IDENTITIES

by

Martin Gilchrist
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Name: Martin Gilchrist

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Examining Committee: Dr. G. A. C. Graham
Chair

____

Dr. A. H. Lachlan, Senior Supervisor

____

Dr. A. R. Freedman

____

Dr. S. K. Thomason

____

Dr. B. R. Alspach

____

Dr. W. A. R. Weiss, External Examiner
University of Toronto

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Abstract

In this work we will study *identities*. Loosely speaking an identity is an equivalence relation on the edges of a finite complete graph. Identities arise naturally in a Ramsey theory setting when one analyzes the finite coloring patterns that must occur when the edges of a large complete graph are colored with a comparatively small set of colors. We restrict ourselves to color sets of size $\leq \aleph_0$, and graphs whose vertex set is of size greater than that of the color set but less than or equal to $\aleph_\omega$. We show that it is consistent that for all $n < \omega$ the identities realized by all ω-colorings of the complete graph on $\aleph_n$ is strictly contained in the set of identities realized by all ω-colorings of the complete graph on $\aleph_{n+1}$ vertices.

We generalize the notion of identity by adjoining an ordering on the set of vertices of the graphs, and an ordering on the set of colors. The objects arising from the generalization are called CV-identities. We consider the CV-identities that arise when coloring complete well-ordered graphs with the color set ω ordered in the usual way. With this color set we consider the CV-identities that arise when the vertex set is one of $\aleph_1$, $\aleph_2$, and $\aleph_\omega$, ordered as a set of ordinals. We also determine the CV-identities that arise when the color set is $\mathbf{Q}$ with the usual ordering and the graph being colored has vertex set $\aleph_1$.

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Contents

Chapter 1

Introduction and Definitions

1.1 Notation and terminology

There are two basic notions that will be studied in this work. They are *identity* and *CV-identity.* Identities arise naturally in a Ramsey theory setting when one analyzes the finite coloring patterns that must occur when the edges of a large complete graph are colored with a comparatively small set of colors. Identities have been studied by Shelah [15] and Gilchrist and Shelah [5, 6]. Their definition will be given here and they will be analyzed in chapter 2. CV-identities are a broad generalization of identity. We will delay their definition until later. They will be analyzed in chapter 3.

An *ω-coloring* is function $f : [B]^2 \longrightarrow \omega$ where B is a set of ordinals ordered in the usual way. The set B is the *field* of f and is denoted $\mathrm{fld}(f)$.

Definition 1.1 Let f, g be ω-colorings. We say that f *realizes* the coloring g if there is a one-one map $k : \mathrm{fld}(g) \longrightarrow \mathrm{fld}(f)$ such that for all $\{x, y\}, \{u, v\} \in \mathrm{dom}(g)$

$$f(\{k(x), k(y)\}) \neq f(\{k(u), k(v)\}) \Rightarrow g(\{x, y\}) \neq g(\{u, v\}).$$

We write $f \simeq g$ if f realizes g and g realizes f. It should be clear that $\simeq$ induces an equivalence relation on the class of ω-colorings. We call the equivalence classes *identities.* The collection of all identities is denoted ID.

Definition 1.2 Let f, g be ω-colorings. We say that f *V-realizes* the coloring g if there is an order-preserving map $k : \text{fld}(g) \longrightarrow \text{fld}(f)$ such that for all $\{x,y\},\{u,v\} \in \text{dom}(g)$

$$f(\{k(x),k(y)\}) \neq f(\{k(u),k(v)\}) \Rightarrow g(\{x,y\}) \neq g(\{u,v\}).$$

We write $f \simeq_V g$ if f V-realizes g and g V-realizes f. Note that $\simeq_V$ induces an equivalence relation on the class of ω-colorings. We call the equivalence classes *V-identities*. The collection of all V-identities is denoted ID_V.

For both types of realization we will call the map $k : \text{fld}(g) \longrightarrow \text{fld}(f)$ an *embedding*. In the above definition the V refers to vertices. In most situations that follow, B will be a cardinal less than or equal to $\aleph_\omega$ ordered in the usual way as a set of ordinals. In the following we will speak of ω-colorings realizing (rather than V-realizing) other ω-colorings whenever the context makes the type of realization clear. If f, g, h, l are ω-colorings, with $f \simeq g$ and $h \simeq l$, then f realizes h if and only if g realizes l. Thus without risk of confusion we may speak of identities realizing colorings and of identities realizing other identities. The same is true of V-identities. If I and J are identities we call J a *subidentity* of I, written $J \hookrightarrow I$ if I realizes J. The notion of sub-V-identity is similarly defined. We say that an identity I is of *size* r if $|\text{fld}(f)| = r$ for some (all) $f \in I$. In the following we will consider only identities of finite size.

An identity can be regarded as a finite structure $\langle A, E\rangle$, where E is an equivalence relation on $[A]^2$, see [15]. The correspondence is given by: $A = \text{fld}(f)$, and $\{x,y\} \simeq_E \{u,v\}$ if and only if $f(\{x,y\}) = f(\{u,v\})$.

A V-identity can be regarded as a finite structure $\langle A, E, <_A\rangle$, where E is an equivalence relation on $[A]^2$ and $<_A$ is a linear ordering of A. The correspondence is given by: $A = \text{fld}(f)$, $\{x,y\} \simeq_E \{u,v\}$ if and only if $f(\{x,y\}) = f(\{u,v\})$, and $x < y$ if and only if $x <_A y$.

We remark that the notion of subidentity does not correspond to that of substructure. To simplify the language in what follows we find it convenient to abuse terminology by referring to ω-colorings and structures $\langle A, E\rangle$ as identities rather than as representatives of identities. Similarly for V-identities. In each case the intended meaning should be clear.

Definition 1.3 Let $f : [B]^2 \longrightarrow \omega$ be an ω-coloring and $I = \langle A, E, <_A\rangle$ be a structure corresponding to a V-identity. Let $k : A \longrightarrow B$ be an order-preserving map such that for all $\{x,y\},\{v,w\}\in[A]^2$,

$$f(\{k(x),k(y)\})\neq f(\{k(v),k(w)\})\Rightarrow\{x,y\}\not\simeq_E\{v,w\}.$$

Then k is called an *embedding* of I into f. A similar definition is given when I is an identity.

Definition 1.4 Let K be a collection of identities. We define

$$\mathcal{S}(K)=\{I:(\exists J\in K)(I \text{ is a subidentity of } J)\}.$$

Similarly, for K a collection of V-identities, we define

$$\mathcal{S}_V(K)=\{I:(\exists J\in K)(I \text{ is a sub-V-identity of } J)\}.$$

Definition 1.5 Let D be a set of ordinals, $m<\omega$ and $f:[B]^2\longrightarrow\omega$ be an ω-coloring.

i) $\mathcal{I}(f,n)$ ($\mathcal{I}_V(f,n)$) is the collection of (V-) identities of size n realized by f.

ii) $\mathcal{I}(D,m,n)=\bigcap\{\mathcal{I}(g,n)\,|\,g:[D]^2\longrightarrow m\}$, $\mathcal{I}_V(D,m,n)=\bigcap\{\mathcal{I}_V(g,n)\,|\,g:[D]^2\longrightarrow m\}$.

iii) $\mathcal{I}(D,n)=\bigcap\{\mathcal{I}(g,n)\,|\,g:[D]^2\longrightarrow\omega\}$, $\mathcal{I}_V(D,n)=\bigcap\{\mathcal{I}_V(g,n)\,|\,g:[D]^2\longrightarrow\omega\}$.

iv) $\mathcal{I}(f)=\bigcup\{\mathcal{I}(f,n):n<\omega\}$, $\mathcal{I}_V(f)=\bigcup\{\mathcal{I}_V(f,n):n<\omega\}$.

v) $\mathcal{I}(D)=\bigcup\{\mathcal{I}(D,n):n<\omega\}$, $\mathcal{I}_V(D)=\bigcup\{\mathcal{I}_V(D,n):n<\omega\}$.

We now define several collections of identities. To do this we establish some notation.

Let σ, $\tau\in{}^{<\omega}2$. We write $\tau\subseteq\sigma$ whenever τ is an initial segment of σ. A *binary tree* is a subset t of ${}^{<\omega}2$ such that for all $\sigma\in t$ and $\tau\in{}^{<\omega}2$, $\tau\subseteq\sigma$ implies $\tau\in t$. Let T be the set of all finite binary trees. For $t\in T$, σ is a *leaf* of t if $\sigma\in t$ and σ is $\subseteq$-maximal in t. We write $B(t)$ for the collection of leaves of t. For $t\in T$ we define an identity $I_t=\langle A_t,E_t\rangle$ in the following manner: A_t is the collection of leaves of t, and for $\{\alpha,\beta\},\{\gamma,\delta\}\in[A_t]^2$ we have $\{\alpha,\beta\}\simeq_{E_t}\{\gamma,\delta\}$ if and only if $\alpha\cap\beta=\gamma\cap\delta$. An identity J which can be represented as I_t for some $t\in T$ is said to be realized by a *binary tree*. We denote by ID_T the collection $\{I_t:t\in T\}$ and by IDT the collection $\mathcal{S}(\mathrm{ID}_T)$. I_m is used to denote I_t when $t={}^{(m+1)}2$. We call I_m the identity realized by the complete binary tree of height m+1. It should be noted that IDT is equal to $\mathcal{S}(\{I_m:m<\omega\})$.

If $J=\langle B,F\rangle$ is an identity and $A\subset B$ we define the *restriction* of J to A to be the identity $I=\langle A,F\cap([A]^2\times[A]^2)\rangle$. This will be written: $I=J{\restriction}A$. Similarly for V-identities.

Definition 1.6 Let $n < \omega$, $I = \langle A, E \rangle$, $J = \langle B, F \rangle$ be identities, $\bar{a} = \langle a_1, \ldots, a_n \rangle \in A^n$ be a sequence of distinct elements from A, and $\bar{b} = \langle b_1, \ldots, b_n \rangle \in B^n$ be a sequence of distinct elements from B. J is obtained from I by *duplication* of $\bar{a}$ to $\bar{b}$ if

i) $B = A \sqcup \bar{b}$

ii) $I = J \restriction A$

iii) the mapping which is the identity on $A \setminus \bar{a}$ and which maps $\bar{a}$ to $\bar{b}$ is is an embedding of I into J as structures

iv) F is the least equivalence relation on $[B]^2$ consistent with i) – iii).

When $n = 1$ we say that J is a *one-point* duplication of I.

Definition 1.7 Let $I = \langle A, E, <_A \rangle$ and $J = \langle B, F, <_B \rangle$ be V-identities. J is obtained from I by end-duplication if there exist final segments $\bar{a}, \bar{b}$ of $\langle A, <_A \rangle$, $\langle B, <_B \rangle$ respectively such that the structure $\langle B, F \rangle$ is obtained from the structure $\langle A, E \rangle$ by duplicating $\bar{a}$ to $\bar{b}$.

Let IDE_V denote the minimal collection of V-identities which is closed under end-duplication, the taking of subidentities, and which contains the trivial V-identity of size one. Let IDE $= \{\langle A, E \rangle : \langle A, E, < \rangle \in IDE_V\}$

Let IDM denote the least class of identities which is closed under duplication, the taking of subidentities and which contains the identity of size one. We will show that IDM = IDT, see §2.5 theorem 2.37. This result is probably known but we have been unable to find it in the literature.

We quote the following results of Shelah:

Theorem 1.8 **([15])** $\mathcal{I}(\aleph_1) \supset$ *IDE.*

Theorem 1.9 **([14])** *There exists* $f : [\aleph_1]^2 \longrightarrow \aleph_0$ *such that* $\mathcal{I}(f) \subseteq$ *IDE.*

Corollary 1.10 $\mathcal{I}(\aleph_1) =$ *IDE.*

Theorem 1.11 **([16])** *If* $\kappa \geq \aleph_\omega$ *then* $\mathcal{I}(\kappa) \supseteq$ *IDM.*

1.2 New results on Identities

We now summarize the original results to be presented in the thesis. We begin with results related to the study of identities.

The most important result presented here says that every model M of ZFC has a generic extension in which cardinals are preserved and in which $I_m \notin \mathcal{I}(\aleph_m)$. This is theorem 2.18 below which appears in the joint paper [5]. The main idea of the proof, in particular the idea of using historical forcing, is due to Shelah. Another result from [5] states that for all m, $0 < m < \omega$, $\mathrm{eh}(\mathcal{I}(\aleph_m)) \subseteq \mathcal{I}(\aleph_{m+1})$. Here eh is a certain operation on sets of identities, see definition 2.20. This appears here as theorem 2.21 and is due to the author. These two theorems imply that, for each $m \geq 1$, the consistency of ZFC implies the consistency of ZFC plus $\mathcal{I}(\aleph_m) \subsetneq \mathcal{I}(\aleph_{m+1})$. In Section 2.3 we sharpen this result showing that Con(ZFC) implies the consistency of ZFC plus $(\forall m \geq 1)(\mathcal{I}(\aleph_m) \subsetneq \mathcal{I}(\aleph(_{m+1}))$.

The most important open problem about identities is to characterize, for each $m < \omega$, the set of identities I such that ZFC $\vdash I \in \mathcal{I}(\aleph_m)$. Below, the set of identities which are in $\mathcal{I}(\aleph_m)$ in every model of ZFC will be denoted IDA^m. We approach this problem by analyzing a certain collection $\mathcal{C}^m$ whose definition is based on the forcing construction used in [5]. For this we must refer to the partial ordering which determines the appropriate notion of forcing. Let m, $2 \leq m < \omega$, be fixed and

$$F = \{f_A : A \in \wp(\aleph_m)\}$$

be an indexed family of mappings such that f_A is a one to one function from A into A such that rng(f) has order type $|A|$. From the pair (F, m) a set $\mathbf{P}^{F,m}$ of finite ω-colorings is defined (see §2.1) whose fields are finite subsets of $\aleph_m$. $\mathbf{P}^{F,m}$ is ordered by inverse inclusion — this is the notion of forcing used in [5] which we discuss in more detail here.

We let $\mathcal{C}^{F,m}$ denote:

$$\{I \in \mathrm{ID} : \exists p \in \mathbf{P}^{F,m}\,(p \text{ realizes } I)\}\,,$$

and $\mathcal{C}^m$ denote the union of the sets $\mathcal{C}^{F,m}$ as F runs through all possible families. Intuitively, ID $\setminus\mathcal{C}^m$ is the set of all identities which are omitted by every possible generic coloring of $\aleph_m$, where "generic" is restricted to the particular sense exploited in [5]. Of course, ideally one would like information about the *intersection* of the sets $\mathcal{C}^{F,m}$ as F runs through all possible families. But we know of no way of working with the intersection rather than the union. We do not know if the union is in fact different than the intersection.

Our principal findings about the sets $\mathcal{C}^m$ are:

i) $\mathcal{C}^m$ is recursively enumerable. (§2.4, theorem 2.26)

ii) $\mathcal{C}^m \subseteq$ IDT. (§2.5, lemma 2.32)

iii) Let t be a binary tree. Then $I_t \in \mathcal{C}^m$ if and only if t does not embed the complete binary tree of height $m+1$. (§2.7, corollary 2.45)

iv) For each m there is no finite set of identities $\mathcal{J}^m$ such that $I \notin \mathcal{C}^m$ if and only if some member of $\mathcal{J}^m$ imbeds in I. (§2.5, theorem 2.36)

v) The tree identities in $\mathcal{C}^2$ do not generate $\mathcal{C}^2$. More precisely $\mathcal{S}(\mathcal{C}^2 \cap \mathrm{ID}_T) \neq \mathcal{C}^2$. (§2.7, theorem 2.52)

vi) It is false that $I \in \mathrm{IDT} \Rightarrow (I \notin \mathcal{C}^2 \Leftrightarrow I_2$ embeds in $I)$. (§2.6, corollary 2.41)

Other new results not concerned with $\mathcal{C}^m$ are the following:

i) For all $k, m < \omega$, $\mathcal{I}(k, m, 4) \neq \mathcal{I}(\aleph_1, \aleph_0, 4)$. This implies that the identities realized by coloring finite graphs with finitely many colors cannot capture the diversity of identities realized by colorings of infinite graphs with infinitely many colors. (§2.9, theorem 2.64)

ii) For all $k < \omega$, if there exists an $\aleph_1$-complete, $(\aleph_k, \aleph_k, < \omega)$-saturated ideal on $\aleph_k$, then $\mathcal{I}(\aleph_k) \supseteq$ IDT. (§2.8, theorem 2.57)

1.3 New results on CV-identities

In Chapter 3 we will investigate a new concept called CV-identity. Let f be an ω-coloring. In passing from f to the identity $f/\simeq$, both the ordering of the vertex set, fld(f), and the ordering of the set of colors, rng(f), are forgotten. The V-identities are what we get when only the color ordering is forgotten. The CV-identities are obtained when neither ordering is forgotten.

An O-coloring is a triple $\langle f, F, <^F \rangle$, where $f : [B]^2 \longrightarrow F$, $B \subset$ On, and $<^F$ is a linear ordering of F. The O-coloring $\langle f, F, <^F \rangle$ *realizes* the O-coloring $\langle g, G, <^G \rangle$ if there exists an order-preserving map $k : \text{fld}(g) \longrightarrow \text{fld}(f)$ such that

$$f(\{k(x), k(y)\}) <^F f(\{k(u), k(v)\}) \Rightarrow g(\{x, y\}) <^G g(\{u, v\})$$

for all $\{x, y\}, \{u, v\} \in \text{dom}(g)$. We say that the O-colorings are *equivalent* , written $f \simeq g$, if f realizes g and g realizes f. The equivalence classes of finite O-colorings are called CV-identities. We denote by ID_{CV} the collection of all CV-identities. For a cardinal κ and an ordering $\langle F, <^F \rangle$, $\mathcal{I}_{CV}(\kappa, F, <^F)$ denotes the set of all CV-identities realized by every O-coloring, $\langle f, F, <^F \rangle$, with $\text{fld}(f) = \kappa$. When F has a natural ordering, such as in the cases when $F = \mathbf{Q}$ and $F = \omega$, $\mathcal{I}_{CV}(\kappa, F, <^F)$ is abreviated $\mathcal{I}_{CV}(\kappa, F)$. In situations where $\langle F, <^F \rangle$ is clear from the context we write $\mathcal{I}_{CV}(\kappa)$ instead of $\mathcal{I}_{CV}(\kappa, F)$.

Since other authors have not considered CV-identities, the only results about them which can be considered as already known are those which follow from the existing literature on identities and V-identities. We now list the results that have been proved concerning CV-identities.

i) We define a collection $\mathcal{C}$ of CV-identities and show that

$$\text{Con(ZFC)} \Rightarrow \text{Con}(\text{ZFC} + \mathcal{I}_{CV}(\aleph_1, \omega) = \mathcal{C}).$$

(§3.1, theorem 3.2)

ii) There exist a CV-identity J whose underlying identity is an element of $\mathcal{I}(\aleph_2)$ yet $J \notin \mathcal{I}_{CV}(\aleph_2, \omega)$. (§3.2 This is contained in a remark following lemma 3.25.)

iii) We define a large class, $\mathcal{E}$, of CV-identities and show that $\mathcal{I}_{CV}(\aleph_\omega, \omega) \supseteq \mathcal{E}$. (§3.3, theorem 3.29)

iv) $\mathcal{I}_{CV}(\aleph_1, \mathbf{Q})$ only contains two CV-identities. They are the ones having field size one and two. (§3.4, theorem 3.41)

v) As an application CV-identities we exhibit a c.c.c notion of forcing which destroys all $\aleph_2$-saturated ideals on $\aleph_1$. The existence of such partial orders is known, but this is a new example. (§3.2, theorem 3.28)

Chapter 2

Identities

In this chapter we will examine $\mathcal{I}(\kappa)$ for κ a cardinal less than or equal to $\aleph_\omega$. The literature on identities up to 1992, discussed on page 5, leaves open the question of how the sets $\mathcal{I}(\aleph_m)(2 \leq m < \omega)$ fit between IDE ($= \mathcal{I}(\aleph_1)$) and IDT ($\subseteq \mathcal{I}(\aleph_\omega)$). Some progress in this direction has been made in the papers [5] and [6] by the author and Shelah. In [5] it is shown that if ZFC is consistent then so is ZFC + $\mathcal{I}(\aleph_{m+1}) \supsetneq \mathcal{I}(\aleph_m)$ for each $m < \omega$. This result will be reproduced here. It will also be expanded and analyzed in greater depth.

We now present some lemmas which describe some general information about the set IDA^m, for $m \geq 2$. The proof of the first is a slight modification of the proof of lemma 6 in [15]. For this reason we offer no proof.

Lemma 2.1 *Let κ be a cardinal, $\aleph_2 \leq \kappa < \aleph_\omega$, and $I \in \mathcal{I}_V(\kappa)$. If J is an end-duplication of I then $J \in \mathcal{I}_V(\kappa)$.*

Lemma 2.2 *Let $2 \leq m < \omega$. Then IDA^m $\subseteq$IDT.*

Proof: Let M be a model of ZFC in which $2^{\aleph_0} \geq \aleph_m$. We show that there exists $f : [\aleph_m]^2 \longrightarrow \omega$ such that $I(f) \subseteq$ IDT. Let $g : \aleph_m \longrightarrow {}^{\omega}2$ and $h : {}^{<\omega}2 \longrightarrow \omega$ be one to one maps. Define $f : [\aleph_m]^2 \longrightarrow \omega$ by $f(\{\alpha, \beta\}) = h(g(\alpha) \cap g(\beta))$. Now let $n < \omega$ and $A \in [\aleph_m]^n$. Define r to be $\max\{|g(a) \cap g(b)| : \{a, b\} \in [A]^2\}$. Define $t \in T$ to be the finite binary tree for which $B(t) = \{g(a) \restriction (r+2) : a \in A\}$. It should be clear that $f \restriction A$ is realized by the identity I_t. $\square$

Lemma 2.3 *For all $k \geq 1$, $\mathcal{I}(\aleph_{k+1}) = ID$ if and only if $2^{\aleph_0} \leq \aleph_k$.*

Proof: To prove the ' if ' direction, let $f : [\aleph_{k+1}]^2 \longrightarrow \omega$. By the Erdos Rado theorem there exists $B \subseteq \aleph_{k+1}$ such that $|B| \geq \aleph_1$ and $|f''B| = 1$. Clearly $f \restriction [B]^2$ realizes all identities in ID. To prove the other direction note that if $2^{\aleph_0} \geq \aleph_{k+1}$ the function defined in the previous lemma does not realize any identity not in IDT. This clearly suffices since IDT $\subsetneqq$ ID. □

We now present a theorem that will simplify the task of characterizing $\mathcal{I}(D)$ for D a set of ordinals.

Theorem 2.4 *Let D be a set of ordinals and $I \in \mathcal{I}(D)$. Then there exists $J \in \mathcal{I}_V(D)$ such that I is the reduct of J. Conversely the reduct of any $J \in \mathcal{I}_V(D)$ is in $\mathcal{I}(D)$.*

Proof: Let I be an identity which is not the reduct of any V-identity in $\mathcal{I}_V(D)$. Let $J_1, \ldots, J_k$ be the V-identities that have I as their underlying identity. For each $1 \leq i \leq k$ there exist $f_i : [D]^2 \longrightarrow \omega$ such that $J_i \notin \mathcal{I}_V(f_i)$. Define $f : [D]^2 \longrightarrow \omega$ by $f(\{\alpha, \beta\}) = g(\langle f_i(\{\alpha, \beta\}) : i \leq k \rangle)$, where $g : \omega^k \longrightarrow \omega$ is any one-one function. Then $\{J_i : 1 \leq i \leq k\} \cap \mathcal{I}_V(f) = \emptyset$. This implies that $I \notin \mathcal{I}(f)$ and so $I \notin \mathcal{I}(D)$. The converse is obvious. □

To explain the work to follow we now sketch the contents of [5] which will occupy sections 1 and 2 of this chapter. First we fix a natural number $m \geq 2$ and a model, M of ZFC. A collection of functions $F = \langle f_A : A \in \wp(\aleph_m) \rangle$ is chosen as a set of parameters for defining a partial order $\mathbf{P}^{F,m}$. We stipulate that $f_A : A \longrightarrow A$ is one-to-one and rng(f) has order type $|A|$. This partial order consists of a collection of pairs $\langle u, c \rangle$ where c is an ω-coloring, fld$(c) = u$ and u is a subset of $\aleph_m$. We choose a $\mathbf{P}^{F,m}$-generic filter G and essentially define f_m to be $\bigcup\{c : \exists u(\langle u, c \rangle \in G)\}$. We show that $M[G] \models I_m \notin \mathcal{I}(f_m)$. This is one of the two key results of [5] and is presented in §2.1. The other is a construction which shows how certain identities in $\mathcal{I}(\aleph_{m+1})$ may be generated from identities in $\mathcal{I}(\aleph_m)$. This construction occupies §2.2.

We extend the above results using an iterated forcing construction and show the existence of a model M such that $M \models (\forall m, 2 \leq m < \omega)(\mathcal{I}(\aleph_\omega) \supsetneqq \mathcal{I}(\aleph_{m+1}) \supsetneqq \mathcal{I}(\aleph_m))$. This result appears in the section §2.3.

Unfortunately, the results of [5], although a step in the right direction, do not resolve, for $m \geq 2$, the central question as to which identities are in $\mathcal{I}(\aleph_m)$ in every model of ZFC. (We recall that this set of identities is denoted IDA^m.) On the other hand it is still open whether the method of [5] resolves this question. Towards identifying how far historical forcing can help us determine IDA^m, in §2.4 - §2.7, we investigate the class of identities $\mathcal{C}^m$ defined as follows. First let $\mathcal{C}^{F,m}$ denote

$$\{I \in ID : \exists p \in \mathbf{P}^{F,m}(p \text{ realizes } I)\}$$

and $\mathcal{F}^m$ denote

$$\{F : F \text{ is a function, } \mathrm{dom}(F) = \wp(\aleph_m), \text{ and}$$
$$(\forall X \subseteq \aleph_m)(F(X) : X \longrightarrow X \text{ is one-to-one and } \mathrm{rng}(F(X)) \text{ has order type } |X|)\}.$$

Then $\mathcal{C}^m$ is defined to be $\bigcup\{\mathcal{C}^{F,m} : F \in \mathcal{F}^m\}$. Since the generic coloring f_m considered in §2.1 realizes at most colors in $\mathcal{C}^{F,m}$ we have

$$\mathrm{IDA}^m \subseteq \bigcap\{\mathcal{C}^{F,m} : F \in \mathcal{F}^m\} \subseteq \mathcal{C}^m.$$

It is an open question whether $\mathcal{C}^m \subseteq$ IDA^m. So it seem worthwhile investigating the set $\mathcal{C}^m$.

