Symmetric Properties of Sets and Real Functions

by

Hongjian Shi

B.Sc., Henan Normal University, China, 1984
M.Sc., Peking University, China, 1987

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Department
of
Mathematics and Statistics

SIMON FRASER UNIVERSITY
February 1995

APPROVAL

Name: Hongjian Shi

Degree: Master of Science

Title of thesis: Symmetric Properties of Sets and Real Functions

Examining Committee: Dr. G. A. C. Graham
Chair

Dr. B. S. Thomson, Senior Supervisor

Dr. P. Borwein

Dr. A. Freedman

Dr. T. Tang, External Examiner

Date Approved: March 1, 1995

Title of Thesis/Project/Extended Essay:

Symmetric Properties of Sets and Real Functions.

Author: ______________________

(signature)

Shi, Hongjian

(name)

Feb. 14, 1995

(date)

Abstract

In this thesis some symmetric properties of real functions are investigated and several problems are solved. The results of Buczolich-Laczkovich and Thomson about the range of symmetric derivatives are extended to a general case. A new type of antisymmetric sets is introduced and discussed, therefore some results of dense Hamel bases are obtained. It is shown that the typical functions of symmetrically continuous functions and symmetric functions have c-dense sets of points of discontinuity. Also an existence proof of a continuous nowhere symmetrically differentiable function and a continuous nowhere quasi-smooth function is given through the application of Baire category theorem.

Acknowledgements

I would like to express my sincere thanks to my supervisor Dr. Brian S. Thomson for his guidance and substantial support, and also for his excellent monograph *Symmetric Properties of Real Functions* which introduced me to the topics in the thesis.

Contents

Chapter 1

Introduction

Symmetric properties of real functions play an important role in many problems. This is particular true in the theory of trigonometric series. For example, the sequence of partial sums $\{s_n(x)\}$ of the trigonometric Fourier series of a function f is transformed by a summability method into a sequence $\{\sigma_n(x)\}$ and the difference $\sigma_n(x) - f(x)$ is essentially the convolution of $f(x+t) + f(x-t) - 2f(x)$ and $(\sin nt)/t$. The famous Riemann first theorem and second theorem in the theory of trigonometric series all are related to the symmetric derivatives and the second symmetric derivatives in entirely natural ways.

By *the symmetric properties* of real functions we mean properties arising from the expression

$$f(x+t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}[f(x+t) - f(x-t)]$$

which defines the even parts and the odd parts of the function f at the point x. The continuity and differentiability properties of f can be in some cases analyzed by studying the corresponding expressions involving

$$f(x+t) - f(x-t) \quad \text{and} \quad f(x+t) + f(x-t) - 2f(x).$$

The investigation of these symmetric properties stretches back a century and a half. It has been among the interests of many famous mathematicians. Now it has become a vigorous area as symmetric real analysis.

In this thesis several problems are investigated and solved. These mainly come from my reading the excellent monograph *Symmetric Properties of Real Functions* written by Dr. Brian S. Thomson. This monograph includes almost all recent developments in this area. The thesis is organized as follows.

In Chapter 2 the range of symmetric derivatives is investigated. The results of Z. Buczolich and M. Laczkovich [3], B. S. Thomson [1, p. 276] are extended to a general case.

In Chapter 3 a new type of sets is introduced. Some properties of these sets are obtained and some results of dense Hamel bases are therefore obtained by showing that the dense Hamel bases are sets of such type.

In Chapter 4 the typical properties of symmetrically continuous functions and symmetric functions are investigated. It is shown that the typical functions of symmetrically continuous functions and symmetric functions have c-dense sets of points of discontinuity. This answers two problems posed in [1. p. 422].

In Chapter 5 an application of the Baire category theorem to the space of continuous functions is given, and therefore the existence of a continuous nowhere symmetrically differentiable function and a continuous nowhere quasi-smooth function is proved.

In Chapter 6 we make a summary of this thesis and pose several interesting, open problems.

Chapter 2

The Range of Symmetric Derivatives

In this chapter we discuss the range of a type of generalized derivatives, i.e. the symmetric derivatives.

An arbitrary function f is said to have *a symmetric derivative* at a point $x \in R$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists or equals to ∞. Also we say that the function f is symmetrically differentiable at the point x (allowing infinite values). We use $SDf(x)$ to denote the symmetric derivative of the function f.

A function f is said to be a Baire 1 function if f is the pointwise limit of a sequence of continuous functions.

It is easy to see that if the ordinary derivative $f'(x)$ of $f(x)$ exists then $SDf(x)$ exists. However if $SDf(x)$ exists $f'(x)$ need not exist. According to the Darboux property the range of the ordinary derivatives is well known. If a function f is continuous and has a derivative, even allowing infinite values, the range of $f'(x)$ must be an interval or a single point. For the symmetric derivatives the range is not as simple as that of the ordinary derivatives. For example the function $f(x) = |x|$ is everywhere symmetrically differentiable with

$$SDf(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The range of the symmetric derivative of $f(x)$ is just a set of three values. Thus a symmetric derivative needs not to have the Darboux property.

In 1983 Larson [14] showed that the range of the symmetric derivative of a bounded function f with a symmetric derivative everywhere is same as that of some continuous function g for which $SDf(x) = SDg(x)$. In the same year the journal of Real Analysis Exchange [20] posed a query asked by Larson how to characterize the situation under which finite symmetric derivatives have the Darboux property. In 1987 Kostyrko [11] gave a characterization to answer the query as follows.

Theorem 1 *Let h be a locally bounded symmetric derivative. Then h has the Darboux property if and only if there exists a function f satisfying that for any $a, b \in R$, there exists a point $z \in [a, b]$ such that $f(b) - f(a) = SDf(z)(b - a)$ and that $SDf = h$.*

In 1991 Buczolich-Laczkovich [3] showed the following theorem.

Theorem 2 *There is no symmetrically differentiable function whose symmetric derivative assumes just two finite values.*

Later, using a completely different method Thomson [1, p. 276] showed the above theorem and the following result.

Theorem 3 *Let $\alpha, \beta, \gamma \in R$ with $\alpha < \gamma < \beta$ and $\gamma \neq \frac{1}{2}(\alpha + \beta)$. Then there is no symmetrically differentiable function whose symmetric derivative assumes just the three values α, β and γ.*

Here we follow Thomson's method in [1,p. 276] to show a general case about the range of symmetric derivatives.

Theorem 4 *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ and $\alpha_i \neq \frac{1}{2}(\alpha_k + \alpha_l)$, $k \neq l$, $1 \leq k, l, i \leq n$, and $n > 2$ be a natural number. Then there is no symmetrically differentiable function whose symmetric derivative assumes just the n values $\alpha_1, \alpha_2, \ldots, \alpha_n$.*

Proof. As mentioned above in [14] Larson showed that the range of the symmetric derivative of a bounded function f with a symmetric derivative everywhere is same as that of some continuous function g for which $SDf(x) = SDg(x)$. Thus the theorem can be reduced to showing that there is no continuous function with this property. We know from Theorem 2 and Theorem 3 that the conclusions are true for the cases $n = 2, 3$. To use mathematical induction we assume that the conclusions are true for all $2 < n \leq p$ and show that the conclusion is also true for $n = p + 1$.

If there is a continuous function whose derivative assumes just $p + 1$ values, then both $f(x) - \alpha_1 x$ and $\alpha_{p+1}x - f(x)$ are nondecreasing. Since the symmetric derivative function $SDf(x)$ is a Baire 1 function there are points of continuity of $SDf(x)$ in every interval. But at such one point of of continuity of $SDf(x)$ there must be an interval in which $SDf(x)$ assumes only one value of $\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_{p+1}$. So in such interval the function f is linear with the slope of this value. Thus there is a maximal open set G so that in every component of G the function f is linear with slope, one of $\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_{p+1}$.

Let P denote the complement of the set G, then P has no isolated points. If not, suppose $b \in P$, (a, b) and (b, c) are contained in G, the function f is linear with slope α_i on (a, b) and with slope α_j on (b, c), α_i, α_j are two numbers of $\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_{p+1}$. If $\alpha_i = \alpha_j$ the function f is linear on (a, c) and so $b \notin P$, a contradiction. If $\alpha_i \neq \alpha_j$ then $SDf(b) = \frac{1}{2}(\alpha_i + \alpha_j)$ and this contradicts the hypothesis.

