Fixed Length Steady-State Crack in Bending Problem for a General Viscoelastic Material

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FIXED LENGTH STEADY-STATE CRACK IN BENDING

PROBLEM FOR A GENERAL VISCOELASTIC MATERIAL

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Abstract

The discrete spectrum model for a general viscoelastic material expresses the shear relaxation and creep functions as a sum of decaying exponentials and a constant term. This idealization of real viscoelastic materials provides an adequate and computationally efficient model for practical problems. The problem of a crack in a viscoelastic material subject to an asymmetric bending moment causing closure at only one end is solved for the case of a discrete spectrum model in that equations for displacement and pressure are obtained. These equations have kernels which are infinite series of integrals involving the creep and relaxation functions of the material. In the case of a standard linear model, evaluation of these kernels can be reduced to summation of geometric series. For the more general spectrum model, this method breaks down. A different method is used to express the kernels of the integral equations for displacement and pressure as solutions of other integral equations which allows them to be determined in closed form. The specific model ($N = 2$) is studied in detail and the results of numerical calculations are presented. The standard linear model ($N = 1$) is recovered as a special case. The method used is of considerable generality, having been previously applied to the three-dimensional normal indentation problem and the problem of a fixed length crack under a fluctuating normal load.

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Chapter 1

Introduction

1.1 The Viscoelastic Model

The very simplistic but yet effective visualization of viscoelasticity as a combination of hookean solid-like and newtonian liquid-like characteristics captures the fundamentals that underlie polymer behaviour on a macroscopic level. Yet, to fully appreciate the molecular basis of viscoelastic phenomena, it is advantageous to be aware of some basic polymer structure.

A polymer is a chain of molecules which contains atoms held together by covalent bonds. When monomer molecules react together by means of a chemical process called polymerization, either linear chains or a three-dimensional network of polymer chains is produced. Some examples of this structure are shown in Fig.(1.1). The main characteristic of such chains is that the chemical bonding is strong and directional along the chains, but bonding sideways is by weak Van DerWaals forces.

There are three main classes of polymers : thermoplastics, elastomers and thermosets. The viscoelastic properties of the rubberlike materials classified as elastomers are the focus of this thesis. As machine elements, perhaps the most important function of elastomers is their ability to attenuate vibrations. Two examples of applications in civil engineering are bridge bearings and earthquake-proof foundation isolation in skyscraper buildings as discussed by Moore [1]. Such resilient layers of viscoelastic material effectively allow movement due to changing temperature as well as static and

(1) (2) (3)

Figure 1.1: Schematically represented polymer molecules : (1) linear; (2) branched; and (3) network.

dynamic loading. Additionally, resilient seatings prevent corrosion, wear, the ingress of water and moisture as well as acting as a barrier against noise transmisssion. Table 1.1 presents a few viscoelastic materials and their more interesting features and applications. A more detailed characterization is given by Hertzberg [2].

The importance of such materials in our daily lives has generated considerable interest in the phenomenological theory of linear viscoelasticity. Summarizing phenomenological theory in an elegant and attractive form offers some mathematical challenges as well as practical benefits. The calculations are of practical value in permitting prediction of the behaviour of a material in a certain situation perhaps inaccessible to direct experiment. In addition, the panorama of time or frequency scale of a function's behaviour can be mapped out. To this end, various assumptions have been made about the form of the relaxation function. Golden and Graham [3], [4] state that there are perhaps three broad categories

1. Continuous and discrete spectrum models and degenerate forms of these;

2. Power law and associated models;

3. Forms deriving from molecular theories.

The development of a discrete spectrum model for the problem of a fixed length crack subject to an asymmetric bending moment is the focus of this thesis.

Material	Major Characteristics	Applications
Polymethyl-methacrylate (Plexiglas)	Amorphous; brittle; general replacement for glass	Signs; canopies; windows
Polyvinyl chloride	Primarily amorphous; fire self-extinguishing; relatively inexpensive	Floor coverings; film; toys; water pipes
Polytetra-fluoroethylene (teflon)	Extremely high molecular weight; extraordinary resistance to chemical attack; nonsticking	Cooking utensil coatings; bearings and gaskets; nonstick, load-bearing pads
Nylon 66	Excellent wear resistance; high strength	Fabric; light machinery components; wheels; pulleys, rollers

Table 1.1: Characteristics of Selected Polymers

Certain molecular theories, for example Rouse theory as described by Ferry [5], do indeed predict discrete spectra. In fact, any experimentally observed stress relaxation curve which decreases monotonically can in principle be fitted with any desired degree of accuracy to such a model by taking a sufficiently large number of terms. In the analysis of experimental data, however, it is difficult to resolve more than a few decay times, and it has been found that most viscoelastic materials do not decay exponentially but instead obey a power law [5].

In spite of these apparent difficulties, the discrete spectrum model is widely used by engineers to solve problems. The main reasons are, first, as was already mentioned, the ease with which data can be fit and second, from the computational view point, the exponential model provides much greater efficiency for a desired degree of accuracy when compared to a corresponding power law function.

1.2 The History of the Crack Problem

The problem under consideration is that of a fixed length crack in a viscoelastic medium subject to an asymmetric bending moment. Asymmetry is arrived at by having the bending moment spend more time and have a greater maximum in one direction than in the other. Plane strain conditions are presumed, and a discrete spectrum model is considered for the viscoelastic material. Equations for the displacement and pressure are obtained, and the "angular advance" in the crack opening and "angular delay" in the crack closing are considered to solve the problem.

This type of problem was considered in the elastic case by Bowie and Freese [6] and Comninou and Dundurs [7]. The crack was found to close over part of its length, including one end, with the end which closed depending on the sign of the applied bending moment. Also, the closures were instantaneous over the entire extent of these regions, rather than gradually starting at the end.

The Classical Correspondence Principle is known to cover the problem of a fixed length or stationary crack in a viscoelastic medium where the crack has always been open [4]. This means that its solution is closely related to the corresponding elastic solution. A less trivial problem which has been considered by Graham and Sabin [8] and Graham [9] is that where crack closure can occur. They present solutions and detailed numerical results for Maxwell and Voigt materials as well as for the more general standard linear model.

The generalized Classical Correspondence Principle has been used [9] to solve the problem of a viscoelastic crack in a field of pure bending for certain bending histories and materials. The solution is obtained from elastic solutions some of which involve material overlap and some of which do not. This solution suggests that a viscoelastic crack can close down gradually when its elastic analogue would close down instantaneously.

This same problem is approached by Golden and Graham [10] more directly by solving the equations of Linear Viscoelasticity rather than using the Correspondence Principle. The solutions emerge naturally and not as a result of an intuitively based, detailed construction. In essence, it is a boundary value problem where conditions on the boundary are not known *a priori*, but must be determined in the course of solving the problem. Two noteworthy physical results are obtained. Firstly, that in contrast to the elastic case, smooth closure occurs. Secondly, that there exists a frequency-dependent class of materials for which the crack does not close (at least initially) at the negative end. It is thus concluded that, if a steady-state phenomenon of this kind exists, then the fracture propagation properties of such materials will exhibit interesting asymmetric features.

The solution of the problem of a crack in a viscoelastic medium subjected to a sinusoidal bending moment is formulated in terms of non-singular integral equations in space and time by Golden, Graham and Trummer [11]. The integral equation method used to solve the problem is of considerable generality having been used previously to solve problems with transversely moving indentors on viscoelastic media. The numerical treatment yields physically meaningful results, most notably, that in contrast to elastic behaviour, the partial closure is not instantaneous. An approximate method of solution which is simple and gives good results is also presented.

Previously, Golden and Graham [12], investigated steady-state solutions which close at only one end of the crack subjected to an alternatively positive and negative asymmetric bending moment. The method developed is in general terms the same as that applied to obtain the steady-state solutions of the normal contact problem [13]. Numerical results are presented for the standard linear model. Viscoelastic behaviour is characterized by the angular delay, which is a measure of the delay in closure once the stress has become compressive on the positive end of the crack, and the angular advance, a measure of how early the crack opens before the stress has become tensile. Both of these are a result of creep effects from previous cycles tending to keep the crack open. The relationships derived provide the basis for the work presented in this thesis.

A more general approach, which is applicable to materials with viscoelastic response given by discrete spectrum models, is presented by Golden and Graham [14] for the case of a fixed length crack under a fluctuating normal load. The relaxation and creep functions are given by a sum of decaying exponentials and a constant term. It is shown that the kernels of the integral equations for pressure and displacement obey integral equations which allow them to be determined in closed form.

In the context of a different problem, namely that of three-dimensional steady-state indentation, Golden, Graham and Lan [15] extend the method to a contact problem for a viscoelastic material described by a discrete spectrum model. Considerable analytical progress is made before resorting to numerical calculation.

The results of both of these works [14], [15] are called upon extensively to obtain the equations describing the fixed length crack subject to an asymmetric bending moment which causes closure only at one, the positive, end. Chapter 2 discusses some of the fundamental background of the problem based on the work of Muskhelishvili [17]. By using complex potentials and conformal maps, some of the most powerful tools of mathematics, the two-dimensional problem is cast in the form of a Hilbert problem from the Kolosov-Muskhelishvili equations. In Chapter 3, the decomposition of hereditary integrals is used to obtain expressions for displacement and pressure which are then completely determined for the general viscoelastic model in Chapter 4 by obtaining the integral equations for the kernels. Finally, Chapter 5 presents numerical results for the case of the crack being subjected to an asymmetric sinusoidal bending moment.

Chapters 4 and 5, except where the standard linear model is described, form the contribution to the acquired body of knowledge in this field. The closed form representation of displacement and pressure as a function of the general viscoelastic model parameters and the applied bending moment is the most significant result. For the special case of an asymmetric bending moment, constraints for the $N = 2$ general viscoelastic model parameters are numerically obtained such that closure of the crack is restricted to the positive end. These constraints are of practical importance in that they determine a specific class of viscoelastic materials to which the analytical results may be applied.

Chapter 2

The Fundamental Equations

To establish a general acquaintance with the nature of crack problems in viscoelasticity, a brief presentation will be made of the constitutive relations, dynamical equations and the form of the boundary conditions that characterize the boundary value problem for the crack subject to an asymmetric bending moment.

2.1 Plane Non-inertial Crack Problems

The non-inertial approximation for viscoelastic materials is analagous to the approximation in the theory of simple harmonic motion through a frictional medium, where the acceleration term is neglected compared with the frictional resistance term. In materials with high internal friction losses, inertial effects depending on ρ, the density of the material, may be neglected compared with viscous effects. This non-inertial approximation results in a major simplification of the equations of motion

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i \tag{2.1}$$

where $u_i(\vec{r}, t)$ are the components of the displacement vector $\vec{u}(\vec{r}, t)$ giving the displacement of the point $\vec{r}$ from its equilibrium position, b_i are the contributions of body forces and σ_{ij} are the Cartesian components of stress. The acceleration term is dropped from (2.1) to obtain

$$\sigma_{ij,j}(\vec{r},t) = 0 \tag{2.2}$$

in the absence of body forces.

Now let us confine our attention to the case where the crack lies along a single line in an infinite viscoelastic medium, taken to be the x-axis. Consider the stress distribution of such a problem. If the applied stresses at infinity are subtracted from this stress distribution, thus giving the distribution for the problem where the stresses tend to zero at infinity and have uniform applied stress, of the same magnitude and opposite sign, on the open crack faces, then this distribution obeys (2.2). This follows from the fact that the applied stresses, which are independent of position, contribute nothing.

For problems involving a crack in a field of bending, at distances far from the origin, the stress is zero on the open face and linear in x along the x-axis. By subtracting this linear term, we obtain a linear stress distribution on the open crack face and no divergent stresses at infinity. Also, the equilibrium equations still hold.

2.2 The Kolosov-Muskhelishvili Equations

A very powerful complex variable technique for solving two-dimensional boundary value problems in elasticity is based on the Kolosov-Muskhelishvili equations [4]. The viscoelastic Kolosov-Muskhelishvili equations also provide a useful starting point for considering two-dimensional viscoelastic boundary value problems.

As in the case of the contact problem, the approach used for the crack problem is based on the viscoelastic Kolosov-Muskhelishvili equations given in general form by

$$\sigma_{11}(\vec{r},t) + \sigma_{22}(\vec{r},t) = 2[\phi(z,t) + \bar{\phi}(\bar{z},t)] = 4Re\{\phi(z,t)\}, \tag{2.3}$$

$$\Sigma(\vec{r},t) = \sigma_{22}(\vec{r},t) - i\sigma_{12}(\vec{r},t) = \phi(z,t) + \bar{\phi}(\bar{z},t) + z\bar{\phi}'(\bar{z},t) + \bar{\psi}(\bar{z},t), \tag{2.4}$$

$$2\int_{-\infty}^{t} dt'\mu(t-t')D'(r,t') = \int_{-\infty}^{t} dt'\kappa(t-t')\phi(z,t') - \bar{\phi}(\bar{z},t) - z\bar{\phi}'(\bar{z},t) + \bar{\psi}(\bar{z},t), \quad (2.5)$$

where

$$D'(\vec{r},t) = \frac{\partial}{\partial x}[u_1(\vec{r},t) + iu_2(\vec{r},t)]. \quad (2.6)$$

The quantity $\kappa(t)$ is related to $\nu(t)$, the generalized Poisson's ratio for the material by

$$\hat{\kappa}(\omega) = 3 - 4\hat{\nu}(\omega) \quad (2.7)$$

where $\hat{\nu}(\omega)$ is given by

$$\hat{\nu}(\omega) = \frac{\hat{\lambda}(\omega)}{2[\hat{\lambda}(\omega) + \hat{\mu}(\omega)]} \quad (2.8)$$

and $\hat{\nu}(\omega)$, $\hat{\lambda}(\omega)$ may be regarded as a generalization of Lamé's constants. The ˆ denotes the Fourier transform. $\Sigma(\vec{r},t)$ and $D'(\vec{r},t)$ are sometimes referred to as the complex stress and the complex displacement derivative, respectively. The causal function $\mu(t)$ [4] is given by

$$\mu(t) = G(0)\delta(t) + \dot{G}(t)H(t) \quad (2.9)$$

where $H(t)$ is the Heaviside step function defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \quad (2.10)$$

and the singular Delta function is

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0, \end{cases} \quad (2.11)$$

$$\int_{-\infty}^{\infty} dt\delta(t) = 1. \quad (2.12)$$

The relaxation function of the material is $G(t)$.