We conjecture that $\mathcal{C}^m$ does not depend on the particular ground model of ZFC being considered. However, the most we have been able to show in this direction is that $\mathcal{C}^m$ is recursively enumerable (see §2.4). It would be helpful to have an explicit criterion for I to be in $\mathcal{C}^m$ in terms of the structure of I. However, at present no such criterion is known.

One might hope that ID $\backslash\mathcal{C}^m$ could be characterized by a finite number of "constraints". In §2.5 we eliminate this possibility by showing that there is no finite $\mathcal{J}^m \subseteq$ ID such that

$$(\forall I \in \mathrm{ID})[I \notin \mathcal{C}^m \Leftrightarrow (\exists J \in \mathcal{J}^m)(J \hookrightarrow I)].$$

Now recall the collection, IDT, of identities generated by finite binary trees. We also prove in §2.5 that IDM = IDT.

In §2.6 we demonstrate the existence of an identity in IDT $\backslash\mathcal{C}^2$ which does not embed I_2. This shows that IDT $\backslash\mathcal{C}^2$ is not characterized by the single constraint I_2.

Now recall the definition of T and the identities I_t for $t \in T$. It was conjectured that for m, $2 \leq m < \omega$ there exists $T_m \subseteq T$ such that $\mathcal{C}^m = \mathcal{S}(\{I_t : t \in T_m\})$. In §2.7 we refute this conjecture for the case $m = 2$. (The question is open for $m > 2$ but seems likely to go the same way.) Three results are needed.

Firstly, we show that, for $2 \leq m < \omega$, $I_t \in \mathcal{C}^m$ if and only if t does not embed the complete binary tree of height $m+1$. Secondly, we show that if t does not embed the complete binary tree of height 3 then $I_t \in \mathrm{eh}(\mathcal{I}(\aleph_1))$. Finally, we exhibit a particular identity $J \in \mathcal{C}^2$ of size 8 such that $J \notin \mathrm{eh}(\mathcal{I}(\aleph_1))$. (The operator eh on sets of identities is defined in §2.2.)

That $J \in \mathcal{C}^2$, is proved by studying a certain operation $I \mapsto \hat{I}$ on $\mathcal{I}(\aleph_m)$. These results are in the section §2.7. This concludes our analysis and extension of the results in [5].

In §2.8 we continue the analysis of $\mathcal{I}(\kappa)$ and show that, if there exists an $\aleph_1$-complete $(\aleph_n, \aleph_n, < \omega)$-saturated ideal on $\aleph_n$, then $\mathcal{I}(\aleph_n) \supseteq$ IDT. Thus the preceding results will demonstrate the existence of a c.c.c. forcing notion that destroys all such ideals.

Finally we consider finite colorings of finite complete graphs. We show that for $n > 3$, there do not exist $k, m < \omega$ such that $\mathcal{I}(k, m, n) = \mathcal{I}(\aleph_1, \omega, n)$.

2.1 The Forcing Construction

For $2 \leq m < \omega$ we are going to define a partial order for which the corresponding notion of forcing will allow us to omit an identity from $\mathcal{I}(\aleph_m)$. The resulting kind of forcing is called historical forcing (see [13]) and first appeared in Baumgartner and Shelah [2]. In this method conditions are allowed into the partial order if they can be constructed from the amalgamation of simpler conditions satisfying certain properties.

2.1.1 The Partial Order

Fix m, $2 \leq m < \omega$. The definition of the partial order depends on a parameter $F = \{f_A : A \in \wp(\aleph_m)\}$, where for each $A \subseteq \aleph_m$, $f_A : A \longrightarrow A$ is a one-to-one function such that $\mathrm{rng}(f)$ has order type $|A|$. We let $\mathcal{F}^m$ denote the set of all such F.

Definition 2.5 Let $2 \leq m < \omega$, $F \in \mathcal{F}^m$, and $1 \leq t \leq m$. For $\bar{\alpha} = \langle \alpha_1, \ldots, \alpha_t \rangle$, a sequence of distinct elements in $\aleph_m$ we define a subset $A_{\bar{\alpha}}$ of $\aleph_m$. The definition is by recursion on t, the length of the sequence $\bar{\alpha}$. If $t = 1$ then $\bar{\alpha} = \langle \alpha_1 \rangle$ and $A_{\bar{\alpha}}$ is defined to be $\{x \in \aleph_m : x < \alpha_1\}$. Now suppose that $\bar{\alpha} = \langle \alpha_1, \ldots, \alpha_{n+1} \rangle$. Let $\bar{\beta}$ denote $\langle \alpha_1, \ldots, \alpha_n \rangle$. We assume that $A_{\bar{\beta}}$ has been defined. If $\alpha_{n+1} \in A_{\bar{\beta}}$, we define $A_{\bar{\alpha}}$ to be $\{x \in A_{\bar{\beta}} : f_{A_{\bar{\beta}}}(x) < f_{A_{\bar{\beta}}}(\alpha_{n+1})\}$. If $\alpha_{n+1} \notin A_{\bar{\beta}}$ we define $A_{\bar{\alpha}}$ to be the empty set.

Definition 2.6 Let $2 \leq m < \omega$, $F \in \mathcal{F}^m$, $b \in \aleph_m$, and $a \in [\aleph_m]^{<\omega}$. We say $bR^{F,m}a$ if and only if there exists $\bar{\alpha} = \langle \alpha_1, \ldots, \alpha_m \rangle \in {}^m a$ such that $b \in A_{\bar{\alpha}}$.

When $m = 2$ we will simplify the notation in the following manner. Let $F \in \mathcal{F}^2$, $b \in \aleph_2$, and $a \in [\aleph_2]^{<\omega}$. As just specified, $bR^{F,2}a$ if and only if there exist $\alpha_1, \alpha_2 \in a$ such that $\alpha_2 < \alpha_1$, $b < \alpha_1$, and $f_{\alpha_1}(b) < f_{\alpha_1}(\alpha_2)$. To simplify the notation we define, for $\gamma \in \aleph_2$, the linear order $<_\gamma^F$ by $\alpha <_\gamma^F \beta$ if and only if $f_\gamma(\alpha) < f_\gamma(\beta)$. With this new notation we have that $bR^{F,2}a$ if and only if there exist $\alpha_1, \alpha_2 \in a$ such that $b <_{\alpha_1}^F \alpha_2$.

Let $\mathbf{R} = \{\langle u, c \rangle : u \in [\aleph_m]^{<\omega}, c : [u]^2 \longrightarrow \omega\}$. The elements of the partial order will be certain pairs in $\mathbf{R}$. The next three definitions will enable us to select from $\mathbf{R}$ the desired subset.

Definition 2.7 $p = \langle u, c \rangle \in \mathbf{R}$ is the *amalgam* of $p^0 = \langle u^0, c^0 \rangle$ and $p^1 = \langle u^1, c^1 \rangle \in \mathbf{R}$ if there exist $h < \omega$ and increasing sequences $i_0^0, \ldots, i_h^0$ and $i_0^1, \ldots, i_h^1$ in $\aleph_m$ such that for all s, t with $(0 \leq s < t \leq h)$, and all $i, j, k, l < \aleph_m$:

i) $u^0 = \{i_0^0, \ldots, i_h^0\}$ and $u^1 = \{i_0^1, \ldots, i_h^1\}$

ii) $c^0(\{i_s^0, i_t^0\}) = c^1(\{i_s^1, i_t^1\})$

iii) $i_t^0 = i_t^1 \vee i_t^0 < i_t^1$

iv) $u = u^0 \cup u^1$

v) $c \supset (c^0 \cup c^1)$

vi) $\{i, j\} \notin [u^0]^2 \cup [u^1]^2$ implies $c(\{i, j\}) \notin \mathrm{rng}(c^0) \cup \mathrm{rng}(c^1)$

vii) $c(\{i, j\}) = c(\{k, l\})$ implies $(\{i, j\} = \{k, l\} \vee \{i, j\}, \{k, l\} \in [u^0]^2 \cup [u^1]^2)$

The amalgamation of p^0 and p^1 to p is *allowed by* F provided that p is the amalgam of p^0 and p^1 and for all t:

viii) $i_t^0 \neq i_t^1$ implies $\neg i_t^1 R^{F,m} u^0$.

Note that, if there is an amalgamation of p^0 and p^1 then there cannot be one of p^1 and p^0. So there is an essential lack of symmetry. It is also worth observing that in terms of the notion of duplication, the amalgam of p^0 and $p^1 \in \mathbf{R}$ may be regarded as being obtained from p^0 by simultaneous duplication of all the elements in $u^0 \setminus u^1$.

Definition 2.8 $q = \langle u^q, c^q \rangle \in \mathbf{R}$ is a *one-point extension* of $p = \langle u^p, c^p \rangle \in \mathbf{R}$ if $u^q = u^p \cup \{r\}$ for some $r > u^p$, $c^p \subset c^q$, and for all $i, j, k, l \in u^q$

i) $\{i,j\} \notin \mathrm{dom}(c^p)$ implies $c^q(\{i,j\}) \notin \mathrm{rng}(c^p)$

ii) $c^q(\{i,j\}) = c^q(\{k,l\})$ implies $(\{i,j\}, \{k,l\} \in \mathrm{dom}(p) \vee \{i,j\} = \{k,l\})$.

Definition 2.9 Let $\mathbf{P}_0^{F,m} = \{\langle u, c\rangle \in \mathbf{R} : |u| = 1\}$ and let $\mathbf{P}_{n+1}^{F,m}$ be the subset of $\mathbf{R}$ which contains $\mathbf{P}_n^{F,m}$, all amalgams of pairs of elements from $\mathbf{P}_n^{F,m}$ allowed by F and all one-point extensions of elements of $\mathbf{P}_n^{F,m}$. Let $\mathbf{P}^{F,m} = \bigcup\{\mathbf{P}_n^{F,m} : n < \omega\}$. Given $p = \langle u^p, c^p\rangle$ and $q = \langle u^q, c^q\rangle$ we define $p \leq q$ to hold if $u^p \supseteq u^q$ and $c^p \supseteq c^q$.

Closing of $\mathbf{P}^{F,m}$ under one-point extensions is necessary to show that our forcing produces a function whose domain is of size $\aleph_m$. Elements of $\mathbf{P}^{F,m}$ are called *forcing conditions.* Conditions $p, q \in \mathbf{P}^{F,m}$ are said to be *compatible* if there exists $r \in \mathbf{P}^{F,m}$ such that $r \leq p$ and $r \leq q$. A condition $p = \langle u, c\rangle$ is said to *realize* an identity $I = \langle A, E\rangle$ just if the ω-coloring c realizes I. In this case the embedding $k : A \longrightarrow \mathrm{fld}(c)$ which witnesses the realization of I in c is called an *embedding* of I into p. Similarly for V-identities.

Lemma 2.10 *Let* $2 \leq m < \omega$, $F \in \mathcal{F}^m$, *and* $a \in [\aleph_m]^{<\omega}$. *Then* $|\{c : cR^{F,m}a\}| \leq \aleph_0$.

Proof: As there are only finitely many $\bar{\alpha} \in {}^m a$ it is sufficient to show that $|\{c : cR^{F,m}\bar{\alpha}\}| \leq \aleph_0$, for each $\bar{\alpha} \in {}^m a$. Fix $\bar{\alpha} = \langle \alpha_1, \ldots, \alpha_m\rangle$. Let B denote the set of non-empty initial segments of $\bar{\alpha}$ and consider the sets $\{A_{\bar{\beta}} : \bar{\beta} \in B\}$. By induction on $|\bar{\gamma}|$, we show that for all $\bar{\gamma} \in B$, $|A_{\bar{\gamma}}| \leq \aleph_t$, where $t = m - |\bar{\gamma}|$.

When $|\bar{\gamma}| = 1$, $A_{\bar{\gamma}} = \{x : x < \alpha_1\}$ and so $|A_{\bar{\gamma}}| \leq \aleph_{m-1}$. Now let $\bar{\delta} \in B$ have length $n+1$ and define $\bar{\eta}$ to be $\bar{\delta} \restriction \{1, 2, \ldots n\}$. If $\alpha_{n+1} \in A_{\bar{\eta}}$ then $A_{\bar{\delta}} = \{x \in A_{\bar{\eta}} : f_{A_{\bar{\eta}}}(x) < f_{A_{\bar{\eta}}}(\alpha_{n+1})\}$.

By the induction hypothesis $|A_{\bar{\eta}}| \leq \aleph_{m-|\bar{\eta}|}$ and since $f_{A_{\bar{\eta}}}$ has range of order type $|A_{\bar{\eta}}|$, $|A_{\bar{\delta}}| \leq \aleph_{m-|\bar{\eta}|-1} = \aleph_{m-|\bar{\delta}|}$. If $\alpha_{n+1} \notin A_{\bar{\eta}}$ then $A_{\bar{\delta}} = \emptyset$. In either case the result follows. We conclude that $|A_{\bar{\alpha}}| \leq \aleph_0$. □

Lemma 2.11 *For m, $2 \leq m < \omega$ and $F \in \mathcal{F}^m$, $\mathbf{P}^{F,m}$ is c.c.c.*

Proof: Let $\langle p_\alpha : \alpha < \omega_1 \rangle$ be a sequence of distinct conditions. By thinning we can suppose that there are $n, l < \omega$ and i_j^α $(\alpha < \omega_1, 0 \leq j \leq n)$ such that for all $\alpha, \beta < \omega_1$ and all j, k, with $0 \leq j < k \leq n$,

i) $u^{p_\alpha} = \{i_0^\alpha, \ldots, i_n^\alpha\}$

ii) $i_j^\alpha < i_k^\alpha$

iii) $c^{p_\alpha}(\{i_j^\alpha, i_k^\alpha\}) = c^{p_\beta}(\{i_j^\beta, i_k^\beta\})$

iv) $p_\alpha \in \mathbf{P}_l^{F,m}$.

Applying a Δ-system argument, see [8] page 49, allows us to thin the sequence of conditions further so that

$$\forall\alpha\forall\beta(i_t^\alpha = i_t^\beta) \vee (\forall\beta < \omega_1)(\forall\alpha < \beta)(i_t^\alpha < i_t^\beta) \quad (0 \leq t \leq n).$$

Let $T = \{t \leq n : i_t^\alpha \neq i_t^\beta \text{ some } \alpha, \beta < \omega_1\}$. For p_0 and p_α to have a common extension in $\mathbf{P}^{F,m}$ we need only show that $i_t^\alpha R^{F,m}\{i_0^0, \ldots, i_n^0\}$ fails for all $t \in T$. By the previous lemma $|\{i \in \aleph_m : iR^{F,m}\{i_0^0, \ldots, i_n^0\}\}| = \aleph_0$. For each $t \in T$, i_t^α is strictly increasing in α, whence $i_t^\alpha R_0^m\{i_0^0, \ldots, i_n^0\}$ fails for all sufficiently large $\alpha < \omega_1$. Since T is finite, p_0 and p_α have a common extension in $\mathbf{P}^{F,m}$ for all sufficiently large α. □

Lemma 2.12 *For each $\alpha < \aleph_m$*

$$D_\alpha = \{\langle u, c \rangle \in \mathbf{P}^{F,m} : \exists\beta(\beta \in u \wedge \alpha < \beta)\}$$

is dense in $(\mathbf{P}^{F,m}, <)$.

Proof: Let $\alpha < \aleph_m$ and $p \in \mathbf{P}^{F,m}$. Choose $\beta > \alpha$ and define $q = \langle u^q, c^q \rangle \in \mathbf{P}^{F,m}$ to be the one-point extension of p such that $u^q = u^p \cup \{\beta\}$. It is clear that $q \in D_\alpha$ and q extends p. □

We now fix m with $2 \leq m < \omega$, a countable model M of ZFC, and $F \in (\mathcal{F}^m)^M$. We use the superscript M to indicate that a notion is being interpreted in the model M. Now we step outside the model M and we choose a subset of $(\mathbf{P}^{F,m})^M$ which will be denoted by G. Note that G will not be a set in the sense of M although all its elements are in M. G will be obtained by the method of forcing, see Chapter VII of [8]. A set X in $(\wp(\mathbf{P}^{F,m}))^M$ is called *dense* if for all $p \in (\mathbf{P}^{F,m})^M$ there exists $q \in X$ with $q \leq p$. We recall that a subset of $(\mathbf{P}^{F,m})^M$ is *generic* if it is closed upwards and directed downwards with respect to $\leq$, and meets every dense subset of $(\mathbf{P}^{F,m})^M$. Because M is countable it is easy to show that generic sets exist. Fix a generic set G. The method of forcing allows us to construct a new model $M[G]$ of ZFC of which M is a submodel and G is an element, see theorem 4.2 of Chapter VII of [8]. It is always the case that the ordinals in $M[G]$ are the same as the ordinals in M. However, in general some cardinals of M will fail to be cardinals in $M[G]$. This is because there may be functions in the model $M[G]$, not in M, which "collapse" some cardinals of M. In the present situation, since $\mathbf{P}^{F,m}$ is c.c.c. the cardinals in $M[G]$ are exactly the same as the cardinals in M, see theorem 5.10 of Chapter VII of [8]. From now on, except where explicitly stated, we will be talking about sets in the model $M[G]$. G is a set of finite ω-colorings, pairwise compatible in the partial order $\mathbf{P}^{F,m}$. (Note that $\mathbf{P}^{F,m}$ is the same whether we interpret it in M or in $M[G]$.) The set $\bigcup\{c : \exists u(\langle u, c\rangle \in G)\}$, denoted g, is a function whose domain is $[B]^2$ for some $B \subseteq \aleph_m$. Since G is generic it intersects each of the sets D_α, $\alpha < \aleph_m$. Thus, for each $\alpha < \aleph_m$ there exists $p = \langle u, c\rangle \in G$ and $\beta > \alpha$ such that $\beta \in u$. Thus $\text{fld}(g)$ is cofinal in $\aleph_m$ whence $|B| = \aleph_m$. Let $h : \aleph_m \longrightarrow B$ be the unique order preserving bijection and define $f_m : [\aleph_m]^2 \longrightarrow \omega$ by $f_m(\{\alpha, \beta\}) = g(\{h(\alpha), h(\beta)\})$. To show that an identity is not realized by f_m it is sufficient to show that it is not realized by any condition in $\mathbf{P}^{F,m}$. The next theorem follows from the above discussion.

Theorem 2.13 *Let $M \models ZFC$, $2 \leq m < \omega$, and $I \in ID$. There exists a cardinal preserving generic extension $M[G]$ of M such that $M[G] \models I \in \mathcal{I}(\aleph_m)$ implies $M \models I \in C^m$.*

2.1.2 Omitting I_m

In this subsection we will not distinguish between the identity I_m and the structure $\langle {}^{(m+1)}2, E\rangle$ where E is the equivalence relation on $[{}^{(m+1)}2]^2$ defined by $\{\eta,\nu\} \simeq \{\alpha,\beta\}$ if $\eta \cap \nu = \alpha \cap \beta$. We will show that f_m does not realize I_m by showing that no condition in $\mathbf{P}^{F,m}$ realizes I_m.

Definition 2.14 $\langle \eta_0, \ldots, \eta_m, \eta_{m+1}\rangle$ is a *special sequence* if

i) $\eta_i \in {}^{(m+1)}2$

ii) $|\eta_i \cap \eta_{i+1}| = i$ for all $i \leq m$.

Lemma 2.15 *Let $p \in \mathbf{R}$ and h be an embedding of I_m into p. There exists a special sequence $\langle \eta_0, \ldots, \eta_m, \eta_{m+1}\rangle$ such that $h(\eta_i)R^{F,m}\{h(\eta_0), \ldots, h(\eta_{m-1})\}$ for $i \in \{m, m+1\}$.*

Proof: We define $\eta_k \in {}^{(m+1)}2$ and $A_k \in \wp(\aleph_m)$, $0 \leq k \leq m-1$, by recursion on k such that for all $i < m-1$

i) $|\eta_i \cap \eta_{i+1}| = i$

ii) $A_{i+1} = \{x \in A_i : f_{A_i}(x) < f_{A_i}(h(\eta_{i+1}))\}$

iii) $A_i \supseteq \{h(\gamma) : \gamma \in {}^{(m+1)}2 \wedge |\gamma \cap \eta_i| = i\}$.

Let η_0 be the unique $\nu \in {}^{(m+1)}2$ such that $h(\nu) = \max(\mathrm{rng}(h))$ and $A_0 = \{x \in \aleph_m : x < h(\eta_0)\}$. Suppose that η_k and A_k have been suitably defined for all $k \leq j$ where $j < m-1$. Let C denote $\{\nu \in {}^{(m+1)}2 : |\nu \cap \eta_j| = j\}$. From iii) of the induction hypothesis, $h''C \subseteq A_j$. Let η_{j+1} be the unique $\nu \in C$ such that $f_{A_j}(h(\delta)) < f_{A_j}(h(\nu))$ for all $\delta \in C \setminus \{\nu\}$. Define A_{j+1} to be $\{x \in A_j : f_{A_j}(x) < f_{A_j}(h(\eta_{j+1}))\}$. Clearly $|\eta_{j+1} \cap \eta_j| = j$. Consider $\gamma \in {}^{(m+1)}2$ such that $|\gamma \cap \eta_{j+1}| = j+1$. Clearly $|\gamma \cap \eta_j| = j$ and thus by the induction hypothesis $h(\gamma) \in A_j$. Also $\gamma \in C$ and $\gamma \neq \eta_{j+1}$ so $f_{A_j}(h(\gamma)) < f_{A_j}(h(\eta_{j+1}))$ by the choice of η_{j+1}. Thus $h(\gamma) \in A_{j+1}$. This completes the induction step and the definition of $\eta_0, \ldots, \eta_{m-1}$ and $A_0, \ldots, A_{m-1}$. Letting η_m, η_{m+1} be the two elements of $\{\nu \in {}^{(m+1)}2 : |\nu \cap \eta_{m-1}| = m-1\}$ completes the construction. By induction, for all $0 \leq i < m$, $\{h(\eta_{i+1}), \ldots, h(\eta_{m+1})\} \subseteq A_i$ and $A_i = A_{\langle h(\eta_0), \ldots, h(\eta_i)\rangle}$, where $A_{\langle h(\eta_0), \ldots h(\eta_i)\rangle}$ is given by definition 2.5. From this it is clear that $h(\eta_i)R^{F,m}\{h(\eta_0), \ldots, h(\eta_{m-1})\}$ for $i \in \{m, m+1\}$. ◻

Lemma 2.16 *Let $\langle \eta_0, \ldots, \eta_{m+1} \rangle$ be a special sequence, $p, q \in \mathbf{R}$, p be a one-point extension of q, and h be an embedding of $J = I_m \restriction \{\eta_0, \ldots, \eta_{m+1}\}$ in p. Then h is an embedding of J in q.*

Proof: Let $p = \langle u, c \rangle, q = \langle v, d \rangle$ and $J = \langle B, F \rangle$. Towards a contradiction suppose that $u \setminus v = \{h(\eta_i)\}$. If $i < m$, then $\{\eta_m, \eta_i\} \simeq_F \{\eta_{m+1}, \eta_i\}$, but $c(\{h(\eta_m), h(\eta_i)\}) \neq c(\{h(\eta_{m+1}), h(\eta_i)\})$ since p is a one-point extension of q. This contradicts h being an embedding. If $i \in \{m, m+1\}$, consideration of the pairs $\{\eta_0, \eta_m\}, \{\eta_0, \eta_{m+1}\}$ leads to a similar contradiction. □

Lemma 2.17 $I_m \notin C^m$.

Proof: Let $I_m = \langle A, E \rangle$. Towards a contradiction let $q = \langle v, d \rangle \in \mathbf{P}^{F,m}$ be a condition into which I_m can be embedded. Let $\langle \eta_0, \ldots, \eta_{m+1} \rangle$ be a special sequence satisfying the conclusion of lemma 2.15. By restriction we have an embedding h of $I_m \restriction \{\eta_0, \ldots, \eta_{m+1}\}$ in q. Choose $p = \langle u, c \rangle \in \mathbf{P}^{F,m}$ such that $p \leq q$, $\{h(\eta_0), \ldots, h(\eta_{m+1})\} \subseteq u$ and $|u|$ is minimal. From lemma 2.16, p is not a one-point extension of $r \in \mathbf{P}^{F,m}$. Therefore there are $p^0 = \langle u^0, c^0 \rangle, p^1 = \langle u^1, c^1 \rangle \in \mathbf{P}^{F,m}$ such that p is the amalgam of p^0 and p^1. Since neither p^0 nor p^1 can replace p, there exist $i, j \leq m+1$ and $a, b \in u$ such that $h(\eta_i) = a \in u^0 \setminus u^1, h(\eta_j) = b \in u^1 \setminus u^0$.

From the definition of amalgamation, $\{a, b\}$ is the only pair in $[u]^2$ which is assigned the color $c(\{a, b\})$ by p. The only pair in $[\{\eta_0, \ldots, \eta_{m+1}\}]^2$ which is in an E-equivalence class of size one is $\{\eta_m, \eta_{m+1}\}$. Thus i, j are $m, m+1$ in some order. Also, for each $k < m$, $h(\eta_k) \in u^0 \cap u^1$. Otherwise one of the pairs $\{a, h(\eta_k)\}$, $\{b, h(\eta_k)\}$ would also be assigned a unique color by c, contradiction. From lemma 2.15 we conclude $bR^{F,m}u^0$ since $\{h(\eta_0), \ldots, h(\eta_{m-1})\} \subseteq u^0$. This contradicts the definition of amalgamation and completes the proof of the lemma. □

From lemma 2.17 it is clear that in $M[G]$, $I_m \notin \mathcal{I}(f_m)$. Hence we have the following theorem.

Theorem 2.18 *In $M[G]$, $I_m \notin \mathcal{I}(\aleph_m)$.*

2.2 Realization of I_m

We will show that $I_m \in \mathcal{I}(\aleph_{m+1})$ for all m, $2 \leq m < \omega$. This will be done by showing that if a given collection of identities is contained in $\mathcal{I}(\aleph_m)$, we can extend the collection in a nontrivial way to a new collection contained in $\mathcal{I}(\aleph_{m+1})$.

