SUPERLINKED FINITE COVERS WITH CENTRAL KERNELS

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Superlinked finite covers with central kernels

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Abstract

We investigate the structure of transitive, untwisted, superlinked finite covers whose kernels are central in their automorphism groups. We introduce the concept of an extended conjugate system for a pair (W, K), where W is a permutation structure and K is a finite abelian group. This concept allows us to characterize the given class of finite covers for structures W which satisfy a fairly general condition; further, the irreducibility of such a cover is equivalent to a simple condition on a corresponding extended conjugate system. Finally, we consider structures with strong types, for which there is a much simpler characterization.

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Dedication

For Dee.

Contents

Chapter 1

Introduction

Among the topics currently attracting much interest from model theorists is the investigation of covers of countable $\aleph_0$-categorical structures. The main reason for this interest is the connection between covers and the programme of cataloguing all countable totally categorical structures. Zil'ber's Ladder Theorem ([8, Theorem 7.2]) tells us that any such structure is built up from a strictly minimal set by a finite sequence of finite and affine covers; so to complete the programme, what is needed is a better understanding of covers.

Our focus will be on finite covers. Intuitively, a finite cover of a structure W is obtained in two steps. First we replace each element of W with a finite set, producing a free cover (whose only relations are those inherited from W). Next we expand the free cover by (possibly) adding new relations which induce no new structure on W. The general problem, one which seems quite difficult, is to describe completely the finite covers of a given structure.

A few specific examples have been studied in detail, most notably the case where W is the projective space of the countably infinite dimensional vector space over the field of size two (see [1, 9]). The general situation is somewhat reduced in [4, 5, 6] by decomposing finite covers into simpler ones. Furthermore, both Ivanov [10] and Evans [4, 5] have made progress using a notion of universal cover: the former classifies finite covers of highly homogeneous structures, and the latter considers structures which have a so-called graphic triple of types. Yet in spite of this work, much more must be done to bring the complete solution within reach.

In this thesis we take another nibble. We concentrate on finite covers that are transitive and untwisted, with finite central kernels. The definitions of these properties are given in Chapter 2, along with the formal definition of a finite cover. Since the structures of interest are $\aleph_0$-categorical, we prefer to think in terms of permutation groups.

In Chapter 3 we present the crucial tool for our investigation, the extended conjugate system. This idea is really just an extension of the conjugate systems that Evans studies in the context of locally transitive finite covers (see [5]). Under a fairly general assumption on W, we show that every transitive, untwisted finite cover of W with finite central kernel is uniquely determined by its extended conjugate system. Futhermore, we present a condition on the extended conjugate system which is equivalent to the irreducibility of the corresponding cover.

We add a stronger assumption on W in the final chapter, namely that W has a strong type. In this situation, every extended conjugate system is determined uniquely by a small part of itself, the conjugate system as defined by Evans. Applying this to the material in Chapter 3, we get a slight improvement of Evans' result in [5].

Although the presentation is fairly self-contained, it is assumed that the reader is well-versed in the usual model-theoretic concepts, for example those found in [3]. In addition, knowledge of the basic concepts of general topology and permutation group theory would certainly be an asset.

As for notation, we use the capital Roman letters W and M both for structures and their bases; the context will clarify this ambiguity. Other sets are assigned the letters X or Y. The set of n-tuples and the set of finite sequences of elements from X are denoted by X^n and $X^{<\omega}$, respectively, and $\wp(X)$ denotes the power set of X. We reserve the letters G and H for groups, all of which will be permutation groups. Finally, $\mathrm{Sym}(X)$ denotes the symmetric group on the set X, and $\mathrm{Aut}(W)$ denotes the automorphism group of the structure W. Remaining notation will be defined as we go along.

This thesis owes a particular debt to the work of Evans [4, 5], which supplied both the background and the inspiration.

Chapter 2

Background

Our aim in this chapter is to transform the intuition behind the concept of a finite cover into a formal definition. It should be clear that when we speak of a structure W, we are not interested in any particular language; what concern us are the relations on W that are 0-definable. When W is $\aleph_0$-categorical, these are exactly the relations that are invariant under $\mathrm{Aut}(W)$ (see [7, Corollary 7.3.4]). So we want our definition to express a relationship between automorphism groups. This definition will be given in Section 2.2.

First we review some well-known results about permutation groups.

2.1 Permutation Groups and Structures

A *permutation group* G on a set X is just a subgroup of the symmetric group $\mathrm{Sym}(X)$. Such a group naturally partitions X into *orbits*: the G-orbit of a point $x \in X$ is the set $[x] = \{gx : g \in G\}$. We say that G is *transitive* on X if it has only one orbit, and *regular* on X if in addition no $g \in G \setminus \{1_X\}$ has a fixed point in X. The *stabilizer* in G of a set $Y \subseteq X$ is the subgroup $G_Y = \{g \in G : \forall y \in Y(gy = y)\}$; when $G = \mathrm{Sym}(X)$, we use the notation $\mathrm{Sym}(X/Y)$ for this stabilizer.

More generally, if H is any group acting on X via a group homomorphism $\eta : H \to \mathrm{Sym}(X)$, we can similarly define the stabilizer in H of a set $Y \subseteq X$ and the H-orbit of a point $x \in X$. H acts transitively (resp. regularly) if its image under η is transitive (resp. regular) on X. The action is *faithful* if η is injective.

Note that any permutation group G on a set X has a natural action on X^n, given by $g(x_0, \ldots, x_{n-1}) = (gx_0, \ldots, gx_{n-1})$ for $g \in G$, $x_0, \ldots, x_{n-1} \in X$. Similarly, G has a natural action on $\wp(X)$, given by $gY = \{gy : y \in Y\}$ for $g \in G$, $Y \subseteq X$. When we speak of G acting on these sets, we always refer to the natural actions.

There is a standard topology on Sym(X), defined in terms of its stabilizers. We take as a basis of open sets all cosets of stabilizers of finite tuples from X, i.e. the sets $g \cdot \text{Sym}(X/\overline{a})$, where $g \in \text{Sym}(X)$, $\overline{a} \epsilon X^{<\omega}$. One of the useful features of this topology is that it picks out the automorphism groups of first-order structures on X.

Proposition 2.1 (i) *A sequence $\langle f_n \rangle$ converges to f in Sym(X) if and only if for any $x \in X$, $f_n x = fx$ for all sufficiently large n, ie: $\langle f_n \rangle$ converges pointwise to f.*

(ii) *Sym(X) is a Polish group, ie: a separable completely metrizable topological group.*

(iii) *A subgroup of Sym(X) is closed if and only if it is the automorphism group of some first-order structure on X.*

Proof: (i) If $\langle f_n \rangle$ converges to f in Sym(X) then for any $x \in X$, $f \cdot \text{Sym}(X/x)$ is an open neighborhood of f, so eventually $f_n \in f \cdot \text{Sym}(X/x)$; that is, $f_n x = fx$ for all sufficiently large n.

Conversely, suppose $\langle f_n \rangle$ converges pointwise to f. Given any open neighborhood U of f, there is some $\overline{a} \in X^{<\omega}$ such that $f \cdot \text{Sym}(X/\overline{a}) \subseteq U$. But since $\overline{a}$ has finite length, $f_n \overline{a} = f\overline{a}$ for all sufficiently large n. So $f_n \in f \cdot \text{Sym}(X/\overline{a}) \subseteq U$ for large n.

(ii) Using (i) it is easy to show that Sym(X) is a topological group. We get a countable dense subset by choosing for each pair $\overline{a}, \overline{b} \in X^{<\omega}$ of the same length a permutation $g_{\overline{a},\overline{b}} \in$ Sym(X) which maps $\overline{a}$ to $\overline{b}$. Finally, if we enumerate X as $x_0, x_1, \ldots$, we can derive the given topology on Sym(X) from the following complete metric:

$$d(g,h) = \begin{cases} 0, & \text{if } g = h \\ 1/2^i, & \text{if } i \text{ is minimal such that } gx_i \neq hx_i \text{ or } g^{-1}x_i \neq h^{-1}x_i. \end{cases}$$

(iii) This is (2.6) from [2]. □

We have already observed that the automorphism group tells us all we need to know about an $\aleph_0$-categorical structure. Now we know which permutation groups are indeed automorphism groups. Stripping away the extraneous information, we are led to the following definition, due to Evans [5].

Definition 2.2 A *permutation structure* is a pair (W, G), where W is a non-empty set and G is a closed subgroup of $\mathrm{Sym}(W)$. W is called the *base* and G is called the *automorphism group* of the permutation structure. The structure is *irreducible* if its automorphism group has no proper closed subgroups of finite index. In practice, we will denote the automorphism group by $\mathrm{Aut}(W)$, and we will refer to the permutation structure simply as W.

By the Ryll-Nardzewski Theorem [7, Theorem 7.3.1], a first-order structure is $\aleph_0$-categorical exactly when its automorphism group has only finitely many orbits on n-tuples, for each $n \in \omega$. Thus, it makes sense to define $\aleph_0$-categoricity of permutation structures. Although the assumption of $\aleph_0$-categoricity is needed to guarantee that a permutation structure corresponds to a unique (up to interdefinability) first-order structure, we will not make this assumption unless explicitly stated. We do, however, assume the countability of all permutation structures (i.e. the base is always assumed to have cardinality $\aleph_0$).

Given a permutation structure W, a subset $X \subseteq W$, and a set Y on which $\mathrm{Aut}(W)$ acts, we denote by $\mathrm{Aut}(X/Y)$ the group of permutations of X which extend to automorphisms of W fixing Y pointwise. In particular, $\mathrm{Aut}(W/Y)$ is the stabilizer of Y in $\mathrm{Aut}(W)$. For a finite sequence $Y_1, \ldots, Y_m$ we write $\mathrm{Aut}(W/Y_1, \ldots, Y_m) = \bigcap_{i=1}^m \mathrm{Aut}(W/Y_i)$, and for a singleton $\{y\}$ we write $\mathrm{Aut}(W/y) = \mathrm{Aut}(W/\{y\})$. If X is a finite subset of W, an *n-type over X in W* is an $\mathrm{Aut}(W/X)$-orbit on W^n. When W is $\aleph_0$-categorical, this corresponds to the set of realizations of a model-theoretic n-type over X in W (see [7, Corollary 7.3.3]).

2.2 Finite Covers

Adding new relations to an $\aleph_0$-categorical structure corresponds to passing to a proper closed subgroup of its automorphism group. So it should be clear how to express the concept of finite cover in the context of permutation structures. Again we take the definition from [5].

Definition 2.3 A *finite cover* of the permutation structure W consists of a permutation structure M together with a finite-to-one surjection $\pi : M \to W$ such that

(i) the *fibres* $M(w) = \pi^{-1}(w)$, $w \in W$, form an $\mathrm{Aut}(M)$-invariant partition of M;

(ii) the induced *restriction map* $\rho : \mathrm{Aut}(M) \to \mathrm{Sym}(W)$, given by $\rho(g)w = \pi(g\pi^{-1}(w))$ for $g \in \mathrm{Aut}(M)$, $w \in W$, has image $\mathrm{Aut}(W)$.

When $f \in \mathrm{Aut}(W)$ and $g \in \mathrm{Aut}(M)$ are such that $\rho(g) = f$, we say that f is the *restriction* of g to W, and g is an *extension* of f.

The *kernel* of a finite cover $\pi : M \to W$ is just the kernel $\mathrm{Aut}(M/W)$ of the corresponding restriction map ρ. If this is finite, the cover is *superlinked*; if it is trivial, the cover is *trivial*. Given $w \in W$, we call $\mathrm{Aut}(M(w)/w)$ the *fibre group* above w and $\mathrm{Aut}(M(w)/W)$ the *binding group* above w. Note that the binding group is always a normal subgroup of the fibre group; the cover is *untwisted* if, for all $w \in W$, the fibre group is the same as the binding group. A *transversal* of the cover is a map $a : W \to M$ such that $a(x) \in M(x)$ for each $x \in W$.

Example 2.4 A vector space covering its projective space provides an easy illustration of these concepts. Explicitly, consider an $\aleph_0$-dimensional vector space V over a finite field F_q, remove the origin to get $M = V \setminus \{0\}$, and let W be the corresponding projective space consisting of the 1-dimensional subspaces of V. For the automorphism groups we set $\mathrm{Aut}(M) = \mathrm{GL}(V)$, the group of all non-singular linear transformations of V, and $\mathrm{Aut}(W) = \mathrm{PGL}(V)$, the quotient of $\mathrm{GL}(V)$ by its center. Mapping each non-zero vector to the subspace that it spans produces a finite cover $\pi : M \to W$ with fibres of size $q - 1$. The kernel, the fibre groups and the binding groups are all isomorphic to F_q^*, the multiplicative group of F_q. In particular, this cover is both superlinked and untwisted.

We end this section with three easy lemmas that appeared in [5]. They will be useful in Chapter 3, when we restrict our attention to finite covers that are transitive, untwisted, and superlinked.

Lemma 2.5 *The kernel of any irreducible, superlinked finite cover $\pi : M \to W$ is central in Aut(M).*

Proof: Let K be the kernel of the cover, and let N be the centralizer of K in $\mathrm{Aut}(M)$. We want to show that $N = \mathrm{Aut}(M)$. Now N is the kernel of the natural group homomorphism $F : \mathrm{Aut}(M) \to \mathrm{Aut}(K)$ which sends $g \in \mathrm{Aut}(M)$ to the map

$$h \mapsto ghg^{-1} \quad (h \in K).$$

So N has finite index in $\mathrm{Aut}(M)$, since $\mathrm{Aut}(M)/N$ is isomorphic to the (finite) image of F. It is easy to see that N is closed: it contains the limit of each of its convergent sequences. The irreducibility assumption now implies that $N = \mathrm{Aut}(M)$. □

Lemma 2.6 *Suppose $\pi : M \to W$ is a transitive finite cover with central kernel. Then Aut($M/a, W$) is trivial for any $a \in M$.*

In particular, if $\pi : M \to W$ is an untwisted, transitive finite cover with central kernel K, then K acts faithfully and regularly on each fibre. Furthermore, given $w \in W$, each $f \in \mathrm{Aut}(W/w)$ has a unique extension in $\mathrm{Aut}(M/M(w))$.