In fact the set P is empty. If not then P is perfect. From the fact that $SDf(x)$ is Baire 1 there is a point of continuity of $SDf(x)$ relative to P. Thus there must be a nonempty portion $P \bigcap (a, b)$ so that $SDf(x)$ assumes just one of the $p + 1$ values $\alpha_1, \ldots, \alpha_p, \alpha_{p+1}$ for all $x \in P \bigcap (a, b)$. This value would not be α_1 or α_{p+1}. In fact if $SDf(x) = \alpha_i$ for all $x \in P \bigcap (a, b)$, α_i is one value of $\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_{p+1}$. Consider some interval (c, d) contiguous to P in (a, b). In the interval (c, d) the function f

is linear with the slope, one value of $\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_{p+1}$ and $SDf(c) = \alpha_i$. Since $f'_+(c)$ is one value of $\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_{p+1}$, it follows that $f'_-(c)$ exists too. Noting that $\alpha_1 \leq f'_-(c) \leq \alpha_{p+1}$, if $\alpha_i < (\alpha_1 + \alpha_{p+1})/2$, we have

$$f'_+(c) = 2SDf(c) - f'_-(c) < 2(\frac{\alpha_1 + \alpha_{p+1}}{2}) - \alpha_1 = \alpha_{p+1}.$$

This means the function f can not have slope α_{p+1} on (c, d), thus in such case in the entire interval (a, b) the symmetric derivative $SDf(x)$ assumes at most p values. If $\alpha_i > (\alpha_1 + \alpha_{p+1})/2$, we have

$$f'_+(c) = 2SDf(c) - f'_-(c) > 2(\frac{\alpha_1 + \alpha_{p+1}}{2}) - \alpha_{p+1} = \alpha_1.$$

This means that the function f can not have slope α_1 on (c, d), and therefore in the entire interval (a, b) the symmetric derivative $SDf(x)$ also assumes at most p values in this case. In any case the symmetric derivative $SDf(x)$ assumes at most p values on the interval (a, b). This contradicts the assumption for the case $n \leq p$. Hence the set P is empty. If the symmetric derivative $SDf(x)$ assumes just one value this also contradicts the assumption. If the symmetric derivative $SDf(x)$ assumes more than one values then $SDf(x)$ must assume some value of the form $(\alpha_i + \alpha_j)/2$ from the construction of the set G. This contradicts the hypothesis. Therefore there is no continuous function whose symmetric derivative assumes just $p + 1$ values. By mathematical induction the theorem follows. □

Chapter 3

A Type of Antisymmetric Sets

A function $f : R \to R$ (where R is the real line) is said to be *exactly locally symmetric* if at each point $x \in R$ there is a $\delta_x > 0$ such that $f(x+h) = f(x-h)$ holds for all $0 < h < \delta_x$ [1, p. 48].

A set is said to be *exactly locally symmetric* if its characteristic function is exactly locally symmetric [1, p. 48].

Davies [5] and Rusza [17] showed the following theorem independently.

Theorem 5 *Let f be a function such that at each point $x \in R$ there is a positive number $\delta_x > 0$ so that*

$$0 < t < \delta \implies f(x+t) - f(x-t) = 0.$$

Then f is constant off a closed countable set.

If f is the characteristic function of a set E in the above theorem, then either E or its complement has countable closure.

S. Marcus in [15] suggested investigating the following exactly locally antisymmetric set.

A set $A \subset R$ is said to be *exactly locally antisymmetric* if for every $x \in R$ there is a $\delta_x > 0$ such that for each $h, 0 < h < \delta_x, x + h \in A$ if and only if $x - h \notin A$.

In [9] Kostyrko showed that there is no exactly locally antisymmetric set. Later K. Ciesielski and L. Larson in [4], P. Komjath and S. Shelah in [12] obtained more results about the exactly locally antisymmetric functions.

Motivated by the interesting structure of the exactly locally antisymmetric sets we introduce here a new type of sets whose structure is in some extent similar to that of the exactly locally antisymmetric sets and discuss some of their properties.

Definition 6 *A nonempty set E is said to be a right antisymmetric set if for each $x \in R$ and each positive $h \in R, x-h \in E$ implies $x+h \notin E$.*

Theorem 7 *A right antisymmetric set is an one element set.*

Proof. Let E be a right antisymmetric set and $x \in E$. If there is a point $x_1 \in E, x_1 \neq x$, then at the point $(x+x_1)/2$, for $h = |x-x_1|/2$,

$$\frac{x+x_1}{2} - h, \frac{x+x_1}{2} + h \in E.$$

This contradicts that E is a right antisymmetric set. Thus the set E is an one element set. ◫

Definition 8 *A set E is said to be a right locally antisymmetric set if for every $x \in R$ there exists a number $\delta_x > 0$ such that $x-h \in E$ implies $x+h \notin E$ if $0 < h < \delta_x$.*

It is easy to see that the set $\{0, -1/2, 1/3, -1/4, 1/5, \cdots\}$ is a right locally antisymmetric set. In the following theorem we will see that any Hamel basis which is dense in the real line R is also a right locally antisymmetric set.

A set $\mathcal{B}$ of real numbers is called a Hamel basis if $\mathcal{B}$ satisfies the following:

1^0 any finite subset $H = \{x_1, x_2, \ldots, x_n\}$ of $\mathcal{B}$ is rationally independent, i.e. if $r_1x_1 + r_2x_2 + \cdots + r_nx_n = 0$ where $r_1, r_2, \ldots, r_n$ are rational numbers, then $r_1 = r_2 = \cdots = r_n = 0$;

2^0 for any real number $x \notin \mathcal{B}$ there exist a finite subset $H_1 = \{y_1, y_2, \cdots, y_m\}$ of $\mathcal{B}$ and m rational numbers such that $x = r_1y_1 + r_2y_2 + \cdots + r_my_m$.

The existence of such a set was established by Hamel in 1905 through an application of the Axiom of Choice.

Theorem 9 *Any Hamel basis $\mathcal{B}$ which is dense in the real line R is a right locally antisymmetric set.*

Proof. A Hamel basis $\mathcal{B}$ which is dense in the real line does exist. See [13, p. 261].

For $x \in \mathcal{B}$ and every $\delta > 0$ and $x - h \in \mathcal{B}$, if $x + h \in \mathcal{B}$ then

$$x - \frac{1}{2}(x+h) - \frac{1}{2}(x-h) = 0.$$

This contradicts the definition of Hamel basis. Thus $x - h \in \mathcal{B}$ implies $x + h \notin \mathcal{B}$. For $x \notin \mathcal{B}$, there must exist a $\delta_x > 0$ such that $x - h \in \mathcal{B}$ implies $x + h \notin \mathcal{B}$ if $0 < h < \delta_x$. If not, since $\mathcal{B}$ is dense in the real line R there exist two positive numbers h_1, h_2 such that $x - h_1, x + h_1, x - h_2, x + h_2 \in \mathcal{B}$. Then

$$\frac{1}{2}(x + h_1 + x - h_1) = \frac{1}{2}(x + h_2 + x - h_2),$$

that is

$$\frac{1}{2}(x+h_1) + \frac{1}{2}(x-h_1) - \frac{1}{2}(x+h_2) - \frac{1}{2}(x-h_2) = 0.$$

This contradicts the definition of Hamel basis. Thus $\mathcal{B}$ is a right locally antisymmetric set. □

A set is said to have *the Baire property* if it can be expressed as the symmetric difference of an open set and a first category set.