Definition 2.19 Let $\langle J_i : 1 \leq i \leq n \rangle$ be a finite sequence of identities. We define the *end-homogeneous amalgam* of the sequence as follows. Choose a sequence of ω-colorings $c_i : [G_i]^2 \longrightarrow \omega$ such that $c_i \in J_i$, $G_i \cap G_j = \emptyset$ for $1 \leq i < j \leq n$, and $\text{rng}(c_i) \cap \text{rng}(c_j) = \emptyset$ for all $1 \leq i < j \leq n$. Let $G = \bigcup\{G_i : 1 \leq i \leq n\}$. Now choose a new ω-coloring $c : [G]^2 \longrightarrow \omega$ such that for all $\{r, s\}$, $\{t, v\} \in [G]^2$ and all i, $1 \leq i \leq n$,

i) $c \supset c_i$

ii) $c(\{r, s\}) \in \text{rng}(c_i)$ if and only if $\{r, s\} \in \text{dom}(c_i)$

iii) if $\{r, s\}$, $\{t, v\}$ are not in $\bigcup\{\text{dom}(c_j) : 1 \leq j \leq n\}$, then

$$c(\{r, s\}) = c(\{t, v\}) \Leftrightarrow \min\{j : r \in G_j \vee s \in G_j\} = \min\{j : t \in G_j \vee v \in G_j\}.$$

The end-homogeneous amalgam of $\langle J_i : 1 \leq i \leq n \rangle$ is the identity realized by c.

Definition 2.20 Let $\mathcal{I}$ be a collection of identities. Define $\text{eh}(\mathcal{I})$ to be the collection of identities produced by forming all end-homogeneous amalgams of finite sequences of identities in $\mathcal{I}$.

Theorem 2.21 *Let $1 \leq m < \omega$. If $\mathcal{I} \subseteq \mathcal{I}(\aleph_m)$ then $eh(\mathcal{I}) \subseteq \mathcal{I}(\aleph_{m+1})$.*

Proof: Let $f : [\aleph_{m+1}]^2 \longrightarrow \omega$ and $\langle J_i : 1 \leq i \leq n \rangle$ be a sequence of identities in $\mathcal{I}$. We will produce by recursion a sequence $\langle \langle A_k, B_k \rangle : 0 \leq k \leq n \rangle$ such that:

i) f induces J_i on the set A_i for $1 \leq i \leq n$

ii) $B_i \supset B_{i+1}$ for $0 \leq i < n$

iii) $|B_i| = \aleph_{m+1}$ for $0 \leq i \leq n$

iv) $A_{i+1} \subset B_i \setminus B_{i+1}$

v) $f(\{a_1, b_1\}) = f(\{a_2, b_2\})$ whenever there exist i, j $(1 \le i \le j \le n)$ such that $\{a_1, a_2\} \subset A_i$ and $\{b_1, b_2\} \subset B_j$.

Define B_0 to be $\aleph_{m+1}$ and A_0 to be empty. By induction suppose that $\langle A_i, B_i \rangle$ have been defined for $i \le k < n$. Let C_k be the first $\aleph_m$ elements of B_k. For each $b \in B_k \setminus C_k$ there exists a subset D_k of C_k and $c_{b,k} < \omega$ such that $|D_k| = \aleph_m$ and $f(\{b, x\}) = c_{b,k}$ for all $x \in D_k$. Now choose a finite set $A_b \subset D_k$ such that f induces I_{k+1} on A_b. There are only $\aleph_m$ finite subsets of C_k and a countable collection of possible values for $c_{b,k}$. Thus we can choose $B_{k+1} \subset B_k \setminus C_k$ of cardinality $\aleph_{m+1}$ and $c_k < \omega$ such that $A_{b_1} = A_{b_2}$ for all $\{b_1, b_2\} \subset B_{k+1}$ and $c_{b,k} = c_k$ for all $b \in B_{k+1}$. We let $A_{k+1} = A_b$ for $b \in B_{k+1}$. It is easy to see that f induces the desired identity on the set $\bigcup\{A_i : 1 \le i \le n\}$. □

Theorem 2.22 *For all m such that $1 \le m < \omega$, $I_m \in \mathcal{I}(\aleph_{m+1})$.*

Proof: We first claim that for all $m < \omega$, I_{m+1} is the end-homogeneous amalgam of $\langle I_m, I_m \rangle$. Let I_{m+1} be denoted by $\langle {}^{(m+2)}2, E \rangle$, and for $i = 0, 1$ denote by S_i the collection $\{\eta \in {}^{(m+2)}2 : \eta(0) = i\}$. For $\eta \in {}^{(m+2)}2$ let $\hat{\eta} \in {}^{(m+1)}2$ be defined by $\hat{\eta}(x) = \eta(x+1)$. Now $\eta \mapsto \hat{\eta} (\eta \in S_i)$ is an isomorphism of $I_{m+1} \restriction S_i$ onto I_m. Note that in the end-homogeneous amalgam of $\langle I_m, I_m \rangle$ all edges between the two copies of I_m receive the same color. The claim then follows because $\{\eta, \nu\} \simeq_E \{\alpha, \beta\}$ whenever $\{\eta, \alpha\} \subseteq S_0$ and $\{\nu, \beta\} \subseteq S_1$.

The proof of the theorem is by induction on $m < \omega$. By the claim I_1 is the end-homogeneous amalgam of $\langle I_0, I_0 \rangle$. Since I_0 is the trivial identity of size two and thus an element of $\mathcal{I}(\aleph_1)$, we apply theorem 2.21 and conclude $I_1 \in \mathcal{I}(\aleph_2)$. Assume the result holds for $m \le j$. By the claim I_{j+1} is the end-homogeneous amalgam of $\langle I_j, I_j \rangle$. By the induction hypothesis $I_j \in \mathcal{I}(\aleph_{j+1})$, whence, by theorem 2.21, $I_{j+1} \in \mathcal{I}(\aleph_{j+2})$. □

We note that this theorem provides a new proof of the fact that $\mathcal{I}(\aleph_\omega) \supseteq \text{IDT}$. This is a corollary of a more general result given in [16]. From theorem 2.18 and theorem 2.22 we obtain:

Theorem 2.23 *For each $m \geq 1$, Con(ZFC) implies that there is a model of ZFC in which $\mathcal{I}(\aleph_m) \subsetneq \mathcal{I}(\aleph_{m+1})$.*

2.3 Iterated Forcing

Let M be a model of ZFC. We follow the notation of [8] and describe an iterated forcing of length ω in the model M. We first define P_n and π_n by induction on $n < \omega$. Fix $F \in (\mathcal{F}^2)^M$. Let P_0 be $\{0\}$ and π_0 be an P_0-name for the partial order $\mathsf{P}^{F,2}$. Now assume that P_n and π_n have been defined for all $n \leq m$, such that P_n is a partial order and π_n is a P_n-name for a partial order. P_{m+1} is defined to be

$$\{p : p = \langle \rho_n : n \leq m \rangle,\ p \restriction m \in \mathsf{P}_m,\ \rho_m \in \operatorname{dom}(\pi_m) \text{ and } p \restriction m \Vdash \rho_m \in \pi_m\}.$$

The ordering given to P_m is defined as follows. For $p = \langle \rho_n : n \leq m \rangle$ and $p' = \langle \rho'_n : n \leq m \rangle \in \mathsf{P}_{m+1}$, $p \leq p'$ if and only if $p \restriction m \leq p' \restriction m$ and $p \restriction m \Vdash \rho_m \leq \rho'_m$.

Let $\phi(F, m+1)$ state that $F \in \mathcal{F}^{m+1}$. We now claim that

$$1_{\mathsf{P}_m} \Vdash \exists F \exists \sigma (\phi(F, m+1) \wedge \sigma = \mathsf{P}^{F,m+1}).$$

The proof is as follows. Take any P_m-generic G. Then in the model $M[G]$ choose $F \in (\mathcal{F}^{m+1})^{M[G]}$ and define the partial order $\mathsf{P}^{F,m+1}$. Since $\mathsf{P}^{F,m+1} \in M[G]$ it must have a P_m-name. Thus in all generic extensions of M the statement has been shown be true, whence it is forced by 1_{P_m}. Thus the claim has been shown to be true.

Applying the *maximal principle*, see Theorem 8.2 Chapter VII of [8] we conclude that there exist P_m-names $\tilde{F}$ and $\tilde{\sigma}$ such that $1_{\mathsf{P}_m} \Vdash \phi(\tilde{F}, m+2) \wedge \tilde{\sigma} = \mathsf{P}^{\tilde{F},m+2}$. Set π_{m+1} equal to such a name, $\tilde{\sigma}$.

The partial order P_ω is defined to be

$$\{p : p = \langle \rho_n : n < \omega \rangle, (\forall n < \omega)(p \restriction n \in \mathsf{P}_n) \wedge |\operatorname{supt}(p)| < \omega\}.$$

(Here supt(p) denotes the set of $n < \omega$ such that $\rho_n \neq 1_{\pi_n}$.) For $p = \langle \rho_n : n < \omega \rangle$ and $p' = \langle \rho'_n : n < \omega \rangle, p \leq p'$ if and only if for all $n < \omega$, $p \restriction n \leq p' \restriction n$.

Theorem 2.24 *Let M be a model of ZFC, and let G be P_ω-generic over M. Then*

$$M[G] \models (\forall m, 2 \leq m < \omega)(\mathcal{I}(\aleph_\omega) \supsetneq \mathcal{I}(\aleph_{n+1}) \supsetneq \mathcal{I}(\aleph_n)).$$

Proof:

Fix n, $2 \leq n < \omega$. We follow the notation of definition 5.10 of [8] and define $i_{n,\eta} : \mathbf{P}_n \longrightarrow \mathbf{P}_\eta$ for $n < \eta \leq \omega$ by defining $i_{n,\eta}$ to be the $p' \in \mathbf{P}_\eta$ such that $p' \restriction n = p$ and $p'(m) = 1_{\pi_m}$ for $n \leq m < \eta$. We define G_n to be $i^{-1}_{n,\omega}(G)$. Then by lemma 5.13 of [8], G_n is $\mathbf{P}_n$-generic over M. In the model $M[G_n]$ we define $\mathbf{Q}_n$ to be $\mathrm{val}(\pi_n, G_n)$ and H_n to be

$$\{\mathrm{val}(\rho, G_n) : \rho \in \mathrm{dom}(\pi_n) \wedge \exists p(p\hat{\ }\langle\rho\rangle \in G_{n+1})\}.$$

By lemma 5.13 of [8], H_n is $\mathbf{Q}_n$ generic over $M[G_n]$. Now $\mathbf{Q}_n$ is the partial order that the model $M[G_n]$ considers to be $\mathbf{P}^{F,n+2}$, (for some $F \in \mathcal{F}^{n+2}$ as defined in $M[G_n]$). Using the methods of §2.1 there exists a function $g_n : [\aleph_{n+2}]^2 \longrightarrow \omega$ such that $I_{n+2} \notin \mathcal{I}(g_n)$. Again by lemma 5.13 $M[G_n] \subseteq M[G]$. This implies that g_n is in the model $M[G]$. As n was arbitrarily chosen it is clear that

$$M[G] \models (\forall n,\, 2 \leq n < \omega)(\exists h_n : [\aleph_n]^2 \longrightarrow \omega)(I_n \notin \mathcal{I}(h_n)).$$

This coupled with the fact that ZFC $\vdash (\forall n,\, 1 \leq n < \omega)(I_n \in \mathcal{I}(\aleph_{n+1}))$, see §2.2, theorem 2.22, gives the desired result. □

2.4 C^m is Recursively Enumerable

In this section we show that, for $2 \leq m < \omega$, C^m is a recursively enumerable collection in the sense that there exists an algorithm which lists all the elements of C^m.

Definition 2.25 Let $p \in \mathbf{R}$. We say that $H \subset \mathbf{R}$ is a *history* of p if H is a collection minimal with respect to the following conditions:

i) $p \in H$

ii) If $q = \langle u^q, c^q\rangle \in H$ and $|u^q| \neq 1$ then there exist $q^0, q^1 \in H$ such that q is the amalgam of q^0 and q^1 or there exists $q^0 \in H$ such that q is a one-point extension of q^0.

Let $F \in \mathcal{F}^m$. If in addition we require that in ii), all of the amalgamations are allowable with respect to F, we say that H is *allowable with respect to F*.

Given $p \in \mathbf{R}$ and a history H of p there is a natural indexing of the elements of H by elements of $\{0,1\}^{<\omega}$. If p^0 and $p^1 \in H$ and they are amalgamated to form p we let $p_{\langle 0\rangle}$ denote p^0 and $p_{\langle 1\rangle}$ denote p^1. If $q = p_\sigma \in H$ and $q^0 = \langle u^0, c^0\rangle$ and $q^1 = \langle u^1, c^1\rangle \in H$ are amalgamated to form q we designate q^0 as $p_{\sigma^\frown\langle 0\rangle}$ and q^1 as $p_{\sigma^\frown\langle 1\rangle}$. In the case that $p_\sigma = q \in H$ is a one-point extension of q' we let q' denote $p_{\sigma^\frown\langle 0\rangle}$ ($p_{\sigma^\frown\langle 1\rangle}$ is left undefined).

Suppose that $p = \langle u, c\rangle \in \mathbf{P}^{F,m}$ for some particular $F \in \mathcal{F}^m$. We show that there exists $G \in \mathcal{F}^m$ and $p^* = \langle u^*, c^*\rangle \in \mathbf{P}^{G,m}$ such that p^* realizes the same identities as p and $u^* \subset \omega$. For $\bar{\alpha} \in {}^{\leq m}u$ let $A^F_{\bar{\alpha}}$ denote the set obtained from definition 2.5. Recall that for any $a \in u$ and $B \subseteq u$, $aR^{F,m}B$ holds if and only if $a \in A^F_{\bar{\alpha}}$ for some $\bar{\alpha} \in {}^mB$. Let $v \subset \aleph_m$ be a finite set such that $u \subseteq v$ and

$$\forall\bar{\alpha}\forall\bar{\beta}[(\bar{\alpha},\bar{\beta} \in {}^{\leq m}u \wedge A^F_{\bar{\alpha}} \subsetneq A^F_{\bar{\beta}}) \Rightarrow (\exists x \in v)(x \in A^F_{\bar{\beta}} \setminus A^F_{\bar{\alpha}})].$$

Let $x \mapsto x^*$ $(x \in v)$ be the unique order-preserving map of v onto an initial segment of ω. For $X \subseteq \aleph_m$ let X^* denote $\{x^* : x \in X \cap v\}$. For $\bar{\alpha} \in {}^{\leq m}u$ we define $g_{B_{\bar{\alpha}}}$, where $B_{\bar{\alpha}}$ denotes $(A^F_{\bar{\alpha}})^*$, to be the unique map from $B_{\bar{\alpha}}$ into itself such that

$$f_{A_{\bar{\alpha}}}(x) < f_{A_{\bar{\alpha}}}(y) \Leftrightarrow g_{B_{\bar{\alpha}}}(x^*) < g_{B_{\bar{\alpha}}}(y^*)$$

for all $x, y \in v \cap A_{\bar{\alpha}}$. By the choice of v it is clear that for all $\bar{\alpha}$, $\bar{\beta} \in {}^{\leq m}u$, $A^F_{\bar{\alpha}} \neq A^F_{\bar{\beta}}$ implies $B_{\bar{\alpha}} \neq B_{\bar{\beta}}$. Thus we may choose $G \in \mathcal{F}^m$ to be any element which assigns the mapping $g_{B_{\bar{\alpha}}}$ to the set $B_{\bar{\alpha}}$. By induction on the length of $\bar{\alpha}$ we have $A^G_{(\bar{\alpha})^*} = (A^F_{\bar{\alpha}})^*$. This implies $xR^{F,m}B$ if and only if $x^*F^{G,m}B^*$ for all $x \in u$ and $B \subseteq u$.

Let $H = \langle p_\sigma : \sigma \in {}^{\leq n}2\rangle$ be the history of p. We define a condition p^* and its history $H^* = \langle p^*_\sigma : \sigma \in {}^{\leq n}2\rangle$ by letting $p^*_\sigma = \langle u^*_\sigma, c^*\rangle$, where $c^* : u^*_\sigma \longrightarrow \omega$ is defined by $c^*(\{x^*, y^*\}) = c(\{x,y\})$ for all $\{x,y\} \in [u]^2$. It is clear that H^* is a history of p^*, allowable with respect to G. That is to say $p^* \in \mathbf{P}^{G,m}$.

Theorem 2.26 *For each $2 \leq m < \omega$ there is an algorithm which lists the elements of C^m.*

Proof: Let I be an identity in C^m. It is easy to see that there exists an algorithm that generates all pairs $\langle H, G_0\rangle$ where $H = \langle p_\sigma : \sigma \in {}^{\leq n}2\rangle$ is a history of an element of $\mathbf{R}$ whose universe is a subset of ω and G_0 is a collection of one-to-one functions $\{g_A : A \subseteq$

$\max(u_\emptyset)$ and $g_A : A \longrightarrow A\}$. We run the algorithm and and upon the output of each pair $\langle H, G_0\rangle$ we determine whether or not H is allowed by those $G \in \mathcal{F}^m$ which contain G_0. If H is allowed we list all subidentities of $p_\emptyset$. By our previous results if there exists $F \in \mathcal{F}$ and $p \in \mathbf{P}^{F,m}$ such that p realizes I there exists $G \in \mathcal{F}^m$ and $p^* = \langle u^*, c^*\rangle \in \mathbf{P}^{G,m}$ such that p^* realizes I and $u^* \subset \omega$. Thus our procedure lists all identities in C^m. That it lists only elements of C^m is obvious. □

We conjecture that C^m is in fact recursive in the sense that there exists an algorithm which determines whether or not a given identity is an element of C^m. The argument above shows that, given m and the isomorphism type of an element of $\mathbf{R}$, one can effectively check whether for some $F \in \mathcal{F}^m$, there exists $p \in \mathbf{P}^{F,m}$ of the given isomorphism type. The difficulty in computing the membership of C^m lies in our inability to compute from the size k of an identity I a number k' such that

$$I \in C^m \Rightarrow (\exists p = \langle u, c\rangle)(I \text{ embeds in } p \text{ and } |u| < k').$$

The details of this problem are quite sensitive to the particular way one sets up the parameters F. For example, instead of the mapping f_A (see the beginning of 2.1) one could substitute a well-ordering $<^A$ of A of order-type $|A|$. One could also restrict A to run through only those sets which can arise in the definition of the sets $A_{\bar{s}}$ (see definition 2.5).

2.5 Finite Bases

Being unable to characterize C^m we try to gather as much information as possible about this set of identities. Here we show that $C^m \subseteq \mathrm{IDT}$ and that for m, $2 \leq m < \omega$ there does not exist a finite set of identities $\mathcal{J}_m$ such that $I \notin C^m$ if and only if J has some identity in $\mathcal{J}_m$ as a subidentity. We require some general lemmas about duplication. The first two follow easily from the definitions, so we offer no proof.

Definition 2.27 Let $\mathcal{E}$ and $\mathcal{D}$ be collections of identities. We say that $\mathcal{E}$ is an *basis* for $\mathcal{D} \setminus C^m$ if for all identities $I \in \mathcal{D}$, $I \in \mathcal{D} \setminus C^m$ if and only if there exists $J \in \mathcal{E}$ which embeds in I.

Lemma 2.28 *Let $I = \langle A, E\rangle$ be a subidentity of $J = \langle A, F\rangle$ and $a \in A$. Let I' and J' be obtained from I and J respectively by duplicating a. Then I' is a subidentity of J'.*

Lemma 2.29 *Let $n < \omega$ and $K_i = \langle A_i, E_i\rangle$ $(1 \leq i \leq 4)$ be identities. Let $a_1, \ldots, a_{n+1} \in A_1$ be distinct, K_2 be obtained from K_1 by duplicating $\langle a_1, \ldots, a_n\rangle$, K_3 from K_2 by duplicating a_{n+1}, and K_4 from K_1 by duplicating $\langle a_1, \ldots, a_{n+1}\rangle$. Then K_4 is a subidentity of K_3.*

Lemma 2.30 *Let $n < \omega$ and $I = \langle B, F\rangle$ be obtained from $J_0 = \langle A_0, E_0\rangle$, by duplicating $\bar{a} = \langle a_1, \ldots, a_n\rangle$ to $\bar{b} = \langle b_1, \ldots, b_n\rangle$. For $1 \leq i < n$ let J_{i+1} be obtained from J_i by duplicating a_i to b_i. Then I is a subidentity of J_n.*

Proof: The proof is by induction on n. When $n = 1$ the result is trivial. Thus suppose that the result holds for $n \leq j$. Let I, $J_0, J_1, \ldots, J_{j+1}, \bar{a} = \langle a_1, \ldots a_{j+1}\rangle$ and $\bar{b} = \langle b_1, \ldots b_{j+1}\rangle$ be as in the hypothesis of the lemma. Let K_1 be obtained from J_0 by duplicating $\langle a_1, \ldots, a_j\rangle$ and K_2 be obtained from K_1 by duplicating a_{j+1}. By induction K_1 is a subidentity of J_j and thus by lemma 2.28, K_2 is a subidentity of J_{j+1}. By lemma 2.29 I is a subidentity of K_2. Thus $I \hookrightarrow K_2 \hookrightarrow J_{j+1}$. □

Lemma 2.31 *IDT is closed under duplication in the sense that, if $I \in IDT$, and J is obtained from I by duplicating $\bar{a}$ to $\bar{b}$, then $J \in IDT$ also.*

Proof: From the previous lemma we may suppose that J is obtained from I by duplicating a to b. Let $I = \langle A, E\rangle$, $J = \langle B, F\rangle$, $t \in T$, and $h : A \longrightarrow B(t)$ witness that $I \hookrightarrow I_t = \langle B(t), E_t\rangle$. Define t' to be $t \cup \{h(a)^\frown\langle 0\rangle, h(a)^\frown\langle 1\rangle\}$. Then $k : B \longrightarrow B(t')$ by $k(x) = h(x)$ for $x \in A \setminus \{a\}$, $k(a) = h(a)^\frown\langle 0\rangle$, $k(b) = h(a)^\frown\langle 1\rangle$, is an embedding of J into $I_{t'}$. □

Lemma 2.32 *For all $m > 1$, $C^m \subseteq IDT$.*

Proof: It is clearly sufficient to prove that, for all $p \in \mathbf{P}^{F,m}$, if I_p is the identity represented by $p \in \mathbf{P}^{F,m}$, then $I_p \in$ IDT. Towards a contradiction suppose that $p = \langle u, c\rangle \in \mathbf{P}^{F,m}$ is such that $I_p \notin$ IDT with $|u|$ least possible. Clearly $|u| \neq 1$. There are now two cases:

Case 1. p is obtained by amalgamating $p_0, p_1 \in \mathbf{P}^{F,m}$. Then I_p is obtained from I_{p_0} by duplication. But $I_{p_0} \in$ IDT implies $I_p \in$ IDT by lemma 2.31, which contradicts the choice of p.

Case 2. p is a one-point extension of $q \in \mathbf{P}^{F,m}$. By choice of p, $I_q \in$ IDT. Let J be obtained from I_q by one-point duplication. Then $J \in$ IDT by lemma 2.31. But $I_p \hookrightarrow J$ by inspection, so $I_p \in$ IDT, contradiction. □

We now define some identities which will be used to show that there does not exist a finite basis for ID $\setminus C^m$.

Definition 2.33 Let $P_n = \langle A_n, E_n \rangle$ be the identity of size n where $A_n = \{1, 2, \ldots, n\}$ and the only equivalence class of E_n that is not a singleton, is the collection $\{\{i, i+1\} : 1 \leq i < n\}$. Let $Q_n = \langle B_n, F_n \rangle$ be the identity of size n where $B_n = \{1, 2, \ldots, n\}$ and the only equivalence class of F_n that is not a singleton, is the collection $\{\{i, i+1\} : 1 \leq i < n\} \cup \{\{1, n\}\}$.

Lemma 2.34 *Let n be odd, $n \geq 3$ and $m > 1$. Then $Q_n \notin C^m$.*

Proof: Fix m and n satisfying the hypothesis. We will show that $Q_n \notin$ IDT and then apply lemma 2.32. Let t be a binary tree, $B(t) = \{\eta_1, \ldots, \eta_n\} \subset {}^{<\omega}2$, and suppose that the identity I_t has Q_n as a subidentity. Without loss of generality $i \mapsto \eta_i$ $(1 \leq i \leq n)$ induces an embedding of Q_n into I_t. Then there exists τ such that $\eta_i \cap \eta_{i+1} = \tau$ for $i = 1, 2, \ldots, n-1$ and $\eta_1 \cap \eta_n = \tau$. This is impossible as n is odd. □

Lemma 2.35 *For all $n > 1$ and $m \geq 2$, $P_n \in C^m$.*

Proof: Fix n and let J_1 be P_n restricted to the even elements of the set $\{1, 2, \ldots, n\}$. Similarly let J_2 be the restriction to the odd elements. It is clear that all the equivalence classes of edges in the structures representing both J_1 and J_2, all have size one. Thus each is an element of $\mathcal{I}(\aleph_1)$. Now observe that P_n is a subidentity of the end-homogeneous amalgam of J_1 and J_2. Thus we may apply 2.21, to see that $P_n \in \mathcal{I}(\aleph_2) \subseteq \mathcal{I}(\aleph_m)$ for all $m \geq 2$. If there were a model M of ZFC in which $P_n \notin C^m$, then from theorem 2.13 there would be a model $M[G]$ of ZFC such that $P_n \notin \mathcal{I}(\aleph_m)$, contradiction. □

Theorem 2.36 *For each m, $2 \leq m < \omega$ there does not exist a finite basis, $\mathcal{J}_m$, for $ID \setminus C^m$.*

Proof: The result follows from lemmas 2.34 and 2.35. Fix m, $2 \leq m < \omega$. Towards a contradiction let $\mathcal{J}_m$ be a basis for ID $\setminus C^m$. For each odd n, $3 \leq n < \omega$, $Q_n \notin C^m$ and thus there exists $J_n \in \mathcal{J}_m$ such that $J_n \hookrightarrow Q_n$. As Q_n is a cycle of length n any subidentity must be a monochromatic path of length n, a collection of disjoint monochromatic paths of length less than n, or Q_n itself. The first two options are not possible as $P_k \in C^m$ for all $k \geq 1$. Thus Q_n must be an element of $\mathcal{J}_m$ for all odd n, $3 \leq n < \omega$. □

The next theorem does not concern itself with bases but the proof involves some of the lemmas we have just proved.

