A CASE STUDY OF A NEW MATHEMATICS CURRICULUM FOR ALTERNATE SCHOOL STUDENTS

by

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A Case Study of a New Mathematics Curriculum

for Alternate School Students

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ABSTRACT

This thesis is an account of my attempt to introduce a radically different style of mathematics education to a traditionally problematic student population. I reflect on the experiences both while engaged in the case study and after its completion.

The curriculum was based on the four cornerstones expounded in the NCTM's (1989) *Standards* document: problem-solving, communication, reasoning, and connections. Discovery learning and cooperative, open-ended problem solving replaced the school's individualistic, arithmetic-based program.

The young people in this study attended a greater Vancouver alternate school because their temperaments and behaviours were not suited to a regular school setting. Eleven students volunteered to participate, but I focus on two girls and three boys in particular. Four salient personal problems characteristic of this population are learning difficulties, poor self-esteem, social disabilities, and chronic, deep unhappiness.

With their insights, discoveries, and verbalizations, it soon became evident that my students were capable of learning many non-traditional and often complex mathematical topics. Individual traits such as perceptiveness, dependability, algebraic acumen, and enthusiasm all contributed to the over-all success of the cohort. Within the class, metacognitive comments became more frequent as individual passivity gave way to group activity.

The project was not without serious difficulties, however. Chronically poor student attendance remained a problem. Student mistrust of a new teacher, personality conflicts and bad feelings made it doubtful at times whether communication and cooperation were achievable aims at all. Initially, the students were very hesitant to talk to each other about mathematics and would frequently give up on problems rather than persist at a cooperative solution. With the passive, individualized nature of their previous math courses, such resistance to active, cooperative learning was understandable. Last, excessive direct instruction and leading by the teacher often precluded inter-student communication and discovery learning.

As the study progressed, student discourse and cooperation became more the norm. Infighting diminished, just as student confidence in the subject grew. The mathematics curriculum had many special benefits for these students. It served as a vehicle for honing social skills and for improving communication. It offered a chance for success in a typically failure-prone area. Finally, it provided a cognitive and emotional substitute for depression. Thus, in a way, the course could be described as a "social curriculum".

DEDICATION

This thesis is dedicated to the memory of Fred Walker,

alternate school educator and teacher of mathematics.

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Introduction

She's leaving home
After living alone
For so many years.
(Lennon and McCartney, 1967)

In September 1983, one of my students asked me a question which, considering the qualifier which followed it, I found unusual and intriguing. Her exact words were, "What do I need math for anyway? I'm a hooker." The setting was a residential treatment centre for emotionally disturbed adolescents. In particular, I was with four students in the "classroom" of a building termed the "closed unit" where unopenable, unbreakable windows and systems of locked metal doors were supposed to preclude the inevitable bolts for "freedom" that many of these youngsters would make. The questioner was Nancy, a fourteen year old girl. Having fled an abusive home for the streets of Vancouver, she had been apprehended by the police, and had been sent by the courts to the treatment centre where she now found herself in my small mathematics class.

As an experienced mathematics teacher, I was accustomed to the perennial question, "Why do we have to study this for?" and all the variants thereof. To be sure, I had a whole repertoire of responses which (in my own mind, at least) justified the existence of my professional specialty. But obviously, Nancy's particular query could not have been answered in the conventional ways. My unrehearsed response was something like, "But you won't be a hooker for all your life."

My consternation arose in part because this was my first alternate school assignment. However, it was not just for want of a ready answer that I was taken aback. From my subsequent discussion with her, I learned that this youngster was not being flippant or cheeky; she had asked a personal question and was expecting a serious answer. Specifically, in spite of debilitating emotional problems, this tragically precocious teenager was seeking a rationale for learning

mathematics. Simply put, Nancy was an "exceptional child" who was asking about a "regular mathematics education."

Ten years and four different alternate schools later, the question that most fascinated me was, "What might mathematics offer to students with emotional and social disabilities (ESD)?" In a sense, this is Nancy's question.

In 1993, after a four year hiatus from alternate settings (spent as a hospital and homebound teacher in strictly one-on-one teaching situations), I was assigned to a half-time position at the Vale Road Learning Facility (VRLF) alternate school. Disgruntled with the mathematics curriculum at VRLF, encouraged by my recent studies in mathematics education, and happy to have the chance to work with a group of students again, I launched into my thesis. My plan was to document and study the process of creating and introducing a new mathematics curriculum for my alternate school students.

This study will attempt to answer the following questions concerning a new approach to the mathematics education of the young people at VRLF.

•Which traits of alternate school students should be of concern to mathematics educators?

•How can the needs of these young people best be met through curriculum modifications?

•What barriers will inhibit or conflict with curriculum revision for alternate school adolescents? For instance, will parents or administrators be hesitant to change the status quo? Do non-specialist teachers have the experience and background necessary to teach the curricular innovations I envisage?.

•Will the process of curricular modification result in changes in students' attitudes and beliefs about mathematics? Can we encourage the students to give up their notion of mathematics as a subject "handed down" by experts for memorization?

•To what degree will the students adapt to a new curriculum? For instance, how can students be taught to work collaboratively when many have not done so since elementary school?

With curricular modifications, can our students be encouraged to abandon their passive learning roles for active engagement in problem-solving? Will we succeed in fostering new cognitive and metacognitive activity in a population of classically "inactive learners"? In other words, with new topics and new approaches to learning, can alternate students be induced to "act like mathematicians at the limits of their own community's (the classroom's) knowledge" (Schoenfeld, 1992, p. 362)?

•What can mathematics offer to alternate school students? For most alternate students, the learning of mathematics has been a negative experience. Can a new curriculum serve to empower them mathematically and to enhance their self-confidence in the subject? In sum, who are these troubled young pupils, what mathematics can they learn, and how can they best learn it?

Chapter One considers the availability of literature pertinent to the mathematics education of adolescents with ESD. Chapter Two investigates the social and emotional problems of the students commonly placed in alternate school programs. Chapter Three provides a rationale for change: I outline why the old curriculum at VRLF needed revision, and the curricular innovations made. Finally, Chapter Four describes the five month case study wherein several alternate school students from Vale Road Learning Facility experienced a new approach to mathematics learning.

At the onset, however, it is important to describe what is meant by the term "curriculum" because a curriculum is more than just a course outline or syllabus. The following definition by the National Council of Teachers of Mathematics (NCTM) is germane to this study:

> A curriculum is an operational plan for instruction that details what mathematics students need to know, how students are to achieve the identified curricular goals, what teachers are to do to help students develop their mathematical knowledge, and the context in which learning and teaching occur. (1989, p.1)

Obviously, curricular reform is a complex task: from the beginning I knew that my thesis would be completed long before any new curriculum itself was finished. Throughout the project, my

feelings towards this case study in curricular change were a mixture of excitement and trepidation. Schoenfeld (1992) sums up well my sentiments:

> ...[T]he imminent implementation of curricula with ambitious pedagogical and philosophical goals will raise a host of unavoidable and fundamentally difficult theoretical and practical issues. It is clear that we have our work cut out for us, but it is also clear that progress over the past decade gives us at least a fighting chance for success. (p. 366)

Chapter I

The availability of pertinent literature

What does the literature have to say about the mathematics education of adolescents with emotional and social disabilities? Unfortunately, the *literature* itself says that writings are scarce indeed. For example, in their analysis of articles on disturbed children, Weaver and Morse (1981) conclude that "relatively little can be identified as explicit to mathematics, to say nothing of focusing principally on mathematics" (p. 112). In the same vein, Wallace and Kauffmann (1986) state that "mathematics has not been as thoroughly studied or researched in regard to handicapped learners as have other academic areas" (p. 265). My search was obviously not hindered by confusion over an excess of pertinent literature.

Literature from related domains is more plentiful. For instance, I found two books featuring the mathematics education of *elementary students* with ESD (e.g., Morgan & Reinhart, 1991; Van Witsen, 1977). There are numerous texts on the mathematics education of *learning disabled* adolescents (e.g., Bley & Thornton, 1989; Cawley, 1984). (I found that a considerable amount of research from the overlapping LD field is pertinent to the alternate school population, and I refer extensively to this literature in Chapters 2 and 3.) Last, it is not at all uncommon to find articles on the *language arts* education of adolescents with ESD (e.g., Long, Morse & Newman, 1965; Wilson & Evans 1980).

Why does such a bias exist in favor of language arts over mathematics in the literature on students with ESD? One possible reason is suggested from a quote in a chapter entitled *Suggestions for an Educational Curriculum* (for children with ESD):

> Can a child attain independence without the ability to read? Without being able to read understandingly? Without being able to express himself orally? In writing? Without being able to calculate at reasonable speed? The answer is he cannot.
> Warnock, 1976, in Wilson & Evans, 1980, p. 171.

One cannot disagree with Warnock's sentiment. But her words illustrate a common bias towards mathematics education: that is, while language arts has several components such as reading comprehension, oral expression, and writing, mathematics is frequently equated with one lone skill: pencil and paper calculation. Meanwhile, Wilson and Evans devote no pages whatsoever in their book to mathematics.

There are also several possible explanations for the unequal weighting within special education categories and, in particular, for the lack of emphasis on the ESD field. For example, behaviour disorders in children have traditionally been studied more in psychology and psychiatry than in education (Glavin & Annesley, 1971). Also, there is an overlap of special education categories: the fact that many students with learning disabilities (LD) also have emotional and social problems, means that the ESD problem cannot be studied as a separate phenomenon. Finally, proponents of a noncategorical approach to special education seriously oppose attempts to consider any of the three traditional handicapped groups [ESD, LD, and mentally retarded (MR)] in isolation (Hallahan & Kauffman, 1977).

However, a few authors do focus on mathematics for the adolescent with ESD. But often, the *mathematical concepts* under consideration are lacking in scope or in meaningfulness. Some researchers appear to use mathematics solely as a convenient measure of academic achievement . For example, Glavin (1973) administered reading and arithmetic tests to determine that students' academic gains made after part-time placement in resource rooms were not maintained following a full time return to regular classes. Page and Edwards (1978) used mathematics classes to demonstrate that "reinforcing academic behavior is an effective method of reducing disruptive classroom behavior" (p. 413). Others are unrealistic as to the mathematics under consideration. Ashlock (1981) suggests that student boredom can be avoided by supplementing basic arithmetic skills with abstract algebra concepts such as "division on whole numbers is an operation which associates a pair of whole numbers with a third unique number"

(p. 285). As I describe later, Cox (1975) trivializes the subject by considering only basic computational algorithms in her comparative study of the mathematical skills of "handicapped" and "normal" student populations.

Several researchers have investigated the low mathematics achievement levels of students with social and emotional problems but offer little in the way of remedial suggestions. These studies are described in the section on academic deficits.

Actually, Weaver and Morse (1981) wrote the only piece I found which is explicitly and entirely devoted to adolescents, to emotional and social problems, and to a meaningful mathematics education. This excellent article provides a thorough depiction of "socially and emotionally impaired" (SEI) children, and I refer to it later. It also includes pedagogic recommendations which are precursors of the NCTM (1989) *Standards*:

> [L]et us not shortchange SEI children and youth by interpreting their mathematics programs too narrowly in terms of an unfortunate myopic interpretation of "basic skills." Our planning of mathematics programs should be done in full cognizance of a more enlightened interpretation of what is "basic"....(Weaver & Morse, 1981, p. 135)

I now describe the students in my alternate school, and the educational strategies and innovations for teaching them mathematics.

Chapter II

The alternate school students

If it wasn't for this school of mine
I'd be in jail doing time.
Juliana

Vale Road Learning Facility is an alternate school on an independent site. It consists of 60 students, aged fifteen to nineteen. Students can receive credit for Grade 10 modified or regular courses (mathematics, English, social studies, science and consumer education) and for English (or Communications), mathematics, science and technology, and social studies at the Grade 11 or 12 level. The school is structured to provide an educational experience for students who, for various reasons, have not been successful in regular secondary settings. Students attend classes for half a day only, either in the morning or in the afternoon. With this feature, they may hold down a part-time job, or participate in work experience programs. The maximum class size is twelve pupils, thereby facilitating greater teacher/student contact, and minimizing the distractions common in larger high school classes. Students are allowed to work at individual rates, and so are not subject to the temporal pressures characteristic of the schools they left. These features allow for academic flexibility, and are meant to provide students with a non-threatening, relaxed learning situation.

Students are referred to the program by regular school personnel, by the junior level alternate school (Tamarac Re-Entry Program), or by district level discipline committees. Students may even apply directly to the program if they are at least fifteen years old. Importantly, as older students, they can consider themselves as "volunteer" alternate school pupils, since no one student is obligated to attend because of Ministry age requirements. Referrals and applications are made to the district special counsellor, and students are selected for admissibility by this counsellor, the head teacher, and the principal of the program.

In fall 1993, the VRLF staff consisted of one full-time teacher, three half-time teachers, a half-time head teacher, and a child care counsellor. The full-time teacher had two groups of

students, and taught five subjects to each group. The three half-time teachers each had one group of students to whom they each taught five subjects. As a half-time teacher, I had one class of twelve students to whom I taught five subjects.

Having taught for three years (from 1986 to 1989) at a previous incarnation of the very same VRLF program, I noticed that between 1989 and 1993, certain changes had occurred. For one thing, the school had grown in size: the number of staff had increased from two to five full-time equivalent teachers, and the number of students had gone from forty-eight to sixty. As I shall discuss later, the nature of the students had changed drastically. What had not changed was the mathematics curriculum which I had taught to so many students in the last decade.

In order to plan an effective curriculum for alternate school adolescents, one must first ask, "What is their nature?" and "What are their needs?" I do not wish to label or define pupils as "emotionally and socially disabled," for as Csapo (1981) points out:

> ...the effects of labels, viewed from the point of view of the labeled, can be harmful and detrimental. Being called emotionally disturbed or any one of its variants implies a negative sanction by society. The child's self-perception is likely to be influenced by this label, and the perception of others is likely to be negatively coloured....The pupil is given a label because he has exhibited certain types of behaviors but why does he exhibit those behaviors? Because he is emotionally disturbed. Why is he emotionally disturbed? Because he behaves that way. The harmful consequences of this circular explanation far outweigh the labeler's intent of helpfulness. (p. 4)

It is more useful to work with a set of descriptors which can be used as the basis for any study. In spite of the incompleteness of any group profile, and of the relative lack of relevant research literature, I believe that a definite and practical portrayal of my alternate school students does emerge.

Important as they are, answers to my first question above (in the form of diagnostic information) should not be considered in strict isolation. Rather, they should be accompanied by

specific answers to the second question (in the form of prescriptive teaching and educational recommendations). For example, the knowledge that a certain student is severely withdrawn

would be of little value to a mathematics teacher who has no classroom strategies for dealing with socialization problems. Thus, educators should be aware not only of the deviant behaviours, disturbed emotional states, and different learning styles to expect from students with ESD, but also of the pedagogic strategies available for teaching them mathematics and even for alleviating their individual problems. Such awareness is also important for any educators who are planning the protocols for, or actually participating in an academic study of alternate school students. In other words, unless the teacher is familiar with the complex characteristics of their disabilities, and with the ways in which their personalities have not meshed properly with their environments, their mathematics education will probably falter and fail. To this end, the mathematics specialist, even though well-versed in the subject and astute in general teaching methods, must seek the wisdom of special educators in the area of emotionally and socially disabled adolescents.

Characteristics of alternate school adolescents

Most of these special students exhibit deep and chronic deficits in mathematics achievement which in many cases do not stem from learning disabilities nor from low native intelligence. As well as learning difficulties, they tend to have low levels of self-esteem, and unsatisfactory social functioning. In particular, socialization problems are manifested by poor social self-image, difficulties in forming personal relationships, language disabilities, and inadequate skills in social cognition. The last, and probably most pervasive trait which underlies many aspects of their lives is their profound unhappiness. There are indeed a preponderance of "damaged" kids at Vale Road Learning Facility.

1) Academic Problems.

Alternate school students usually display academic deficits and have experienced actual course failures.

Academic deficits

A student with an *academic deficit* is one whose level of achievement, as measured by a standardized achievement test, is not at a level commensurate with that student's age. The actual *deficit* is defined as a number which equals the quantitative difference between the student's measured achievement (in grade levels), and the expected grade-level corresponding to that student's age. Thus the term has meaning only within the context of a school and means that a pupil is behind her age group in specific content which the school considers important and age-appropriate.

Seven out of the eleven VRLF students in the study agreed to take the Wide Range Achievement Test (WRAT), and each scored below his or her expected grade level. The average deficit for these students was 5.1 grades. (Oddly enough, the average deficit of 59 students who attended the previous alternate school from 1986 to 1987 was also 5.1 grades.) In comparison, the average deficit for a population of 72 Grade 8, 9, and 10 students enrolled in the regular mathematics program of a district secondary school was only 1.9 grades.

The fact that the alternate school students had the larger average mathematics deficit is not surprising, since one of their most common behaviour problems was school performance related. As well, because they were older than their peers from the other school populations, the alternate program students would naturally have "bigger" grade deficits than their younger counterparts. The average age of the 11 students in the VRLF study group was 17.4 years, and their average expected grade level was 11.9. While the expected average grade level for the 1986-1987 alternate school students was 11.0, the corresponding average expected levels for the regular class students was only 9.7 grades respectively. Finally, and most importantly, many alternate school students had still not mastered the so-called "basic skills," even after repeated attempts over the years at remediation. The issue of whether they should be subjected to yet another round of rote mathematics will be discussed later.

It is probably no surprise that these pupils generally fare poorly on mathematics achievement tests. Indeed, the research literature is consistent with this observation. In the summary which follows, I have retained the same descriptors (such as *disturbed, normal,* and *troubled*) which the writers used in their reports.

Stone and Rowley (1964) discovered that of their group of 116 "disturbed" children, 86% fell below expected achievement levels in mathematics.

In their longitudinal study, Feldhusen, Thurston and Benning (1970) compared the academic performance of students designated as "aggressive" with that of peers rated as "socially approved." The researchers found that after five years, the same aggressive children still performed significantly lower in mathematics achievement than did their regular peers.

Glavin and Annesley (1971) provided an intricate statistical analysis of their results concerning the mathematics performance of "normal and troubled" boys. They found that 73% of the behaviour problem population were underachieving in arithmetic, compared to 34% of the total normal group. In all, 38% of the disturbed boys were classified as extreme underachievers compared to only 6% of the normal boys. Further, just 24% of the handicapped group achieved at the expected level compared to 59% of the normal group. Finally, no child from the special group could be described as an "extreme overachiever".

In her detailed analysis of arithmetic errors in "handicapped" and normal populations, Cox (1975) found that her special education group had a much higher incidence of systematic errors (14.5%) compared to an average of only 6.3% for the normal population. The highest frequency of errors for the special education students occurred at the intermediate (pre-adolescent) level, with an average of 20%. (The "handicapped" category consisted of children placed in classrooms for the "mentally retarded", "emotionally disturbed" and "learning problem" classrooms.)

Finally, for Epstein and Cullinan (1983), "behaviourally disordered" children scored higher in reading and spelling than their peers with LD, but in mathematics, the two groups tested equally low. The same authors comment:

> [b]ack-to-basic programs generally feature extensive drill exercises in which students provide rote responses to arithmetic computation facts, and little explicit attention is given to teaching underlying principles and relationships among mathematical concepts. Capable and motivated students may sooner or later discover such relationships, but there is virtually no reason to believe that most mildly handicapped pupils will do so. In fact, research indicates that pupils with learning disabilities, behavior disorders, and mental retardation generally show substantial achievement deficits in arithmetic computation or reasoning or both. (Cullinan, Lloyd, & Epstein, 1981, p. 42).

Indeed, as Morse, Cutler and Fink (1964) state in their now classic study, "it is a matter of common clinical observation that many such children are academically retarded" (p. 33). There is even evidence for the corollary of this observation: Ross (1964) and Kauffman (1985) both note that it is difficult to find a child with ESD who is academically advanced.

Interestingly enough, however, the highest score I have ever seen in both regular and special education (12.0) was achieved by Danielle, a very disturbed grade 9 girl in the residential treatment centre. She had been sexually abused, and her problems included chronic sexual acting out, poor impulse control, and other delinquent behaviours. Her case would merit qualitative study as an example of a young person whose environmental traumas, behavioural problems and severe emotional conflicts certainly had no negative effect on her academic performance in mathematics.

In other words, one should not jump to the conclusion that all pupils with ESD tend to fare poorly in mathematics. Nor should one confuse mathematics *achievement* (as measured on standardized tests) with mathematics *ability*. For example, my VRLF students generally scored poorly on the WRAT, but as I point out later, they constantly impressed me with the level of mathematics which they were capable of doing. Also, consider the brilliant mathematicians who experienced emotional and social problems in their lives. Georg Cantor suffered from mental

depression and nervous breakdowns. The great Newton himself was "a neurotic and misanthropic adult, one who rarely experienced the warm glow of human friendship" (Dunham, 1990, p. 160). The names Cardano and Galois also spring to mind when considering the combination of mathematical genius and mental instability. And would it not be entirely reasonable if any one of the above-mentioned geniuses experienced negative school experiences in his youth?

But the real issue is not mathematics deficits per se, but rather the relative extent and size of these deficits. For the recorded achievement levels among alternate school students are *profoundly* low, and educators definitely need to consider the implications of such scores for this population.

One may object to the validity of the particular assessment tools by which institutions determine academic deficits. As well, one may find the mathematics on such tests meaningless. Indeed, all my previous alternate schools used the results of each individual's entry test to partly determine the individual math program for that student. In other words, the rote-level mathematics which a student missed on the achievement test was the same mathematics which made up that student's curriculum. More importantly, the constant repetition over the years of poor achievement test results by low-achieving students can be devastating to their self-image. Perhaps it was fear of yet another failure which prompted one VRLF student's outright refusal to write the WRAT with the rest of the study group. Minna's reasons were stated succinctly: "I can't do that. I forget how to do that. So I won't do it!" Another student, Bob, asked me, "What does WRAT stand for? Does it mean I'm stupid?" These negative reactions suggest that such tests are hardly innocuous, even if the results are kept confidential and are not used to create individualized mathematics programs.

This conclusion obviously begs the question, "Why bother with achievement tests at all then?" My purpose in giving the WRAT to the VRLF group was to hopefully prove a point: that

is, low achievement test scores do not necessarily correlate with mathematical ability. But since my students would not have believed this conjecture a priori, I decided that this was the last WRAT test I would ever administer.

The WRAT scores of the VRLF study group were not good indicators of which students would accept, or which students would reject, my new mathematics topics. For instance, I put tremendous stock in Ron, who scored the highest grade level of the group (9.8). Unfortunately, after only three sessions, he was the first (and last) student to quit the project. On the other hand, Dale, who had the lowest grade level (5.3) of the cohort went on , as I shall describe later, to become a "star" performer in topics such as probability and logic.

Course failures

It is understandable why many pupils with LD continue to underachieve: they have a "disorder due to identifiable or inferred central nervous system dysfunction" (Wong, 1988, p. 15) which can defy even the most heroic and innovative interventions. Similarly, teachers expect that most mentally retarded children will be chronic under-achievers. But why do the majority of alternate students, even those of average or above average intelligence, and those with no discernible learning disabilities, have such high failure rates in school? In order to answer this question, one must look at the extent of the problem, and then at the dynamics of course failures in the lives of these young people.

Out of eleven participants in the study, only three (Minna, Ron and Chad) had not failed a secondary level mathematics course before entering VRLF. In fact, Ron and Chad were the only students of the group who completed regular Mathematics 8 and 9. Shawn had received only *Standing Granted* (SG) in Mathematics 6 and 7, and had no secondary credit whatsoever, in spite of having spent the last two years in learning assistance centres (LAC's) and on the hospital homebound program. Likewise, Nina had not completed a single secondary level mathematics course, mainly because of intermittent periods of incarceration. Without exception, the students

showed what has become an almost classic pattern in special and regular education alike: that is, an abrupt downturn in academic performance occurred around Grade 8 or Grade 9. (See Table 1.1 .)

With the exception of Dale and Alonzo, each of my students did at least satisfactorily in mathematics up to Grade 7. Then, every student but one (Adie) went into an irreversible nosedive in mathematics. Adie enjoyed success, but in Mathematics 9B and Mathematics 10B which are the ultra-modified, locally developed "basic skills" courses peculiar only to my school district. (For years, VRLF and its predecessors have used these two courses for Grade 10 mathematics equivalency; these are the same courses which my curriculum was designed to replace). It should also be noted that in most cases, grades in other subjects fell in a similar fashion.

Table 1.1 below gives a more complete picture of my students' academic records prior to entering VRLF. The occurrence of two letter grades under a given course indicates that the course was attempted twice. (Records for Nina and Shawn were unavailable.)

Table 1.1   Student Mathematics Records Before Entry to VRLF

MATHEMATICS COURSE

	5	6	7	8	9	10	11
Donny	C	C+	C	F			
Chad	A	B	B	C	P		
Adie	C	C	P	F,P	B	C+	
Juliana	B	C	C+	F			
Dale	C	C	F	F,P			
Alim					C	F	
Alonzo	D	D	C-	W,C+	F		
Minna	P	C-	SG	P	P		
Ron			A	A	P		

Conventional wisdom explains this sudden turn for the worse in part as student reaction to unfamiliar, bigger, more impersonal secondary schools and also as the result of incipient adolescence. However, I believe that these explanations are not sufficient to explain the difficult

plight of our alternate school students. Unfortunately, mathematics underachievement and course failure is not a simple, independent symptom which can be treated with a quick-fix motivational scheme and a few curricular modifications. Nor will the problem simply "go away" as students with ESD mature or become better adjusted to high schools. As I shall argue later, one important contributory factor to student failure in high school is the nature of the present mathematics curriculum, especially the courses offered to many troubled students under the rubric "general mathematics." More seriously, alternate school teachers are faced here with the phenomenon Eli Bower (1980) called "an inability to learn that cannot be explained by intellectual, sensory, or health factors" (p.22). Although this description is quite similar to the primary characteristic of learning disabilities, most of the alternate school students at VRLF do not have LD. Instead, their underachievement is a chronic problem which is inter-related with at least two other characteristics of ESD: lack of self-esteem and poor social skills. And as I shall point out later, educators must consider this inter-connectedness between different temperaments as they plan interventions and innovations to help students learn mathematics.

2) Lack of self-esteem

Self-esteem is defined by Webster's Ninth Dictionary (1987) as "a confidence and satisfaction in oneself." Thus if one lacks self-esteem, one has a low opinion of oneself. Now self-concept is the perception one has of one's roles, talents, and identity. For adolescents, roles impinging on self-concept could be student, son, daughter, boy-friend, girl-friend, hockey player, figure skater, and so on. For the alternate school student, low self-esteem could involve self-judgements such as incompetent, stupid, and inadequate.

One can easily speculate how chronic academic failure in mathematics might contribute to lack of self-esteem, and vice versa. After repeated failures and eventual relegation to the nether world of general mathematics, some students blame themselves for being "stupid." They become disillusioned, constantly predict failure for themselves in mathematics, and refuse to make any

further efforts. These students become caught in a vicious circle of failure, self-criticism, anger, and withdrawal from the learning situation. Their thoughts of failure soon become self-fulfilling prophecies; they will indeed fail, and the cycle will start anew, to end only when they drop mathematics or drop out of school entirely.

Other students' reaction to repeated academic disasters can vary slightly, but the result is the same: loss of self-esteem. For instance, faced with the prospect of having university personnel observing the case study at VRLF, Minna responded negatively: "I'm so lousy at math. I missed a lot of school." Together, this fear of failure and anticipation of failure often have tragic results: eventually, she withdrew herself completely from learning at VRLF. So for Minna the self esteem/failure cycle came full swing.

Finally, some alternate school students react to chronic failure with comments like the one Juliana made when provoked by Ron:

Ron: Men are better at math. I'm better at math than you are.
Juliana: Math is memorization. What's so important about math anyway?

Instead of reducing their self-esteem, they chose to change their *self-concepts* with regards to mathematics. This school subject now logically makes no sense to the role they have conveniently chosen for themselves. However, their attitude ("I can't do mathematics but who needs it anyways?") results in the same refusal behaviour as the two cases outlined above: from the teacher's point of view, they are all withdrawing from their learning situations. As Schoen and Hallas (1993) comment, "repeated failure and endless frustrations lead them to a sense of powerlessness, fear, and shame" (p. 113).

A link between self-esteem and attitude to learning mathematics was found in a study conducted by the American Association of University Women (1991). In a survey of 2400 girls and 600 boys aged nine to fifteen, these researchers discovered that students who enjoyed mathematics and science generally had a positive sense of self-worth. But they found alarming

implications for the math education of adolescent girls. In elementary school, 67% of boys and 60% of girls stated "I always feel happy the way I am." However by secondary school, 46% of the boys and only 29% of the girls reported that they still felt that way. Paralleling this decrease in self-esteem was a decrease in the girls' attraction to mathematics. In the elementary school, 84% of the boys and 81% of the girls stated that they enjoyed mathematics. But by secondary school, these numbers declined to 72% for the boys and only 61% for the girls. Thus, as well as being more at risk than boys for poor academic achievement in mathematics (Feldhusen et al., 1970), adolescent girls seem less inclined to enjoy the subject than their male counterparts.

There are many emotionally healthy students who fit the "I can't do math, I'm afraid of math, or the I refuse to do math" categories of mathematics withdrawal. However, for the student with ESD, the symbiotic relationship of reinforced failure and downward spiralling self-esteem usually permeates most other school subjects and activities. For example, according to Canfield and Wells (1976), a negative self-concept is a better predictor of failure in reading than is low IQ. Finally, as Weaver and Morse (1981) write,

> ....[these] youngsters live mostly in anxiety, defensive negativism, and failure and are continually grappling with a sense of inadequacy. The positive effect from learning is seldom theirs. They are behind their potential ability in actual achievement. School often connotes failure, and the teachers are the taskmasters who "set you up to fail again." For most of them school is far from a happy place.
> (p. 99)

And unfortunately, although indications of low self-esteem usually come into sharper focus in the learning environment, these feelings can permeate the lives of students with ESD outside of school as well. Troubled teenagers' internal feelings of inadequacy and failure coexist with a perception of the *whole external world* as a "hostile environment in which they will seek to cope rather than [as] an exciting, challenging field in which they will be able to demonstrate their skills and abilities" (Larcombe, 1985, p.32). In particular, pervasive negative feelings can also be turned *outwards*, in the direction of family members, peers and authority figures such as mathematics teachers (as will be reported in Chapter 4). Thus, in concert with self-efficacy and

self-esteem, there exists a third interdependent aspect of the alternate school adolescent's profile: the social dimension.

3) Socialization problems

Maslow (1987) posits that achievement and social recognition are essential factors in the development of self-esteem. Given that the mathematics achievement of alternate school adolescents is low, one might ask, "To what degree does lack of social recognition or of social contact in general, place these students at risk emotionally?"

Unfortunately, just as the sine qua non of the learning disabilities field is inappropriately low academic achievement, the ESD domain is characterized by deep, chronic unhappiness and by *serious socialization problems*. These latter problems can be considered under the following four general headings: social self-image, personal relationships, language problems and social cognitions.

Social self-image

Social self-image, which is an extension of self-esteem, is the perception people have of themselves socially. Just as students with ESD generally have a poor concept of their academic abilities, they also have negative feelings about their own social status. The "I can't do mathematics" statements are paralleled by declarations like, "I'm unpopular," and "I can't get along with other kids." More seriously, the remarks "I hate mathematics" and "Who needs math anyway?" have their respective social counterparts: "I dislike other people" and "Who needs friends anyway?" Thus social self-image for children with ESD has two aspects: the first is a feeling of not being accepted by other people, and the second is a reciprocal feeling of dislike or indifference to others. If these sentiments are directed towards the mathematics teacher, we can see yet another factor to explain alternate school students' poor performance in the subject. That is, not only do they hate mathematics, but they might hate the mathematics teacher as well.

In their study, Roberts and Peterson (1992) examined the relationship between academic achievement and social self-image during early adolescence. Surprisingly, they found that for non-academic girls, there was an inverse relationship between social self-image and mathematics achievement; in other words, relatively *large* gains in social self-image were associated with *low* mathematics scores in these students. The authors suggest that popularity among girls is based on peer norms: girls who are more successful than their peers are not sought out as friends. Or perhaps girls are concerned with the negative social implications of success in mathematics: girls who are "brainy" in math must also be "nerds" or "egg-heads." For boys, the same conflict appeared only during the changeover from sixth to seventh grade. No conflict between social self-esteem and achievement in other academic subjects was found.

This study raises several questions about the mathematics education of alternate school girls. Might they attempt to boost their low social self-images by deliberately sabotaging their own achievement in mathematics? Do many bright but troubled girls give up mathematics careers or mathematical excellence as a trade-off for peer acceptance? Or, already faced with a record of failure in the subject, might they seek out as friends those peers with similar academic records? If the answer to any of these questions is "yes," then the implications for the reinforced failure of alternate school girls in mathematics are grave. Clearly, a replication of the study is warranted, including as subjects boys and girls with emotional and social disabilities.

Alternate school adolescents' feelings of academic inadequacy are usually well-rooted in the stark reality of repeated and well-documented (e.g., report cards) failures. But how well-founded are their typically negative social self-images? Combs and Snygg (1959) state,

> [f]rom birth to death man is continually engaged in the search for greater feelings
> of adequacy. Whether or not he is successful in this quest will be determined by
> the perceptions he is able to make in the course of his lifetime....Inadequate
> people have grave doubts about their capacities to deal with events. Their
> experience has taught them that they are more often than not unliked, unwanted,
> [and] unacceptable....(in Larcombe, 1985, p. 55)

This has implications for the social experience of alternate school youth: how they act with other people, how others react to them, and the reasons behind their social isolation.

Personal relationships

Alternate school adolescents have difficulties in forming and maintaining social relationships. In the search for causes of this phenomenon, one must be cautious because causality, especially in the social/emotional realm, is difficult to establish. As Hallahan and Kauffman (1982) remark about emotional and learning problems, "attempting to find out which causes which can be a frustrating and fruitless endeavor" (p. 47). Again, one will not discover *causes*, but rather possible *correlates* which affect each other in a "spiral of reciprocal negative interactions" (Hallahan & Kauffman, 1982, p. 190).

A number of research studies indicate that poor peer relationships are associated with low academic achievement (Bursuck, 1989; Lindzey & Byrne, 1978, in Bryan, 1976; Lilly, 1970, in Bruininks, 1978). Indeed, Wilchesky and Reynolds (1986) state, "In our clinical work with LD children, it is difficult to remember any child who was not suffering in that way" (p. 411).

One can easily speculate how academic failure might contribute to social ostracism. First, an adolescent's poor achievement could result in his being perceived as "different" by peers and adults (Hoyle & Serafica, 1988). Likewise, gaining adequate grades is *usually* valued by students, but performing poorly could earn a child with ESD the social disdain of his peers.

Repeated academic failure can result in frustration and lack of self-esteem. From these negative feelings, undesirable behaviours such as social withdrawal, distractibility and hostility can develop (Lerner, 1985). As well, a failure-ridden student might be hesitant to take an active part in learning experiences with his/her classmates. Thus opportunities for social as well as academic practice are missed. As Tanis Bryan (1983) and her colleagues of the Chicago Institute found, a failing child "avoids new challenges and engages in maladaptive behaviour rather than

risk failure or public disclosure" (p. 17). This research also discovered that such a student usually attributes his/her own failures to personal lack of ability, and his/her successes to luck or to task easiness. So, devoid of any opportunity to raise his/her self-esteem, or to impress others through personal academic success, the youngster gets caught in a vicious cycle of academic failure and social problems.

Research has also shown that hyperactivity, another common ESD trait, was positively correlated with social rejection in an elementary school population (Flicek & Landau, 1985, in Kistner & Gatlin, 1989). The latter researchers found that 71% of hyperactive students were classified as "rejected" socially, whereas only 30% of non-hyperactive students were so classified. Indeed, such were the social plights of Dale and Juliana, two hyperactive members of the VRLF study group whose interesting behaviours are discussed in Chapter 4.

In mathematics classes, youngsters with ESD frequently have trouble paying attention to what is important, and this inattention extends to the social realm (Osman, 1979). Also, as hyperactive adolescents, they have been known to impulsively blurt out rude, shocking and inappropriate comments such as "Boy, are you ugly!" They do not see their behaviour as inappropriate (Hallahan, Kauffman, & Lloyd, 1985), nor do they care about the consequences of their behaviour. Thus, a normal child might describe a classmate with attention deficit disorder (ADD) in the following way:

> She can't stick to one math topic for too long. She doesn't pay attention when you talk to her, and she can't wait her turn to talk in our group. She makes the teacher mad by calling out in class, and she says the weirdest things right out of the blue.

Understandably, such a student's behaviours in a mathematics class would probably not win her many friends.

Likewise, adolescents with more severe mental illnesses such as neuroses and psychoses usually display behaviours which could result in the alienation of peers and adults alike. For

instance, neurotic teenagers are given to diverse symptoms such as phobias, obsessions, compulsions, fatigability, hysteria, hypochondria, depression, depersonalization, and anxiety. Although these conditions are mainly subjective, they often result in behaviours which society finds unwarranted and unpleasant. In the same way, psychoses such as schizophrenia are characterized by bizarre, regressive or withdrawn behaviour, misinterpretation of reality, and loss of empathy to others. Students with this serious and tragic affliction are not only at risk socially, but also suffer from reduced intellectual functioning. In sum, their presence in an alternate school mathematics classroom is not rare, and their teacher should be forewarned of their enrollment and should be prepared for the inevitable learning and socialization problems these students will bring with them.

Two other dimensions of social disabilities under Quay's (1979) classification system are the *anxiety-withdrawn* group and the *immaturity* group. The first dimension includes students who at first appear to cause no problems whatsoever in the classroom. Shy, self-conscious and withdrawn, they often fade into the background of any adolescent learning group, but lacking social confidence, these students are easily identifiable as friendless loners. In the classroom, these aloof young people are very content to sit passively and listen quietly to the mathematics teacher lecture. In Chapter 4, I describe one student with many of the above characteristics.

On the contrary, students in the immaturity dimension do attract attention to themselves, but in rather unique ways. Although in my opinion this classification describes nobody in the VRLF cohort, it deserves mention here because its occurrence in alternate school mathematics classes is otherwise quite common. Passive, sluggish, and inattentive, these pupils typically have poor coordination, and are slow to finish their work. Also, they can lack initiative, and as such can be easily led by peers. Seemingly preoccupied, the immature students lack interest in both individual and group school activities and thereby are at risk both mathematically and socially.

The last classification in Quay's (1979) scheme is the *socialized aggression* group. Commonly known as juvenile delinquents, adolescents in this category have been romanticized as James Dean-like "rebels without a cause." Typically, one hears of these students because they run afoul of the law through such activities as rowdy behaviour, stealing, vandalism, fighting or drug use.

Of the eleven students in my project, six had past histories of violence or aggressive behaviour towards peers. Three were highly suspected by school board officials of gang involvement . As well, three had been convicted under the Young Offenders Act of theft. Eight of my students had records of drug use or dependency.

Oddly enough, these behaviours often occur in a peer group context, hence the label *socialized*. For although these young people frequently belong to gangs, engage in cooperative stealing, and keep "bad company," they do keep company nonetheless. Loyal to delinquent friends, and accepted by delinquent sub-groups, they can hardly be called loners. Indeed, one might question whether they should be classified as *socially deviant* at all.

Nonetheless, the label *anti-social* can be applied to those behaviours which seriously violate society's norms and legal sanctions. And in any mathematics classroom, if there are any instances of disruptive behaviour and foolishness, the socialized-aggressive teenager will be in there with both sleeves rolled up. For the alternate school mathematics teacher, there are two important considerations with this kind of student. First, the classroom atmosphere can be at risk, since through negative peer pressure, other students often tend to admire and to imitate a boisterous and aggressive personality. Second, the problem teenager is at serious personal risk not only for quitting or failing mathematics, but also for dropping out of or being expelled from school entirely. The future bodes poorly for these adolescents: undereducated and socially maladroit, they can face chronic unemployment, poverty, health problems, and careers as criminals and prisoners.

Language difficulties

Language difficulties in children with ESD can be considered in two parts: the first involves actual linguistic skill deficits involving forms and codes, and the second concerns problems in pragmatics which is the performance of language in social situations.

Linguistic skill deficits can mean that some students, especially the learning disabled, cannot find the right words to communicate. They can experience problems with syntax, semantics and morphology which in turn cause their listeners to become uncomfortable and hesitant about engaging in further conversation (Kronick, 1976). Likewise, students who use strange or incorrect language might cause their peers to think of them as unintelligent or incompetent. Also, poor receptive language skills could mean that adolescents with ESD will miss out on the meaning (and inevitably the use) of idioms and figures of speech which are conducive to bonding in young people. Last, these pupils could chose to withdraw from embarrassing situations where language skills are required, and thus removed from oral practice, their linguistic problems are further worsened. Thus we see another potential spiral of deleterious interactions in the realm of emotional and social disabilities.

The consideration of such language difficulties is important as more and more teachers plan to engage their students in mathematical *discussion*. Indeed, as Pimm (1987) points out , "discussion has quickly become a new panacea, part of the new orthodoxy....[However], despite the apparently innocuous request to increase the amount of discussion in mathematics teaching, this has not been achieved" (pp. 45-46). Perhaps one of the reasons for the lack of successful implementation of discussion in mathematics classrooms is that some students lack the linguistic ability to converse generally, let alone mathematically. In particular, closer attention should be paid to students with LD and ESD who enter alternate school mathematics classrooms with serious linguistic deficits. How teachers might better prepare such students linguistically will be discussed later.

Closely related to linguistic competence is the idea of pragmatics or communicative competence, which "involves knowing how to use and comprehend styles of language appropriate to particular social circumstances" (Pimm, 1987, p. 4). To investigate problems in pragmatics, educators of children with ESD can turn to research in the overlapping field of learning disabilities.

Two of the students in the VRLF study group (Dale and Alonzo) had been clinically diagnosed with learning disabilities in written and oral expression. Similarly, a third student (Shawn) had long been recognized with severe LD in both mathematics and language skills. In Chapter 4, I consider in detail the implications of learning disabilities as these three students learned mathematics in a classroom where writing and speaking were encouraged.

It is now universally acknowledged that children with LD frequently experience social and emotional problems. Indeed, there has recently been considerable discussion to promote deficits in communicative performance as *primary* learning disabilities (Newcomer, 1989a; Gresham & Eliot, 1989a). The following symptoms have been noted regarding the pragmatic deficits of youngsters with LD. First, they may violate the "personal space" of others during conversations (Valletutti, 1987). They are poorly skilled at initiating a dialogue (Donahue & Bryan, 1983) and they are less able to maintain a conversational flow than their normal peers (Donahue, Bryan & Pearl, 1980). To maintain a conversation, they use unsatisfactory strategies which differ from those of their peers (Mathinos, 1988). Last, children with LD may be unable to modify their communications when speaking to different audiences (Bryan & Pflaum, 1978). It is obvious that these students can have difficulties in social expression which cannot be remedied by extra grammar exercises in a learning assistance centre.

Many progressive and well-intentioned mathematics teachers are trying to replace written tests with assessments of oral performances. As they plan the logisitics of oral testing, they should also consider the deficits in communicative competence that will hinder many of their

students. And as I shall later point out, the poor verbal skills of my students with LD had to be taken into consideration as I relied more and more on speak-aloud protocols as a means of assessing student learning during the study.

Social metacognitions

In a manner similar to communicative incompetence, many students show definite problems in social awareness. For example, they may not perceive such subtleties of social interaction as facial expressions, body gestures, and tone of voice (Osman, 1979; Little, 1993). They often have difficulties reading how others feel (Lerner, 1983) and how others feel towards them (Hoyle & Serafica , 1988). Just as they fail to learn academic mathematics, adolescents with ESD do not seem to learn from social experience: they make repeated mistakes in social-interpretation situations. Unable to recognize social cues, they appear strange because they cannot make their behaviour fit the social circumstances (Hallahan, Kauffman & Lloyd, 1985). Furthermore, with the rejection resulting from this lack of social awareness, they will be denied the normal contact they need to grow socially and emotionally. Their future often looks poor: coupled with poor social skills and an angry temperament, they are at risk for dropping out of high school, juvenile crime, and for adult psychopathology (Gresham & Eliot, 1989b). Such students must be identified by our schools, so that in academic situations such as mathematics classes, they can be given the opportunity "to learn to be normal."

The socialization problems discussed here take on an even greater importance when considered in the light of modern constructivism. The constructivist view of mathematics education posits that

> learning mathematics requires construction, not passive reception, of knowledge, and to know mathematics requires constructive work with mathematical objects *in a mathematical community*....The role of the community is to provide the setting, pose the challenges, and offer the support that will encourage mathematical construction. (Davis, Maher, & Noddings. 1990. pp.2-3). (Italics are mine.)

Instead of considering knowledge as absolute, formal and "out there," constructivists see it as individually constructed and "socially mediated". As well, learning is not viewed as "getting knowledge from the outside," but as "constructing knowledge in a social context" (Jaworski, 1992). According to Cobb, Wood and Yackel (1990), "social interaction...constitutes a crucial source of opportunities to learn mathematics...and as such, mathematical learning is, from our perspective, an interactive as well as constructive activity" (p. 127). Thornton, Tucker, Dossey and Bazik (1983) write of the "vital role of social interaction in the growth of the ability to deal with knowledge" (p. 48). Finally, the NCTM (1991) explains that

> mathematics is learned in a social context, one in which discussing ideas is valued.
> Classrooms should be characterized by conversations about mathematics among
> students and between students and the teacher. (p. 96)

For these reasons, I have called my project a "social curriculum" and throughout the study have tried to focus on student talk as paramount as my alternate school students did their mathematics.

The educational implications for those students with poor social interactions and low levels of linguistic communication are clear: these pupils are at far greater risk for failure in mathematics than we ever imagined ten or fifteen years ago. For unless proper considerations are given to these students, and unless appropriate interventions are made, they could fare just as poorly under a constructivist teaching modality as they did under a lecture/example/practice routine. Teachers are faced with a conflict: some students are not social, communicative people, whereas "mathematics is, among other things, a social activity, deeply concerned with communication" (Pimm, 1987, p. xvii).

4) Pervasive mood of unhappiness.

I turn now to the common trait which underlies all the aforementioned academic, self-esteem and socialization problems: alternate school students are often unhappy. As Hallahan et al. (1985) state, "their lives seem filled with sadness that squelches pleasure in nearly every

situation" (p. 141). Unfortunately, society lets many of these adolescents remain in place until they get "bad" enough to treat.

And for some unfortunate youngsters, the sadness develops into a chronic and pervasive form called clinical depression. Known as melancholia in centuries past, this psychiatric illness is manifested by painful dejection, low self-esteem, difficulty in thinking, psychomotor retardation, and social isolation (Maag & Forness, 1991). Here again is an instance where poor intellectual functioning, low self-worth and poor social skills can appear simultaneously in the same individual.

At Vale Road Learning Facility, the incidence of mental depression has increased alarmingly since 1988 when I last taught in the alternate program. At least three of my study group students were receiving psychiatric counselling for depressive illnesses My colleague, who has taught full time at VRLF for the last four years, and who has extensive experience in special education, estimated that fully 72% of the students who enrolled in his class this year were suffering from serious depression.

So what can mathematics do for alternate school students for whom the learning of mathematics is indeed tied into an intricate web including personal emotions and social relationships? My hope in developing a new approach to mathematics education was to turn what for many students is a boring, threatening, and anxiety-causing subject into an exciting, satisfying, and ego-boosting experience. In the next chapter, I consider how alternate schools might best serve students mathematically.

Chapter III

Teaching mathematics to alternate school adolescents

Given the rather negative descriptions of the last chapter, it is not difficult to understand why many mathematics teachers resist the placement of pupils with ESD in their classes. Indeed, teachers often maintain that emotional and social problems are not part of their mandate or expertise. In frustration, they may be heard to bemoan, "I deal with mathematics, not with psychological problems!" In more than just a few cases, over-zealous mainstreaming by some districts has resulted in the regular classroom *mis*-placement of some severely disturbed youngsters. However, I believe that for *any* students who are in *any* mathematics class, "psychological problems" cannot be separated from mathematics teaching and learning. Also, I submit that through pedagogical innovations and modifications, the alternate school mathematics teacher in particular can avoid adding to the problems of students, and conversely, can contribute to their therapeutic and learning processes. Although I am making these proposals with special education in mind, some of them are derived from certain modern theories in regular education and indeed, most are not radically foreign to regular teaching.

In this chapter, then, I discuss curricular considerations for alternate school adolescents under the following headings: the theory of interactionism, desirable characteristics of the teacher, government policies, the educational locale, a rationale for a new curriculum, what mathematics should be taught, and finally, how it should be taught.

The theory of interactionism.

Underlying my ideas on mathematics interventions is the theory of interactionism. This concept posits that behaviours are a result of the interaction of a child's own characteristics (referred to as his/her temperament), and the demands, stresses and expectations of his environment. According to this theory, consonance between the child's capacities, behavioural styles and motivations on the one hand, and the environment on the other, results in what is called

a "goodness of fit." However, where dissonance between child and environment exists, there is a "poorness of fit" which can lead to maladaptive behaviour and poor development (Chess & Thomas, 1984).

The following are examples of how poorness of fit could occur in the classroom. A hyperactive girl has difficulty when she is forced to sit still for long teacher presentations. A pupil who by nature is "slow to warm up" reacts poorly when the teacher regularly begins the class with lively opening drills. And a learner with low persistence displays inappropriate behaviours when he is given an overly drawn-out mathematics problem.

Unlike behaviourism, which manipulates the consequences of behaviour, or the psychoanalytic approach, which attempts to uncover underlying mental pathology, interactionism proposes that teachers handle the child in a manner appropriate to his/her temperamental characteristics. For instance, the teacher in the first example above could help accommodate his hyperactive student by lecturing less, or by allowing her to get up and stretch once in a while. The second teacher might tone down her drills, conduct them less frequently, or allow her pokey pupil to sit them out. And shorter mathematics projects would suit the temperament of the third learner. Note that the teacher need not necessarily alter the child's entire learning environment. Rather, in considering an individual student's temperament, the teacher makes minor adjustments which allow the student to behave appropriately, remain in the regular classroom, and have the opportunity to mature in temperament through peer modeling.

In other cases, more radical changes must be made to suit the individual. For instance, resource rooms and isolated alternate schools like VRLF are sometimes necessary as learning environments for certain students. For many, separate sites away from the bells and regimens of regular high schools offer a haven where learning can eventually occur. Finally, there are students for whom temperamental considerations are not enough; often disturbances need to be

treated from the point of view of genetic, biochemical, neurological, perceptual, or cognitive factors. As Thomas, Chess, and Birch (1976) state,

> [t]he child's temperament is only one of the many issues to be considered by professional workers concerned with the prevention of pathology in psychological development, though often an important one. As our findings have demonstrated, the degree to which parents, teachers, pediatricians, and others handle a youngster in a manner appropriate to his temperamental characteristics can significantly influence the course of his psychological development. The oft-repeated motto, "Treat your child as an individual," achieves substance to the extent that the individuality of a child is truly recognized and respected. (pp. 386-387)

Alternate school mathematics teachers: desired traits

At times, there can be a poorness of fit between a student and a teacher. Commonly called a "personality conflict," its occurrence prompts the question, "What teacher traits constitute a *goodness* of fit between a mathematics educator and an alternate school student?" In other words, what are the characteristics of an alternate school mathematics teacher which result in a positive teacher/student relationship and in the optimum learning of mathematics?

In general, research cannot provide definite answers to the above questions. Kaufmann and Wong (1991) conducted a research review to determine whether generic (i.e., regular education) skills are sufficient for dealing with students with behavioural disorders. They found that "at best...the distinctive features of effective teachers of students with behavioural disorders remain hypothetical" (p. 226). In a review of research on the preparation and characteristics of teachers of students with ESD, Zabel (1987) found that "most existing literature consists of opinion" (p. 172). The following, therefore, is my *opinion* of valuable teacher traits in the mathematics education of alternate school youths.

First, I wish to stress the importance of the teacher's relationship to mathematics itself. By relationship, I mean specifically the teacher's concept of, his competence in, and his enthusiasm for mathematics. These three attributes are important factors in education, because

they determine which mathematics is taught, how it is taught, and inevitably, if the students themselves will value it. And the value students place on the mathematics presented to them will to a great extent affect their decision to learn it or not.

Primarily, a teacher of alternate school pupils must see more to mathematics than arithmetic computation, right and wrong answers, and the rote learning of rules and facts . On the contrary, his concept of the subject should include the view that mathematics involves people tending to relationships between things, and not just the things themselves. With this broader perspective, the teacher can provide mathematical opportunities which transcend the usual humdrum content of most curricula for special students.

As well, like all mathematics educators at the secondary level, alternate school teachers should have the competence to acquire new mathematical knowledge, to appreciate the concepts and processes inherent in mathematics, to see the connections between the various branches of mathematics, to recognize the relationships of mathematics to other disciplines, and to feel confident in their ability to do mathematics. These competencies are especially important in light of the new topics for "general" mathematics such as those chosen for my VRLF project. Proficiency in mathematics is required for enhancing and modifying instruction in difficult special education situations. In other words, how else can "the subject matter be developed from different perspectives and in several alternative ways so as to accommodate students with different backgrounds and learning styles" (Committee on the Mathematical Education of Teachers, 1991, p. 3) ?

Just as important as competencies are the *attitudes* of teachers towards mathematics. Simply put, qualified teachers love their subject. They not only appreciate the beauty and fascination of mathematics, but are enthusiastic about knowing, learning, using, and teaching it. Such attitudes are important for students and teachers alike. For alternate school youths, with their typically negative or neutral feelings towards math, it is important that love of the subject be

demonstrated by their teacher as they learn. For teachers, with all the concomitant pressures of the profession, the enjoyment of what they teach is a vital part of their psychological well-being. Thus, I believe that positive learning atmospheres are impossible without teachers' personal high regard for mathematics.

The second important aspect I wish to consider involves teachers' attitudes towards methodologies in mathematics education, or more importantly, towards *changes* in methodologies. Mathematics teachers are being asked to adopt a new "curriculum and environment...that are very different from much of current practice" (NCTM, 1991, p.1). As I shall discuss shortly, these advocated changes are very compatible with my proposals for teaching techniques to serve the special needs of alternate school pupils. To be effective, teachers of these students should be willing to experiment with new curricular methods. As Hadley (1992) writes, "the teacher must be flexible and confident enough to become the facilitator of learning instead of the dispenser of information" (p. 263).

In this new role, an educator must also be willing to relinquish the traditional teacher-centred authority of a lecture-based classroom to the students, who are invited to engage in open-ended problem solving and to converse freely in pairs or groups about the mathematics they are doing. For some teachers, the decision to give more control of the classroom agenda to students will be difficult indeed. But the potential gains in terms of increased inter-pupil communication and pupil self-direction render the status quo unsatisfactory.

However, teachers cannot be expected to just step into the new roles advocated by the NCTM. As Schoen and Hallas (1993) ask,

> [w]hat new management techniques will a teacher have to learn in order to
> implement such changes as using technology, more challenging content, and
> small-group teaching methods? Will the risks be worth it for the teacher? Will the
> necessary teacher education and administrative support be provided? How?
> (p.116)

I shall look closer at the notion of funding for teacher inservice training in the next section on government policies.

Many teachers of modified mathematics courses do not possess the traits of competence and positive attitude: BC schools routinely force non-mathematics specialists to teach mathematics to homogeneous groups of low-attaining students. Now, a school board would not hire a non-mathematics specialist to teach Honours Mathematics 12. In my opinion, the placement of inexperienced and often unwilling teachers in modified mathematics courses is indicative in itself of a myopic administrative view of mathematics and also of a low premium placed by school districts on the worth of pupils with ESD. Thus, to ensure equal "opportunity for all" (NCTM, 1989, p.4), I contend that *all* students should receive their mathematics education from teachers with mathematical expertise.

A third area of concern is how teachers should relate to alternate school students themselves. Just as it is necessary that a teacher love her subject, it is equally imperative that she have a sense that all children, even the most obstreperous or obnoxious, are equally deserving of a quality mathematics education. (The idea of a "quality" mathematics education for alternate school children will be addressed shortly.) As Weaver and Morse (1981) comment, "to care even when the child is bad is the difficult remedial stance for the teacher" (p. 103).

Such a stance requires *empathy* for, or at least understanding of, alternate school individuals. By empathy is meant an "identification of oneself with another" and the resulting capacity for one to answer self-questions of the form,"What would I be like if the same things happened to me that happened to John?" This ability often requires that a teacher be informed of psychological phenomena such as the usual consequences of grievous wrongs inflicted on a young child, or the unfortunate symptoms of neurochemical imbalances beyond a student's control.

For example, consider a teacher who knows that his new student just came from a home where she was sexually abused. This teacher also understands that verbal rudeness to adult males is a common reaction of abused children. Thus informed, the teacher finds it easier to dig deeper into his reservoir of patience in dealing with the new girl's behaviour, and the student is thereby given the time to adapt to her new teacher and learning environment.

Therefore, as a necessary condition for student empathy and understanding, mathematics teachers' knowledge of the special needs of alternate school students is important. And obviously, such knowledge is also required in planning curricula for these pupils. School districts are thereby faced with the serious obligation of providing in-service training in special education for regular teachers assigned to alternate sites. Alternate school students deserve nothing less than specialist teachers who are also well informed about special education.

We all tend to remember courses taken in the past not so much by the content covered, but more by the teacher who taught it. My alternate school students, in particular, chose to lay the blame for their poor academic records not upon themselves, or upon the curriculum, but rather upon their former secondary school mathematics teachers. In a survey conducted relatively early in the project, students were given the following task:

> *Try to visualize yourself in the last math course you took. Write down two things which could have made that math experience better than it actually was (in your opinion).*

Seven students responded to the task, and out of the fourteen possible remarks, nine were critical of the mathematics teacher. The following were their comments:

- The teacher could have been more charismatic.
- If the teacher wasn't such a bore it would have been better.
- If the teacher did more with the class.
- If the teacher made math much more enjoyable and if he helped you out when you were stuck.
- The teacher was stiff, with no sense of humor didn't really take alot of time to explain things.
- No teacher to teach me anything therefor [sic] Math was done on my own which was boring.
- my teacher was an idiot.

•If he would have helped us out more.
•it would be better if he wasn't such a socialist.

My students, at least, appeared to place high value on the teaching styles and levels of commitment of their former mathematics educators (not to mention political affiliations).

Finally, I wish to discuss perhaps the most important single trait (after empathy or understanding) that a teacher of alternate school students should have: a sense of humour. I am not referring to the stand-up comic type of humour which in my experience, shy pupils withdraw from, and rowdy students try to outdo. Nor do I mean the Bonzo the Clown style of goofiness which most adolescents find immature. To be effective, a teacher's sense of humour should be relaxed and natural: used properly it should convey to disturbed students the message that nothing, especially mathematics, is really worth being unhappy over for too long. After all, no person is going to get sick and die if the mathematics syllabus is not covered to Ministry specifications. With a mildly self-deprecating wit, teachers can demonstrate to alternate school pupils that it is all right to laugh at oneself. A teacher can use laughter for the prevention of boredom, and comic relief for the neutralization of potentially difficult interactions with teenagers. Humour can also serve to motivate learning as students discover that mathematics and fun can occur simultaneously. Indeed, research by Bryan and Bryan (1991) has produced evidence that positive mood can increase students' feelings of self-efficacy and improve mathematical performance. Alternate school teachers should also enjoy much of their students' sense of humor: mutual laughter makes for a healthy mathematics class. Last, a sense of humour can go a long way to promoting staff morale, and is indispensable for the individual teacher's own mental health, especially in special education settings. In sum, teachers must convey to alternate school youngsters that although the world is truly mad, the easiest way to stay sane oneself is to laugh at it.

Government policies and pronouncements

While on the topic of humour, this might be a suitable point to ask the question, "What does the Ministry of Education have to say about the mathematics education of youth with ESD and alternate school students in particular?" Here, I trace the BC government's official record on the subject for the last seven years.

The Ministry of Education's (1988) Mathematics Curriculum Guide (7-12) contains no mention whatsoever of young people with emotional and social disabilities, nor even special education students. In a one page "rationale" which includes "affective domain goals," it does briefly mention positive attitudes, successful experiences, and "content and presentation ...appropriate to the increasingly diverse needs of all students" (p. viii). But how mathematics teachers are to promote success and positive attitudes is not discussed. Nor is mention made about how they are to provide "appropriate content" out of the very detailed "intended learning outcomes" which follow. Obviously, this is not a curriculum guide in the true sense of the word *curriculum*. Rather, the writers (who are mathematics teachers) should have named their tome a *syllabus* or *course content guide*. And finally, one gets the impression here of mere lip service to the affective needs of students.

In the draft document for its mammoth educational reform program (called Year 2000), the Ministry of Education (1989a) finally broadens its definition of *curriculum* to include "the set of planned learning experiences, and...the experiences themselves" (p. 10). However, it barely mentions the topic of exceptional children, and relegates the particular topic of alternate school children to the category of *"etc"* :

> Special needs children (learning disabled, mentally handicapped etc.) will be enrolled in one of the Graduation Program options but the school will be able to modify the Common Curriculum for these students....(p. 41)

As I shall point out later, the school system has been modifying the mathematics curriculum for exceptional children for years, and the poor way in which it has done so has in part contributed to their dilemma.

In its second publication of the same year, titled *Working Plan #1*, the Ministry of Education (1989b) set a timetable for school districts to "develop policies and procedures to ensure coordination and delivery of special needs services to students" (p. 45). My first reaction to these vague directives was, "Surely such policies and procedures are already in place." In 1989, the reaction of the BC Association of Special Education Teachers was a concerted submission to the Ministry asking not only for clarification of vague statements, but also for recognition of exceptional children. To my knowledge, a *Working Plan #2* has not yet been published.

However, in its *Graduation Program Response Draft*, the Ministry of Education (1990) fleshes out its philosophy on special needs students, and among other things calls for "common learning activities that favor social interaction within heterogeneous settings" (p. 58). The writers go on to state the following:

> The learning outcomes of the Graduation Program are applicable to all students; within this framework, it is possible for all students to experience success and a sense of achievement through appropriately designed educational programs. The reporting of achievements is based on...standards of achievement that are appropriate for the student. For a small number of these learners, there may be a need to adjust the specific intended learning outcomes. (p. 59)

I interpret the above comments to mean that to insure student success, educators should shift achievement standards to suit individual learners, rather than to shift the syllabus to suit learner needs. Does this signify that students with LD or ESD should receive different tests and evaluation measures than their normal peers? Or does it mean that exceptional children should undergo the same evaluation procedures, but with different norms or criteria set for reporting results? Is the Ministry declaring that the learning outcomes of the "Common Curriculum" are sacrosanct and immutable? I maintain that, along with methodologies, it is the *specific intended*

learning outcomes which must be appropriately adjusted to suit the needs and abilities of special students. Further, I feel that if the current syllabus (as outlined in the Mathematics "Curriculum" Guide) remains rigidly in place, then a large number of these learners will be short-changed.

The Mathematics Curriculum/Assessment Framework was the first Year 2000 document dealing exclusively with mathematics and was not issued by the Ministry until early in 1992. Remarkably close to the *Curriculum and Evaluation Standards* (NCTM, 1989) in language, it "does not impose a change in subject content or in the designation of key topics" (p.6) of the 1988 Curriculum Guide. Nor does it offer any new directions for the mathematics education of special need students. For teachers are told that

> ...learning opportunities should be designed to help learners...work in environments that accommodate the learner's social, emotional, physical, and intellectual needs. (p. 22)

But as usual, no explanation of terms nor elaboration is provided.

In June 1992, educators received the latest instalment of the Year 2000 initiative, called *The Intermediate Program: Foundations* (Ministry of Education, 1992b). Concerned with the education of students in grades four through ten, this document addresses rather well the interests and needs of special students. It purports to promote development in six areas: intellectual, artistic and aesthetic, physical, social responsibility, emotional, and social. In the section labeled *emotional development*, the writers express concern that some learners "may not value their own abilities, appearances, or experience, especially when compared to those of their peers" (p. 39), and that "young people who develop a strong sense of self, however, are also capable of seeing other people's perspectives and incorporating these perspectives into their understanding and behaviour" (p. 41). This section also discusses *self-esteem* in considerable detail, and describes it as "the combination of self-confidence and self-respect" (p. 41). Under the heading *social development*, it is acknowledged that some "learners may need guidance in understanding the effects of their personal actions on others. As well, they may need to learn means of solving their conflicts constructively" (p. 47). Expressing an obviously constructivist viewpoint, the writers

posit that "learning is not a passive act...[and that] it is critical that students have opportunities to compare and contrast their understanding with that of others" (p. 11). In the same vein, the document says, "The intellectual development of Intermediate learners is [thereby] related to emotional and social maturity" (p. 34). In other words, "learning...is a social process...and is enhanced by interaction with others" (p. 60).

I was gratified to see that this document gives social and emotional development equal status with intellectual development, and that it acknowledges the interdependency of the intellectual, emotional and social goals. However, its writers give no hint of how they envisage the implementation of these goals. And although educators are told that "incorporating students with special needs into classroom settings means that teachers will require support" (p. 82), the Ministry has not explained how it plans to provide such support. Finally, no mention is made of the role of alternate schools as solutions for students who cannot function at all within regular school.

Nor does *The Graduation Program Policy* (Ministry of Education 1993a) refer to alternate learning sites for students at the Grade 11 and 12 levels. However, in a continuance of the government's mainstreaming message, educators are told that "schools must be prepared... to support students with special needs. This includes...students with intellectual, emotional, behavioural or learning disabilities" (p. 2). Again, this document echoes strongly the sentiments expressed in the NCTM's (1989) *Curriculum and Evaluation Standards* : according to the government, "...graduates will...understand connections between areas of study, be competent in problem solving, critical thinking and decision making...[and]...communicate effectively" (p. 1). In particular, "[s]tudents are taught to reason logically, use various mathematical models to solve problems, and meet standards of numeracy" (p.9). In sum,

[t]hree important principles support the Graduation Program framework. These
are: • learning requires the active participation of the student.
        • learning is both an individual and a group process
        • people learn in a variety of ways and at different rates.   (p. 2)

Unfortunately, no mention is made of alternate settings as one means of providing such a "variety" of learning opportunities.

By the end of 1993, it was obvious that the political/educational winds had changed considerably since the initiation of the Year 2000 program of reform in 1988. Reacting to parental and public pressure, and nearing the end of its political mandate, the government backtracked on many of its progressive proposals. In its most current document, *Improving the Quality of Education in British Columbia*, the Ministry of Education (1993b) re-introduced "basic skills" as its new educational buzzword: "By the end of Grade 10, [students] are expected to have mastered the basics..." (p. 3). I am certainly in favour of students learning basic arithmetic skills but I object mainly to how such skills are taught, in particular to older adolescents who have failed to "master the basics" throughout their school lives. Also, I fear the introduction of new government-imposed curricula which are nothing but reformulations of what is presently taught to alternate school students.

There are a few positive signs coming from the government, however. For instance, the same document purports that

> [e]nsuring that students learn to read, write and do basic mathematics is a primary goal of schools. These traditional basics are at the centre of an expanded set of essential skills that students need in today's world. The most important of these are the ability to solve problems and to use computer-based technology. (p. 2)

As well, the government has recently proclaimed that BC schools may obtain all current NCTM *Standards* documents by the same bureaucratic procedure whereby textbooks are allocated. By providing what is tantamount to free distribution of the NCTM publications, the Ministry of Education has not only apparently endorsed the Council's leadership in mathematics education, but has also provided an effective means of communicating the *Standards'* call for reform. And importantly, in spring 1994, the Minister of Education announced an additional $30 million in grants to school districts for special education. Whether or not any of this money will be allocated

for curricular reform in alternate education, and in particular for inservice training for mathematics teachers, remains to be seen. As O'Shea writes,

> If we are serious about curriculum change, we must learn the lessons of history. Real change will require time and money spent on well-conceived and -executed plans for teacher re-education. There are no easy alternatives. (1981, p. 23)

Educational settings

According to Crawford (1992), a setting is "the relationship between persons acting and the arena in which they act." In regard to my topic, there are perennial questions regarding the appropriate choice of arena in which the players (i.e., special students and their educators) can best carry out the drama of mathematics education. In general, these questions have taken the form of a long-running debate featuring supporters of mainstreaming and supporters of segregating special learners. In other words, is it wise to place students with ESD in regular classrooms, or should they be taught apart from their "regular" peers? And if they should be taught apart, what special settings are appropriate for them?