Lemma 10 *Let B be a nonempty set with the Baire property, then the set $\frac{1}{2}(B+B)$ contains an interval. Here $\frac{1}{2}(B+B) = \{\frac{1}{2}(x+y) : x, y \in B\}$.*

Proof. Since B is a nonempty set with the Baire property then

$$B = G \triangle P$$

where G is an open set and P is a first category set. Hence there exists an interval $(a,b) \subset G$,

$$B \supseteq (a,b) \triangle P \supseteq (a,b) \setminus P.$$

For any point $x \in (a,b)$ there must exist a real number h, $0 < h < \min\{x-a, b-a\}$ such that $x - h, x + h \in (a,b) \setminus P$. If not then the interval $(x - \min\{x-a, b-x\}, x + \min\{x-a, b-x\})$ would be of first category. This contradicts the fact that any interval can not be of first category. See Theorem 25. Thus

$$x = \frac{1}{2}(x - h + x + h) \in \frac{1}{2}[(a,b) \setminus P + (a,b) \setminus P].$$

So

$$x \in \frac{1}{2}(B+B) \quad \text{and} \quad (a,b) \subset B.$$

The lemma is proved. □

Theorem 11 *If a set E is a right locally antisymmetric set. Then it has inner measure zero, and can contain no second category set with the Baire property.*

Proof. We use $m_i(A)$ to denote the inner measure of a set A, $m(A)$ to denote the Lebesgue measure of a Lebesgue measurable set A.

First we show that $m_i(E) = 0$. Suppose $m_i(E) > 0$, then there exists a closed set $F \subset E$ with $m(F) > 0$. Thus there exists at least one point $x_0 \in F$ such that for every $\delta > 0$,

$$m[(x_0 - \delta, x_0 + \delta) \bigcap F] > 0.$$

Hence for each positive integer n,

$$m[(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \bigcap (F \setminus \{x_0\})] > 0.$$

Set

$$A_n = \frac{1}{2}[(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \bigcap (F \setminus \{x_0\}) + (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \bigcap (F \setminus \{x_0\})]$$

where $A + A = \{x + y : x \in A, y \in A\}$. Then $\{A_n\}$ is a decreasing sequence of sets. From the measure-theoretic theory [19, p. 250] each set A_n contains a closed interval J_n. By mathematical induction we can require that

$$J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots \supseteq J_n \supseteq J_{n+1} \supseteq \cdots$$

and then

$$A_n \supseteq J_n, \quad m(J_n) \le m(A_n) \le \frac{4}{n} \to 0.$$

Since J_n are all closed, obviously $\bigcap_{n=1}^{\infty} J_n = \{x_0\}$. Therefore there exists a sequence $\{h_n\}$ with $h_n \downarrow 0$ and

$$x_0 - h_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \bigcap F, \qquad x_0 + h_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \bigcap F.$$

Hence we get a sequence $\{h_n\}$ with $h_n \downarrow 0$ such that $x_0 - h_n \in E$ and $x_0 + h_n \in E$. This contradicts that E is a right locally antisymmetric set. Thus $m_i(E) = 0$.

We now show that the set E can contain no second category set with the Baire property. Suppose that E contains a second category set B with the Baire property. Then B is the symmetric difference of an open set and a first category set. So there exists at least one point $x_0 \in B$ such that for every $\delta > 0$, $(x_0 - \delta, x_0 + \delta) \bigcap B$ is a second category set which has the Baire property and contains a set of symmetric difference of an interval and a first category set. For every positive integer n, set

$$B_n = \frac{1}{2}[(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \bigcap (B \setminus \{x_0\}) + (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \bigcap (B \setminus \{x_0\})].$$

Then by Lemma 10 each set B_n contains a closed interval J_n. The following is same as in the case of the inner measure and we can find a contradiction. Thus the set E can contain no second category set with the Baire property. ◫

Combining Theorem 9 and Theorem 11 yields the following corollary.

Corollary 12 *A Hamel basis which is dense in the real line has inner measure zero and can contain no second category set with the Baire property.*

Kuczma in [13, p. 225, p.227] obtained more general results about a Hamel basis that each Hamel basis has inner measure zero and can contain no second category set with the Baire property by using different methods.

We can define a left locally antisymmetric set similarly and obtain the similar results for such sets.

Chapter 4

Some Typical Properties

In this chapter we study the properties of symmetrically continuous functions and symmetric functions.

A function $f : R \rightarrow R$ is said to be *symmetrically continuous* at $x \in R$ if

$$\lim_{h \to 0}[f(x+h) - f(x-h)] = 0.$$

A function $f : R \rightarrow R$ is said to be *symmetric* at $x \in R$ if

$$\lim_{h \to 0}[f(x+h) + f(x-h) - 2f(x)] = 0.$$

A natural question is to ask for the continuity properties of such functions. From the following equality

$$f(x+h) - f(x) = \frac{f(x+h) - f(x-h)}{2} + \frac{f(x+h) + f(x-h) - 2f(x)}{2}$$

it is easy to see that if a function f is continuous at a point $x \in R$ then f is both symmetrically continuous and symmetric at the point x. If a function is symmetrically continuous or symmetric at one point, such point need not be a point of of continuity. For example, the function $f(x) = x^{-2}$ is symmetrically continuous everywhere but discontinuous at $x = 0$. This example also shows that symmetric continuity has properties quite distinct from ordinary continuity: while the function $f(x) = x^{-2}$ is symmetrically continuous at $x = 0$ it is not bounded near that point nor is it defined at the point. Moreover for any countable set C we can construct an everywhere symmetrically continuous function that is discontinuous precisely on the set C: let $x_1, x_2, \cdots, x_n, \cdots$ be an enumeration of C, define $g(x) = 0$ for $x \notin C$ and $g(x_n) = 2^{-n}$.

Similar statements can be said for symmetric functions.

In 1964 Stein and Zygmund [1, p. 25-27] first showed the most important continuity properties of symmetrically continuous functions and symmetric functions as follows.

Theorem 13 *If $f : R \to R$ is Lebesgue measurable and is symmetrically continuous on a Lebesgue measurable set E, then f is continuous a.e. on E.*

Theorem 14 *If $f : R \to R$ is Lebesgue measurable and is symmetric on a Lebesgue measurable set E, then f is continuous a.e. on E.*

From the Stein and Zygmund theorems the set of points of discontinuity of a symmetrically continuous function and that of a symmetric function are first category. In 1935 Hausdorff [8] posed in *Fundamenta Mathematicae* the problem of whether the set of points of discontinuity of a symmetrically continuous function can be uncountable. In 1971 Preiss [16] answered this problem through the use of uncountable N-sets. Also see [1, p. 52]

A set E is said to be an N-set if there is a trigonometric series

$$a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

with

$$\sum_{n=1}^{\infty}(|a_n| + |b_n|) = +\infty$$

converges absolutely at every point of E.

Lusin and Denjoy showed that N-sets must have measure zero and be of the first category. See [1, p. 52]. Zygmund [19, Vol. I, p. 250] gave an example of a trigonometric series

$$\sum_{n=1}^{\infty} n^{-1} \sin n! x$$

which converges absolutely on an uncountable set.

Theorem 15 (Preiss) *Let E be an N-set. Then there is a bounded, 2π-periodic function f that is everywhere symmetrically continuous and discontinuous at each point in E.*

In 1989 Tran in [18] constructed a bounded measurable symmetric function whose set of points of discontinuity is uncountable, and also showed that the absolute value function of this function is symmetrically continuous and its set of points of discontinuity is uncountable.

In a Banach space of functions a property of functions is said to be typical if the set of all functions possessing this property is residual. The best known example is nondifferentiability: the set of functions in $C[0,1]$, the Banach space of continuous functions on $[0,1]$ with the supremum norm, that do not possess a derivative at any point is residual. Hence nowhere differentiability is a typical property of continuous functions.

In 1964 Neugebauer first studied typical properties of symmetric functions and showed the following theorem. See [1, p. 151].

Theorem 16 *Let $BS[a,b]$ be the set of all bounded measurable, symmetric functions equipped with the supremum metric. Then the typical function $f \in BS[a,b]$ has a dense set of points of discontinuity.*

By using his methods we can easily get a typical result for symmetrically continuous functions as follows.