Theorem 2.37 *IDT = IDM.*

Proof: " $\supseteq$ ". This containment follows from lemma 2.31. " $\subseteq$ ". Recall that I_m is the identity realized by the binary tree of height $m+1$. By induction on $m < \omega$ we show that $I_m \in$ IDM. This suffices as IDT $= \mathcal{S}(\{I_m : m < \omega\})$. When $m = 0$ the result is trivial. Suppose the result holds for $m \leq j$. Consider $I_j = \langle A_j, E_j \rangle$. By definition $A_j = {}^{(j+1)}2$ and by the induction hypothesis there exists $J = \langle B, F \rangle \in$ IDM and an embedding $f : {}^{(j+1)}2 \longrightarrow B$. Let $B = \{b_0, \ldots, b_k\}$ for some $k < \omega$ and $\{c_0, \ldots, c_k\}$ distinct elements not in B. We define a sequence $\langle K_0, \ldots, K_k \rangle$ of identities in IDM, ($K_i = \langle C_i, G_i \rangle$). Set K_0 equal to J and for $0 \leq i < k$, let K_{i+1} be obtained from K_i by duplicating b_i to c_i. Now define $g : {}^{(j+2)}2 \longrightarrow C_k$ by $g(\eta) = b_i$ if $\eta = \nu^\frown\langle 0 \rangle$ and $f(\nu) = b_i$ and $g(\eta) = c_i$ if $\eta = \nu^\frown\langle 1 \rangle$ and $f(\nu) = b_i$. It is clear that g is an embedding of I_{j+1} into K_k. □

2.6 Bases Relative to IDT

Since the last section shows that it is impossible to characterize ID $\setminus C^m$ by a finite basis the question naturally arises whether IDT $\setminus C^m$ has a finite basis. The question remains open.

What we show here is that $\{I_2\}$ is not a basis for IDT $\setminus C^2$. In fact we will show that there exists $J \in$ IDT $\setminus C^2$ which does not embed I_2.

We begin with a proposition that will be used to prove that $J \notin C^2$. Let $K = \langle A, E, <_A \rangle$ denote the CV-identity for which $A = \{0,1,2\}, 0 <_A 1 <_A 2$, and the only E-equivalence class which is not a singleton is $\{\{0,1\},\{1,2\}\}$.

Proposition 2.38 *Let* $2 \le m < \omega, F \in \mathcal{F}^2$, *and* $p \in \mathbf{P}^{F,2}$. *Then* p *does not realize* K.

Proof: Towards a contradiction let $p = \langle u, c \rangle$ realizing K be chosen so that $|u|$ is minimal. Let $H = \langle p_\sigma : \sigma \in {}^{\le n}2 \rangle$ be the history of p and $f : A \longrightarrow u$ witness the realization. By the minimality of $|u|$ we can assume that neither $p_{\langle 0 \rangle}$ nor $p_{\langle 1 \rangle}$ realize K and that there exist distinct x, y such that $x \in \text{rng}(f) \cap u^{(0)} \setminus u^{(1)}$ and $y \in \text{rng}(f) \cap u^{(1)} \setminus u^{(0)}$. Also note that there exists $z \in \text{rng}(f) \cap u^{(0)} \cap u^{(1)}$, otherwise all edges in $[\text{rng}(f)]^2$ would receive distinct colors which contradicts f being an embedding. By the definition of amalgamation $c(\{x,y\}) \notin \{c(\{x,z\}), c(\{y,z\})\}$. Therefore $f(1) = z$ and so either $x < z < y$ or $y < z < x$. Without loss of generality suppose $x < z < y$. Let y' be the element of $u^{(0)}$ corresponding to y. Then the mapping $\{(0,x),(1,y),(2,z)\}$ witnesses that $p_{\langle 0 \rangle}$ realizes K, contradiction. □

Definition 2.39 Let $J = \langle A, E \rangle$ be the identity characterized by setting $A = \{1,2,3,\ldots,8\}$ and letting the E-equivalence classes that are not singletons be:

i) $E_1 = \{\{1,8\},\{8,2\},\{2,5\},\{5,1\},\{1,7\},\{5,3\},$
$\{3,7\},\{3,6\},\{4,6\},\{4,7\},\{2,6\},\{4,8\},\{5,4\},\{2,7\}\}$

ii) $E_2 = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$

iii) $E_3 = \{\{5,6\},\{6,7\},\{7,8\},\{8,5\}\}$.

The identity J can be visualized as a coloring of the edges of the complete graph on the vertex set $\{1,2,\ldots,8\}$. The coloring has the following properties. The edges on $\{1,2,3,4\}$ are colored so that there is a monochromatic cycle from 1 to 2 to 3 to 4 and then back to 1. The remaining two edges get two colors appearing nowhere else. This is the equivalence

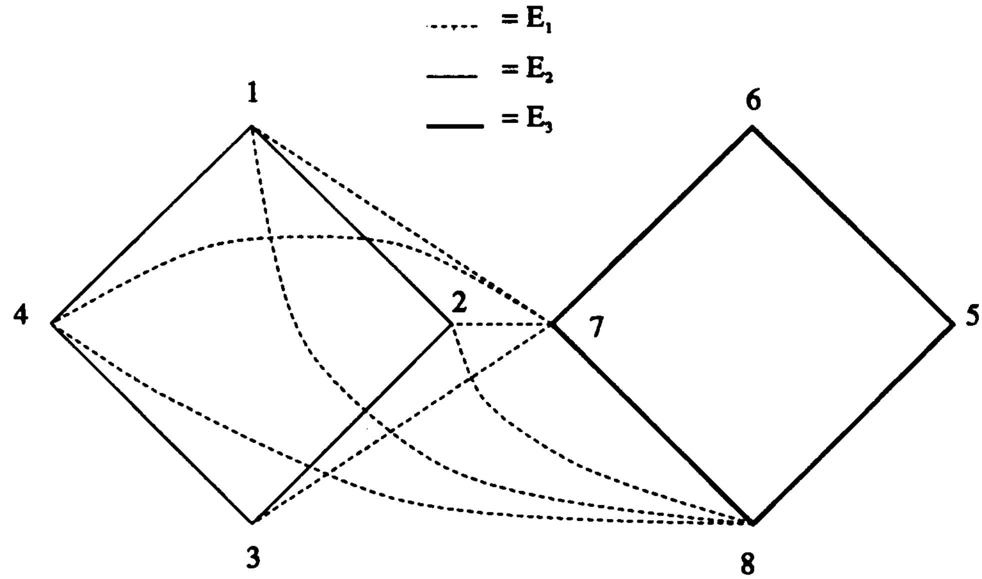

Figure 2.1: The E-equivalence classes E_2, E_3, and a portion of E_1.

class E_2. The edges on $\{5,6,7,8\}$ are colored in a similar fashion. More precisely there is a monochromatic cycle from 5 to 6 to 7 to 8 and back to 5. This color is distinct from all other colors, as are the two colors appearing on the two remaining edges. This is equivalence class E_3. All other edges, except $\{1,6\}$ and $\{8,3\}$, are given a color distinct from all other colors. This is equivalence class E_3. (In order to refer to this color we will name it blue.) The two remaining edges between the cycles, $\{1,6\}$ and $\{8,3\}$, are also given two distinct colors appearing nowhere else. In figure 2.1 we have indicated only the pairs in E_1 which contain either 7 or 8; 5 is E_1 related to all of $1,2,3,4$, amd 6 to $2,3,4$.

The following symmetries should be noted. Each point lies on one of two monochromatic cycles of length 4. Each monochromatic cycle contains two points such that each connects to every other point in the other monochromatic cycle with an edge that gets colored blue. Each monochromatic cycle contains two points such that each connects to exactly three points in the other monochromatic cycle with an edge that gets colored blue. It is easy to see that J is an element of IDT: by coloring the edges $\{1,6\}$ and $\{3,8\}$ blue we obtain the identity I_2.

Theorem 2.40 $J \in IDT \setminus \mathcal{C}^2$.

Proof: Towards a contradiction let $F \in \mathcal{F}^2$, and choose $p = \langle u, c\rangle \in \mathbf{P}^{F,2}$ and $f : A \longrightarrow u$ witnessing that p realizes J. We simplify the notation and use $<_\alpha$ instead of $<^F_\alpha$. Let $H = \langle p_\sigma : \sigma \in 2^{\leq n}\rangle$ be a history of p. Choose $x \in A$ such that $f(x)$ is maximum in $f''A$. Because of the symmetries of the identity, by relabeling the points we may assume that x is one of 7 or 8. We show that each of these possibilities implies the existence of an amalgamation in H that is not allowed by F.

Case 1. $f(1), f(2), f(3), f(4) < f(7)$.

Fix $y \in \{1,2,3,4\}$ such that $f(y)$ is the $<_{f(7)}$-greatest element of $\{f(1), f(2), f(3), f(4)\}$. Observe that the points $1,2,3,4$ all lie on the same monochromatic cycle and that for all $y \in \{1,2,3,4\}$, $\{y, 7\}$ is in the equivalence class E_1. Using the symmetries of J again, we may assume that $y = 4$ or $y = 3$. Since the proof is similar in both cases we examine only the first. We conclude that $f(1)R^{F,2}\{f(4), f(7)\}$ and $f(3)R^{F,2}\{f(4), f(7)\}$. Let $J_1 = \langle A_1, F_1\rangle$ denote $J \restriction \{1,3,4,7\}$. Observe that the only F_1-equivalence class which is a singleton is $\{1,3\}$. Choose $\sigma \in {}^{\leq n}2$ of maximal length so that $\{f(1), f(3), f(4), f(7)\} \subset u^\sigma$. By the maximality of $|\sigma|$ and the properties of amalgamation, $f(4), f(7) \in u^{\sigma^\frown\langle 0\rangle} \cap u^{\langle 1\rangle}$ and either $f(1)$ or $f(3) \in u^{\sigma^\frown\langle 1\rangle} \setminus u^{\sigma^\frown\langle 0\rangle}$. This contradicts the definition of p_σ being allowed by F. This concludes the first case.

Case 2. Otherwise.

Clearly $x = 8$. First note that $\{y, 7\} \in E_3$ for all $y \in \{1,2,3,4\}$. Thus, by the previous proposition there do not exist $i, j \in \{1,2,3,4\}$ such that $f(i) < f(7) < f(j)$. Therefore $f(7) < f(i)$ for all $i \in \{1,2,3,4\}$. This is impossible as p then realizes K on the set $\{f(7), f(1), f(8)\}$. This concludes the second case. □

Corollary 2.41 *$\{I_2\}$ is not a basis for $IDT \setminus \mathcal{C}^2$.*

2.7 Identities Realized by Binary Trees

In this section we will show that $\{I_m\}$ is a basis for $\mathrm{ID}_T \setminus \mathcal{C}^m$, see corollary 2.45. One should contrast this result with the corollary 2.41 of the previous section. We will also show that there does not exist a collection $T_2 \subseteq T$ such that $\mathcal{C}^2 = \mathcal{S}(\{I_t : t \in T_2\})$, see theorem 2.52.

Recall the definition of T, the collection of finite binary trees. For $\eta \in t \in T$ we define $t \restriction \eta$ to be the set $\{\nu \in t : \eta \subseteq \nu \vee \nu \subseteq \eta\}$ and for $B \subseteq {}^{<\omega}2$ we define $\bar{B}$ to be the set $\{\eta \in {}^{<\omega}2 : \exists \nu \in B(\nu \supseteq \eta)\}$.

Lemma 2.42 *Let t be a binary tree not embedding the complete binary tree of height 2. Then $I_t \in \mathcal{I}(\aleph_1)$.*

Proof: We show that I_t is the underlying identity of an ordered identity which is produced by a sequence of one-point end-duplications. The result then follows from theorem 1.10. Let B denote the set of leaves of t. Since t does not embed the complete binary tree of height 2, we can define by recursion, $\langle \nu_1, \dots, \nu_n \rangle$ and $\langle \delta_1, \dots, \delta_{n-1} \rangle$ such that $\{\nu_1, \dots, \nu_n\} = B$ and for all $1 \leq i < j \leq n$, $\nu_i \cap \nu_j = \delta_i$. We define a sequence of ordered identities $\langle J_1, \dots, J_{n-1} \rangle$. $J_1 = \langle A_1, E_1, <_1 \rangle$ is obtained by setting $A_1 = \{a_1, a_2\}$ and $a_1 <_1 a_2$. For $1 \leq j \leq n-2$ we define J_{j+1} to be the V-identity obtained from J_j by duplicating a_{j+1} to a_{j+2}. It is clear that the mapping $\nu_i \mapsto a_i$, $1 \leq i \leq n$ is an embedding of I_t into the identity underlying J_{n-1}. □

Theorem 2.43 *Let $1 \leq m < \omega$ and $t \in T$ be a finite binary tree not embedding the complete binary tree of height $m+1$. Then $I_t \in \mathcal{I}(\aleph_m)$.*

Proof: The proof is by induction on m. When $m = 1$ the conclusion follows from the previous lemma. Now suppose that the result is true for $m < j$. We claim that if a binary tree t does not embed the complete binary tree of height $j+1$ there exists $w < \omega$ and a sequence of binary trees $\langle t_v : v < w \rangle$ such that $I_t = \mathrm{eh}(\langle I_{t_v} : v < w \rangle)$ and for all $v < w$, t_v does not embed the complete binary tree of height j. We then apply the induction hypothesis and theorem 2.21 and conclude that $I_t \in \mathcal{I}(\aleph_j)$. Let t be a binary tree that does not embed the complete binary tree of height $j+1$.

Let B denote the set of leaves of t. By recursion, for $v = 1, 2, \dots$ we define $\tau_v \in {}^{<\omega}2$, $B_v \subseteq B$, and $l_v \in \{0, 1\}$ such that:

i) $\bar{B}_v$ does not embed the complete binary tree of height j

ii) $B_v \subseteq B \cap t \restriction \tau_v{}^\frown\langle l_v \rangle$

iii) For $1 \leq r < s \leq w, \tau_r \subset \tau_s$ and $\tau_r\hat{}\langle l_r \rangle \not\subseteq \tau_s$.

The recursion ends when we reach w such that $B = \bigcup\{B_v : 1 \leq v \leq w\}$. Let $\tau_1 = \bigcap\{\eta : \eta \in B\}$. Since t does not embed the complete binary tree of height $j+1$ there exists $l_1 \in \{0,1\}$ such that $t \restriction (\tau_1\hat{}\langle l_1 \rangle)$ does not embed the complete binary tree of height j. Choose such an l_1 and define $B_1 = B \cap t \restriction (\tau\hat{}\langle l_1 \rangle)$. Now suppose that $\langle \tau_v : v < k \rangle, \langle B_v : v < k \rangle, \langle l_v : v < k \rangle$ have been defined with the above properties. Define $\tau_k = \bigcap\{\eta : \eta \in B \setminus \cup\{B_v : v < k\}\}$. If $\tau_k \notin B$ we again apply the hypothesis that t does not embed the complete binary tree of height $j+1$ to show there exists $l_k \in \{0,1\}$ such that the complete binary tree of height j does not embed in $t \restriction \tau_k\hat{}\langle l_k \rangle$. Choose such an l_k. Let $B_k = B \cap t \restriction \tau_k\hat{}\langle l_k \rangle$. If $\tau_k \in B$, we define B_k to be $\{\tau_k\}$. This completes the construction.

For $1 \leq v \leq w$ we let t_v denote the binary tree $\bar{B}_v$. It is easy to prove by induction that for $1 \leq r < s \leq w$, $\alpha \in B(t_r)$ and $\beta \in B(t_s)$, $\alpha \cap \beta = \tau_r$. Thus I_t is the end-homogeneous amalgam of the sequence $\langle I_{t_1}, \ldots, I_{t_w} \rangle$. Each t_v does not embed the complete binary tree of height j. Applying the induction hypothesis gives $I_{t_v} \in \mathcal{I}(\aleph_{j-1})$. Thus by theorem 2.21, $I_t \in \mathcal{I}(\aleph_j)$. □

Corollary 2.44 *Let t be a finite binary tree. If t does not embed the complete binary tree of height 3, then $I_t \in eh(\mathcal{I}(\aleph_1))$.*

Proof: Let $t \in T$ be a binary tree not embedding the complete binary tree of height 3. By the claim in the proof of the previous theorem there exists a sequence of trees $\langle t_v : v \leq w \rangle$ such that for $1 \leq v \leq w$, t_v does not embed the complete binary tree of height 2, and $I_t = \text{eh}(\langle I_{t_v} : 1 \leq v \leq w \rangle)$. By lemma 2.42, each $I_{t_v} \in \mathcal{I}(\aleph_1)$. □

Corollary 2.45 *Let $I_t \in ID_T$. Then for m, $2 \leq m < \omega$, $I_t \in C^m$ if and only if t does not embed the complete binary tree of height $m+1$.*

Proof: The "if" part is immediate from lemma 2.43 and the "only if" part from theorem 2.17. □

We now define an identity J, and show that it is an element of $\mathcal{C}^2$. Our interest in J is that it is an example of an identity in $\mathcal{C}^2 \setminus \mathcal{S}(\mathcal{C}^2 \cap \mathrm{ID}_T)$. To obtain J we define a ternary function $(I,m,n) \mapsto \hat{I}(m,n)$ from ID $\times \mathbf{Z}^+ \times \mathbf{Z}^+$ to ID such that J is the value at $(I_0, 2, 2)$. The mapping will have the property that for all $j \geq 2$, and $m,n \geq 1$, $I \in \mathcal{I}(\aleph_j)$ implies $\hat{I}(m,n) \in \mathcal{I}(\aleph_j)$. Since $I_0 \in \mathcal{I}(\aleph_2)$ we conclude that $J \in \mathcal{I}(\aleph_2)$.

Definition 2.46 Let $J = \langle A, E \rangle$ be the identity obtained by setting $A = \{1,2,3,\ldots,8\}$ and letting the E-equivalence classes that are not singletons be:

i) $E_1 = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$

ii) $E_2 = \{\{1,6\},\{1,8\},\{3,6\},\{3,8\}\}$

iii) $E_3 = \{\{2,5\},\{2,7\},\{4,5\},\{4,7\}\}$

iv) $E_4 = \{\{1,5\},\{1,7\},\{3,5\},\{3,7\}\}$

v) $E_5 = \{\{2,6\},\{2,8\},\{4,6\},\{4,8\}\}$.

See figures 2.2 and 2.3.

We now define the operation $(I,m,n) \mapsto \hat{I}$. Let $k,m,n < \omega$ and I be an identity of size k. Let $G = \{1,,2,\ldots,k\} \subset \omega$, $g : [G]^2 \longrightarrow \omega$, be such that $g \in I$. Let H be the set $\{1,2,\ldots,k(m+n)\}$. We define a new ω-coloring h, from g,m,n, by letting $h : [H]^2 \longrightarrow \omega$ be any function such that for $i_1 < j_1$ and $i_2 < j_2$

$$h(\{i_1,j_1\}) = h(\{i_2,j_2\})$$

if and only if either

$$j_1, j_2 \geq km+1 \wedge i_1, i_2 \leq km \wedge k \text{ divides } |j_1 - j_2| \wedge k \text{ divides } |i_1 - i_2|$$

or there exists $r,s < \omega$ and $\{p_1,p_2,p_3,p_4\} \subseteq \{1,2,\ldots,k\}$ such that:

i) $p_1 = i_1 - sk, p_2 = j_1 - sk, p_3 = i_2 - rk, p_4 = j_2 - rk$ and

ii) $g(\{p_1,p_2\}) = g(\{p_3,p_4\})$.

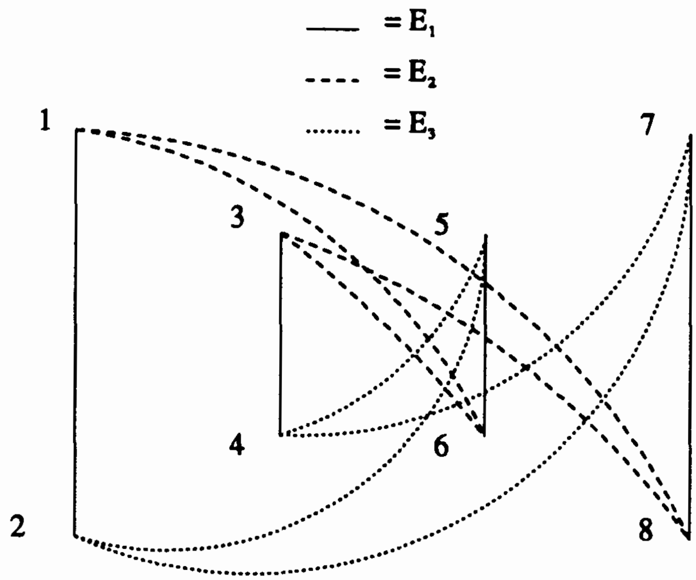

Figure 2.2: The E-equivalence classes E_1, E_2, and E_3.

We call $\langle h, H\rangle$ the (m,n)-*iterate* of $\langle g, G\rangle$ and the identity $\hat{I}$, realized by h, the (m,n)-*iterate* of I. Note that $\hat{I}$ depends only on I, m, n. The coloring h has the following intuition. For $1 \leq i \leq m+n$ a copy of the coloring g is defined on the set $\{ki+j : 1 \leq j \leq k\}$. Now fix a pair $(k_1, k_2) \in \{1,2,\ldots k\} \times \{1,2,\ldots,k\}$. We choose the k_1th vertex from each of the first m copies and the k_2th vertex from each of the last n copies. We then put a single new color on all the edges between the set of vertices so chosen in the first m copies to all the vertices chosen in the last n copies. We do the same thing for all such (k_1, k_2). All edges for which we have not specified a color are assigned a new and distinct color. See figure 2.4 for a partial diagram of the (2,4)-iterate of Q_5.

Theorem 2.47 *Let I be an identity and $i \geq 2$. For all $m, n < \omega$, $I \in \mathcal{I}(\aleph_i)$ implies the (m,n)-iterate of I is an element of $\mathcal{I}(\aleph_i)$.*

Proof: Let $i \geq 2$ and $f : [\aleph_i]^2 \longrightarrow \omega$. Let $I = \langle A, E\rangle$ with $|A| = k$. Since $I \in \mathcal{I}(\aleph_i)$, for $l < \aleph_i$ we can choose $\{\beta_{l,j} : 1 \leq j \leq k\}$ such that for all $l < \aleph_i$, f induces I on $\{\beta_{l,j} : 1 \leq j \leq k\}$, $1 \leq j_1, j_2 \leq k$ and $l_1 < l_2$ imply $\beta_{l_1,j_1} < \beta_{l_2,j_2}$ and $j_1 < j_2$ implies $\beta_{l,j_1} < \beta_{l,j_2}$. For $l_1, l_2 < \aleph_i$ let $g_{l_1,l_2} : \{\beta_{l_1,j} : 1 \leq j \leq k\} \longrightarrow \{\beta_{l_2,j} : 1 \leq j \leq k\}$ be the order preserving bijection. Without loss of generality we can assume that g_{l_1,l_2}

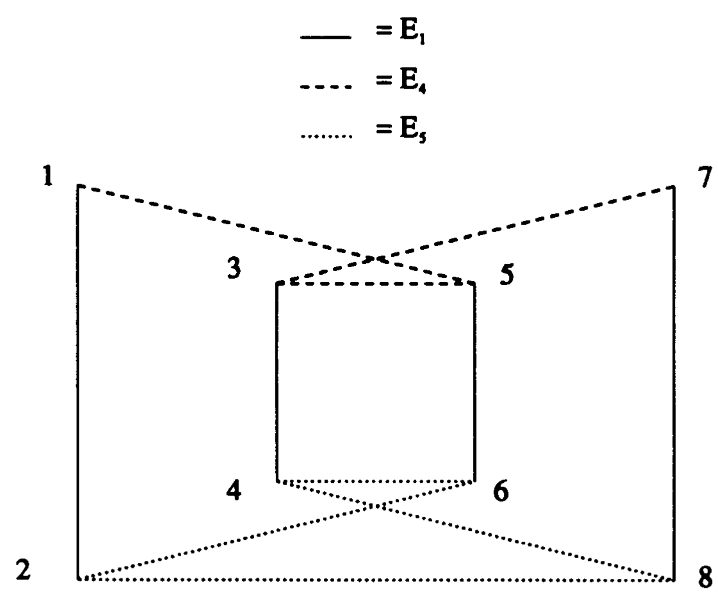

Figure 2.3: The E-equivalence classes E_1, E_4 and E_5.

is an f-isomorphism. Thus we assume that $\{j_1, j_2\} \subseteq \{1, \ldots, k\}$ and $l_1, l_2 < \aleph_i$ imply $f(\{\beta_{l_1,j_1}, \beta_{l_1,j_2}\}) = f(\{g_{l_1,l_2}(\beta_{l_1,j_1}), g_{l_1,l_2}(\beta_{l_1,j_2})\})$. For $l \geq \aleph_{i-1}$ we define a sequence of sets B^l_r and elements of ω, $c_{l,j,r}$ $(1 \leq r, j \leq k)$ such that:

i) $B^l_r \subseteq \aleph_{i-1}$ and $|B^l_r| = \aleph_{i-1}$

ii) $r_1 > r_2 \Rightarrow B^l_{r_1} \subseteq B^l_{r_2}$

iii) for all $l_1, l_2 \in B^l_r$ for all $1 \leq j \leq k$ and $1 \leq s \leq r$

$$f(\{\beta_{l,s}, \beta_{l_1,j}\}) = f(\{\beta_{l,s}, \beta_{l_2,j}\}) = c_{l,s,j}.$$

Using only cardinality considerations the construction should be clear. Again by considering cardinalities there exist $C \subset \aleph_i \setminus \aleph_{i-1}$ of cardinality $\aleph_i$ and $c_{r,j} < \omega\,(1 \leq r, j \leq k)$ such that for all $l \in C$, $c_{l,r,j} = c_{r,j}$.

