Quadrant Constructions and Applications in Western Europe During the Early Renaissance

by

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Abstract

During the Middle Ages and the early Renaissance in the Latin West the quadrant under its many guises was an important scientific and mathematical instrument for a number of scientific disciplines. The earliest use of the quadrant, as R.T. Gunther in his *Early Science in Oxford* suggests, was in surveying and then subsequently in the service of astronomy when horary lines were added. However, the quadrant was not limited just to the computational and mensurational needs of astronomers and surveyors. Cartographers, navigators and militiamen and bombardiers also adopted the quadrant for their work in the years to follow.

Not only was the quadrant useful in the work of the astronomer *et al.*, but in the hands of these specialists the quadrant was adapted and modified (either in the manner in which it was employed or through changes to various incorporated scales). The aim of this work will be to examine developments in the ways in which the quadrant was constructed and used during the early Renaissance from the early sixteenth century to the mid-seventeenth century in Western Europe by specialists in the areas of astronomy, surveying, navigation and military science.

Acknowledgements

I would like to express my sincerest thanks to Marjorie Nelles from Interlibrary Loans who assisted me with the numerous texts which I sought to examine in this study. Her advice and suggestions in tracking down older texts available through microforms was most helpful. I must also express my gratitude to Elaine Fairey who later took over for Marjorie Nelles for her help, and indeed all the staff in the inter-library loans division of the Simon Fraser University library deserves much thanks.

I might not have been able or inspired to construct a Gunter's quadrant from brass without the help of Dennis Michaelson from the School of Engineering Science. Despite his busy schedule, he happily obliged me with his assistance, knowledge and his precious time which he gladly gave up for me.

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Chapter 1

Introduction and Preliminaries

1.1 Central Problem

In the course of their studies, historians of science have often contemplated the mutual relationship and influence which science and technology have upon each other. If one observes the growth of Western Europe from the Late Middle Ages onward, for example, one sees a society struggling to understand its relationship to the rest of the world and the roles which science and technology play in this struggle, disciplines which allow them to investigate and affect the physical properties of the world in which they live. Today, we can look back upon that history to examine the roles of science and technology and to observe those mutual developments.

It is with this hindsight that contemporary historians of science can see influences that certain scientific instruments have had upon particular areas of science. A case in point is the scientific instrument known as the quadrant, which enjoyed a fruitful life as a practical tool for Renaissance practitioners of astronomy, surveying and other applications.

The theoretical developments of the quadrant can be broken down into three stages. The first is the acquisition of the contents of various Arabic texts; second is a period of exposition which corresponds to the university tradition of using this instrument as a pedagogical tool; the third, a time of innovation in the design and use of the quadrant, is the final stage in the development and use of the quadrant. It is this last stage which we shall examine in this work to see the professional influences on quadrant constructions and the incorporation of various scales and tables, as well as, the various uses for the quadrant. The various stages are not so clearly defined; however, the first is approximately from the 10^{th} to the 13^{th} century; the second is during the 13^{th} and 14^{th} centuries; and the final stage is from the 15^{th} century to the middle of the 17^{th} century.

The first two stages in the development of the quadrant, although not the focus of this study, are of course also important; we will provide a brief account of important developments during this time so as to set the stage for the examination of this instrument as it was used in the early renaissance.

The quadrants which will be examined here are ultimately derivations of medieval Islamic quadrants. David King describes in an article on Islamic astronomical instruments four classes of quadrants.[1] The four classes, all of which were invented by Muslim astronomers, are : (i) the sine quadrant, (ii) the horary quadrant, (iii) the astrolabic quadrant and (iv) the *shakkāziyya* quadrant. Those quadrants which will concern us here will be the horary and astrolabic quadrants with other simpler variants as the various professions borrowed different ideas from these two quadrants. In general, we will consider only hand-held vertical quadrants. As such, we will not examine, for example, the mural or large azimuthal quadrants of Tycho Brahe or horizontal quadrants used for triangulation.

Each of these quadrants functions to perform a particular set of tasks. The horary quadrant allows one to determine the seasonal daylight hours,[2] and the astrolabic quadrant appears to embody the same operations as an astrolabe. The astrolabe bears a stereographic representation of certain stars and celestial circles such as the equator and tropics, the ecliptic and altitude circles onto the plane of the equator. Astronomers discovered that they could utilize symmetry about the meridian to allow

[1] King, *Some remarks on Islamic astronomical instruments*, p. 16.

[2] The seasonal daylight hour differs from the 24th equal part of a day or a complete revolution of the earth in that it is the 12th equal part of the total daylight for a given day. Consequently, under the assumption that one makes all observations from the northern hemisphere, the observer will notice that a seasonal hour is shorter for a day when the sun has a southerly declination and longer when the sun has a northerly declination.

them to work with half of this celestial projection folded in half again, to produce a quadrant.

Each quadrant is based upon ideas related to plane and spherical trigonometry and projective geometry, the mathematical tools employed by the practitioners who constructed these quadrants. In the sections to come, we will examine briefly these ideas behind the construction of these various instruments; in the chapter on quadrant constructions, these ideas will come together in geometrical demonstrations of the validity of the various scales that appear on these quadrants.

1.2 Original Sources

The study and appreciation of scientific instruments in general is not a new phenomenon. Indeed, even during the early Middle Ages people were just as likely to collect instruments like the astrolabe and quadrant for their intrinsic value and beauty as for their utility; however, we hope here to extend this interest in scientific instruments, and more specifically in hand-held quadrants, to the investigation of original texts in the early Renaissance which demonstrate quadrant constructions and applications in various scientific disciplines. These texts, like the instrument itself, can be viewed as important sources representative of scientific enquiry and progress within fields concerned with making more accurate measurements to allow for certain theoretical and practical advances.

These original texts for the most part remain dispersed throughout various museums, universities and private collections. Hence, viewing these works, given our locality and availability of sources, has been not only difficult but next to impossible. I have, therefore, relied primarily on sources which have been made available in recent decades on microform in North America. This, however, has not provided a complete and thorough means for examining these original works. In such cases, where available, later translations and editions were used and sometimes modern commentaries. In only a few instances were we able to examine original sources first hand. These sources which we have considered were for the most part written between the early 16^{th} and mid-17^{th} centuries.

Maddison[3] suggests that the place with the largest collection of works on astronomical and mathematical instruments is the Museum of the History of Science in Oxford. Of immense importance to scholars in the history of science, the museum is a treasury for significant collections of printed works, numerous scientific instruments and information on collections found elsewhere. We note also that the Deutsches Museum in Munich possesses an important collection of Latin texts on microfilm. However, the systematic examination of printed texts and early manuscripts which are scattered throughout the world's libraries and museums awaits more researchers in this field, which itself depends on generous research funding and further interest in this field.

The texts which we have examined fall primarily in two groups. The first group consists of original works which discuss explicitly the construction of a given quadrant. (In some cases, these works also provide examples of quadrant applications in the fields of astronomy and surveying.) In the second group, we have collected texts which demonstrate various applications of the quadrant within a given discipline. Those disciplines which form the backbone of this study include astronomy, cartography, navigation and ordnance work.

After we have surveyed some of the various works available to us on the quadrant, pausing occasionally to examine more closely some important developments in quadrant applications, we will discuss in some detail an advanced form of an astronomer's quadrant known as Gunter's quadrant. Edmund Gunter's influence can be seen in many of the works which we have examined, as well as the numerous examples of quadrants which exist today. We ourselves have constructed such a quadrant based upon Gunter's instructions from *The description and use of the sector, cross-staff, and other instruments*, and a description and a printed reproduction of our quadrant is to be found in the final chapter of this work.

[3]Francis Maddison, *Early Astronomical and Mathematical Instruments*, pp. 21-22.

1.2.1 Important Scientific Works Examined Within the Context of Quadrant Developments

We will now briefly outline, by profession, some primary works which are relevant to this study. They represent those works which were influential for decades and sometimes centuries, and demonstrate the principles behind these scientific arts, and the application and construction of a quadrant best suited to these practices.

Gunnery

The first work to be written from a mathematical point of view on military ordnance which studies the motion of projectiles and their trajectories was written by Nicolo Tartaglia in *Nova Scientia* in 1537. This work by Tartaglia begins a clearer understanding of the motion of a projectile and breaks away from earlier philosophies and in some sense from the authority of Aristotle. The break from Aristotle, however, is not one of complete abandonment as is evident of other important works on military ordnance and the trajectory of projectiles. These other related works by Lucar, Smith, Santbech and Norton will provide a greater sense of the problems faced by gunners in practice and theory and how the quadrant was employed.

Astronomy

We have found most of our primary works to be astronomical in nature, and by and large works which were written in the late 16^{th} and the first half of the 17^{th} centuries. We mention Robert Recorde as an important scientific, literary and education figure who did much to aid the spread of astronomy throughout England. It is known that he intended to write a work on the quadrant, however, he must have used one to perform observations and tutor his students. The quadrant does arise in the discussion between a Master and a Scholar in his *The Castle of Knowledge.*

Two other figures who were prominent astronomers in the 16^{th} century included Peter Apian and Oronce Finé. Both men, the former of German descent and the latter of French origin, wrote works on various forms of the quadrant. In his *Cosmographicum Caesareum*, Apian describes the construction and use of a trigonometric quadrant, and in another work he describes the construction of a newly constructed astronomical quadrant. Likewise, Finé wrote a treatise on an astronomical quadrant.

The 17^{th} century ushers forth a collection of new texts on quadrants - on their construction, but mostly descriptive approaches to the arrangement of the various curves on the quadrant. Edmund Gunter's *The description and use of the sector* stands as the most popular work on the quadrant during that century. Its influence can be seen in the work of William Leybourn and John Collins, and by the various examples of extant quadrants which were constructed in the 17^{th} century.

Surveying

In the field of surveying, the quadrant played a relatively small role since surveyors possessed many other instruments for performing whatever measurements they were doing. Nonetheless, early texts show how the quadrant could be applied to this science. There also existed texts which examined the art of surveying in addition to other sciences like geography, navigation and astronomy. George Atwell's *The Faithfull Surveyour*, Anthony Fitzherbert's work on surveying, *Here begynneth a ryght frutefull mater*, Leybourn's *The compleat surveyor* and John Norden's *The Surveyor's Dialogue* represent works on surveying which are relevant here within the context of quadrant applications.

Navigation

Lastly, we have examined a few works on navigation to gain some understanding of this art and how instruments like the quadrant were used to sail the seas. At this point, we mention William Barlow's *The Navigator's Supply*, Martin Cortes' *The arte of nauigation* and Pedro de Medina's *The arte of navigation* as some texts which we have examined.

1.3 Modern Literature

Unfortunately, much of the modern literature has not examined the quadrant within the same context in which we have examined it. Indeed, we are tempted to say that scholars have neglected to study the quadrant, and scientific instruments in general, until rather recently. Even then, the relative number of studies is quite small.

Scholars like Emmanuel Poulle in "Le quadrant noveau médiéval", Lynn Thorndike in "Who wrote *quadrans vetus*?" and David King in various works on astronomical instruments have focussed primarily on the quadrant in medieval times in the Latin West and medieval Islam. Others have produced works which provide insight into substantial collections of instruments. We mention, as an example, Anthony Turner and Harriet Wynters, two authors who have written on various collections of scientific and astronomical instruments. Works by Francis Maddison, Edmund Kiely and E.G.A. Taylor have been quite useful for their bibliographies which have provided us with much of those primary works on the construction and use of the quadrant which we have consulted.

To our knowledge, little, if anything at all, has been done to study the quadrant as a scientific instrument in the context of its use by practitioners of several different professions. In the modern times, the quadrant has by and large only been looked upon as an astronomical instrument used for telling time. In this work, we have attempted to fill some of the void, particularly by looking at the mathematical tools behind the construction of astronomical quadrants.

1.4 The Mathematical Tools

In this section, we introduce two mathematical tools which were necessary for the construction of the quadrant: (i) the stereographic projection and (ii) spherical trigonometry. We will discuss the former from an historical point of view, taken as a tool developed in antiquity and applied to areas like cartography, the mapping of the heavens and, in this instance, the construction of particular curves on the quadrant.

Spherical trigonometry, for the most part, plays an important role in the construction of curves significant in astronomical matters. Again we have provided a description of these tools for our readers within the historical context from which they developed, a description of trigonometry "functions" in the early Renaissance and how these functions were calculated.

1.4.1 Stereographic Projections

The idea of representing points on the celestial sphere by those on a plane perpendicular to the axis of the celestial sphere by a projection from a pole of the sphere has been of great importance in practical astronomy since antiquity. This idea, which today is known as a "stereographic projection", was also employed later in the Late Middle Ages in the construction of certain types of quadrants (for astronomers), and indeed in the construction of the more commonly used instrument known as the astrolabe. Although the earliest known example of this projection is in the *Planisphaerium* of Claudius Ptolemeus[4] (fl. 127-151 A.D.), it is the opinion of Neugebauer that there is sufficient evidence that the invention of stereographic projections predates Apollonius. Evidence for its invention indicates that it may be attributed to Hipparchus of Nicaea (ca. 180-125 B.C.). [5]

A stereographic projection exhibits two important mathematical properties. The first property is the mapping of circles to circles and the second is the preservation of angles, or conformality. The second of these properties, however, does not appear to have been known in ancient times.

In Apollonius' *Conics* I, 5, he demonstrates the existence of two families of circles which arise from taking sections of an oblique cone.[6] The first family consists of those sections taken parallel to the circular base of the cone. In fig. 1.1, let ABC represent an axial triangle[7] of an oblique cone. Then any section DE which is made parallel to BC is also a circle.[8] The second family, known as the "subcontrary sections", is produced in the following manner. If an axial triangle, ABC, of the cone is cut by any line GF such that $\angle AGF = \angle ACB$,[9] then the plane through GF and perpendicular

[4]In a recent work by J.L. Berggren on Ptolemy's work with celestial and terrestrial maps [Berggren, 1991, p. 3], he cautions readers against interpreting the *Planisphaerium* as a work on pointwise stereographic projections *per se*, rather than as simply describing a means for constructing images of celestial circles.

[5]Otto Neugebauer, *A History of Ancient Mathematical Astronomy*, p. 858.

[6]Apollonius further shows the uniqueness of these two families of circles in proposition 9 of Book 1.

[7]An axial triangle is the triangle which results by taking the intersection of an oblique cone with a plane containing the axis of the cone.

[8]Apollonius proves this result in I, 4 of the *Conics*.

[9]The equality of $\angle AFG$ and $\angle ABC$ follows from the similiarity of $\triangle ABC$ and $\triangle ADE$.

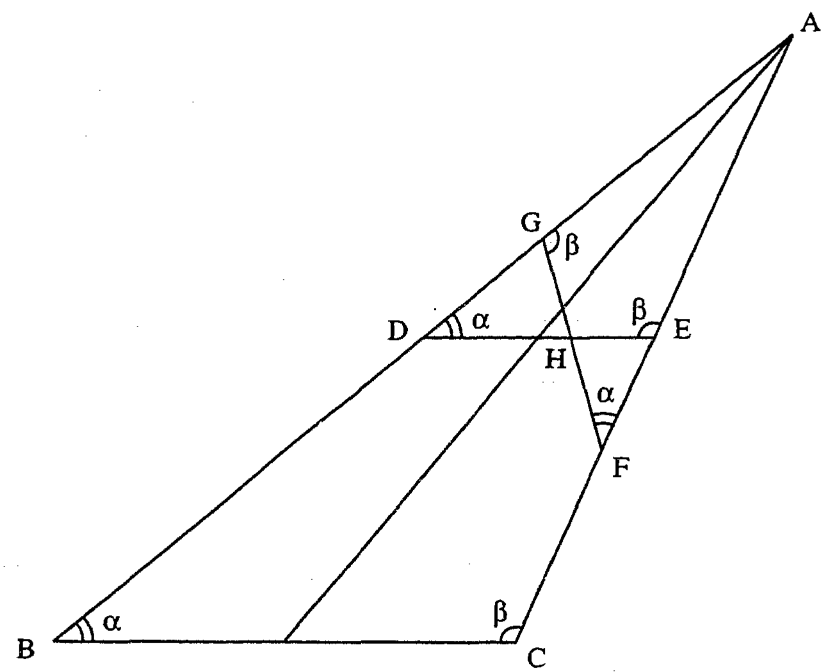

Figure 1.1: Two families of conic sections which are circles.

to the axial triangle creates a subcontrary section. (In the case of a right circular cone, these two families of circles are one and the same.) In *Conics* I, 9, Apollonius shows that no other section can be a circle.

Apollonius' *Conics* provides us with the necessary elementary propositions needed to demonstrate the first of the aforementioned properties of a stereographic projection. Let us consider in fig. 1.2 a sphere AD with centre O and let $B'C'$ be the diameter of a circle on the sphere. A stereographic projection maps the points B' and C' to the points B and C in the plane of projection respectively by the straight lines $AB'B$ and $AC'C$ where A is the point of projection and the axis AO extended meets the plane of projection at D at right angles. $\triangle AB'C'$ represents an axial triangle of a cone with vertex A and circular base whose diameter is $B'C'$. It remains to be shown that BC represents the diameter of a circle in the plane of projection. $\triangle AB'D$ is a right triangle since it subtends the diameter AD. The equality of $\angle AC'B'$ and $\angle ADB'$ follows from the fact that they subtend the common chord AB'. Given that $\triangle ABD$ is a right triangle, and $B'D$ is perpendicular to the other side of that triangle, then $\triangle AB'D$ is

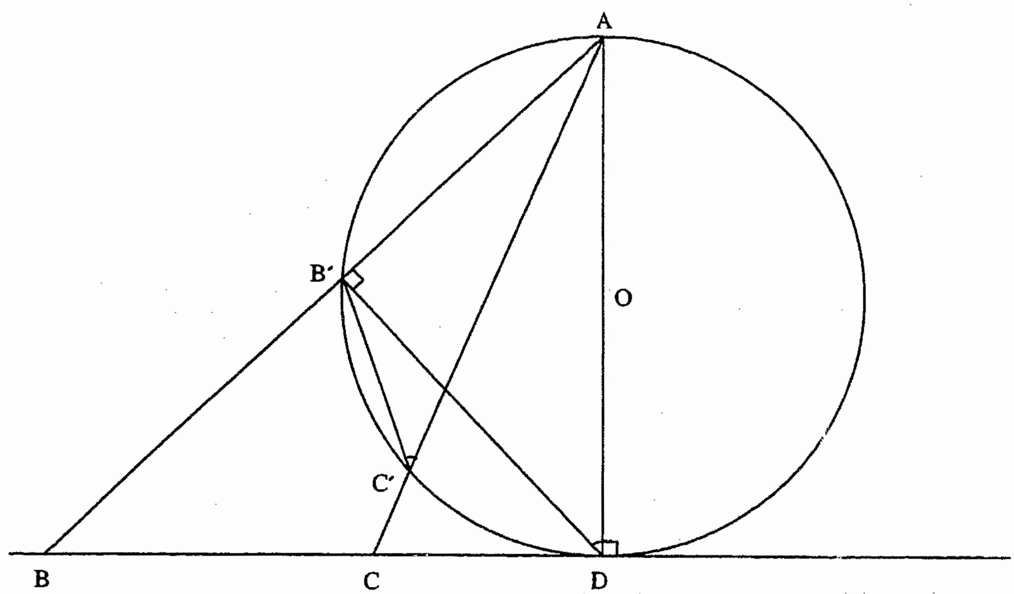

Figure 1.2: Under a stereographic projection circles map to circles.

similar to $\triangle ADB$. Since $\angle A$ is common to both triangles, $\angle ABD = \angle ADB'$ and thus $\angle ABD = \angle AC'B'$. Thus, the plane of projection cuts the cone in a subcontrary circle and BC is the diameter of that circle in this plane.

The second property of a stereographic projection is of no interest here since it never was employed by practitioners in the construction of the curves on an astrolabe or quadrant.

1.4.2 Trigonometry

The development of trigonometry has been a long and fruitful one especially in the service of astronomy.[10] Indeed, from the Late Middle Ages up to the present we find trigonometry employed in the work of surveyors and navigators. What will concern us here is the use of spherical trigonometry in the application and construction of the quadrant, and some theorems on plane triangles used in the calculations by mathematical practitioners.

[10] *Cf.*, *e.g.*, Edward S. Kennedy, "The History of Trigonometry," in *Studies in the Islamic Exact Sciences.*

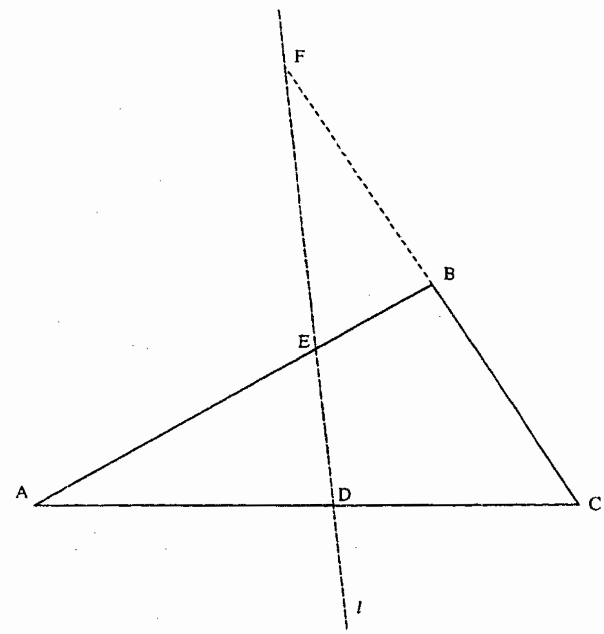

Figure 1.3: Menelaos' theorem for the plane.

It would appear that spherical trigonometry arose from a need to identify the position of a heavenly body in the celestial sphere, and theorems such that of Menelaos, provide us with a method for solving spherical triangles. This theorem of Menelaos for the plane states that if the sides of a triangle are cut by a transversal, then the product of three non-adjacent segments is equal to the product of the remaining segments. For example, if ABC is a triangle and l is a transversal which cuts the three sides at D, E, F respectively as in fig. 1.3, then it follows that

$$\overline{CF} \cdot \overline{DE} \cdot \overline{AB} = \overline{BC} \cdot \overline{DF} \cdot \overline{AE} \tag{1.1}$$

or simply as,

$$\frac{\overline{CF}}{\overline{BC}} = \frac{\overline{DF}}{\overline{DE}} \cdot \frac{\overline{AE}}{\overline{AB}} \tag{1.2}$$

where we may write two "outer segments" in this configuration in terms of "inner segments". Likewise, we may write two inner segments in terms of outer segments as in

$$\frac{\overline{AE}}{\overline{EB}} = \frac{\overline{AD}}{\overline{DC}} \cdot \frac{\overline{CF}}{\overline{BF}} \tag{1.3}$$

Given these two expressions for the ratio of two inner and outer segements respectively, the following trigonometric formulae for right spherical triangles arise from the trigonometry of chords and their relationship to our modern trigonometric functions:

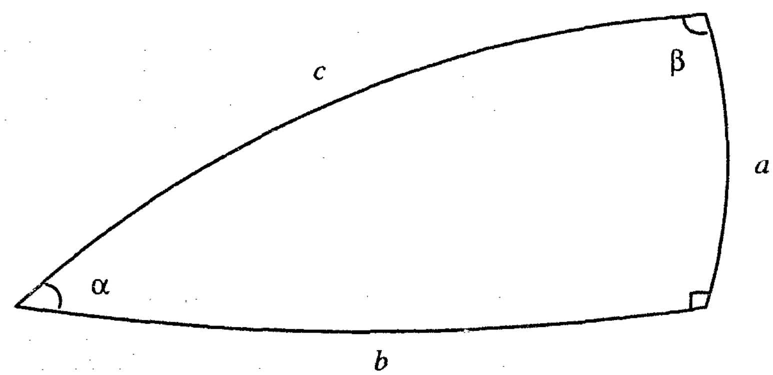

Figure 1.4: Trigonometric relations for spherical triangles.

$$\sin(\alpha) = \frac{\sin(a)}{\sin(c)} \tag{1.4}$$

$$\cos(\alpha) = \frac{\tan(b)}{\tan(c)} \tag{1.5}$$

$$\tan(\alpha) = \frac{\tan(a)}{\sin(b)} \tag{1.6}$$

$$\cos(c) = \cos(a) \cdot \cos(b) \tag{1.7}$$

Early trigonometry owes its existence and development to many different cultures and the astronomers thereof, especially to the Hindus, and not just the ancient Greeks; and it was expressed in geometric language using the lengths of chords which subtend angles or arcs in a circle. In the case of plane triangles, Menelaos assumed that the "theorem of Menelaos" was well known to his peers.