Theorem 17 *Let $BSC[a,b]$ be the set of all bounded measurable, symmetrically continuous functions equipped with the supremum metric. Then the typical function $f \in BSC[a,b]$ has a dense set of points of discontinuity.*

By using the Preiss and Tran constructions we give an elementary proof to show that the typical functions of symmetrically continuous functions and symmetric functions have c-dense sets of points of discontinuity. This answers two open problems posed by Thomson in [1, p. 422]:

- *In the space of bounded, symmetric functions with supremum norm does the typical function have a c-dense set of discontinuities?*
- *In the space of bounded, symmetrically continuous functions with supremum norm does the typical function have a c-dense set of discontinuities?*

Throughout this chapter, $BSC[a, b]$ denotes the set of all bounded measurable, symmetrically continuous functions defined on the interval $[a, b]$ and equipped with the supremum metric ρ, and $BS[a, b]$ denotes the set of all bounded measurable, symmetric functions defined on $[a, b]$ and equipped with the supremum metric ρ. $D(f)$ denotes the set of points of discontinuity of function f. A^c denotes the complement of a set A. Let

$$C_1 = \{f \in BSC[a, b] : f \text{ has continuum points of discontinuity on } [a, b]\}$$

$$C_2 = \{f \in BS[a, b] : f \text{ has continuum points of discontinuity on } [a, b]\}$$

Lemma 18 (Tran [18]) *There are functions $g_1 \in BSC[a, b]$ and $g_2 \in BS[a, b]$ both of which have continuum points of discontinuity in every subinterval of $[a, b]$.*

Proof. Tran gave a construction of a function $g \in BS[a, b]$ for which $D(g)$ is uncountable and constructed g_2 from g. In the same way we can constructed g_2 from the absolute value function of the function g. We can also use the Preiss construction, Theorem 15 to construct a function g_1 as in the lemma. In fact, let $\{(a_n, b_n)\}$ be the set of all intervals with rational endpoints. For every n there are a uncountable N-set $E_n \subseteq (a_n, b_n)$ and a function f_n such that $0 \leq f_n \leq 1$, $f_n(x) > 0$ for all $x \in E_n$ and f_n is discontinuous precisely on E_n. Since E_n is a uncountable set of type F_σ, E_n is of power c. Set

$$g_1 = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$$

Since any N-set is of measure zero it is easy to see that g_1 is discontinuous precisely on the set $\{x \in [a, b] : g_1(x) > 0\}$ from Stein and Zygmund theorem, Theorem 13. Note

$$\{x \in [a, b] : g_1(x) > 0\} = \bigcup_{n=1}^{\infty} E_n.$$

For every interval (c,d) there is an interval $(a_n,b_n) \subseteq (c,d)$ such that the set $D(f_n)\bigcap(a_n,b_n)$ is of power c. So the set

$$D(g_1)\bigcap(c,d) \supseteq D(f_n)\bigcap(a_n,b_n)$$

is of power c. Therefore $D(g_1)$ is a c-dense set. □

Lemma 19 *For any interval $[a,b]$ there are an interval $[c,d]$ contained in $[a,b]$ and a bounded measurable, symmetrically continuous function f on the real line such that the function f has a c-dense set of points of discontinuity on $[c,d]$, $f(x)$ is continuous at each point of $[a,b]\setminus(c,d)$ and $f(x)=0$ if $x \notin (a,b)$.*

Proof. For any interval $[a,b]$, apply Lemma 18 to obtain a function $f_1 \in BSC[a,b]$ such that f_1 has a c-dense set of points of discontinuity on $[a,b]$. By the Stein-Zygmund theorem, Theorem 13 we know that f_1 is continuous almost everywhere on $[a,b]$. Then we can choose points c and d such that $a<c<d<b$ and f_1 is continuous at the points c and d. Set

$$f(x)=\begin{cases} f_1(x) & \text{if } c<x<d \\ \text{linear segment connecting } (a,0) \text{ and } (c,f_1(c)) & \text{if } a\le x\le c \\ \text{linear segment connecting } (d,f_1(d)) \text{ and } (b,0) & \text{if } d\le x\le b \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that the function f satisfies our requirements. □

Lemma 20 *For any interval $[a,b]$ there are an interval [c,d] contained in $[a,b]$ and a bounded measurable symmetric function f such that f has a c-dense set of points of discontinuity on [c,d], $f(x)$ is continuous at each point of $[a,b]\setminus(c,d)$ and $f(x)=0$ if $x\notin(a,b)$.*

Proof. Using a similar method to that in Lemma 19 and Theorem 14, we can obtain the result easily. □

Theorem 21 *The sets C_1, C_2 are open dense sets in $BSC[a,b]$ and $BS[a,b]$ respectively.*

Proof. We first show that C_1 is an open dense set in $BSC[a,b]$. Let $\{f_n\} \subseteq C_1^c$ be a Cauchy sequence. Then there is a function $f \in BSC[a,b]$ such that $f_n \longrightarrow f$ uniformly. Let e_n denote the set of points at which f_n is discontinuous, then the power of e_n is less than c. Thus the power of $\bigcup_{n=1}^{\infty} e_n$ is less than c. Since f_n converges to f uniformly on $[a,b]$ we know that f is continuous at each point $x \in [a,b] \setminus \bigcup_{n=1}^{\infty} e_n$, so $f \in C_1^c$. Hence C_1^c is closed and C_1 is open.

Now we show that C_1 is dense in $BSC[a,b]$. For every ball $B(f,\epsilon) \subseteq BSC[a,b]$, if $f \in C_1$ there is nothing to prove. We assume $f \in C_1^c$, since each uncountable Borel set has power c and the set of points of discontinuity of the function f is of type F_σ, then f has at most countable points of discontinuity. From the Lemma 19 there is a function $g \in BSC[a,b]$ such that g has a c-dense set of points of discontinuity on $[a,b]$. Let M be a constant M such that $|g(x)| \leq M$ for all $x \in [a,b]$ and set

$$h = f + \frac{\epsilon}{2M} g.$$

Then $h \in BSC[a,b]$, h has uncountable many points of discontinuity on $[a,b]$ and

$$\rho(h,f) = \rho(f + \left[\frac{\epsilon}{2M}\right] g, f) = \rho(\left[\frac{\epsilon}{2M}\right] g, 0) < \epsilon$$

where ρ is the supremum metric on $BSC[a,b]$. Thus $h \in C_1$ and hence C_1 is dense.

We now show that C_2 is an open dense set in $BS[a,b]$. Let $\{f_n\} \subseteq C_2^c$ be a Cauchy sequence. Then there is a function $f \in BS[a,b]$ such that $f_n \longrightarrow f$ uniformly. Let e_n denote the set of points at which f_n is discontinuous, then the power of e_n is less than c. Thus the power of $\bigcup_{n=1}^{\infty} e_n$ is less than c. Since f_n converges to f uniformly on $[a,b]$ we know that f is continuous at each point $x \in [a,b] \setminus \bigcup_{n=1}^{\infty} e_n$, so $f \in C_2^c$. Hence C_2^c is closed and C_2 is open.

Now we show that C_2 is dense in $BS[a,b]$. For every ball $B(f,\epsilon) \subseteq BS[a,b]$, if $f \in C_2$ there is nothing to prove. We assume $f \in C_2^c$, since each uncountable Borel set has power c and the set of points of discontinuity of the function f is of type F_σ, then f has at most countable points of discontinuity. From the Lemma 20 there is a function $g \in BS[a,b]$ such that g has a c-dense set of points of discontinuity on $[a,b]$. Let M be a constant M such that $|g(x)| \leq M$ for all $x \in [a,b]$ and set

$$h = f + \frac{\epsilon}{2M} g.$$

Then $h \in BS[a,b]$, h has uncountable many points of discontinuity on $[a,b]$ and

$$\rho(h,f) = \rho(f + \left[\frac{\epsilon}{2M}\right] g, f) = \rho(\left[\frac{\epsilon}{2M}\right] g, 0) < \epsilon$$

where ρ is the supremum metric on $BS[a,b]$. Thus $h \in C_2$ and hence C_2 is dense. □

From Theorem 16, Theorem 17 and Theorem 21 we have the following corollary.

Corollary 22 *The typical functions in $BSC[a,b]$ and $BS[a,b]$ have dense sets of points of discontinuity with power c.*

We now show some stronger results than those of Corollary 22.