The third and final result of this section uses the fact that the restriction map of any finite cover is closed and continuous (see [5, Lemma 1.1]).

Lemma 2.7 *If Aut(W/w) is irreducible for each $w \in W$ then every finite cover of W is untwisted.*

Chapter 3

Main Results

From now on we concentrate exclusively on finite covers that are transitive, untwisted and superlinked, with central kernels. Part of the motivation for studying this class of covers comes from the following conjecture, which appeared in [5]. Here, $\mathrm{Aut}^0(W/X)$ denotes the intersection of the closed subgroups of finite index in $\mathrm{Aut}(W/X)$, where X is a finite subset of the structure W.

Conjecture 3.1 *Let W be a countable, irreducible $\aleph_0$-categorical structure, and suppose that for every finite $X \subseteq W$, $Aut^0(W/X)$ is of finite index in $Aut(W/X)$. Then there is a natural number r such that the kernel of any irreducible superlinked finite cover of W is generated by a set of size no greater than r.*

We know that the kernel of any irreducible superlinked finite cover $\pi : M \to W$ is central; further, if W is transitive then by passing to an $\mathrm{Aut}(M)$-orbit on M we can always find a *transitive* irreducible finite cover of W with the same kernel as π. So at least when the stabilizer in $\mathrm{Aut}(W)$ of an element of W is irreducible, the situation reduces to determining the possible finite central kernels of transitive, untwisted finite covers.

We will obtain a partial answer to the conjecture, for W such that

$$(\forall x, y, z \in W)(\exists w \in W)(\mathrm{tp}(w, x) = \mathrm{tp}(w, y) = \mathrm{tp}(w, z)), \tag{3.1}$$

where $\mathrm{tp}(u, v)$ denotes the 2-type of $(u, v) \in W^2$ over the empty set, i.e. the $\mathrm{Aut}(W)$-orbit of (u, v). In the process, we will characterize the transitive, untwisted, superlinked finite covers (with central kernels) of such structures W.

3.1 Extended Conjugate Systems

Let W be a transitive permutation structure, and suppose that $\pi : M \to W$ is an untwisted, transitive, superlinked finite cover with central kernel K. By Lemma 2.6, any automorphism f of W which fixes a point $x \in W$ has a unique extension $\widehat{f} \in \mathrm{Aut}(M)$ fixing the fibre $M(x)$ pointwise. If f also fixes $y \in W$ then there is a unique $\alpha \in K$ with the same action on $M(y)$ as $\widehat{f}$. As in [5, Section 3.2], this defines for each pair $(x, y) \in W^2$ a map

$$\phi_{x,y} : \mathrm{Aut}(W/x, y) \to K$$

with the following properties.

Proposition 3.2 *The maps $\phi_{x,y}$ are continuous homomorphisms satisfying:*

(i) *if $f \in Aut(W/x, y)$ and $g \in Aut(W)$ then $\phi_{gx,gy}(gfg^{-1}) = \phi_{x,y}(f)$;*

(ii) *if $f \in Aut(W/x, y, z)$ then $\phi_{x,y}(f)\phi_{y,z}(f) = \phi_{x,z}(f)$.*

Proof: Given $x, y \in W$ and $f, g \in \mathrm{Aut}(W/x, y)$, let $\widehat{f}$ and $\widehat{g}$ be the unique extensions of f and g in $\mathrm{Aut}(M/M(x))$, so that $\widehat{f}\widehat{g}$ acts on $M(y)$ as $\phi_{x,y}(f)\phi_{x,y}(g)$. But $\widehat{f}\widehat{g} \in \mathrm{Aut}(M/M(x))$ extends fg, hence acts on $M(y)$ as $\phi_{x,y}(fg)$. So $\phi_{x,y}(fg) = \phi_{x,y}(f)\phi_{x,y}(g)$, and $\phi_{x,y}$ is a homomorphism.

Next suppose $f \in \mathrm{Aut}(W/x, y)$ and $g \in \mathrm{Aut}(W)$ have extensions $\widehat{f} \in \mathrm{Aut}(M/M(x))$ and $\widehat{g} \in \mathrm{Aut}(M)$. Then $\widehat{g}\widehat{f}\widehat{g}^{-1} \in \mathrm{Aut}(M/M(gx))$ extends gfg^{-1}, and so has the same action as $\phi_{gx,gy}(gfg^{-1})$ on $M(gy)$. It follows that $\phi_{gx,gy}(gfg^{-1}) = \widehat{g}\phi_{x,y}(f)\widehat{g}^{-1} = \phi_{x,y}(f)$, since K is central in $\mathrm{Aut}(M)$. This proves (i).

For (ii), suppose $f \in \mathrm{Aut}(W/x, y, z)$ has extension $\widehat{f} \in \mathrm{Aut}(M/M(x))$. Then $\phi_{x,y}(f)^{-1}\widehat{f} \in \mathrm{Aut}(M/M(y))$ also extends f, hence acts on $M(z)$ as $\phi_{y,z}(f)$. So $\widehat{f}$ acts on $M(z)$ as both $\phi_{x,y}(f)\phi_{y,z}(f)$ and $\phi_{x,z}(f)$, and the result follows.

It remains to prove continuity. For this, we need only show that the kernel of $\phi_{x,y}$ is closed. So consider any sequence $\langle f_n \rangle \in \mathrm{Ker}\,\phi_{x,y}$ with limit f. Let $\widehat{f_n} \in \mathrm{Aut}(M/M(x))$ extend f_n for each n, and enumerate W as $w_0, w_1, \ldots$. Since for each m the sequence

$\langle f_n(w_m)\rangle$ is eventually constant and the fibre $M(w_m)$ is finite, we can find a chain of successive subsequences $\langle \widehat{f}_{0,n}\rangle \supseteq \langle \widehat{f}_{1,n}\rangle \supseteq \cdots$ of $\langle \widehat{f}_n\rangle$ such that, for each $m \in \omega$ and $a \in M(w_m)$, $\langle \widehat{f}_{m,n}(a)\rangle$ is a constant sequence. So $\langle \widehat{f}_{n,n}\rangle$ is a subsequence of $\langle \widehat{f}_n\rangle$ converging to some $\widehat{f} \in \mathrm{Aut}(M/M(x), M(y))$ which restricts to f. So we must have $\phi_{x,y}(f) = 1$, i.e. f is in the kernel of $\phi_{x,y}$. □

In some sense, the homomorphisms $\phi_{x,y}$ tell us how the cover $\pi : M \to W$ behaves with respect to pairs of points in W. But in general, we need more information to characterize the cover.

Consider now any pair (f, g) of automorphisms of W, both mapping y to z, and suppose f has fixed point w and g has fixed point x. Again, f and g have unique extensions $\widehat{f} \in \mathrm{Aut}(M/M(w))$ and $\widehat{g} \in \mathrm{Aut}(M/M(x))$. Since both $\widehat{f}$ and $\widehat{g}$ map the fibre above y onto the fibre above z, their actions on $M(y)$ must differ by a unique element of K.

So for each 4-tuple $(w, x, y, z) \in W^4$ we get a map

$$\psi_{w,x,y,z} : \mathrm{Aut}(W/w)_{y\mapsto z} \times \mathrm{Aut}(W/x)_{y\mapsto z} \to K,$$

where $\mathrm{Aut}(W/x)_{y\mapsto z}$ denotes the coset of $\mathrm{Aut}(W/x, y)$ consisting of all $f \in \mathrm{Aut}(W/x)$ such that $fy = z$. Explicitly, $\psi_{w,x,y,z}(f, g) = \alpha \in K$ if and only if α and $\widehat{g}^{-1}\widehat{f}$ have the same action on $M(y)$, where $\widehat{f} \in \mathrm{Aut}(M/M(w))$ extends f and $\widehat{g} \in \mathrm{Aut}(M/M(x))$ extends g. Using the product topology on $\mathrm{Aut}(W/w)_{y\mapsto z} \times \mathrm{Aut}(W/x)_{y\mapsto z}$, it is easy to show that $\psi_{w,x,y,z}$ is continuous; the proof is similar to the one given for the homomorphisms $\phi_{x,y}$.

Proposition 3.3 *The continuous maps $\psi_{w,x,y,z}$ satisfy the following properties:*

(i) *if $f \in Aut(W/w)_{y\mapsto z}$, $g \in Aut(W/x)_{y\mapsto z}$ and $k^{-1}h \in Aut(W/y, z)$ then* $\psi_{hw,kx,hy,hz}(hfh^{-1}, kgk^{-1}) = \psi_{w,x,y,z}(f, g)\phi_{y,z}(k^{-1}h)$;

(ii) *if $f \in Aut(W/w)_{y\mapsto z}$, $g \in Aut(W/w)_{y\mapsto z}$, $h \in Aut(W/w, y)$, and $k \in Aut(W/x, y)$ then* $\psi_{w,x,y,z}(fh, gk) = \psi_{w,x,y,z}(f, g)\phi_{w,y}(h)\phi_{x,y}(k)^{-1}$;

(iii) *if $f_i \in Aut(W/v_i)_{y\mapsto z}$ for $1 \le i \le 3$ then* $\psi_{v_1,v_2,y,z}(f_1, f_2)\psi_{v_2,v_3,y,z}(f_2, f_3) = \psi_{v_1,v_3,y,z}(f_1, f_3)$;

(iv) *if* $f_1 \in Aut(W/w)_{y\mapsto z}$, $f_2 \in Aut(W/w)_{z\mapsto v}$, $g_1 \in Aut(W/x)_{y\mapsto z}$, *and* $g_2 \in Aut(W/x)_{z\mapsto v}$ *then* $\psi_{w,x,y,z}(f_1,g_1)\psi_{w,x,z,v}(f_2,g_2) = \psi_{w,x,y,v}(f_2f_1,g_2g_1)$;

(v) *if* $f \in Aut(W/w)_{y_1\mapsto z_1,y_2\mapsto z_2}$ *and* $g \in Aut(W/x)_{y_1\mapsto z_1,y_2\mapsto z_2}$ *then* $\psi_{w,x,y_2,z_2}(f,g) = \psi_{w,x,y_1,z_1}(f,g)\phi_{y_1,y_2}(g^{-1}f)$;

(vi) *if* $f \in Aut(W/w)_{y\mapsto z}$ *and* $g \in Aut(W/x)_{y\mapsto z}$ *then* $\psi_{w,x,y,z}(f,g)^{-1} = \psi_{x,w,y,z}(g,f) = \psi_{w,x,z,y}(f^{-1},g^{-1})$;

(vii) *if* $f \in Aut(W/w,y)$ *and* $g \in Aut(W/x,y)$ *then* $\psi_{w,x,y,y}(f,g) = \phi_{w,y}(f)\phi_{x,y}(g)^{-1}$;

(viii) *if* $f \in Aut(W/w,x)_{y\mapsto z}$ *then* $\psi_{w,x,y,z}(f,f) = \phi_{w,x}(f)$;

(ix) *if* $x, v_1, \dots, v_4, w_1, \dots, w_4, w'_1, \dots, w'_4, y_1, \dots, y_4, y'_1, \dots, y'_4 \in W$ *and* $f_i \in Aut(W/v_i)_{y_i\mapsto y_{i+1},y'_i\mapsto y'_{i+1}}, g_i \in Aut(W/w_i)_{x\mapsto y_i}, h_i \in Aut(W/w_i)_{y_i\mapsto y_{i+1}}, g'_i \in Aut(W/w'_i)_{x\mapsto y'_i}, h'_i \in Aut(W/w'_i)_{y'_i\mapsto y'_{i+1}}$ *for* $1 \le i \le 4$ *(where* $y_5 = y_1$, $y'_5 = y'_1$, $g_5 = g_1$, *and* $g'_5 = g'_1$*), then*

$$\prod_{i=1}^{4}\psi_{v_i,w_i,y_i,y_{i+1}}(f_i,h_i)\cdot\prod_{i=1}^{4}\psi_{w_i,w_{i+1},x,y_{i+1}}(h_ig_i,g_{i+1}) = \prod_{i=1}^{4}\psi_{v_i,w'_i,y'_i,y'_{i+1}}(f_i,h'_i)\cdot\prod_{i=1}^{4}\psi_{w'_i,w'_{i+1},x,y'_{i+1}}(h'_ig'_i,g'_{i+1}),$$

provided $f_4f_3f_2f_1 = 1_W$.

Proof: We will prove only (i),(ii),(iv), and (ix); the rest are easy.

Suppose f,g,h,k satisfy the hypotheses of (i). Let $\widehat{f} \in \mathrm{Aut}(M/M(w))$ and $\widehat{g} \in \mathrm{Aut}(M/M(x))$ extend f and g, let $\widehat{h},\widehat{k} \in \mathrm{Aut}(M)$ be extensions of h and k such that $\widehat{k}^{-1}\widehat{h} \in \mathrm{Aut}(M/M(y))$, and write $\alpha = \psi_{hw,kx,hy,hz}(hfh^{-1},kgk^{-1})$. Then $\widehat{h}\widehat{f}\widehat{h}^{-1} \in \mathrm{Aut}(M/M(hw))$ extends hfh^{-1} and $\widehat{k}\widehat{g}\widehat{k}^{-1} \in \mathrm{Aut}(M/M(kx))$ extends kgk^{-1}, so $\widehat{k}\widehat{g}^{-1}\widehat{k}^{-1}\widehat{h}\widehat{f}\widehat{h}^{-1}$ acts on $M(hy)$ as α. Conjugating by $\widehat{k}$, we see that $\widehat{g}^{-1}\widehat{k}^{-1}\widehat{h}\widehat{f}$ acts on $M(y)$ as α (since $\widehat{h}^{-1}\widehat{k}$ fixes $M(y)$ pointwise). Conjugating now by $\widehat{g}^{-1}$, it follows that $\widehat{k}^{-1}\widehat{h}\widehat{f}\widehat{g}^{-1}$ acts on $M(z)$ as α. But $\widehat{k}^{-1}\widehat{h}$ acts on $M(z)$ as $\phi_{y,z}(k^{-1}h)$, so $\widehat{f}\widehat{g}^{-1}$ acts on $M(z)$ as $\phi_{y,z}(k^{-1}h)^{-1}\alpha$, and $\widehat{g}^{-1}\widehat{f}$ acts on $M(y)$ as $\phi_{y,z}(k^{-1}h)^{-1}\alpha$. Thus, $\alpha = \psi_{w,x,y,z}(f,g)\phi_{y,z}(k^{-1}h)$, proving (i).

