Modal Logics of Hyper-relational Frames

by

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Abstract

The notion of a frame using hyper-relations is introduced to generalize Kripkean binary relational frames and Jennings' and Schotch's n+1-ary relational frames. A more general truth-condition for modal formulas in the hyper-relational setting is defined by a theory of strictness. The modal formula $\Box\alpha$ is true at point x iff α satisfies x's strictness measure at all relata of x. For each $i \leq n$, logic $\mathcal{K}_n^i$ is defined and shown to be determined by the class of hyper-relational frames. I illustrate some interesting and distinctive features of the logic by proving a few correspondence-theoretic results for logic $\mathcal{K}_n^1$ and a few completeness and incompleteness results for its extensions.

In memory of my father, Lu Jian Min

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Chapter 1

Introduction

The standard semantics of modal logic is based on the notion of a binary relational frame consisting of a non-empty universe, U, and a binary relation defined over U. A natural generalization is to allow n+1-ary relational frames — that is to say, frames based on n+1-ary relations. A benefit of this approach is that philosophically significant distinctions obscured or unavailable in the strong modal logics of binary relational frames emerge in the weaker modal logics determined by the frames of the generalized semantic idiom. (cf. [Jennings and Schotch, 1984] and [Schotch and Jennings, 1980]).

In this thesis, which is based on Jennings' and Schotch's work, we build up a semantics adopting two further generalizations. The first involves the introduction of hyper-relational frames. Within this semantic idiom, the arities of the tuples to which a point is related are permitted to vary in width. The second is the adoption of a formally more general truth-condition for modal formulas. No doubt many intriguing philosophical questions may rise from the proposed semantics. However, our interest lies exclusively in formal questions, in particular, which formal properties of normal modal logic, conventionally conceived, can be preserved in the more general semantic setting. For this we need make no apology. It is not always possible for us to foresee how a formal theory will find applications. Non-euclidean geometry is a case in point, and this thesis must be another. So, for example, we propose no philosophical readings for $\Box$, but rather focus on proving metatheorems after defining the semantics and its base logic.

The structure of the work is as follows. In the first chapter, we introduce the notion of hyper-relational models and their modal logics. Actually, for each i ≥ 1, we define an infinite sequence of logics { $\mathcal{K}_n^i$: n ≥ i }, where $\mathcal{K}_m^m$ is the base normal modal logic $\mathcal{K}$.

In chapter 2, we prove soundness and completeness for $\mathcal{K}_n^i$ with respect to the class of hyper-relational frames.

In chapter 3, we deal with logic $\mathcal{K}_n^1$, the largest in the sequence of logics $\mathcal{K}_n^i$ (n > i). Some interesting and distinctive features of the logic such as incompleteness and definability are examined.

In an appendix we deal with the connections between $\mathcal{K}_n^i$ logics.

1.1 Basic Syntax

A propositional modal language usually has three elements:

(1) *At*:

An infinite sequence of propositional variables (atomic sentences), namely

$p, p_1, p_2, p_3, \ldots$

$q, q_1, q_2, q_3, \ldots$

$r, r_1, r_2, r_3, \ldots$

...

(2) *k*:

A set of connectives, namely

$\neg$ (negation), $\rightarrow$ (implication), and $\Box$ (necessity).

(3) Φ:

A recursively (inductively) defined set of sentences.

The sentences of Φ are calied well-formed formulas (wffs).

To define Φ we introduce metalogical variables, α, β, γ, δ etc. which range over the members of Φ. We use "$\Rightarrow$" to abbreviate English "if ... then ...", use "&" to abbreviate English "and".

(a) Every propositional variable is in Φ.

(b) $\alpha \in \Phi \Rightarrow \neg\alpha \in \Phi$.

(c) $\alpha \in \Phi \ \& \ \beta \in \Phi \Rightarrow \alpha \rightarrow \beta \in \Phi$.

(d) $\alpha \in \Phi \Rightarrow \Box\alpha \in \Phi$.

(e) Nothing is in Φ except as prescribed by (a), (b), (c) and (d).

Other connectives ($\vee$, $\wedge$, $\leftrightarrow$, $\top$, $\bot$ and $\Diamond$) are metalinguistically defined by $\neg$, $\rightarrow$, $\Box$ as usual:

$(\alpha \wedge \beta) =_{df} \neg(\alpha \rightarrow \neg\beta)$.

$(\alpha \vee \beta) =_{df} (\neg\alpha \rightarrow \beta)$.

$\top =_{df} (\alpha \rightarrow \alpha)$.

$\bot =_{df} \neg(\alpha \rightarrow \alpha)$.

$\Diamond\alpha =_{df} \neg\Box\neg\alpha$.

1.2 Basic Semantics

Before looking at the semantics of propositional modal logic, let us recall what a model $\mathcal{M}$ for propositional logic is. A model $\mathcal{M}$ for propositional logic is a pair $< U, V >$ where U is a non-empty set of objects called points, and V is a function from *At* into $\wp(U)$.

It is usual to introduce a three place relation $\vDash$ to give an account of the truth-conditions for wffs of Φ in a model $\mathcal{M} = < U, V >$.

(1) $\forall p \in At, \forall x \in U, \vDash^{\mathcal{M}}_{x} p$ if $x \in V(p)$, else, $\nvDash^{\mathcal{M}}_{x} p$;

(2) $\forall \alpha \in \Phi, \forall x \in U, \vDash^{\mathcal{M}}_{x} \neg\alpha$ if $\nvDash^{\mathcal{M}}_{x} \alpha$, else, $\nvDash^{\mathcal{M}}_{x} \neg\alpha$;

(3) $\forall \alpha, \beta \in \Phi, \forall x \in U, \vDash^{\mathcal{M}}_{x} \alpha \rightarrow \beta$ if $\nvDash^{\mathcal{M}}_{x} \alpha$ or $\vDash^{\mathcal{M}}_{x} \beta$, else, $\nvDash^{\mathcal{M}}_{x} \alpha \rightarrow \beta$;

$\models^{\mathcal{M}}_x \alpha$ is read "α is true or holds at point x in model $\mathcal{M}$"; $\not\models^{\mathcal{M}}_x \alpha$ is read "α is false at point x in model $\mathcal{M}$".

With these notions in hand, we proceed to define models for propositional modal logic, by specifying a structural element and, in terms of that structure, the truth-conditions for modal formulas. We define models for Kripke semantics, n+1-ary relational semantics and hyper-relational semantics respectively.

1.2.1 Kripke Semantics

In [Kripke 1963], Saul Kripke made an important contribution to our understanding of modal logic by defining a truth-condition for modal formulas in terms of one binary relation. The truth-condition given here is one inspired by that work.

Definition 1. *A binary relation R on a non-empty set U is a subset of U^2.*

We write $y \in R(x)$ or xRy, or say x is related to y, if $< x, y > \in R$.

Definition 2. *A binary relational frame, or a Kripke frame $\mathcal{F}$ is an ordered pair $< U, R >$ where U is a non-empty set and R is a binary relation on U.*

Definition 3 *A Kripke model $\mathcal{M}$ is a triple $< U,R,V >$ where $< U,R >$ is a binary frame and V is a function from At into $\wp(U)$. The truth-condition for each propositional formula is defined according to the customary inductive definition. The truth condition for modal formulas is given by:*

$\models^{\mathcal{M}}_x \Box\alpha$ *iff* $\forall y \in R(x)$, $\models^{\mathcal{M}}_y \alpha$, *or*

$\models^{\mathcal{M}}_x \Box\alpha$ *iff* $\forall y, xRy \Rightarrow \models^{\mathcal{M}}_y \alpha$.

That is, $\Box\alpha$ is true at a point x iff α is true at every x-relatum.

1.2.2 n+1-ary Relational Semantics

A fairly obvious generalization of Kripke semantics is to allow n+1-ary relational frames — frames based on n+1-ary relations. A truth-condition for modal formulas in the n+1-ary relational setting was suggested by R. E. Jennings and P. K. Schotch in 1970's. The base logic determined by this semantics is weaker than that determined by Kripke semantics. As a result, formulas $\Box p \rightarrow \Diamond p$ and $\neg\Box\bot$, for example, which have quite different deontic readings, are no longer equivalent.

Definition 4. *An n+1-ary relation R on a non-empty set U is a subset of U^{n+1} where $n \geq 1$.*

Let $\tau = \langle y_1, \ldots, y_n \rangle$ be an n-tuple. If $\langle x, y_1 \ldots y_n \rangle \in R$, we write $\tau \in R(x)$ or $xR\tau$ and say x is related to τ or τ is an x-relatum.

Definition 5. *An n+1-ary relational frame $\mathscr{F}$ is an ordered pair $\langle U, R \rangle$ where U is a non-empty set and R is an n+1-ary relation on U.*

Definition 6 *A Jennings-Schotch model $\mathscr{M}$ is a triple $\langle U,R,V \rangle$ where $\langle U,R \rangle$ is an n+1-ary relational frame and V is a function from At into $\wp(U)$. The truth-condition for each propositional formula is defined according to the customary inductive definition. The truth condition for modal formulas is given by:*

$\models_x^{\mathscr{M}} \Box\alpha$ *iff* $\forall \tau \in R(x), \exists z \in \tau : \models_z^{\mathscr{M}} \alpha$, *or*

$\models_x^{\mathscr{M}} \Box\alpha$ *iff* $\forall y_1 \ldots y_n,\ xRy_1 \ldots y_n \Rightarrow \exists z \in \{ y_1 \ldots y_n \} : \models_z^{\mathscr{M}} \alpha$.

That is, $\Box\alpha$ is true at a point x iff α is true at some point in every n-tuple that x is related to.

1.2.3 Strictness Theory

Consider Jennings-Schotch's truth-condition again:

$\models_x^{\mathscr{M}} \Box\alpha$ iff $\forall \tau \in R(x), \exists z \in \tau : \models_z^{\mathscr{M}} \alpha$.

The truth-condition says, $\Box\alpha$ is true at a point x iff α is true at at least *one* point in each n-tuple that x is related to. However, under different strict requirements, it is quite natural to state that $\Box\alpha$ is true at a point x iff α is true at distinct i ($i \leq n$) points in each n-tuple that x is related to. Intuitively, i is the number of places that a necessitation needs to be true as required by a semantic agent. If we think of the i distinct points in each n distinct points in each tuple τ that x is related to as x's strictness measure, then $\Box\alpha$ is true at x if α satisfies x's strictness measure at every x-relatum.

In order to state the truth-condition, we need a language including $\forall$, $\exists$, $\wedge$, $\Rightarrow$, $=$, $\neq$, $\{$, $\}$, $\in$, $\subseteq$, etc. Although $\wedge$ is used in both here and propositional modal language, there will be no confusion in particular contexts.

For convenience, we use

(1) $(\neq, y_1 \ldots y_n)$ to abbreviate sentence $(y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge \ldots \wedge y_{n-1} \neq y_n)$.

(2) $\{\neq, y_1 \ldots y_n\}$ to abbreviate the set $\{y_1 \ldots y_n\}$ for which $(\neq, y_1 \ldots y_n)$,

(3) $\exists\{\neq, z_1 \ldots z_n\} \subseteq \{y_1 \ldots y_m\}$ to abbreviate the sentence:

there is a set $\{\neq, z_1 \ldots z_n\}$ such that $\{z_1 \ldots z_n\} \subseteq \{y_1 \ldots y_m\}$, where m and n are finite.

(4) $\exists\{\neq, z_1 \ldots z_n\} \subseteq \{y_1 \ldots y_m\}\alpha$ to abbreviate the sentence:

$\exists z_1 \ldots \exists z_n ((\{z_1 \ldots z_n\} \subseteq \{y_1 \ldots y_m\}) \wedge (\neq, z_1 \ldots z_n) \wedge \alpha)$, where $m \geq n$ is finite,

(5) $\forall\{\neq, z_1 \ldots z_n\} \subseteq \{y_1 \ldots y_m\}\alpha$ to abbreviate the sentence:

$\forall z_1 \ldots \forall z_n, ((\{z_1 \ldots z_n\} \subseteq \{y_1 \ldots y_m\}) \wedge (\neq, z_1 \ldots z_n) \Rightarrow \alpha)$, where $m \geq n$ is finite.

We can now state the truth-condition for modal formulas in a n+1-ary relational frame as follows:

$$\models^{\mathcal{M}}_x \Box\alpha \text{ iff } \forall\tau \in R(x), \exists\{\neq, z_1 \ldots z_i\} \subseteq \tau : \forall z_{j(1 \leq j \leq i)}, \models^{\mathcal{M}}_{z_j} \alpha, \text{ where } i \leq n.$$

It is easy to see that if x is related to a tuple of $|\tau|<i$, all necessitations would be false at x. In order to ignore those tuples of cardinality less than i, we may have the following revised truth-condition:

$$\models^{\mathcal{M}}_x \Box\alpha \text{ iff } \forall\tau \in R(x),\ |\tau| \geq i \Rightarrow (\exists\{\neq, z_1 \ldots z_i\} \subseteq \tau : \forall z_{j(1 \leq j \leq i)}, \models^{\mathcal{M}}_{z_j} \alpha), \text{ where } i \leq n.$$

Formally, the truth-condition says, $\Box\alpha$ is true at a point x if α is true at i distinct points in each n-tuple τ ($|\tau|\geq i$) that x is related to.

Let i=1, then the above truth-condition turns out to be the same as that in a Jennings-Schotch model, since $|\tau|\geq 1$.

1.2.4 Hyper-relational Semantics

n+1-ary relational semantics is really a generalization of Kripke semantics. However, n+1-ary relational frames still have the assumption that each point in the universe is either a deadend, i.e. it is related to nothing at all, or the arity of the tuples that it is related to is n. Actually, in both n+1-ary relational and Kripkean binary frames, the size of tuples is fixed. The two differ only in where they fix it. In what follows, we relax this fixity in hyper-relational frames where the arities of the tuples that a point is related to may vary in width.

Definition 7. *A hyper-relation R on a non-empty set U is a subset of $U^2 \cup \ldots \cup U^k$, where $k \geq 2$.*

Let $\tau = < y_1, \ldots, y_m >$ be an m-tuple ($1\leq m\leq k-1$). If $< x, y_1 \ldots y_m > \in R$, we write $\tau \in R(x)$ or $xR\tau$ and say x is related to τ or τ is an x-relatum.

When $R \subseteq U^k$ for some $k\geq 2$, we say R is a trivial hyper-relation. n+1-ary relations discussed earlier are n+1-ary trivial hyper-relations.

Definition 8. *A hyper-relational frame $\mathscr{F}$ is an ordered pair $< U, R >$ where U is a non-empty set and R is a hyper-relation on U. We also can put a hyper-relational frame as a triple $< U,R,k >$ if we know that there is a finite number such that $k \geq 2$ and $R \subseteq U^2 \cup \ldots \cup U^k$.*

In the strictness-theoretical setting, the truth-condition for modal formulas says, $\Box\alpha$ is true at a point x iff α is true at i distinct points in each n-tuple τ ($|\tau|\geq i$) that x is related to. Formally, the basic idea is that we check the truth of $\Box\alpha$ by checking whether α is true at i distinct points in each n points in each x-relatum. In a hyper-relational frame, however, arities of tuples that a point is related to may be greater than n. It is reasonable to say that if $\Box\alpha$ is true at x, then α is true at some i distinct points in each n distinct points in each tuple τ that x is related to. In other words, x is true at $|\tau|$-n+i distinct points in each tuple that x is related to, if $|\tau|\geq n$[1].

Now we are ready to define the model in the hyper-relational setting.

