}/_{27}}$. On the interval ($\sqrt{2}$,$\sqrt{2\,{}^{10}/_{27}}$), both al-Kūhī and al-Ṭūsī can guarantee two roots, but al-Khayyām cannot. On the interval ($\sqrt{{}^{8}/_{27}}$, $\sqrt{2}$), both al-Kūhī and al-Ṭūsī can guarantee no roots, whereas al-Khayyām cannot. Al-Kūhī's and Al-Ṭūsī's conditions are identical except that the former identifies the condition $R = 2$ which is specific to his problem and which does not enter into the considerations of the general cubic equation. The condition for $R > 2$ does not arise in the general cubic equation because *a* and *c* are in no way related. On the other hand, the quantities k, R and ρ are interdependent as can be seen from the relation $R = {}^{2}/_{3}k/\rho$. Table 1 shows immediately where al-Khayyām errs in his conditions (6′) and (7′).

Both al-Kūhī and al-Ṭūsī invoke the 'Lemma of Archimedes' in finding their respective conditions for a solution. The lemma states that if a line GA be divided at B such that BG = 2BA, then $BG^2.\ BA > GD^2.\ DA$ for any other point D on the line GA (Fig. 82). This lemma first appeared in the solution Eutocius gave of Archimedes' problem, where he states the same result, and gives a proof of it by means of conic sections[135].

Figure 82

Al-Ṭūsī draws on this result directly in his determination of condition (10); he finds that the maximum value attained by the expression $ax^2 - x^3 = x^2(a - x) = BG^2.BA$ occurs when BG = 2AB. Al-Kūhī, on the other hand, uses the result when he is finding the minimum value for the ratio R. Since the numerator of R depends only on ρ, this occurs when the denominator is maximum. The expression on the denominator is the volume of the cone AGT which equals $BZ.ZE^2$. Al-Kūhī states that this expression attains a maximum when ZE = 2BZ. Neither al-Kūhī nor al-Ṭūsī states how he came to the result in the lemma of Archimedes. Perhaps we are to assume that it was accessible to them through the transmission of Eutocius' commentary on Archimedes' *On the Sphere and Cylinder*; however, they both provide a proof of the result without

[135]See pp 21-22, above.

using conic sections.

Both proofs are divided into two parts; the first part shows that (Fig. 82)

(1) $BG^2 . BA > GD^2 . DA$ for any other point D between A and B, and the second part shows that (2) $BG^2 . BA > GF^2 . FA$ for any other point F between B and G. Of course, the proofs of part (1) and (2) are very similar for each geometer. The following is al-Kūhī's proof based on Berggren 1996. (Fig. 83)

Extend AG to E such that EG = BG, then

$BG^2 = AB.BE$.

Now, since B is nearer than D to the middle of segment AE, we have

(*) $AB.BE > AD.DE$, whence

$BG^2 > AD.DE$.

Therefore, we can write $DE.DB/BG^2 < DE.DB/AD.DE$, so $DE.DB/BG^2 < BD/AD$, or

$(BD.DE).DA < BD.BG^2$, and by adding $BG^2.DA$ to both sides, we obtain

$BD^2.DA < BG^2.AB$.

Al-Ṭūsī's proof is almost identical except that he supplies details of the argument resulting in (*) that al-Kūhī assumes without proof. This result can be found in Euclid's *Elements* II, 6; it applies directly to al-Kūhī's proof using the line segment EA. However, it is not as easily seen in al-Ṭūsī's proof since he does not follow al-Kūhī's procedure of extending AG to E.