There is no shortage of research on the efficacy of various settings for special students. There is, however, a remarkable lack of consensus as to which model stands above the rest. For instance, in regard to the self-esteem of students, there are studies favoring special class over regular class placement (Battle & Blowers, 1982; Yauman, 1980), and studies favoring regular classes over resource rooms (McKinney & Feagans, 1984; Wang & Birch, 1984). Similarly, research reviews (Leinhardt & Pallay, 1982; Sindelar & Deno, 1978), and meta-analyses (Carlberg & Kavale, 1980; Wang & Baker, 1986), offer conflicting results concerning the benefits of various placement models for children with special needs. Indeed, Leinhardt & Pallay (1982) postulate that setting is *not* a primary variable in education. Rather, we should study the complex interactions between placement options, teaching processes, and individual student temperaments and needs.

In the absence of solid empirical evidence one way or the other, I accept on a humanistic basis that the regular classroom is the desired educational setting in which to teach mathematics to students with ESD. Nonetheless, I feel that indiscriminate mainstreaming is the current vogue in many BC school districts, a phenomenon Newcomer (1989b) calls the "new, improved holy grail". A basic tenet of mainstreaming is that every student's education "must be provided in the *least restrictive environment* that is consistent with the child's educational needs" (Hallahan, Kauffman & LLoyd, 1985, p.19). However, the regular classroom might be the "*most* restrictive environment" in which to educate certain pupils (Cruickshank, 1977; Gresham, 1982; Newcomer, 1989b). In other words, individual differences in the behaviour and temperament of a student might result in a poorness of fit between that student and the characteristics of an integrated mathematics classroom. For instance, some students might respond more effectively to small group instruction in a relatively less complex or threatening resource room. Others adolescents might not "fit" into the regular high school scene with its vice-principals, bells, and crowded hall-ways and be better suited to the smaller classes and flexible time-tables of alternate schools like VRLF. In more serious cases, still others might need to be pulled completely from malignant home situations and educated in residential treatment centres. Thus, saying that mainstreaming in the regular mathematics classroom is the desired situation does not imply that mainstreaming will result in successful learning for every child.

Whether in an isolated setting or in a regular school, effective mathematics education requires two conditions: a meaningful curriculum and a social context in which to teach it. I now turn to the final three considerations of this chapter: why a new curriculum is needed, what mathematics one should teach to alternate school students, and finally, how one should teach it to them.

Rationales for a new curriculum

In general, the choice of mathematical topics for secondary curricula is determined by two factors, the first of which is the education community's concept of mathematics. In the words of mathematician René Thom, "all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (in Pimm, 1987, p. 47). In particular, the topics which are traditionally taught to alternate students result, in part, from a very narrow view of mathematics. For instance, some educators still value paper-and-pencil computations, manipulation of symbols, geometric formulae, and mechanical algorithms as meaningful mathematics. Invariably, low achievers are presented such topics in high school not only once, but time and time again in a spiralling fashion. John Goodlad (1984) made the following observations from his study of thirty-eight American schools.

> The impression I get from the topics, materials, and tests of the curriculum is of mathematics as a body of fixed facts and skills to be acquired, not as a tool for developing a particular kind of intellectual power in the student. One might expect to see by the upper elementary years activities designed to use basic skills previously acquired; instead, these skills reappear as ends in themselves....Why [do] so few mathematics teachers...appear to get much beyond a relatively rote kind of teaching...? (pp. 209-210)

Goodlad's question applies especially to teachers of special needs students. In other words, why are such mundane and boring mathematical topics generally reserved for low achievers? Why do we see such striking differences between mathematics courses for low attainers and courses for the university-bound?

These questions bring me to the second factor which determines the actual mathematics taught: that is, educators' beliefs about students' abilities and needs. In particular, we underestimate our special students' abilities and misinterpret their needs. Educators equate low achievement with low ability, and also attempt to fulfil alternate school students' need for success with low-level mathematics.

To illustrate the problem, I will outline the mathematics curriculum as it existed at the Vale Road alternate school before I began my project. Invariably, students new to VRLF were given the Wide Range Achievement Test (WRAT) in mathematics as soon as they entered the classroom. Based on the results of this entrance test, students were placed on individualized mathematics programs which were then driven by their performance on pre- and post-tests. The mathematics content of the program, for those who were weak in computational algorithms, consisted essentially of computational algorithms. Students could eventually work their way to more "advanced" topics such as percentage, manipulation of algebraic expressions, and the use of geometric and trigonometric formulae.

The use of an entry-level achievement test and the choice of such mundane mathematical topics are both based on the special education maxim "Above all, avoid failure." Educators are convinced that the safest way to ensure success in alternate school students is through "fail-safe" topics such as elementary arithmetic. Thus, the temptation is to think, "Easy makes for success." Indeed as Pimm (1992) points out, "there is a force in education which says that everything should be easy and unchallenging." It appears that the creators of the original VRLF mathematics program were influenced by that force.

Well-intentioned as they were, these educators did not realize that a better adage would have been "Avoid failure *and* frustration." The following quote from the NCTM (1989) *Curriculum Standards* sums up well the mathematical situation of alternate pupils:

> If students have not been successful in "mastering" basic computational skills in previous years, why should they be successful now, especially if the same methods that failed in the past are merely repeated? In fact, considering the effect of failure on students' attitudes, we might argue that further efforts towards mastering computational skills are counterproductive. (p. 66)

And even if these special students are successful in their post-tests, more often than not they forget their rote-learned knowledge within days.

It is not difficult to recapitulate two main traditional teaching methodologies found to this day in most mathematics classrooms (NCTM, 1989, p.1). First, there is the *teacher talks, pupils listen, pupils practice* system whose ubiquity prompted Welch in his 1978 study to remark, "The most noticeable thing about math classes was the repetition of this routine" (p. 6). Then there is the individualized approach common in many special education settings (my alternate school included). Here, the student writes a pre-test, which is marked by the teacher. Then, seeking individual help at times from the teacher, the student practices the mathematics missed on the pre-test. After writing a post-test on the same material, the student, if successful, goes on to write the next pre-test. And so on. As O'Shea (1987) asks,

> What is there in this room that would attract a student to mathematics? Where is the sense of inquiry, triumph, and joy in doing mathematics? (p. 33)

Apart from the tedious nature of both the mathematics and the methods by which it is taught, there is one obvious feature which both the above methodologies have in common: they preclude any meaningful oral communication between students. In the special education situation, this lack of inter-student math talk means that students sorely miss needed opportunities for structured social interaction. And as I will argue, the students in both settings are, from a constructivist viewpoint, missing a valuable educational experience. For as Cobb, Wood and Yackel (1990) state,

> social interaction...is the process by which individuals create interpretations of situations that fit with those of others for the purposes at hand. In doing so, they negotiate and institutionalize meanings, resolve conflicts, mutually take each others' perspectives and, more generally, construct consensual domains for coordinated activity. (p. 127)

Thus, in as far as they are being denied the occasion to interact, alternate school students in the above situations would be in double jeopardy: social and academic.

Mathematics for alternate school students

My challenge was to create mathematical experiences which the special needs student perceives as comprehensible and workable, but not as "childish" nor irrelevant. As I shall describe later, I lost many of my students intellectually during a rather complex algebraic derivation of the Golden Ratio. As well, my students felt that one of my very first planned activities (with pentominoes) was infantile. As Mason (1992) suggests, there is a very delicate tension for the teacher to maintain as low attaining pupils succeed with complex mathematical concepts. For instance, tasks which are open-ended work well, but if these are too open and do not allow some form of closure for distractible students, then frustration can easily occur. Problems must be chosen so that even those students with limited mathematical or linguistic repertoires can participate in the solutions. In other words, teacher expertise in mathematics and pedagogy (special and regular education) is essential for the judicious choice of learning activities for alternate school youngsters.

Ideally, mathematical experiences and situations should be planned which are of sufficient complexity and multiplicity that the learners' focus is drawn from the teacher to the mathematics. For students with emotional disabilities, particular benefit is also obtained when their concentration shifts from their own troubled inner thoughts, to the topics at hand. If the task is sophisticated enough, and if opportunities for co-operative learning are provided, the students will gain socialization experience as they draw on the expertise of fellow classmates.

Having noticed that the present curriculum insisted to a high degree on rote memorization, and on the rehashing of previously learned mathematics, I decided to develop a course with the following characteristics. First it was not to be an elementary school level curriculum. In particular, arithmetic computation would not be the direct object of instruction in this curricular model; rather, number sense and computation would be discussed and strengthened in the context of problem solving and discovery learning. I would choose topics which were complex enough

to challenge secondary students, interesting enough to hold their attention and yet within the academic abilities of most low achieving students. The last criterion, of course, was that any new topic would have to interest me as well.

My curriculum was also designed to be relevant to the educational needs of alternate school adolescents. My goal was to foster a positive attitude to mathematics. In short, I wanted unmotivated pupils to come to enjoy doing mathematics. In consideration of the social needs of special students, I designed my curriculum from a cultural perspective. That is, students should get a sense that *people* do mathematics and that mathematics grows and develops as does any other aspect of culture. Hopefully, students would thereby come to appreciate the role that mathematics plays in so many aspects of society.

However, as Mason (1991) concludes in his article *Questions About Geometry*, "attributes of pleasure, engagement, or mathematical thinking are significant, but not primary" (p. 83). More important are the conceptual themes of the curriculum: namely, problem solving, reasoning, communication, connections, and mathematical discovery. Included are units on pentominoes, probability, Pascal's triangle, scientific notation and large numbers, inductive and deductive reasoning, the Golden Ratio, fractals, and computer geometry. (See Appendix 7 for a discussion of the mathematical topics chosen.) Since my goal was to promote student awareness of mathematical concepts and methodologies, my pedagogical aim was not trivial.

Contrary to my expectations, it was not difficult to find mathematical topics for my study group. Suggestions for open-ended mathematical experiences abounded: for instance, the NCTM's Addenda Series was an excellent source of well-developed lesson ideas. In effect, my curricular activities were designed so that generally

> [d]ifferentiation in learning outcomes occurs by blending core lessons for all students with extended activities that students can complete to different levels of abstraction. Each lesson begins with a core activity ideally launched from some significant question that the new concepts address. The initial activity is generally at a concrete or semiconcrete level. This activity is followed by questions and discussion leading to key ideas and

applications....[This] allows individuals or groups to move ahead to their own highest level of abstraction for a given lesson. (NCTM, 1992, p. 73)

What proved difficult, however, was to monitor and control the depth to which such lessons could be taken without losing the students academically and motivationally.

The aim of my research was not to one-sidedly create a "successful" and "impressive" mathematics curriculum to be imposed upon the students at Vale Road Learning Facility. Rather, I planned to use a collaborative approach which would involve the students from the beginning of the undertaking. The reasons for this approach are as follows. First, I hoped that the students would develop a sense of ownership of the new mathematics program. For the past four years, the students have been encouraged to become involved in decision making around the school and to develop and share with the staff a sense of responsibility for the program's success.

Second, I did not pretend to know what mathematics was "best" for my new students. My experience has shown on numerous occasions that topics I felt to be interesting and stimulating were of no interest to special needs students. Indeed, these students are so accustomed to being spoon-fed mathematical pabulum, that they often feel threatened by new topics (such as math art, probability, open-ended problems) and often reject these as a matter of course. My plan was that, if allowed to make some choices of mathematical topics themselves, they would not develop such apprehensions about strange, new mathematics. So the focus of this thesis is on the *process* of alternate school curricular reform as well as on the end results.

Having written about new school mathematics for alternate school students, I now consider how it might be taught.

New methodologies for teaching alternate school pupils: a constructivist alternative.

Cooperative learning

The alternative to teacher-centered learning is clear, and is fast gaining acceptance amongst mathematics educators. As Steffe (1990) comments, "the change in world view that has been suggested shifts the focus of 'teaching' from the activity of the teacher to the activity of the students *as understood by the teacher*" (p. 177). From a classroom dominated by teacher-talk, teacher choice, and teacher evaluation, we are moving to a setting where inter-pupil communication is an integral part of learning. That is, within a framework of social interactions, the teacher establishes a mathematical community in which concepts can be discussed and learned by students in a way that is meaningful to them. Indeed, in her consideration of constructivism, Jaworski (1989) comments,

> Linguistic communication becomes supremely important - teachers encouraging pupils to talk, and listening to them; providing opportunity for pupils to talk and listen to each other, encouraging open negotiation of meanings... (p. 295)

Pimm (1992) points out a rather global benefit to be derived from learning in a group context as opposed to teacher centered instruction. In the group mode, the presence of five students can be considered a *strength*. For instead of relying on *her* expertise alone to attempt to convey *her* personal understanding of a mathematical concept, the teacher can draw on the collective wisdom of the whole group. That is, in addition to the teacher's point of view, there are five other unique perspectives of the same concept. As a result, the teacher can "use pupils' apparent misconceptions in order to gain insight into [her] own images and constructions" (Jaworski, 1989, p. 295). Also, there will be a greater likelihood that students will succeed in negotiating conceptual meaning out of five points of view as opposed to just one. In short, "there are sound cognitive reasons for allowing students to work together" (Noddings, 1990, p. 17). In the case of the mathematics classroom, one could argue that the interactive whole is greater than the sum of its inactive parts.

For the alternate school pupil turned off for years by the drone of the teacher's voice,
concept formulation through peer interaction is indispensable for improved academic achievement.
Also, as McEntire (1984) writes,

> [t]he teacher who deliberately creates a learner centered communicating atmosphere
> manages a classroom that promotes student growth in the following: assuming
> responsibility, feeling accepted and respected, self-motivation, active participation,
> human interaction, feeling secure enough to respond and inquire, feeling
> understood, becoming self-disciplined, verbalizing with ease, achieving insights,
> becoming aware of appropriate attitudes, changing values, responding to
> genuineness in others, and respecting and valuing interaction with the teacher.
> (p. 258).

(Learning through group communication can be problematic too, because of poor interpersonal
relations among students with social disabilities.) In view of the previous lack of success for
alternate students in traditional mathematics classrooms, and in consideration of the crucial
benefits to be derived from the interactive model, educators must commit now to change the way
mathematics is taught.

Such a commitment is not a light one. There is considerable evidence that most secondary
teachers have not implemented any form of cooperative learning in their classrooms (Cooney,
1992; NCTM, 1989; Robitaille, Schroeder, & Nicol, 1990). The switch from a lecture
methodology to an interactive learner centered approach cannot be achieved simply by substituting
tables for rows of desks, asking students to communicate, and hoping for the best. Many
teachers have no training whatsoever in the management of cooperative groups, inquiry learning,
open-ended problems, learning contracts, and projects. The way most educators teach is the way
they were taught mathematics, and the way in which they were trained to teach mathematics. In
Chapter 4, I recount my own difficulties in abandoning my habitual teacher-centred role as I
endeavored for the first time in my career to establish a learner-centred classroom.

History has taught that educational innovations cannot be simply hoisted on teachers
"from above." In explaining why reforms often fail to obtain teacher support, O'Shea (1981)
writes:

Put simply, change has always been imposed on the teacher. Although the teacher is almost universally recognized as the most important factor in the educational process, little importance has been assigned to the preparation of teachers as change agents. (p. 22)

Thus, the implementation of cooperative learning will require a firm commitment to teach complex new methodologies to a large number of inservice mathematics teachers. Governments, school districts, administrators, and tax-payers have to be convinced of the urgency of the situation, for such an undertaking will be expensive. Teachers will have to come to realize the value of such an enterprise, for in the last analysis, it is they who implement the changes. Moreover, as Noddings (1990) concludes, "there is a large part for teacher education to play" (p. 18).

The mathematics teacher will have to be aware of several pitfalls which can occur when introducing cooperative learning and small group instruction for alternate school students, . For instance, students who lack self-reliance will be prone to merely copying results, or relying on others to make decisions. They also may be hesitant to express their opinions over the majority view (Larcombe, 1985). Withdrawn pupils may at first refuse to be drawn in to any form of discussion, whereas boisterous ones may tend to offer rude or disruptive comments. Last, alternate and regular students alike can be cruel and offensive to each other, easily choosing to ignore any directives to cooperate.

With alternate students in mind, I will offer a few practical suggestions for implementing cooperative learning groups in mathematics classrooms. Research suggests that four or five students is the optimum size for cooperative learning groups (Serra, 1989a). Since the average daily size of my mathematics cohort at VRLF was five, I can attest to this notion. The presence of four or five students usually meant one or two members of the group could become distracted or temporarily lose attention without losing the whole group's momentum. On the other hand, students found that they could "disappear" in groups larger than five and withdraw from learning activities. As well, lack of class focus and difficulties in students assessment were common results of large learning groups.

Although random grouping can be practiced in the beginning, rather judicious matching of students according to needs and abilities is imperative once the teacher comes to know the pupils. To avoid disturbances, it is only logical to avoid placing more than one overly aggressive teenager in any one group. Likewise, a withdrawn student should not be included at a table consisting mainly of overly-loquacious peers. Pimm (1987) offers the following *gambit:* encourage talking in pairs at first. As a result, in larger group or whole-class contexts, "many more pupils may have something to contribute, having rehearsed its expression in the less threatening context of conversation with a neighbour" (p. 50).

Noddings (1990) points out that educators should not assume that work groups and cooperative learning are an automatic pedagogical panacea. Rather, since the loftiest aim of all is effective mathematics education, inter-student communication which is task-focused, accurate, and clear can be construed as a separate end unto itself.

Student talk

For some time now, mathematics educators have known of the advantage to be derived from encouraging students with ESD to talk in mathematics classrooms. For example, Amidon and Flanders (1961) found that dependency-prone students learned more in a class where the teacher asked more questions and encouraged verbal participation than they did in a direct lecture-style situation. These researchers argue that under a direct teaching style, this type of student finds more satisfaction in compliance than in understanding the mathematics. But when a student is free to express her doubts, ask questions, and gain reassurance, her understanding keeps pace with teacher compliance. The implications of this study are clear. All too commonly we recommend close supervision and direct teaching as an antidote to lower achievement in mathematics, and this practice may be more harmful than helpful.

In their 1973 study, Grimm, Bijou and Parsons found that when students with ESD were encouraged to verbalize their thinking during problem-solving, students' accuracy rates went up.

The writers explain that overt language made possible the teacher monitoring of students' mistakes. Pimm (1987) extends this notion: such externalization of students' thoughts allows teachers to discover their pupils' ideas and beliefs as well. And for alternate pupils, I would argue for the therapeutic benefit of speaking their troubled minds out loud. Such rendering of private thoughts to the public realm not only could serve as a psychological release valve for the student, but also could pave the way for academic and emotional mediation by the teacher.

However, effective and meaningful pupil talk does not just occur spontaneously. Rather, the teacher must encourage and foster it, and the students must regularly practice it. Initially, the teacher has to accept student comments without criticism or correction, in order to establish that it is the act of self-expression which is of worth, as well as the act of saying the correct things. The teacher must be aware of his own body language as well, since grimaces and looks of boredom convey rejection of students' comments just as easily as words. Once these formerly reticent students feel comfortable in expressing their mathematical thoughts in public, the teacher can begin to demand greater fluency and more accurate expression.

But as Pimm (1987) warns, students should be made aware that improvement in oral fluency is an appropriate aim, lest the teacher appear to be requesting things which the mathematical situation does not apparently require. This last caveat is particularly important for alternate school students, who need to understand clearly the demands which are being placed on them. Left in the dark as to the teacher's agenda, they will be given over to the all too familiar frustrations and insecurities which in turn will quickly preclude any further self-expression.

Assessment

Perhaps the single most frequent cause of frustration amongst alternate students is written examinations. One reason for this angst is that students are unaware of assessment criteria, or else such criteria are not consistently applied by the teacher. Other pupils have real problems with written expression, or else they experience test phobia; hence the frequently heard comment, "I

can do the math all right, but when it comes to the test I just freeze up." Indeed, such problems with examinations are understandable, since written tests are the epitome of the many failures these students have experienced. Having had no other forms of assessment, they equate failing examinations with personal failure and ineptitude. To many of these pupils, the world is a confusing, inconsistent and harsh reality of which school mathematics examinations form a salient part.

This situation is quite ironic in that written examinations in alternate school mathematics often measure nothing of worth anyway. Testing whether a student can perform isolated irrelevancies such as "40 is 23% of ___ " or can factor 20 different trinomials tells the teacher little about the pupil's degree of concept acquisition, reasoning skills, problem solving abilities or facility with heuristics. As von Glasersfeld (1990) writes, "understanding can ...never be demonstrated by the presentation of results that may have been acquired by rote learning" (p. 26). I wonder the following: why do teachers continue to base their judgments only on what the pupil is able to write down?

The alternatives consist in both what and how mathematics teachers assess. First, they must have reasonable standards and expectations for special students. For instance, success for some students may be interpreted as one month's complete attendance or the handing in of an assignment for the first time in years (Vatter, 1992). For other pupils, it might mean contributing to class discussion or remaining on task within a group. Evaluation criteria could include use of class time, preparedness for class, and neatness of work. The NCTM (1989) recommends that teachers assess students' mathematical *disposition* which "refers not simply to attitudes but to a tendency to think and act in positive ways" (p. 233). By attaching importance to indicators like interest, confidence, perseverance, and willingness to engage, educators naturally encourage students to internalize these traits.

A variety of assessment methods is possible to suit the needs of alternate pupils. These include behaviour checklists, reports, essays, oral presentations, and projects. As I describe in Chapter 4, I awarded common marks to individual group members based on that whole group's effort or achievement. To give encouragement, I gave marks for various stages of assignments, as well for the finished project. Since the students were asked to give opinions on and suggestions for different grading methods, they became part of the entire assessment process. Students' involvement in evaluation can become an important activity in itself as they develop a keen awareness of the assessment criteria. In these ways, then, I attempted to convey to pupils that mathematics is more than just right or wrong answers, and that in secondary schools, wrong answers are "all right."

Metacognition in mathematics education

Metacognition may be defined as cognition which deals critically with one's own cognitive activities. To be metacognitive is to think about one's own thinking, to know about one's own knowledge, or as Gattegno (1974) puts it, to be "aware of one's own awareness" (p. 80). Schoenfeld (1987) outlines the following three aspects of metacognition:

a) Your knowledge about your own thought processes. How accurate are you in describing your own thinking?

b) Control, or self-regulation. How well do you keep track of what you're doing when (for example) you're solving problems, and how well (if at all) do you use the input from these observations to guide your problem solving actions?

c) Beliefs and intuitions. What ideas about mathematics do you bring to your work in mathematics, and how does that shape the way that you do mathematics?
(p.190)

Thus metacognition refers to one's self-knowledge, self-orchestration, and beliefs as one engages in cognitive activity. Metacognitive *strategies* are processes, tactics or rules which one uses to plan and to monitor one's own cognitive progress in a task. As Flavell (1976) succinctly writes, "cognitive strategies are invoked to make cognitive progress, metacognitive strategies are invoked

to monitor it" (p. 232). Finally, a further aspect of the concept which transcends all the above is one's ability to reflect on and to understand the utility of self-awareness itself.

In mathematics education, metacognition involves one's self-awareness as a learner of mathematics. Such awareness could include for example, one's mathematical strengths or weaknesses, and one's preferences for various mathematical topics and learning modes. As mathematical behaviour, metacognition refers to monitoring one's progress in a task, and to making strategic decisions based on one's self-observations. It involves the learner's beliefs about mathematics and mathematics learning, and the effects those beliefs have upon mathematical achievement. Last, metacognition in math entails the learner's appreciation not only of the value of heuristics and cognitive strategies, but also of the value of metacognitive strategies themselves.

Many pupils fail to use cognitive and metacognitive strategies as spontaneously, frequently, and efficiently as their peers. For some students, the absence of metacognitive behaviours can mean failure just as the presence of such processes may promote good problem solving (Schoenfeld, 1983). Some students possess the necessary talent to solve a problem, but nonetheless fail in their attempts. In these cases, their failure can stem from attitudinal problems or from metacognitive ineptitude. (Lester, Garofalo & Kroll, 1989)

To explain this lack of metacognitive behaviours, some researchers posit actual strategic disabilities or deficits. However, others also consider certain students, especially those with LD, as "passive learners" who are unwilling or unmotivated to behave in a metacognitive (or cognitive) way (Cherkes-Julkowski, 1985; Torgesen, 1977). This trait stands in direct contrast to the constructivist viewpoint which "implies a way of teaching that acknowledges learners as active knowers" (Noddings, 1990, p. 10).

Alternate school students become inactive problem solvers quite easily. With relatively few academic successes of any kind, they show little or no faith that any strategic action on their

part will result in a solution to a problem. These pupils may develop a "learned helplessness" which causes them to rely on others for help or else to give up. From their point of view, heuristics and metacognitive plans are "for the brainy guys in the class."

There is abundant support for the recognition of metacognition as an integral part of mathematics education (Crosswhite, 1987). For example, Gattegno (1974) posits that educating students means striving to make them aware of certain powers they already possess which they can use in the same way as mathematicians.

> Among the many different functionings of man's mind we can see two which go to make the mathematician and these are the awarenesses, first, of relationships as such, and second, of the dynamics of the mind itself as it is involved in any functioning. Knowing this, teachers can serve their students best by bringing them to the state of watchfulness in which they perceive how one becomes aware of relationships and of the dynamics of the mind. (p. 81)

Likewise, Dawson (1992) states that

> being aware of what one is doing is the only way in which learning will occur. The additional step of being aware of oneself as one is learning is a challenge that requires working with students to help them develop tools for accomplishing this task. (p. 24)

Garofalo (1987) maintains that if we want our students to become active learners and doers of mathematics rather than mere knowers of mathematical facts and procedures, we must design our instruction to help develop their metacognition. Rohwer and Thomas (1989) call for instruction which will explicitly aid pupils to acquire the metacognitive strategies necessary for planning and monitoring during problem solving.

Research literature from the LD field provides ample evidence that metacognitive strategies in mathematics can and should be taught to special students. First, pupils *can* be trained to use and maintain metacognitive skills (Borkowski et al., 1989; Bos, 1988; Garofalo & Lester, 1985). Second, there is a positive relationship between success in mathematics and metacognitive strategy instruction (Garofalo & Lester, 1985; Loper, Hallahan, & Ianna, 1982; Montague, 1992). Training in metacognitive skills leads to a decrease in impulsivity and distractibility in students with ESD (Davis & Hajicek, 1985; Frith & Armstrong, 1985) and to

increases in on-task behaviour (Hallahan, 1978; Hallahan & Lloyd, 1987; Snider, 1987). Using metacognitive strategies can have positive effects on the self-esteem of students (Borkowski, 1992; Labercane & Battle, 1987). Indeed, the positive value of metacognitive training for students with learning disabilities appears to be unequivocal. In regard to these successes, Garofalo and Lester (1985) offer the following comment.

> As mathematics educators and teachers, not only should we incorporate such metacognitive supplements in our efforts to train students to be proficient in applying algorithms and heuristics, but we should also help our students adopt a metacognitive posture toward mathematical performance in general. (p. 173)

To promote metacognition in my alternate school study group, I adhered to the following precepts derived from research in both mathematics education and learning disabilities. Initially, the instructor must introduce tasks and strategies which result in success for the students, thereby inducing "the belief in the general utility of being strategic" (Borkowski et al., 1989, p. 59). It is also beneficial to give praise (or marks) for all attempted solutions (rather than just for correct answers) so the students will come to appreciate that even a failed or dead-end cognitive strategy can add positively to his/her metacognitive repertoire. Self-expression can be encouraged by having each learner complete questionnaires or written response logs about both negative and positive mathematical experiences.

The teacher must also be a co-problem solver with students. This "team effort" approach will help actively involve socially withdrawn students and help to eliminate low attainers' negative view of "teacher as omnipotent possessor of all knowledge." Then, the teacher herself must demonstrate metacognitive problem solving as she teaches. Questions such as "Have we seen this situation before?" and "How well do you feel this approach working so far?" can be posed by the teacher as examples of metacognitive behaviour.

The effective promotion of metacognition might require attitudinal changes on the part of the teacher. A generation later, I still recall my beginner's trepidation whenever the

superintendent or principal visited my classroom. My main fears focused on the possibility that this particular class of all classes would become unruly and/or confused about the lesson of the day. However, in contrast to the "Avoid bewilderment at all costs" maxim of twenty years ago, there is today a new perspective on students' befuddlement in mathematical tasks. John Mason (1992) speaks of how a "beautiful confusion" in a student's mathematical thought can provide opportunities for group learning. In other words, there is relatively little that the mathematics class can extract from the neat situation when a student gives a correct answer, or professes to understand a concept. But there is pedagogical value in an individual's ambiguity or consternation. As Mason and Davis (1991) suggest, something can be learned from being *stuck*::

> If and when you get stuck, acknowledge that fact by writing STUCK!, (and if you recognise negative feelings such as anger or frustration arising inside you, put that into the writing of the word STUCK!). Then accept the fact that being stuck is an honourable state, a state from which much can be learned about yourself, and ultimately, about helping pupils. Only you can decide when to take a break, when to ponder a question as you go to sleep and upon waking, and when to conclude that you have made as much progress as you are likely to.... (p. 5)

As I relate in Chapter 4, the above comments are examples of metacognitive strategies which I introduced to my failure-prone students as they attempted to do mathematics.

For the development of self-regulatory skills, Schoenfeld (1987b) advocates that students learn mathematics in small groups. He maintains that such cooperative learning emulates the metacognitive activity which occurs in actual collaborative communities of mathematicians. Schoenfeld talks of a "society of mind" in which metacognition comes alive naturally through the dialogues between active learners.

Finally, metacognitive expertise develops through extensive practice and experience. After all, one learns to read well by reading a lot, and one learns to write expertly by writing a lot. The difficult task for teachers is not to provide extensive metacognitive experience. Rather, it is to provide the meaningful mathematical context for such experiences to occur. Students cannot be expected to use metacognitive strategies about, for example, adding fractions or irrelevant word

problems. As Lester, Garofalo and Kroll (1989) aptly put it, "teachers who expect their students to be metacognitive must insure that their students have something to be metacognitive about" (p.86).

In planning a curriculum with cooperative learning and challenging mathematical topics, it was thus my hope to foster self-awareness and self-monitoring in my students. From the point of view of the inactive learner concept, I wanted to motivate my alternate school students to take chances, and to awaken them metacognitively to become "good strategy users" (Pressley, 1986, in Goldman, 1989, p.44). In Chapter 4, I relate my own attempts in this regard, and the results observed.

Beliefs and mathematics education

Beliefs, the third aspect of metacognition, have as strong an influence on mathematical behaviour as do self-awareness and self-orchestration. This point was driven home to me after hearing a former VRLF student (Jag) reveal his beliefs about the nature of mathematics learning:

> The teacher teaches you at the beginning of the class, and gives you a bunch of questions to practice on, and at the end of the chapter gives you a test. That's the way I've been doing it since grade school. That's the way you do math.

He goes on to explain how he began to fail mathematics:

> About half way through grade 10, I started looking in the back of the book and copying the answers from the back, 'cause it would take too long, and I would just copy the answers. And that was why I didn't understand what I was doing. I just started failing tests....Sometimes I wouldn't even go to math class because I didn't understand. I thought there's no point in going to math. So I'd just walk out to the store or something. I don't know...I didn't understand it. If didn't copy the answers out of the back of my book I'd get in trouble, so I'd just, I'd say forget it and go.

Jag believed that school mathematics meant listening to a teacher, practising questions from the text, and writing chapter tests. For him, Mathematics 10 involved getting the correct answers to homework exercises. This last belief was so strong, he considered it sufficient to placate the teacher with a semblance of completed homework. However, copying from the answer key precluded conceptual understanding which led to test failures, frustration, and to the

inevitable truancies for which he was eventually suspended. Although aware that he lacked the understanding necessary to pass exams, he nonetheless puts his beliefs into action, or rather inaction.

It is not difficult to ascertain the origins of Jag's beliefs: he was merely following the rules of the game called Mathematics 10. In other words, his beliefs about mathematics are derived from the current curricular structure which gives individual homework and seatwork such prominence.

There is a strong consensus amongst mathematics educators about the nature of many students' beliefs towards mathematics. (Baroody & Ginsberg, 1990; Borasi, 1992; Lester, Garofalo & Kroll, 1989; Schoenfeld, 1992; Schoenfeld, 1989; Silver, 1987; Silver, 1982) For instance, students can have mistaken beliefs about mathematics such as:

• Mathematics is mostly memorization.

• There is usually only one way to solve any mathematics problem.

• If one really understands the mathematics, then problems should only take a few minutes.

• Mathematics is based entirely on rules.

• Being stuck in mathematics is ignominious and means one is stupid.

Unfortunately, as Schoenfeld (1989) comments, "they practice what they claim to believe" (p.349).

Many students also have *conflicting* beliefs about mathematics and mathematics learning. For instance, in a questionnaire adapted from Borasi (1992), I asked my students to rate different statements as either DEFINITELY TRUE, SORT OF TRUE, NOT VERY TRUE, or NOT TRUE AT ALL. (See Appendix 1.) Derek qualified the statement "There is always a rule to follow in solving mathematical problems" as *definitely true*, while he rated the declaration "Every mathematical question has only one right answer" as *not true at all*. When I pointed out how his thinking seems contradictory to me, Derek replied, "I don't personally see any conflict." Likewise, Schoenfeld (1989) describes students with apparently contradictory beliefs:

Despite their assertions that mathematics helps one to think logically and that one can be creative in mathematics, they claim that mathematics is best learned by memorization. (p. 348)

For some students, lack of mathematical sophistication may account for their inability to see such conflicts in their beliefs. In other words, they may not have the cognitive maturity or competence to discuss the complex philosophical underpinnings of mathematics. For others, claims about creativity and discovery in mathematics might simply be regurgitations of the favorable rhetoric they hear from "progressive" mathematics teachers. In Derek's case, I suspect this latter reason.

There is, however, virtual agreement amongst educators as to the main cause of beliefs such as those listed above (Schoenfeld, 1992). As McLeod (1989) comments, "There is nothing wrong with the students' mechanism for developing beliefs about mathematics" (p. 247). Indeed, Borasi (1992) points out that such beliefs are "quite justified by the mathematical experiences students are likely to have had during their years of schooling" (p. 208). In other words, while they are not true reflections of mathematics, such beliefs are often accurate descriptions of the mathematics offered in schools. Schoenfeld (1987) sums up the situation well:

> As a result of their instruction, many students develop some beliefs about what math is all about that are just plain wrong - and those beliefs have a strong negative effect on their mathematical behaviour. (p. 195)

The effects of such erroneous convictions are legion. For instance, students will not attempt to understand mathematics which they believe is only to be memorized. They will forego strategy monitoring and self-regulation if they believe that there is only one way to solve a problem. They will quickly give up on problems that they believe are typically solvable in a few minutes. Students' mathematical inquiries will be limited to the search for rules which they believe govern all mathematical situations. Given such negative repercussions, educators cannot disregard such pre-existing beliefs.

Research findings by Lee and Wheeler (1986) provide a good example of the effects of beliefs on the behaviour of algebra students. These researchers discovered that many students rarely turn spontaneously to algebra to solve problems, even when judged capable of performing the necessary algebraic manipulations. It was found that these students attributed an extremely low status to algebra; in particular, they saw algebra "as an irrelevant activity but which for some mysterious reason seems to please teachers and researchers" (p. 101). Thus, faulty belief structures can inhibit desirable metacognitive behaviours (e.g., strategy choice) as well as cognitive ones.

What steps can be taken to promote more accurate and productive belief systems? First, it should be appreciated that beliefs, unlike emotions, are mainly cognitive in nature, and build up slowly over long periods of time (McLeod, 1989). They are perhaps more difficult to alter than students' faulty cognitions or weak mathematical understandings. As well, beliefs have a strong affective component, and thus rooted in deep emotional layers, they may be extra resilient to modification. Finally, as Borasi (1992) warns, shifts in beliefs are dependent upon an institution's paying more than mere lip service to new educational approaches.

As an important first step, Borasi (1992) suggests that teachers encourage students "to reflect on their beliefs...to engage them in activities that require them to state what they think and feel about mathematics" (p. 209). In this way, what Silver (1987) calls the "hidden curriculum" of deleterious beliefs can be brought out in the open.

Finally, true credal change will come only after modifications to the source of those student beliefs: the mathematical content and methodologies of our present curricula. In the words of Silver (1987),

[i]n designing mathematics curricula or planning instructional activities, we need to be mindful that students will integrate their experience with the activity, unit, or course that we are preparing with their prior experiences to form or to modify attitudes toward and beliefs about mathematics and mathematical problem solving. Let us teach toward this hidden curriculum to allow our students to develop attitudes and beliefs that reflect a view of mathematics as vibrant, challenging, creative, interesting, and constructive. (p. 57)

Indeed, my task was to give my students something to believe in!

Peer pressures

While promoting mathematics learning among alternate school youth, the teacher should bear in mind the following fact: many such youngsters will be beyond the help of their mathematics teacher, in spite of well-considered and persistent interventions. For some unfortunate ones, their social and emotional wounds are so grave as to be beyond immediate cure. For others, deleterious or non-existent home situations make learning in school an impossibility. But perhaps the strongest force of all which hinders (or encourages) their participation in learning activities is peer pressure.

The need for peer group acceptance is one of the salient features of adolescent passage. Thus, it is not surprising that socially bereft teenagers feel an especially strong exigency for friends. And in view of that intense wish for acceptance, we can see why alternate school adolescents are especially susceptible to the pressures exerted by their peers.

Of course, there are pupils who are not influenced by fellow classmates. Some youngsters, having endured chronic social rejection, have concluded that they will never be accepted and have given up trying. Other individuals, sometimes termed psychopathic or anomic, have no normal need nor tendency for friends. Still others will have strong social affiliations outside the mathematics classroom, and are therefore not subject to influences by fellow classmates. And since these students are beyond even *positive* group pressures, they will be a special challenge to motivate.

Individual companions, cliques, and sub-groups can influence mathematics students in many negative ways. For example, there is a common phenomenon in our culture

> whereby it is socially quite acceptable...to confess ignorance about [mathematics].
> to brag about one's incompetence at doing it, and even to claim that one is
> mathophobic! (Bishop, 1988, p. xi)

Likewise, although it is quite common to hear students say they are useless in mathematics, as if such a comment were no reflection on their intelligence, it is unusual to hear them tell their friends that they are useless in reading (Laslett, 1977). Also, amongst teenagers there has always been a very pervasive group attitude which rejects academic success in general as obsequious or in the current vernacular, "sucky". Finally, a group can take a collective dislike to a teacher or to a teacher's educational agenda. When such attitudes gain hold in the mathematics classroom, alternate students are among the first to jump on the anti-math bandwagon. And when these attitudes become the dominant value of the peer group, the teacher has a serious motivational problem. This was my greatest fear in initiating the project at Vale Road Learning Facility.

There are manoeuvres which can prevent any anti-mathematics backlash from becoming a contagion. For instance, at the beginning of a course, one of my colleagues tries to search out any student she deems to be the "dominant" member of any educationally subversive clique. Having identified the "rebel" leader, she relentlessly endeavors to woo him/her over to her cause. With this student "on side" as she puts it, the rest of the group settles in to do some mathematics. Another technique works quite well in setting up and maintaining a positive peer culture in resource rooms or in other special education settings. Essentially, only two students are registered on the first day of the school year. After these two undergo a suitable period (usually just two or three days) of acclimatization to the math classroom and to the teacher, a second pair of students is brought in. The process continues until a whole class (usually ten to twelve students) is registered. The procedure is effective, since with the very first two pupils, the teacher can more easily "create" a positive learning ambience which thereafter serves as the status quo to be imitated by any newcomers who follow. (This process was to be followed in September 1994 at VRLF:

given the chaos that reigned during my first few months there. I frequently wished that it had been done a year previously as well.) Finally, there are effective intervention programs which can turn around a negative youth subculture so that peer power can be mobilized in a educationally productive way (Vorrath & Brendtro, 1985).

In general though, alternate school teachers must develop their own repertoire of schemes to ensure that peer pressure remains positive. Simply put. if the kids are on your side, your job will be a joy. And when, against tremendous odds, alternate students chose to form a team with the teacher and the rest of the class, and when with a sense of mathematical community they enjoy learning mathematics. then teaching mathematics becomes even more rewarding.

In the next chapter I recount in detail the new curricular experiences at VRLF.

Chapter IV

A social curriculum

The prelude
"...this year I'm not too sure about this school."

My task in implementing a new curriculum began on my first day at VRLF. However, no actual innovations in mathematics were introduced for quite some time. During my first six weeks as a teacher there, the challenge had less to do with teaching mathematics than with winning the trust and respect of the students in the program.

From the beginning, the students were disillusioned by the loss of two teachers and by the unexpected departure of their counsellor on the first day of school. As well, they felt that their building had been invaded by strangers: five multicultural workers, two district counsellors, and two hospital/homebound teachers had just been assigned new offices at VRLF. Finally, they were mistrustful of the two new teachers (myself and Katey who doubled as head-teacher).

On opening day, students revealed their apprehensions when asked to make brief summaries of their feelings about the start of the school year. The following are excerpts from their writings:

(All student compositions and conversations have been accurately transcribed: students' errors in grammar, pronunciation, and spelling are reproduced exactly from the original tapes or writings.)

> • My feelings from the past year are the same but I think they should have kept the old teachers.
> •I like this scooll but i don t licke the changes with the teacherr. and i havant changed over the sumer.
> •I want Wendy Back. I don't like that all the New teachers are gone. I like VRLF but all the changes are going to be Hard to Handell.
> •If you realy want to know how I feel about Vale Road I used to love it here I really got along with Wendy last year and I felt good about coming to school but this year Im not to sure about this school New teachers, extra people. I just want to get out of here as soon as posible.

Not all students expressed misgivings about the change in staff:

> •I don't care about having new teachers. I've had new teachers every year I've been here.
> •My feelings towards the new teachers are the same as they were for Wendy. I

have great respect for my teachers because they are giving me my education.
•I really enjoyed working with Wendy and I miss her already but life is full of
changes and I can adapt quickly. So may we have a good time and I hope we can
establish a good functional teacher student relationship

Unfortunately, the sentiments expressed by the last student were not to come to fruition

within the next few months at least. The situation worsened tremendously when Eric, the only

teacher with experience at VRLF, suddenly received a court summons which took him on a week-

long trip to Toronto. Uncertain as to the present regulations and practices, I vainly attempted to

establish some of the same logistical classroom guidelines that were in place five years ago when I

last taught in the alternate program. Soon tantrums, arguments, disagreements and strife set the

general tone as our students came together in resentment. Obviously, a new mathematics

curriculum could not have been parachuted in or imposed top-down upon such a unwilling group

of students.

Mercifully, for all concerned, the situation improved. With the help of a core group of

sympathetic students (Adie, Juliana and Alonzo in particular), and with a better feel for the

existing tone of the school, our staff began to feel more and more accepted by our students as

rapport and trust slowly grew. So, buoyed by the increasing positive mood at VRLF, I decided to

try a few preliminary activities before the formal commencement of the pilot project.

My first attempt at encouraging cooperative learning involved Shawn and Dinah, both

students with severe learning disabilities. I gave them each a series of decoding puzzles involving

arithmetic questions, and asked them to split the questions, each contributing answers for

collaborative solutions. Shawn had many of the characteristics of Quay's (1979) anxiety-

withdrawn dimension described in Chapter 2. Extremely lethargic and depressed, he appeared

barely able to lift his pencil. His comment to Dinah was "Who cares. I don't care what this is."

While Shawn sat and did nothing, Dinah did four or five puzzles, and even asked for more.

There was, however, absolutely no communication between them. In reaction to her fellow

student's inaction. Dinah commented, "He's too slow." The next day. I talked to her about how difficult it was to involve Shawn in most activities. (The initials *DT* in all transcripts are mine.)

> Dinah: I ask him if he wants help and he just withdraws into his shell.
> DT: Instead of asking him whether or not he wants help, maybe ask him to share the workload.
> Dinah: Yeh maybe.

In any event, Dinah never got the chance to try out a new approach. Shawn did not attend that afternoon; he was already establishing a pattern of chronic absenteeism which would later land him in jeopardy with the entire alternate program.

Soon after, I tried to actively involve Minna and Juliana in a mould growing project in science. Neither student was interested: Juliana was distracted by a boy passing by in the hallway; Minna expressed disdain at the whole idea of physically *doing* something for a change. For the moment, my hopes plunged. I thought. "What will their reaction be to *mathematics* in a lab setting? Man oh man!"

These preliminary attempts at innovation served a useful purpose: that is, I quickly came to understand that my students would not automatically jump at the chance to engage in cooperative or active learning situations. This realization should not have been surprising: after months or even years spent alone, passively doing seatwork, the students at VRLF could not be expected to metamorphose instantly into interactive learners. If it occurred at all, the transformation would be slow and gradual. And it would require patience, as well as a great deal of judicious lesson planning on my part.

Certain comments made by students also served to remind me early on about the nature of VRLF students. For instance, one student expressed well his frustration with the solitary, learn-at-your-own pace style of education: "In the morning I wake up and think I have to go to Vale Road and work by myself. It sucks man. I'd rather be in a regular school." Finally, the

following remarks by Juliana drove home for me the negative self-concept of many alternate school students: "Everybody looks at us as if we're bad kids. Most of us are different. Most of us don't care." These few comments were forewarnings of what would be two common features of our mathematical experiences in the months to come: poor student self-image, and lack of self esteem.

I was relieved that official reaction to my plans for reform were positive. For instance, the district's Director of Instruction wrote, "[The proposal]...is well designed and current in theory and methodology/ pedagogy....It is the kind of research that will contribute significantly to the 'body of knowledge' that constitutes our profession." Ever the pragmatist, my own principal termed the proposal "interesting and potentially useful." Another principal and former mathematics department head approved of my suggested reforms, and expressed dismay that there had not yet been any changes in general mathematics curricula.

As part of the "informed consent" forms for parents and guardians, and for the students, I provided an information sheet describing my plans in detail. (See Appendix 2.) All parents who were asked gave permission for their children to partake in the project. Also, at no time during or after the case study, did one parent ever ask for clarifications, make comments to me, or withdraw permission for student participation.

In many ways, the alternate school setting afforded me considerable freedoms as a researcher and as a teacher that I probably would not have enjoyed in a regular secondary setting. For one thing, there were no school-mandated restrictions on my pilot course such as cross-grade exams or rigid departmental regulations. With only eleven students, I found that it was much easier to experiment with group learning and with different evaluation techniques. At Vale Road Learning Facility, I could also incorporate students who were actually taking different levels of mathematics: in particular, Adie, Alim and Juliana sought credit for Mathematics 11A, Alonzo for Intro Mathematics 11, and Ron for Mathematics 10. The others, Chad, Donny, Minna, Shawn,

Nina and Dale were enrolled in Mathematics 10B. This multi-grade, multi-level situation could not have occurred in a regular secondary class, at least during normal school hours.

Finally, the very fact that Mathematics 10B was a locally developed course precluded any official opposition against my "tampering" with its content. Indeed, when Katey asked me earlier if I intended to follow the provincial guidelines in planning my course, I was able to respond that I intended to throw out Mathematics 10B entirely, and to replace it using guidelines from the NCTM.

Nor was there any initial opposition or hesitancy from my students about their participation in the project. Early in January 1994, I met with my class to ask for volunteers for the case study. We discussed how the mathematics they would learn, and the method whereby they would learn it, would be different from what they were used to. Also, they were informed about research protocols such as questionnaires, group discussions and attitude surveys. Last, I told the students that they would receive whole or partial credit for the courses in which they were originally enrolled. In effect, the only real obstacle I encountered at this time was the students' persistent forgetfulness in returning their parental consent forms.

I vividly recall this first meeting because for one thing, it was such a unusual configuration to have all my students seated together around one table. I also remember that their main concerns at the time centered around their poor social self-image and the revelation that my professors from Simon Fraser University would probably want to visit VRLF:

Bob: Will they know that we are social misfits?
Minna: If we go to this school we have to be.
Juliana: Will they want to know our names?

As I write, my answer to the last query is, "Considering the mathematics that your group would do over the next five months, yes, Juliana, they probably would want to know your names."

Topic 1: Pentominoes

"It can't be done! That's my answer!"

On the final day of January 1994, the last consent form was finally submitted, and the first session of the pilot project was accordingly scheduled for Tuesday 3 February. At the start of this initial meeting, I stipulated the main course expectation: students must participate in no less than thirty-two mathematics sessions to receive course credit. These sessions would be held twice a week, on Tuesdays and Thursdays, in the latter part of the afternoon, from 1:35 (after their break) until dismissal time (2:45). Any sessions missed would have to be "made up" at some other time. The students agreed with these attendance stipulations.

Such attendance requirements were new for the participants because previously, there had been no consequences for "excused" absenteeism. Students are *"absent excused"* at VRLF if, before the start of the class, they phone the school with a valid excuse for their expected absence. This distinction will be important later as I discuss the net effects of chronic "unexcused" absenteeism upon the fates of several members of the study group. This early on at least, I was hoping that these policies, together with intrinsic student desire to do some interesting mathematics, would improve my students' already dismal attendance records.

I announced my intention to introduce formal marks and grades levels as part of student assessment. (Generally, students received only an arbitrary "C+" on their pupil records when they finished a course at VRLF.) Again, the students were in agreement with this curricular change.

My rationales for introducing both course credit and course grades for the pilot project were the following First, there is a well-known precedent for this procedure: in her study of mathematical inquiry, Borasi (1992) proposed that her two students earn academic credit for her mini-course on mathematical definitions. Second, I was interested to see if the students would

respond to the challenge of earning an "A" or a "B" with greater effort and involvement in the mathematics. Finally, and most importantly, I believe that all students, from elementary grades through to university, expect and need some sort of *formal* recognition from their schools for their efforts.

Before introducing the first topic, I gave the students the mathematics questionnaire to complete. (See Appendix 1) I imposed no time limit on its completion, and the students worked away at it diligently. I decided to forestall any discussion of the results until I had a close look at their responses. Also, time was running out for the period, and I wanted the students to launch themselves well into the series of pentomino activities. Therefore, the results of the questionnaire and subsequent group discussions about it will be brought in at various points during Chapter III.

Pentominoes are the shapes one can make by aligning five squares edge-to-edge. (See Figure 3.1.) Mathematicians have created a myriad of problems with pentominoes, many of which I judged suitable in difficulty level for older adolescents. Also, I liked the idea of starting the course with a concrete experience: with no real idea yet of my student's ability levels, I thought that they all would be equal to the challenge created by my planned manipulative activities. An excellent source of materials and problems was *Pentomino Lessons* by Creative Publications (1986).

Figure 3.1   The 12 pentominoes.
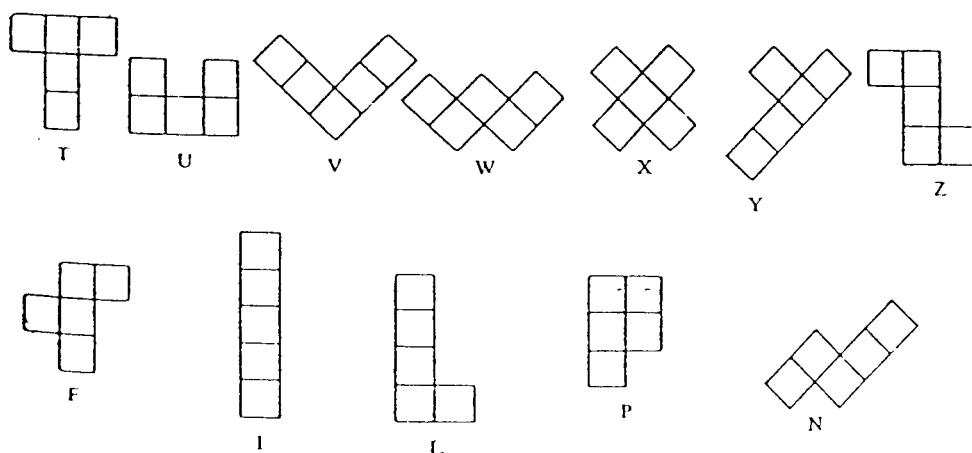
For the first activity, I divided the seven students present into two pairs, and one group of three; to each group I gave about a hundred wooden one inch squares. Their instructions were to construct as many shapes as possible by joining five squares along their edges. Adie, Juliana and Minna finished first, with exactly 12 pentominoes and they were satisfied that there were no more distinct pentominoes possible. Alonzo and Donny constructed more than twelve, and when I explained that some of their constructions were derived from others by means of a flip, they were able to narrow down their constructions to 12. Finally, I asked each student to draw the 12 pentominoes on a grid. At this point, they discovered how to distinguish each piece as a letter of the alphabet; the sequence of letters F L I P N T U V W X Y Z could be easily remembered and would provide an effective means of referring to each pentomino.

While the groups were working on their constructions, students passing by the classroom door noticed the seven students sitting around one table. I heard them ask, "What are you guys doing, art?" and "What's going on in there?" For them, and for myself as well, this constituted an unusual sight: this was the first time all year my class worked together on one thing at the same time! So far, I was pleased with the way the class was progressing.

However, the next activity was not a popular one with the students for one reason: they deemed it too simple. They were asked to create "pentomino creatures" (such as faces and animals) and "pentomino numbers and letters" with the laminated paper sets of pentominoes which they had just cut out. They experimented with their pieces for a short while, with comments like "This is too easy." (Adie) and "What's your point? Do we get to fingerpaint next?" (Ron). At this point, a sort of panic overtook me. Recalling the student revolt of the previous fall, I immediately began to fear the worst scenario: the students would decide right now, as a group, that they would not participate further with me in the case study. Where would become of my thesis? Worse yet, what would happen to my own self-credibility as a reformer in

mathematics education? So when I saw the obvious general lack of interest in this phase of the lesson, I immediately dumped it.

In retrospect, I appreciate that such obvious student rejection of a particular activity can be useful. Since this was a pilot project as well as a case study, I now know better: I would never introduce this apparently puerile assignment to any future alternate school groups. In other words, lessons learned by the teacher on the first run-through of any course can only serve to benefit future instruction.

I knew that the next activity would be more challenging: the very same puzzle had been presented to my Masters program cohort at SFU as part of our mathematics modelling course. The problem is simply put: can you arrange all 12 pentominoes in a 6 by 10 rectangle? In the beginning, I tried to inject an element of doubt into the proceedings, but Juliana was astutely sceptical that the problem might have no solution:

DT: Is this task possible?
Adie: Everything's possible.
Juliana: You wouldn't be asking us to do it if it wasn't possible.

Spurred on in their newfound certitude that the puzzle did indeed have a solution, the students remained well on task for the remainder of the session.

As they worked and pondered the problem, their comments made me appreciate that the pentominoes activities contained a gold mine in mathematical concepts. For instance, consider Dale's quick response to my question

Dale: To do a 6 by 10, you obviously have to use all of them.
DT: How many squares would be left over if you used only 11 pentominoes?
Dale: Five obviously.

The concept of area was being brought into play without the usual formality of introducing formulae like $A = l w$ to be practised and memorized. As well, this was one of my first inklings

of his sharp ability in mental arithmetic. Although their attempts were mainly trial and error, instances of student intuition, strategic thinking, and problem-solving abilities appeared.

> Dale: I always have trouble with leftovers like the F, X, and L.
> DT: How could you try something different?
> Dale: We can do it if we could use the same pentominoes twice.
> Juliana: Obviously certain pieces can't be used in the corners.
> Dale: I could use these left-overs first, I guess. *(He places the F, X, and L on the rectangle.)* There, now I got all the troublesome ones out first.

By the end of the first session, however, no solution had been discovered.

Dale persisted at the 6 by 10 rectangle problem during all of Friday's regular class, attempting a solution by dogged trial and error. On Monday, he entered the room with the comment, "That puzzle drove me crazy all weekend" and continued to work away at it for another two hours. This tenacity was quite remarkable because Dale, as a student with diagnosed attention deficit disorder, could not usually sit still or concentrate on a task for more than ten minutes. As I shall later describe, Dale's shift from distractibility to work ethic would continue to amaze me and the other VRLF staff over the next few months.

On the following Monday, I held a "mini-session" for Chad, a new student to my class. Chad had recently returned to VRLF after a suspension for chronic absenteeism. In the beginning, I did not hold much hope for him in the school, and frankly, I was even apprehensive about having such an apparently unpredictable student join the case study. But within days, my misgivings about Chad would disappear, to be replaced by appreciation for his contributions to the group.

While Chad worked silently (and even morosely in my opinion) on the activities of the first session, Dale wondered if he should return to the wooden squares the group started with. After a few attempts with these, he abandoned the strategy. Finally, after some time, he blurted out, "It can't be done! That's my answer! Tell me right out. It can't be done, can it!" I enjoyed

hearing this new opinion, because in effect it was the first instance of one of my students refuting the commonly held belief that all questions in mathematics always have an answer.

However, my optimistic mood was shattered when, from the back of the class, came Minna's quick, sarcastic reply: "Yes it can. Obviously!" Then, just before the start of Session 2 on Tuesday, Dale came up with a solution to the 6 by 10 puzzle. The pride was evident in his smile and I enthusiastically shook his hand. But later, when he was exuberantly and confidently helping his partner Ron, Adie told him to "shutup." Then, Minna hit him with the comment, "And it took you like three days, Dale?" Although he was doing well with his unusually persistent and focused problem solving behaviour, it appeared even this early in the study that Dale was being socially ostracized by his peers, and in particular by the girls in the group. Such peer rejection was not surprising in view of the research cited earlier which correlates hyperactivity with social rejection. (See page 23.)

SESSION 2   Tuesday 8 February   (Adie, Chad, Dale, Juliana, Minna, and Ron)
"Can you get the computer to solve it?"

For the first part of the session, the students worked individually on the 6 by 10 problem. I was amazed at the great concentration and calm which which they applied themselves to the task; there were none of the curses or oaths which were usual for this group. Ostensibly in disbelief at what they saw, passersby in the hallway stared at the cohort seated purposefully together around a table.

After Dale's verbal altercation with Minna and Adie, I asked him to compose a brief account of how he solved the puzzle. He wrote,

> I don't know how I did it I just kept on going at it then finally got It the time I got
> I started in the middle the hardest part that was hit was my nerves very fusterating
> the think that kept me going was went I start something I want to finish it

I recognized the syntax errors, lack of punctuation and spelling errors as familiar symptoms of Dale's learning disabilities in written expression. It was becoming evident to me, however, that

he certainly appeared to have no disabilities in the learning of mathematics. His deep concentration on this first open-ended problem was fascinating to behold, for in effect, I had never before observed him live up to his claim,"...when I start something I want to finish it." Indeed, the majority of his assignments in English and social studies went unfinished because he simply did not have the perseverence. Importantly, I was excited at the prospect of watching in Dale the opening up of a latent talent which no one, including himself, knew existed.

When Dale asked if there was more than one answer, I told the students to guess how many different solutions they thought existed for the problem. Adie estimated twelve, Ron six, Dale four, and Chad speculated three. I told them that a computer had found over 2300 solutions, and Dale responded, "No way. I don't believe that. Show me where you read that!"

During session two, it was encouraging that the students initiated dialogues on some of the problem solving heuristics that I was planning to introduce in the study. For instance, Dale inquired about the possibility of using an Apple IIE (our only computers in the room so far) to come up with a solution to our problem:

Dale: Can you get the computer to solve it?
DT: I don't think our computers here can do that.
Dale: When you go to SFU, why can't you get the computer there to solve it?

Then Adie hit on the pentominos' connection to combinatorics with her comment, "It's like if you have five numbers, how many combinations are there?" Finally, when I asked them why this was such a hard puzzle, Chad answered, "Because it's such a large rectangle." Remarkably, the suggestion to reduce (or specialize) the problem to a simpler case had come from the group, and not from me.

In response to my question, "What would be an easier puzzle?", the students offered many answers, but I chose their suggestion "3 by 5" for an obvious reason: if one could cover up four 3 by 5 rectangles using all twelve different pentominoes simultaneously, then one would have a

solution to the larger 6 by 10 rectangle. Together, we listed all five possible 3 by 5 solutions, but alas, no four used all 12 pentominoes. (See Figure 3.2)

Figure 3.2 The five 3 by 5 solutions.

FPU   LNV   LPV   LTY   NPU   PUV   PUY
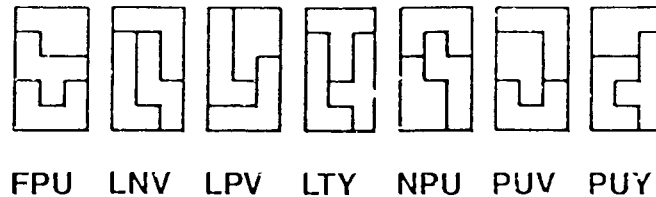
I then asked the student pairs to work on 5 by 6 puzzles; they understood that two such simultaneous solutions with no duplication would result in a solution to the 6 by 10 rectangle. In the time that was left, they came up with several solutions, but no two were simultaneous. In the meantime, by 2:30 PM, Juliana had her head down on her desk, and Minna was evidently disgruntled, muttering quietly something about "not wanting to do this shit any more." With student interest obviously waning, and not wanting to belabour what was supposed to be a simplified version of the original problem, I decided not to pursue the solution the next day.

Nonetheless, the session ended on a very positive note for me. After the class, Chad told me his opinion of his first session with the group:

> Chad: Time has really flown by today. I think this is a good approach to math.
> DT: Why do you say so?
> Chad: (*pausing to think*) Because it's fun...and interesting.

I was delighted that Chad, formerly so negative towards education, was expressing his approval of my approach to doing mathematics. It was fortunate for my sense of well-being that he did so, because by the end of the week other students would have reacted in quite the opposite way.

SESSION 3   Thursday   10 February   (Dale, Juliana, Donny, and Ron)
"It was enjoyable because it was a group effort."

I was extremely disappointed that only four students (out of a possible eight at this point) showed up for session 3. Even though Chad, Adie, Minna and Alonzo phoned in with legitimate excuses, I was upset, and decided to cancel the mathematics session. But Dale objected. He said to me, "We have math scheduled on Tuesdays and Thursdays, and there should be no exceptions. If you cancel, they will think they can stay away and have the math postponed." I could not help but agree with his reasoning, and the session proceeded. Thereafter, true to Dale's proposal, the mathematics group met regardless of attendance.

I had planned three activities for this session. First,the students would complete a questionnaire about their past experiences in mathematics. They would then work on their first assignment, which was essentially to write a recapitulation of their experiences with pentominoes. (See page 85.) Last, I planned to introduce topic 2: probability.

The class started off poorly, with great underlying negative friction and frequent abusive outbursts between Dale and Juliana. Eventually, I had to intervene, with the promise to send them both home if the verbal taunts continued. I told them that their antics were starting to give me "a pain in the gut." This comment seemed to quell their feisty spirits for a time.

The four students then separated into pairs with instructions to collaborate on their assignment. Juliana and Donny worked quietly behind a screen at the back of the class. There was little dialogue between them, and in the end, contrary to instructions, they handed in completely separate summaries for part A. Ron and Dale finished both the survey and the assignment, but Ron was sarcastic throughout the exercise, calling it at one point, "pointless bullshit." Dale contributed greatly to the completion of the summary, dictating extemporaneously while Ron mainly transcribed Dale's words. Juliana returned to the larger group table, and had suddenly become friendly and chatty. Ron joined in the verbal abuse, calling her a "feminazi."

My first attempt at Pimm's (1987) aforementioned "talking in pairs gambit" was obviously unsuccessful.

At this point, I realized two things: The students would need encouragement and direction if I expected inter-student communication, and the ability of this group to get along well enough to do mathematics was in question. Although I believed my main mandate at VRLF was to provide successful academic experiences for my students, without social order it would be impossible to teach mathematics. As Csapo (1986) comments,

> [h]uman survival, development, learning, and accomplishment in life depend on the acquisition and maintenance of skills necessary for successful social interaction. Individuals who exhibit some degree of social incompetence may find themselves at a disadvantage when compared to socially competent peers. (p.1)

With no formal training in abnormal group psychology, I still believed that mathematics could be a vehicle where social skills are developed. A "social curriculum" is a two-way street: the best mathematics education should occur in a group context, and conversely, the social order of the group can be enhanced through its deliberations of mathematics.

In spite of my expectations of social growth for my students, I realized that they were simply practising behaviours to which they were accustomed: that is, their educational experiences had them working silently, alone, and with little opportunity, at the secondary level at least, to express themselves mathematically. I also knew that my students had often been subjected to circle meetings and encounter sessions where they had been encouraged to "bare their souls" in public. I was certain that numerous interviews with psychiatrists, social workers, parole officers and counsellors had left many VRLF students reluctant to divulge any further their inner feelings, even if these feelings were about mathematics. Some students even had a sort of "psycho-smartness" about them; in other words, familiar with current psychological jargon, they were often adept at answering personal questions. For instance, in answer to part C-2 of Assignment 1, Juliana stated

|o|bviously this assignment had some educational value or we would have not been subjected to such material. These kind of problems not only have to do with math but social interaction, whether or not the subject can function well within a group. This is probably observed as well as the math problems presented.

For these reasons, then, I was hesitant to engage the students in an immediate follow-up group discussion of Assignment 1 which quizzed their affective reactions to the last three sessions.

PENTOMINOES: ASSIGNMENT 1.

For this project:
  1) Work with a partner on Part A ("Summary of Events"). Hand in one summary between the two of you.
  2) Work alone on Part B and Part C. Hand in your own responses to these sections.
  3) Time for the assignment: 1/2 an hour. Total marks = 20)

  A) SUMMARY OF EVENTS (9 marks)
    *Write a summary of what you did with the pentominoes, starting with your **introduction** to them (as wooden blocks). Describe the **problem** you were asked to consider. Describe your **efforts** to solve it, and the **results** you obtained.*

  B) IMPRESSIONS OF THE EXPERIENCE (8 marks)
    1) Did you find the experience enjoyable? Not enjoyable? WHY??
    2) Were you ever frustrated? Describe the feeling.
    3) Were you bothered that, as a group, we have not yet found a solution to the
        problem? Explain why you were (or were not) bothered.
    4) Do you feel like coming back to this problem in the future? (eg today, next week, next year) Why or why not?

  C) WHAT DID YOU LEARN? (3 marks)
    1) Name 2 aspects of this experience which might be useful in solving other problems.
    2) Name 1 thing from this experience that you will remember for the rest of your life.

Out of the eight starting students in the cohort, only Alonzo did not attempt the assignment; he was away on work experience at the time. Since Chad started late with the group, he completed a solo effort on Part A, a day after the rest of the group. As well, Adie and Minna were absent the day that the in-class assignment was completed; therefore, their commentaries were done separately as well. The group wrote the following summaries for part A.

  Juliana: First we were asked to make shapes with 5 blocks requiring all the edges put together to produce various shapes, we came up with 12.
    We then cut the shapes out from paper and asked to fill a 6 x 10 grid filling each space with a pentaminoe shape.
    We were then asked to fill 3 x 5 grids that required 3 pentaminoe shapes per grid, and

again but with 6 x 5 grids this time.

Donny: We were first ask to make as many shapes with 5 blocks and all the edge touches so we came up with 12 different shapes. Then we were to fill the 12 shapes into a 6 x 10 grids which was really hard because we would always be short by 1 shape. Our goal in this particular activity was to be the first group to solve this puzzle.

Dale and Ron: We used blocks to make pentominoes. We then tried making numbers, letters and animals out of these pentominoes that we had constructed from the blocks and then from paper cut-outs. We then attempted to put these pentominoes in question, into a 6 x 10 sheet of paper with a rectangular graph on it. Dale K. after an extensive amount of aggravating time finally figured out the puzzle. Our teacher, Mr. David Tambellin [a lover not a fighter] then forced us to pursue making smaller 3 x 5 puzzles to see if we could put four 3 x 5's together and make a six by ten puzzle. Through our testings we found this be impossibe. So we then moved on to trying to put two 6 x 5's together to make a 6 x 10 rectangle. And in our efforts we are very close to doing such a task. Now we are currently writing this assignment to recap our findings.

Adie: We started out with the blocks and made a few pentominoes. Then we cut out pentominoes and tried a few puzzles. We had started with the harder puzzle and then down to the easier ones. First we did a 6 x 10 rectangle and found 1 way to do it with 12 pentominoes, then we 3 x 5's and 6 x 5's. We found quite a lot of ways to do it.

Minna: We started with the wooden blocks and made pentominoes. Then we cut out pentominoes and tried a few puzzles.