Definition 9. *An i-n-model (i≤n) $\mathcal{M}^i_n$ — i is a strictness measure — on a frame $\mathcal{F}$ = < U,R,k > is a pair < U,V > where V is a function from At into $\wp(U)$. The truth-condition for each propositional formula is defined according to the customary inductive definition. The truth-condition for modal formulas is given by:*

$$\models_x^{\mathcal{M}^i_n} \Box\alpha \text{ iff } \forall\tau \in R(x),\ (\ |\tau|\geq n \Rightarrow (\ \exists\{\neq, z_1 \ldots z_{|\tau|-n+i}\} \subseteq \tau : \forall z_{j(1\leq j\leq|\tau|-n+i)}, \models_{z_j}^{\mathcal{M}^i_n} \alpha\ )),$$

or

$$\models_x^{\mathcal{M}^i_n} \Box\alpha \text{ iff } \forall m(k>m>0)\forall y_1 \ldots \forall y_m\ (xRy_1 \ldots y_m \wedge g\geq n \Rightarrow (\ \exists\{\neq, z_1 \ldots z_{g-n+i}\} \subseteq \{y_1 \ldots y_m\} : \forall z_{j(1\leq j\leq g-n+i)} \models_{z_j}^{\mathcal{M}^i_n} \alpha\ )),$$ *where g is the cardinality of { $y_1, \ldots, y_m$ }.*

By the definition of $\Diamond$, the truth-condition for $\Diamond\alpha$ would be:

$$\models_x^{\mathcal{M}^i_n} \Diamond\alpha \text{ iff } \exists\tau \in R(x) : |\tau|\geq n \wedge (\ \forall\{\neq, z_1 \ldots z_{|\tau|-n+i}\} \subseteq \tau \Rightarrow \exists z_{j(1\leq j\leq|\tau|-n+i)} : \models_{z_j}^{\mathcal{M}^i_n} \alpha\ ), \text{ or}$$

$$\models_x^{\mathcal{M}^i_n} \Diamond\alpha \text{ iff } \exists m(k>m>0)\exists y_1 \ldots \exists y_m : (xRy_1 \ldots y_m \wedge g\geq n \wedge (\ \forall\{\neq, z_1 \ldots z_{g-n+i}\} \subseteq \{y_1 \ldots y_m\}, \exists z_{j(1\leq j\leq g-n+i)} \models_{z_j}^{\mathcal{M}^i_n} \alpha\ ),$$ where g is the cardinality of { $y_1, \ldots, y_m$ }.

We write $\models^{\mathcal{M}^i_n} \alpha$ to abbreviate: $\forall x \in U, \models_x^{\mathcal{M}^i_n} \alpha$;

$\mathcal{F} \models^i_n \alpha$ to abbreviate: $\forall\mathcal{M}^i_n,\ \models^{\mathcal{M}^i_n} \alpha$, where $\mathcal{M}^i_n$ is a model on $\mathcal{F}$;

We say $\mathcal{F}$ = < U,R,k > is a frame for a wff α if $\mathcal{F} \models^i_n \alpha$.

Note that a hyper-relational frame < U,R > together with a valuation function V from *At* into $\wp$(U) is not a model in any sense at all. It is neutral or independent before we apply to it a truth-condition with different parameters i and n for modal formulas.

---

1. If α is true at some i distinct points in each n distinct points in a tuple τ, then there are at most n-i distinct points in which α is false. Therefore α is true at at least $|\tau|$-(n-i) distinct points.
The assumption of $|\tau|\geq n$ will always make $|\tau|$-n+i a positive number.

From the truth-condition for modal formulas, we can see that a hyper-relational model is really a very general one. Assume that frames are restricted within (n+1-ary) trivial hyper-relational frames and that i =1, then:

$\models_x^{\mathcal{M}'_n} \Box\alpha$ iff $\forall y_1 \ldots \forall y_n\, (xRy_1 \ldots y_n \wedge (\neq, y_1 \ldots y_n) \Rightarrow \exists y_{i(1\leq i\leq n)} : \models_{y_i}^{\mathcal{M}'_n} \alpha)$.

The truth-condition for modal formulas above is the same as that of n+1-ary relational semantics except that it has the restriction of inequality $(\neq, y_1 \ldots y_n)$. If we let $n = 1$ in the above truth-condition, then we restore the Kripkean truth-condition.

1.2.5 Base Logic

The base logic for Kripke semantics is defined as follows:

(a) all substitution instances of tautologies. (We name this set as PL)

(b) all substitution instances of axiom [K]:

$\Box p_1 \wedge \Box p_2 \rightarrow \Box(p_1 \wedge p_2)$,

and is closed under the rules:

(c) [RM]

$$\frac{\alpha \rightarrow \beta}{\Box\alpha \rightarrow \Box\beta},$$ and

(d) [RN]

$$\frac{\alpha}{\Box\alpha.}$$

We call this base logic $\mathcal{K}$.

The base logic for n+1-ary relational semantics is the same as $\mathcal{K}$ except that (b) axiom [K] is replaced by [K_n]:

$\Box p_1 \wedge \Box p_2 \wedge \ldots \wedge \Box p_{n+1} \rightarrow \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_n \wedge p_{n+1}))$,

We call the base logic for n+1-ary relational semantics $\mathcal{K}_n$. Actually, we have defined an infinite sequence of logics $\{ \mathcal{K}_n : n>1 \}$ where $\mathcal{K}_1$ is $\mathcal{K}$.

The base logic for hyper-relational semantics is the same as $\mathcal{K}$ except that (b) axiom [K] is replaced by [K_n^i]:

$\Box p_1 \wedge \Box p_2 \wedge \ldots \wedge \Box p_{t+1} \rightarrow \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_t \wedge p_{t+1}))$, where $t = C(n,i)$ and $1 \leq i < n$.

As [K_n^i] has two parameters, actually, for each $i \geq 1$, we have defined an infinite sequence of logics $\{ \mathcal{K}_n^i : i \leq n \}$.

It is easy to see that if $i = 1$, then axiom [K_n^1] is [K_n], since $C(n,1) = n$; and that if $i = n$, then axiom [K_n^n] is [K], since $C(n,n) = 1$. In other words, logic $\mathcal{K}_n^1$ is the same as $\mathcal{K}_n$, and $\mathcal{K}_n^n$ is the well-known normal logic $\mathcal{K}$.

We write $\mathcal{K}_n^i\mathcal{A}$ as the smallest logic containing $\mathcal{K}_n^i \cup \{ A \}$ for a formula A.

A proof or derivation of α in logic $\mathcal{K}_n^i$ is a finite sequence of wffs $\gamma_1 \ldots \gamma_m$ where γ_m is α and for each γ_j $(1 \leq j \leq m)$,

either γ_j is a substitution instance of a propositional theorem,

or γ_j is a substitution instance of [K_n^i],

or γ_j is obtainable from a previous theorem in the sequence by [RN],

or γ_j is obtainable from a previous theorem in the sequence by [RM].

Each γ_j $(1 \leq j \leq m)$ is called a theorem of the logic.

We write $\Gamma \vdash_{\mathcal{K}_n^i} \alpha$ if α is provable from Γ in logic $\mathcal{K}_n^i$.

Chapter 2

Logic $\mathcal{K}_n^i$

In [Brown 1993], it was showed that $\mathcal{K}_n^i$ logic is complete with respect to a semantic setting which is equivalent to the n+1-ary relational semantics. Here we show at one stroke that $\mathcal{K}_n^i$ logic is complete with respect to the class of hyper-relational frames.

Claim: *If* $\exists\tau \in R(x)$, $(|\tau| \geq n \wedge (\forall\{\neq, z_1 \ldots z_{|\tau|-n+i} \} \subseteq \tau \Rightarrow \exists z_{j(1\leq j\leq|\tau|-n+i)} : \not\models_{z_j}^{\mathcal{M}_n^i} \alpha))$, *then,* $\exists\tau \in R(x) : |\tau| \geq n \wedge (\exists\{\neq, z_1 \ldots z_n \} \subseteq \tau : (\forall\{\neq, u_1 \ldots u_i \} \subseteq \{\neq, z_1 \ldots z_n \}, \exists u_{j(1\leq j\leq i)} : \not\models_{z_j}^{\mathcal{M}_n^i} \alpha))$.

Proof.

Assume $\exists\tau \in R(x)$, $(|\tau| \geq n \wedge (\forall\{\neq, z_1 \ldots z_{|\tau|-n+i} \} \subseteq \tau \Rightarrow \exists z_{j(1\leq j\leq|\tau|-n+i)}, \not\models_{z_j}^{\mathcal{M}_n^i} \alpha))$.

Let $w_1, \ldots, w_{|\tau|}$ be $|\tau|$ distinct points such that points falsifying α are put in front of those verifying α.

Then, by the assumption, in the most right hand side $|\tau|-n+i$ distinct points, there is a point w falsifying α.

Since all points before w falsify α, there are at least n-i+1 distinct points falsifying α.

Therefore, in the most left hand side n distinct points, there are at most i-1 distinct points verifying α.

Therefore, $\exists\{\neq, z_1 \ldots z_n \} \subseteq \tau : (\forall\{\neq, u_1 \ldots u_i \} \subseteq \{\neq, z_1 \ldots z_n \}, \exists u_{j(1\leq j\leq i)} : \not\models_{z_j}^{\mathcal{M}_n^i} \alpha)$.

■

Theorem 1. *$\mathcal{K}_n^i$ is sound with respect to the class of hyper-relational frames.*

Proof:

We need only show that rules [RM] and [RN] preserve validity and axiom $[K_n^i]$ is valid:

Let $\mathcal{M}_n^i = < U,R,k,V >$ be an arbitrary i-n-model and x an arbitrary point in U.

(1) The validity of [RM]:

Assume that $\models_n^i \alpha \rightarrow \beta$ and that $\models_x^{\mathcal{M}_n^i} \Box\alpha$.

Then, from the truth-condition, $\forall\tau \in R(x)$, ($|\tau| \geq n \Rightarrow$ ($\exists\{\neq, z_1 \ldots z_{|\tau|-n+i}\} \subseteq \tau$: $\forall z_{j(1\leq j\leq|\tau|-n+i)}, \models_{z_j}^{\mathcal{M}_n^i} \alpha$)).

But $\forall\tau \in R(x)$, ($|\tau| \geq n \Rightarrow$ ($\exists\{\neq, z_1 \ldots z_{|\tau|-n+i}\} \subseteq \tau$: $\forall z_{j(1\leq j\leq|\tau|-n+i)}, \models_{z_j}^{\mathcal{M}_n^i} \alpha \rightarrow \beta$)), since $\models_n^i \alpha \rightarrow \beta$.

Then $\forall\tau \in R(x)$, ($|\tau| \geq n \Rightarrow$ ($\exists\{\neq, z_1 \ldots z_{|\tau|-n+i}\} \subseteq \tau$: $\forall z_{j(1\leq j\leq|\tau|-n+i)}, \models_{z_j}^{\mathcal{M}_n^i} \beta$)).

Therefore, $\models_x^{\mathcal{M}_n^i} \Box\beta$.

But $\mathcal{M}_n^i$ and x are arbitrary.

So if $\models_n^i \alpha \rightarrow \beta$, then $\models_n^i \Box\alpha \rightarrow \Box\beta$.

(2) The validity of [RN]:

Assume that $\models_n^i \alpha$.

Then it is easy to see from the truth-condition that $\models_n^i \Box\alpha$.

(3) The validity of $[K_n^i]$:

$[K_n^i]$ is ($\Box p_1 \wedge \Box p_2 \wedge \ldots \wedge \Box p_{t+1} \rightarrow \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_t \wedge p_{t+1}))$, where $t=C(n,i)$.

Assume that $\models_x^{\mathcal{M}_n^i} \Box p_1 \wedge \Box p_2 \wedge \ldots \wedge \Box p_{t+1}$.

Then, by the truth-condition, $\forall p_{k(1\leq k\leq t+1)}$, $\forall\tau \in R(x)$, ($|\tau| \geq n \Rightarrow$ ($\exists\{\neq, z_1 \ldots z_{|\tau|-n+i}\} \subseteq \tau$: $\forall z_{j(1\leq j\leq|\tau|-n+i)}, \models_{z_j}^{\mathcal{M}_n^i} p_{k(1\leq k\leq t+1)}$)).

We need to prove that $\models_x^{\mathcal{M}_n^i} \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_{t-1} \wedge p_{t+1}))$.

Assume for reductio that $\not\models_x^{\mathcal{M}_n^i} \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_{t-1} \wedge p_{t+1}))$.

Then by the truth-condition, $\exists\tau \in R(x)$, ($|\tau| \geq n \wedge$ ($\forall\{\neq, z_1 \ldots z_{|\tau|-n+i}\} \subseteq \tau \Rightarrow$ $\exists z_{j(1\leq j\leq|\tau|-n+i)}, \not\models_{z_j}^{\mathcal{M}_n^i} ((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_t \wedge p_{t+1}))$)).

Then by our claim, $\exists\tau \in R(x), |\tau|\geq n \wedge (\exists\{\neq, z_1 \dots z_n \} \subseteq \tau : \forall\{\neq, u_1 \dots u_i \} \subseteq \{\neq, z_1 \dots z_n \}, \exists u_{j(1\leq j\leq i)}, \not\models_{z_j}^{\mathcal{M}_n^i} ((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \dots \vee (p_t \wedge p_{t+1}))))$.

Then, it cannot be the case that $\exists p_l, \exists p_m, \exists\{\neq, u_1 \dots u_i \} \subseteq \{\neq, z_1 \dots z_n \}, \exists u_{j(1\leq j\leq i)}$: $\models_{u_j}^{\mathcal{M}_n^i} (p_i \wedge p_j)$, where $1\leq l<m\leq t+1$.

In other words, the maximal number of propositional variables that satisfy

$\forall\{\neq, u_1 \dots u_i \} \subseteq \{\neq, z_1 \dots z_n \}, \exists u_{j(1\leq j\leq i)} : \not\models_{z_j}^{\mathcal{M}_n^i} ((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \dots \vee (p_t \wedge p_{t+1}))$)

is $C(n,i)$.

This means that there are at most $C(n,i)$ necessitations of propositional variables in $\{p_1 \dots p_{t+1}\}$ which are true at x.

This contradicts the assumption that $\models_x^{\mathcal{M}_n^i} \Box p_1 \wedge \Box p_2 \wedge \dots \wedge \Box p_{t+1}$, where $t = C(n,t)$.

Therefore, $\models_x^{\mathcal{M}_n^i} \Box p_1 \wedge \Box p_2 \wedge \dots \wedge \Box p_{t+1} \rightarrow \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \dots \vee (p_t \wedge p_{t+1}))$, where $t = C(n,i)$.

But $\mathcal{M}_n^i$ and x are arbitrary.

Therefore, $\models_n^i \Box p_1 \wedge \Box p_2 \wedge \dots \wedge \Box p_{t+1} \rightarrow \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \dots \vee (p_t \wedge p_{t+1}))$, where $t = C(n,i)$.

■

Before we prove the completeness for $\mathcal{K}_n^i$ logic, we need some definitions and lemmas.

Lemma 2. *If* $\vdash_{\mathcal{K}_n^i} (\zeta \wedge \Box\alpha) \rightarrow \Box\delta$, *then* $\vdash_{\mathcal{K}_n^i} (\zeta \wedge \Box(\alpha \vee \beta)) \rightarrow \Box(\delta \vee \beta)$, *where* ζ *is a conjunction of necessitations.*

Proof.

Assume that $\vdash_{\mathcal{K}_n^i} (\zeta \wedge \Box\alpha) \rightarrow \Box\delta$.

Then we have a proof of $(\zeta \wedge \Box\alpha) \rightarrow \Box\delta$, that is, a finite sequence of theorems:

$$\gamma_1 \dots \gamma_j \dots \gamma_m$$

where $\gamma_m = (\zeta \wedge \Box\alpha) \rightarrow \Box\delta$

and for each γ_j $(1\leq j\leq m)$,

either γ_j is a substitution instance of propositional theorem, or γ_j is a substitution instance of $[K_n^i]$,

or γ_j is obtainable from a previous theorem in the sequence by [RN],

or γ_j is obtainable from a previous theorem in the sequence by [RM].