For the problem of a crack in an infinite plane, assuming that stresses and rotations vanish at infinity and the resultant of all the forces acting on the crack face is assumed to cancel to zero, then the complex potentials $\phi(z,t)$, $\psi(z,t)$ behave as

$$\phi(z,t) \sim O(\frac{1}{z^2}), \qquad \psi(z,t) \sim O(\frac{1}{z^2}) \tag{2.13}$$

at large z. The functions $\phi(z,t)$ and $\psi(z,t)$ are analytic everywhere except on some or all of the real axis.

It is interesting to note that in the case of a half-plane under load, assuming again that the stresses and rotations vanish at infinity and the boundary stresses along the x-axis fall off as $\frac{1}{x^2}$ or faster at large distances from the origin, then at large $|z|$

$$\phi(z,t) \sim -\frac{Q}{2\pi z}, \qquad \psi(z,t) \sim \frac{\bar{Q}}{2\pi z} \tag{2.14}$$

where $Q = X + iY$ is the resultant of the external forces acting on the x-axis [4]. This implies that $\Phi(z,t)$, $\Psi(z,t)$ defined by

$$\Phi'(z,t) = \phi(z,t), \qquad \Psi'(z,t) = \psi(z,t) \tag{2.15}$$

have logarithimic singularities at infinity and, it means physically, that absolute (as opposed to relative) displacements cannot be calculated. Clearly, from (2.13) and (2.14), this difficulty will not arise in crack problems.

The method generally employed to solve the non-inertial crack problem is, in the first instance, to reduce (2.3), (2.4), (2.5) to a Hilbert problem in precisely the manner developed by Muskhelishvili (1963), and then to handle the specifically viscoelastic aspects.

2.3 The Hilbert Problem

With the following definitions, we first explicitly state the boundary conditions. Let the region of the x-axis, on the crack face, be $F(t)$, made up of two disjoint sets, $O(t)$ and $C(t)$, $O(t)$ being the region on which the crack face is open and $C(t)$ being the region on which it is closed. All stresses are zero at infinity and off the crack face, and the displacements are continuous everywhere, in particular along the x-axis. Then, on $O(t)$, we have

$$u_2(x, 0^+, t) - u_2(x, 0^-, t) > 0, \tag{2.16}$$

$$\sigma_{22}(x, 0^+, t) = \sigma_{22}(x, 0^-, t) = -p(x, t), \tag{2.17}$$

$$\sigma_{12}(x, 0^+, t) = \sigma_{12}(x, 0^-, t) = -s(x, t), \tag{2.18}$$

where $0^\pm$ denotes y approaching zero from above (+) and below (−). The quantities $p(x, t)$, $s(x, t)$ are the specified pressure and shear on the crack face. On $C(t)$, frictional forces between the faces will be neglected. Therefore, on $C(t)$,

$$u_2(x, 0^+, t) - u_2(x, 0^-, t) = 0, \tag{2.19}$$

$$\sigma_{22}(x, 0^+, t) = \sigma_{22}(x, 0^-, t) < -p(x, t), \tag{2.20}$$

$$\sigma_{12}(x, 0^+, t) = \sigma_{12}(x, 0^-, t) = -s(x, t). \tag{2.21}$$

From (2.4), this has the consequence that

$$\begin{aligned} &\phi^+(x,t) + \bar{\phi}^-(x,t) + x\bar{\phi}^{-\prime}(x,t) + \bar{\psi}^-(x,t) \\ &\quad = \phi^-(x,t) + \phi^+(x,t) + x\bar{\phi}^{+\prime}(x,t) + \bar{\psi}^+(x,t) \end{aligned} \tag{2.22}$$

at every point on the x-axis, where $\phi^\pm(x, t)$ are the limits of $\phi(z, t)$ from above and below the real axis. We write (2.22) as

$$\begin{aligned} &\phi^+(x,t) - \bar{\phi}^+(x,t) - x\bar{\phi}^{+\prime}(x,t) - \bar{\psi}^+(x,t) \\ &\quad = \phi^-(x,t) + \bar{\phi}^-(x,t) - x\bar{\phi}^{-\prime}(x,t) - \bar{\psi}^-(x,t), \end{aligned} \tag{2.23}$$

so that the function $\phi(z,t) - \bar{\phi}(z,t) - z\bar{\phi}'(z,t) - \bar{\psi}(z,t)$ is analytic everywhere. By virtue of (2.13) it is also zero at infinity. It follows from Liouville's Theorem that it is zero everywhere so that [4]

$$\bar{\psi}(\bar{z},t) = \phi(\bar{z},t) - \bar{\phi}(\bar{z},t) - \bar{z}\bar{\phi}'(\bar{z},t), \tag{2.24}$$

and we write (2.3), (2.4), (2.5) as

$$\sigma_{11} + \sigma_{22} = 2[\bar{\phi}(\bar{z},t) + \phi(z,t)] = 4Re\{\phi(z,t)\} \tag{2.25}$$

$$\sigma_{22} - i\sigma_{12} = \phi(z,t) + \phi(\bar{z},t) + (z-\bar{z})\bar{\phi}'(\bar{z},t) \tag{2.26}$$

$$2d(\vec{r},t) = [\kappa * \phi](z,t) - \phi(\bar{z},t) - (z-\bar{z})\bar{\phi}'(\bar{z},t) \tag{2.27}$$

where

$$d(\vec{r},t) = \int_{-\infty}^{t} dt'\mu(t-t')D'(\vec{r},t') \tag{2.28}$$

and the convolution notation

$$\int_{-\infty}^{\infty} dt'\kappa(z,t-t')\phi(z,t') = \int_{-\infty}^{\infty} dt'\phi(z,t-t')\kappa(z,t') = [\kappa * \phi](z,t) \tag{2.29}$$

has been used. Approaching the x-axis from above and below in (2.27), and subtracting gives

$$2[d(x,0^+,t) - d(x,0^-,t)] = (1+\kappa) * [\phi^+(x,t) - \phi^-(x,t)] \tag{2.30}$$

or

$$\frac{1}{2}\int_{-\infty}^{t} dt'l(t-t')\Delta'(x,t') = \phi^+(x,t) - \phi^-(x,t), \tag{2.31}$$

where $l(t)$ is defined by

$$\hat{l}(\omega) = \frac{4\hat{\mu}(\omega)}{1+\hat{\kappa}(\omega)} \tag{2.32}$$

and $\Delta'(x,t)$ is the x derivative of the complex displacement across the gap, given by

$$\Delta'(x,t) = D'(x,0^+,t) - D'(x,0^-,t). \tag{2.33}$$

In the case of fully open cracks that are stationary or growing, $C(t)$ is empty and $F(t)$ is expanding or stationary. We expect to obtain solutions closely related to the corresponding elastic solutions since this is the type of problem covered by the Extended Correspondence Principle [4]. For the crack problem, whenever $\phi(z,t)$ is known, all the quantites of interest can be evaluated from the Kolosov-Muskhelishvili equations.

From (2.26) we obtain

$$\Sigma(x,t) = -p(x,t) + is(x,t) = \phi^+(x,t) + \phi^-(x,t), \qquad x \in F(t). \tag{2.34}$$

Also, for $x \in F'(t)$ (the complement of $F(t)$ on the x-axis) off the crack face, the left-hand side of (2.31) is known to be zero for all t, since if $x \in F'(t)$ it follows that $x \in F'(t')$, $t' \leq t$ and the complex displacement $\Delta(x,t')$ is zero for all $t' \leq t$. Therefore, the function $\phi(z,t)$ is continuous on $F'(t)$. This condition together with (2.34) constitutes a Hilbert problem, the solution of which is

$$\phi(z,t) = \frac{X(z,t)}{2\pi i}\int_{F(t)} dx' \frac{\Sigma(x',t)}{X^+(x',t)(x'-z)} + P(z,t)X(z,t) \tag{2.35}$$

where $P(z,t)$ is a polynomial, as yet undetermined, and $X(z,t)$ is given by

$$X(z,t) = \{\prod_{i=1}^{n}[z-a_i(t)][z-b_i(t)]\}^{-\frac{1}{2}}, \tag{2.36}$$

the union of intervals $[a_i(t), b_i(t)]$ being $F(t)$. We choose the branch of $X(z,t)$ such that $z^n X(z,t) \rightarrow 1$ as $|z| \rightarrow \infty$. Note that singularities have been allowed at the end points since it is not clear *a priori* whether they can be excluded.

For the case of a single crack, the correct behaviour at infinity is obtained by choosing $P(z,t) = 0$.

2.4 The Problem of a Crack Subject to an Asymmetric Bending Moment

The constitutive equations will be written in the form

$$\sigma_{ij}(r,t) = 2\int_{-\infty}^{t} dt'\mu(t-t')\epsilon_{ij}(r,t') + \delta_{ij}\int_{-\infty}^{t} dt'\lambda(t-t')\epsilon_{kk}(r,t'), \quad i,j = 1,2,3 \quad (2.37)$$

where $\sigma_{ij}(r,t)$ and $\epsilon_{ij}(r,t)$ are the stress and strain tensors at position $r = (x_1, x_2, x_3) = (x, y, z)$ and time t. Both $\mu(t)$ and $\lambda(t)$ are zero for negative t and

$$2\mu(t) = \frac{d}{dt}[H(t)G_1(t)] = \delta(t)G_1(0) + H(t)\dot{G}_1(t) \quad (2.38)$$

$$3\lambda(t) = \delta(t)(G_2(0) - G_1(0)) + H(t)(\dot{G}_2(t) - \dot{G}_1(t)) \quad (2.39)$$

where $H(t)$ is the Heaviside step function and δ(t) is the singular Delta function described by equations (2.10), (2.11) and (2.12). The relaxation functions of the material for shear and dilation are G_1 and G_2. The integrals are taken up to t^+ so the entire contribution of the Delta function is included.

The Fourier transform of the constitutive equations, using the Faltung theorem yields

$$\bar{\sigma}_{ij}(r,\omega) = 2\bar{\mu}(\omega)\bar{\epsilon}_{ij}(r,\omega) + \delta_{ij}\bar{\lambda}(\omega)\bar{\epsilon}_{kk}(r,\omega), \quad (2.40)$$

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} dt e^{-it\omega} f(t). \quad (2.41)$$

Consider the problem of a crack under plane strain conditions of which we consider the particular cross-section of the medium lying in the xy plane. The crack is assumed to be along the x-axis occupying the fixed interval $[-c, c]$. The neutral axis of the crack in a field of pure bending is the y-axis. The ratio of the bending moment to the moment of inertia of the cross section about the y-axis will be termed $\eta(t)$. The sign convention is that when $\eta(t)$ is positive, there will be compression across the positive

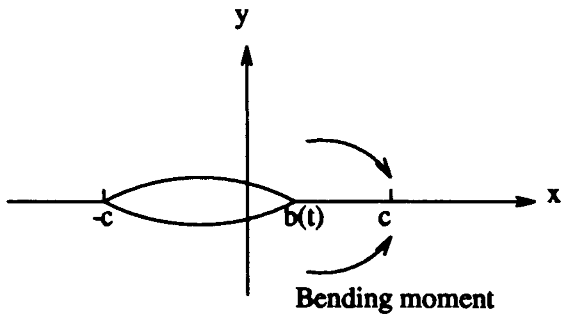

Figure 2.1: Partially closed crack along the x-axis.

x-axis, at least at points far from the origin. It will be assumed that closure will occur only at the positive end. Then on the open portion of the crack

$$p(x,t) = -\sigma_{22}(x,t) = -\eta(t)x. \tag{2.42}$$

The boundary conditions, neglecting friction between the faces, may be stated as follows : on the open portions of the crack from (2.16), (2.17), (2.18)

$$u_2(x,0^+,t) - u_2(x,0^-,t) > 0, \tag{2.43}$$

$$\sigma_{22}(x,0^+,t) = \sigma_{22}(x,0^-,t) = \eta(t)x, \tag{2.44}$$

$$\sigma_{12}(x,0^+,t) = \sigma_{12}(x,0^-,t) = 0, \tag{2.45}$$

and on the closed portions from (2.19), (2.20), (2.21)

$$u_2(x,0^+,t) - u_2(x,0^-,t) = 0, \tag{2.46}$$

$$\sigma_{22}(x,0^+,t) = \sigma_{22}(x,0^-,t) < \eta(t)x, \tag{2.47}$$

$$\sigma_{12}(x,0^+,t) = \sigma_{12}(x,0^-,t) = 0 \tag{2.48}$$

in terms of the stress tensor $\sigma_{ij}(x,y,z)$, $i,j = 1,2,3$ and the normal displacement $u_2(x,y,z)$. The contribution of $\eta(t)$ is included explicitly in the boundary conditions which is equivalent to adding a term $\eta(t)x$ to the conventional stress component σ_{22}. From the symmetry of the problem $u_2(x,0^+,z) = -u_2(x,0^-,z)$.

Denote $u_2(x,0^+,t)$ by $u(x,t)$ and its derivative with respect to x by $w(x,t)$. Then the relationship between displacement and pressure [10] valid in the quasi-static approximation is

$$\int_{-\infty}^{t} dt' l(t-t') w(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx' \frac{p(x,t')}{x'-x}; \qquad p(x,t) = -\sigma_{22}(x,0,t) \tag{2.49}$$

where l is the inverse Fourier transform of

$$\bar{l}(\omega) = \frac{\bar{\mu}(\omega)}{1-\bar{\sigma}(\omega)} \tag{2.50}$$

in which $\bar{\nu}(\omega)$ is the viscoelastic generalization of Poisson's ratio, given in terms of $\bar{\mu}(\omega)$, $\bar{\lambda}(\omega)$ by the standard formula.

Under the Correspondence Principle, the above relationship can be seen as the viscoelastic generalization of the standard relationship between displacement and pressure on the half-plane. If the variable x is in a region where $p(x,t)$ is non-zero, the integral on the right-hand side is understood to be a principal value integral.