Theorem 23 *The typical function $f \in BSC[a,b]$ has a c-dense set of points of discontinuity.*

Proof. Let

$$A_n = \left\{ f \in BSC[a,b] : \begin{array}{l} \text{there exists an interval } (c,d) \subseteq (a,b) \text{ with} \\ d - c \geq 1/n \text{ such that } f \text{ has a set of points of} \\ \text{discontinuity on } (c,d) \text{ with power less than } c \end{array} \right\}$$

We will prove first that A_n is closed. If $\{f_k\}$ is a Cauchy sequence from A_n there is a function $f \in BSC[a,b]$ such that f_k converges to f uniformly. Since $f_k \in A_n$ there is an interval $(c_k, d_k) \subseteq [a,b]$ with $d_k - c_k \geq 1/n$ such that f_k has a set of points of discontinuity on (c_k, d_k) with power less than c. An elementary compactness argument shows that there is a subsequence $\{(c_{k_i}, d_{k_i})\}$ of $\{(c_k, d_k)\}$ such that the sequences $\{c_{k_i}\}$ and $\{d_{k_i}\}$ are both monotonic. For convenience we also use $\{(c_k, d_k)\}$ to denote the above subsequence $\{(c_{k_i}, d_{k_i})\}$. Then there are $c, d \in [a,b]$ such that $c_k \longrightarrow c$, $d_k \longrightarrow d$ and $d - c \geq 1/n$. It is easy to check that

$$(c,d) \subseteq \bigcup_{k=1}^{\infty} (c_k, d_k).$$

Let us denote $e_k \subseteq (c_k, d_k)$ as the set of points at which f_k is discontinuous. Then the power of e_k is less than c and thus the power of $\bigcup_{k=1}^{\infty} e_k$ is less than c. It is obvious that f is continuous at each point of the set

$$(c,d) \setminus \bigcup_{k=1}^{\infty} e_k.$$

So $f \in A_n$ and therefore A_n is closed.

We now show that A_n^c is dense. Let $B(f,\epsilon)$ be an arbitrary ball contained in $BSC[a,b]$. If $f \in A_n^c$ then there is nothing to prove. We can assume $f \in A_n$. Suppose $I_1, \ldots, I_k$ contained in $[a,b]$ are disjoint maximal intervals with length at least $1/n$ such that f has a set of points of discontinuity on each I_i with power less than c. Then these intervals have no common endpoints. For each $I_i = (c,d)$, apply Lemma 19 to obtain a function $g_i \in BSC[a,b]$ such that g_i has a c-dense set of points of discontinuity on an subinterval J_i of I_i, g_i is continuous at each point of $I_i \setminus J_i$ and $g_i(x) = 0$ if $x \notin I_i$. Set

$$g = \sum_{i=1}^{k} g_i.$$

Then $g \in BSC[a,b]$ and $g = 0$ if $x \notin \bigcup_{i=1}^{k} I_i$. Let M be a constant such that $|g(x)| \leq M$ and set

$$h = f + \frac{\epsilon}{2M} g.$$

Then $h \in BSC[a,b]$ and $h(x) = f(x)$ if $x \notin [a,b] \setminus \bigcup_{i=1}^{k} I_i$.

We now show that the set $D(h) \bigcap I$ is of power c for any interval I with length at least $1/n$. For any interval I with length at least $1/n$ if the set $D(g) \bigcap (I \bigcap I_i)$ is of power c for some $I_i \in \{I_1, \ldots, I_k\}$, then the set $D(h) \bigcap (I \bigcap I_i)$ is of power c since the set $D(f) \bigcap (I \bigcap I_i)$ is of power less than c. Thus the set $D(h) \bigcap I$ is of power c. If the set $D(g) \bigcap (I \bigcap I_i)$ is of power less than c for every I_i ($i = 1, 2, \ldots, k$), then the function g is continuous at each point of the set $I \bigcap \bigcup_{i=1}^{k} I_i$. Hence in such case if the set $D(f) \bigcap (I \setminus \bigcup_{i=1}^{k} I_i)$ is of power c then the set $D(h) \bigcap I$ is of power c since $g(x) = 0$ if $x \notin \bigcup_{i=1}^{k} I_i$. If the set $D(f) \bigcap (I \setminus \bigcup_{i=1}^{k} I_i)$ is of power less than c then the set $D(f) \bigcap I$ is of power less than c since the set $D(f) \bigcap I_i$ is of power less than c for every $i = 1, \ldots, k$. Therefore I is contained in some I_i of $\{I_i, i = 1, \ldots, k\}$ from the maximality of $\{I_i, i = 1, \ldots, k\}$. This contradicts the assumption that the function g is continuous at each point of the set $I \bigcap \bigcup_{i=1}^{k} I_i$. In any case the set $D(h) \bigcap I$ is of power c and so $h \notin A_n$, i.e. $h \in A_n^c$. Note that

$$\rho(h, f) = \rho(f + \left[\frac{\epsilon}{2M}\right] g, f) = \rho(\frac{\epsilon}{2M} g, 0) < \epsilon.$$

This means that $h \in B(f, \epsilon)$ and A_n^c is dense. Therefore A_n is nowhere dense.

Since the set of all functions which have c-dense sets of points of discontinuity is the set

$$BSC[a, b] \setminus \bigcup_{n=1}^{\infty} A_n,$$

the result follows. □

Theorem 24 *The typical function $f \in BS[a, b]$ has a c-dense set of points of discontinuity.*

Proof. Let

$$A_n = \left\{ f \in BS[a,b] : \begin{array}{c} \text{there exists an interval } (c,d) \subseteq (a,b) \text{ with} \\ d - c \geq 1/n \text{ such that } f \text{ has a set of points of} \\ \text{discontinuity on } (c,d) \text{ with power less than } c \end{array} \right\}$$

We will prove first that A_n is closed. If $\{f_k\}$ is a Cauchy sequence from A_n there is a function $f \in BS[a, b]$ such that f_k converges to f uniformly. Since $f_k \in A_n$ there is an interval $(c_k, d_k) \subseteq [a, b]$ with $d_k - c_k \geq 1/n$ such that f_k has a set of points of discontinuity on (c_k, d_k) with power less than c. An elementary compactness argument shows that there is a subsequence $\{(c_{k_i}, d_{k_i})\}$ of $\{(c_k, d_k)\}$ such that the sequences $\{c_{k_i}\}$ and $\{d_{k_i}\}$ are both monotonic. For convenience we also use $\{(c_k, d_k)\}$ to denote the above subsequence $\{(c_{k_i}, d_{k_i})\}$. Then there are $c, d \in [a, b]$ such that $c_k \longrightarrow c$, $d_k \longrightarrow d$ and $d - c \geq 1/n$. It is easy to check that

$$(c, d) \subseteq \bigcup_{k=1}^{\infty} (c_k, d_k).$$

Let us denote $e_k \subseteq (c_k, d_k)$ as the set of points at which f_k is discontinuous. Then the power of e_k is less than c and thus the power of $\bigcup_{k=1}^{\infty} e_k$ is less than c. It is obvious that f is continuous at each point of the set

$$(c,d) \setminus \bigcup_{k=1}^{\infty} e_k.$$

So $f \in A_n$ and therefore A_n is closed.

We now show that A_n^c is dense. Let $B(f,\epsilon)$ be an arbitrary ball contained in $BS[a,b]$. If $f \in A_n^c$ then there is nothing to prove. We can assume $f \in A_n$. Suppose $I_1, \ldots, I_k$ contained in $[a,b]$ are disjoint maximal intervals with length at least $1/n$ such that f has a set of points of discontinuity on each I_i with power less than c. Then these intervals have no common endpoints. For each $I_i = (c,d)$, apply Lemma 20 to obtain a function $g_i \in BS[a,b]$ such that g_i has a c-dense set of points of discontinuity on an subinterval J_i of I_i, g_i is continuous at each point of $I_i \setminus J_i$ and $g_i(x) = 0$ if $x \notin I_i$. Set

$$g = \sum_{i=1}^{k} g_i.$$

Then $g \in BS[a,b]$ and $g = 0$ if $x \notin \bigcup_{i=1}^{k} I_i$. Let M be a constant such that $|g(x)| \leq M$ and set

$$h = f + \frac{\epsilon}{2M} g.$$

Then $h \in BS[a,b]$ and $h(x) = f(x)$ if $x \notin [a,b] \setminus \bigcup_{i=1}^{k} I_i$.