Chad: I had to discover the 12 pentominoes. We were asked to try to fit them into a 6 x 10 rectangle. I attempted to solve it by trial and error but it was difficult. Then we tried examining the problem on a smaller scale. This brought to light new, smaller patterns that could be fit together on a slightly larger scale. These brought out many different combinations of 6 pentominoes. We tried to find 2 combinations that didn't use the other's pentominoes.

I was generally impressed with individual efforts at writing a summary of events (Part A). The students took obvious care to be precise and clear in their descriptions of the pentominoes activities. In particular, Chad's account was not only articulate, but also it captured well the spirit of the "specialization to simpler cases" heuristic.

In answering part B, most of the students wrote that they enjoyed the experience, and were quite forthright in describing any negative feelings towards it:

Juliana: I found the experience entertaining because it was a challenge, but an easy one at that. When I couldn't fill the spaces with the shapes properly I became easily frustrated and disinterested with the activity.
There is a solution to this problem that we have not yet solved, it doesn't bother me, but as for the others I do not know. It wouldn't bother me to return to the problem given but I wouldn't be choked up if I didn't.

Donny: I found this experience enjoyable because you need to do alot of thinking to come up with 12 shapes and the filling the 6 x 10 grid was even harder. I was never frustrated because it was fun, easy and it seem like a game. I was not bothered because even though it was not solve, we put alot of effort and teamwork into it. Maybe once in a while I would come back to it to see what I could have done different.

Dale: B(1) Yes I found it enjoyable because it got my mind wondering and gave me something to look forward to: ie: finishing it.
(2) Yes I was frustrated I was getting eratated and annoyed.
(3) Yes I was and I will be bothered until we find a solution.
(4) Yes I do in the near future so I could figure out the problem.

Minna: B(1) It was alright I didn't pay too much attention so I wasn't shure of what to do.
(2) Yes. Didn't know what to do.
(3) No
(4) Yes. Next time I'll try and pay more attention.

Chad: B(1) It was enjoyable because it was a group effort. There was active participation + learning which is a lot better than the work we do alone here, in my opinion.
(2) No. If I was then it was only the natural frustration of not being able to do something.
(3) No I am not bothered because I suspect it is part of the techniques being used to teach the class.
(4) I don't think I'll make a concious effort.

Adie: B(1) I found the experience not entirely enjoyable. It was fun at first but then I got bored of tring.
(2) Yes, I was pretty frustrated. I got pretty pissed off when I was trying to figure out the 6 x 10, so I gave up a couple times, but it was ok.
(3) For a while I was bothered that us as a group didn't find a solution to the problem but after a while I didn't care. I was bothered because it was taking too long.
(4) I don't really think that I would want to come back to this problem because I'd get really sick of it and not bother doing it.

Ron: B(1) Well it passed the time, I'll give it that. But it was no trip to playland.
(2) Yes I was frustrated. Hence I felt frustrated. It was agonizing but pleasant.
(3) I was and am not bothered because its only a math course its not the entire world to me.
(4) I'm flexible on returning or not returning to this problem in the future.

The students' comments about their frustration levels verified my earlier hunch that the experience was too drawn out for many of them. Indeed, open-ended problems are problematic in this way: if a solution is too long in the finding, and if the students' activities or heuristics are not varied enough, tedium or loss of interest can rapidly occur. At this point, I made a note to myself that extra care would have to be taken to control the pace of future group experiences.

Although I was concerned that Ron's remarks were written in a derisive vein, I was pleased at the reactions of Chad and Donny to their work. In particular, their expressions of satisfaction as active, team problem-solvers was music to my ears. I could place credence in Chad's remarks especially, because earlier in the year he had openly expressed to me an obvious dislike for the way the courses were taught at VRLF. Indeed, I had reason to believe all my students, since they were, for the most part, well past the stage of trying to impress any authority figures in the educational hierarchy.

Importantly, Chad was able to recapitulate the two heuristics which the group practised during the pentominoes lessons:

> *Name 2 aspects of this experience which might be useful in solving other problems.*
> **Chad**: The patterns and the concept of scaling problems down in size.

Finally, the answers provided to the last part of the assignment varied greatly from student to student:

> *Name one thing from this experience that you will remember for the rest of your life.*
> **Dale**: The one thing that I will remember is the fusteration.
> **Chad**: The word "pentominoes."
> **Adie**: One thing from this experience that I will remember for the rest of my life is the teacher who gave it to me!
> **Ron**: There is nothing from this experience that I will remember except for maybe the boredom.
> **Donny**: The grid and all the effort I put into it is what I would remember for the rest of my life. If I don't solve it I would probably remember it for the rest of my life.

I was particularly happy with all of Donny's responses, not only for their content, but also for the extra effort he put into them; as an ESL student, Donny could benefit not only mathematically, but also linguistically from the new emphasis on communication. Hopefully, the practice derived from such written assignments would help all my students overcome any linguistic skills deficits.

I marked the students' assignments mainly on the basis of the depth of the answers given. Thus Dale, who was the only student to actually solve the 6 x 10 puzzle, did not receive any extra credit for his feat. (In retrospect, I feel he should have been recognized with a few bonus marks.) I hoped that the high marks given would serve two purposes. First, I felt the students should receive a boost in self-confidence by obtaining a relatively good grade on the very first assignment. Second, I wished to promote the type of self-expression which was asked for in the assignment: my students needed all the encouragement they could get to open up lines of communication.

I felt that the pentominoes experience was successful. With the exception of Ron, all the students had participated actively in the two sessions, and generally had given positive reports of their experiences. Pleased with Dale's new demeanor, Chad's expressed interest in group learning, and with the general contributions of Adie, Juliana and Donny, I was looking forward to presenting the next topic: probability.

Topic 2: Probability

SESSION 3 continued   (Dale, Juliana, Donny and Ron)

I planned to introduce probability with a "motivating question" which hopefully would have the following qualities. It should be of sufficient complexity as to allow the students to engage in open-ended problem solving for at least two or three sessions. It should relate strongly enough to the topic at hand to serve as a starting point for student investigation. It should not be too difficult for the students to comprehend, and to work at. Last, it should appeal to the interests of adolescents.

Accordingly, I adapted the following problem from the NCTM's (1992) *Addenda Series* publication *A Core Curriculum*:

<u>Motivating question:</u>   *On average, there are 10 babies born per week at Grace Hospital. An obstetrician at the hospital, Dr. Mat Ernity, observed that over the last week (31 January -6 February), out of 10 births there were 4 girls born in a row. The other staff thought that having 4 girls born in a row was a rather unusual event.*

  Dr. Ernity, however, made the following bet with several of his colleagues. He said, "I will wager $100 that this event (4 girls in a row) will occur not only once, but 5 times again within the next year.

Was this a good bet for Dr. Ernity to make?

Two students volunteered the following initial comments:

  Juliana: Out of 52 weeks, there have to be 4 girls born in a row.

  Dale: But the staff said that this was an "unusual event".

In other words, Juliana proffered that this would be a good wager, whereas considering what he thought was a clue in the presentation, Dale voiced the opposite opinion. It was encouraging to have these students "go out on a limb" mathematically: as pointed out in Chapter 2 (pp. 17-19), alternate school students are often fearful of public failure, and so are timorous about publicly committing themselves to a question.

Soon after, Dale continued brainstorming the situation. He asked, "What is 52 divided by 5? There would have to be 4 girls in a row about every 10 weeks or so." Not only was he again practising quick mental arithmetic, but also, with his question/self answer, he was evidencing comprehension of the problem situation. Indeed, given the small group size, I soon came to rely on such extemporaneous questions and comments for their value in contributing to student assessment.

I then told the students to write what they thought would be a typical pattern for ten births. I said, "For example, record a boy followed by two girls by writing BGG. Continue on until you have ten births. Then, repeat this process four more times, until you have five lists of sequences for ten births each." In retrospect, the class should have completed this activity first, before hearing the motivating question. In this way, the results would have been more natural and intuitive, and not biased to the "four girls in a row" aspect of the problem. Nonetheless, the group quickly finished the preliminary task, and concluded that four girls in a row is probably an odd occurence.

When asked how we could model the births of ten girls in a row, they were quick with their suggestions:

Juliana: We should do a draw out of a hat. Use ten different patterns.
Dale: We should get pieces of paper with B or G written on them and draw them out of a hat.
DT: That's a good idea, Dale.

In a show of one-upmanship, Dale then teased Juliana about her comment. It was becoming evident that Dale was not entirely blameless for the jibes and abuse which were directed against him by others in the class. Such thoughtless behaviour, which is often typical of ADD adolescents, would only acerbate fellow students, and exacerbate Dale's already poor peer status.

I told Juliana that her idea was a good one too, because we would look at these "patterns" very soon. Her suggestion indeed had some merit, although she was off-base with the idea of putting only *ten* different patterns in a hat. (The correct suggestion of 2^{10} permutations would have been right on the mark, but difficult to handle at this point.)

Since most of the third session had been taken up by Assignment 1, and by the survey, time ran out at this point, and the school day was concluded. I left that day with a strong sense that the students would be willing participants in the activities to come on the following Tuesday.

SESSION 4   Tuesday 15 February   (Adie, Chad, Donny, Dale, Minna, Ron, and Alonzo)
"I belong to a family of five children."

Ron was present, but chose to work on his Social Studies 11 course instead. Alonzo returned from work experience but Juliana was absent.

Before the session began, Dale could not resist snooping in my shopping bag which contained recently purchased decks of playing cards and pairs of dice. The idea immediately came to him to use cards to simulate births: a red card would represent the birth of a boy, and a black card that of a girl.

We first reviewed some basic concepts of probability using the cards and the dice. The students had absolutely no problems with questions like, "What is the probability of drawing a red King out of a full deck of cards?"

Since Minna, Adie, Chad and Alonzo were absent from the last session, I had to present the motivating problem again. But unlike last day, I first instructed the new students to write down some examples of ten births in a row. Interestingly, each of their creations included at least one stretch of four or more girls. Then I asked the group for suggestions on how to model the births of ten children. Dale eagerly volunteered his idea to the group, and all three groups began to use the cards, with one person dealing, and one person recording 13 strings of 10 cards each.

Without saying anything, I brought out the new dice and placed them on the group table. After Adie and Minna took a pair, and began to experiment: on their own, they decided to let a throw of 1, 2, or 3 represent a boy's birth, and a roll of 4, 5, or 6 a girl's. Finally, Minna and Adie then switched to a new model: they did the last four runs by flipping coins. I was most encouraged to see first Dale, and now Adie and Minna invent their own models of this real-world situation. But when Dale offered to help them, Adie's disdainful reaction was ,"Yeh I know!" With the increase in verbal put-downs, I could sense the friction escalating between Adie and Dale.

So far, model results were rather atypical: for example, out of a total of 13 trials of 10 "births" each, Alonzo and Don recorded *no* runs of 4 or more girls. The group ran a total of 52 simulations of 10 births in a row. The final experimental probability calculated from their efforts was 6/52 or .12, which was also low. (For the answer to the motivating question, "What is the probability that a series of 10 births will have 4 or more girls in a row?", see page 106.) So at this point, hoping for less divergence between the theoretical and the experimental, I was eager to have the students try some computer modeling.

The prospect of using the computer was exciting for me, since this was the first real use my students made of it outside of computer assisted instruction and word processing. I explained that the computer could "flip a coin" through its random number generator, and that essentially it would print a "B" or a "G" according to whether a "1" or "0" respectively turned up. Using an overhead projector, I attempted to show the students the actual step-by-step logic of the small program being used. However, after trying to make sense out of the lines of Applesoft Basic, the students expressed consternation. I realized right away that a foundation in elementary programming should have preceded exposure to loops and random number functions.

Each pair of students ran fifty-two computer simulations of ten births each. The total experimental probability of four girls in a row recorded by the whole group was 25/158 or 0.16. I explained that the fraction obtained by the group was experimental, whereas their answers for the earlier card and dice questions were termed "theoretical" probabilities.

Almost as if in anticipation of a problem I was going to present to the group, Adie then set things rolling with a very astute question:

> Adie: Is it possible to get two identical runs of 10?
> DT: That is one of the questions I was going to ask you.

I wrote Adie's question on the blackboard as question number one along with the second question "What is the probability of obtaining 10 girls in a row?" I then tried to make a connection with a heuristic used in the pentomino problem:

> DT: Instead of trying to cover a 6x10 rectangle what did we do?
> Dale: Tried to do 3x5's and 6x10, no I mean 5x6.
> DT: So instead of solving a difficult problem we choose to solve a...? [*This was a leading question!*]
> Chad: A simpler one.
> DT: Great. What is it about our original question here which makes it a difficult one?"
> Chad: The four.

DT. Yeh, OK. What else? [*In retrospect this should have been recognized as a useful answer too.*]

Adie: The ten.

DT: What would be a simpler problem be then?

Chad: Make it out of 5 births.

DT: Alright let's consider five births. I'd like you to look at this question number three as well.

Rejoicing inwardly over Chad's suggestion of five, I wrote on the blackboard, "How many different birth orders are there for a family of 5 children?"

Dale: I belong to a family of five children.

DT: What's the birth order, Dale?

Dale: GGGBG

DT: So I want you to find out how many other different birth orders like this one there are.

I was most pleased with the group's problem solving. First. Adie wondered if two identical permutations were feasible. Then Dale and Chad remembered the technique of reducing a problem to a simpler one. Finally, Dale made the connection from the question at hand to the real world: that is, he brought in his own family permutation to the problem. Such comments were indeed useful in assessing student understanding.

Finally, I asked the students to work individually at listing permutations for a series of five births. Chad hit on a pattern of sorts by reducing the number of "B"'s each time He wrote. "BBBBB, BBBBG, BBBGG. BBGGG. BGGGG. GGGGG."

Chad: Is there a formula or an equation or something to compute this?

DT: If there is, I'm not going to tell you. You have to do this question first, and then maybe you will see it.

Dale: Is it 5 times 5 times 2 equals 50?

DT: Too big, Dale. You don't think I'd make you come up with that many do you? How many do you have Adie?

Adie: [*Her head is on the table by now; there are only ten minutes left in the period*] Fourteen. I could come up with a bunch more, but I'm out of it right now.

Alonzo: Is twenty-two close?

DT: You're getting there.

Chad: [*He draws five boxes, and attempts to reason.*] There's 5 choices of 2 each.

Dale: [*Now wandering away from the table*] I'll work on this at home where it's quieter.

The loss of attention by Adie and Dale was not a concern to me, since it happened so late in the period. As at the end of the last session, I felt good about the way the students were responding to the open-ended problem situation. Inter-student rapport seemed improved. Little did I suspect, however, that a major emotional storm would erupt at VRLF the very next day.

INTERSESSION: Wednesday 16 February
"I find your program ridiculous."

At the beginning of the class I explained last session's activities to the recently returned Juliana. She did thirteen simulations with cards and I made a point of listing her results with the rest of the group's. Interestingly, while she copied last day's questions from the board, Dale hovered in the background, offering ideas and suggestions to the two of us. Even though no formal mathematics session was scheduled, he wanted to continue on with the probability topic.

Whilst in the school office during the afternoon recess (commonly called "break" at VRLF), several staff members and I were startled by a great commotion coming from the hallway. Looking up, I witnessed books flying, classroom doors slamming, and students chasing each other: this was an altercation between three members of my mathematics cohort: Minna, Dale, and Juliana. In the aftermath, Minna was rude and verbally abusive to Melvin: she was suspended for the day. Dale and Juliana were also sent home to cool down. In protest over what she deemed excessive penalties imposed by the staff, Adie threatened to quit the program altogether. Questions of self-doubt abounded in me: "Would I have a math group to work with tomorrow? For the next four months?" But the biggest bomb to shake my mathematics educator's ego that day was yet to drop.

Following the brouhaha, Melvin asked Ron for his impressions of the events. We discussed the school and the students involved, and the topic soon turned to education in general, and to the mathematics case study in particular. Ron was extremely forthright in his comments:

Ron: To be completely honest, I find your math program ridiculous. It wastes my time. I find the math problems to be quite boring. He had us putting together wooden squares, and then making figures or faces out of them.

Melvin: Are there any problems that you could bring in to the rest of the class?

Ron: Not really. Also, the way you are doing it is wrong. You said "Let's make a list of the ways to solve this problem". *[Laughs]* I can understand those ways without your explanation. You over-explain everything. You should get rid of those stupid questions like "What did you feel about the experience?" Those written questions are just like an assignment out of a book. The students do them only because they want to get them over with. It would be better if you asked them orally. Also, you give too much explaining about how to answer them. I know what they mean, you don't have to go over each one and re-explain in detail. It's really patronizing. I learn more when the teacher lectures, and tells me what I have to know, and then the teacher asks all the students questions about the lecture. I will say this: I find your program so far wastes my time. I will grant you the "so far."

I then explained to Ron my own feelings on lectures; in particular I outlined my negative student experiences at university with that style of teaching. Although he did not buy into this argument, he did make one concession to me:

DT: Maybe then it's not the math problems, but the way they're presented.

Ron: Yeh, that could be.

Trying to appeal to his interest in drawing, I showed him a math-art poster. He commented that he was not interested in that kind of art, that the poster was actually quite simple and that he could copy it in ten minutes if he wanted. Finally, I gave him a list of three probability questions hoping that their relatively high degree of difficulty might trigger his curiosity:

1) How many different "birth orders" (permutations) of 10 children are there?

2) How many different permutations (in birth order) are there for n children?

(Find a formula!)

3) What is the probability for a family of 13 children to consist of all males?

And in view of his avowed reluctance to work in groups, I suggested he try these problems on his own.

Soon after the shock of Ron's denial of my pedagogical agenda wore off, I realized that many of his criticisms were valid. As mentioned earlier, I knew that the pentomino activity involving letter and animal creation was definitely "too elementary" for older adolescents. In addition, I was beginning to doubt the efficacy of formal attitude surveys with these kids; my suspicion was that over their educational careers, they had been over-subjected to probes of their personal feelings. Finally, I was recently noticing a recurring negative feature in my teaching; put simply, I was talking too much in the mathematics class. I presumed that this excess of "teacher talk" might be stifling student self-expression and creative thought; now I had been served notice that as well, my loquacity was turning some students off the program. Well, mea culpa. It was time to regroup and perhaps make some changes to my teaching.

But the day was not entirely a negative experience for me. Before leaving for home, Adie announced her decision to remain in the program. She actually agreed to mediate between the embattled study group members. Indeed, throughout the project, Adie would often withdraw from learning situations; however, soon after such spells of academic negativity, she would return to the fold, and in effect, lift the spirits of her teacher.

SESSION 5   Thursday 17 February 1994   (Dale, Adie, Chad, Donny)
"I have a little book from elementary school with numbers like that in it. I'm going to check them out at home."

At our lunchtime team meeting, I brought up Ron's negative comments towards the new math program. Eric's advice was to avoid pressuring him to participate and to permit him to observe on the periphery, in the hope that he might choose to join the group. Katey commented that some kids feel "safe" working out of a book; in other words, it is non-threatening for them to be able to study on their own with no pressures to actively communicate with others. My colleagues' advice was entirely just and accurate; indeed, the program permitted its students

students to remain inactive learners, and the students were only too well habituated to this mode of education. In the months to come, my challenge would be to rouse my passive study participants to an active mode of mathematics learning.

At this point, I began to worry about the negative effects of poor attendance upon the case study: if such absenteeism became chronic, it would become more and more difficult to maintain group momentum and pedagogical continuity.

Before the mathematics class started, I posed a few questions about the last session to Dale and Chad. In particular, we were discussing the students' results with thirteen experimental trials each:

> DT: Would you want to base your answer on each experiment, or on the computer trying it one hundred times?
>
> Dale: On the computer.
>
> DT: Why?
>
> Dale: Because they *[the students]* didn't do it fifty-two times.
>
> DT: Why do you say fifty-two times?
>
> Dale: OK, because the computer is more accurate.
>
> Chad: The more times they do it the more accurate it would probably get.

In this exchange, both Chad and Dale hit upon the concept of a larger sample space producing a more accurate experimental probability. Interestingly, my third question seemed to make Dale unsure of his answer and so he tried a different tack: I wish now that I had asked him to explain his comment "the computer is more accurate".

I began the session by distributing the second assignment to the students.

PROBABILITY: ASSIGNMENT 2

Main problem: What is the probability that a series of 10 births will have 4 or more girls born in a row? Questions:
> *1) How many different "birth orders" of 10 children are there?*
> *2) What is the probability of having 10 girls in a row?*
> *3) What are the possible "birth orders" for a family of 5 children?*
> > *-List them all.*
> > *-How many are there?*

My only comment to the group was that the questions on the assignment were written in reverse order of difficulty.

The class began with directions to continue on from where we ended last day; that is, work out all the possible permutations of five births. I told them that they would have five minutes, and then we would look at their results. All the students had their attempts from session four at hand, and earnestly began to work from them. Finally, Chad voluntarily listed his results on the board: BBBBB, BBBBG, BBBGG, BBGGG, BGGGG etc. Dale's results were the same, except he started with GGGGG, GGGGB etc. Donny and Adie sat quietly and developed their own lists. To my chagrin, Ron remained in the back, scarcely looking up during the whole hour. He did social studies, and did not mention his personal "assignment" from yesterday.

Before the five minutes expired, the students began to discuss their results.

Dale: I think I got them all. [*He had a total of sixteen permutations*].
Chad: What's the formula for this? You have a two because there are two choices, and you have a five.
DT: Some good thinking going on here.
Dale: Is the formula five times five times two? ?

Chad's question concerning a possible formula and Dale's reasonable guess were exactly the type of inquiring behaviour I was hoping for. However, the students began to have difficulty keeping track of their permutations past twenty-two. Remembering last session's ennui, I wrote the first combination on the board: BBBBB.

DT: What is another possible combination ?
Adie: You could have four boys and one girl.

From here, the students picked up the pattern, and quickly identified the remaining combinations. Then, we wrote down all the permutations of each combination; the process involved the active, vocal participation of Chad, Dale, and in particular Adie. Don, always the silent one, remained on task by writing the results down. They easily found the five permutations of the 4 boys-1 girl

combination, and while the task of listing the permutations of 3 boys-2 girls took a while longer, student interest remained high, until we had listed all ten. At this point, we had

5 boys	4 boys 1 girl	3 boys 2 girls	2 boys 3 girls	1 boy 4 girls	5 girls
1	5	10			

Chad surmised that the next combinations would be 10, 5, 1, "since they are identical to the first three." To this point, the group had discovered the following:

5 boys	4 boys 1 girl	3 boys 2 girls	2 boys 3 girls	1 boy 4 girls	5 girls
1	5	10	10	5	1

Enthused now with the knowledge that the total number of permutations for 5 births was 32, the students spontaneously began to guess for ten births. Chad offered sixty-four; Dale suggested one hundred thirty-eight, and Adie surmised two hundred eight. Hoping that they would opt for an even simpler case than five births, I asked:

DT: Do you want to go through the same procedure for ten now? We could start like this [*writing on the blackboard*]:

10 boys	9 boys 1 girl	8 boys 2 girls

Dale: [*rising, and going to blackboard*] I can tell what numbers go in the first two. [*writing below my combinations.*]

10 boys	9 boys 1 girl	8 boys 2 girls
1	10	

DT: Do you want to continue on? It'll look like this: [*writing on board*]

10 boys	9 boys 1 girl	8 boys 2 girls
1	10	
BBBBBBBBBB	BBBBBBBBBG	BBBBBBBBGG

What will the total number of permutations be for the eight boys-two girls combination? Do you really want to try it?

Adie: No way.

DT: So instead of ten, what could we try?

Dale: Try four.

Adie: Oh no!

Recalling that she was worn out at the end of session 4 while doing the same thing, I tried to placate her.

DT: Don't worry Adie, it won't be too tedious. Just try it. First, write down the combinations you're going to work with. What are they?

The group came up with the following:

4 boys   3 boys 1 girl   2 boys 2 girls   1 boy 3 girls   4 girls

Looking back, I now realize that my questions, "Do you want to continue on?" and "Do you really want to try it" were rather leading in their attempt to bring the students to try a simpler case. However, at the time, I was extremely worried about student frustration and boredom, and wanted to keep things moving quickly. Such interference on my part, coupled with excessive teacher-talk, was continuing to concern me.

From this point on, the interchanges were lively, and the suggestions came fast : this situation was unusual for so near the end of the day!

Dale: [*right away*] I know what the first two are. The first number is 1, and the second number will always equal the number of babies.
Chad: Would the third one be double the second number? [*Observing the n = 5 example*]. One thing we've established is that the number of combinations will be one greater than the total number of the family. [*i.e The nth row of Pascal's triangle has n + 1 entries.*]

Together, we found that the total number of permutations of two boys and two girls is 6. Chad finished off the row as follows:

4 boys	3 boys 1 girl	2 boys 2 girls	1 boy 3 girls	4 girls
1	4	6	4	1

Observing that time was running out, I asked the group to work out the permutations for $n = 3$. As the students called out the various entries, we built up the three columns in the following table on the blackboard:

Figure 3.3

Number (*n*) of Children	Total permutations	Permutations of each combination
1	2	1, 1
2	4	1, 2, 1
3	8	1, 3, 3, 1
4	16	1, 4, 6, 4, 1
5	32	1, 5, 10, 10, 5, 1
6	64	
7	128	
8	256	
9	512	
10.	1024	

As we approached the 256 mark, there was general excitement about converging on the answer. Finally, when the answer (1024) came out, I could sense group's the satisfaction. I then asked, "Can you tell me how many permutations there will be for a family of thirteen children?" Adie, Dale, and Chad, all computed the same answer on their calculators; Donny was still very quiet, but appeared to be listening intently.

DT: So, what is the probability that a family of 10 children will have all girls?

Adie: One in 1024.

DT: What's the most likely configuration in a family of 4?

Adie and Chad: Two boys, two girls.

Dale: One thing you know for sure. [*Goes to the board and adds 1, 6; 1, 7; 1, 8; 1,9; 1,10 to the thrid column for n = 6, 7, 8, 9, and 10 respectively.*]

DT: What numbers will go on the extreme right side?

Chad: 6,1; 7,1; 8,1; 9,1; 10,1.

DT: If there are 20 children in the family, can you tell how many permutations there will be without doing the doubling process?

Chad: Should we be able to tell right away?

DT Yes. You can give the answer instantly if you know the formula.

Chad: We have to find out how to go from the numbers {1,2,3,4...} and get {2,4,8,16...}

DT. Well said, Chad. Repeat that please.

Chad: Take any given number, and get the number in the middle.

DT: These numbers in the middle are special numbers, I'll tell you that. If you're into computer science, those numbers will come up a lot.

Chad: I have a little book from elementary school with numbers like that in it. I'm going to check them out at home.

This was a timely place for the session to end. I was pleased that the students were able to answer each of the three questions on their original assignment. Furthermore, the contributions and inductive insights offered by Dale, Adie, and Chad gave me cause for joy: here was the mathematical awareness which I hoped would grow in my students. Already I was looking forward to the next session and one of my favorite topics: Pascal's Triangle and its myriad of patterns.

INTERLUDE: Friday 18 February 1994 (Dale and Alonzo)
(Dale as mathematics teacher)

Returning after a two day absence, Alonzo asked me what he had missed in Thursday's session. Hoping for an opportunity to observe some student interaction, I asked Dale if he would explain the last day's activities to Alonzo. He agreed. Importantly, this mini-session would be the first instance of inter-student communication with no teacher direction whatsoever.

Chad was at the next table, seemingly engrossed in a book and when I asked him if he would like to participate, he responded, "Not really; not unless I have to." His tone was neither flippant nor annoyed, but rather apologetic. Juliana was also present, and I purposely did not ask her if she wanted to sit in also. (This decision was based on the uproar between Dale and Juliana two days earlier. Also, I was interested in whether or not Juliana would show any personal initiative by participating voluntarily.) She spent the time engaged in gathering new social studies modules from the filing cabinet, and paid no attention whatsoever to Dale and Alonzo.

Dale and Alonzo's activity was unique in another way as well: this was the first time I used recording instruments to audiotape the voices of my students. (All transcripts up to this point were done in a personalized form of shorthand which I learned whilst attempting to keep apace of fast lecturers at university.) I asked both if they would mind if I recorded their tutorial session. They agreed, although somewhat reluctantly. It is not difficult to understand their hesitancy: this first attempt at using a tape recorder involved no less than three rather imposing microphones

placed on their table, and a comcomitant tangle of cords. (Soon after, I learned that this technological overkill was completely unnecessary; one unobtrusive piezo-electric microphone would suffice to pick up the conversations of all my students sitting around the group table, as well as the voices of any people standing close by at the blackboard.)

Dale's behaviours while tutoring Alonzo impressed me. In his explication of the permutations of the births of five children, he sought to involve Alonzo in the process with questions like "And then what do you um, think you do afterwards?" and "Can you think of any other ones we could do?" In other words, rather than give a straight out explanation to Alonzo, he demonstrated some good teaching techniques. In addition to the direct questioning, his frequent use of the rhetorical mini-question "right?" indicated a social awareness of his classmate. Finally, his explanation evidenced thorough understanding of the original mathematical aim of session five: that is, find a pattern or formula which will allow us to calculate the total number of permutations of n births. Here, Dale was referring to the chart which was left on the board from the last session. (See Figure 3.3.)

> Dale: And then for the next one."What is the probability of having 10 girls in a row?" right, and...it...we'd have to do it this way, right? Or, we could find out, ah, some type of formula that we could do for it right? Cause it'd make it...it'd take forever I mean if you do it this way, right? So, like what we did, we put...we tried...we made a chart, up on the board, right? See up on the board? We have 1,2,3,4,5,6,7,8,9,10 right? Then we did number 4 right? [*writing*] Um, did 4 right? And we did exactly the same. Four boy, 1 girl...uh sorry. Four boys, no girl. Three boys, 2 girl, 3 boys, 1 girl, and then 2 boys, 2 girls, and then 3 girls, 1 boy, and 4 girls, right? When we did that, we got 16 altogether right?
> Alonzo: Uh huh.
> Dale: So we put 16 up on the chart. And then we did 3 and we got 8, and then we did 4 and we got 2 right? So like, the main thing that we wanted to uh, find out is uh...
> Alonzo: A sequence?
> Dale: Yeh, like uh...
> Alonzo: Like how they're cut in half each time?

Dale: A method, a method that you could do it more easily, like straight out of your head, right? So like what we figured out is, if you look up on the board....Now, can you find a sequence out of like, how it goes 2, 4, 8, 16, 32, 64, 128 ?

Alonzo: It doubles each time.

Dale: Yeh, it doubles each time, and how we got the numbers on there...we just added up, like all the ones that were on the board there right? And it doubles each time. And now it'll be a lot easier to figure it out right?

Alonzo: Uh huh.

Dale: Then for, 10, as you can see, it was 120. Or 1024, right? So it's one out of 1024 that you could get ten girls in a row.

Alonzo: So....

Dale: Then number one, "How many different birth orders are there of 10 children?"....

Alonzo: [writing] One thousand and twenty-four....

Dale: Just basically like a ...[unintelligible].

Alonzo: So you have the same question then.

Dale: And now, and now you can figure out how you get 30 right? Just by doing the same thing. Right? Like, what we...where we ended we were trying on how to get, uh, to figure out how to get 30 without knowing any of the other numbers, right? And that's where we're going to continue tomorrow.

I welcomed the changes that seemed to be occurring in Dale since he was last in my class the previous fall: once barely able to sit still for longer than five minutes, he was now assuming an active, lasting interest in mathematics. More importantly, he was showing signs of skills which I once feared were lacking in his social repertoire: Dale was the central figure in the cohort's first really sustained episode of cooperative learning.

However, the group's general lack of collaboration still concerned me: at this point, I was pondering whether or not to award points on assignments for participation in group efforts. On the one hand, I wanted to promote coactive problem solving. On the other hand, I felt that cooperative energy should flow naturally out of a desire to solve the problem. I was thus engaged in a classic debate with myself: do I emphasize intrinsic or extrinsic rewards for desired social behaviour? Then, recalling my earlier commitment to involve the kids in this curriculum, I

decided that one day soon, I should seek *their* reactions to this very question in particular, and to other concerns about evaluation in general.

Postscript: The theoretical probability of recording 4 or more girls [or boys] in a row in a series of 10 births is .25.

Topic 3: Pascal's Triangle

SESSION 6: Tuesday 22 February, 1994 (Dale, Donny, Alonzo)

Juliana was present, but in the first hour prior to the mathematics session, she did nothing but work on her favorite topic: astrology. This time, she was investigating the numerology of winning Lotto 649 numbers. Although she had promised to do social studies over the weekend, there was no sign of any completed homework. ("It's at home.") Soon, she wondered over to visit another classroom but was soon asked to leave. Finally, I discovered her conversing on the pay phone in the hallway, and gave her two minutes to get back to class. ("But I'm talking to a friend.") When she didn't get off the phone, I sent her home. I was in quandry: she was a bright kid, with much to offer the cohort. But her behaviour was far too disruptive for the rest of my class, her attendance was poor (fourteen days missed out of thirty-six since January), and she had completed only two units of school work in the same period. Consequently, in consultation with my colleagues, we suspended Juliana from school until she arranged a meeting with us to "reassess the goals and objectives of the program, and develop a student contract." Too bad, I thought, because the Lotto 649 would be a mathematics topic at our Thursday session. However, Juliana did not present herself on Thursday to work things out; on the contrary, she voluntarily stayed away from VRLF, on suspension, for nine more school days.

Chad was absent, and he left the following rather vague excuse on the school's answering machine: "I had other commitments." Upon hearing this explanation, I phoned him and told him that he should make VRLF his first commitment or go elsewhere for an education. It was disappointing not to have him here because yesterday (after I had offered considerable praise for

his contributions), he said he would show up for sure. At this point, the future for both Chad and

Juliana as participants in the mathematics project seemed dim.

Nonetheless, with the three students who did show up, and with three newly purchased

calculators to facilitate their learning, I was eager to explore with them the connections between

their previous work in probability, and Pascal's Triangle. The partly completed table from last

day was rewritten on the board:

Figure 3.4

Number (n) of Children	Total permutations	Permutations of each combination
1	2	1, 1
2	4	1, 2, 1
3	8	1, 3, 3, 1
4	16	1, 4, 6, 4, 1
5	32	1, 5, 10, 10, 5, 1
6	64	1, 6, __ __ 6, 1
7	128	1, 7, __ __ 7, 1
8	256	1, 8, __ __ 8, 1
9	512	1, 9, __ __ 9, 1
10.	1024	1, 10, __ __ 10, 1

My plan was to initiate a search for patterns in Pascal's Triangle which would allow the students

to induce the missing numbers in the permutations column above. So, after reviewing the

questions in Assignment 2, I asked them two questions which were answerable from the table:

"What is the probability of having 10 girls in a row?" and "What is the probability of 9 boys and 1

girl?" Finally, I asked the questions which I thought would provide a segue into Pascal's

Triangle: "What is the probability of having 8 boys and 2 girls?" and "What are the missing

numbers in our table?"

However, as the result of the faulty layout of the table, Dale surprised me with a different

means of calculating the numbers in the 6th row:

DT: Can you see any pattern which will help you determine the missing numbers today?
Dale: [*Referring to the 6th row*] One plus 6 is 7; 6 plus 1 is, is another 7. So 14 take
away 64 is 50 . So that makes this 25. And twenty-five. Right? [*Dave writes 25's in the*

6th row to give the following: 1, 6, 25, 25, 6, 1 /

DT: You're guessing 1, 6 , 25. How did you get 25 again?

Dale: How?

DT: Yep.

Dale: Fifty divided by 2. There's only 2 spots left open.

DT: Alright. I put 2 spots there ah, because I didn't have enough room on the blackboard.[*I should have admitted that this was just a sloppy layout.*] Let's look at the number of spots that we have going down there and see if you can predict how many there should be.

Dale: [*after a few seconds*] There should be another one. One more.

DT: How many total?

Dale: There should be one more.

DT: [*Erasing the 25's and writing in another blank to obtain 1,6,_ _ , _ , _ , 6,1*] For a total of?

Dale: For a total of 7.

DT: Seven, OK.

Dale: OK, I've got something wrong in that one.

DT: How did you get that?   How did...how did you come up with 7?

Dale: Huh? How'd I come up with 7? [*Goes to the board*] You have 2 here, 3,4,5,6,7....So it'll be 8 for 7, and 9 for 8.   So on and so on.

DT: OK That's one of the patterns you're going to be looking at today. The trouble with the way it's on the board right now is that it's all cramped together. And the fact is, I put two little bars there, and that was confusing. That made you think there were two numbers. OK, you said 24?

Alonzo: 25.
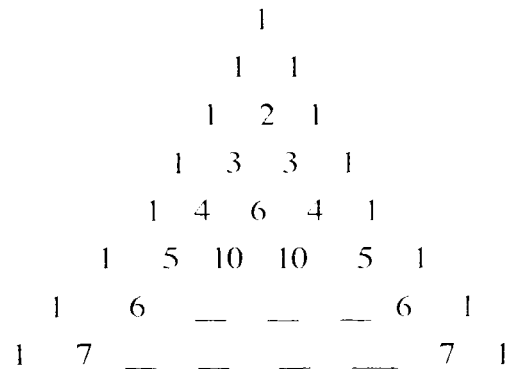
Dale: 25, but that's wrong.

DT: OK But it's a good guess. [*A better word would have been "deduction."*] It's the sort of pattern we're looking for.

Dale: I think once we find out...like up to 7 maybe, we can start to figure out...a pattern.

Given my faulty presentation of the problem, Dale was able to provide an answer which, although technically wrong, was obtained in a logically correct fashion. In other words, he made a great deduction based on the information provided. Throughout my teaching career, such mathematical behaviour by my students has always been a source of satisfaction to me: the members of the VRLF study cohort would prove no exception to this phenomenon.

Wanting to introduce Pascal's Triangle to the proceedings, I suggested that we rearrange the numbers into a blank triangular grid which I projected on the blackboard. (See Figure 3.5 .) I chalked in the first three rows, and Dale volunteered to write in the remaining ones, down to row seven:

<u>Figure 3.5</u>

```
               1
             1   1
           1   2   1
         1   3   3   1
       1   4   6   4   1
     1   5   10  10   5   1
   1   6   __  __  __   6   1
 1   7   __  __  __  __   7   1
```

Dale was the first of the three to remark on a pattern, but it was Donny who finally conjectured the third number in the sixth row:

> DT: These numbers are now in a slightly...slightly more logical pattern here. And we'll see what happens when we look at it a little closer. There's a 1 actually at the top of this. So that forms a perfect triangle, a triangular shape. Now, can we come up with some sort of pattern here to generate, say the numbers which come after the 6? This triangle consists of numbers that are in rows, and we can call (1,1) the first row.
>
> Dale: By any chance....[*Jumping out of his seat and heading back to the board*]
>
> DT: Dale?
>
> Dale: Yeh?
>
> DT: Have a seat.

Here I was concerned that what he had to say would be out of range of the microphone which was placed on the table in front of the three students. In the past, Dale was constantly being told to "sit down." Now, his chair-leaving behaviour resulted not from his ADD, but rather from his increasing motivation to do mathematics. Indeed, such behaviour was to be fostered, not reprimanded.

Dale: [*pointing to the 1,3,6,10...sequence*] Could this be...going down by 4's? No I don't think so.

DT: Going down by 4's?

Dale: Could the next one be 14?

Alonzo: That's what I was thinking too. One, six, sixteen, fourteen...

DT: One, six, fourteen?

Donny: Fifteen. [*Danny has seen the triangular number pattern 1, 3, 6, 10, 15.....!*]

Alonzo: Sixteen, fourteen.

DT: [*writes suggestions in*] Uh huh. Why? [*pause*] Why 14?

Dale: It's just the way that the pattern goes. The pattern goes 6, 10, 14.

DT: Ah!

Dale: And it'll probably go to 20 next.

DT: You're looking at a pattern going down this way? (*Pointing to triangular numbers*)

Alonzo: Uh huh.

DT: OK. That's an interesting pattern going down there. What's the first number in that sequence?

Dale: One.

DT: [*Writes 1,...*] Donny?

Dale: 3, 6, 10, 14.

DT: Why 14?

Donny: 2, 3, 4, 5.

DT: [*Unintelligible*]....Don? I think you said it right there. You saw something.

Dale: It keeps on doubling.

Donny: 2, 3, 4, 5. Change it to 15.

DT: Change the 14 to ...?

Donny : 15.

DT: Why?

Donny : 'Cause it goes 2,3,4,5.

This exchange was typical of Donny's behaviour in our sessions: often he would be the first to become aware of mathematical patterns, but he would inform us of his discoveries in at best one or two syllable communiqués. Given his ESL situation, one would expect such reticence; therefore, I especially tried to encourage self-expression on Donny's part:

DT: Explain...That's correct; it does. Explain that to ah, Alonzo and Dale.

Donny: It goes 1,2,3,...

DT: OK, where was the 2 and the 3 that you said?

Donny: Between 2 and 3 and 4 and 5 and 6.

DT: Ah! Do you see that? See what Donny came up with there? I pointed here [*between the 1 and the 3*] and Donny said...[*pause*] What did you say Donny?

Donny: 2 .

DT: Two. Right? [*Pointing to the 1 and 3*]

Dale: Four and five.

DT: Uh, careful. What's the difference here Donny? [*revealing the answer!*]

Dale: 2.

Donny: 2

DT: You're talking about the difference between the 2 numbers.

Often, in my desire to preserve the class's momentum, I was guilty of putting words or answers into my students' mouths. Here, I volunteered the key word "difference" for Donny and the others. I was indeed aware of my behaviour at this early stage, but my over-loquaciousness was most difficult to stifle.

The group had now established that the sixth row of the triangle consisted of the numbers 1, 6, 15, ___, 15, 6, 1. It was Dale who discerned the remaining entry:

Dale: OK, so the middle one would be ah....[*pause*]

Alonzo: 17.

Dale: It'd be 20.

DT: How'd you get 20?

Dale: How'd I get 20? I added 1, 6, 15, 15, 6, 1 and that equals 44. and then 64 which we already know, take away 44 is 20.

DT Well done. There's lots of ways to come up with 20, and Dale found one I hadn't thought of. See what he did? Explain it one more time, Dale.

Dale: I added all....

DT [*Interrupting*] This is the mystery number here now.

Dale: I added all the numbers up...

DT Yes.

Dale: ...and took away that from the ah average and I subtracted it.

DT: You added all the numbers up and got....?

Dale: 44.

DT: Where did 64 come from?

Dale: 64 came from...that was the total number of the possibilities.

Alonzo: In 6.

DT: In the 6. Very good. Good. You've got the whole number 6. Well done. All right.

At this point, I handed the pupils blank templates of Pascal's Triangle and they each filled in the entries gleaned so far. Sensing the students' restlessness so close to the end of the session, I told them to look closely at what they had so far. This time it was Alonzo who had the "Aha!" or "Eureka!" experience.

DT: Sometimes if you look at shapes or patterns from a slightly different angle the solution jumps out at you. Look at the number 15 in the 6th row. Look around it, below it, above it. To the right and left of it. Look at the 15. Look at the number 20 next door to it.

Alonzo: Five and 10 equal 15. Ten and 10 equal 20.

DT Voilà!

Alonzo: Fifteen and 6 equal 21.

DT: Well done. That is but one pattern.

Dale: 21 and 7 equals 28. And the next one down...28 and 8 is 36. Thirty-six and 9 is 45.

DT: Do you think that discovery might help you generate the rest of the triangle? The 10th row?

Dale: David, so put 1,9 on your big triangle there. It's 1,9,36, ... 35,35,56. *[writing]* No the next one's 56...70...

DT: Finish of the triangle to the 10th row.

This was a tough call. Regretfully, I chose to lead the students to the point where they "fell over" the pattern. On the other hand, they seemed to enjoy their "discovery," and left the school that afternoon feeling good about themselves.

Alonzo ably demonstrated his understanding of the pattern by explaining it aloud to Donny, who was seemingly stuck as the other two computed the remaining entries to row ten.

Later, while working on their assignment, the students were asked to explain precisely how the triangle was generated. I called on Alonzo for a contribution:

> DT: Alonzo, you came up with what I thought was the quickest way to generate the triangle.
>
> Alonzo: So adding the top two numbers.
>
> DT: Right. So put that in precise words. How would you say the triangle is generated?
>
> Dale: By adding...(*pause*)
>
> DT: How do you get any one number in Pascal's Triangle?
>
> Alonzo: By adding those two numbers.
>
> Donny: What was that again?
>
> Alonzo: Add the numbers right above.
>
> DT: How many numbers are right above?
>
> Alonzo: Two.

I was pleased when such vocalizations occurred: they not only demonstrated understanding, but they also served to promote self confidence in the speaker. But most importantly, they laid a foundation for the social interactions which I was eager to promote amongst the members of the study group.

Dale's suggestion for checking our results also evidenced a good conceptual grasp of the situation.

> DT: Now theoretically, do you have to do any more arithmetic to get the rest of the 10th row?
>
> Dale: No.
>
> DT: Why not?
>
> Dale: 'Cause it's just...the same as the other one. 1, 10, 45, 120, 220, 252...
>
> DT: What's the probability in a family of 10 children of having 8 girls...8 boys and 2 girls?
>
> Dale: I got a question here.
>
> DT: Answer my question first..
>
> Dale: Wait. Why does it go 120 and then 110?
>
> Alonzo: 210.[*Correcting Dale's slip*]
>
> Dale: OK. OK. Never mind.
>
> DT: What's the probability of 8 boys and 2 girls?

Dale: The probability of 8 girls?

DT: 8 boys 2 girls.

Dale: 45.

DT: Out of?

Dale: 1024.

DT: What's the probability of 4 boys 6 girls?

Dale: 4 boys 6 girls, that would be 210 over 1024.

DT: In any family of 10, what's...what combination would have the highest probability of occuring?

Dale: Ah....

Alonzo: 4 boys 6 girls.

Dale: No....

Alonzo: 5 boys 5 girls.

Dale: It'd be 5 girls 5...5 girls 5 boys...252. I know a good way you could find out if our thing is right, Dale.

DT: [Ignoring Dale's comment] What does that reduce to? 252 out of 1024? Fraction? [pause] It's about...what? Roughly, 1 out of ...?

Dale: One out of every 4.

DT: One out of every 4 families, yeh. Half and half, does that make sense? In a family of 10 the most likely combination will be....?

Alonzo: Five boys, 5 girls.

Dale: Uh....

DT: These are probabilities and it's come true with the triangle, right?

Dale: For making sure, when you added it up you could just add up all the numbers in the 10th row and see if it equals 1024.

DT: Check it out. Does it?

Dale: [Using calculator] Yeh.

In my preoccupation with achieving closure to the lesson, I ignored Dale's first suggestion. That night, listening to the tape of the session, I realized that the first step to inculcate student participation was easy: I should listen more and talk less.

The students spent the rest of the session working on and discussing Assignment 3. (See Appendix 3.) They were asked to work in teams of three people each, and to submit one

completed project per group. Dale perceived that the eighth and final question was a difficult one: *"What are the first four terms of the 100th row of Pascal's Triangle?."*

> Dale: [unintelligible]...how tough these questions are, Dale. They're pretty tough. (pause) What are all the probabilities for 100?
>
> DT:: The back page?
>
> Dale: Yeh.
>
> DT: Well question 8 is pretty tough.
>
> Dale: For 100, can you just...times...like have...[Goes to board]...1, 20? This would be 85, this would be 240, this would be 520, this would be 304, and this would be so on, so on, and so on, you know. [Mentally doubling entries in 10th row to get 100th row]
>
> DT: You're doubling?
>
> Dale. Yeh.
>
> DT: To go to the 100th row?
>
> Dale: Yeh, for the first couple of them.

After an interlude of five minutes or so, when it became apparent that the students were stuck on question eight, I decided an intervention was in order.

> DT: Look at the back page for a sec. Question 8. It's worth 2 points right?
>
> Dale: Uh huh.
>
> DT: So I'm assuming that you're going to get half a point for every number you come up with. "What are the first 4 terms in the 100th row of this triangle?" What kind of size paper would you need to get down to the 100th row if you were going to....
>
> Dale: Yeh. Yeh. But wait....
>
> DT: No, wait a sec. [Interrupting again] First of all, what sort of size paper would you need do you think?
>
> Dale: A large fucking paper. [laughter]
>
> DT: A *very* large paper. Right, so....
>
> Dale: The first two terms we already know is 1 and a hundred.
>
> DT: Alright, you got 2: you got 1 out of 2 already. [sarcastically] That's not tough eh? Do you see that Donny?
>
> Donny: Yeh. [weakly]
>
> DT: In the 100th row, Dale, the first 2 terms would be....?
>
> Dale: One and a hundred.
>
> DT: What would the last terms be?

Dale: One and a hundred.

DT: Right

Dale: There we go, we got 2 terms.

DT: Out of 4. Now, find the 3rd and the 4th in the 100th row. That's the challenge question for today.

Alonzo: The 3rd and the 4th.

Dale: OK we'll worry about that one last.

My intention here was to preclude student discouragement by acknowledging that this question was indeed difficult. In other words, they should not be dismayed if an answer was not soon forthcoming. And importantly, I wanted to instill in the students the attitude that not all questions in mathematics are readily answerable.

The three students recognized the challenge presented by the third question in the assignment, which asked for the sum of all the numbers in the 11th, 15th, 30th, 100th, and finally rth row of Pascal's Triangle. Having recognized the pattern of doubling the previous results to compute the sum of entries in each of the 11th through to the 15th rows, they saw the difficulty of doubling in computing the sums for the higher 30th and 100th rows. Soon, they expressed frustration with the process. Dale asked , "Okay! But is there a faster way? Instead of fucking adding all...instead of adding all the way down?" Donny came up with the suggestion "Thirty to the power of...thirty to the power of...whatever!" Alonzo's suggestion was a realistic one: "We need a different calculator." However, what the students really needed was a brush-up on exponents and powers. As Dale succinctly commented: "I don't know how to do powers, though."

I wrote several expressions like 3^2 and 5^4 on the board, and asked them if they knew what each example equalled. Donny responded quickly and accurately. But Dale, although he indicated that he remembered the symbolism, obviously needed some tutoring. Alonzo fared poorly as well. But finally, after several examples, Dale and Alonzo were able to compute simple powers, and to write repeated multiplications as powers. However, they were still not able to

convert the row sums from Pascal's Triangle (ie. 2, 4, 8, 16, 32, etc.) to powers of two. I decided another hint was in order.

Dale: Ah gee! I had something going there. OK, if we go 30 to the....no. Yeh, 30 to the 15th power. That be it? [*pause*] Is that it?

DT: You're making it too complex.

Dale: That wouldn't be it? 30 to the 15th power?

DT Here's a clue. Let's go back to the basic probability thing. We're flipping a coin, heads or tails. We're looking at births of children B or G. How many choices did you have? When you flipped a coin? How many choices were there when an infant was born?

Alonzo: Two.

DT: Start thinking 2. Two.

Donny: You just double it?

DT: Certainly doubling is coming into it there.

Dale: Okay.

Donny: Something to the power 5...

DT:: Oh. Something to the power 5.

Donny: ...equals 32.

DT : [*writing on board*] $\square^5 = 32$, $\square^4 = 16$, etc.]

Dale: OK. Something to the power 4 equals 16. Something to the power 3 equals 8. So for r [row] 30, we don't know yet! [*Laughs*]

DT: No but it's getting...can you fill in the blanks there?

Dale: OK, for r 30, something to the power 30 equals something. For 3c [*question 3c*] something to the power 30 equals something. Am I right? Then for a hundred, something to the power 100 equals something.

Donny: Ah, 2 to the power 5 equals 32. Right?

Dale: 2 to the power of 3 equals 8; 2 to the power 4 equals 16.

Donny: Yeh.

Dale: 2 to the power 30 equals whatever we're getting. 2 to the power 100 equals whatever we're getting. It's 2 to the power...for everything.

Donny: Yeh.

Dale: [*Celebrating*] Yeahhh!

DT: What about r?

Dale: r ? 2 to the power of r

It was heartening to witness such moments of triumph when discovery resulted from the inductive process of observing patterns which was made possible in turn by a simple review of the "basics".

When I asked the students if they could "figure out" 2^{100}, Dale suggested that the answer would not fit on the calculator. When asked to discover how far the machine would take them, they found that their calculators overflowed at about 2^{25}. I told them that mathematicians would readily accept the representation 2^{100} as a final answer, explaining that 2^{100} had as much meaning as any huge number they might have somehow worked out. Then, when queried for the sum of all the numbers in the 500th row of Pascal's Triangle, they all agreed with Dale's answer "2 to the 500th power." This, I said, is the value of a generalization: if you know that the answer for the rth row is given by the formula 2^r, where r is any number you like, then you could give an answer for the 2 millionth row of the triangle. Dale's laughing response was, "You know what? If we knew this a long time ago, it [the assignment] would have been history." Dale, at least, seemed to appreciate the value of a general formula.

The students had only two days to complete a rather difficult assignment. Although I felt that they had enough understanding to answer most of the questions, they would obviously need further instruction on question 6 ["*What is the third number in each of the 15th row, and the 30th row?*"] and on question eight ["*What are the first four terms in the 100th row of Pascal's Triangle?*"]. I hoped for some real collaboration on their part, and in particular that Donny, Dale and Alonzo would readily agree to tutor Adie, Chad and Minna who missed session 6.

SESSION 7 Thursday, 24 February, 1994 (Donny, Adie, Chad)
"Is there an easier way to do that?"

For this session, I planned to have the students centre their investigation of combinatorics around the mathematics of lotteries. I hoped that the students would discover the relationship between combinations and Pascal's Triangle, and that equipped with the combinatorics formula

$C^n_r = \dfrac{n!}{r!\,(n-r)!}$, they would be empowered to answer the more difficult questions on

Assignment 3.

The fourth assignment centered around the following project:

Devise a Lottery system for Vale Road. Assume there are 75 potential ticket-buyers in the building (including students and staff).

    a) What will you call the lottery? In other words, instead of 649, what numbers will you use to describe it?

        b) What is be the probability of winning this lottery?

        c) What is the cost of the tickets?

        d) What is the winning prize?

        e) Given your answers to b,c, and d above, explain how your lottery would make money.

The rest of the assignment included the following questions:

Use Combination Notation $\quad C^n_r = \dfrac{n!}{r!\,(n-r)!} \quad$ *eg.* $\quad C^6_4 = \dfrac{6!}{4!\,(6-4)!}$

(Read " 6 choose 4 ", or 6 things taken 4 at a time)

1) What is the probability of winning the Lotto 649?

2) How many different hands of poker are there?

3) What is the 3rd term in the 30th row of Pascal's Triangle? What is the 3rd term in the 100th row?

4) A woman's purse is snatched by 2 men. A day later, the police show her a lineup of 12 suspects. How many different ways can the woman pick 2 people out of the lineup? If the 2 perpetrators are among the 12, how many ways can she pick out 2 people so that at least one of the two is guilty?

5) Griesy Pizza Parlour offers 14 different toppings for its pizzas. If only 3 of the toppings are allowed on any 1 pizza, how many different pizzas can be made?

After describing our staff's abysmal luck in the Lotto 649 pool, I outlined the project to the three students present:

DT: What I'd like to get you guys to work on is to devise a lotto for Vale Road. Let's say you are the lottery organizers, and of course you want to make money at it. You want to sell lottery tickets to the people in the building, including staff. So you want to figure out how

you can devise a lotto system that makes you money. If it's going to make you money, people are going to have to buy tickets. How are you going to make it attractive to them? What's going to be the "pull"?

Don: How much money will they win?

Chad: Or you could have it like, everyone in the kingdom...you know, has their name in a box, and whoever gets pulled gets sacrificed to the gods. (*He laughs.*)

DT: I'm thinking more of a reward type lottery. What is the setup with the 649? What is it about the 649 that makes it so difficult to win?

Chad: I don't know; I don't play it.

DT: What is it you have to do?

Adie: (*unintelligible*)

DT: How many numbers?

Adie: Six.

DT: Right Donny? Do you know about the system? Playing the lotto? Do you know how it works?

Donny: No.

DT: You have 49 numbers, and you have to pick 6. Why would that be difficult here?

Donny: The 649?

DT: Yeh.

Chad: Because here there's not that many people.

DT: Yeh. What would happen? What would be the result of using it here?

Chad: No one would win.

DT: Yeh. Probably not. Probably not. We're looking at a probability of winning which is probably pretty small. Question 1 on your assignment today is "What is the probability of winning the 649?"

Adie. Ohh!

DT: But your major assignment will be, devise a system (*unintelligible*) The problem with the 649 is that 49 numbers makes it quite cumbersome.

Chad: (*singing the 649 ditty*) Big big big big...big big big big....

Chad's quips in this exchange are characteristic of the humour which we all frequently enjoyed during the project. While comic relief is a most desirable ingredient in any secondary classroom, the spontaneous occurence of laughter and good fun was an especially welcomed component of

our alternate school. Not only does humour serve to lubricate the oft grating wheels of education, it also elevates all of our spirits and helps create the positive atmosphere necessary for learning.

Also evident in the above dialogue is my tendency to over-explain things for the students. For instance, instead of asking Adie or Chad to describe the workings of the Lotto 649 to Donny (and thereby encouraging the inter-student communication I so desired), I offered the explanation myself. And with the comment "The problem with the 649...", I effectively cut short the students' investigation of my original question with a spoon-fed answer. As mentioned in the discussion of session 6, the review of sessional transcripts has continually made clear to me this preponderance of teacher talk in my mathematics class. At this stage of the study, I wanted the students to converse more, to express themselves more, and to become reliant on each other rather than exclusively on their teacher.

Adie and Chad then suggested that 6-30, (that is, "30 choose 6") 6-20 or 6-10 lotteries would be more appropriate for our school. Finally, I suggested we investigate a game with a total of only five numbers, and the group worked out the probabilities of winning a 1-5 (that is, "5 choose 1") and a 2-5 lottery. When we compared the respective win probabilities of each system (1 in 5, and 1 in 10), Chad made a conjecture:

> Chad: 10. So all you did was times the 5 by the 2.
> DT: Oh. Good guess. What happened here? What do we have for the pick 1?
> Adie: Five.
> DT: That was the easy one. There were 5 of them right?
> Chad: Yeh. Ohh! And so I would assume that um...
> DT: *(Writes 5 choose 3 on board; begins to list combinations)*
> Adie: That would be 15 wouldn't there?
> DT: Good guess.
> Chad: I think it's 15.

After listing the combinations of drawing numbers in a 3-5 lotto, the group was surprised to discover that the probability of winning it was the same as for a 2-5 (1 in 10). Finally, without any prompting from me, Chad made the connection to Pascal's Triangle:

Chad: Well, they go...they go along this row *(pointing to Pascal's triangle)*

DT: Which row?

Chad: The 5 row.

DT: Oh!

Chad: 1,5,10,10. So I would assume that if you're gonna do 4 fives [4-5], there would only be 5, right?

DT: A 4-5, right.

Chad: And if you were to do a 5-5, there's only one possibility. And it goes along the row of 5. Yep!

DT: Think that fits?

Chad: I think so.

DT: How can we make a test of it?

Chad: By doing a 4-5.

The students agreed to check Chad's conjecture by listing the combinations for a 4-5 lotto. But Chad balked at my idea of verifying the connection to Pascal's Triangle with a 6 number situation: "Well, I'd stake bets on this that...that ah you know, since we learned the Pascal's Triangle....*(laughs)*." I asked them which game would correspond to the number 15 in the 6th row of the triangle. When Chad answered incorrectly (3-6), I told them to find the probability of winning a 3-6 game. When they found that the total number of combinations for a 3-6 was actually 20 (the *fourth* number in the sixth row), Adie was the first to discover the pattern: in essence, to find C_r^n, take the $(r+1)$st number in the n th row of Pascal's Triangle. With their triangles in front of them, all three students were able to answer questions like "What's the probability of winning a 3-6 Lotto?"

Chad: Uh, one out of 20. So, the probability for a um 649 would be 1 out of...a lot of numbers.

DT: Sure. How are you going to find that number out?

Chad: Well...you.... OK, I have an idea. *(pause)*

DT: *(Unintelligible)* To do a ...to do a 5 model, what row do you go to in Pascal's?

Chad: Umm, we go to the 5th row. We have to go to the 49th row.

DT: *(did not hear Chad's answer)* To go to...to do a 6 model,

Adie: You have to go to the 6th row.

DT: The 6th row. To do a 49 model?

Adie: 49th.

DT: The 49th row of Pascal's.

Chad: Is there an easier way to do that?

Adie: *(unintelligible) (laughs)*

Chad: Can you take the....

DT: Oh I hope there's an easier way.

Chad: Can you take the 7 row, and times every number on there by 7?

DT: Interesting. There is a bulk way to do this. Or a 'bulky' way to do this.

Donny: Continue the triangle on.

DT: Yeh. How wide would the triangle be?

Chad: Big!

I then told the students that I would show them a "little formula" which would allow them to compute any term in the triangle. I also let them know that this would be a rare occurence: for the most part, in this course they would discover such things for themselves. There were two reasons why I decided to present the combinatorics formula to them directly. The derivation of the combination notation formula $C^n_r = \dfrac{n!}{r!\,(n-r)!}$ is so complex, it would simply have confused them at this point. Second, the formula would enable us to compute many interesting, real-life probabilities. In view of the good use we made of the formula, I would not hesitate to present it to other classes in a similar fashion.

The students caught on quickly to the relationship of combination notation to Pascal's Triangle and to the lotto problems at hand. However, they needed tutoring on two subskills: calculator multiplication, and reducing fractions. For instance, Adie asked, "When you use a calculator can you go 6 times 5 times 4 times 3 times 2, or do you have to do 6 times 5 is...times 4.... Can you do it faster?" Then, when the numbers became too large for pocket calculators, the

need arose for a refresher on simplifying fractions like $\frac{32!}{4!}$. This would certainly not be the last time that mini-lessons on "basics" would be required. That evening, in reaction to this relatively new situation I wrote: "How much more sense it makes to teach or review arithmetic and other skills in context rather than in complete isolation!"

Throughout the practice in reducing fractions, Adie was anxious to apply the new formula to the 649 question: "Try the 649 now!" and "Now let's do the 649!" Mainly silent to this point, Donny was quick to master the "cancellation" procedures necessary to calculate C_6^{49} . He rose to the task and was the first to come up with the astronomical odds of a ticket being a winner in our government's Lotto 649: 1 in 13 983 816!

Although student attendance was a worry, I was happy with the performance of Donny, Adie, and Chad in this session. Ironically, the small group size made it easy to assess their understanding of the probability concepts: their active involvement in the lesson, their comments, their questions, and even their expressed misconceptions made me feel that perhaps exams and quizzes would not be required, at least for the time being. Since we were approaching the quarter mark in the study, I felt that a cohort meeting would be in order to address concerns such as assessment, attendance, and future mathematics topics.

Inter-Session Monday 28 February, 1994 (Dale, Chad, Don, Adie, and Minna)
"Why don't we all just work in one group and... and do it?"

I presented the students with the following proposal which would tie their attendance into course evaluation. Essentially, my idea was to award each student five points for being at a regular Tuesday or Thursday session. If a student missed a scheduled session, then there would be no five points, no matter what the excuse. As established at the onset of the study, anyone absent for a regular session must somehow make up that time in what we would now call a "make-up" session. No attendance points would be awarded for such sessions.

I informed the students that I wanted their input so that in the end, it would be partly their course as well. Since this was a pilot project, I informed them that their ideas would probably be implemented in a mathematics course eventually designed for the whole school. In other words, if there was an idea that would not work with them, my conjecture was that it probably would not work with a course in the larger school context.

My recommendation for an attendance policy was generally accepted by the cohort. But Dale had an alternative suggestion for attendance points: "Yeh, add it on to your, like...put the attendance points...add up all the attendance points and then, ah, once you do an assignment, if you're say a couple of marks off, like say you have fifteen out of twenty, and you got all these bonus points, so you pop it up to twenty out of twenty." I responded, "The only problem with that is if you get twenty out of twenty on an assignment, then there's no real incentive for you to come to the next session." Minna asked, "Can we just do it at the end, when we're finished it if we are like, a bit off? Like in...say we have around sixty percent or something like that...(*unintelligible*)...just add it up and maybe it'll go up to another grade or something. That's better than adding it up every time you do something, you know?" I promised the students that we would discuss it further to see what we could decide on.

The implementation of such an attendance system was troublesome to me. On the one hand, I had to ask myself, "What about intrinsic motivation? Should not student interest in the course be enough of a motivating factor for attendance?" On the other hand, I was teaching in a vacuum: no official VRLF attendance policy existed, short of provisions for punishment for unexcused absences. I decided to take the risk and implement the policy. I rationalized, "Of what motivational value is an excellent mathematics program if the students are not there in the first place to experience how good it is?"

The second topic concerned assignments. I explained that I had not been too forceful about the due dates on them. (For instance, assignment 3 was due on Thursday last, but I extended the date until the Friday.) I asked the students if they thought I should be stricter with deadlines.

Dale: I don't think you should be.

DT: You don't think I should be? Why?

Adie: Because, this is a "work at your own pace" thing.

DT: That's true. In other words....?

Dale: Like, like, give us like a deadline, but not for the next day. Like, give it to us on Tuesday, don't say it has to be done on Thursday. Say it has to be done a week from Thursday.

DT: OK.

Dale: Like, give us a week to do it, right?

DT: There seem to be some extremes here. Adie says that it's not...this is a "work at your own rate" school and I'm saying....

Chad: Yeh, but this is a course.

Minna: But still, you're going to get behind in your assignments and stuff. *(very true)*

Dale: ....*(unintelligible)* a course where you work at your own pace. Give us a week and....

Chad: But we had the option not to take it. *(i.e. the pilot project course)*

DT: Alright. Maybe the uh compromise is to put a deadline on it, but to extend it, and not make it, like Dale says, within the next couple of days. It may be more app...appropriate to make it a coup...within the next couple of weeks.

Chad: Yeh, maybe more leeway.

DT: I run into a bit of a problem, and that is sometimes you need to get the results of an assignment done before you can go on to the next topic and understand it. So in that case, would ah...would it be alright to say this has to be done for next time so that we can go on. That make sense?

Minna: Uh huh.

DT: It's putting a constraint on you that you never had before in this course. But, like you say, we are running a course. Sometimes, you need to...to have what's called the pre-requisites before you can go on. OK? So, I'm willing to listen to that argument about not being so exacting and so strict with the time. I'm also thinking of bringing in a little bit of testing as well.

Adie: Uh huh.

DT: Not yet, but there will be a certain unit or two where you should do probably a written test on your own.

Adie: That's cool.

It was interesting to see the students take sides in this debate about assignments. Minna and Chad argued that this was after all, an academic course and that without deadlines, students would fall behind. Dale and Adie lobbied for the status quo in which there were no deadlines in any courses at VRLF. I was pleased that a consensus was achieved, and that through this consensus, a course policy was implemented.

Then Chad raised a point which I had contemplated a great deal myself.

Chad: Well, it's just a concern I've been having. Like I don't understand like, um, my math that I took before?

DT: Yeh?

Chad: I was learning things like uh there was this group of numbers with the N , and it had like a line on it. [*He is refering to the symbol $\bar{N}$ for the set of natural numbers.*] And it was this group where you were doing all sorts of weird stuff like that and algebra. And now all of a sudden, and then that math (*pointing to the filing cabinet containing materials from the regular VRLF course.*) it's like divide this and multiply that. And now this course is like...pentominoes, and probability. and I'm wondering how it all ties in. And I'm wondering if my...my math is going to suffer when I get out of here and get to a regular school.

DT: OK.

Chad: I'm concerned about that.

Adie: Smart. (*laughs*)

Chad: Because it's already suffered a lot, this transition. And only for myself, from French to English and...and...and ah I mean, it seems to me more preferable (*laughs*) that this course would be better or more preparing for the rigors ahead in regular school (*laughs*) Because it's more advanced: like other...that stuff like I mean...you know that you can do it. You know what I'm saying?

DT: No, it's a valid point, and that's why I was pretty particular about what I wrote down in the description for your parents. Because a lot of parents would look at what we're doing and say "You're not learning how to divide, or..."

Adie: I already know how to divide though!

DT: Some of you do!

Chad: Speak for yourself!

DT: Yeah, right.

(*laughter*)

Dale: Dividing is boring. This is interesting.

DT: Your point is uh taken, Chad. And, I've thought about...a lot about that in the sense that I'd like to give you guys a modicum of algebra before you finish this course.

Chad: What?

DT: Algebra. It's such an important tool.

Chad: Yeh.

DT: But in my opinion it's become overemphasized, in particular in the academic courses.

Chad: Could...will.....