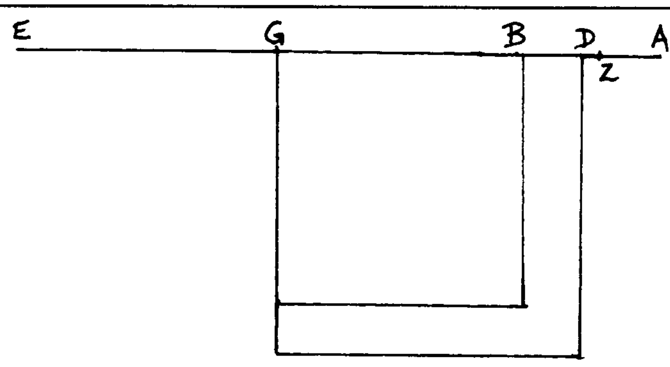


Figure 83

Al-Khayyām did not recognize the condition (10) which follows from the lemma of Archimedes, although he did realize that the conic sections would be tangent when BG = 2AB, that is, when $x = \frac{2}{3}c$. It is especially noteworthy that al-Kūhī, who worked before both al-Khayyām and al-Ṭūsī, not only correctly used the lemma of Archimedes, but found the simplest proof of it[136]. Moreover, he also proves that if D, Z are on AB (Fig. 83) and if D is nearer to B than Z is then $AD.DG^2 > AZ.ZG^2$.

[136]Both are significantly easier than the proof via conic sections.

Conclusion

The ruler and compass were the first tools that the ancient Greeks had to construct solutions to geometric problems. Euclid's *Elements* provided both a basis and a method for these constructions; the first three Postulates acted as postulates of construction, and the Propositions themselves, such as inscribing rectilineal figures in a circle and the problem of the application of areas, provided a method that would constantly be used by Greek mathematicians in their solutions to geometric problems. This already rich tradition of geometry as exemplified in the *Elements* grew considerably with the introduction of conic sections as a means of problem-solving.

The solutions to special geometric problems such as the cube duplication, the angle trisection and the problem of Archimedes served to open a new class of possible constructions, which included neusis constructions, mean proportionals, and certain line segments satisfying certain relations[137]. All these types of constructions would be called upon by many Islamic geometers for solutions to other geometric problems such as the construction of the heptagon, which could also be reduced to the construction of a certain triangle. Geometers could approach any problem, and seek to reduce it to some type of construction in this new class created by the use of conic sections. A prime example is the approach Ibn al-Haytham took in the problem of finding a number divided into two parts such that one is the cube of the other[138], which he transformed into one of finding certain mean proportionals. Another is al-Khayyām's reduction of the division of the quarter circle into that of the construction of a certain triangle[139].

This new class of constructions required the geometer to derive expressions which were in the form of the symptoms of conic sections relative to certain lines from the given relations in the problem. Naturally, this necessitated a thorough knowledge and understanding of the conic sections and their properties. In the problems we have surveyed, an overwhelming majority were solved by means of parabolas and hyperbolas; the parabolas were defined by the relation

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[137]Equivalent, of course, to cubic equations.

[138]See p. 72 for the discussion of this problem.

[139]See p. 73 for the discussion of this problem.

$y^2 = px$, and the hyperbolas were often defined by the relation $xy = ab$. The ellipse was only used once, by Diocles, most likely because its symptom is a rather complicated ratio (as is the hyperbola's) and, since it does not have asymptotes, there is no property such as the above-mentioned one for the hyperbola that a geometer could appeal to. Indeed, this property for the hyperbola, as proved in *Conics* II.4, made it much easier for geometers to describe a hyperbola in their analyses.

One wonders whether the geometers thought it possible to construct their solutions by means of intersecting different conic sections. In the case of the cube duplication, we have four solutions by means of four different combinations of conic sections; it seems as though there was a motivation to find different combinations in this case. However, in other problems, the goal seemed to have been to discover simpler solutions rather than explore different combinations of conic sections. In fact, Abū'l-Jūd wrote a letter to argue that his construction of the heptagon was a simpler than the methods used by al-Kūhī and al-Ṣaghānī. One of his arguments is that, according to him, it is *known* that the parabola is simpler than the hyperbola, so his solution by means of a parabola and a hyperbola is simpler than al-Ṣaghānī's by means of three hyperbolas.

The task of identifying conic sections in the anaysis of a problem was not always straightforward, as we saw in al-Kūhī's problem, and the methods used by geometers varied from problem to problem. That is, there was no pattern, no general procedure that could be applied to all such constructions. It was perhaps the search for such a pattern that motivated both Greek and Islamic geometers to discover alternate solutions to a given problem. Was there a certain symbiosis between the problems and the methods for solving them? The answer to this question would come through the process of generalizing the problems themselves.