Define $k(t)$, zero for negative t, to be the inverse of $l(t)$ in the sense that

$$\int_0^t dt' k(t-t') l(t') = \int_0^t dt' l(t-t') k(t') = \delta(t) \tag{2.51}$$

so that the above relationship can be rewritten as

$$w(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx' \frac{q(x',t)}{x'-x} \tag{2.52}$$

where

$$q(x,t) = \int_{-\infty}^{t} dt' k(t-t') p(x,t'). \tag{2.53}$$

If the open portion of the crack at time t is $[-c, b(t)]$, $b(t) \leq c$ we have

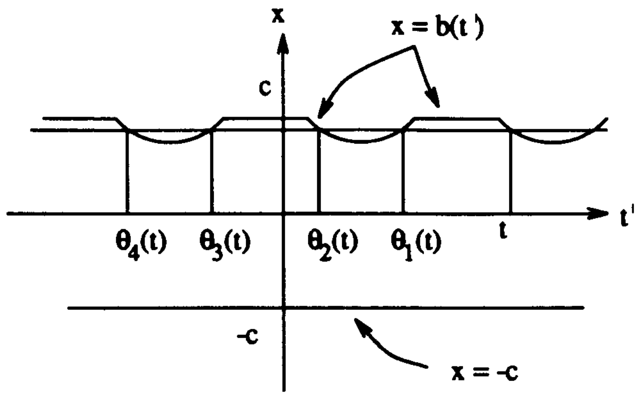

Figure 2.2: *A priori* possible behaviour of the quantity $b(t)$ schematically portrayed, with the quantities $\theta_r(t)$ indicated.

$$q(x,t) = -\frac{1}{\pi}\int_{-c}^{b(t)} dx' \frac{w(x',t)}{x'-x}. \tag{2.54}$$

Times t for which $b(t)$ is decreasing will be considered.

Chapter 3

Decomposition of Hereditary Integrals

It is of interest to observe that the Generalized Partial Correspondence Principle [4] may be used to generate the results of this chapter, but an approach based on the decomposition of hereditary integrals is taken to deduce rather than assume that closure of the crack is instantaneous. As such, the unknown quantity is $\Sigma(x', t)$ as given by (2.34) with $s(x, t)$ being equal to zero. In this chapter the problem will be transformed to determine $p(x, t)$ based on the decomposition of $q(x, t)$. Note that if $b(t)$ was considered to be increasing at time t, then a somewhat different set of equations could be constructed based on a decomposition of $p(x, t)$.

3.1 Decomposition Method

Now $q(x, t)$ will be decomposed analagously to [4], [13] as

$$q(x,t) = \int_{W_q(t)} dt'\Gamma_q(t,t')q(x,t') + \int_{W_p(t)} dt'\Gamma_p(t,t')p(x,t') \tag{3.1}$$

where $W_q(t)$ is the set of all times $t' \leq t$ such that $b(t') < b(t)$ and $W_p(t)$ is the set $t' \leq t$ such that $b(t') \geq b(t)$. Also

$$\Gamma_p(t,t') = N_0(t,t')R(t';\theta_1(t),t) + N_2(t,t')R(t';\theta_3(t),\theta_2(t)) + \dots \tag{3.2}$$

$$\Gamma_q(t,t') = N_1(t,t')R(t';\theta_2(t),\theta_1(t)) + N_3(t,t')R(t';\theta_4(t),\theta_3(t)) + \dots \tag{3.3}$$

where

$$R(t;t_2,t_1) = \begin{cases} 1, & t \in [t_2,t_1] \\ 0, & t \notin [t_2,t_1] \end{cases} \tag{3.4}$$

and the $N_r(t,t')$ are defined by

$$N_0(t,t') = k(t-t') \tag{3.5}$$

$$N_r(t,t') = \begin{cases} \int_{t'}^{\theta_r(t)} dt' N_{r-1}(t,t'')l(t''-t'), & r \text{ odd} \\ \int_{t'}^{\theta_r(t)} dt' N_{r-1}(t,t'')k(t''-t'), & r \text{ even} \end{cases} \tag{3.6}$$

where the $\theta_l(t)$, $l = 1,2,\dots$ are the earlier times, taken in order, at which $b(t') = b(t)$.

The second term on the right-hand side can be written as

$$\int_{W_p(t)} dt'\Gamma_p(t,t')p(x,t') = B(t)x, \qquad -c \le x < b(t), \tag{3.7}$$

$$B(t) = -\int_{W_p(t)} dt'\Gamma_p(t,t')\eta(t') \tag{3.8}$$

using the definition of $W_p(t)$ and the expression for $p(x,t)$.

Using the expression for $q(x,t)$ in the above and the definition of $W_q(t)$, we obtain

$$\frac{1}{\pi}\int_{-c}^{b(t)} dx' \frac{w(x',t) - v_c(x',t)}{x'-x} = -B(t)x, \qquad -c \le x < b(t), \tag{3.9}$$

where

$$v_c(x,t) = \int_{W_q(t)} dt'\Gamma_q(t,t')w(x,t'). \tag{3.10}$$

When the Hilbert transform is inverted, the following integral equation for $w(x,t)$ is obtained :

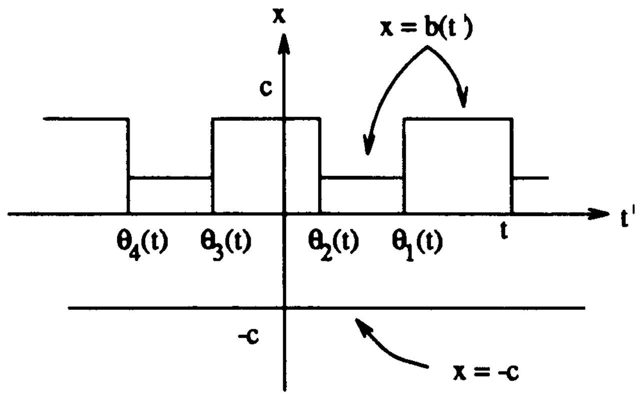

Figure 3.1: Behaviour of $b(t)$ for one-sided partial closure.

$$w(x,t) = \int_{W_q(t)} dt'\Gamma_q(t,t')w(x,t') + B(t)w_e(x,t), \quad -c \le x \le b(t) \tag{3.11}$$

where

$$w_e(x,t) = \frac{1}{m(x,t)}\left\{\frac{1}{2}\left(\frac{b(t)+c}{2}\right)^2 + x\left(\frac{b(t)-c}{2} - x\right)\right\} \tag{3.12}$$

$$m(x,t) = \{(b(t)-x)(x+c)\}^{\frac{1}{2}}. \tag{3.13}$$

The arbitrary constant that arises on inversion of finite Hilbert transforms can be put to zero because $u(x,t)$ vanishes at both ends of the open region and because, for $t' \in W_q(t)$, $b(t') < b(t)$. Let some partial closure be present at time t by considering $b(t) < c$. Then for $t' \in W_q(t)$ partial closure will also be present. It follows that in the above $w(x,t)$ and $w(x,t')$ will have no singularities at the positive end. Therefore, $w_e(x,t)$ can have no singularities either, at $x = b(t)$. So if partial closure occurs at the positive end only

$$b(t) = \frac{c}{3}, \tag{3.14}$$

$$w_e(x,t) = w_o(x) = \frac{1}{3}\left(\frac{\frac{c}{3}-x}{x+c}\right)^{\frac{1}{2}}(2c+3x)d_1(x), \tag{3.15}$$

$$d_1(x) = \begin{cases} 1, & -c < x \leq \frac{c}{3} \\ 0, & \frac{c}{3} \leq x < c. \end{cases} \tag{3.16}$$

The only other option permitted by the equation is a singularity at $x = c$, corresponding to a fully open crack. Therefore, we can only have instantaneous partial closing and reopening, of the kind that occurs in elastic theory, for a crack that is closing at only one end.

Since the choices of $W_p(t)$ and $W_q(t)$ adopted are not unique, emphasis will be shifted to $P(t)$ and $O(t)$ which will be defined as the sets of times $t' \leq t$ such that partial closure is present and for which the crack is fully open, respectively.

Let the crack be partially closed at time t so that the times immediately preceding time t must be regarded as belonging to $W_p(t)$. This suggests that $P(t) \subset W_p(t)$ and $O(t) \subset W_p(t)$. Thus $W_p(t) = (-\infty, t]$ and $W_q(t)$ is empty. Then during times of partial closure

$$w(x,t) = -M(t)w_0(x) \tag{3.17}$$

$$M(t) = \int_{-\infty}^{t} dt' k(t-t')\eta(t'). \tag{3.18}$$

When the crack is fully open it is natural [12] to put $W_p(t) = O(t)$, $W_q(t) = P(t)$ so that

$$w(x,t) = -A(t)w_0(x) + B(t)w_1(x) \tag{3.19}$$

$$w_1(x) = w_e(x,t)|_{b(t)=c} = \frac{1}{2}\frac{(c^2-2x^2)}{(c^2-x^2)^{\frac{1}{2}}} \tag{3.20}$$

and

$$A(t) = \int_{P(t)} dt' \Gamma_q(t,t')M(t'), \tag{3.21}$$

$$B(t) = -\int_{O(t)} dt'\Gamma_p(t,t')\eta(t'), \qquad t \in O(t), \tag{3.22}$$

where $\theta_r(t)$ are now the times of opening and closing. If $M(t), t \in O(t)$ is decomposed, we obtain

$$M(t) = \int_{P(t)} dt'\Gamma_q(t,t')M(t') + \int_{O(t)} dt'\Gamma_p(t,t')\eta(t') \tag{3.23}$$

$$= A(t) - B(t). \tag{3.24}$$

Let (3.19), (3.20) hold at all times but where during times of partial closure we have, instead of (3.21), (3.22), $A(t) = M(t)$; $B(t) = 0$; $t \in P(t)$. Note that it is always true that $A(t) - B(t) = M(t)$.

3.2 Equations for Displacement and Pressure

It follows from (3.19), (3.20) that the displacement is given by

$$u(x,t) = \frac{1}{2}A(t)(\frac{c}{3} - x)^{\frac{3}{2}}(x+c)^{\frac{1}{2}}d_1(x) + \frac{1}{2}B(t)x(c^2 - x^2)^{\frac{1}{2}}. \tag{3.25}$$

The condition that $u(x,t)$ remains non-negative and that no closure occurs at $x = -c$ is [12] for all t

$$A(t) \geq 0; \qquad B(t) \geq 0; \qquad \frac{A(t)}{B(t)} \geq \frac{3}{4}(\frac{3}{2})^{\frac{1}{2}}. \tag{3.26}$$

We then have

$$v(x,t) \equiv \int_{-\infty}^{t} dt'l(t-t')w(x,t') = -C(t)w_0(x) + D(t)w_1(x) \tag{3.27}$$

where

$$C(t) = \int_{-\infty}^{t} dt'l(t-t')A(t'); D(t) = \int_{-\infty}^{t} dt'l(t-t')B(t'). \tag{3.28}$$

Then by extending the integral in (2.54) over all space and transferring the hereditary integral from $q(x,t)$ to $w(x',t)$ we obtain

$$p(x,t) = -\frac{1}{\pi}\int_{-c}^{c} dx' \frac{v(x',t)}{x'-x} \tag{3.29}$$

$$= C(t)[-x + g(x)d_2(x)] + D(t)x, \qquad |x| < c, \tag{3.30}$$

where

$$g(x) = \frac{(\frac{2c}{3}+x)(x-\frac{c}{3})^{\frac{1}{2}}}{(x+c)^{\frac{1}{2}}} \tag{3.31}$$

$$d_2(x) = \begin{cases} 0, & x \in [-c, \frac{c}{3}] \\ 1, & x \in [\frac{c}{3}, c] \end{cases} \tag{3.32}$$

with the aid of the Hilbert transforms in [10], [4].

Since (2.42) must hold on the open part of the crack

$$C(t) - D(t) = \eta(t). \tag{3.33}$$

The boundary conditions require that

$$C(t) > 0, \qquad t \in P(t). \tag{3.34}$$

Note that

$$C(t) = 0, \qquad t \in O(t). \tag{3.35}$$

Decomposing the hereditary integral for $D(t)$, we obtain

$$D(t) = \int_{P(t)} dt' \Pi_c(t,t') B(t') + \int_{O(t)} dt' \Pi_0(t,t') D(t), \qquad t \in P(t) \tag{3.36}$$

where

$$\Pi_0(t,t') = \sum_{r=1,3,5\ldots} T_r(t,t') R(t'; \theta_{r+1}, \theta_r) \tag{3.37}$$

with $T_r(t,t')$ defined by

$$T_0(t,t') = l(t-t') \tag{3.38}$$

$$T_r(t,t') = \begin{cases} \int_{t'}^{\theta_r} dt'' T_{r-1}(t,t'')k(t''-t'), & r \text{ odd} \\ \int_{t'}^{\theta_r} dt'' T_{r-1}(t,t'')l(t''-t'), & r \text{ even} \end{cases} \tag{3.39}$$

and where $\theta_1, \theta_2, \ldots$ are the previous times of closing, opening etc.. Thus it can be deduced [12] that

$$D(t) = -\int_{O(t)} dt' \Pi_0(t,t')\eta(t') = -\sum_{r=1,3,5\ldots} \int_{\theta_{r+1}}^{\theta_r} dt' T_r(t,t')\eta(t'), \qquad t \in P(t). \tag{3.40}$$

Now the general strategy for solving the problem will be outlined.

Let the crack be open at time t and let the previous times of opening, closing etc., be $\theta_1, \theta_2, \ldots$ which are presumed to be known. Then from (3.2), (3.3), (3.8), (3.23), (3.33) and (3.35)

$$B(t) = -\sum_{r=0,2,4,\ldots} \int_{\theta_{r+1}}^{\theta_r} dt' N_r(t,t')\eta(t'), \tag{3.41}$$

$$A(t) = M(t) + B(t), \tag{3.42}$$

$$C(t) = 0, \tag{3.43}$$

$$D(t) = -\eta(t), \tag{3.44}$$

where $\theta_0 = t$. The time of the next partial closing, t_c, is determined by

$$B(t_c) = 0. \tag{3.45}$$

During the closed phase

$$A(t) = M(t), \tag{3.46}$$

$$B(t) = 0, \tag{3.47}$$

$$C(t) = \eta(t) + D(t), \tag{3.48}$$

where $D(t)$ is given by (3.40). The time of next opening, t_0, is determined by

$$C(t_0) = 0. \tag{3.49}$$

As shown in [12], the various coefficients are continuous at times of opening and closing.