DT: But I'm...I'm ah trying to find a system to give you some algebra.

While I appreciated Dale's comment about our course, I nonetheless had, in a sense, some of Chad's misgivings about pedagogic accountability. Would my students suffer academically for lack of formal algebra instruction? Should they not be receiving some sort of formal exposure to plane geometry? One unassailable fact gave me renewed courage: few of my students in past alternate schools retained their "traditional" learning beyond a few weeks anyway. After all, my only worry would be if and when parents expressed serious misgivings about their children's new "radical" mathematics course.

Anxious for their input on the topics covered so far, I asked the group, "Do you find this math that we're doing difficult?"

Adie: Sort of.

Chad: I find the math that we're doing in this course really good.

DT: Is it difficult though?

Adie: Yeh. 'Cause....

Chad: It's...it's difficult , yeh. But it's a good difficult. Not like an excruciating difficult. (*laughs*)

Adie: (*laughs*)

Chad: It's like, you know, just a comfortable difficult. It's...it's like a rewarding difficult.

DT: OK. (*laughs*) For instance, um, I would like to incorporate some testing when we do the algebra. And what's really important is that everybody have a good foundation on how to do integers. How to add and subtract integers.

Chad: See? What's an integer? I know it in French, but I have no idea, no clue what it is in English.

DT: Les chiffres négatifs.

Chad: Huh?

DT: Les chiffres négatifs et positifs.

Chad: Negative numbers?

DT: Yeh. So you're adding and subtracting them. It has to be 100%. And it's an important skill. And I would...I would include that in testing. And...and some algebra too.

Dale: Why do you need algebra?

Chad: Algebra's cool!

Dale: If you're just going to do...do a normal job?

DT: What do you mean by a normal job?

Dale: Why do you need algebra if you're just going to do....

Chad: Why do you need social studies?

Dale: You don't need algebra...

Chad: Why do you need phys ed?

Dale: ... in everyday jobs.

Chad: Part of the flaws in our school system.

Dale: You need phys ed so you won't be a fat slob. But why do you need uh algebra? You don't use algebra hardly ever.

DT: I would disagree with that...

Chad: Same reason.

Adie: You do use algebra.

DT: ...because it is useful. And second, second point, you don't know what job you're going to be in.

Dale: I know what job I'm going to be in.

DT: And third...

Dale: I'm going to be a cop.

DT: OK.

Adie: (*laughing*)

DT: What about the algebra needed to work out whether a car's speeding or not?

Dale: It's all done on a computer.

DT: Mathematics. And thirdly, uh, we'll talk about this at length some other time. My idea of math is that it's not a subject that necessarily has to be useful, in the sense that it's useful like a shovel or some kind of tool. What...what use is studying art? What use is a painting? What use is a poem or a short story? So these are the questions you can ask about math. But ironically, algebra is a very useful tool, and that's why I want to include it. I'm...I'm still trying to work out a system to present it to you though, because I'm not really happy with the way it's done normally. So that's my challenge: to come up with a (*unintelligible*).

Dale: I don't need algebra in the military.

DT: You don't know.

Dale: I know.

Chad: *x* equals the number of rounds fired by *y* number of machine guns used, divided by *z* number of people killed.

(*laughter*)

Chad: ....*a* amount of foreign countries. (*laughs*) Yeh, of course you need algebra.

I have always found such debates enjoyable. But discussions like the one above on the value of mathematics serve added purposes. They provide the students with a non-threatening forum for revealing their opinions of my subject. The air gets cleared: the students have a reinforced idea of my enthusiasm for the subject, yet they understand that I will not "come down" on them for expressing themselves. And hopefully, they will come closer to my way of viewing mathematics.

After a few words on the problem of chronic student lates, I gave them some final thoughts on attendance: "It disappoints me when you don't show up and when you...especially when you don't show up for one of our math sessions. It's nice to have as many people as we can. There are eight formally enrolled in the group, and if there's eight people here, it means that there's eight different ideas that are coming out most of the time. The more ideas the better." These comments were followed by the usual student committments to do better in the future.

I then asked the students to spend the rest of the day trying to catch up on late assignments. Adie expressed concern about the lack of calculators.

Adie: We need calculators though. How are we going to do it ?

DT: You can share mine, and I'll try and see what I can find down in my room.

Chad: You can use the computers for calculators...

DT: You can, but it's fairly awkward (*referring to Apple II's*)

Adie: Why don't we all just work in one group and...and do it?

I missed Adie's last comment in the classroom, but later, reviewing the tape, I thought, "Yes! From now on, we will all just work in one group."

For the remainder of the day, the students persisted well on their assignment tasks. I suggested to Minna (who just returned from a suspension and had missed the last three sessions) that she work at constructing Pascal's Triangle, with Adie's help. Since Donny and Adie were working on the Lotto 649 question of Assignment 4, I suggested to Donny that he move to the table with Adie and Minna. After considerable help from me to recapitulate the connections to Combination Notation, they were able to answer questions 2 through 5 with no difficulty.

Donny seemed to need constant reinforcement that what he was about to attempt was "correct." Once he had been encouraged, he would proceed just fine. For instance, Adie found it difficult to explain the generation of Pascal's triangle to Minna. When Adie asked me to help Minna, I asked Donny if he could express the pattern to her. With a little prompting to be precise in his English, he did so. Minna caught on instantly, worked on her triangle, and then went on to Assignment three.

At the beginning of the afternoon, Dale had inquired what he missed on Thursday, and expressed interest at the combination notation work still left on the board. I asked Chad to help Dale with Session 7 work. They worked on listing combinations (with my prompting) for 5 choose 1, 5 choose 2, and 5 choose 3 lottos. Dale saw that the results corresponded to row 5 of Pascal's triangle. He expressed confusion, and I told him that it is much easier to learn in a regular group session than at a "compressed" lesson like this one. (This was an obvious plug for attendance!) With five minutes to go, he asked me to delay the explanation of combination notation until tomorrow. Finally, Chad wondered where the combination notation formula came

from. He was relieved when I reminded him that we did not derive it or prove it last day, but rather, that I had just presented it to the class.

Still worried about too much teacher-talk and explicit teacher-explanations, I told the students of my concern as they left for the day. They also heard that one way to change this situation was to have more."student talk" (their phrase). That evening, I decided that session eight would be devoted to completing the Lotto assignment, and that I will not offer one word of conversation to the process. The students would be told that between all of them, there should expertise enough to complete the assignment by the end of the day.

The fractal package which I had ordered in January arrived this day from the NCTM. A few of the students had a look at it, and expressed interest; indeed, it looked ideal for this group. Since the recently ordered Macintosh computer was not yet on site, I would have to wait to introduce fractals. This situation was unfortunate, especially since the time was opportune to make the beautiful connection between fractals and Pascal's triangle.

Five of the eight students formally registered in the case study were present for this extra session. A sixth (Alonzo) was absent ; a seventh (Juliana) was under suspension ; and an eighth (Ron) was in the back of our classroom, away from the group. At the beginning of the meeting, I arranged a table and microphone for the students present. After calling them (more or less individually) to sit around the table, Ron had asked me, "Aren't you going to ask me too, David?" I responded, "I didn't think you wanted to participate, Ron. Do you want to join us?" He replied, "I'll do so when I feel the need." I told him that he was certainly welcome to rejoin us any time. During the one-half hour work session which followed our meeting, Ron doodled on a cartoon; this behaviour was unusual for him, because normally, he wasted very little class time.

My last words of the day went to Ron. Attempting to appeal to his artistic side, I explained how at UBC, a person could study mathematics either in the Faculty of Arts, or in the

Faculty of Science. Also, I recounted how most of my mathematics professors were either quite science/practical or else arts/humanities oriented. He offered no comment whatsoever. I then asked him why he hadn't started his grade ten level mathematics yet. (Since November, he had completed all the 10 courses here, plus English 11 and Social Studies 11). He seemed embarrassed, and stated something about wanting to save it for last, while he concentrated on one course at a time.

Ron never did rejoin the cohort. Instead, he went on to finish the program at VRLF and ironically set a school record for the fastest completion of Grade 10 and 11. His last course was the regular academic Mathematics 10 which he did virtually alone.

Looking back, I see that at this quarter point in the case study, it was becoming obvious who the major and minor players in the study were. Already on the periphery were Alonzo and Shawn. Minna's temperament would eventually be her demise at Vale Road. The remaining five students--Adie, Dale, Chad, Juliana and Donny would now constitute the central focus of the remainder of this narrative.

SESSION 8 Tuesday 1 March 1994 (Adie, Minna, Dale, Chad)

Adie and Minna arrived at school early, but at 12:30 there with no other cohort arrivals. Even though the session was not scheduled to start until 1:35, I decided to phone Dale, Chad and Donny.

Dale's mom reported that he had left for school. She asked how he was faring at VRLF, and was pleased to hear of his excellent behaviour and progress in the mathematics study. Mentioning that his ADD had been a problem, she added that his disorder came in cycles. Asked if the last four weeks have been a "low" point, she responded yes. She claimed that the cycles, both up and down, are of indeterminant lengths, and are difficult to predict. At that point, I hoped

that Dale's success to date in the mathematics group was more than just the "down side" of a hyperactivity cycle.

By 12:45, I decided to phone Chad and Donny. The latter was not at home. But Chad, having just woken up, answered the phone and promised to head in to school. Meanwhile, with the prospect of poor attendance looming large once again, my plans for a "group" problem solving session this day seemed to be in vain.

While Adie and Minna spent the first hour of the afternoon working on annual pictures, Dale wanted to finish his assignment 3. After I helped him with combination notation, he worked diligently. It should be noted that this extra, concerted effort took place *outside* the regularly scheduled mathematics class.

At 1:35, Adie and Minna had to be coaxed back to the room. (They were taking annual pictures of the multi-cultural workers.) Dale was already on task with assignment 4, and was asking for help on the first question. Soon, he had mastered the combination notation (that is, he had memorized the technique), and was having success with the questions. I suggested to Adie and Minna that they work with Dale, since he and Adie were at exactly the same point on Assignment 4. Minna refused outright . Adie responded, "Minna won't work cooperatively with Dale".

Sometime later (around 1:50) I found Minna doing science. When I asked her if she wanted to try some math (after all, this was the scheduled time for it), she responded adamantly, "No, I don't want to do it! I don't know how, so I don't want to try!" Finally, around 2:15, Adie begin to coach Minna in combination notation and they worked steadily for half an hour. I wanted to tape their efforts: they seemed engrossed in the process, and Minna kept on task the whole time. However, I feared that plopping a tape recorder on their desk at that point would

have stifled their learning/tutoring session. Adie whispered to me when I approached the desk: "I think she might have...(*pointing to Minna*)...got it!"

Chad finally arrived at 2:15 and I informed him that I was glad he made the effort at least. I found it difficult to berate students for tardiness since they were in effect showing up for the mathematics sessions. On the contrary, I was quite happy whenever students arrived for mathematics, late or otherwise.

I asked Dale (who by this time was enjoying success with the combination notation) to bring Chad up to date. At this point in the transcript, the two students were pondering the large answers derived from combination notation and Chad was confused about the special readout on my scientific calculator. Here, the number 3.4561×10^{10} was displayed as $\boxed{3.4561 \quad 10}$. The following is a transcript of Dale and Chad attempting to make sense out of scientific notation. (The microphone was inadvertently shut off for the first part of their dialogue.)

Dale: The decimals are this far apart. That's the...that's the uh power.
Chad: What's the decimal point mean?
Dale: That's...the decimal point means...the decimal point is....
Chad: These (*unintelligible*) calculators. You can't read them.
Dale: The decimal point is the decimal for it.

On the board, without saying anything, I wrote their number in scientific notation: 3.4561×10^{10}.

Dale: OK. OK. Yeh, I'll explain it to him that way.
Chad: Times 10-10? Why is it times?
Dale: That's to the....
Chad: Because....
DT: It's called scientific notation.
Dale: The number's to the power of 10.
Chad: Exactly, but you don't....
DT: The calculator merely gives you the exponent of 10.

Chad: But...OK, wait a second. What's the difference between that and...and a number like uh you know, uh 184 10? Why is it 10 10?

DT: It's the standard.

Chad: Huh?

DT: The standard is to use 10 because we're in base 10.

Here I realized that 10^{10} was a poor example because of the extra 10 as the exponent: any another exponent would have been better.

DT: Here's an example. I'll just put two huge numbers into the calculator and multiply them together. *(The answer turns out to be 1.1744×10^{13})*

Chad: Uh huh.

DT: It's going to guarantee an overflow.

Dale: So it would be 10 to the 13th power.

DT: The calculator's taking the answer, and putting the number part as a number between 1 and 10. And it's saying take that, and multiply it by 10 to the 13th power.

Chad: Multiply it by the 10? How...why is...where's the 10? *(A good question!)*

DT: The 10 is not there. The 10 is assumed to be there.

Chad: Ohh!

DT: That's why we call it standard notation. That's a good question. Why can't it be 10 084 to the 10?

Chad: So it's not 1.1744 to the power of thirteen.

DT: No. It's times 10 to the 13. Which means you take the decimal point and shift....

Chad: So that's actually a decimal point, and that's only 1....

DT: Uh huh. You want to work out the true version of it, you take the point here....

Dale: That would be...it would be 1 to the 1, 2, or 1, 7 , 4 , 4, ...

DT: That's 4, 5, ...*(counting decimal places along with Dale)*

Dale: *(adding zeros to the number 1744)* ... 5, 6, 7, 8, 9, 10, 11, 12, 13.

DT: Thirteen.

Chad: Yeh, see, I learned to do commas in French. *(In French immersion, Chad evidently learned the European custom of using commas for decimal points.)* That's another big difference.

Dale: So...so that'd be 11 trillion, right? That trillion? Hundreds, thousands , billion...

DT: Millions....

Dale: Billions, trillions, ...so that'd be 11 trillion, 744 billion....

DT: It's tricky, Chad. We should probably do a little primary scientific notation before....

Chad: Calculators!

In retrospect, I should not have suggested that Chad's problem with the calculator readout was caused by the lack of a preliminary session on scientific notation. Traditionally, we teach pre-requisites in anticipation of a given concept or skill. But here, his problem naturally set the stage for some sort of exploration of scientific notation. Thus, my choice of large numbers as the next topic now had a context. In other words, given the reality of the calculator's readout, the students would hopefully see a need for studying scientific notation next.

Chad and Dale tackled the Lotto 649 problem, and then went on to the second question of assignment 4: "How many different hands of poker are there?" Their dialogue picks up where they are just computing the 649 probability. It is interesting to note not only their collaboration, but also Dale's new assumed role as a teacher.

Dale: (*to Chad*) So do you understand how we got this and then when we divided by 720 (*6!*), we got that answer? Do you understand now?
Chad: And that's not a power then.

Here, Chad was confused by the fact that my calculator showed the answer with a slight space between two of the digits. In other words, because of a quirk in the LCD readout, the answer 13983816 was displayed as 139838 16 , which resembles scientific notation. No wonder he was exasperated; his comment about calculators was totally understandable.

Dale: No. No because it's.....
Chad: Then what is it?
Dale: That's the number.
Chad: And that's in...in place of a comma or a decimal point.
DT: That's the way...that's just the way the calculator is portraying the answer.
Chad: Oh.
DT: Yeh, that's not a...that's not scientific notation.
Chad: Ohh!
Dale: But if there's a great spot out um in front of it....
Chad: Because that's only 1 LCD there right?
DT: Yeh.

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Chad: OK. Well now it makes sense. But it's the confounded calculator that looks like that from when you're looking at it from over there.

Dale: It's a farther spot away right? *(i.e. scientific notation)*

Chad: Stupid, stupid, stupid!

Dale: OK if it's a long way around...a long way away, that's power.

Chad: I thought...it looked like it was a long way away.

Dale: *(laughs)* So, that's the answer to that one. *(the 649 question)*

Chad: There's that many different possibilities. So it would be 1,2,

Dale: Do the formula first. *(i.e. write* $C^n_r = \dfrac{n!}{r!\,(n-r)!}$ *)*

Chad: Huh?

Dale: Just write out the formula first. Like just do...*(Dale shows Chad in writing)*

Chad: OK.

Dale: *(unintelligible)*...do the formula.

Chad: That is a 9 isn't it? *(pause in conversation as Chad writes the formula out)*

Dale: There.

Chad: *(on to the next question)* How many different hands are there in poker? Huh? Is there 12 hands of poker? It's been so long since I've played poker.

Dale: OK, in poker there's 52 cards. Right?

Chad: Uh huh.

Dale: And you got 5 hands...and you got 5 cards in your hand . How many different hands are there? You get the same as....

Chad: No, there's the royal flush, there's the straight flush. there's the...

Dale: No!

Chad: ...4 of a kind, 2 of a kind, there's 3 of a kind, there's a straight....

Dale: Yes, but how many different ways can you have 2 of a kind? How many different kinds...different ways can you have a straight? How many different ways can you have a flush? Know what I mean?

Chad: So what it's really asking...how many different ways can you hold 5 cards in your hand? *(exactly!)* Yeh.

Dale: Basically.

At this point, through their dialogue alone, Chad and Dale had successfully made their own sense out of the question. Then, Chad began to deviate from the assignment, only to be pulled back on task by Dale.

Chad: You could disguise that as a light switch. (*referring to the microphone*) Oh God, that's kind of scary.

Dale: Number 2 would be the same thing as....

Chad: You know they can look at you from space and identify you. That's nuts. Nuts.

Dale: And they don't have a cure for AIDS.

Chad: Yeh right.

Dale: Number 2 would be the same thing. Alright?

Chad: What you're really (*unintelligible*) is how many different combinations can you hold 5 cards in your hand? Out of 52. So. Is that counting jokers?

Dale: OK. OK.

Chad: Do you have to write it all out?

Dale: Yeh. Yeh. You have to show your work. You see I had to write it all out.

Chad: OK. (*writes* $\frac{52!}{5!\,(52-5)!}$)

Dale: Actually...and the difference between 52 and 5? You didn't want to write down 52•51•50....

Chad: Well, no but I wrote it down as 52 (*factorial*)....

Dale: Is that good enough? Now I just have to use the calculator? If I didn't have a calculator right? (*cancels the 47•46•45•44... in the numerator and denominator*)

Chad: Exactly.

Dale: I'd...I'd...use a calculator for these ones right? (*meaning using the factorial key directly*) So I wouldn't use it for number 3, right? (*Question 3 on the assignment*) What is the 3rd term in the 30th row of Pascal's triangle, right?

Chad: Done it.

Dale: You do...what you do....(*writing* C^{30})

Chad: Two! Underneath it. (*writes 2!*)

Dale: 30 and 2.

Chad: 30 and 2.

Dale: ....and 2. For 3rd term of the 100th, do a hundred, and two. (*writes* C^{100}_{2})

Chad: I see.

Dale: Two.

Chad: Two. Yeh.

Dale: Very good. He understands how to do the third one.

Chad: Thank you, Dale.

This session ended on a positive note for me. First, Dale's second stint as a tutor was a great success. Not only did he keep Chad focused, he also appeared successful in communicating the lesson to his classmate (for which Chad expressed his gratitude). Also, it was a pleasure to hear such a large recording segment without my voice! This interchange between Chad and Dale was exactly what I was hoping for: pure inter-student communication.

Topic 4: Large numbers and scientific notation

SESSION 9   Thursday 3 March 1994   (Adie, Dale, Chad, Donny, Minna, Alonzo)
"*I came the closest!*"

For this session, I hoped that by creating their own "analogies" for certain huge magnitudes, the students would develop an intuitive "feel" for such numbers. Later, in session 10, I planned to teach scientific notation.   The core of their new assignment (#5) was the following four-part problem:

> Devise an analogy to explain or make clear the size of each of the following numbers:
> 1) *one billion*
> 2) *the number of possible 649 tickets*
> 3) *100 billion (the number of stars in the galaxy)*
> 4) *one googol.*

This problem was followed by a set of fifteen exercises which were adapted from Jacob's (1982) excellent text, *Mathematics: a Human Endeavour.* I first introduced them to a googol (by means of a *Peanuts* cartoon), and we reviewed the full decimal representations (i.e., with all the zeros) of the powers of ten from 10^1 to 10^{12}. When we reached one trillion, the students agreed that the large number of zeros in the numbers was becoming problematic. We investigated a system for writing the numbers in a "nicer" way; that is, using powers. The students seemed to recall effectively the notion of power, and were able to come up quickly with the power notation for the numbers 10^1 through to 10^{33} (or 1 decillion).

While I was discussing the formal names for large numbers (such as octillion and nonillion), it became apparent that Dale was not listening to my explanation:

DT: Will you ever use these words very much? Probably not.

Adie: No? What's the....?

DT: Will you be required to memorize them? No. But uh...they're fun to use because today we're going to be looking at some huge numbers and to be able to name them is...

Dale: David?

DT: ...it's not only fun but uh necessary.

Dale: I was looking at this, OK? And you wanted us to devise a model for it right?

DT: I'm going to explain that in a sec.

Dale: But just wait. I'm confused. You wouldn't be able to write 10 to a power for this There's so many different numbers in it. It's not all zeros.

Dale was wondering how to write our 649 probability number in scientific notation, compared to a number like 3000000 which would be simply 3.0×10^6.

DT: It's OK. We'll look at that in a sec. (*pause*) Now, never mind one dec...one decillion. Do we have any idea how large say one million is? What is one million? Certainly, to have a million dollars is beyond our comprehension.

Dale: I see what you want.

At first, I was somewhat annoyed at Dale for not being attentive to my discourse. Actually, his "interruption" was a manifestation of his enthusiasm for the assignment. I recognized a great opportunity to use his confusion as a segue into an example analogy for a large number. My comment about a million dollars was all Dale needed for an explanation: as I shall soon describe, he launched into the assignment whole-heartedly and was the only student who completed the four-part assignment.

As an example of an analogy, I had the students guess how long it would take to count to one million, if one number were named per second. While Donny worked away silently on his calculator, the students were quick to offer suggestions:

Alonzo: A million seconds.

DT: A million s....OK, what is...but that doesn't tell me anything about the...the size of....

Alonzo: Converting it into like minutes.

DT: Yeh. That's what I want you to do.

Dale: It'd take you about an hour and a half.

DT: One and one half hours? (writes Dale's guess on board)

Donny: (working away with calculator) ..

Dale: Sixty seconds divided by a million.

Dale's reversal of dividend and divisor is typical of errors made by students with LD. In fact, since Dale uttered such "reversals" so frequently, and since his meaning was clear, I learned to largely ignore his errors of syntax.

After receiving guesses ranging from 12 to 60 hours, I missed a great pedagogical opportunity: instead of having the students work out the answer, I gave it to them directly. Nonetheless, they were surprised to discover how long counting to a million would in fact take:

DT: Anyway, it was kind of nice to see you want to reach for a calculator, or to ask how many seconds there are in an hour. Because that's how you would work out the answer. That's how you would come up with a model of one million. The answer is 12 days.
Adie: Oh no!
Dale: It'd take you 12 days!
DT: Yeh.
Donny: I came the closest.

In spite of his reticence, Donny continued to surprise me with his answers. He was so soft spoken, that the whole group did not hear his contribution which was nonetheless picked up by the microphone and not appreciated until I reviewed the tape of the session later that night:

I recommended to the students that they first tackle the questions on large numbers (from *Mathematics: a Human Endeavour*, Jacobs, 1982), before tackling the assignment. In general, the group worked well at questions such as "What does 10^0 mean?" and "Which of these numbers is the largest? 100^4 , 1000^3, 10000^2." However, faced with the following question, the students seemed stymied, and would not commit even to initiating a cooperative effort to solve it:

The human brain cell contains about 10^{10} brain cells, called neurons, and is far more complex than any computer that has ever been built. If each cell was represented by a dot

in a line of dots like this ⌐. . . . .⌐,   how far would the line stretch?

I was dismayed by what I perceived to be a general trend: while the students were willing to take on simple, one-step questions, they were often hesitant to tackle multi-faceted or open-ended problems.

Although the session went satisfactorily, I was rather despondent after school that day. The following is an excerpt from the diary entry I wrote soon after the students left.

> So far, the students are still very dependent on the teacher for help, direction, explanations, and ultimately, correct answers. I'm concerned with the preponderance of utterances like, "Show me how to do it," or "I'm stuck, what do I do next?" and "Is this the right answer?" Donny, for one, seems constantly to be seeking advice and help. However, at least he tries, and for the most part, tries to follow through to the end of the assignment.
>
> The students are sometimes prone to discouragement (eg Adie, Minna) when they don't understand, or aren't able to proceed. Adie's comment, "I don't want to do any more of this stuff man. I'm just going to be in the normal regular math." (But, in the end, the two stopped their note-passing, and went back to the exercises.)
>
> The students are reluctant to work collaboratively. Today, for the most part, they worked silently and individually on their exercises. After all, they've been conditioned to be inactive learners.
>
> My tendency is still to try to over-explain, to talk too much; it's difficult to back-off when students are frustrated or stuck. But I shouldn't be getting "down" on things when students don't "perform". This should be an opportunity for exploration! Don't get depressed at their behaviour. Document these behaviours. Learn from them, and experiment with new approaches which will, over time, encourage them to develop positive attitudes towards a subject which they've learned to consider boring and loathsome. Of course, they're going to balk at open-ended activities where they are for the first time being forced to think.
>
> Student attention and participation during my "presentation" is always good. They listen and respond to me. By their responses, I think that they follow the logical flow. This is a system they are used to, perhaps from elementary school. But when directed to work in pairs, they withdraw into themselves, regressing to how they worked under the *Math in Life* ABE Math 10B regimen.
>
> The exception appears to be Dale, who dives head first into the problem, sticks with it, and discusses with classmates, (or me) possible avenues of attack. Today, he worked with Dehlia, [*teacher on call*] collaborating on a solution to the "brain cell" question. He could easily be the subject of the thesis. Is this but a lull in his ADD circuitry? Or has he found a forte? Or both? If he becomes hyperactive again, will he be able to carry on in class as well as he has been? This guy deserves an A on his report card.

Obviously, I was trying to balance my negative post-session feelings with positive observations and explanations. But my doubts about the project were of sufficient strength that I often considered abandoning the project entirely.

Perhaps my negative emotions that afternoon resulted from an after-class conversation with Chad, who I was convinced would soon be suspended from school over his poor attendance:

DT: *(to Chad alone)* We have to work out something for you.

Chad: Yeh.

DT: Cause it ain't working .

Chad: It almost worked today. My friend's been calling me up every morning.

DT: And I phoned you too, and you were up and at 'em this morning. You can't get in.

Chad: I...I...it's my sister, damn it!

DT: No, no, no, no.

Chad: *(laughs)* It is. Well...yeh.

DT: The problem rests with one person, man.

Chad: I'll get here. I'll get my uh...my uh shit together.

DT: That's what I want to hear, yeh. And then I want to see you do it.

Chad: Yeh.

DT: Because you can...you can do well here.

Chad: Yeh.

DT: If you come.

Chad: Yeh.

Not that there was anything obviously "magical" in it, but this conversation represented a turning point for Chad at VRLF. From this day on, he did not miss a single session of mathematics. His punctuality improved to such an extent that he was a common fixture in the hallways one-half hour before classes began in the afternoon. Indeed, Chad went on to win awards the following June for "most improved attendance" and "most improved punctuality." And in the weeks to follow, he was to become an ace performer, particularly in algebra.

However, the student of the moment was still Dale, whose mathematical star was still in ascendency. During the next week, Dale would astound the teaching staff at VRLF with his perserverence in mathematical problem solving.

To occupy the next two sessions, I planned work on scientific notation, and closure for Assignment 5, and for the many questions on large numbers. I needed to establish clearer deadlines and expectations on all assignments: since report cards were due out at the end of next week, I would set a Tuesday deadline for Assignments 3 and 4 as well. Following the write-up of session 10, therefore, I will provide an analysis of student performance on assignments in particular, and on my choice of assignments as the major assessment tool for the curriculum to date.

Interlude: Monday 7 March 1994
"Do you want me to show you?"

At the beginning of the regular class, I asked the students to have assignments 3 and 4 ready for the next day (Tuesday). To his credit, Chad worked hard to finish his, and handed them in. To my disappointment, however, he eschewed my suggestion to do his mathematics cooperatively with the other cohort members, and worked alone. Adie and Minna chose to work together, but their task was cutting and pasting for the school annual, not doing mathematics. I approached them well after break about doing some work on the assignments:

DT: Will you have your assignments ready for tomorrow?
Minna: I don't know. Maybe.
Adie: We already spend two days a week on math.

Adie's retort chagrined me particularly, especially in context of the previous comment she recently made about quitting the project and taking the "regular" mathematics.

As was often the case during the study, another individual's behaviour served to buoy my spirits which had been recently depressed by the antics of other students. In this case, Dale

teamed up with Dehlia, who had accepted my invitation to visit the classroom at any time. They worked together the whole afternoon, mainly on the analogy part of the assignment.

Dale's first task was to come up with an analogy for the number 1 billion. He decided that he would compute how far 1 billion of his size 10 shoes would stretch. The two quickly discovered that 1 billion of his shoes would go around the world almost three times. Now although Dehlia helped him considerably with the metric conversions, Dale alone came up with the idea for the analogy.

Having completed the fist question, Dale was eager to go on to the next task of making clear the magnitude of the Lotto 649 probability.

Dale: *(to DT)* What was the number of possible 649 tickets again?

DT: *(points to answer on blackboard)*

Dale: Oh yeh.

Dehlia: How did you arrive at that? I know it's...it's calculating the possibilities: 13 million 9 eighty -three eight...

Dale: Do you want me to show you?

Dehlia: Yeh, show me.

Dale: I'm gonna get a piece of paper.

Dale eagerly demonstrated to Dehlia how the number 13, 983,816 was derived from the formula
$$C_6^{49} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \ldots}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 43 \cdot 42 \cdot 41 \ldots} .$$

With Dehlia on the scene, I decided to "test" Dale's grasp of probability. I made his task open-ended, asking him to show her a probability example. He proposed the problem of computing the probability of having four girls in a family of 20, and proceeded to work out
$$C_4^{20} = \frac{20!}{4! \, (20 - 4)!} . \quad \text{(Figure 3.6 is a copy of his work.)}$$

Figure 3.6
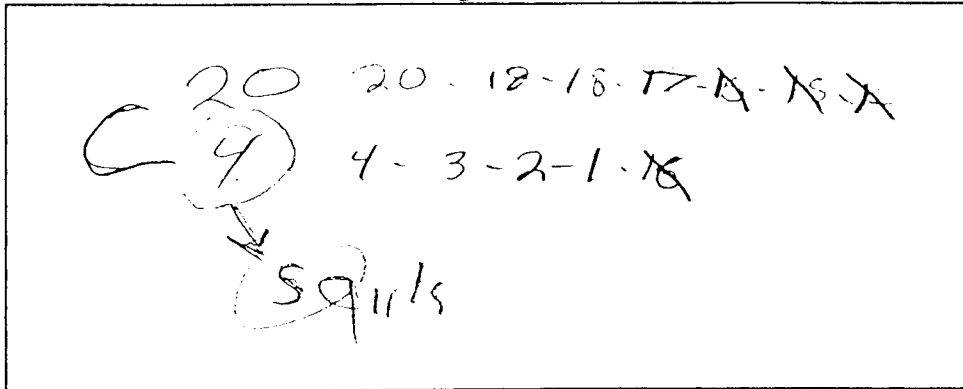
DT: Explain to Dehlia how...how you could work out another probability Dale. Other than 649. Show that example, but with smaller numbers.

Dale: OK.

DT: Another case of probability.

Dale: OK ah...how many possibilities are there of...the possibilities of...a hund....no out of 20 babies, that 4 will be girls. That'd be an OK one to do?

DT: That's certainly another application of combination notation, yeh.

Dale: (*unintelligible*) combination notation of 20 and....

Dehlia: Four.

Dale: Four, right. So it'd be 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 and then you....

Dehlia: Take 4 from 20.

Dale: Yeh, so it'd be 16, OK? And then you'd start cancelling out. OK, I'll do them long form this time. Twenty times 19 times 18 times 17 equals....

Dehlia: 116,280 .

Dale: 116 thousand, 2 hundred and eighty. Four times 3 times 2 times 1 equals 24.

Dehlia: It's still going to be a pretty big number.

Dale: 1 1 6 2 8 0 divided by 24. Possibilities are...

Dehlia: So the chances of only having 4 girls out of....

Dale: 4845.

Dale successfully worked out the number of permutations of four girls in twenty births. But by not bringing in the number 2^{20} as the denominator of the probability, he demonstrated that he did

not internalize well his recent experience involving the connections between permutations, Pascal's Triangle, combination notation, and probability.

After this side trip proposed by his teacher, Dale was anxious to get back to the problem at hand: how could one create an analogy for N = 13,983,816? Again, he came up with the idea by himself: how many times can 13,983,816 people fill BC Place? By dividing N by 60 000, he came up with the solution: 213 sell-outs.

As soon as he had he read the answer from his calculator, he was ready to proceed to the next problem: devise an analogy for the number 10^{11}.

> Dale: One hundred billion stars in the galaxy. Damn! (*laughs*) A hundred billion.
> Dehlia: A hundred billion.
> Dale: What's something that's huge that could hold like a lot...a lot of water?
> Dehlia: A lot of water?
> Dale: How many gallon jugs of water are in the....
> Dehlia: Pacific Ocean?
> Dale: No. Ah....
> Dehlia: Oh, the tank at the aquarium? Or Kits pool, or...ah what's another big pool?
> Dale: Hey uh David?
> Dehlia: We don't know the depths of those.
> Dale: David, is there a way that we could find out the uh amount of water that's in Lake Superior or something?

Although Dale's idea could have been developed into a feasible analogy, Dehlia suggested at this point that they look at mountain heights. Assuming that it takes two weeks to climb Mount Everest (height 8848 metres), the two went on to claim that it would take 54 lifetimes of steady ascent to climb 100 billion metres. (Dale suggested that one lifetime should not include the retirement years, since "because people aren't going to be climbing when they're eighty.") Later, during session 11, I would revisit Dale's attempt at describing 100 billion, and point out the small (but significant) arithmetic error which prevented them from discovering the true answer of 5400 lifetimes.

Having finished question three, Dale immediately expressed eagerness to attack the last problem: how can one conceive of a googol? At this point, Chad began to complain that the other class had just gone on its break. Remarkably, Dale (who in the past would have been eager to leave the room) ignored the mention of recess. The students in my class finally left for their break, but he stayed behind and pressed on with the problem at hand. The duo returned to Dale's original idea of bodies of water. As a hint, I gave them the fact that 18 mL of water (1 mole) contained 6.23×10^{22} molecules. Then, while I was explaining that the labels "cubic centimetre" and "millilitre" were one and the same, Dale left the room. I thought, "Finally, it's happened. He has lost interest, he has become discouraged, and his attention span has ended." Much to my surprise (and delight!), he returned to the classroom (while the other students were still on their break) with the comment, "So, where were we?"

After Dale consulted an atlas to find the total volume of water in the Earth's oceans, Dehlia led the way in calculating the number of millilitres (1.37×10^{24}) and the number of molecules (4.6×10^{46}) therein. Dale expressed dismay that they had reached only an order of magnitude not even equal to one half that of a googol. He asked me an appropriate question, "Why would they even have a googol?" (Here, Dehlia claimed erroneously that if they could get to 10^{50}, they would have reached "one half a googol.") Dale's final answer on his assignment was that in 10^{53} worlds of oceans, there would be 1 googol of molecules. Next to his answer he wrote, "That number is still pretty high, so my conclusion is there is no realistic way to make a model for a googol."

Although Dale and Dehlia worked as co-problem solvers, Dehlia often tended towards her natural role as teacher. She often questioned in a Socratic fashion, and generally lead the way in matters of computation. However, Dale was able to come up with original ideas for analogies in three out of the four questions. Though this day he seemed content in his role of follower, he was

equally up to assuming the leadership role, as demonstrated in his tutoring sessions with his peers, Chad and Alonzo.

What is truly remarkable in this session, however, is Dale's new-found perserverence. Typically, a student with ADD would have lost interest after five minutes. On the contrary, Dale worked with Dehlia for over two straight hours. Indeed, some special educators would be completely satisfied with Dale's progress based solely on his improved attention span and positive task behaviours.

At the time, I was disappointed in one aspect of Dale's performance. For even though he evidenced a considerable degree of creativity in the problems, his poor number sense seemed to handicap him somewhat in his deliberations. But this begs the question, "How else is he going to acquire number sense?" My answer was and still is, "Certainly not through a math course which teaches in a rote manner those arithmetic or algebraic skills which he appears to lack." In retrospect, perhaps I was being too hard on him, and on myself. I would venture to say that most high school graduates would not have the "good number sense" I expected from my students.

Session 10 Tuesday 8 March (Adie, Chad, Dale, Donny, Minna and Alonzo)
"She can't do it because I can't explain to her properly."

This session was devoted to exercises on scientific notation. In the first set of activities, the students were given a paragraph containing astronomical facts such as the distance from Earth to Mars (56 million kilometres) and the speed of light. Here, I was trying to avoid having the students manipulate large magnitudes which textbook exercises often present with no context whatsoever. I asked the students to enter the large numbers into scientific calculators, and to record the results in a table. When Dale entered 300000 into his calculator and pressed EE for scientific notation, he forgot the significance of the resulting readout $\boxed{3 \quad 05}$ (i.e. 3×10^5) .

> Dale: OK. This is 3 to the power 5.
> DT: Is that what that is? Is that 3 to the power 5? Think now. What does 3 to the power 5 mean?

Dale: But you said...you said it's not 10 to the power 5.

DT: I didn't say that.

Dale: It's ten to the power 5, just like up there, right? (*referring to an example on the blackboard*). Three to the power 5.

DT: That's the first number. Why is it three to the power 5? If you use your logic you'll see that 3 to the power 5 is probably not what you're after either. (*writes 3^5 on board*) What does that mean?

Dale: 3 times 3 times 3 times 3 times 3.

DT: You can work that out in your head. Three times 3 is nine, times 3 is 27. See if you can work that out in your head.

Dale: It would be 10 to the power 5 , and that three has to fit in somewhere.

DT: Yes it does. From the three how do you....

Dale: (*excited*) Three times 10 to the power 5.

DT: Right.

Dale: Right?

DT: Right.

Dale, although initially off in left field with his understanding of "$\boxed{3 \quad 05}$", conjectured that "that three has to fit in somewhere." When reminded of what he already knew about powers and exponents (that is, the meaning of 3^5), he enjoyed his success in coming to terms with the meaning of the calculator readout. Indeed, when students are initially confused over various mathematical concepts, an appeal to what they know is all that is often necessary to bring them around to apparent understanding.

The exercise also provided a context for reviewing the basic terminology of powers: the students themselves came up with the terms *base* and *exponent*. This is the power of small group learning: although individuals may have forgotten (or never even learned) definitions or concepts, usually at least one person will come up with the required knowledge or perspective.

Now the students' task was to write the scientific notation version of the numbers in a column beside the calculator readout versions. Adie was initially confused when the calculator displayed only $\boxed{3 \quad 02}$ instead of $\boxed{3.00 \quad 02}$ for the number 300.

Adie: I didn't get three point zero zero.

DT: What did you get, just 3?

Adie: Yeh.

DT: OK That's alright. Sometimes the calculator won't put in the zeros. These are just space fillers.

Adie: Yeh.

DT: What does this mean, 3 times...? What is that number "10 to the 2" after all?

Adie: A hundred

DT: Hundred. Sure. Three times a hundred is....?

Adie: Three hundred.

DT: Three hundred. Three times ...? *(points to 3×10^5 on board)*

Adie: 100 thousand.

Here, Adie's initial confusion provided a good opportunity for me to quiz her orally on her understanding of scientific notation. I was now becoming more and more convinced that in this small group setting, formal exams might not be necessary for evaluation. Instead, I had the opportunity to directly ask each student a few questions. In the much larger classrooms of mainstream schools, efficient oral examination on a daily basis is probably impossible.

The last exercise in Part A asked the students to enter two large numbers into the calculators: 150 000 000 (the distance from the Earth to the Sun in kilometres) and 9 500 000 000 000 (the number of kilometres in one light year). When it was found that ordinary scientific calculators were too small to display these numbers directly, I asked Alonzo (who was using my Texas Instrument TI 81) how he managed in his attempts:

DT: Try...uh enter the numbers below on your calculator: A hundred and 50 million. And the second one. Try...just try it. Even try it on this one. *(i.e. the TI 81)* What happens?

Dale: You could enter 150. *(Dale is close to the notion of scientific notation.)*

Adie: Can't fit it in there.

DT: Will it fit in on the TI ? Did you try it on the big one?

Alonzo: What are we supposed to enter here?

DT: Ah number 1 there.

Alonzo: Yeh. I've entered that.

DT: Did it enter?

Alonzo: Yeh, OK.

DT: It's a bigger calculator and it can handle big numbers. Most calculators have how many spaces?

Adie: Seven or eight.

DT: Yeh. Eight usually. Seven or eight.

Alonzo: This is a cool calculator.

DT: What would happen if you had to work with these numbers though?

Dale: You'd have to enter into...in from scientific notation.

Finally, it was time to see if the group noticed the pattern for scientific notation: that is, numbers in SN are expressed as the product of a number between 1 and 10, and a power of 10. Here, we were using a list of numbers which the students had already converted to SN using their calculators.

DT: In the...in the final form what do you notice about the scientific notation that's consistent? In particular, what do you notice about the 1st number? Each time?

Adie: They're all under ten.

DT: Yes. They're all under 10. And they're all greater than...?

Alonzo: Five.

Donny: One.

Dale : One.

DT: They're all greater than one. In other words, the first part of your scientific notation should be a number between ah one and less than 10. And I'll put...make a sign "less than" like that OK? (*writing* $1 \leq N < 10$ *on board*) And greater than or equal to. And that's the number that goes here. What number would we use here to express 150 million?

Dale: One?

Adie: One point five.

DT: One point five.

Dale: Yeh. So we put that in. (*i.e. into the calculator*)

DT: (*interrupts*) Times...?

Donny: Ten.

Adie: Seven.

Dale: Ten to the power 7.

DT: Alright if we put a...

Adie: Eight...ten to the eight.

DT: One , 2, 3, 4, 5, 6, 7, 8...very good. You notice that you had to move...you have to

move the decimal over 8 places here to get this one back again. That's scientific notation.

The eight tells you how many places to move this decimal to get the number you're talking

about.

Dale: So we have to enter it in our calculator.

This was the first time that I had attempted to teach scientific notation using calculators. I was

pleased that the technique was working. First the calculators provided examples of scientific

notation at the push of a button. Also, they provided a context for learning SN: the group seemed

to appreciate that to enter large numbers into calculators required a working knowledge of

scientific noation. .

After several examples on the board, I felt that the students were ready to reinforce their

understanding of scientific notation with some practice. But instead of having the students work

on scores of meaningless examples from a text, I assigned two sets of exercises from

Mathematics: a Human Endeavour. Although the problems were not metric, in each case an

interesting context was provided. In the first set, they were asked to convert numbers to scientific

notation:

> *Inflation of the value of money is a serious economic problem. In 1946, inflation of the*
> *currency was so bad in Hungary that the gold pengo was worth 130 quintillion paper*
> *pengos. Write this number in scientific notation. (The pengo was replaced that year by*
> *another unit of money.)* (Jacobs, 1982, p. 105)

Finally, in the second set, the students actually multiplied and divided numbers in scientific

notation:

> *Every minute more than 8.4×10^{11} drops of water flow over Niagara Falls. Every drop*
> *contains 1.7×10^{21} molecules. How many molecules of water pass over Niagara Falls?*
> (Jacobs, 1982, p. 105)

The students worked diligently on their exercises, but there was very little inter-personal

collaboration in spite of my pleas. At one point, after several students had asked my for help, or

for verification of their work, I told them, "Instead of asking me, compare notes. You're allowed

to that. You don't have to work by yourselves. Compare notes to see if you're getting the right

answers." For the most part, there were long pauses with no conversation whatsoever. The students worked, but they worked silently, on their own.

Verbal insults directed at Dale continued over the last two sessions. On Thursday, during our mathematics session, Adie call him "a fucking goof" and this day, she made fun of his occasional stammering. Referring to his perserverence, Minna commented, "What a nerd!" (This latter comment was reminiscent of the aforementioned research findings by Roberts and Peterson (1992) concerning girls' negative attitudes towards high mathematics achievement.) At this point, I felt that I was in a dilemma. I was hesitant to step in and directly discipline Minna and Adie, because I wanted them to change their treatment of Dale voluntarily. However I felt a responsibility for the tone of the class, and importantly, for Dale's feelings. Since my mild admonitions directed at Minna, Adie and Juliana had not remedied their social behaviours, I felt that there were two challenges for the future: to insist on tolerance between students, and also to foster social metacognition. How I would go about encouraging such changes I had absolutely no idea!

Minna's comportment of late caused me concern. This day, for instance, she initially refused to join the group and merely sat at her desk for the first while and doodled away. She became so easily discouraged with mathematics that I feared she would soon decide to quit the cohort. At the end of the class, I asked Adie if she would help Minna:

> Adie: I can't...I can't do tutoring. You'll just have to help her and show her how to do it because I can't.
> DT: If I do it, she kind of backs off .
> Adie: She doesn't...she won't do it.
> DT: Pardon?
> Adie: She can't do it because I can't explain to her properly.
> DT: Yeh but give it a try.
> Adie: I already did.
> DT: Not this part though.

Adie; Yeh I did.

DT: Can you just take...take half an hour on Thursday and...and work with her? Go through each question together? That's all you can do. You'll succeed more than I do. I know that. OK? I appreciate it.

I left that day thinking that if I cannot reach Minna, and that if even her classmates and friends cannot help her, then her tenure at VRLF was precariously hanging by a slender thread indeed.

On the positive side, I was as confident as a teacher can ever be, I suppose, that learning had occurred in my mathematics classroom this day. The smallness of my class meant that I could monitor each student's learning in a way I had never experienced as a teacher. Because of the small student/teacher ratio, I felt that both direct observation and dialogue continued to provide a thorough and meaningful gauge of my students' progress.

SESSION 11   Thursday 10 March 1994   (Adie, Dale, Donny, Minna)

This session was devoted to allow the students to try to catch up on their assignment (#5) on scientific notation, so that it could be handed in by Friday for report cards. My other motive in allowing a "free" period was to encourage cooperation between students in their work. Alas, such teamwork still proved elusive.

Adie, Dale and Donny worked alone on their assignments, but Minna chose to do science instead. When I asked her if she needed help, or if she was going to work on the assignment, she responded that she was going to do "the other math" in any event. Her final remark to me was indicative of the negative feelings she still harboured about mathematics: " I really don't understand it so I'm not going to bother."

Rather than ask each other for help, the students were still turning to their teacher. For instance, although any fellow student could have provided the answer, Donny directed the simple informational question to me: "How many days in a year again?" Adie solicited my help with some of the SN problems. Dale spent most of the session investigating the error in his and

Dehlia's previous computation of 100 billion divided by 8848 (the height of Mt. Everest). With a great deal of help from his teacher, he realized that by mistakenly entering 1 billion (10^9) initially instead of 100 billion (10^{11}), his answer was out by a factor of 100. At this point, however, he was able to correct his previous answer quite quickly:

> DT: Can you tell me the answer right away?
>
> Dale: It was 54 before, I think.
>
> DT: What's the key operation here?
>
> Dale: 54 times 2.
>
> DT: What's the key? No.
>
> Dale: Sorry. Fifty four times ten to the power of 2, so 54 times 100.
>
> DT: Exactly. 54 times 100. Which is....?
>
> Dale: *(reaches for calculator)*
>
> DT: No, don't do it on there!
>
> Dale: Five thousand and 4 hundred.
>
> DT: Five thousand four hundred and...it's a big difference eh?
>
> Dale: Yeh. Holy shit! OK, 5400 lifetimes to climb a hundred billion meters!

Each student had much to offer to the rest of the group, but such energies were still not being shared. That evening, I thought that individual assignments might be precluding the cooperative learning which I desired.

To his credit, Dale was the only student in the cohort to create analogies of all four large numbers. Donny and Adie both concluded that one billion seconds was approximately 32 years, and that 100 billion seconds was accordingly, 3200 years. Neither of the latter two attempted to investigate one googol, or the number of possible 649 tickets. Chad and Minna never did turn in assignment 5.

In retrospect, the results, although disappointing, are not surprising. Since these students have been subjected to a lifetime of only "short answer" mathematics, it should not be unusual that they balk at such open-ended assignments. The results of previous assignments and exercises during the project are consistent with this theory. On the one hand, all the students were ready to

tackle questions like "How many different hands of poker are there?" On the other hand, in Assignment 4, I found it impossible to motivate the group to work on creating a "customized" Lotto for VRLF. Similarly, Assignment 3 problems like "What is the 20th term of the sequence {1, 3, 6, 10, 15...} ?" were devoured eagerly by my students. However, their response to the open-ended question "What are the first four terms in the 100th row of Pascal's Triangle" was rather less than enthusiastic. Once they had the combinatorical formula, the students were quite happy to work out the answer. In sum, open-ended problem solving, like co-operative learning, is not an activity which alternate school students will automatically adopt just because progressive mathematics educators think that it is time for change.

With all the assignments in, I computed student averages for the first eleven sessions of the mathematics pilot project. In the past, student's report cards at VRLF were mainly anecdotal, with a nominal C+ given for completed courses. At this time, I was happy to depart from this custom and my students were eager to receive actual "regular" school letter grades. The group's positive expectation of report cards stemmed from two factors: first, there was the novelty of a new grading system, and second, individual marks (computed from attendance and from five assignments) were actually quite good. Accordingly, Adie, Chad, Dale and Donny were all awarded A grades on their report cards. But regretfully, Alonzo, Minna and Juliana did not complete enough assignments to warrant any marks in mathematics.

As long as the group size remained small enough for individual monitoring, I would continue with student dialogue during open-ended problem solving as an evaluation modality. Assignments would still serve as the main means of formal assessment, but henceforth, these would be completed and marked as group projects. Perhaps this new assessment procedure would also foster the students' interest in complex problem solving and the cooperative learning that I had been aspiring for.

Topic 5   Inductive and deductive logic

<u>SESSION 12   Tuesday 15 March 1994</u>   (Dale, Chad, Donny, Shawn, Alonzo)

This day marked the return of Shawn who months previously had been formally removed from the VRLF register because of poor attendance. As well as having learning disabilities, he suffered from deep and chronic depression. Having just witnessed the transformation of Dale, I was particularly curious to see what a new approach to mathematics might do for Shawn. In particular, I thought his return timely, since we were about to begin a new topic which I felt might suit many of my students' temperaments.

My decision to include inductive and deductive reasoning in the curriculum was an easy one. First, like all the mathematical topics I chose, this one was of great interest to me. I was thus eager to explore an important subject which was emphasized relatively little in regular school mathematics courses. As noted in the NCTM's (1989) *Standards* document, "[i]nductive and deductive reasoning are required individually and in concert in all areas of mathematics (p. 143)." Likewise, I felt that this topic would be particularly useful to students weak in basic life skills:

> The potential for transfer between mathematical reasoning and the logic needed to resolve issues in everyday life can be enhanced by explicitly subjecting assertions about daily affairs to analysis in terms of the underlying principles of reasoning. (p. 145)

Last, many of the experiences I planned were of the so-called "easy-entry" variety; in other words, little formal mathematics background would be necessary for participation. It was for this last reason in particular that I felt that an exploration of logical reasoning might appeal to students like Shawn.

Overall, this topic was so successful with the cohort that we eventually extended it to cover ten whole sessions. (Of course, with the new curriculum, we enjoyed the freedom to delve as deeply as we liked into any topic.) Attendance improved, and even students from other classes began to attend our mathematics classes out of interest. And importantly, certain other students now took their turns to blossom mathematically just as Dale had done earlier on in the project.

Session twelve was rushed, as I had forgotten about a special school event which necessitated moving back the afternoon start time. Also, my handouts were not collated, and I somehow misplaced the "lesson plan" sheet containing previously thought-out examples of inductive reasoning. Simply put, confusion at the onset was rampant. Certainly, if a "repeat" class had followed this one, the second lesson would have proceeded much more smoothly.

Chad interrupted my bungled beginning with an important question:

DT: Alright, guys. Inductive reasoning. (*pause*) Inductive reasoning everybody is ah....
Chad: I have a question. Um...every time we do...we do the math thing, we do something new.
DT: Yes?
Chad: Are they going to tie together? Or is that just the way it is?
Dale: That's just the way it is.
DT: My prediction is – yes in answer to your first question.
Chad: Yeh?
DT: And in answer to your question...your second question – yes. That's the way it is.
Chad: Cool.

Chad's query was important one, as I was looking forward to having the students discover through their investigations the inter-relatedness of all mathematics. And ironically, it would be Chad himself who would impress me again and again with his quick insights into mathematical connections.

When I asked the students for an examples of inductive and deductive reasoning, Dale asked, "Inductive is adding stuff right?" A little later, I understood his confusion. Using several examples, I went on to explain that inductive reasoning entailed three activities: 1) observing data, 2) recognizing patterns, and 3) making generalizations. Dale wanted clarification on a different meaning of the term:

Dale: I uh...OK. OK. Induction...induction means to make ob...observations right?
DT: (*Fumbling with students' handouts*) Right.

Dale: Inducted, right? How about "inducted into the Hall of Fame?" Would that mean make an observation there?

When I explained the Latin roots of the word, he was content with the idea that *induction* meant to "bring in" or "lead in", whether it be sports heros or experimental data.

Chad was able to give the following example of deductive reasoning from Sherlock Holmes:

> He'd find an envelope, and he'd have to find where the envelope came from. So he looked at the type of paper it was, so then he figured out it came from this place. So then he'd look at the stamp or something like that, or the handwriting, and he figured out (*unintelligible*) and then he knew he had a composite of the person who sent the envelope. That sort of thing. I think that's the way it worked right?

Still, the students were uncertain about deduction in mathematics, and in spite of any definition I could give, they would naturally remain so until they all experienced examples of both types of reasoning for themselves.

For this topic, I was not wanting for materials: Serra's (1989a) *Discovering Geometry* provided a plethora of excellent activities and examples. In fact, any educator in search of a text for any regular or alternate geometry courses at the secondary level need look no further than this book. Together with my own problems and puzzles, Serra's textbook was invaluable in providing the framework for most of the geometric activities throughout the project.

The text was also useful in that it allowed me to hear my students read aloud for the first time. For instance, I was pleased to hear Dale read so fluently: he proceeded at a good clip, with no stuttering, and a type of intonation which indicated understanding of the text. As well, he laughed at the punchline of Serra's example, and added his own apropos comments in the middle of the paragraph. During the next few sessions, I would have the chance to listen to the oral reading of all my students as we all took turns with the examples from the book. Indeed, I was soon most impressed with the reading skills of *every* member of the cohort.

Gradually, after doing many examples (which the students all found enjoyable), I sensed that there was general understanding of the concepts of inductive and deductive reasoning. The following is an example of one such exercise from *Discovering Geometry:*

> *Julio, sitting in the last row of a bus, notices that six girls in a row get on the bus wearing jeans, and each girl sits down on the left side of the bus. He conjectures that each girl that gets on the bus wearing jeans will sit on the left side of the bus. What is wrong with his conjecture?* (Serra, 1989a, p. 43)

This was the last exercise of the session, and I was elated that Shawn (who had said virtually nothing to that point) eventually contributed a key idea to the group:

DT: If you were on a bus and you saw this happen, every...every girl that came on the bus wearing jeans sat on the left.

Shawn: There's no reason to think that.

DT: Why not Shawn? Very good.

Shawn: Just because that's like incorrect.

DT: It's an incorrect conjecture. Why?

Shawn: Because...(*unintelligible*)

DT: (*interrupts*) Does wearing jeans have anything to do with where you sit on the bus?

Shawn: No.

DT: What could have been the explanation for the fact that all the girls who wore jeans sat on the same side of the bus?

Dale: Maybe....

Shawn: They were friends?

DT: OK. But these are people coming on at different stops; they aren't even talking to each other.

Dale: It could simply be a...

Shawn: Coincidence.

DT: Exactly! Very good, Mr. Shawn.

Chad: It's a coincidence.

DT: You have to distinguish between what's a coincidence...

Dale: How do you spell that?

DT: Coincidence, (*writing on board for Dale*) or what we call commonly...or you just say that that's just a....?

Shawn: Fluke.

Dale: Luck of the draw.

Shawn demonstrated a well-known truth about the minds of people with learning disabilities: there is often undetermined intelligence lurking just beneath the surface. At the end of this session, I was already hoping that given the chance, he would surprise us all with glimpses of just how profound that intelligence can be.

SESSION 13   Wednesday 16 March 1994   (Juliana, Dale, Adie, Chad, and Alonzo)

Rather than cancel Thursday's session because of a previously scheduled school ski trip , I decided to move the session to Wednesday. The start time also had to be changed to 12:30, since "rec" normally takes place after the Wednesday afternoon recess. As a result, students were late arriving for the mathematics class. At first, only Adie, Dale, and Juliana were on hand. Having just received notice from Chad that he would be late, I waited until 12:35 before starting. There was also no word from Donny, and Alonzo had not yet appeared. Tardiness and attendance were relatively new worries to me as an alternate school teacher (and major ones at that) since prior to this group project, such concerns meant little to the running of individualized programs.

Before the afternoon session even began, Adie arrived at school in a foul mood, and it was all I could do to convince her to stay for mathematics:

Adie : Can you give me the math I missed on Tuesday?
DT: We're going to do math today, and you can easily pick up on Tuesday's work.
Adie: Why? I thought it was on Tuesdays and Thursdays.
DT: Well, it's because of the ski trip tomorrow.
Adie: I'm not staying at school today.
DT: Why can't you stay?
Adie: I don't like this school.
DT: Well, stay for the math anyway.
Adie: OK. I'll stay for the math, and then I'm going home.

It was not until after school that I learned the reason for her petulance: according to Melvin, Adie had been having a "personality conflict" with Juliana, and wanted to move to Eric's class in the

afternoons. "Great!" I thought. "This is just what the group needs: another conflict to parallel the cantankerousness between Dale and the girls."

Frankly, Juliana's unexpected return from a three week suspension that afternoon caused me an uncertain apprehension and malaise. It was difficult to logically explain my feelings towards her. Was I worried that she would disrupt whatever inter-student rapport I had struggled to build so far? Or was she simply too much like a new student infringing on my established routine? In any event, my concerns were unfounded, because that day at least, she fit in quite nicely with the group, contributed to the activity, and related quite well to other group members.

My apprehensions around the shaky beginning to the session were soon swept aside as the students engrossed themselves fully in Serra's (1989a) often humorous exercises on inductive logic. The following was the last problem posed in the lesson entitled "*What is Inductive Reasoning?*":

1. Give an example of a situation outside of school where inductive reasoning is used correctly.
2. Give an example of a situation outside of school where inductive reasoning is used incorrectly. (p. 44)

Since I was eager to diminish my influence in the classroom, I thought that such an open-ended problem would afford an excellent opportunity to observe student interaction. Accordingly, I left the classroom, and did not hear the results until I previewed the tape later that same evening.

Juli: (*sighs*)
Adie: (*sighs*) We want to give an example of one that is used incorrectly.
Juli: Huh?
Adie: Let's give an example of one that's used incorrectly.
Juli: Um....
Adie: I don't know.
Chad: What? What are we supposed to do?
Juliana: Give an example outside of school where inductive reasoning is used incorrectly. Um...
Adie: Don't we have to pick one, right?

Chad: Give an example. OK, but what's inductive reasoning?

Alonzo: Well if...like say if you're driving down the road, and it takes you ten meters to stop your car...

Chad: Uh huh.

Alonzo: ...and I don't know how to explain that.

Juli: OK say....

Chad: You could conjecture that at that speed, every time I try to stop, it will take 10 meters.

Alonzo: Ten meters to stop your car.

Juli: Or what about every time you get to a stoplight, you're standing there, and um...OK if you don't press the WALK signal, right, you don't press that little button....

Adie: That car one of yours...ten meters. That's probably the best one to go with.

Alonzo: Yeh.

(Dale returns from the office)

Dale: What's that?

Chad: I agree with that.

Adie: When you stop your car....

Alonzo: If it takes...if you're going a certain speed, and it takes you ten meters to stop your car, every time. I don't even know how to explain it man. I'm still trying to figure out how to explain it.

Dale: It takes ten minutes *(sic)* to stop your car.

Alonzo: *(to Dale)* Meters.

Chad: I think you explained it quite....

Dale: Not necessarily though, because some brakes may be....

Juli: Ten meters.

Chad: Oh yeh. What happens if your brakes failed one time?

Juli: Aghh! *(exasperated at the interruption)*..To stop your car...if every time you do that then you can....

Chad: OK, we can say....

Juli: Assume.that the next time...

Chad: *(interrupting)* OK. But wait....

Juli: ...you do that it'll take another ten meters to stop your car. *(raising her voice over Chad's)*

Chad: It might not be correct.

Juli: *(Laughs)*

Adie: That's incorrect. Let's just use that one.

Chad: You could say every...if you're going at, you know, so...if you're going so fast, and you try and stop, and it takes you ten meters, you can conjecture that every time you're going that speed, and you try and stop, it'll take you 10 meters unless your brakes fail, or you have a heart attack and aren't able to move.

Juli: Ohh!

This was the first time that all the pupils related so well to each other in their mathematics. Also remarkable is the fact that although they still tended to interrupt each others' comments, the students treated their peers' ideas with respect and courtesy throughout. Only at one point was there any sign of exasperation or impatience; however, this incident with Juliana appeared to occur only out of eagerness to get on with the assignment. Importantly, even with their teacher out of the room, they were constantly striving to obtain closure or consensus on their assignment. Finally, there was constant cooperation between five students for whom individualized progress had been the norm during their entire careers at VRLF.

What was different about this session? First, I feel that the larger class size contributed to the group's accomplishment. In his teachers' guide for *Discovering Geometry*, Serra (1989b) offers the following suggestions for the size of cooperative groups:

> Research has shown...that four or five is the optimum size for most cooperative learning situations....If groups are too small, then there is not enough dialogue and someone is left out. If there are too many in a group, then there is too much happening, and managing group cooperation becomes very difficult. (p. 16)

Second, when I was present, I made a great effort this time not to answer every student question, deflecting them back to the group instead. Finally, my decision to leave the classroom altogether was obviously a good one.

Buoyed by the students' latest cooperative comportment , I was motivated to work out some sort of evaluative procedure for their group work. I began to entertain the idea of not necessarily rewarding them so much for a finished group project, but more for the collaborative process whereby they resolved the problem. Also, I wondered if the automatic five points for

merely attending a session should not be earned by 1) being on time; 2) staying on task; 3) social proprieties; and finally, 4) actively cooperating in mathematical activities.

However, after the up-coming spring break, I would have to discuss my ideas on evaluation with the students before changing anything. The worst thing I could do at this point, I felt, would be to turn their new-found behaviours around by imposing restrictions which they would almost surely take as chastisements.

Inter-session.   Monday 28 March

Upon returning to school, I was ecstatic to find that CWEC's new Macintosh computer had been delivered during the holidays. Since I was familiar with Quick Basic, Geometers' Sketch Pad, and various graphing utilities, the pilot course could take several different paths. However, there was now a considerable onus on me to master each of these wonderful computer applications.

SESSION 14   Tuesday March 29 1994   (Dale, Juliana, Adie, Chad, Alim, and Naomi)
"Oh, I don't like fractions."

Two other surprises awaited me at the start of our first lesson following spring break. First, Naomi, a student from another class who had been sitting in my room for some time, declared that she now regretted not having started mathematics with our group. I asked her if she had finished any mathematics, and since she already had passed Mathematics 10B, I told her she could possibly get some credit for Mathematics 11A if she chose to join the group. She remained in the back of the room, but was attentive throughout the group session, and even offered suggestions for solutions. In subsequent sessions, Naomi actually moved to our group table and participated.

Surprise number two was Alim. He had credit for Mathematics 10B, so he chose not to participate regularly in the study. However, he also was interested in the proceedings, and began

to join in. He kept moving closer to the front where the group was sitting, and eventually pulled up a chair to the table.

The interest that both these students professed this day was extremely gratifying to me. First, it was a recognition that I was doing something right. Second, it was an indication of intrinsic student motivation: this is what we should be aiming for. Grades or performance incentives, fear of failure or punishment, negative consequences for not participating, and so on, ideally should not be the major motivating forces behind student participation and attendance.

At this date, I was still struggling with the dilemma of poor attendance. Dale was a no-show today. Alonzo phoned to say he had found work, and would not be returning in the near future. And Shawn (poor Shawn) would be suspended for the absolute last time this week. (His performance with the group in the last session was thus a one-time event!) I asked myself, "Should I install a 'consequence' for absences? If secondary motivators exist at all, should they not be positive?" In any event, I felt that in the end, it would be interesting to compare students' overall attendance to their attendance on Tuesdays and Thursdays.

To further encourage the type of cooperative behaviour which occurred in the last session, I told the students that they would each receive ten points for their work in the previous session: one point for the answer, and nine points out of ten for their contributions to the group. I informed them that in other words, the *processes* here were at least as important as any final results they turned in.

For this session, I had the students work on *number patterns*: in particular, they would be given a logical progression of four or more numbers (or letters), and would have to make a conjecture about the next number (or letter) in the sequence. We used Serra's lesson on number patterns (1989a, p. 45), but I had the students keep their texts closed while one student at a time wrote the sequences on the blackboard. This procedure was meant to prevent students like Dale

from working on ahead of the group. After all, the aim of the session was as much to promote cooperative learning as it was to come up with the answers.

Juliana remarked that such sequence problems were often included on IQ tests. The pupils were interested to hear that by practising such number puzzles, they could actually enhance their performance on future intelligence tests. These kids were forever school-wise!

Having just answered the by now familiar triangular number problem {1, 3, 6, 10, ___}. they chose the following interesting route for the fourth sequence in the exercise set:
{1, 4, 9, 16, 25, 36. ___ } :

Juli: OK   The n...the difference between these is 3, between 4 and 9 is 5, between 9 and 16 is 7, between 16 and 25.....

Chad: It would be 9.   It is 9.

Adie: And the next one is 11.

Juli: So 3, 5, 7, 9....

Dale: Eleven.

Adie: Eleven.

Juli: Eleven.

Chad: And then?

Adie: Thirteen.

Chad: So....

Dale: Forty nine.

Chad: Hey, right on!

Juli: Forty-nine?

Adie: Forty nine.   OK.

At the time, I was surprised that they did not recognize the progression as perfect squares at all. They relied instead on the difference patterns which they had investigated earlier with the triangular numbers. In retrospect, my reaction was probably due to my expectation that the students should have had the same mind-set of the pattern as I did. Later I discuss further the implications of the attempts to impose my particular "awarenesses" upon the students.

The seventh progression { 1/6, 1/3, 1/2, 2/3, _____ , ____ } evoked a familiar response from the students before they settled in to solve it:

Adie: Oh I don't like fractions.

Juli: Oh I hate fractions! (*pause*) Can I have a text?

DT: No it's OK. I don't want you to have a text.

Juli: Oh OK.

DT: Just go from off the board.

Juliana's abhorrence of rationals actually prompted her to request a copy of the problems in the textbook so she could go on to the next non-fraction example.

According to Serra's rules for classroom deliberation, any item with an asterisk (*) beside it meant that students could ask the teacher for a clue.

Dale: OK, David! Clue!

DT: No. Copy it down first.

Juli: OK.

DT: Work on it for a couple of minutes, and then (*unintelligible*)

Juli: One sixth, one third, one half, two thirds. OK. What do these all have in common?

Naomi: They're fractions.

(*laughter*)

Dale: Yeh. They're all fractions. Good job.

Juli: Ah, thank you. OK.

Chad: Look at the bottom numbers: they go 6, 3, 2, 3. So, I would say that the last number may be...might be a 6. On just the bottom.

Adie: So 2 sixths?

Chad: I don't know.

Juli: OK maybe the top. The next one. Because there's been 1, 2, 3 "one's". Three "one's", so there should be....

Adie: Three "twos."

Juli: Three "twos."

Adie: Right.

Juli: So the next one will be 2 over....?

Adie: Two over (*unintelligible*) something.

Juli: OK. How many numbers are there over all? OK, that's it? OK, wait. Hold on. OK, that's not right. OK, Hold on.

DT: Hang on. Another question? Try to observe a pattern in there...

Juli: Oh Jesus!

DT:...for about ah half a minute. See what you can come up with.

Dale: We've been observing patterns *(unintelligible)*

Juli: OK quiet quiet quiet quiet quiet. OK. One sixth to one third.