In Menelaos' treatise, the *Sphaerica*, which survives only in an Arabic translation, the analogous cases for spherical triangles are proven by their planar counterparts. These theorems, as we shall discover, played an important role in the development of observational astronomy and the construction of the quadrant.

Today we use and teach trigonometry to students as functions which represent ratios or numbers. Prior to this recent development in the history of mathematics, trigonometry for the Greeks, the Hindus, the Arabs, the Latin medievalists and early Renaissance practitioners meant trigonometric lines or lines which represented the values of trigonometric functions for various arguments. Discussing developments in Western European mathematics, Dirk Struik says that the use of the unit circle and the expression of trigonometric functions as ratios is attributable to Leonard Euler.[11] The evidence for this lies in Euler's *Introductio in analysin infinitorum*.

Though the conceptual change in trigonometric functions did not take place until the middle of the 18^{th} century, changes had already begun to take place about two centuries prior to Euler's work. Trigonometry was becoming a separate science from astronomy and was being applied to other areas of mathematics like algebra and disciplines like geography. Nonetheless, trigonometry meant the study of trigonometric lines which were expressed as a ratio of the diameter of some circle with a given radius.

In the 16^{th} century, Georg Rheticus, a student of Copernicus, broke with the view which traditionally held that trigonometric functions were defined with respect to the arcs of circles, and turned to the right triangle to express all six trigonometric functions, *i.e.*, sine, cosine, tangent, cotangent, cosecant, and secant. Instead of expressing the trigonometric functions as the ratio of chords, tangent lines, secant lines and parts of the diameter of a circle with a very large radius, they were expressed using a right triangle with a hypotenuse or base given as a given number of parts.[12]

This convention for describing the trigonometric functions was encountered by many scholars on the continent, but not in England. The use of trigonometric lines remained in use throughout England during the 17^{th} century as can be observed through the works of people like Edmund Gunter and Richard Norwood in their respective works, *Use of the sector* and *Trigonometrie*.

[11] Struik, *A Concise History of Mathematics*, p. 90.

[12] Rheticus in his *Opus palatinum de triangulis* uses 10,000,000 for a radius. *Cf., e.g*, Boyer and Merzbach, *A History of Mathematics*, p. 326-7.

Chapter 2

The Quadrant Prior to the Early Renaissance

2.1 Transmission and Translation – From Medieval Islam to the Latin West

Astronomy has occupied a place of great nobility within various cultures for millennia, long before most (if indeed not all) of the other sciences were established. This discipline has shown much utility in the pursuit of scientific knowledge and religious truth; however, much of its progress depended on the development of scientific instruments. From the earliest use of the gnomon to the use of dials, quadrants and astrolabes and developments in optics (which gave rise to the telescope), astronomy has helped us better to understand the universe and has sharpened our faculties for reflection on the nature of the world in which we live. It is not our intention to examine the history of astronomical instruments here, since many other scholars have already done so. We will, however, provide a brief discussion on the role of the quadrant in positional astronomy in Medieval Islam and in the Latin West before we discuss its use during the early Renaissance from the early 16^{th} to mid-17^{th} centuries.

The earliest known example of an instrument resembling a quadrant is described in Ptolemy's *Almagest* I, 12 where he describes the construction of an instrument for determining the distance of the sun from the zenith when it crosses the meridian. Ptolemy's quadrant, which is drawn on the face of a square slab of wood or stone, is a quarter circle with the arc graduated into 90 degrees and further subdivisions; at the centre of the quadrant is fixed a peg from which the shadow cast at noon would provide the zenith distance of the sun in the meridian. From these observations made over time, the meridian altitude of the sun in the summer and winter solstices could be determined; hence, the obliquity of the ecliptic and the local latitude could be known.[1] Ptolemy's work served as the authoritative text on mathematical astronomy, furthering the astronomical and religious needs of both the Medieval Islamic and Latin worlds.

The acceptance of astronomy, as well as other matters foreign to the Muslim world, occurred only after much debate over its utility for this theocentric society. Sayili has described in some detail the problems of acceptance of the secular sciences, and indeed astronomy, into the ideologies and religious doctrines of Medieval Islam; however, in time Islamic scientists adopted that framework established by the Greeks - and some influence from India - working to correct, extend and complete the astronomical corpus. (A scientist from later in the "golden age", al-Bīrūnī, aptly reflects these ideas by saying that the job of Muslim scientists was "to confine ourselves to what the ancients have dealt with and endeavour to perfect what can be perfected".[2]) The quadrant (as well as other observatory instruments) served within the development of Islamic astronomy to correct astronomical tables and tables of geographical coordinates, and to calculate the obliquity of the ecliptic with a much greater precision than ever achieved before the Arabs, as well as to be used in the distinctive branch of Islamic astronomy known as the science of time measurements (useful in the conduct of religious duties) and the determination of the direction of the *qibla*, the sacred direction toward Mecca, which Muslims face during prayer.

[1] *Cf.*, G.J. Toomer's translation and annotation of *Ptolemy's Almagest* where Ptolemy in Book I has described the construction of two instruments which were used for determining the arc between the two solstices, pp. 62-63.

[2] This appears quoted in [Lindberg, 1992, p. 176] from Carra de Vaux, "Astronomy and Mathematics" (in Arnold, Thomas, and Guillaume, Alfred, eds. *The Legacy of Islam*, pp. 376-97. London: Oxford University Press, 1931).

To perform these tasks required the use of large quadrants (among other instruments) with graduated arcs constructed to provide the greatest amount of precision in the data collected from celestial observations. Such large quadrants were created at or for some given observatory. The Islamic observatory is a "product and part of the Islamic society and civilization".[3] Most were short-lived, and two were even intentionally demolished.[4] The short life span of some observatories (as opposed to observation posts) seem to have arisen, in part, from technical difficulties with the construction of such large astronomical instruments. While the quadrant was not the only instrument which the Muslims used, it seems quite possible, though difficult to show here, that the construction of large quadrants (with radii measuring several feet) may have resulted in the early closure of some observatories due to an inability to construct these instruments accurately enough.

A number of texts and commentaries have come down to us on the nature of the observatory and the use of scientific instruments, as well as the quadrant[5]. We have commentaries from a number of astronomical observers which include remarks made by al-Battānī, al-Bīrūnī and al-'Urdī to name three whose work is discussed in some detail by Sayili.

Al-Bīrūnī is said to have cited the location - just recently discovered - of the Qāsīyūn Observatory near the Dayr Murrān monastery which Sayili concludes was built between the Fall of 830 and the Summer of 831 on the order of al-Mamūn. Like most other observatories, Qāsīyūn had a number of prominent astronomers, astrologers and instrument makers; most likely, like many of the other Islamic observatories there were also others, such as engineers, architects, mechanics, treasurers, librarians and other clerks, who formed a part of this impressive and elaborate organization.

Though the text is not clear to Sayili, in his *Determination of the Coordinates of Localities* al-Bīrūnī mentions a wall quadrant made of marble with an inner radius of 10 *dhira'* or about five metres. Fixed to the arc was a sliding device with an aperture

[3]Sayili, "Introduction," *The Observatory in Islam* by Aydin Sayili, p. 4.
[4]*Ibid.*, p. 4.
[5]The work *The Observatory In Islam* by Aydin Sayili is the authoritative study on this subject.

to allow an observer to take either the sun's altitudinal or zenith distance by aligning the sun and the spike located at the centre of the quadrant as viewed through the aperture. The graduation of the arc of this quadrant was done by 'Alī ibn 'Īsā al-Usṭurlābī who is cited by Ibn Yūnus with great praise in his *'Alīī, Kitāb al Zīj al Kabīr al-Ḥākimī* as the instrument maker.[6]

Al-Battānī is known to have made numerous solar observations to determine amongst other things the obliquity of the ecliptic at his private observatory in Raqqa. Moreover, it is known that al-Battānī possessed a number of astronomical instruments which included an astrolabe, a gnomon, various sun clocks, parallactic rulers and a mural quadrant. Of the latter, al-Battānī notes in his *Zīj al-Ṣābī* that such a quadrant should be at least one metre in radius and that any increase in the radius would enhance the precision in the instrument - and hence effect an improvement in making an observation. To use the quadrant, al-Bīrūnī says in his *Taḥdīd* that al-Battānī had used an alidade with his quadrant and was the first to use such a sighting mechanism on a quadrant.[7]

Al-Bīrūnī also gives an account in this same work of Sulaymān ibn 'Isma who made observations in Balkh with a mural quadrant with a radius of 8 *dhira'* or approximately 4 metres. This quadrant, too, al-Bīrūnī claims was equipped with an alidade and was used for determining the obliquity of the ecliptic.[8] Al-Bīrūnī also claims that Abū'l Ḥasan Aḥmad ibn Sulaymān had used a large quadrant with a radius of 20 *dhira'* (approximately 33 feet) which he used to calculate the latitude of the city of Zarnaj in Sijistan.[9]

Beside these citations by al-Bīrūnī of the work and astronomical instruments of other astronomers, al-Bīrūnī himself made observations of the heavens, some of his earliest ones being made in Khwarazm. He also is known to have made numerous astronomical observations in various cities of Khurasan and is reported to have used a quadrant with a radius of 3 metres. From these observations he was able to calculate

[6]Sayili, *The Observatory in Islam*, p. 72.
[7]*Ibid.*, p. 96-7.
[8]*Ibid.*, p. 98.
[9]*Ibid.*, p. 111.

the obliquity of the ecliptic and the geographical coordinates of a number of cities.[10]

The 13^{th} century observatory in Marāgha, its foundation still extant, was probably one of the most important observatories in Medieval Islam. The observatory possessed a fine library containing 400,000 manuscripts[11] many of which would have been written by the many prominent scientists attached to this observatory, including Naṣīr al-Dīn al-Ṭusī, Mu'ayyad al-Dīn al-'Urḍī and Muḥyī al-Dīn al-Maghribī. Al-'Urḍī, the chief instrument maker at Marāgha, is known to have constructed or suggested the construction of various instruments for this observatory one of which includes a mural quadrant. He mentions that this quadrant had a radius of more than 14 feet and was graduated in degrees and minutes. With the instrument which probably was the first instrument constructed at Marāgha, the latitude for Marāgha could be obtained and the obliquity of the ecliptic could be calculated. Undoubtedly, al-'Urḍī had a number of astronomical instruments at his disposal, and this quadrant was not the only quadrant at Marāgha. Pannekoek states that al-'Urḍī also constructed a quadrant with a radius of 10 feet.[12]

Though the above helps to show the importance of both the observatory and the quadrant in astronomical and geographical matters, most observatories and astronomers would have had access to hand-held quadrants. As we have mentioned earlier in this study, David King has made numerous studies of Medieval Islamic quadrants of this nature, work which is part of an increasing trend of studying scientific instruments and their relationship to science.

King talks about the horary quadrant which would have been used for time-telling either at any latitude using seasonal hour lines which serve to divide the length of daylight into twelve equal parts or at a specific latitude using equinoctial hour lines. Without providing any information on the source of the particular manuscript, he mentions a 9^{th} century text from Baghdad where a horary quadrant is described which includes a fixed and a moveable cursor and shadow square. This poses a problem for historians of science, as King notes, who previously thought that such an

[10] *Ibid.*, p. 127.

[11] Pannekoek, *The History of Astronomy*, p.169.

[12] *Ibid.*, p. 169.

instrument was a latter European innovation known as the *quadrans vetus.* We should not necessarily construe this to mean that a European astronomer did not re-discover such an instrument, and it seems risky to conclude any Islamic influence in Europe on the basis of one text alone. (King, however, emphasizes that we have only begun to scratch the surface in our study of Islamic texts dealing with instruments.) If its appearance in Europe does turn out to be due to Islamic influence, then the horary quadrant will turn out to be but one of a number of scientific instruments to have found its way into medieval Europe from the Islamic world. In any case, it will be discussed later.

One other quadrant which interests us here is the astrolabic quadrant or as it is known in the Latin tradition *quadrans novus* or more appropriately as *quadrans astrolabicus.* It was used by Oronce Finé in 1534 in his work by the same name, but its origin is again very much a mystery. We will also discuss this form of quadrant shortly, but here it will suffice to say that it incorporates those symmetrical markings which are found on an astrolabe. The symmetry of the lines engraved on the tympans of the astrolabe about the meridian allows one to use just half of the astrolabe (on either side of the meridian) which is then folded in half again to form a quarter circle. The rete of the astrolabe is replaced by a plumbline and bead which represents the sun or some star. Its popularity as an astronomical instrument, which we shall shortly examine, is evident from the greater appeal (over the astrolabe) which it had to practitioners in the Late Middle ages and the early Renaissance in Western Europe.

Medieval Islam played an important role in the collection, the correction and the perfection of astronomy from other cultures around them, and in particular from the Hellenistic (*e.g.*, Euclid, Apollonius and Ptolemy). Although without its contribution, it is difficult to predict how European science would have developed; astronomy in the Latin West, as a cultural region in its own right, is also important to our understanding of the evolution of the purpose of the quadrant within astronomy. We shall examine this evolution by briefly examining the role which the quadrant played in the area of astronomy in Late Medieval Europe.

On the whole it is now accepted that the Middle Ages were not the "Dark Ages" of popular imagination. As in Medieval Islam, astronomy in the Latin West was of great relevance to the human condition and of practical importance both to the church (in regulating the civil and liturgical calendars) and to the secular world. And beyond its mundane benefits, astronomy was to enlighten man, free his soul from secular affairs and shift his attention to the more sublime. However, astronomy owes its existence on the whole to its utility and usefulness for time-reckoning and as a theoretical introduction to astrology. Moreover, astronomy formed part of the educational organization found in the *universitas* of Medieval Europe.

The early history of Latin astronomy from the time of Isodore, Bishop of Seville, in the early 7^{th} century and for the next four or five centuries seems to point clearly toward cosmology rather than mathematical astronomy as its greatest concern. Fragments of mathematical astronomy do appear, although they are rare.[13] The fact that the medievals were working with the summaries of Martianus Capella (5^{th} century A.D.) rather than someone like Ptolemy reveals much about what was available. The fact that there was no significant link between theory and observations of heavenly phenomenon is emphasized by the fact that the best instrument of the time, which only became available in the 9^{th} century, was a crude sundial.

2.2 Quadrants in the Latin West - *Quadrans vetustissimus, vetus* and *novus*

Recent discussions on the introduction of the quadrant to the Latins from the Arabs, and on the quadrant in general, have evolved into a few studies by scholars such as Paul Tannery, Henri Michel, J. Millás Vallicrosa, Abbé A. Anthiaume and Jules Sottas, Emmanuel Poulle, Lynn Thorndike and Nan Hahn.[14] Here our discussion on the quadrant in the late Middle Ages will draw upon these works to provide a brief sketch of the types of quadrants which appeared during this time.

Millás Vallicrosa has written about the history of the *quadrans vetus*, which we

[13]Pedersen, Astronomy in *Science in the Middle Ages*, pp. 306-7.

[14]*Cf.*, Those texts by these authors to which we refer are listed in our bibliography.

shall describe shortly; however, he has also established the existence of an older quadrant found in the Latin West, which is similar to the *quadrans vetus*, and which he calls *quadrans vetustissimus*. The *quadrans vetustissimus*, Millás Vallicrosa claims, was introduced as early as the 10^{th} century during the transmission of Arabic science to the Latin West. Poulle cautions that certain people (he does not say who) have claimed incorrectly that the quadrant with a cursor was a Latin invention. This invention, these same people claim, began in the 13^{th} century with the *quadrans vetus* and a text written by one Johannes or Robertus Anglicus of Montpellier.[15]

There is a significant difference between the *quadrans vetus* and the *quadrans vetustissimus*. A series of parallel lines (parallel to one of the sides of the quadrant) is constructed on the latter instrument, in the space, which is itself a quarter circle, above the graduated limbus, the cursor and a representation of the ecliptic. These lines, Millás Vallicrosa says, suggest the use of two trigonometric functions, the sine and cosine, for some corresponding angle. Poulle suggests that the use of these lines may have been beyond the comprehension of 10^{th} century scholars for calculating the seasonal hours. This would suggest that the invention of the *quadrans vetus*, which replaced this series of parallel lines with the construction of seasonal hour lines, is an invention of the Latin West. The cursor, however, is derived from Arabic sources. Of course, it is quite speculative to say that the *quadrans vetus* is a derivative of the older, but the use of hour lines is clearly easier than the use of this series of parallel lines for time-telling. Lorch even suggests that the development of horary quadrants comes after the *quadrans vetustissimus*. He does not discuss how the sine-quadrant might have been related to the *quadrans vetustissimus*.

The earliest reference to a quadrant, which Millás Vallicrosa qualifies as *quadrans vetustissimus*, appears in the extant Latin work *De operatione vel utilitate astrolabii* translated from an Arabic work by the Jewish astronomer Māshā'allāh (Messahala) who flourished around 815-820 A.D. As the title suggests, the work is on the astrolabe; however, those parts of the text which describe a double quadrant on the dorsal side of

[15]There has been some debate as to who the author is of this text. Thorndike, for example, takes up this problem in his "Who wrote quadrans vetus?". Other scholars, like Hahn, have made attempts to re-appraise the evidence gathered together to determine the author of the *Quadrans vetus*.

the astrolabe have been interpreted as describing a separate instrument - the *quadrans vetustissimus.* Despite the problems of authorship with this work, Hahn, who has investigated this text, states that the appearance of an ordered description of the parts of this quadrant could be an artifact of the text.

Today there survive numerous tracts on the *quadrans vetus.* These early treatises were not concerned with the construction and use of the *quadrans vetus*, *per se.* Early treatises were known by the title *Tractatus quadrantis.* The qualification of *vetus* only came into being when Jacob ben Machir ben Tibbon presented another quadrant - not neccessarily of his own invention - in the late 13^{th} century, known as the *quadrans novus.*

The importance of the *quadrans vetus* during the late Middle Ages can be determined by a number of factors. It is likely that treatises on the *quadrans vetus* were used in universities as pedagogical tools for teaching mathematics, astronomy and astrology. Rashdall says that these subjects thrived in Italian universities like Bologna. Those who studied astrology at Bologna for instance, were required to read various works which included Māshā'allāh's treatise on the astrolabe and an anonymous *Tractatus Quadrantis.*[16]

In the 12^{th} century, we find the *universitas* emerging as a centre of knowledge, learning and creative scientific developments. The universities, which were more than just places of learning, were places where knowledge was gathered, studied and shared by students and masters alike. As such, we find treatises and a few examples of quadrants which were employed pedagogically to further knowledge in the areas of astronomy and geometry as it might have been applied to surveying, for example.

Of the numerous tracts on the construction and applications of the quadrant during this period (though certainly less abundant than the number of treatises on the astrolabe), we will mention only a few relevant works here.

The first of these appears around 1140 and is entitled *Practica geometriae.* The work which is attributable to Hugh of St. Victor is divided into abstract and practical geometry. It is similar to another work analyzed by Hahn where geometry is partitioned in the same manner ("*Geometri due sunt partes principales theorica et*

[16] Rashdall, *The Universities in the Middle Ages*, p. 248-249.

practica").[17] Here, the description of the essential parts of a quadrant is for the construction of the quadrant on the dorsal side of an astrolabe, and the practical applications are the usual mundane routines for obtaining measurements for altimetry, planimetry and stereometry.

In the early 13^{th} century, we find a Latin treatise by Sacrobosco, the Parisian teacher and author of *De Sphaera Mundi*, which discusses this device. Sacrobosco named this much smaller treatise on the *quadrans vetus* simply *Tractatus magistri Ioannis de Sacrobosco super compositione quadrantis simplicis et compositi et utilitatibus utriusque.*[18] The treatise which describes the construction of a quadrant with and without a cursor and a few applications does not seem to have enjoyed a fruitful existence, though it must have been used frequently as a textbook by Sacrobosco in Paris if nowhere else. The treatise which provides instructions for finding the sun's declination, the time of day and the height of objects is important mathematically and stylistically. Stylistically, it differs from earlier works;[19] mathematically, Sacrobosco remarkably describes how to construct equal hour lines.

The extent to which people outside of academia knew about the *quadrans vetus* cannot be determined entirely, although Hahn has been able to locate 4 extant quadrants. They are found in museums in Florence, London, Cambridge and Oxford, and date from the 13^{th} century until approximately 1600.[20] In addition, Hahn also suggests some reasons for the rarity of existing instruments, and suggests that broken cursors, the potentially short life span of wooden quadrants and melted down brass quadrants could account for relatively few extant examples of quadrants.

Of course, the biggest case for the quadrant's popularity is the number of extant treatises and references to the *quadrans vetus*. Hahn has examined over 60 Latin

[17] *Cf.*, Nan L. Hahn, *Medieval Mensuration*, 1982. Hahn discusses and analyses this work by an unknown writer in this critical edition of two works which have many textual similarities in an effort to answer questions about authorship, textual sources and developments of the quadrant.

[18] *Cf.*, Olaf Pedersen, "In Quest of Sacrobosco," *Journal for the History of Astronomy*, 16 (3, 1985), pp. 185-186. In this article Pedersen provides an excellent study of the works of Sacrobosco (which also include his *Computus* and *Algorismus*) and insight into the enigmatic character of Sacrobosco. The manuscript examined by Pedersen who attributes this text on the construction of the quadrant to Sacrobosco is a Parisian manuscript (Paris BN Lat. 7196, 25r-27v, seac. XIII.).

[19] Hahn, "Introduction," *Medieval Mensuration*, p. xxxiii.

[20] *Ibid.*, *Medieval Mensuration*, p. xi-xi.

manuscripts from various libraries across Europe from which he has composed a critical Latin edition. Translations also exist in German, Hebrew and Greek, and Paul Tannery has written a critical edition on the *quadrans vetus* in an attempt to determine the origin of a Greek translation of *Geometrie due sunt partes.* The treatise has also been printed in Gregory Reisch's *Margarita Philosophica.* The entire work also appears in *De elementis geometrie* by Johannes Grüniger, who erroneously attributed the work to Martin Waldzemüller.

The *quadrans novus*, to distinguish it from the *quadrans vetus*, was known only as such in Latin circles. Its original name, given by Jacob ben Machir ben Tibbon, was the *quadrant of Israel.* The *quadrans novus* was quite different from its predecessors. In essence, it was derived from the astrolabe by a double abatement of the face of the astrolabe along two axes of symmetry. Oftentimes, the cursor was omitted, though the shadow square and the equal hour lines remained intact.

The *quadrans novus* was quite a novel invention; however, the diurnal motion of the sun and the stars, as they were represented on the quadrant, became more complex despite the fact that a greater precision can be achieved with this instrument versus the astrolabe. Although the quadrant's size accounts for its greater accuracy, performing a calculation now takes a few steps and only presents an opportunity for errors to accumulate. Extant examples of the *quadrans novus* have examined by E. Poulle in *Le quadrant nouveau médiéval* and *L'astrolabe-quadrant du Musée des antiquités de Rouen* by Anthiaume and Sottas. R.T. Gunther's *Early Science in Oxford* also provides some insight into scientific instruments at Oxford.

All of these quadrants or their various constructed parts would appear again over the next 400 years in Western Europe. For instance, Oronce Finé's *Quadrans astrolabicus omnibus Europae regionibus deserviens* was a very important treatise on the *quadrans novus* which was published in the 16^{th} century; Edmund Gunter would modify the *quadrans novus* by including a cursor and reducing the construction of certain curves to two stereographic projections.

Together, these extant works and surviving instruments provide some insight into the Arabo-Latin roots of the quadrant during the late Middle Ages.

Chapter 3

Renaissance Practitioners and the Quadrant

3.1 The Astronomers

By the middle of the 15^{th} century astronomy was beginning to play to the practical needs of the people of western Europe. The numerous treatises on calendar reform prompted by the church and various councils serve to show the usefulness which astronomy was seen to possess; in addition, navigation and cartography made high demands on astronomy during a time when merchants, sailers and wayfarers, amongst others, needed assistance to travel abroad. Also, yearly astrological prognostications soon became the order of the day and were a major impetus for the development of astronomy in the Latin West.

The spread of astronomical information had much of its roots predominantly in Paris until the mid-14^{th} century.[1] Soon, other universities in Europe strengthened in this area, and knowledge of advanced astronomers eventually made its way to England. However, Pedersen notes that the best known astronomical tradition for the early 15^{th} century lies in Vienna with a succession of students and teachers beginning with John of Gmunden, Georg Peurbach and Regiomontanus.[2] We shall begin here by looking

[1] Pedersen, Astronomy in *Science in the Middle Ages*, p. 329.
[2] *Ibid.*, p. 330.

at the work of these three men, their influence on astronomy and use of the quadrant in their work.