3.2.1 Steady-State Limit

Consider $\eta(t)$ to be periodic with period Δ and assume that sufficient time has passed to establish steady-state conditions. Let $[t_0, t_c]$ be a given time period for which the crack is completely open and $[t_c, t_0 + \Delta]$ be the subsequent period when it is partially closed. For $t \in (t_0, t_c]$, we have

$$\theta_1 = t_0; \qquad \theta_2 = t_c - \Delta; \qquad \theta_3 = t_0 - \Delta, \qquad etc., \tag{3.50}$$

while for $t \in (t_c, t_0 + \Delta]$

$$\theta_1 = t_c; \qquad \theta_2 = t_0; \qquad \theta_3 = t_c - \Delta, \qquad etc.. \tag{3.51}$$

The coefficients $A(t)$, $B(t)$, $C(t)$, $D(t)$ will all have period Δ. Then $B(t)$, as given by (3.41), becomes

$$B(t) = -\int_{t_0}^{t} dt' k(t-t')\eta(t') - \int_{t_0}^{t_c} dt' \Gamma(t,t')\eta(t'), \qquad t \in (t_0, t_c], \tag{3.52}$$

where

$$\Gamma(t,t') = \sum_{k=1}^{\infty} N_{2k}(t, t' - k\Delta) \tag{3.53}$$

while (3.40) becomes

$$D(t) = -\int_{t_0}^{t_c} dt'\Pi(t,t')\eta(t'), \qquad t \in (t_c, t_0 + \Delta], \tag{3.54}$$

where

$$\Pi(t,t') = \sum_{k=0}^{\infty} T_{2k+1}(t, t' - k\Delta). \tag{3.55}$$

$A(t)$ and $C(t)$ are given by (3.41) and (3.48), respectively.

Chapter 4

Integral Equations

It will now be shown that the kernels of (3.52), (3.54) obey certain integral equations, which at least for spectrum models of viscoelastic behaviour, allow them to be determined in closed form. Formulas for these kernels are then presented for both the standard linear model and the general viscoelastic model.

4.1 Integral Equations for Kernels

The viscoelastic material will be assumed to have a unique Poisson ratio ν. (The viscoelastic functions characterizing bulk and shear deformation in the material are proportional.) Consider shear relaxation and creep functions of the form [15]

$$G(t) = G_0 + \sum_{i=1}^{N} G_i e^{-\frac{t}{\tau_i}}; \qquad J(t) = J_0 + \sum_{i=1}^{N} J_i (1 - e^{-\frac{t}{\tau_i'}}). \tag{4.1}$$

Then the singular functions $l(t)$ and $k(t)$ have the form

$$l(t) = l_0 \delta(t) + \sum_{i=1}^{N} l_i e^{-\alpha_i t}; \qquad k(t) = k_0 \delta(t) + \sum_{i=1}^{N} k_i e^{-\beta_i t} \tag{4.2}$$

where

$$l_0 = \frac{1}{k_0} = h[G_0 + \sum_{i=1}^{N} G_i] = \frac{h}{J_0}; \qquad l_i = \frac{-hG_i}{\tau_i}; \qquad k_i = \frac{J_i}{\tau_i' h}; \tag{4.3}$$

$$\alpha_i = \frac{1}{\tau_i}; \qquad \beta_i = \frac{1}{\tau_i'}; \qquad h = (1-\nu)^{-1}. \tag{4.4}$$

It follows from the inverse function relationship of $l(t)$ and $k(t)$ given by

$$\int_{t_1}^{t_2} dt' l(t_2 - t')k(t' - t_1) = \int_{t_1}^{t_2} dt' k(t_2 - t')l(t' - t_1) = \delta(t_2 - t_1) \tag{4.5}$$

that the coefficients $l_i, k_i, i = 0, 1, 2, \ldots, N$ are related by

$$l_0 k_0 = 1; \tag{4.6}$$

$$l_0 + \sum_{i=1}^{N} \frac{l_i}{\alpha_i - \beta_j} = 0, \qquad j = 1, 2, \ldots, N; \tag{4.7}$$

$$k_0 + \sum_{i=1}^{N} \frac{k_i}{\alpha_j - \beta_i} = 0, \qquad j = 1, 2, \ldots, N; \tag{4.8}$$

$$l_i = -\{\sum_{j=1}^{N} \frac{k_j}{(\alpha_i - \beta_j)^2}\}^{-1}, \qquad i = 1, 2, \ldots, N; \tag{4.9}$$

$$k_i = -\{\sum_{j=1}^{N} \frac{l_j}{(\alpha_j - \beta_i)^2}\}^{-1}, \qquad i = 1, 2, \ldots, N; \tag{4.10}$$

First consider $\Pi(t, t')$ in (3.54), as given by (3.55). According to the definition of $T_n(t, t')$, we have

$$T_n(t, t') = \int_{t'}^{\theta_n} dt''' \int_{t'''}^{\theta_{n-1}} dt'' T_{n-2}(t, t'') l(t'' - t''') k(t''' - t') \tag{4.11}$$

for odd numbers $n \geq 3$.

In fact the integral over t'' can be extended at the lower limit to t' since $l(t'' - t''')$ vanishes over this interval. Thus the integrals can be interchanged to obtain

$$T_n(t, t') = \int_{t'}^{\theta_{n-1}} dt'' T_{n-2}(t, t'') G_n(t'', t'), \tag{4.12}$$

where

$$G_n(t'',t') = \int_{t'}^{\theta_n} dt''' l(t''-t''')k(t'''-t'). \tag{4.13}$$

From the inverse relationship (4.5) between $k(t)$ and $l(t)$, it can be deduced that

$$G_n(t'',t') = \delta(t''-t'), \qquad t'' < \theta_n \tag{4.14}$$

so that

$$T_n(t,t') = T_{n-2}(t,t') + \int_{\theta_n}^{\theta_{n-1}} dt'' T_{n-2}(t,t'')G_n(t'',t'), \qquad t' < \theta_n. \tag{4.15}$$

By making the subscript explicitly odd, we have [14]

$$T_{2n+1}(t,t'-n\Delta) = T_{2n-1}(t,t'-n\Delta) + \int_{\theta_{2n+1}}^{\theta_{2n}} dt'' T_{2n-1}(t,t'')G_{2n+1}(t'',t'-n\Delta) \tag{4.16}$$

where

$$\begin{aligned} G_{2n-1}(t'',t'-n\Delta) &= \int_{t'-n\Delta}^{\theta_{2n+1}} dt''' l(t''-t''')k(t'''-t'+n\Delta) && (4.17) \\ &= \int_{t'}^{\theta_1} du\, l(t''-u+n\Delta)k(u-t') && (4.18) \\ &= G(t''+n\Delta,t') && (4.19) \end{aligned}$$

on transforming the variable of integration according to $u = t''' + n\Delta$. The function $G(t'',t')$ given by

$$G(t'',t') = \int_{t'}^{\theta_1} du\, l(t''-u)k(u-t') \tag{4.20}$$

has the same functional form as $T_1(t,t')$ with t'' replaced by t. Now (4.16) and (4.19) yield (using a further change of variable $u = t'' + (n+1)\Delta$)

$$T_{2n-1}(t,t'-n\Delta) = T_{2n-1}(t,t'-n\Delta) + \int_{\theta_3}^{\theta_2} du T_{2n-1}(t,u-(n-1)\Delta)G(u+\Delta,t') \tag{4.21}$$

so that $\Pi(t,t')$, given by (3.55), obeys the equation

$$\Pi(t,t') = T_1(t,t') + \Pi(t,t'-\Delta) + \int_{\theta_3}^{\theta_2} du\Pi(t,u)G(u+\Delta,t'). \tag{4.22}$$

Thus

$$\Pi(t,t'-\Delta) = T_1(t,t'-\Delta) + \Pi(t,t'-2\Delta) + \int_{\theta_3}^{\theta_2} du\Pi(t,u)G(u+\Delta,t'-\Delta). \tag{4.23}$$

Repeated substitution of (4.23) and its successors into (4.22) together with the assumption that

$$\lim_{n\to\infty} \Pi(t,t'-n\Delta) = 0, \tag{4.24}$$

which is justified in [15], gives an integral equation for $\Pi(t,t')$ of the form

$$\Pi(t,t') = K(t,t') + \int_{\theta_3}^{\theta_2} du\Pi(t,u)K(u+\Delta,t') \tag{4.25}$$

where

$$K(u,t') = \sum_{n=0}^{\infty} G(u,t'-n\Delta). \tag{4.26}$$

Recall that $G(t,t'-n\Delta) = T_1(t,t'-n\Delta)$.

The integral equation (3.52) with kernel $\Gamma(t,t')$ given by (3.53) can be solved in a similar fashion. $\Gamma(t,t')$ obeys [12] the integral equation

$$\Gamma(t,t') = J(t,t') + \int_{\theta_2}^{\theta_1} du\Gamma(t,u)L(u,t') \tag{4.27}$$

where

$$L(u,t') = \sum_{n=1}^{\infty} H(u,t'-n\Delta) \tag{4.28}$$

$$H(t'',t') = \int_{t'}^{\theta_2} dul(t''-u)k(u-t') \tag{4.29}$$

and

$$J(t,t') = \int_{\theta_2}^{\theta_1} du k(t-u)L(u,t') + \sum_{n=1}^{\infty} k(t-t'+n\Delta) \tag{4.30}$$

provided that

$$\lim_{n\to\infty} \Gamma(t,t'-n\Delta) = 0. \tag{4.31}$$

4.2 Solutions of Integral Equations for Kernels

In the previous section, the steady-state crack problem was reduced to the solution of two integral equations. The two kernels in these equations are infinite sums of terms involving integrals of the viscoelastic functions. In section 4.2.1, we present formulas for these kernels for the standard linear solid given by Golden and Graham [12]. Then we proceed to solve the integral equation (3.54) with kernel $\Pi(t,t')$ for the general viscoelastic case in section 4.2.2.

4.2.1 Standard Linear Model

As before, a unique Poisson's ratio ν will be assumed. Consider

$$l(t) = l_0\delta(t) + l_1 e^{-\alpha t}; \qquad k(t) = k_0\delta(t) + k_1 e^{-\beta t}, \tag{4.32}$$

where

$$k_0 l_0 = 1; \qquad k_1 = -\frac{l_1}{l_0^2}; \qquad \beta = \alpha - \frac{k_1}{k_0}. \tag{4.33}$$

Note that k_0, l_0, k_1, α and β are positive, while l_1 is negative. Under steady-state conditions, by adapting the results of [13],

$$N_{r+2}(t,t'-\Delta) = N_r(t,t')E, \qquad t \neq t' \tag{4.34}$$

$$T_{r+2}(t,t'-\Delta) = T_r(t,t')E; \qquad E = \exp\{-(\alpha-\beta)(t_0-t_c) - \alpha\Delta\}. \tag{4.35}$$

Also for $t \in (t_c, t_0 + \Delta]$

$$T_1(t,t') = l_1 k_0 \exp\{-\alpha(t - t_c) - \beta(t_c - t')\}. \tag{4.36}$$

Using these results in (3.52), (3.54), we obtain

$$B(t) = -k_0\eta(t) + k_1 e^{-\beta t} I(t, t_c) - \frac{k_1 e^{-\beta t} I(t_0, t_c)}{1 - E}, \qquad t \in (t_0, t_c] \tag{4.37}$$

$$D(t) = -\frac{l_1 k_0 \exp\{-\alpha(t - t_c) - \beta t_c\} I(t_0, t_c)}{1 - E}, \qquad t \in (t_c, t_0 + \Delta] \tag{4.38}$$

where

$$I(t_1, t_2) = \int_{t_1}^{t_2} dt' e^{\beta t'} \eta(t'). \tag{4.39}$$

The second term on the left-hand side of (4.37) is a consequence of the fact that the non-singular part of $k(t - t') = N_0(t, t')$ in (3.52) is included in the infinite summation while the integration range of this term stops at t.

4.2.2 General Viscoelastic Model

For the discrete spectrum model, equation (4.20) becomes

$$G(u, t') = \sum_{i,j=1}^{N} \frac{l_i k_j}{\alpha_i - \beta_j} \exp\{-\alpha_i(u - \theta_1) - \beta_j(\theta_1 - t')\}, \qquad u > \theta_1 \tag{4.40}$$

and (4.26) can be expressed as

$$K(u, t') = \sum_{i,j=1}^{N} K_{ij} \exp\{-\alpha_i(u - \theta_1) - \beta_j(\theta_1 - t')\}, \tag{4.41}$$

$$K_{ij} = \frac{l_i k_j}{\alpha_i - \beta_j} \frac{1}{1 - e^{-\beta_j \Delta}}. \tag{4.42}$$

To solve (4.25), we make the ansatz for $\Pi(t, t')$ of the form

$$\Pi(t,t') = \sum_{i,j=1}^{N} P_{ij}(t) e^{\beta_j t'}, \tag{4.43}$$

which clearly obeys (4.24). When substituted into (4.25), an algebraic equation is obtained [15]

$$\sum_{i=1}^{N} P_{ij} = \sum_{i=1}^{N} K_{ij} e^{-\alpha_i(t-\theta_1)-\beta_j\theta_1} +$$
$$\sum_{i,m,n=1}^{N} \frac{P_{im}K_{nj}}{\beta_m - \alpha_n}\{e^{\beta_m\theta_2-\alpha_n(\theta_2+\Delta-\theta_1)-\beta_j\theta_1} - e^{\beta_m\theta_3-\beta_j\theta_1}\} \tag{4.44}$$

which will certainly be satisfied if the stronger condition that term by term cancellation takes place in the variable i. This may be expressed as a matrix equation

$$P = K_1 + PAK_2 \tag{4.45}$$

where P is a square matrix formed by P_{ij} and

$$(K_1)_{ij} = K_{ij} e^{-\alpha_i(t-\theta_1)-\beta_j\theta_1}, \tag{4.46}$$

$$(K_2)_{nj} = K_{nj} e^{-\beta_j\theta_1}, \tag{4.47}$$

$$A_{mn} = \frac{e^{-\alpha_n(\Delta+\theta_2-\theta_1)+\beta_m\theta_2} - e^{\beta_m\theta_3}}{\beta_m - \alpha_n}. \tag{4.48}$$

The formal solution of (4.45) is

$$P = K_1(I - AK_2)^{-1}. \tag{4.49}$$

Similarly, for the integral equation (4.27)

$$\sum_{n=1}^{\infty} k(t - t' + n\Delta) = \sum_{i=1}^{N} \frac{k_i e^{-\beta_i\Delta}}{1 - e^{-\beta_i\Delta}} e^{-\beta_i(t-t')}, \qquad \theta_1(t) \le t' \le \theta_2(t) + \Delta, \tag{4.50}$$

and

$$L(u,t') = \sum_{i,j=1}^{N} L_{ij} \exp\{-\alpha_i(u-\theta_2) - \beta_j(\theta_2 - t')\}, \tag{4.51}$$

$$L_{ij} = \frac{l_i k_j}{\alpha_i - \beta_j} \frac{e^{-\beta_j \Delta}}{1 - e^{-\beta_j \Delta}}. \tag{4.52}$$