Adie: How about one sixth to 2 thirds?

Juli: I wish I knew my fractions better.

In her frustration, Juliana was actually motivated at this point to recall a forgotten fact about fractions. I doubt whether such a desire would ever have come from the drills of the old mathematics course at VRLF. But undaunted by their lack of "basic skills," Juliana and her classmates carried on.

Juli: OK, one sixth to one third is....OK,what if you double one sixth?

Adie: It'd be 2 sixths.

Alim: It just goes small. It gets small. Then it goes big, and then as soon as it hits half it goes up again.

Juli: One sixth. No. Yeah!

Adie: One sixth would be two sixths if you double it right?

Dale: Twelves.

Adie: Twelves. Sorry. Two twelves.

Alim: OK You see? It goes one sixth, one third, and then a one half. After one half it goes up again.

Here was an example of poor arithmetic skills not affecting the momentum of a student investigation. Adie was initially correct in stating that twice one sixth was two sixths, but she stood corrected by Dale's erroneous assertion that twice one sixth equals two twelfths. Although this misconception did not lead to the discovery of a pattern, they maintained interest in the question, and their drive towards a solution was not impeded.

With his comment "It gets small. Then it goes big", I thought that Alim was referring to the relative magnitudes of the fractions. Reviewing the transcript, I saw that he was actually

referring to only the denominators of the fractions; luckily his teacher did not interrupt at the time with a word of reproachment.

Chad finally hit upon a valid visual pattern:

Chad: OK Wait. Look at this here. Hey! I got it I think! Look at it this way. If you put...if you draw them, right? See, they get bigger by...they get bigger by one sixth! (*Draws fraction pies on board*) Each time. So two thirds....
Juli: By one sixth!
Chad: So two thirds. The next one is....OK, two thirds is....Well, let's just draw it here. Um. Two thirds is that much, by one more sixth. So another one would be 3 quarters. Three quarters is the answer.
Dale: Three quarters?
Chad: Three quarters.
DT: Explain how you got that again?
Chad: Well, I drew them.
Adie: If you drew them out.
Chad: And they get bigger by one sixth each time.
DT: And your answer?
Chad: Three quarters.
Alim: Is that right?

The only fault in Chad's answer stemmed from his careless pie diagram . At this point, taking a cue from Dale that the students were eager to move on, and believing that they were close enough to a solution, I felt that a clue was in order.

DT: I like that concept of one sixth. Before you go on Dale, here's a little clue. Change each of those fractions to a common denominator.
Chad: Ah!
Adie: Oh God! Get out of here! I don't know how to do that.
Alim: The common denominator is 6.
DT: If you don't know how to do it, ask the person next to you.
Juli: The common denominator is 2.
Chad: Well that's what I just did basically.
Alim: No 6.

Juli: It's two.

DT: Write the fractions out again.

Juli: No it's two. It's smaller.

Alim: Oh it's two right, because 2 goes into 3?

Juli: Whoops. Six.

Under each term of the original progression 1/6, 1/3, 1/2, 2/3, ____ , ____ I had one of the students write the appropriate equivalent fraction with denominator 6.

Dale: OK That'd be 2 out of 6.

Adie: Two thirds. I mean 2 sixths. Yeh!

Dale: This one will be three sixths, and this one will be four...

Adie: Four sixths.

Dale: ...over six. And this one would be....

Juli: Five sixths.

Dale: Five....

Chad: Ohh! Whoops! Whoops!

Alim: Yeh that's pretty easy.

Adie: Six sixths would be the last one. That would be one.

Juli: Horrrahh! That's very simple.

Chad: Sorry about that.

Dale: No the next one would be five over six. Yeh yeh. and the next one would be 6.

Juli: No but what's the one on top of it? Five sixths. OK, then six sixths.

Dale: That's the answer too.

Adie: OK! Sure.

Juli: (*singing*) Joy to the World...

Again, we had an excellent opportunity to review basic skills in context: instead of being obliged to do pages and pages of meaningless examples, the students sensed a real need to work with common denominators.

I was most impressed with the cohort's effort during this session. For one thing, Adie and Juliana were kinder to Dale on this day: there were no unkind remarks when he made his slip (very usual for students with LD) about "common factors" instead of common denominators. The

students displayed great perseverance in their task, perhaps because they actually enjoyed the exercises! I was beginning to feel that I was actually observing a developing work ethic as well as a social change in the students. These changes would hopefully continue to manifest themselves and grow in the weeks to come.

Inter-session. Wednesday 30 March.

Juli, Chad and Alim made my day! They all announced that they wanted to resume inductive mathematics when they arrived and were disappointed when I said not until Thursday.

Thursday 31 March.

Another positive sign that the students were valuing their mathematics classes occurred this morning. Dale phoned me bright and early to excuse himself from the session because of his job. When I asked him if there was no way out of it, he replied no, but then he asked for some work to do at home. I gave him the remainder of last day's work, plus the new assignment on picture patterns. I thought that it would be interesting to see what he would do at home, on his own.

At noon, Katey complimented me for doing "wonders" with Dale, recalling "how off the wall he was last fall." Since collegial feedback was exceedingly rare, it was always good to receive input (especially positive) on my new program. In spite of some new desirable behaviours from the students, I felt at times that I was operating in a collegial vacuum at VRLF. Did any other teachers elsewhere really care about a new approach to mathematics? Perhaps it was time to communicate with mathematics educators in other alternate schools.

In any event, I had to cancel Thursday's session. Adie and Chad were already in another room watching a movie. When I went to remind them that we had mathematics this day, they pleaded with me to let them watch the movie ("Oh, come on David!") so I agreed to let them stay. Originally I thought this would be a good "test" of student motivation: would they pick movies or

mathematics? But I did not feel too disappointed when they opted for the video, since in my classroom, they would have spent the first part of the afternoon in the usual passive learning mode anyway, doing English, science or social studies.

Juliana arrived one hour late. She had one look at the video, and decided to return to the class. ("It's in black and white, and it's dubbed.") Surprisingly, she asked for some mathematics to do and I informed here that we would have to re-schedule the class. In retrospect, I should have given her something: such an impromptu session could have been interesting! Whenever in the future, the students ask to do mathematics (like Juliana today; Chad, Alim and her yesterday), I would oblige them. Why stick to an arbitrary schedule? Take advantage of their interest! But above all, I thought, "These are good signs."

SESSION 15   Tuesday 5 April 1994   (Adie, Dale, Chad, Donny, Alim)
"This isn't going to get us anywhere."

This lesson continued the logical sequences exercise set begun in the last session. To my great satisfaction, the students' attention span and interest in the problems remained high throughout. After successfully solving four progressions (such as 3, 5, 11, 29, 83, 245, ___), the students were presented with the following problem to tackle: O, T, T, F, F, S, S, E, N, ___ . Their efforts in this last case are remarkable for three reasons. They collaborated well in a group effort to try to solve the puzzle. They worked largely on their own, without soliciting very much help from me. Last, they refused to give up, even when, noticing they were stuck, I suggested they proceed to the next question.

Having experienced success with patterns through finite differences, the students' main line of attack involved assigning to each letter a number corresponding to its position in the alphabet. Alim wrote the following on the blackboard:

$$O, \quad T, \quad T, \quad F, \quad F, \quad S, \quad S, \quad E, \quad N$$
$$15 \quad 20 \quad 20 \quad 6 \quad 6 \quad 19 \quad 19 \quad 5 \quad 14$$

Consternation set in almost immediately.

Adie: Holy shit!

Dale: Now what?

Alim: This isn't...looks impossible.

Dale and Adie: (*laugh*)

Alim: It does though. Cause look there's like single....

Chad: And you can't read it backwards

Alim: Single double double double double single single.

Dale: OTTFF....

Chad: Umm....

Dale: Wait a second. I see something there. OK, you know how the 20's right? They go...(*goes to board*) I see something here. I see something here. Fifteen, and fourteen, they go down by 1. The twenties and nineteens, they go down by 1. The sixes and the five they go down by 1. So....

Alim: Yeh but why is there double numbers though?

Dale: Is that sort of close, David? What I was talking about there?

Alim: It's a pattern, but....

DT: It's a pattern but....

Adie: But....!

Alim: I don't think you're getting nowhere.

Chad: I don't...hmm. This is straaaaange.

Dale: OK David, we don't know this one.

Adie: Yeh David. We don't know this one.

Alim: This one's really puzzling.

Alim's comments "This one's really puzzling"; "This...looks impossible"; and "I don't think you're getting nowhere" are metacognitive in nature. With the first two utterances, he is estimating the relative degree of difficulty of the problem at the onset. Then, his last observation is an evaluation of the group's problem solving strategy so far. In the same vein, Chad commented a little later, "Maybe we're going at it the wrong way by putting numbers to the letters." It was good to hear students engaging in spontaneous metacognitive activity during problem solving, especially since the behaviour was a result of being stuck.

Although the students had reached a dead end in their strategy, they were not ready to give up on the problem. Even when I suggested, "Let's leave that one!", Dale retorted, "I want to

figure this out." This state of affairs presented an excellent opportunity for an introduction to Mason and Davis' (1991) the state of "stuck" and the emotions involved therein.

DT: I want to ask you guys to keep a record of how it feels to be stuck...

Alim: It's frustrating.

DT: ...in a situation. So right now you're stuck on (*unintelligible*) .

Alim: We have to come together as a team.

Adie: Oh.

Alim: Let's be a family..

Adie: (*laughs*)

DT: Well it helps...it helps to come together as a team. But at times, a team can get stuck. And we're going to have to devote some time to the word "stuck". What it's like to be stuck, and what you can do about it.

Alim: Nothing's impossible.

Dale: You can do it.

DT: So Alim said one...one aspect of being stuck is frustration for him. What about you Don?

Adie: Just give up.

Chad: Well, yeh, that's...I think that's dead on the button. When you're stuck you're frustrated.

Alim: I think that's what everyone feels.

DT: What other emotion do you feel? Like Adie said, "I just give up."

Dale: How do I feel? Just abandon it.

Alim: My frustration turns to anger.

Adie: Pissed off

Chad: I feel like giving up, or working harder to solve it. It depends.

Adie: Just quit.

DT: What does it depend on?

Chad: Well, whether...how interested I am in it.

DT: Right. How you feel?

Chad: Yeh, cause I'm not going to...you know...go on and on trying to solve a problem I really couldn't care less about. You know. It's not worth my time. But....

Alim: I hate it when I'm trying to solve a problem that (*unintelligible*).

Chad: Since we're all at school, and we're all working on this, and we have the group thing going, and we all seem to be directing our energies towards the same goal, it's...it makes it interesting, and therefore I want to.

Chad's comment summarized one of the main themes that I had been concerned with since the beginning of the project: namely, I believe that there is a positive correlation between cooperative learning and student motivation. Of course, the third variable involved in the pedagogical equation is mathematical content of sufficient complexity to stimulate the students. In the main, I feel that their comments about giving up or abandoning a problem refer largely to their past experiences when they were working alone on meaningless curricular content.

Realizing that the pupils were on the wrong path, and feeling rather guilty that this was, in fact, a trick question, I asked them about their strategy so far:

DT: And if you're using inductive reasoning, and you see no visible pattern, what can you do ? What can you say about this problem then? Or what can you say about your approach?
Adie: Wrong approach.
Chad: It's not working.
DT: Good. It's not working. So you'd better do what?
Adie: Change your approach.
DT: Dale? Adie?
Adie : Do it different. Something different.

However, the students still pressed on with the same strategy by proceeding to the second level of finite differences. After I provided them with the eleventh letter in the sequence, they had the following on the blackboard:

```
O, T, T, F, F, S, S, E, N, T, E ___
15 20 20 6 6 19 19 5 14 20 5
  5  0 14 0 13 0 14 9 6 15
```

Undaunted, they decided to continue to look for a pattern :

Chad: Wait. I think I've got something here.
Adie: Write it up.
Chad: Go back to the last one, and write the numbers up there again. And then maybe if we all look at the numbers we can see something.

Adie: OK.

DT: Look. If you decide you want to stay with question 26, that's fine with me.

Dale: Does everyone want to stay with 26?

Alim: Yeh!

Adie: Yeh!

Chad: OK.

DT: I only wanted to go on so you wouldn't get frustrated by being stuck.

Although Dale came close to the solution ("Maybe it's trying to say a word."), the group made a final metacognitive judgement on the futility of its approach:

Alim: This isn't going to get us anywhere

Adie: (laughs)

Dale: I know. It ...it won't get us anywhere. Right ?

I finally encouraged them to go on to the next question:

DT: One part of getting stuck is to admit you're stuck. Say, let's leave number 26, and do a few more.

Alim: Come back.

Adie: OK. Let's do number 30.

DT: And then come back to it. Maybe on the bus home you could have a look at it or something like that. When you're in a different environment sometimes these things will jump out at you.

Alim: I think we should do number 30.

With such a group consensus for change, I was comfortable to have the students finally move on from the problem O, T, T, F, F, S, S, E, ___ (whose solution I leave to the reader).

Finally, the students went on to consider the sequence {4, 8, 61, 221, 244, 884, ___ }. Dale quickly showed me the next number (8671) in the progression, and for the rest of the session, enjoyed the role of "teacher". Soon Adie began to notice the pattern:

Adie: Half of 8 is 4, half of 4 is 2.

Chad: But it won't....we can't get into....

Adie: Half of 44...half of 44 is 22; half of 2 is 1. Right? Well when you look at that those numbers are doubled, but....

Chad: They're also opposite.

Adie: Yeh, they're doubled. 88 and 44; 4 and 2; 44 and 22; and 1 and 2

Chad: Huh?

Adie: Huh? (laughs) 88, right? Half of 88 is 44.

Chad: Ohh!

Adie: Four...half of 4 is 2.

Chad: Ohh! By Jove, I think she's got it.

Dale: Explain how you're doing it.

Dale was anxious to show the group what he already knew:

Dale: Here. Here. Can I show them?

DT: Yeh. Yeh. Go ahead.

Alim: Here's the chalk.

DT: You...you guys are close. You'll be...you're on the verge of getting it.

Dale: You were really close. It was 884 times 2.

Adie: Oh my God.

Chad: What?

Adie and Dale: 884 times 2.

Chad: Why?

Dale: Which was hundred and seven. (sic)

Adie: One thousand 7 6 8 (1768). Whatever.

Dale: So switch it around: 8, 6, 7, 1

Chad: Ohh! OK.

Adie: OK. Umm....

Alim: We're stupid eh?

Adie: Well, we almost had it. (laughs)

Chad: Yeh, I see how you get it.

Dale: Do you want me to do number 20 David?

I was happy that finally the group was solving problems with very little input from me. In fact, it was heartening to have one of the students take over the role of teacher for a change.

The last problem of the session { 6, 8, 5, 10, 3, 14, 1, ___ } provided an excellent opportunity to review some number theory with the cohort. The group immediately wrote the finite differences above the given sequence:

$$2 \quad 3 \quad 5 \quad 7 \quad 11 \quad 13$$
$$6, \quad 8, \quad 5, \quad 10, \quad 3, \quad 14, \quad 1, \quad \_\_\_\_$$

They deliberated for a while on possible patterns, and since we were nearing the end of the class, I asked them to consider the set of numbers that they had written above the originals:

Adie: I know! Those are all prime numbers. Those are all numbers that....

Chad: They can't be divided by anything.

Adie: They can't multiply and get an answer. Or something. You know what I mean?

Chad: They can't be divided by something.

Adie: (*laughs*) Like you can't...they can't go uh....

DT: Why isn't 22 a prime?

Chad: Because they have 2....(*pause*)

DT: Adie, you explain it. Why it's not.

Adie: Because 2 times 11 is 22.

DT: Why isn't 21 a prime?

Adie: Because 7 times 3 is 21.

DT: Why is 19 a prime?

Adie: Because they don't have multiply factors or whatever.

Chad: Yeh! (*claps*)

DT: Exactly. They only have...how many factors does each prime have?

Two unidentified voices: One ...one

Chad: Two.

I ended the class by writing a formal definition of prime numbers on the board. Normally, the definition would be written at the beginning of the class, and would be followed by numerous examples. Here, the concept was reviewed in the context of a problem, and the students themselves were involved in a way which had meaning for them.

Again, I was pleased with the group's conduct during the mathematics class. They seemed to like the inductive puzzles, as evidenced by their willing participation in the solutions. And even though there was very little overt contribution from Donny, he seemed to follow the group, and spoke out when errors were committed.

I was concerned that Dale showed some of the anti-social behaviours associated with learning disabilities. For instance, at the beginning of the class, he found it humorous to hold a lit cigarette lighter close to Alim's arm. Then, he offered Adie some candies, and after she accepted a few, he informed her laughingly that he had just retrieved them off the floor. Such antics, which were consistent with the ADD behaviours discussed in Chapter 2, only served to alienate Dale from his peers. During this session I gained a new perspective on Dale's social ostracism: from now on, I would not be so quick to condemn only those students who mistreated him. Peer relationships were after all multi-faceted phenomena, and perhaps were far out of my pedagogical league to control.

At the beginning of the afternoon class, Dale submitted the "homework" assignment I had given him on the phone. Essentially, it involved discerning geometric patterns, and actually drawing the next logical shape in each of the various visual progressions. Dale did poorly: he left many unfinished, and obviously had difficulty in making conjectures and/or in translating his conclusions into pictures. However, the full implications of his disappointing performance would not hit me until much later in the project when we began to investigate fractals.

By the fifteenth session, Chad's attendance had improved drastically. However, his tardiness remained a chronic problem: today he was an hour and a quarter late for the afternoon class. Again, my sentiments were, "Well, at least he showed up on time for the math lesson." I had difficulty chastising him because he was a likeable lad, and was becoming a very active participant in the mathematical considerations. I could sense that Chad's level of contribution was increasing, and in this regard, he was beginning to eclipse Dale.

INTER-SESSION Wednesday 6 April

Two nurses just assigned to VRLF were visiting the classrooms and chatting with the students. I happened to overhear Chad's opinions of the school which he expressed to them. He said, "Really, I want to go back to regular school with my friends. The courses here are boring,

except for the math, because it involves open discussions." Such validations of what I was attempting at Vale Road were instrumental in driving my desire for curricular change. Indeed, positive reinforcement is a two-way phenomenon; it is as important for teachers to receive it as it is for students.

SESSION 16   Friday 8 April 1994   (Adie, Chad, Donny, Alim)
"Powers of the patterns."

Prior to the mathematics session, Alim asked, "Are we doing math today?" When I responded in the affirmative, he expressed his opinion of his experiences so far:

Alim: I like doing math in this class. It's fun. Also I understand it. It's better doing math in groups.

Adie: What are we doing today? Finishing off that exercise?

DT: We'll be doing Exercise C, and then going on to deductive logic

Adie: What's the answer to that problem? *(ie O T T F F S S E N T E ___)*

DT: I'm not telling.

Adie: Aghh!

DT: I'll just keep it on the board here for you to look at.

Alim: Can we work on it today?

It was good to hear that despite not achieving closure on last day's problem, the students were eager to continue their quest for a pattern.

Serra's (1989a) exercise C in inductive logic involved making conjectures on patterns such as the following:

$$8 + (9 \bullet 0) = 8$$
$$7 + (9 \bullet 9) = 88$$
$$6 + (9 \bullet 8 \bullet 7) = -?-$$
$$5 + (9 \bullet 8 \bullet 7 \bullet 6)... = -?-$$
$$------- ? ------- = 88888 \quad (p. 47)$$

Taking his turn, Chad easily discovered the pattern.

Chad: The first one is 3 eights. And the next one is 4 eights.

DT: Three eights.

Chad: And then...oh yeh. No doubt. *(laughs)* And uh...um...and the last one is 4 plus bracket nine times 9876.

DT: If anything, just realize the arithmetic that you've done here, without even a calculator.

Chad: Powers of the patterns.

DT: Powers of the patterns. Very good.

An important aim of mathematics education expressed in the NCTM's (1989) Standards is for students to gain "mathematical power" (p. 5). As discussed in Chapter 2, alternate school students often have poor self esteem and low self confidence. It was my hope that through such experiences, they would turn from these previous feelings of inadequacy and hopelessness to an awareness of their own abilities and powers to do mathematics.

We then turned to consider examples of *deductive* reasoning. The following problem from Serra's (1989a) text provided good practice and resulted in some extended group problem-solving.

> *Edith, Ernie, and Eva have careers as economist, electrician, and engineer, but not necessarily in that order. The economist does consulting work for Eva's business. Ernie hired the electrician to rewire his new kitchen. Edith earns less than the engineer but more than Ernie. Match the names with the occupations.* (p. 563)

Alim immediately commented, "We have to draw this out." I thought this would be an opportune time to introduce them to a deductive logic grid, and so I drew the following on the board.

	Edith	Ernie	Eva
Economist			
Electrician			
Engineer			

After I explained what "consulting work" was, the students began their task. Following considerable debate, they arrived at the stage where the solution fell into place, thanks largely to the chart.

Alim: Ernie's not the engineer for sure.

Adie: How do you know?

Alim: Because look....

Chad: I think we've just been guessing, unless we knew....

Alim: (*at same time, reading*) Edith earns less than the engineer but more than Ernie.

DT: Say that again Alim?

Alim: Edith earns less than the engineer but more than Ernie.

DT: What do you know from that statement?

Alim: So Ernie's not the Engineer.

Adie: (*at the same time*) He's not the engineer.

DT: Ah! OK. So?

Chad: Why do we know that?

DT: Explain.

Alim: Because uh...it says here, "Edith earns less than the engineer." Right?

Adie: (*interrupting Alim*) So obviously he's not....

Alim: But more than Ernie. So obviously he's not the engineer.

Chad: Ohh!

Alim: Ernie's the economist.

DT : Right. OK.

Chad: OK. Way to go.

Alim: OK.

Adie: (*laughs*) So Edith is the electrician.

Don: No Edith's the....

Adie: No, no. (*unintelligible*)...because of the big cross over the engineer.

Alim: OK Now that...that we know (*unintelligible*)

Adie: OK. Edith is the electrician. And Eva is the engineer.

Alim: How do you know?

Adie: Because...look.

Don: Yeh.

Adie: There's nobody else there: you can put a cross underneath the economic thing...economist. Right there. 'Cause....

Alim: Yeh yeh yeh.

Adie: So she's obviously the electrician.

Chad: Oh ho!

Alim: I get it!

Chad: Cool!

Alim: Cool! That's wicked!

Chad: And that would leave...(*unintelligible*) engineer.

Alim: Hey this stuff's kind of fun, eh?

Adie and Chad: (*laugh*)

DT: That's deduction.

I enjoyed observing the students work together for two reasons. First, it was good to see them reach a conclusion through cooperative problem-solving. Second, their enthusiasm for the problem and the process was clearly evident. Now that they were getting a taste of group learning, I hoped that their willingness to participate in this modality would continue to grow.

There was one aspect of their group behaviour which needed modification: that is, they continually interrupted each other during the debating process. (Such behaviour is not restricted to alternate school students: even graduate students are known to talk out of turn during their classes.) Even though such talking out of turn could probably be attributed partially to their enthusiasm for the problem, the students would obviously need some guidance in group decorum in general, and in forum speaking in particular.

After this introduction to deductive reasoning, I asked them to consider what the term meant:

DT: What 's the difference between deduction...deductive logic or reasoning, and inductive reasoning?

Alim: It's when you take things out. When you see what...what...it's hard to explain. OK. Like you throw things away...you throw them out.

Chad: You get the actual answer for sure. With no doubt about it .

DT: True.

Chad: You....

Alim: You go step by step and you...like...like what we just did. Like...you marked off the things that they weren't.

Chad: No, but we did that with inductive reasoning too. Didn't we?

DT: What was the hallmark of inductive reasoning?

Chad: Patterns.

DT: Yeh. And before that?

Chad: Observing the data.

DT: I think you said, Alim, I think you said you go step by step. Those were your words?

Alim: Yeh but it's hard to explain. Like...you know what I mean though. Like you throw....

DT: Yeh, I know what you mean.

Alim: ....throw things away to find out the answer.

DT: You reject certain things as being not true, as you go.

Alim: Yeh, you deduct (*sic*) them.

In spite of his imaginative interpretation of the root of the word "deduce," Alim was accurate in his assessment that with deductive reasoning, we eliminate possibilities to arrive at the truth.

For the final exercise of the session, I presented the students with the classic series problem $S = 1 + 2 + 3 + 4 + 5 + 6 + ...+ 98 + 99 + 100$. Adie came up with an approach which I had not seen before: she proposed adding 99 plus 1, 98 plus 2, 97 plus 3, etc. so that the result was 100 in each case.

Chad: That still leaves you with another 100 at the end though, so you have to add that hundred on.

Adie: Yeh yeh..

Chad: And you do that 50 times.

Adie: 50 times, right. Which gives you 5...5 thousand right?

Chad: And you add the other hundred on, and it's 51 hundred, right?

Even though their first estimate of the answer was incorrect, Adie came up with a beautiful phrase to describe her process: she remarked, "What we're doing...we're kind of like folding it in half you know?". Right away, the situation became clear to Chad, and he corrected their original mistake:

Chad: There's not two 50's in there. It's just 50, so....

Adie: Leave the 50 out, and you....

Don: Get four thousand nine.

Chad: Five thousand fifty.

I shall never forget Adie's "folding in half" depiction of the original sum, and I shall use it in future presentations of the famous Gauss arithmetic problem:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + ... 48 + 49 + 50$$

$$\underline{100 + 99 + 98 + 97 + 96 + 95 + 94 + 93 + 92 + 91 + ... + 52 + 51 +}$$

Indeed, as they teach, there is always so much that mathematics educators can learn from their students .

SESSION 17   Monday 11 April 1994 (Chad, Donny, Juliana)
"Boy, it must be tough being a teacher, eh?"

I scheduled this extra session to make up for the class lost two weeks back. Unfortunately, student attendance was typical of a Monday at VRLF. Alim and Adie phoned in their excuses, but Dale, who failed to notify the school for this session and for the last one on Friday, was causing me some concern.

I first presented the students with the following problem, and let them work on it on their own for a time:

> *A class of 32 students was surveyed about their preferences for pizza toppings.*
> *-1 student likes onions only*
> *-5 students like pepperoni only*
> *-6 students like mushrooms only*
> *-20 students like pepperoni*
> *-8 students like pepperoni and mushrooms*
> *-3 students like pepperoni and onions*
> *-3 students don't like any of the toppings*
> *How many students like all of the toppings?*
> *How many students like mushrooms and onions only?*

Chad: The numbers that they give us adds up to more than 32. So some of them are like changing their minds or something.

Juli: No! Because some of the...if you read the question properly, some of the students like pepperoni, some of the students like pepperoni only.

Chad: (*laughs*)

Juli: So 20 of those students who like pepperoni could like something else too.

Chad: Not if they like pepperoni only.

Juli: (*raising her voice: seemingly impatient and agitated*) Here it says, "20 students like the pepperoni, and 5 students like the pepperoni only."

Chad: Ohh, I see!

Juli: So there's...understand?

Chad: Yeh. Hmm.

Juli: How many students like all of the toppings? Um....

Chad: Well, we can cross that out. (*pause*)

DT: I'm just going to run off a couple of photocopies, so share information that you're...that you're working through.

Juli: Uh huh.

Chad: Hmm.

Juli: Hmm. Hmm. I understand. (*pause*)

Chad: Does it mean...like mushrooms and onions. Does it mean that they like them together? Or to they...they...they like one?

Juli: They like them separate.

Chad: Yeh.

Juli: I'm not sure. I doesn't say, does it?

Chad: No it doesn't say. I hate questions like that that ask you a question.

At this point, I left the room. The two students considered the problem for another minute, then began talking of non-mathematical things. Even though little mathematics was done in my absence, the students were quick to get back on topic upon my return to the classroom. Chad sought clarification on the wording of the problem, and then posed a key question:

DT: What have you got?

Juli: OK. This is difficult

DT: Have you got an answer?

Chad: OK. First of all, it's inconcise. No, I've got a question.

Juli: Inconcise?

Chad: Maybe that's not the right word, but hey um...(*unintelligible*)

DT: What's your question?

Chad: OK. It says, "How many students like mushrooms and onions?" Do they like...how many students like mushrooms and onions together? Or how many students like mushroom pizzas, but they like onions?

DT: Together. Yeh, you're right.

Chad: OK Shit. Well, are we supposed to draw a diagram?

DT: Well, do you think that will help?

Chad: I can't think of how to do it though.

Chad's question about drawing a diagram provided the perfect segue into Venn diagrams. I then drew two non-intersecting circles on the blackboard.

DT: Pretend that this is the mushroom circle. (*pointing to first circle*) If you're in this circle, you like mushrooms.

Juli: Uh huh.

DT: (*pointing to second circle*) If you're in this circle, you like onions. Where would you be if you like both?

Juli: Right here. (*draws a connecting "corridor" between the two circles*)

DT: Yeh but if you're in here (*pointing to where Juli drew*) you're out of both circles. That means you like none.

Chad: (*laughs*)

Juli: OK. OK. Umm....

DT: See?

Chad: Uh huh. Overlap them.

Don: Overlap them.

Thus the students themselves came up with the basic idea for the Venn diagram: intersecting circles represent the intersections of sets. To accommodate the other "topping", Chad suggested that we "make the...like the Olympic sign. Overlap them". The group then came up with a nearly completed Venn diagram: (See Figure 3.7)

Figure 3.7
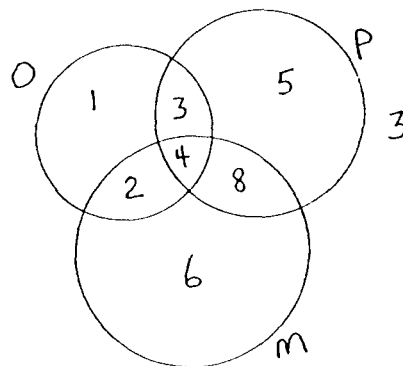
Juliana was the first to deduce the answer to the first question.

> Juli: So 4 of these people have to like everything.
>
> Chad: I don't understand how you....
>
> Juli: It says 20 people like pepperoni, right?
>
> Don: Yeh.
>
> Chad: Oh ho!
>
> Juli: So you add these up. It's only...it comes to 16 people who like pepperoni, so there's 4 left. So where are you going to put them?

Finally, following Chad's metacognitive question, Don proposed a way to answer the second part of the problem.

> Chad: Are we on the right track?
>
> Juli: Yes.
>
> Don: If you add all them up and then find out the left over....
>
> Chad: Oh wait a second. Yeh. (adding) 6, 4, 10, 15, 16, 17, 18, 19. 19 and 8 is 27. We're missing...and there's a 3 over there. We're missing 2. So I guess the 2 would go in the other one right? Yeh, right there.
>
> Juli: Oh!
>
> Chad: Cool.
>
> Juli: There!
>
> Chad: Yeh!
>
> DT: (unintelligible) Juliana?
>
> Juli: Yeh..
>
> Don: So how many like onions and mushrooms!
>
> Juli: Well, that was pretty good.
>
> DT: How long do you think it would have taken you just fooling around with it...with the words?
>
> Chad: Days.
>
> Juli: Probably a long time.
>
> Chad: Wow!
>
> DT: This is called a Venn diagram, named after the mathematician who invented it. And Venn diagrams are usually used in set theory, to make things clearer. The circles represent

sets. Each circle is a set of people, and where they intersect, we call those intersections of sets.

Chad: Yeh.

The curriculum again provided a situation where the problem at hand necessitated the theory. Usually, the teacher would introduce Venn diagrams, and would then follow up with numerous examples of their applications. But here, in order to effectively solve the mathematical question, the students came up with the Venn construct largely on their own.

This session also resulted in some interesting social dynamics between Chad and Juliana. As an intelligent individual, Juliana seemed used to setting her own academic agenda, whether in a group or individual setting. But Chad reacted several times to her attempt to dominate the flow of the lesson, and in particular to her tendency to interrupt others. For instance, at one point, Juliana tried to silence her peer:

Juli: Ssh!

Chad: Quit telling me to keep quiet!

At three different points in the session, Juliana directed the following rather bossy comments to Chad: •"Can you just be quiet? I'm trying to read this."

•"Ssh! Do you mind? I can't just study this."

•"Ssh! Do you mind? This is my answer!"

Chad responded with the following comments: "You shouldn't put people down so much!" and "You've done the same to me. Now you know how it feels. Reflect upon that for a second!" Considering Juliana's vehemence, I felt that Chad handled the situation with relative constraint. Yet he said to Juliana what she needed said to her. In retrospect, I realize that any reaction against such inimical comments was far more effective coming from a peer than it was coming from me. Perhaps peer pressure was the solution to my dilemma over how to foster positive inter-student relationships.

Nonetheless, I took the opportunity to talk to Donny, Juliana and Chad about classroom propriety and orderliness:

DT: Let's reflect upon the situation here.

Chad: (*laughs*)

DT: I was going to bring this up with the whole group. And that is, if you're working in a group, we all have to allow each other space to complete sentences. And I'm the number one guilty party for that. Because I listen to these tapes after, and I'm constantly interrupting you guys.

Chad: (*laughingly to DT*) There you go again! Geez!

Juli: Oh God!

Chad: You're talking.

DT: I'm just waiting for you to finish your sentence Chad.

Chad: Oh !

DT: It's difficult to do because you get excited about having an idea, and you want to inject it into the group. But we're always cutting each other off.

Juli: OK.

DT: And like I said, I'm probably the number one user of that technique: cutting off.

Chad: Really?

DT: Yeh, I do it all the time. I don't allow you to complete your sentences.

Chad: Hmm.

In truth, I was as guilty as any of my students of interrupting others' oral contributions. I hoped that by including myself in any culpability, the students would be more receptive to my observations about our group dynamics.

At the end of the day, Juliana and Chad appeared to part on good terms. The session ended on a rather humorous note with the following discussion of the teaching profession:

DT: I've always said that being a teacher has its drawbacks in...in physical health. One of them I thought was...since I use a pen so much in my job...that I get a little groove in my finger right there?

Chad: Oh yeh!

Juli: Hmm.

DT: So that's a health hazard, right? I've got a groove there.

Chad: Yeh, teachers should get danger pay.

DT: Danger pay, or or just workman's compensation.

Chad: (*laughs*) Yeh!

DT: So I've got a little groove. And the other thing is all this dust, eh? Chalk dust I'm breathing all the time. So I should get compensation when I retire for damage to my lungs...

Chad: Yeh.

DT: ...after all these years.

Juli: Hmm.

Chad: Boy it must be tough being a teacher, eh David?

DT: It is. It really is. *(laughs)*

Juli: Well, they don't even pay as much as a teacher's worth do they?

DT: Not as far as I'm concerned.

Juli: Hmm.

Chad: Yeh.

DT: No one gets what they're worth as far as they're concerned.

Chad: But you get all the rest that they don't pay you in money out of other things in your job don't you?

Juli: Do they pay you while you're on holiday? While the school's out on holiday?

Chad: I mean you get us!

That's right, Chad. I do have students like you who come shining through enough of the time to make my profession well worth it all.

SESSION 18   Tuesday 12 April 1994   (Adie, Dale, Chad, Juliana)
"Now I feel like a real mathematician." "

Since we had passed the halfway point in the mathematics project, I decided to pause and obtain some student feedback. To that end, I handed each student a list of questions about past sessions, and about where we should head from this point. (See Appendix 4.)

The first part of the questionnaire read as follows:

Have a look through your "folders", and consider the following questions.
We've covered the following topics: 1) Pentominoes 2) Probability 3) Pascal's Triangle
4) Scientific Notation & large numbers 5) Reasoning

1) Should all of these topics be included in future Math courses for VRLF? Should some not be included? Why not?
2) Were you bothered that the some of the problems you were presented with were not solved? Why or why not?
eg. "What is the probability that a series of 10 births will have 4 or more girls born in a row?"   or "Cover a 6 x 10 rectangle with pentominoes."

Would you like to revisit these problems in Part II of the course?
3) How do you feel about working in a group, as opposed to working by yourself?

Student opinion was quite varied in response to question 1. For instance, Adie cited boredom as the principal reason for her dislike of the large numbers topic: "The thing was with me ...after I tried, and didn't want to do it any more? Then I got bored." Dale's initial response to the question shocked me at first:

DT: OK Dale? I heard you...I think you said the same...(*as Adie*)
Dale: Everything!

Immediately I thought, "Oh no! He disliked everything." But to my relief, Dale meant the opposite of what I perceived initially.

DT: The same chapter?
Dale: No everything. I thought about everything. I think everything should stay.
DT: Should stay?
Dale: Yeh Cause I...I...I actually had fun doing the large numbers so....
Chad: Yeh I didn't mind large numbers.
Dale: Cause I got most of the answers, right? So I didn't mind large numbers.

Finally, Chad expressed dislike of the probability unit for philosophical reasons, commenting, "I'm not a probability kind of guy."

In response to the second question, Dale suggested "that we keep the last three sessions for getting all the ones that were unsolved." While Adie expressed disinterest in the idea of a future look at unsolved problems, Chad articulated a comparison between open-ended problem solving, and the classic format of other mathematics courses:

DT: Well, what was the feature of other math courses that's different?
Chad: Well, they always solved the problem. They...well....Well, you know, I remember at Moscrop how it went was...the teacher would go through a thorough explanation, we'd do problems together...
DT: Uh huh...
Chad: ...then we'd uh...then we'd do the problems in the book. And then the next day we'd mark 'em all on the board. We'd all take turns going up doing the questions, and...so

that was...that was OK. You know. It was done good. This is...this is kind of more fun though. Because um...it might be the small number that we have, or else it might just be the uh...I don't know. It's more of a group effort, and that kind of contributes to the good uh...I don't know. It's fun, and it's...it's a lot more....I guess because we're...we're taking time to explore it, and come to our own conclusions, and work through it that way. So therefore, we're teaching ourselves. Kind of. In a way. Something like that. (*laughs*) I don't know.

At this point, I was pleased that Chad had not only succeeded in describing the process of discovery learning through cooperative effort, but also had given another endorsement of the idea.

During a discussion of the role of mathematicians at the university level, Juliana and Dale became involved in an altercation reminiscent of small children:

Juliana: (*to Dale*) Do you mind?
Dale: (*to Juliana*) Juliana would you....!
Juli: Can you tell him, David?
DT: (*irritated*) Keep to yourself!
Dale: (*angry*) I've got my foot here, on this bar...
Juli: (*shouting*) He keeps on kicking my feet !
Dale: ...and she keeps on kicking my foot. Right?
DT: You guys are like grade 5 kids....
Dale: She's a bitch!
Juli: (*laughs*)....Well...(*unintelligible*)...stupid.
Dale: David, I wasn't doing nothing.
Juli: Oh no, not at all.
Dale: David, look under the table.
DT: I'm not here to be a referee. I'm here to teach math. I don't have time for this nonsense. Alright. Which brings us to question three: "How do you feel about working in a group as opposed to working by yourself?"
Chad: (*laughs at the irony*)

More and more I began to feel that it was futile for me to intervene in a major way with such interpersonal conflicts. Surely many well-intentioned teachers and youth workers before me had tried personally to effect change in the social behaviours of both Juliana and Dale, and obviously had

failed at any major correction. I assumed by now that all I could do was keep the basic peace in the classroom, and provide a structured and interesting enough program to keep the students from each others' throats.

I noticed that Chad never got involved in personal conflicts or in petty arguments. Also, he mainly kept his distance from such rifts and fits of ill temper involving others. With such objectivity, I welcomed his rare comments directed at his peers' comportments For instance, during round two of the on-going battle between Juliana and Dale, Chad finally spoke out:

> Chad: Why don't you give it a rest Juliana?
> Juli: I don't feel like it.
> Chad: I don't like sitting around hearing it like all day.
> Juli: So?
> Chad: So there's no point to it.
> Juli: Sure there is.
> Chad: I mean you know....You complain when...when there's a ...when...when something happens, but you're the one instigating it all day.

I hoped at the time that I would be able to rely further on peer intervention such as Chad's to alter negative behaviours within the group.

Whereas Juliana's response to question 3 was typical of her mood of the day, the others expressed their satisfaction with cooperative over individualized learning:

> Juli: I like working in a group that is smart, not stupid (*laughs*)
> DT: No. Divorce yourselves from personalities. But working in a group as opposed to working by yourselves on these problems?
> Adie: It's better.
> Juli: Better.
> DT: Better?
> Adie: Because you get more ideas, and more....
> Juli: And you can figure out things easily.
> Adie: Yeh.
> DT: Get more ideas and....?

Adie: More...you feel other points of views of everybody.

DT: OK.

Chad: Uh huh. I like working in a group more than (*alone*) too.

At this point, I sought some feedback from the students regarding *my* perception of how they were faring in a group setting:

DT: What I want you to do though, as we're getting into the second half, is to let me know when this becomes a routine, and when this becomes boring. Because there...sometimes a change is good too. Maybe you should go back to working by yourself for a day or two..

Dale: Dale?

DT: I thought about that. Just a change. Maybe working in pairs. Now here's what I perceive....

Dale: I think when we worked in pairs at the uh beginning was better.

Adie: No!

DT: I perceive just the opposite of that. I started off in the beginning having you guys work kind of paired up. And I find it didn't go as well as having a group of five or four. Did you find that? That was my observation. I was a little disappointed in the way it worked, because I knew you weren't going to work by yourselves, but that you would work together with someone else.

Chad: Umm....

DT: But I found that uh...in your pairs you tended to get bogged down, and give up rather easily. There's not been one session here in this group that the group let down. And I commend you guys. It's been great! The way you guys have stuck to these problems for an hour, an hour fifteen minutes.

Chad: Hmm.

DT: And I'm not just talking attention span. I'm talking about persevering, and keeping an interest in the problem.

Chad: Uh huh.

DT: My inkling (and see if you agree with this) is that the group has something to do with that.

Chad: I would agree.

DT: My observation, being in a group myself, is that if I decided I want to take a couple of minutes off, I can kind of retreat and let someone else take over

Chad: Uh huh.

DT: I could pay attention, but I don't have to always be performing.

Chad: Yeh.
DT: Someone's going to carry the ball for a while, then it's my turn. What do you think?
Adie: Uh huh.

Here, I was trying to obtain what Ball (1988) calls "respondent validation" of my observations so far. I initiated the above exchange hoping that by reporting my interpretations to the students, they in turn would verify or validate my views as accurate from their points of view.

I was sceptical about the results of this research tactic. First, I did not want to put my words into their mouths, only to have them thrown back at me and then reported as a correspondence between my ideas and theirs. Second, the short, terse responses obtained from the students in the above dialogue suggested that at this point, the students were perhaps content just to agree with me and to get on with the session. I was therefore leery about seeking "respondent validation" again before the end of the project.

I was nonetheless confident that these students would not hold back their feelings about the program from me, nor try to impress me with what they thought I wanted to hear. Thus, I was eager to hear their suggestions about where the project should go from here. In answer to the question, "What things are working okay?", the students responded that both group work and open-ended problem solving should continue. But in answer to the question, "What things should be done differently?" their answers centered on the theme of being stuck:

Adie: Could take too long on one thing. Like spend too much time on one....
Dale: (interrupts) If we're stuck, help us out.
Adie: Like, don't waste too much time on one thing. It'd be boring, right?

Their message was clear. Continue on with complex problems, but do not let us languish too long in trying to find solutions. So, I would continue to walk the fine line between the mundane and the interesting, and between the challenging and the intimidating. But I welcomed their challenge: this dichotomy, after all, was one of the foundations of the new approach to a mathematics curriculum.

The mid-project survey then asked the students for their input on future curricular topics:

What are some topics that you would like to experience in Part II?
Here are some areas I'd like to include:
- *Algebra (& integers)*
- *Geometric Art (eg Escher)*
- *Statistics*
- *The number π*
- *Percentage & Finance*
- *Trigonometry*
- *Fractals*
- *Area, Volume & Pythagoras (on the computer)*

The only negative reactions to the above suggested topics came out of students' confusion about the meanings of some of the content areas. For instance, Adie confused the word "fractals" with fractions and immediately rejected the suggestion outright. Chad expressed disinterest in area, volume and Pythagoras until I explained that this geometry would be done on the computer. Also, he expressed the same dislike for statistics that he had for probability:

Chad: I've hated all my life to hear about statistics you know, like...um...with social services or whatever?

Dale: Do you like sports though?

Adie: Yeh.

Chad: No I don't.

Dale: Well, a lot of statistics are in sports.

Chad: I know, but you see....

Dale: That's what I want to do.

Chad: I think it just kind of...of abolishes the concept of uh...individuality, and I don't like statistics.

DT: You might want to think about statistics, in that...in an objective way. Like, you might want to know about it so that you...when you see a statistic, you can make an educated opinion.

Chad: Well....

DT: You need to know how people can use statistics to make you think differently. So that you know the pitfalls and dangers of statistics. And statistics can be very dangerous.

Chad: Uh huh.

DT: And they can be used wildly by people who want to convince you that they're right.

Chad: Uh huh.

DT: So, my idea in putting that in there is to say, "How can statistics mislead you?" And I thought maybe we could spend half of a session looking at wild, inaccurate misleading statistics.

Chad: Hmm....

DT: People use numbers....

Chad: Well it's just...every time I hear a statistic you know, it's like oh....

DT: You say there's lies?

Chad: Either, well, it's either that, you know, that statistic is apparent in myself, or it's not, you know. That applies to me or it doesn't, you know and so, what's the point of a statistic? I just don't like them because it groups people together and makes people statistics. and that's how it just ....

DT: It's very powerful.

Chad: It's really powerful.

In spite of his reluctance, I was confident that with the application of statistics I had in mind for the group, Chad would be a willing participant.

The last part of the survey was an open-ended question about the nature of mathematics itself:

> *Let's try to formulate a **definition** of mathematics.*
>   *Eg. History is the study of our past.   Biology is the study of life.   Chemistry is the study of matter.*
> *Mathematics is the study of _____ (Don't restrict yourself to just one or two words)*

Student answers were as interesting as they were varied:

Juli: Time? No.

DT: Time. (*writes on board*)

Adie: Numbers.

DT: Numbers. (*writes on board*)

Adie: And how to use them.

Juli: Of logic?

DT: Numbers and how to use them Adie?

Adie: How to use them and how to live with them.

DT: And how to what?

Adie: Live with them.

DT: Juliana. Logic?

Juli: Um hum.

DT: You said time as well, right?

Chad: Harmony. (pause) In the material world.

DT: Any others?

Adie: Answers. It gives the answers to everything, doesn't it?

DT: Does it?

Adie: Yeh.

DT: Do we have the answers to everything in this course?

Adie: Oh just that one thing. (pointing to the problem O T T F F S S __ on the board)

When reminded, Adie eventually agreed that indeed, we had tried several problems for which

mathematics had not provided us answers. This time, however, the students insisted on knowing

the answer: in other words, how would I define mathematics myself?

Dale: What do you think it is?

DT: I'm not going to say.

Dale: Why?

Adie: Because he's probably right.

DT: I have my own formulation, but I want to see if you can come up with different things
than I think. If I tell you what I think, you're gonna...you might say, "Oh good. That's
over and we don't have to worry about it."

Chad: Oh no. No I don't care how you see it. I'm just curious, because we're all putting
down our answers here.

DT: I...I'd prefer to leave it for now.

Chad: Oh, OK.

That evening, I recognized that I should have told them my version of the definition (that

mathematics is the study of patterns). After all, I was a member of the group as well, and as

such, should have been wary of establishing myself as some omniscient owner of the correct

answer. Chad's comment was just: all the answers we were "all putting down" were equally

valid.

To end the session, I presented the students with the following problem.

Consider the set of triangular numbers 1, 3, 6, 10, 15, 21, ...
1) What is the 15th triangular number?
2) What is the hundredth triangular number?
3) What is the nth triangular number?

Dale immediately remembered a similar version of the problem from our work with Pascal's triangle. When he asked why we were revisiting the same numbers, I stated that the third question was not part of the earlier deliberations. I then explained how the nature of this assignment was different than that of assignment 3. Primarily, this was to be a group problem; the students would be turning in a "group solution" to it. The mark that the students got would not necessarily be based on the correct answer, but rather on the number of different ways they could come up with the correct answer. I recommended that the students utilize the following list of heuristics which I had just posted on the wall above the blackboard.

Heuristics for Solving Problems
1) *Make a guess.*
2) *Work it out.*
3) *Use a calculator.*
4) *Use a computer.*
5) *Make a diagram or draw a picture.*
6) *Make a chart or table.*
7) *Look for patterns.*
8) *Is there a simpler problem?*
9) *Have we done something similar?*

Again, I stressed that the heuristics or "ways" were just as important as any final answers.

DT: Now you might not be able to use the way to come up with the answer. But if it's a feasible way, list it. Put it...jot it down on the list. And then in the group, either divide up the list, or tackle the problem together one at a time. I'll leave it up to you.

Chad: So we can't just start working on it?

DT: Sure.

Dale: Well one of the ways....

DT: (*interrupting Dale*) Start with number 1, and make a guess.

Dale: One of the ways is to just keep adding up the numbers, and to add an extra number every time.

DT: OK If you put that down on the final solution you hand in to me, you'll get credit for it. If you're listing that, write "work it out." And then add a sentence or two to explain...Is that a good or a bad way to work it out?

Dale first hit on the idea of using the numbers 15 and 100 to come up with the answers:

You could probably go 15 times something equals something, 100 times something equals something, *n* times something equals something. And you'll get the answer for the 15th one.

A little later, he was also the first to recall that the group had already worked with a formula of sorts which would indeed give the 15th, 100th, or *n*th triangular number. He commented, "Don't we already have that formula for the triangle? For Pascal's Triangle? With the C? Combination?" It was gratifying to have the students recall mathematics which had been covered a relatively long time before.

Chad was the first to articulate the way of working out the question "longhand:"

"OK, I thought of another way. You can take the number like 15, and then the number is 15 plus fourteen, plus thirteen plus eleven plus nine plus eight plus 5 plus 3 plus 1. And that's the number whatever that is. But how do you figure that out right? Well that's not so hard to do. And so with a hundred it would be 100 plus 98 plus 96..."

Finally, he made the important connection to a topic which we had just recently investigated:

Chad: Hmm. I guess that would work for 100 too. We already did a hundred. It's five thousand fifty (5050). Remember?

Adie: For the 100th number would you have to go a hundred plus 99 plus...?

Chad: Yeh ,we already did that though, remember? That big question?

Adie: The alphabet (*unintelligible*) ?

Chad: No, it was uh...I might have it here.

Adie: Yesterday (*unintelligible*) last week right?

Chad: Yeh it was the big question.

Adie: 5-50. Right. Yeh.

Chad: 5 thousand 50.

Adie: But how do you know the answer to that one already?

Chad: Because it is. You know, um....

Adie: There's gotta be a quicker way to do it.

Chad: Yeh here it is. Compute the sum of 1, 2, 3, 4, all the way up to 100 right? We...we got 5 thousand 50 out of it, so....

Adie: That's wrong though.

Chad: Huh?

Adie: That's wrong.

Chad: It's not wrong: it's right.

Adie: No, that's right, but it's not what we're trying to do here.

Chad: But it works. Well it just worked for 15. 'Cause think about it. The first number is just 1. The second number is 2 on the bottom and one on the top. The third number there's 3 on the bottom, 2, and then 1, and that makes 6. The 4th number there's 4 on the bottom, 3, 2, and 1. Right? And...and so...so on and so forth.

Juli: Umm.

Chad: The fifth...or the fifth one, there's 5 on the bottom, 4, 3, 2, 1, and so it must be...there's a pattern. Right? So....

Juli: Hmm.

Adie: (*laughs*) Hmm. So what is the nth number?

Chad: Oh that's we need to find: the formula.

The above exchange is important for two reasons: it shows Chad's reasoning in deriving a solution, and it outlines his successful attempt to communicate his awareness to his peers. Importantly, these two phenomena occurred with no outside help from his teacher.

He was able to determine the formula by considering Adie's description of "folding" the series $S = 1 + 2 + 3 + 4 + 5 + \ldots + 98 + 99 + 100$. The following is a transcript of Chad's moment of discovery:

Chad: I've got it! It's...with yours (*Adie's*) it came with a hundred and one. Which is one more than the number. So maybe it's...and then you do it that...and then you do it. Oh, I've almost got it. I'm right there. Hmm. It'd be like n....Didn't we already come up with the equation to do that though? It was like a hundred times...plus a hundred and....No, a hundred times a hundred and one, divided by 2. So maybe it's like n plus n plus 1, or n plus 1 divided by 2. No. N times n plus 1 divided by 2. Cool. OK. Right on.

I then asked Chad to explain his reasoning to Adie. His first comment was noteworthy to say the least:

Chad: It's n ... (*laughs*) Now I feel like a mathematician! Hey I got it! N times n plus 1 divided by 2.

DT: Explain how you got that.

Adie: How did you get n?

Chad: That's what we did with the hundred, remember? It'd be...when he showed

that...the way that the kid (*Gauss*) came up with it was a hundred times a hundred and one?

Adie: Yeh.

Chad: Remember how that....? So, it's the same here.

DT: I don't want you to stop with the problem right now though. That's one way...that's the right answer, but I want to see if you guys can work with the algebra next time, and use exactly what Adie did, which was folding the thing in half.

Chad: OK.

DT: Alright? And manipulate these symbols a bit. I think there'll be something you can learn from that next time.

Adie: Um hum.

DT: OK?

Chad: OK.

DT: Well done! I thought it would take you guys a period and a half to do this. And you did it.

Chad: Well, we're smart!

Chad felt like a mathematician because like Adie, he was performing like one: he again made relatively original discoveries and he effectively communicated the mathematics involved to his peers. We had reached a new high point in the study. I had great reason to be optimistic about how the students would handle the new topics which we would investigate between then and 31 May.

SESSION 19   Friday 15 April 1994   (Adie, Juliana, Chad, Alim, and Rosalie)
"Can we get a soccer ball now? Please?"

Unfortunately, my new-found enthusiasm was short-lived. This day, Donny was suspended for breaching the terms of an attendance contract which he had signed a few weeks previously. I regretted this loss because in his own quiet way, he contributed much to the mathematics project. I hoped he would opt to return to VRLF after his two-week suspension. Also, Chad did not show up on time to resume work on his equation. Originally, I included the nth triangular number problem as the first question (out of 4) of the new assignment (number 7) for the Friday group. But since he wasn't there to carry the ball, I substituted another problem for it. In retrospect, I should have left it, because the question had a lot of potential for connections to

other areas such as the handshake problem and the distributive principle [ie. $n(n+1) = n^2 + n$].
"Oh well," I thought, "maybe there will be another chance to return to it."

This day, a student from another class, Rosalie, asked to participate in our mathematics class. I was happy that a new person was on the scene, and also that perhaps word was spreading through the school about what we were doing.

Rosalie did not wait to contribute to the group. I presented a "10 Minute Mystery" to them, and she was the first to cut through the red herrings in the problem to arrive at the "solution" to the crime. During a recapitulation of the nth triangular number question, Rosalie was at first lost, since she was not present for the last session. However, after a brief explanation of the meaning of "the nth term of a sequence", she was well able to participate in the following lesson concerning triangular numbers.

Together, we wrote the prime factorization of each triangular number in the following way:

1,	3,	6,	10,	15,	21,	28,	36,
1x1	1x3	2x3	2x5	3x5	3x7	2x2x7	2x3x2x3

This exercise also provided the opportunity for a good review of factors. Juliana wondered why we did not include all the factors of 6, and seemed to accept the idea that to include x and 1 as factors of any number x would be trivial:

Juli: Isn't one and 6 a factor too?
Alim: 3 and 5? (*moving on to the factors of 15*)
Juli: Isn't one....?
DT: Yeh, we can include that, but we're not going to, because it's kind of...
Juli: Yeh.
DT: ...the same for all of them.
Juli: Right, right.

Here, I missed the opportunity with Juliana's query to make a connection to prime numbers: in that case, the consideration of x and 1 as the *only* factors of x would hardly be trivial.

A lively search for a pattern followed:

DT: Now look at those and see if there's a way to predict this number from the prime factors. Is there any pattern to these numbers?

Alim: Yeh.

Adie: Those numbers are written twice.

Alim: It goes uh....

Juli: Yeh, that's it! I get it! I get it! I get it!

Alim: Every second number is written twice.

Juli: The next one would be....

Alim: Like 3 3 5 5 7 7 .

Adie: Are they all odd numbers after? So the next one's going to be....

Alim: 2 2.....3 3....

Adie: OK, the next one's going to be....

DT: OK, one at a time.

Adie: The next one is going to be 2 2 2 times 3 3 3 .

DT: So you're saying nine times 27 ?

Adie: No.

DT: Or 8 times 27?

Alim: Nah, that's not it.

Adie: No.

DT: You know what we need here?

Alim: I don't think there's...there's a pattern but it's too hard to figure out..

DT: Well, actually there is no pattern there.

Adie: Oh.

Juli: Oh. Thanks a lot! (*laughs*)

Alim: (*laughs*)

I was initially concerned that after such an effort, the students would become discouraged when informed that they had reached a dead end. However, they were able to accept that no pattern was discernible in the prime factorization as depicted on the blackboard.

I told the students that in order to start again from a dead end, it sometimes helps to change the format of the numbers, and that often things will then "pop out". I suggested that since Chad's formula $\frac{n(n+1)}{2}$ has a two in the bottom, we might try multiplying each of the prime

factorizations by 2. Essentially, the new representation involved doubling the smallest factor in each case.

1,	3,	6,	10,	15,	21,	28,	36,
1x1	1x3	2x3	2x5	3x5	3x7	2x2x7	2x2x3x3
1x2	2x3	3x4	4x5	5x6	6x7	7x8	8x9

Now, while I was writing the numbers on the board, they went completely off task, jokingly discussing their social lives at VRLF. However, by the time I reached 10, the students saw the pattern:

> Juli: 5 times 6
>
> DT: Ha!
>
> Alim: 6 times 7, 7 times 8, 8 times 9, 9 times 10. Got it!
>
> Adie: Yeah!
>
> DT: That's the 8th term. The 9th term will be...but not quite. I doubled these to get....
>
> Alim: Uh huh.
>
> Adie: Now divide it by two.
>
> DT: Divide it by 2. So 9 times 10, divided by 2 is....?
>
> Juli: 60.
>
> DT: What?
>
> Juli: No.
>
> DT: Not quite. That's OK.
>
> Juli: Thirty...forty!
>
> Alim: Nine times 10...forty...forty five.
>
> Juli: 45.
>
> The group: Yeh! yeh! yeh!

I appreciated the students' enthusiasm, especially in light of the fact that they brought themselves back to the mathematics at hand. It is, however, questionable here as to what they actually "discovered." I was leading them very directly, and they became aware of one small thing: the pattern which resulted from my suggestion of multiplying by 2. This discovery itself is not important; what it did for the students' attitude is.

Finally, I handed out Assignment 7 to the group. Its preamble read as follows:

This is a group assignment. You will hand in one finished product only. It should be a summary of all your individual and group efforts.
Everyone will receive the same mark (out of 40). Your mark will be based on:
    **the clarity of your presentation*
    **evidence of co-operation in trying to solve the problem*
    **the number of different techniques (heuristics) you used*
I am not so much interested in final answers for these problems, as I am in the ways you go about trying to solve them. So,
         • *try as many different heuristics as possible*
         • *record your attempts in detail*
Suggestion: For each problem, choose one person from the group to do the final write-up. Choose a different person for each problem.
DUE DATE: FRIDAY 22 APRIL

The first problem read as follows:

The design of a soccer ball is based on a solid figure called a truncated icosahedron. Three different views of this solid are shown below.

How many black pentagons does the solid have?
How many white hexagons does the solid have?

The group was quick to get under way:

DT: Problem number 1 there is appropriate for Alim (*an avid soccer player*). What's one obvious heuristic that you can use to solve this problem?

Juli: Get a real soccer ball. . . . .

Rosie: Find a ball in the gym.

Adie: Yeh.

DT: Before you go to the soccer ball in the gym, see what you can come up with here, OK? Go for it. (*leaves class*)

Alim: I know how to do this.

Rosie: There's 28 white, and 16 black.

Alim: (*counting*) 1, 2, 3, 4, 5, 6 . That's straight up from the top, right?

Juli: Yeh.

Alim: 1, 2, 3, 4, 5, 6, on the top. And there's 6 on the bottom too, right? Which is what?

Juli: 6.

Alim: 12.

Juli: 12.

Alim: Then all you do is, you're adding these ones.

Adie: What? Do that again?

Alim: (*unintelligible*)...going around 1, 2, 3. And this side there'll be 3 more.

Juli: Um hum.

Alim: 12, 13, 14, 15...18! (*counting white pentagons*)

Juli: 18!

Alim: Possibility. Go grab a ball!

Juli: OK!

Alim had discovered the correct answer for the number of white pentagons (12), but was uncertain about the group's calculation for the black ones. At this point, I arrived back in the room, and advised them to try a different method of attack before looking at an actual ball.

DT: Heuristic number 7. Number 7.

Juli: Look for patterns.

DT: What are the patterns?

Juli: Is there a pattern here? Hmm. Oh wait. Wait. I know. I know. I know. OK. Around every black thing, there's....

Rosie: 5 white petals.

Using this observed pattern, the group was able to come to a conclusion:

Alim: Well, the black ones have been...so then all you do is you go 12....

Juli: Times 5!

Alim: No, because they're used...they're used. They're used double times.

Juli: Well....

Alim: 'Cause, look.

Juli: OK.

Alim: You know what I mean?

Juli: Um hum.

Rosie: 'Cause there's 5 around. No, it would be 12 times 5.

Juli: Yeh.

Alim: No....

Rosie: No, it wouldn't.

Adie: It would be double this. You double that, so you double this.

Alim: No, no.

Juli: 1, 2, 3, 4, 5,...

Alim: No, these.... Look. 1, 2, 3, 4, 5 around this one.

Rosie: 5 around this one, and 5 around this one.

Alim: But you can't count this. You can't count the ones you've already counted

Juli: 'Cause you can't...because this is a part of that too.

Alim: Yeh.

Juli: Oh!

Alim: This is a part of that, and a part of that

Rosie There would be a lot of those copies

Juli: Yeh. So how...OK 1, 2,3,4, 5. There's 5 on the top, and 5 on the bottom, so that's at least 10. And there's 5 around...1, 2, 3, 4, 5,...

Alim: So 15.

Adie: 15

Rosie: And 5 more on the bottom which is 20.

Juli: But no...Oh yeh, right. 1, 2, 3, 4, 5, around the back, and 1, 2, 3, 4, 5 . 'Cause there's 3 up here, and there's 2 up here, so that's 3 on the back and 3 on the top.

Alim: No.

Juli: Or 3 on the...so that's...that is 15.

Alim: Look! There's 5 around the top, right?

Juli: Right.

Alim: There's 5 around the bottom right?

Juli Um hum.

Alim: Now all you have to do is count these ones...1, 2, 3, 4, 5.

Juli: 4, 5. And then there's those on the back.

Alim: On the back there's 5 more so that's 10, 20

Juli: So there's 20.

Alim: Yeh but....

Although the students had arrived at the correct answer, they were still sceptical. After this effort, I felt that they were completely justified in asking to see the real thing in order to test the accuracy of their deliberations:

> Rosie: Can I just see a soccer ball?
>
> Adie: Can we get a soccer ball now ? Please?
>
> Alim: We're going to (*unintelligible*) seriously, David.
>
> Adie: Yeh please?

When Adie returned with the ball, they were pleased to actually discover that their thinking was correct. Such immediate verification is important for students, and I made a mental note to allow for it more often in our problem-solving.

Juliana brought the group back into focus with her exhortation, "On to the second problem!" This proved to be the most difficult part of the assignment. It read as follows:

> *On the Island of Zombies, there is no easy way to distinguish the zombies, who always lie, from the human beings, who always tell the truth. Life is further complicated by the fact that all yes or no questions are answered with either BAL or DA. But we do not know which means YES and which means NO.*
>
> *Suppose you meet a native. In just one yes-or-no type question, it IS possible to find out whether the speaker is a human being or a zombie.*
> *a) What is the question?*
> *b) Explain how this question enables you to find out whether the speaker is a human or zombie.*
> *c) How can you then discover the meanings of BAL and DA?*
>   *(Hint: Use heuristic number 6.)*

The group made several stabs at a solution:

> Adie: I know! I know! You ask the native if he's a zombie or not.
>
> Rosie: But they always lie.
>
> Juli: That's right. Oh yes. . . . . Or ask him something obvious about his body like...say he had big nostrils. Or like say... I know! I know! You just say um..."Do you have a big nose?" and he'd be like, "BAL", and then...and then if he lied, um....
>
> Alim: How do you know which one means yes and which one means no? If he had a big nose, and he said BAL, and if BAL meant yes, and if he said BAL, right, and then he'd be telling the truth because he knows he has a big nose. BAL would mean yes.

Chad: You could ask him a series of questions that would contradict...lead him in a circle. Get him to contradict himself, or something. Ask him...I don't know.

Adie: Yeh but in just one (*question*), you have to know.

The preceding dialogue is noteworthy because each suggestion is in turn refuted by another member of the group, but not in a rude or insulting manner. Also, the dialogue involves no input from their teacher.

In spite of Juliana's reminder of the hint to use heuristic number 6, the group decided democratically not to make a chart or table, but to go on to the next problem.

Adie: Let's do the next one.

Juli: No.

Alim: Yes.

Chad: Oh...

(*laughter*)

DT: Make a group decision. You can go on to any question you want.

Juli: All in favor say "aye."

Alim: Say BAL or DA!

(*laughter*)

Adie: Let's do this one.

Juli: No.

Adie: Look at it . Read it.

Juli: No, do the second one first. Yeeoow!

Adie: OK. You guys do the second one. We'll do the third one.

Juli: No you can't do that. We're supposed to do it as a group.

Chad: Well....

Juli: OK. If you want to, we'll do it.

(*laughter*)

Now in agreement with her peers, Juliana spontaneously began to read the third problem aloud. The students pondered it for a short while, but since the end of the class was near, their conversations soon turned to more personal topics. Considering their efforts for the period, I was content to let them take the last five minutes off. However, I was looking forward to the next

session when we would take a fresh look at the two unsolved problems which were discussed this day.

"Is this an emotional support class, or what?"

A few days previously, I had placed a small poster of the sequence {O, T, T, F, F, S, S } above the door. Alim, Dale and Juliana considered it briefly at the beginning of session 20; I was pleased that the students were willing to return to previously unsolved problems instead of just resigning themselves to failure, as they were previously wont to do.

I had no agenda for the day, except to have the group work on its latest assignment. Dale, Juliana and Alim argued about which problem the class would tackle first:

Juli: Hey Alim! We gotta figure out this problem now. We gotta figure out this problem!
Alim: (*reading*) "On the Island of Zombies, BAL or DA...."
Juli   BAL or DA.
Alim: But we do know...(*reading*) "explain how the question"....
Juli: (*interrupts*) We're not doing that one dummy.
Alim: No, we are, (*unintelligible*).
Juli: (*laughs*) Oh that's a good one. We're doing this. We're doing problem number 3. I'm going to go up to the blackboard and do it.
Alim: No let's do number 2 first!
Juli: No, we all agreed last time that we were going to do number 3.
Alim: We decided to come back to number 2.
Dale: We're going to to number 2 next.
DT: I"ll leave that here. You just turn the switch on.
Alim: OK. Thanks a lot.
Juli: OK Number 2.

I decided to bow out of the students' agenda planning completely, leaving a "logic grid" for the overhead projector in case they wanted to tackle problem 3. And eventually, they agreed on which problem to consider first with no outside interference.

The students found the zombie problem difficult from the onset. To set them on a course, I asked the following questions:

DT: What are the possibilities from this scenario here?

Alim: What's the possibility?

DT: Right. What are the different possibilities?

Alim: A zombie or...

Juli: A human!

Alim: ...a human

DT: OK.

Alim: BAL means... BAL or DA. We don't know what it is.

DT: OK. But what are the possibilities there?

Alim: If he 's a...if he says....

Juli: Bal....

DT: No. OK. Go ahead. I'm sorry.

Alim: If ah...I don't know. The possibilities are....

Dale: He doesn't know.

Alim: He'll lie. Or he'll tell the truth.

Dale: Zombie, human.

Juli: Right.

DT: That's...that's a possibility. What are the possibilities for BAL and DA? And I'm going to leave you with it.

Alim: What's the possibilities?

DT: What could BAL mean?

Alim and Dale: Yes or no.

DT: What could DA mean?

Group: Yes or no.

Alim: Oh.

DT: Those are your possibilities.

Alim: So here we go.