John of Gmunden (ca. 1380-1442) is known to have left a document behind which states that his sizable library of books and instruments were to be bequeathed to the university's new Faculty of Arts library in Vienna.[3] This document even hints at the extensive work which John of Gmunden did to correct and update various astronomical tables, and to teach his students on theological, mathematical and astronomical matters using various important texts of the day and scientific instruments like the quadrant.

John of Gmunden possessed at least two quadrants and is known to have written two tracts on the use and construction of the quadrant. The first, *Profacius Judaeus, De compositione novi quadrantis et de eiusdem utilitatibus*, was a discussion of a treatise written by Jacob ben Machir ibn Tibbon, known as Profatius Judaeus, in the 13^{th} century. The second tract that John of Gmunden wrote is also based upon the same work. Evidence for the year in which they were written indicates that John of Gmunden may have written the first of these tracts as early as 1425. The third tract (and other smaller tracts with uncertain dates) show further interest in the quadrant - possibly written by some of John of Gmunden's students.[4] John of Gmunden did not limit himself to the discussion of various texts (which show his familiarity with scholars like Campanus, Richard of Wallingford and Robert Grosseteste), nor just the correction and calculation of old and new astronomical tables. His knowledge and use of various instruments like Campanus' equatorium, Richard of Wallingford's Albion, the astrolabe, sundials and Jacob ben Machir's *novus quadrantus* were applied for the purposes of performing certain calculations in conjunction with given observations,

[3]Mundy, "John of Gmunden," p. 198. Mundy supplies in his study of John of Gmunden the original text which Gmunden wrote describing his gift to the library. John of Gmunden says, "*Ego Mag. Johannes de Gmunden, baccalarius formatus in theologica, canonicus s. Stephani Wienn. et plebanus in Laa augere cupiens utilitatem ac incrementum inclite facultatis artium studii Wienn. volo et dispono, quod libri mei ... et similiter instrumenta astronomica post obitum meum maneant apud facultatem arcium ....*"

[4]*Ibid.*, p. 200.

and not for purposes of performing observations of celestial bodies for the mere purpose of obtaining data through new observations.[5] This would seem to support John of Gmunden's working as a teacher of astronomy.

A student of John of Gmunden, Georg Peurbach (1423-61) is most known for his work, *Theoricae novae planetarum* (1454), which is derived from his lecture notes on planetary theory to replace the older *Theorica planetarum* by Campanus of Novara.[6] Peurbach is also known to have worked with Johannes Muller or Regiomontanus, as he is better known, to improve upon observational instruments like the quadrant in attempts to make improvements in astronomical tables, and in particular the *Alphonsine Tables.* Unfortunately, Peurbach passed away in his prime, and while Peurbach's ambitions to write a new translation of Ptolemy's *Almagest* based upon Greek texts which he had hoped to obtain with assistance from Cardinal Bessarion of the Byzantine church were not accomplished fully, his student, Regiomontanus, continued in his mentor's steps to devote the rest of his life to reforming astronomy.

Regiomontanus (1436-76) is known for his work in performing celestial observations and the faithful restoration of texts some of which include the *Commentariolus* written with Peurbach, the *Ephemerides* and the completed *Epitome* of Ptolemy's *Almagest* which he and Peurbach were to complete together. While we cannot say with any great certainty (based upon our sources), it is quite likely that both Peurbach and Regiomontanus used quadrants for their astronomical observations. Peurbach and Regiomontanus are seen as supporters of the mathematical astronomy of Ptolemy. However, to make Ptolemy's models for the sun, the planets, the moon, *etc.*, work requires the knowledge of various parameters. To obtain corrected values for the parameters would have required Peurbach and Regiomontanus to do accurate observations with instruments large enough and accurate enough to improve upon those provided by Ptolemy and tables like the *Alphonsine Tables* and possibly the *Toledan Tables.*

[5] *Ibid.*, p. 201.

[6] Pedersen, Astronomy in *Science in the Middle Ages*, p. 330. Pedersen remarks that this text by Peurbach was not a text written in the tradition of the 16^{th} century humanists when classical texts were restored to their original form, in this case Ptolemy's *Almagest*, but rather as a corrected edition of Campanus' work with some additions.

By the middle of the 16^{th} century, England witnessed the beginning of a number of important developments and traditions in astronomy. These developments included the vernacular movement in scientific writing, original compositions and the improvement of scientific instruments. While we cannot offer any proof, we may hypothesize that this growth in the number of English treatises on the construction and application of quadrants is due primarily to the later developments in English universities in the fields of mathematics, astronomy, the other mathematical sciences and eventually a growing organization of instrument makers; certainly prior to this time period, the language of scholarship was Latin, and moreover, original scholarship was limited to writing commentaries on those authorities whose work was used to teach young scholars subjects like astronomy. Hence, the appearance of English works on the quadrant was a relatively late development. This can be seen by the number of English treatises on the quadrant in our bibliography. We shall now examine approximately a dozen mathematical practitioners and instrument makers from the middle of the 16^{th} to the middle of the 17^{th} century who were major figures in further developing astronomy, mathematics and improving scientific instruments - primarily those who made quadrants.

It can be argued that Robert Recorde's *Castle of Knowledge* (1556) represents the greatest textbook on astronomical science written in the 16^{th} century. Robert Recorde (1510-58) is known to have studied physics at Oxford and Cambridge where at the latter he completed his Doctor of Medicine in 1545. He became widely known by many for his teaching abilities and his talents as a writer. Recorde was also a master of the Latin and Greek languages. The *Castle of Knowledge* (1556) is the first comprehensive and original treatise on the elements of astronomy written in the English language; the arrangement of this 300 page text shows Recorde's masterful ability as a teacher. Moreover, this book reflects his ability to impart to his students information in a precise and logical manner, teaching first principles and concepts and then having the students apply such knowledge to real problems, *e.g.*, constructing their own celestial spheres.

We must also note that during this century, we can see a growing number of scientists who are producing works which are anti-Aristotelian and embrace the Copernican system over the Ptolemaic view of the universe. Recorde, who introduced Copernicus' ideas in English, clearly falls into this category; moreover, he indicates his willingness to introduce his students to the idea of the earth rotating about its own axis, as well as its revolving about the sun when they were more fully capable of understanding Copernicus' system . He says,

> but an other time, as I sayd, I will so declare his [Copernicus] supposition, that you shall only wonder to hear it, but also peraduenture be as earnest then to credite it, as you [the young scholar] are now to condemne it".[7]

We should note that Recorde was not really a staunch anti-Aristotelian nor greatly against Ptolemy, however, he does say that

> all men take heed, that both in him [Ptolemy] and in al mennes workes, you be not abused by their autoritye, but euermore attend to their reasons, and examine them well, euer regarding more what is saide, and how it is proued, then who saieth it: for aoutoritie often times deceaueth many menne.[8]

We have discussed Recorde's views on teaching and the Copernican system not merely for the sake of revealing his own views on teaching and astronomy. Indeed, both are quite important here to the use of scientific instruments in general and the influence which Recorde, and others as we shall soon see, had upon the community of mathematical practitioners and instrument makers. Recorde is known to have used astronomical instruments to demonstrate various principles and concepts in astronomy. In his work the *Gate of Knowledge*, which would otherwise be unknown if he had not made reference to it in his *Castle of Knowledge*, Recorde says that the construction and use of the astronomer's quadrant was discussed. In the unpublished parts of his *The Pathway to Knowledge*, Recorde also intended to expound the description of a newly designed geometrical quadrant.

[7]Recorde, *Castle of Knowledge*, p. 165.
[8]*Ibid.*, p. 127.

Although the use of instruments designed at a time when the idea of a geocentric cosmos held sway might seem to be impossible once that view was replaced by a heliocentric cosmos, such is not the case. Francis R. Johnson discusses the nature of this apparent dilemma in his *Astronomical Thought in Renaissance England.* Johnson asserts that the change brought about by Copernicus' *De revolutionibus* was theoretical only, and did not affect the place of the traditional instruments within observational astronomy.[9] For purposes of observation, an observer of the heavens must use the geocentric model of the heavens since an observer must necessarily be at the centre of his astronomical coordinate system. Even today, astronomical instruments are based upon the geocentric model of the universe since an astronomer looks out upon the starry heavens from some place on the earth and not the sun.[10] Indeed, we cannot assume anything about an author's beliefs on the true nature of the heavens based upon his use of astronomical instruments. Recorde serves to demonstrate this point of view. Indeed, even Copernicus realized this problem and demonstrated this using Jupiter as an example to show that one could determine the position of the planet relative to the earth and then relate this to the determination of the planet in its orbit about the Sun.

The popularity of Recorde's work on the Copernican model of the cosmos and the influence of the *Castle of Knowledge* did not overshadow the work of another early 16^{th} century scholar. Leonard Digges (1510-58), also an early Copernican, helped to propel the principles and concepts of astronomy to a much larger audience. Digges' *A Prognostication of Right Good Effect* (1555) and later editions printed by his son, Thomas, which Thomas titled *A Prognostication euerlasting* (1576) were popular works, which Johnson claims were the best of the "perpetual almanacks".[11] This work by Digges, however, was not a textbook like Recorde's *Castle of Knowledge*, but it is a good indication of the tendency of scholars of the time, like Digges and Recorde, to write in the vernacular for much wider audiences who lacked an appropriate knowledge of Latin. Digges also wrote a work called *A geometrical practise named Pantometria* in

[9] Johnson, *Astronomical Thought in Renaissance England*, p.117.
[10] *Ibid.*, p. 117.
[11] *Ibid.*, p. 123.

which he describes a number of instruments, including a quadrant which resembles most other quadrants of the time, possessing a shadow square and a graduated limbus with three scales marking every degree, every fifth degree and every tenth degree.[12]

One astronomer who undoubtedly represents a great scientist in the 16^{th} century is Peter Apian (1495-1552), who made numerous improvements in observational methods, scientific instruments and scientific theories. Apian is also quite well known for his many contributions to mathematics, and particularly as an early German *Reichenmeister* or calculator. Peter Bennewitz, who took the Latinized form of his name, Petrus Apianus, is known for a number of important texts in astronomy and mathematics. His *Astronomicum Caesareum* which he wrote in 1540 is a *magnum opus* on astronomy and represents a true gem in early printed works. The work includes a number of volvelles which served to ease the burden of those who were not so mathematically gifted, but wished to know the positions of the planets and the moon.

Apian was truly gifted as an astronomer. To his credit is attributed the discovery that the tails of comets point away from the sun. He also constructed sundials, and is known to have constructed one in 1524 which was mounted on the southern wall of the Trausnick castle near Landshut.[13] Many of his astronomical publications were on astronomical instruments including quadrants, the torquetum, the cross-staff and other measuring devices.

In 1519, he had already written a work on the horary quadrant (*Quadrantum horarium*), and in 1532 he wrote *Quadrans Apiani Astronomicus* which describes the use of the quadrant in astronomical and surveying matters.[14] In the latter, he describes a quadrant with equal hour lines. His *Instrument Büch* of 1533 also discusses the use of a "new" quadrant, and in the second part of his *astronomicum Caesareum*, Apain describes the use of another quadrant called *Meteoroscopion planum Apiani*

[12] An illustration of Digges' geometric quadrant from his work *A geometrical practise named Pantometria* is pictured in plate 9 in the English translation of Maurice Daumas' *Scientific Instruments of the* 17^{th} *and* 18^{th} *centuries*. *Cf.*, [Daumas, 1953].

[13] A commentary on Apian's work and the *Astronomicum Caesareum* accompanies the 1969 facsimilie of the *Astronomicum Caesareum*. *Cf.*, p. 45.

[14] *Ibid.*, p. 48-9.

which was used to perform calculations involving right spherical triangles.

Apian's *Astronomicum Caesareum* may have been a masterpiece, but it did not please everyone. Rheticus, for example, mocked it, calling the work "an art of threads" - a reference to the use of threads with the volvelles for performing calculations. Also Kepler, in his *Astronomia nova*, criticizes Apian's work which he calls "misdirected" and a poor reflection of the physical world, and he shows no appreciation for Apian's "automatons" which use many "wheels in order to reproduce the figments of the human imagination".[15] Despite these barbs, Apian's scholastic career brought him many honours.

Of course, we should not forget at least to mention the Danish astronomer, Tycho Brahe (1546-1601). Like Copernicus, Tycho Brahe and his work have been examined extensively[16], almost to the point where his work (and that of Copernicus) have clouded the achievements of others during the 16^{th} century. Science in the 16^{th} century was very strongly related to astronomical and astrological matters to which mathematics was inextricably tied. Brahe's work and the work of his contemporaries reflect this general feeling.

Tycho Brahe used many observational instruments including many different types of quadrants. Unfortunately, very little is told about his use of hand-held instruments. He did, however, possess many large quadrants for performing various observations, most of which were done at Uraniborg on the island of Hveen.

In the 16^{th} century, we see a rising number of instrument makers in England, as well as practitioners, teachers and tutors of astronomy and mathematics who used scientific instruments like the quadrant. John Cheke (1514-57), first Regius Professor of Greek at Saint John's College at Cambridge (1530-44), and Provost of King's College (1548-57), played an important role in teaching mathematics to others, and in particular to Prince Edward who later became Edward VI. He designed a quadrant for the young prince in 1551 which was engraved (most likely) by William Buckley (1519-71).[17] Other makers and designers of scientific instruments during this century include

[15] *Ibid.*, p. 62.

[16] See the biographical work by Victor Thoren on Tycho Brahe called *The Lord of Uraniborg* (Cambridge University Press: New York, 1990).

[17] Taylor, *The Mathematical Practitioners of Tudor and Stuart England*, p. 168-9. This instrument

John Dee (1527-1608), William Cunning (1531-86) who designed a new quadrant,[18] William Bourne (fl. 1565-88), the Cambridge mathematician from Christ's Church, Oliver Thomas (fl. 1569-1624), whose patrons included Lord Petre of Ingatestone who was interested in Oliver's work on the making and use of simple instruments like the quadrant, and Thomas Hood (fl. 1577-96) who concerned himself primarily with nautical astronomy and the practical uses for astronomical instruments in his mathematical lectures.

The 17^{th} century seems to have assembled quite a large number of mathematical practitioners in England. Taylor briefly describes the lives and major works of approximately 400 mathematicians, astronomers, navigators, surveyors and instrument makers in her *The Mathematical Practitioners of Tudor and Stuart England.* Her work in this area served as an important tool in identifying those practitioners who were associated with quadrant developments and applications in the service of astronomy and also other sciences.

Many of these practitioners were quite familiar with Edmund Gunter (1581-1626). We will not say much about this man here since we will briefly discuss him and his work in the final chapter of this thesis. Gunter belonged to a few academic circles at Gresham Collge and in London which included people like Henry Briggs (1561-1630), William Oughtred (1575-1660), George Atwell (*ca.* 1588-1659), Elias Allen (*fl.* 1606-54), Samuel Foster (*fl.* 1619-52), Henry Sutton (*fl.* 1637-65) and John Collins (1625-83) to name but a few with whom he was familiar. Based upon the number of texts and extant examples of his quadrant, Gunter obviously influenced many practitioners of the mathematical sciences;[19] however, while Gunter may have influenced many people, Taylor tells us an anecdotal story about the relationship between Gunter and Henry Savile, and a further story about Gunter and Oughtred.[20] Apparently, Gunter had been sent for by Savile who was about to appoint him as the

was in the British Museum when Taylor wrote this book.

[18] *Ibid.*, p. 172.

[19] Gunter quadrants are known to have been constructed by people like Samuel Flower, Walter Hayes, Elias Allen. *Cf.*, [Gunther, 1923] for other extant examples of Gunter's quadrant. Gunter also influenced people like Leybourn and Collins, as is shown in the final chapter of this work, as a source for their own writings.

[20] Taylor, *The Mathematical Practitioners of Tudor and Stuart England*, p.60- 62.

first Savilian Chair of Geometry at Oxford; however, upon his arrival, Savile felt that Gunter's knowledge of geometry was nothing more than a "showing of tricks" with his sector and quadrant in hand, and, showing no appreciation for Gunter's work, dismissed Gunter and sent for Briggs at Cambridge. We also know that Oughtred was not a great admirer of Gunter after Gunter had copied from memory Oughtred's 'horizontal instrument' and inserted this plane projection and other projections into his *Description and Use of the Sector* without acknowledgment. To make matters worse, Elias Allen insisted that Oughtred include in a short book on navigation one of Gunter's tables, which only outraged Oughtred further.

3.2 The Surveyors and Engineers

The art of surveying is one laden with numerous scientific instruments and tools for mensuration and discovery. As Edmond Kiely says in his work, *Surveying Instruments*, the well known adage that "Necessity is the Mother of invention" holds much truth when held up to the developments in surveying, and many other sciences.[21] The same would hold true for developments in astronomy and navigation, two disciplines which we also examine in this work. Instruments have served practitioners since the times of the ancient Babylonians, Egyptians and Greeks, and continued to do so through the Middle Ages in Islam and the Latin West.

The need for surveying and surveyors developed early from various needs, and in particular the need for calculating taxes according to the size of one's estate, the construction of buildings and for irrigation developments. Plumb-bob levels, sighting instruments and ropes and chords of various lengths were used to perform these tasks. These instruments were employed for matters of levelling, sighting and direct measurement; nor can we forget the use of right-angled instruments from which our modern day carpenter's square was ultimately derived.

During the Middle Ages, the Arabs and the Latins continued to use some of the same surveying instruments which existed in antiquity. For levelling, however, water levels do not seem to be so common, and instruments like the geometric square,

[21] Kiely, *Surveying Instruments*, p. 8.

astrolabe and quadrant were used for such tasks. Besides testing whether a stretch of land, for example, was level using a quadrant, the Muslims also described methods for testing if some object were straight or plane. To test the straightness of some object, a taut string would be used, and once one straightedge was established, this straightedge could be used to verify the straightness of other devices. To determine the planeness of a given surface, a straightedge would be applied to the surface in all possible directions, and one would observe whether any light would pass between the straightedge and the plane surface.[22]

In the world of academia, surveying was a practical geometric application. Geometry for the Latins took on quite a literal meaning, *i.e.*, measuring the earth. The rediscovery of the *corpus agrimensorum* in the 11^{th} century, due in part to Gerbert, signalled a new approach to geometry.[23] In universities, geometry was approached by those who taught it as a practical science. Not only was it useful for surveying or taking measurements, but it had a place in erecting buildings and geographical matters. In the quadrivium of the university, the emphasis shifted from the early distinction of Boethius - the philosophical ponderings of arithmetic, and continuous magnitudes of geometry (continuous magnitudes without motion) and heavenly bodies (continuous magnitudes with motion) - to applications useful in the lives of all people.[24]

Despite the fact that most geometrical knowledge consisted of nothing more than the study of right angle triangles and the use of proportions, Hugh of St. Victor, who wrote *Practica geometriae*, does describe the use of the astrolabe. In the late 13^{th} century, Kiely says that the quadrant was introduced as a tool to be used in surveying, and in the first half of the 14^{th} century the geometrical square was also introduced to this discipline.

Leonardo Fibonnaci of Pisa (*ca.* 1180-1250) was a man learned in both practical and theoretical mathematics. He was the first of medieval writers to mention the use of the plumb-bob in surveying and its use in measuring the distance between two

[22] *Ibid.*, p. 57.

[23] Brian Stock, "Science, Technology, and Economic Progress in the Early Middle Ages," *Science in the Middle Ages*, p. 36.

[24] Pearl Kibre and Nancy G. Siraisi, "The Institutional Setting: The Universities," *Science in the Middle Ages*, p. 122.

points on the slope of a hill.[25] He also describes in his *Practica geometriae* how the quadrant could be use to determine the height of a tower. But this work does not just pertain to practical geometry. He also includes some theoretical work based on Euclid's *Elements* and work by Archimedes.[26] Fibonnaci's quadrant was a relatively simple quadrant which consisted of a graduated arc and a shadow square. Considering the types of exercises which Fibonnaci describes for this quadrant, as Kiely suggests, we may be inclined to conclude as Kiely suggests, that his quadrant was used primarily for surveying.[27]

Examples of these instruments which we have noted above were used; however, we have very little evidence that land surveying was done during the Middle Ages. There was often little need for measuring estates since the boundaries of the property were denoted by land markers like trees, boulders and buildings. Land was owned by nobles and monastic figures, and was leased on a yearly basis to tenants. Land was arbitrarily assigned to people at the whim of the nobleman who owned the land. Hence, there was no need for accurate surveying, and areas were determined only approximately.[28]

During the Renaissance, developments in surveying were due to five movements. These five, Kiely says, involved the disintegration of feudalism, a growing interest in cartography through Ptolemy's work in geography and the discovery of new places, the art of shooting ordnance and need for protective fortifications, navigation upon the seas and the need for more accurate astronomical instruments for performing observations.[29] Numerous instruments were used during the Renaissance including water levels, plumb-bob levels, right angle surveyor squares, the surveyor's cross, various derivatives of the astrolabe and the quadrant, theodolites, triangulation instruments and the magnetic compass.

We have observed that the quadrant was used primarily for astronomical work until about the beginning of the 16^{th} century when it was used more and more by

[25]Kiely, p. 57.
[26]Michael S. Mahoney, "Mathematics," *Science in the Middle Ages*, p. 159.
[27]Kiely, p. 79.
[28]*Ibid.*, p. 97.
[29]*Ibid.*, p. 102.

surveyors and navigators. These quadrants were relatively simple compared to their astronomical counterparts. They consisted primarily of graduated arcs and shadow squares and used plumb lines for levelling. Some quadrants used alidades, but these were primarily used for triangulation and were oriented horizontally on a staff. The first large scale attempt at triangulation was done with a quadrant by Willebrod Snellius in Holland in 1617.[30]

Clearly, the quadrant did play a role in surveying in the early Renaissance; however, it is difficult to say how popular it was as an instrument among surveyors. Numerous texts, such as the *Quadrans vetus* and *The compleat surveyor* by William Leybourn, do show how the quadrant would be used to take the height of a tower, or the depth of a well or the height of an inaccessible object. Texts seem to demonstrate basic Euclidean principles disguised as various methods for surveying. Gunners' quadrants and astronomical quadrants do exist, as do simple quadrants with graduated arcs and shadow squares; however, it would be difficult to say that the quadrant was a commonly used instrument among surveyors, especially when surveyors possessed many other instruments suitable for surveying. It does seem possible that the quadrant was superseded by other instruments, since it would be best suited for taking the elevation or height of some object. The quadrant could, of course, be used for levelling, however, many other instruments existed for this task as well, *e.g.* water levels and plumb bobs. Moreover, texts like George Atwell's *The Faithful Surveyour* and *The Surveyors Dialogue* by John Norden demonstrate that one can do more easily with other instruments the same tasks for which one might be tempted to use a quadrant. We are inclined to believe that the quadrant did not play an important role to advance the art of surveying, but was only employed to demonstrate simple relationships with similar triangles disguised in relatively ordinary surveying problems.

[30] *Ibid.*, p. 168n. Kiely also notes that Jean Picard also attempted to do a similar survey in the north of France.

3.3 The Navigators

In the section, "Signs in the Sky" from her book *The Haven-Finding Art*, Eva Taylor says that "even today, of course, since the ultimate sources of time-keeping and position-finding are the heavenly bodies, the sailor must look up at the sky".[31] This is true only to a certain extent. For the earliest navigators relied just as much and perhaps even more upon other skills and knowledge of the seas and coastal waters. These sea-going explorers and merchants relied on the knowledge of the winds, water currents, various forms of marine life, marine bed deposits and the recognition of diverse ports and landmarks from sea.

So it is without the aid of scientific instruments and the observations of celestial bodies that the first navigators left for the open bodies of water. Nonetheless, it would appear that the first use of any instruments for navigating, namely lead and line, was very early. Sailors used the lead and line to study marine sediment on the bottom of the sea to give an idea of the nature and distance of the shoreline being approached.

With regards to early navigational techniques prior to the Late Middle Ages, it would have been difficult for northern navigators to apply techniques related to time-telling and such when they relied upon the sun for determining the time and position of their ship. The low altitude of the sun at higher latitudes must have made accurate solar observations difficult to perform. Likewise, sailing by the stars would be difficult as well since the stars would appear to move around the Celestial Pole in a more oblique plane to the local horizon when an observer moves closer to the North Pole. While this, in the opinion of Taylor,[32] may explain why northern sailors lagged behind their counterparts in the South, people still sailed with the aid of the sun in the more northern latitudes throughout the seasons. Southern sailors who were unexperienced in such matters, however, might have found sailing more difficult while attempting to observe the sun's position in the sky.

With the arrival of texts from the Islamic world also came certain attitudes toward science. The Arabs advanced not only science, but technology as well, through the

[31]Taylor, *The Haven-Finding Art*, p. 3. The term "heavenly bodies" here also includes artificial ones such as geopositional satellites.