Indeed, the process of generalizing in the field of geometric studies often stimulates by raising new questions for solution not posed in the special problems. This was quite apparent in the generalization of Archimedes' problem, which required an investigation into the conditions necessary and sufficient for the existence of a solution. This investigation proved to require a high level of geometrical rigour, as did similar investigations made by al-Kūhī and later on by al-Ṭūsī.

Gradually, some geometric problems, the first being Archimedes' problem, were translated into algebraic form and were solved as special cubic equations also by means of conic sections.

Islamic geometers realized that they could use the same methods for constructing these special cubic equations as they had in their constructions of geometric problems. Once again, the process of generalization led Islamic geometers to consider different forms of cubic equations, and this led to the creation of some kind of general theory of cubic equations.

Al-Khayyām interpreted these algebraic cubic equations geometrically by constructing their roots by means of intersecting conic sections. The roots, of course, were the line segments that would make the two solids represented by each side of the cubic equation equal to each other. In a sense then, the construction of roots was simply the construction of two different solids equal in volume to one another, very much then the kind of problem Pappus identified as "solid". Although al-Khayyām implies in his *Algebra* that cubic equations in general cannot be constructed by means of ruler and compass, and Descartes repeats this in 1637, it was not until 1837 that P.L. Wantzel proved it[140]. Actually, it is interesting to note, in light of our earlier discussion on combinations of conic sections, that in his *Geometria* Descartes gave unique constructions of roots of cubic equations exclusively by means of a parabola and a circle.

We have seen how early solutions of problems such as the cube duplication and the construction of the heptagon were attempted by 'planar' methods, that is, by means of ruler and compass only. It would almost seem natural, for geometers experienced in problem-solving via conic sections, also to attempt solving 'planar' problems by means of 'solid methods'. In fact, Pappus mentions such a case in Book IV of the *Collection*; he criticizes Apollonius for using conics to solve 'plane' problems in *Conics* V, 58 and 62. Although Pappus is correct, since Apollonius could have used a circle instead of a hyperbola for solving these two problems, Apollonius' constructions apply to a wider range of problems (Conics V, 58 - 63), all of which could not have been solved by 'planar' methods. It would be interesting to discover other such cases, and to determine whether the 'solid' construction is either more elegant or more general.

In a sense, the cubic equations solved by al-Ṭūsī in the second part of his *Algebra* are examples of problems which do not require 'solid' constructions. Al-Khayyām had given them separate constructions by means of intersecting conic sections, but al-Ṭūsī showed that they could

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[140]There a useful method to determine whether a problem is constructible by means of ruler and compass described by Kaplanski, I. *Fields and rings*. Chicago - London, 1972.

all be reduced to the cubic equations found in the first part of the *Algebra*.

Over a span of almost 1500 years, conic sections became another tool not only in problem-solving, but in the development of the theory of cubic equations. During this time, ancient Greek and Islamic geometers alike were brilliantly able to turn many difficult problems into the construction of two simple curves.