Then

$$\Gamma(t,t') = \sum_{i,j=1}^{N} Q_{ij}(t) e^{\beta_j t'} \tag{4.53}$$

satisfies (4.31) and will satisfy (4.27) if

$$Q = L_1 + QBL_2 \tag{4.54}$$

where Q is a square matrix formed by Q_{ij} and [15]

$$(L_1)_{ij} = \frac{k_i k_j}{1 - e^{-\beta_j \Delta}} D_{ij} \exp\{\beta_i(\theta_1 - t) - \beta_j(\theta_2 + \Delta)\} \tag{4.55}$$

where

$$D_{ij} = \sum_{n=1}^{N} \frac{l_n e^{\alpha_n(\theta_2 - \theta_1)}}{(\beta_i - \alpha_n)(\alpha_n - \beta_j)}, \tag{4.56}$$

$$(L_2)_{nj} = L_{nj} e^{-\beta_j \theta_2}, \tag{4.57}$$

$$B_{mn} = (e^{(\beta_n - \alpha_n)\theta_1 + \alpha_n \theta_2} - e^{\beta_m \theta_2})/(\beta_m - \alpha_n). \tag{4.58}$$

The formal solution of (4.54) is

$$Q = L_1(I - BL_2)^{-1}. \tag{4.59}$$

Substituting the kernels into the periodic equations (3.52) and (3.54) we have

$$B(t) = -k_0 \eta(t) - \sum_{i=1}^{N} k_i \int_{t_0}^{t} dt' e^{-\beta_i(t-t')} \eta(t') - \sum_{i,j=1}^{N} Q_{ij}(t) I(t_0, t_c) \quad \text{for} \quad t \in (t_0, t_c] \tag{4.60}$$

and

$$D(t) = -\sum_{i,j=1}^{N} P_{ij}(t)I(t_0, t_c) \quad \text{for} \quad t \in (t_c, t_0 + \Delta] \tag{4.61}$$

where

$$I(t_1, t_2) = \int_{t_1}^{t_2} dt' e^{\beta_j t'} \eta(t'). \tag{4.62}$$

The equations for displacement and pressure given by (3.25) and (3.30) thus become for the closed phase

$$u(x,t) = \frac{1}{2} M(t)(\frac{c}{3} - x)^{\frac{3}{2}}(x+c)^{\frac{1}{2}} d_1(x), \tag{4.63}$$

$$p(x,t) = \eta(t)[-x + g(x)d_2(x)] - \sum_{i,j=1}^{N} P_{ij}(t)I(t_0, t_c)[g(x)d_2(x)], \qquad |x| < c \tag{4.64}$$

and for the open phase

$$\begin{aligned} u(x,t) = \frac{1}{2}[M(t) - k_0\eta(t) - \sum_{i=1}^{N} k_i \int_{t_0}^{t} dt' e^{-\beta_i(t-t')}\eta(t') \\ - \textstyle\sum_{i,j=1}^{N} Q_{ij}(t)I(t_0,t_c)](\frac{c}{3} - x)^{\frac{3}{2}}(x+c)^{\frac{1}{2}} d_1(x) \\ + \tfrac{1}{2}[-k_0\eta(t) - \textstyle\sum_{i=1}^{N} k_i \int_{t_0}^{t} dt' e^{-\beta_i(t-t')}\eta(t') - \sum_{i,j=1}^{N} Q_{ij}(t)I(t_0,t_c)]x(c^2 - x^2)^{\frac{1}{2}}, \end{aligned} \tag{4.65}$$

$$p(x,t) = -\eta(t)x, \qquad |x| < c. \tag{4.66}$$

Chapter 5

Asymmetric Sinusoidal Moment

A numerical determination of the limits of material behaviour to which the preceding formulae apply is of importance for their effective application. Results concerning the standard linear model and the general viscoelastic model for the case $N = 2$ are developed in terms of dimensionless quantities. Essentially, the restriction that closure must occur only on the positive end of the crack places limits on the dimensionless parameters describing the viscoelastic material. These material limits are presented in the form of a series of plots obtained by solving equations developed in Chapters 3 and 4.

5.1 Standard Linear Model

The results presented in this section are those obtained by Golden and Graham [12].

The following asymmetric sinusoidal moment will be considered for the case of a standard linear model

$$\eta(t) = \eta_0(d + \sin(\omega t)), \qquad 0 < d \leq 1 \tag{5.1}$$

$$\Delta = 2\pi/\omega. \tag{5.2}$$

A plot of this is shown in Fig.(5.1). Notice that

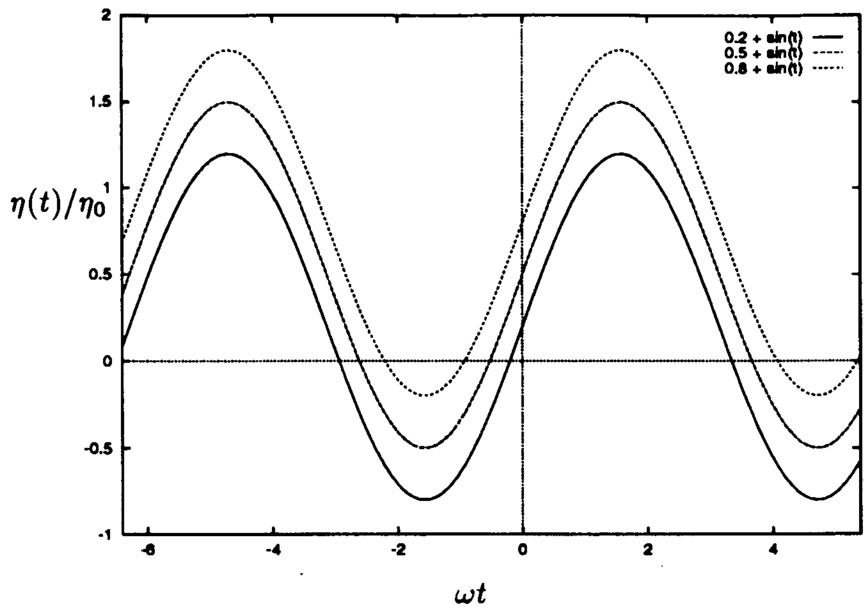

Figure 5.1: Plots of $\eta(t)/\eta_0$ for three values of d.

$$\eta(t)\begin{cases} \geq 0 & 2\pi - \gamma \leq \omega t \leq (2n+1)\pi + \gamma \\ < 0 & (2n-1)\pi + \gamma \leq \omega t \leq 2n\pi - \gamma \end{cases} \tag{5.3}$$

$$\gamma = \sin^{-1}(d) \tag{5.4}$$

for all integers n. By virtue of (2.46), (2.47) and (2.48) partial closure can occur at the positive end only when $\eta(t)$ is positive. We consider a period of closure $[t_c, t_0 + \Delta]$ in the time interval $[-\frac{\gamma}{\omega}, \frac{(\pi+\gamma)}{\omega}]$. The quantity $I(t_1, t_2)$ given by (4.62) has the form

$$I(t_1, t_2) = \eta_0[\frac{d}{\beta}(e^{\beta t_2} - e^{\beta t_1}) + S(t_1, t_2)] \tag{5.5}$$

$$S(t_1, t_2) = \int_{t_1}^{t_2} dt' e^{\beta t'} \sin(\omega t') = \frac{e^{\beta t'} \sin(\omega t' - \psi)}{(\beta^2 + \omega^2)^{\frac{1}{2}}}\Big|_{t_1}^{t_2} \tag{5.6}$$

where the phase ψ is defined by

$$\tan\psi = \frac{\omega}{\beta}. \tag{5.7}$$

By virtue of (3.45), (3.49), (4.37) and (4.38), the equations determining t_c, t_0 are

$$k_0\eta_0(d+\sin(\omega t_c)) + \frac{k_1 e^{-\beta t_c} I(t_0,t_c)}{1-E} = 0, \tag{5.8}$$

$$\eta_0(d+\sin(\omega t_0)) - \frac{l_1 k_0 \exp\{-\alpha(t_0-t_c+\Delta)-\beta t_c\} I(t_0,t_c)}{1-E} = 0. \tag{5.9}$$

The quantity $M(t)$, defined in (3.18), has the form

$$M(t) = \eta_0\{d\hat{\gamma}_1(0) + Im[\hat{\gamma}_1(\omega)e^{i\omega t}]\} \tag{5.10}$$

$$= \eta_0\{d\hat{\gamma}_1(0) + |\hat{\gamma}_1(\omega)|\sin(\omega t-\phi)\} \tag{5.11}$$

where $\hat{\gamma}_1(\omega)$ is a quantity proportional to the complex modulus for creep and has the form

$$\hat{\gamma}_1(\omega) = k_0 + \frac{k_1}{\beta + i\omega} \tag{5.12}$$

and ϕ is the loss angle for the material [12] defined by

$$\tan\phi = \frac{k_1\omega}{k_0(\beta^2+\omega^2)+k_1\beta}, \qquad 0 \le \phi \le \frac{\pi}{2}. \tag{5.13}$$

Thus

$$M(t) = \eta_0\{d(k_0+\frac{k_1}{\beta}) + (k_0 + \frac{k_1\beta}{\beta^2+\omega^2})\sin(\omega t) - \frac{k_1\omega}{\beta^2+\omega^2}\cos(\omega t)\} \tag{5.14}$$

$$= \eta_0\{d(k_0+\frac{k_1}{\beta}) + [(k_0+\frac{k_1\beta}{\beta^2+\omega^2})^2 + (\frac{k_1\omega}{\beta^2+\omega^2})^2]^{\frac{1}{2}}\sin(\omega t-\phi)\}. \tag{5.15}$$

If we set

$$\Theta = \gamma - \pi - \omega t_0, \qquad \Phi = \omega t_c + \gamma, \tag{5.16}$$

where

$$\Psi = \pi + 2\gamma - \Theta - \Phi \geq 0, \tag{5.17}$$

then Θ is a measure of the "angular advance" in the crack opening, before the effect of the applied moment on the positive x-axis changes from compression to tension, while Φ is the "angular delay" in the crack closing, once the effect of the applied moment has become compressive on the positive x-axis [12]. The quantitiy Ψ measures the amount of time that the crack is closed down on the right-hand side in each cycle. In terms of these quantities and the angles ψ, ϕ defined by (5.7), (5.13) the equations (5.8), (5.9) become

$$\begin{aligned}(1-E)(\sin(\gamma)+\sin(\Phi-\gamma)) + f\sin(\gamma)(1-e^{(\Psi-2\pi)\cot(\psi)}) \\ +f\cos(\psi)(\sin(\Phi-\gamma-\psi)+e^{(\Psi-2\pi)\cot(\psi)}\sin(\gamma-\psi-\Theta)) = 0\end{aligned} \tag{5.18}$$

$$\begin{aligned}(1-E)(\sin(\gamma)-\sin(\gamma-\Theta)) + f\sin(\gamma)e^{-(1+f)\Psi/g}(1-e^{(\Psi-2\pi)\cot(\psi)}) \\ +f\cos(\psi)e^{-(1+f)\Psi/g}[\sin(\Phi-\gamma-\psi)+e^{(\Psi-2\pi)\cot(\psi)}\sin(\gamma-\psi-\Theta)] = 0\end{aligned} \tag{5.19}$$

where E is given as in [12] to be

$$E = \exp(\frac{-\Psi\sin(\phi)}{\sin(\psi)\sin(\psi-\phi)} - 2\pi\cot(\psi)) \tag{5.20}$$

and the dimensionless quantities f, g are defined as

$$f = \frac{k_1}{k_0\beta}, \qquad g = \frac{\omega}{\beta} = \tan(\psi). \tag{5.21}$$

Thus we have that

$$f = \frac{\sin(\phi)}{\cos(\psi)\sin(\psi-\phi)}; \qquad \frac{1+f}{g} = \cot(\psi) + \frac{\sin(\phi)}{\sin(\psi-\phi)\sin(\psi)}. \tag{5.22}$$

Equations (5.18) and (5.19) may be combined to give

$$\sin(\gamma) + \sin(\Phi - \gamma) = e^{(1+f)\Psi/g}(\sin(\gamma) - \sin(\gamma - \Theta)). \tag{5.23}$$

Certain particular solutions of the above are discussed by Golden and Graham [12] for Voigt and Maxwell materials. For a Voigt material $\psi = \phi$, and thus from (5.19) or (5.23) $\Theta = 0$. Then (5.18) implies that Ψ satisfies

$$\begin{aligned}(\cos(\phi)\sin(\gamma - \phi) - \sin(\gamma))e^{(2\gamma - \Phi - \pi)\cot(\phi)} \\ + \cos(\phi)\sin(\Phi - \gamma - \phi) + \sin(\gamma) = 0.\end{aligned} \tag{5.24}$$

A Maxwell material is characterized by $\phi = \frac{\pi}{2}$. In this case (5.18), (5.23) reduce to

$$\begin{aligned}(1 - e^{-\Psi\tan(\phi)})(\sin(\gamma) + \sin(\Phi - \gamma)) \\ +(2\pi - \Psi)\tan(\phi)\sin(\gamma) - 2\tan(\phi)\cos(\tfrac{\Theta - \Phi}{2})\sin(\tfrac{\Psi}{2}) = 0,\end{aligned} \tag{5.25}$$

$$\sin(\gamma) + \sin(\Phi - \gamma) = (\sin(\gamma) - \sin(\gamma - \Theta))e^{\Psi\tan(\phi)}. \tag{5.26}$$

5.1.1 Numerical Results

The condition that closure takes place only at one end is (3.26). This may be written as

$$\min_{t_o \le t \le t_c}\left[\frac{M(t)}{B(t)}\right] \ge \frac{3}{4}\left(\frac{3}{2}\right)^{\frac{1}{2}} - 1 = -0.08144 \tag{5.27}$$

together with the non-negativity requirements on $A(t)$ and $B(t)$. Equality in this condition implies a relation between the dimensionless parameters f and g for each value of d. A sample of a plot of $M(t)/B(t)$ is shown in Fig.(5.2). The equations determining the relation between the dimensionless parameters are (5.8), (5.9) and (5.27), which in terms of f and g yield a system of four equations, namely