We now show that the set $D(h) \bigcap I$ is of power c for any interval I with length at least $1/n$. For any interval I with length at least $1/n$ if the set $D(g) \bigcap (I \bigcap I_i)$ is of power c for some $I_i \in \{I_1, \ldots, I_k\}$, then the set $D(h) \bigcap (I \bigcap I_i)$ is of power c since the set $D(f) \bigcap (I \bigcap I_i)$ is of power less than c. Thus the set $D(h) \bigcap I$ is of power c. If the set $D(g) \bigcap (I \bigcap I_i)$ is of power less than c for every I_i $(i = 1, 2, \ldots, k)$, then the function g is continuous at each point of the set $I \bigcap \bigcup_{i=1}^{k} I_i$. Hence in such case if the set $D(f) \bigcap (I \setminus \bigcup_{i=1}^{k} I_i)$ is of power c then the set $D(h) \bigcap I$ is of power c since $g(x) = 0$ if $x \notin \bigcup_{i=1}^{k} I_i$. If the set $D(f) \bigcap (I \setminus \bigcup_{i=1}^{k} I_i)$ is of power less than c then the set $D(f) \bigcap I$ is of power less than c since the set $D(f) \bigcap I_i$ is of power less than c for every $i = 1, \ldots, k$. Therefore I is contained in some I_i of $\{I_i, i = 1, \ldots, k\}$ from the maximality of $\{I_i, i = 1, \ldots, k\}$. This contradicts the assumption that the function g is continuous at each point of the set $I \bigcap \bigcup_{i=1}^{k} I_i$. In any case the set $D(h) \bigcap I$ is of power c and so $h \notin A_n$, i.e. $h \in A_n^c$. Note that

$$\rho(h, f) = \rho(f + \left[\frac{\epsilon}{2M}\right] g, f) = \rho(\frac{\epsilon}{2M} g, 0) < \epsilon.$$

This means that $h \in B(f, \epsilon)$ and A_n^c is dense. Therefore A_n is nowhere dense.

Since the set of all functions which have c-dense sets of points of discontinuity is the complement of the first category set $\bigcup_{n=1}^{\infty} A_n$, the result follows. □

Chapter 5

An Application of The Baire Category Theorem

In 1899 R. Baire first formulated his famous category theorem on the real line. It can be extended to complete metric spaces as follows. See [2, p. 377].

Theorem 25 (Baire Category Theorem) *Let (X, ρ) be a complete metric space, and S be a countable union of nowhere dense sets in X. Then the complement of S is dense in X.*

The principal use for the notion of category is in the formulation of existence proofs. The Baire category theorem has been a basic tool for us to see mathematical objects which otherwise may be difficult to see. By using Baire Category Theorem S. Banach and S. Mazurkiewicz first gave an existence proof of continuous functions that have no points of differentiability.

Let us use the following expressions,

$$D^1 f(x,h) = [f(x+h) - f(x-h)]/h,$$

$$D^2 f(x,h) = [f(x+h) + f(x-h) - 2f(x)]/h.$$

In 1969 Filipczak in [7] constructed a continuous function f defined on [0,1] which satisfies for each $x \in (0,1)$,

$$\limsup_{h \to 0} D^1 f(x,h) = +\infty.$$

In 1972 Kostyrko in [10] used this example to show that the typical function $f \in C[0,1]$, the set of all real continuous functions with the supremum metric, satisfies for each $x \in (0,1)$,

$$\limsup_{h\to 0} D^1 f(x,h) = +\infty, \qquad \liminf_{h\to 0} D^1 f(x,h) = -\infty.$$

In 1987 Evans [6, Theorem 1] constructed a function $f \in C[0,1]$ which satisfies that for each $x \in (0,1)$,

$$ap\limsup_{h\to 0^+} D^1 f(x,h) = +\infty, \qquad ap\liminf_{h\to 0^+} D^1 f(x,h) = -\infty,$$

$$ap\limsup_{h\to 0^+} |D^2 f(x,h)| = +\infty.$$

He used this example to show that such functions are typical in $C[0,1]$.

In this chapter we directly show that the typical function $f \in C[0,1]$ satisfies for each $x \in (0,1)$,

$$(1) \quad \limsup_{h\to 0} |D^1 f(x,h)| = +\infty, \qquad (2) \quad \limsup_{h\to 0} |D^2 f(x,h)| = +\infty$$

without using the constructions of Filipczak and Evans. An application of the Baire category theorem to the space $C[0,1]$ yields the existence of a function $f \in C[0,1]$ satisfying (1) and (2) for each $x \in (0,1)$.

Lemma 26 *Let $f \in C[0,1]$, n be a positive integer, m and ϵ be two given positive constants. Then there exists a finite piecewise linear function $g \in C[0,1]$ such that for each $x \in [0,1]$, $|f(x) - g(x)| < \epsilon$ and for each $x \in [1/n, 1-1/n]$, $|D^1 g(x,h)| > m$ for some h with $0 < |h| < 1/n$.*

Proof. Since the function f is continuous on [0,1], it is uniformly continuous on [0,1]. For $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$|f(x_1) - f(x_2)| < \epsilon/16$$

whenever $x_1, x_2 \in [0,1], |x_1 - x_2| < \delta_1$. Take

$$\delta = \min\{\frac{\epsilon}{4m}, \frac{\delta_1}{5}, \frac{1}{5n}\}$$

and partition [0,1] as

$$0 = x_0 < x_1 < \cdots < x_k = 1.$$

Here $x_i - x_{i-1} = \delta$ if i is an odd number, $x_i - x_{i-1} = 3\delta$ if i is an even number except $i \neq k$. If k is an even number, $x_k - x_{k-1} = \delta$ or 2δ or 3δ depending on how many subintervals we get if we partition [0,1] into subintervals with length δ.

Let g be a finite piecewise linear function which connects the following points $a_0, a_1, \ldots, a_{k-1}, a_k$. Here

$$a_i = (x_i, f(x_i) + (-1)^i \epsilon/2), i = 0, 1, \ldots, k-1.$$

If k is an odd number,

$$a_k = (x_k, f(x_k) - \epsilon/2).$$

If k is an even number, a_k is the intersection point of the line $x = 1$ with the half line starting from a_{k-1} and parallel to the segment $a_{k-3}a_{k-2}$. See the figure (i). In the figure $r = \epsilon$.

We now verify that the function g satisfies our requirements. Obviously g is a finite piecewise linear, continuous function and for each $x \in [0,1]$,

$$|f(x) - g(x)| < \epsilon.$$

For any i, noting that $\delta \leq \frac{\epsilon}{4m}$ we have

$$|f(x_i) + (-1)^i \epsilon/2 - (f(x_{i-1}) + (-1)^{i-1}\epsilon/2)|$$

$$\geq \epsilon - |f(x_i) - f(x_{i-1})| > (15/16)\epsilon$$

and

$$|f(x_i) + (-1)^i \epsilon/2 - (f(x_{i-1}) + (-1)^{i-1}\epsilon/2)|$$

$$\leq \epsilon + |f(x_i) - f(x_{i-1})| < (17/16)\epsilon,$$

and therefore

$$|\text{slop}a_{i-1}a_i + \text{slop}a_i a_{i+1}| = ||\text{slop}a_{i-1}a_i| - |\text{slop}a_i a_{i+1}||$$

$$\geq \frac{(15/16)\epsilon}{\delta} - \frac{(17/16)\epsilon}{3\delta} = \frac{28\epsilon}{48\delta} > m$$

where slopab denotes the slope of the segment connecting the points a and b. Hence for $0 < i < k$, if $x = x_i$ or x_{i+1} choose $h = \delta$ then $|D^1 g(x,h)| > m$. If $x \in (x_i, x_{i+1})$, we can choose $h = \min\{x - x_i, x_{i+1} - x\}$ then

$$|D^1 g(x,h)| = |\frac{g(x+h)-g(x)}{h}| + |\frac{g(x-h)-g(x)}{h}|$$

$$\geq (2)\frac{(3/4)\epsilon - (1/16)\epsilon}{3\delta} = \frac{11\epsilon}{24\delta} > m.$$

If $x \in (x_{i+1}, x_{i+2})$ choose $h = \min\{x - x_{i+1}, x_{i+2} - x\}$ the same method for $x \in (x_i, x_{i+1})$ yields that $|D^1 g(x,h)| > m$. Thus the function g satisfies our requirements and the lemma follows. □

Lemma 27 *Let $f \in C[0,1]$, n be a positive integer, m and ϵ be two given positive constants. Then there exists a finite piecewise function $g \in C[0,1]$ such that for each $x \in [0,1], |f(x) - g(x)| < \epsilon$ and for each $x \in [1/n, 1-1/n], |D^2 g(x,h)| > m$ for some h with $0 < |h| < 1/n$.*