[32]*Ibid.*, p. 65.

making of scientific instruments and the construction of observatories where these instruments were used.[33] One of the first mathematical instruments at this time to have been possibly used at sea was the astrolabe - an instrument which embodies many of the features of the quadrant. We cannot say with any certainty how early the astrolabe was employed as part of the equipment on board a ship; whenever that might have been, sailors still relied upon more traditional sailing techniques and when the astrolabe was used it most likely was used on land since any attempt to use the instrument on less than calm seas would have given inaccurate results.

Until this time, uneducated navigators could still (in some sense) sail a ship, but to use navigational aids the sailor required training and education, and the more numerous illiterate sailors continued to use the magnetic needle and windrose as navigational aids, for the use of such instruments as the astrolabe would be difficult to comprehend by anyone untrained in mathematics. Moreover, the mathematicians did not have the uneducated sailor in mind when they themselves were struggling to master these new mathematical ideas and instruments.

It seems quite plausible then that an astronomer or astrologer would have been employed on a sailing vessel. There certainly would have been the need for such a person to operate an astrolabe or an astronomer's quadrant at sea. Taylor suggests that the astronomer's quadrant does not antedate the 13^{th} century. We are inclined to accept the truth of this though the same principles had been embodied on the dorsal sides of astrolabes for a few centuries prior to the invention of the quadrant.

It is not until the mid-15^{th} century, with the invention of the mariner's astrolabe, the frequent employment of the cross-staff and the use of the magnetic needle and compass, that we find the first record of an observation taken with a quadrant.[34] The adoption of the quadrant (and indeed the astrolabe) for sailing purposes saw modifications in the features of the quadrant. No longer present were the complex lines used for time-telling and determining planetary positions. The lines for the tropics and the ecliptic were no longer preserved, and the shadow square was removed (only to be restored later). The quadrant as first used by sailors in the 15^{th} century,

[33] *Cf.*, *e.g.*, Sayili, *The Observatory in Islam*. This is the standard reference on this subject.
[34] Taylor, p. 159.

was constructed from boxwood or brass and consisted only of a sighting apparatus, a plumb line and a graduated limb divided from 0–90 degrees.

At the time of Henry VI in the mid-15th century, sea captains tended to use the quadrant not to determine unknowns such as angular measurements, but, rather, their position relative to some port. On the quadrant would be inscribed, in addition to the graduated arc on the limbus of the quadrant, the names of various ports and important landmarks. The position on the quadrant where, at a particular site, the plumbline crosses the limbus represented the ship's latitude determined by altura (*i.e.*, by the instruction and observation of solar or stellar altitude), and the various alturas provided the method by which ports were sought. By travelling up or down through the various latitudes while sighting the Pole Star when the Guards were in an east-west position until the plumb line fell across the name of the port of interest, the vessel would then be in an east-west position with that particular port.

The first text on navigation which describes uses for the quadrant at sea appears in 1518.[35] Taylor does not give any indication of who the author is though she says that the work was written as a compilation of early observations made by Portuguese sailors and printed by a "scholarly German printer in Lisbon". In this work, the author describes the relationship between a given number of degrees of longitude and the equivalent number of leagues. Also described is the method of using the altura for determining the latitude of the port of destination, which remained a standard practice for many years, as well as the use of the sun in its meridional transit at noon to determine a ship's terrestrial position.

To this point we have discussed briefly points which reflect the work which the Portuguese did to develop the art of sailing and the role which the quadrant played. Indeed, the Portuguese laid down the foundations for navigational techniques based on astronomical observations; however, the Spanish and the French soon shared centre

[35] *Ibid.*, p. 160. This appears to contradict what Taylor says two pages later, that the oldest extant manuscript of a navigation manual is the *Regimento do Astrolabio e do Quadrante* which was printed in Lisbon in 1509 with the likelihood of an earlier printed version being possible. Without consulting this work, we cannot resolve this question though perhaps the reference to the quadrant would be to the quadrant on the back of the astrolabe, rather than the independent instrument mentioned in the 1518 treatise.

stage with their Portuguese counterparts. The travels of Christopher Columbus and his interest in and utilization of Portugese navigational techniques quickly piqued the interests of Spanish navigators. Likewise, a work entitled *Le Grant* [sic] *Routier et Pilotage* by Pierre Garcie written in the late 15^{th} century proved to be quite popular in France and in an English translation for many decades.

A development in the early 16^{th} century was the mandatory training of all Spanish navigational pilots in the use of the mariner's astrolabe and the quadrant. On the instructions of Queen Joanna of Castile in 1508, Amerigo Vespucci began to carry out her majesty's decree to train and test all Spanish merchants and shipmasters. Soon a school of teachers and navigational practioners emerged to educate all of Spain's navigators-to-be. The formation of marine pilot educators, however, did not make a big splash in other countries, even if the English Arctic explorer, Stephen Borough, did try (to no avail) to convince Queen Elizabeth I to adopt a similar program to educate marine pilots.

3.4 The Military Gunners and Bombardiers

During the Middle Ages, the "science of motion" was a significant part of natural philosophy. Indeed, "the division of natural philosophy reflect[ed] the distinctions between mobile bodies".[36] In medieval times, Aristotle was considered one of the great authorities, and his *Physics* represents the most influential work on motion in that epoch. For Aristotle the idea of motion encompassed more than just locomotion, but in terms of projectiles, the "science of motion" concerns only locomotion. In this section on military gunners, bombardiers and the quadrant, we will examine a few works by mathematical practitioners who were influenced by Aristotle, then Tartaglia and also Galileo.

Aristotelian motion can be understood through two basic principles: a moving

[36]Geoffrey of Haspyl provides us with this bit of insight into the medieval arrangement of Aristotle's works which defined nature as being a principle of motion. Here, Geoffrey of Hespyl's thought on motion comes from his introduction on his commentary on Aristotle's *On Generation and Corruption* which is found in "The Science of Motion" by John Murdoch and Edith Sylla in *Science in the Middle Ages* by David Lindberg, p. 206.

object requires something to move the body, and the distinction between "natural" motion and "violent" motion.[37] The force which moves a body under natural motion, however, really was not a force as we understand forces, but an internal characteristic of the body which "is responsible for its tendency to move [the body] toward its natural place as defined by the ideal spherical arrangement of the elements".[38] And a body would cease to move under natural motion when the body reached its natural place within the order of the elemental spheres. When a body is moved under violent motion an external force is the mover. This force violated the heavy object's natural preference to remain in or move toward its most natural resting place; such motion would continue so long as the external force was continually applied to the object.

This seems like a very plausible framework; however, it could be problematic under certain circumstances. For instance, why does an object projected horizontally not suddenly stop in its path when the external mover of he object ceases to move it and fall downward to the ground? Aristotle answers this problem by indicating that the medium through which the object travels also acts upon the object since the external mover of the object also acts upon the medium.[39] The medium also acts as a resistant to the motion of the object. In other words, air, for example, plays a role in how fast an object moves through it; however, Aristotle was not concerned about the quantitative aspects of a body in motion, *i.e.*, space-time relationships, but rather the qualities characteristic of all types of motions.

Aristotle's theory of local motion (*motus localis* in a medieval context), represents only one part in his theory on change. His theory on local motion, which we shall henceforth call motion, seems quite odd when we view it within the context of dynamics; however, Aristotle's theories and his medieval commentators' views on motion play a very different role within the whole conception of change. We must not forget this, even though we are only examining Aristotle's influence on medieval minds and

[37] Lindberg, *The Beginnings of Western Science*, p. 58. Motion in this context applied only to sublunary motion.

[38] *Ibid.*, p. 58. The arrangement of the elemental spheres in Aristotelian cosmology puts at the centre of this arrangement earth, and then moving outwards towards the celestial firmament the elements water, air, fire, the lunar sphere, the celestial region for the planets and then the sphere of the stars.

[39] *Ibid.*, p. 59.

thinkers of the early Renaissance as it pertains to projectiles and ultimately the use of the quadrant when it is used to mount any piece of ordnance.

Lindberg talks about the science of motion in the Middle Ages rather briefly. In fact, few people are really capable of seeing and comprehending the entire picture of this science in medieval times. He says, "The intellectual framework of medieval theories of motion is a conceptual jungle, suitable only for hardened veterans and certainly no place for day-trips from the twentieth century".[40] Nonetheless, we must take a brief look at late medieval and early Renaissance views of motion in order to understand the context in which projectile motion was developed in the late 16^{th} and early 17^{th} centuries.

The mathematical descriptions of motion more or less belong to a medieval process developed by scholars over a number of centuries beginning in the Late Middle Ages. It never was a part of the purely qualitative analysis of the theory of change. Mathematics, after all, has nothing to do with healthy individuals becoming sick, a pot of water brought to a boil on the stove and the quality of virtue in a human being who becomes corrupt. But these are the sort of qualitative changes which Aristotle and medieval thinkers pondered.

Discussions on motion look at issues related to kinematics and dynamics. The former is primarily concerned with mathematically-based discussions without any reference to causation and the latter is taken up by questions on the nature of the causative forces behind a body in motion. In essence, motion was studied either from the position of motion as a cause or an effect. Both views had strong supporters, but in terms of ordnance in the late 16^{th} century, a new tradition based on these two different perspectives was used to explain this type of motion.

This new tradition had its beginnings in the first half of the 14^{th} century with Jean Buridan who saw violent motion as an impressed motion which he called an *impetus*. This theory of impetus remained intact until the 17^{th} century when new attempts were made to describe unresisted motion as a change in position which required neither an internal nor an external motive force.[41] This theory says that an *impetus* by the

[40] *Ibid.*, p. 291.
[41] *Ibid.*, p. 303.

projector is transmitted to the object being moved, and the impetus then "continues to act as an internal cause of its continued motion".[42] Critics of the theory of impetus, like those of Aristotle, found it difficult to explain how this impetus, which comes from nowhere, could possibly decrease over time.

Early in the 16^{th} century, mathematicians like Nicolo Tartaglia were making attempts to explain mathematically the motion of a projectile. With the help of mathematics and natural philosophy, cannons or other ordnance could be mounted to the required angle above or below the horizon for a successful shot using a gunner's quadrant. The quadrant was used primarily for setting the cannon at the proper angle for discharging the shot. It was, however, used for other purposes as well, as we shall see.

Tartaglia's *Nova Scientia*, which he wrote in Venice in 1537, is the first work which treats the art of projectiles within a mathematical framework. The impetus for this work, Tartaglia says in his *Letter of Dedication to the Duke of Urbino*[43] was a question directed to Tartaglia by a friend who was an "expert bombardier" on how one ought to aim a piece of artillery to obtain the furthest distance. Tartaglia indicates that he took up this problem despite not ever having operated a piece of artillery or guns.[44] Tartaglia presented his findings, based upon physical and geometrical reasoning, to his doubtful friend, showing him how to mount a piece of artillery. He says,

> And to do this most expeditiously, you must have a square made of metal or hard wood that includes a quadrant with its vertical pendant ... and placing a part of its longer leg in the barrel or mouth of the piece lying straight along the bottom of the tube, elevate the said piece so that the pendant, [*i.e.*, plumb line] cuts the curved side of the quadrant [*i.e.*, the graduated circular arc or limbus] in two equal parts.[45]

The graduated arc of the quadrant is divided into 12 equal parts as Tartaglia says and shows in an illustration of erecting a cannon at 45^o above the horizon using a

[42]Murdoch and Sylla, p. 212.

[43]The Duke of Urbino, Francesco Maria della Rovere was an employed member of the Venetian government who organized the defense against the Turks who threatened to invade Italy and particularly Venice.

[44]Drake, *Mechanics in Sixteenth-Century Italy*, p. 63-4.

[45]*Ibid.*, p. 64.

quadrant. And he then explains how this artillery piece makes a 45^o angle with the horizon.

Tartaglia continues to tell another story where two other bombardiers challenge Tartaglia to the same question, but one of the bombardiers had claimed that "the gun would shoot much farther at two points lower on the square".[46] A bet was made, and the men set out to a field near Santa Lucia to determine which hypothesis was correct. Naturally, as Tartaglia recounts the story, Tartaglia wins. He describes how

> Each man shot according to his proposition, without any advantage in the powder or in the ball;[47] and he that used our determination shot a distance of 1972 perches of six feet per perch, while the other, who aimed two points lower, shot only 1872 perches. By this trial all the bombardiers and other people saw the truth of our determination though before this experiment they were in disagreement....[48]

The *Nova Scientia* sought, as Tartaglia says, to investigate the kinds of motion that can take place in a "heavy body".[49] The influence of Aristotle on Tartaglia's work is quite clear since Tartaglia's physical description of a projectile is done by considering natural and violent motion. But Tartaglia realized that the knowledge of how a shot is fired and moves under these physical constraints was not sufficient. A bombardier required practical experience and the knowledge of

> the extent of his artillery shots according to their various elevations. Knowing both these [*i.e.*, also the ability to judge the distance to one's target], he will not err much in his shots; but lacking either of them, he can never shoot by reasoning but only by his judgement. And if by chance on the first shot he hits the place or comes close to the place he wishes, it is rather by luck than by science, especially in long shots.[50]

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[46] "Two points lower on the square" would have meant that the quadrant would have been set for an elevation of $\frac{4}{12}$ of 90^o (and not $\frac{6}{12}$ of 90^o) which is 30^o.

[47] They used a 10 foot long, 4300 pound cannon called a 20 pound culverin.

[48] *Ibid.*, p. 65.

[49] *Ibid.*, p. 65.

[50] *Ibid.*, p. 67.

The problem at hand is merely being able to determine the distance to some inaccessible place using various inventions (like the quadrant). This, of course, was nothing which could not have been done decades and centuries before this time. In fact, Tartaglia cites Johannes Stoeffler, Oronce Finé and Peter Lombard as practitioners who knew how to perform such measurements; however, Tartaglia claims he knows a better way for determining such distances. He does not appear to say anything more about this in his letter of dedication.

Tartaglia almost did not publish this work. He felt that working to improve such a "damnable exercise, destroyer of the human species, and especially of Christians in their continual wars" was "deserving of no small punishment by God".[51] He acknowledges how he once destroyed all of his work on this subject so that no one could profit from this sort of wrong doing; however, when threats of danger from the Turks under the emperor, Suleiman, loomed over Venice, Tartaglia realized that it was not permissible for him to keep this information from others who would defend Venice.

The structure of Tartaglia's work is Euclidean in nature. The first book is an axiomatic approach to the study of motion of a heavy body comprising 14 definitions, 5 suppositions, 4 axioms and 5 propositions. His second book, which he writes in a similar manner, provides a descriptive approach to the nature of the trajectory of a projectile.

In definitions 5, 6 and 7, Tartaglia describes the meaning of motion and its two types. "The movement of a uniformly heavy body *is that transmutation which it makes occasionally from one place to another, the endpoints of which [movement] are two* instants".[52] Tartaglia notes that some scholars of his time say that there are 6 types of movement or transmutation; however, Aristotle cites only three: a change in quantity, quality and location. Naturally, as Tartaglia says, the first two transmutations are irrelevant here. He then describes natural and violent motion as motion governed by no force and some "motive power" respectively. In the case of artillery, the motive power comes from the cannon and the use of gun powder.

The first proposition deals with the much debated question of the nature of an

[51] *Ibid.*, p. 68.
[52] *Ibid.*, p. 72.

accelerating body in natural motion. Tartaglia, however, says that the swiftness of some object is related to the distance traversed. In other words, he considered acceleration to be a change in distance and not a change with respect to time, a view which was held until Galileo demonstrated the contrary. Tartaglia also discusses the nature of the motion of a heavy body travelling through a tunnel passing diametrically through the earth, as Galileo later discussed in his *Dialogue*.[53]

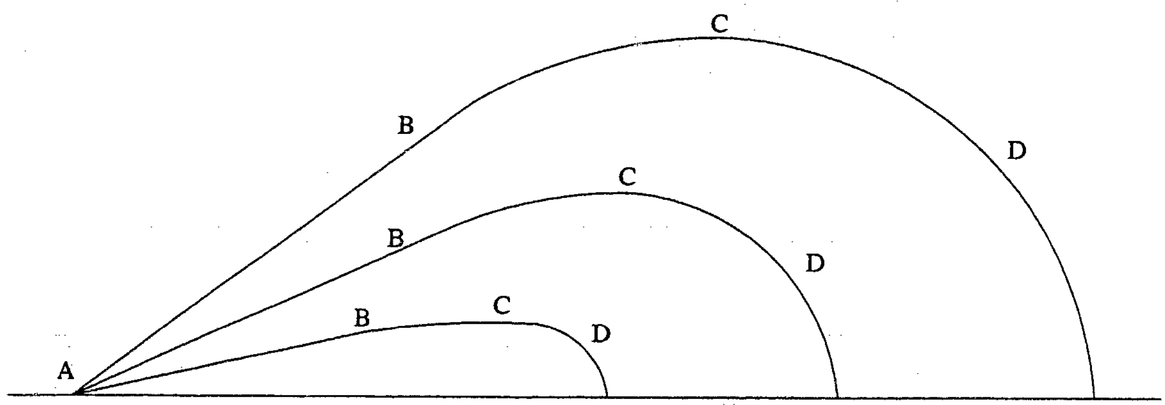


Figure 3.1: The trajectory of a projectile.

Tartaglia describes the trajectory of a projectile in supposition II in the second book as that which "will always be partly straight and partly curved, and the curved part will form part of the circumference of a circle",[54] and upon the termination of any such violent motion "will move with natural motion which will be tangent to the curved part of the violent motion",[55] *i.e.*, in a direction which is ultimately perpendicular to the apparent plane of the horizon.

In fig. 3.1, AB represents a straight line, and BC represents an arc of a circle which together represent the projectile under violent motion. CD, as mentioned above, is tangent to the arc BC and represents the natural motion of a heavy body back to its natural resting place; however, Tartaglia did not believe that the motion from A to B was along a straight line "because of the weight residing in that body, which

[53] *Ibid.*, p. 75-76.
[54] *Ibid.*, p. 84.
[55] *Ibid.*, p. 85.

continually acts on it and draws it toward the center of the world".[56] To him, this part of the trajectory was insensibly curved. As a final note on Tartaglia, we observe that in proposition 8 of his second book, he addresses the fact that a projectile travels farthest if mounted with a quadrant to an elevation of 45°. His proof uses a physical argument which is "anything that passes from the less to the greater, and through everything in between, necessarily passes through the equal".[57]

However, an example presented later is somewhat problematic. He gives the example from astronomy where the day can sometimes be longer or shorter than the night. It does follow, as he says, that there will be a point in time during the year when the day shall be equal to the night and can be "verified to the senses and to the intellect".[58] Tartaglia seems almost a little unsure of himself; he says that if his reasoning is not correct here, then our senses can still verify the validity of this statement. But this theorem is constructed primarily on physical arguments and what is sensibly so to an observer. Tartaglia, however, realizes this and notes that phenomena like the motion of the sun may provide us with situations where things are not quite as the seem, but for all intents and purposes such things are not so problematic.

Soon, works on the art of projectiles by other practitioners appeared more or less in the same vein as Tartaglia until Galileo tackled this problem. Daniel Santbech, for example, wrote a section on *sphaeras tormentarias* in his *Problematum astronomicorum et geometricorum sectiones septem* where he demonstrates the use of a *quadrantus geometricus.*

The trajectory which Tartaglia describes as consisting of three parts was not always treated in those terms by later writers. Santbech is indeed one of these writers. The method which he describes in his *Problematum astronomicorum et geometricorum* is similar to the work of Sebastian Münster in his *Rudimenta Mathematica.*[59] The problem no longer includes the circular arc of the course of the projectile, thus reducing the determination of the range of a shot to the solution of a right triangle.

[56] *Ibid.*, p. 84.

[57] *Ibid.*, p. 91. The proof of this theorem is based upon a law of continuity and physically demonstrates the intermediate value theorem taught in a first year calculus course.

[58] *Ibid.*, p. 92.

[59] Kiely, *Surveying Instruments*, p. 115.

To determine the path of the projectile of the trajectory, according to Münster, we need to know the force with which the cannon can discharge a known weight. Once this is known along with the distance to the mark which the gunner wishes to target, the artillery piece may be mounted to its correct elevation since this amounts to knowing the hypotenuse and base of a right angle triangle. Hence, the angle which to elevate the cannon is known, and is done using a quadrant. Thus, the shot may travel along the hypotenuse, and upon reaching the end of its violent motion falls under natural motion upon its target.

William Bourne (fl. 1565-88) was a very able instructor and mathematical practitioner despite not having a formal education. Indeed, his steadfastness in instructing others on the most important subjects to be learned earned much criticism from other practitioners and scholars.

Bourne himself had been a gunner on the ramparts of Gravesend and Tilbury in the 1560's under Sir William Winter.[60] He felt that there were no texts available in English for gunners and that there certainly was a grave need for such a text. Bourne writes in *The Arte of Shooting in great Ordnance*,

> the cause that hath moued me to write this rude volũe, is this, for that we English men haue not beene counted but of late daies to become good Gunners, and the principall point that hath caused English men to be counted good Gunners, hath been, for that they are hardie or without fear about their ordnance: but for the knowledg in it, other nations and countries haue tasted better therof, as the Italians, French and the Spaniardes, for that English mens haue had but little instructions but that they haue learned of the Doutchmen or Flemings in the time of King Henry the eight.[61]

He adds in chapter 6 of this work that

> I [*i.e.*, Bourne] haue not seen any such book, although it hath been very neer two hundred yeeres since the first inventiō of Ordnance: and excepte

[60] Taylor, *The Mathematical Practitioners of Tudor and Stuart England*, p. 176.

[61] Bourne, *The Arte of Shooting in great Ordnance*. The pages of the preface are unnumbered in this edition. Further citations from the preface will be cited as "Bourne, *Preface*".

> there bee any better booke in some mens hands, such as I haue not seene, as it is like ynough that there may be, there is no Arte in any of them.[62]

In other words, the "Arte" to which Bourne is referring is the "describing of a way or methode, how to atteyne to the certayntie of any matter".[63] The art of shooting ordnance had not yet reached the status of a mathematical science, but Bourne would help to change that.

Bourne explains the one reason why English gunners are thought to be the best. He says that "they are handsome [*i.e.*, skillful or adroit] about their Ordnance in ships, on the sea, &c.".[64] However, on the other hand Bourne explains those things with which gunners have difficulties. He shows great concern that

> those prooues that haue beene made then were no proofes, but to cause those Gunners that did see the experience of those profes, to committe a further errour as touching the Shooting in great Ordnance.[65]

The reason for the errors pertains to the use of the quadrant. Bourne notes that the quadrant has a graduated arc of 90 degrees and that "the principall use of the quadrant, is to know what any peece will cast at the mount of euerie Degree, and so from degree unto degree, unto the best of the Rander".[66] The problem lies in the gunner's ability to determine the distance as the crow flies to that point the gunner wishes to strike. He says the problem has to do with the fact that on the most part

> there is seldome any ground that you shall find leuell, but it will be higher or lower then the ground that the peece standeth upon ... and yet in the time of service there is no using of the Quadrant but in some cases, and then take a great large one [*i.e.*, quadrant], for in a small [one] you soone commit errour.[67]

[62]Bourne, *The Arte of Shooting in great Ordnance*, p. 21.

[63]*Ibid., p. 21.*

[64]Bourne, *Preface.*

[65]Bourne, *Preface.*

[66]Bourne, *Preface.* The "best of the rander" is when the piece of artillery would be elevated to shoot a mortar or shell the farthest distance, *i.e.*, an angle of 45°.

[67]Bourne, *Preface.*

Bourne also reiterates the age-old problem of accuracy which small hand-held quadrants lack for an observer who performs observations with such instruments. Because such instruments were most likely never used in practice, gunners relied upon their own experience. Experience, however, could still not help to determine the effects which the combination of different cannons, shots, powder, and ramming of the shot, *etc.*, could have upon the distance a shot travels, and the height of the gunner's target.