Next suppose f, g, h, k satisfy the hypotheses of (ii), and let $\widehat{f} \in \mathrm{Aut}(M/M(w))$, $\widehat{g} \in \mathrm{Aut}(M/M(x))$, $\widehat{h} \in \mathrm{Aut}(M/M(w))$, and $\widehat{k} \in \mathrm{Aut}(M/M(x))$ extend f, g, h, and k, respectively. Then $\widehat{k}^{-1}\widehat{g}^{-1}\widehat{f}\widehat{h}$ acts on $M(y)$ as $\psi_{w,x,y,z}(fh, gk)$, hence so does $\widehat{g}^{-1}\widehat{f}\widehat{h}\widehat{k}^{-1}$, since k fixes y. The result follows, since $\widehat{g}^{-1}\widehat{f}$ and $\widehat{h}\widehat{k}^{-1}$ act on $M(y)$ as $\psi_{w,x,y,z}(f,g)$ and $\phi_{w,y}(h)\phi_{x,y}(k)^{-1}$, respectively.

For (iv), suppose f_1, f_2, g_1, g_2 satisfy the hypotheses, and let $\widehat{f_i} \in \mathrm{Aut}(M/M(w))$ and $\widehat{g_i} \in \mathrm{Aut}(M/M(x))$ extend f_i and g_i $(i = 1, 2)$. Then $\widehat{g_2}^{-1}\widehat{f_2}$ acts on $M(z)$ as $\psi_{w,x,z,v}(f_2, g_2)$ and $\widehat{f_1}$ maps $M(y)$ onto $M(z)$, so $(\widehat{g_2}\widehat{g_1})^{-1}(\widehat{f_2}\widehat{f_1}) = \widehat{g_1}^{-1}(\widehat{g_2}^{-1}\widehat{f_2})\widehat{f_1}$ acts on $M(y)$ as $\psi_{w,x,z,v}(f_2, g_2)\widehat{g_1}^{-1}\widehat{f_1}$, which in turn acts on $M(y)$ as $\psi_{w,x,z,v}(f_2, g_2)\psi_{w,x,y,z}(f_1, g_1)$. Since $\widehat{f_2}\widehat{f_1} \in \mathrm{Aut}(M/M(w))$ extends f_2f_1 and $\widehat{g_2}\widehat{g_1} \in \mathrm{Aut}(M/M(x))$ extends g_2g_1, we thus have $\psi_{w,x,y,v}(f_2f_1, g_2g_1) = \psi_{w,x,y,z}(f_1, g_1)\psi_{w,x,z,v}(f_2, g_2)$.

Finally, assume the hypotheses of (ix), and for $1 \leq i \leq 4$ write $\alpha_i = \psi_{v_i,w_i,y_i,y_{i+1}}(f_i, h_i)$, $\beta_i = \psi_{w_i,w_{i+1},x,y_{i+1}}(h_ig_i, g_{i+1})$, $\alpha'_i = \psi_{v_i,w'_i,y'_i,y'_{i+1}}(f_i, h'_i)$, and $\beta'_i = \psi_{w'_i,w'_{i+1},x,y'_{i+1}}(h'_ig'_i, g'_{i+1})$, and let $f_i, g_i, h_i, g'_i, h'_i$ have extensions $\widehat{f_i} \in \mathrm{Aut}(M/M(v_i))$, $\widehat{g_i}, \widehat{h_i} \in \mathrm{Aut}(M/M(w_i))$ and $\widehat{g'}_i, \widehat{h'}_i \in \mathrm{Aut}(M/M(w'_i))$, respectively. Then for each i, $\widehat{g_{i+1}}^{-1}\widehat{h_i}\widehat{g_i}$ acts on $M(x)$ as β_i, so $\widehat{g_{i+1}}^{-1}\widehat{h_i}\alpha_i\widehat{g_i}$ acts on $M(x)$ as $\alpha_i\beta_i$. But α_i and $\widehat{h_i}^{-1}\widehat{f_i}$ have the same action on $M(y_i)$, and $\widehat{g_i}$ maps $M(x)$ onto $M(y_i)$, so $\widehat{g_{i+1}}^{-1}\widehat{h_i}\alpha_i\widehat{g_i}$ and $\widehat{g_{i+1}}^{-1}\widehat{h_i}\widehat{h_i}^{-1}\widehat{f_i}\widehat{g_i}$ have the same action on $M(x)$. Thus, $\widehat{g_{i+1}}^{-1}\widehat{f_i}\widehat{g_i}$ acts as $\alpha_i\beta_i$ on $M(x)$. Multiplying these factors together shows that $\widehat{g_1}^{-1}\widehat{f_4}\widehat{f_3}\widehat{f_2}\widehat{f_1}\widehat{g_1}$ and $\prod_{i=1}^{4}\alpha_i\beta_i$ have the same action on $M(x)$, so $\widehat{f_4}\widehat{f_3}\widehat{f_2}\widehat{f_1}$ acts on $M(y_1)$ as $\prod_{i=1}^{4}\alpha_i\prod_{i=1}^{4}\beta_i$. Similarly, $\widehat{f_4}\widehat{f_3}\widehat{f_2}\widehat{f_1}$ acts on $M(y'_1)$ as $\prod_{i=1}^{4}\alpha'_i\prod_{i=1}^{4}\beta'_i$. So we get the desired result by noticing that $\widehat{f_4}\widehat{f_3}\widehat{f_2}\widehat{f_1}$ is in K. □

Propositions 3.2 and 3.3 motivate the following definition.

Definition 3.4 Let W be a transitive permutation structure and let K be a finite abelian group (with discrete topology). An *extended conjugate system* for (W, K) is a pair (Φ, Ψ) such that

- $\Phi = \langle \phi_{x,y} : x, y \in W \rangle$ is a system of continuous homomorphisms $\phi_{x,y} : \mathrm{Aut}(W/x, y) \to K$ satisfying Properties (i) and (ii) of Proposition 3.2;
- $\Psi = \langle \psi_{w,x,y,z} : w, x, y, z \in W \rangle$ is a system of continuous maps $\psi_{w,x,y,z} : \mathrm{Aut}(W/w)_{y \mapsto z} \times \mathrm{Aut}(W/x)_{y \mapsto z} \to K$ satisfying Properties (i) through (ix) of Proposition 3.3.

The extended conjugate system corresponding to a transitive, untwisted, superlinked finite cover $\pi : M \to W$ (with central kernel) is called its *associated* extended conjugate system.

Note that we can rewrite the properties in Proposition 3.3 in terms of Ψ alone, for example by using $\psi_{x,x,y,y}(f, 1_W)$ in place of $\phi_{x,y}(f)$. Then given Ψ we can recover Φ by *defining* $\phi_{x,y}(f) = \psi_{x,x,y,y}(f, 1_W)$. So the Φ in this definition is somewhat redundant. We include it for two reasons: first, for notational simplicity, and second, to show explicitly the connection between this concept of an extended conjugate system and Evans' concept of a conjugate system in [5].

Compared with the others, Property (ix) of Proposition 3.3 at first seems rather complicated. In the context of a transitive, untwisted, superlinked finite cover $\pi : M \to W$ with central kernel K, it says that if $\widehat{f} \in \mathrm{Aut}(M/W)$ is a product of automorphisms of M, each pointwise fixing some fibre of π, then $\widehat{f}$ acts as the same element of K on every fibre. Since $\mathrm{Aut}(M/W) = K$, this is trivially the case (in fact, property (ix) is true for every $m \in \omega$, not just for $m = 4$). In the next section, where we *construct* a transitive, untwisted, superlinked finite cover from a given extended conjugate system for (W, K), we will need Property (ix) to guarantee that K is indeed the kernel (cf. Lemma 3.15).

3.2 The Characterization

In the previous section we showed how to associate an extended conjugate system with any transitive, untwisted, superlinked finite cover with central kernel. For structures W satisfying (3.1), there is a converse result. That is, given any extended conjugate system (Φ, Ψ) for (W, K), where K is finite abelian, there is a unique (up to isomorphism) transitive, untwisted, superlinked finite cover $\pi : M \to W$ with central kernel K and associated extended conjugate system (Φ, Ψ). We begin with the uniqueness assertion.

First we need to define the concept of an isomorphism between finite covers of a structure W. We say that permutation structures M and M' are isomorphic if there is a bijection $\mu : M \to M'$ which carries $\mathrm{Aut}(M)$ to $\mathrm{Aut}(M')$, in the sense that the induced map $\lambda : \mathrm{Aut}(M) \to \mathrm{Sym}(M')$ defined by $\lambda(f) = \mu f \mu^{-1}$ for $f \in \mathrm{Aut}(M)$ is a group isomorphism between Aut(M) and Aut(M'). The pair (μ, λ) is called a *permutation structure isomorphism*; since λ is induced by μ, we will often refer to such an isomorphism simply as μ. An *isomorphism of finite covers* $\pi : M \to W$ and $\pi' : M' \to W$ is a permutation structure isomorphism $\mu : M \to M'$ such that $\pi'\mu = \pi$.

Now fix W satisfying (3.1) and suppose $\pi : M \to W$ and $\pi' : M' \to W$ are transitive, untwisted, superlinked finite covers with the same central kernel and the same associated extended conjugate system. Formally, we assume that there is an isomorphism η from the kernel K of π to the kernel K' of π' such that $\psi'_{w,x,y,z} = \eta\psi_{w,x,y,z}$ and $\phi'_{x,y} = \eta\phi_{x,y}$ for every $w, x, y, z \in W$, where $(\Phi, \Psi) = (\langle\phi_{x,y}\rangle, \langle\psi_{w,x,y,z}\rangle)$ and $(\Phi', \Psi') = (\langle\phi'_{x,y}\rangle, \langle\psi'_{w,x,y,z}\rangle)$ are the extended conjugate systems associated with π and π', respectively. We want to construct an isomorphism (μ, λ) between these covers such that λ extends η.

Note that if such an isomorphism exists, then for any $\alpha \in K$ we have $\mu\alpha = \eta(\alpha)\mu$. Since K acts regularly on each fibre in M, μ will therefore be completely determined if we know how it maps a single point in each fibre. Fixing transversals $a : W \to M$ and $a' : W \to M'$, we will construct μ by determining $\mu(a(x))$ for every $x \in W$.

For any fixed $x \in W$, we should be able to find an isomorphism $\mu = \mu_x$ which maps $a(x)$ to $a'(x)$. In fact, if it exists, μ_x is uniquely determined as follows:

> Given $y \in W$, pick $w \in W$ such that $\mathrm{tp}(w, x) = \mathrm{tp}(w, y)$ and pick $f \in \mathrm{Aut}(W/w)$ such that $fx = y$. Let $\widehat{f} \in \mathrm{Aut}(M/M(w))$ and $\widetilde{f} \in \mathrm{Aut}(M'/M'(w))$ extend f, and let $\beta \in K$ be such that $\widehat{f}a(x) = \beta a(y)$. Then we define
>
> $$\mu_x(a(y)) = \eta(\beta)^{-1}\widetilde{f}a'(x).$$

It is easy to see why we must define $\mu_x(a(y))$ this way. For if μ_x does exist, then $\mu_x\widehat{f}\mu_x^{-1}$ also extends f and fixes $M'(w)$ pointwise, so we must have $\mu_x\widehat{f}\mu_x^{-1} = \widetilde{f}$, and thus $\mu_x(a(y)) = \widetilde{f}\mu_x\widehat{f}^{-1}a(y) = \widetilde{f}\mu_x\beta^{-1}a(x) = \eta(\beta)^{-1}\widetilde{f}a'(x)$. We claim that μ_x is indeed an isomorphism.

Lemma 3.5 *For any $x \in W$, μ_x is a well-defined bijection from M onto M'.*

Proof: We must show that the definition of $\mu_x(a(y))$ does not depend on the choice of $w \in W$ and $f \in \mathrm{Aut}(W/w)_{x \mapsto y}$. The fact that μ_x is a bijection will then be obvious.

So suppose we also have $v \in W$ and $g \in \mathrm{Aut}(W/v)_{x\mapsto y}$. Let $\widehat{g} \in \mathrm{Aut}(M/M(v))$ and $\widetilde{g} \in \mathrm{Aut}(M'/M'(v))$ extend g, and let $\gamma \in K$ be such that $\widehat{g}a(x) = \gamma a(y)$. Then $\widehat{g}^{-1}\widehat{f}a(x) = \gamma^{-1}\beta a(x)$ so $\psi_{v,w,x,y}(f,g) = \gamma^{-1}\beta$, hence $\psi'_{v,w,x,y}(f,g) = \eta(\gamma)^{-1}\eta(\beta)$ so $\widetilde{g}^{-1}\widetilde{f}a'(x) = \eta(\gamma)^{-1}\eta(\beta)a'(x)$. The result follows, since K' is central in $\mathrm{Aut}(M')$. □

Lemma 3.6 *For any $x, w \in W$ and $f \in Aut(W/w)$, if $\widehat{f} \in Aut(M/M(w))$ and $\widetilde{f} \in Aut(M'/M'(w))$ extend f then $\mu_x\widehat{f}\mu_x^{-1}a'(x) = \widetilde{f}a'(x)$.*

Proof: This is immediate from the definition of μ_x. □

Lemma 3.7 *For any $x, y \in W$ there is some $\alpha \in K$ such that $\mu_x = \eta(\alpha)\mu_y$. In particular, if $\widehat{f} \in Aut(W)$ then $\mu_x\widehat{f}\mu_x^{-1} = \mu_y\widehat{f}\mu_y^{-1}$.*

Proof: Let $\alpha \in K$ satisfy $\mu_x a(y) = \eta(\alpha)a'(y)$, and consider any $z \in W$. Using (3.1) we can find $w \in W$ such that $\mathrm{tp}(w,x) = \mathrm{tp}(w,y) = \mathrm{tp}(w,z)$; thus, there are automorphisms $f \in \mathrm{Aut}(W/w)_{x\mapsto y}$ and $g \in \mathrm{Aut}(W/w)_{y\mapsto z}$. As usual, let $\widehat{f} \in \mathrm{Aut}(M/M(w))$ and $\widetilde{f} \in \mathrm{Aut}(M'/M'(w))$ extend f, let $\widehat{g} \in \mathrm{Aut}(M/M(w))$ and $\widetilde{g} \in \mathrm{Aut}(M'/M'(w))$ extend g, and let $\beta, \gamma \in K$ satisfy $\widehat{f}a(x) = \beta a(y)$ and $\widehat{g}a(y) = \gamma a(z)$. Then $\widetilde{f}a'(x) = \eta(\alpha)\eta(\beta)a'(y)$ and $\widehat{g}\widehat{f}a(x) = \beta\gamma a(z)$, so $\mu_x a(z) = \eta(\beta)^{-1}\eta(\gamma)^{-1}\widetilde{g}\widetilde{f}a'(x) = \eta(\alpha)\eta(\gamma)^{-1}\widetilde{g}a'(y) = \eta(\alpha)\mu_y a(z)$. Since $z \in W$ was arbitrary, $\mu_x = \eta(\alpha)\mu_y$. □

So the bijections μ_x all have the same induced map $\lambda : \widehat{f} \mapsto \mu_x\widehat{f}\mu_x^{-1}$. It remains to show that λ maps $\mathrm{Aut}(M)$ onto $\mathrm{Aut}(M')$.