For each $\gamma_{j(1\leq j\leq m)}$, we prove that if $\vdash_{\underline{K}_n^i} \gamma_j \leftrightarrow ((\zeta \wedge \Box\alpha) \rightarrow \Box\delta)$, then $\vdash_{\underline{K}_n^i} (\zeta \wedge \Box(\alpha \vee \beta)) \rightarrow \Box(\delta \vee \beta)$.

The proof is by induction on the length of the proof of γ_m.

Basis:

There are two cases:

(1) Assume γ_1 is a substitution instance of a propositional theorem.

Then $\vdash_{\underline{K}_n^i} \gamma_1 \leftrightarrow (\Box\alpha \rightarrow \Box\alpha)$.

But $\Box(\alpha \vee \beta) \rightarrow \Box(\alpha \vee \beta)$ is also a propositional theorem.

Therefore, $\vdash_{\underline{K}_n^i} \Box(\alpha \vee \beta) \rightarrow \Box(\alpha \vee \beta)$.

(2) Assume γ_1 is a substitute instance of $[K_n^i]$. Then,

$\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_{t+1} \rightarrow \Box((\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3) \vee \ldots \vee (\alpha_t \wedge \alpha_{t+1}))$, where $t = C(n,i)$ and $i \leq n$.

Let $\alpha = \alpha_k$ and $\alpha_k \in \{ \alpha_1 \ldots \alpha_{t+1} \}$.

By PL, $\forall\alpha_h \in \{\alpha_1 \ldots \alpha_h \ldots \alpha_{t+1}\}$, $\vdash_{\underline{K}_n^i} \alpha_h \rightarrow \alpha_h \vee \beta$.

Then, $\vdash_{\underline{K}_n^i} \Box\alpha_h \rightarrow \Box(\alpha_h \vee \beta)$, by [RM].

Then by PL, $\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_{k-1} \wedge \Box(\alpha_k \vee \beta) \wedge \Box\alpha_{k+1} \wedge \ldots \wedge \Box\alpha_{t+1} \rightarrow$

$\Box(\alpha_1 \vee \beta) \wedge \ldots \wedge \Box(\alpha_k \vee \beta) \wedge \ldots \wedge \Box(\alpha_{t+1} \vee \beta)$. (I)

But by $[K_n^i]$, $\vdash_{\underline{K}_n^i} \Box(\alpha_1 \vee \beta) \wedge \ldots \wedge \Box(\alpha_k \vee \beta) \wedge \ldots \wedge \Box(\alpha_{t+1} \vee \beta) \rightarrow$

$\Box(((\alpha_1 \vee \beta) \wedge (\alpha_2 \vee \beta) \vee ((\alpha_1 \vee \beta) \wedge (\alpha_3 \vee \beta)) \vee \ldots \vee ((\alpha_t \vee \beta) \wedge (\alpha_{t+1} \vee \beta)))$.

Then $\vdash_{\underline{K}_n^i} \Box(\alpha_1 \vee \beta) \wedge \ldots \wedge \Box(\alpha_k \vee \beta) \wedge \ldots \wedge \Box(\alpha_{t+1} \vee \beta) \rightarrow$

$\Box(((\alpha_1 \wedge \alpha_2) \vee \beta) \vee ((\alpha_1 \wedge \alpha_3) \vee \beta)) \vee \ldots \vee ((\alpha_t \wedge \alpha_{t+1}) \vee \beta)))$, by PL.

And by PL, $\vdash_{\underline{K}_n^i} \Box(\alpha_1 \vee \beta) \wedge \ldots \wedge \Box(\alpha_k \vee \beta) \wedge \ldots \wedge \Box(\alpha_{t+1} \vee \beta) \rightarrow$

$\Box(((\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3) \vee \ldots \vee (\alpha_t \wedge \alpha_{t+1})) \vee \beta)$. (II)

Then, from (I) and (II),

$\vdash_{K_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box(\alpha_k \vee \beta) \wedge \ldots \wedge \Box\alpha_{t+1} \rightarrow \Box(((\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3) \vee \ldots \vee (\alpha_t \wedge \alpha_{t+1})) \vee \beta)$.

Inductive Step:

Assume that the lemma holds for each γ_h if h<j.

We just need to consider the following [RM]:

Assume $\gamma_j = \Box\alpha \rightarrow \Box\delta$ is obtainable from a previous theorem in the sequence by [RM]:

Then $\vdash_{K_n^i} \alpha \rightarrow \delta$.

But $\vdash_{K_n^i} \alpha \vee \beta \rightarrow \delta \vee \beta$.

Then $\vdash_{K_n^i} \Box(\alpha \vee \beta) \rightarrow \Box(\delta \vee \beta)$, by [RM].

■

Lemma 3. *If* $\vdash_{K_n^i} (\zeta \wedge \Box\alpha) \rightarrow \Box\delta$ *and* $\vdash_{K_n^i} (\zeta \wedge \Box\beta) \rightarrow \Box\delta$, *then* $\vdash_{K_n^i} (\zeta \wedge \Box(\alpha \vee \beta)) \rightarrow \Box\delta$, *where* ζ *is a conjunction of necessitations.*

Proof:

Assume $\vdash_{K_n^i} (\zeta \wedge \Box\alpha) \rightarrow \Box\delta$ and $\vdash_{K_n^i} (\zeta \wedge \Box\beta) \rightarrow \Box\delta$.

Then by Lemma 2,

$\vdash_{K_n^i} (\zeta \wedge \Box(\alpha \vee \beta)) \rightarrow \Box(\delta \vee \beta)$ and $\vdash_{K_n^i} (\zeta \wedge \Box(\beta \vee \delta)) \rightarrow \Box(\delta \vee \delta)$.

Then, $\vdash_{K_n^i} (\zeta \wedge \Box(\alpha \vee \beta)) \rightarrow (\zeta \wedge \Box(\beta \vee \delta))$ and $\vdash_{K_n^i} (\zeta \wedge \Box(\beta \vee \delta)) \rightarrow \Box(\delta \vee \delta)$.

Therefore, $\vdash_{K_n^i} (\zeta \wedge \Box(\alpha \vee \beta)) \rightarrow \Box(\delta \vee \delta)$.

Therefore, $\vdash_{K_n^i} (\zeta \wedge \Box(\alpha \vee \beta)) \rightarrow \Box\delta$.

■

Definition 4. *Let* Γ *be a set of wffs and* $\Delta \subseteq \wp(\Gamma)$. Δ *is an* <u>*i-n-distribution*</u> *(*$1 \leq i \leq n$*) of* Γ *iff* Δ *is an n-decomposition*[1] *of* Γ *such that* $\forall\alpha \in \Gamma, \exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta : \alpha \in \cap\{\Theta_1 \ldots \Theta_i\}$. *If* $i = 1$, *we say that an i-n-distribution of a set* Γ *is an* <u>*n-partition*</u> *of* Γ.

---

1. Δ is an n-decomposition of Γ if Δ is a n-member set such that $\Delta \subseteq \wp(\Gamma)$ and $\cup\Delta = \Gamma$.

For example, let $\Gamma = \{ p_1, p_2, p_3, p_4 \}$, then $\Delta = \{ \{p_1, p_3, p_4\} \{p_1, p_2, p_4\} \{p_2, p_3\} \}$ is a 2-3-distribution of Γ.

Lemma 5 (Distributional Compactness). *If Σ is a set of finite sets of wffs having the property that for each i-n-distribution Δ of $\cup\Sigma$, $\exists\Psi \in \Sigma$, $\exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta$ such that $\Psi \subseteq \cap\{\Theta_1 \ldots \Theta_i\}$, then there is a finite subset Σ_0 of Σ having the property that for each i-n-distribution Δ of $\cup\Sigma_0$, $\exists\Psi \in \Sigma_0$, $\exists\{\neq, \Theta_1 \ldots \Theta_i\}\subseteq \Delta$ such that $\Psi \subseteq \cap\{\Theta_1 \ldots \Theta_i\}$.*

Proof:

We prove that if it is not the case that for every finite subset Σ_0 of Σ, for each i-n-distribution Δ of $\cup\Sigma_0$, $\exists\Psi \in \Sigma_0$, $\exists\{\neq, \Theta_1 \ldots \Theta_i\}\subseteq \Delta$ such that $\Psi\subseteq \cap\{\Theta_1 \ldots \Theta_i\}$, then it is not the case that for each i-n-distribution Δ of $\cup\Sigma$, $\exists\Psi \in \Sigma$ $\exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta$ such that $\Psi \subseteq \cap\{\Theta_1 \ldots \Theta_i\}$.

Assume that for every finite subset Σ_0 of Σ, it is not the case that for each i-n-distribution Δ of $\cup\Sigma_0$, $\exists\Psi \in \Sigma_0$, $\exists\{\neq, \Theta_1 \ldots \Theta_i\}\subseteq \Delta$ such that $\Psi\subseteq \cap\{\Theta_1 \ldots \Theta_i\}$.

Then for every finite subset Σ_0 of Σ, there is an i-n-distribution Δ of $\cup\Sigma_0$ such that $\forall\Psi \in \Sigma_0, \forall\{\neq, \Theta_1 \ldots \Theta_i\}\subseteq \Delta,\ \Psi\not\subseteq \cap\{\Theta_1 \ldots \Theta_i\}$. (*)

Assume that Σ is a set of finite sets of wffs.

Let $\Sigma = \{ \Psi_1 \ldots \Psi_j, \ldots : j \in I^+ \}$ and $\forall j$, let $\Psi_j = \{ \alpha_{j1} \ldots \alpha_{jk_j} \}$.

For simplicity, assume that Σ is a countable infinite set.

We start with a first-order language ***L*** which has:

(1) n unary predicate letters: $P_1 \ldots P_n$.

(2) $\forall j$, $\forall l$, such that $1\leq l\leq k_j$, a distinct individual constant c_{jl} representing the elements of Ψ_j.

Now let $A = D \cup C$ where

$$D = \{ \forall x, \exists\{\neq, m_1 \ldots m_i\} \subseteq \{1 \ldots n\}, (\forall m_{k(1\leq k\leq i)}, P_{m_k}(x)) \wedge (\forall(1\leq q_1\neq q_2\leq i), \exists y, P_{m_{q1}}(y) \wedge \neg P_{m_{q_2}}(y)) \}$$ and

$$C = \{ \forall\{\neq, m_1 \ldots m_i\} \subseteq \{1 \ldots n\}, \exists(1\leq q\leq k_j), \forall m_{p(1\leq p\leq i)}, \neg P_{m_p}(c_{j_q}) : j \in I^+ \}.$$

By (*), every finite subset of A has a model.

Therefore, by first-order Compactness, A has a model.

Therefore, there is an i-n-distribution Δ of $\cup\Sigma$, $\forall\Psi \in \Sigma$, $\forall\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq D$ such that $\Psi \not\subseteq \cap\{\Theta_1 \ldots \Theta_i\}$.

Therefore, it is not the case that for each i-n-distribution Δ of $\cup\Sigma$, $\exists\Psi \in \Sigma$, $\exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq D$ such that $\Psi \subseteq \cap\{\Theta_1 \ldots \Theta_i\}$.

■

Lemma 6. *Let γ be a wff and $\Gamma = \{ \alpha_1, \ldots, \alpha_j$: for some finite j $\}$. If, for each i-n-distribution Δ of Γ, $\exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta$ such that $\cap\{\Theta_1 \ldots \Theta_i\} \vdash_{\underline{K}_n^i} \gamma$ then $\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_j \rightarrow \Box\gamma$.*

Proof:

The proof is by induction on the number of the members of Γ, Card(Γ).

Base step:

Let $j < C(n,i)+1$.

Since $j < C(n,i)+1$, then there is an i-n-distribution Δ of Γ such that

$\neg(\exists\alpha_p, \alpha_q \in \Gamma, \exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta : (\{ \alpha_p, \alpha_q \} \subseteq \cap\{\Theta_1 \ldots \Theta_i\}))$, where $1 \leq p \neq q \leq k$.

Let Δ_0 be the i-n-distribution of Γ such that $\neg(\exists\alpha_p, \exists\alpha_q \in \Gamma, \exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta_0 : (\{ \alpha_p, \alpha_q \} \subseteq \cap\{\Theta_1 \ldots \Theta_i\}))$, where $1 \leq p \neq q \leq k$.

Then $\forall\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta_0$, $\cap\{\Theta_1 \ldots \Theta_i\}$ is a unit set.

But by the assumption, $\exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta_0, : \cap\{\Theta_1 \ldots \Theta_i\} \vdash_{\underline{K}_n^i} \gamma$.

That is, $\exists\alpha \in \Gamma : \{ \alpha \} \vdash_{\underline{K}_n^i} \gamma$.

Let α_k be the wff in $\Gamma : \{ \alpha_k \} \vdash_{\underline{K}_n^i} \gamma$.

Then $\vdash_{\underline{K}_n^i} \alpha_k \rightarrow \gamma$.

By {RM}, $\vdash_{\underline{K}_n^i} \Box\alpha_k \rightarrow \Box\gamma$.

Therefore, $\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_j \rightarrow \Box\gamma$.

Inductive Hypothesis: Assume that the lemma holds for all sets of Card(Γ)<j.

Inductive step:

Let $j \geq C(n,i)+1$.

Let $\Gamma_0 = (\Gamma - \{\alpha_p, \alpha_q\}) \cup \{\alpha_p \wedge \alpha_q\}$, where $1 \leq p \neq q \leq j$.

Consider any i-n-distribution Δ of Γ such that $\forall\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta$ such that $(\alpha_p \subseteq \cap\{\Theta_1 \ldots \Theta_i\} \Leftrightarrow \alpha_q \subseteq \cap\{\Theta_1 \ldots \Theta_i\})$.

Then by the assumption, for each i-n-distribution Δ of Γ, $\exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta$ such that $\cap\{\Theta_1 \ldots \Theta_i\} \vdash_{\underline{K}_n^i} \gamma$.

Then for each i-n-distribution Δ_0 of Γ_0, $\exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq D_0$ such that $\cap\{\Theta_1 \ldots \Theta_i\} \vdash_{\underline{K}_n^i} \gamma$.

But, $\mathrm{Card}(\Gamma_0) = j - 1$.

$\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box(\alpha_p \wedge \alpha_q) \wedge \ldots \wedge \Box\alpha_j \rightarrow \Box\gamma$, by inductive hypothesis.

Therefore, $\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_j \wedge \Box(\alpha_p \wedge \alpha_q) \rightarrow \Box\gamma$, where $1 \leq p \neq q \leq j$. (*)

By applying Lemma 3 to (*) $C(t,2) - 1$ times,

$\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_j \wedge \Box((\alpha_1 \wedge \alpha_2) \vee \ldots \vee (\alpha_t \wedge \alpha_{t+1})) \rightarrow \Box\gamma$, where $t = C(n,i)$.

But $\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \Box\alpha_2 \wedge \ldots \wedge \Box\alpha_{t+1} \rightarrow \Box((\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3) \vee \ldots \vee (\alpha_t \wedge \alpha_{t+1}))$, where $t = C(n,i)$.

Therefore, $\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \Box\alpha_2 \wedge \ldots \wedge \Box\alpha_j \rightarrow \Box((\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3) \vee \ldots \vee (\alpha_t \wedge \alpha_{t+1}))$, since $j \geq t$.

Therefore, $\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_j \rightarrow \Box\gamma$.

■

Lemma 7. *Let $\Sigma = \{\Psi_1 \ldots \Psi_h\}$ be a finite set of finite sets of wffs and $\forall s(1 \leq s \leq h)$, ζ_s be the conjunction of all the elements of Ψ_s. Let $\cup\Sigma = \{\alpha_1 \ldots \alpha_j\}$. If, for each i-n-distribution Δ of $\cup\Sigma$, $\exists\Psi \in \Sigma$, $\exists\{\neq, \Theta_1 \ldots \Theta_i\} \subseteq \Delta$ such that $\Psi \subseteq \cap\{\Theta_1 \ldots \Theta_i\}$, then $\vdash_{\underline{K}_n^i} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_j \rightarrow \Box(\zeta_1 \vee \ldots \vee \zeta_h)$.*

Proof:

Assume that for each i-n-distribution Δ of $\cup\Sigma$, $\exists\Psi \in \Sigma$, $\exists\{\neq, \Theta_1 \dots \Theta_i\} \subseteq \Delta$ such that $\Psi \subseteq \cap\{\Theta_1 \dots \Theta_i\}$.