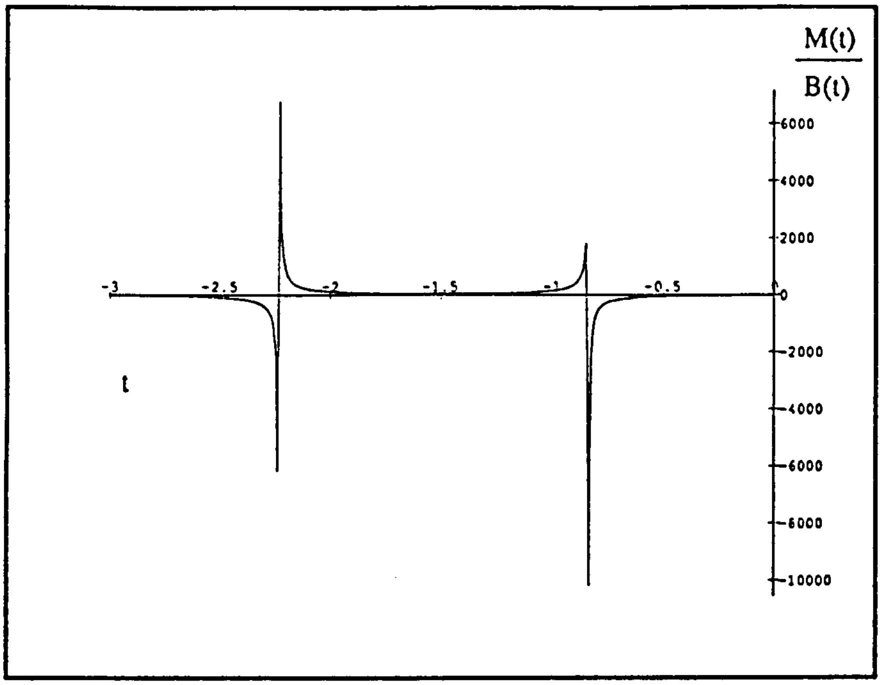

Figure 5.2: $M(t)/B(t)$ for dimensionless parameters $f = 20$, $g = 70$ and $d = 0.8$, $t_o = -2.2$, $t_c = -0.8$. Note that $M(t)/B(t)$ is greater than -0.08144 for $t_o \leq t \leq t_c$.

$$d + \sin(\omega t_c) + \frac{f\omega e^{\frac{-\omega t_c}{g}} I(t_o, t_c)}{g\eta_0(1-E)} = 0, \tag{5.28}$$

$$d + \sin(\omega t_o) + \frac{f\omega \exp(\frac{f(\omega t_c - \omega t_o - 2\pi) - \omega t_o - 2\pi}{g}) I(t_o, t_c)}{g\eta_0(1-E)} = 0, \tag{5.29}$$

$$\frac{d(1+f) + [(1+\frac{f}{1+g^2})^2 + (\frac{fg}{1+g^2})^2]^{\frac{1}{2}} \sin(\omega t - \phi)]}{d + \sin(\omega t) + \frac{f\omega}{g\eta_0} e^{\frac{-\omega t}{g}} [I(t, t_c) + \frac{I(t_o, t_c)}{1-E}]} + \frac{3}{4}(\frac{3}{2})^{\frac{1}{2}} - 1 = 0, \tag{5.30}$$

and the fourth equation being the derivative of (5.30). Here

$$E = \exp[\frac{f}{g}(2\pi - \omega t_o - \omega t_c) - \frac{2\pi}{g}] \tag{5.31}$$

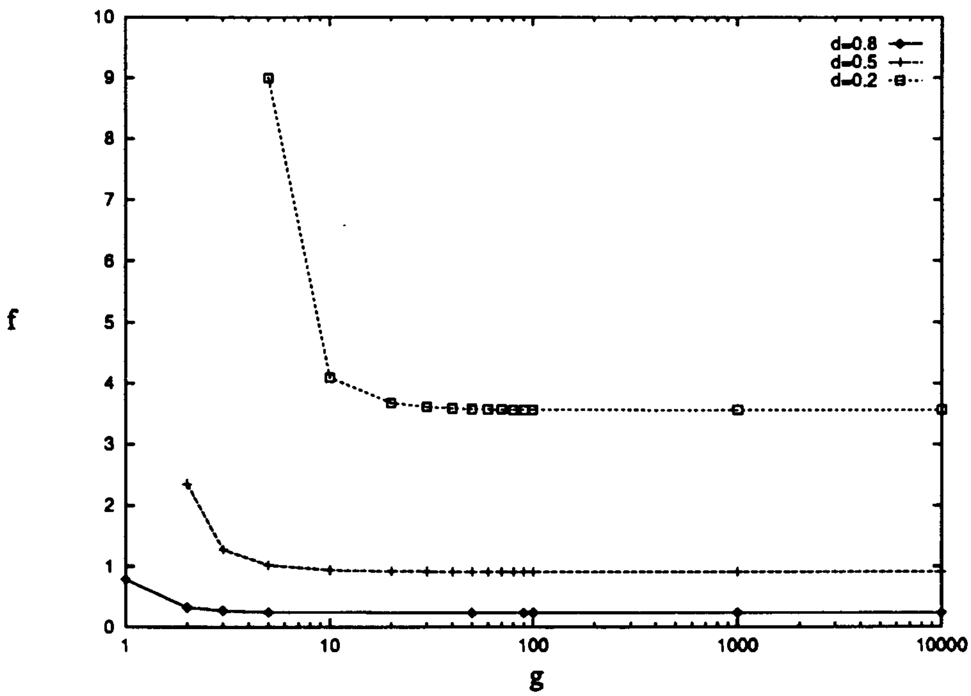

Figure 5.3: Curves outside of which partial closure on one side only is possible.

and

$$I(t_1, t_2) = \eta_0[\frac{d\omega}{g}(e^{\frac{\omega t_2}{g}} - e^{\frac{\omega t_1}{g}}) + \frac{e^{\frac{\omega t'}{g}} \sin(\omega t' - \psi)}{\omega(\frac{1}{g^2} + 1)^{\frac{1}{2}}}|_{t_1}^{t_2}], \quad (5.32)$$

as follows from (4.35), (5.5) and (5.6).

Equation (5.30) is the form of

$$\frac{M(t)}{B(t)} - \frac{3}{4}(\frac{3}{2})^{\frac{1}{2}} + 1 = 0 \quad (5.33)$$

in terms of f and g. The derivative of this equation is used to determine the value of t such that (5.30) is a minimum.

Equations (5.28), (5.29) are equivalent to (5.8), (5.9) and consequently help determine t_c and t_o.

Curves of the relationship between values of f and g which satisfy (5.28)-(5.30) and the derivative of (5.30) are shown on Fig.(5.3). These results are in agreement with those presented by Golden and Graham [12].

5.2 General Viscoelastic Model

Following a brief discussion of the $N = 1$ case of the general model (equivalent to the standard linear model), a procedure analogous to that employed for the standard linear model is applied to the general viscoelastic model for the case $N = 2$. Generalization to $N = 3, 4, \ldots$ follows easily.

5.2.1 Case N = 1

The asymmetric sinusoidal moment will now be considered for the case $N = 1$ for the equations obtained for the general model and then compared to the standard linear model.

As before let (5.1) be the form of this asymmetric sinusoidal moment. Equations (5.3) - (5.7) hold as for the standard linear model.

By virtue of (4.61), (4.60), (3.45) and (3.49) the equations determining t_c, t_0 are

$$k_0\eta_0(d+\sin(\omega t_c)) + k_1\int_{t_0}^{t_c} dt' e^{-\beta_1(t_c-t')}\eta(t') + Q_{11}(t_c)I(t_0,t_c) = 0, \tag{5.34}$$

$$\eta_0(d+\sin(\omega t_0)) - P_{11}(t_0+\Delta)I(t_0,t_c) = 0. \tag{5.35}$$

The quantity $M(t)$ is defined as in (3.18) with the loss angle given by (5.13).

Defining Θ, Φ and Ψ as in (5.16) - (5.17) we have

$$\begin{aligned}&\sin(\gamma)+\sin(\Phi-\gamma)+f\sin(\gamma)[1-e^{(\Psi-2\pi)\cot(\psi)}](1+\tilde{Q}_{11})\\&\quad+f\cos(\psi)[\sin(\Phi-\gamma-\psi)+e^{(\Psi-2\pi)\cot(\psi)}\sin(\gamma-\psi-\theta)](1+\tilde{Q}_{11})=0,\end{aligned} \tag{5.36}$$

$$\begin{aligned}&\sin(\gamma)-\sin(\gamma-\Theta)-\tilde{P}_{11}f\sin(\gamma)[1-e^{(2\pi-\Psi)\cot(\psi)}]\\&\quad+\tilde{P}_{11}f\cos(\psi)[e^{(2\pi-\Psi)\cot(\psi)}\sin(\Phi-\gamma-\psi)+\sin(\gamma-\psi-\Theta)]=0,\end{aligned} \tag{5.37}$$

where $Q_{11}(t_c) = k_1 e^{-\beta_1 t_c}\tilde{Q}_{11}$, $P_{11}(t_0+\Delta) = -\frac{k_1}{k_0}e^{-\beta_1 t_0}\tilde{P}_{11}$ and the dimensionless quantities f and g are defined as in (5.21).

The above equations (5.36), (5.37) can be compared to those given by (5.18), (5.19) to find that since

$$1+\tilde{Q}_{11} = \frac{1}{1-E} \tag{5.38}$$

and

$$\tilde{P}_{11} = \frac{e^{-(1+f)\Psi/g}e^{(\Psi-2\pi)\cot(\psi)}}{1-E} = \frac{E}{1-E} \tag{5.39}$$

where E is given by (5.20), the results given in [12] follow directly.

The relationship between $\tilde{P}_{11}$ and $\tilde{Q}_{11}$ can then be expressed as

$$\tilde{P}_{11} = E(1+\tilde{Q}_{11}). \tag{5.40}$$

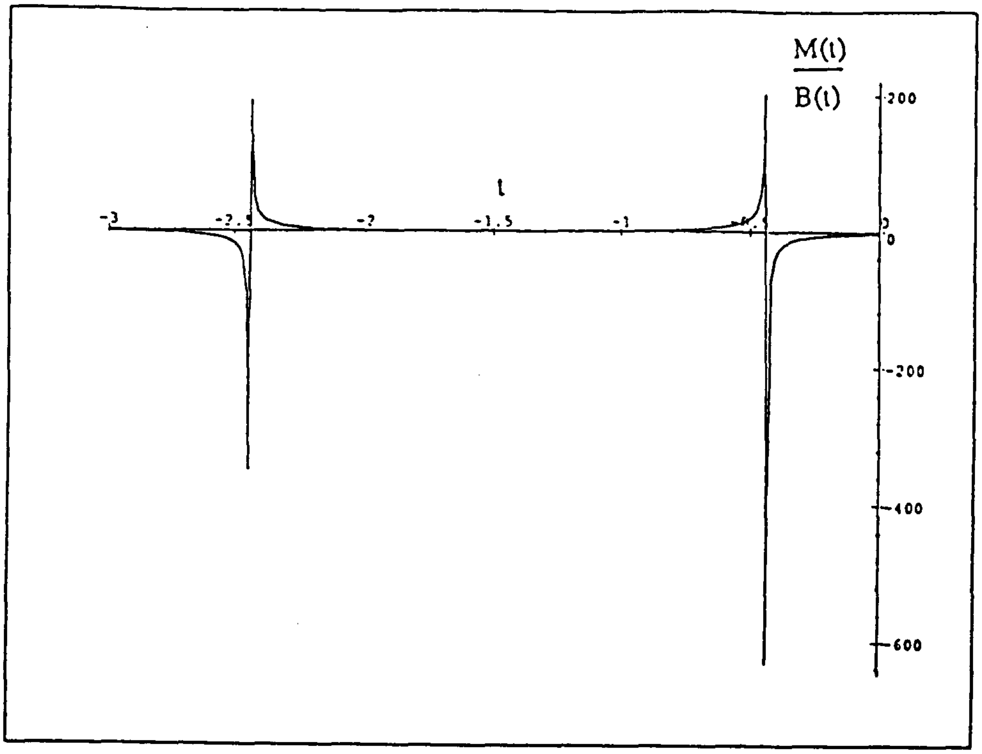

Figure 5.4: $M(t)/B(t)$ for dimensionless parameters $g_1 = 150$, $f_1 = 1.5$, $g_2 = 100$ and $f_2 = 0.5$ with $d = 0.5$, $t_o = -2.0$ and $t_c = -0.44$.

5.2.2 Numerical Results for $N = 2$

For the $N = 2$ case, again consider the asymmetric sinusoidal moment to be of the form given by (5.1). Now, by virtue of (3.45), (3.49), (4.60) and (4.61), the equations determining t_c,t_o are

$$k_0\eta(t_c) + \sum_{i=1}^{2} k_i \int_{t_o}^{t_c} dt' e^{-\beta_i(t_c-t')}\eta(t') + \sum_{i,j=1}^{2} Q_{ij}(t_c)I(t_o,t_c) = 0, \qquad (5.41)$$

$$\eta(t_o+\Delta) - \sum_{i,j=1}^{2} P_{ij}(t_o+\Delta)I(t_o,t_c) = 0. \qquad (5.42)$$

Note that the definitions for the dimensionless parameters and related quantities are like those of the standard linear model :

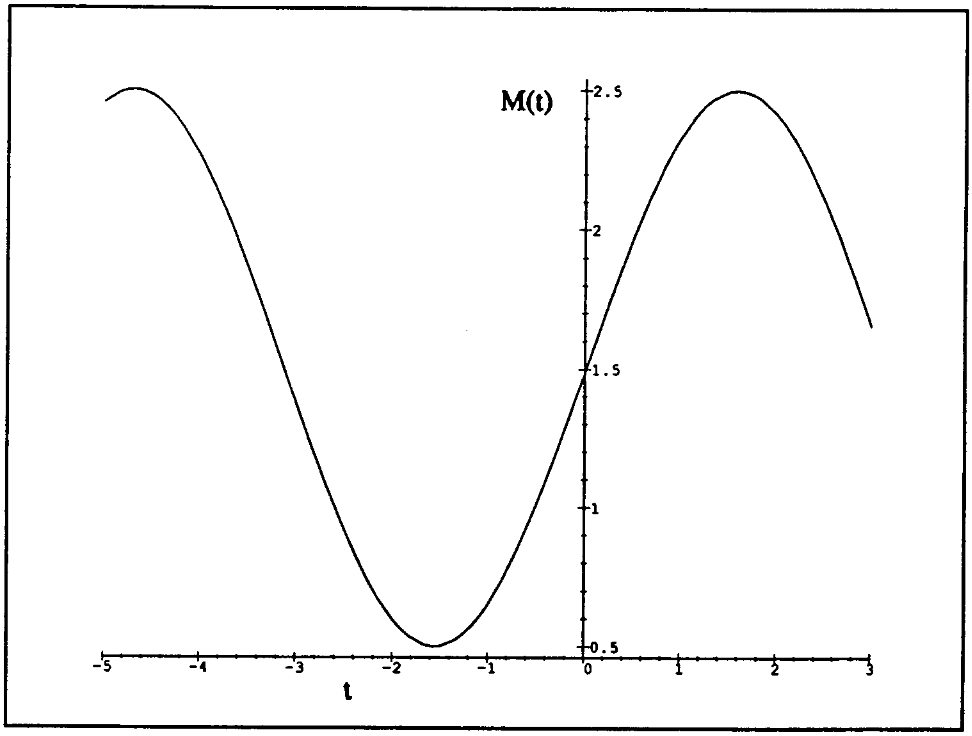

Figure 5.5: $M(t)$ for dimensionless parameters $g_1 = 150$, $f_1 = 1.5$, $g_2 = 100$ and $f_2 = 0.5$ with $d = 0.5$, $t_o = -2.0$ and $t_c = -0.44$.