Lemma 3.8 *Given $f \in Aut(W)$ and $w \in W$ there are $g, h \in Aut(W)$ such that $f = gh$, g fixes w, and h fixes some $u \in W$.*

Proof: Let $v = f^{-1}w$ and choose u satisfying $\mathrm{tp}(u,v) = \mathrm{tp}(u,w)$. Then we can find $h \in \mathrm{Aut}(W/u)$ mapping v to w, so that $g = fh^{-1} \in \mathrm{Aut}(W/w)$. □

Lemma 3.9 *For every $x \in W$, (μ_x, λ) is an isomorphism.*

Proof: We must show that $\lambda(\widehat{f}) \in \text{Aut}(M')$ whenever $\widehat{f} \in \text{Aut}(M)$, and that each $\widetilde{f} \in \text{Aut}(M')$ is in the image of λ.

By the previous lemma, any $\widehat{f} \in \text{Aut}(M)$ can be written as a product $\widehat{g}\widehat{h}$, where $\widehat{g} \in \text{Aut}(M/w)$ and $\widehat{h} \in \text{Aut}(M/u)$ for some $w, u \in W$, and any $\widetilde{f} \in \text{Aut}(M')$ can be written as a product $\widetilde{g}\widetilde{h}$, where $\widetilde{g} \in \text{Aut}(M'/w')$ and $\widetilde{h} \in \text{Aut}(M'/u')$ for some $w', u' \in W$. It suffices then to consider automorphisms which have a fixed point in W.

So suppose $\widehat{f} \in \text{Aut}(M/w)$, where $w \in W$. Since there is some $\alpha \in K$ such that $\alpha\widehat{f}$ fixes $M(w)$ pointwise, and since $\lambda(\alpha\widehat{f}) = \eta(\alpha)\lambda(\widehat{f})$, we can assume that $\widehat{f} \in \text{Aut}(M/M(w))$. Let $\widetilde{f} \in \text{Aut}(M'/M'(w))$ have the same restriction to W as $\widehat{f}$. Then from what has already been proved, $\lambda(\widehat{f})a'(y) = \mu_y\widehat{f}\mu_y^{-1}a'(y) = \widetilde{f}a'(y)$ for every $y \in W$, and so $\lambda(\widehat{f}) = \widetilde{f} \in \text{Aut}(M)$.

Similarly, if $\widetilde{f} \in \text{Aut}(M'/w)$ acts on $M'(w)$ as $\eta(\alpha)$, then $\widetilde{f} = \lambda(\widehat{f})$, where $\widehat{f} \in \text{Aut}(M/w)$ has the same restriction to W as $\widetilde{f}$ and acts on $M(w)$ as α. □

We have proved the following result.

Theorem 3.10 *Let W be any permutation structure satisfying Property (3.1). Then every transitive, untwisted, superlinked finite cover of W with central kernel is determined up to isomorphism by its associated extended conjugate system. That is, if $\pi : M \to W$ and $\pi' : M' \to W$ are transitive, untwisted, superlinked finite covers of W with central kernels K and K' and associated extended conjugate systems $(\langle\phi_{x,y}\rangle, \langle\psi_{w,x,y,z}\rangle)$ and $(\langle\phi'_{x,y}\rangle, \langle\psi'_{w,x,y,z}\rangle)$, respectively, and if there is an isomorphism $\eta : K \to K'$ such that $\phi'_{x,y} = \eta\phi_{x,y}$ and $\psi'_{w,x,y,z} = \eta\psi_{w,x,y,z}$ for all $w, x, y, z \in W$, then there is an isomorphism of finite covers $(\mu, \lambda) : M \to M'$ such that λ extends η.*

Example 3.11 Let W be the countable, homogeneous local order. So W has base $\mathbb{Q} \cap [0, 2\pi)$ and a single binary relation R defined by:

$$xRy \text{ if and only if } 0 < y - x < \pi \text{ or } 0 < (y + 2\pi) - x < \pi.$$

Given any finite cyclic group K of order n, following [4, Chapter 5] we get a transitive, untwisted, superlinked finite cover $\tau : M \to W$ with central kernel K and trivial extended conjugate system by setting

$$M = \bigcup_{k=0}^{n-1} [(\mathbb{Q} \cap [0, 2\pi)) + 2k\pi]$$

and

$$\tau(q + 2k\pi) = q \text{ for } q \in \mathbb{Q} \cap [0, 2\pi), 0 \leq k < n,$$

where the structure on M consists of a single binary relation R' defined by:

$$xR'y \text{ if and only if } 0 < y - x < \pi \text{ or } 0 < (y + 2n\pi) - x < \pi.$$

On the other hand, there is another obvious transitive, untwisted, superlinked finite cover of W with central kernel K and trivial extended conjugate system: its base is $W \times K$ and its automorphism group is $\mathrm{Aut}(W) \times K$, where K acts on itself by left multiplication. Provided n is at least two, these two covers are not isomorphic, since only the first is irreducible. So the conclusion of Theorem 3.10 fails for this W. Of course, W does not satisfy (3.1), but for any *pair* $x, y \in W$ there is a $w \in W$ such that $\mathrm{tp}(w, x) = \mathrm{tp}(w, y)$.

Analyzing the proof of Theorem 3.10, we note that Property (3.1) is used only in the proof of Lemma 3.7; for the rest it is enough to assume that for any $x, y \in W$ there is some $w \in W$ such that $\mathrm{tp}(w, x) = \mathrm{tp}(w, y)$. So with a little more work we get the following corollary.

Corollary 3.12 *Let W be a permutation structure with a finite number m of 3-types, and suppose that for any $x, y \in W$ there is a $w \in W$ such that $tp(w, x) = tp(w, y)$. If K is a finite abelian group of order n then any extended conjugate system (Φ, Ψ) for (W, K) is the associated system of at most (up to isomorphism) nm transitive, untwisted, superlinked finite covers of W with central kernel K.*

Proof: Consider any two transitive, untwisted, superlinked finite covers $\pi : M \to W$ and $\pi' : M' \to W$ with central kernel K and associated extended conjugate system (Φ, Ψ). (Formally we should assume that the kernels of π and π' are isomorphic to K via isomorphisms which send their associated systems to (Φ, Ψ), but we prefer to keep notation as simple as possible.) By the proof of the theorem, fixed transversals $a : W \to M$ and $a' : W \to M'$ provide us with bijections $\mu_x : M \to M'$ for each $x \in W$, which are isomorphisms provided Lemma 3.7 holds.

We can measure the extent to which Lemma 3.7 fails for π and π' as an m-tuple of elements of K. Given any triple $(x, y, z) \in W^3$, let $\gamma_{x,y,z} \in K$ be such that $\mu_x a(z) = \gamma_{x,y,z}\alpha\beta a'(z)$, where $\mu_x a(y) = \alpha a'(y)$ and $\mu_y a(z) = \beta a'(z)$. An easy (but very tedious) calculation shows that $\gamma_{x,y,z}$ depends only on the 3-type of the triple (x, y, z), i.e. $\gamma_{fx,fy,fz} = \gamma_{x,y,z}$ for any $f \in \mathrm{Aut}(W)$. So with each 3-type in W we have associated an element of K. Since W has exactly m 3-types, this associates with π and π' an m-tuple of elements of K, which is the identity in K^m if and only if Lemma 3.7 holds.

Now suppose we have $nm+1$ transitive, untwisted, superlinked finite covers $\pi_j : M_j \to W$ $(0 \le j \le nm)$ with central kernel K and associated extended conjugate system (Φ, Ψ), and fix transversals $a_j : W \to M_j$. We must find $j \ne k$ such that π_j and π_k are isomorphic. For any pair (i, j) we get bijections $\mu_x^{i,j} : M_i \to M_j$ and a corresponding m-tuple $\bar\gamma^{i,j} \in K^m$. It is enough to show that $\bar\gamma^{i,k} = \bar\gamma^{j,k}\bar\gamma^{i,j}$ whenever $0 \le i, j, k \le nm$; then since K^m has only nm elements, there are j, k such that $0 \le j < k \le nm$ and $\bar\gamma^{0,k} = \bar\gamma^{0,j}$, hence $\bar\gamma^{j,k}$ is the identity and each $\mu_x^{j,k}$ $(x \in W)$ is an isomorphism.

Let $x, y \in W$ and pick $w \in W$ and $f \in \mathrm{Aut}(W/w)_{x\mapsto y}$. For each i let $\widehat{f^i} \in \mathrm{Aut}(M_i/M_i(x))$ extend f and let $\widehat{f^i}a_i(x) = \delta_i a_i(y)$, where $\delta_i \in K$. If $0 \le i, j, k \le nm$ then

$$
\begin{aligned}
\mu_x^{j,k}\mu_x^{i,j}a_i(y) &= \mu_x^{j,k}\delta_i^{-1}\widehat{f^j}a_j(x) \\
&= \delta_i^{-1}\delta_j\mu_x^{j,k}a_j(y) \\
&= \delta_i^{-1}\widehat{f^k}a_k(x) \\
&= \mu_x^{i,k}a_i(y).
\end{aligned}
$$

Since x and y were arbitrary, $\mu_x^{j,k}\mu_x^{i,j} = \mu_x^{i,k}$ whenever $x \in W$ and $0 \le i, j, k \le nm$.

Now let $x, y, z \in W$ and for each i, j write $\mu_x^{i,j}a_i(y) = \alpha^{i,j}a_j(y)$, $\mu_y^{i,j}a_i(z) = \beta^{i,j}a_j(z)$ and $\mu_x^{i,j}a_i(z) = \gamma_{x,y,z}^{i,j}\alpha^{i,j}\beta^{i,j}a_j(z)$. For any i, j, k we have $\alpha^{i,k} = \alpha^{j,k}\alpha^{i,j}$ and $\beta^{i,k} = \beta^{j,k}\beta^{i,j}$ since $\mu_x^{i,k} = \mu_x^{j,k}\mu_x^{i,j}$ and $\mu_y^{i,k} = \mu_y^{j,k}\mu_y^{i,j}$, so

$$
\begin{aligned}
\mu_x^{i,k}a_i(z) &= \mu_x^{j,k}\mu_x^{i,j}a_i(z) \\
&= \gamma_{x,y,z}^{j,k}\alpha^{j,k}\beta^{j,k}\gamma_{x,y,z}^{i,j}\alpha^{i,j}\beta^{i,j}a_k(z) \\
&= \gamma_{x,y,z}^{j,k}\gamma_{x,y,z}^{i,j}\alpha^{i,k}\beta^{i,k}a_k(z),
\end{aligned}
$$

hence $\gamma^{i,k}_{x,y,z} = \gamma^{j,k}_{x,y,z}\gamma^{i,j}_{x,y,z}$. But again x, y, z were arbitrary, so $\bar{\gamma}^{i,k} = \bar{\gamma}^{j,k}\bar{\gamma}^{i,j}$, as required. □

Turning now to the existence question, we again fix a permutation structure W satisfying Property (3.1), a finite abelian group K, and an extended conjugate system $(\Phi, \Psi) = (\langle\phi_{x,y}\rangle, \langle\psi_{w,x,y,z}\rangle)$ for (W, K); we seek a transitive, untwisted, superlinked finite cover $\pi : M \to W$ with central kernel K and associated conjugate system (Φ, Ψ).

It is clear what we should use for the base of M and the map π. For we know that K must act regularly and faithfully on each fibre of such a cover, so we just let M be the disjoint union $\bigsqcup_{v\in W} M(v)$, where for each $v \in W$, $M(v)$ is a set on which K acts regularly and faithfully. Then K has an obvious action on M, so we can identify it with a subgroup of $\mathrm{Sym}(M)$. What is not so obvious is how to construct the rest of $\mathrm{Aut}(M)$.

Let $S = \bigcup_{v\in W} \mathrm{Aut}(W/v)$. Given $f \in S$ with fixed point $v \in W$, there should be exactly one $\widehat{f}^v$ in $\mathrm{Aut}(M)$ extending f and fixing $M(v)$ pointwise. It will be enough to determine each of these automorphisms $\widehat{f}^v$, since together with K they must generate all of $\mathrm{Aut}(M)$ (by Lemma 3.8).