DT: Now use logic. OK? The question is not a trick question, and it's not a long question. What are two questions that you could ask here?

Juli: Are you a human, or are you a zombie?

DT: OK And, what else could you ask?

Juli: Um.....

Dale: What does yes...or what does DA mean and what does...uh....

DT: Right!

Juli: Ahh!

I thought immediately that the clues I had given were too obvious. Even so, the trio continued to spin its wheels and went nowhere towards a solution. Dale finally offered to relate a problem which he thought was similar to the zombie plexer:

Dale: It's the same as ah...when you get to the um...crossroads. Which way's right, and there's two people there? Um...have you ever heard that story about...

Alim: (*Interrupting Dale*) OK. Next...(*unintelligible*) Are you a zombie?

Juli: And he says BAL.....

Dale: ...the crossroads, and there's two people there?

Juli: And he says BAL.

Dale: Or a fork in the road...(*unintelligible*)...who's telling the truth, and the person who lies? How do they figure that one out?

Alim Wait!

DT: What was the question?

Alim: You're not meeting a zombie, you're meeting....

Juli: A thing. Wait!

Alim: Use logic.

Juli: Wait. Let me think.

DT: (*to Dale*) That's right. It's the same.

Dale: How did I figure that one out?

DT It's the same kind of idea.

Dale: How did I figure that one out?

DT: Using logic.

At the time, I was pleased at Dale's apparent attempt to use heuristic #9: "Have we done something similar?" (See p. 203.) Looking back, I feel he contributed nothing to the solution of the problem except to come up with a similar problem to which he did not recollect a solution. At least Dale was not held to task here for his failure to remember. As I point out later, there were instances where I placed too much emphasis on the function of memory in doing mathematics.

My main criticism of the last few sessions was that whereas the students were finally working together in problem-solving, they still lacked the group skills necessary for effective communication. For instance, they frequently talked out of turn, interrupting other students who were speaking. Like Juliana and Alim in the above dialogue, they were not used to listening to each others' comments and would habitually ignore ideas which were put forth in the process. However, I knew by this point in the study that changes would come slowly. With reminders from me, and more importantly, peer guidance from listeners such as Chad, I was confident that positive changes in group decorum would eventually occur.

Finally, the students admitted they were perplexed, and I thought this moment of discouragement to be a good opportunity to further explore the "stuck" phenomenon.

Juli: We just don't know.
Alim: This question's stumped me.
Juli: We just don't know.
DT: Alright. Just...just pause here for a second. This is called being stuck, or stumped. How do you feel?
Alim: Frustrated.
Juli: Bored.
DT: OK. Frustrated, number one?
Juli: Angry. Ha!
DT: Really? An emotion like anger?
Juli: Well no. I just feel really frustrated.
DT: That could be. I feel...I get angry too when I get stuck sometime.
Juli: It makes me feel helpless. Or not....kinda... cause I don't know it, and I want to know the answer and I can't figure it out.
Alim: It makes me think this thing's impossible.
DT: Dale?
Dale: Confused.
Alim: Is this an emotional support class, or what?
DT: No, but the only thing you should realize from that is that those emotions are completely natural. And if you allow them to be the main theme of your work right now,

then you won't get anywhere. But if you accept them as natural, then you can actually use them to your advantage.

Alim: How?

DT: By accepting them first of all.

Alim: I could....

DT: What is this, a touchy-feely group? Accept your feelings? No, just accept that the fact of being stuck is a natural thing, and then it happens to everybody, and you don't want to wear that as part of your personal thing.

Juli: Um hum So does....

DT: (*interrupting*) What are you asked to find? If you're stuck, go back to the original question.

Alim's comment about "emotional support" served to remind me of a crucial truth about these kids: they have been subjected to all sorts of psychological theories and practices, including anger management training, psychiatric sessions, support groups, psychological counselling, and peer intervention therapy. I frequently needed to remind myself that I am mainly a teacher of mathematics, and as such, should be aware of their sensitivities towards attempts to manipulate them psychologically or emotionally.

I did not realize until I heard the tapes of the session exactly how I interrupted both Juliana and Alim at the end of the above dialogue. This was unfortunate for two reasons. First, there could have been some interesting comments there, but now they are lost forever. Second, I should practise what I preach, especially after the following sermonette which I gave at the beginning of this very session:

DT: Now, from this session as well, we have to develop sort of a plan for doing work in groups. Some sort of...almost like a sort of code which we'll go by for the rest of the sessions.

Alim: Code?

DT: Yeh, a sort of set of guidelines that will guide group behaviour. And as we go today, they'll probably spring up, and we can start making a list.

Admittedly, at the top of this list should have been the example of how the teacher just interrupted two students in a row.

At this point, Rosie entered the classroom, and asked, "Are you guys still doing math?" I was pleased at her arrival, hoping that a fresh viewpoint on the zombie problem would help the group out its dilemma. Alas, no such enlightenment occurred, although Dale hit on the right question:

DT: What are you trying to get the answer to?

Juli: To whether or not the guy...the person or thing is a human or zombie.

Alim: What are you?

DT: Number 1, yeh. And what about...what else are you trying to find out?

Dale: I think you should ask....

Juli: *(interrupting Dale)* What's BAL and DA mean?

Rosie: What is...what does BAL mean, and what does DA mean.

DT: OK. Those are questions. What is...your answer has to be yes or no, so how could you rephrase that?

Dale: Does BAL mean yes?

Rosie: What does this mean?

Dale: Does BAL mean yes?

Juli: OK. *(writing on board)*

Dale: Does DA mean yes? Could you ask each? OK. There's only 1 person, right? So...

Juli: Does...does....

Dale: DA mean yes?

Unfortunately, no one was able to discern how to interpret the various responses which a human or zombie would give to Dale's question.

Finally, Alim suggested, "Wait til Chad gets here. He'll help us." Almost as if on cue, Chad walked in the door, one-half hour late, with the comment, "OK. It's all quite elementary, you see." Eventually, not even having heard my previously given clues, he asked, "You can use the words BAL and DA in your question, right?" Unfortunately, the students did not follow through on his suggestion, and ignoring my previous comments, considered questions like "Do zombies lie?" and "Do you have any children?" So, fearing that frustration would soon set in, I again asked the key question:

DT: Hang on for a second. Since this is question of logic, you have to be aware of all the different possibilities. What are the possibilities, when you approach this native?

Juli: He could be human or a zombie.

DT: (*writing on board*) What are the other possibilities? In this question? That you're not sure about?

Juli: BAL means yes and DA means no.

DT: What's a third possibility?

Rosie and Dale: BAL means no.

Mainly under my direction, we created a table which resembled the following:

	BAL = yes	BAL = no
Zombie		
Human		

I asked the group to test out the questions which Juliana had previously written on the board. We were able to reject the questions "Are you a zombie?" and "Are you human?" as inconclusive. Now, I was hoping that the students would see the solution from either of the two remaining questions:

Dale You just ask him one question.

DT: You ask him if BAL means yes. And what will the answer be if he's human?

Chad: BAL.

Rosie: It'll be yes.

DT: No. (*to Rosie*)

Chad: Ohh! It'll be BAL, right?

DT: Yeh. You can see from the chart there. Isn't that simple?

Chad: We came up with that question on the first day.

DT: What will...what will the answer be if you asked a person "Does BAL mean yes?" and he says DA, what is he?

Dale: He's a zombie.

Rosie: Zombie.

DT: He's gotta be. He's got no other choice. No matter what the possibilities are, he's going to say the same word.

Dale: So the question is, "Does BAL mean yes?"

Rosie: That was easy.

Dale: Could you have done it with this question here? *(ie Does BAL mean no?)*

Group: Yeh.

DT: Oh I'm sure.

Rosie: Yeh.

Alim: We've got it!

DT: Well you've got part one.

The truth was, the students would never have "got it" without my rather explicit help and direction. Chad was correct; they did come up with that question on the first day. However, they could not pick it up and run anywhere with it. In fact, I had to practically explain the solution to the third question of the zombie problem as well. And when in the end, one student asked another, "BAL means yes right? Or does it mean no?", I knew that certain members of the group had no comprehension of the solution as presented in the logic table.

Perhaps I was expecting too much "discovery" in their mathematical experiences. After all, under the classic regime, the teacher would simply state the problem, work through its solution with a modicum of student help, and then assign "similar" problems for the class to solve. So, it was probably unrealistic to expect the group to resolve every problem thrown at it. All I could do at this point was offer advice from my own experience as a confused undergraduate mathematics student:

> And you have to kind of take it home. Or when you're riding home on the bus, look at it, and if it's a little bit foggy right now, it'll click. It's one of those type of things.

SESSION 21   Thursday 21 April 1994   (Adie, Dale, Chad, Juliana)
"This is confusing, but it's getting there."

For this session, I hoped that the students would work cooperatively on their assignment 7, which was due the next day. Contrary to my expectations however, this class turned into a bizarre occasion where the group fell apart.

At the start, Chad explained the solution to the zombie problem to Adie and Juliana:

Chad: The question is,"Does BAL mean yes". Because...you weren't here yesterday were you?

Adie: Tuesday, I wasn't here.

Chad: Because if BAL was yes, then a human would say BAL. Right? And if BAL was no, he'd say BAL as well. So if he was a zombie, his answers would always be DA. And if he was human, his answers would always be BAL, no matter what they meant.

Juli: Get it? Got it. Good.

Chad: (*laughs*) Well, the....

Adie: (*laughs*) OK. You ask the human...

Chad: Uh huh...

Adie: ...if BAL means yes, and he'd say BAL, right?

Chad: Uh huh. And he'd say it, if you ask him that question, no matter what BAL meant, he'd always say BAL.

As far as I was concerned, Chad's spoken explanation was as good as any written exam could be in determining for me the state of *his* understanding.

However, in spite of their indications of comprehension, both Adie and Juliana did not yet grasp the meaning of the solution. Chad was in the process of re-explaining the answer when hell broke loose:

Chad: And then, you know, he'd say BAL because BAL means yes. In this column. So he's telling the truth. If BAL was no, and you asked him "Does BAL mean yes?", he'd say BAL because BAL would mean no.

Adie: Wait, wait!

Juli: Right!

Adie: Wait. OK. OK. (*unintelligible*) the last one again? If we were asking a human, "Does BAL mean...

Chad: ...yes?

Adie: ...yes?", he'd say, "BAL."

Chad: Yeh.

Adie: And if we asked him, "Does BAL mean no?"....

Chad: No! Just this one question. Nothing else. Here....

Juli: (*Interrupting*) If BAL means yes...if BAL means yes....

Chad: Hey, I'm explaining something!

Juli: Shut up! If Bal means yes....

Chad: Holy fuck!

Juli: Fuck you! If BAL means yes, but...and if it meant no, right? Here. BAL, BAL, BAL, BAL. OK. Does BAL mean yes? Right? And the human says BAL, but if it...if BAL meant no, he'd say DA. (*nonsense*) Right? Get it?

Adie: Oh, OK.

Juli: And if it was a zombie, he always lies, right? And "Does BAL mean yes?", and the zombie'd say DA. Right? 'Cause he always lies, so obviously....

Adie: Oh, OK.

Juli: So DA means no, and BAL means yes.

Chad's composure in the face of such outrageous insults by Juliana was remarkable. Without a word, he left the table, and went over to where Dale and I were considering the next problem.

Chad's explanation was correct, and Juliana's was nonsense; yet ironically, Adie expressed comprehension of Juliana's meaningless ramblings. For the rest of the period, Juliana, who by now was in an irascible state, sat by herself and spoke only to tell others to shut up. After a few minutes, Adie asked to speak to me alone:

Adie: I don't....I don't want to do this math today.

DT: Well then, don't do it.

Adie: Can I do something else?

DT: Well you already are.

Adie: Well I want to do my résumé.

DT: But I appreciate you asking. Well, it's up to you.

Adie: Well, I want to get your permission. You know (*unintelligible*).

DT: Well I'm not going to say you have to do math. That's against the whole spirit of the thing.

Adie: I just don't feel like doing it, because today I don't want to do it. Can I do my....

DT: I'm not going to come down and say...I'm not going to force anyone to sit here.

Adie: OK. Can I do my résumé then?

DT: You've got an assignment to hand in.

Adie: Yeh. I want to take that home, and work on my own.

Shortly after, Adie left the classroom for five minutes. When she returned she sat quietly and worked on her résumé. However, after the above exchange I thought that Adie at least appreciates how important I feel the mathematics program is.

This was the first time in any of the sessions that the group fell apart, and I was disappointed. It was especially disconcerting for me to see Adie beg off since she knew the assignment was due the next day.

In hindsight, I feel that my expectations for the group were unrealistically high. With a difficult task like the zombie puzzle, I should not have been surprised at such student frustration and withdrawal. Every teacher has classes where continuity is lost and pupil interest disappears, yet I was upset at the first instance of such an occurrence. In the future, I should perhaps subscribe more to the following adage: "Always expect your students to do well. Don't expect your students to do well always."

In the meantime, Dale and Chad had immersed themselves in a consideration of the musicians problem:

> *Musicians Joe, Jim, John, Jake, and Josh recently finished in the top five positions of a music poll. Each plays in a different type of group-soul, jazz, country & western, reggae, and rock- and each performer plays a different instrument. From the clues given, try to determine for each position on the poll, the man, his type of group, and the instrument he plays.*
> *1) The bass-player's reggae group finished third.*
> *2) The rock and country & western groups do not have keyboard instruments in them, and neither does Josh's group.*
> *3) Jake's group placed higher in the polls than the bass-player's group which placed higher than the soul group but none of these groups finished first.*
> *4) The drummer's group finished behind John's group but ahead of the jazz group; however, none of these groups finished last.*
> *5) The guitar player's group does not play country & western.*
> *6) Joe does not play his piano in the soul group.*
> (Henderson, 1982, p. 3)

The problem came with a logic grid which I projected onto the blackboard. (See Figure 3.8.)

Figure 3.8

(Henderson, 1982, p. 3)

As if in compensation for the behaviours of Juliana and Adie, Chad and Dale worked wonderfully well together for the rest of the class. They adapted themselves quickly to working with the logic table, agreeing that without it, a solution to the problem would be "next to impossible." They also engaged in some spontaneous metacognitive language at times, monitoring their progress and commenting on the degree of difficulty of the problem.

Dale: OK. Wait a sec. We know that the bass and the soul group did not finish first, right?

Chad: Um hum.

Dale: OK The soul group did not finish first.

Chad: We do? We know that?

Dale: And didn't finish third.

Chad: We know that. We know that Jake's band finished second.

Dale: OK. Jake's not.....

Chad: We know that the reggae band finished third.

Dale: Jake's not in the soul group.

Chad: Huh?

Dale: Jake's not with the soul. Jake is not in the soul group. Know why?

Chad: Oh yeh.

Dale: Jake's not....

Chad: Jake's not in the reggae band either.

Dale: How do you know that?

Chad: Well...well, because he finished higher than the bass player's group, and the bass player's reggae group finished third.

Dale: Oh yeh.

Chad: So, Jake's not in reggae, and he's not in....

Dale: ....soul.

Chad: Soul? Oh yeh.

Dale: (unintelligible) OK.

Chad: This is con-fus-ing. (pause)

Dale: This is confusing, but it's getting there.

Chad even thought a change of approach would be beneficial when at one point, he suggested, "How about we work on it quietly for a second?"

Remarkable also was the discourse evidenced by the two students during their deliberations. Respectful of each other's opinions and offerings, they were at times even humble in admitting error and gracious in praising good logic:

Dale: So we know that jazz did not finish first. We know that.

Chad: No, we don't.

Dale: Yes we do.

Chad: How do we know that ?

Dale: Because the drummer's group finished behind John's group, but ahead of the jazz group. OK?

Chad: We don't know who John is.

Dale: I...I know. But they finished ahead of the jazz group. Alright? And if they...the jazz group can't be first.

Chad: Ahh!

Dale: You know?

Chad: Yeh, yeh. That's...that's right

Dale: Understand?

Chad: Uh huh. Sorry. That's good.

Although there was no final resolution to the musicians problem, top marks had to go to Dale and Chad for their persistence, and their cooperative problem-solving.

Inter-session. Friday, 22 April 1994

Much to my delight, upon arriving at the school at 11:30, I found Adie in the classroom, already working on mathematics. In particular, she was writing up problem 4, which was her contribution to Assignment 7. Still hard at work at 12:00 when I popped in to see her, she said that the problem was arduous. I told her what was expected in the assignment: I did not want to necessarily see a correct answer. Rather, she should use as many heuristics as possible, and simply write the word "STUCK" whenever she got bogged down. For the remainder of the afternoon, she, Chad, and Alim each worked on their respective write-ups. As I have recounted before in this narrative, Adie would often disappoint me, only to come back soon after to rejoin the fold.

The students handed in their contributions to Assignment 7 by the end of the class on Friday. Although problems 1 and 2 were dealt with by the group, Adie's solution to problem 4 was hers alone, since she was the only one to work on it. Her writeup was rather anemic, consisting only of a logic table and a diagram. (See Appendix 5) In sum, Adie did not follow my suggestion about including as many different heuristics as possible. Then again, deliberating alone, she obviously did not try a variety of approaches to the problem.

Chad's contribution was fleshier, since in addition to the completed zombie problem, he and Dale had almost solved the music poll conundrum. The following is his written description of their efforts to find a solution to the latter:

> We started off by filling in all the given information onto the chart then we started cross-examining things like "Well if this is true then this would be true as well." Soon we had X's and √'s all over the graph. It got kind of overwhelming. We wanted a way to simplify

it. We thought of writing things down in sentences. We tried making a new type of table. It was frustrating but we kept interested.

He also provided a good account of the group's struggles with the zombies:

We thought of a lot of questions that didn't work. We kept trying to think of the right question, not thinking of other factors. In fact, I think we even came up with the right question once, but we didn't realize it. We decided to make a chart and assume that BAL meant yes. So then if we asked a human if BAL meant yes, he would answer BAL. A zombie would say DA. Then we assumed that BAL meant no. If we asked a human if BAL meant yes, and it meant no, he would answer BAL (no). A zombie would answer DA again.

Chad's writing was always a pleasure to read: he used well-constructed, complete sentences with few if any spelling mistakes. Indeed, I looked forward to working with him in the English course.

Alim's description of the cohort's efforts was closer to what I wanted, since he actually listed several "heuristics that we used":

1- we guessed how many black and white pentagons there were
2- we worked it out by counting and multiplying
3- we used the three diagrams of the soccer ball.
8- we used a simpler problem by counting how many black and white pentagons on a real soccer ball.
9- we looked at a real ball which is similar to the diagrams.

He went on to describe their experience:

On the first diagram you can see that the view is from the top of the ball. There were six black pentagons so we doubled it by looking at the second diagram which is a view from the side. It clearly shows that there are 6 on top and 6 on the bottom. We used the same format in finding how many white pentagons there are.

Clearly, Alim also demonstrated good composition skills.

I was generally happy with the assignments, and awarded each of the students a group mark of 30 out of 40. However, I was not satisfied with the fact that problem 3 involved only Dale and Chad, and that essentially, problem 4 received attention from only one student: Adie. Hence, these last two questions were in no way group efforts. In the future, I would have to take steps to insure that each group assignment indeed received a *group* effort.

<u>SESSION 22   Tuesday 26 April 1994   (Adie, Dale, Chad)</u>
"That was well done. I like that."

In view of the "incompleteness" of the last assignment, I decided to devote session 22 to

achieving closure. I felt that before we went on to the next topic, the whole group should have a

chance to consider solutions to the last two problems.

Unfortunately, I accidentally erased about half of session 22 by recording the next class

over top. This error was unfortunate, because first of all, there was some great problem-solving

this day, as well as good attention to task and some good insights. Unfortunately, much of the

recording of session 23 was merely background material .

In the following exchange between the three students, Dale took the lead in the reasoning

for the musicians problem:

> Dale: John didn't play the drums.
>
> Chad: Oh cool.
>
> Dale: John played the guitar, right?
>
> DT: We don't know yet.
>
> Adie: So, John played...Jake played the drums.
>
> Chad: John was first . We know that.
>
> Dale: What did John play? And then we'll find out...
>
> Chad: We don't know. We know that Jake plays...
>
> Adie: The drums?
>
> Chad: ...the drums.
>
> Dale: OK. Through...through trial and error, right? Since there's only one and two left.
> Um...we know it says here, "Rock and country and western groups do not have keyboard
> ins...instruments.", so that means since John came in first, John didn't play the piano or the
> organ.
>
> Chad: Joe plays the piano. And he doesn't play it in the soul group.
>
> Dale: OK.
>
> Chad: Wait a second here. The stupid (*unintelligible*)....
>
> Dale: He either plays bass or the guitar. Do...do we know who played the bass?
>
> Adie: The bass player's reggae group....

Dale: Yes! The bass player's reggae group. The bass is out. So that means John played the guitar. You guys understand how I got that? OK There's only one or two left, right? So that means...and in one of the questions it says the rock and country and western group did not play the piano or the organ? Piano or the organ, right?

Not only did Dale lead the way, but he also paused to explain his reasoning to his peers. I believe that such patience and consideration for the status of others during problem-solving is the hallmark of a natural teacher.

Dale's leadership continued on throughout their deliberation of the problem, and eventually he received a thank-you of sorts for his efforts.

Adie: But we still have to figure out who played all these instruments though don't we?

Dale: Uh huh. . . . .

Chad: Yep. The guitar player's group does not play country and western. So he doesn't play country and western. (*laughs*)

Dale: Yes, but we figured out that John plays the guitar.

Adie: We....

Dale: Right?

Adie: Guitars...guitar...country and western.

Dale: Understand?

Chad: No I missed it. I'm sorry.

Dale: OK, here.

Adie: Oh boy!

Dale: This question right here...uh number 2. Rock and country and western groups do not have keyboard instruments, right? That'd mean organ and the piano are out, right?

Chad: Uh huh.

Dale: OK.

Chad: Bass is out.

Dale: Bass was out. Why? Because....

Chad: The drums are out too.

Dale: And drums are out, so that means he plays the guitar! Right? Right?

Chad: How do you get his name? What....?

Dale: OK. Because Jake or John...John finished first, right?

Chad: Yes.

Dale: We've already established that. And the only places that we had open...rock and country and western right? And so, we figured...we already figured out,right? He came...that he played the guitar. And it said the guitar group...the guitar player does not play country and western.

Chad: Ohh!

Dale: John's the guitar player, so that means John's with rock. He came in first.

Chad: Yeh, I see it. Right on. Way to go.

I felt that such peer acknowledgement was very important for Dale, especially in view of his poor relationships with other group members in the past.

Also noticeable from the above two exchanges is the students' frequent use of the first person plural pronouns. Importantly, I thought this indicative that a group "mentality" or mindset was indeed taking hold. I was particularly pleased to witness the cooperative relationship this day between two erstwhile antagonists: Adie and Dale.

Finally, there was almost a sense of celebration as the resolution of the problem came to all three students

Dale: OK. Joe is not in soul. OK. We know that John is in the rock group, right?

Adie: Uh huh.

Dale: John is in the rock group.

Adie: Right.

Dale: We know that...we know that Jake is in the country and western group.

Adie: Yeh.

Dale: Do we know which group Josh is in?

Chad: Josh plays reggae.

Adie: Reggae, because reggae came in last.

Chad: No, reggae came in third.

Adie: Oh yeh. Soul came in last.

Dale: Do we know which group Jim's in?

Chad: Working on it.

Dale: OK. For country and western. OK. Could ah Josh for number 3? OK. We just have to find out who played jazz.

DT: Who played soul? Joe does not play his piano in the soul group.

Chad: Yeh.

DT: So where does he play his piano?

Chad: Either in jazz or rock. Oh but not rock. So he plays his piano in the jazz group. Right on! (*howls*)

Dale: So...so the piano is in jazz?

Chad: Yep.

Dale: That means the organ is in soul. OK. But we just have to find out if Joe... OK. Joe played piano right? Joe played in the jazz. (*writing on grid*) That means that Jim was in the soul. Jim...soul.

DT: And I think that's uh....

Dale: That's it!

Chad: Yeh!

DT: And you guys did it. Well done!

Chad: OK! Right on job! Narf!

On account of their successful problem-solving effort, I was certain that at this point, my good feeling equalled their own in intensity.

I then questioned Chad and Dale about how they felt during this mathematical experience, as opposed to their last effort.

DT: Compare...compare what happened to you two on Thursday compared to what happened today. What...how did it...what was the difference?

Chad: I don't know.

Dale: I don't know.

DT: How was it different?

Chad: I don't know.

Dale: We related more to the question.

Chad: Well....

DT: Obviously. How was it different than Thursday?

Dale: You know (*unintelligible*) (*laughs*) We got the answer.

Chad: Hmm?

DT: How...how did the flow go on Thursday for you two?

Dale: Back and forth, back and forth...(*unintelligible*).

Chad: We got stuck in a rut.

DT: Stuck in a rut. Right.

Chad: Yeh.

Dale: Stuck between a rock and a hard place. . . . .

Chad: I guess maybe it was looking at it like brand new...like without all the...

DT: Fresh ?

Chad: ...X's all over the.place.

DT: Right. Sometimes it pays to just throw out what you were doing. Just put it aside.

In other words, one of the best techniques for problem-solving is simply to take a break: go for a walk, read a book, do something different. As Mason and Davis (1991) suggest, a fresh outlook or different viewpoint is often all that is needed to lift the problem-solver out of the rut. This tactic has worked on many occasions for me, and perhaps should be included in the list as heuristic number 10: "Leave the problem for a while."

The discussion lead to the renewal of a theme about group work that had been brought up by Adie much earlier on in the project.

Chad: One thing that I found that um...was that...what was confusing is that...was that it got confusing when you're on a train of thought, right? And then, all sorts of other people get into different trains of thought? When the problem's so complex as this? I mean we figured it out, so it worked, but still it kind of destroys your train of thought.

DT: What does? What destroys your train of thought?

Chad: When someone else starts having a train of thought. When we're working on something. Especially with all these different things.

DT: Right.

Chad: And, I don't know....

DT: Too much happening at once?

Chad: Huh?

DT: Too much happening...

Chad: Yeh.

DT: ...sometimes?

Chad: That's the only thing I found kind of irritating at times. But we figured it out. And that's good.

Adie: Well, you know I think the....Gosh. (laughs) I don't know how to say it though. Like, you'll be saying something, right? If I wait for you to finish, then I'm totally going to forget what...

Chad: Um hum.

Adie: ...you know, the...the problem is. What I spotted, you know?

Chad: Yeh.

DT: OK.

Chad: Yeh. I mean, in other problems it's easy, you know. With lots of people, it's better that we throw our ideas out because they're not as complex as this. But this one is so complex, and you're trying to remember all sorts of different things.

Adie: Yeh.

The students described an interesting dilemma for people working in small groups. If the problem at hand is perplexing enough, or of sufficient sophistication, then it is often difficult for learners to remember or keep track of their own ideas whilst they listen to and consider the offerings of others in the group. Hence, there is a tendency for everyone to blurt out his/her ideas as soon as they are formed. Perhaps after listening to their explanation, I would in the future be a little more tolerant when the classroom "forum" was lost because of simultaneous outbursts or interruptions.

With undiminished energy, the students then began to deliberate problem 4 for the first time. (See Appendix 5.) Dale finally came up with the last bit of deductive thinking necessary to solve the puzzle:

Dale: Wait a second! This guy's the Japanese, and he has a Philips TV. This guy can't be Japanese because he has an Hitachi.

Adie: Right! That's right!

Chad: Hey!

Dale: Yes?

Chad: Right on!

DT: Good deduction.

Chad: Yeh.

Dale: What does this guy have? The uh...Phillips.

Chad: Hey that's cool. That means the other guy's got the Sony.

Dale: That means that this guy has the snails.

Chad: No!

Dale: Yeh.

Adie: Yes! No!

Chad: The Sony TV has the snails. The guy in the red house has snails.

Dale: Sony.

Chad: Boy, I tell you man. Those English people are weird.

Dale: No shit. Escargots! OK. So that means this guy is the uh...

Adie: Spaniard.

Dale: ...Spaniard.

Adie: And this guy has the dog.

Dale: Dog. And that means this guy has the uh zebra.

Adie: Zebra.

Chad: We already got the orange juice. Cool! Right on! (*howls*)

DT: Wait a minute! Don't go!

Chad: Don't go.

DT: Excellent (*unintelligible*)! Good effort!

Chad: That was well done. I like that!

Group problem-solving simply does not get much better than this. There was good logical thinking, cooperation, and sustained effort by everyone. Finally, the loud enthusiasm accompanying the solution was not individual, but rather collective. It was a celebration by the group which, after all, had produced the solution.

The timing of the final resolution also was excellent: they made their discovery just as the clock hit 2:45. However, I needed to keep them from leaving the school, since I wanted to send them away with a survey which needed to be completed for the introduction to the next topic.

I gave each student two separate pages, each containing a set of different sized rectangles. (See Figure 3.9.) The group's task was to survey as many people as possible, asking for each set, "Which rectangle looks the best?" or "Which rectangle is more pleasing to your eye?" Now VRLF students are not used to having homework, so I hoped rather nervously that the cohort would show up on Thursday with enough data to begin our consideration of the golden rectangle.

Figure 3.9

Set 1   Set 2
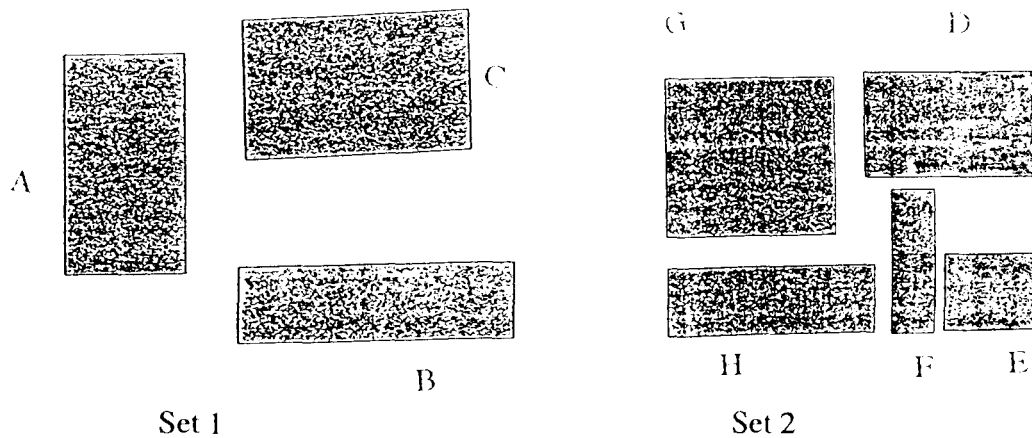
INTER-SESSION Wednesday 22 April 1994

Just as I was hoping for a full turn-out of students on Thursday, I received news that Alonzo would be away for an extended period to work up north. However, a new student, Nina, would be attending my class starting Thursday. Since she and her guardian had already signed the study consent form, she would be joining our mathematics group immediately, just in time for the next session. Nina, of Grade 9 age, promised to be an interesting addition to the project, since she reportedly had not had any formal mathematics education since Grade 6. I was certainly glad that we would be starting a new topic for her first day, so that she would not be confused by entering the study group in the middle of some confusing mathematical deliberations. Also, there would not be any "heavy" mathematics to discourage her on her first day at VRLF.

Topic 6   The Golden Ratio

SESSION 23   Thursday 28 April 1994   (Adie, Dale, Chad, Juliana, Nina, Lee-Ann)
"That's fascinating!"

I had no plan of action whatsoever in case the students' survey came out adversely. In other words, I was counting heavily on the majority of those polled actually favoring the two

golden rectangles out of all the other choices. So, it was with considerable trepidation that I asked Chad (who was serving as tally-master for the group) for the results of the poll.

With a sigh of relief, I listened to Chad report that the two golden rectangles, (C and D) were indeed the most popular choices. (See Figure 3.9.) After giving the students a brief history of the golden ratio in architecture, I explained their next group activity. The following dialogue concerning ratios resulted.

DT: Uh...the next thing to do is measure the lengths of the sides of these rectangles. And we're going to calculate what's called a ratio of the length of the longer side to the length of the shorter side for each, C and D. Now what is a ratio? Going back into your math? Remember what a ratio is?

Juli: Uh....

DT: Another word for it?

Chad: It's how many took to something else.

DT: OK. You could say a ratio of 3 to 2, you mean?

Dale: Stats? Stats?

DT: It comes out in statistics, yeh. But what is...what is it? Give me another example of one.

Chad: A report.

Annalia: 5 to 6?

Chad: The time.

Juli: Do they have the 2 little dots in between?

DT: Yeh. That's another way to write a ratio. And you could write 5 dot dot 6. Like that. (*writing on board*) OK? Third way? Simply a....?

Dale: Division.

DT: Yeh. You could write 3 divided by 2. Now when you get up in this level of mathematics, you tend not to use the dividing sign so much.

Chad: The line.

DT: Yeh. Use the line. Instead of 3 divided by 2, we say....

Chad: 3 over 2

Juli: 3 to 2.

DT: What's another name for this?

Juli: Fraction!

DT: Fraction. So a ratio is just a fraction. Just a way of writing 2 numbers. Comparing 2 numbers. What we're going to do is measure the length of both C and D, and compare it to what we call the width. Alright? So, do that. Grab a ruler. Let's do it all in centimetres, so we're....

Juli: We can do it together.

Again, the topic at hand provided another opportunity for the group to recall the nomenclature for so-called basics like ratios, which, in the former course, would have been a subject unto itself, devoid of context.

The students exhibited no problems in their measuring; in fact, they were even quibbling about whether they should report their findings to the nearest tenth or hundredth of a centimetre. Finally, they computed the length:width ratios of rectangles C and D as 1.63 and 1.61 respectively. This finding led to a discussion of similarity, and to a review of area and perimeter:

DT: We say that those rectangles that you see in front of you, C and D, have the same what?

Don: (*quietly*) Shape.

DT: You said it. I heard you.

Annalia: Shape. They're the same, only bigger.

Annalia was the third student from another class who asked to sit in on our mathematics proceedings. She then asked, "Is that the same with [*rectangles*] B and H?"

After her question, the pupils set out measuring the rectangles of the non-Golden variety. "Hopefully," I thought, "none of these other rectangles will measure anwhere near to the golden ratio, and thereby complicate things immensely." By this time, the students had assumed roles in the process. Juliana operated the calculator, Chad wrote down the data, and the others actively came to a consensus about the accuracy of their measurements. Luckily, not one of the newly calculated ratios for the remaining rectangles was close to 1.6.

I had asked Katey to speak briefly to the class, since she doubled as a specialist in interior design, and possibly could address the aesthetic side of our consideration of the Golden Ratio. She described an excellent connection between mathematics and art:

> The one thing I always find is there's also almost a mathematical basis to how you decorate, and to...or how you design windows and that sort of thing. As to what looks pleasing? Now here's a thing you might not have heard. I don't know if this works out with the one to six (*referring to the number 1.6, the Golden Ratio*) but in design they call it the one to three rule. If you were making a little collection of things, like a little collection of pictures on the wall? Or of uh something...baskets sitting on a table? You always find that the odd numbers work, and are more pleasing to the eye. And the even numbers don't work, and are not as pleasing. So if you put two things on a table, it's going to be unbalanced. If you have 3, a group of 3, it's more pleasing to your eye.

Katey was willing to put her theory to the test, since we had photos of Greek architecture handy:

Katey: Yeh. Anyhow, did you look at the Acropolis, or the Parthenon? How many pillars are there? Was it an odd number? Just see for fun. Probably prove me wrong.

Dale: There's eight.

Katey: 1 2 3 4 5 6...Oh god! OK. It doesn't work.

Chad: Yeh, but there's 1, 2, 3, 4, 5...there's 7 spaces in between, so 7 rectangles.

Katey: You're right. That's true, too. When you're doing that, you have to think of looking at spaces.

Chad's observation was doubly astute, because it was these "rectangles" in between Greek columns that the class would next investigate as instances of the Golden Ratio.

Dale wondered about the relationship between human anatomy and mathematics:

Dale: OK. Well...um...this Doryphoros one...type statue. How does that stand out?

DT: We'll look at that in a sec....

Katey: Yeh. Even....

DT: ....'cause that's going to tie into anatomy.

Katey: I was just going to say, that with anatomy, there's the balance for the bones in your body. Every bone is um...balanced to the next bone. Like this uh...this is longer, and this is shorter, they're not the same? And your whole body's made up that way. You have a longer leg, then you have a shorter part of your leg. Then you have a shorter part of your foot, so that proportion is in your body too, or in most bodies.

Chad: That's fascinating!

It was good for the students to hear about such mathematical connections from a visitor like Katey whom they knew as their art teacher and specialist. In fact, any curriculum of this type should actively involve not only mathematicians from outside the school, but people like her who could establish connections in a way that the regular mathematics teacher never could.

Her last comments about the human body were timely, since the students were about to investigate their own anatomies for evidence of the golden ratio. First, from a photograph, they measured one of the Parthenon rectangles, and found that the length:width ratio was 1.7. At the same time, I measured the Parthenon photo which was projected onto the blackboard, and Dale came up with the same answer. This provided a good opportunity for a discussion of similarity:

> Dale: (*computing on the calculator*) 1.7 for the one you got too. (*i.e. the one projected on the board*) Dale?
>
> DT: This one?
>
> Dale: Yeh.
>
> DT: Now I just measured mine on the board up here, with this magnification. Why do I get the same result as you guys, with your pieces of paper?
>
> Chad: 'Cause the ratios are the same.
>
> DT: Why?
>
> Chad: Because it doesn't matter how big the rectangle is, as long as the proportions are the same. Because it could be small, and you could get the same ratio. As long as you keep the proportions right when you go like that. (*gestures with his arms*) And then....
>
> DT: What's "going like that" called?
>
> Chad: Uh...expanding?
>
> DT: Or?
>
> Dale: Enlarging.
>
> DT: Enlarging or....? We do that in photography.
>
> Chad: Magnifying.

Unfortunately, the girls were talking away in the background during this exchange. According to theory, they were supposed to be listening and making their own "constructions" out of Chad's and Dale's verbalizations. Nevertheless, I was satisfied with the comments of these two members of the group.

After discovering the Golden Ratio on the photograph of Doryphoros (and after the inevitable joking about the spearbearer's anatomy), the students went on to the lab type activity of measuring aspects of their own bodies. Here, we used Serra's (1989a) well-designed exercise (p. 478) for measuring and formulating the data from each student. I divided the class into two, with the boys and girls in different groups. For the most part, the students kept on task. This was a particularly good introduction for Nina, for although she said nothing in our group discussions, she actively participated in taking and recording the measurements asked for.

The real challenge in this topic would come during the next session, however, when I planned to take the cohort through an algebraic discovery of the Golden Ratio. Hopefully, enough of the students would have the expertise to follow the lesson. And I feared losing or discouraging students like Nina, who simply did not have any mathematical background to speak of.

SESSION 24   Tuesday 3 May 1994   (Donny, Dale, Chad, Juliana)
"I learned this in Grade 6."

During the pre-session time, Juliana's invective towards Dale suddenly increased in frequency and intensity. Accordingly, I took Juliana outside after one particularly derogatory appellation.

> DT: You use these terms far far too often in class: I've noticed that listening to tapes.
> Juli: They aren't swear words.
> DT: That doesn't matter. You have to edit your choice of words, and to refrain from calling people names.

Perhaps I had no ultimate control over Juliana's negativity towards others, but I felt responsible for the tone of language and for outright put-downs of others in my classroom. By this time, I was becoming extremely protective of our successful group dynamics, and I was prepared to evict students lest such anti-social behaviours poison what we had achieved socially since our start in

February. After all, if the schools could adapt a "zero tolerance" policy towards physical violence, then I felt justified in establishing the same consequences for vituperative behaviours.

At the start of the session, Dale provided the group with a real-world example of the Golden Ratio. As it turned out, he had actually measured the Golden Ratio in renovations being done by a relative:

> DT: What was it you measured again?
> Dale: Ah...his windows.
> DT: Windows.
> Dale: And he has a door, like going into his den.
> DT: Right.
> Dale: Right? And it measured 1.6.
> DT: Is that uh...is this house ah...custom built?
> Dale: Yeh.
> DT: Is he designing it himself?
> Dale: No. Um...it's an old house. But he uh...tore out the whole living room and all the bedrooms and that, right? So it was nothing, right, except blank.
> DT: Yeh.
> Dale: And then he put up the other....
> DT: So is it he who's doing that, or was it originally there? These ratios?
> Dale: The ratios weren't there because he took like the windows out and he moved the windows around. He....
> DT: You might want to ask him if that's what he's up to. Because a lot of people are quite aware of that.
> Dale: No, I asked him and that's what he had done.
> DT: He did? Oh, OK. The Golden Ratio, right?
> Dale: He...he's smart so....

Following Katey's real-world contribution, his revelation seemed to provide a wonderful continuity from the previous session.

Dale's example also served as an excellent segue into the lesson I had planned for this session. In my presentation of the problem, Chad actually provided the key question:

DT: Let's say you're building a new house. And you want to construct a window. And let's say you know the length of the window is going to be 2 meters. Alright? The length has to be 2 metres, alright? But you want a Golden Rectangle for a window. (*draws diagram on board*)

Dale: Oh.

DT: You want this for architectural design. So the question is, if the length is 2....

Chad: What's the width?

That Chad understood the problem is evidenced also by his subsequent formulation of the solution:

Chad: So, 1.6 times what equals 2 metres, right?

However, with my own agenda in mind, I ignored Chad's suggestion $1.6w = 2$. Instead, I presented the equation $\frac{2}{w} = 1.6$ and took the students through a whole recapitulation of cross products. I reasoned that the latter equation was preferable because it resembled the way in which the students computed the golden ratio in the last session: that is, dividing length by width. I also wanted to prepare the students algebraically for a derivation of the golden ratio which would involve cross products. Ironically, we ended up with Chad's equation anyway which we proceeded to solve formally.

In retrospect, I should have gone with Chad's initial solution. After all, the cross product is a gimmick which focuses on memory, not understanding. And Chad understood! If at the time this episode had taught me about the folly of insisting on my own pedagogic agenda, I could have avoided a disastrous session a few days later.

When I asked the students to recollect how to solve such the equation $1.6\,w = 2$, Chad came up with the solution.

Chad: The answer's 1 point 25.

DT: How'd you get that?

Chad: I....

Don: Divide.

Juli: Is it?

Chad: 2 divided by 1.6

Juli: It can't be.

DT: Correct.

Juli: Is it?

Chad: Yeh.

DT: But how do you reason that?

Chad: It's backwards.

Chad's last answer was a natural enough description of the classic equation-solving technique taught to beginning algebra classes. It was good when students provided such apt transitions into what I wanted to cover next:

DT: What is the operation in here between the 1.6 and the w ?

Chad: Dale: Donny: Times. *(All three respond at once.)*

DT: Multiplying, right. What's the opposite of multiplying?

Don: Chad: Division.

DT: How can I get rid of this 1.6? After all I don't want 1.6 here. I want w all by itself. Well, what's 1.6 doing to the w?

Chad: Multiplying.

DT: Multiplying. What's the opposite of multiplying again?

Chad: Donny: Division.

DT: So what happens if I divide by 1.6 like this? *(writes* $\frac{1.6}{1.6} w = 2$ *)*

Chad: Huh?

Dale: It takes away the w.

Chad: It takes away the 1.6.

DT: What does it leave?

Chad: The w.

Then finally, in student vernacular again, Dale was able to recall the division property of equality concept:

DT: But I can't just go dividing one side of the equation by 1.6.

Dale: You've got...you got to do the other side of it.

Following my explanation of the properties of equality using the teeter-totter analogy, the students were able to recapitulate the concept.

DT: What you do to one side....
Dale: ....you must do to the other.
DT: To keep it in....?
Chad: Dale: Balance.
Dale: Fuck, I learned this in Grade 6.

As described in chapter 3, alternate school students often harbour bad feelings about having to review so-called elementary mathematics. Their reasoning goes something like this: "We must be stupid if we can't remember such easy stuff." In order to alleviate any such possible sentiments here, I related how quickly I forgot some of the math which I had just learned in one of my mathematics courses in the graduate program.

DT: The nature of math, everyone, is that you're going to have to go over stuff, you know, in a way like this, to try to click it back into your usable memory. And I'm sure for all of you it's there, but you haven't...if you haven't used to for 2 years, 3 years? You've got to kind of reach into your mind to pull it out. OK?
Chad: Yeh, but you never forget how to ride a bike.

Perhaps we should be teaching alternate school students the mathematical equivalent of riding a bike. Cross products and equation-solving techniques are rapidly forgotten if not practiced and making students dependent upon memory only leads to frustration. But once learned, problem solving skills and mathematical awareness will probably be retained for life.

Having fulfilled my agenda to this point, I felt that the students were ready to consider Euclid's twenty-three centuries old question:

What are the dimensions of a rectangle that has the property that when it is divided into a square and a rectangle, the smaller rectangle has the same shape as the original?

[Here, we were using the text *Mathematics 9* by Kelly, Atkinson, and Alexander (1987).]

Looking at the diagram projected on the blackboard, *(See Figure 3.10 below.)* Chad was able to come up with the correct algebraic expression.

Figure 3.10
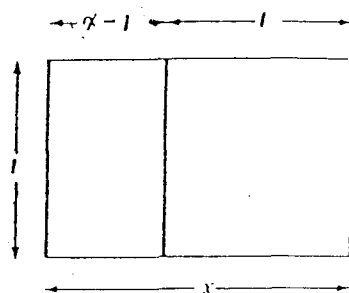
DT: Just for easiness' sake, let's let the width of the rectangle be 1. OK? We can let it be anything we want, but one's a nice number for simplicity, alright? And let's let...we don't know what the length is, so let's call the length x. What does x over 1 equal?

Dale: We know it's bigger than 1.

DT: And this will be the answer to Euclid's question. What are the proportions of length to width which allow us to do this?

Chad: To do what?

DT: Well first of all, if we let the length of the rectangle be x, and its width be 1, this little length in here is...see your square? Whoops. Here's your 1 by 1 square, right?

Chad Um hum.

DT: OK Chad. If this length here is 1....

Chad: Um hum.

DT: And this big length up here is x....

Chad: 2?

DT: No. What is this length here?

Chad: That's uh...x minus 1.

Again, Chad demonstrated his talent in matters algebraic. Indeed, any alternate school class could benefit from a key student who is as good at generalizing as Chad.

With color shading on the diagram, and with the word equation below, the students were able to come up with the quadratic equation whose solution was the Golden Ratio.

$$\frac{\text{Length of original rectangle}}{\text{Width of original rectangle}} = \frac{\text{Length of smaller rectangle}}{\text{Width of smaller rectangle}}$$

$$\frac{x}{1} = \frac{1}{x-1}$$

Chad spontaneously attempted to "cross-multiply" the recently derived equation. So at that point, after I suggested we should use brackets, we had $x(x - 1) = 1$. When we tried substituting $x = 1$ and $x = 2$ into the equation, Chad was able to interpolate: "So it'd be like somewhere between 1 and 2. It'd be a decimal." Finally Don suggested $x = 1.5$ as a trial, and the group was under way. Dale led the way, and using successive approximations, they did not quit until dismissal time, and they had come up with an approximation for the Golden Ratio of $x = 1.618033$.

I was pleased that the last two lessons went according to my plan. In the last session, the class used the survey and the actual measurements of the chosen rectangles to produce an approximation for the Golden Ratio of 1.6. This session, using formal geometry and algebra, we ended up with a very accurate version of the same number. Thus we had come full circle from induction to deduction. All this was accomplished by a cohort of alternate school students with an average WRAT score of 6.6 .

SESSION 25   Thursday 5 May 1994   (Donny, Dale, Chad, Rosie, Karl)
"Where is this getting us?"

Encouraged by the group's accomplishments in the last session, I was eager to explore the golden ratio in greater depth with the students. By the end of the class, however, I would be wishing that I had left golden rectangles and had just gone on to the next topic. The fiasco of session 25 was the result of many factors: having new students arrive midway through the development of a topic; teaching too many new concepts at once; introducing subject matter which was beyond the scope of the students; and last, simply insisting on my own agenda.

I have stated before that any teacher in an open-ended learning situation has the ultimate challenge of choosing learning experiences which are challenging but, at the same time, not hopelessly difficult for the students. The results of my failing that challenge were obvious this day: general student disinterest, lack of focus, and discouragement were the central themes to arise

out of this lesson. And regretfully, I fell back completely into a teacher-centred, lecture style lesson.

My plan for the session was essentially to have the students investigate the zeros of the golden equation $y = x(x - 1) - 1$ from its graph. Later, I reasoned, we could look at the computer graph of the same function, and using the trace feature of the graphing program, come up with an accurate approximation of the golden ratio. So, before the class began, I was enthusiastic about using the new Macintosh computer for the first time in the study.

I predicted that in order to accomplish this task, the students would first need to review the distributive principle. In particular, I felt that the equation could be graphed more easily by converting it to the form $y = x^2 - x - 1 = 0$. It would also have to be in polynomial form to enter it in the computer graphing program.

In retrospect, all of the above activities could have been accomplished without insisting on the algebraic background necessary to manipulate the original equation $\frac{x}{1} = \frac{1}{x - 1}$. After all, in the last session Dale and the others had no trouble evaluating the golden ratio from this form. But nothing at this point could have dissauded me from my agenda.

We first went over the results of the last two sessions, mainly for the benefit of Alim, Rosie and Karl who were not in attendance. (Karl was another student from a different class who asked to sit in on our mathematics class.) The summary started off well enough. Chad recapitulated the results of the rectangle survey from the first lesson, and Dale made the following observation:

Dale: That blackboard will probably be a golden ratio too then.
DT: What's that?
Dale: The blackboard...
Rosie: The blackboard?
Dale: ...will probably be one too. Just by looking at it.

DT: I don't know. It'd be interesting to see. We'll get the tape in here and measure it. We never did measure the blackboard. However, we should have spent the whole period measuring it and other objects instead of forging ahead into algebra and coordinate geometry. With a little planning, I could have organized an interesting field trip entitled "In Search of the Golden Ratio."

It soon became obvious that the reasoning behind how we obtained the equation $x(x - 1) = 1$ seemed lost on even those students who were present last day.

> Rosie: Does it matter what size the triangle is? I mean rectangle?
> DT: That's what we're trying to find out. So we said in order to do that, let's let the width of this big rectangle be 1. Make things simple.
> Dale: And the reason it has to equal 1 is because of...the total dimensions of the square is 1. That's why it has to equal 1. Is that right David?

It is clear to me now that with such evidence of my students' lack of understanding, I should have immediately abandoned my ambitious lesson plan at this point.

But undauntedly (and unwisely), I descended further into the abyss of mass student confusion. I explained that to simplify matters, we would now change the equation $x(x - 1) = 1$. First came the actual distribution of $x(x - 1)$.

> DT: Before we can go on and look at this equation, we have to alter its form a little bit. You need to pick up a little algebra here, right? So pay attention to this. There'll be a little worksheet on it. There's another way to write this, without the brackets. And it's going to produce an equation which will allow us to use the computer, and allow you to use graph paper to look at this problem of the golden ratio.
> Dale: Is that computer set up to do that right now?
> DT: Well, it's set up to graph. And that's what we're going to do later. OK? We'll perform an operation on the brackets here, in the following way.
> Dale: Joy.

There would be no such emotion in our class this day. Had I allowed the students to graph sooner rather than later, disaster could have been avoided.

By plunging into an explanation of how to distribute through the brackets of $x(x-1)$, I made my first critical mistake. Instead of starting with the expression $x(x-1)$, I should have used purely numerical examples. By now, I had lost all the students. This fact should have been painfully obvious when even Chad expressed dismay at the result $x(x-1) = x^2 - x$.

Also, the students were completely off task, talking to each other sotto voce. They were not paying attention because they did not understand a single thing which had occurred so far this session (with the possible exception of the students' rectangle survey). The inevitable unpleasant teacher/student exchange about their lack of focus was but a prelude to what was to follow.

Although the group handled numerical examples such as $2 \cdot (8-5)$ successfully, only Chad seemed able to distribute expressions like $a\,(a-5)$ which contained symbols. Distributivity is indeed an important algebraic concept, but so far, given my pedagogic approach, the students had almost no chance to comprehend it.

From here, matters only worsened. I handed out a worksheet with twenty or so items such as $3(\,2x-5y\,) = $ _____ and $x(\,3x^2 - 2x + 6) = $ _____ . The exercise was a disaster. Actually, the true picture of the group's algebraic inexpertness emerged only after I went over the papers that evening. Of all the students, only Chad enjoyed a modicum of success with his answers. Dale managed to complete only three questions. Alim, Karl, and Rosie were confused and frustrated. And Nina, with no background whatsoever in algebra, or even in integers, was completely lost.

This day, I was following a general teaching scheme which had worked well up to this point in the study group. That is, so-called basic skills in arithmetic and algebra are best presented in the context of an open-ended problem. But my presentation of the distributive principle as part of the golden ratio problem was doomed for several reasons. First, it was not a concept that can be presented and practised in one hour alone. Second, such an algebra lesson should have

proceeded from the concrete (e.g. algebra tiles) to the familiar (e.g. numbers) to the abstract (e.g. symbols). And last, the students were lacking a strong foundation in integers and basic algebra. For instance, I should never have presented Nina with such relatively sophisticated concepts and skills. In sum, as an experienced teacher, I should have known better.

Our foray into graphing was almost as catastrophic. To begin with, on the students' worksheet, I made an error in algebraic format. The question asked them to draw the graph of the equation $y = x^2 - x - 1 = 0$. Needless to say, they found that "zero" an initial source of confusion. And as they set about filling in a range of values corresponding to the integral domain $-4 \le x \le 5$, Chad's explanation fell on deaf peer ears. And my own patience level was, by this time, on empty.

> Chad: Supposing x is 5, then that would mean 5 times 5 is 25. Minus x...5...20. Minus 1 is 19.
> DT: Yeh. (*then to Dale who was talking out of turn*) I don't want to have to explain everything again.
> Alim: Just kick him out.
> Dale: Sorry, David.
> DT: You could learn from what Chad's saying here. You're not into it? You may as well go home.
> Dale: I'm listening.
> DT: Maybe it's just this afternoon.
> Chad: It's the hot day that it is.
> DT: I just get concerned because if you just paid attention to what he's saying, you could learn from what he's saying.
> Rosie: So shut up and listen!
> Dale: So....
> Rosie: Sorry.
> DT: We're taking this equation and we're going to put different values in for x. Chad just put 5 in. So what did you get Chad?
> Chad: I got 19.
> DT: 5 squared....
> Chad: Minus 5......

(*students talking in background*)

DT: OK. Look, if you don't want to be here, then go. And if there's two people left, I'll work with them. OK? Alim?

Alim: What?

DT: I mean if you don't want to participate, then don't. It's completely voluntary.

Alim: I say one thing and you freak out.

DT: I'm not freaking out. I just...I just talked to Dale about it and I turn around and you're not listening to what he's saying. You've got to listen to what people are saying here. OK, what's happening here Chad?

Chad: Got 19.

DT: How'd you get 19?

Chad: 5 squared is 25, minus 5 is 20, minus 1 is 19.

DT: What he's doing is substituting the number 5 into that equation.

Dale: David, can I ask you one question?

DT: Yeh, go ahead. You can ask me several questions.

Dale: Where is this getting us?

Dale's question was entirely à propos. Where indeed was this getting us? Tempers were flaring. Comprehension was low. Student interest was zero. The temperature was 30°. At this point, there was nothing "golden" about x.

Mercifully, the students were able to complete their tables of ordered pairs, and plot the points on the grid provided. Karl was the first to notice the symmetry of the resulting parabola, but Chad was the first to finish the graph of $y = x^2 - x - 1$. Of all the pre-requisite skills needed to participate in my lesson this day, graphing was the only one which the students appeared to have remembered from mathematics classes past.

As time was just running out, I asked Chad to show me the golden ratio on his graph. He circled the positive zero of $y = x^2 - x - 1$. And when I asked about the negative zero, he answered that the negative answer "didn't count". In the last analysis, I felt that Chad was probably the only student in the group that day who would be able to answer Dale's last question meaningfully.

That evening, listening to the recording of this last session, I came to the conclusion that my students' lack of basic algebraic skills was the main cause of the lesson's failure. In retrospect, however, I appreciate that the fault was not with my students, but rather with my approach to teaching the golden ratio. The topic is rich and has many other facets which do not require an algebraic background. Thus judiciously chosen activities on my part could have resulted in successful experiences for the group. As Gattegno (1974) comments, teachers "should make people aware of certain powers they already possess which they can use in the same way that they are used by mathematicians (p. 82). Algebra is but one of many "methods" included in the NCTM's (1989) delineation of the term "mathematical power":

> This term denotes an individual's abilities to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems. (p. 5)

In sum, most alternate school students would benefit from well-planned lessons which make them aware of the mental powers which they already have.

I intended to do some serious planning for the next session. Perhaps I would retreat to the concrete and have the students actually draw golden rectangles. The computer graph could serve as a good recapitulation of the just completed graphing exercise. Or, perhaps I would forget the golden ratio entirely, recoup our losses, and move on to fractal geometry.

SESSION 26   Tuesday 10 May 1994   (Adie, Dale, Chad, Donny)
"I have precision German instruments."

At the start of this session, I felt much better about continuing on with the Golden Ratio. For one thing, there promised to be nowhere near the confusion of the last class, since each of the students in attendance this day was familiar with the topic. As well, I had in mind an activity which was well within their abilities, and which at the same time would hopefully contribute to their understanding of golden rectangles.

First, I asked the students to determine which of four given rectangles were golden. After measuring the dimensions of the rectangles, and verifying their measurements within the group, they computed the various length:width ratios. The group then verified that the rectangles labelled A and D were golden.

Each student was provided with a compass and straightedge. Following my example on the blackboard, they divided each of the four rectangles into a square and smaller rectangle. The students then measured the dimensions of each of the recently constructed smaller rectangles, and worked out the length:width ratio for each. They were then ready to make conjectures:

> DT: Which of the smaller rectangles are golden rectangles, obviously....?
> Don: Chad: A and D.
> DT: The same as before. OK. What...what conjecture can you make about golden Rectangles? And start off your conjecture with the following. If you take a golden rectangle, and subdivide it, or divide it into a square and a littler rectangle....
> Adie: It's always going to be a golden.
> DT: Then?
> Adie: It's going to be the same thing. The golden...it's going to be a golden rectangle.
> DT: Good. OK. So here, let's formulate this conjecture in exact ...as exact terms as we can. You were right. (*writing on board*)
> Adie: The result is...um...that the smaller rectangle is also a golden rectangle. (*Laughs*)
> DT: Good for you. Well done. Well done.

In view of my leading questions, I obviously needed to remind myself that teacher silence could also be golden. Adie and Chad were immediately able to extend the reasoning:

> DT: What can we do from here?
> Adie: Do it again.
> Chad: Um....
> DT: With what?
> Adie: With the rectangle.
> DT: What one?
> Adie: The smaller one.

DT: The little one. What will happen if we divided *it* into a rectangle and a square?

Chad: We'll have another golden rectangle.

Adie: We'll have another golden....

DT: Why?

Chad: Because of our conjecture.

DT: Because the conjecture is true?

Adie: Yeh.

Chad: Yes.

Adie's comment "Do it again" was a good deduction: if the conjecture is true, then the next generation of smaller rectangles would also be golden. Thus, aided by actual geometric constructions, the students seemed to have arrived at an understanding of Euclid's original problem which eluded them in the theoretical considerations of last day.

When I asked the students what would happen if they kept going with their "subdivisions", Chad answered, "You'd get infinity. That's an interesting concept, infinity. I like infinity." Immediately, I thought that this excellent answer could provide an introduction to fractals. I hoped also that such enthusiasm for the idea of the infinite would persist during the next topic.

Chad questioned the practicality of an infinite number of constructions. Accordingly, I referred to the idea of performing such a task on a computer:

Chad: How...if it gets so small that you're dealing with like one millimetre by one, you know?

DT: Right.

Chad: The ratio's always going to be 1.6 .

DT: Yeh. No matter how small this is, if you have something like....

Chad: Oh, 'cause the ratio doesn't need to be necessarily in centimetres or millimetres or whatever.

DT: It could be...it could be in microns. Theoretically. It could be...could be as small as you want.

Chad: Oh.

DT: For as long as you have a sharp compass point. With a computer, we could do it. Set up a program to do it. What would the restrictions of the computer be?

Chad: The resolution.

DT: Yeh. In other words how many....?

Chad: Pixels.

Chad's acuity made me eager to investigate whether the Geometer's Sketchpad could be utilized at some future point in this regard.

Unfortunately, the students' constructions were soon limited by the finite nature of their geometric instruments. After the third sub-division, their rectangles no longer measured up as golden. (See Figure 3.11.)

Figure 3.11
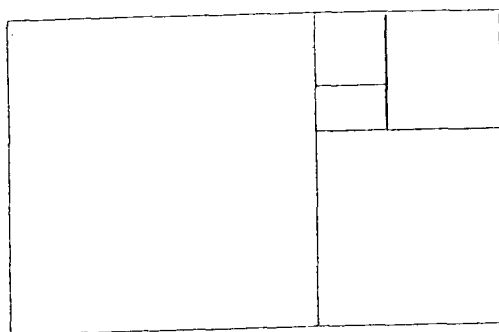
Chad: Wait a second here. I didn't get a golden rectangle. Therefore our conjecture isn't true.

Don: Do it again?

DT: Just a sec.

Chad: I got 2 squares. Uh huh.

DT: It doesn't look too golden.

Chad: Yeh, that just doesn't cut it. That doesn't work.

The fault here was not in the students' constructions. Rather, the original rectangles which I provided were too small, so that by the third sub-division, the resulting rectangles were so tiny that the width of the compass lead was resulting in inaccuracies in measurement.

This hitch was a good example of the importance of planning and trials of student activities for such open curricular projects. I know that the next time I try this activity with students, the original rectangles will be as huge as possible, thereby ensuring that the third or fourth generation of golden rectangles will still be large enough for accuracy.

To make things worse, I missed a "golden" opportunity when Chad commented that his third sub-division resulted in two squares. Immediately, I should have asked him, "When can a rectangle be divided completely into two squares?" The resulting discussion probably would have served to point out that his discrepancy was due to construction and measurement errors, and not due to a false conjecture. In any event, the students seemed willing to accept the explanation that my original rectangles were too small. After all, why doubt your own reasoning any further once the *teacher* accepts blame for the situation?

At the beginning of the class, I had remarked about the high quality compasses which I was providing for each of the students. Then, in mock defence of his own constructions, Chad poked fun at my earlier comments:

Adie: You're off by a little bit.
Chad: Hell, no. I have precision German instruments.
DT: You're off though. I can't understand it. But uh....
Adie: Chad's off.
Chad: Are you doubting the precision of these German instruments?

Indeed, the tone of this session was remarkably different from that of the previous day.

Finally, to test their understanding of Euclid's definition, I asked the students about making larger constructions. (See Figure 3.12.)

DT: How would you build another one?..Bigger one? (*ie. starting with golden rectangle ABCD*)
Adie: Double it.
Chad: Add a square.

DT: Uh huh.

Adie: Oh yeh. Add a square.

They were even able to direct my model construction on the blackboard.

Chad: Use the compass.

DT: To do what?

Adie: To measure the top of the square.(*i.e. side AB*)

Don measured the new, larger rectangle on the blackboard, and the group verified that it was indeed golden. From there, the students were able to visualize the construction of the next rectangle, and we ended up with the following diagram on the board:

<u>Figure 3.12</u>
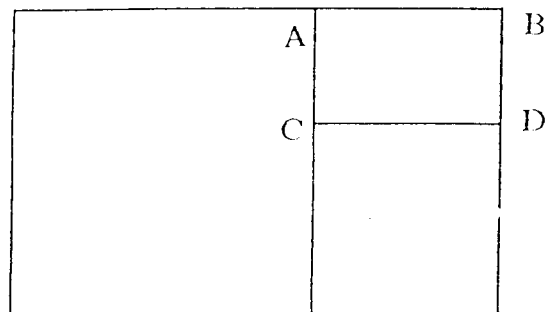
At this point, I felt as satisfied as possible that the students had a well-founded geometric understanding of the concept underlying the golden rectangle.

I then quickly outlined our previous derivation of the quadratic $x^2 - x - 1 = 0$ from the original equation $\frac{x}{1} = \frac{1}{x - 1}$ which in turn was obtained from the similar rectangles in Figure 3.10 . Hopefully, the students would now see the reason why we spent so much (futile) time and energy in converting it from rational to standard form: we now had the equation in the form necessary for entry into the computer.

I was excited about this, our first real use of the new Macintosh computer. I was also hopeful that the graphing utility would help shed more light on the solution of the golden quadratic equation. The question was, "For what values of x will the function $y = x^2 - x - 1$ equal 0 ?" To begin with, we were presented with a choice of different equation types. This menu resulted in a

brief discussion of the differences between quartic, cubic and quadratic equations. Then, we considered how to enter our quadratic, given the standard quadratic form $y = ax^2 + bx + c$. There was some confusion about how b and c would both equal -1, but this was not the time for such considerations. After all, this was to be a wondrous moment with a graphing utility! Chad entered the numbers 1, -1, and -1, pressed GRAPH, and instantly we had our result. (See Figure 3.13.)

Figure 3.13

$$y = x^2 - x - 1$$
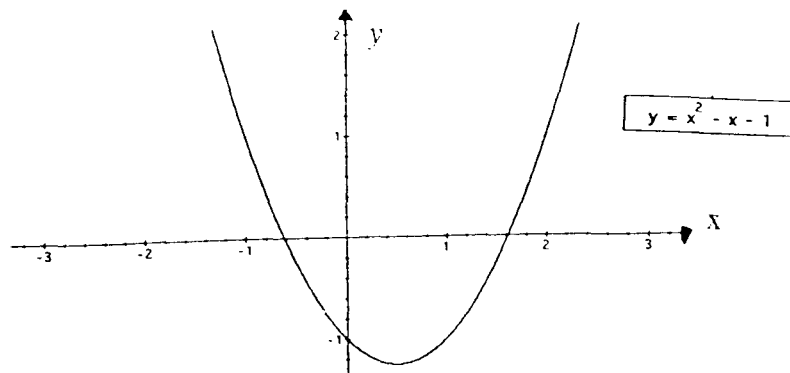
At first, the students were hesitant to modify the grid, so I gave them the following directive to encourage experimentation with the graph, "Go ahead. It won't destroy anything."

Finally, came the big question relating to the golden ratio.

DT: But we're interested in where does $x^2 - x - 1$ ...where does it equal....?
Dale: Zero.

Following a quick review of ordered pairs, and the characteristics of the axes, it was time to use the *trace* feature of the graphing program to give a computer approximation of the golden ratio. Each student was thus able to locate the zero of the function to three decimal accuracy.

Chad posed two interesting questions regarding the graphing utility. He asked, "Isn't there some way you can type in to ask it where...where that is...instead of using the mouse?" The answer to his question is, of course, yes. There are indeed computer programs and calculators which will determine the zeros of functions on command. Then he asked, "How come

that number didn't match the one we had on the board there?" (i.e. the group's extimation of the golden ratio taken to 5 decimal places.) I explained that such approximations would be more accurate if we had a more powerful computer program.

His questions led me to try to explain that there was also a formula which would allow us to compute an extremely accurate solution to $x^2 - x - 1 = 0$. However, I was loathe to introduce the quadratic formula to the group for three reasons. I felt that my students (with the possible exception of Chad) did not have the foundations in algebra to understand its derivation. Also, I was hesitant to simply drop another formula down upon them from out of nowhere; at that point, I wanted the combinatorics formula to be the only "magic" equation which I presented without any sort of logical derivation. Most importantly, I did not want a repeat of session 25 where I needlessly complicated the golden ratio topic for my students.

I was content with the tone and the results of this session. In particular, I felt our first use of graphing technology was a tremendous success. It provided a review of graphing techniques and elementary coordinate system conventions and it demonstrated the power of the computer to expedite a solution to a problem. I left that day with more evidence that with even the most elementary of mathematical foundations, my cohort was capable of some very profound mathematical discoveries.

SESSION 27   Thursday 12 May 1994   (Adie, Donny, Chad,Juliana and Alim)
"Holy! That's another Golden thing, right?"

I devoted this session for the students to work together on Assignment 8. In order to encourage cooperation, I asked for only one submission from the whole group, with an individual student responsible for the final write-up of each question. Since Dale led the way with successive approximations in computing the solution to the golden equation, we assigned him questions one and two in absentia. (His written response to the topic, along with those of the other students will be covered in my description of Session 28.)