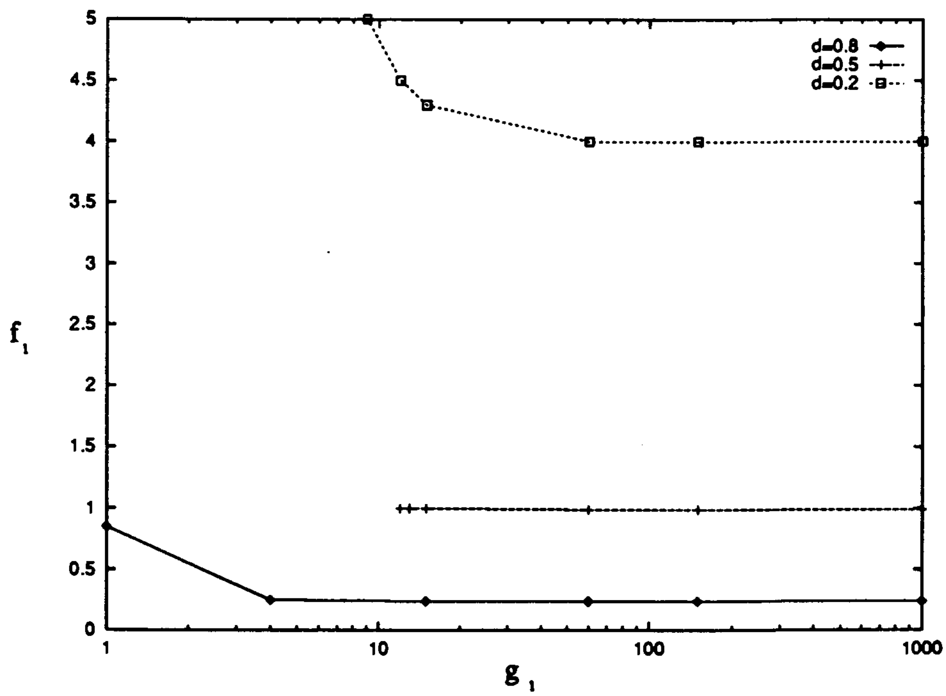

Figure 5.6: Curves outside of which partial closure on one side only is possible for $f_2 = 0.02$ and $g_2 = 100$.

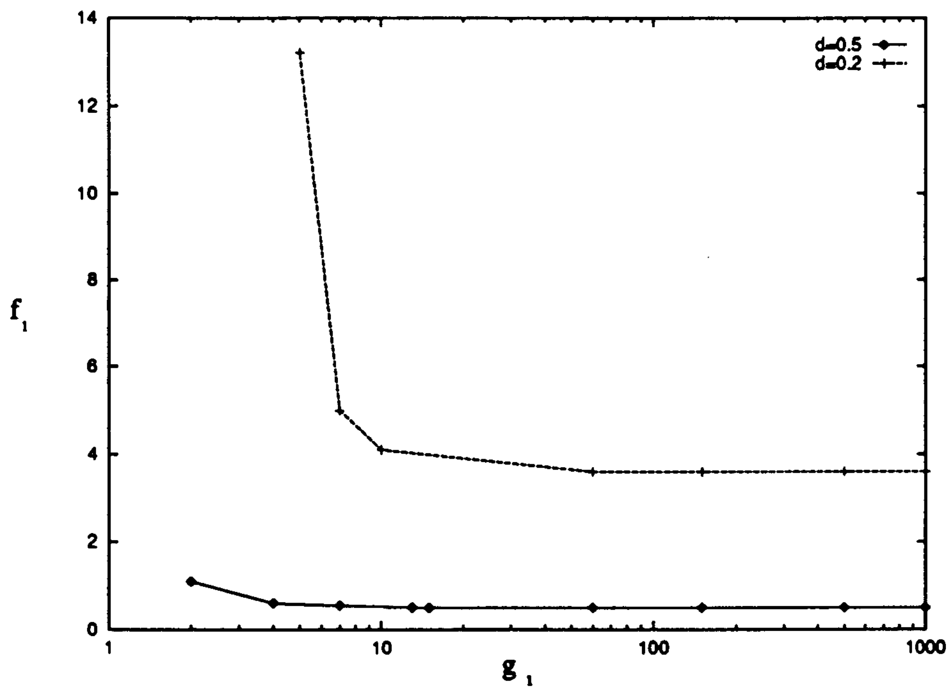

Figure 5.7: Curves outside of which partial closure on one side only is possible for $f_2 = 0.5$ and $g_2 = 100$.

$$f_i = \frac{k_i}{k_0\beta_i}, \qquad g_i = \frac{\omega}{\beta_i} = \tan(\psi_i), \tag{5.43}$$

$$\tan(\phi_i) = \frac{f_i g_i}{1 + g_i^2 + f_i} \qquad i = 1, 2. \tag{5.44}$$

The quantity $M(t)$, defined by (3.18), has the form

$$\begin{aligned} M(t) &= \eta_0 k_0[d(1 + f_1 + f_2) + [(1 + \frac{f_1}{1+g_1^2})^2 + (\frac{f_1 g_1}{1+g_1^2})^2]^{\frac{1}{2}} \sin(\omega t - \phi_1) \\ &+ [(1 + \frac{f_2}{1+g_2^2})^2 + (\frac{f_2 g_2}{1+g_2^2})^2]^{\frac{1}{2}} \sin(\omega t - \phi_2) - \sin(\omega t)], \end{aligned} \tag{5.45}$$

and from (4.60),

$$B(t) = -k_0\eta(t) - \sum_{i=1}^{2} k_i \int_{t_o}^{t} dt' e^{-\beta_i(t-t')}\eta(t') - \sum_{i,j=1}^{2} Q_{ij}(t) I(t_o, t_c). \tag{5.46}$$

The values of f and g which correspond to partial closure at only one end must thus satisfy (5.41), (5.42) expressed in terms of these dimensionless quantities as well as the equation determined by (5.33) and its derivative for the $N = 2$ case. Alternately, it is possible to approximate equation (5.33) by noticing that the constant on the right of (5.27) is small. If it is replaced by zero the condition that

$$\min_{t_o \le t \le t_c} M(t) = 0 \tag{5.47}$$

may be used instead. The quantity $M(t)$ is plotted in Fig.(5.5) for sample parameters. Consequently, Fig.(5.6) and Fig.(5.7) are curves determined by either condition (5.47) or (5.27) outside of which partial closure on one side only is possible.

It is important to note that f_2 must be chosen relatively small and g_2 relatively large if the $N = 2$ term is to act as a correction to the $N = 1$ model. If the dimensionless parameters are not chosen in this manner, then (5.41), (5.42) need not converge to a feasible solution. For Fig.(5.6), f_2 and g_2 are chosen to provide minimal correction, and thus the curves are virtually identical to those obtained for the standard linear model in Fig.(5.3). For Fig.(5.7), they are chosen for a larger correction. The results correspond to intuition since the larger correction permits a wider range of f_1 and g_1 values which satisfy condition (5.27). Since the larger f_1 is the more viscoelastic the material, this would indicate that the correction allows a tendency towards more elastic materials. If in addition d is sufficiently large, all physically reasonable (f_1, $g_1 > 0$) values of f_1 and g_1 are acceptable.

5.3 Energy Loss

The average rate of dissipation of mechanical energy per cycle will now be considered for the cracked viscoelastic body subject to an asymmetric bending moment. As was shown by Golden and Graham [18], the average energy loss per cycle under steady-state conditions is derivable from the rate of work done by the boundary forces. Since the isothermal approximation was assumed in the derivation, the results remain valid for a sufficiently small rate and magnitude of deformation when considered in the context of the thermal conductivity and specific heat capacity of the material.

5.3.1 The General Viscoelastic Model

The average rate of dissipation of mechanical energy in heat per cycle, D_c, may be determined for a general viscoelastic model by the procedure shown in the Appendix. Using partial integration and periodicity, D_c can be written [18] in the form

$$D_c = -\eta_0 \omega \int_{\omega t_o}^{\omega t_o + 2\pi} dt \{K_{p1} M(t) - (K_{f1} - K_{p1}) B(t)\} \cos(\omega t). \tag{5.48}$$

The equations for M(t) and B(t) for the case $N = 2$ are given by (5.45) and (5.46), respectively. The times of partial closing and opening are determined by (3.45), (3.49) and the condition expressed by (5.27), together with the non-negativity requirements on $A(t)$ and $B(t)$, which must be satisfied at all times to ensure that partial closure occurs only at the positive end. The resulting integral may be evaluated to determine the quantity $D_c/(\eta_o^2 k_0 \omega c^4)$ in terms of dimensionless parameters. The reason for the incompleteness of the curves for lower values of ψ in Fig.(5.8) is the need to apply condition (5.27) to ensure that closure takes place at only one end.

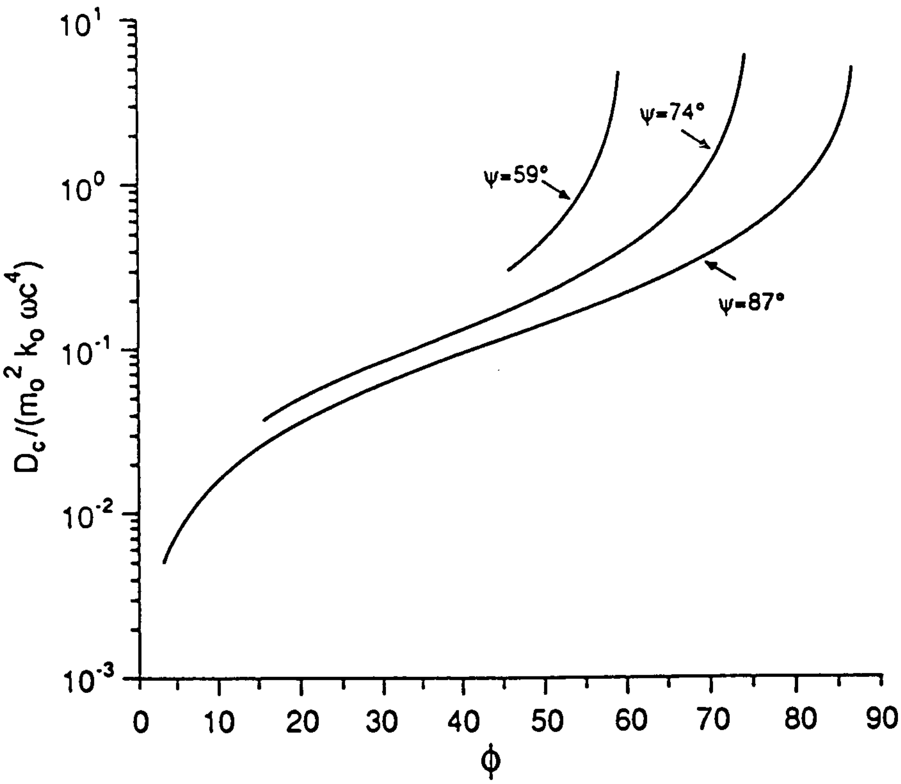

Figure 5.8: The quantity $D_c/(\eta_o^2 k_0 \omega c^4)$ plotted as a function of ϕ for various values of ψ [18].

Chapter 6

Conclusion

The Kolosov-Muskhelishvili equations, a form of the Papkovich-Neuber solution of the elastostatic equations under plane strain conditions, form the basis for the solution of the two-dimensional boundary value problem of the crack subject to an asymmetric bending moment. Underlying this formulation is the non-inertial approximation for viscoelastic materials which lends itself to transferring the applied stresses at infinity to the open crack faces. Although not described in detail, the Correspondence Principles permeate aspects of the solution which may be ascribed to their Papkovich-Neuber foundation.

The method of solution relies on a decomposition of hereditary integrals. This approach is taken to deduce, rather than assume, that closure of the crack is instantaneous. An area-based decomposition method, as opposed to point-based methods where it is necessary to trace the history of each point on the boundary, is undertaken due to the special time behaviour, characterized by the parameter $b(t)$, of the boundary region. Thus, in the steady-state limit, expressions for the kernels of the integral equations defining the decomposed quantities are found to obey integral equations which, for spectrum models of viscoelastic behaviour, allow them to be determined in closed form. Equations for the displacement and pressure are consequently available.

The numerical results obtained for the $N = 1$ or standard linear model are in agreement with the work done by Golden and Graham [12]. It is found that the general viscoelastic model for the case $N = 2$ supports the same type of solutions as the standard linear model.

The average rate of dissipation of mechanical energy in heat per cycle may be arrived at by introducing a Spencer [19] modification to the Kolosov-Muskhelishvili equations. The resulting functional form describing the rate of dissipation of mechanical energy is such that there is a dependence on certain constants incorporating the Spencer modification and the appropriate expressions for $M(t)$ and $B(t)$.

The techniques and methods of solution described must be attributed chiefly to the pioneering work in the field of viscoelasticity of Golden and Graham. Novelty lies in the application of these procedures to the problem of a fixed length crack subject to an asymmetric bending moment for the case of a general viscoelastic material. This work contributes to understanding the fracture characteristics of viscoelastic materials.