We will need a fixed reference point $x \in W$ and a transversal function $a : W \to M$. In addition, for each $y \in W$ we fix $u_y \in W$ such that $\mathrm{tp}(u_y, x) = \mathrm{tp}(u_y, y)$ and $k_y \in \mathrm{Aut}(W/u_y)$ such that $k_y x = y$, making sure that $k_x = 1_W$. Since there is not yet any structure on the fibres of π, we should be able to demand that $\widehat{k_y}^{u_y} a(x) = a(y)$ for each $y \in W$. Then there is only one possible way to define extensions of the automorphisms in S:

> Let $f \in S$ have fixed point $v \in W$. Given $y \in W$, let $z = fy$ and choose $w \in W$ such that $\mathrm{tp}(w, x) = \mathrm{tp}(w, y) = \mathrm{tp}(w, z)$ and $g, h \in \mathrm{Aut}(W/w)$ such that $gx = y$ and $hy = z$. Then for any $\alpha \in K$ we define
>
> $$\widehat{f}^v(\alpha a(y)) = \alpha\psi_{v,w,y,z}(f, h)\psi_{u_y,w,x,y}(k_y, g)\psi_{u_z,w,x,z}(k_z, hg)^{-1}a(z).$$

Properties (iii) and (iv) of Proposition 3.3 guarantee that this definition does not depend on the choice of w, g and h: given another suitable triple w', g', h', we have

$$\begin{aligned}
&\psi_{v,w',y,z}(f,h')\psi_{u_y,w',x,y}(k_y,g')\psi_{u_z,w',x,z}(k_z,h'g')^{-1} \\
&= [\psi_{v,w,y,z}(f,h)\psi_{w,w',y,z}(h,h')][\psi_{u_y,w,x,y}(k_y,g)\psi_{w,w',x,y}(g,g')] \\
&\quad \cdot[\psi_{u_z,w,x,z}(k_z,hg)\psi_{w,w',x,z}(hg,h'g')]^{-1} \\
&= [\psi_{v,w,y,z}(f,h)\psi_{u_y,w,x,y}(k_y,g)\psi_{u_z,w,x,z}(k_z,hg)^{-1}] \\
&\quad \cdot[\psi_{w,w',y,z}(h,h')\psi_{w,w',x,y}(g,g')\psi_{w,w',x,z}(hg,h'g')^{-1}] \\
&= \psi_{v,w,y,z}(f,h)\psi_{u_y,w,x,y}(k_y,g)\psi_{u_z,w,x,z}(k_z,hg)^{-1}.
\end{aligned}$$

Lemma 3.13 *For any $v \in W$ and $f \in Aut(W/v)$, $\widehat{f}^v$ is a well-defined permutation of M which preserves the partition of M into fibres, restricts to f on W, commutes with every element of K, and fixes the fibre $M(v)$ pointwise.*

Proof: To see that $\widehat{f}^v$ fixes $M(v)$ pointwise, choose $w \in W$ such that $\mathrm{tp}(w,x) = \mathrm{tp}(w,v)$ and $g \in \mathrm{Aut}(W/w)$ such that $gx = v$, and let $h = 1_W$. Then since $\psi_{v,w,v,v}(f, 1_W) = 1 = \psi_{u_v,w,x,v}(k_v, g)\psi_{u_v,w,x,v}(k_v, 1_W g)^{-1}$, we have $\widehat{f}^v(\alpha a(v)) = \alpha a(v)$ for each $\alpha \in K$, as required. The remaining assertions are obvious. □

Let G be the subgroup of $\mathrm{Sym}(M)$ generated by $\{\widehat{f}^v : v \in W \mathrm{and} f \in \mathrm{Aut}(W/v)\}$ together with K. Clearly every permutation in G preserves the partition of M into fibres and restricts to an automorphism of W, since this is true for the permutations generating G. We want to use G as the automorphism group of M, but first we must check that it is closed.

Lemma 3.14 *If $v \in W$ and $f \in Aut(W/v)$ then $\widehat{f^{-1}}^v = (\widehat{f}^v)^{-1}$.*

Proof: Given y and $z = fy$ in W, let $w \in W$, $g \in \mathrm{Aut}(W/w)_{x \mapsto y}$ and $h \in \mathrm{Aut}(W/w)_{y \mapsto z}$. Then by Property (vi) of Proposition 3.3,

$$\begin{aligned}
\widehat{f^{-1}}^v a(z) &= \psi_{v,w,z,y}(f^{-1},h^{-1})\psi_{u_z,w,x,z}(k_z,hg)\psi_{u_y,w,x,y}(k_y,h^{-1}hg)^{-1}a(y) \\
&= [\psi_{v,u,y,z}(f,h)\psi_{u_y,w,x,y}(k_y,g)\psi_{u_z,w,x,z}(k_z,hg)^{-1}]^{-1}a(y) \\
&= (\widehat{f}^v)^{-1}a(z).
\end{aligned}$$

□

Lemma 3.15 *The stabilizer of W in G is K.*

Proof: By the previous lemma, any $g \in G$ can be written as a product $\alpha \widehat{f_m}^{v_m} \cdots \widehat{f_1}^{v_1}$, where $\alpha \in K$ and $f_i \in \mathrm{Aut}(W/v_i)$ for $1 \leq i \leq m$. So it suffices to show that $\widehat{f_m}^{v_m} \cdots \widehat{f_1}^{v_1} \in K$ whenever $f_m \cdots f_1 = 1_W$. We do this by induction on m.

For the base step we prove the result for $m = 4$; since by direct computation $\widehat{1_W}^v = 1_M$ for any $v \in W$, this base step implies the result for $1 \leq m \leq 3$. Fix $y_1, y_1' \in W$, and let $y_{i+1} = f_i y_i$ and $y_{i+1}' = f_i y_i'$ for $1 \leq i \leq 4$, so that $y_5 = y_1$ and $y_5' = y_1'$. Next choose $w_i, w_i' \in W$, $g_i \in \mathrm{Aut}(W/w_i)_{x \mapsto y_i}$, $h_i \in \mathrm{Aut}(W/w_i)_{y_i \mapsto y_{i+1}}$, $g_i' \in \mathrm{Aut}(W/w_i')_{x \mapsto y_i'}$, and $h_i' \in \mathrm{Aut}(W/w_i')_{y_i' \mapsto y_{i+1}'}$ for $1 \leq i \leq 4$, and let $w_5 = w_1$, $w_5' = w_1'$, $g_5 = g_1$, $h_5 = h_1$, $g_5' = g_1'$, and $h_5' = h_1'$. Then $\widehat{f_4}^{v_4} \cdots \widehat{f_1}^{v_1}$ acts on $M(y_1)$ as

$$\prod_{i=1}^{4} \psi_{v_i,w_i,y_i,y_{i+1}}(f_i, h_i) \prod_{i=1}^{4} \psi_{u_{y_i},w_i,x,y_i}(k_{y_i}, g_i) \prod_{i=1}^{4} \psi_{u_{y_{i+1}},w_i,x,y_{i+1}}(k_{y_{i+1}}, h_i g_i)^{-1}$$
$$= \prod_{i=1}^{4} \psi_{v_i,w_i,y_i,y_{i+1}}(f_i, h_i) \prod_{i=1}^{4} \psi_{u_{y_{i+1}},w_{i+1},x,y_{i+1}}(k_{y_{i+1}}, g_{i+1}) \psi_{u_{y_{i+1}},w_i,x,y_{i+1}}(k_{y_{i+1}}, h_i g_i)^{-1}$$
$$= \prod_{i=1}^{4} \psi_{v_i,w_i,y_i,y_{i+1}}(f_i, h_i) \prod_{i=1}^{4} \psi_{w_i,w_{i+1},x,y_{i+1}}(h_i g_i, g_{i+1}),$$

and $\widehat{f_4}^{v_4} \cdots \widehat{f_1}^{v_1}$ acts on $M(y_1')$ as

$$\prod_{i=1}^{4} \psi_{v_i,w_i',y_i',y_{i+1}'}(f_i, h_i') \prod_{i=1}^{4} \psi_{u_{y_i'},w_i',x,y_i'}(k_{y_i'}, g_i') \prod_{i=1}^{4} \psi_{u_{y_{i+1}'},w_i',x,y_{i+1}'}(k_{y_{i+1}'}, h_i' g_i')^{-1}$$
$$= \prod_{i=1}^{4} \psi_{v_i,w_i',y_i',y_{i+1}'}(f_i', h_i') \prod_{i=1}^{4} \psi_{u_{y_{i+1}'},w_{i+1}',x,y_{i+1}'}(k_{y_{i+1}'}, g_{i+1}') \psi_{u_{y_{i+1}'},w_i',x,y_{i+1}'}(k_{y_{i+1}'}, h_i' g_i')^{-1}$$
$$= \prod_{i=1}^{4} \psi_{v_i,w_i',y_i',y_{i+1}'}(f_i, h_i') \prod_{i=1}^{4} \psi_{w_i',w_{i+1}',x,y_{i+1}'}(h_i' g_i', g_{i+1}').$$

Using property (ix) of Proposition 3.3, we see that $\widehat{f_4}^{v_4} \cdots \widehat{f_1}^{v_1}$ acts as the same element of K on both $M(y_1)$ and $M(y_1')$. Since y_1 and y_1' were arbitrary, $\widehat{f_4}^{v_4} \cdots \widehat{f_1}^{v_1} \in K$.

Before proceeding with the induction, note that $\widehat{g_2 g_1}^v = \widehat{g_2}^v \widehat{g_1}^v$ whenever $g_1, g_2 \in \mathrm{Aut}(W/v)$. For by what we have already proved, if $g_3 = (g_2 g_1)^{-1}$ then $\widehat{g_3}^v \widehat{g_2}^v \widehat{g_1}^v$ belongs to K and fixes $M(v)$ pointwise, so $(\widehat{g_2 g_1}^v)^{-1} \widehat{g_2}^v \widehat{g_1}^v = \widehat{g_3}^v \widehat{g_2}^v \widehat{g_1}^v = 1_M$, since K acts faithfully on $M(v)$.

Now suppose the result holds for $m = r \geq 4$ and let $f_i \in \mathrm{Aut}(W/v_i)$ $(1 \leq i \leq r+1)$ satisfy $f_{r+1} \cdots f_1 = 1_W$. By Lemma 3.8, we can write $f_r f_{r-1} = g_r g_{r-1}$, where $g_r \in \mathrm{Aut}(W/v_{r+1})$ and $g_{r-1} \in \mathrm{Aut}(W)$ fixes some $v'_{r-1} \in W$. Applying the base step to $g_{r-1}^{-1} g_r^{-1} f_r f_{r-1}$, we get $\alpha \widehat{g}_r^{v_{r+1}} \widehat{g}_{r-1}^{v'_{r-1}} = \widehat{f}_r^{v_r} \widehat{f}_{r-1}^{v_{r-1}}$ for some $\alpha \in K$. Let $f'_i = f_i$ and $v'_i = v_i$ for $1 \leq i \leq r-2$, $f'_{r-1} = g_{r-1}$, $f'_r = f_{r+1} g_r$, and $v'_r = v_{r+1}$. Then by the induction hypothesis,

$$\widehat{f}_{r+1}^{v_{r+1}} \cdots \widehat{f}_1^{u_1} = \alpha \widehat{f}_{r+1}^{v_{r+1}} \widehat{g}_r^{v_{r+1}} \widehat{g}_{r-1}^{v'_{r-1}} \widehat{f}_{r-2}^{v_{r-2}} \cdots \widehat{f}_1^{v_1} = \alpha \widehat{f'}_r^{v'_r} \cdots \widehat{f'}_1^{v'_1} \in K.$$

So the result holds for $m = r + 1$. □

Lemma 3.16 *G is a closed subgroup of Sym(M).*

Proof: Suppose that $\langle \widetilde{f}_n \rangle$ is a sequence in G converging to $\widetilde{f} \in \mathrm{Sym}(M)$. Let $\langle f_n \rangle$ be the corresponding sequence in $\mathrm{Aut}(W)$, with limit $f \in \mathrm{Aut}(W)$. Since f can be written as a product of maps in $S = \bigcup_{v \in W} \mathrm{Aut}(W/v)$, there is some $\widetilde{g} \in G$ extending f. So we can assume without loss that $f = 1_W$; otherwise we just consider the sequence $\langle \widetilde{f}_n \widetilde{g}^{-1} \rangle$ with limit $\widetilde{f}\widetilde{g}^{-1}$.

Throwing away a finite number of terms in the sequence and multiplying through by an element of K if necessary, we can further assume that for some fixed $v \in W$, every $\widetilde{f}_n$ fixes $M(v)$ pointwise. Then by Lemma 3.15, $\widetilde{f}_n = \widehat{f}_n^v$ for each n, since both $\widetilde{f}_n$ and $\widehat{f}_n^v$ restrict to f_n and fix $M(v)$ pointwise.

Given $y \in W$, let m be large enough that f_n fixes y for every $n \geq m$. Picking $w \in W$ such that $\mathrm{tp}(w, x) = \mathrm{tp}(w, y)$, we have $\widetilde{f}_n a(y) = \widehat{f}_n^v a(y) = \psi_{v,w,y,y}(f_n, 1_W) a(y) = \phi_{v,y}(f_n) a(y)$ for $n \geq m$. Since the sequence $\langle \phi_{v,y}(f_n) \rangle$ converges to $\phi_{v,y}(1_W) = 1$ by continuity, it follows that $\widetilde{f}_n$ fixes $M(y)$ pointwise for all sufficiently large n, hence so does $\widetilde{f}$. So $\widetilde{f} = 1_M \in G$. □

We are now justified in defining $\mathrm{Aut}(M) = G$. Note that the extensions in $\mathrm{Aut}(M)$ of any $f \in S$ fixing $v \in W$ are exactly the maps $\alpha \widehat{f}^v$ for $\alpha \in K$. So the next result is clear.

Lemma 3.17 *$\pi : M \to W$ is a transitive, untwisted, superlinked finite cover with central kernel K.*

All that is left is to check that $\pi : M \to W$ has the correct associated extended conjugate system.