For each $l \in C$ choose $D_l \in [B^l_k]^m$. Since $D_l \in [\aleph_{i-1}]^m$ and $[\aleph_{i-1}]^m$ has cardinality $\aleph_{i-1}$, the pigeon-hole priciple tells us that there exists $D = \{l_1, \ldots, l_m\}$ and $C' \subseteq C$ of cardinality $\aleph_i$ such that $D_l = D$ for all $l \in C'$. Let $l_{m+1}, \ldots, l_{m+n} \subset C'$ be distinct. The (m, n)-iterate of I is then induced by f on the set $\{\beta_{l_s,j} : 1 \leq s \leq m+n, 1 \leq j \leq k\}$. $\square$

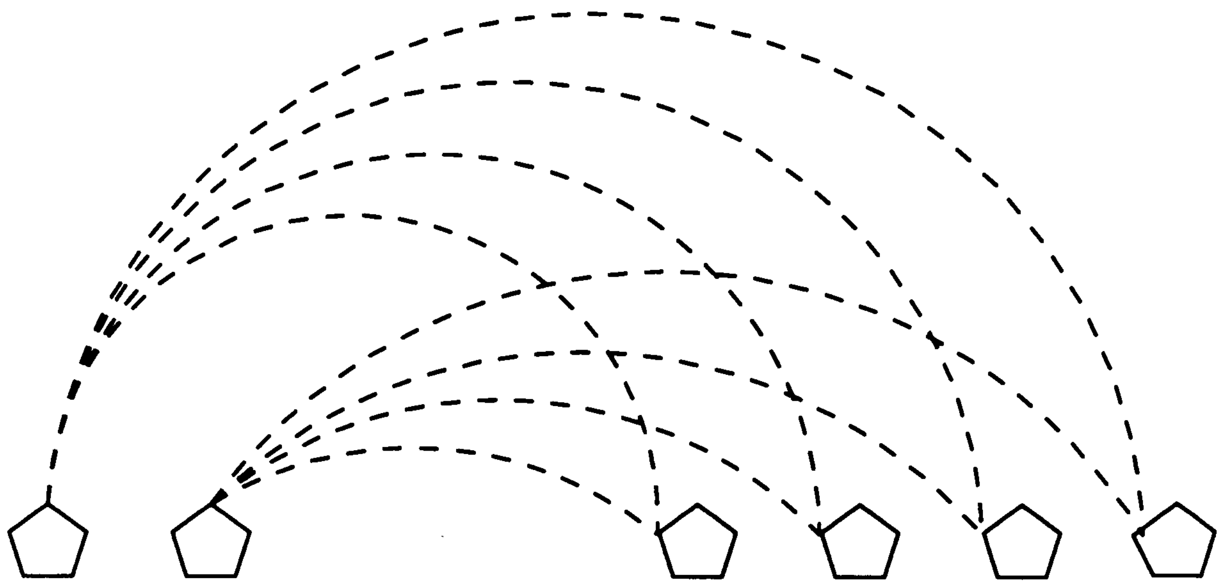

Figure 2.4: One new equivalence class in the (2,4)-iterate of Q_5.

Corollary 2.48 $J \in C^2$.

Proof: J is equal to the (2,2)-iterate of I_0. Note that I_0 is the unique identity of size 2. □

We now prove a proposition and a lemma which will be used to show that $J \notin \mathrm{eh}(\mathcal{I}(\aleph_1))$. Let $L = \langle C, G, <_C \rangle$ denote the ordered identity for which, $C = \{0,1,2\}, 0 <_C 1 <_C 2$, and the only G-equivalence class which is not a singleton is the set $\{\{0,2\},\{1,2\}\}$.

Proposition 2.49 $L \notin IDE_V$.

Proof: Towards a contradiction let $K = \langle B, F, <_B \rangle \in \mathrm{IDE}_V$ realizing L be such that $|B|$ is minimal. Let $f : C \longrightarrow B$ witness the realization. Define $K_1 = \langle B_1, F_1, <_1 \rangle$ to be the restriction of K to the range of f. By the minimality of $|B|$ there exist $M = \langle A, E, <_A \rangle \in \mathrm{IDE}_V$, $\bar{a} \in {}^{<\omega}A$, a final segment of A and $\bar{b} \in {}^{<\omega}B$ such that K is obtained from M by duplicating $\bar{a}$ to $\bar{b}$. By the minimality of B, $\mathrm{rng}(f) \cap \bar{b} \neq \emptyset$ and $\mathrm{rng}(f) \cap \bar{a} \neq \emptyset$. Since f is order-preserving this implies $f(2) \in \bar{b}$. Now $|\mathrm{rng}(f) \cap (\bar{a} \cup \bar{b})| \neq 3$. Otherwise, the definition of end-duplication implies that F_1 has three distinct equivalence classes, a contradiction to f being an embedding. Thus $f(0) \in B \setminus (\bar{a} \cup \bar{b})$, $f(1) \in \bar{a}$, and $f(2) \in \bar{b}$. Now $\{f(1), f(2)\}$

lies in an F_1-equivalence class of size one by the definition of end-duplication. But since f is an embedding $\{f(1), f(2)\} \simeq_{F_1} \{f(0), f(2)\}$, contradiction. □

Lemma 2.50 $I_1 \notin \mathcal{I}(\aleph_1)$.

Proof: Let $\hat{I}_1$ denote an ordered identity whose underlying identity is I_1. By theorem 1.10 it suffices to show that no $J \in \text{IDE}_V$ realizes $\hat{I}_1$. Towards a contradiction let $J \in \text{IDE}_V$. It is clear that $\hat{I}_1$ realizes L and thus J must also realize L. This contradicts the previous proposition. □

Theorem 2.51 $J \notin eh(\mathcal{I}(\aleph_1))$.

Proof: Towards a contradiction choose $K = \langle D, G\rangle \in \text{eh}(\mathcal{I}(\aleph_1))$ and $f : A \longrightarrow D$ witnessing the fact that $J = \langle A, E\rangle$ is realized as a subidentity of K. Let K be the end-homogeneous amalgam of $J_1, J_2, \ldots, J_n$ for some $n < \omega$, where $J_i = \langle B_i, F_i\rangle$ is an element of $\mathcal{I}(\aleph_1)$. Since $I_1 \hookrightarrow J$ and $J_i \in \mathcal{I}(\aleph_1)$, by the previous lemma, $\text{rng}(f)$ is not contained in a single B_i. From the definition of end-homogeneous amalgamation an edge in $[B_i]^2$ cannot be G-equivalent to an edge not in $[B_i]^2$. Every $a \in A$ belongs to an element of E_1. Thus if any edge is mapped by f into $[B_i]^2$, then A is mapped into B_i, contradiction. In the end-homogeneous amalgamation of $J_1, \ldots, J_n$, the color classes, i.e. the G-classes, are those from $J_1, \ldots, J_n$ respectively, together with new classes, one for each i, $1 \le i < n$. An edge belongs to the i-th new class just if it links B_i with B_j for some j, $i < j \le n$. So in the present context there exists a particular i such that if $\{a, b\} \in E_1$, then one of $f(a), f(b)$ is in B_i and the other is in $\bigcup\{B_j : i < j \le n\}$. Without loss of generality $i = 1$ and f is the identity. So each of $\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}$ meets B_1 in a singleton. The identity J is invariant under the permutations (12)(34), (56)(78), (13)(24) and (57)(68) so it suffices to consider the three cases:

Case 1. $\{1,3,5,7\} \subseteq B_1$.

This is impossible as $J \restriction \{1,3,5,7\} = I_1$ and by the previous lemma $I_1 \notin \mathcal{I}(\aleph_1)$.

Case 2. $\{2,3,5,7\} \subseteq B_1$.

The largest equivalence class in $K \restriction \{f(1), f(3), f(5), f(7)\}$ has three edges yet $J \restriction \{1, 3, 5, 7\}$ has an equivalence class with four edges.

Case 3. $\{2, 3, 6, 7\} \subseteq B_1$.

Here $\{f(3), f(5)\} \not\simeq_G \{f(3), f(7)\}$ yet $\{3, 5\} \simeq_E \{3, 7\}$.

□

Theorem 2.52 *There does not exist $T_2 \subseteq T$ such that $C^2 = \mathcal{S}(\{I_t : t \in T_2\})$.*

Proof: Towards a contradiction let $T_2 \subseteq T$ be such that $C^2 = \mathcal{S}(\{I_t : t \in T_2\})$. $J \in C^2$ so there must exist $t \in T_2$ such that $J \hookrightarrow I_t$. Now t cannot embed the complete binary tree of height 3 by corollary 2.45 and thus $I_t \in \mathrm{eh}(\mathcal{I}(\aleph_1))$ by corollary 2.44. This provides a contradiction as $J \notin \mathrm{eh}(\mathcal{I}(\aleph_1))$ by theorem 2.51. □

2.8 Saturated Ideals

In this section we will show that the existence of a certain kind of ideal on $\aleph_m$ implies that $\mathcal{I}(\aleph_m) \supseteq \mathrm{IDT}$. Throughout the section λ denotes an uncountable cardinal and, for any $\mathfrak{J} \subseteq \wp(\lambda)$, $\mathfrak{J}^+$ denotes $\wp(\lambda) \setminus \mathfrak{J}$. For $X, Y \in \wp(\lambda)$ we write $X \sim_{\mathfrak{J}} Y$ if $X \triangle Y \in \mathfrak{J}$. This is clearly an equivalence relation on $\wp(\lambda)$. We denote by $[X]$, the equivalence class of X and by $\wp(\lambda)/\mathfrak{J}$ the collection of all equivalence classes. For equivalence classes $[X]$ and $[Y]$ we say that $[X] \leq [Y]$ if $X \setminus Y \in \mathfrak{J}$ and note that the relation is independent of the choice of representatives, X and Y. We define four kinds of ideal on λ.

Definition 2.53 Let κ be a cardinal. A collection $\mathfrak{J} \subset \wp(\lambda)$ is a *κ-complete ideal on λ* if the following conditions are satisfied:

i) $X \subseteq Y \subset \lambda$ and $Y \in \mathfrak{J}$ imply $X \in \mathfrak{J}$

ii) $\{X_\alpha : \alpha < \eta\} \subset \mathfrak{J}$ and $\eta < \kappa$ imply $\bigcup\{X_\alpha : \alpha < \eta\} \in \mathfrak{J}$

iii) $\emptyset \in \mathfrak{J}$

iv) $\lambda \notin \mathfrak{J}$

v) $\alpha \in \lambda$ implies $\{\alpha\} \in \mathfrak{I}$

Definition 2.54 Let κ and μ be cardinals and $\mathfrak{I}$ be a κ-complete ideal on λ. We say that $\mathfrak{I}$ is *μ-saturated* if for every collection $\{X_\alpha : \alpha < \mu\} \subseteq \mathfrak{I}^+$ there exist $\alpha, \beta < \mu$ such that $X_\alpha \cap X_\beta \in \mathfrak{I}^+$.

Definition 2.55 Let κ, μ, and ν be cardinals. A κ-complete ideal $\mathfrak{I}$ on λ is said to be *$(\mu, \mu, < \nu)$-saturated* if for every $X \in [\mathfrak{I}^+]^\mu$ there exists $Y \in [X]^\mu$ so that for every $\delta < \nu$ and $Z \in [Y]^\delta$ $(\bigcap Z \in \mathfrak{I}^+)$.

Definition 2.56 Let κ and ν be cardinals and $\mathfrak{I}$ be a κ-complete ideal on λ. $\mathfrak{I}$ is *μ-dense* if there exists $\mathcal{D} \in [\mathfrak{I}^+]^\mu$ such that for all $[X] \in \wp(\lambda)/\mathfrak{I}$ there exists $D \in \mathcal{D}$ such that $[D] \leq [X]$.

Throughout this section we will assume that all ideals under discussion are $\aleph_1$-complete. It has recently been shown that it is consistent relative to the consistency of the existence of certain large cardinals that there exists an $\aleph_1$-dense ideal on $\aleph_2$, see [3]. This has two consequences of interest to us. The first is the consistency of the existence of an $(\aleph_2, \aleph_2, < \omega)$-saturated ideal on $\aleph_2$. This is the set theoretic hypothesis of the main theorem of this section. That the existence of an $\aleph_1$-dense ideal on $\aleph_2$ implies the existence of an $(\aleph_2, \aleph_2, < \omega)$-saturated ideal on $\aleph_2$ follows from the same result for ideals on $\aleph_1$, mentioned by Laver in [10]. The second consequence is a result of Woodin (private communication): the existence of an $\aleph_1$ dense ideal on $\aleph_2$ implies CH. From CH and the Erdos-Rado theorem it follows that $\mathcal{I}(\aleph_2) =$ ID. Thus the existence of an $\aleph_1$-dense ideal on $\aleph_2$ trivially implies that $\mathcal{I}(\aleph_2) =$ ID. The same proof cannot be given when one merely hypothesizes the existence of an $(\aleph_2, \aleph_2, < \omega)$-saturated ideal on $\aleph_2$ since the existence of such an ideal does not imply CH. One can add $\aleph_2$ Cohen reals to a model in which there exists an $\aleph_1$-dense ideal on $\aleph_2$ to produce a model in which CH does not hold and there is an $(\aleph_2, \aleph_2, < \omega)$-saturated ideal on $\aleph_2$. This is also a result of Woodin (private communication).

We now establish some notation to be used below. For $\mathfrak{I}$ an ideal on $\aleph_2$, and B a subset of $\aleph_2$ of cardinality $\aleph_2$, let $f : \aleph_2 \longrightarrow B$ denote the unique order-preserving bijection from $\aleph_2$ onto B. By $\mathfrak{I}^+(B)$ we denote the collection $\{Y \subset B : f^{-1}(Y) \in \mathfrak{I}^+\}$.

Theorem 2.57 *Let $k \geq 2$. If there exists an $(\aleph_k, \aleph_k, < \omega)$-saturated, $\aleph_1$-complete ideal $\mathfrak{I}$ on $\aleph_k$ then $\mathcal{I}(\aleph_k) \supseteq IDT$.*

Proof: For each $n, 1 \leq n < \omega$ we will show that $I_{n-1} \in \mathcal{I}(\aleph_k)$. Let $f : [\aleph_k]^2 \longrightarrow \omega$. We fix n, $1 \leq n < \omega$. For each $\eta \in {}^{<n}2$, $\nu \in {}^{\leq n}2$, and $\zeta \in {}^{n}2$ we define subsets I_ν, J_ν, M_η, D_η of $\aleph_k$, a sequence of sets $\langle K_\eta^\alpha : \alpha \in M_\eta \rangle$, and elements $c_\eta < \omega$ and $\alpha_\zeta \in \aleph_k$ such that

i) $I_\nu \cap J_\nu = \emptyset$

ii) $J_\eta \supseteq M_\eta = I_{\eta\hat{}\langle 0\rangle} \cup J_{\eta\hat{}\langle 0\rangle}$

iii) $I_\eta \supseteq D_\eta = I_{\eta\hat{}\langle 1\rangle} \cup J_{\eta\hat{}\langle 1\rangle}$

iv) for all $\alpha \in M_\eta$, $K_\eta^\alpha \in \mathfrak{I}^+(I_\eta)$

v) for all $\alpha \in M_\eta$ and $x \in K_\eta^\alpha$, $f(\{x, \alpha\}) = c_\eta$

vi) if $\zeta \supseteq \eta\hat{}\langle 0\rangle$ then $\alpha_\zeta \in M_\eta$

vii) $D_\eta = \bigcap\{K_\eta^{\alpha_\beta} : \beta \supseteq \eta\hat{}\langle 0\rangle, \beta \in {}^{n}2\}$.

viii) $\nu \supseteq \eta \Rightarrow (J_\nu \cup I_\nu) \subseteq (J_\eta \cup I_\eta)$.

To start the construction we partition $\aleph_k$ into two disjoint subsets $I_\emptyset, J_\emptyset$ each of cardinality $\aleph_k$. There are three procedures to follow.

Procedure 1. If I_η and J_η have been defined we define $I_{\eta\hat{}\langle 0\rangle}$ and $J_{\eta\hat{}\langle 0\rangle}$ as follows. For $\alpha \in J_\eta$ choose $K_\eta^\alpha \in \mathfrak{I}^+(I_\eta)$ and $c_\eta^\alpha < \omega$ so that $f(\{\alpha, x\}) = c_\eta^\alpha$ for all $x \in K_\eta^\alpha$. Choose $c_\eta < \omega$ and $L_\eta \subseteq J_\eta$ of cardinality $\aleph_k$ such that $c_\eta^\alpha = c_\eta$ for all $\alpha \in L_\eta$. By $(\aleph_k, \aleph_k, < \omega)$ saturation let $M_\eta \subseteq L_\eta$ be of cardinality $\aleph_k$ with the property

$$\forall X[X \subset M_\eta \wedge |X| < \omega \Rightarrow \cap\{K_\eta^\alpha : \alpha \in X\} \in \mathfrak{I}^+(I_\eta)]. \tag{2.1}$$

Choose $I_{\eta\hat{}\langle 0\rangle}$ and $J_{\eta\hat{}\langle 0\rangle}$ to be subsets of cardinality $\aleph_k$ which partition M_η. This completes the first procedure.

Procedure 2. If I_η has been defined and α_δ has been chosen for each $\delta \supseteq \eta\hat{}\langle 0\rangle$, we define $I_{\eta\hat{}\langle 1\rangle}$ and $J_{\eta\hat{}\langle 1\rangle}$ as follows. Let H_η denote $\{\alpha_\delta : \eta\hat{}\langle 0\rangle \subseteq \delta,\ \delta \in {}^{n}2\}$ and $D_\eta = \cap\{K_\eta^\alpha : \alpha \in H_\eta\}$. From equation (2.1) and vi), $D_\eta \in \mathfrak{I}^+(I_\eta)$. Choose $I_{\eta\hat{}\langle 1\rangle}$ and $J_{\eta\hat{}\langle 1\rangle}$ to be subsets of cardinality $\aleph_k$ which partition D_η. This completes the second procedure.

Procedure 3. If $\ln(\zeta) = n$ and I_ζ has been defined, choose $\alpha_\zeta \in I_\zeta$.

We construct the sets I_η, J_η ($\eta \in {}^{\leq n}2$) and the ordinals α_η ($\eta \in {}^n2$) by repeatedly applying the three procedures above. It is easy to check that properties i) to viii) hold. Let η and ν be distinct elements of ${}^n2$. We claim that if $\eta \cap \nu = \gamma$ then $f(\{\alpha_\eta, \alpha_\nu\}) = c_\gamma$. Without loss of generality we may assume that $\eta \supseteq \gamma^\frown\langle 0\rangle$ and $\nu \supseteq \gamma^\frown\langle 1\rangle$. Recall the sets M_γ and $K_\gamma^{\alpha_\bullet}$ defined during the construction of $J_{\gamma^\frown\langle 0\rangle}$ and $I_{\gamma^\frown\langle 0\rangle}$. It is clear from the construction that $\alpha_\eta \in I_\eta \subseteq J_{\gamma^\frown\langle 0\rangle} \cup I_{\gamma^\frown\langle 0\rangle} \subseteq M_\gamma$ and thus $f(\{\alpha_\eta, x\}) = c_\gamma$ for all $x \in K^\gamma_{\alpha_\eta}$. Now recall the set D_γ defined during the construction of $J_{\gamma^\frown\langle 1\rangle}$ and $I_{\gamma^\frown\langle 1\rangle}$. It is clear that $\alpha_\nu \in I_\nu \subseteq I_{\gamma^\frown\langle 1\rangle} \cup J_{\gamma^\frown\langle 1\rangle} \subseteq D_\gamma \subseteq K^\gamma_{\alpha_\eta}$. This implies that $f(\{\alpha_\nu, \alpha_\eta\}) = c_\gamma$. Thus for all $\{\nu, \eta\}, \{\beta, \delta\} \in [{}^n2]^2$

$$\eta \cap \nu = \beta \cap \delta \Rightarrow f(\{\alpha_\eta, \alpha_\nu\}) = f(\{\alpha_\beta, \alpha_\delta\}).$$

This suffices to prove the theorem since $f \restriction \{\alpha_\eta : \eta \in 2^n\}$ realizes I_{n-1}. $\square$

2.9 Finite Analogs

The question addressed here is whether $\mathcal{I}(\aleph_1, \omega, t)$ can be generated by finite colorings of finite graphs. Clearly we cannot hope to find some m and n such that $\mathcal{I}(m, n, t) = \mathcal{I}(\aleph_1, \omega, t)$ for all t simultaneously, since t must be less than or equal to m. Thus we can ask whether, for fixed t, there exist $m, n < \omega$ such that $\mathcal{I}(\aleph_1, \omega, t) = \mathcal{I}(m, n, t)$. This question is not a simple application of Ramsey theory. It is true that for a fixed finite number of colors and desired finite monochromatic complete subgraph, we can find sufficiently large complete graphs such that no matter how they are colored they must have the desired monochromatic complete subgraph. But this is clearly too strong, since for all t there exist identities that are not in $\mathcal{I}(\aleph_1, \omega, t)$ and a monochromatic subgraph of size t realizes all identities of size t. To answer the question we determine the identities of size 2,3, and 4 in $\mathcal{I}(\aleph_1, \omega)$. We show for $t = 2, 3$ that there exist $m, n < \omega$ (depending on t) such that $\mathcal{I}(\aleph_1, \omega, t) = \mathcal{I}(m, n, t)$. Once four points are used the situation gets more complex and it turns out that the identities in $\mathcal{I}(\aleph_1, \omega, 4)$ cannot be written in the form $\mathcal{I}(m, n, 4)$ for any $m, n < \omega$. We define a collection of identities $M_i = \langle A_i, E_i\rangle$. In the following we represent E_i by its associated partition of $[A_i]^2$.

$M_1 : A_1 = \{1,2\}, E_1 = \{\{12\}\}$

$M_2 : A_2 = \{1,2,3\}, E_2 = \{\{12,13\},\{23\}\}$

$M_3 : A_3 = \{1,2,3,4\}, E_3 = \{\{12,13,14\},\{23,24\},\{34\}\}$

$M_4 : A_4 = \{1,2,3,4\}, E_4 = \{\{12,34\},\{23\},\{24\},\{13\},\{14\}\}$

$M_5 : A_5 = \{1,2,3,4\}, E_5 = \{\{12,23,34\},\{14\},\{13\},\{24\}\}$

$M_6 : A_6 = \{1,2,3\}, E_6 = \{\{12,13,23\}\}$.

For obvious reasons we call M_6 a *monochromatic triangle*. The following two lemmas are immediate. They show that if there exists t such that for all $m, n < \omega$, $\mathcal{I}(\aleph_1, \omega, t) \neq \mathcal{I}(m, n, t)$ then $t \geq 4$.

Lemma 2.58 $\mathcal{I}(2,1,2) = \{M_1\} = \mathcal{I}(\aleph_1, \omega, 2)$

Lemma 2.59 $\mathcal{I}(3,2,3) = \mathcal{S}(\{M_2\}) = \mathcal{I}(\aleph_1, \omega, 3)$

We now prove a series of lemmas which show that there are no m, n such that $\mathcal{I}(\aleph_1, \omega, 4) = \mathcal{I}(m, n, 4)$.

Lemma 2.60 $M_5 \notin \mathcal{I}(\aleph_1, \omega, 4)$.

Proof: Recall the V-identity K, which was defined at the beginning of §2.6. We first claim that $K \notin$ IDE. The proof is similar to the proof of proposition 2.38 so we omit it. By corollary 1.10 $\mathcal{I}(\aleph_1, \omega) =$ IDE. Thus it suffices to show that there does not exist a V-identity J^*, produced by a sequence of end-duplications such that M_5 embeds in the identity J underlying J^*. Towards a contradiction choose $J^* = \langle A, E, <_A \rangle \in \text{IDE}_V$ with $|A|$ least possible such that there exists an embedding $f : A_5 \longrightarrow A$, of M_5 into the identity (denoted by $J = \langle A, E \rangle$) underlying J^*. Let $L^* = \langle B, F, <_B \rangle, \bar{b} \subseteq B$, and $\bar{a} \subset A$ be chosen so that J^* is obtained from L^* by duplicating $\bar{b}$ to $\bar{a}$. By the minimality of $|A|$, $\text{rng}(f) \cap \bar{a}$ and $\text{rng}(f) \cap \bar{b}$ are both non-empty. We now note that each pair, $\{x, y\}$ of consecutive elements in the sequence $\langle f(1), f(2), f(3), f(4) \rangle$ lie in the same E-equivalence class. This implies that $\{x, y\} \subseteq B$ or $\{x, y\} \subseteq A \setminus \bar{b}$. Since $B \setminus \bar{b} < \bar{b} < \bar{a}$, by inspection we see that J^* must realize K, contradiction. □

Lemma 2.61 *For $m \leq 2n+1$, $M_3 \notin \mathcal{I}(m,n,4)$.*

Proof: Define $f : [2n+1]^2 \longrightarrow \{1,\ldots,n\}$ by $f(\{a,b\}) = i$ whenever $b-a = i \pmod{2n+1}$ or $b-a = -i \pmod{2n+1}$. In effect we place $2n+1$ points symmetrically on the circumference of a circle and give the edges the same color if they have the same length. Each vertex belongs to exactly 2 edges of each color. The identity M_3 has a vertex which belongs to three edges all in the same equivalence class. Thus M_3 is not induced by this coloring. □

Definition 2.62 Let $m, n < \omega$, $i < m$, $k < n$, $f : [m]^2 \longrightarrow n$. Define $T \subseteq [m]^2$ to be the *k-star of f at i* if $T = \{\{i,j\} \in [m]^2 : f(\{i,j\}) = k\}$.

The following lemma can be found in [7] theorem 3.6. We give our own proof.

Lemma 2.63 *If $m > 2n+1$ and $m \geq 4$ then $M_5 \in \mathcal{I}(m,n,4)$.*

Proof: The idea is that relatively few colors are used so that from some vertices we get large k-stars for some $k < n$. If any of the k-stars overlap, M_5 will be realized.