Proof. Again the function f is uniformly continuous on [0,1]. For $\epsilon > 0$ there exists $\delta_1 > 0$ such that

$$|f(x_1) - f(x_2)| < \epsilon/16$$

whenever $x_1, x_2 \in [0,1], |x_1 - x_2| < \delta_1$. Take

$$\delta = \min\{\frac{\epsilon}{6m}, \frac{\delta_1}{10}, \frac{1}{10n}\}$$

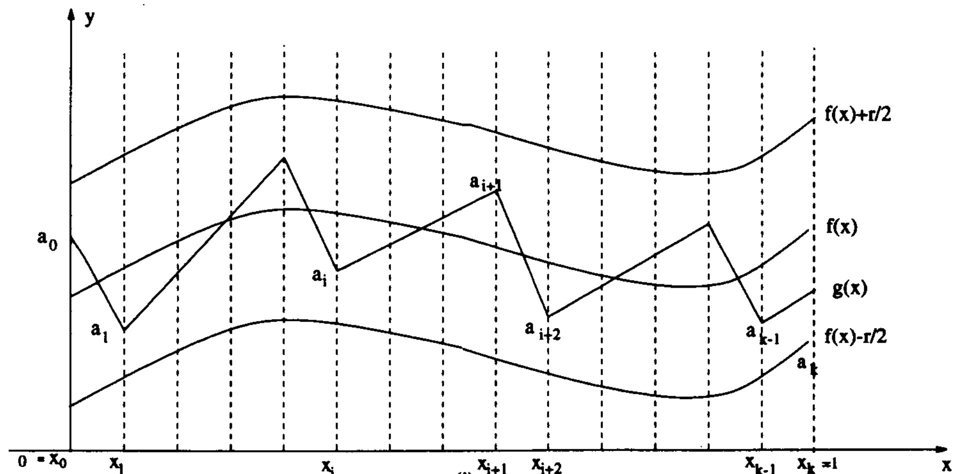
and partition [0,1] as

$$0 < x_0 < x_1 < \cdots < x_k = 1.$$

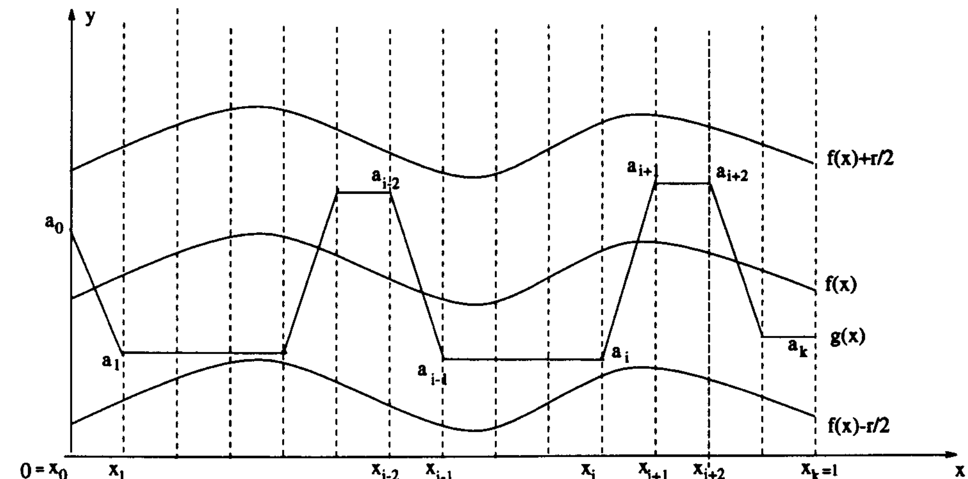
Here $x_i - x_{i-1} = \delta$ if i is not a number of the form $4l+2$, l is a nonnegative integer. If i is a number of the form $4l+2$, $x_i - x_{i-1} = 3\delta$ except $k = 4l+2$. If k is a number of form $4l+2$, $x_k - x_{k-1} = \delta$ or 2δ or 3δ depending on how many subintervals we get if we partition [0,1] into subintervals with length δ.

Let g be a finite piecewise linear function which connects the following points $a_0, a_1, a_2, \ldots, a_k$. Here

$$a_0 = (x_0, f(x_0) + (3/8)\epsilon), \quad a_1 = (x_1, f(x_1) - (3/8)\epsilon).$$



(i)



(ii)

The point a_2 is the intersection point of the line $x = x_2$ with the half line starting from the point a_1 and parallel to the x-axis,

$$a_3 = (x_3, f(x_3) + (3/8)\epsilon),$$

a_4 is the intersection point of the line $x = x_4$ with the half line starting from the point a_3 and parallel to the x-axis,

$$a_5 = (x_5, f(x_5) - (3/8)\epsilon).$$

Similarly as for a_2 we can define a_6, and continue in this way to get $a_0, a_1, a_2, \ldots, a_k$. See the figure (ii). In the figure $r = \epsilon$.

We now verify that the function g satisfies our requirements. Obviously g is a finite piecewise linear, continuous function and for each $x \in [0,1]$,

$$|f(x) - g(x)| < \epsilon.$$

For the left we need to verify that for each $x \in [x_{i-2}, x_{i+2}]$ as indicated in the figure (ii), $|D^2 g(x,h)| > m$ for some h with $0 < |h| < 1/n$. We can assume $3 < i < k-3$ since $x \in [1/n, 1-1/n]$ and $\delta \leq \frac{1}{10n}$. For $x \in [x_{i-1}, x_i]$, choose $h = \min\{x - x_{i-2}, x_{i+1} - x\}$ and note $\delta \leq \frac{\epsilon}{6m}$,

$$|D^2 g(x,h)| = |\frac{g(x+h) - g(x)}{h}| + |\frac{g(x) - g(x-h)}{h}|$$
$$\geq \frac{(3/4)\epsilon - (1/16)\epsilon}{(5/2)\delta} = \frac{11\epsilon}{40\delta} > m.$$

Partition $[x_i, x_{i+1}]$ into three equal subintervals $[x_i, x^1], [x^1, x^2], [x^2, x_{i+1}]$. For $x \in [x_i, x^1]$, choose $h = x_{i+1} - x$ then

$$|D^2 g(x,h)| = |\frac{g(x+h) - g(x)}{h}| - |\frac{g(x) - g(x-h)}{h}|$$
$$\geq (1 - 1/3)\frac{(3/4)\epsilon - (1/16)\epsilon}{\delta} = \frac{11\epsilon}{24\delta} > m.$$

For $x \in [x^1, x^2]$, choose $h = x_{i+3} - x$ then

$$|D^2 g(x,h)| = |\frac{g(x+h) - g(x)}{h}| + |\frac{g(x) - g(x-h)}{h}|$$
$$\geq 2[\frac{(1/3)((3/4)\epsilon - (1/16)\epsilon)}{(2 + (2/3))\delta}] = \frac{11\epsilon}{64\delta} > m.$$

For $x \in [x^2, x_{i+1}]$, choose $h = x - x_i$ then

$$|D^2 g(x,h)| = |\frac{g(x-h) - g(x)}{h}| - |\frac{g(x+h) - g(x)}{h}|$$
$$\geq (1 - 1/3)\frac{(3/4)\epsilon - (1/16)\epsilon}{\delta} > m.$$

For $x \in [x_{i+1}, x_{i+2}]$, choose $h = \min\{x - x_i, x_{i+3} - x\}$ then

$$|D^2 g(x,h)| = |\frac{g(x+h) - g(x)}{h}| + |\frac{g(x-h) - g(x)}{h}|$$

$$\geq \frac{(3/4)\epsilon - (1/16)\epsilon}{2\delta} = \frac{11\epsilon}{32\delta} > m.$$

For $x \in [x_{i-2}, x_{i-1}]$ using the same method for $x \in [x_i, x_{i+1}]$ we can show that the function g satisfies our requirements. Hence the lemma follows. □

Theorem 28 *The typical function $f \in C[0,1]$ satisfies (1) for all $x \in (0,1)$.*