Lemma 3.18 *(Φ, Ψ) is the associated extended conjugate system of the cover $\pi : M \to W$.*

Proof: We have to show that for any $v_1, v_2, y, z \in W$, if $f_i \in \mathrm{Aut}(W/v_i)_{y \mapsto z}$ for $i = 1, 2$ then $(\widehat{f_2}^{v_2})^{-1}\widehat{f_1}^{v_1}$ acts on $M(y)$ as $\psi_{v_1,v_2,y,z}(f_1, f_2)$. Let $w \in W$ satisfy $\mathrm{tp}(w,x) = \mathrm{tp}(w,y) = \mathrm{tp}(w,z)$ and pick $g \in \mathrm{Aut}(W/w)_{x \mapsto y}$ and $h \in \mathrm{Aut}(W/w)_{y \mapsto z}$. Then

$$\begin{aligned}
(\widehat{f_2}^{v_2})^{-1}\widehat{f_1}^{v_1}a(y) &= [\psi_{v_2,w,y,z}(f_2,h)\psi_{u_y,w,x,y}(k_y,g)\psi_{u_z,w,x,z}(k_z,hg)^{-1}]^{-1} \\
&\quad \cdot [\psi_{v_1,w,y,z}(f_1,h)\psi_{u_y,w,x,y}(k_y,g)\psi_{u_z,w,x,z}(k_z,hg)^{-1}]a(y) \\
&= \psi_{v_1,w,y,z}(f_1,h)\psi_{v_2,w,y,z}(f_2,h)^{-1}a(y) \\
&= \psi_{v_1,v_2,y,z}(f_1,f_2)a(y),
\end{aligned}$$

by Property (iii) of Proposition 3.3. □

The following characterization summarizes the results of this section.

Theorem 3.19 *Suppose that (Φ, Ψ) is an extended conjugate system for (W, K), where W is a permutation structure satisfying Property (3.1) and K is a finite abelian group. Then there is a unique (up to isomorphism) transitive, untwisted, superlinked finite cover $\pi : M \to W$ with central kernel K and associated extended conjugate system (Φ, Ψ).*

3.3 Irreducibility

Our motivation for studying the class of transitive, untwisted, superlinked finite covers with central kernels was in part due to Conjecture 3.1, since the irreducible superlinked covers at the heart of the conjecture belong to this class. Given the characterization in Section 3.2, it would therefore be nice to have a condition which picks out exactly those extended conjugate systems whose corresponding covers are irreducible. We assume throughout this section that W is irreducible, in addition to satisfying Property (3.1), since non-irreducible structures cannot have irreducible finite covers.

Closely related to irreducibility is the concept of a split cover. We say that a finite cover $\pi : M \to W$ with kernel K *splits* if there is a closed subgroup $H \leq \mathrm{Aut}(M)$ such that $H \cdot K = \mathrm{Aut}(M)$ and $H \cap K = 1$, i.e. H is a closed complement to K in $\mathrm{Aut}(M)$.

Lemma 3.20 *Let W be an irreducible permutation structure satisfying Property (3.1), and suppose $\pi : M \to W$ is a transitive, untwisted, superlinked finite cover with central kernel K and associated extended conjugate system $(\Phi, \Psi) = (\langle \phi_{x,y} \rangle, \langle \psi_{w,x,y,z} \rangle)$. Then π is irreducible if and only if for any non-trivial surjective group homomorphism $\eta : K \to A$, the transitive, untwisted, superlinked finite cover $\pi' : M' \to W$ with central kernel A and associated extended conjugate system $\eta(\Phi, \Psi) = (\langle \eta\phi_{x,y} \rangle, \langle \eta\psi_{w,x,y,z} \rangle)$ is non-split.*

Proof: Given η we can easily describe π' using π. By uniqueness, if $N = \mathrm{Ker}\,\eta$ then M' is the permutation structure consisting of the N-orbits on M, with automorphism group $\mathrm{Aut}(M)/N$ (corresponding to the action of $\mathrm{Aut}(M)$ on M'), and $\pi' : M' \to W$ is the restriction of π to M'. Suppose that this cover splits, say H' is a closed complement to the kernel K/N in $\mathrm{Aut}(M)/N$. Let H be the inverse image of H' under the quotient map $\mathrm{Aut}(M) \to \mathrm{Aut}(M)/N$, so that $H' = H/N$. Then $H \cap K = N$ and $H \cdot K = \mathrm{Aut}(M)$ since $(H \cap K)/N = (H/N) \cap (K/N) = N/N$ and $(H \cdot K)/N = (H/N) \cdot (K/N) = \mathrm{Aut}(M)/N$, so H is a proper subgroup of finite index in $\mathrm{Aut}(M)$ (note that $\mathrm{Aut}(M) \cap K = K \neq N$ since η is non-trivial). Furthermore, H is closed since the quotient map is continuous. So π is not irreducible.

Conversely, if $\mathrm{Aut}(M)$ has a proper closed subgroup H of finite index then the restriction to W of H is all of $\mathrm{Aut}(W)$, thus $H \cdot K = \mathrm{Aut}(M)$. So if $\eta : K \to K/N$ is the quotient homomorphism, where $N = H \cap K$, then H/N is a closed complement to the kernel K/N of the corresponding cover $\pi' : M' \to W$. □

So to get an irreducibility condition for extended conjugate systems we need only find out which systems correspond to split covers. Since these covers are in some sense degenerate, the solution should somehow be related to the degeneracy of extended conjugate systems.

Definition 3.21 An extended conjugate system $(\Phi, \Psi) = (\langle \phi_{x,y} \rangle, \langle \psi_{w,x,y,z} \rangle)$ for (W, K) is *degenerate* if there are continuous homomorphisms $\phi_x : \mathrm{Aut}(W/x) \to K$ for $x \in W$ such that

(i) if $f \in \mathrm{Aut}(W/x)$ and $g \in \mathrm{Aut}(W)$ then $\phi_{gw}(gfg^{-1}) = \phi_w(f)$;

(ii) if $f \in \mathrm{Aut}(W/w)_{y \mapsto z}$ and $g \in \mathrm{Aut}(W/x)_{y \mapsto z}$ then $\psi_{w,x,y,z}(f,g) = \phi_w(f)^{-1}\phi_x(g)\phi_y(g^{-1}f)$.

The system (Φ, Ψ) is *fully non-degenerate* if for any non-trivial surjective homomorphism $\eta : K \to A$, the system $\eta(\Phi, \Psi)$ is non-degenerate.

Theorem 3.22 *Suppose W is an irreducible permutation structure satisfying Property 3.1 and $\pi : M \to W$ is a transitive, untwisted, superlinked finite cover with central kernel K and associated extended conjugate system (Φ, Ψ). Then*

(i) *π splits if and only if (Φ, Ψ) is degenerate;*

(ii) *π is irreducible if and only if (Φ, Ψ) is fully non-degenerate.*

Proof: By Lemma 3.20 it suffices to prove (i).

So suppose that π splits, say H is a closed complement to K in $\mathrm{Aut}(M)$. Then each $f \in \mathrm{Aut}(W)$ has a unique extension $\widehat{f}$ in H. This provides us with an obvious definition of the maps $\phi_x : \mathrm{Aut}(W/x) \to K$, namely if $f \in \mathrm{Aut}(W/x)$ then $\phi_x(f)$ is the unique element of K with the same action as $\widehat{f}$ on the fibre $M(x)$.

These maps ϕ_x are homomorphisms since $\widehat{fg} = \widehat{f}\widehat{g}$ for any $f, g \in \mathrm{Aut}(W)$; they are continuous since $\mathrm{Ker}\,\phi_x = \rho(H_{M(x)})$ is a closed subgroup of $\mathrm{Aut}(W)$, where ρ is the restriction map. Given $f \in \mathrm{Aut}(W/x)$ and $g \in \mathrm{Aut}(W)$ we have $\widehat{(gfg^{-1})} = \widehat{g}\widehat{f}\widehat{g}^{-1}$, so $\phi_{gx}(gfg^{-1}) = \widehat{g}\phi_x(f)\widehat{g}^{-1} = \phi_x(f)$. Finally, if $(f,g) \in \mathrm{Aut}(W/w)_{y \mapsto z} \times \mathrm{Aut}(W/x)_{y \mapsto z}$ then $\phi_w(f)^{-1}\widehat{f} \in \mathrm{Aut}(M/M(w))$ and $\phi_x(g)^{-1}\widehat{g} \in \mathrm{Aut}(M/M(x))$, so $\widehat{g^{-1}f} = \widehat{g}^{-1}\widehat{f}$ acts on $M(y)$ as $\psi_{w,x,y,z}(f,g)\phi_w(f)\phi_x(g)^{-1}$, hence $\psi_{w,x,y,z}(f,g) = \phi_w(f)^{-1}\phi_x(g)\phi_y(g^{-1}f)$. So the maps ϕ_x satisfy the properties of Definition 3.21, and (Φ, Ψ) is degenerate.

For the converse we assume that (Φ, Ψ) is degenerate, say $\phi_x : \mathrm{Aut}(W/x) \to K$ $(x \in W)$ is a system of homomorphisms satisfying (i) and (ii) in Definition 3.21. Given $w \in W$ and $f \in \mathrm{Aut}(W/w)$, there is a unique $\widehat{f}^w \in \mathrm{Aut}(M)$ extending f and acting on $M(w)$ as $\phi_w(f)$. If f also fixes $x \in W$ then $\phi_w(f)^{-1}\widehat{f}^w$ acts on $M(x)$ as $\phi_{w,x}(f) = \psi_{w,w,x,x}(f, 1_W) =$

$\phi_w(f)^{-1}\phi_x(f)$, so $\widehat{f}^w = \widehat{f}^x$. Thus, for each $f \in S = \bigcup_{w\in W}\mathrm{Aut}(W/w)$ we can define $\widehat{f} \in \mathrm{Aut}(M)$ to be the unique extension of f acting as $\phi_w(f)$ on $M(w)$ for some (every) fixed point $w \in W$ of f. Let H be the subgroup of $\mathrm{Aut}(M)$ generated by the maps $\widehat{f}$ for $f \in S$. We claim that H is a closed complement to K in $\mathrm{Aut}(M)$.

Clearly $H \cdot K = \mathrm{Aut}(M)$, since every $f \in \mathrm{Aut}(W)$ has an extension in H by Lemma 3.8. It is slightly harder to show that $H \cap K = 1$. Consider any $\widehat{f_m}\cdots\widehat{f_1} \in H \cap K$, where f_i fixes some $v_i \in W$ for each i. Writing $\widetilde{f_i} = \phi_{v_i}(f_i)^{-1}\widehat{f_i} \in \mathrm{Aut}(M/M(v_i))$, we must show that $\widetilde{f_m}\cdots\widetilde{f_1} = \phi_{v_m}(f_m)^{-1}\cdots\phi_{v_1}(f_1)^{-1}$. Let $x, y_1 \in W$, and for $1 \le i \le m$ let $y_{i+1} = f_i y_i$, so that $y_{m+1} = y_1$. Now pick $w_i \in W$ and $g_i, h_i \in \mathrm{Aut}(W/w_i)$ such that $g_i x = y_i$ and $h_i y_i = y_{i+1}$ for $1 \le i \le m$, and let $w_{m+1} = w_1$ and $g_{m+1} = g_1$. Then as in the proof of Proposition 3.3, $\widetilde{f_m}\cdots\widetilde{f_1}$ acts on $M(y_1)$ as

$$\begin{aligned}
&\prod_{i=1}^{m}\psi_{v_i,w_i,y_i,y_{i+1}}(f_i,h_i)\prod_{i=1}^{m}\psi_{w_i,w_{i+1},x,y_{i+1}}(h_ig_i,g_{i+1})\\
&= \prod_{i=1}^{m}\phi_{v_i}(f_i)^{-1}\phi_{w_i}(h_i)\phi_{y_i}(h_i^{-1}f_i)\phi_{w_i}(h_ig_i)^{-1}\phi_{w_{i+1}}(g_{i+1})\phi_x(g_{i+1}^{-1}h_ig_i)\\
&= \prod_{i=1}^{m}\phi_{v_i}(f_i)^{-1}\phi_{y_i}(h_i^{-1}f_i)\phi_x(g_{i+1}^{-1}h_ig_i)\\
&= \prod_{i=1}^{m}\phi_{v_i}(f_i)^{-1}\phi_x(g_i^{-1}h_i^{-1}f_ig_i)\phi_x(g_{i+1}^{-1}h_ig_i)\\
&= \prod_{i=1}^{m}\phi_{v_i}(f_i)^{-1}\phi_x(g_{i+1}^{-1}f_ig_i)\\
&= \phi_x(g_1^{-1}f_m\cdots f_1g_1)\prod_{i=1}^{m}\phi_{v_i}(f_i)^{-1} = \prod_{i=1}^{m}\phi_{v_i}(f_i)^{-1},
\end{aligned}$$

since $f_m\cdots f_1 = 1_W$. But y_1 was arbitrary, so $\widetilde{f_m}\cdots\widetilde{f_1} = \phi_{v_m}(f_m)^{-1}\cdots\phi_{v_1}(f_1)^{-1}$, as required.

It remains to show that H is closed. Let $\langle\widetilde{f_n}\rangle$ be any sequence in H with limit $\widetilde{f} \in \mathrm{Aut}(M)$, and for each n let f_n be the restriction of $\widetilde{f_n}$ to W. Without loss of generality we can assume that $\langle f_n\rangle$ has limit 1_W. Given $w \in W$, we can find $m \in \omega$ such that $f_n \in \mathrm{Aut}(W/w)$ for every $n \ge m$. Then $\widetilde{f_n} = \widehat{f_n}^w$ for such n, so $\widetilde{f_n}$ acts on $M(w)$ as $\phi_w(f_n)$; since ϕ_w is continuous and $\langle f_n\rangle$ has limit 1_W, eventually $\widetilde{f_n}$ fixes $M(w)$ pointwise. Since this is true for every $w \in W$, we have $\widetilde{f} = 1_M \in H$. So H is closed. □

The *image* of an extended conjugate system (Φ, Ψ) for (W, K), denoted by $\mathrm{Im}(\Phi, \Psi)$, is the subgroup of K generated by the union of the images of the maps $\psi_{w,x,y,z}$. Since the quotient map $\eta : K \to K/\mathrm{Im}(\Phi, \Psi)$ produces a (trivial) degenerate conjugate system $\eta(\Phi, \Psi)$, a necessary condition for the full non-degeneracy of (Φ, Ψ) is that $\mathrm{Im}(\Phi, \Psi) = K$. If the stabilizer of a point in W is irreducible then this condition is also sufficient, for in this case a degenerate system must have trivial image (the only continuous homomorphism with domain $\mathrm{Aut}(W/x)$ and finite image is the trivial map). Combining this with the previous theorem produces the following result.