But $\forall\Psi \in \Sigma$, $\Psi \vdash_{K_n^i} \zeta$, where ζ is the conjunction of Ψ.

Then for each i-n-distribution Δ of $\cup\Sigma$, $\exists\Psi \in \Sigma$, $\exists\{\neq, \Theta_1 \dots \Theta_i\} \subseteq \Delta$ such that $\cap\{\Theta_1 \dots \Theta_i\} \vdash_{K_n^i} \zeta$, where ζ is the conjunction of Ψ.

Then for each i-n-distribution Δ of $\cup\Sigma$, $\exists\{\neq, \Theta_1 \dots \Theta_i\} \subseteq \Delta$ such that $\cap\{\Theta_1 \dots \Theta_i\} \vdash_{K_n^i} (\zeta_1 \vee \dots \vee \zeta_h)$.

By Lemma 6, $\vdash_{K_n^i} \Box\alpha_1 \wedge \dots \wedge \Box\alpha_j \rightarrow \Box(\zeta_1 \vee \dots \vee \zeta_h))\}$.

■

Now we are ready to prove the fundamental theorem for $\mathcal{K}_r^i$.

Let $\Box(x) = \{\alpha : \Box\alpha \in x\}$.

Definition 8. *Let $\mathscr{L}$ be a $\mathcal{K}_n^i$ logic. A* <u>*canonical*</u> <u>*model*</u> *for a consistent $\mathscr{L}$ logic is an i-n-model $\mathscr{M}^{\mathscr{L}} = < U^L, R^L, V^L >$ in which:*

(1) U^L is the set of $\mathscr{L}$-maximal consistent sets of wffs.

(2) $xR^Ly_1 \dots y_n \Leftrightarrow (\neq, y_1 \dots y_n) \Rightarrow (\forall\alpha \in \Box(x) \Rightarrow (\exists\{\neq, u_1 \dots u_i\} \subseteq \{y_1 \dots y_n\} : \alpha \in \cap\{u_1 \dots u_i\}))$.

(3) $V^L(p) = \{x \in U^L : p \in x\}$.

As a canonical model is actually based on a trivial hyper-relational frame, the truth-condition for modal formulas will be:

$\models_x^{\mathscr{M}^L} \Box\alpha$ iff $\forall y_1 \dots \forall y_n\, (xRy_1 \dots y_n \wedge (\neq, y_1 \dots y_n) \Rightarrow \exists\{\neq, z_1 \dots z_i\} \subseteq \{y_1 \dots y_n\} : \forall z_{j(1 \leq j \leq i)}, \models_{z_j}^{\mathscr{M}^L} \alpha)$, and

$\models_x^{\mathscr{M}^L} \Diamond\alpha$ iff $\exists y_1 \dots \exists y_n\, (xRy_1 \dots y_n \wedge (\neq, y_1 \dots y_n) \wedge \forall\{\neq, z_1 \dots z_i\} \subseteq \{y_1 \dots y_n\}, \exists z_{j(1 \leq j \leq i)} : \models_{z_j}^{\mathscr{M}^L} \alpha)$.

Fundamental Theorem 9. *For each point x in* $\mathcal{M}^{\mathcal{L}}$, $\alpha \in x$ *iff* $\models_x^{\mathcal{M}^L} \alpha$.

Proof:

The proof is by induction on the length of α.

We prove only the inductive step for α of the form $\Box\beta$.

$\Rightarrow$ Assume that $\Box\beta \in x$.

By the definition of R_L,

$\forall y_1 \ldots \forall y_n\ (xR_L\ y_1 \ldots y_n \Rightarrow (\ (\neq, y_1 \ldots y_n) \Rightarrow (\forall \Box\alpha, \Box\alpha \in x \Rightarrow \exists\{\neq, u_1 \ldots u_i\} \subseteq \{y_1 \ldots y_n\} : \alpha \in \cap\{u_1 \ldots u_i\}\ )).$

Since $\Box\beta \in x$, then

$\forall y_1 \ldots \forall y_n,\ (xR_L y_1 \ldots y_n \Rightarrow (\ (\neq, y_1 \ldots y_n\ ) \Rightarrow \exists\{\neq, u_1 \ldots u_i\} \subseteq \{y_1 \ldots y_n\} : \alpha \in \cap\{u_1 \ldots u_i\}\ )).$

Then $\models_x^{\mathcal{M}^L} \Box\beta$ (by inductive hypothesis and the truth-condition).

$\Leftarrow$ Assume that $\Box\beta \notin x$.

We must prove that $\not\models_x^{\mathcal{M}^L} \Box\beta$.

By the truth-condition, we must prove that

$\exists y_1 \ldots \exists y_n\ (xRy_1 \ldots y_n \wedge (\neq, y_1 \ldots y_n\ ) \wedge (\forall\{\neq, z_1 \ldots z_i\ \} \subseteq \{y_1 \ldots y_n\}, \exists z_{j(1 \leq j \leq i)} : \not\models_{z_j}^{\mathcal{M}^L} \beta\ )).$

By the definition of R_L and inductive hypothesis, it is sufficient to prove that

$\exists y_1 \ldots \exists y_n\ ((\forall\alpha \in \Box(x) \Rightarrow (\exists\{\neq, u_1 \ldots u_i\} \subseteq \{y_1 \ldots y_n\} : \alpha \in \cap\{u_1 \ldots u_i\}\ )) \wedge (\forall\{\neq, z_1 \ldots z_i\} \subseteq \{y_1 \ldots y_n\}, \exists z_{j(1 \leq j \leq i)} : \beta \notin z_j\ )).$

Assume that it is not the case.

Then $\forall y_1 \ldots \forall y_n\ (\forall\alpha \in \Box(x) \Rightarrow \exists\{\neq, u_1 \ldots u_i\} \subseteq \{y_1 \ldots y_n\}\ \alpha \in \cap\{u_1 \ldots u_i\}\ ) \Rightarrow \exists\{\neq, z_1 \ldots z_i\} \subseteq \{y_1 \ldots y_n\} : \beta \in \cap\{z_1 \ldots z_i\}\ ).$

Then, for each i-n-distribution $\Delta = \{\ y_1 \ldots y_n\ \}$ of $\Box(x)$, $\exists\{\neq, z_1 \ldots z_i\ \} \subseteq \Delta$ such that $\beta \in \cap\{\neq, z_1 \ldots z_i\ \}$.

Then, for each i-n-distribution $\Delta = \{\ y_1 \ldots y_n\ \}$ of $\Box(x)$, $\exists\{\neq, z_1 \ldots z_i\ \} \subseteq \Delta$ such that $\cap\{\neq, z_1 \ldots z_i\ \} \vdash_{\mathbf{K}_n^i} \beta$.

Then, for each i-n-distribution $\Delta = \{ y_1 \dots y_n \}$ of $\Box(x)$, $\exists\{\neq, z_1 \dots z_i \} \subseteq \Delta$ such that there is a finite subset Ψ of $\cap\{\neq, z_1 \quad . z_i \}$, $\Psi \vdash_{K_n^i} \beta$.

Let $\Sigma = \{ \Psi$: for each i-n-distribution Δ of $\Box(x)$, $\exists\{\neq, z_1 \dots z_i \} \subseteq D$ such Ψ is a finite subset of $\cap\{\neq, z_1 \dots z_i \}$ and $\Psi \vdash_{K_n^i} \beta \}$.

Then, by the definition of Σ, for each i-n-distribution Δ of $\cup\Sigma$, $\exists\Psi \in \Sigma$, $\exists\{\neq, z_1 \dots z_i\} \subseteq \Delta$ such that $\Psi \subseteq \cap\{z_1 \dots z_i\}$.

But, by Lemma 5, there is a finite Σ_0 of Σ such that for each i-n-distribution Δ of $\cup\Sigma_0$, $\exists\Psi \in \Sigma_0$, $\exists\{\neq, z_1 \dots z_i\}\subseteq \Delta$ such that $\Psi \subseteq \cap\{z_1 \dots z_i\}$.

Let $\Sigma_0 = \{\Psi_1 \dots \Psi_h\}$, $\cup\Sigma = \{\alpha_1 \dots \alpha_j\}$ and for each $1 \leq s \leq h$, ζ_s is the conjunction of the elements of Ψ_s.

By the definition of Σ_0:

$\vdash_{K_n^i} \zeta_1 \vee \dots \vee \zeta_h \rightarrow \beta$.

$\vdash_{K_n^i} \Box(\zeta_1 \vee \dots \vee \zeta_h) \rightarrow \Box\beta$, by [RM].

$\vdash_{K_n^i} \Box\alpha_1 \wedge \dots \wedge \Box\alpha_j \rightarrow \Box(\zeta_1 \vee \dots \vee \zeta_h)$, by Lemma 7.

$\vdash_{K_n^i} \Box\alpha_1 \wedge \dots \wedge \Box\alpha_j \rightarrow \Box\beta$, by PL.

But $\Box\alpha_1 \wedge \dots \wedge \Box\alpha_j \in x$.

Therefore $\Box\beta \in x$, contrary to hypothesis.

■

Corollary 9. *Logic $\mathcal{K}_n^i$ is determined by the class of hyper-relational frames..*

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Chapter 3

Logic $\mathcal{K}_n^1$

In this chapter, we look first at some correspondence-theoretic results for logic $\mathcal{K}_n^1$, then illustrate some completeness and incompleteness results for some of its extensions. Those results can be easily extended mutatis mutandis to the $\mathcal{K}_n^i$ logics.

While doing this, however, we will confine our attention to the class of the so-called *[n+1,k]-ary* hyper-relational frames which has the property that the arity of a tuple to which a point may be related is never less than n. With this in mind, an *[n+1,k]-ary* hyper-relational frame is defined as follows:

Definition: *An <u>[n+1,k]-ary</u> <u>hyper-relation</u> on a non-empty set U is a subset of* $U^{n+1} \cup \ldots \cup U^k$ *where* $k \geq n+1$ *is a natural number.*

Definition: *An <u>[n+1,k]-ary</u> <u>hyper-relational frame</u> is an ordered pair* $< U, R >$ *where U is a non-empty set and R is an [n+1,k]-ary hyper-relation. We can put an [n+1,k]-ary hyper-relational frame as a triple* $< U,R,k >$ *where* $U \neq \varnothing$ *and* $k \geq n+1$ *is a natural number and* $R \subseteq U^{n+1} \cup \ldots \cup U^k$.

An $\mathcal{M}_n$ model on an [n+1,k]-ary hyper-relational frame is defined as usual where the truth-condition for modal formulas is:

$$\models_x^{\mathcal{M}_n} \Box\alpha \text{ iff } \forall\tau \in R(x),\ |\tau| \geq n \Rightarrow (\exists\{\neq, z_1 \ldots z_{|\tau|-n+1} \} \subseteq \tau : \forall z_{j(1 \leq j \leq |\tau|-n+1)}, \models_{z_j}^{\mathcal{M}_n} \alpha),$$

or

$$\models_x^{\mathcal{M}_n} \Box\alpha \text{ iff } \forall\tau \in R(x),\ \forall\{\neq, z_1 \ldots z_n \} \subseteq \tau, \exists z_{j(1 \leq j \leq n)} : \models_{z_j}^{\mathcal{M}_n} \alpha).$$

For $\Diamond\alpha$,

$\models_{x}^{\mathcal{M}_n} \Diamond\alpha$ iff $\exists\tau \in R(x),\ |\tau| \geq n \wedge (\ \forall\{\neq, z_1 \ldots z_{|\tau|-n+1}\} \subseteq \tau \Rightarrow \exists z_{j(1\leq j\leq|\tau|-n+1)}, \models_{z_j}^{\mathcal{M}_n} \alpha\ )$, or

$\models_{x}^{\mathcal{M}_n} \Diamond\alpha$ iff $\exists\tau \in R(x), |\tau| \geq n \wedge (\ \exists\{\neq, z_1 \ldots z_n\} \subseteq \tau, \forall z_{j(1\leq j\leq n)} : \models_{z_j}^{\mathcal{M}_n} \alpha\ )$.

A point in an 1-n-model $\mathcal{M}_n$ will make $\Box p$, needless to say, make $[K_n^i]$, true trivially if it is related to only tuples of arities less than n. It is quite obvious that, by the truth-condition for modal formulas, logic $\mathcal{K}_n^i$ is still determined by the class of [n+1,k]-ary hyper-relational frames.

3.1 Correspondence

3.1.1 R-second-order Definability

In hyper-relational models, the truth-condition of modal formulas is actually a second-order statement. However, when the relation is restricted to trivial hyper-relations, the truth-condition turns out to be first order. We call sentences of the former kind R-second-order sentences.

In correspondence theory, we need to find the class of R-second-order frames on which a wff (in propositional modal language) is valid. Especially, we need to find the class of R-second-order frames for a wff such that the restricted class of trivial hyper-relational frames are also frames for the wff.

Definition. *A wff α is first-order definable if there is a first-order sentence δ with predicates R and = such that for any frame $\mathcal{F}$, $\mathcal{F} \models_n^1 \alpha$ iff $\mathcal{F} \models_n^1 \delta$.*

Definition. *A wff α is R-second-order definable if there is a R-second-order sentence δ with predicates R and = such that for any frame $\mathcal{F}$, $\mathcal{F} \models_n^1 \alpha$ iff $\mathcal{F} \models_n^1 \delta$, then α is first-order definable.*

From the definition, it is easy to see that if a wff is not first-order definable, then it is not R-second-order definable.

Definition. *An $[n+1,k]$-ary frame $< U, R, k >$ is n+1^k-transitive iff R satisfies the condition:*

$\forall x, \forall \tau \in R(x), \forall\{\neq, y_1 \dots y_n\} \subseteq \tau, \forall \tau_1 \in R(y_1) \dots \forall \tau_n \in R(y_n), \forall\{\neq, z_{11} \dots z_{1n}\} \subseteq \tau_1 \dots \forall\{\neq, z_{n1} \dots z_{nn}\} \subseteq \tau_n, \exists\{\neq, w_1 \dots w_n\} \subseteq \{\neq, z_{k1} \dots z_{kn} : 1 \leq k \leq n\}, \exists \tau_0 \in R(x) (\{w_1 \dots w_n\} \subseteq \tau_0)$.

Theorem 1. *The formula [4], $\Box p \rightarrow \Box\Box p$, is valid on an $[n+1,k]$-ary frame $< U, R, k >$ iff R is $n+1^k$-transitive.*

Proof.

$\Rightarrow$ Suppose that $< U,R,k >$ is any $[n+1,k]$-ary frame which is not $n+1^k$-transitive.

Then $\exists x, \exists \tau \in R(x), \exists\{\neq, y_1 \dots y_n\} \subseteq \tau, \exists \tau_1 \in R(y_1) \dots \exists \tau_n \in R(y_n), \exists\{\neq, z_{11} \dots z_{1n}\} \subseteq \tau_1 \dots \exists\{\neq, z_{n1} \dots z_{nn}\} \subseteq \tau_n, \forall\{\neq, w_1 \dots w_n\} \subseteq \{\neq, z_{k1} \dots z_{kn} : 1 \leq k \leq n\}, \forall \tau_0 \in R(x) (\{w_1 \dots w_n\} \not\subseteq \tau_0)$.

Let the above existential variables be the actual points in U.

We define a model $\mathcal{M}$ on $< U,R,k >$ such that $V(p) = U - \{ z_{j_{k1}} \dots z_{j_{kn}} : 1 \leq k \leq n \}$.