Bourne shows a great deal of indignation toward the habits and knowledge of gunners in his time. He says that although gunners use quadrants, which some call "a Triangle and other fond and foolishly names", and "haue not knowledge what a degree signifieth".[68] Bourne's outrage is not merely directed to gunners, but to those writers before him who lacked any mathematical training. And to make right some of the ignorance of gunners and writers on artillery matters, he describes what a degree is geometrically and tells how a quadrant is divided into 90 degrees. On the quadrant, he says that it is like

> ȳ fourth part of the heavens, for the Zeneth or pricke in the heauens (ouer the Crowne of your head, downe to the horizon) is deuided into 90. equall partes, according unto the Quadrant.[69]

Bourne tackles the problem of setting an artillery piece level with the horizon to fire a shot at point blank range. To begin he says,

> Repaire unto a very leuell ground, as a plaine marrish, that is iust water-leuell, and then to finde the right line or point blanke, rayse a butte or banke in that plain ground, and then sette uppe a marke the iust height of the peece that lyeth in the carriage, and take a quadrant, with a rule fast thereunto, and put the rule into the mouth of the peece, and coyne the breech of the peece up and downe, untill the plummet hang at the corner of the Quadrant, and then ... shall the peece, lye right with the horyzon.[70]

[68]Bourne, *The Arte of Shooting in great Ordnance*, p. 21.
[69]*Ibid.*, p. 22.
[70]*Ibid.*, p. 24.

Now that the mouth of the piece of artillery has been adjusted, a shot is discharged; by noting whether the shot hits the embankment below, above or at the mark on the embankment, the artillery piece is moved closer to, further away from or at the same distance from the embankment. Under the same conditions, the cannon is discharged again until the target is hit. The distance for this cannon, and type of shot and powder for which the shell has been shot at point blank range can be determined. With this information now known, the cannon can be elevated to each degree, and the distance of the shot determined.

In chapter 9, Bourne considers the trajectory of a shot through the air when the artillery piece is mounted at any degree. Bourne's work here departs slightly from the theory which Tartaglia laid out, and may be viewed as an attempt to approximate a parabola by a series of straight lines and circular arcs. When the piece is elevated between 0^o and 45^o, all trajectories are similarly described in terms of 4 piece-wise curves in space which describes the motion of a shot propelled by the "violence of the blast of the powder".[71] The first part of the trajectory is a straight line (OA). While the projectile still travels under violent motion, the second part of the trajectory is circular in nature (AB). The third part ends when the projectile reaches the highest part of its trajectory (at C). This part of the trajectory is also circular. The fourth and final part is downwards circular (D). The entire trajectory is completed under violent motion.

If we consider mounting an artillery piece at an elevation higher than 45^o, the trajectories will be described by 5 piece-wise curves. The trajectories are similar to these with an elevation less than 45^o, except the final course of the projectile is in a "perpendicular line downe to the earth";[72] the final course of this trajectory is done under natural motion, whereas the shot otherwise travels under violent motion. The trajectory is such that

> firste it is driuen violently by the blast of he powder up into the ayre by a ryght lyne, and then secondlye, as the violent drifte doth decay, so it flyeth circularly, and thirdly, the force of the drifte being all decayed, it

[71] *Ibid.*, p. 38.
[72] *Ibid.*, p. 40.

> muste needes haue hys naturall course, and all things that be of earthly substance, muste needes returne to the earth agayne.[73]

For the remaining sections in Bourne's work, he discusses tasks like shooting mortar, discharging shot from a hill top or a valley and shooting from a coastline "at the brode side of a Shippe that is under sayle, and going".[74] Mortar pieces are mounted above an elevation of 45° and "those peeces are used at the seege of Townes, for the annoyance of their enimies, ȳ is to say, to the intent to beat downe their lodgings or houses, with diueres other purposes more".[75]

To shoot a mark on a hill or in a valley, the artillery is mounted as if the mark were at the same level as the mouth of the cannon, but then to this is added the angle that the gunner and the mark make with the horizon. And lastly, the discharging of a shot from shore at a ship at sea requires being able to aim the artillery piece so as to hit the ship broadside under calm seas with the best possible gunpowder.

In 1588, Lucar Cyprian (b. 1544) wrote an important work which was based upon the work of Tartaglia. His work, *Three Bookes of Colloquies Concerning the Arte of Shooting in Great and Small Peeces of Artillerie*, was a translation and augmentation of Tartaglia's work to which Lucar added an additional work which he called *A Treatise Named Lucar Appendix, Collected by Cyprian Lucar Gentleman, Out of Divers Good Authors in Divers Languages.* The work, as the title suggests, is a collection of dialogues which Lucar wrote with Tartaglia, the Duke of Urbino, Gabriel Tadino, the prior of Barletta, a bombardier and a gun founder *et al.* as the participants in the various discussions.

In the first dialogue, Tartaglia and the Duke are discussing the construction and use of a quadrant for mounting and shooting various ordnance. Lucar's description of the quadrant is slightly different from Tartaglia's description. The quadrant, as is shown in Tartaglia's work, is indeed a quarter circle; however, Lucar describes the same quadrant with a few additions. Each of the twelve divisions is divided further into twelve parts; however, Lucar does not show this in his diagram of the quadrant.

[73] *Ibid.*, p. 41.
[74] *Ibid.*, p. 51.
[75] *Ibid.*, p. 41.

He says, "although I have not diuided this figure into so many parts, because they would heere marre the same [*i.e.*, the figure of the quadrant pictured in this book]".[76] Moreover, the quadrant also bears a shadow square, the sides of which are marked with "right shadow" and "contrarie shadow", and also a sighting mechanism. All of this suggests that Lucar was familiar with the shadow square as a tool for determining altitudes and areas in surveying.

Lucar continues with the dialogue between Tartaglia and the Duke. They discuss the nature of a projectile shot at various elevations. Tartaglia concludes that at an angle of 45°, the shot will travel the farthest distance; if the ordnance is mounted higher than this, then the ordnance will be mounted for mortar. The colloquy concludes with their discussion of creating a table for specific ordnance for all elevations. Tartaglia then concludes by saying that the whole art of shooting ordnance is not merely to be understood, but rather to be tested by the senses, *i.e.*, "seeing is believing".

The second dialogue again involves Tartaglia and the Duke who discuss the nature of the projectile. Lucar says a couple of times in commentaries at the side of the main text, that Tartaglia "by these words in a right line, meaneth an insensible crooked line".[77] Tartaglia draws upon Aristotle and the science of weights to describe how a piece of ordnance when it is mounted horizontally "flyeth more heauily out of a peece ... than it wil doe out of the same peece any whit eleuated." Moreover,

> a pellet shot out of a peece lying leuell rangeth in a more crooked line, and more sooner beginneth to decline downwards to the ground than it will do when it is shot out of a peece somewhat eleuated, & it striketh with lesse force than it wil do out of the same any whit eleuated.[78]

The remaining colloquies discuss the effects of a warm cannon on shooting pellets. A warmer cannon will shoot farther since the powder will not be as wet, and hence

[76]Lucar, *Three Bookes of Colloquies*, p. 2.

[77]*Ibid.*, p. 6.

[78]*Ibid.*, p. 8-9. Tartaglia does not, to our own knowledge, say this in his own work, and therefore it must be an addition made by Lucar. The text does appear as part of Lucar's running commentary on the discussion between Tartaglia and the Duke. In fact, in Tartaglia's own work, he never discussed the effect of the weight of a body on its speed in free fall. He does, however, discuss the effect of the medium as a resistance on the motion of a body. That is, a body will appear to travel faster, if it makes a greater effect, *i.e.* a greater penetration through the air, on the resistant, *i.e.* the air.

ignite with a greater force. Other dialogues discuss how by sighting along a particular line and given elevation, the trajectory of the projectile is determined. The effect of damaged ordnance or the use of hand guns finishes book one of Lucar's work.[79]

In *A Treatise Named Lucar Appendix*, Lucar lists a number of authors whose work Lucar had drawn upon to write his work. He cites 9 Italian writers, 11 Latin and 5 English authors. Some of these include Tartaglia, Girolamo Cataneo, Cosimo Bartoli, Daniel Santbech, Sebastion Münster, Hieronymus Cardano, Gemma Frisius, Robert Recorde and Leonard and Thomas Digges.

In this work, Lucar discusses such things as checking with a quadrant to see that the ground under a piece of ordnance is level. The mounting of a piece of ordnance is also examined, and Lucar notes that while a piece of ordnance mounted at 45^o shoots the farthest, as Tartaglia says in his own work, Lucar also says that William Bourne was quite aware that the wind could have a significant affect on the distance travelled by some shot or pellet. Lucar considers various scenarios under which a piece of ordnance may be shot, including at a ship on a river. He ends this short treatise by showing how one may determine the heights of buildings, inaccessible points and the altitude of the sun with a gunner's quadrant, a geometric square and a semicircle.

The Gunner: The Making of Fire Works by Robert Norton was printed in 1628. Norton does not claim to know much about artillery; however, he indicates that he wrote this work to represent the truth according to experience. Rhetoric had no place in his work. Moreover, he wrote this work to show others of the errors found in the works of Tartaglia, Rosselli, Cataneo and a Mr. Smith.[80]

Norton begins to describe the trajectory of the projectile at theorem 28 in his work. In theorem 28, he introduces natural and violent motion. Theorem 32 indicates that there are three parts to the trajectory of a projectile, namely as straight line, followed by a curved declining arc and then a straight line; however, the curved arc is not the arc of a circle. In theorem 40, Norton says that the middle part of the trajectory has "a very great resemblance of the Arkes Conicall. And in the Radons aboue 45.

[79]We have not been able to examine books two and three which consist of another 25 pages.

[80]We do not know who Mr. Smith is; however, Norton indicates that Smith wrote a work called the *Art of Gunnery* which was later called the *Complete Souldier.*

[*i.e.*, when the ordnance is mounted at an angle of 45°] they doe much resemble the Hyperbole, and inall vnder the Ellipsis: But exactly they neuer accord...".[81]. Theorem 42 says that the apparently straight line which is the final stage of the trajectory is still helical in nature and consists of both violent and natural motion. Just as surprising, Norton says in theorem 43 that when a piece of ordnance is mounted at 45°, the greatest distance travelled by a shot is not achieved; however, he does not say anything further.

In Chapters 1 and 2, Norton provides some interesting historical detail on the art of shooting ordnance in ancient and "modern" times, as well as, various types of ordnance. Norton talks about various authors who attribute the creation of gunpowder to the Chinese, or Archimedes or even a King Vitey who summoned an evil spirit to show Vitey how to defeat the Tartars, and others still.

Norton, of course, describes how to mount a piece of ordnance, but he also tells how a particular type of gunners' quadrant can be used to tell if the "peece is then truly bored".[82] That is, the quadrant is used to determine if the bore of the gun is smooth and evenly so around the entire inside of the bore. Norton also shows a number of different types of gunner's quadrant.[83] The gunner's quadrant set inside the mouth of the cannon is not only graduated with the usual 12 points, but it is also graduated into 90°.

The final work which we shall examine here is a work called *The Compleat Gunner in Three Parts* written in 1672. The author is not known to us, but the work does appear listed in short title catalogues (STC's). Although the work was written in 1672, one can see from the work the progress which had been made over the previous centuries. Instruments needed for the gunner to do his job included calibers, compasses, gunners scales and quadrants, geometrical squares, various levels and weights, various tables, ladles, sponges and rammers and much more. The work also discusses how these instruments were to be made.

In the second part of the work, various chapters outline how the quadrant could

[81] Norton, *The Gunner: The Making of Fire Works*, p. 12.
[82] *Ibid.*, p. 81-2.
[83] *Ibid.*, p. 90-1. In the text, the diagram is titled "La facon forme de repartement des quadrants".

be used for determining the heights of towers and inaccessible points. The author, however, still uses Tartaglia as the authority on the nature of the trajectory of a projectile which is described according to violent and natural motion, but also includes mixed motion which Tartaglia said could not exist.

Appended to this work is a short treatise entitled *The Doctrine of Projects Applyed to Gunnery By those late famous* Italian *Authors Galilaeus and Torricellio.* In this work we find references to Galileo's studies on motion where the trajectory of a projectile is described as parabolic in nature. In Proposition 2, the author describes how "The impetus and Amplitude being given, to find the direction according to which the Parabola was made; as also to find the Altitude". The text is difficult to read and the accompanying diagram is unclear.

The work concludes with the description of a quadrant with unequal divisions. The divisions are not meant to represent elevations, but rather "the lengths of the Ranges". It's construction is based upon the theory which Galileo laid out on the parabolic nature of the trajectory of a projectile. The divisions on the quadrant marked 1 through 12. The author says,

> Thus we shall be assured, that the Gun, if it shall be elevated to one point of the said Quadrant, shall carry such a distance, whatever it be: and elevated to two points, shall precisely double that Range: and if to three, it shall carry three of those spaces...[84]

At the sixth point, the greatest distance is achieved; this occurs when the ordnance is elevated to 45°. From the points 6 through to 12, the distance achieved "shall go in the same manner decreasing". Their construction is then described according to Galileo's theory.

In a span of approximately 200 years, the theory of projectiles developed quite considerably from the point of view of kinematics and dynamics. The theory of natural and violent motion was replaced by a theory of impetus and then eventually by the ideas of momentum and a gravitational force. The characteristics of the trajectory were studied, and eventually the trajectory was described parabolically. And lastly,

[84] *The Doctrine of Projects*, p. 17.

we find a newly constructed quadrant used for determining the distance which a shot would travel over a level piece of ground.

Chapter 4

Constructing and Using a Quadrant

In this chapter, we will describe the construction of a number of curves found on most astronomical quadrants; however, we will examine how these curves are constructed in light of those constructed on Gunter's quadrant. Edmund Gunter (1581-1626), a Queen's Scholar at Westminster School and Christ Church, Oxford, is best known for his work, *Description and Use of the Sector*, which was originally written in Latin and published in 1607 as *De Sectore. Descriptio et Usus.* Gunter possessed a keen interest in mathematics and sundials and was a Professor of Astronomy at Gresham College from 1619-1626. He constructed various types of dials and is known for various other inventions.[1]

Gunter's *Description and Use of the Sector* was originally circulated in the form of copies of his own hand-written notes, which he made available to his students and friends. The tedium of writing each copy of the text by hand finally prompted him to produce an English translation for printing in 1623.[2] In 1624, he wrote *The Use of*

[1]Taylor, *The Mathematical Practitioners*, p. 196. She notes that Gunter also created a new cross-staff and a surveying chain based on the decimal principal. Also, Oughtred states in his *Circles of Proportion* that "[t]he honour of the invention of logarithms, next to the Lord of Merchiston [John Napier] and our Mr. [Henry] Briggs, belongth to Master Gunter, who exposed their numbers upon a straight line."

[2]*Ibid.*, p. 339.

the Quadrant to accompany his quadrants which were constructed by the well-known mathematical instrument maker, Elias Allen (*fl.* 1606-54).

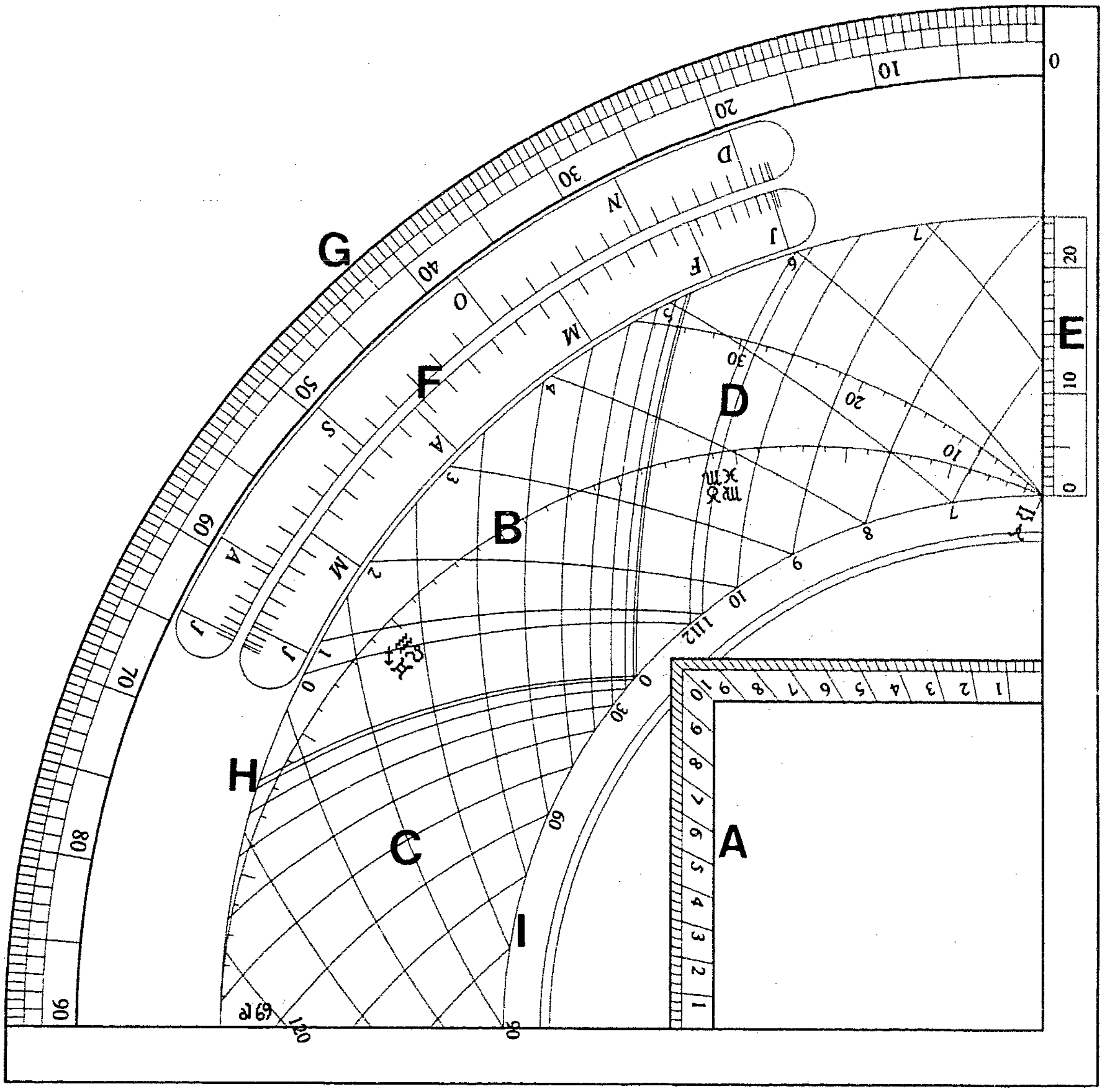


Figure 4.1: Gunter's Quadrant.

Before we proceed to elaborate on the construction of the various parts of the quadrant, we shall briefly describe those parts common to most quadrants as they appear on Gunter's quadrant as in fig. 4.1. A plumb line with a sliding bead is attached to the vertex (loosely called the centre) of the quadrant. The face of the quadrant is etched with a **shadow square** (A) used for trigonometrical calculations, and planar projections of the **ecliptic** (B), **azimuth lines** (C) and **equal hour lines** (D). The quadrant is also fitted with a scale for determining the sun's **declination** (E), a fixed **cursor** (F) which provides the user with the sun's noon altitude for a given day and latitude, and a graduated 90 degree arc on the **limbus** (G). The **Tropics of Cancer and Capricorn** are represented by the quarter circle (H) and the equator is represented by the quarter circle (I).

The parts of Gunter's quadrant are noted here to provide an easy reference for the reader when the individual parts are discussed later in this section. Technical words which have been mentioned thus far and later on in this chapter and are printed in bold type appear in a glossary at the end of this study.

4.1 On the Graduated Limbus

As the name suggests, the shape of the quadrant is one quarter of a circle. It is this physical feature and the division of the curved outer limbus of the quadrant into degrees and oftentimes fractions thereof that appear to be the only common features in all forms of vertically hand-held quadrant used by all practitioners. As is apparent, the graduated limbus allows the user to determine an appropriate angular measurement; however, the answer to the question of who thought of using a device like the quadrant to take such a measurement perhaps will remain out of our reach.

Nonetheless, the procedure used to take an angular measurement can be explained in the light of Euclid's *Elements* VI, 8. Whether the Euclidean proposition was the original idea for the invention of such a device will not be answered here, although it provides an exact demonstration for this operation with the quadrant.

Proposition VI, 8 says that

> If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.

Since the quadrant is a quarter of a circle, the angle at A is a right angle as is shown in fig. 4.2. The plumb line from A falls perpendicularly to the plane of the apparent local horizon at D. Thus, by proposition 8, $\triangle ABC \sim \triangle DBA \sim \triangle DAC$. Therefore,

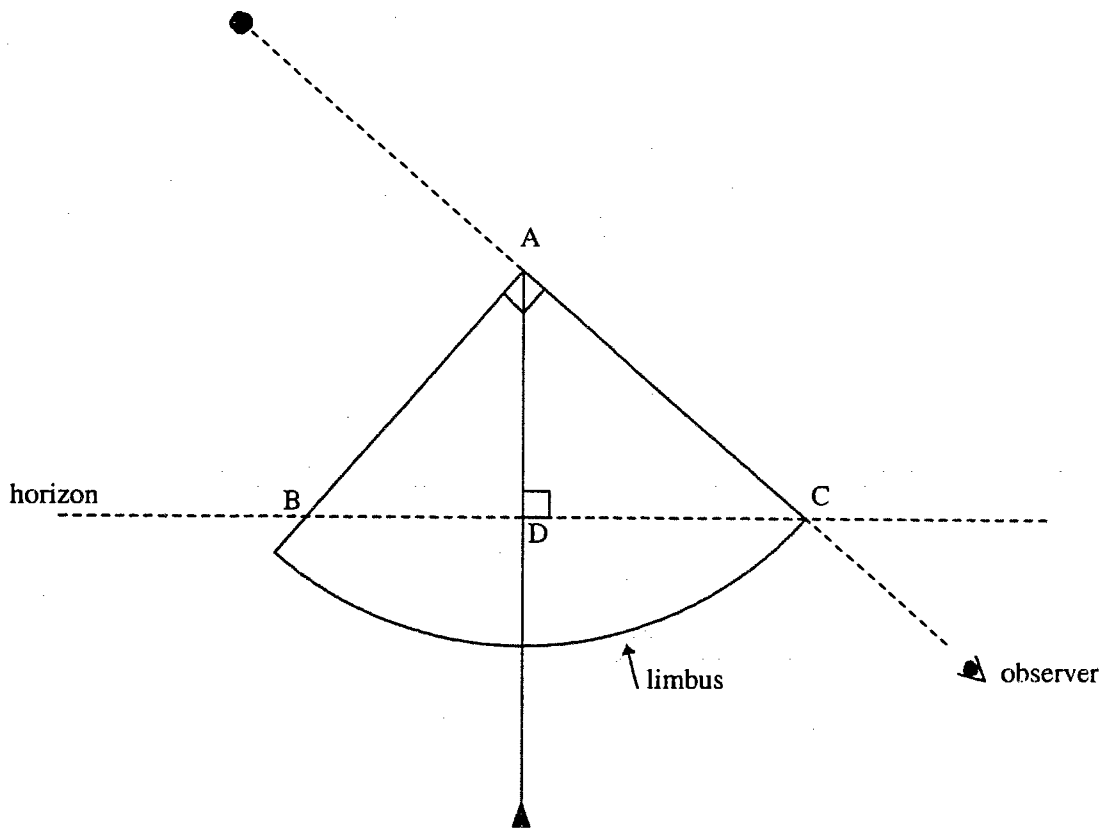


Figure 4.2: Sighting an object to give the angular measurement made with the local horizon.

the plumb line which crosses the graduated limbus provides the angle that the sighted object makes with the plane of the local horizon since $\angle BAD = \angle ACB$. (In Euclid's proof of this theorem, he also identifies the corresponding angles.)

With the limbus graduated from 0° to 90° starting from the radial edge AB, a user may perform a number of tasks. An astronomer might use a quadrant, with additional tables, to carry out calculations requiring the knowledge of one's latitude or the sun's declination, both of which may be obtained by taking the altitude of the sun in its meridional transit. One may also determine the latitude of one's locality by sighting the Pole Star. Navigators, as well, used the same method of sighting the Pole Star to determine their ship's latitude or the altura for a given port or landmark. In the Middle Ages, many manuscripts of the *corpus agrimensorum* tradition, like

Boethius' *Geometria* show numerous examples of the sorts of tasks a surveyor might perform. Though other instruments like the theodolite, the torquetum and the sector became available in the early Renaissance, the quadrant remained an important tool in demonstrating the underlying geometric principles to the surveyor. Problems related to altimetry, planimetry and stereometry were of great importance, for example, in determining the height of a building, or the depth of a well. And a military gunner would use the quadrant in this manner to mount his gun at a predetermined angle to strike his enemy.

4.2 On the Shadow Square

The shadow square as it was known by the medieval and early Renaissance practitioners originated in Medieval Islam. The shadow square is one element which was also constructed on the dorsal side of astrolabes. The square is placed with two of its sides along the two radial edges of the quadrant at the vertex of the quadrant. The remaining two sides, the *umbra versa* and the *umbra recta*, are divided into an equal number of divisions, usually twelve in number, though sometimes they are divided into nine or ten divisions and sometimes into further equal subdivisions.[3] Each of these two sides is divided from zero to twelve, starting from each radial edge of the quadrant to where the two scales meet at the twelfth division. On the quadrant, the side of the square known as the *umbra recta* ($B'O$) is perpendicular to the side used for sighting objects; the remaining side is designated as the *umbra versa* ($C'O$) in fig. 4.3.