Proof. Let

$$A = \left\{ f \in C[0,1] : \begin{array}{l} \text{there exist some point } x \in (0,1) \text{ and constant } C \\ \text{such that } \limsup_{h\to 0} |D^1 f(x,h)| \leq C \end{array} \right\},$$

$$A_{nm} = \left\{ f \in C[0,1] : \begin{array}{l} \text{there exists some } x \in [1/n, 1-1/n] \text{ such that} \\ |D^1 f(x,h)| \leq m \text{ whenever } 0 < |h| < 1/n \end{array} \right\}.$$

Then

$$A = \bigcup_{n,m=1}^{\infty} A_{nm}.$$

For any pair (n,m) and any Cauchy sequence $\{f_k\}$ from the set A_{nm} there exists a function $f \in C[0,1]$ such that f_k converges to f uniformly. For each f_k there exists a point $x_k \in [1/n, 1-1/n]$ such that for all $0 < |h| < 1/n$, $|D^1 f(x_k, h)| \leq m$. An elementary compactness argument shows that there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that x_{k_i} converges to some point $x \in [1/n, 1-1/n]$. Thus it is easy to see that for all $0 < |h| < 1/n$, $|D^1 f(x,h)| \leq m$. Hence $f \in A_{nm}$ and A_{nm} is closed. We now show that the complement of A_{nm} is dense in $C[0,1]$. For any function $f \in A_{nm}$ and $\epsilon > 0$, by Lemma 26 we can find a function g such that for all $x \in [0,1]$,

$$|f(x) - g(x)| < \epsilon$$

and $g \notin A_{nm}$. Thus the complement of A_{nm} is dense in $C[0,1]$ and the theorem follows. □

Theorem 29 *The typical function $f \in C[0,1]$ satisfies (2) for all $x \in (0,1)$.*

Proof. Let

$$A = \left\{ f \in C[0,1] : \begin{array}{l} \text{there exist some point } x \in (0,1) \text{ and constant } C \\ \text{such that } \limsup_{h \to 0} |D^2 f(x,h)| \le C \end{array} \right\},$$

$$A_{nm} = \left\{ f \in C[0,1] : \begin{array}{l} \text{there exists some } x \in [1/n, 1-1/n] \text{ such that} \\ |D^2 f(x,h)| \le m \text{ whenever } 0 < |h| < 1/n, \end{array} \right\}.$$

Then

$$A = \bigcup_{n,m=1}^{\infty} A_{nm}.$$

For any pair (n,m) and any Cauchy sequence $\{f_k\}$ from the set A_{nm} there exists a function $f \in C[0,1]$ such that f_k converges to f uniformly. For each f_k there exists a point $x_k \in [1/n, 1-1/n]$ such that for all $0 < |h| < 1/n$, $|D^2 f(x_k, h)| \le m$. An elementary compactness argument shows that there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that x_{k_i} converges to some point $x \in [1/n, 1-1/n]$. Thus it is easy to see that for all $0 < |h| < 1/n$, $|D^2 f(x,h)| \le m$. Hence $f \in A_{nm}$ and A_{nm} is closed. We now show that the complement of A_{nm} is dense in $C[0,1]$. For any function $f \in A_{nm}$ and $\epsilon > 0$, by Lemma 27 we can find a function g such that for all $x \in [0,1]$,

$$|f(x) - g(x)| < \epsilon$$

and $g \notin A_{nm}$. Thus the complement of A_{nm} is dense in $C[0,1]$ and the theorem follows. □

Corollary 30 *The typical function $f \in C[0,1]$ satisfies (1) and (2) for all $x \in (0,1)$.*

Proof. Since the intersection set of two residual sets in $C[0,1]$ is also residual in $C[0,1]$, the corollary follows. □

Corollary 31 *There exists a function $f \in C[0,1]$ satisfying (1) and (2) for all $x \in (0,1)$.*

Proof. By applying Corollary 30 and the Baire category theorem, Theorem 25, the existence of functions in $C[0,1]$ satisfying (1) and (2) for all $x \in (0,1)$ follows. □

Chapter 6

Conclusion

In Chapter 2 the results of Buczolich-Laczkovich [3] and Thomson [1, p. 276] were extended to a general case. Up to now the natural structure of the range of the symmetric derivatives is unknown. As in our mind the symmetric derivatives should have some similar but weaker property than the Darboux property of the ordinary derivatives. To this we pose a open problem as follows.

Open Problem 32 *Let A be a infinite set contained in R for which no element can be the average of any other two elements. Does there exist a function whose symmetric derivative assumes values exactly the set A?*

In Chapter 3 we discussed the properties of the right locally antisymmetric sets. We know from [13, p. 82, p. 261] that every Hamel basis has power of *continuum* and that there exists a Hamel basis which is of first category and measure zero. Thus under the *Axiom of Choice* there exists a right locally antisymmetric set which has power of *continuum*. We want to know the following.

Open Problem 33 *Does there exist an uncountable, right locally antisymmetric set without the Axiom of Choice?*

In Chapter 4 we showed that the typical functions of symmetrically continuous functions and symmetric functions have c-dense sets of points of discontinuity. It is an interesting, prospective topic for us to look for the typical properties of some classes of real functions and discuss the properties of typical functions. In Chapter 5 we gave an existence proof of nowhere symmetrically continuous functions and nowhere quasi-smooth functions. It is unknown that the methods used can be applied to the study of other questions. This is the thing we want to try in the near future. Here we give two interesting problems.

Open Problem 34 *Is the typical function $f \in BS[a,b]$ nowhere quasi-smooth in (a,b)? or smooth in (a,b)? Here $BS[a,b]$ denotes the set of all bounded measurable, symmetric functions defined on the interval $[a,b]$ and equipped with the supremum metric.*

Open Problem 35 *Does the typical function $f \in BSC[a,b]$ have a symmetric derivative at no points in $[a,b]$? Here $BSC[a,b]$ denotes the set of all bounded measurable, symmetrically continuous functions defined on $[a,b]$ and equipped with the supremum metric.*

Bibliography

[1] B. S. Thomson, *Symmetric Properties of Real Functions*, Marcel Dekker Inc., New York, 1994.

[2] A. M. Bruchner, J. B. Bruckner and B. S. Thomson, *Real Analysis*, to appear.

[3] Z. Buczolich and M. Laczkovich, Concentrated Borel measures, Acta Math., Hungar., 57 (1991), 349-362.

[4] K. Ciesielski and L. Larson, Uniformly antisymmetric functions, Real Analysis Exchange, 19 (1993-94), 226-235.

[5] R. O. Davies, Symmetric sets are measurable, Real Anaalysis Exchange, 2 (1987-89), 87-89.

[6] Michael J. Evans, Approximate smoothness of continuous functions, Colloquium Mathematicum 54 (1987), 307-313.

[7] L. Filipczak, Example d'un fonction continue privée de dérivée symétrique partout, Colloquium Mathematicum 20 (1969), 249-253.

[8] F. Hausdorff, Probléme No. 62, Fund. Math., 25 (1935), 578.

[9] P. Kostyrko, There is no strongly antisymmetric set, Real Analysis Exchange 17 (1991), 423-426.

[10] P. Kostyrko, On the symmetric derivative, Colloquium Mathematicum 25 (1972), 265-267.

[11] P. Kostyrko, The symmetric derivative and the Darboux property, Real Analysis Exchange 13 (1987-88), 305-309.

[12] P. Komjath and S. Shelah, On uniformly antisymmetric functions, Real Analysis Exchange 19 (1993-94), 218-225.

[13] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Uniwersytet ĺaski, Warsaw, 1985.

[14] L. Larson, The symmetric derivative, Trans. American Math. Soc., 277 (1983), 589-599.

[15] S. Marcus, Symmetry in the simple case: the real line, Comput. Math. Appl., 17 (1989), 103-115.

[16] D. Preiss, A note on symmetrically continuous functions, Časopis Pěst. Mat., 96 (1971), 262-264.

[17] I. Z. Ruzsa, Locally symmetric functions, Real Analysis Exchange 4 (1978-79), 84-86.

[18] T. C. Tran, Symmetric functions whose set of points of discontinuity is uncountable, Real Analysis Exchange 12 (1986-87), 496-509.

[19] A. Zygmund, *Trigonometric Series*, Cambridge University Press, London, 1968.

[20] Real Analysis Exchange 9 (1983-84), 295, Query 172.