Since $\forall\{\neq, w_1 \dots w_n\} \subseteq \{\neq, z_{k1} \dots z_{kn} : 1 \leq k \leq n\}, \forall \tau_0 \in R(x) (\{w_1 \dots w_n\} \not\subseteq \tau_0)$, by the definition of truth-condition, $\models^{\mathcal{M}}_{x} \Box p$.

But, on the other hand, since $\forall z \in \{\neq, z_{k1} \dots z_{kn} : 1 \leq k \leq n\}, \not\models^{\mathcal{M}}_{z} p$, then $\forall \tau_{j(1 \leq j \leq n)} \in R(y_j), \exists z \in \{\neq, z_{j1} \dots z_{jn}\} \subseteq \tau_j, \not\models^{\mathcal{M}}_{z} p$.

Therefore, $\forall y_{j(1 \leq j \leq n)}, \not\models^{\mathcal{M}}_{y_j} \Box p$.

Therefore, $\not\models^{\mathcal{M}}_{x} \Box\Box p$.

Therefore, $\not\models^{\mathcal{M}}_{x} \Box p \rightarrow \Box\Box p$.

$\Leftarrow$ Suppose that $\mathcal{M} = < U,R,k,V >$ is an arbitrary $[n+1,k]$-ary model which is $n+1^k$-transitive.

Let x be an arbitrary point in U.

Suppose that $\models^{\mathcal{M}}_{x} \Box p$. We must show that $\models^{\mathcal{M}}_{x} \Box\Box p$.

Suppose that $\nvDash_{x}^{\mathcal{M}_n} \Box\Box p$.

Then, $\exists\tau \in R(x), \exists\{\neq, y_1 \ldots y_n\} \subseteq \tau, \forall y_{k(1\leq k\leq n)}, \nvDash_{y_k}^{\mathcal{M}_n} \Box p$.

As a result, $\forall y_{j(1\leq j\leq n)}, \exists\tau_j \in R(y_j), \exists\{\neq, z_{j1} \ldots z_{jn}\} \subseteq \tau_j$ such that $\forall z_{jk(1\leq k\leq n)}, \nvDash_{z_{jk}}^{\mathcal{M}_n} p$.

But, by transitivity, $\exists\{\neq, w_1 \ldots w_n\} \subseteq \{\neq, z_{j1} \ldots z_{jn} : 1\leq j\leq n\}, \exists\tau_2 \in R(x) : (\{w_1 \ldots w_n\} \subseteq \tau_2)$.

By the truth definition, $\vDash_{x}^{\mathcal{M}_n} \Box p$, which contradicts the assumption that $\vDash_{x}^{\mathcal{M}_n} \Box p$.

■

Definition. *An [n+1,k]-ary frame < U, R, k > is <u>n+1^k-euclidean</u> iff R satisfies the condition:*

$\forall x, \forall\tau_1 \in R(x), \forall\tau_2 \in R(x), \forall\{\neq, u_1 \ldots u_n\} \subseteq \tau_1, \forall\{\neq, v_1 \ldots v_n\} \subseteq \tau_2, \exists u_{j(1\leq j\leq n)} : (\exists\tau \in R(u_j) : \{v_1 \ldots v_n\} \subseteq \tau)$.

Theorem 2. *The formula [5], $\Diamond p \rightarrow \Box\Diamond p$, is valid on an [n+1,k]-ary frame < U, R, k > iff R is <u>n+1^k-euclidean</u>.*

Proof.

$\Rightarrow$ Suppose that < U,R,k > is an arbitrary [n+1,k]-ary frame which is not $n+1^k$-euclidean.

Then $\exists x, \exists\tau_1 \in R(x), \exists\tau_2 \in R(x), \exists\{\neq, u_1 \ldots u_n\} \subseteq \tau_1, \exists\{\neq, v_1 \ldots v_n\} \subseteq \tau_2, \forall u_{j(1\leq j\leq n)}, (\forall\tau \in R(u_j), \{v_1 \ldots v_n\} \not\subseteq \tau)$.

Let the above variables be the actual points in U.

We define a model $\mathcal{M}_n$ on < U,R,k > such that $V(p) = \{v_1 \ldots v_n\}$.

Then, $\vDash_{x}^{\mathcal{M}_n} \Diamond p$.

But $\forall u_{j(1\leq j\leq n)}, \nvDash_{u_j}^{\mathcal{M}_n} \Diamond p$, since $\forall\tau \in R(u_j), \{\neq, v_1 \ldots v_n\} \not\subseteq \tau$.

Hence $\nvDash_{x}^{\mathcal{M}_n} \Box\Diamond p$.

So $\nvDash_{x}^{\mathcal{M}_n} \Diamond p \rightarrow \Box\Diamond p$.

$\Leftarrow$ Suppose that $\mathcal{M}_n$ = < U,R,k,V > is an arbitrary [n+1,k]-ary model which is $n+1^k$-euclidean.

Let x be an arbitrary point in U.

Suppose that $\models_x^{\mathcal{M}} \Diamond p$. We must show that $\models_x^{\mathcal{M}} \Box\Diamond p$.

Suppose that $\not\models_x^{\mathcal{M}} \Box\Diamond p$.

Then $\exists\tau_1 \in R(x), \exists\{\neq, u_1 \ldots u_n\} \subseteq \tau_1 : \forall u_{i(1\leq i\leq n)}, \not\models_{u_i}^{\mathcal{M}} \Diamond p$.

But, by assumption, $\models_x^{\mathcal{M}} \Diamond p$.

Then $\exists\tau_2 \in R(x), \exists\{\neq, v_1 \ldots v_n\} \subseteq \tau_2, : \forall v_{i(1\leq i\leq n)}, \models_{v_i}^{\mathcal{M}} p$.

But R is $n+1^k$-euclidean.

Therefore, $\exists u_{j(1\leq j\leq n)} (\exists\tau \in R(u_j) : \{v_1 \ldots v_n\} \subseteq \tau)$.

Then, $\exists u_{i(1\leq i\leq n)} \models_{u_i}^{\mathcal{M}} \Diamond p$.

So we have a contradiction.

■

Definition. *An $[n+1,k]$-ary frame $< U, R, k >$ is* <u>$n+1^k$-degenerate</u> *iff R satisfies the condition:*

$\forall x, \forall\tau \in R(x), \neg\exists\{\neq, z_1 \ldots z_n\} \subseteq \tau$, *where* $1\leq h<n$.

Theorem 3. *Each of the formulas $\Box p$, $[K_h^1]$ and $[B]$ $(p \to \Box\Diamond p)$ is valid on an $[n+1,k]$-ary frame $< U, R, k >$ iff R is* <u>$n+1^k$-degenerate</u>.

Proof.

We prove the theorem for $\Box p$ first.

(a) $\Rightarrow$ Suppose that $< U,R,k >$ is an arbitrary $[n+1,k]$-ary frame that is not $n+1^k$-degenerate.

Then $\exists x, \exists\tau \in R(x), \exists\{\neq, z_1 \ldots z_n\} \subseteq \tau$

Let the above variables be the actual points in U.

We define a model $\mathcal{M}$ on $< U,R,k >$ such that $V(p) = U - \{z_1 \ldots z_n\}$.

Then $\not\models_x^{\mathcal{M}} \Box p$.

$\Leftarrow$ Suppose that $\mathcal{M} = < U,R,k,V >$ is an arbitrary $[n+1,k]$-ary model which is $n+1^k$-degenerate.

Let x be an arbitrary point in U.

By the definition of truth-condition, it is easy to see that $\models_x^{\mathcal{M}_n} \Box p$.

(b) $\Rightarrow$ Suppose R is not $n+1^k$-degenerate.

Then $\exists x, \exists \tau \in R(x), \exists\{\neq, z_1 \ldots z_n\} \subseteq \tau$.

Let the above variables be the actual points in U.

We define a model $\mathcal{M}_n$ on $< U,R,k >$ such that $\forall 1 \leq h \leq n$, $V(p_h) = U - \{z_1 \ldots z_{h-1}, z_{h+1} \ldots z_n\}$. (The valuation is possible since $h<n$).

Then, $\models_x^{\mathcal{M}_n} \Box p_1 \wedge \Box p_2 \wedge \ldots \wedge \Box p_{h+1}$.

But $\forall(1 \leq h \leq n)$, $\not\models_{z_h}^{\mathcal{M}_n} ((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_h \wedge p_{h+1}))$.

so $\not\models_x^{\mathcal{M}_n} \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_h \wedge p_{h+1}))$.

$\Leftarrow$ Suppose that $\mathcal{M}_n = < U,R,k,V >$ is an arbitrary $[n+1,k]$-ary model which is $n+1^k$-degenerate.

Let x be an arbitrary point in U.

By the definition of truth-condition, $\Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_h \wedge p_{h+1}))$ is true at x trivially.

Hence, $\models_x^{\mathcal{M}_n} \Box p_1 \wedge \ldots \wedge \Box p_{h+1} \rightarrow \Box((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee \ldots \vee (p_h \wedge p_{h+1}))$.

(c) $\Rightarrow$ Suppose R is not $n+1^k$-degenerate.

Then $\exists x, \exists \tau \in R(x), \exists\{\neq, z_1 \ldots z_n\} \subseteq \tau$

Let the above variables be the actual points in U.

We define a model $\mathcal{M}_n$ on $< U,R,k >$ such that $V(p) = \{x\}$.

This will make $\Diamond p$ false everywhere, needless to say, at point $z_1 \ldots z_n$.

So $\not\models_x^{\mathcal{M}_n} \Box\Diamond p$.

But $\models_x^{\mathcal{M}_n} p$.

So $\not\models_x^{\mathcal{M}_n} p \rightarrow \Box\Diamond p$.

$\Leftarrow$ Suppose that $\mathcal{M}_n = < U,R,k,V >$ is an arbitrary $[n+1,k]$-ary model which is $n+1^k$-degenerate.

Let x be an arbitrary point in U.

Suppose that p holds at x.

It is easy to see that $\Box\Diamond p$ is true at x also.

■

Theorem 4. *Formula $\Diamond p \rightarrow \Box p$ is valid on an [n+1,k]-ary frame $F = < U, R, k >$ iff R satisfies the condition: $\forall x$, ($\forall \tau_1 \in R(x)$, $\forall\{\neq, y_1 \ldots y_n\} \subseteq \tau_1 \wedge \forall \tau_2 \in R(x)$, $\forall\{\neq, z_1 \ldots z_n\} \subseteq \tau_2 \Rightarrow |\{y_1 \ldots y_n, z_1 \ldots z_n\}| \leq 2n-1$).*

Proof.

$\Rightarrow$ Let < U, R, k > is an arbitrary [n+1,k]-ary frame without the required condition.

Then $\exists x$ ($\exists \tau_1 \in R(x)$, $\exists\{\neq, y_1 \ldots y_n\} \subseteq \tau_1 \wedge \exists \tau_2 \in R(x)$ $\exists\{\neq, z_1 \ldots z_n\} \subseteq \tau_2 \wedge |\{y_1 \ldots y_n, z_1 \ldots z_n\}| > 2n-1$).

We define a model $\mathcal{M}_n$ on < U,R,k > such that p true at each point of $\{y_1 \ldots y_n\}$ and false at each point of $\{z_1 \ldots z_n\}$.

Then by the definition of truth-condition, $\models_x^{\mathcal{M}_n} \Diamond p$ and $\not\models_x^{\mathcal{M}_n} \Box p$.

So we have a model falsifying $\Diamond p \rightarrow \Box p$.

$\Leftarrow$ Assume that $\mathcal{M}_n$ = < U,R,k,V > is an arbitrary [n+1,k]-ary model with the required condition.

Let x be an arbitrary point in U.

Assume that $\models_x^{\mathcal{M}_n} \Diamond p$. We need to prove that $\models_x^{\mathcal{M}_n} \Box p$.

Assume for reductio that $\not\models_x^{\mathcal{M}_n} \Box p$.

Then, by the definition of truth-condition, $\exists \tau_1 \in R(x)$, $\exists\{\neq, y_1 \ldots y_n\} \subseteq \tau_1$: $\forall y_{j(1 \leq j \leq n)}$, $\models_{y_j}^{\mathcal{M}_n}$ p; and

$\exists \tau_2 \in R(x)$, $\exists\{\neq, z_1 \ldots z_n\} \subseteq \tau_2$: $\forall z_{j(1 \leq j \leq n)}$, $\not\models_{z_j}^{\mathcal{M}_n}$ p

But, by the definition of the frame, $|\{y_1 \ldots y_n \ldots z_1 \ldots z_n\}| \leq 2n-1$.

So we have a contradiction.

■

Theorem 5. *The formula [D], $\Box p \rightarrow \Diamond p$, is valid on an [n+1,k]-ary frame < U, R, k > if R satisfies the condition:* $\forall x, \exists \tau \in R(x) : |\tau| \geq 2n\text{-}1$.

Proof.

Assume that $\mathcal{M}_n = $ < U,R,k,V > is an arbitrary [n+1,k]-ary model with the required condition.

Let x be an arbitrary point in U.

Then $\exists \tau \in R(x) : |\tau| \geq 2n\text{-}1$.

Assume that $\models_x^{\mathcal{M}_n} \Box p$.

Then by truth-condition, $\forall \{\neq, y_1 \ldots y_n\} \subseteq \tau, \exists y_{j(1 \leq j \leq n)} : \models_{y_j}^{\mathcal{M}_n}$.

Therefore, $\exists \{\neq, y_1 \ldots y_n\} \subseteq \tau, \forall y_{j(1 \leq j \leq n)} : \models_{y_j}^{\mathcal{M}_n}$, since $|\tau| \geq 2n\text{-}1$.

By truth-condition, $\models_x^{\mathcal{M}_n} \Diamond p$.

Hence $\models_x^{\mathcal{M}_n} \Box p \rightarrow \Diamond p$.

■

Definition. *An [n+1,k]-ary frame < U, R, k > is* <u>*n+1^k-convergent*</u> *iff R satisfies the condition:*

$\forall x, \forall \tau_1 \in R(x), \tau_2 \in R(x), \forall \{\neq, u_1 \ldots u_n\} \subseteq \tau_1, \forall \{\neq, v_1 \ldots v_n\} \subseteq \tau_2, \exists \{\neq, w_1 \ldots w_{2n-1}\},$
$((\exists u \in \{\neq, u_1 \ldots u_n\}, \exists \tau_3 \in R(u)\ \exists \{\neq, w_1 \ldots w_{2n-1}\} \subseteq \tau_3)) \wedge (\exists v_0 \in \{\neq, v_1 \ldots v_n\},$
$\exists \tau_4 \in R(v_0), \exists \{\neq, w_1 \ldots w_{2n-1}\} \subseteq \tau_4)$), *where* $n>1$.

Theorem 6. *The formula [G], $\Diamond\Box p \rightarrow \Box\Diamond p$ is valid on an [n+1,k]-ary frame < U, R, k > if R is* <u>*convergent*</u>.

Proof.

Assume that $\mathcal{M}_n = $ < U,R,k,V > is an arbitrary [n+1,k]-ary model with the required condition.

Let x be an arbitrary point in U.

Assume that $\models_x^{\mathcal{M}_n} \Diamond\Box p$.

Then $\exists \tau \in R(x), \exists \{\neq, u_1 \ldots u_n\} \subseteq \tau : \forall u_{j(1 \leq j \leq n)} \models_{u_j}^{\mathcal{M}_n} \Box p$.

Assume that $\not\models_x^{\mathcal{M}_n} \Box\Diamond p$.

Then $\exists\gamma \in R(x), \exists\{\neq, v_1 \ldots v_n\} \subseteq \gamma : \forall v_{j(1\leq j\leq n)} \models_{v_j}^{\mathcal{M}_n} \Box\neg p$.