Both analytical and numerical results allude to future projects which may be of interest to orientations either applied or theoretical in nature. For example, further insight into the analytical equations might be gained from a detailed description of the relationships to the Correspondence Principles. A substantially less trivial matter lies in investigating the effects of relaxing the non-inertial approximation. Experimentally, optimal problem formulation, implying how many terms of the general model must be retained to accurately determine desired quantities, would be of great practical interest. Also, determining parameters and bending moments likely to cause alternate forms of crack closure or lack of closure based on the techniques described in this work might prove useful. Finally, the formulation of the stress intensity factors related to this problem would make the results more accessible to engineering applications.

Appendix: Energy Considerations

The developments presented here are ascribed to the work of Golden and Graham [18]. They formulated an expression for the average rate of dissipation of mechanical energy per cycle in terms of $M(t)$ and $B(t)$, quantities for which formulae were derived in Chapter 4. Consequently, these energy considerations are an important practical application of the equations for the fixed length crack in a general viscoelastic material allowing closure at only the positive end.

When inertial effects are neglected, the rate of work done by the boundary stresses is

$$R = \int_B ds n_i \sigma_{ij} \dot{u}_j = \dot{E} + \dot{D}, \tag{A.1}$$

where the integral is over the boundary of the body and the dot indicates time differentiation. The quantity $\dot{E}$ is the rate of increase of stored energy in the viscoelastic body, and $\dot{D}$ is the rate of dissipation of mechanical energy into heat. Of interest is the plane strain configuration for which ds reduces to a line element.

The quantity E is a unique function of the state of the system and is thus periodic whereas the rate of dissipation is not. Therefore, averaging over a cycle, we obtain

$$< R >= \frac{1}{\Delta} \int_t^{t+\Delta} R(t) dt = \frac{1}{\Delta} \int_t^{t+\Delta} dt \dot{D}(t) \equiv D_c \tag{A.2}$$

where D_c is the average rate of dissipation of mechanical energy in heat, per cycle. This means that D_c is equal to the rate of work done by the boundary forces, averaged over a cycle.

The rate of input of energy by the boundary forces (for plane strain configurations the rate of input per unit length) is

$$R = \int_O \sigma_{ij}\dot{u}_j n_i ds = \int_O (\sigma_{ij}^{(0)} + \sigma_{ij}^{(1)})(\dot{u}_j^{(0)} + \dot{u}_j^{(1)}) n_i ds \tag{A.3}$$

where O is the outer boundary of the body, presumed to be very distant from the crack, and where the second integral results from considering the boundary stresses transferred to the crack face with $\sigma_{ij}^{(0)}$, $\epsilon_{ij}^{(0)}$ and $u_i^{(0)}$, $i,j = 1,2$, referred to as the associated problem first described in section 2.1. This is true if there is total or partial contact on the crack face because either the stress component $\sigma_{ij}n_i$ or the displacement derivative component $\dot{u}_j$ is always zero on the crack face. The term $\sigma_{ij}^{(0)}\dot{u}_j^{(0)}$ will give a negligible contribution compared with the others.

The quantity

$$R_1 = \int_O \sigma_{ij}^{(1)}\dot{u}_j^{(1)} n_i ds \tag{A.4}$$

is the work done on the body in the absence of the crack and will be infinite for an infinite body. The contribution due specifically to the presence of the crack is given by

$$R_c = \int_O [\sigma_{ij}^{(0)}\dot{u}_j^{(1)} + \sigma_{ij}^{(1)}\dot{u}_j^{(0)}] n_i ds. \tag{A.5}$$

This is a finite quantity. As was pointed out by Spencer [19] in the context of elastic crack problems, one must introduce a modified solution which falls-off rapidly at large distances from the crack. The first term of (A.5) becomes negligible with the Spencer modification. It emerges that R_c is the same as the expression for the rate of work done by the boundary forces that would be obtained in the associated problem - with the stresses and displacements determined on the open crack face.

To evaluate R_c, it is necessary to consider the Kolosov-Muskhelishvili equations (2.3)-(2.5) with an introduced Spencer modification. A convenient way of ensuring that the modified stresses and displacements obey the field equations is to add a polynomial to the standard form of $\phi(z,t)$. Consequently, $\phi(z,t)$ will not vanish at infinity although, formally speaking, it diverges as z^n where n is the degree of the polynomial. This is not of concern since the solution is considered to apply within a large but finite circle.

One now obtains

$$\bar{\psi}(z,t) = \phi(\bar{z},t) - \bar{\phi}(\bar{z},t) - \bar{z}\bar{\phi}'(\bar{z},t) + \bar{q}(\bar{z},t) \tag{A.6}$$

where $\bar{q}(\bar{z},t)$ is a polynomial in $\bar{z}$, and the Kolosov-Muskhelishvili equations become

$$[\sigma_{11}^{(0)} + \sigma_{22}^{(0)}](x,y,t) = 2[\phi(z,t) + \bar{\phi}(\bar{z},t)], \tag{A.7}$$

$$\begin{aligned} \Sigma^{(0)}(x,y,t) &= [\sigma_{22}^{(0)} - i\sigma_{12}^{(0)}](x,y,t) \\ &= \phi(z,t) + \phi(\bar{z},t) + (z-\bar{z})\bar{\phi}'(\bar{z},t) + \bar{q}(\bar{z},t), \end{aligned} \tag{A.8}$$

$$\begin{aligned} 2[\mu * \frac{d}{dx}U^{(0)}](x,y,t) &= [\kappa * \phi](z,t) - \phi(\bar{z},t) \\ &\quad -(z-\bar{z})\bar{\phi}'(\bar{z},t) - \bar{q}(\bar{z},t). \end{aligned} \tag{A.9}$$

To obtain an explicit form of $\phi(z,t)$, results presented earlier for the crack subject to an asymmetric bending moment such that sudden partial closure occurs at the positive end only will be modified according to the work of Golden and Graham.

Consider

$$p(x,t) = -\eta(t)x, \tag{A.10}$$

$$u_1^{(1)}(x,y,t) = [g_1x^2 - h_1y^2]/2, \quad u_2^{(1)}(x,y,t) = h_1xy, \tag{A.11}$$

so that

$$U^{(1)} = Kz^2 + L\bar{z}^2 + 2Lz\bar{z} \tag{A.12}$$

where

$$K = (g_1 + 3h_1)/8, \qquad L = (g_1 - h_1)/8. \tag{A.13}$$

Then, from (3.25), the displacement on the crack face has the form

$$u(x,t) = (A(t)u_0(x))/2 + (B(t)u_1(x))/2, \tag{A.14}$$

$$u_0(x) = \begin{cases} (\frac{c}{3} - x)^{\frac{3}{2}}(x+c)^{\frac{1}{2}} & ,x \in [-c, \frac{c}{3}] \\ 0 & ,x \in [\frac{c}{3}, c], \end{cases}$$

$$u_1(x) = x(c^2 - x^2)^{\frac{1}{2}} \tag{A.15}$$

where the explicit forms of $A(t)$ and $B(t)$ are given by (4.37), (4.38), respectively, for the standard linear model and by (4.60), (4.61), respectively, for the general viscoelastic model. Also,

$$v(x,t) = -C(t)w_o(x) + D(t)w_1(x), \tag{A.16}$$

$$w(x,t) = \frac{d}{dx}u(x,t), \tag{A.17}$$

$$w_0(x,t) = \begin{cases} (\frac{1}{3}(\frac{\frac{c}{3}-x}{x+c})^{\frac{1}{2}}(2c+3x) & ,x \in [-c, \frac{c}{3}] \\ 0 & ,x \in [\frac{c}{3}, c], \end{cases}$$

$$w_1(x) = \frac{1}{2}\frac{c^2 - 2x^2}{(c^2 - x^2)^{\frac{1}{2}}}. \tag{A.18}$$

The discontinuity in $\phi(z,t)$ across the open crack face is related [4] to $v(x,t)$ by

$$v(x,t) = Im[\phi^+(x,t) - \phi^-(x,t)] \tag{A.19}$$

where $\phi^{\pm}(x,t)$ are the limits of $\phi(z,t)$ from above and below the real line. Also, from (2.34)

$$\sigma_{22}(x,0,t) = \phi^+(x,t) + \phi^-(x,t). \tag{A.20}$$

By virtue of these two relations and certain Hilbert transform formulae, we deduce that

$$\phi(z,t) = \frac{-C(t)(\frac{z-b}{z+c})^{\frac{1}{2}}}{2\pi} \int_{-c}^{c/3} dx'(\frac{x'+c}{\frac{c}{3}-x'})^{\frac{1}{2}} \frac{x'}{x'-z}$$
$$-\frac{D(t)}{2\pi(z^2-c^2)^{1/2}} \int_{-c}^{c} dx' \frac{x'(c^2-x'^2)^{1/2}}{x'-z} + B_1 + B_2 z + B_3 z^2 + B_4 z^3. \quad \text{(A.21)}$$

In the large z limit

$$\phi(z,t) \sim \frac{1}{z^2}[N_0 + \frac{N_1}{z}] + B_1 + B_2 z + B_3 z^2 + B_4 z^3, \quad |z| \to \infty \quad \text{(A.22)}$$

where

$$N_0 = -K_{p0}C(t); \qquad N_1 = K_{f1}D(t) - K_{p1}C(t) \quad \text{(A.23)}$$

where

$$K_{p0} = -\frac{1}{2\pi}\int_{-c}^{c} dx x^2 (\frac{x+c}{\frac{c}{3}-x})^{\frac{1}{2}} = -(\frac{c}{3})^3, \quad \text{(A.24)}$$

$$K_{f1} = \frac{1}{2\pi}\int_{-c}^{c} dx x^2 (c^2 - x^2)^{\frac{1}{2}} = (\frac{c}{2})^4, \quad \text{(A.25)}$$

$$K_{p1} = -\frac{1}{2\pi}\int_{-c}^{c/3} dx x^3 (\frac{x+c}{\frac{c}{3}-x})^{\frac{1}{2}} - \frac{2}{3}cK_{p0} = \frac{c^4}{3^3}. \quad \text{(A.26)}$$

In the partially closed case, the leading term, varying as z^{-1}, vanishes.

To evaluate R_c, it is convenient to consider O to be a large circle centered at the origin and to express the various quantities in polar coordinates (r, θ). The displacements in polar and Cartesian coordinates are related by

$$u_1 + iu_2 = e^{i\theta}(u_r + iu_\theta). \quad \text{(A.27)}$$

The stress tensor in polar coordinates is determined by the relations [17]

$$\sigma_{rr} + \sigma_{\theta\theta} = \sigma_{11} + \sigma_{22}, \quad \text{(A.28)}$$

$$\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta} = e^{2i\theta}(\sigma_{22} - \sigma_{11} + 2i\sigma_{12}). \quad \text{(A.29)}$$

Thus R_c can be written as

$$R_c = R_c \int_0^{2\pi} r d\theta [\Sigma_r^{(0)} \dot{U}^{(1)} + \Sigma_r^{(1)} \dot{U}^{(0)}] e^{-i\theta} \tag{A.30}$$

where

$$\begin{aligned} \Sigma_r^{(0)} &= (1 + e^{2i\theta})\phi(z,t) + \bar{\phi}(z,t) - e^{2i\theta} \\ &\quad [\bar{\phi}(z,t) - (z - \bar{z})\phi'(z,t) + B_5 + B_6 z + B_7 z^2 + B_8 z^3], \end{aligned} \tag{A.31}$$

$$\Sigma_r^{(1)} = -\frac{1}{4}\eta(t) r (e^{-i\theta} - e^{3i\theta}). \tag{A.32}$$

In this case we take $q(z,t)$ to be a third degree polynomial. As mentioned in the previous section, it is required that the $1/r^2$ and $1/r^3$ terms in $\Sigma_r^{(0)}$ vanish on a large circle $r = r_o$. For $\phi(z,t)$ given by (A.22), this condition may be satisfied by taking

$$\begin{aligned} &B_1 = \frac{N_0}{r_0^2}; \quad B_2 = \frac{3N_1}{r_0^4}; \quad B_3 = \frac{-3N_0}{r_0^4}; \quad B_4 = \frac{-4N_1}{r_0^6}; \\ &B_5 = \frac{4N_0}{r_0^2}; \quad B_6 = \frac{12N_1}{r_0^4}; \quad B_7 = \frac{-6N_0}{r_0^4}; \quad B_8 = \frac{-12N_1}{r_0^6}. \end{aligned} \tag{A.33}$$

From (A.9) we obtain

$$U^{(0)}(x,y,t) = \frac{1}{2} \left\{ \begin{array}{l} -\frac{1}{z}[\gamma * \kappa * (N_0 + \frac{N_1}{2z})] + [\gamma * \kappa * T](z,t) \\ +\frac{1}{\bar{z}}[\gamma * (N_0 + \frac{N_1}{2z})] - [\gamma * T](\bar{z},t) \\ -\frac{(z-\bar{z})}{z^2}[\gamma * (N_0 + \frac{N_1}{z})] - (z - \bar{z}) \\ (\gamma * (B_1 + B_2\bar{z} + B_3\bar{z}^2 + B_4\bar{z}^3)) - [\gamma * W](\bar{z},t) \quad , \end{array} \right.$$

where

$$T(z,t) = B_1 z + (B_2 z^2)/2 + (B_3 z^3)/3 + (B_4 z^4)/4, \tag{A.34}$$

$$W(\bar{z},t) = B_5 \bar{z} + (B_6 \bar{z}^2)/2 + (B_7 \bar{z}^3)/3 + (B_8 \bar{z}^4)/4. \tag{A.35}$$

Thus it can be obtained that

$$R_c = -2\pi\eta(t)\frac{d}{dt}\{\kappa * (N_1)\}(t) = -2\pi\eta(t)\frac{d}{dt}\{-K_{p1}A(t) + K_{f1}B(t)\}. \qquad \text{(A.36)}$$

If we consider a time period $[t_o, t_c]$ when the crack is completely open and a subsequent period $(t_c, t_o + \Delta)$ when it is partially closed, we have that

$$R_c = -2\pi\eta(t)\frac{d}{dt}\{-K_{p1}M(t) + (K_{f1} - K_{p1})B(t)\}, \quad \text{for all t.} \qquad \text{(A.37)}$$

Thus, from (A.2), D_c can be calculated by evaluating

$$D_c = \frac{1}{\Delta}\int_t^{t+\Delta} dt R_c(t). \qquad \text{(A.38)}$$

Detailed solutions for the standard linear model are presented by Golden and Graham [18].

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