Chad's previous recapitulation of the graph of $y = x^2 - x - 1$ strengthened my conviction that he had a good grasp of the concept of a function's zeros. When I asked how we knew from the graph that there was actually only one solution, he answered, "Well, there's two places where it goes over zero and we don't use the negative one." Accordingly, I suggested that Chad should handle the group's submission of the Assignment 8 question on the graph of the golden equation. As for the other questions, I advised the group to decide later which particular students would co-ordinate the final drafts of questions 3 through 6.

We went on to consider Question 3 from Assignment 8:

Simplify this expression.

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + 1}}}}$$

How close is this value to the value of the Golden Ratio?
(Kelly, Atkinson & Alexander, 1987, p. 198)

Chad was the first to express dismay at the sight of the continued fraction, and he offered the following metacognitive comment as if in explanation:

> Two! Plus 1 is three, divided by 1 is 3, plus 1. That's four. Where am I now? Four plus 1...four divided by 1 is four plus...it'd be five. What the hell? I don't know! I've never done an equation like that before!

After the group deliberated in vain for a few minutes, Juliana finally asked, "Well, shouldn't we start from the bottom then?" The students agreed to her good strategic suggestion, and what followed was essentially a review of fraction arithmetic. After computing the denominator 3/2 , the division technique still eluded the group. Chad asked, "But how do you do something like that? First, they thought that the numerator 1 (corresponding to the denominator 3/2) would have to be converted to 2/2 to obtain common denominators. But all they needed was a reminder of the rule that they no doubt had heard many many times: :

DT: Now you need to go back and ask yourself, "How do you divide one fraction by another?"

Chad: You times it.

Adie: You gotta switch the things around, and multiply.

Juli: Ahh!

Chad: Aha!

DT: So what do I do?

Adie: Two thirds...

DT: And....?

Adie: One...times two thirds.

DT: And what do we get here?

Adie: Hmm.

Chad: Two thirds. Two thirds. Aha!

Normally, working alone, a student would find such a misunderstanding in arithmetic an insurmountable obstacle in the path towards simplifying such a complex expression. But working together, the students were able to reconcile what turned out to be a communal misconception and carry on.

After computing the second fraction from the bottom to be 3/5, they recognized a pattern in computing the successive fractions.

Chad: So when we want to work it out...3/5 again...

DT: OK

Chad: ...all we need to do is invert that....

DT Ahh!

Chad: Ah hah! .... Wait! One plus....That'll be 3 thirds plus....No, 5 fifths. (*laughs*) Five fifths plus that is 8 fifths.

DT: Now....now you did it right. Now you....

Juli: Ha!

Chad: Now all you do is invert, and add 1. So you have 5 eighths.

Juli: Plus 1.

Chad: No, 8 eighths plus 5. (*ie.* 8/8 + 5/8)

Juli: 13.

Chad: 13.

Juli: Over....?

DT: Calculator please?

Chad: 13 over 8.

DT: What does that equal?

Chad: Well that would be....

Juli: It's one plus...one....

Chad: One and....

Juli: 5 over 8.

Chad:.What's 13 minus 8? Five eighths.

Juli: It's one and 5 over 8.

DT: What does that equal as a decimal ?

Adie : One and five divided by 8.

DT: Calculator. Yeh. Don't (*unintelligible*). Write out the answer here. She's going to give it to you. You got the answer Chad. Well done.

Adie: Shit.

DT: Write it as a decimal.

Chad: Hey, I couldn't have done it without you wonderful people.

Adie: One point 6 2 5. Is it? One point 6 2 5. Holy! That's another Golden thing, right?

Two aspects of Chad's contribution to the cohort are again manifested here. First, there is his ability to discern and apply patterns. And second, there is his contrasting modesty and appreciation of the efforts of all the other "wonderful people" behind the group's problem solving success . For the triumph of discovery expressed by Adie's last exclamation above is truly a group triumph.

While planning this session, I was hesitant to discuss the reasoning behind the connection between the Golden Ratio and the continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + 1}}}}$$

However, buoyed by the group's most recent enthusiasm, I decided to go ahead with the presentation.

DT: What equation did you use to get it? (*ie. the golden ratio*)

Adie: *x*...

Chad: Squared....

Adie: ....squared, minus *x* minus one equals zero.

DT: OK. Now uh....Let's see. Just trying to (*unintelligible*). (*writing on board*) (*pause*)

Here, I recall being unsure about how to exactly proceed in order to produce the continued fraction from the golden equation $x^2 - x - 1 = 0$. Panic was about to set in because I had no readily available notes on the process, and I knew that there was essentially only one "trick" which would lead us down the fruitful path.

However, my memory served me well this day. The first step in the process, dividing both sides of the equation by *x*, produced the following: $\frac{x^2}{x} - \frac{x}{x} - \frac{1}{x} = \frac{0}{x}$. We then had a brief review of the meanings of the terms $\frac{0}{x}$, $\frac{x}{x}$ and $\frac{x^2}{x}$. Normally, such generalizations would appear almost out of nowhere, usually at the beginning of a basic course in algebra. Here, dwelling on these algebraic laws in a context hopefully made them meaningful to the students.

At this point, I tried to explain to the students the significance of our new-found equation $x - 1 - \frac{1}{x} = 0$. Chad provided the opportunity for an interesting conversational diversion:

DT: It's the same as the original because we haven't done anything illegal in it. And if you go and try to plug values in like you did over here, you're still going to come up with....? One point....?

Adie: Six.

DT: You're going to come up with the golden ratio. So this is still a valid formula.

Chad: Why is it called the golden ratio? Why isn't it the silver ratio?

Adie: Yeh.

DT: Well, it's as good as gold.

Chad: Why isn't it the purple ratio, or the blue?

DT: This thing comes up so often, that mathematicians have to glorify it.

Chad: But gold is ugly.

Juli: Are you a mathematician?

DT: I think so, yes. And you are too, when you're acting as mathematicians.

Juli: Yah! (*said in a "Oh sure!" tone*)

DT: Mathematicians are people who do....?

Juli and Chad: Math.

DT: Mathematics.

Juli: It seems so easy for you.

DT: Well, I mean, I'm in a different element. If I was up at Simon Fraser, taking a course from a prof up there, I'd be, you know....

Juli: (*laughs*)

DT: It's all relative.

I welcomed such exchanges because in a sense, they served to level the educational playing field.

With the exception of Alim, who had said very little up to this point, the students seemed comfortable with the process of isolating the variable x in the equation $x - 1 - \frac{1}{x} = 0$. The students also seemed comfortable with each other this day: cooperation and good-natured ribbing had taken over from the individual efforts and the name-calling of past sessions.

When we ended up with $x = 1 + \frac{1}{x}$ as the final variation of the original equation, Chad asked an interesting question:

Chad: Wait. Didn't we start out with that in the first place?

Adie: No.

DT: Ah...it's a variation of the theme.

Chad: No, I thought we really did start out with exactly that. Well, practically.

DT: I wouldn't be surprised if it comes around to that because...

Chad: What?

DT: ...this is the same equation as the original, and that's the same as this guy here. (*Pointing to* $x^2 - x - 1 = 0$). They're all the same, but just have different...they just look different. They're just in disguise.

Chad: OK.

Chad's observation had merit, for although our new equation was not identical to the rational form derived from similar rectangles, they were indeed the same equation. Here, I missed the

opportunity for Chad to see how easily the equation $x = 1 + \frac{1}{x}$ could be transformed into the original equation involving the ratios of the sides of two golden rectangles, namely, $\frac{x}{1} = \frac{x + 1}{x}$.

We had just established that the golden ratio equals 1 plus its inverse. I then replaced the variable x on the right hand side of the equation with a box: $x = 1 + \frac{1}{\boxed{}}$

Chad: You're going to put 1 plus 1 over x under the....uh huh!

DT: I'm going to put 1 plus 1 over x in its place. (*writing* $1 + \frac{1}{x}$ *in the box to obtain*

$1 + \dfrac{1}{\boxed{1 + \frac{1}{x}}}$)

Chad: Ah ha! Oh ho!

Adie: Ah!

Chad: Yeh but x....then you erase the x, and you do it all over again.

DT: Yeh.

Chad: Until you get sick and tired of doing it.

DT: (*adding to the equation to obtain* $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{x}}}$)

Chad: But that can go on forever.

Chad appreciated the infinite nature of the continued fraction representation of the golden ratio.

Then, Juliana made a connection to another famous irrational number.

Juli: And it's like pi. Is that....(*pause*)

DT: It's like pi, exactly.

Juli: Oh!

DT: We call it phi, in mathematics. P-H-I.

Juli: Why?

DT: They said, "Let's give it a name. It's used so much, it keeps cropping up, let's call it a Greek letter." And that...that stuck, so phi is the golden ratio. Other people use that. We haven't used it , this phi. It's just like pi in geometry. Alright?

Adie: OK.

This was another opportunity to expand on a connection suggested by a student. Certainly, if I was unable to explain a bit more about π on the spot, I could have introduced it during the next

session. Ideally, with a fully developed curriculum, the teacher could simply reach into a filing cabinet for a folder containing student materials for an instant lesson on a topic such as the irrational number π.

However, there was an agenda for this session, and I felt obliged to leave π for another day. We now turned to consider the fifth question of Assignment 8:

> *The Golden Ratio is hidden in the famous Fibonacci Sequence {1,1,2,3,5,8,13,21,34...}. Try to discover how.*

Initially, I tried to encourage the students to "fool around" with the Fibonacci numbers for a while with their calculators. But inertia and frustration quickly set in, and the students turned to other topics of conversation such as Letterman and the lack of male nudity in movies. Sensing the mathematical wheels grinding to a halt, I felt obliged to direct the group in the right direction:

DT: Now here's another clue. You've got to use a calculator. You've used the calculator already for that. But what are you asked to find here? What do you want? What does the question ask?

Juli: We want that phi number.

DT OK. What is phi? Another word for it?

Adie: Golden ratio.

DT: The golden what?

Adie and Juli: Ratio.

DT: What is a ratio?

Juli: Two dots in between....

Chad: (*laughs*)

DT: That's one way to put it. That's another way to represent it.

Juli: I dunno. Fraction!

DT: Fraction?

Chad: So many to so many.

Juli: Decimal?

DT: Or a decimal, yeh. Or a fraction. What operation is involved when you say....

Juli and Chad: Division.

DT: OK.

Juli: Hmm. Let's brainstorm now.

In review of this exchange, I feel that I was far to explicit in my clue-giving. Again, the spectre of too much teacher influence had returned to haunt me. Would I allow the students to follow Juliana's suggestion, or would I choose to interfere again? Alas, I could not keep quiet.

DT: How is the sequence generated? What operation is involved in going from number to number there?

Juli: Uh...adding!

Chad: It's that number, plus the number before it.

DT: Right. Adding. Plus. That's right. But what does a ratio want you to do?

Chad: Oh. It wants to go the differences between the numbers.

Juli: Divide.

DT: Right.

Juli: OK, a ratio. Oh! OK. OK. Divide. OK.

From here, the students no longer needed my "help," so mercifully, I remained quiet.

Adie: So, 1 divided by 2; 3 divided by 2....(*laughs*)

Juli: 3 divided by 2 is....

Adie: One point 5?

Juli: Hold on.

Chad: 3 divided by 2 is 1.5.

Juli: (*working on calculator, thinking out loud*) OK. One point 5. One point....no. Five. No. Oh.

Adie: You go 5 divided by 3 now.

Chad: (*loudly*) One point 6! I got it. Look at that!

It was gratifying to see the excitement generated from a discovery, especially considering how off-task the students were just moments ago. Also, this was the second "Eureka" phenomenon of the session. And blessedly, my own voice did not figure in that great moment.

The students set to work, and using the nineteenth and twentieth Fibonacci numbers, came up with an approximation for the golden ratio correct to 9 decimal places. Donny, who unfortunately was working alone, apart from the group, was the first to arrive at such an accurate estimation. And although I would have liked more contribution from him and Alim, I was

satisfied with today's session. We had dealt with three different complex problems related to the golden ratio, and with the exception of a brief span of inattention towards the end of the class, the students had remained remarkably on task. Furthermore, they had just spent one hour and fifteen minutes together in almost complete harmony and cooperation. And although I knew that the major antagonists were not together during this session, I remained optimistic that the last five sessions of the project would be as fruitful.

SESSION 28   Tuesday 17 May 1994   (Adie, Dale, Chad, Nina)
"...the negative number...didn't work with the geometry we were doing."

This session would be the last dedicated to Assignment 8, and to the golden ratio topic. The school day started off with tremendous disappointment. At the beginning of the afternoon class, Nina was the only student present. By 1:00 PM, my discouragement had turned into despair: none of the "regulars" were there. I had thoughts like, "After twenty-seven sessions, this is the end of the program, because of the no-shows." I asked myself, "Isn't my math program enough of an attraction to get them here?"

But then, my hopes elevated slightly when, at 1:05, Adie appeared, fifty minutes late. Finally, Chad (1:55) and Dale (1:58) drifted in, twenty minutes late for the mathematics session itself. In a regular school, such habitual tardiness would have landed most students in trouble. Here, however, in the alternate school, I was able to modify the expectations considerably, thereby providing the "goodness of fit" between school and temperament described in Chapter 3. I was hard pressed to express anger at three students who showed up to do some mathematics.

Reconsidering the continued fraction question from the last session, Chad was able to recollect and verbalize the pattern discovered: "That's all we're doing: one plus a fraction, invert it, and add one. Invert it and add one. Invert it and add 1." However, the group decided that the responsibility of the written solution for question 3 should be left to Juliana (again in absentia). Accordingly, Chad completed the following answer to question 4 by the end of the session:

Question 4. Does the equation $x^2 - x - 1 = 0$ have more than one solution? In other words, could there be 2 different values for the golden ratio x? Hint: To find out, sketch the graph of the equation $y = x^2 - x - 1$. In your report, explain how the graph enabled you to come up with an answer.

Chad's Answer:

In an attempt to figure out the Golden Ratio on a graph we used the equation $y = x^2 - x - 1$ We gave x values of 5 to –4 and plotted the results onto a graph. We had two results that landed on 0 of the y-axis. One was a negative number which wasn't what we were looking for. The other number was 1.61. We verifyed that by making a graph with the same parabola on the computer. We entered in the values of 1, –1, and –1 into the equation and it gave us the curve. We could not use the negative number as it didn't work with the geometry we were doing.

Chad's solution was excellent for two reasons. It was well-articulated and grammatically correct, and his mathematical language, although unconventional, was accurate in its own right. For instance, phrases such as "we had two results that landed on 0 of the y-axis," and "it didn't work with the geometry we were doing" were completely original utterances which nonetheless made mathematical sense. Chad's submission was given five points out of five towards the total mark for Assignment 8.

Although his writing was less articulate than that of Chad, Dale's account of the solution for questions 1 and 2 was just as mathematically accurate.

Question 1 Use your calculator and the method of guess and check to solve the equation $x^2 - x - 1 = 0$. Question 2 Write down the value of the golden ratio to three decimal places.

Dale's answer. We started of trying to solve the equasion $x(x - 1) = 1$ so we got the calculator out and tried to figure out the equasions by trial and error. We started of by $x = 1.61$ and that was way off. The closest we came of the calculator was $x = 1.61805$ witch = to 1.000635 to the place value of ten tousanth. We didn't go any further when we finished we were satisfied with our efforts so we stoped but you can go further.

It took me a few minutes to discern what Dale was referring to with the number 1.000635. This was the closest that the group came on the calculator to the right hand side of the equation $x(x - 1) = 1$. Clearly, Dale's language and spelling are not untypical of what one would expect from a student with LD. However, his high mathematical acumen and attentiveness to the problems at hand continued to astound me. Dale's effort earned four marks out of five for question 1. (Question 2 was not given a mark.)

This time, Adie realized that I wanted more than just an answer to the problem. At the start of the session, she asked Chad, "Is this how we're supposed to do it? Just write it out like this?" Chad gave her a good answer: "I guess so. Just brain storm. Just write everything." Adie's consultation paid off, for she turned in the following description of the group's efforts in trying to solve problem 5.

Question 5. The golden ratio is hidden in the famous Fibonacci Sequence. {1,1,2,3,5,8,13,21,...}. Try do discover how. For full marks, use several heuristics. In your write-up, comment on how well each heuristic worked for your group.

Adie's answer. 1) We went out to the 20th number by adding the number before a number to get the next number.
2) We found the pattern in the Fibonacci Sequence and it was found by adding, like I said in no.#1.
3) Then we took a few guesses and came up with dividing the bigger no# by the no# right before it and we are still trying to see how long it will take to get to the most acturate golden ratio.
4) We got $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3} = 1.6$ and $\frac{6765}{4181} = 1.6228175$
We used heuristics no# 1, 2, 3, 6, 7, 9.

divide	Answer:
$1 \div 1$	1
$2 \div 1$	2
$3 \div 2$	1.5
$5 \div 3$	1.6667
$8 \div 5$	1.6
$13 \div 8$	1.625
$21 \div 13$	1.6153846
$34 \div 21$	1.6190476
$\frac{317811}{196418}$	1.618033989

Even though Adie failed to comment on the heuristics which she listed so fleetingly, I was happy with her attempt to verbalize the group's efforts. Thus, so as not to discourage this behaviour in the future, I gave her a mark of five out of five for her contribution to Assignment 8.

Finally, Nina handed in a response to question 6 which asked for the results of the students' investigation of the golden ratio in human anatomy.

Question 6. Report on the measurements you made within your group. Include the names of each group member, along with the measurements made, and the computed ratios. Make a summary or conclusion of the results.

Nina's answer was merely a table listing anatomical ratios for three group members only. Her response was disappointing since there were neither conclusions nor summary. In view of her work ethic during the actual measurement taking, and considering her newness to the program, I gave her meagre effort a rather generous three marks out of five.

Finally, Juliana turned in a solution for question 3, the continued fraction. Absent for the Thursday session, she went to work at it on Friday afternoon. However, she was immediately blocked in her efforts to recreate the solution from Tuesday's session, and had to be reminded to start at the bottom of the expression. She even had difficulty with arithmetic tasks like $1 + 3/2$. Frustrated and angry, she said she wanted to go and work elsewhere. Finally she shouted with consternation, "I don't know how to do this!" Angrily, I replied, "You might as well go to the other room then!"

Juliana's emotions are typical of the malignant self-esteem/failure cycle discussed in Chapter 2 (p. 17). Feelings of inadequacy are often directed to the outside world, and I happened to be the closest person at whom to direct hostility. For my part, I immediately regretted losing my patience with Juliana, but I was weary that day of coping with students' lack of basic skills, and lack of motivation. Unfortunately, Juliana was the person who had to bear the brunt of one of my rare outbursts of anger.

The next time we met, we made amends, and I worked with her a while on the continued fraction problem. She then made the following excellent write-up of the problem. (See Figure 3.14 which is a photocopy of her rather neat layout of the solution.) She wrote:

Starting with the bottom of the question start dividing Q.1A then taking each fraction and dividing step by step. On Thursday I felt frustrated because I got it in the beginning and I forgot how to do it, but I managed now.

I could only hope that the next time I became angry with a student, it would not be caused by a lapse in his/her memory about an arithmetic algorithm.

Figure 3.14   274
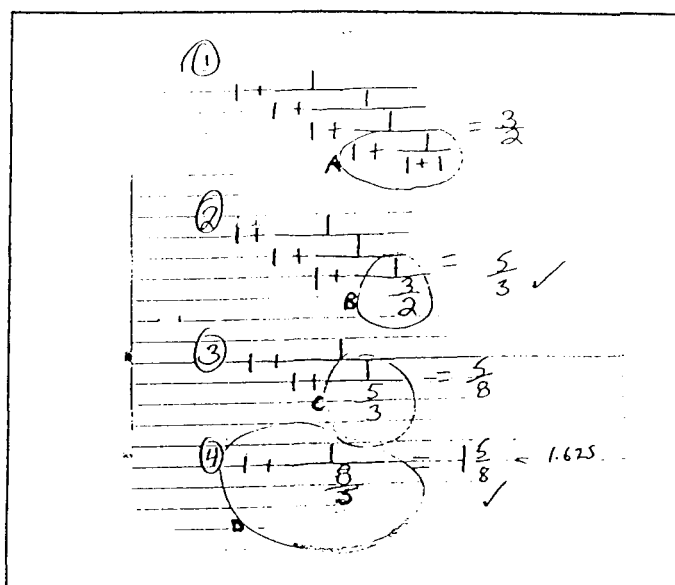
Since Juliana did not mention any of the group's efforts in solving the problem earlier, I gave her four marks out of five for her answer.

So in total, the cohort received 21 marks out of 25 for the whole of assignment 8. I feel that the experiment with group assignments was successful. First, I was reasonably confident that with the exception of Nina, each member of the class contributed fairly to the solutions of the questions. I observed that for the most part, the accounts provided by individual students matched closely enough the actual group problem-solving efforts. But with larger classes to observe, or with more groups to monitor, a teacher's attempt to implement group assignments might be considerably more difficult or even outright impractical.

For the last topic, I switched to solo assignments.since the graphic activities for fractals were intrinsically highly individualized. One student in particular would suffer the effects of such solitary activities.

Topic 5: Fractals

"It has the same pattern that everywhere else has."

From the onset, I was enthusiastic about studying fractals with the group. The topic was relatively new to me, and I looked forward to learning about the concept along with my students. Our investigations would also make good use of computer technology. Furthermore, fractals would be a good diversion for the students: this topic would represent our first real excursion into geometry since pentominoes. And importantly, I felt that fractals would be an "easy entry" topic: in other words, no real formal algebra or geometry was necessary to initiate a program of study in the area.

For materials, I used the excellent workbook by Peitgen, Jürgens, Saupe, Maletsky, Perciante and Yunker (1991) titled *Fractals for the Classroom: Strategic Activities Volume I.* Each unit is a series of well-planned, classroom-ready worksheets. By progressing through such a series, students should be able to discover for themselves various important concepts and properties of fractal geometry. Connections to other branches of mathematics are also stressed; the authors point out that in Unit I alone, connections are made to the following mathematical topics: congruency, ratios, numerical patterns, geometric patterns, geometric sequences, Pascal's triangle, similarity, area, visualization, modular arithmetic, coordinate systems, logic and truth tables, exponential functions, and the limit concept. Thus, the workbook can be described as "strategic" in that discovery learning and connections are its goals. Fortunately, permission is given to photocopy material from this publication for educational purposes. For the fractal topic, I am thus able to provide exact copies of my students' work.

We also used several fractal programs from public domain software, and we had recourse to fractal art gleaned from various textbooks and posters. As a bonus, the publications by Peitgen et al. (1991) came with a set of color slides for classroom use.

We covered the following topics: the Sierpinsky triangle, tree fractals, Pascal's triangle, the snowflake fractal, and the Mandelbrot set. For their first activity, the students set to work constructing a Sierpinsky triangle on isometric grid paper. (See Appendix 6 and Figure 3.15 below.) I felt that the activity would be a relatively simple one, since it essentially involved bisecting the sides of any upward pointing triangle to create three new triangles. The first sign that all was not well came in a comment from Dale: "Do you know what, David? I played connect the dots in school when I was in grade four." A while later, as the other three students progressed effortlessly in their constructions, Dale complained, "David, this is really a bummer. Especially when you're tired, you know." Finally, after his peers at already finished the construction down to the last bisection possible, Dale announced, "OK. I'm not going any further." He then put his paper away.

Dale's frustration in constructing the Sierpinsky triangle was consistent with his previous difficulty in drawing geometric patterns in the logic unit. My suspicions about another aspect of his learning disabilities were heightened; that is, he might have had problems in visual-motor coordination. Subsequent fine motor activities in the fractal unit would eventually confirm my diagnosis.

<u>Figure 3.15</u>
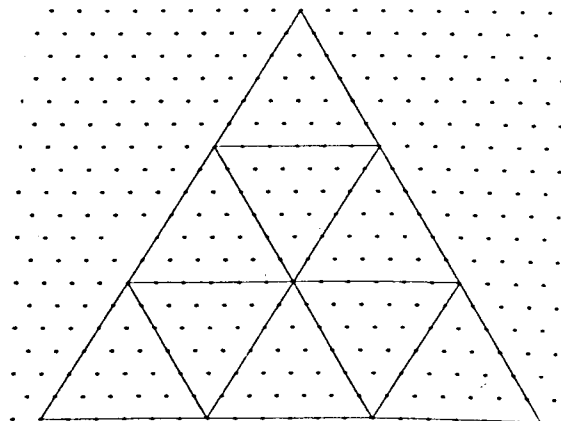
When the students had completed five divisions or "stages" of the triangle, I showed them a computer program which was capable of ten stages. My first question was regarding the limits of the computer.

DT: Look at how fast it's doing it. What's going to happen though eventually to the computer? Theoretically, we should be able to go to 11 or 12, right? What will eventually happen?

Dale: You won't be able to see nothing.

Juli: It'll go pure dark.

DT: You won't ...you won't even what?

Juli: See anything.

DT: Why? What's the limit of the computer? Or let's say, of what you're watching here?

Juli: Space?

Unfortunately, I missed the opportunity to ask the students what the effective limit of their own drawings was. And I later regretted having "told" the students one of the salient features of fractals; that is, they are generated much more quickly and efficiently by computers than by human hands. Such a property would have been easily discerned by any student using pencil and paper to construct fractals.

Next, I turned to the concept of self-similarity within the fractal.

DT: Take this top triangle right here. OK? What does it look like compared to the...to the big triangle?

Dale: A lot smaller?

DT: It's smaller, but it's...what?

Juli: The same.

DT: The same what, Juliana? You're right.

Juli: It's the same pattern that everywhere else has.

DT: Sure. Sure. We say it's similar. It has the same...it has the same shape and the same pattern.

Later, I regretted providing the word "similar" for the students: with a few more questions, I was sure that they could have come up with the term: ironically, these were answers which they were just on the verge of discovering themselves.

Finally, I led the students in an investigation of patterns in the various stages of construction of the Sierpinski triangle. The following series of questions dealt with the number of triangles present after each stage:

Figure 3.16

| Stage 0 | Stage 1 | Stage 2 | Stage 3 | Stage 4 |
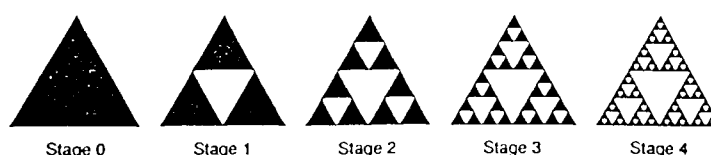
NUMBER OF TRIANGLES

1. Count the number of shaded triangles at each stage 0 through 4.

STAGE	0	1	2	3	4	5	...	n
NUMBER	1							

2. Extend the pattern to predict the number of triangles at stage 5.
   What constant multiplier can be used to go from one stage to the next?

3. Generalize to find the number of triangles for level n.
   As n becomes large without bound, what happens to the number of triangles?

(Peitgen et al., 1991, p.13)

Dale was quick to pick up on the pattern. In effect, he answered my next question before I even asked it.

DT: So stage 1. (*Stage 0 has one.*) How many shaded triangles?
Dale: You times it...you times it by 3 all the time.
DT: OK 3. And stage 2, Dale?
Dale: Say again?
Juli: Nine

DT: Nine, yeh.

Dale: 27.

DT: 27.

Dale: What's 27 times 3? 27 times 3? (*unintelligible*)

Don: 81.

DT: What Don?

Don: 81

DT: 81.

Dale: Yeh, yeh. I had 27 times 3 is 81.

DT: 81, OK? And the next one?

Dale: 81 times 3 is....

Don: 163.

DT: No.

Don: 243.

It was good to see Donny responding verbally for a change. Since he seemed to respond to direct questioning, I would have to call upon him by name more often.

When asked to generalize for n stages, Dale made a useful connection to a past activity.

DT: Now question 2. What constant multiplier can be used to go from one stage to the next then?

Dale: Three times.

DT: Three! Multiplier of 3, OK? Generalize now to find the number of triangles for level n. So, the pattern goes 1, 3, 9,....

Dale: We basically do what we did with the uh...fractal triangle.

DT: Not fractal triangle. Can you remember what it was called?

Dale: Fractal.

DT: Dale?

Dale: That's what I said.

DT: Not fractal.

Dale: Fractal. No?

DT: The other triangle we dealt with?

Dale: ...was the fractal triangle.

DT: Hey? *(writing P -A - .on board)*

Dale: Fractal triangle. P... Let me see! Move! Not a fractal triangle! Oh fuck.this! *(unintelligible)*. *(Throws his pencil down.)*

Juli: Pascal?

Dale: Pascal. That's it.

DT: Yeh, that's right. It's the same idea.

Dale was astute in comparing the group's searches for algebraic generalizations in the "fractal" triangle and "Pascal's" triangle. However, his difficulty in verbal expression got the best of him again. His behaviour was understandable: already frustrated with an intricate geometric construction, he did not need the additional problem of verbal confusion. In retrospect, I should not have kept prompting him to come up with the correct word. A simple "You mean Pascal, don't you?" from me at the beginning would have avoided all the negative emotions he experienced trying to verbalize the appropriate term.

With minutes to go in the session, I decided to dismiss the class for the day. I felt that a break and a fresh start would prove beneficial in their consideration of an algebraic formula for the nth stage of the Sierpinski gasket. Besides, Chad would be in attendance next Tuesday, and I looked forward to challenging him to find more algebraic generalization formulae.

The day previous, Chad had announced to me that he had to work and would miss session 29. Thus, I had the idea that presenting the Sierpinsky triangle to him early would accomplish two aims: it would serve as a "practice run" for me on a topic I had never taught before, and it would keep Chad up to date on a topic with which I hoped he would lead the way.

He worked well at questions 1 through 3. Although he had some difficulty coming up with 3^n as the generalization for the number of shaded triangles after n stages, he was able to predict what happens to the number of triangles as n becomes "large without bound." He also needed coaching to arrive at a suitable generalization for the total area of shaded triangles at stage

n. I thought that if Chad needed help here, then certainly the rest of the group would have to be guided carefully as we considered questions 4 through 6 of the Sierpinski triangle assignment.

SESSION 30   Tuesday 24 May 1994   (Adie, Donny, Chad, Juliana)
"It's the same as our triangles."

The first task facing the students was as follows:

Figure 3.17

Stage 0   Stage 1   Stage 2   Stage 3   Stage 4
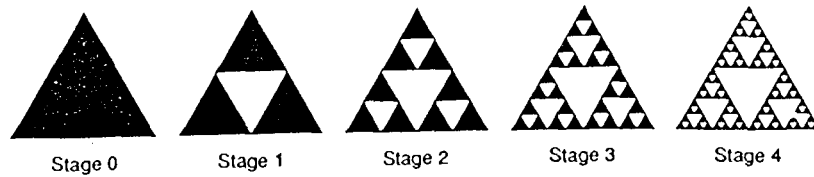
AREA OF TRIANGLES

4. Let the area at stage 0 be 1.   Find the total shaded areas at stages 1 through 4.

STAGE	0	1	2	3	4	5	...	n
AREA	1							

5. Extend the pattern to predict the total area at stage 5.
   What constant multiplier can be used to go from one stage to the next?

6. Generalize to find the total area at stage *n*.
   As *n* becomes large without bound, what happens to the shaded area?

(Peitgen et al., 1991, p.13)

Since Chad had been through the exercise with me, he agreed to keep relatively quiet while the other students considered the questions. Observing that one fourth of the large triangle had been "taken away", leaving a white area in the middle, Juliana and Adie stated that 3/4 of the original triangle was left shaded at stage one. When the girls experienced some difficulty in counting the total number of small triangles at stage 2, Donny came to their rescue, and the total shaded areas at this stage was found to be 9/16. At this point, I commented on how tedious it would be to total the small triangles at stage 3. Finally, Adie came up with a fast way of counting:

Adie: There's 16 here, and 16 here, and 16 here.

DT: Yeh. How many 16's are there?

Adie: Four 16's

Juli: Oh, four. Right!

DT: And total?

Adie: 64.

DT: Alright

Juli: That's right.

DT: And shaded?

Adie: 27.

Finally, using the same congruence properties, Don and Juliana came up with the answer for stage 4, namely $\frac{81}{256}$. They appreciated that it would be easier to make a conjecture to predict the solution for stage 5. Don was the first to answer correctly, and he responded in his typical mono-syllable mode:

Don: 1024.

Juli: How'd you get that?

Don: Well, 256 times 4.

Adie: Times 4?

Don: OK Well, 64 times 4 is 256.

DT: 16 times 4...?

Adie: 64.

Juli: 4 times 4 is 16.

DT: OK So what....?

Adie: What'd you get? A thousand what?

Don: 24.

DT: What about the top one?

Don: Well it's 243.

DT: How'd you get that, Don?

Adie: How'd you get that? Yeh.

Don: From here. *(i.e. question 1)*

Juli: From where?

Adie: Oh yeh. 245. *(sic)* It's written out at stage 5. OK It's the right answer, eh? 245?

DT: Yeh but how did you go from 81 to this one here?

Don: Times by 3.

DT: By 3. See the tops? (*i.e. the numberators*)

Adie: Oh yeh. .

DT: So the top has to be multiplied by 3. That's 243...So what would stage 6 be?

Adie: Um, 120...1024 times 4? That'll do it. (*unintelligible*) calculator. And 243 times 3....

Don: 729 over that.

It was heartening to see Don become involved with his peers, and actually lead the way towards a solution. During Chad's agreed-to hiatus, it seemed Don took over the leadership role in finding patterns.

Now it was time to make a generalization for stage *n*. Donny continued to carry the ball:

DT: So question 5 says, "What constant multiplier can be used to go from one stage to the next?" What are you multiplying by? On each stage?

Don: 4 .

DT: 4 on the bottom.

Don: And 3 on the top.

DT: What is the constant multiplier that you use to go from one to the other?

Juli Mm....3 fourths.

After a brief review of exponents, Adie was able to come up with the generalization: at stage *n*, the shaded area would be $\left(\frac{3}{4}\right)^n$. As well, she was able to accurately describe the area as *n* increased in size.

DT: What happens to the area now?

Adie: It's smaller and smaller.

DT: Generalize to find the total area at stage n. There's the area at stage n. At stage 100 the area would be....? How would you write that? Stage 100?

Adie: To the 100.

DT: Three quarters to the....?

Adie: 100th power.

DT: What happens to the shaded area? The shaded areas become....?

Chad: Smaller.

Adie: Tinier. More triangles.

DT: The shaded area gets....?

Adie: Smaller.

DT: Smaller. It approaches what?

Chad: 0.

Here were my Mathematics 10 modified students actually working with the Mathematics 12 concept of limits. They had indeed come a long way from their individualized arithmetic programs of fractions, decimals and percent.

While discussing limits, Chad asked the interesting question: "How can we prove that infinity actually exists?" I explained that in mathematics, we cannot let infinity equal any quantity per se, but we can talk about what happens as something approaches infinity. I thought this was a good point to review the concept of infinity in relation to the Sierpinski fractal:

DT: What are you restricted by in these divisions that you're doing there?

Chad: Pencils.

Juli: Space.

DT: Pencils and paper. You're restricted by the physical thickness of your pencil. And the fact that...how much longer could you shade like Adie's doing, right? (*Having missed the last session, Adie was still shading her Sierpinski triangle*) What is the computer restricted to in this?

Juli: Umm....

DT: I mean this has gone down pretty small.

Chad: Pixels.

DT: What's a pixel, Chad?

Chad: It's a point on the screen.

DT: That's right. So the computer's restricted too, by the tiny dot which is as far down as the computer will go. OK? In your mind though, what are you restricted to, as far as these divisions go?

Chad: Umm....

DT: What's the restriction?

Chad: Well you're not.

DT: You're not at all?

Chad: No you're not.

DT: So can you conceive of this process going on forever?

Chad: Yes.

DT: How small a triangle do you want?

Adie: Not too small. (*laughs*)

DT: Not too small. But you could get a triangle as small as you....?

Don: Want.

DT: Don said it. As small as you want. And that's the idea of infinity.

At the time, I was happy with each student's active participation in our deliberations. The students were obviously no longer the "inactive learners" as described in Chapter 3 (p. 59).

In hindsight, however, I had again established myself at the centre of our investigation. In the above exchange, "Don said it" alright, but only after I had put the words I wanted to hear into his mouth. With this Socratic mode of questioning, I was leading the students again. Not satisfied with Chad's comment "Well you're not", I gave him no opportunity to elaborate on it. So instead of listening to Chad's interpretation of infinity, I chose to ask more questions in the hope of eliciting the phrase "as small as you want."

Unfortunately, I continued in my teacher-centred role for the next activity. Each student proceeded to draw a fractal tree using an isometric dot grid. (See Figure 3.18.) Basically, they had no problems in the construction, and were able to answer easily the questions which followed:

DT: What are these lengths here, if this is 1? (*pointing to the initial vertical segment*)

Juli: Half.

DT: Yeh.

Don: Oh, a half.

DT: OK? And then, how many branches have a length of 1/4? This is the question here. How many branches have a length of 1/4?

Chad: Oh...4.

DT: Yeh. These ones here. So watch what I'm doing on the board here. I'm doing it in stages. So these ones here are 1/4 right? (*putting table on board*)

Juli: And then 1/8.

DT: Right.

Juli: And then a sixteenth!

Figure 3.18
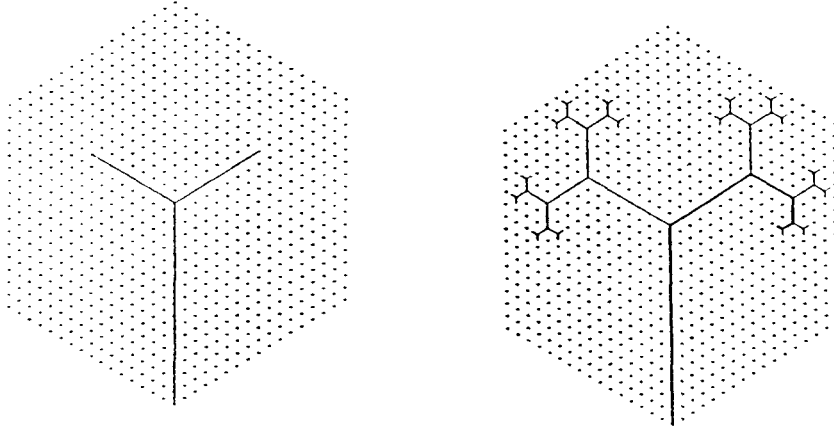
Construction   From the endpoint of each branch, draw two new branches
half as long growing off at 60° in opposite direction.

Suppose the tree starts with an initial vertical segment of 1 unit as the trunk. Imagine further that the tree continues growing branches, over and over by the process given, until fully grown. Visualize this completed tree.

How many branches have lengths of 1/4? of 1/16? What is the sum of the lengths of all branches 1/4 long? 1/16 long?

What is the total length of all branches of the completed tree?

One interesting shape found on the completed tree is a *spiral*. Start at the base of the tree and turn right at each and every junction point. Note how these particular branches trace out a spiral.

(Peitgen et al., 1991, pp. 19-20)

DT: So, we have the length and the number of branches. (*referring to table on board*) The first branch is a length of 1. There's 2 branches of length 1/2, 4 branches of length 1/4, right? How many branches have a length of 1/8?

Adie: 8

DT: Right Adie. So its 8, OK? Sixteen?

Adie: 16.

DT: What's the total length of all the branches of the completed tree now?

Chad: Oh....oh! That's easy then. If we went down to what...1/16th?

DT: I think you did, didn't you? We went 1, 2, 3, 4, 5....(*counting branches*) 1/32.

Chad: Well then there'd be 1, 2, 3, 4, 5. There'd be 5. That'd be the length.

DT: (*Writing on board chart*) What would the total length be if you did this 100 times?

Chad and Adie: 100.

By this point, the students had ample experience in constructing tables for inductive reasoning. Rather than trust their abilities to become aware of a pattern, I drew a table for them with the comment, "Watch what I'm doing on the board here." Although the students all contributed to the process with spirit, the heuristic they used came from their teacher.

The students then traced spirals on their completed trees. Without any prodding from me, Chad made the interesting connection: "That makes a spiral like the one on the Golden Rectangle." Whether his conjecture was exactly true or not, I did not know at the time. But I recall thinking, "I would not be surprised if they were indeed the same." I also recall being caught up in the spirit of discovery with the rest of the group: this was mathematics at its best.

Finally, we went on to consider the connections between fractals and Pascal's triangle.

Figure 3.19

```
Row
 0                               1
 1                             1   1
 2                           1   2   1
 3                         1   3   3   1
 4                       1   4   6   4   1
 5                     1   5  10  10   5   1
 6                   1   6  15  20  15   6   1
 7                 1   7  21  35  35  21   7   1
 8               1   8  28  56  70  56  28   8   1
 9             1   9  36  84 126 126  84  36   9   1
10           1  10  45 120 210 252 210 120  45  10   1
11
12
13
14
15
```

(Peitgen et al., 1991, p. 29)

The students were well able to recollect the basic features the triangle:

DT: Remember how it's generated?

Don: Yeh.

DT: How'd you get the 15 in the 6th row? Remember? What did you do to get the (*entry*) 15?

Juli: You add the 5 and the 10.

DT: Great. OK How many numbers are there in row 8?

Don: 9.

DT: How many in row 9?

Don: 10.

DT: In row 10?

Don: 11.

DT: How many will be in row *n*?

Don: Row *n*?

Juli: *n* .

Don: *n* plus 1.

DT: *n* plus 1 right. OK (*reading directions*) "Enter the numbers needed for rows 11 and 12." Just put rows 11 and 12 in there.

Don: OK.

Chad: I can't remember how to do it.

DT: Now remember back. Do you have to do every single addition there?

Adie: No.

DT: Why not?

Adie: Because it doubles.

DT: Right.

Chad: Oh yeh, I remember how to do this now.

DT: Remember that image you had Adie? What did you say? Something about it....?

Adie: Folding in half.

I was pleased that the students remembered the patterns which they found earlier. However, I should not have been disappointed if their memories had failed them here: after all, a teacher's purpose is to promote not memorization of mathematical facts, but rather awareness of the ways mathematical patterns can be discovered.

Adie remembered her imagery from the arithmetic series, and was able to see its usefulness in another context. This was the second beautiful "connection" of the day!

Now a third connection was about to be established: that is, the one between Pascal's triangle and fractals. Instead of completing rows 13, 14, and 15 of the triangle, the students were directed to simply enter the letters E for an even entry, and O for an odd one. They quickly recollected the rule for adding odds and evens and before long, the triangle was completed to row 15.

In the next activity the students were asked to fill in Pascal's triangle with the numbers 1 or 0 depending on whether the corresponding entries were originally odd or even respectively. (See Figure 3.20.) This step created a bit of confusion, since in the previous exercise, the students had entered the letter O for odd, and now they were being asked to enter the number 0 for even. As Chad remarked, "This is confusing, because I was just writing O's for odd, and now they're even." (In the future, I would simply skip the step with the letters.)

Figure 3.20
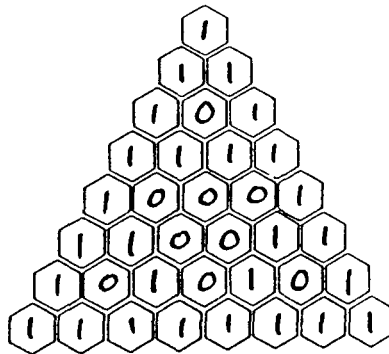
(Peitgen et al., 1991, p.30)

At this point, the directions were easy enough. "Shade in the entries with 1's (odds) and leave unshaded the entries with 0's (evens)." Soon, the final connection was made. (See Figure 3.21)

Figure 3.21
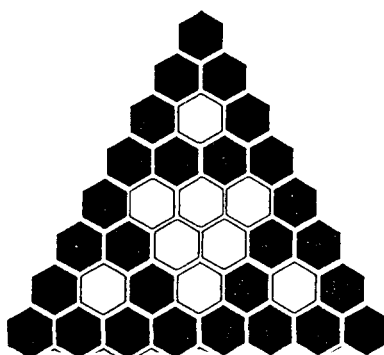
(Peitgen et al., 1991, p. 30)

Chad: Oh I see what's happening. I see!

Chad: Ah ha!!

DT: Ah ha!

Adie: Yep.

Chad: I can't say I'm surprised. (*laughs*)

DT What do you notice?

Chad: It's the same as our triangles!

Adie: Yeh.

DT: What triangle?

Chad: The ones we were just doing

Juli: These! (*pointing to her Sierpinski triangle*)

DT: Sierpinski and Pascal must have known each other, right?

Adie: Yeh, they must have.

These were indeed special moments when powerful mathematical connections were made and appreciated.

Finally, the students were asked to "give a rule for coloring each cell based upon the coloring of the two cells immediately above it." Chad was able to generalize for the group:

Chad: Um, well, if they're both black...we already went over that.

DT: OK. Go ahead. Say that.

Chad: They're...if they're the same....

DT: Yeh?

Chad: If they're the same, then it's white. And if they're not...then it's black.

DT: If they're the same....?

Chad: Well if they're both black, or both white, then the one underneath it's going to be white. If one of them is white, and one of them is black, it's going to be black.

DT: OK When do you color a cell? Only if....?

Adie: When the two on top are the same color.

Chad: I think when they're different.

Don: If the numbers are even or odd....

Adie: Color the cell if the two at the top are plain. Whatever. Filled in.

DT: Look at your diagram again there.

Adie: Aghh!

DT: You're close.

Chad: If they're black, and white.....

DT: Chad just said it right there, but he....

Chad: I said it three or four times.

DT: But you used a lot of words. Try and condense that. When do you color a cell in?

Chad: If the two cells above it are of alternating colors.

DT: Say it...instead of saying alternating colors, you can say....?

Chad: Different!

Chad was correct in exclaiming "I said it three or four times!" Nonetheless, when things were rolling along smoothly as they were here, I felt that the students could be pressed to be extremely accurate in their verbalizations. Also, I welcomed the opportunity to have a student express the answer in as many different ways as possible. In this way, the chances would be good that the speaker's peers would comprehend at least one version of the solution.

I then handed out a sheet containing the rule for coloring the cells of Pascal's triangle, and a practice triangle. The students had just enough time to complete the 16-row triangle. (See Figure 3.22) Given the high spirits in which they left the class that afternoon, and given their success with the Pascal connection so far, I was confident that in the next session, they would easily complete the Sierpinsky array in a much larger triangle.

Figure 3.22
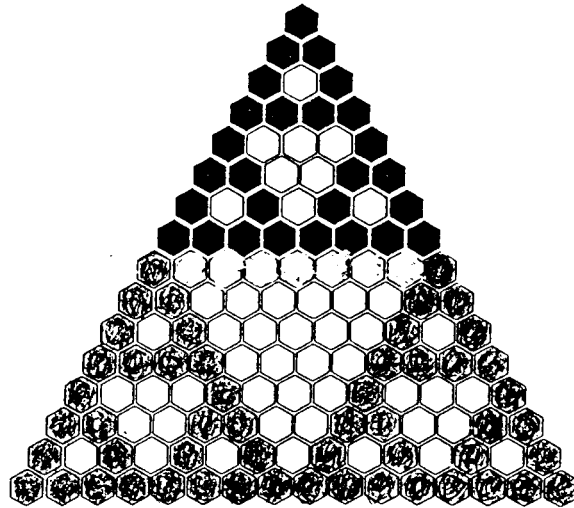
(Pietgen et al., 1991, p. 31)

SESSION 31 Thursday 26 May 1994 (Adie, Dale, Chad, Don)
"Oh God! I can see it forming!"

For this session, I had the students work on two different fractals: the Sierpinski/Pascal triangle from last day, and a snowflake curve. While they worked, I demonstrated various magnifications of the Mandelbrot set. This arrangement worked well, since the computer program took a few minutes to complete each stage.

After the first magnification of the Mandelbrot "Seahorse Valley", I wanted to discuss the appearance of one of the infinite number of miniature islands, all similar to the main set. The students, however, noticed other aspects of the enlarged section. (See Figure 3.23 .)

Dale: That's wild. It looks like different colors of a snowflake eh?
DT: I think it's close to finished.
Adie: Yeh.
DT: Remember one thing: what the original looked like. What do you notice here?
(pointing to a midget island)
Dale: Same.
DT: And a little one here.

Figure 3.23
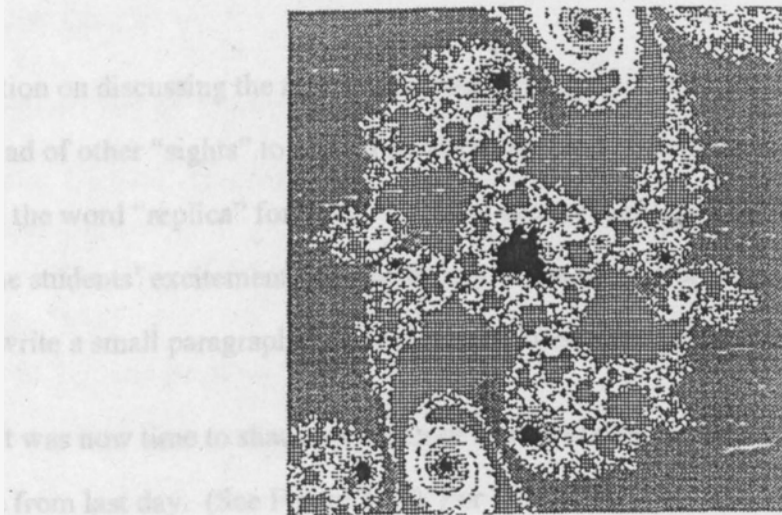
Dale: Looks like little inlets.

DT: But what about in terms of the shape of the original?

Adie: That looks like....

DT (*Interrupts*) We just magnified a little part of the coastline.

Dale: What that looks like now....it looks like the sun if it was a complete circle. It would look like the sun, you know if the sun has jagged tips on it?

DT: Like flares? But see this little part in here?

Adie: Um hum.

DT: What does that remind you of when you look at the original one you started with? See this black part?

Adie; Yeh.

DT: It's almost a replica.

Adie: Yeh.

DT: Right? There's another little replica. Maybe that black part there we might want to look at later.

Dale: It's almost a replica of the whole thing.

DT Exactly.

Dale: Ohh!

DT: It's a self-replica

Dale: I see now...those 3 points.

DT: OK?

Dale: Holy shit! That's cool!

My fixation on discussing the mini-island had two results. I stifled student verbalizations about the myriad of other "sights" to behold on the screen and instead of waiting for a group response, I supplied the word "replica" for them. Looking back, I realize what a great opportunity I missed. Given the students' excitement over the beauty of the Mandelbrot set, I should have asked them each to write a small paragraph describing the richness they beheld.

It was now time to shade a version of Pascal's triangle which was much larger than the example from last day. (See Figure 3.24) For Don, Adie and Chad, the activity was largely a pleasant one and it was gratifying to observe their reactions when the Sierpinski pattern began to emerge in Pascal's triangle.

Figure 3.24
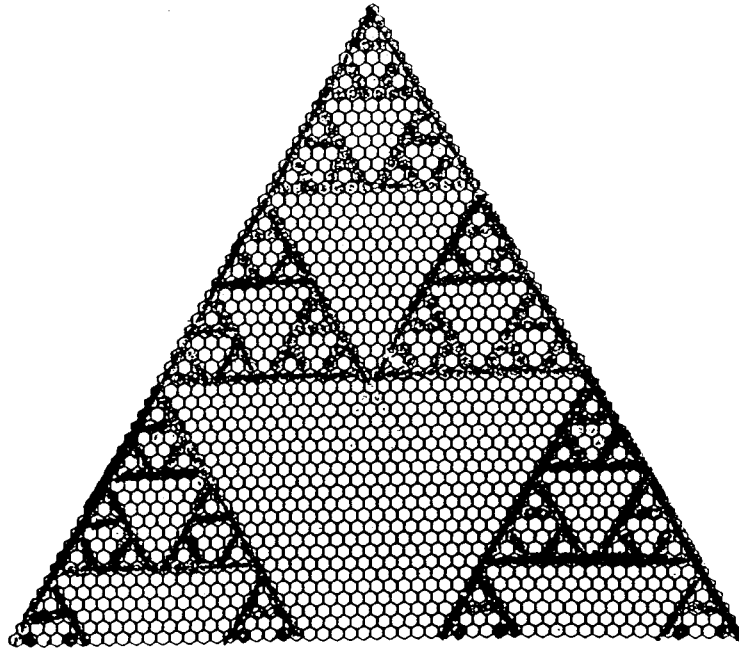
Adie: Yeh! (*laughs*) Kind of trippy!

Don: You see triangles inside.

Adie: Yeh, you see triangles inside.

Don: Is that what it is?

DT: Well, it's like the one you did the other day, except it's just taking the triangle down.

Don: Yeh.

Adie: Ohh!

A bit later, the triangle began to take on its grand form:

Adie: Oh God! (*pause*) I can see it forming!

Dale: Me too.

Don: This...it's like that...a little triangle in each one have a whole in it (*sic*)

Adie: Yeh!

Don: It could go down forever.

In his own words, Don was describing the concepts of self-similarity and infinity. And with her exclamations, Adie was expressing the joy of mathematical discovery.

Dale, however, was not amused with the task at hand. The following is a chronological list of his comments as he attempted to shade in the appropriate dots of the triangle.

- I don't ever want to see that triangle again.
- I don't want to do these David, it's stupid.
- Do you know what this reminds me of? Like when you were in grade 6 and 7? How the last week of school, you got to color and that. And now we're at the last couple days of this math course, and we get these stupid games. I'm nineteen, David. I don't have to be filling in dots.
- This makes your eyes go all wacky.
- Man, this is going to be stupid.
- Bogus.
- My eyes are dead. I'm seeing dots.
- Can you shut the lights off? It'll probably be easier.
- Can you shut this one off for me up here?
- David, my eyes man!
- That's shit!
- My eyes are like turning into dots.

Dale's difficulty in performing the task resulted in several types of comments. For instance, he questioned the validity and importance of the activity itself. As well, he described exactly what the activity is like for him visually. Finally, there were expressions of the sheer frustration he experienced as the result of his perceptual/motor disability.

Unfortunately, there was no sympathy forthcoming from any of his peers:

Dale: I fucked up.

Adie: Who cares. Just do it. Why don't you stop complaining?

(*Dale rips up his work and throws it away.*)

Adie: I saw you screwed up! (*laughs*)

Dale: I'm too bloody tired to be f..fooling around with...dots!

Adie: What are you doing here if you're tired?

Dale: Is that all we have to do today?

This last comment was so untypical of Dale, since in the past, he always strived for closure on his assignments. Clearly, this sort of activity was not suited for all students. Provisions must be made in geometry curricula to allow for the shortcomings of students with learning disabilities. All I could hope was that Dale would fare better with the assistance of a computer and Geometer's Sketchpad.

At the end of the day, I reflected how variable the experiences had been in this pilot project. From the depths of academic dismay, to the lofty experiences of discovery; from almost cruel indifference to sublime cooperation, this class ran the gamut, not only this day, but quite often since the beginning of the study. For me, many of these high and low points centred around the themes of attendance, student discord, cooperative versus individual learning, and my own teaching style.

There were many instances when I thought poor individual student attendance would be the demise of the project. In the long run, however, the group did not let me down. Although the school eventually suspended Nina, Minna, and Shawn because of unexcused absenteeism, these students (along with Alim and Alonzo) were present enough of the time to provide a quorum whenever any of the five "regulars" were away. In other words, because of the group, our sessions never skipped a beat and student truancy did not pose the threat to the study which I feared. The curriculum was thus able to provide a certain "goodness of fit" between the mathematics classes and the students' tendency to irregular attendance.

Inter-student strife was another oft described area of concern. Anxious to promote harmony and cooperation, I initially took it upon myself to admonish students whenever fighting or other outbursts occurred. Before too long, I began to appreciate Chad's cool and well-put chastisements of insults and anti-social behaviours. Thus the most effective means of achieving peace in the mathematics class was probably positive peer pressure.

In spite of my efforts to encourage cooperative learning, it was at first difficult to awaken the students from the passive learning mode to which they had become accustomed. With the exception of several instances where students helped classmates who had missed school, they seemed to prefer working alone. Gradually, given more and more opportunities for group problem-solving, the students began to enjoy the benefits of working together. Old habits are often difficult to change, and in the case of individual versus group learning, it was only a matter of time before they gave up the former modality to embrace the latter.

As it turned out, there were some old habits which I had difficulty relinquishing as well. The silence (or lack thereof) of the teacher has been a constant theme running through this narrative. From the start I knew that I was asking leading questions, putting words into students' mouths, intervening on their deliberations, spoon-feeding answers, offering explicit explanations, interrupting, giving obvious clues, and in general talking too much. I was also aware that my teacher-centred approach stifled creative activity and student expression and promoted reliance on me rather than on the group. For a long stretch, during the logic unit in particular, I was able to maintain silence, and the students proved themselves quite adept at doing some great mathematics. Even towards the end, however, I tended to lapse back into a teacher-focussed Socratic style of questioning.

At the time, I rationalized that such teacher interventions were necessary lest student frustration and boredom set in and class momentum be lost. I feel in hindsight that such concerns were valid, especially given the nature of the students as delineated in Chapter 2. There was,

however, a far simpler reason for my pedagogic behaviour. Simply put, this was the way I had

taught mathematics for over twenty years and old habits such as these die hard. Part of this

teaching style involved a tendency to impose my mathematical agenda on my students. For

instance, I knew of certain relationships in Pascal's triangle, and my plan was that my students

should become aware of these very same relationships. The point is, there are many discoverable

patterns and connections inherent in all the topics we covered here, so there was no need to insist

that the ones I had in mind should form our agenda.

My problem was not "teacher talk" per se, but rather an excess of it. To be sure, a certain

amount of direct instruction is part of the teacher's mandate:

> What we present to students of any age is what they cannot possibly find within
> themselves; these are the specific labels and notations whose exact forms are neither
> necessary nor held in common in all parts of the world....Therefore it is the teacher's
> job to introduce to students the technical words of mathematics, words, like
> "subtraction," "logarithm," "isomorphism," and the various notations, and make
> them clear through examples or definitions. (Gattegno, (1974), p. 81)

In promoting metacognition in mathematics, it is also desireable for a teacher to demonstrate to the

students not only his/her own awarenesses, but also how he/she became aware. But if a teacher is

to truly educate their awareness, he/she must allow the students ample opportunity to explore, to

observe, and to conjecture on their own. From the many wonderful instances where such

discovery learning occurred at VRLF, I know first hand that if given the chance, alternate school

students will readily fill the vacuum resulting from teacher silence.

Topic 8:   The Geometer's Sketchpad

SESSION 32   Tuesday 31 May 1994   (Chad, Don)
"Can we go?"

Session 32, the final mathematics class of the project, was ironically quite anti-climactic.

For one thing, it was the most poorly attended session of all. Adie and Juliana phoned in sick,

and Dale asked to be excused because he was working. Nina and Alim had become chronic non-

attenders who were on the verge of suspension from the school. Minna and Shawn had both been suspended, and Alonzo was on extended work experience.

I spent the whole time guiding Chad and Don through the basics of the Geometer's Sketchpad. Since communication between the two students was non-existent, the transcript of this session contains virtually nothing of note except teacher explanations. By the time I handed out Assignment 10, they were bored and were asking, "Can we go?"

The session ended quickly with an unanticipated early dismissal for a staff meeting. I knew that this was not really the absolute end: the students all owed extra time to the project because of days missed. So there would be a second chance to observe the students' reactions to the Geometer's Sketch Pad.

EXTRA SESSIONS.
"I think that was ingenious of you."

9 June 1994 (Don and Chad); 22 June (Adie and Juliana); 22 June (Dale)

In creating Assignment 10, my intention was twofold: first, I wanted the students to learn how to use Geometer's Sketchpad. Second, through judicious questioning, I hoped that they would also learn some rather traditional geometry, albeit in a non-traditional way. Question 8 read as follows:

> *Use the line tool to construct a segment, a ray, and a line. In your own words, explain the difference between a segment, a ray, and a line.*

As well as teaching themselves to perform the constructions on the computer, the students would hopefully discover the subtle distinctions between segment, ray and line. In view of my dominance of the last session with Chad and Donny, I vowed to leave the students alone as much as possible for this assignment.

Unfortunately, the session with Adie and Juliana was much like that earlier experience: since this was their first time with the Sketchpad, they sought (and received) considerable teacher

assistance. At one point, thinking they were under way, I left the classroom to see if they would proceed on their own. To my disappointment, they did little in my absence, and eventually finished to question three only. Before leaving the class, Adie agreed to arrange for a final session on the computer before the end of the school year. To my chagrin, however, I still had not heard from her when school ended on 30 June.

Bt this time, Dale had finished all his courses at VRLF, and was holding down a full-time job. So I appreciated that he agreed to an after-school session to finish up the last assignment of the mathematics course. Regretfully, he was by himself, so instead of inter-student communication on the transcript, there was considerable direct instruction from me. However, his responses to Question 10 were memorable. The students were asked to construct five separate circles, and to have the computer measure the radius, circumference and area of each. Using the built-in calculator, they were to compute the *circumference/diameter* and *area/square of radius* ratios for each circle. The following are Dale's two responses, the first of which was oral:

> My conjecture is that the circumference divided by the diameter always equals π.
> So A (*area*) divided by *r* times itself...*r* squared, equals pie."

Even though Dale spelled π as "pie", I was pleased that he was able to make such articulations as the result of an inductive approach to geometry on the computer. With the computer, he was able to persevere and to enjoy success in a task which, with compass and straightedge alone, his learning disabilities would probably have rendered impossible.

Given their previous experience with Geometer's Sketchpad, Chad and Donny were capable of considerably more independence. They soon established problem-solving roles: Chad operated the computer keyboard, and Donny served as scribe for the proceedings. The following was their response to question 8 just quoted above:

> Chad: OK Here's a segment. Which one's a ray, this one? That's a ray. There's a ray. And that'll be a line.

Don: What's the difference? (*reading*) "In your own words explain the difference between a segment, a ray and a line."

Chad: OK. A segment has two ends, a ray has one end, and a line has no ends.

Don: (*transcribes Chad's words*)

Now usually, the teacher would make the drawings, explain the concept, and write the definitions for the class to copy. But with this powerful geometry program, the students can probably discover most everything by themselves.

For Chad and Donny, the true moment of triumph came with the most difficult question of the assignment. Their instructions were simply to construct the form shown below in Figure 3.25, given that the big triangle is equilateral, with each side bisected. After several trials with circles and midpoints, they were still unable to construct the original equilateral triangle.

Figure 3.25
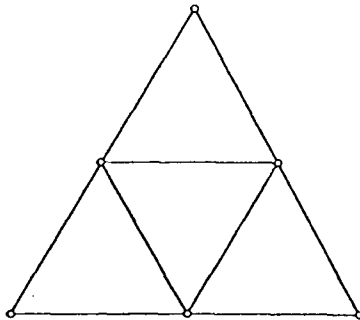
Chad was getting nowhere with the following construction:

Figure 3.26
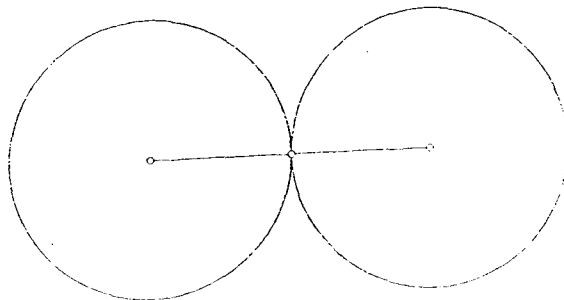
Finally, Donny hit upon the construction necessary:

> Donny: Make a circle go around.
> DT: What was your suggestion Don?
> Donny: Make a circle go around that. Instead of two, just.make one.
> Chad: Yeh, OK. Why don't we do that?

Donny's suggestion was simply to make a larger circle with radius equal to the initial segment, and centre on the end of the segment. In other, words, he was proposing the first step in the classic construction of an equilateral triangle. (See Figure 3.27.)

Figure 3.27
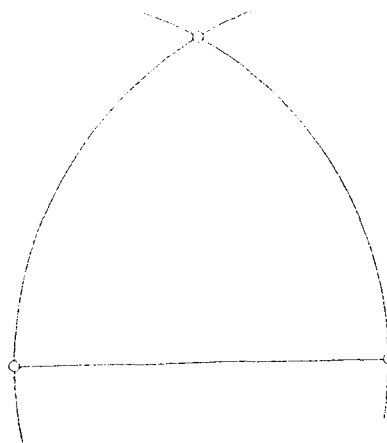
Chad eagerly attempted the construction with a flurry of positive metacognitive comments and a complement to Donny as well:

> Chad: OK. Ah, how would we do that? Oh, I think I know. There. Would that make a big circle? Oh wait! I think we're on the right track here. I think we are. I think that was ingenious of you. There. Now, it'll work! We've got to put a point there, don't we? Yep. So we have to rely on....
> DT: Pull the menu down. The construct menu.
> Chad: Point on an object?
> DT: Next one down?
> Chad: Huh? Point of intersection!
> DT: Right.

Chad: Oh! Ah ha!

DT: What do you need to do to get a point of intersection?

Chad: Donny, you're a genius.

DT: Just select...that's right!

Chad: Ah ha ha ha!

Reviewing the transcript of Chad and Donny's experience with the Geometer's Sketchpad, I realized that again, there were far too many teacher interjections into the proceedings. However this last time, I was so caught up in the general excitement, I could not help but participate.

This was Donny's and Chad's last session in the project and it was a memorable one. For this was a true "Ah ha!" moment achieved partly through individual ingenuity, but in the last analysis, mainly through cooperation between students. Indeed, the many memorable moments such as this last one had me already convinced to continue the program in the fall.

COLLEGIAL SESSION 27 June 1994 (Mel, Eric, Katey, Mavis, Maureen, and Edith)

Only a very small group of people in our building had any idea of what had transpired mathematically in our classroom: myself, and the cohort consisting of Dale, Donny, Chad, Adie and Juliana. Now, the personnel at VRLF remarked from time to time on certain obvious innovations to my classroom, such as the group scenario and the regularly scheduled mathematics sessions. My colleagues also noticed certain behavioural changes in individual students, such as Dale and Chad. However, with only my occasional rhetoric concerning open-ended problem solving and cooperative learning, the staff had virtually no knowledge of the project. So at this late point in the school year, I decided that the teachers and support staff should have at least one first-hand experience of the new curriculum.

My motivation in planning a sample session for them was based on more than purely informational motives. Every teacher at VRLF taught all five academic courses. So I realized that if this curriculum were to become a permanent fixture at this alternate school, I would have to sell it to the *entire* VRLF team. Each of my colleagues would have to be convinced that the old course

was not worth continuing. Each teacher would have to feel competent and comfortable enough in the discipline before agreeing to teach it. And each person would have to agree to the concept of discovery learning in a group context. In sum, I hoped that staff participation in an actual problem-solving session would promote my cause.

Invitations went out, and all three teachers (Maureen, Eric, and Katey) agreed to attend. And I was happy that Edith (secretary), Mavis (multi-cultural worker) and Melvin (youth care worker) accepted my invitation as well. Indeed, six people would make a group of ideal size. My plan was to give the adults a sample of what the students experienced; accordingly, I decided to revisit Pascal's triangle, and the probability topic covered earlier with the cohort. Realizing that the future of the curriculum was at stake, I planned carefully and thoroughly for the upcoming session.

The session began with one of my all-time favorite topics, namely, patterns in Pascal's Triangle. The group's challenge was to articulate the generating pattern of the triangle, and to evaluate the sum of all the entries in the 100th row. With the enthusiasm typical of teachers, they rose to the occasion in a way reminiscent of how my students dealt with the situation earlier.

However, my aim was not to compare learning styles, but rather to convince my colleagues of the pedagogical value of a new approach to the teaching and learning of mathematics. Their initial reactions to the process were favorable:

DT: Mathematics is simple if you can make it simple. (*referring to patterns as a heuristic*)
Katey: But also it can be fun. I've never had fun before.
Eric: Always see it as an IQ test.
Katey No, I'm just seeing it as fun, then I don't get uptight.
Mavis: It's not fun. It's math.
Katey: It's math, but it's more fun than usual.
Maureen: That's because it doesn't count for anything. If this counted for marks, I'd be...it'd no longer be fun.

> Katey: It's cooperative, because normally in a math class we were never allowed to talk. You just did your own thing. Because we can talk, it makes it much more interesting. Because I have this fascinating person on my right, (*Mel*) who helped me. And then listening to other person's comments....
>
> Mavis: We're all on your right too, you know, if you go around.
>
> DT: Well, let's get on to the main topic. Let's roll.
>
> Katey: That wasn't the main topic?
>
> DT: That was just the introduction.
>
> Katey: Oh my gosh!