Towards a contradiction let $f : [m]^2 \longrightarrow n$ which does not realize M_5. Fix $k < n$ and consider the following matrix.

$$a_{ik} = \begin{cases} 0 & \text{if } |\{j : f(\{i,j\}) = k\}| \in \{0,1\} \\ 1 & \text{if } |\{j : f(\{i,j\}) = k\}| = 2 \\ \text{r+1} & \text{if } |\{j : f(\{i,j\}) = k\}| = r > 2 \end{cases}$$

We claim that if $\sum_{i=0}^{m-1} a_{ik} > m$ then f realizes M_5 in color k. The proof is by induction on m. Let $m = 4$ and assume that $\sum_{i=0}^{m-1} a_{ik} > m$. There must exist $j < 4$ such that $a_{j,k} > 1$. Thus vertex j is connected to all 3 remaining vertices with edges of color k. Now $a_{i,k} \leq 4$ for all i so there must exist $i \neq j$ such that $a_{i,k} \geq 1$. Thus there are at least two edges leaving vertex i is color k. By inspection M_5 is realized in color k. Now suppose the result holds for all $t < m$. We first eliminate all monochromatic triangles of color k. To this end, choose a $\subseteq$-maximal collection S of subsets of m satisfying the properties that $s \in S$ implies $|s| = 3$ and $f''[s]^2 = \{k\}$. Define $Z = m \setminus \bigcup S$. Clearly $|Z| < m$ if $S \neq \emptyset$. Note that for all $s \in S$ there do not exist $l \in s$ and $m \notin s$ such that $f(\{l,m\}) = k$, otherwise M_5 is realized by f. So $a_{i,k} = 1$ for each $i \in \bigcup S$. Thus $\sum_{i \in Z} a_{i,k} > |Z|$ and we may apply the induction hypothesis. Thus without loss of generality we may assume that $S = \emptyset$.

Let T be a star at j, and $|T| \geq 2$. Since f does not realize a a monochromatic triangle in color k by an earlier reduction, and since f does not embed M_5, every edge e of color k which meets $\bigcup T$ belongs to T. So the points of $\bigcup T$ can be deleted from fld(f) leaving the inequality intact. We are left with stars of size 0 and 1 and thus $\sum_{i=0}^{m-1} a_{ik} = 0$, contradiction. Thus the claim is proved.

We now claim that for all i,

$$\sum_{k=0}^{n-1} a_{ik} \geq m - n + 1.$$

which may be see as follows: Since each i belongs to $m-1 \geq 2n+1$ edges, more than two of the edges have some color k and thus for this k we have $a_{ik} \geq 4$. Fix such k. Now consider a color $k' \neq k$. If more than one of the edges through i has color k', recoloring all but one of these edges with k leaves $\sum_{k=0}^{n-1} a_{i,k}$ unchanged or diminished. So $\sum_{k=0}^{n-1} a_{i,k}$ is least when $n-1$ of the edges through i have a unique color and the remaining $m-n$ have a common color. This completes the proof of the second claim.

We now complete the proof of the lemma. Since $m > 2n+1$, $m(m-n+1) > mn$. Thus

$$\sum_{i=0}^{m-1} \sum_{k=0}^{n-1} a_{i,k} \geq m(m-n+1) > mn.$$

This implies that $\sum_{i=0}^{m-1} a_{i,k} > m$ for some k. By the first claim M_5 is realized by f, contradiction. □

Theorem 2.64 *For all m, n, $\mathcal{I}(\aleph_1, \omega, 4) \neq \mathcal{I}(m, n, 4)$*

Proof: Towards a contradiction choose $m, n < \omega$ such that $\mathcal{I}(\aleph_1, \omega, 4) = \mathcal{I}(m, n, 4)$. $M_3 \in \mathcal{I}(\aleph_1, \omega, 4)$ so by lemma 2.61, $m > 2n+1$ and thus, by lemma 2.63, $M_5 \in \mathcal{I}(m, n, 4)$. This contradicts lemma 2.60 which says that $M_5 \notin \mathcal{I}(\aleph_1, \omega, 4)$. □

Chapter 3

CV-identities

One should now recall the definitions of CV-identity, and $\mathcal{I}_{CV}(D)$, for D a set of ordinals, given in §1.3. Let $\langle f, F, <^F\rangle$ be an O-coloring. Denote by $\mathcal{I}_{CV}(f)$ the collection of all finite CV-identities that are realized by f.

In a manner analogous to that for identities and V-identities, a CV-identity can be represented by a structure $I = \langle u, <, \preceq\rangle$ where u is a set linearly ordered by $<$ and $\preceq$ is a pre-order of $[u]^2$. Given such a structure let $B \subset \omega$ be the initial segment of ω such that $|B| = |u|$ and $k : u \longrightarrow B$ be the order preserving bijection. Define $f : [B]^2 \longrightarrow \omega$ to be the unique function whose range is an initial segment of ω such that for all $\{a,b\}, \{x,y\} \in [u]^2$, $\{a,b\} \preceq \{x,y\}$ if and only if $f(\{k(a),k(b)\}) \leq f(\{k(x),k(y)\})$. The CV-identity corresponding to the structure is the $\simeq_{CV}$-class which contains the O-coloring $\langle f, \omega, <\rangle$. It is easy to see that every CV-identity can be represented in this manner.

As a notational convenience, for $\{x,y\}, \{a,b\} \in [u]^2$ we will write $\{x,y\} \sim \{a,b\}$ whenever $\{x,y\} \preceq \{a,b\}$ and $\{x,y\} \succeq \{a,b\}$. We also write $\{x,y\} \prec \{a,b\}$ whenever $\{x,y\} \preceq \{a,b\}$ and $\{x,y\} \not\sim \{a,b\}$.

In this chapter we study the collection $\mathcal{I}_{CV}(\kappa)$ for various cardinals κ. In section 3.1 we examine the case when κ equals $\aleph_1$. We define a collection C, of CV-identities and show that ZFC $\vdash C \subseteq \mathcal{I}_{CV}(\aleph_1)$. We then construct a model M of ZFC via a forcing argument such that $M \models C = \mathcal{I}_{CV}(\aleph_1)$. We have been unable to show that this result is provable in ZFC, or that there exist models of ZFC in which $C \neq \mathcal{I}_{CV}(\aleph_1)$. A few more words concerning this problem will be given at the end of this introduction.

Section 3.2 concerns itself with CV-identities in $\mathcal{I}_{CV}(\aleph_2)$. We exhibit two CV-identities, R_1 and R_2, which have the same underlying identity. We show that $R_2 \in \mathcal{I}_{CV}(\aleph_2)$ in all models of ZFC. In contrast to this result we produce two models of ZFC that show that $R_1 \in \mathcal{I}_{CV}(\aleph_2)$ is independent of ZFC. The first model is constructed by a c.c.c. forcing over a ground model which contains a $\Box_{\omega_1}$-sequence. It satisfies $R_1 \notin \mathcal{I}_{CV}(\aleph_2)$. Thus not all CV-identities, whose underlying identity is in $\mathcal{I}(\aleph_2)$, occur as elements of $\mathcal{I}_{CV}(\aleph_2)$. The second model is one in which there exists an $\aleph_2$-saturated, $\aleph_1$-complete ideal on $\aleph_1$. The ideal is used to show that $R_1 \in \mathcal{I}_{CV}(\aleph_2)$. We deduce that the c.c.c. forcing notion used to construct the first model destroys all $\aleph_2$-saturated, $\aleph_1$-complete ideals on $\aleph_1$. For some history and ramifications of this result see [18].

In section 3.3 we define a collection $\mathcal{E}$, of CV-identities and show that in all models of ZFC, $\mathcal{I}_{CV}(\aleph_\omega) \supseteq \mathcal{E}$.

In the final section we study $\mathcal{I}_{CV}(\aleph_1, \mathbf{Q})$, where $\mathbf{Q}$ is ordered in the usual way. The reader should note that up to this point the color set has had order type ω. We define a collection of CV-identities, $\mathcal{D}$, and prove that $\mathcal{I}_{CV}(\aleph_1, \mathbf{Q}) = \mathcal{D}$ in all models of ZFC. The method of proof is as follows. By a forcing construction we produce a model in which the desired equality holds. We then show for each CV-identity I, if there exists $f : [\aleph_1]^2 \longrightarrow \mathbf{Q}$ in the generic extension such that $I \notin \mathcal{I}_{CV}(f)$ then there exists $g : [\aleph_1]^2 \longrightarrow \mathbf{Q}$ in the ground model such that $I \notin \mathcal{I}_{CV}(g)$. The reasoning is as follows. The coloring f can be coded as an (ω_1, ω)-model of a certain first order theory T_I. The existence of such a model is equivalent to the consistency of another first order theory T_I'. We then note that the consistency of T_I' is absolute for models of ZFC. Finally, the consistency of T_I' in the ground model allows us to construct the coloring g.

We present a few remarks concerning the problem left open in section 3.1. The method of proof used in section 3.4 to show ZFC proves $\mathcal{I}_{CV}(\aleph_1, \mathbf{Q}) = \mathcal{D}$ cannot be used to show that ZFC proves $\mathcal{I}_{CV}(\aleph_1, \omega) = \mathcal{C}$. Suppose that we try to mimic the methods used in section 3.4. As shown in section 3.1 we can force the existence of a model of ZFC in which $\mathcal{I}_{CV}(\aleph_1, \omega) = \mathcal{C}$. For each CV-identity I for which there exists $f : [\aleph_1]^2 \longrightarrow \omega$ such that $I \notin \mathcal{I}(f)$, with f in the generic extension, we can construct an (ω_1, ω)-model, code it as a consistent first order theory, and show that this theory is consistent in the ground model.

This is the procedure used in section 3.4. Unfortunately, the consistency of the theory does not allow us to construct, at least directly, a function $g : [\aleph_1]^2 \longrightarrow \omega$ that omits I. The most we are able to do is construct a function $g : [\aleph_1]^2 \longrightarrow B$ that omits I, where B is a countable set that is linearly ordered. One hope was that it would be possible to show that there exists a countable subset C of B of order type ω and an uncountable subset D of $\aleph_1$ such that $g(\{d_1, d_2\}) \in C$ for all $\{d_1, d_2\} \in [D]^2$. We conclude the section with a theorem that shows that this is not always possible. The theorem says that if CH holds then there exists a function $f : [\aleph_1]^2 \longrightarrow \omega + \omega$ such that for all $B \subseteq \aleph_1$, for all $S \subseteq \omega + \omega$

$$|B| = \aleph_1 \wedge \ \mathrm{ot}(S) = \omega \Rightarrow \exists \{\alpha, \beta\} \in [B]^2 (f(\{\alpha, \beta\}) \notin S).$$

We conclude this introduction by noting that the results for identities proved in lemma 2.3 are also true for CV-identities. We offer no proof as it is similar to the one already given in lemma 2.3.

Lemma 3.1 *For all* $k \geq 1$, $\mathcal{I}_{CV}(\aleph_{k+1}) = ID_{CV}$ *if and only if* $2^{\aleph_0} \leq \aleph_k$.

3.1 CV-identities at $\aleph_1$

The classification of the collection of identities $\mathcal{I}(\aleph_1)$ has been demonstrated in [15]. We produce a similar classification for CV-identities. In this section we will define a collection $\mathcal{C}$ of CV-identitiess and show that there is a model of ZFC in which $\mathcal{I}_{CV}(\aleph_1) = \mathcal{C}$. Towards this result we define $f_n : [\{0, 1, \ldots, n\}]^2 \longrightarrow \omega$ for $1 \leq n < \omega$ by $f_n(\{x, i\}) = i$ for all $0 \leq i < x \leq n$. Define $\mathcal{C} = \bigcup \{\mathcal{I}_{CV}(f_i) : i < \omega\}$.

Theorem 3.2 *There exists a model of ZFC in which* $\mathcal{I}_{CV}(\aleph_1) = \mathcal{C}$.

Proof: This theorem follows immediately from lemmas 3.3 and 3.4 to be proved in the next two subsections. □

3.1.1 Inducing CV-identitiess

Lemma 3.3 *Let* $f : [\aleph_1]^2 \longrightarrow \omega$. *Then* $\mathcal{I}_{CV}(f) \supseteq \mathcal{C}$.

Proof: By recursion on $n < \omega$ we define a decreasing sequence $\langle B_n : n < \omega \rangle$, of subsets of $\aleph_1$. To start choose $B_0 \subseteq \aleph_1 \setminus \{0\}$ of cardinality $\aleph_1$ such that $\lambda x f(\{0, x\})$ is constant on B_0. Given B_n of cardinality $\aleph_1$ choose $B_{n+1} \subseteq B_n \setminus \{\min(B_n)\}$ of cardinality $\aleph_1$ such that $\lambda x f(\{\min(B_n), x\})$ is constant on B_{n+1}. This completes the recursion. Clearly $f \restriction [\{\min(B_n : n < \omega\}]^2$ realizes every member of $\mathcal{C}$. □

3.1.2 Omitting CV-identities

Throughout this subsection we will let M be a model of ZFC. The main result of this section is the following lemma.

Lemma 3.4 *There exists a cardinal preserving forcing extension N of M and functions $f^a, f^b, f^l : [\aleph_1] \longrightarrow \omega$ in N such that for every $I \notin \mathcal{C}$, one of f^a, f^b, f^l does not realize I.*

We first establish some definitions and notation. We define 2 linear orders of $\omega_1 \times \omega_1$ called the *lexicographic* and the *antilexicographic* orders respectively by:

$$(m_1, n_1) <^l (m_2, n_2) \Leftrightarrow (m_1 < m_2 \vee (m_1 = m_2 \wedge n_1 < n_2))$$

and

$$(m_1, n_1) <^a (m_2, n_2) \Leftrightarrow (m_1, n_1) >^l (m_2, n_2).$$

Towards defining the partial orders that will be used in forcing we let

$$\mathbf{R} = \{g : g \text{ is a mapping from } [B]^2 \text{ into } \omega, |B| < \omega, B \subset \aleph_1\}.$$

For $f \in \mathbf{R}$ define $\text{fld}(f) = \{x, y \in \aleph_1 : \{x, y\} \in \text{dom}(f)\}$. We say that $f, g \in \mathbf{R}$ are *isomorphic*, denoted $f \simeq g$, if $|\text{fld}(f)| = |\text{fld}(g)|$ and $f(\{x, y\}) = g(\{k(x), k(y)\})$ for all $\{x, y\} \in \text{dom}(f)$, where k is the unique order preserving map from $\text{fld}(f)$ onto $\text{fld}(g)$. We say that the pair $\langle f, g \rangle$ is *proper* if $f \simeq g$ and $\text{fld}(f) \setminus \text{fld}(g) < \text{fld}(g) \setminus \text{fld}(f)$. (Here and in the remainder of the thesis, for sets of ordinals A and B, we write $A < B$ whenever $a \in A$ and $b \in B$ implies $a < b$.) For $F \in \{l, a\}$ an *F-amalgam* of f and g, where $\langle f, g \rangle$ is proper, is an extension $h \in \mathbf{R}$ of f and g such that $\text{fld}(h) = \text{fld}(f) \cup \text{fld}(g)$, $\text{rng}(f) \cup \text{rng}(g) < \text{rng}(h \setminus f \cup g)$, and

$$h(\{x_1, y_1\}) < h(\{x_2, y_2\}) \Leftrightarrow (x_1, y_1) <^F (x_2, y_2)$$

for all $x_1, x_2 \in \text{fld}(f) \setminus \text{fld}(g)$ and $y_1, y_2 \in \text{fld}(g) \setminus \text{fld}(f)$.

Definition 3.5 For $F \in \{l, a\}$ define the partial ordering $\mathbf{P}^F$ as follows. Let

$$\mathbf{P}^F_1 = \{f : f \text{ is a mapping from } [B]^2 \text{ into } \omega, |B| = 1\}.$$

Define

$$\mathbf{P}^F_{n+1} = \mathbf{P}^F_n \cup \{g \in \mathbf{R} : \exists f_1, f_2 \in \mathbf{P}^F_n \text{ such that } g \text{ is their F-amalgam }\}.$$

Let $\mathbf{P}^F = \bigcup\{\mathbf{P}^F_n : n < \omega\}$, with the ordering of inverse inclusion.

Lemma 3.6 *Let $F \in \{l, a\}$. Then $\mathbf{P}^F$ is c.c.c.*

Proof: Let $\mathcal{A} = \{f_\alpha : \alpha < \omega_1\} \subseteq \mathbf{P}^F$. For each $\alpha < \omega_1$ let B_α denote $\text{fld}(f_\alpha)$.Without loss of generality there exist $m, n < \omega$ such that $|B_\alpha| = n < \omega$ and $f_\alpha \in \mathbf{P}^F_m$ for all $\alpha < \omega_1$. We may also assume that $f_\alpha \simeq f_\beta$ for all $\alpha, \beta < \omega_1$. By a Δ-system argument we may assume that there exist $B \subset \omega_1$ and a collection $\{D_\alpha : \alpha < \omega_1\}$ such that

$$B_\alpha = B \cup D_\alpha \wedge B \cap D_\alpha = \emptyset \wedge [(x \in D_\alpha \wedge y \in D_\beta \Rightarrow x < y)]$$

whenever $\alpha < \beta < \omega_1$.

Choose distinct α, β, with $\alpha < \beta$. Clearly the F-amalgam of f_α and f_β is in $\mathbf{R}$: denote it by g. Since both f_α and f_β are elements of $\mathbf{P}^F_m$, g is an element of $\mathbf{P}^F_{m+1}$. Thus $\mathcal{A}$ is not an antichain. □

Let G^l be $\mathbf{P}^l$ generic over M and G^a be $\mathbf{P}^a$-generic over $M[G^l]$. Define N to be the model $M[G^l][G^a]$ and in N denote by g^a and g^l the functions $\bigcup G^a$ and $\bigcup G^l$ respectively.

Lemma 3.7 *The cardinalities of fld(g^a) and fld(g^l) are both $\aleph_1$.*

Proof: It is sufficient to show that $\{f \in \mathbf{P}^F : \text{fld}(f) \not\subseteq \alpha\}$ is dense in $\mathbf{P}^F$ for all $\alpha < \omega_1$. This is easy and we leave it to the reader. □

For $F \in \{a, l\}$ let h^F be the unique order-preserving bijection from $\aleph_1$ onto $\text{fld}(g^F)$, and define $f^F : [\aleph_1]^2 \longrightarrow \omega$ by $f^F(\{\alpha, \beta\}) = f^F(\{h^F(\alpha), h^F(\beta)\})$. Our next goal is to define $f^b : [\aleph_1]^2 \longrightarrow \omega$, the third coloring we need for the conclusion of lemma 3.4. For $\sigma, \tau \in {}^{<\omega}2$ we write $\sigma <_l \tau$ if

$$[(\sigma \cap \tau)^\frown\langle 0\rangle \subseteq \sigma \wedge (\sigma \cap \tau)^\frown\langle 1\rangle \subseteq \tau \wedge |\sigma| = |\tau|] \vee |\sigma| < |\tau|.$$

Let $h : {}^{<\omega}2 \longrightarrow \omega$ be the unique bijection satisfying $\sigma <_l \tau \Rightarrow h(\sigma) < h(\tau)$. Choose an injective map $g : \omega_1 \longrightarrow {}^{\omega}2$. Now $f^b : [\aleph_1]^2 \longrightarrow \omega$ is defined by $f^b(\{\eta, \nu\}) = h(g(\eta) \cap g(\nu))$.

We now outline the proof of lemma 3.4. Let I be a CV-identity which is not in $\mathcal{C}$. It will be shown that one of f^a, f^l, f^b does not realize I. To this end we name some particular CV-identities: $\{K_i : 1 \leq i \leq 10\}$, $\{J_i : 1 \leq i \leq 4\}$. Then, towards a contradiction, we suppose that I is realized by each of f^a, f^l, f^b. We first show that none of K_1 to K_{10} is realized in I. From this we deduce that I must realize one of J_1 to J_4. We finish by showing that f^b does not realize each of J_1 to J_4.

We name the ten CV-identities that are not elements of $\mathcal{C}$ having field size three. For each i, $1 \leq i \leq 10$ let $K_i = \langle u_i\ <_i, \preceq_i\rangle$ be such that $u_i = \{0, 1, 2\}$ and $<_i$ is the usual ordering. To complete the definition we define the order on the edge colors.

i) $K_1 : \{1,2\} \prec \{0,1\} \prec \{0,2\}$

ii) $K_2 : \{1,2\} \prec \{0,2\} \prec \{0,1\}$

iii) $K_3 : \{0,1\} \prec \{1,2\} \prec \{0,2\}$

iv) $K_4 : \{0,2\} \prec \{1,2\} \prec \{0,1\}$

v) $K_5 : \{0,2\} \prec \{0,1\} \sim \{1,2\}$

vi) $K_6 : \{0,1\} \sim \{1,2\} \prec \{0,2\}$

vii) $K_7 : \{1,2\} \prec \{0,1\} \sim \{0,2\}$

viii) $K_8 : \{0,2\} \sim \{1,2\} \sim \{0,1\}$

ix) $K_9 : \{0,2\} \sim \{1,2\} \prec \{0,1\}$

x) $K_{10} : \{0,1\} \prec \{1,2\} \sim \{0,2\}$.

As a notational convenience we use $\prec$ instead of $\prec_i$ and $\sim$ instead of $\sim_i$ since the context makes the intention clear. It should be noted that a CV-identity $K = \langle u, <, \preceq \rangle$ for which $u = \{0,1,2\}$ and $0 < 1 < 2$, is an element of C if and only if $\{x,y\} \prec \{1,2\}$ for all $\{x,y\} \in [u]^2 \setminus \{\{1,2\}\}$.

We now name four of the CV-identities that occur on four vertices. For each i, $1 \leq i \leq 4$ let $J_i = \langle u_i, <_i, \preceq_i \rangle$ be such that $u_i = \{0,1,2,3\}$ and $<_i$ is the usual order. To complete the definition we define the order on the edge colors:

i) $J_1 : \{0,2\} \prec \{0,1\} \prec \{1,2\} \prec \{0,3\} \prec \{1,3\} \prec \{2,3\}$

ii) $J_2 : \{0,1\} \prec \{0,2\} \prec \{1,2\} \prec \{0,3\} \prec \{1,3\} \prec \{2,3\}$

iii) $J_3 : \{0,3\} \prec \{0,1\} \prec \{1,3\} \prec \{0,2\} \prec \{1,2\} \prec \{2,3\}$

iv) $J_4 : \{0,1\} \prec \{0,3\} \prec \{1,3\} \prec \{0,2\} \prec \{1,2\} \prec \{2,3\}$.

We denote the collections $\{K_i : 1 \leq i \leq 10\}$ and $\{J_i : 1 \leq i \leq 4\}$ by $\mathcal{K}$ and $\mathcal{J}$ respectively.

Lemma 3.8 *In N, f^l does not realize any member of $\mathcal{K} \setminus \{K_1\}$ and f^a does not realize any member of $\mathcal{K} \setminus \{K_2, K_3\}$.*

Proof: We will show that f^l does not realize K_2 and that f^a does not realize K_1. The remaining arguments are very similar and are left to the reader.

Lemma 3.9 *f^l does not realize K_2.*

Proof: It is enough to show that no $p = \langle u^p, c^p \rangle \in \mathbf{P}^l$ realizes K_2. Suppose otherwise. Choose m minimal such that there exists $p \in \mathbf{P}^l_m$ for which there exists an order-preserving map $g : \{0,1,2\} \longrightarrow u^p$ showing that K_2 is realized. Let $p_0 = \langle u^0, c^0 \rangle$ and $p_1 = \langle u^1, c^1 \rangle$ be such that p is their amalgam. Since m is minimal $\text{rng}(g) \cap u^0 \setminus u^1$ and $\text{rng}(g) \cap u^1 \setminus u^0$ are non-empty.

Suppose $|\text{rng}(g) \cap u^0 \Delta u^1| = 2$. Then $c(\{g(1), g(2)\}) < c(\{g(0), g(1)\})$ because $\{1,2\} \prec_2 \{0,1\}$. Now note that $g(0) \in u^0 \cap u^1$, $g(1) \in u^0 \setminus u^1$, and $g(2) \in u^1 \setminus u^0$ so that by the definition of amalgamation we have $c(\{g(1), g(2)\}) > c(\{g(0), g(1)\})$, contradiction.

Hence, $|\text{rng}(g) \cap u^0 \Delta u^1| = 3$. There are two cases. The first is $\{g(0), g(1)\} \subseteq u^0 \setminus u^1$ and $g(2) \in u^1 \setminus u^0$. The second occurs when $g(0) \in u^0 \setminus u^1$ and $\{g(1), g(2)\} \subseteq u^1 \setminus u^0$. In the first case $c(\{g(1), g(2)\}) < c(\{g(0), g(1)\})$ since $\{1,2\} \prec_2 \{0,1\}$. This contradicts the definition of amalgamation as new edges get colors larger than all colors previously used. In the second case $c(\{g(0), g(2)\}) < c(\{g(0), g(1)\})$ because $\{0,2\} \prec_2 \{0,1\}$. This contradicts the fact that the lexicographic order is used when coloring new edges. □

Lemma 3.10 *f^a does not realize K_1.*

Proof: It is enough to show that no $p = \langle u^p, c^p \rangle \in \mathbf{P}^a$ realizes K_1. Suppose otherwise. Choose m minimal such that there exists $p \in \mathbf{P}^a_m$ for which there exists an order-preserving map $g : \{0,1,2\} \longrightarrow u^p$ verifying that K_1 is realized. Let $p_0 = \langle u^0, c^0 \rangle$ and $p_1 = \langle u^1, c^1 \rangle$ be such that p is their amalgam. Since m is minimal $\text{rng}(g) \cap u^0 \setminus u^1$ and $\text{rng}(g) \cap u^1 \setminus u^0$ are non-empty.