Its name is derived from its original function of using the length of the shadow to calculate the height of some erect object. A similar example involves replacing the gnomon by a building, for example, and the shadow cast by the building is used to calculate the height of the building. When an object like the sun is sighted at an angle with the horizon of less than 45°, the plumb line on the quadrant passes through the *umbra versa*. When the angle made is between 45° and 90°, then the plumb line will

[3] *Cf.*, *e.g.*, Sutton, *A description and use of a large quadrant*, 1669. In this work, Sutton describes the construction of a shadow square with 10 equal divisions.

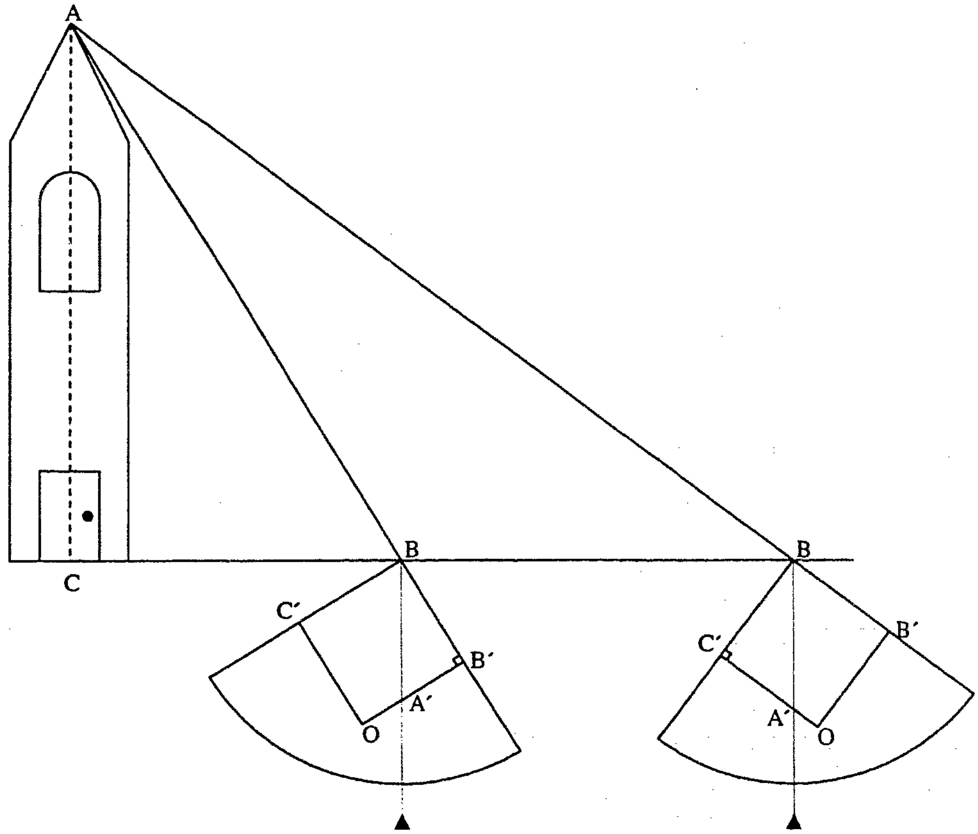


Figure 4.3: Using the shadow square to find the height of an accessible object like the top of a steeple.

fall on the *umbra recta.*

A practitioner taking the altitude of a steeple on a building, for example, as in fig. 4.3, would perform the following task. The length BC would be measured, and then the the top of the steeple would be sighted along the side BB'. The plumb line would then fall across OC' or OB' at A' depending on whether $\angle CBA$ was less or greater than 45°. In the case when the angle is less than 45°, $\triangle ABC$ is similar to $\triangle A'BC'$ and

$$\frac{AC}{BC} = \frac{A'C'}{BC'} \tag{4.1}$$

and hence

$$AC = \frac{A'C' \cdot BC}{BC'}. \tag{4.2}$$

BC', since it is a side of the square, is equal to the number of equal divisions which are graduated on both the *umbra recta* and *umbra versa.* The plumb line crosses the *umbra versa* at some known point A' such that the length of $A'C'$ can be determined;[4] hence, the height of the steeple is known. Specifically, if the scale has 12 divisions and the plumb line falls at graduation mark n, then the height is $\frac{n}{12}$ of the distance to the base point C.

If on the other hand $\angle ABC$ should be greater than 45° then $\triangle ABC$ is similar to $\triangle BA'B'$. Hence,

$$\frac{A'B'}{B'B} = \frac{BC}{AC} \tag{4.3}$$

and hence

$$AC = \frac{B'B \cdot BC}{B'A'}, \tag{4.4}$$

and hence the height of the steeple is known.

In the first case, when $\angle ABC$ is less than 45°, the ratio $\frac{A'C'}{BC'}$ is equivalent to

$$tan(\angle ABC) = tan(\angle A'BC').$$

In the other case, $\frac{A'B'}{BB'}$ is equivalent to

$$cot(\angle ABC) = cot(\angle BA'B').$$

The remaining scales and lines found on an astronomer's quadrant are the stereographically projected lines of the horizon, ecliptic, equator and tropics, a set of azimuth lines, the equal and unequal hour lines and the cursor. We shall demonstrate here the underlying mathematical constructions which are needed to produce these various lines on the quadrant.

4.3 On the Stereographic Projection of Certain Celestial Circles

We shall describe the construction on the plates of an astrolabe of the horizon, ecliptic, equator and the tropics of Cancer and Capricorn as shown in fig. 4.4 in a representation of these circles on a plane. Here the underlying idea rests upon the principle

[4]The length corresponding to this point may need to be interpolated if the plumb line $A'B$ falls between two consecutive graduated points on the scale.

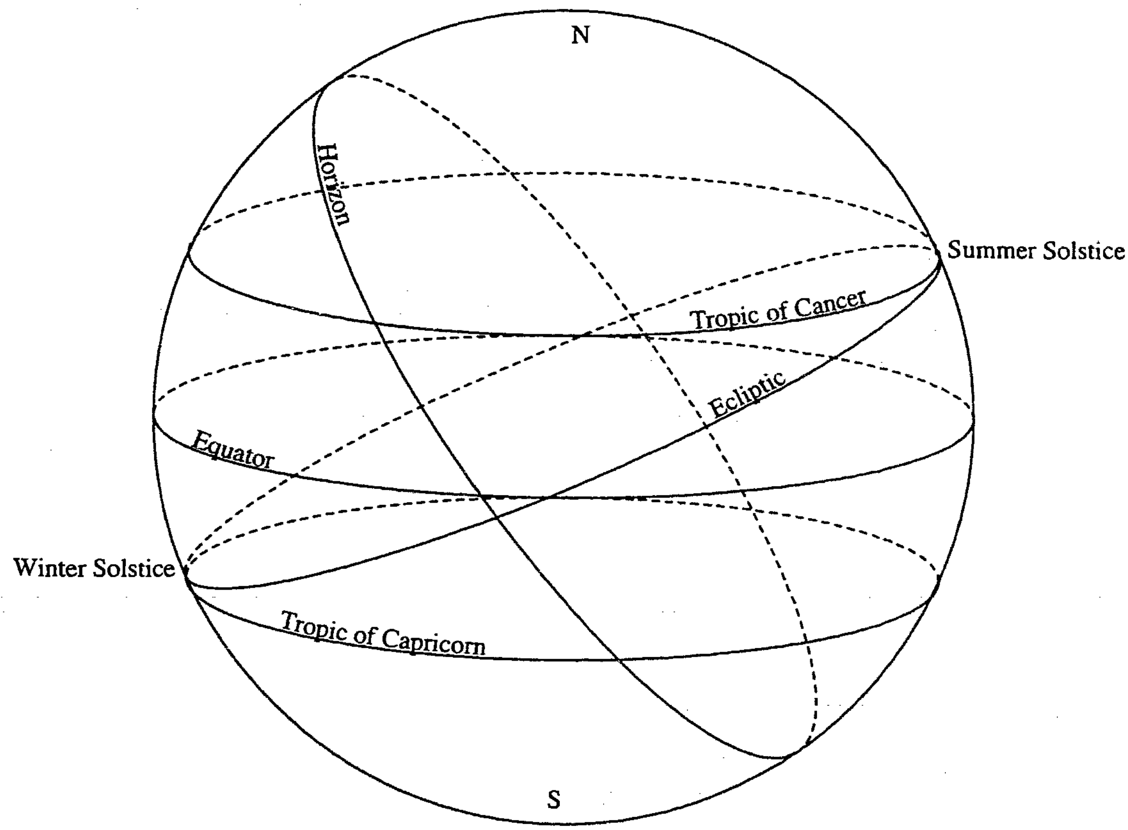


Figure 4.4: The celestial sphere.

of the stereographic projection. As is noted in (1.4.1), a stereographic projection is a one-to-one relation which maps circles on a celestial sphere onto circles on some plane (usually the plane of the equator) from a point of projection (usually a pole of the celestial sphere). The projection of the equator is nothing more than the equator itself since it lies in the projection plane. All other circles which lie in planes parallel to the plane of the equator are projected as circles which are concentric with the equator.[5]

Although we have not discussed the second property of a stereographic projection which deals with conformality or the preservation of angles in any detail, since there is no evidence of this being known, we can say that writers on the quadrant and the astrolabe certainly realized that the projections of two tangent circles were also tangent. With this in mind, the construction of oblique circles can be explained. To construct the projection of an oblique circle on the plane required knowing the

[5] *Cf.*, Apollonius, *Conics* I, 4.

projection of two other circles, namely the two circles tangent to it that are parallel to the equator.

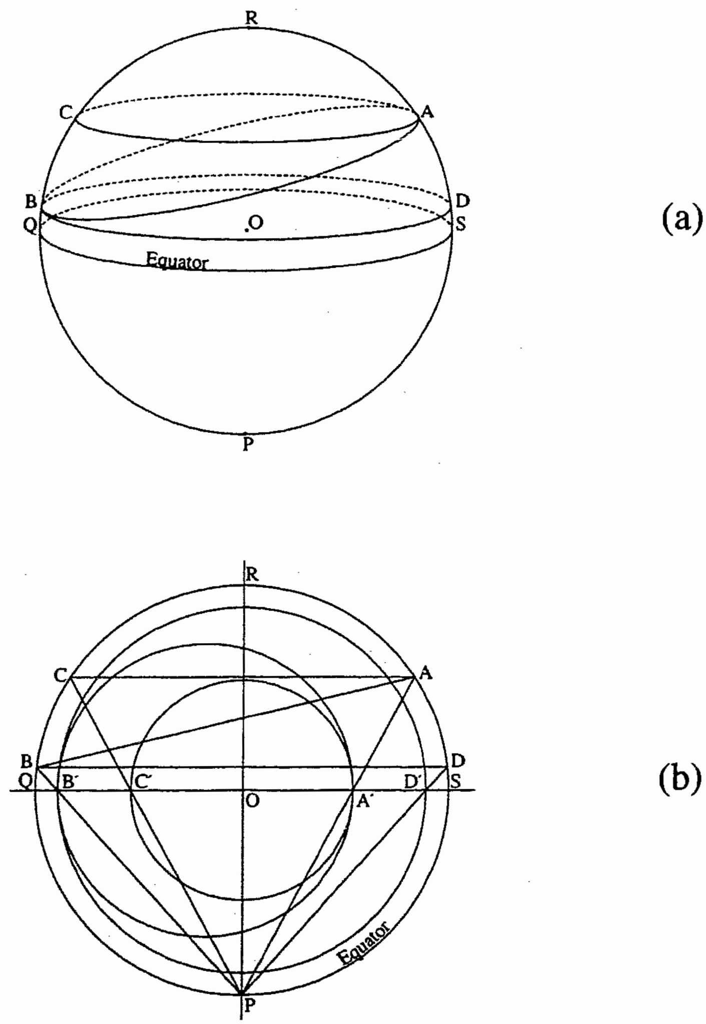


Figure 4.5: Stereographic projection of an oblique circle.

In fig. 4.5, let $PQRS$ represent the celestial sphere with O as centre and the equatorial plane QOS, as well as the stereographic projection of the equator. Let AB be the given oblique circle. Constructing the stereographic projection of this circle requires two tangent circles which lie on planes parallel to the plane of the equator QOS, *i.e.* the circles BD and AC. These circles BD and AC, which are called **circles of declination**, are mapped to the circles $B'D'$ and $A'C'$ respectively and are concentric with the equator $PQRS$. Hence, the stereographic projection of the

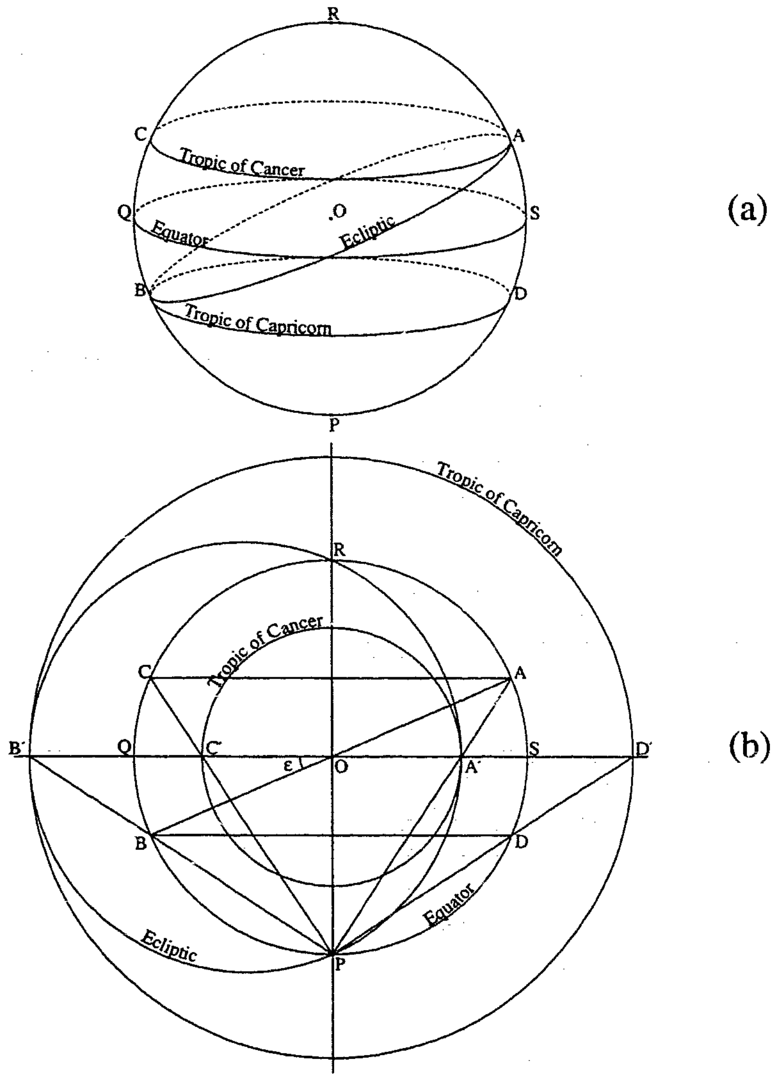


Figure 4.6: Stereographic projection of the ecliptic, tropics, and equator.

circle AB, *i.e.* circle $A'B'$, will be tangent to the two aforementioned circles $B'D'$ and $A'C'$.

This gives us a method for constructing, for example, the ecliptic. The ecliptic is a great circle which intersects the plane of the equator at two diametrically opposite points which correspond to the spring and autumn **equinoxes**. The **obliquity of the ecliptic** is approximately equal to $23°30'$ and varies insensibly over one person's lifetime. To construct the stereographic projection of the ecliptic, we require two circles tangent to the ecliptic which lie in planes parallel to the plane of the equator:

namely the **tropics of Cancer** and **Capricorn**. They touch the ecliptic at those points which correspond to the **solstices** or the beginnings of the zodiacal signs of Cancer and Capricorn and lie in planes parallel to the celestial equator.

Again, let $PQRS$ represent both the celestial sphere with centre O and equatorial plane QOS, as well as the stereographic projection of the equator as shown in fig. 4.6(b). Let AB be the ecliptic; AC and BD represent the tropics of Cancer and Capricorn respectively. The stereographic projections of the tropics of Cancer and Capricorn result in the concentric circles $A'C'$ and $B'D'$. Hence, the ecliptic will be mapped to the circle $A'RB'P$ which is tangent to the projections of both tropics at A' and B'. And the desired ecliptic circle can be constructed, as the one with diameter $A'B'$.

Any great circle in the celestial sphere will cut the equator at diametrically opposite points, so that when it is projected, its image will cut the equator at the same diametrically opposite points. In the case of the ecliptic, it cuts the equator at the points P and R.

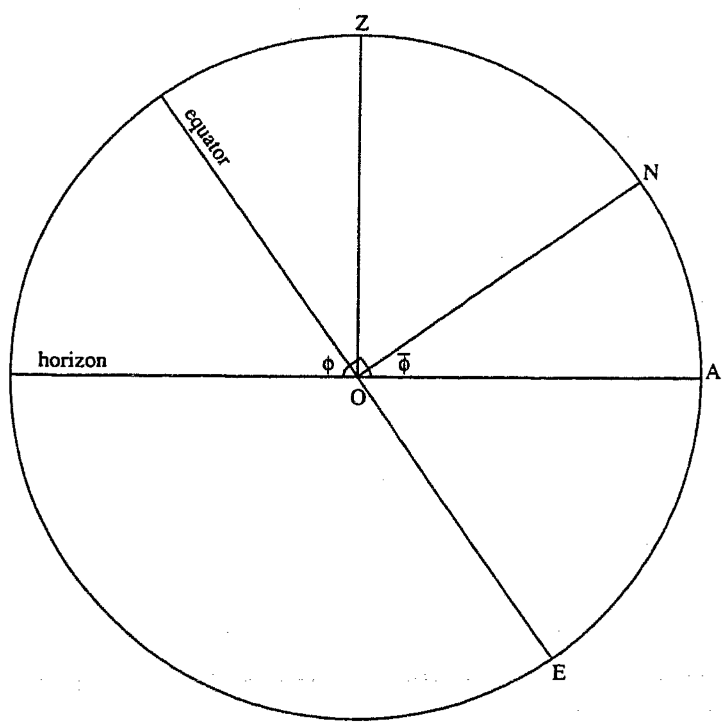


Figure 4.7: The plane of the horizon.

Likewise, the horizon for a given locality will cut the equator at diametrically opposite points and its projected image will cut the equator at the same diametrically opposite points since it, too, is a great circle. (Note that on the astrolabe, the rete which has the projection of the ecliptic is not on the same plate as the horizon.) The horizon is like the ecliptic except that it lies in a plane with an obliquity equal to the complement of the local latitude. In fig. 4.7, ZO, where Z is the zenith of the given locality, makes a right angle with the local horizon through A and B. The local latitude which is ϕ and the obliquity of the horizon, namely $\angle EOA$, together are a right angle. Hence, the obliquity of the horizon is $90° - \phi$ or the complement of the latitude which we represent here by $\overline{\phi}$. The construction of the stereographic projection of the local horizon is therefore similar to that of the ecliptic.

4.4 Constructing Certain Circles by Stereographic Projections

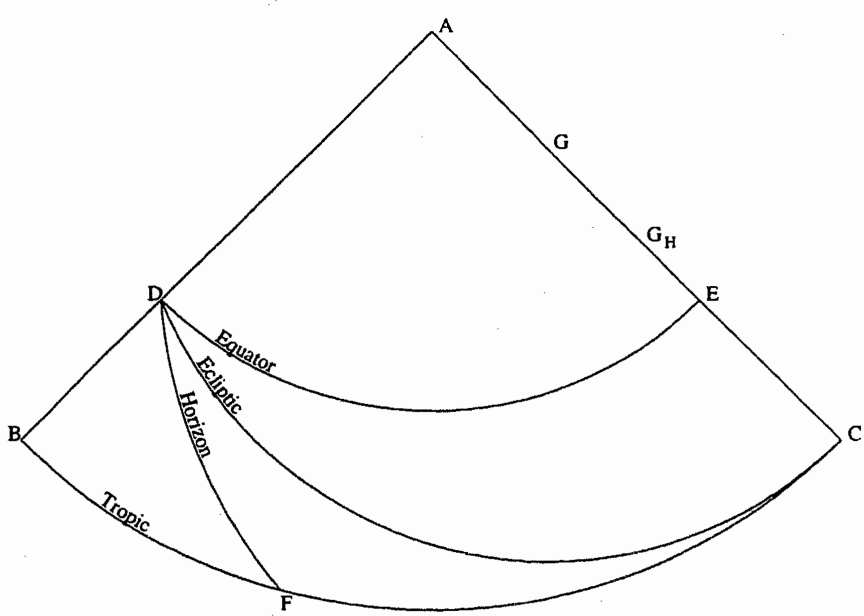


Figure 4.8: The constructed curves of the equator, tropic, ecliptic and horizon.

Thus far we have provided a theoretical basis from which we may now actually construct these circles, *i.e.*, the ecliptic, horizon, and the tropics of Cancer and Capricorn. To begin, a quarter circle is constructed with a given radius, AB as in fig. 4.8 . Let A be the centre and the radius AB sweep out a 90^o arc, BC. This arc, as Gunter says, will represent either tropic. Some texts like John Collin's *The sector on a quadrant* indicate the length of the radius *e.g.* "from the Center is placed a Line of equal parts, of 5 inches in length...".[6] On the other hand, works like Gunter's *Description and Use of a Portable Instrument* [known by the name of Gunter's Quadrant] describe the construction of the various curves in relation to an arbitrary radius.

The next curve which can be constructed easily is the equator. This arc is concentric with the arc representing either of the tropics. Gunter describes the construction of this arc by dividing the radius AB at some point D such that

$$\frac{AB}{AD} = \frac{10000}{6556}; \tag{4.5}$$

then through D with A as centre, construct the arc DE. This ratio is just an approximation to four correct decimal places of the ratio

$$\frac{1 + sin(\varepsilon)}{cos(\varepsilon)} \tag{4.6}$$

where ε is the obliquity of the ecliptic.[7]

The construction of the ecliptic requires calculating the centre of the projected circle representing the ecliptic. The theory behind the stereographic projection of the ecliptic indicates that the projection of the ecliptic will touch the tropic (at C) and meet the equator (at D).

To calculate the center of this circle, we shall use fig. 4.6(b) to determine the location of this point. First, we calculate the radius of the ecliptic, *i.e.*, the circle $A'RB'P$. The diameter is $A'B'$, and $A'B' = B'O + OA'$ where $B'O$ and OA' are the radii of the circles which are stereographic projections of the tropics of Capricorn and Cancer, respectively. Also

$$OB' = OQ \cdot \frac{1 + sin(\varepsilon)}{cos(\varepsilon)} \tag{4.7}$$

[6] John Collins, *The sector on a quadrant*, p. 1.

[7] Relatively few texts, *e.g.*, Collins, Gunter, Leybourn, describe the actual construction of these lines and are more apt to provide only a description of the curves.

and

$$OA' = OQ \cdot \frac{cos(\varepsilon)}{1 + sin(\varepsilon)}. \tag{4.8}$$

Thus, the radius of the circle $A'RB'P$ is

$$OQ \cdot \frac{1}{cos(\varepsilon)} \tag{4.9}$$

and the centre is

$$OQ \cdot \frac{1 + sin(\varepsilon)}{cos(\varepsilon)} - OQ \cdot \frac{1}{cos(\varepsilon)} \tag{4.10}$$

or

$$OQ \cdot tan(\varepsilon) \tag{4.11}$$

from O. Thus, in fig. 4.8 we can now construct a new point G on AC such that

$$AG = AD \cdot tan(\varepsilon) \tag{4.12}$$

since AD is the radius of the projection of the celestial equator. Now with G as centre and GD as radius construct the arc CD. This describes that part of the ecliptic which lies within the quadrant bounded by OR and OQ in fig. 4.6(b). The symmetry about the meridian, $B'OD'$ allows us to concern ourselves with only a half of the stereographic projection.