By convergence, $\exists\{\neq, w_1 \ldots w_{2n-1}\}, (\exists u \in \{\neq, u_1 \ldots u_n\}, \exists\tau_3 \in R(u), \exists\{\neq, w_1 \ldots w_{2n-1}\} \subseteq \tau_3) \wedge \exists v_0 \in \{\neq, v_1 \ldots v_n\}, \exists\tau_4 \in R(v_0), \exists\{\neq, w_1 \ldots w_{2n-1}\} \subseteq \tau_4)$.

This means that p holds at n distinct points in $\{\neq, w_1 \ldots w_{2n-1}\}$ and that p fails at n distinct points in $\{\neq, w_1 \ldots w_{2n-1}\}$. So we have a contradiction.

■

Theorem 7. *The formula [M] $\Box\Diamond p \rightarrow \Diamond\Box p$ is valid on an [n+1,k]-ary frame F = $\langle U, R, k \rangle$ iff R satisfies the condition: $\forall x, \forall\tau_1 \in R(x), \forall\{\neq, y_1 \ldots y_n\} \subseteq \tau_1, \forall\{\neq, z_1 \ldots z_n\} \subseteq \tau_1, \forall y_{i(1\leq i\leq n)}, \forall\tau_2 \in R(y_i), \forall\{\neq, u_1 \ldots u_n\} \subseteq \tau_2, \forall z_{j(1\leq j\leq n)}, \forall\tau_3 \in R(z_j), \forall\{\neq, v_1 \ldots v_n\} \subseteq \tau_3, Card(\{\neq, u_1 \ldots u_n\} \cup \{\neq, v_1 \ldots v_n\}) \leq 2n-1$.*

Proof:

Assume that $\mathcal{M}_n = \langle U,R,k,V \rangle$ is an arbitrary [n+1,k]-ary model with the required condition.

Let x be an arbitrary point in U.

Assume that $\models_x^{\mathcal{M}_n} \Box\Diamond p$ and for reductio that $\not\models_x^{\mathcal{M}_n} \Diamond\Box p$.

By the truth-condition, $\forall\tau_1 \in R(x), \forall\{\neq, y_1 \ldots y_n\} \subseteq \tau_1 \Rightarrow \exists y_{i(1\leq i\leq n)} : \models_{y_i}^{\mathcal{M}_n} \Diamond p$, and $\forall\tau_2 \in R(x), \forall\{\neq, z_1 \ldots z_n\} \subseteq \tau_2 \Rightarrow \exists z_{j(1\leq j\leq n)} : \models_{z_j}^{\mathcal{M}_n} \Diamond\neg p$.

Since $\models_{y_i}^{\mathcal{M}_n} \Diamond p$ and $\models_{z_j}^{\mathcal{M}_n} \Diamond\neg p$, by the truth-condition,

$\exists\tau_2 \in R(y_i), \exists\{\neq, u_1 \ldots u_n\} \subseteq \tau_2, \forall u_{k(1\leq k\leq n)}, \models_{u_k}^{\mathcal{M}_n} p$, and

$\exists\tau_3 \in R(z_j), \exists\{\neq, v_1 \ldots v_n\} \subseteq \tau_3, \forall v_{k(1\leq k\leq n)}, \models_{v_k}^{\mathcal{M}_n} \neg p$.

But by the property of the frame, $Card(\{\neq, u_1 \ldots u_n\} \cup \{\neq, v_1 \ldots v_n\}) \leq 2n-1$. We have a contradiction.

■

Theorem 8. *Formula [T], $\Box p \rightarrow p$ has no [n+1,k]-ary relational frames, where n>1.*

Proof:

Assume that $< U,R,k >$ is an arbitrary [n+1,k]-ary frame, where $n>1$.

Let x be a point in U.

We define a model on $< U,R,k >$ such that p false at x, and true everywhere else.

By the truth-condition, this makes $\Box p$ true at x.

Hence $\Box p \rightarrow p$ is false at x.

Therefore, formula [T] has no [n+1,k]-ary relational frames, if $n > 1$.

■

Theorem 9. *(1) Formula $\Box\alpha \rightarrow \Diamond\beta$, especially formula [D], has no $n+1^k$-degenerate model, if $n > 1$.*

(2) Formula [D] is not R-second-order definable, if $n > 1$.

Proof:

(1) For any $n+1^k$-degenerate model such that $n > 1$, by Theorem 3, $\Box p$ is true at the model and $\Diamond q$ has no $n+1^k$-degenerate model.

So the formula of the sort $\Box\alpha \rightarrow \Diamond\beta$ has no $n+1^k$-degenerate model.

(2) Let $n > 1$. First we define an [n+1,k]-ary frame $\mathscr{F}_i = < U_i, R_i, n\text{-}2 >$ for each $i \in N = \{1,2,3,\ldots\}$ as follows:

(a) U_i has exactly $i(2n\text{-}2)+2$ distinct points.

(b) Every point x_j in U_i is related to all the convex 2n-2 tuples in a cycle of the set $U_i - \{y_j\}$[1].

It is easy to see that sentence (a) is first-order sentence. And, as the frame is finite and the arity of the tuples is fixed, sentence (b) is also first-order sentence.

Figure 1 illustrates how a point is related to each convex 2n-2 tuple in frames $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{F}_3$ respectively, for $n = 2$ and $n = 3$.

---

1. The cycle of $U_i - \{y_j\}$ is defined as follows:
   (i) each point in $U_i - \{x_j\}$ has a unique successor.
   (ii) x_t is the successor of x_s, if $x_s, x_t \in U_i - \{x_j\}$ and (a) $s = t\text{-}1$ or (b) $s+1 = j = t\text{-}1$ or (c) $\forall x_k \in U_i - \{x_j\}$, $t \leq k \leq s$.

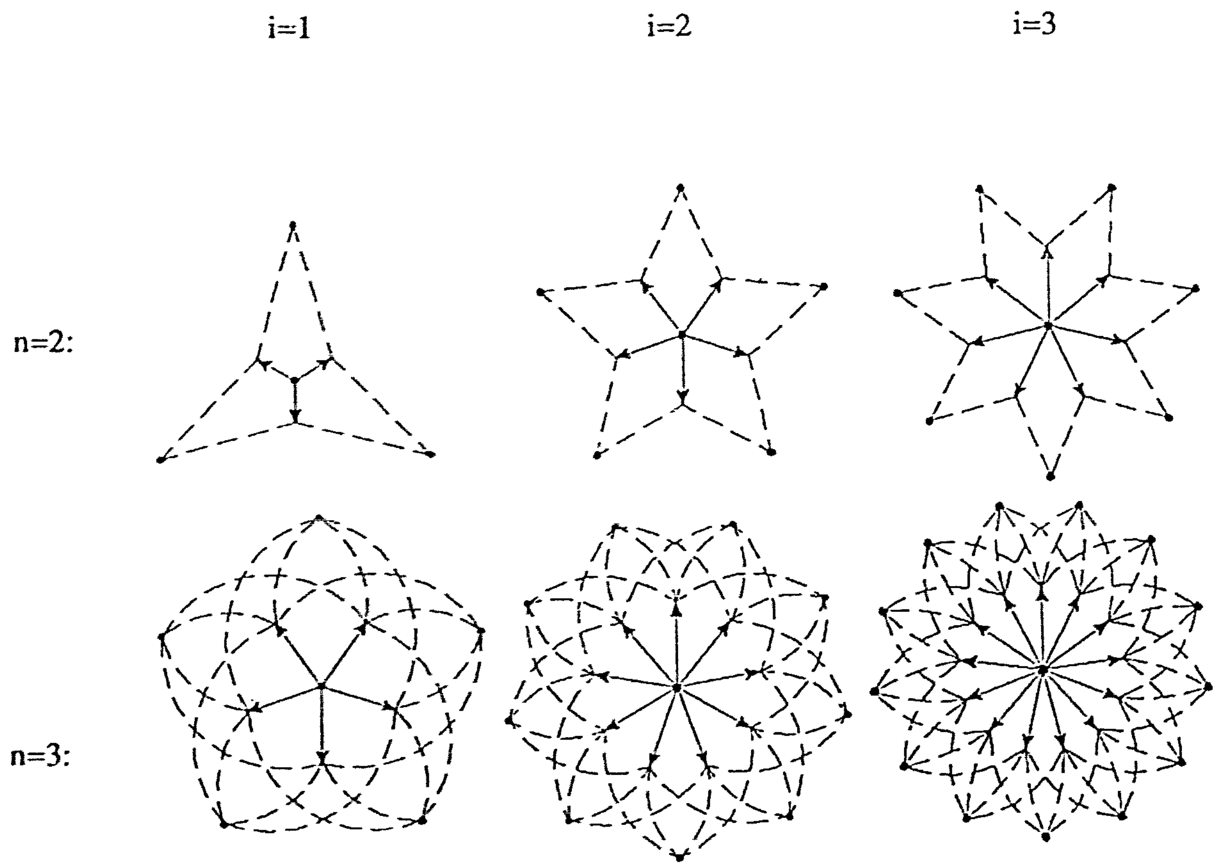

Figure 1.

But [D] is valid on each $\mathscr{F}_i$:

For an arbitrary model on $\mathscr{F}_i$ and an arbitrary point x in U_i, assume that $\models_x^{\mathcal{M}} \Box p$.

Since $|\tau|=2n\text{-}2$, by the truth-condition,

$\forall \tau \in R(x), \exists\{\neq, y_1 \dots y_{n-1}\} \subseteq \tau : \forall y_{j(1\leq j\leq n)}, \models_{y_j}^{\mathcal{M}} p.$

Assume for reductio that $\not\models_x^{\mathcal{M}} \Diamond P$. By the truth-condition,

$\forall \tau \in R(x), \exists\{\neq, y_1 \dots y_{n-1}\} \subseteq \tau : \forall y_{j(1\leq j\leq n)}, \models_{y_j}^{\mathcal{M}} \neg p.$

If i=1. Then we have a contradiction, since the cycle has only 2n-1 points.

If i>1.

The cycle has i(2n-2)+1 points. Thinking of the cycle as the result of adding (i-1)*(2n-2) points in the cycle of 2n-1 points.

Since we get a contradiction in the cycle with 2n-1 points, and $\forall \tau \in R(x)$,

$(\exists\{\neq, y_1 \dots y_{n-1}\} \subseteq \tau : \forall y_{j(1\leq j\leq n)}, \models_{y_j}^{\mathcal{M}} p, \ \& \ \exists\{\neq, y_1 \dots y_{n-1}\} \subseteq \tau : \forall y_{j(1\leq j\leq n)}, \models_{y_j}^{\mathcal{M}} \neg p).$

We still get a contradiction in the cycle of i(2n-2)+1 points.

Now assume that [D] is first-order definable.

Then there is a first-order formula α such that for each frame $\mathscr{F} \models_n^i [D]$ iff $\mathscr{F} \models_n^i \alpha$.

Let β_i be the first-order property for $\mathscr{F}_i$ i.e. (a) $\wedge$ (b).

And let $\Sigma = \{ \beta_i \wedge \alpha : i \geq 1 \}$.

Since each finite subset of Σ has a model, by first-order Compactness, Σ has a model.

But by the definition of Σ, the universe U of the model for Σ is infinite. For simplicity, let $U = \{ z_1, z_2 \dots \}$, and each $z_j \in U$ be related to each convex 2n-2 tuple in the ordered cycle of $U - \{ z_j \}$.

But if we put p true at z_i if i is even, and put p false at z_i if i is odd, then both $\Box p$ and $\Box\neg p$ are true at each z_j.

Hence [D] fails in this model. Contradiction.

Hence [D] is not first-order definable.

Hence [D] is not R-second-order definable.

■

Theorem 10. *The formula [G] $\Diamond\Box p \to \Box\Diamond p$ is not R-second-order definable.*

Proof:

The argument is similar to that in Theorem 9.

First we define an [n+1,k]-ary frame $\mathcal{F}_i = < U_i, R_i, n\text{-}2 >$ for each $i \in N = \{1,2,3,\ldots\}$ as follows:

(1) U_i has exactly i(2n-2)+2 distinct points, among which only one, say x, is an initial point and the others, say $y_1 \ldots y_{i(2n-2)+1}$, are non-initial points.

(2) Let the set of non-initial points $\{y_1 \ldots y_{i(2n-2)+1}\}$ form an ordered cycle, and each point in $\{y_1 \ldots y_{i(2n-2)+1}\}$ be related to each convex 2n-2 tuple in the cycle of $\{y_1 \ldots y_{i(2n-2)+1}\}$ like in Theorem 9.

(3) Let x be related to each n distinct points in $\{ y_1 \ldots y_{i(2n-2)+1} \}$.

To prove that the formula [G] is valid on each of the $\mathcal{F}_i$ we need results (i) and (ii).

(i) $\models^{\mathcal{M}}_{x} \Diamond\Box p \to \Box\Diamond p$.

Assume for reductio that $\not\models^{\mathcal{M}}_{x} \Diamond\Box p \to \Box\Diamond p$.

Then $\models^{\mathcal{M}}_{x} \Diamond\Box p$ and $\not\models^{\mathcal{M}}_{x} \Box\Diamond p$.

Then $\exists y_j, \models^{\mathcal{M}}_{y_j} \Box p$ and $\exists y_k, \not\models^{\mathcal{M}}_{y_k} \Diamond p$, $1 \leq j,k \leq i(2n-2)+1$.

From the proof of Theorem 9, if $\models^{\mathcal{M}}_{y_j} \Box p$, then $\models^{\mathcal{M}}_{y_j} \Diamond p$.

But by the definition of the frame $\mathcal{F}_i$ the tuples that y_j and y_k are related to are exactly same.

So $\models^{\mathcal{M}}_{y_k} \Diamond p$, contradicting the fact that $\not\models^{\mathcal{M}}_{y_k} \Diamond p$.

(ii) $\forall y \in \{ y_1 \ldots y_{i(2n-2)+1} \}, \models^{\mathcal{M}}_{y} \Diamond\Box p \to \Box\Diamond p$.

Assume $y \in \{ y_1 \ldots y_{i(2n-2)+1} \}$ and $\models^{\mathcal{M}}_{y} \Diamond\Box p$ and $\not\models^{\mathcal{M}}_{y} \Box\Diamond p$.

Then there are points y_j and y_k in $\{ y_1 \ldots y_{i(2n-2)+1} \}$ such that $\models^{\mathcal{M}}_{y_j} \Box p$ and $\not\models^{\mathcal{M}}_{y_k} \Diamond p$.

By the same argument as that in (i), $\models^{\mathcal{M}}_{y} \Diamond\Box p \to \Box\Diamond p$.

Now assume for reductio that the [G] is first-order definable.

Then there is a first-order formula α such that for each frame $\mathcal{F} \models^{i}_{n}$ [G] iff $\mathcal{F} \models^{i}_{n} \alpha$.

Let β_i be the first-order property for $\mathcal{F}_i$

And let $\Sigma = \{ \beta_i \wedge \alpha : i \geq 1 \}$.

Since each finite subset of Σ has a model, then, by first-order compactness, Σ has a model.

But by the definition of Σ, the model for Σ should contain an initial point, say x, and infinite non-initial points, say $U = \{ z_1, z_2 \ldots \}$, and x is related to each n points in U and each $z_j \in U$ is related to each convex 2n-2 tuple in the ordered cycle of $U - \{ z_j \}$.

But if we put p true at z_i if i is even, and put p false z_i if i is odd, then both $\Box p$ and $\Box\neg p$ are true at each z_j.

Therefore $\Diamond\Box p$ is true and $\Box\Diamond p$ is false at x. Hence [G] fails on this model.

Hence [G] is not first-order definable.

Hence [G] is not R-second-order definable.

■

Corollary 11. *If M and N are sequences of $\Box$ and $\Diamond$'s, then the formula $M\Box p \rightarrow N\Diamond p$ is not R-second-order definable.*

Proof:

The frame $\mathscr{F}_t$ for the formula [G] in theorem 10 is also a frame for $M\Box p \rightarrow N\Diamond p$.