Suppose $|\text{rng}(g) \cap u^0 \Delta u^1| = 2$. Then $c(\{g(1), g(2)\}) < c(\{g(0), g(2)\})$ because $\{1,2\} \prec_1 \{0,2\}$. Now note that $g(0) \in u^0 \cap u^1$, $g(1) \in u^0 \setminus u^1$, and $g(2) \in u^1 \setminus u^0$ so that by the definition of amalgamation we have $c(\{g(1), g(2)\}) > c(\{g(0), g(2)\})$, contradiction. Hence $|\text{rng}(g) \cap u^0 \Delta u^1| = 3$. There are two cases. The first is $\{g(0), g(1)\} \subseteq u^0 \setminus u^1$ and $g(2) \in u^1 \setminus u^0$. The second occurs when $g(0) \in u^0 \setminus u^1$ and $\{g(1), g(2)\} \subseteq u^1 \setminus u^0$. In the first case $c(\{g(1), g(2)\}) < c(\{g(0), g(1)\})$ as $\{1,2\} \prec_1 \{0,1\}$. This contradicts the definition of amalgamation since new edges get colors larger than all colors previously used. In the second case $c(\{g(0), g(1)\}) < c(\{g(0), g(2)\})$ because $\{0,1\} \prec_1 \{0,2\}$. This contradicts the fact that the anti-lexicographic order is used when coloring new edges. □

This completes the proof of lemma 3.8. □

Lemma 3.11 *Let $K = \langle u, <, \preceq \rangle$ be a CV-identity in which no member of $\mathcal{K}$ is realized such that $K \notin \mathcal{C}$ and $|u| \geq 4$. Then some member of $\mathcal{J}$ is realized in K.*

Proof: We first note that since $K \notin C$ there must exist $\{a,b,c,d\} \subset u$ such that $a < b,c,d$, $b < d$, and $\{a,c\} \succeq \{b,d\}$. The ordering of the set $\{a,b,c,d\}$ determines the cases we must examine.

Case 1. $a < c < b < d$. $K \upharpoonright \{c,b,d\} \notin \mathcal{K} \Rightarrow \{c,b\} \prec \{b,d\}$. Thus $\{a,c\} \succeq \{b,d\} \succ \{c,b\}$. This implies that $K \upharpoonright \{a,b,c\} \in \mathcal{K}$, contradiction. Thus the first case is vacuous.

Case 2. $a < b < d < c$. Since $K \upharpoonright \{b,d,c\} \notin \mathcal{K}$, $\{b,c\} \prec \{d,c\}$. Similarly, restricting K to the set $\{a,b,d\}$ we get $\{a,d\}, \{a,b\} \prec \{b,d\}$. Restricting to $\{a,b,c\}$ gives $\{a,c\} \prec \{b,c\}$.

One of the following orders on edge colors must occur:

i) $\{a,d\} \prec \{a,b\} \prec \{b,d\} \preceq \{a,c\} \prec \{b,c\} \prec \{d,c\}$

ii) $\{a,b\} \prec \{a,d\} \prec \{b,d\} \preceq \{a,c\} \prec \{b,c\} \prec \{d,c\}$.

iii) $\{a,b\} \sim \{a,d\} \prec \{b,d\} \preceq \{a,c\} \prec \{b,c\} \prec \{d,c\}$.

Each of these has J_1 or J_2 as subidentity.

Case 3. $a < b < c < d$.

Since $K \upharpoonright \{a,b,c\} \notin \mathcal{K}$, $\{a,c\} \prec \{b,c\}$. Similarly, restricting K to the set $\{a,b,d\}$ we get $\{a,d\}, \{a,b\} \prec \{b,d\}$. Restricting K to $\{b,c,d\}$ gives $\{b,c\} \prec \{c,d\}$. Thus one of the following orders must occur:

i) $\{a,d\} \prec \{a,b\} \prec \{b,d\} \preceq \{a,c\} \prec \{b,c\} \prec \{d,c\}$

ii) $\{a,b\} \prec \{a,d\} \prec \{b,d\} \preceq \{a,c\} \prec \{b,c\} \prec \{d,c\}$.

iii) $\{a,b\} \sim \{a,d\} \prec \{b,d\} \preceq \{a,c\} \prec \{b,c\} \prec \{d,c\}$.

Each of these has J_3 or J_4 as a subidentity. □

We establish some notation and definitions. Let $I = \langle u_I, <_I, \preceq_I \rangle$ and $J = \langle u_J, <_J, \preceq_J \rangle$ be CV-identities. We say that I is a *reordering* of J if $u_I = u_J$ and $\preceq_I = \preceq_J$. Let $\mathcal{L}$ be the collection of CV-identities, $\{L_1, L_2\}$, where, for $i = 1,2$, $L_i = \langle u_i, <_i, \preceq_i \rangle, u_i = \{\alpha, \beta, \gamma, \delta\}$ and $\alpha <_i \beta <_i \gamma <_i \delta$. We complete the definition by defining the order on the edge colors. To simplify notation we use $\prec$ instead of $\prec_i$ as the context makes the usage clear. The colors are ordered according to:

i) $L_1 : \{\alpha,\beta\} \sim \{\alpha,\gamma\} \sim \{\alpha,\delta\} \prec \{\beta,\gamma\} \sim \{\beta,\delta\} \prec \{\gamma,\delta\}$.

ii) $L_2 : \{\alpha,\gamma\} \sim \{\alpha,\delta\} \sim \{\beta,\gamma\} \sim \{\beta,\delta\} \prec \{\alpha,\beta\} \prec \{\gamma,\delta\}$.

Lemma 3.12 *Let K be an CV-identity of size four. If K is an element of $\mathcal{I}_{CV}(f^b)$ then K is realized in a reordering of L_1 or a reordering of L_2.*

Proof: Let $\{a_i : 1 \le i \le 4\} \subset \aleph_1$ and K be the identity realized by $f^b \restriction [\{a_i : 1 \le i \le 4\}]^2$. We will show that K is a reordering of L_1 or of L_2. Recall the function g used in the definition of f^b. Define G to be the set $\{g(a_i) \cap g(a_j) : 1 \le i,j \le 4\}$. Now there exists a $\subsetneq$-preserving map from G onto the binary tree whose set of leaves is $\{\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1\rangle\}$ or the complete binary tree ${}^{\le 2}2$. It should be observed that the color assigned to $\{a_i, a_j\}$ by f^b is primarily determined by the length of $g(a_i) \cap g(a_j)$. Thus in the first case K is a reordering of L_1. In the second case K is a reordering of L_2. □

Lemma 3.13 *No member of $\mathcal{J}$ is realized by a reordering of a member of $\mathcal{L}$.*

Proof: The following two propositions clearly suffice.

Proposition 3.14 *No J in $\mathcal{J}$ is realized by a reordering of L_1.*

Proof: Towards a contradiction suppose that there exists i, $1 \le i \le 4$ and a linear ordering $<^*$ of $\{\alpha,\beta,\gamma,\delta\}$ such that $K = \langle\{\alpha,\beta,\gamma,\delta\}, <^*, \preceq_1\rangle$ is a reordering of L_1 which realizes J_i. Since, in J_i, $\{2,3\}$ is the unique edge which is assigned the greatest color, γ and δ are the two largest elements with respect to $<^*$. Within these constraints we have four orderings that $<^*$ can impose on the set $\{\alpha,\beta,\gamma,\delta\}$. Since the permutation (γ,δ) is an automorphism of $\langle\{\alpha,\beta,\gamma,\delta\},\preceq_1\rangle$ it is sufficient to consider the orderings

i) $\alpha <^* \beta <^* \gamma <^* \delta$

ii) $\beta <^* \alpha <^* \gamma <^* \delta$

Case 1. In this case, $\{\beta,\gamma\} \succ \{\alpha,\delta\}$ in L_1, yet $\{1,2\} \prec \{0,3\}$ in J_1 and J_2. Also $\{\alpha,\gamma\} \prec \{\beta,\delta\}$ in L_1, yet $\{0,2\} \succ \{1,3\}$ in J_3 and J_4.

Case 2. In this case, $\{\alpha,\gamma\} \prec \{\beta,\gamma\}$ in L_1 but $\{1,2\} \succ \{0,2\}$ in J_i.

Thus no reordering of L_1 realizes any of the CV-identities in $\mathcal{J}$. $\square$

Proposition 3.15 *No J in $\mathcal{J}$ is realized by a reordering of L_2.*

Proof: Towards a contradiction suppose that there exists i, $1 \leq i \leq 4$ and a linear ordering $<^*$ of $\{\alpha,\beta,\gamma,\delta\}$ such that $K = \langle\{\alpha,\beta,\gamma,\delta\}, <^*, \preceq_2\rangle$ is a reordering of L_2 which realizes J_i. Since, in J_i, $\{2,3\}$ is the unique edge which is assigned the greatest color, γ and δ are the largest elements with respect to $<^*$. Thus α,β are the smallest. In K the edge between the second and third elements in the $<^*$ ordering is one of $\{\alpha,\gamma\}$, $\{\alpha,\delta\}$, $\{\beta,\gamma\}$, or $\{\beta,\delta\}$. Since the color assigned to all these edges is less than the color assigned to $\{\alpha,\beta\}$, the color assigned to the edge between the second and third elements in the $<^*$-order is less than the valued assigned to the edge between the first and second elements in the $<^*$-ordering. By inspection, in J_i, the color of the edge between the second and third elements is greater than that of the edge between the first and second, contradiction. $\square$

This completes the proof of lemma 3.13. $\square$

Lemma 3.16 *No member of $\mathcal{J}$ is realized by f^b.*

Proof: Towards a contradiction let $J \in \mathcal{J}$ be realized by f^b. Now J has size four and thus by lemma 3.12 it is realized in a reordering of L_1 or of L_2. This contradicts lemma 3.13. $\square$

We now prove lemma 3.4. It is easy to see that all CV-identities that are not elements of $\mathcal{C}$ and have field size three are elements of $\mathcal{K}$. By lemma 3.8, in $N = M[G^a][G^l]$, each member of $\mathcal{K}$ is not realized by some coloring of $[\aleph_1]^2$. Thus the inclusion $\mathcal{I}_{CV}(\aleph_1) \subseteq \mathcal{C}$ is valid for CV-identities of field size three. Let $I \notin \mathcal{C}$ have field size greater than three. By lemma 3.11 one of the CV-identities in $\mathcal{J}$ is realized in I. But lemma 3.16 shows that each CV-identity in $\mathcal{J}$ is not realized by f^b. So f^b does not realize I. This completes the proof of lemma 3.4.

3.2 CV-identities at $\aleph_2$

Two particular CV-identities, called R_1 and R_2, play a role in this section. We will consider two models of ZFC. In one $R_1 \in \mathcal{I}_{CV}(\aleph_2)$ (theorem 3.22) and in the other $R_1 \notin \mathcal{I}_{CV}(\aleph_2)$ (theorem 3.24). We demonstrate that in all models of ZFC, $R_2 \in \mathcal{I}_{CV}(\aleph_2)$ (lemma 3.25) and $R_1 \in \mathcal{I}_{CV}(\aleph_3)$ (lemma 3.26). This provides a proof of the consistency of $\mathcal{I}_{CV}(\aleph_2) \neq \mathcal{I}_{CV}(\aleph_3)$ (theorem 3.27). The CV-identities R_1 and R_2 have the same underlying identity, called I_1, and ZFC $\vdash I_1 \in \mathcal{I}(\aleph_2)$. This answers in the negative, the question as to whether all reorderings of a CV-identities in $\mathcal{I}_{CV}(\aleph_2)$, whose underlying identity is in $\mathcal{I}(\aleph_2)$, are themselves elements of $\mathcal{I}_{CV}(\aleph_2)$. The forcing notion used in this section provides a new way of finding c.c.c. forcing extensions of models of ZFC in which there is no $\aleph_2$-saturated, $\aleph_1$-complete ideal on $\aleph_1$. A previous construction of such forcing extensions is found in [18].

We define $R_i = \langle u_i, <_i, \preceq_i \rangle$ by setting $u_i = \{0,1,2,3\}$ and $<_i$ to be the usual ordering. The order of the edge colors is given by:

i) $R_1 : \{0,2\} \sim \{0,3\} \sim \{1,2\} \sim \{1,3\} \prec \{2,3\} \prec \{0,1\}$

ii) $R_2 : \{0,2\} \sim \{0,3\} \sim \{1,2\} \sim \{1,3\} \prec \{0,1\} \prec \{2,3\}$.

We need the function $\rho : [\omega_2]^2 \longrightarrow \omega_1$ defined in [17] by Todorcevic. Fix a model M of ZFC in which there exists a $\square_{\omega_1}$ sequence $\langle C_\alpha : \alpha < \omega_2 \wedge \alpha \text{ a limit ordinal } \rangle$. Extend the definition of C_γ for $\gamma < \omega_2$ by setting $C_{\alpha+1} = \{\alpha\}$. Define $\rho : [\omega_2]^2 \longrightarrow \omega_1$ by

$$\rho(\{\alpha,\beta\}) = \sup\{\rho(\alpha, \min(C_\beta \setminus \alpha)), \mathrm{ot}(C_\beta \cap \alpha), \rho(\psi,\alpha) : \psi \in C_\beta \cap \alpha\}.$$

Following the notation of [17] and [18] we write $\rho(\alpha,\beta)$ for $\rho(\{\alpha,\beta\})$.

The following are properties of ρ, see [17].

i) $\rho(-,\alpha) : \alpha \longrightarrow \omega_1$ is countable to one,

ii) If $\alpha < \beta < \gamma < \omega_2$ then

(a) $\rho(\alpha,\beta) \leq \max\{\rho(\alpha,\gamma), \rho(\beta,\gamma)\}$

(b) $\rho(\alpha,\gamma) \leq \max\{\rho(\alpha,\beta), \rho(\beta,\gamma)\}$

We quote the following lemma from [18].

Lemma 3.17 *Let an uncountable family F of finite subsets of ω_2 and an ordinal $\mu < \omega_1$ be given. Then there are distinct $x, y \in F$ such that for every $\alpha \in x \setminus y$, $\beta \in y \setminus x$, $\gamma \in x \cap y$*

$$\rho(\alpha,\beta) \geq \ max\{\mu, min\{\rho(\alpha,\gamma), \rho(\beta,\gamma)\}\}.$$

For $b \in \aleph_2$, $\bar{a} \in [\aleph_2]^{<\omega}$ we define $bR\bar{a}$ to hold if there exists $a_1, a_2 \in \bar{a}$ with $a_1 < a_2$ such that $\rho(b, a_2) \leq \rho(a_1, a_2)$.

We note that for a fixed $\bar{a} \in [\aleph_2]^{<\omega}$ there are only countably many c such that $cR\bar{a}$. We let $\mathbf{R}$ denote $\{\langle u, c\rangle : u \in [\aleph_2]^{<\omega}, c : [u]^2 \longrightarrow \omega\}$. We say that $p = \langle u, c\rangle \in \mathbf{R}$ is the *amalgam* of $p^0 = \langle u^0, c^0\rangle$ and $p^1 = \langle u^1, c^1\rangle \in \mathbf{R}$ if there exist $h < \omega$ and increasing sequences $i^0_0, \ldots, i^0_h$ and $i^1_0, \ldots, i^1_h$ in $\aleph_2$ such that for all s, t, $0 \leq s < t \leq h$ and all $i, j, k, l < \omega_2$

i) $u^0 = \{i^0_0, \ldots, i^0_h\}$ and $u^1 = \{i^1_0, \ldots, i^1_h\}$

ii) $c^0(\{i^0_s, i^0_t\}) = c^1(\{i^1_s, i^1_t\})$

iii) $i^0_t = i^1_t \vee i^0_t < i^1_t$

iv) $u = u^0 \cup u^1$

v) $c \supset (c^0 \cup c^1)$

vi) $(\{i,j\} \notin [u^0]^2 \cup [u^1]^2) \Rightarrow c(\{i,j\}) > \mathrm{rng}(c^0) \cup \mathrm{rng}(c^1)$

vii) $c(\{i,j\}) = c(\{k,l\}) \Rightarrow (\{i,j\} = \{k,l\} \vee \{i,j\}, \{k,l\} \in [u^0]^2 \cup [u^1]^2)$

viii) $i \in u^0$, $j \in u^1$, $k \in u^0 \cap u^1$ and $k < i < j$ imply $\rho(i,j) \geq \ \max\{\rho(k,i), \rho(k,j)\}$

ix) $i^0_t \neq i^1_t \Longrightarrow \neg i^1_t R u^0$.

Now recall the notions of *one-point extension* and *history* as given in the previous chapter. Also recall the natural indexing of elements of a history

Definition 3.18 We define a sequence of subsets of $\mathbf{R}$. Let $\mathbf{P}_0 = \{\langle u, c\rangle \in \mathbf{R} : |u| = 1\}$. Given $\mathbf{P}_n$, let $\mathbf{P}_{n+1}$ be the subset of $\mathbf{R}$ which contains $\mathbf{P}_n$, all amalgams of pairs of elements of $\mathbf{P}_n$, and all one-point extensions of elements of $\mathbf{P}_n$. Let $\mathbf{P} = \bigcup\{\mathbf{P}_n : n < \omega\}$. For $p = \langle u^p, c^p\rangle$ and $q = \langle u^q, c^q\rangle$ let $p \leq q$ mean that $u^p \supseteq u^q$ and $c^p \supseteq c^q$.

Lemma 3.19 *P is c.c.c*

Proof: Let $\langle p_\alpha : \alpha < \omega_1\rangle$ be a sequence of conditions. By thinning we can suppose that there are $n, l < \omega$ and $i^\alpha_j\,(\alpha < \omega_1, 0 \le j \le n)$ such that for all $\alpha, \beta < \omega_1$ and all j, k, $(0 \le j < k \le n)$

i) $u^{p_\alpha} = \{i^\alpha_0, \ldots, i^\alpha_n\}$

ii) $i^\alpha_j < i^\alpha_k$

iii) $c^{p_\alpha}(\{i^\alpha_j, i^\alpha_k\}) = c^{p_\beta}(\{i^\beta_j, i^\beta_k\})$

iv) $p_\alpha \in \mathsf{P}_l$.

Applying the Δ-system argument allows us to thin the sequence of conditions further so that

$$\forall\alpha\forall\beta(i^\alpha_t = i^\beta_t) \vee (\forall\beta < \omega_1)(\forall\alpha < \beta)(i^\alpha_t < i^\beta_t) \quad (0 \le t \le n).$$

Since $\rho(-, \alpha)$ is countable to one we can inductively define a sequence $\langle \alpha_\beta : \beta < \omega_1\rangle$ such that $\beta < \gamma < \omega_1 \wedge b \in u^{p_{\alpha_\gamma}} \setminus u^{p_{\alpha_\beta}}$ implies

$$\neg\exists a_1\exists a_2(a_1, a_2 \in u^{p_{\alpha_\beta}} \wedge \rho(b, a_2) \le \rho(a_1, a_2)).$$

By Velickovic's lemma 3.17 we may choose $\gamma, \delta \in \{\alpha_\beta : \beta < \omega_1\}$ such that for all $i, j \in u^{p_\gamma} \cup u^{p_\delta}$,

$$i \in u^{p_\gamma},\ j \in u^{p_\delta}, \text{ and } k \in u^{p_\gamma} \cap u^{p_\delta} \text{ implies } \rho(i,j) \ge \min\{\rho(k,i), \rho(k,j)\}.$$

Using the quoted properties of ρ we conclude that, if $i \in u^{p_\gamma}$, $j \in u^{p_\delta}$, $k \in u^{p_\gamma} \cap u^{p_\delta}$, and $k < i < j$, then

$$\rho(i,j) \ge \max\{\rho(k,i), \rho(k,j)\}.$$

So the conditions p_γ and p_δ have a common extension in P. □

Lemma 3.20 *Let $p \in \mathsf{P}$. Then p does not realize R_1.*

Proof: Towards a contradiction suppose the lemma fails. Let m be minimal such that there exists $p \in \mathbf{P}_m$ realizing R_1. Let $H = \langle p_\sigma : \sigma \in 2^{\leq m} \rangle$ be a history of p. Let $g : \{0,1,2,3\} \longrightarrow u^p$ be the order preserving map witnessing the realization. Since m is minimal $\mathrm{rng}(g) \cap u^{(0)} \setminus u^{(1)}$ and $\mathrm{rng}(g) \cap u^{(1)} \setminus u^{(0)}$ are nonempty. Note that in R_1 each edge of the cycle (02130) has the same color. Thus for any two consecutive elements x, y, of the sequence $\langle g(0), g(2), g(1), g(3), g(0) \rangle$, either $x, y \in u^{(0)}$ or $x, y \in u^{(1)}$. On the other hand, from the minimality of m, $\mathrm{rng}(g)$ intersects both $u^{(0)} \setminus u^{(1)}$ and $u^{(1)} \setminus u^{(0)}$. It follows that either $g(0), g(1) \in u^{(0)} \cap u^{(1)}$ or $g(2), g(3) \in u^{(0)} \cap u^{(1)}$. The former contradicts $\{0,1\}$ being the unique edge of R_1 with the maximum color. Thus $g(2), g(3) \in u^{(0)} \cap u^{(1)}$ and $g(0) \in u^{(0)} \setminus u^{(1)}$, $g(1) \in u^{(1)} \setminus u^{(0)}$, or vice-versa. Without loss of generality the first possibility occurs.

We conclude from the above that $\{g(2), g(3)\} \subseteq u^{(0)} \cap u^{(1)}$. Towards a final contradiction note that $\rho(g(1), g(3)) > \rho(g(2), g(3))$ since $\neg g(1) R \{g(2), g(3)\}$. Let R^* denote the CV-identitiy $R_1 \restriction \{1,2,3\}$. Choose $\sigma \in 2^{\leq m}$ such that $|\sigma|$ is maximal subject to $\{g(1), g(2), g(3)\} \subseteq u^{\sigma}$. This implies $|u^{\sigma^\frown(0)} \setminus u^{\sigma^\frown(1)} \cap \{g(1), g(2), g(3)\}| \geq 1$ and $|u^{\sigma^\frown(1)} \setminus u^{\sigma^\frown(0)} \cap \{g(1), g(2), g(3)\}| \geq 1$. Now note that the color of the edge $\{g(2), g(3)\}$ is maximal in the collection of colors occuring on the edge set $[\{g(1), g(2), g(3)]^2$. Thus $g(2) \in u^{\sigma^\frown(0)} \setminus u^{\sigma^\frown(1)}$ and $g(3) \in u^{\sigma^\frown(1)} \setminus u^{\sigma^\frown(0)}$, or vice-versa. If $g(1) \notin u^{\sigma^\frown(0)} \cap u^{\sigma^\frown(1)}$ some new edge receives a color previously used, a contradiction to the definition of amalgamation. Using property viii) of the definition of amalgamation we conclude that $\rho(g(1), g(3)) \leq \rho(g(2), g(3))$, a contradiction to a previous note. ☐

Let M be a model of ZFC and G be $\mathbf{P}$-generic over M. Define $g = \bigcup\{c : \exists u(\langle u, c \rangle \in G)\}$.

Lemma 3.21 *In $M[G]$, $|\mathit{fld}(g)| = \aleph_2$.*

Proof: The set $\{\langle u, c \rangle \in \mathbf{P} : u \not\subset \alpha\}$ is dense in $\mathbf{P}$ for all $\alpha < \omega_2$ since $\mathbf{P}$ is closed under one-point extensions. So $\mathrm{fld}(g)$ is cofinal in $(\aleph_2)^M$. But, since $\mathbf{P}$ is c.c.c. $(\aleph_2)^M = (\aleph_2)^{M[G]}$. Hence the conclusion. ☐

Theorem 3.22 *Let M be a model of ZFC and G be $\mathbf{P}$-generic. In $M[G]$ there is a function $f : [\aleph_2]^2 \longrightarrow \omega$ such that $R_1 \notin \mathcal{I}_{CV}(f)$.*

Proof: In $M[G]$, $|\text{fld}(g)| = \aleph_2$ so there exists a unique order-preserving bijection $h : \aleph_2 \longrightarrow \text{fld}(g)$. Define $f : [\aleph_2]^2 \longrightarrow \omega$ by $f(\{\alpha, \beta\}) = g(\{h(\alpha), h(\beta)\})$. It is clear from lemma 3.20 that $R_1 \notin \mathcal{I}_{CV}(f)$. □

We are now going to show that there exists a model of ZFC in which $R_1 \in \mathcal{I}_{CV}(\aleph_2)$ and $\mathcal{I}_{CV}(\aleph_2) \neq \text{ID}_{CV}$. The model will be one in which there exists an $\aleph_2$-saturated, $\aleph_1$-complete ideal on $\aleph_1$. It has been shown that the existence of such ideals is consistent relative to the consistency of the existence of certain large cardinals, see [9]. It is shown in [1] and in [10] that when one adds Cohen reals to a model of ZFC the existence of an $\aleph_2$-saturated, $\aleph_1$-complete ideal on $\aleph_1$ is preserved. Since CH fails in this model we may apply lemma 3.1 to conclude $\mathcal{I}_{CV}(\aleph_2) \neq \text{ID}_{CV}$.

Lemma 3.23 *Let $\kappa > \omega$ be a cardinal such that $\kappa \longrightarrow (\kappa, \omega)^2$, $f : [\kappa]^2 \longrightarrow \omega$, and $n < \omega$. Then either $\mathcal{I}_{CV}(f) = ID_{CV}$ or there exists $D \subseteq \kappa$ such that $|D| = \kappa$ and $f(\{\alpha, \beta\}) > n$ for all $\{\alpha, \beta\} \in [D]^2$.*

Proof: Define $g : [\kappa]^2 \longrightarrow \{0, 1\}$ by $g(\{\alpha, \beta\}) = 0$ if and only if $f(\{\alpha, \beta\}) > n$. Now use the fact $\kappa \longrightarrow (\kappa, \omega)^2$. If there is a homogeneous set of size κ in color 0 we are done. Thus we may assume that there is $D \subset \kappa$ of order type ω such that $f(\{\alpha, \beta\}) \leq n$ for all $\{\alpha, \beta\} \in [D]^2$. Now apply $\omega \longrightarrow (\omega)^2_{n+1}$ to find $E \subseteq D$ and $k \leq n$ such that D has order type ω and $f(\{\alpha, \beta\}) = k$ for all $\{\alpha, \beta\} \in [E]^2$. Clearly $f \restriction [E]^2$ realizes all CV-identities. □

Theorem 3.24 *If there exists an $\aleph_1$-complete, $\aleph_2$-saturated ideal $\mathfrak{J}$ on $\aleph_1$ then, $R_1 \in \mathcal{I}_{CV}(\aleph_2)$.*

Proof: Let $f : [\aleph_2]^2 \longrightarrow \omega$. For $\alpha \geq \aleph_1$ choose $C_\alpha \subseteq \aleph_1$ and $d_\alpha < \omega$ so that $C_\alpha \in \mathfrak{J}^+$ and $f(\{x, \alpha\}) = d_\alpha$ for all $x \in C_\alpha$. Choose $D \subseteq \aleph_2 \setminus \aleph_1$ and $c_1 < \omega$ so that $|D| = \aleph_2$ and $d_\alpha = c_1$ for all $\alpha \in D$. Applying the previous lemma we may assume that there exists $D_1 \subseteq D$ such that $|D_1| = \aleph_2$ and $f(\{\alpha,\beta\}) > c_1$ for all $\{\alpha,\beta\} \in [D_1]^2$. By $\aleph_2$-saturation there exists $\{\alpha,\beta\} \in [D_1]^2$ such that $C_\alpha \cap C_\beta \in \mathfrak{J}^+$. Let $c_2 = f(\{\alpha,\beta\})$ and $C = C_\alpha \cap C_\beta$. Note that $c_1 < c_2$ and $|C| = \aleph_1$. We again apply the previous lemma to find distinct $a, b \in C$ and $c_3 < \omega$ such that $f(\{a,b\}) = c_3$ and $c_2 < c_3$.