Corollary 3.23 *Let W, $\pi : M \to W$, K, and (Φ, Ψ) be as in Theorem 3.22, and suppose that Aut(W/x) is irreducible, for $x \in W$. Then*

(i) *π splits if and only if $Im(\Phi, \Psi) = 1$;*

(ii) *π is irreducible if and only if $Im(\Phi, \Psi) = K$.*

This brings us back to our starting point, answering Conjecture 3.1 for structures W satisfying Property (3.1) in which the stabilizer of a point is irreducible.

Corollary 3.24 *Let W be a permutation structure satisfying the hypotheses of Conjecture 3.1. Suppose in addition that W satisfies property (3.1) and that Aut(W/x) is irreducible, for $x \in W$. Then there is an $r \in \omega$ such that the kernel of any irreducible superlinked finite cover of W is generated by a set of size at most r.*

Proof: Since W is $\aleph_0$-categorical, it has only finitely many 2-types. So there is an $m_1 \in \omega$ such that for any $x, y \in W$, $\mathrm{Aut}^0(W/x, y)$ has index at most m_1 in $\mathrm{Aut}(W/x, y)$. Let $m_2 \in \omega$ be the number of 4-types in W. We will prove the result with $r = m_1^2 m_2$.

By the previous corollary, it suffices to show that for any finite abelian group K, the image of any extended conjugate system $(\Phi, \Psi) = (\langle \phi_{x,y} \rangle, \langle \psi_{w,x,y,z} \rangle)$ for (W, K) is generated by a set of size at most $m_1^2 m_2$. Given $x, y \in W$, $\mathrm{Ker}\, \phi_{x,y}$ contains $\mathrm{Aut}^0(W/x, y)$, thus it has index no greater than m_1 in $\mathrm{Aut}(W/x, y)$. So the image of any $\phi_{x,y}$ has size at most m_1. It follows from Property (ii) of Proposition 3.3 that for any $w, x, y, z \in W$, the image of $\psi_{w,x,y,z}$ has size no greater than m_1^2. But $\psi_{w,x,y,z}$ and $\psi_{fw,fx,fy,fz}$ have the same image for $f \in \mathrm{Aut}(W)$ (by Property (i) of Proposition 3.3), so the union of the images of the maps $\psi_{w,x,y,z}$ has size at most $m_1^2 m_2$. This proves what we want, since this union generates $\mathrm{Im}(\Phi, \Psi)$. $\square$

Chapter 4

Structures with Strong Types

Most of our results have been presented in a way that closely parallels [5], in which Evans characterizes the (locally transitive) untwisted, irreducible, superlinked finite covers of transitive $\aleph_0$-categorical structures with strong types. There the fundamental concept is that of a conjugate system for a triple (W, K, R), where R is a 2-type in the permutation structure W and K is a finite abelian group. Although our characterization applies to a more general class of structures, it also seems to use more information to describe each cover, so it is not obvious that we have improved [5, Corallary 3.9]. Our goal in this chapter is to show that we have indeed generalized Evans' result.

Definition 4.1 (i) Let W be a permutation structure with 2-type R, and let K be a finite abelian group. A *conjugate system* for (W, K, R) is a system of continuous homomorphisms $\phi_{w,x} : \mathrm{Aut}(W/w, x) \to K$, where $(w, x) \in R$, satisfying Properties (i) and (ii) of Proposition 3.2. As in Section 3.1, we associate a conjugate system for (W, K, R) with each transitive, untwisted, superlinked finite cover $\pi : M \to W$ with central kernel K.

(ii) A *strong type* over a transitive structure W consists of a non-constant map p which assigns to each finite subset $X \subseteq W$ a 1-type $p|X$ over X in such a way that

- if $Y \subseteq X$ then $p|X \subseteq p|Y$;
- if $f \in \mathrm{Aut}(W)$ then $p|fX = f(p|X)$.

The associated 2-type of a strong type p is the type $R = \{(w, x) \in W^2 : w \in p|\{x\}\}$.

Note that this definition of a conjugate system for (W, K, R) is slightly more general than in [5]: we do not require that the maps $\phi_{x,y} : \mathrm{Aut}(W/x, y) \to K$ be surjective. A cover $\pi : M \to W$ whose associated conjugate system for $(W, \mathrm{Aut}(M/W), R)$ consists of surjective maps is said to be *locally transitive* with respect to R.

Clearly every transitive structure with a strong type satisfies (3.1), so the results of Chapter 3 apply. We will show that if W is a (not necessarily $\aleph_0$-categorical) transitive permutation structure with a strong type p and associated 2-type R, and if K is a finite abelian group, then every conjugate system for (W, K, R) extends uniquely to an extended conjugate system for (W, K). Since we are not assuming that W is $\aleph_0$-categorical, this leads to a direct improvement of [5, Corollary 3.9].

The proof will be given in two steps. First we will show how to extend a conjugate system $\langle \phi_{w,x} : (w, x) \in R \rangle$ for (W, K, R) to a system $\Phi = \langle \phi_{x,y} : x, y \in W \rangle$ satisfying Proposition 3.2.

Lemma 4.2 *If X is a finite subset of W, $f \in Aut(W)$ and $w \in p|(X \cup fX)$ then there are $g \in Aut(W/X)$ and $h \in Aut(W/w)$ such that $f = hg$. In particular, h and f fix the same elements of X.*

Proof: Since $w \in p|fX$ we have $f^{-1}w \in p|X$, so w and $f^{-1}w$ are in the same type over X. Thus we can find $g \in \mathrm{Aut}(W/X)$ mapping $f^{-1}w$ to w, so that $h = fg^{-1} \in \mathrm{Aut}(W/w)$. □

So there is clearly only one possible way to extend $\langle \phi_{w,x} : (w, x) \in R \rangle$ to Φ:

Given $x, y \in W$ and $f \in \mathrm{Aut}(W/x, y)$, choose $v \in p|\{x, y\}$ and $w \in p|\{x, y, v, fv\}$. By Lemma 4.2 we can find $g \in \mathrm{Aut}(W/x, y, v)$ and $h \in \mathrm{Aut}(W/x, y, w)$ such that $f = hg$. Then we define

$$\phi_{x,y}(f) = \phi_{w,x}(h)^{-1}\phi_{w,y}(h)\phi_{v,x}(g)^{-1}\phi_{v,y}(g).$$

Note that the maps $\phi_{w,x}, \phi_{w,y}, \phi_{v,x}, \phi_{v,y}$ are already known, since $(w, x), (w, y), (v, x), (v, y) \in R$.

Lemma 4.3 *For each $x, y \in W$, $\phi_{x,y}$ is well-defined.*

Proof: We must show that the definition of $\phi_{x,y}(f)$ does not depend on our choice of v, w, g, and h. Keeping v and w fixed, it is easy to see that the definition does not depend on g and h. For if we pick another pair $g' \in \mathrm{Aut}(W/x, y, v)$, $h' \in \mathrm{Aut}(W/x, y, w)$ such that $f = g'h'$, then the two definitions of $\phi_{x,y}(f)$ differ by

$$\phi_{w,x}(g'g^{-1})\phi_{w,y}(g'g^{-1})^{-1}\phi_{v,x}(g'g^{-1})^{-1}\phi_{v,x}(g'g^{-1}) = \phi_{w,v}(g'g^{-1})\phi_{w,v}(g'g^{-1})^{-1} = 1.$$

Suppose now that we have another pair $v' \in p \mid \{x, y\}$ and $w' \in p \mid \{x, y, v', fv'\}$. We can assume that (v', w') "refines" (v, w), in the sense that $v' \in p \mid \{x, y, v, w, f^{-1}w\}$ and $w' \in p \mid \{x, y, v, w, v', fv, fv'\}$; otherwise we just work with a common refinement $v'' \in p \mid \{x, y, v, w, v', w', f^{-1}w, f^{-1}w'\}$ and $w'' \in p \mid \{x, y, v, w, v', w', v'', fv, fv', fv''\}$. If we can use the same pair (g, h) with both (v, w) and (v', w') to define $\phi_{x,y}(f)$ then we will get the same result, since the definitions will differ by $\phi_{w',w}(h)\phi_{w',w}(h)^{-1}\phi_{v',v}(g)\phi_{v',v}(g)^{-1} = 1$. So we will be done if we can find $g \in \mathrm{Aut}(W/x, y, v, v')$ and $h \in \mathrm{Aut}(W/x, y, w, w')$ such that $f = hg$. Picking $g_1 \in \mathrm{Aut}(W/x, y, v)_{f^{-1}w \mapsto w}$ we have $v', g_1v' \in p \mid \{x, y, v, w\}$, so we can find $g_2 \in \mathrm{Aut}(W/x, y, v, w)$ mapping g_1v' to v'; hence $w', g_2g_1f^{-1}w' \in p \mid \{x, y, v, w, v'\}$ and there is some $g_3 \in \mathrm{Aut}(W/x, y, v, w, v')$ mapping $g_2g_1f^{-1}w'$ to w'. The maps $g = g_3g_2g_1$ and $h = fg^{-1}$ have the required properties. □

Lemma 4.4 *The system $\Phi = \langle \phi_{x,y} : x, y \in W \rangle$ satisfies Proposition 3.2.*

Proof: Properties (i) and (ii) are easy to prove using the corresponding properties of $\langle \phi_{w,x} : (w, x) \in R \rangle$. Assuming the maps $\phi_{x,y}$ are homomorphisms, it is also easy to prove continuity. For given a sequence $\langle f_n \rangle$ in $\mathrm{Aut}(W/x, y)$ with limit 1_W and an element $w \in p \mid \{x, y\}$, eventually $f_n \in \mathrm{Aut}(W/x, y, w)$, so by Property (ii) and the continuity of $\phi_{w,x}$ and $\phi_{w,y}$, $\phi_{x,y}(f_n) = \phi_{w,x}(f_n)^{-1}\phi_{w,y}(f_n)$ has limit 1.

The hard part is showing that the maps $\phi_{x,y}$ are homomorphisms. Fix $x, y \in W$ and let $f_1, f_2 \in \mathrm{Aut}(W/x, y)$. We must show that $\phi_{x,y}(f_2f_1) = \phi_{x,y}(f_2)\phi_{x,y}(f_1)$. We consider two cases.

Case 1. *There are $v, w \in W$ such that $v \in p \mid \{x, y\}$ and $w \in p \mid \{x, y, v, f_1v, f_2v, f_2^{-1}v, f_2^{-1}w\}$.*

Choose $g_1, g_2 \in \mathrm{Aut}(W/x,y,v)$ and $h_1, h_2 \in \mathrm{Aut}(W/x,y,w)$ such that $f_i = h_i g_i$, $i = 1,2$. We can do this in such a way that $g_2^{-1}f_2$ fixes w. For if we pick $g_2' \in \mathrm{Aut}(W/x,y,v)$ mapping $f_2^{-1}w$ to w then $g_2'w, f_2w \in p \mid \{x,y,v,w\}$, so there is some $g_2'' \in \mathrm{Aut}(W/x,y,v,w)$ mapping $g_2'w$ to f_2w, and $g_2 = g_2''g_2'$, $h_2 = f_2g_2^{-1}$ have the required properties. Since $(w, g_2^{-1}w), (g_2^{-1}w, x), (g_2^{-1}w, y) \in R$, we have

$$\begin{aligned}
\phi_{x,y}(f_2)\phi_{x,y}(f_1) &= \prod_{i=1}^{2}[\phi_{w,x}(h_i)^{-1}\phi_{w,y}(h_i)\phi_{v,x}(g_i)^{-1}\phi_{v,y}(g_i)] \\
&= \phi_{g_2^{-1}w,x}(g_2^{-1}h_2g_2)^{-1}\phi_{g_2^{-1}w,y}(g_2^{-1}h_2g_2)\phi_{w,x}(h_1)^{-1}\phi_{w,y}(h_1)\phi_{v,x}(g_1g_2)^{-1}\phi_{v,y}(g_1g_2) \\
&= [\phi_{w,x}(g_2^{-1}f_2)^{-1}\phi_{w,g_2^{-1}w}(g_2^{-1}f_2)][\phi_{w,y}(g_2^{-1}f_2)\phi_{w,g_2^{-1}w}(g_2^{-1}f_2)^{-1}] \\
&\quad \cdot[\phi_{w,x}(h_1)^{-1}\phi_{w,y}(h_1)\phi_{v,x}(g_1g_2)^{-1}\phi_{v,y}(g_1g_2)] \\
&= \phi_{w,x}(g_2^{-1}f_2f_1g_1^{-1})^{-1}\phi_{w,y}(g_2^{-1}f_2f_1g_1^{-1})\phi_{v,x}(g_1g_2)^{-1}\phi_{v,y}(g_1g_2) \\
&= \phi_{x,y}(g_2^{-1}f_2f_1g_2).
\end{aligned}$$

But we already know that Φ satisfies Property (i) of Proposition 3.2, so $\phi_{x,y}(f_2)\phi_{x,y}(f_1) = \phi_{x,y}(f_2f_1)$.