This does not explain, however, why the projection of the ecliptic on Gunter's quadrant is merely one arc instead of two since the projection would be folded yet again along POR. This is most perplexing since Gunter does not address this. It can be explained by considering two stereographic projections: one from the north pole and the other from the south pole. The projection from the north pole projects all celestial points in the northern hemisphere onto the equinoctial plane. The same is done for those points in the southern hemisphere, except the point of projection is the south pole. Therefore, circles of declination equidistant from both sides of the celestial equator will coincide when projected onto the plane of the equinoctial. Hence, the tropics will coincide with each other, and together the two projections provide a four-fold symmetry along the prime meridian and the celestial meridian through the east and west cardinal points.[8]

In the same manner, to construct the stereographic projection of the horizon for one's latitude, ϕ, we determine a point G_H on AC such that

$$\frac{AG_H}{AD} = tan(\phi); \tag{4.13}$$

then we construct the arc DF with centre G_H and radius G_HD. This arc DF represents that part of the horizon that falls between one of the tropics and the equator.[9]

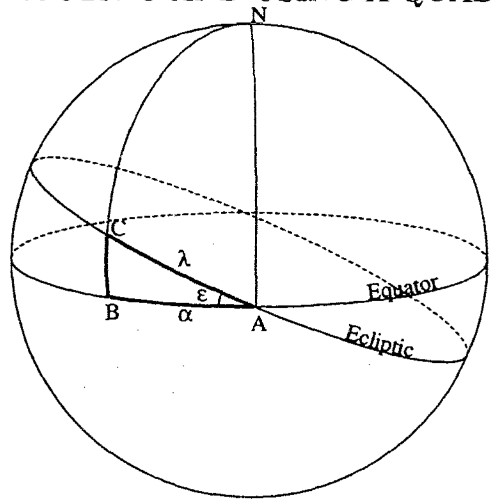
4.5 On Graduating the Ecliptic and the Horizon

In the case of the ecliptic and the horizon, we must now discuss the process by which we graduate both circles into degrees and minutes. However, due to the symmetry which already exists within the stereographic projection of these celestial circles we need only consider doing the divisions for a part of each of these circles, namely a quarter of the ecliptic and only that part of the horizon that falls between the equator and one tropic.

In order to accomplish these unequal divisions on the projected circles, we use the trigonometric properties of right spherical triangles. To divide the ecliptic, the **right**

[8]Gunter describes the construction of the ecliptic by dividing AC at G such that $AG : AD = 4343 : 10000$ (the labels are mine). The true ratio of $AG : AD = tan(23°30') \approx 0.4348$. Quadrants like Gunter's would have been relatively small, *i.e.*, approximately 6-8 inches; and given the error for constructing such curves with engraving tools, an error of five ten-thousandths would have been negligible. However, Gunter still does not provide the reader with the knowledge of how he calculated such a result. In the first book of *The description and use of the sector*, he describes how to divide a line into a given ratio. Still Gunter would have had to rely upon tables of trigonometric lines to approximate the value of $tan(23°30')$, if he had not used a sector (as he describes in chapter one) to determine the length of the tangent line in a circle with a radius of 10000 units corresponding to the angle $23°30'$.

[9]Gunter describes the construction of the horizon in a slightly different manner. For purposes of using the quadrant, the horizon is constructed so that it passes through the equinox. With this arrangement of curves, Gunter calculates the point of intersection which the horizon makes with the tropic. The point of intersection cuts the tropic at $\arcsin(tan(\varepsilon)/cot(\phi))$, *i.e.*, F in fig. 4.8 where ε is the obliquity of the ecliptic and ϕ is the latitude of the given locality for which we are constructing the horizon. Gunter then says: "[a]nd if you finde a point at [G_H], in the line AC, whereon setting the compasses [sic], you may bring the point at [D], and this point in the tropique [*i.e.*, F] both into the circle, the point [G_H] shall be the horizon". In essence, Gunter finds the centre of the projected circle and its radius to construct that part of the horizon between the tropic and the equator.



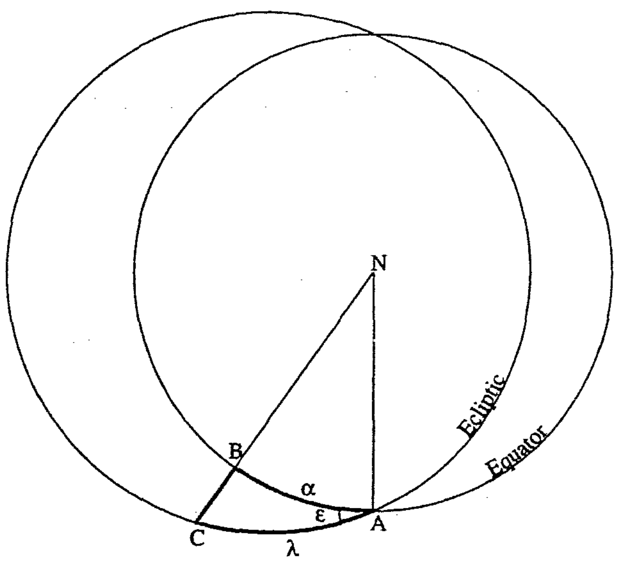


Figure 4.9: Dividing the ecliptic.

ascension for each degree of **longitude** on the ecliptic is found. The same holds true for dividing the horizon - a corresponding right ascension is found for a given **azimuth** on the horizon. In each case, the longitude or azimuth is measured from one of the two points of intersection with the equator.

Let us consider the right spherical triangle ABC in fig. 4.9(a) where A is the vernal equinox and AC is the given longitude/azimuth (λ) on the ecliptic/horizon we wish to divide. Given the obliquity (ε) of the ecliptic/horizon, the right ascension (α) is then uniquely determined according to the formula

$$cos(\varepsilon) = \frac{tan(\alpha)}{tan(\lambda)}, \tag{4.14}$$

which is equivalent to (1.7).

Tables of right ascensions for each integral value of longitude were quite common in most texts on quadrant constructions; a writer would either copy such a table from some other known work or construct a table using (4.14), but in the manner of computation for the early Renaissance.[10] Armed with such a table, the right ascensions for each of the signs in the ecliptic and any intermediate divisions, for example, could be used to easily divide the ecliptic as we shall demonstrate here.

Gunter provides the following table of right ascensions as shown in fig. 4.10.

With the exception of one entry, all entries are correct to the nearest minute.[11]

In fig. 4.9(a), the arc BC meets the celestial equator at right angles since BC extended passes through the pole of the equator. Since the image of any circle which passes through the pole of projection will be a straight line, the points C and B when projected will lie on a straight line where it crosses the ecliptic and the equator respectively as in fig. 4.9(b). Hence, one only needs to mark off on the equator a point corresponding to the right ascension associated with the given longitude on the ecliptic. Join the vertex and this point with a straight line (extending the straight line if necessary). The straight line meets the ecliptic in the plane at that point which corresponds to the given longitude in the ecliptic. The same holds true for dividing the horizon since it is also a great circle.

We will demonstrate how the ecliptic can be graduated into degrees and minutes. For example, let us mark $\lambda = 30^o$ - the beginning of Taurus - on the ecliptic. The right ascension for $\lambda = 30^o$ can be calculated most accurately using (4.6) where $\varepsilon = 23^o30'$ or using the calculated value from Gunter's table where he gives $\alpha = 27^o54'$. With the

[10]We find such examples of tables in Gunter's *The description and use of the sector*, Finé's *Quadrans Astrolabicus* and in the appendix of Foster's *The Art of Dialling*. The table in Foster's work is in fact a table of declinations for every 25′ and the corresponding longitude. The table could have been used to solve this problem, but would have required an auxiliary table relating the solar declination with the right ascension.

[11]For $\lambda = 85^o$ or 25^o in Gemini, $\alpha \approx 84^o33'1''$ which is $1'1''$ larger than Gunter's calculated entry.

A Table of Right Ascensions						
Gr.	[Aries]		[Taurus]		[Gemini]	
	Gr.	M.	Gr.	M.	Gr.	M.
0	0	0	27	54	57	48
5	4	35	32	42	63	3
10	9	11	37	35	68	21
15	13	48	42	31	73	43
20	18	27	47	33	79	7
25	23	9	52	38	84	32
30	27	54	57	48	90	0

Figure 4.10: A table of right ascensions from Gunter's *The description and use of the sector.*

vernal equinox at A, we construct $\angle ANB = \alpha$ and extend NB to meet the ecliptic at C. This procedure then gives $AC = \lambda = 30°$, and a continuation of this procedure divides the ecliptic correctly.

Save for circles which lie in a plane parallel to the plane of the equator, all other circles which are divided into equal intervals which are projected onto the plane reveal divisions which are unequally spaced. Dividing into degrees those projected circles which correspond to circles of declination result in divisions which are equally spaced. Discussions of how to mark off the divisions of an oblique circle go back to Ptolemy. (For some medieval writers on the astrolabe, the problem of dividing the ecliptic was either considered unimportant or was simply not well understood.[12]) Moreover, while Islamic writers provided a number of variations for dividing the ecliptic, early Renaissance writers in Europe appear to have used only the method we have explained for dividing the ecliptic on the quadrant.

[12]Thomson (ed.), *Jordanus de Nemore and the mathematics of astrolabes: De plana Sphera*, p. 62.

In the following sections (4.6 – 4.10) we explain only how to construct the various curves on the quadrant. The reader who wishes to know how to *use* these curves should consult (4.11).

4.6 On the Construction of the Hour Lines

We can now turn our attention to the construction of the hour lines on the quadrant. There are two types of hour lines: (1) unequal or seasonal hour lines and (2) equal hour lines. The seasonal hour lines define twelve equal intervals of time during which the sun lies above the local horizon (daylight) or below the horizon (night). Naturally, if an observer were in the northern hemisphere, then a seasonal hour would be longer the more northerly is the declination of the sun. Moreover, there are two days in the year during which the seasonal hours of day and night are equal, namely the spring and fall equinoxes. Otherwise, the seasonal hours for a given day of the year will differ in length from day to night depending on the declination of the sun and the latitude of the observer.

4.7 Constructing Unequal or Seasonal Hour Lines

The seasonal hour lines are represented by a set of six circular arcs drawn within a quarter circle which represents the time for half of the diurnal arc of the sun in its transit across the sky. These are not shown in fig. 4.1; however, some Gunter quadrants did possess them either within the shadow square on the front of the quadrant or on the dorsal side. To construct the seasonal hour lines is relatively easy to perform. Let AB be a quarter arc of some circle with radius CB as in fig. 4.11. Divide AB into six equal parts at M, N, O, P and Q. Now construct circular arcs through C and each of the points M, N, O, P and Q one at a time, such that the centres of these circular arcs lie on CB or CB extended. As Lorch demonstrates, [13] this procedure for constructing the hour lines provides a graphical means for determining solutions

[13] Lorch, "A note on the horary quadrant," p. 117.

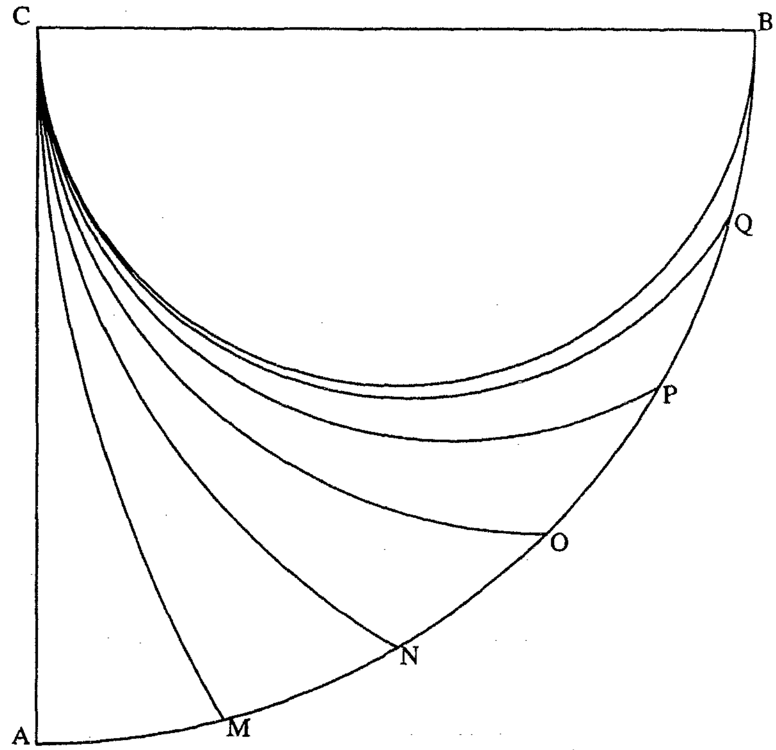


Figure 4.11: The constructed lines of unequal hours.

to the formula

$$sin(h) = cos(t) \cdot sin(H) \tag{4.15}$$

where t corresponds to the seasonal hour before or after the sun's meridional transit at noon and h is the solar altitude at hour t on that day for which the sun's altitude at noon is H as is shown by a graphical demonstration based on fig. 4.12.[14]

We can show that fig. 4.12 provides a graphical representation for (4.15). The hour line, OF, is the arc of a circle with diameter OY. The semi-circle on OB represents the hour line for noon, and the point M' is the intersection of the plumbline with this hour circle. (In practice, a bead would be positioned on the plumb line where it meets this hour line for noon.)

We know that $OM = OM'$ since M and M' represent the fixed position of the sun in the ecliptic on a given day when it falls on the hour line OF and noon, respectively. $OF = OB$ by construction since they are radii of the same circle with

[14] *Ibid.*, p. 116. This modified figure and proof as outlined here appear in Lorch.

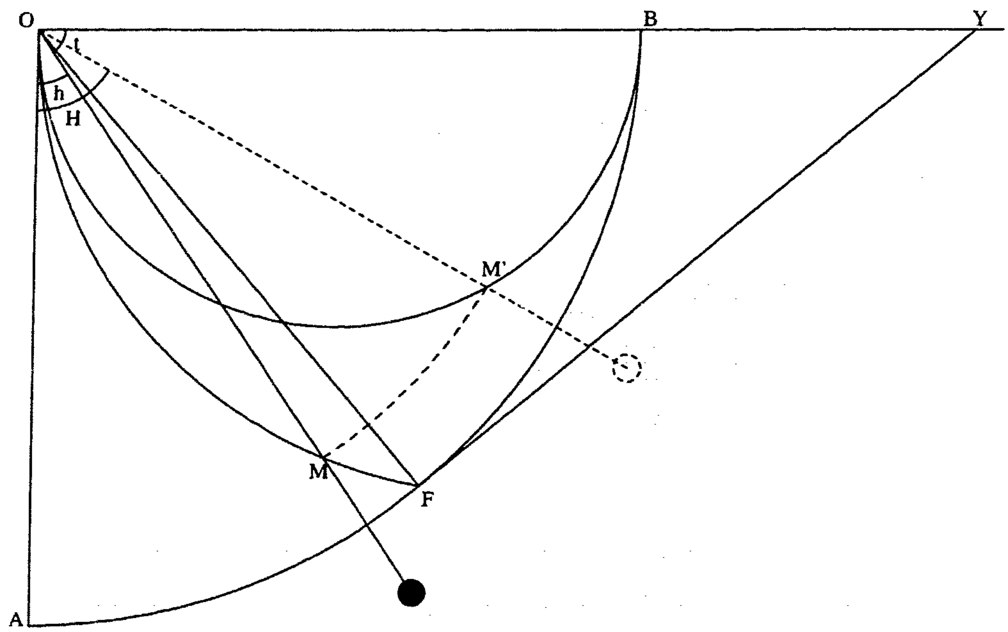


Figure 4.12: Determining the unequal hour for a given sighting.

centre O. But $\triangle OM'B$ is a right triangle since $\angle OM'B$ subtends the diameter of a semi-circle on OB. Both $\angle AOM'$ and $\angle OBM'$ are complimentary angles of $\angle M'OB$. Hence, $\angle AOM' = \angle H = \angle OBM$. Therefore,

$$\frac{OM}{OF} = \frac{OM'}{OB} = sin(H) \tag{4.16}$$

since OB is the hypotenuse and OM' is the side opposite $\angle OBM'$.

Moreover, $\angle OMY$ and $\angle OFY$ are right angles since they subtend the diameter OY of the arc OMF. Now $\frac{OM}{OY} = sin(\angle OYM)$ and $\frac{OF}{OY} = sin(\angle OYF)$ Hence,

$$\frac{\frac{OM}{OY}}{\frac{OF}{OY}} = \frac{OM}{OF} = \frac{sin(\angle OYM)}{sin(\angle OYF)}, \tag{4.17}$$

but $\angle OYM$ and $\angle AOM$ are complementary angles of $\angle MOY$. Hence, $\angle OYM = \angle AOM = \angle h$, and $\angle OYF$ is complementary to $\angle YOF = \angle t$. Therefore, $sin(\angle OYM) = sin(h)$ and $sin(\angle OYF) = cos(t)$. Thus,

$$\frac{OM}{OF} = \frac{sin(h)}{cos(t)}, \tag{4.18}$$

and (4.16) and (4.18) together imply

$$\frac{sin(h)}{cos(t)} = sin(H).^{15} \tag{4.19}$$

The construction of these curves is not the result of a stereographic projection of azimuth lines on the celestial sphere. Such celestial circles would meet at two points when they were projected onto the plane of the equator; however, their very construction indicates that they are not azimuth lines since they meet at only one point, namely O.

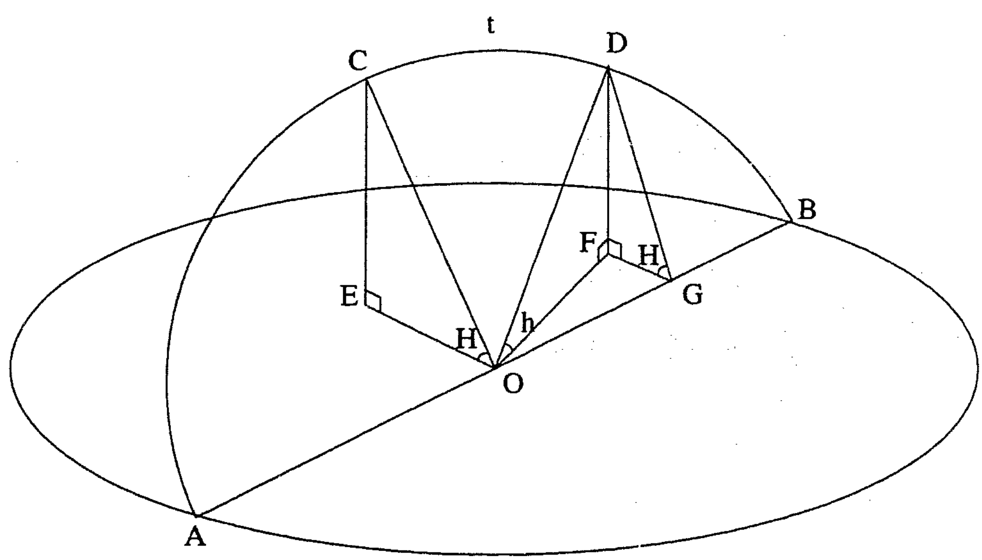


Figure 4.13: Finding the sun's altitude in the sky using equal hours.

This provides a graphical means for calculating H or h in terms of t (measured in degrees, *i.e.*, 15° for each hour) when the other is known, but why does (4.15) provide us with the means for calculating the seasonal hours? Let fig. 4.13 represent the diurnal arc of the of the sun. We assume that the dirunal arc is a semi-circle; however, this would only be true if the sun were at the celestial equator. Let C be the culmination point of the sun at noon, so $\angle COE = \angle H$ represents the sun's altitude. Let $\angle FOD = \angle h$ be the sun's altitude at some other time. Now $\triangle CEO \sim \triangle DFG$. Hence,

$$\frac{CE}{CO} = \frac{DF}{DG}. \tag{4.20}$$

But

$$\frac{CE}{CO} = sin(H) \tag{4.21}$$

and

$$\frac{DF}{DO} = sin(h). \tag{4.22}$$

But $CO = DO$ since they are radii of the semi-circle $ACDB$. Therefore,

$$\frac{CE}{DF} = \frac{sin(H)}{sin(h)}. \tag{4.23}$$

Now the arc CD represents the arc in which the sun travels $\frac{t}{15}$ hours from the prime meridian (*i.e.*, the plane through C, E and O). Hence, $\angle COD = t$.

Given that $\angle COD = \angle ODG$, $CO = DO$ and $\frac{DG}{DO} = cos(\angle ODG)$, we have

$$\frac{DG}{CO} = cos(t), \tag{4.24}$$

and (4.20), (4.23) and (4.24) imply that

$$\frac{sin(H)}{sin(h)} = \frac{1}{cos(t)} \tag{4.25}$$

or simply (4.15).

4.8 Constructing Equal Hour Lines

We may now turn to the construction of the equal hour lines. In this case, an hour corresponds to one twenty-fourth part, or 15°, of an hour circle or circle of declination along which the sun appears to travel in a given day.[16] To construct the equal hour lines, writers like Gunter describe a method whereby the altitude of the sun is determined for a given hour of three different days, which gives three points, one on each of three different circles of declination, on which the sun lies at a given hour on three different days. These three points are calculated to construct a given hour line when the sun has either a strictly positive or negative declination for all three points. This

[16] An alternate definition would be to say that an hour corresponds to one twelfth part of the time between midnight and midday.

allows us to construct two sets of hour lines: one for the winter hours and the other for summer hours. These three points would be plotted, and then a circle would be drawn through these three points. To plot the points requires only the intersection of the stereographically projected circle of declination with a straight line from the vertex of the quadrant to the point on the graduated limbus equal to the altitude.

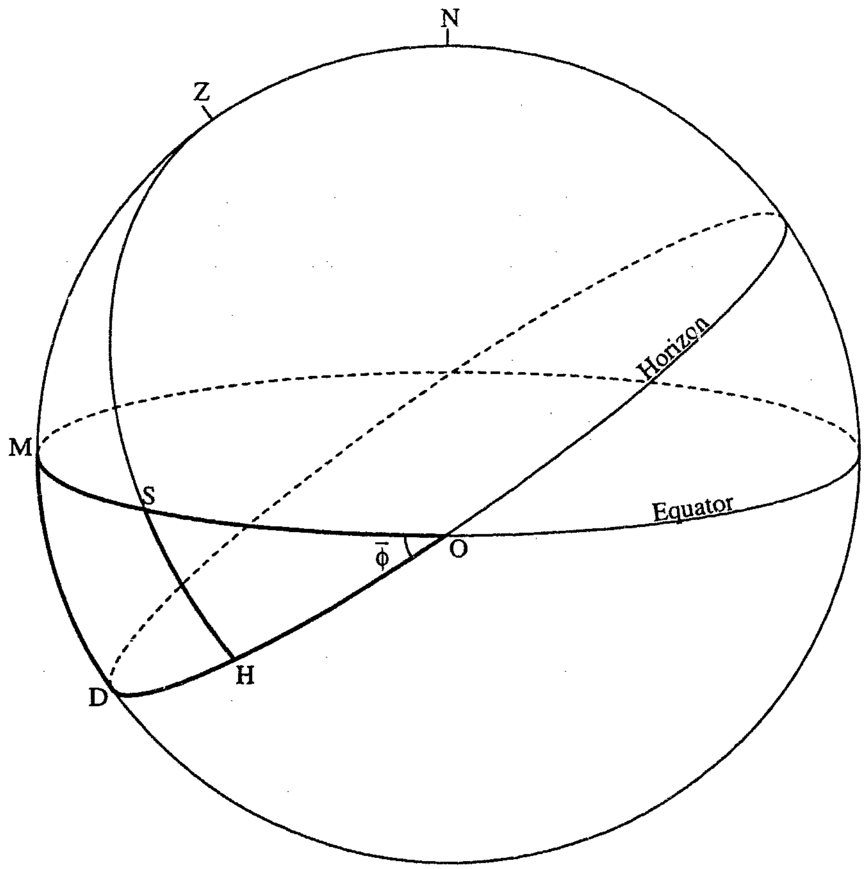


Figure 4.14: Determination of the solar altitude when the sun lies in the equator.

4.8.1 Finding the Sun's Altitude When the Sun Travels Along the Equinoctial

When the sun has no declination, it lies in one of the two equinoxes on the ecliptic. These correspond to the first days of spring and fall. Moreover, the day-circle or the circle of declination in which the sun appears to travel during these two particular days is the equator. Consider fig. 4.14 where the celestial sphere is given with the sun (S) shown above the horizon in its position at one of the equinoxes. The figure is labelled with the equator, the north pole (N), the local horizon, its zenith point (Z), and ZM is equal to the latitude (ϕ) of the locality. S is the sun at some given time of the day, and SH represents the altitude of the sun as measured in a plane vertical to the plane of the horizon through the local zenith and the sun. Since we are concerned with the equinoxes here, the diurnal motion of the sun describes half a great circle which cuts the horizon, which is also a great circle, at two diametrically opposite points, one of which is O. MO is $90°$, and MD represents the meridional solar height at noon.