Indeed, my colleagues' comments provided potential grist for future staff discussion. For instance, how do tests and marks impact on the traditional mathematics education of alternate school students? How might group learning be superior to the completely individualized programs as exist at VRLF? If the group continued to enjoy the experience, and to generate questions such as these, my goal would be accomplished.

After a basic review of probability using cards and dice, I presented the motivating question: "What is the probability that in a series of 10 births, there will be 4 or more girls born in a row?" Before presenting the binomial theory behind the problem, I asked my colleagues to model the situation with the dice and playing cards. Following this activity, they paired up to run computer simulations. After writing on the board the sibling birth orders of all our families, the group totalled all the birth permutations for families of 5, 4, 3, 2, and 1. The connection to Pascal's Triangle was established:

> DT: Do these numbers look familiar to you?
>
> Eric: It's the original pyramid.
>
> Mavis: Pascal's little thing. Is it Pascal?
>
> Maureen: Here we thought that triangle was just fun from the beginning!
>
> DT: It's all connected. You can't get away from it.

Hopefully the staff could now question how such beautiful connections between mathematical topics could ever be experienced by students engaged in rehashing basic arithmetic algorithms.

Finally, with only the denominator of the answer (1024), they discussed the open-ended situation.

> DT: So we end this session without an answer.
> Maureen: I don't mind math without an answer when you get into this kind of stuff.
> Mavis: Well, I'm slower on a lot of these things, but it definitely does make it interesting.
> Edith: I liked your approach. I liked how it evolves.

From their comments, and from their enthusiasm for the experience, I judged the session a success. So at this point, I was looking forward to discussing mathematics education with my colleagues during individual interviews which I set up for the next day.

My main concern was, "Would they agree that change was absolutely necessary?" Maureen's reaction was unequivocal:

> DT: What are the possibilities in the school of doing math like this all the time?
> Maureen: A lot better than doing those stupid pre tests. (*referring to the current ABE program*)
> DT: Because I foresee a need, not only in this, but also in social studies and English. And yet. I can't picture myself using the collegial or sort of communication model for me to be teaching Canadian history.
> Maureen: Was your whole pilot program done like this?
> DT: Yeh. This was one of the things we ran through.
> Maureen: Throw out the other program.

Maureen ended with a strong endorsement for a new program: "What I would say is, any math you can teach that way, you should."

As a non-mathematics specialist, Katey was concerned with the prospect of having to teach the subject using a new approach:

> DT: How would you feel about teaching math in this style yourself?
> Katey: Well, I think I'd have to learn a lot more about the topics, you know, to feel comfortable with it. But I like the style. And I think it's a style that applies to all subject areas: in the basics of cooperative learning, teacher led discovery, that kind of thing. So I

begin to see how it could be done. That way. It would just take more getting comfortable with some of the basics. Or more of the basics.

DT: For me...like the prospect of doing something like this in social studies is frightening.

Katey: Oh is it?

DT: I don't think I could do it.

Katey: Well you c....

DT: I don't think I have the depth, for instance, to talk about the Louis Riel rebellion... I know the date of the thing, and five causes, but...but to....

Katey: Well you probably feel like I do about math then, because I don't feel...like I just don't feel that I know the.... Like I said, I'd have to get a lot more familiar with the material.

For Katey, the recent cooperative learning experience was preferable to what she herself had experienced in the past:

> Once it got going I actually thought it was really fun, which was amazing to me. And I really felt that I learned something? One of the few times in my life about that topic. I thought it was terrific. And I really...you know what was tremendous was the cooperative learning. If I'd been in a class as a kid where you were actually allowed to talk about math, to laugh about it, to work together on it, I would have felt so much better. So that made a big difference yesterday. We had humor about it. And we could team up. And I liked your practical examples too, because they were fun and they were appropriate to real life.

Since both she and Maureen described personal secondary school experiences in mathematics which were negative, I was hopeful that they would be even more in favor of curricular change at VRLF.

Eric was the only one of the three teachers with any post-secondary mathematics. I asked him how he felt non-specialists would fare in an open-ended problem-solving situation:

> Eric: I don't think all people will be able to do it. Uh, or would want to do it. I mean you're going to have some resistance..... Maybe because they themselves don't understand, or maybe because their process....
>
> DT: But again, if...if you don't have the background....
>
> Eric: Right. The more of a background that you have, the more...the greater the resources and repertoire that you can draw from, and that you can...uh pick up in uh...empathic moments while teaching or while (*unintelligible*)

DT: Because a lot of this stuff is unplanned. In math as well. The kids can lead you down avenues, and if you don't know where you're going, you might be alright, but you certainly might not be able to take as full advantage of it.

Eric: And that's what I'm saying about this for non-specialists. Some will be able to do it 'cause they have that flexibility of mind and that they can draw and can make relationships. Others don't.

When asked how he would feel about running a group oriented mathematics course, he answered, "I think it'd be great. I mean it's uh fun. And it's again...it's one of the elements that a place like this needs. We need more variety." This was change for the sake of change, but I welcomed his approval nonetheless.

So, with expressed collegial support for change, I had good reason to be optimistic. For one thing, I might no longer have to teach in pedagogic isolation as I had since February. And, with no negative reactions from parents, school district administrators, or Ministry of Education officials, it seemed feasible now that the mathematics project could now become a permanent feature of the entire school.

On two separate occasions, I tried to obtain feedback from teachers in alternate schools outside my school district. At a February meeting of Lower Mainland alternate school personnel, I outlined my program at VRLF, and distributed a questionnaire asking for input about programs in other schools. Out of fifteen forms handed out, four were returned. All four indicated usage of the ABE program. Finally, hoping to reach as broad an audience as possible, I submitted a description of the VRLF pilot project for publication in the June newsletter of the *BC Alternate Education Professional Specialists' Association* (PSA). I wrote, "If you have any ideas, or innovations in your mathematics program that you feel have worked, I would like to hear from you. " As I write, nearly nine months later, I have not received one reply to my article. I assume that reform in alternate school mathematics education is rare, or at best a well-kept secret.

With the project finished, and the school year coming to a close, I felt positive about the prospects of teaching mathematics again at VRLF in the fall. Importantly, I was confident that new students would buy into the program; indeed, my efforts would be successful only through their voluntary participation in what I have to offer.

The "aftermath"....

Final student comments (Adie and Juliana, Donny and Chad)

Originally I had planned to have final interviews with the students. In fact, I intended to have them complete an attitude survey identical to the one they had done at the beginning of the study. However, I thought better of engaging in such formal final activities. As mentioned previously, they were weary of any form of psychoeducational probing. My pupils were also more than familiar with my pronouncements on mathematics education; I suspected that any answers to such formal questioning might be echos of my own pedagogical pontifications. So I abandoned any ideas of a before/after comparative analysis of their questionnaire responses in favor of using the survey as a starting point for informal discussions of the mathematics course.

One item to be rated by the students on the survey was, "*Every mathematical question has only one right answer.*" Donny's response (*Definitely true*) was the exact opposite of his previous opinion in February (*Not true at all*). Initially I was surprised by this obvious reversal in viewpoint; however in retrospect I see that his response was not so unusual. With the exception of the pentominoes puzzle, most of the problems in the course had only one "correct answer."

This same item invited comments on the relative importance given to "correct answers" in some mathematics curricula:

> DT: Where does this emphasis come from that says you have to get the right answer, or else you fail? Or else you get it wrong. Where does that come from?
> Juliana: From teachers.

> Adie: High school teachers. If it's wrong, it's wrong. They don't let you get a mark for it. In high school (*unintelligible*)
>
> DT: What was the main objective in math courses in the past?
>
> Adie: To try to figure out the answer.

I asked how a student could theoretically receive a good mark on an problem in mathematics without necessarily obtaining the right answers. Chad answered, "By showing an effort, and by attempting to solve it. By coming up with different ways to try and solve it."

To be sure, our case study curriculum stressed problem-solving strategies over correct answers. So I was not surprised when four of the five students rated the statement "There are usually several equally correct ways to work math problems" as "Definitely true," and Dale commented "Sort of true." Chad and Don reflected on what the course meant to them in this regard:

> Chad: But we did find lots of different ways to do things. You know....
>
> Don: Yeh.
>
> Chad: Um...like ah...hmm...I don't know....
>
> Don: Like Pascal triangle.
>
> Chad: Yeh. Some people took a longabout way, and some people took an abstract way, and some people took a kind of curved way, and others took a square way, and....

In other words, there are many ways to solve problems, and in a group setting, this diversity is manifested through individual preferences and talents. In paraphrase of Chad, Gattegno (1974) writes,

> [a] true spirit of yielding to problems brings about the emergence in one's mind of a wealth of alternate routes....Some can be called economical, some elegant, brilliant or far-fetched, some awkward, clumsy, long or tortuous, although all are correct and mathematically acceptable. (p. 85)

When asked how our mathematics course differed from courses taken in high school, Chad stressed the importance of group effort:

> And this was much more uh...group oriented. Right? It was good in that way It was...it was also...instead of explaining the formulas to you, we had to figure them out for ourselves, and explore the problem. In all sorts of ways we could think of. And therefore

come to our own conclusions about it. You know. It was just like it was a different viewpoint.

Having already taken mathematics in another alternate school, Don had a different perspective on the curricular differences:

> Don: More words than numbers.
> DT: OK. What do you say Don?
> Don: Um...I guess um...the other one...there is only like one way answer?
> Chad: Yeh.
> Don: And there's like not really a pattern?
> DT: Not really what?
> Don: Like a pattern in it? There's no pattern?
> DT: In what? No pattern to what?
> Don: The regular math.
> DT: In the other math?
> Don: There's like more numbers than words.
> DT: More numbers than words. OK. That's interesting.
> Don: This one, there was like a lot of words, like inductive reasoning, solving problems....

Don's comments are interesting in light of his status as an ESL student. Many recent immigrants experience little difficulty with regular mathematics courses because these indeed entail "more numbers than words." To his credit, Don did not retreat from our situation where oral and written expression in mathematics were stressed over basic arithmetic and universal algebraic symbols.

Finally, when I asked about evaluation, Don and Chad were not adverse to the idea of exams (which we never had in our course):

> DT: Did you miss not having any tests? Do you think this course should have actual exams?
> Chad: Well, it might have been a good idea to have a test.
> DT: Would you rather have a test that you do by yourself, or would you rather work in a pair?
> Chad: Both.
> DT: OK.
> Don: The questions you don't understand, do together. And the ones you do, do alone.

In retrospect, I realize that my curriculum emphasized group activity almost exclusively over individual performance. Ideally, a mathematics curriculum should have a balance between individual and group efforts. And how expedient it would be to divide the curriculum according to the criterion which Don suggested. Whenever individual failure through miscomprehension seems probable, the curriculum should be flexible enough to switch its emphasis, and allow for the potential of group success through mutual concept construction. This is, after all, the modus operandi of professional mathematicians.

Course results.

At the end of the school year, Don, Dale and Chad received credit for Mathematics 10B as a result of their participation in the project. Their letter grades were as follows: Don (C); Adie (C+); Dale (C+) and Chad (B). Likewise, Adie was given a C+ grade for Mathematics 11A. But for their attendance, Dale, Donny and Adie all would have received B letter grades. Juliana accepted that she did not attend enough sessions to warrant any credit in mathematics. And regretfully, the cohort's absenteeism rate for the mathematics sessions was no better than its general, overall rate. Thus, in spite of the incentive of marks for session attendance, the mathematics program was not as strong a "draw" as I hoped it would be.

My students' school records will show official credits for remedial mathematics courses which typically stress arithmetic, geometric formulae and rote algebra skills. But these high school transcripts will never reflect the rich mathematical experiences for six months in 1994.

Postscript.

With the successful experiences of the last five months, a growing repertoire of activities, and the prospect of over a thousand dollars worth of mathematics materials arriving over the summer, I was excited about the fall. There was no doubt that in my classroom at least, mathematics education based on problem-solving, communication, reasoning, and connections would be a permanent feature.

However, fate decreed otherwise. Over the summer, I was offered a position to teach Mathematics 12 at a district secondary school. After ten years in special education, I knew that the time was right for change. Simply put, this was an offer I could not refuse. So in September 1994, the mathematics program at Vale Road Learning Facility was out of my hands.

Since my hospital/homebound office is still at VRLF, I am able to monitor the mathematics being taught there quite closely. Old teachers have left and new teachers have arrived. But the contents of two whole shelves in the school supply room speak volumes. For here sits the largely unused repository of my one thousand dollar mathematics order of the previous spring. I rue the sight of untouched posters, manipulative kits, computer programs, games, and books on fractals, Pascal's Triangle, patterns, and problem-solving all gathering book-room dust. But my regret is especially deep because of the good use the students there could now be making of these materials.

One day early in September, there was a message for me at my VRLF office to phone Adie. Thinking immediately that there was probably some problem with the transcript of her final marks, I phoned her back. In reality, she was concerned that her last assignment was still incomplete, and wondered when she could come in and finish it. So, as she had done in the past after letting me down, Adie made restitution, and thereby validated once again the confidence I already had her. In a sense, Adie's behaviour was symbolic of all my students who indeed disappointed me many times during the course of our mathematical experiment. But overall, they performed far beyond my expectations, and left me with the certain knowledge of what is possible in the mathematics education of alternate school students.

Conclusions

The purpose of this final segment of my thesis is twofold. First, in a summary of my story of curricular change, I will attempt to answer the question, "What was learned from the study?" Second, in a consideration of my research results, I will ponder, "Where can we go from here?"

From the onset, my task was to introduce a radically different style of mathematics education to a traditionally problematic student population. My curriculum was based on the four cornerstones expounded in the NCTM's (1989) *Standards* document: problem-solving, communication, reasoning, and connections. Thus, as discussed in Chapter Three, discovery learning through cooperative, open-ended problem solving was my choice to replace the individualistic, arithmetic-based program which was the alternate school norm. As predicted in the literature, mathematical topics of sufficient challenge served to awaken my students from a state of learned passivity to an active mode of learning.

At first, I considered this aim ambitious in the light of the nature of the alternate school students. Simply put, these young people attended Vale Road Learning Facility because their temperaments and behaviour were not suited to a regular school setting. In Chapter Two, I outlined the four salient personal problems of this population: learning difficulties, poor self-esteem, social disabilities, and chronic, deep unhappiness. These characteristics were consistent with the available literature on students with emotional and social disabilities. In particular, research findings on the socialization problems of students with attentional deficit disorder were born out by my experiences with the study participants. Although study after study found that children with ESD generally had backgrounds of chronic academic failure, my group experienced success with the new mathematics curriculum.

From the beginning, it was evident that this particular group of students was capable of learning the non-traditional and often complex mathematical topics which I had chosen. Time after time, the cohort impressed me with its questions, insights, discoveries, and verbalizations. As long as the mathematics was not deeply entrenched in algebra, the group never faltered or gave up on any one topic.

There is not one topic out of the eight which I would exclude from a future curriculum. As described in Chapter 4, I would, however, modify my approaches to some of the mathematics (e.g., the golden ratio and pentominoes). An alternate school curriculum could include many other topics such as tessellations, finance, perspective drawing, and episodes from the history of mathematics. Since students enter and leave alternate programs at various times during the year, these topics could all be offered by a school on a continual, rotating basis.

I feel that topics such as probability, fractals, and reasoning would not have been successful if taught on an individual basis. Often, one or two students in the class would be in the dark during a lesson. It seemed, however, that at the group level, there were always enough individuals "in the know" to keep the learning momentum going in a positive direction. Thus, general discouragement never did really take hold at any one time.

Each individual brought personal strengths to the group. Chad was excellent at seeing patterns, and was the group's strength in algebra. Dale's insights and remarkable persistence helped the class overcome difficult moments. Adie's inevitable dependability, Juliana's enthusiasm and Donny's laconically expressed observations all contributed to the over-all success of the cohort. Within the group, metacognitive comments became more frequent as individual passivity gave way to group activity. This last result should hardly be surprising: after all, the students finally had something to be metacognitive about, and also had the group setting in which to overtly express themselves.

Formal student assessment was done mainly through assignments. I found that whole group assignments had the added benefit of encouraging the students to work cooperatively. Marks were also given for attendance, although this policy seemed to have no positive effect on student absenteeism. At no time did I feel the need nor desire for formal individualized tests (which usually test memory only). With such a small class size, I felt confident that by listening to student dialogue and by occasional oral questioning, I could informally monitor and guage each student's learning.

The project was not without its share of serious difficulties, however. For instance, chronically poor student attendance was a problem which I was never able to fully rectify. At times, absenteeism was so high that I thought the program itself was doomed to failure. Student personality conflicts, teasing, name-calling, and bad feelings at times had me doubting whether communication and cooperation were achievable aims at all. In the beginning, the students were very hesitant to talk to each other about mathematics, and they would frequently give up on problems rather than persist at a cooperative solution. But after all, given the passive individualized nature of their previous mathematics courses, I was not surprised at their resistance to active, cooperative learning.

Another area of concern was the preponderance of "teacher talk". Unable to shake off the habit of over a generation of direct classroom teaching, I frequently interrupted the students with overt explanations, leading questions, obvious clues, and outright answers to some mathematical problems. However, in most of the sessions where I remained silent, group dialogues and collaborative efforts actually resulted in discovery learning.

As the study progressed, student cooperation and communication became more the norm. Infighting diminished, just as student confidence in the subject grew. I feel that the mathematics curriculum had many special benefits for these students. It served as a vehicle for honing social skills and for improving discourse. It offered many chances for success in a typically failure-

prone area. Judging from the students' positive reactions to their achievements, I feel it served to improve their self-image regarding mathematical performance. Finally, it provided a cognitive and emotional substitute for depression: we had many good laughs even as we contemplated some very mentally challenging problems. Thus, in a way, the course could be described as a "social curriculum".

Whether or not the new curriculum resulted in changes in student attitudes and beliefs about mathematics is difficult to ascertain. Although no formal attitudinal survey was taken, the cohort's final statements at least evidence recognition of profound differences between the new approach and previous courses. According to the students, their activities over the previous five months stressed the importance of perceiving relationships, recognizing patterns, developing a repertoire of strategies, and communicating their new-found awarenesses. In as much as I designed the course to reflect my own notions of what mathematics is, and in as much as the students bought into it, I venture to conclude that the curriculum altered the students' perceptions of the subject in a positive way.

According to Robitaille and Dirks (1982), a mathematics curriculum may be judged from three different viewpoints:

> [W]e may distinguish among the curriculum as intended, the curriculum as implemented, and the curriculum as attained. By the Intended Curriculum is meant the curriculum as planned at the national, provincial, or local levels by curriculum committees and consultants, and as codified in curriculum guides. The Implemented Curriculum is the curriculum as contained in the various texts and materials which are selected and approved for use in the schools and as communicated to students by teachers in their classrooms. The Attained Curriculum is the curriculum as learned and assimilated by students. (p. 17)

The normal sequence of curricular development would appear to be intended, then implemented, followed (hopefully) by attained. In my case, the order was quite the opposite. With no selection or approval committees involved, I feel that the VRLF curriculum belongs solely in the Attained category. Thus, Chapter 4 is my testimony as to what I believe my students accomplished over five months in 1994.

What can the future bode for mathematics in alternate schools? At the onset, extreme care must be taken in applying the results from VRLF to other special education settings.

In many ways, this setting was unique. My particular efforts at curricular reform met with no opposition from parents, and the school board administration welcomed it. Since the course I discarded at VRLF was completely locally developed, I was free to replace it with any topics I desired. With students from an older secondary population, I could convincingly argue that any further attempts at teaching them "basic skills" would be counterproductive. Furthermore, blessed with small class sizes, I felt that my ability to carefully observe each student eliminated the need for formal written exams. Last, my students were not enrolled in an alternate school mathematics program because of intellectual weaknesses; rather, they attended VRLF as the result of behavioural problems which precluded attendance in regular schools.

In other settings, however, it might be difficult to initiate or sustain such curricular change. Some might argue that many general mathematics students in regular school settings would not be intellectually capable of open-ended discovery learning. Others might maintain that younger students should continue further "training" in basic arithmetic skills. Large class sizes in secondary schools would make evaluation of individuals in a cooperative learning situation extremely difficult. Finally, parents, department heads, administrators, government officials, and even colleagues could erect immense barriers to the well-intentioned educational reformer.

In other words, since this is a case study, its results should not be used in an inductive or experimental sense. As Stake (1980) comments,

> [i]n most other studies, researchers generalize beyond particular instances to search for what is common and pervasive. The case study may or may not be an ultimate interest in the generalizable. The search instead is for an understanding of the particular case. (p. 67)

And certainly, VRLF was a most particular situation.

As an alternate school teacher, I agree with collegial generalizability which "involves leaving the extent to which a study's findings apply to other situations up to the people in those situations" (Merriam, 1988, p. 176). In that spirit, I offer the following suggestions for curricular change in alternate schools at the Grade 10, 11, 12 level:

- Courses in remedial arithmetic should be discarded in favor of a curriculum offering open-ended problem-solving in cooperative learning situations. Inductive and discovery learning should complement direct instruction.

- "Basic" skills in arithmetic and algebra can and should be taught or reintroduced in the context of mathematical topics.

- Evaluation should not be restricted to formal individual written exams: it can also be accomplished through group and individual assignments. If classes are small, then direct observation of students by the teacher can help assess student learning.

- Computers should be a mainstay to promote inductive learning in alternate school mathematics classes. Computer assisted geometry programs such as Geometer's Sketchpad can be useful for learning disabled students who have visual/motor difficulties.

- Teachers searching for a replacement geometry curriculum need look no further than Serra's (1989a) *Discovering Geometry* text. Hopefully, there will be comparable books on algebra available soon.

- School districts should allow release time for teachers to develop curricula.

- Teachers should also have the opportunity for inservice training in cooperative learning and methodologies for learner-centred classrooms.

- For alternate school students, mathematics and social interaction can be mutually enhancing. In other words, students can construct mathematical concepts better through peer interaction than through individual paper work. And importantly, mathematics can be a means by which social improvement is realized.

Educators who decide to change the way they teach mathematics in alternate schools should realize that setbacks, disappointments and moments of despair are probably inevitable. But I can

testify to the professional rewards which can result from such an undertaking. The premiums will extend to the students as well, when they learn that mathematics is after all, a worthwhile social activity.

APPENDIX 1

MATHEMATICS QUESTIONNAIRE FOR STUDENTS

PART A

Please indicate your like or dislike for the following academic subjects. For each of the subjects, say whether you

LIKE IT ALL OF THE TIME   (by writing 4)
LIKE IT MOST OF THE TIME   (by writing 3)
LIKE IT SOME OF THE TIME   (by writing 2)
LIKE IT NEVER   (by writing 1)

ENGLISH _4_   SOCIAL STUDIES _2_

MATHEMATICS _2_   SCIENCE _3_

PART B

Please complete each of the following sentences by writing whatever comes to mind first:

Doing mathematics make me feel _tense_

The problem with mathematics is _for me, I've missed some basic concepts. So it can be difficult_

Mathematics is totally different from _Nothing. Everything is Math_

When I make a mistake in mathematics I _fix them and go over what I've corrected_

What I especially like in mathematics is _It never ends._

PART C.

For each of the following statements, say whether you think it is:

- DEFINITELY TRUE (by writing 4)
- SORT OF TRUE (by writing 3)
- NOT VERY TRUE (by writing 2)
- NOT TRUE AT ALL (by writing 1)

_4_ There is always a rule to follow in solving mathematical problems.
_2_ Mathematics is a very good field for creative people to enter.
_2_ Every mathematical question has only one right answer.
_1_ There is little place for originality in mathematics.
_1_ Only people with a very special talent can learn mathematics.
_4_ There are usually several equally correct ways to work math problems.
_1_ If I understood x, then I could do algebra.
_4_ Everything in mathematics is either right or wrong.

(Adapted from *Learning Mathematics Through Inquiry* by Rafaella Borasi)

APPENDIX 2

SFU MATHEMATICS PROJECT
INFORMATION SHEET FOR STUDENTS

To: My students
From: Dave Tambellini

As a student at _____, you may choose to participate in a mathematics project which I am conducting as part of my Masters program at SFU. The purpose of this information sheet is to describe the project for you, and to inform you about your rights if you decide to partake.

WHAT MATH WILL YOU LEARN?

In part, you will be introduced to many mathematical topics which are regularly taught in Grade 10 math. Thus, you will still work on basic skills (working with whole numbers, fractions, decimals, and per cent), algebra, and geometry. However, the way in which you learn these topics will be different from the regular program. For instance, you will review percentage, but instead of doing pages of routine questions, you will work on percent problems which you and fellow students will actually bring to class yourselves.

As well, you will be introduced to many new topics, and to different types of problems and activities which will challenge you and your classmates.

HOW WILL YOU LEARN MATH?

At present, you do almost all of your work in math alone. But as part of the project, you will do much of your learning in pairs or in small groups. For example, you will still learn geometry, but I will not merely give you formulas and theorems to memorize by yourself. Instead, working in pairs or in small groups, you will discover many of the geometric ideas for yourselves through inquiry, discussion, and experimentation.

WHAT WILL YOU BE ASKED TO DO?

First, you will complete a questionnaire about your thoughts on math. As well as doing the math, you will be asked to comment on your work in interviews. You will asked to discuss (in groups and by yourself) both your attitudes to the math your are doing, and your beliefs about math in general. Some of the time, I will record our conversations. I will also ask each of you to keep a daily journal of your thoughts about your experiences.

WHAT CHOICES WILL YOU HAVE?

First, your participation will be completely voluntary. You will be able to withdraw from the project at any time. (If you quit, you will then resume your regular math course and will receive course credit for the work you did.) Your voice will not be recorded if you do not wish. At any time, you may ask to see what I have written about you. You have the freedom to decide for yourself what past or present information about you will be used in my report. You will be told how to obtain a copy of the research results. Finally, your name will NOT be used in my thesis. In other words, you will have complete anonymity.

WHAT CREDIT WILL YOU GET?

If you are enrolled in Math 10B, I will give you partial or complete credit for that course. If you are taking Math 10, Math 10A, Math 11 or Math 11A, you will receive partial credit for your course. (How much credit you receive depends on the amount of time and effort you spend in the project.)

I encourage you to ask questions about this study.

APPENDIX 3

$$\left(\frac{17\frac{1}{2}}{20}\right)$$
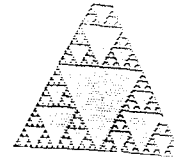
Assignment 3. Pascal's Triangle (20 pts. total)

Due: Thursday 24 Feb.   Names _____

In collaboration with a partner, try the following questions about Pascal's Triangle. We shall consider {1,1} to be the FIRST row of the triangle.

1 Explain how the triangle is generated. (2 pts),

add the two numbers above ✓

2 Describe five different patterns in the triangle. (5 pts)

a) 1 1 1 1 1 1 1 1 1 1 1   ✓

b) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10   ✓

c) 1, 3, 6, 10, 15, 21, 28, 36, 45   ✓

d) *The numbers are the same on the other side.* ✓

e) *add the top two numbers for the answer at the bottom.* ✓

3 Find the sum of all the numbers   (5 pts)

A) in the 11th row. 2048 ✓

B) in the 15th row 32768 ✓

C) in the 30th row $2^{30} = r$ ✓

D) in the 100th row $2^{100} = r$ ✓

E) in the r th row. $r =$ ✗

4 What is the 2nd number in the 99th row? (1 pt) 99 ✓

5 What is the next to last number in the 200th row? 2... (1 pt) ✓

6 What is the third number in the 10th row?   (3 pts) 45 ✓

In the 15th row? 120 ✗

In the 30th row? 465 ✗

7 Find the sequence { 1, 3, 6, 10, 15, ...} in Pascal's triangle.   (4 pts. total)

a) Explain how the sequence is generated. *subtracting the difference between each no. the and creates a pattern* ✓

b) What is the 20th term of this sequence? (Use a calculator)

210 ✓   −3

7 (continued)   { 1, 3, 6, 10, 15, ...}

c) Explain why the numbers in this sequence are called TRIANGULAR numbers.
   (Hint: Use dots to represent each number)   (2 pts.) *The numbers of this sequence called triangular numbers because when we use dots to represent each no. in the sequence it forms a triangle*

−1

3 ·· 6 (∴) 10 (∴) 15 ····· ?

8 (***) What are the first 4 terms in the 100th row of Pascal's Triangle?   (2 pts)
   1, 100, 4950, 15 176. ✗

*** Question #8 is a difficult one. So is the last part of question #6. You may want to get together as a class to solve these.

$-\frac{1}{2}$

APPENDIX 4

How's it goin folks?

Congratulations! We've reached the half-way point in our math program!!

Before we continue, let's pause and reflect a bit on what's gone on, and discuss what we could be doing for the last half.

Part A

Have a look through your "folders", and consider the following questions.
We've covered the following topics:
1) Pentominoes 2) Probability 3) Pascal's Triangle 4) Scientific Notation & large numbers 5) Inductive Reasoning

1) Should all of these topics be included in future Math courses for CWEC? Should some not be included? Why not?

2) Were you bothered that the some of the problems you were presented with were not solved? Why or why not?
eg. What is the probability that a series of 10 births will have 4 or more girls born in a row? or Cover a 6 x 10 rectangle with pentominoes. *NO!!!*
Would you like to revisit these problems in Part II of the course? *NO....*

3) How do you feel about working in a group, as opposed to working by yourself? *feels good. more ideas.*

Part B For the future sessions:
1) What things are working okay? *group work, open ended problems.*

2) What things could be done differently? *not work too long on one problem*
accept others differences
3) What are some topics that you would like to experience in Part II?
Here are some areas I'd like to include:
 •Algebra (& integers) ✓   •Percentage & Finance
 •Geometric Art (eg Escher) ‿   •Trigonometry ✓
 •Statistics ✓   •Fractals
 •The number π ✓   •Area, Volume & Pythagoras (on the computer)

Part C Let's try to formulate a definition of mathematics.
Eg. History is the study of our past. Biology is the study of life. Chemistry is the study of matter. *Harmony Time, logic*
Mathematics is the study of *learning how to live with and use numbers.* (don't restrict yourself to just one or two words)

APPENDIX 5
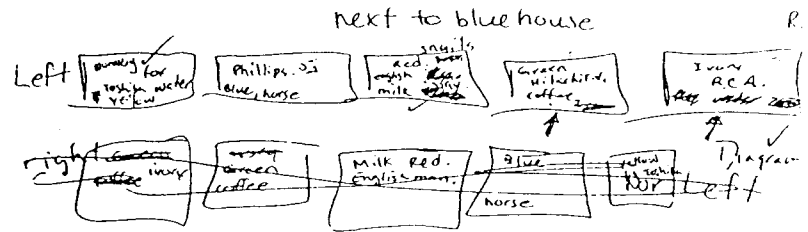
Assignment 7   Problem 4

PROBLEM 4

By combining deduction, analysis, and sheer persistence, the answers to the questions below can be found. (Also, use the same hint given in problem 3) The facts essential to solving the problem are listed below.

1) There are 5 different houses, each of a different colour and inhabited by men of different nationalities, with different pets, drinks, and television sets.

2) The Englishman lives in the red house.

3) The red house is not next to the ivory house.

4) The Spaniard owns the dog.

5) Coffee is drunk in the green house.

6) The Ukrainian drinks tea.

7) The green house is immediately to the right of the ivory house.

8) The owner of the Sony TV also has snails for pets.

9) A Toshiba TV is watched in the yellow house.

10) Milk is drunk in the middle house.

11) The Norwegian lives in the first house on the left.

12) The man who has an RCA TV also lives in the house next to the man with the fox.

13) The Toshiba TV is in the house next to the house where the horse is kept.

14) The owner of the Hitachi TV drinks coffee.

15) The Japanese man has a Phillips TV

16) The Norwegian lives next to the blue house.

17) The man who owns the fox drinks water.

Now, use logic to answer the following:
Who drinks water?
Who owns the zebra?
Who drinks orange juice?
Who owns the fox?

APPENDIX 6

SIERPINSKI TRIANGLE
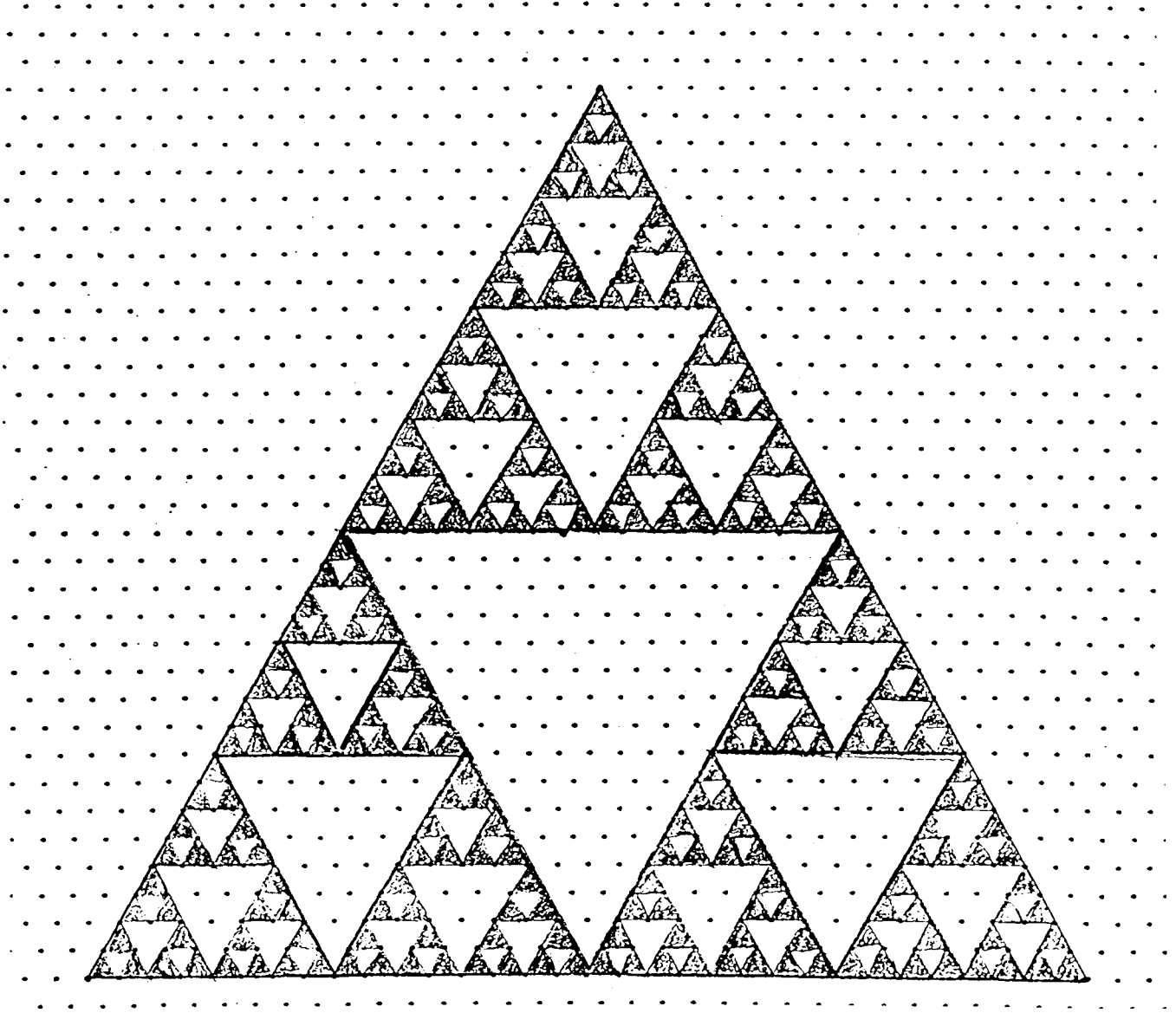
APPENDIX 7

Mathematical Topics from the Curriculum

The following is a reflection on the mathematics which I chose for the alternate school curriculum. With the exception of certain comments on logic, I do not refer here to any of the study group's experiences with the mathematical topics. Rather, the purpose of this appendix is to discuss the mathematics in and of itself, and to point out some of the many connections between the topics.

The reader should not assume that I had predetermined all the mathematical content before the onset of the study. Rather, I decided on each of the themes at different points in the project. Finding appropriate topics was not a difficult task since these sprang quite naturally from their inter-connectedness. For instance, as I point out in this appendix, Pascal's triangle was a natural subject to follow probability, and certain aspects of fractals flowed beautifully from Pascal's triangle. Thus the choice of topics was mediated more by the connections inherent in all mathematics than by any deliberate master planning on my part.

Inductive and deductive logic.

Together, deductive and inductive logic (or reasoning) are the means by which mathematical progress is made.

Inductive logic is the process of observing data, recognizing patterns, and making generalizations (or conjectures) about those patterns. Also called empirical reasoning, inductive logic is the basis of the scientific method.

Here is an example of inductive logic. A person observes that Scare BC Airlines has twice as many crashes as other passenger airlines. In view of such a poor pattern of safety, she conjectures that Scare BC is an unsafe airline. Accordingly, she makes the decision not to fly

with Scare BC. She reasons that based on the company's past record, the probability of her dying on one of its flights is relatively high.

The astronomer Johannes Kepler (1571 - 1630) discovered his laws of planetary motion through inductive reasoning. By observing and recording the motions of many planets over twenty years, he noticed patterns, and by generalizing from this data, he was able to formulate laws which are true for every planet.

With modern, high speed computers, mathematicians can effectively use inductive methods to experiment with numbers or to test hypotheses. In other words, machines are now replacing deductive logic in pure mathematics research: the "clean and clever" arguments of the past are being replaced by computer techniques which systematically test and discard huge numbers of possible answers until the correct one is discovered.

Since this study emphasized discovery learning through experimentation and observation of patterns, my students used inductive reasoning extensively. Generalizing about patterns, the students became aware of certain properties and underlying principles of probability, Pascal's triangle, the golden ratio, and fractals. With repeated simulations, they were able to approximate the probability of having 4 or more girls born in a row in a sequence of 10 births. Observing patterns, they became aware of the relationship between binomial probabilities and Pascal's triangle. Measuring the dimensions of certain rectangles, they found a close approximation to the golden ratio. Through experimentation, the students discovered a relationship between Pascal's triangle and fractals. Using the *Geometer's Sketchpad*, they used the computer to measure the circumferences (C) and diameters (d) of several circles, and through induction, they "discovered" the relationship $C = \pi d$. Finally, with a dedicated unit on logic which provided practice in pure inductive reasoning, the students became aware of the nature of inductive logic itself, and of its place in mathematical inquiry.

Although new mathematical truths can be discovered by experiment or intuition, the ultimate goal of mathematicians is to support inductive discoveries with convincing, mathematically sound arguments. Such arguments are set forth using the rules of deductive logic, which is the underlying structure or foundation of mathematics.

Consider the following three statements as an example of deductive reasoning.

- Frank plays the piano and no other instrument.
- The musical group Green Onions does not have a piano player.
- Therefore Frank is not a member of the Green Onions.

Each of the first two statements above is termed an *hypothesis* of the deductive argument. The last statement is called the *conclusion*. The conclusion is derived from the hypotheses which are assumed to be true.

The renowned Sherlock Holmes was portrayed as a master of deductive logic. For instance, he might have assumed the following two statements to be true:

- Anyone with wet boots must have been afoot last night .
- Mr. Moriarty has wet boots.

He would then be in a position to make a conclusion from the hypotheses: Mr. Moriarty must have been afoot last night.

Isaac Newton (1642 - 1727) used deductive reasoning to derive the same astronomical laws which took Kepler a generation to formulate. To prove the astronomer's first law, Newton initially assumed as true his own law of universal gravitation and his second law of motion. From these hypotheses, he was able to prove deductively that for any planet, the ratio of the square of its period (T) of revolution around the sun to the cube of its average distance (r) from the sun. (i.e., the ratio T^2/r^3 is a constant for all planets in our solar system.)

My students were exposed to a similar deductive exercise. From Euclid's geometric description of a golden rectangle, we deduced algebraically the equation $x^2 - x - 1 = 0$ whose solution is the exact value of the golden ratio. However, feeling that a deductive derivation would

have been too cumbersome at the time. I presented the combinatorics formula $C^n_r = \dfrac{n!}{r! \, (n - r)!}$

to the group without formal proof. (Later in this appendix, I elaborate more on the golden ratio and on combinatorics.) Within a separate deductive logic unit, the students experienced the concept firsthand with the problems outlined on pages 184 and 188 and in appendix 5.

Pentominoes.

A pentomino is a shape which can be made from 5 squares joined along their edges. By experimentation, one can discover that there are 12 distinct pentominoes; that is, there are 12 different shapes each of which cannot be derived from a rotation or flip (or combination of these transformations) of any other pentomino.

By orienting the 12 pentominoes, one can name each shape by its resemblance to a letter of the alphabet. (See Figure A below.)
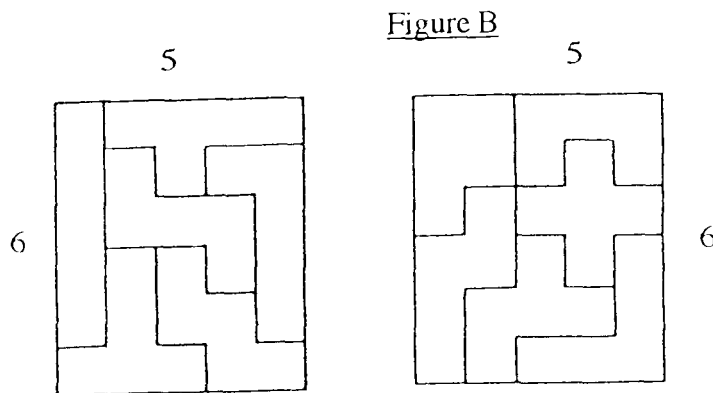
Figure A
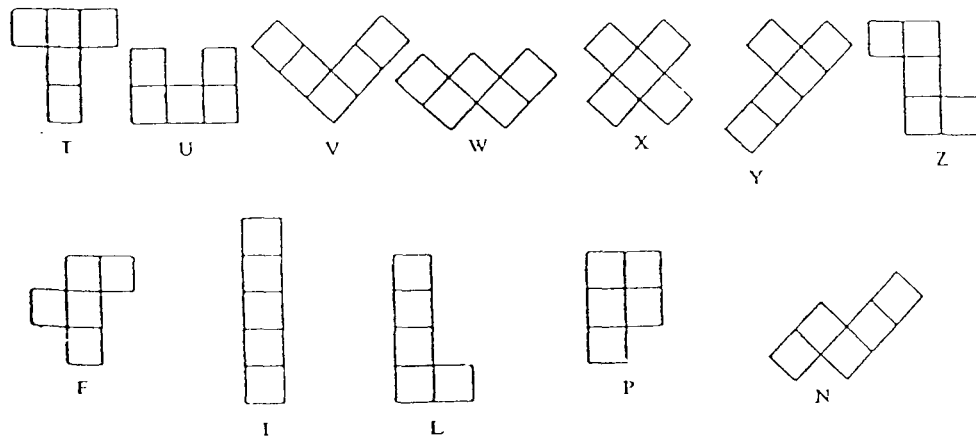
Pentomino puzzles are part of a branch of mathematics called combinatorial geometry. Their popularity in recreational mathematics is probably due in part to the fact that little

mathematical background is necessary to understand the nature and demands of the various puzzles.

One of the most famous pentomino challenges is the 6 x 10 rectangle puzzle. Given that each pentomino consists of 5 unit squares, can you arrange the 12 pentominoes in a 6 by 10 rectangle? Although over 2300 solutions have been found by computer, the reader might have trouble finding even one. Through dogged perseverance and trial and error, only one student in the study group came up with a solution.

A problem in mathematics is often solvable by considering a simpler version of the original. For instance, by considering two 5 x 6 rectangles instead of one 6 x 10, one might discover solutions such as the following: (See Figure B.)

Figure B

Note that these two 5 x 6 rectangles provide a solution for a 5 x 12 puzzle as well. Note also that a solution for the latter can be obtained by combining a 5 x 7 and a 5 x 5 rectangle: (See Figure C.)

Figure C
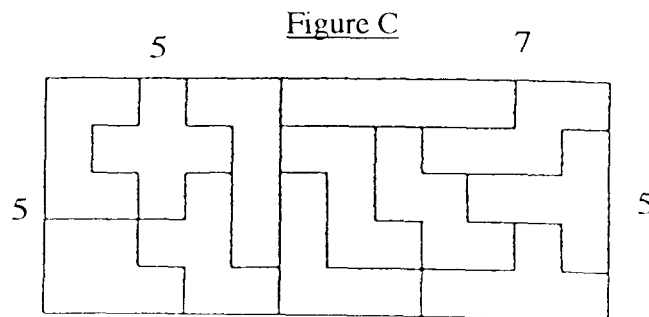
There are valuable mathematical lessons to be learned from pentominoes. After laboring long and hard to find just a single answer, one might be surprised (and humbled) to learn that there are thousands of solutions to the 6 x 10 rectangle problem. (One might also be curious about the power of the computer to conduct investigations into mathematical problems.) In contrast, there are only 2 distinct ways to cover a 3 x 20 rectangle with 12 pentominoes. For certain other puzzles, there are no solutions. Finally, it would not be surprising for a second grader to solve a pentomino puzzle before a university graduate.

Probability

Consider the following question. What is the likelihood that in an obstetrics ward, a list of 10 consecutive births contains 4 (or more) girls in a row? To answer the question, there are two possible routes. One could use simulations (or models) to determine inductively the experimental probability of 4 or more girls in a row. Or, using mathematical theorems, one could use deductive logic to compute the theoretical probability of the event.

It is not difficult to devise models for births: any experiment with a "binomial outcome" would suffice. Thus coin-flipping (head/tail) and card-picking (black/red) are examples of models which can simulate a birth outcome (boy/girl). In calculators and computers, random number generators (one/zero) are accurate and quick ways of modeling a large number of births. The following program (written in Quick Basic) is a computer model of 100 sequences of 10 births. Note that the program also counts the instances of 4 or more girls in a row and computes the resulting experimental probability.

```
REM Simulation for 10 births (x) where n = number of sequences
RANDOMIZE TIMER
DIM P(10)
Counter = 0
g = 0

FOR n = 1 to 100

Foundfour = 1
```

```
g = g + 1

FOR x = 1 to 10
t = 0
LET y = INT( 2*RND)
P(x) = y
IF y = 0 THEN PRINT "G"
IF y = 1 THEN PRINT "B"
IF x >= 4 AND Foundfour = 1, THEN GOSUB Counter

NEXT x
NEXT n

PRINT "The computer model has produced 4 girls in a row"; PRINT " a total of";
Counter; "times out of "; g; ". The experimental probability is"; Counter/g
END
REM This is the counter subroutine
Counter:
t = P(x) + P(x -1) + P(x-2) + P(x -3)
IF t = 0 THEN COUNTER = COUNTER + 1
IF t = 0 THEN Foundfour = 0
Return
```

Of course there will be discrepancies between different simulative outcomes, but generally, the more trials run, the closer any experimental result will be to the corresponding theoretical probability. Using the above computer program, a larger sample space can easily be obtained by replacing the 100 in the line *FOR n = 1 to 100* with larger numbers such as 1000 or 10000.

The theoretical probability of our question can be investigated without intensive formal background in mathematics. By looking at simpler cases (e.g., 5 or fewer children instead of 10), and by observing patterns, accurate conjectures can be made. Consider the possible outcomes (or sample space) for the births of 5 children. The *combinations* are as listed in the first row of Table A. Below each combination are listed the different birth orders (or *permutations*) for that combination. For instance, for the combination {4 boys, 1 girl} there are 5 permutations: BBBBG, BBBGB, BBGBB, BGBBB, and GBBBB. For the combination {3 boys, 2 girls} there are a total of 10 permutations. The total number of permutations for each combination is listed in the third row of the table. Thus there is a total of 32 different birth orders, or permutations, for 5 children (i.e. 1 + 5 + 10 + 10 + 5 + 1 permutations).

Table A

Combinations	5 boys	4 boys 1 girl	3 boys 2 girls	2 boys 3 girls	1 boy 4 girls	5 girls
Permutations	BBBBB	BBBBG BBBGB BBGBB BGBBB GBBBB	BBBGG BBGBG BBGGB BGBGB BGGBB GBBBG GBBGB GBGBB GGBBB BGBBG	GGGBB GGBGB GGBBG GBGBG GBBGG BGGGB BGGBG BGBGG BBGGG GBGGB	GGGGB GGGBG GGBGG GBGGG BGGGG	GGGGG
Total number of permutations	1	5	10	10	5	1

By totalling all the permutations for each case less than 5 children, one may become aware of a pattern. (See Table B.)

Table B

Number of Children	Number of Permutations
1	$2 = 2^1$
2	$4 = 2^2$
3	$8 = 2^3$
4	$16 = 2^4$
5	$32 = 2^5$

One could thus conjecture that there are a total of 2^{10} (or 1024) possible birth orders for 10 children. With the denominator of the theoretical probability known, all that remains is to determine the numerator: i.e., the total number of cases where 4 or more girls in a row occur. This exercise is left to the reader.

Pascal's Triangle

From the previous example, consider Table C which lists the numbers of permutations per combination for different numbers of children.

Table C

Number of children	Permutations
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

So for 4 children, the numbers {1 4 6 4 1} mean that we can list only 1 birth order for 4 boys in a row; 4 birth order permutations of 3 boys and 1 girl; 6 permutations of 2 boys and 2 girls; 3 permutations of 1 boy and 3 girls; and finally, only 1 birth order of 4 girls in a row.

Shifting the positions of the rows in the second column of Table C, we end up with the following array, which is called Pascal's triangle. (See Figure D.)

Figure D

```
Row   0                    1
      1                 1     1
      2              1     2     1
      3           1     3     3     1
      4        1     4     6     4     1
      5     1     5    10    10     5     1
```

This triangle is a virtual treasure trove of patterns and is a source of a myriad of connections to other branches of mathematics.

One of the most obvious patterns is the property by which the triangle is generated: each number or entry in the triangle (except for the 1's on its edges) is obtained by adding the 2 numbers above it. Thus the next row of the triangle is 1, 6, 15, 20, 15, 6, 1.

The rows of the triangle have a connection to a branch of mathematics called combinatorics. Take, for example, the entries in the fourth row: {1, 4, 6, 4, 1}. These numbers indicate the number of ways of chosing 0, 1, 2, 3 or 4 objects respectively out of a set of 4 objects. Assume that the 4 objects are the Ace (A), Queen (Q), King (K), and Jack (J) of hearts. There are 4 ways to choose 1 card, 6 ways to choose 2 cards (KQ, KJ, QJ, AK, AQ, and AJ) and 4 ways to chose 3 cards (AKQ, AKJ, AQJ, and KQJ). There is only 1 way to choose 4 cards (i.e., pick all 4), and 1 way to choose 0 cards (i.e., don't pick any).

Likewise, the numbers in the 7th row (1, 7, 21, 35, 35, 21, 7, 1) represent respectively, the number of ways of choosing 0, 1, 2, 3, 4, 5, 6, or 7 elements out of a set of 7 elements. There are thus 35 ways to choose a committee of 3 people from a club of 7 members, and 35 ways of choosing a committee of 4. The last number, 1, means that there is only 1 way to form a committee of 7 people: choose them all.

The 4th term of the 10th row of the triangle (1, 10, 45, 120, 210, 252, 210, 120, 45, 1) tells us that there are 120 different ways of choosing 3 different pizza toppings from a list of 10 possible toppings. We can state the total number of possible bridge hands (13 cards out of 52) by looking at the 14th entry of the 52nd row of Pascal's triangle. Or, by locating the 7th term of the 49th row, we will know the total number of possible Lotto 649 tickets (each created by choosing 6 numbers from a total of 49).

The last two tasks are clearly formidable. Assuming that the 10th row of Pascal's triangle takes up 10 centimetres of typed space, the 52nd row would be proportionally over half a metre long! As well, the numbers in the triangle quickly become so huge that computing each entry by adding the two numbers above is next to impossible.

Formal combinatorics provides a formula whereby each number in the triangle can be computed relatively easily. (I shall not give its complex derivation here.) The number of ways of

choosing r elements out of a total set of n elements (commonly called "n choose r") is given as $C^n_r = \frac{n!}{r!\,(n-r)!}$. ($n!$ or n factorial indicates the product $n\,(n-1)(n-2)(n-3)....$and so on, down to 1. So $6! = 6\bullet5\bullet4\bullet3\bullet2\bullet1 = 720$.) Thus, the number of ways to choose 3 different pizza toppings from a total of 10 (or 10 choose 3) is also given by the combinatorics formula:

$$C^{10}_3 = \frac{10!}{3!\,(10-3)!} = \frac{10\bullet9\bullet8\bullet7\bullet6\bullet5\bullet4\bullet3\bullet2\bullet1}{3\bullet2\bullet1\bullet(7\bullet6\bullet5\bullet4\bullet3\bullet2\bullet1)} = 120$$

The number of possible Lotto 649 ticket combinations (49 choose 6) can be computed as follows:

$$C^{49}_6 = \frac{49!}{6!\,(49-6)!} = \frac{49\bullet48\bullet47\bullet46\bullet45\bullet44\bullet43\bullet42\bullet41\bullet40....3\bullet2\bullet1}{6\bullet5\bullet4\bullet3\bullet2\bullet1\bullet(43\bullet42\bullet41\bullet40....3\bullet2\bullet1)} = 13,983,816$$

In a similar fashion, one can discover that there are exactly 635 013 559 600 different bridge hands.

Large numbers

Numbers such as 13 983 816 and 635 013 559 600 are so huge that they are probably meaningless unless related somehow to our everyday experience. For instance, the number 1 billion can be given meaning with the answer to the question "How long would it take you to count to 1 billion, assuming that you count at the rate of 1 number per second?" The answer (32 years) surprises most people, and also helps develop an awareness of the size of the number 10^9. How big is our national debt of over \$500 billion? If we paid it down at the rate of \$1 million per day, Canada would be out of debt in approximately 1370 years.

Speaking of getting out of debt, how small is your chance of winning the Lotto 649? With one ticket, your probability of winning Canada's national lottery is 1 out of 13 983 816. To put this number in perspective, imagine the following scenario. Standard tennis balls are laid side-by-side along the Trans Canada highway all the way from Vancouver to Calgary. One (and only one) of those tennis balls has a red interior. For the price of \$1, you can pick one tennis ball

anywhere along the route. If the tennis ball you pick has a red interior, you win $2 million. Such is the size of the number 13 983 816.

How could you ever fully appreciate the infinitesimal odds against being dealt a bridge hand consisting of 4 Aces, 4 Kings, 4 Queens and the Jack of spades? I leave it to the reader to create an appropriate analogy which could help one fathom magnitudes such as 635 013 559 600.

The golden ratio

Over 2000 years ago, Euclid posed the following question. "What are the dimensions of a rectangle that when it is divided into a square and a smaller rectangle, the smaller rectangle has the same shape as the original?" Let us assume that the rectangle in question has a length x, and for convenience, a width equal to 1. (See Figure E below.)

Figure E
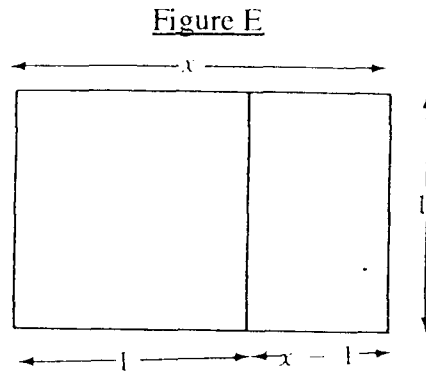
The smaller rectangle will thus have length 1 and width $(x - 1)$. If the shapes are the same, we must have the following similarity situation:

$$\frac{\text{Length of original rectangle}}{\text{Width of original rectangle}} = \frac{\text{Length of smaller rectangle}}{\text{Width of smaller rectangle}}$$

or, $$\frac{x}{1} = \frac{1}{x - 1}$$

This equation can be rewritten as $x^2 - x - 1 = 0$, and its solution x (or $\frac{x}{1}$) is the answer to Euclid's question. This is a quadratic equation (i.e., of the form $ax^2 + bx + c = 0$), so its solution is given immediately by the quadratic formula (where $a = 1$, $b = -1$, and $c = -1$):

$$x = \frac{-b \pm \sqrt{b^2 - 2ac}}{2a} = \frac{-(-1) \pm \sqrt{1^2 - 2(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

The positive value $x = \frac{1 + \sqrt{5}}{2}$, which is the length to width ratio of Euclid's special rectangles, has been termed the golden ratio and has been labelled φ (phi) by some mathematicians. Such rectangles have an apparent aesthetic or harmonious quality which for centuries has made them the shapes of choice for painters, architects, and sculptors.

The golden ratio has many curious properties. A decimal approximation of this irrational number is 1.61803398. Oddly, its reciprocal is 0.61803398: phi is the only positive number that turns into its own reciprocal by subtracting 1. This result becomes obvious when we alter the original defining equation $x^2 - x - 1 = 0$ to the form $x - 1 = \frac{1}{x}$.

As with the irrational numbers π and e, the golden ratio can be expressed as an infinite series. From the equation above, we know that $\varphi = x = 1 + \frac{1}{x}$. By repeatedly substituting $(1 + \frac{1}{x})$ for x into the same expression $(1 + \frac{1}{x})$, we obtain the following continued fraction as an equivalent for the golden ratio:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

Like the ubiquitous π, the golden ratio phi crops up in some surprising places. Consider, for instance, the famous Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ wherein each term (with the exception of the first two) is the sum of its two predecessors. It can easily be shown that the ratio of any two consecutive terms of the Fibonacci sequence approximates the value of φ. The farther out in the progression we go, the closer the quotient of consecutive Fibonacci numbers is to the golden ratio. So 5/3 (or 1.66) is a fairly close approximation; 8/5 (1.6) is closer, and 21/13 (1.6153846) is closer yet. By the time we compute the ratio of the 20th Fibonacci number to the 19th, (i.e., $\frac{6765}{4181}$), we have arrived at a very close approximation (1.618034) to the irrational golden ratio $\frac{1 + \sqrt{5}}{2}$. Actually, the golden ratio can be derived from

any recursive sequence (such as 4, 5, 9, 14, 23, 37...) wherein each term is the sum of the two preceding terms.

There is even a connection between the golden ratio and Pascal's triangle, since the Fibonacci sequence is hidden in the triangle. Its exact location I leave to the reader to discover.

Consider the golden rectangle ABCD in Figure F. By definition, the smaller rectangle LBCM is also golden. Dividing this latter rectangle in turn by Euclid's prescription, we obtain a third rectangle of the golden variety, namely, LBON. This process of dividing each rectangle into a square and smaller rectangle can theoretically be carried on indefinitely, producing an infinite array of golden rectangles.

Figure F
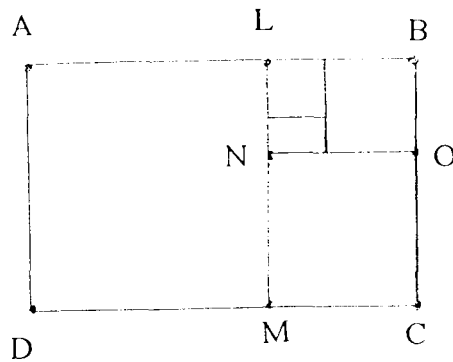
By joining two opposite corners of each square with a quarter circle of radius equal to the side of the square, we end up with a geometric form appropriately called a golden spiral. (See Figure G.)

Figure G
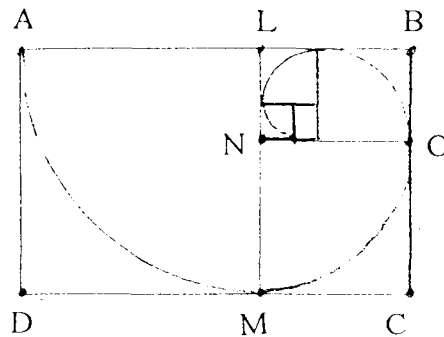
This spiral coils inwards to infinity, and by drawing successively larger squares (starting with a square of dimension DC), we can make it coil outwards to infinity as well. The golden spiral is often found in nature: both mollusk shells and galaxies expand in spirals which do not alter their shapes as they grow. It is also closely related to the Fibonacci progression which is used to describe certain patterns in natural growth. For instance, the spiral arrangement of seeds in a sunflower and the spacing of leaves or petals on certain plants can be described using Fibonacci numbers. Try to spot the Fibonacci number pattern on the stem shown in Figure H below.

<u>Figure H</u>
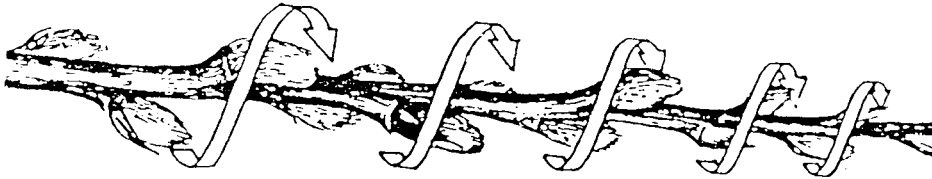
In sum, the golden ratio is second probably only to π in its frequency of occurrence in mathematics. I leave the reader with the diagram in Figure J. Your challenge is to demonstrate that this construction is indeed a golden rectangle.

<u>Figure J</u>
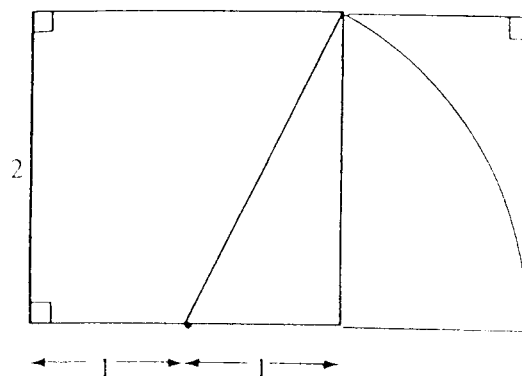
Fractals

The idea of carrying on a geometric process indefinitely (i.e., of performing an "infinite number" of geometric constructions upon an object) is fundamental to the concept of a fractal. Consider the equilateral triangle (stage 0) on the left in Figure K below.
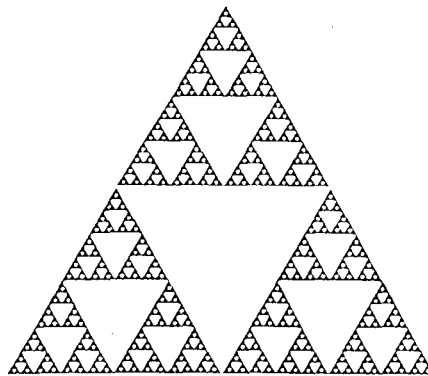
Figure K

Stage 0   Stage 1   Stage 2   Stage 3   Stage 4
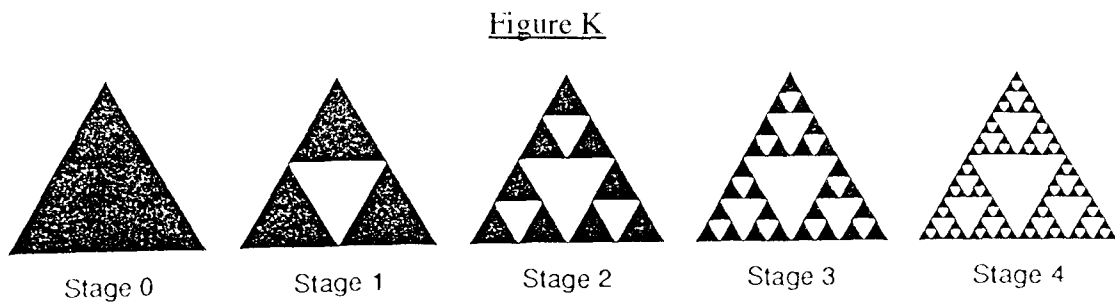
Bisecting its sides and joining the midpoints, we end up with 4 smaller triangles as shown at stage 1. Bisecting the sides of the 3 *upward pointing* triangles in turn, we are left with the 9 smaller upward pointing triangles shown in black at stage 2. We then repeat this procedure: bisecting the sides of these 9 triangles results in 27 new upward pointing black triangles (stage 3). At this stage, it is possible to make a conjecture. How many triangles will there be at stage 4? At stage 5? At stage n?

Assume that there are 3^n black triangles at stage n. Now imagine what happens to the figure as n, the number of stages, becomes larger and larger (or in mathematical terms, *large without bound*). The ultimate figure which results from all of those bisections is called the *Sierpinski triangle*. (See Figure L.) It has also been coined the *Sierpinski gasket*, since by mentally removing all "downward pointing" white triangles, we can consider the triangle to be full of holes .

Let us assume that we have removed an infinite number of white triangles from the original. The resulting Sierpinski triangle has a remarkable feature: it is *self-similar*. Imagine that

we have a microscope which can enlarge any part of the original triangle. Let us focus on the lower right triangle which was formed at stage 1 of the process. If we magnify this triangle by a factor of 2, we will see an exact replica of the entire Sierpinski triangle. Note that we have also "removed" infinitely many white triangles from this smaller triangle. Likewise, train the microscope on one of the even smaller upward-pointing triangles formed at stage 2. Magnifying this triangle by a factor of 4, we see yet another exact copy of the original. This is self-similarity: no matter how deeply we peer into the Sierpinski triangle, we will always find triangles which are exactly the same as the original.

Figure L

Several characteristics of fractals are illustrated by the Sierpinski triangle. First, an iterative process (usually involving a large number of steps which we can imagine approaches infinity) results in self-similar figures. Both Figures K and L were drawn by computers, which with their speed in calculation and efficiency in drawing, have resulted in the recent discovery of many complex fractals such as the Mandelbrot set. Another feature of fractals is their dimensionality. This concept will not be dealt with here except to ask the reader, "In how many planar directions can one move in the Sierpinski triangle? Are there essentially 2 directions as there are in a square, or is there only 1 as on a line? Oddly enough, the true answer lies somewhere between 1 and 2.

I finish Appendix 7 with a connection between the Sierpinski triangle and the seemingly omnipresent Pascal's triangle. On a copy of the grid below (Figure M), shade in all the cells containing entries which are odd. If the entry is even, leave it unshaded or white. You will soon discern a pattern which will obviate the task of considering the oddness or evenness of each number. The final result, the work of one of my students on an extended triangle, is shown in Appendix 5.

<u>Figure M</u>

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5   10  10  5   1
    1   6   15  20  15  6   1
  1   7   21  35  35  21  7   1
1   8   28  56  70  56  28  8   1
1   9   36  84  126 126 84  36  9   1
1   10  45  120 210 252 210 120 45  10  1
```
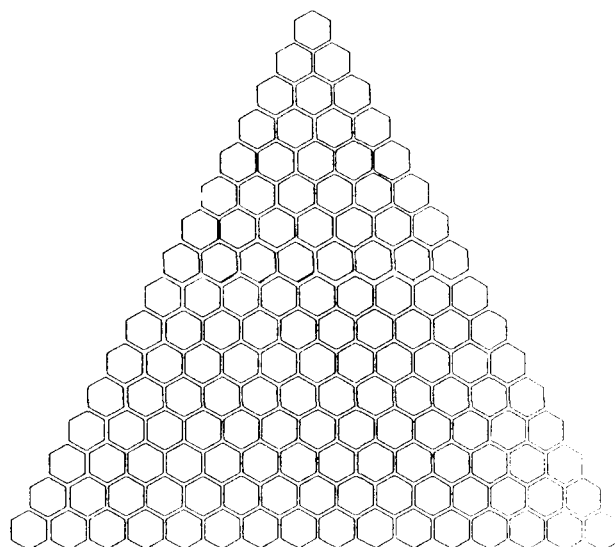
<u>Conclusion</u>

In addition to discussing the mathematics offered in the case study, my aim in writing this appendix is to present my own appreciation of both the beauty and the challenge inherent in mathematics. All too often, mathematics is considered a collection of techniques. Doing mathematics is frequently equated with the recall of memorized formulae and theorems, the manipulation of algebraic symbols, and the calculation of correct answers to problems. For me, mathematics is the branch of human thought which deals with patterns and relationships (or the lack thereof) between things. Learning mathematics means learning to become aware not only of

such relationships, but also of the "dynamics of the mind itself as it is involved in any functioning" (Gattegno, 1974, p.81). Doing mathematics is the art and science of using these awarenesses to explore and discover new relationships and patterns. Finally, by no means should the learning and doing of mathematics be considered as separate activities.

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