It is then easy to verify that $f(\{a,b\}) = c_3$, $f(\{\alpha,\beta\}) = c_2$, and $f(\{x,y\} = c_1$ for $x \in \{\alpha,\beta\}$ and $y \in \{a,b\}$. This together with the fact that $a < b < \alpha < \beta$ implies that f realizes R_1 on $\{a,b,\alpha,\beta\}$. □

Lemma 3.25 $R_2 \in \mathcal{I}_{CV}(\aleph_2)$.

Proof: Let $f : [\aleph_2]^2 \longrightarrow \omega$. For $\alpha \in \aleph_2 \setminus \aleph_1$ choose $C_\alpha \subseteq \aleph_1$ and $d_\alpha < \omega$ such that $|C_\alpha| = \aleph_1$ and $f(\{x,\alpha\}) = d_\alpha$ for all $x \in C_\alpha$. Let $D \subseteq \aleph_2 \setminus \aleph_1$ and $c_1 < \omega$ be chosen so that $|D| = \aleph_2$ and $d_\alpha = c_1$ for all $\alpha \in D$. By lemma 3.23 we can choose, for $\alpha \in D$, $d_{\alpha,1}, d_{\alpha,2} \in C_\alpha$ and $e_\alpha < \omega$ such that $f(\{d_{\alpha,1}, d_{\alpha,2}\}) = e_\alpha > c_1$ By the pigeon-hole principle there must exist $E \subseteq D$, $c_2 < \omega$, and $a, b \in \aleph_1$ such that $|E| = \aleph_2$ and for all $\alpha \in E$, $d_{\alpha,1} = a$, $d_{\alpha,2} = b$ and, $c_2 = e_\alpha$. We may assume that $a < b$. Again by lemma 3.23 we may choose two elements $c < d \in E$ such that $f(\{c,d\}) = c_3 > c_2$. As $c_1 < c_2 < c_3$ and $a < b < c < d$ it is clear that f realizes R_2. □

We now remark that it is not true that if I and J are CV-identities such that I is a reordering of J and $I \in \mathcal{I}_{CV}(\aleph_2)$ then $J \in \mathcal{I}_{CV}(\aleph_2)$. The identities R_1 and R_2 provide an immediate counterexample. We also note that the identity underlying both R_1 and R_2 is I_1 which is an element of $\mathcal{I}(\aleph_2)$, see theorem 2.22.

Lemma 3.26 $R_1 \in \mathcal{I}_{CV}(\aleph_3)$.

Proof: Let $f : [\aleph_3]^2 \longrightarrow \omega$. For $\alpha \in \aleph_2$ choose $C_\alpha \subseteq [\aleph_2, \aleph_2 + \aleph_1)$, $d_\alpha < \omega$ such that $|C_\alpha| = \aleph_1$ and $f(\{x,\alpha\}) = d_\alpha$ for all $x \in C_\alpha$. Let $D \subseteq \aleph_2$, $c_1 < \omega$ be chosen so that $|D| = \aleph_2$ and $d_\alpha = c_1$ for all $\alpha \in D$. We now follow the construction process in the previous theorem, producing three colors, c_1, c_2, and c_3 and four elements $a, b, c, d \in \aleph_2$ such that $c_1 < c_2 < c_3 < \omega$ and $c < d < a < b$. It is clear that f realizes R_1. □

Theorem 3.27 *There is a model of ZFC in which $\mathcal{I}_{CV}(\aleph_2) \neq \mathcal{I}_{CV}(\aleph_3)$.*

Proof: This is immediate from theorem 3.22 and lemma 3.26 □

Theorem 3.28 *Let M be a model of ZFC such that $M \models \Box_{\omega_1}$. Let $\mathbf{P} \in M$ be the partial order defined in this section. Then for all $\mathbf{P}$-generic G,*

$$M[G] \models \not\exists \mathcal{J}(\mathcal{J} \text{ is an } \aleph_2\text{-saturated }, \aleph_1\text{-complete ideal on } \aleph_1).$$

Proof: Theorem 3.22 shows that $M[G] \models R_1 \notin \mathcal{I}_{CV}(\aleph_2)$ which contradicts theorem 3.24 if there exists an $\aleph_2$-saturated ideal on $\aleph_1$. □

3.3 CV-identities at $\aleph_\omega$

We start with some definitions. Let $n < \omega$. Denote by $<_l$ the ordering of ${}^n2$ where $\alpha <_l \beta$ if and only if $\alpha \supseteq (\alpha \cap \beta)^\frown\langle 0\rangle$. Denote by k_n, the unique one-to-one function from 2^n (the ordinal) to ${}^n2$ such that for all $0 \leq i, j < 2^n, i < j$ if and only if $k_n(i) <_l k_n(j)$. For $f \in {}^{({}^{<n}2)}2$ let $g_f : {}^{<n}2 \longrightarrow \omega$ be the unique function such that

i) $\text{rng}(g_f)$ is an initial segment of ω

ii) $\forall \eta, \gamma, \delta \in {}^{<n}2\,((\eta^\frown\langle 0\rangle \subseteq \gamma \wedge \eta^\frown\langle 1\rangle \subseteq \delta) \Rightarrow (g_f(\gamma) < g_f(\delta) \Leftrightarrow f(\eta) = 1))$

iii) $\gamma \subseteq \delta \Rightarrow g_f(\gamma) < g_f(\delta)$.

For $f \in {}^{({}^{<n}2)}2$ define $h_f : [2^n]^2 \longrightarrow \omega$ by $h_f(\{i, j\}) = g_f(k_n(i) \cap k_n(j))$.

We define $\mathcal{E} = \cup\{\mathcal{I}_{CV}(h_f) : f \in {}^{({}^{<n}2)}2 \wedge n < \omega\}$.

Theorem 3.29 $\mathcal{I}_{CV}(\aleph_\omega) \supseteq \mathcal{E}$.

Proof: We will show that for each n, $1 \leq n < \omega$,

$$\mathcal{I}_{CV}(\aleph_{2n}) \supseteq \bigcup\{\mathcal{I}_{CV}(h_f) : f \in {}^{({}^{<m}2)}2 \wedge 1 \leq m \leq n\}.$$

The initial stage of $n = 1$ needs no proof. Thus assume the result for $n \leq k$ and let $f \in {}^{({}^{<k+1}2)}2$. Let $g : [\aleph_{2k+2}]^2 \longrightarrow \omega$. For $i = 0, 1$ define $f_i : {}^{<k}2 \longrightarrow \{0,1\}$ by $f_i(\eta) = f(\langle i \rangle^\frown \eta)$. We have two cases to consider $f(\emptyset) = 0$ and $f(\emptyset) = 1$. We analyze the first case and leave the second since it is similar.

For $\alpha \in \aleph_{2k+1}$ choose $C_\alpha \subseteq [\aleph_{2k+1}, \aleph_{2k+1} + \aleph_{2k})$ and $b_\alpha < \omega$ such that $g(\{\alpha, x\}) = b_\alpha$ for all $x \in C_\alpha$. By the pigeon-hole principle there exists $D \subseteq \aleph_{2k+1}$ and $b < \omega$ such that $b_\alpha = b$ for all $\alpha \in D$. Now let $\alpha \in D$. By lemma 3.23 we may assume without loss of generality that there exists $E_\alpha \subseteq C_\alpha$ such that $|E_\alpha| = |C_\alpha|$ and $\mathrm{rng}(f \restriction [E_\alpha]^2) \subseteq \omega \setminus b + 1$. For $\alpha \in D$, by the induction hypothesis there exists a set $G_\alpha \subset E_\alpha$ such that $g \restriction [G_\alpha]^2$ realizes $\mathcal{I}_{CV}(h_{f_1})$. For each $\alpha \in D$ choose such a set. Again by the pigeon-hole principle there is $D_1 \subseteq D$ of cardinality $\aleph_{2n+1}$ and a set G such that $G_\alpha = G$ for all $\alpha \in D_1$. Let $c = \max(\mathrm{rng}(g \restriction [G]^2))$. By lemma 3.23 we may assume without loss of generality that there exists $D_2 \subseteq D_1$ of cardinality $\aleph_{2k+1}$ such that $g \restriction [D_2]^2 \subseteq \omega \setminus 1 + c$. By the induction hypothesis there exists $H \subset D_2$ such that $g \restriction [H]^2$ realizes $\mathcal{I}_{CV}(h_{f_0})$. It is clear that $g \restriction [H \cup G]^2$ realizes $\mathcal{I}_{CV}(h_f)$. $\square$

3.4 CV-identities with color set ordered like the rationals

In the section we will classify $\mathcal{I}_{CV}(\aleph_1, \mathbb{Q})$. Towards this end we let $\mathcal{D}$ be the set consisting of the two CV-identities which have field sizes one and two. We show $\mathcal{I}_{CV}(\aleph_1, \mathbb{Q}) = \mathcal{D}$. We outline the proof. Let $\mathcal{R}$ be the collection of all CV-identities that have field size three. Let M be a model of ZFC. We first construct c.c.c. partial orders $\mathbb{P}^{(F,i)}$, for $F \in \{a, l\}$ and $i \in \{0,1\}$. In $N = M[G^{a,0}][G^{a,1}][G^{l,0}][G^{l,1}]$ we define $f^{F,i} : [\aleph_1]^2 \longrightarrow \mathbb{Q}$ using $G^{F,i}$. We then show that each $K \in \mathcal{R}$ is omitted by one of the $f^{F,i}$. The final step is to show that a copy of each $f^{F,i}$ can be constructed in the ground model M. This is accomplished by considering 2-cardinal models.

We establish some definitions and notation. From §3.1 recall the *lexicographic* and *antilexicographic* orders of $\omega_1 \times \omega_1$, denoted $<^l$ and $<^a$ respectively. From the same section recall the definitions of *isomorphic* and *proper*. Let

$$\mathbf{R} = \{g : g \text{ is a mapping from } [B]^2 \text{ into } \mathbf{Q},\ |B| < \omega,\ B \subset \aleph_1\}.$$

For $F \in \{l, a\}$ and $i = 0$, an (F, i)-*amalgam* of f and g, where $\langle f, g\rangle$ is proper is an extension $h \in \mathbf{R}$ of f and g such that $\mathrm{fld}(h) = \mathrm{fld}(f) \cup \mathrm{fld}(g)$, $\mathrm{rng}(f) \cup \mathrm{rng}(g) < \mathrm{rng}(h \setminus f \cup g)$, and

$$h(\{x_1, y_1\}) < h(\{x_2, y_2\}) \Leftrightarrow (x_1, y_1) <^F (x_2, y_2)$$

for all $x_1, x_2 \in \mathrm{fld}(f) \setminus \mathrm{fld}(g)$ and $y_1, y_2 \in \mathrm{fld}(g) \setminus \mathrm{fld}(f)$. For $F \in \{l, a\}$ and $i = 1$, an (F, i)-*amalgam* of f and g, where $\langle f, g\rangle$ is a proper pair, is defined exactly as above except $\mathrm{rng}(f) \cup \mathrm{rng}(g) > \mathrm{rng}(h \setminus f \cup g)$

Definition 3.30 For $F \in \{l, a\}$ and $i \in \{0, 1\}$ define the partial ordering $\mathbf{P}^{(F,i)}$ as follows. Let $\mathbf{P}_1^{(F,i)} = \{f : f \text{ is amapping from } [B]^2 \text{ into } \mathbf{Q},\ |B| = 1\}$. For $n \geq 1$ let $\mathbf{P}_{n+1}^{(F,i)}$ denote the union of $\mathbf{P}_n^{(F,i)}$ and

$$\{g \in \mathbf{R} : \text{ there exist } f_1, f_2 \in \mathbf{P}_n^{(F,i)} \text{ such that } g \text{ is their } (F, i)\text{-amalgam } \}.$$

Let $\mathbf{P}^{(F,i)} = \bigcup\{\mathbf{P}_n^{(F,i)} : n < \omega\}$ with the ordering of inverse inclusion.

Let M be a model of ZFC. Let $G^{a,0}$ be $\mathbf{P}^{a,0}$-generic over M. Define $N^{a,0}$ to be $M[G^{a,0}]$. Let $G^{a,1}$ be $\mathbf{P}^{a,1}$-generic over $N^{a,0}$ and define $N^{a,1}$ to be $N^{a,0}[G^{a,1}]$. Let $G^{l,0}$ be $\mathbf{P}^{l,0}$-generic over $N^{a,1}$ and define $N^{l,0}$ to be $N^{a,1}[G^{l,0}]$. Finally let $G^{l,1}$ be $\mathbf{P}^{l,1}$-generic over $N^{l,0}$ and define N to be $N^{l,0}[G^{l,1}]$. In other words $N = M[G^{a,0}][G^{a,1}][G^{l,0}][G^{l,1}]$ where $G^{F,i}$ is $\mathbf{P}^{F,i}$ generic over the appropriate model. In N define $g^{F,i} = \bigcup G^{F,i}$ for $F \in \{a, l\}, i \in \{0, 1\}$. The following two lemmas are proved in a manner similar to lemmas 3.6 and 3.7.

Lemma 3.31 *Let $F \in \{l, a\}$ and $i \in \{0, 1\}$. Then $\mathbf{P}^{(F,i)}$ is c.c.c.*

Lemma 3.32 *The cardinality of fld$(g^{F,i})$ is $\aleph_1$ for each $i \in \{0, 1\}$ and $F \in \{l, a\}$.*

For $F \in \{l, a\}$ and $i \in \{0, 1\}$ let $h^{F,i} : \aleph_1 \longrightarrow \mathrm{fld}(g^{F,i})$ denote the uniqe order-preserving bijection and define $f^{F,i} : [\aleph_1]^2 \longrightarrow \mathbf{Q}$ by $f^{F,i}(\{\alpha, \beta\}) = g^{F,i}(\{h^{F,i}(\alpha), h^{F,i}(\beta)\})$. Recall the collection, $\mathcal{K}$, of CV-identities given in section 3.1. We now define three additional CV-identities which have field size three. Together with the collection $\mathcal{K}$, they constitute the complete set of all CV-identities with field size three. For $i = 11, 12, 13$ let $K_i = \langle u_i, \preceq_i, <_i\rangle$ be such that $u_i = \{0, 1, 2\}$, $<_i$ is the usual ordinal ordering and $\prec_i$ is as follows:

i) $K_{11} : \{0,1\} \prec \{0,2\} \prec \{1,2\}$

ii) $K_{12} : \{0,2\} \prec \{0,1\} \sim \{1,2\}$

iii) $K_{13} : \{0,1\} \sim \{0,2\} \prec \{1,2\}$.

The following lemmas are all proved in a manner similar to the proofs of lemmas 3.9. They will be omitted.

Lemma 3.33 *In N, $f^{l,0}$ omits $\mathcal{K} \setminus \{K_1\}$.*

Lemma 3.34 *In N, $f^{a,0}$ omits $\mathcal{K} \setminus \{K_2, K_3\}$.*

Lemma 3.35 *In N, $f^{l,1}$ omits $\{K_{12}, K_{13}\}$.*

Lemma 3.36 *In N, $f^{a,0}$ omits $\{K_{11}, K_{13}\}$.*

Lemma 3.37 *Let M be a model of ZFC. There exists a cardinal preserving forcing extension N of M and functions $f^{F,i} : [\aleph_1]^2 \longrightarrow \mathbb{Q}$ for $F \in \{a, l\}$, $i \in \{0, 1\}$ in N such that for every CV-identity $I \notin \mathcal{D}$, one of the O-colorings $f^{F,i}$ omits I.*

Proof: Every CV-identity that is not an element of $\mathcal{D}$ has field size 3 or realizes a CV-identity of field size 3. The four lemmas listed show that every CV-identity of field size three is omitted by one of the four colorings $f^{F,i}$. □

We continue with some lemmas needed to prove the main result. The following theorem is essentially proved in [11] on pages 126-133.

Lemma 3.38 *Let L be a countable language containing a designated unary predicate. Let T be an L-theory. There is an extension T' of T such that there is an (ω_1, ω)-model of T if and only if T' is consistent.*

Definition 3.39 Let $L = \{p, <, f, g\}$ be a language, where p is a unary predicate, $<$ a binary predicate and f, g are binary function symbols. Let T_0 be the L-theory which has axioms saying that $<$ is a linear ordering of the universe as well as

i) $\forall x \forall y (x \neq y \Leftrightarrow p(f(x,y)))$

ii) $\forall x \forall y \forall z [(z, y < x) \Rightarrow [p(g(y,x)) \wedge (g(y,x) = g(z,x) \Rightarrow y = z)]]$.

Let $n < \omega$, I be a CV-identity of size n, K a model of T_0, and $A \subset K$ be of cardinality n. Let $h : n \longrightarrow A$ be the order-preserving bijection. Denote by $g_A : [n]^2 \longrightarrow \omega$, the unique function such that $\mathrm{rng}(g_A)$ is an initial segment of ω and for all $\{r,s\}$, $\{t,v\} \in [n]^2$,

$$g_A(\{r,s\}) \leq g_A(\{t,v\}) \Leftrightarrow f^M(h(r),h(s)) \leq^M f^M(h(t),h(v)).$$

Let $\psi_I(x_1, \ldots x_n)$ be the formula in the language L such that for all K such that $K \models T_0$, for all $A = \{a_1, \ldots, a_n\} \in [K]^n$, $K \models \psi_I(a_1, \ldots, a_n)$ if and only if g_A does not realize I. Define T_I to be the theory $T_0 \cup \{\forall x_1, \ldots, \forall x_n \psi_I\}$ and T_I' to be the extension of T_I as given in lemma 3.38.

Lemma 3.40 *Let M be a model of ZFC and I be a CV-identity of size n. In M there exists an (ω_1, ω)-model K of T_I if an only if there exists a function $f : [\aleph_1]^2 \longrightarrow \mathbf{Q}$ that does not realize the CV-identity I.*

Proof: The proof of the ' if ' part is clear. We now prove the ' only if ' part. Since we have a two-cardinal model of T_0, the first axiom shows that the collection of colors, p^K, is a countable set. Thus there exists an order preserving bijection $h : p^K \longrightarrow \mathbf{Q}$. The second axiom shows that there is a subset R of K of order-type $\aleph_1$. Let $k : \aleph_1 \longrightarrow R$ be the unique order-preserving bijection.

Define $f : [\aleph_1]^2 \longrightarrow \mathbf{Q}$ by $f(\{\alpha, \beta\}) = h(f^K(k(\alpha), k(\beta)))$. Since $K \models T_0$, for all $A \in [K]^n$, g_A does not realize I. This suffices to show that f does not realize I since for all $B \in [\aleph_1]^n$, $f \restriction B$ realizes I if and only if g_A realizes I, where $A = \{k(\alpha) : \alpha \in B\}$. □

Theorem 3.41 $\mathcal{I}_{CV}(\aleph_1, \mathbf{Q}) = \mathcal{D}$.

Proof: Let M be a model of ZFC and I be an CV-identity not in $\mathcal{D}$. By lemma 3.37 there exists an extension N of M and a function $f : [\aleph_1]^2 \longrightarrow \mathbf{Q}$ in N such that f does not realize I. By lemma 3.40 there is an (ω_1, ω)-model of T_I in N. It follows from lemmas 3.38 that T_I'

is consistent in N. If T_I' were inconsistent in M, a formal proof of a contradiction from the axioms of T_I' would exist in M and thus also in N, showing inconsistency in N, contradiction. Thus T_I' is consistent in M. By lemma 3.38, this consistency shows the existence, in M, of an (ω_1,ω)-model of T_I. Applying lemma 3.40 we obtain a function $f : [\aleph_1]^2 \longrightarrow \mathbf{Q}$ that omits I. Thus the theorem is proved. □

An attempt at proving that ZFC $\vdash \mathcal{I}_{CV}(\aleph_1,\omega) = \mathcal{C}$ in a manner analogous to that in theorem 3.41 would take the following form. First, using theorem 3.2 we find a model N such that $N \models \mathcal{I}_{CV}(\aleph_1,\omega) = \mathcal{C}$. Let $I \notin \mathcal{C}$ be a CV-identity. By lemma 3.40 there exists an (ω_1,ω)-model of T_I, and so by 3.38, T_I' in consistent in N. Any proof of the inconsistency of T_I' in M would give a proof of the inconsistency in N. Thus T_I' is consistent in M, the ground model. Again, 3.38 shows that T_I has an (ω_1,ω)-model in M. The proof now fails. This (ω_1,ω)-model only shows that there exists in M, a function $f : [\aleph_1]^2 \longrightarrow B$ such that $I \notin \mathcal{I}(f)$, where B is a countable linearly ordered set. One might hope to find a set $C \subseteq B$ of order type ω and an uncountable $D \subseteq \aleph_1$ such that $f(\{d_1,d_2\}) \in C$ for all $\{d_1,d_2\} \in [D]^2$, and thus rescue the theorem. The following combinatorial argument show that this will not always be possible.

Lemma 3.42 *(CH) There exists a function $f : [\aleph_1]^2 \longrightarrow \omega+\omega$ such that for all $B \subseteq \aleph_1$, for all $S \subseteq \omega+\omega$,*

$$(|B| = \aleph_1 \wedge \ ot(S) = \omega) \Rightarrow \exists\{\alpha,\beta\} \in [B]^2(f(\{\alpha,\beta\}) \notin S).$$

Proof: List the subsets of $\omega+\omega$ of order type ω as $\langle S_\alpha : \alpha < \omega_1\rangle$. List the countable subsets of ω_1 as $\langle C_\alpha : \alpha < \omega_1\rangle$. We define f in ω_1 stages. At stage $\alpha < \omega_1$ we define $f(\{\alpha,\gamma\})$ for $\gamma < \alpha$.

We define the construction at stage α. List the set $\{S_\beta : \beta < \alpha\}$ in order type ω as $\langle T_n : n < \omega\rangle$ and the set $\{C_\beta : \beta < \alpha \wedge C_\beta \subset \alpha\}$ as $\langle D_n : n < \omega\rangle$. We now define the values $f(\{\alpha,\gamma\})$ where $\gamma < \alpha$. This is done in ω stages. List the pairs $\{(m,n) : m,n < \omega\}$ in order type ω by defining a bijection $h : \omega \longrightarrow \omega\times\omega$. At stage $k < \omega$ let $h(k) = (m,n)$ and choose $\alpha_k \in \omega+\omega \setminus T_n$ and $\gamma_k \in D_m \setminus \cup\{\gamma_j : j < m\}$. As D_m is countable this is possible. Define $f(\{\alpha,\gamma_k\}) = \alpha_k$. For $\gamma \notin \{\gamma_k : k < \omega\}$ define $f(\{\alpha,\gamma\} = 0$. The construction is now complete.

We now show that f indeed does what we have claimed. Towards a contradiction choose $B \subseteq \aleph_1$ of cardinality $\aleph_1$ and $S_\beta \subset \omega + \omega$ of order type ω and suppose $f(\{b,\gamma\}) \in S_\beta$ for all $b, \gamma \in B$. Let $B \cap \beta = C_\delta$ for some $\delta < \omega_1$. Choose $b \in B$ such that $b > \max\{\delta, \beta\}$. By hypothesis $f(\{b,c\}) \in S_\beta$ for all $c \in C_\delta$ but as $C_\delta \subseteq b, \delta < b$ and $\beta < b$ we have, by the construction, that there exists $\gamma \in C_\delta$ such that $f(\{b,\gamma\}) \in \omega + \omega \setminus S_\beta$.

More precisely, C_δ occurs as D_m for some $m < \omega$ in our listing of $\{C_\eta : \eta < b \wedge C_\eta \subseteq b\}$. Also S_β occurs as T_n for some $n < \omega$ in the listing of $\{S_\eta : \eta < b\}$. Consider $k = h(m,n) < \omega$. By construction we have chosen $\gamma_k \in D_m = C_\delta$ and $\alpha_k \in \omega + \omega \setminus T_n = \omega + \omega \setminus S_\beta$ and then defined $f(\{b,\gamma_k\}) = \alpha_k$. □

Chapter 4

Open Questions

The following are some open questions.

i) Are the methods used in this thesis sufficient to characterize IDA^m? In other words, is it true that $\mathrm{IDA}^m = \mathcal{C}^m$ for $m \geq 2$?

ii) Does $\bigcap\{\mathcal{C}^{F,m} : F \in \mathcal{F}^m\} = \bigcup\{\mathcal{C}^{F,m} : F \in \mathcal{F}^m\}$?

iii) Recall the definition of $f_m : [\aleph_m]^2 \longrightarrow \omega$ given in §2.1. This function depended upon the choice of $F \in \mathcal{F}^m$ used to define $\mathbf{P}^{F,m}$ and the choice of the $\mathbf{P}^{F,m}$-generic G. Is the set of identities realized by f_m independent of the choice of F and G?

iv) Does there exist an algorithm which determines membership in $\mathcal{C}^m$?

v) Does $\mathrm{ZFC} \vdash \mathcal{I}_{CV}(\aleph_1, \omega) = \mathcal{C}$?

vi) Does $\mathrm{ZFC} \vdash \mathcal{I}_{CV}(\aleph_\omega, \omega) = \mathcal{E}$?

vii) Is $\mathcal{I}_{CV}(\aleph_\omega, \omega)$ closed under end-duplication?

viii) Does $\mathcal{I}_{CV}(\aleph_\omega, \omega) = \bigcup\{\mathcal{I}_{CV}(\aleph_n, \omega) : n < \omega\}$?

ix) Does there exist $\kappa > \aleph_\omega$ such that $\mathcal{I}_{CV}(\aleph_\omega, \omega) \subsetneq \mathcal{I}_{CV}(\kappa, \omega)$?

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