(i) The formula holds at x:

Assume that $M\Box p \rightarrow N\Diamond p$ fails at x.

Then $M\Box p$ is true and $N\Diamond p$ is false at x.

We define a operator * as follows: if α is a modal formula, i.e. $\alpha = \Box\beta$ or $\Diamond\beta$ for some wff β, then $*(\alpha) = \beta$.

Since $M\Box p$ is true at x, and x is related to each n tuple in $\{ y_1 \ldots y_{i(2n-2)+1} \}$, then there is a $y_s \in \{ y_1 \ldots y_{i(2n-2)+1} \}$ such that $*(M\Box p)$ is true at y_s. But each $y \in \{ y_1 \ldots y_{i(2n-2)+1} \}$ is related to each 2n-2 convex tuple in the cycle of $\{ y_1 \ldots y_{i(2n-2)+1} \}$, then at the end, there must be a $y_j \in \{ y_1 \ldots y_{i(2n-2)+1} \}$ such that $\Box p$ holds at y_j.

On the other hand, let N' be the dual modality of N ($\Box$ and $\Diamond$ are dual modalities one of the other).

Since $N\Diamond p$ is false at x, $N'\Box\neg p$ is true at x.

Then $*(N'\Box\neg p)$ is true at some $y_t \in \{ y_1 \ldots y_{i(2n-2)+1} \}$, and at the end, there must be a y_k such that $\Box\neg p$ holds at y_k.

Then by the same argument as that in Theorem 10, the formula holds at x.

(ii) The formula holds at each point in $\{ y_1 \ldots y_{i(2n-2)+1} \}$.

Assume that y is an arbitrary point in $\{ y_1 \ldots y_{i(2n-2)+1} \}$, and that $M\Box p$ is true and $N\Diamond p$ is false at y.

Then there is a $y_s \in \{ y_1 \ldots y_{i(2n-2)+1} \}$ such that $*(M\Box p)$ is true at y_s, and $*(N'\Box\neg p)$ is true at some $y_t \in \{ y_1 \ldots y_{i(2n-2)+1} \}$. By the truth-condition, there must be a y_k such that $\Box\neg p$ holds at y_k at the end.

By the same argument as that in Theorem 10, $M\Box p \rightarrow N\Diamond p$ is true at y.

By a similar argument as that in the proof of Theorem 10, $M\Box p \rightarrow N\Diamond p$ is not R-second-order definable, if n>1.

■

3.1.2 Modal Definability

On the other hand, some R-second-order sentences are not modally definable.

Definition 12. *An $[n+1,k]$-ary frame $< U_1, R_1, k_1 >$ is a p-morphic image of an $[n+1,k]$-ary frame $<U_2, R_2, k_2>$ if there is a function f from $<U_2, R_2, k_2>$ to $<U_1, R_1, k_1>$ such that*

(1) f is onto, i.e. $\forall x \in U_2, \exists y \in U_1 : f(x) = y$.

(2) $\forall \tau_2 \in R_2(x), \forall \{\neq, z_1 \ldots z_n\} \subseteq \tau_2, \exists \tau_1 \in R_1(f(x)) : (\{ f(z_1) \ldots f(z_n) \} \subseteq \tau_1) \wedge (\neq, f(z_1) \ldots f(z_n))$.

(3) $\forall \tau_1 \in R_1(f(x)), \forall \{\neq, z_1 \ldots z_n\} \subseteq \tau_1, \exists \tau_2 \in R_2(x), \exists \{\neq, v_1 \ldots v_n\} \subseteq \tau_2 : (f(v_1) = z_1 \wedge \ldots \wedge f(v_n) = z_n)$.

Definition 13. $\mathcal{M}_1 = < U_1, R_1, k_1, V_1 >$ is a *p-morphic image* of $\mathcal{M}_2 = < U_2, R_2, k_2, V_2 >$ if there is a function f from $< U_2, R_2, k_2, V_2 >$ to $< U_1, R_1, k_1, V_1 >$ such that $< U_1, R_1, k_1 >$ is a p-morphic image of $< U_1, R_2, k_2 >$ and $\forall p \in \boldsymbol{At}, \forall x \in U_2, \models_x^{\mathcal{M}_1} p$ iff $\models_{f(x)}^{\mathcal{M}_2} p$.

Theorem 14. *If $< U_1, R_1, k_1, V_1 >$ is a p-morphic image of $< U_2, R_2, k_2, V_2 >$ then $\forall \alpha \in \Phi, \models_x^{\mathcal{M}_2} \alpha$ iff $\models_{f(x)}^{\mathcal{M}_1} \alpha$.*

Proof:

The proof is by induction on the construction of a wff.

Basis:

If α is a propositional variable, the theorem holds by Definition 12.

Inductive step:

We prove only the inductive step for α of the form $\Box\beta$.

$\Rightarrow$ Suppose that $\models_x^{\mathcal{M}_2} \Box\beta$.

Then $\exists \tau_2 \in R_2(x), \exists\{\neq, z_1 \ldots z_n\} \subseteq \tau_2 : \forall z_{i(1 \le i \le n)}, \models_{z_i}^{\mathcal{M}_2} \beta$.

But, by the definition of p-morphism, $\exists \tau_1 \in R_1(f(x)), ((\{f(z_1) \ldots f(z_n)\} \subseteq \tau_1) \wedge (\{\neq, f(z_1) \ldots f(z_n)\}))$.

But by induction assumption, $\forall z_{i(1 \le i \le n)}, \models_{f(z_i)}^{\mathcal{M}_1} \beta$.

Therefore, $\models_{f(x)}^{\mathcal{M}_1} \Box\beta$.

$\Leftarrow$ Suppose that $\models_{f(x)}^{\mathcal{M}_1} \Box\beta$.

Then $\exists \tau_1 \in R_1(f(x)), \exists\{\neq, z_1 \ldots z_n\} \subseteq \tau_1 : \forall z_{i(1 \le i \le n)}, \models_{z_i}^{\mathcal{M}_1} \beta$.

But, by the definition of p-morphism, $\exists \tau_2 \in R_2(x), \exists\{\neq, v_1 \ldots v_n\} \subseteq \tau_2 : (f(v_1) = z_1 \wedge \ldots \wedge f(v_n) = z_n)$.

But by the inductive hypothesis, $\forall v_{i(1 \le i \le n)}, \models_{v_i}^{\mathcal{M}_2} \beta$.

Hence $\models_x^{\mathcal{M}_2} \Box\beta$, by the definition of truth-condition.

■

Theorem 15. *Suppose that* $< U_1, R_1, k_1 >$ *is a p-morphic image of* $< U_2, R_2, k_2 >$. *Then for any wff* α, *if* α *is valid on* $< U_2, R_2, k_2 >$, α *is also valid on* $< U_1, R_1, k_1 >$.

Proof:

Assume that there is a wff α is not valid on $< U_1, R_1, k_1 >$.

This means there is a model $< U_1, R_1, k_1, V_1 >$ which falsifies α.

We now define a model $< U_2, R_2, k_2, V_2 >$ such that $\forall p \in \boldsymbol{At}$, $\forall x \in U_2$, $\models_x^{\mathcal{M}_2} p$ iff $\models_{f(x)}^{\mathcal{M}_1} p$.

By Definition 12, $< U_1, R_1, k_1, V_1 >$ is a p-morphic image of $< U_2, R_2, k_2, V_2 >$.

By Theorem 14, $< U_2, R_2, k_2, V_2 >$ falsifies α.

■

Theorem 16. *Each of the the following first-order sentences is not modally definable, if R is at least a ternary predicate:*

(1) $\forall x\ xRx...x$

(2) $\exists x\ xRx...x$

(3) $\forall x\ \neg xRx...x$.

Proof:

(1) The frame $< \{x,y\},\{<x,x...x>, <y,y...y>\}, k >$ is reflexive for some finite k, but its p-morphic image $< \{x\}, \varnothing, n+1 >$ or $< \{x,y\}, <\{x,y...y\}>, n+1 >$ is not. By Theorem 14, reflexivity is not modally definable.

(2) and (3) The proof is similar to (1).

■

Theorem 17. *Let* α *be a R-second-order formula with an n+1-ary predicate R (n>3) or a binary predicate =. Let* β *be the result of exchanging the positions of the two individuals variables of predicates = or the positions of the individual variables except the first one of R in* α. *Then* $\alpha \rightarrow \beta$ *is not modally definable.*

Proof:

It is easy to see from the definition of p-morphic image. ■

Here is a R-second-order formula that is not modally definable:

$\forall x, \forall \tau \in R(x), \forall\{\neq, y_1 \dots y_n\} \subseteq \tau, \beta \rightarrow \forall x, \forall \tau \in R(x), \forall\{\neq, y_n \dots y_1\} \subseteq \tau, \beta.$

3.2 Completeness and Incompleteness

In this section, we first prove that $\mathcal{K}_n^1 5$ and $\mathcal{K}_n^1 4$ are complete with respect to the class of all $n+1^k$-euclidean frames and $n+1^k$-transitive frames respectively, and then show some incompleteness results.

Theorem 1. *$\mathcal{K}_n^1 5$ is complete with respect to the class of all $n+1^k$-euclidean frames.*

Proof:

We just need to show that the canonical frame for $\mathcal{K}_n^1 5$ is $n+1^k$-euclidean, that is,

$\forall x, \forall\{\neq, y_1 \dots \forall y_n\}, \forall\{\neq, z_1 \dots \forall z_n\}, (xR_L y_1 \dots y_n \wedge xR_L z_1 \dots z_n \Rightarrow (y_1 R_L z_1 \dots z_n \vee \dots \vee y_n R_L z_1 \dots z_n)).$

By the definition of R_L, what we have to show is that if $\Box(x) \subseteq \cup\{\neq, y_1 \dots y_n\} \wedge \Box(x) \subseteq \cup\{\neq, z_1 \dots z_n\}$, then $\exists y \in \{\neq, y_1 \dots y_n\} : \Box(y) \subseteq \cup\{\neq, z_1 \dots z_n\}$.

Assume the antecedent and the negation of the consequence.

Then $\Box(x) \subseteq \cup\{\neq, y_1 \dots y_n\} \wedge \Box(x) \subseteq \cup\{\neq, z_1 \dots z_n\}$ and $\forall y \in \{\neq, y_1 \dots y_n\}, \Box(y) \not\subseteq \cup\{\neq, z_1 \dots z_n\}$.

Let y be an arbitrary element in $\{\neq, y_1 \dots y_n\}$.

Then, $\exists \alpha \in \Box(y), \alpha \notin \cup\{\neq, z_1 \dots z_n\}$.

Then $\alpha \notin \Box(x)$, since $\Box(x) \subseteq \cup\{\neq, z_1 \dots z_n\}$.

Then $\Box\alpha \notin x$.

But $\Diamond\Box\alpha \rightarrow \Box\alpha$ is theorem.

So $\Diamond\Box\alpha \notin x$.

Then $\Box\Diamond\neg\alpha \in x$.

Then $\Diamond\neg\alpha \in \cup\{\neq, y_1 \dots y_n\}$, since $\Box(x) \subseteq \cup\{\neq, y_1 \dots y_n\}$.

Then $\Box\alpha \notin \cup\{\neq, y_1 \dots y_n\}$.

Therefore, $\Box\alpha \notin y$.

Therefore $\alpha \in \Box(y)$. Contradiction.

■

Theorem 2. *$\mathcal{K}_n^1 4$ is complete with respect to the class of all n+1^k-transitive frames.*

Proof:

We just need to show that the canonical frame for $\mathcal{K}_n^1 4$ is n+1^k-transitive, that is, $\forall x$, $\forall y_1 \dots \forall y_n$, $\forall z_{11} \dots \forall z_{1n} \dots \forall z_{n1} \dots \forall z_{nn} \in \mathcal{W}_L$, $(xR_L y_1 \dots y_n \wedge y_1 R_L z_{11} \dots z_{1n} \wedge \dots \wedge y_n R_L z_{n1} \dots z_{nn} \wedge (\neq, y_1 \dots y_n) \wedge (\neq, z_{11} \dots z_{1n}) \wedge \dots \wedge (\neq, z_{n1} \dots z_{nn}) \Rightarrow \exists\{\neq, w_1 \dots w_n\} \subseteq \{z_{11} \dots z_{1n} \dots z_{n1} \dots z_{nn}\} \wedge (\{w_1 \dots w_n\} \subseteq R(x)))$.

By the definition of R_L, what we have to show is that

if $\Box(x) \subseteq \cup\{\neq, y_1 \dots y_n\} \wedge \Box(y_1) \subseteq \cup\{\neq, z_{11} \dots z_{1n}\} \wedge \dots \wedge \Box(y_n) \subseteq \cup\{\neq, z_{n1} \dots z_{nn}\}$,

then

$\exists\{\neq, w_1 \dots w_n\} \subseteq \{z_{11} \dots z_{1n} \dots z_{n1} \dots z_{nn}\} \wedge \Box(x) \subseteq \cup\{w_1 \dots w_n\}$.

Assume the antecedent and $\forall\{\neq, w_1 \dots w_n\} \subseteq \{z_{11} \dots z_{1n} \dots z_{n1} \dots z_{nn}\}$, $\Box(x) \not\subseteq \cup\{w_1 \dots w_n\}$.

Then $\Box(x) \not\subseteq \cup\{\neq, z_{11} \dots z_{1n}\} \wedge \dots \wedge \Box(x) \not\subseteq \cup\{\neq, z_{n1} \dots z_{nn}\})$, that is, $\exists\alpha_1 \in \Box(x) \wedge \alpha_1 \notin \cup\{\neq, z_{11} \dots z_{1n}\} \wedge \dots \wedge \exists\alpha_n \in \Box(x) \wedge \alpha_n \notin \cup\{\neq, z_{n1} \dots z_{nn}\}$.

Then $\alpha_1 \wedge \dots \wedge \alpha_n \in \Box(x) \wedge \alpha_1 \wedge \dots \wedge \alpha_n \notin \cup\{z_{11} \dots z_{1n} \dots z_{n1} \dots z_{nn}\}$.

So $\alpha_1 \wedge \dots \wedge \alpha_n \notin \cup\{\Box(y_1), \dots, \Box(y_n)\}$, since $\cup\{\Box(y_1), \dots, \Box(y_n)\} \subseteq \cup\{z_{11} \dots z_{1n} \dots z_{n1} \dots z_{nn}\}$.

Then formula $\Box(\alpha_1 \wedge \dots \wedge \alpha_n) \notin \cup\{\neq, y_1 \dots y_n\}$.

Then $\Box(\alpha_1 \wedge \dots \wedge \alpha_n) \notin \Box(x)$, since $\Box(x) \in \cup\{\neq, y_1 \dots y_n\}$.

Then $\Box\Box(\alpha_1 \wedge \dots \wedge \alpha_n) \notin x$.

But $\Box(\alpha_1 \wedge \dots \wedge \alpha_n) \rightarrow \Box\Box(\alpha_1 \wedge \dots \wedge \alpha_n)$ is a theorem.

Then $\Box(\alpha_1 \wedge \ldots \wedge \alpha_n) \notin x$, that is, $(\alpha_1 \wedge \ldots \wedge \alpha_n) \notin x$ contradicting the assumption that $(\alpha_1 \wedge \ldots \wedge \alpha_n) \in x$.

■

In [Boolos 1985], logic $\mathcal{K}^1_1\mathcal{H}$([H]: $\Box(\Box p \leftrightarrow p) \rightarrow \Box p$) is shown to be incomplete with respect to Kripke (binary relational) semantics. Formula [4] is not a theorem of logic $\mathcal{K}^1_1\mathcal{H}$, but each binary frame for [H] is also a frame for [4]. To see that for each $n \geq 2$, $\mathcal{K}^1_n\mathcal{H}$ is still an incomplete logic with respect to the class of [n+1,k]-ary relational frames, we need to prove that formula [4] is not a theorem of $\mathcal{K}^1_n\mathcal{H}$, and each [n+1,k]-ary frame for [H] is also an [n+1,k]-ary frame for [4].