CHART 1 - Comparison of Greek and Islamic methods

PROBLEM	METHOD	GREEK SOL'N	Corresponding ISLAMIC SOL'N	Corresponding CUBIC EQ'N
Cube Duplication	$\mathcal{P}_1, \mathcal{P}_2$	Menaechmus	al-Muʿtaman	$x^3 = c$
		Diocles via focus-directrix	---------------	
	$\mathcal{P}, \mathcal{H}$	Menaechmus	al-Muʿtaman	
	$\mathcal{H}, \mathcal{C}$	Inspired by Apollonian neusis	Abū Bakr, al-Khāzin, al-Ṭūsī, al-Muʿtaman	
	Conchoid	Nicomedes , sliding ruler	al-Khāzin via $\mathcal{H}, \mathcal{C}$	
	$\mathcal{P}, \mathcal{C}$	-----------------------	al-Muʿtaman	
Angle Trisection	Neusis via $\mathcal{H}, \mathcal{C}$	Pappus	Aḥmad and Thabit	$x^3 + ch(\alpha) = 3x$ where $x = ch(\frac{1}{3}\alpha)$, and α is the angle to be trisected.
	Directly via $\mathcal{H}$	Pappus	al-Kūhī and al-Sizjī	
		Pappus, focus directrix	----------------	
Heptagon	Archimedean (?) *Lemmata*, Prop. 17		al-Ṣaghānī, $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$	$x^3 + a^3 = 2ax^2 + a^2x$
			Ibn al-Haytham $\mathcal{P}_1, \mathcal{P}_2$	
			Abū'l-Jūd $\mathcal{P}, \mathcal{H}$	
			Kamāl al-Dīn $\mathcal{H}_1, \mathcal{H}_2$	
	Archimedean (?) *Lemmata*, Prop. 18		al-Kūhī $\mathcal{P}, \mathcal{H}$ and $\mathcal{H}_1, \mathcal{H}_2$	$x^3 + ax^2 = 2a^2x + a^2$
			Ibn al-Haytham $\mathcal{P}, \mathcal{H}$	
	Construction of triangle based on a partition of 7.	--------------------	Naṣr ibn Abdallāh $\mathcal{H}_1, \mathcal{H}_2$	$x^3 + 2ax^2 = a^2x + a^3$
			Ibn al-Haytham	
	Two parts; line segment and triangle	--------------------	Ibn Sahl $\mathcal{P}, \mathcal{H}$	$x^3 + ax^2 = 2a^2x + a^2$
From Archimedes' Sphere and Cylinder II, 4	$\mathcal{P}, \mathcal{H}$	Eutocius with diorismos	unknown geometer	$x^3 + c = ax^2$
		Dionysorus, directly	Ibn al-Haytham	
	$\mathcal{H}, \mathcal{C}$	Diocles	-----------------------	
	Translation into algebraic expression by al-Māhānī	-------------------------	al-Khāzin	
			Abū'l-Jūd attempt	

CHART 2 - Survey of Cubic Equations

[ ] = page # in our thesis where topic is discussed.

Problem	Cubic Equation	Name and Method of Geometer	Al-Khayyām and al-Ṭūsī
Archimedes' Problem [81]	$x^3 + c = ax^2$	Abū'l-Jūd, incomplete, using $\mathcal{H}$ and $\mathcal{P}$	$\mathcal{H}$ and $\mathcal{P}$, requires diorismos.
Al-Kūhī's problem [67]	$x^3 + c = ax^2$	Al-Kūhī, using $\mathcal{H}$ and $\mathcal{P}$, complete diorismos.	$\mathcal{H}$ and $\mathcal{P}$, requires diorismos.
Heptagon [72]	$x^3 + p = qx^2 + rx$	Kamāl al-Dīn, using $\mathcal{H}$ and $\mathcal{P}$	$\mathcal{H}_1$ and $\mathcal{H}_2$, requires diorismos.
Nonagon [61]	$x^3 + 1 = 3x$	Abū'l-Jūd	$\mathcal{H}$ and $\mathcal{P}$, requires diorismos.
	$x^3 = 1 + 3x$	al-Bīrūnī	$\mathcal{H}$ and $\mathcal{P}$, always has a solution.
Division of a number into two parts...[72]	$x^3 + x = k$	Ibn al-Haytham, uses $\mathcal{H}$ and $\mathcal{P}$	$\mathcal{C}$ and $\mathcal{P}$, always has a solution.
Division of the Quarter Circle [74]	$x^3 + 200x = 20x^2 + 2000$	al-Khayyām, uses $\mathcal{H}$ and $\mathcal{C}$	$\mathcal{H}$ and $\mathcal{C}$, always has a solution.
Division of 10 into two parts such that...[77]	$x^3 + bx + c = ax^2$	Abū'l-Jūd, uses $\mathcal{C}$ and $\mathcal{P}$	$\mathcal{H}$ and $\mathcal{C}$, requires diorismos.

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