Case 2. *Otherwise.*

Pick $u \in p \mid \{x,y\}$, $v \in p \mid \{x,y,u,f_1u,f_2u,f_2^{-1}u\}$ and $w \in p \mid \{x,y,u,v,f_2u,f_2f_1u,f_2^2u,f_2v\}$. Then there is some $g \in \mathrm{Aut}(W/x,y,u,f_2u,f_2f_1u,f_2^2u)$ mapping f_2v to w, and so $v \in p \mid \{x,y,u,f_1u,gf_2u,(gf_2)^{-1}u,(gf_2)^{-1}v\}$ since $gf_2v = w \in p \mid \{x,y,gf_2u,gf_2f_1u,(gf_2)^2u,u,v\}$. We can therefore apply Case 1 to the pair f_1, gf_2 to get $\phi_{x,y}(gf_2f_1) = \phi_{x,y}(gf_2)\phi_{x,y}(f_1)$. Furthermore, w is in both $p \mid \{x,y,u,f_2u,gu,g^{-1}u,g^{-1}w\}$ and $p \mid \{x,y,u,f_2f_1u,gu,g^{-1}u,g^{-1}w\}$, so we can apply Case 1 to the pair f_2, h and the pair f_2f_1, h, giving us $\phi_{x,y}(hf_2) = \phi_{x,y}(h)\phi_{x,y}(f_2)$ and $\phi_{x,y}(hf_2f_1) = \phi_{x,y}(h)\phi_{x,y}(f_2f_1)$. We get the desired result by combining these three equations. □

The second step is to extend Φ to an extended conjugate system (Φ, Ψ) for (W, K). Using the properties of Proposition 3.3 we can again see that there is only one possibility for the maps $\psi_{w,x,y,z}$. For if $u \in p \mid \{w,x,y,z\}$ then (assuming $\mathrm{tp}(w,y) = \mathrm{tp}(w,z)$ and $\mathrm{tp}(x,y) = \mathrm{tp}(x,z)$) we can find $s \in \mathrm{Aut}(W/w,u)_{y\mapsto z}$ and $t \in \mathrm{Aut}(W/x,u)_{y\mapsto z}$, so that $\psi_{w,x,y,z}$ is completely determined from Φ by applying Properties (ii), (v) and (vii) to $\psi_{w,x,y,z}(s,t)$.

More precisely, we have the following definition:

Given $w, x, y, z \in W$, $f \in \mathrm{Aut}(W/w)_{y \mapsto z}$ and $g \in \mathrm{Aut}(W/x)_{y \mapsto z}$, choose $u \in p \mid \{w, x, y, z\}$, $s \in \mathrm{Aut}(W/w, u)_{y \mapsto z}$ and $t \in \mathrm{Aut}(W/x, u)_{y \mapsto z}$. Then we define:

$$\psi_{w,x,y,z}(f, g) = \phi_{w,y}(s^{-1}f)\phi_{x,y}(g^{-1}t)\phi_{w,u}(s)\phi_{x,u}(t^{-1})\phi_{u,y}(t^{-1}s).$$

Lemma 4.5 *For each $w, x, y, z \in W$, $\psi_{w,x,y,z}$ is well-defined and continuous.*

Proof: It is easy to see that the definition of $\psi_{w,x,y,z}(f, g)$ does not depend on our choice of s and t, provided u is fixed. On the other hand, given another $u' \in p \mid \{w, x, y, z\}$ and a map $k \in \mathrm{Aut}(W/w, x, y, z)_{u' \mapsto u}$, let $s' = k^{-1}sk$ and $t' = k^{-1}tk$; then a simple calculation shows that we get the same result for $\psi_{w,x,y,z}(f, g)$ whether we use u', s', t' or u, s, t in the definition.

Continuity is obvious since $\psi_{w,x,y,z}$ is a product of continuous maps. (Note that we can use the same u, s, t to define $\psi_{w,x,y,z}(f, g)$ for every pair $f \in \mathrm{Aut}(W/w)_{y \mapsto z}$, $g \in \mathrm{Aut}(W/x)_{y \mapsto z}$.) □

Lemma 4.6 *The system $\Psi = \langle \psi_{w,x,y,z} : w, x, y, z \in W \rangle$ satisfies Properties (i)–(ix) of Proposition 3.3.*

Proof: The proofs of (ii) and (vi) are trivial. Properties (vii) and (viii) are clear since we can pick $s = t$ in the definitions of $\psi_{w,x,y,y}(f, g)$ and $\psi_{w,x,y,z}(f, f)$, respectively, and Property (v) is clear since we can use the same $u \in p \mid \{w, x, y_1, z_1, y_2, z_2\}$, $s \in \mathrm{Aut}(W/w, u)_{y_1 \mapsto z_1, y_2 \mapsto z_2}$ and $t \in \mathrm{Aut}(W/x, u)_{y_1 \mapsto z_1, y_2 \mapsto z_2}$ to define both $\psi_{w,x,y_1,z_1}(f, g)$ and $\psi_{w,x,y_2,z_2}(f, g)$. If we use u, s_1, s_2 to define $\psi_{v_1,v_2,y,z}(f_1, f_2)$, u, s_2, s_3 to define $\psi_{v_2,v_3,y,z}(f_2, f_3)$ and u, s_1, s_3 to define $\psi_{v_1,v_3,y,z}(f_1, f_3)$, where $u \in p \mid \{v_1, v_2, v_3, y, z\}$ and $s_i \in \mathrm{Aut}(W/v_i, u)_{y \mapsto z}$ for $1 \leq i \leq 3$, then (iii) is obvious. Similarly, (iv) is obvious if we use u, s_1, t_1 to define $\psi_{w,x,y,z}(f_1, g_1)$, u, s_2, t_2 to define $\psi_{w,x,z,v}(f_2, g_2)$ and u, s_2s_1, t_2t_1 to define $\psi_{w,x,y,v}(f_2f_1, g_2g_1)$, where $u \in p \mid \{w, x, y, z, v\}$, $s_1 \in \mathrm{Aut}(W/w, u)_{y \mapsto z}$, $t_1 \in \mathrm{Aut}(W/x, u)_{y \mapsto z}$, $s_2 \in \mathrm{Aut}(W/w, u)_{z \mapsto v}$, and $t_2 \in \mathrm{Aut}(W/x, u)_{z \mapsto v}$. This leaves to be proved Properties (i) and (ix).

For (i) let $f \in \mathrm{Aut}(W/w)_{y \mapsto z}$ and $g \in \mathrm{Aut}(W/x)_{y \mapsto z}$, and suppose $k^{-1}h \in \mathrm{Aut}(W/y, z)$; we want to show that $\psi_{hw,kx,hy,hz}(hfh^{-1}, kgk^{-1}) = \psi_{w,x,y,z}(f, g)\phi_{y,z}(k^{-1}h)$. The proof is divided into three cases.

Case 1. $k = h$.

In this case the result is clear, since we can use hu, hsh^{-1} and hth^{-1} to define $\psi_{hw,hx,hy,hz}(hfh^{-1}, hgh^{-1})$, where $u \in p \mid \{w, x, y, z\}$, $s \in \mathrm{Aut}(W/w, u)_{y \mapsto z}$ and $t \in \mathrm{Aut}(W/x, u)_{y \mapsto z}$.

Case 2. $k = 1$.

Let $u \in p \mid \{w, hw, x, y, z\}$. Then by Lemma 4.2 we can find $h_1 \in \mathrm{Aut}(W/w, y, z)$ and $h_2 \in \mathrm{Aut}(W/y, z, u)$ such that $h = h_2 h_1$. Next pick $s \in \mathrm{Aut}(W/w, u)_{y \mapsto z}$ and $t \in \mathrm{Aut}(W/x, u)_{y \mapsto z}$ and let $s' = h_2 s h_2^{-1} \in \mathrm{Aut}(W/hw, u)_{y \mapsto z}$. Then

$$\begin{aligned}
\psi_{hw,x,y,z}(hfh^{-1}, g) &= \phi_{hw,y}((s')^{-1}hfh^{-1})\phi_{x,y}(g^{-1}t)\phi_{hw,u}(s')\phi_{x,u}(t^{-1})\phi_{u,y}(t^{-1}s') \\
&= \phi_{w,y}(s^{-1}h_1 f h_1^{-1})\phi_{x,y}(g^{-1}t)\phi_{w,u}(s)\phi_{x,u}(t^{-1})\phi_{u,y}(t^{-1}h_2 s h_2^{-1}) \\
&= [\phi_{w,y}(s^{-1}f)\phi_{w,y}(h_1)^{-1}\phi_{w,z}(h_1)]\phi_{x,y}(g^{-1}t)\phi_{w,u}(s)\phi_{x,u}(t^{-1}) \\
&\quad \cdot[\phi_{u,y}(t^{-1}s)\phi_{u,y}(h_2)^{-1}\phi_{u,z}(h_2)] \\
&= \psi_{w,x,y,z}(f, g)\phi_{y,z}(h_1)\phi_{y,z}(h_2) \\
&= \psi_{w,x,y,z}(f, g)\phi_{y,z}(h).
\end{aligned}$$

Case 3. *Otherwise.*

By Cases 1 and 2,

$$\psi_{hw,kx,hy,hz}(hfh^{-1}, kgk^{-1}) = \psi_{k^{-1}hw,x,y,z}(k^{-1}hfh^{-1}k, g) = \psi_{w,x,y,z}(f, g)\phi_{y,z}(k^{-1}h),$$

as required.

It remains to prove (ix). So suppose $x, v_i, w_i, w_i', y_i, y_i', f_i, g_i, g_i', h_i, h_i'$ $(1 \le i \le 4)$ satisfy the given hypotheses. Then we can choose $u \in p \mid \{x, v_i, w_i, w_i', y_i, y_i' : 1 \le i \le 4\}$ and $k_i \in \mathrm{Aut}(W/v_i, u)_{y_i \mapsto y_{i+1}, y_i' \mapsto y_{i+1}'}$, $s_i \in \mathrm{Aut}(W/w_i, u)_{x \mapsto y_i}$, $t_i \in \mathrm{Aut}(W/w_i, u)_{y_i \mapsto y_{i+1}}$, $s_i' \in \mathrm{Aut}(W/w_i', u)_{x \mapsto y_i'}$, $t_i' \in \mathrm{Aut}(W/w_i', u)_{y_i' \mapsto y_{i+1}'}$. Using u, k_i, t_i to define $\psi_{v_i,w_i,y_i,y_{i+1}}(f_i, h_i)$, $u, t_i s_i, s_{i+1}$ to define $\psi_{w_i,w_{i+1},x,y_{i+1}}(h_i g_i, g_{i+1})$, u, k_i, t_i' to define $\psi_{v_i,w_i',y_i',y_{i+1}'}(f_i, h_i')$, and

$u, t'_i s'_i, s'_{i+1}$ to define $\psi_{w'_i, w'_{i+1}, x, y'_{i+1}}(h'_i g'_i, g'_{i+1})$, we can easily calculate

$$\begin{aligned}
&\prod_{i=1}^{4} \psi_{v_i,w_i,y_i,y_{i+1}}(f_i,h_i) \prod_{i=1}^{4} \psi_{w_i,w_{i+1},x,y_{i+1}}(h_i g_i, g_{i+1}) \\
&= \prod_{i=1}^{4}[\phi_{v_i,y_i}(k_i^{-1}f_i)\phi_{w_i,y_i}(h_i^{-1}t_i)\phi_{v_i,u}(k_i)\phi_{w_i,u}(t_i^{-1})\phi_{u,y_i}(t_i^{-1}k_i)] \\
&\quad \cdot \prod_{i=1}^{4}[\phi_{w_i,x}(s_i^{-1}t_i^{-1}h_i g_i)\phi_{w_{i+1},x}(g_{i+1}^{-1}s_{i+1})\phi_{w_i,u}(t_i s_i)\phi_{w_{i+1},u}(s_{i+1}^{-1})\phi_{u,x}(s_{i+1}^{-1}t_i s_i)] \\
&= \prod_{i=1}^{4}[\phi_{v_i,y_i}(k_i^{-1}f_i)\phi_{v_i,u}(k_i)] \prod_{i=1}^{4}[\phi_{w_i,y_i}(h_i^{-1}t_i)\phi_{w_i,x}(s_i^{-1}t_i^{-1}h_i g_i)\phi_{w_{i+1},x}(g_{i+1}^{-1}s_{i+1})] \\
&\quad \cdot \prod_{i=1}^{4}[\phi_{w_i,u}(t_i^{-1})\phi_{w_i,u}(t_i s_i)\phi_{w_{i+1},u}(s_{i+1}^{-1})] \prod_{i=1}^{4}[\phi_{u,y_i}(t_i^{-1}k_i)\phi_{u,x}(s_{i+1}^{-1}t_i s_i)] \\
&= \prod_{i=1}^{4}[\phi_{v_i,y_i}(k_i^{-1}f_i)\phi_{v_i,u}(k_i)] \prod_{i=1}^{4}[\phi_{u,y_i}(t_i^{-1}k_i)\phi_{u,x}(s_{i+1}^{-1}t_i s_i)] \\
&= \prod_{i=1}^{4}[\phi_{v_i,y_i}(k_i^{-1}f_i)\phi_{v_i,u}(k_i)] \prod_{i=1}^{4} \phi_{u,x}(s_{i+1}^{-1}k_i s_i) \\
&= \phi_{u,y_1}(k_4k_3k_2k_1) \prod_{i=1}^{4} \phi_{v_i,y_i}(k_i^{-1}f_i)\phi_{v_i,u}(k_i),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\prod_{i=1}^{4} \psi_{v_i,w'_i,y'_i,y'_{i+1}}(f'_i,h'_i) \prod_{i=1}^{4} \psi_{w'_i,w'_{i+1},x,y'_{i+1}}(h'_i g'_i, g'_{i+1}) \\
&\quad = \phi_{u,y'_1}(k_4k_3k_2k_1) \prod_{i=1}^{4} \phi_{v_i,y'_i}(k_i^{-1}f'_i)\phi_{v_i,u}(k_i).
\end{aligned}$$

But $\phi_{y'_1,y_1}(k_4k_3k_2k_1)\prod_{i=1}^{4}\phi_{y'_i,y_i}(k_i^{-1}f_i) = \phi_{y'_1,y_1}(f_4f_3f_2f_1) = 1$, so the result follows. □

We have accomplished our goal.

Theorem 4.7 *If W is a transitive structure with a strong type p and associated 2-type R then given a finite abelian group K, every conjugate system for (W,K,R) has a unique extension to an extended conjugate system for (W,K).*

Using the obvious definitions of degeneracy and full non-degeneracy for conjugate systems (as in Definition 3.21), it is clear that a conjugate system for (W, K, R) is fully non-degenerate if and only if its unique extension to an extended conjugate system for (W, K) is fully non-degenerate. So in the present context, the results of Chapter 3 simplify as follows.

Corollary 4.8 *Let W be a transitive structure with a strong type p and associated 2-type R, and let K be a finite abelian group. Then there is a one-to-one correspondence between conjugate systems for (W, K, R) and transitive, untwisted, superlinked finite covers of W with central kernel K. Further, such a cover is irreducible if and only if its corresponding conjugate system is fully non-degenerate.*

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