Lemma 3. *Formula $[K^1_m]$ is not a theorem of $\mathcal{K}^1_n$ for any $1<m<n$.*

Proof:

By soundness, it is enough to show that formula $[K^1_m]$ is false in a model for $\mathcal{K}^1_n$.

Let $\mathcal{F} = < U,R,k >$ be a frame such that $\exists x \in U$ and $\exists\{\neq, y_1 \ldots y_n\} \subseteq U : xRy_1\ldots y_n$.

Let $\mathcal{M}_n$ be a model based on $\mathcal{F}$ such that $\forall i(1 \leq i \leq m)$, $V(p_i) = \{y_i\}$.

Then $\models^{\mathcal{M}_n}_x \Box p_1 \wedge \ldots \wedge \Box p_{m+1}$, since $m<n$.

But $\not\models^{\mathcal{M}_n}_x \Box((p_1 \wedge p_2) \vee \ldots \vee (p_m \wedge p_{m+1}))$.

Therefore $\not\models^{\mathcal{M}_n}_x \Box p_1 \wedge \ldots \wedge \Box p_{m+1} \rightarrow \Box((p_1 \wedge p_2) \vee \ldots \vee (p_n \wedge p_{m+1}))$.

■

Since formula [4] is not a theorem of $\mathcal{K}^1_1\mathcal{H}$, by Lemma 3, formula [4] is not a theorem of $\mathcal{K}^1_n\mathcal{H}$ when n>1.

Lemma 4. *Formula [4] is valid in each [n+1,k]-ary frame in which the formula [H] $\Box(\Box p \leftrightarrow p) \rightarrow \Box p$ is valid.*

Proof:

We show that for an arbitrary frame, if formula [4] is not valid on a frame, then formula [H] is false on it.

Assume that formula [4] is not valid on a frame $\mathscr{F}$, then, by Theorem 3.1.1, $\mathscr{F}$ is not $n+1^k$-transitive.

Then $\exists x, \exists\tau \in R(x), \exists\{\neq, y_1 \ldots y_n\} \subseteq \tau, \exists\tau_1 \in R(y_1) \ldots \exists\tau_n \in R(y_n), \exists\{\neq, z_{11} \ldots z_{1n}\} \subseteq \tau_1 \ldots \exists\{\neq, z_{n1} \ldots z_{nn}\} \subseteq \tau_n, \forall\{\neq, w_1 \ldots w_n\} \subseteq \{\neq, z_{k1} \ldots z_{kn} : 1 \leq k \leq n\}, \forall\tau_0 \in R(x)\ (\{w_1 \ldots w_n\} \not\subseteq \tau_0)$.

Let the above existential variables be the actual points in U.

Now let us construct a model $\mathscr{M}_n$ on $\mathscr{F}$ that falsifies formula [H] $\Box(\Box p \leftrightarrow p) \rightarrow \Box p$.

Put p false at x, $z_{ij(1 \leq i,j \leq n)}$ and false at all points in set $\{\neq, u_1 \ldots u_n\}$ that satisfies condition

(*): $\exists\tau' \in R(x), \{\neq, u_1 \ldots u_n\} \subseteq \tau' \wedge \forall u_{l(1 \leq l \leq n)}, \exists\tau_l \in R(u_l), \exists\{\neq, w_1 \ldots w_n\} \subseteq \{z_{k1} \ldots z_{kn} : 1 \leq k \leq n\}, \{\neq, w_1 \ldots w_n\} \subseteq R(u_l)$.

Put p true everywhere else.

Obviously, $\{\neq, y_1 \ldots y_n\}$ satisfies the condition *(*)*.

Then $\forall y \in \{\neq, y_1 \ldots y_n\}, \not\models_y^{\mathscr{M}_n} p$.

Then $\not\models_x^{\mathscr{M}_n} \Box p$.

But on the other hand, $\forall\tau' \in R(x), \forall\{\neq, v_1 \ldots v_n\} \subseteq \tau'$,

(1) if $\{\neq, v_1 \ldots v_n\}$ satisfies the condition *(*)*, then both p and $\Box p$ are false, that is $\Box p \leftrightarrow p$ is true at some point in $\{\neq, y_1 \ldots y_n\}$;

(2) if $\{\neq, v_1 \ldots v_n\}$ doesn't satisfy the condition *(*)*, then both p and $\Box p$ are true, that is $\Box p \leftrightarrow p$ is true at each point in $\{\neq, y_1 \ldots y_n\}$.

Therefore, $\forall\tau' \in R(x), \forall\{\neq, v_1 \ldots v_n\} \subseteq \tau', \exists v_{j(1 \leq j \leq n)}, \models_{v_j}^{\mathscr{M}_n} \Box p \leftrightarrow p$.

Hence $\models_x^{\mathscr{M}_n} \Box(\Box p \leftrightarrow p)$.

Hence $\not\models_x^{\mathscr{M}_n} \Box(\Box p \leftrightarrow p) \rightarrow \Box p$.

So $\mathscr{F}$ is not a frame for [H].

■

Here we show that $[K_1^1]$ is not provable from logic $\mathcal{K}_n^1\mathcal{H}$, i.e. $\mathcal{K}_n^1\mathcal{H}$ is not a normal logic.

Lemma 5. *For each $n \geq 2$, $\mathcal{K}_n^1\mathcal{H}$ is not a normal logic.*

Proof.

We just need to construct a model for $\mathcal{K}^1_n\mathcal{H}$, which falsifies $[K^1_1]$, $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$.

Consider a model $< U,R,n+1,V >$ where $U = \{x, y, z\}$, $R = \{<x, y, z\ldots z>\}$, $V(p) = \{y\}$ and $V(q) = \{z\}$.

It is easy to see that formula [H] is true at points x,y,z.

But $[K^1_1]$ is false at point x.

■

Theorem 6. *For each $n \geq 1$, $\mathcal{K}^1_n\mathcal{H}$ is incomplete with respect to the class of $[n+1,k]$-ary relational frames.*

Proof.

It follows immediately from lemmas 4 and 5.

■

$\mathcal{K}^1_1\mathcal{H}$ is the simplest incomplete logic with respect to the class of binary relational frames where the degree of [H] is two and it has only one propositional variable (cf.[van Benthem 1978]). But when n>1 and h<n, logic $\mathcal{K}^1_n\mathcal{B}$ and $\mathcal{K}^1_n\mathcal{K}^1_h$ are incomplete and the degree of [B] and $[K^1_m]$ is one.

Theorem 7. *$\mathcal{K}^1_n\mathcal{B}$ is incomplete with respect to the class $[n+1,k]$-ary relational frames, where $n>1$.*

Proof.

Assume that n>1.

By Theorem 3.1.3, each [n+1,k]-ary frame for [B] is also an [n+1,k]-ary frame for formula $\Box p$.

On the other hand, [B] is valid on a binary frame iff it is symmetrical.

But, for any 1-1-model $\mathcal{M} = < U,R,V >$ and any $x \in U$, $\models^{\mathcal{M}}_x \Box p$ iff x is a deadend.

Therefore, $\not\models^1_1 [B] \rightarrow \Box p$.

Therefore, by soundness, $\nvdash_{K^1_1} [B] \rightarrow \Box p$.

Hence, $\nvdash_{K^1_n} [B] \rightarrow \Box p$.

■

Theorem 8. *$\mathcal{K}^1_n\mathcal{K}^1_h$ is incomplete with respect to the class of [n+1,k]-ary relational frames where n>1 and h<n.*

Proof.

The proof is similar to the argument in Theorem 7.

■

Moreover, let $\Sigma = \{\Psi_1 \ldots \Psi_h\}$ be a finite set of finite sets of wffs such that for some n-partition[2] π of $\cup\Sigma$, $\forall\Psi \in \Sigma$, $\forall\Theta \in \pi$, $\Psi \not\subseteq \Theta$. Let ζ_s be the conjunction of elements of Ψ_s $(1 \leq s \leq h)$, and $\cup\Sigma$ the set $\{ \alpha_1 \ldots \alpha_j \}$, and [Q] the wff $\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_j \rightarrow \Box(\zeta_1 \vee \ldots \vee \zeta_h)$. Then, based on the n-partition, we can define a non-n+1^k-degenerate model $\mathscr{M}$ falsifying [Q]. But by a similar proof to that of Theorem 3.3, Q is valid on an [n+1,k]-ary frame iff it is n+1^k-degenerate. But on the other side, $\models^1_1$ [Q] and $\not\models^1_1 \Box p$. Therefore, $\nvdash_{K^i_n} [Q] \rightarrow \Box p$. Then we have:

Theorem 9. *$\mathcal{K}^1_n Q$ is incomplete with respect to the class of [n+1,k]-ary relational frames.*

But we know that there are uncountable many such different [Q] like formulas. So there are uncountable many incomplete logics.

Since there are incomplete logics with respect to [n+1,k]-ary relational frames, [n+1,k]-ary relational semantical consequence is stronger than its correspondent logical consequence. As expected, a weaker semantical consequence which is based on a general [n+1,k]-ary realtional frames is defined as follows:

Definition 10. *A general [n+1,k]-ary relational frame is < U, R, n+1, W > where < U, R, n+1 > is an [n+1,k]-ary relational frame, and W is a set of sets of points of U satisfying the following conditions:*

---

2. See Definition 2.4.

(1) If $A \in W$*, then* $W - A \in W$*,*

(2) If $A \notin W$ *or* $B \in W$*, then* $W - (A \cup B) \in W$*, and*

(3) If $A \in W$*, then* $\{ x \in U : \forall \tau \in R(x), \forall \{\neq, z_1 \ldots z_n\} \subseteq \tau, \exists z_{1 \leq j \leq n} \in A \} \in W$.

Theorem 11. *Each* $\mathcal{K}_n^1$ *logic is characterized by a class of general [n+1,k]-ary relational frames.*

Proof:

Let $< U^L, R^L, n+1, V^L >$ be the canonical model for any $\mathcal{K}_n^1$ logic. We define a subset of the power set of U^L, W such that $A \in W$ iff $A = | \alpha |_L$ for some α. It is sufficient to see that $< U^L, R^L, n+1, W >$ is a general frame. But because of the properties of maximal consistent set, the general frame satisfies (1) and (2) of definition 10. By the definition of the canonical model, it satisfies (3) of definition 10.

■

Appendix A

Some Connections Between $\mathcal{K}^i_n$ Logics

In this appendix, we deal with the connections between $\mathcal{K}^i_n$ logics. But first of all, we come back to our original definitions of hyper-relations and hyper-relational frames in Chapter 1 and 2. Here are the definitions again:

A *hyper-relation* on a non-empty set U is a subset of $U^1 \cup \ldots \cup U^k$, where k is a natural number. A *hyper-relational frame* is an ordered pair < U, R > where U is a non-empty set and R is a hyper-relation.

By examining $[K^i_n]$, we can show that $\{ \mathcal{K}^i_n : n \geq i \} = \{ \mathcal{K}^1_m : m \geq 1 \}$.

Theorem 1. *(1) For each $\mathcal{K}^i_n$ there is a $\mathcal{K}^1_m$ such that $\mathcal{K}^i_n = \mathcal{K}^1_m$, where $m = C(n,i)$.*

(2) For each $\mathcal{K}^1_m$ $(m \neq 2)$ there is a $\mathcal{K}^i_n$ such that $\mathcal{K}^1_m = \mathcal{K}^i_n$ where $i \neq 1$ and $n \neq m$.

(3) For each $\mathcal{K}^i_n$ there is a $\mathcal{K}^j_n$ such that $\mathcal{K}^i_n = \mathcal{K}^j_n$, where $j = n\text{-}i$.

Proof:

(1) It is easy to see from the axiomatizations of $\mathcal{K}^i_n$ and $\mathcal{K}^1_m$.

(2) If m = 1, then $\mathcal{K}^1_m$ is $\mathcal{K}^1_1$.

Let i = n.

Then, by the definition, $\mathcal{K}^i_n$ is $\mathcal{K}^1_1$.

If m>2.

Let i = n-1.

Since C(n,1) = C(n,i), $\mathcal{K}^1_n = \mathcal{K}^i_n$.

But n ≠ 2. Then i ≠ 1.

(3) Let j = n-i.

Then C(n,i) = C(n,j).

Then $[K^i_n] = [K^j_n]$.

Then $\mathcal{K}^i_n = \mathcal{K}^j_n$.

■

Theorem 2. *(1) For all i, if m>n, then* $\mathcal{K}^i_m$ *is a proper sublogic of* $\mathcal{K}^i_n$

(2) For all n, if j>i, then $\mathcal{K}^j_n$ *is a proper sublogic of* $\mathcal{K}^i_n$.

Proof. (1) Construct an i-m-model in which $[K^i_m]$ is true but $[K^i_n]$ is false.

(2) Construct a j-n-model in which $[K^j_n]$ is true but $[K^i_n]$ is false.

■

Although $\mathcal{K}^i_n$ and $\mathcal{K}^1_m$ (m = C(n,i)) are same logic, and they are complete with respect to both the class of $\mathcal{M}^1_m$ and the class of $\mathcal{M}^i_n$, their models are not equivalent.

Let m = C(n,i) and i>1. We can falsify that $\forall\mathcal{M}^i_n \forall\mathcal{M}^1_m \forall\alpha$, if $\models^{\mathcal{M}^1_m} \alpha$ then $\models^{\mathcal{M}^i_n} \alpha$, where $\mathcal{M}^i_n$ and $\mathcal{M}^1_m$ are based on the same frame and the same valuation.

Consider a model $\mathcal{M}^i_n = < \{x, y_1 \dots y_n\}, \{<x, y_1 \dots y_n>\}, V >$ such that $\{\neq, y_1 \dots y_n\}$ and $V(p) = \varnothing$.

Then by the truth definition, $\not\models^{\mathcal{M}^i_n}_x \Box p$.

But i>1. Then m>n.

By the truth definition, $\models^{\mathcal{M}^1_m}_x \Box p$ trivially.

Theorem 3. *Let m = C(n,i) and i>1. Then* $\forall\mathcal{M}^i_n \forall\mathcal{M}^1_m \forall\alpha$, $\models^{\mathcal{M}^i_n} \alpha \Rightarrow \models^{\mathcal{M}^1_m} \alpha$, *where* $\mathcal{M}^i_n$ *and* $\mathcal{M}^1_m$ *are on the same frame and valuation.*

Proof. The proof is by induction on the construction of wffs. Here we just show that the lemma holds when α is the form of $\Box\beta$.

Assume that $\not\models^{\mathcal{M}^1_m} \alpha$.

Then $\exists x$, $\not\models^{\mathcal{M}^1_m}_x \Box\beta$.

By the truth-condition, $\exists\tau \in R(x), \exists\{\neq, z_1 \dots z_m\} \subseteq \tau : \forall z_{j(1\leq j\leq m)}, \not\models^{\mathcal{M}^1_m}_{z_j} \beta$.

Then $\exists\tau \in R(x), \exists\{\neq, z_1 \ldots z_n\} \subseteq \tau : \forall\{\neq, u_1 \ldots u_i\} \subseteq \{\neq, z_1 \ldots z_n\}, \forall u_{j(1\leq j\leq i)}, \not\models_{u_j}^{\mathcal{M}b_n^i} \beta$, since n<m and i<m.

Then $\not\models_x^{\mathcal{M}b_n^i} \Box\beta$.

Then $\not\models^{\mathcal{M}b_n^i} \Box\beta$.

■

Since the semantic consequence of $\mathcal{M}b_m^1$ is stronger than that of $\mathcal{M}b_n^i$, there is a wff A such that $\mathcal{K}_n^i\mathcal{A}$ is complete with respect to some class of $\mathcal{M}b_n^i$ but $\mathcal{K}_m^1\mathcal{A}$ is incomplete with respect to the class of $\mathcal{M}b_m^1$, where m=C(n,i). We leave it to readers to find such a formula A.

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