With the problem described geometrically as above, we can determine the altitude of the sun for any hour when it lies in either of the two equinoxes by considering the two spherical triangles, $\triangle MOD$ and $\triangle SOH$ which share the angle $\angle MOD = \overline{\phi}$. By (1.4), we know that

$$sin(\overline{\phi}) = \frac{sin(MD)}{sin(MO)} = \frac{sin(SH)}{sin(SO)} \tag{4.26}$$

and $sin(MO) = 1$ since MO is a quarter arc. ZM is the complement of MD, and hence $sin(MD) = sin(\overline{\phi}) = cos(\phi) = cos(ZM)$. Likewise SO is the complement of MS; thus, $sin(SO) = cos(MS)$. Hence, (4.26) implies that

$$cos(MZ) = \frac{sin(SH)}{cos(MS)} \tag{4.27}$$

or

$$cos(\phi) = \frac{sin(altitude)}{cos(hour)} \tag{4.28}$$

which gives the relation of the sun's altitude with its corresponding hour angle measured from the prime meridian when the sun lies in the equator.

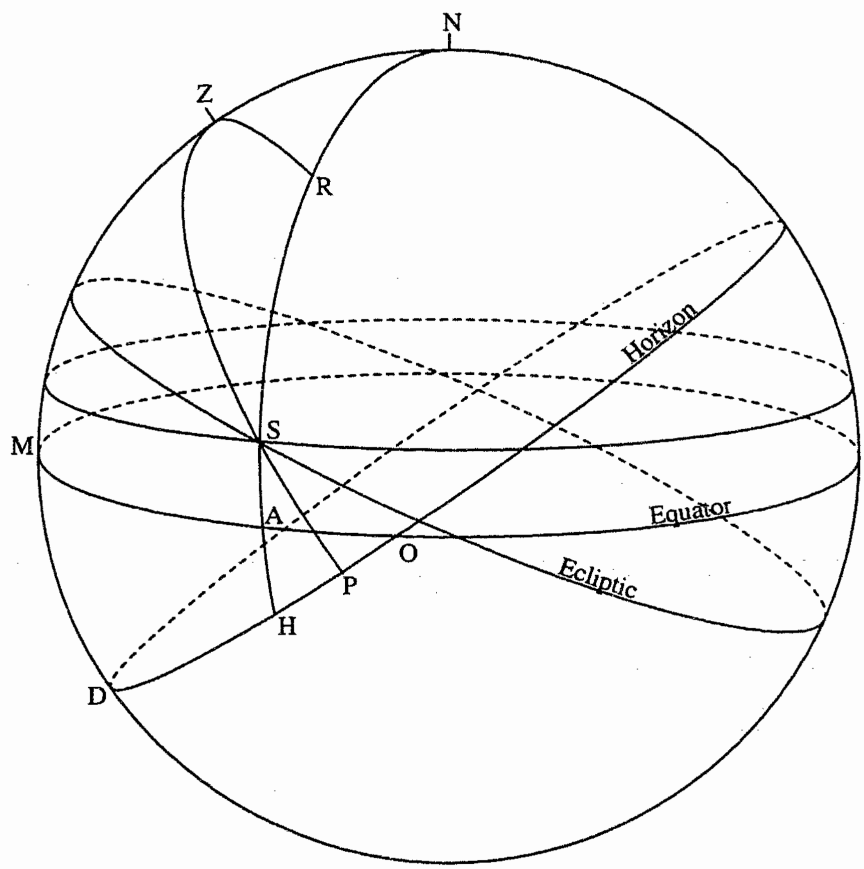


Figure 4.15: Determining the hour lines when the sun has a non-zero declination.

4.8.2 Finding the Sun's Altitude When the Sun Travels Along a Day-Circle Different from the Equator

We may now consider the case when the sun lies upon some day circle other than the celestial equator. Here, as in the case when the sun had no declination, the sun's altitude is determined for the particular hour of interest. To explain the construction of the equal hour lines when the sun has a non-zero declination, we shall refer to fig. 4.15.

If the hour is six in the morning or six at night, we may easily calculate the altitude of the sun since the sun will lie 90° to the east or west of the meridional transit point of the sun in the diurnal arc. Hence, the $\angle MNA$ is a right angle and $\triangle ZNS$ is a right spherical triangle. Hence, by (1.7),

$$cos(SZ) = cos(ZN) \cdot cos(SN), \tag{4.29}$$

where SZ is the complement of the sun's altitude PS, SN is the complement of the sun's declination AS and ZN is the complement of the local latitude MZ. Hence, (4.29) becomes

$$sin(PS) = sin(MZ) \cdot sin(AS), \tag{4.30}$$

i.e., $sin(h) = sin(\phi) \cdot sin(\delta)$, where δ is the declination of the sun, ϕ is the latitude of the observer and h is the latitude of the sun. At noon, the sun's altitude is given by $\overline{\phi} \pm \delta$. The declination ($\delta$) will be added to the complement of the latitude ($\overline{\phi}$) in the case when the sun has a northerly declination; otherwise, it is subtracted. (Medieval texts do not show negative declinations.)

To determine the sun's altitude (SP) for any other hour, we first must calculate the magnitude of arc AH. $\triangle HAO$ is a right spherical triangle and $\angle AOH$ is equal to the complement of the local latitude. Furthermore, arc AO is the complement of the hour away from noon or $90° - MA$. Thus, by (1.6), we have

$$tan(\overline{\phi}) = \frac{tan(AH)}{sin(AO)} \tag{4.31}$$

or equivalently

$$tan(AH) = tan(\overline{\phi}) \cdot cos(AM). \tag{4.32}$$

Given that ϕ and the number of hours until noon (arc AM) are known, the arc AH is then uniquely determined.

With the arc AH calculated, we will mark off on arc NSH a point R such that arc $NR =$ arc AH. Hence, arc AR is the complement of both the arcs HA and NR.

It can now be shown that the great circle containing ZR makes a right angle with SN, since (1.4) implies

$$\frac{tan(MD)}{sin(OM)} = \frac{tan(AH)}{sin(OA)} \tag{4.33}$$

or equivalently

$$cos(AM) = \frac{tan(RN)}{tan(ZN)}. \tag{4.34}$$

Hence,

$$cos(\angle MNA) = \frac{tan(RN)}{tan(ZN)} \tag{4.35}$$

since $cos(AM) = cos(\angle MNA)$. And (4.35) implies that $\angle ZRN$ is a right angle by (1.5).

Thus, $\triangle ZRS$ and $\triangle ZRN$ are both right spherical triangles with a common side ZR. Therefore, by (1.7), we obtain both

$$cos(ZR) = \frac{cos(SZ)}{cos(SR)} \tag{4.36}$$

and

$$cos(ZR) = \frac{cos(ZN)}{cos(NR)}. \tag{4.37}$$

Together, equations (4.36) and (4.37) imply

$$\frac{cos(ZN)}{cos(NR)} = \frac{cos(SZ)}{cos(SR)}, \tag{4.38}$$

by Euclid's *Elements* V, 11, or by considering the complements of these arcs, we may write

$$\frac{sin(MZ)}{sin(AR)} = \frac{sin(PS)}{sin(HS)}. \tag{4.39}$$

The arcs AR, MZ, and HS are all known since they are respectively the complements of the calculated value for AH, the latitude for the given locality, and the calculated value of the sun's declination AS added or subtracted to the previously calculated value AH. Hence, PS is uniquely determined and yields a calculated value for the sun's altitude for a given locality for any hour required. The formula given in (4.39) and the calculation for AH from (4.32) provide a two step calculation for this problem. The construction of the point P on the arc NSH is not so obvious; one might opt for an additional calculation to determine the altitude, arc PS. Instead of (4.39), we can determine $\angle AHO$ by a formula which provides a relationship of the sides and the angles in the right spherical triangle $\triangle HAO$. With this angle known, we may then determine the altitude of the sun (arc SP) by using the right spherical triangle $\triangle HSP$ and the sine law for spherical triangles where

$$\frac{sin(\angle HPS)}{sin(HS)} = \frac{sin(\angle SHP)}{sin(SP)}, \tag{4.40}$$

and then since $\angle HPS$ is a right angle,

$$sin(SP) = sin(\angle SHP) \cdot sin(HS). \tag{4.41}$$

4.8.3 Determining the Time of Sunrise and Sunset

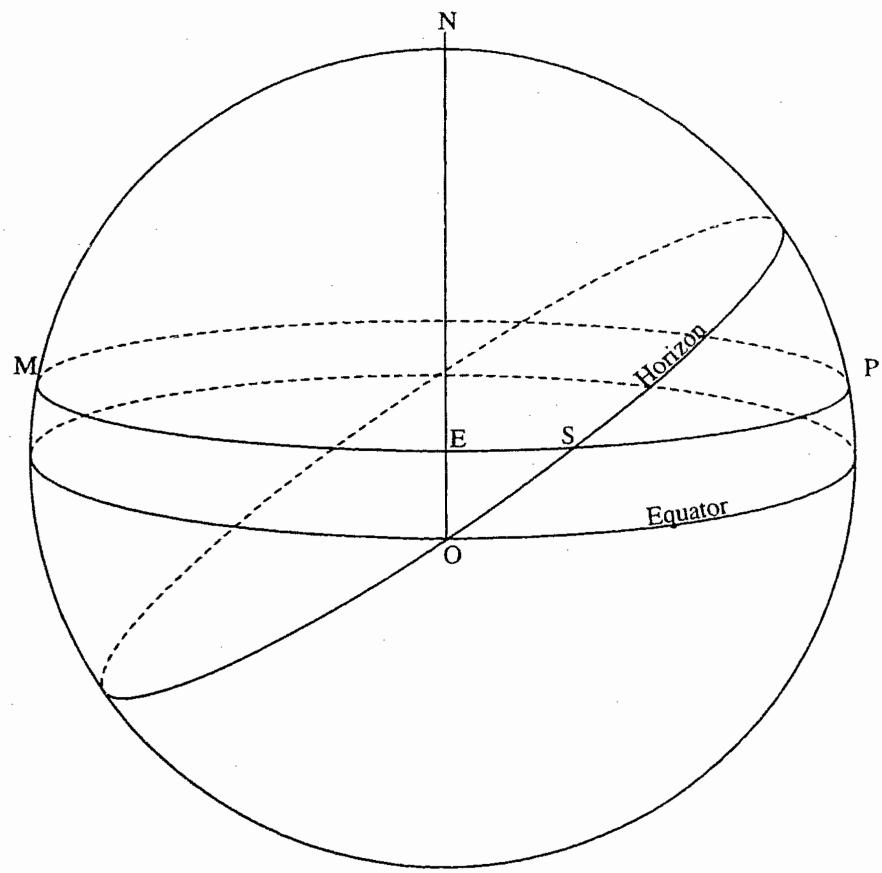


Figure 4.16: Determing the time of sunset and sunrise.

We may also determine the hour when the sun rises and sets for any given declination which the sun may have, though this is not pertinent to construct any of the lines on the quadrant. With the hour lines and the horizon constructed, one may simply read off the time when sun sets or rises when its position in the ecliptic is known by either its declination, right ascension or longitude. Nonetheless, one may ascertain a more precise time. With reference to fig. 4.16, the sun (S) lies in the hour circle MSN where it has reached the horizon. The point E represents the position of the sun at six in the morning or at night; hence, arc ES is the hour away from the hour of six. EO is the sun's declination in the right spherical triangle $\triangle OES$. And $\angle ESO$ is the complement of the latitude of the given locality; thus, by (1.6)

$$tan(\overline{\phi}) = \frac{tan(EO)}{sin(ES)} \tag{4.42}$$

or equivalently

$$tan(\overline{\phi}) = \frac{tan(\delta)}{sin(hour)}. \tag{4.43}$$

4.9 On the Construction of the Lines of Azimuth

To complete our survey of the various curves constructed on a quadrant, we will examine the lines of azimuth which are closely related to the equal hour lines. An azimuth circle is a great circle of the sphere which goes through the zenith of a given locality and the azimuth line is the stereographic projection of that circle onto the quadrant. The azimuth circle cuts the horizon at some point whose angular measurement from the intersection of the local meridian and the horizon is called the azimuth of the celestial body. In our case, the celestial body is the sun. The appearance of the azimuth lines is a result of a stereographic projection, and there are a few methods which one might use to construct them. One geometrical method requires projecting stereographically the zenith point of the given locality and two diametrically opposite points on the intersection of the horizon with the azimuth circle. Once these are known, a circle may be constructed through these three points, or more correctly the arc of this circle which lies between the tropics of Cancer and Capricorn. This method was certainly known in medieval Islam in addition to other graphical methods;[17] however, it appears that in the texts which we consider here,

[17] *Cf., e.g.*, J.L. Berggren, *Medieval Islamic Methods for Drawing Azimuth Circles on the Astrolabe* for a survey of graphical methods for constructing the azimuth circles on the astrolabe. These methods used on the astrolabe hold equally as well for the quadrant though the nature of the construction of the azimuth lines on quadrants in medieval Islam should be considered in some other study.

numerical methods were used to construct the lines of azimuth. For example, for a particular azimuth circle and three different declinations of the sun when it lies either northerly or southerly of the equator for all three declinations, the sun's position on that azimuth circle might be projected stereographically onto the plane of a quadrant. Through these points an arc of a circle could then be drawn between one of the tropics and the equator. (Typically the points where the azimuth circle crosses the two tropics and the equator would be taken as the three points through which one would construct the circle to represent the azimuth circle, but in the case of Gunter's quadrant, we need two other points to construct the entire azimuth circle between both tropics and the celestial equator: one when the sun's declination is positive and the other negative.)

When the sun is at either of the equinoxes and hence has no declination, *i.e.* the sun is on the equator, we may easily determine its azimuth and use fig. 4.14 to demonstrate the spherical geometry and trigonometry needed to perform this calculation. In this figure of the celestial sphere, N is the north pole, Z is the zenith of the given locality and S represents the sun. Arc SH is the sun's altitude and its corresponding azimuth is the arc HO on the horizon. The angle at O or $\angle SOH$ is the complement of the local latitude. Hence, in the right spherical triangle $\triangle SOH$,

$$tan(\overline{\phi}) = \frac{tan(SH)}{sin(HO)} \tag{4.44}$$

by (1.6) or equivalently

$$sin(HO) = tan(\phi) \cdot tan(SH). \tag{4.45}$$

To determine the azimuth of the sun when it has some declination requires first knowing the sun's altitude at the equator which we have just discussed. So now let us assume the sun has a non-zero declination, say arc SA in fig. 4.17, and let us examine the right spherical triangles $\triangle ZMB$ and $\triangle SAB$. By (1.4), we know that

$$sin(\angle ZBM) = \frac{sin(MZ)}{sin(ZB)} \tag{4.46}$$

and

$$sin(\angle SBA) = \frac{sin(AS)}{sin(SB)}. \tag{4.47}$$

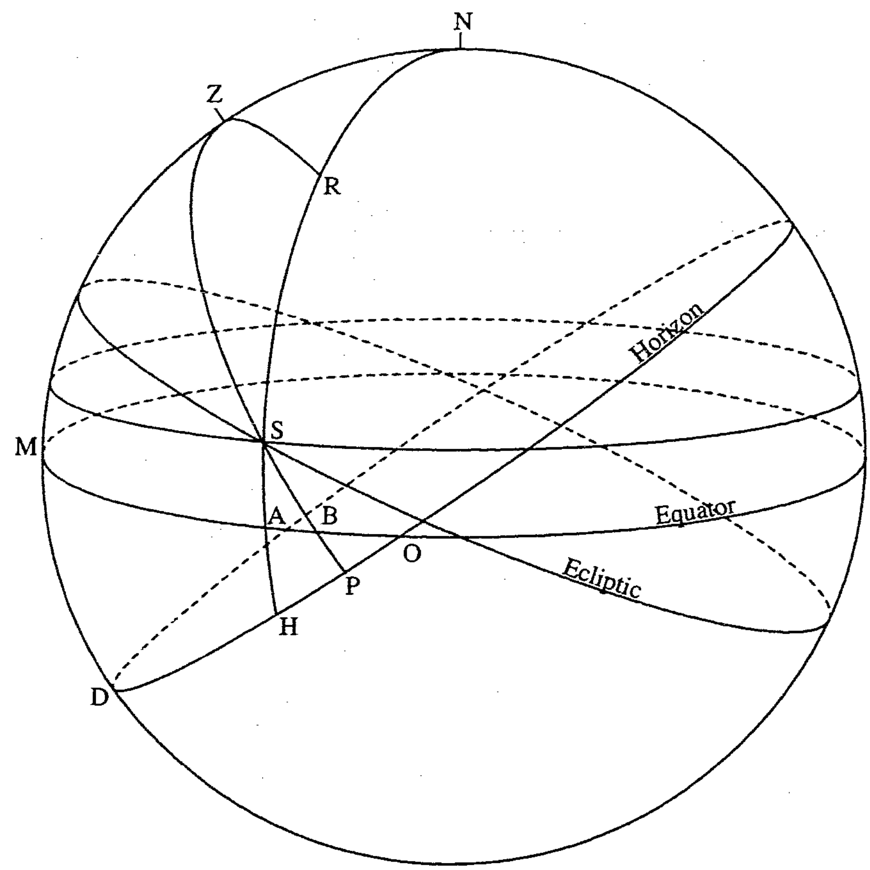


Figure 4.17: Determining the azimuth of the sun from a given solar altitude when the sun has some declination.

But arc ZB is the complement of arc BP, the altitude of the sun were it on the same azimuth but at the equator; thus, $sin(ZB) = cos(BP)$. Since both triangles share a common angle at B, $\angle ZBM = \angle SBA$ and hence $sin(\angle ZBM) = sin(\angle SBA)$. It then follows that

$$\frac{sin(MZ)}{sin(AS)} = \frac{cos(BP)}{sin(SB)}. \tag{4.48}$$

Now that we have the calculated the values of the sun's altitude when it is on the celestial equator (BP) and the arc of the altitude circle from the equator to the sun (BS) when it is not, we must now consider three cases to determine the sun's altitude given its azimuth and its declination (AS). If the latitude of the given locality and the sun's declination are both northerly or southerly , then, if the azimuth lies between 0° and 90° of the prime meridian as measured from the southernmost point of the horizon, the sun's altitude is determined by adding BS to the altitude of the sun's projection onto the celestial equator. If the azimuth is greater than 90° and the sun lies above the horizon, then the sun's altitude is equal to the depression of the sun as if it were on the same azimuth circle at the equator subtracted from the sun's declination (δ). If, however, one of ϕ or δ is northerly and the other southerly, then the sun's altitude when it lies above the horizon is calculated also by taking the depression of the sun as it lies in the same azimuth circle and subtracting it from the sun's declination.

Given the sun's altitude for any prescribed declination and any azimuth, we can construct the corresponding lines of azimuth on the quadrant. Texts which we have examined, describe the construction of the azimuth lines numerically instead of geometrically. The sun's altitude was calculated from three different solar declinations for a given azimuth with all declinations chosen to be positive or negative, and these points would be plotted on the quadrant. An arc of a circle would then be constructed through these points between one of the tropics and the equator.

4.10 On the Cursor

Lastly, we will describe the construction of the cursor which appears on many astronomer's quadrants. In some cases it appears as a fixed engraving on the quadrant, but in some late medieval examples, it is movable and can slide about an arc so as to function for any latitude. The texts that we have examined on quadrant constructions, however, possess only instructions for constructing and using a fixed cursor. In one way this is quite sensible since the other sets of lines, *i.e.* the azimuth lines and the equal hour lines, are constructed for only one given latitude; however, a practitioner who possessed tables for such things could still use a quadrant with a movable cursor. Without such tables, the usefulness of such a quadrant for telling time would not be so great.

To construct the cursor for a given latitude (ϕ), we need to know the declination of the sun for every day of the year; however, typically, only every fifth or tenth day of the month is used in the construction of the cursor. During the 16^{th} and 17^{th} centuries, tables would be calculated for or sometimes by the instrument makers who constructed quadrants. In the few works which explicitly give tables for the sun's declination or longitude,[18] none explain how the tables were derived. They could have been obtained from earlier texts or through observation or through calculation using the mean solar motion to name just a few possibilities. In the end, the construction of the cursor will have a error so insensibly small for such a small quadrant that we need not be too concerned here. Only larger quadrants on the order of several feet in radius will require more careful attention in the construction of the tables for the construction of the cursor.

With a table of solar declinations in hand, we now need to calculate the solar meridional altitude, *i.e.* the sun's altitude at noon for every day of the year. This is easy to compute since the sun's altitude for a given day is equal to $\overline{\phi} \pm \delta_n$ where δ_n is the declination for day n, and the sun's declination is added in the case when the sun's declination and the latitude of the given locality lie in the same hemisphere as is defined by the equator; otherwise, the sun's declination is subtracted.[19]

The cursor is usually placed between the tropic and the graduated limbus of the quadrant. To inscribe the days and the months of the year in the quadrant, a straight edge would be placed through the vertex of the quadrant and through the graduated division on the outer limbus corresponding to the solar meridional altitude for that day. At that point a line would be drawn. One must remember that for every possible declination, there will be two positions of the sun with that declination; or at least there will be two days whose solar declinations will be quite close to one another. If one were to draw these lines they would appear so close together as to be indistinguishable especially on a hand-held vertical quadrant. To avoid this problem, half of the space

[18] *Cf.*, *e.g.* Edmund Gunter's *The description and use of the sector, the crosse-staff and other instruments*, and William Leybourne's transcriptions of Gunter's works on quadrants.

[19] The construction of the cursor in my representation of the quadrant, however, was constructed using the equations of time. The greater accuracy attained by using these equations on the other hand would not necessarily produce better results when one actually used the quadrant in practice because of its size.

reserved for the cursor would be used for marking the solar meridional altitudes when the sun lies between the summer and winter solstice; the other half would be reserved for marking the solar meridional altitudes when the sun traverses the remaining half of the ecliptic on its journey from the winter solstice back to the summer solstice. Normally, every fifth day of the month would be marked off on the quadrant, and the first day of each month would also be noted.

We have thus described the construction of the major curves which were found on a quadrant from the early Renaissance until about the mid-17^{th} century. Quadrants were still constructed after this time, but slowly fell out of fashion perhaps for want of more accurately computed astronomical parameters. Practitioners also worked on other nomographic representations of some of the various functions considered here and others as well. Gunter's quadrant represents a truly advanced form of the more traditional quadrants, and we shall stop here with our description of the methods for constructing the various curves as we have done in this chapter.

4.11 Employing the Quadrant and Its Curves in Astronomical Matters

We have now constructed those curves necessary to perform observations and calculations; however, as in many texts and manuscripts which we have studied in this work, we need to complete this study with an examination of how the quadrant is used.

To use the quadrant, two sights need to be added on the side of the quadrant which meets the limbus at the 90^o mark - one sight near the vertex and the other nearer the limbus. Lastly, a plumb line needs to be attached to the quadrant at its center. A small sliding bead must also be placed upon this plumb line.

4.11.1 Finding the Solar Altitude

One obvious use of the quadrant is to take the altitude of the sun, moon, or stars. To take the altitude of the sun on the $22nd$ day of November, for example, a sighting of the sun is taken, "shooting the sun" through both sights, and the plumb line cuts the limbus at that angular measurement which corresponds to the altitude of the sun at the time of the observation. Hence, on the 22*nd* day of November at noon, the plumb line crosses the limbus at $20^o47'$ for an observer at the latitude of 49^o, *e.g.*, Vancouver. Likewise, if the time is 3:00 *p.m.*, the solar altitude for that same day and observational latitude is $10^o3'$.

4.11.2 On the Sun's Position in the Ecliptic

If the sun's position is known in the ecliptic, then we may easily determine either the declination or the right ascension of the sun. To determine the right ascension, we lay the plumb line over the point in the ecliptic, and where the plumb line crosses the limbus, the right ascension is then determined; however, if the sun, as it lies on the ecliptic, has a longitude of more than 90^o, then the right ascension will be more than 90^o. For example, Gunter says

> As if the place of the Sunne given be the fourth degree of [Gemini], the thread laid on this degree shall cut 62 degrees in ther Quadrant, which is the right ascension required. But if the place of the Sunne be more then 90 gr. from the beginning of [Aries], there must be more then 90 gr. allowed to the right ascension; for this instrument is but a quadrant [*i.e.*, a quarter circle]: and so if the Sunne be in 26 gr. of [Cancer], you shall finde the thread to fall in the same place, and yet the right ascension to be 118 gr.[20]

Just as we can determine the sun's right ascension by knowing its position in the ecliptic, so, reversing the procedure, we can calculate the sun's position in the ecliptic from its right ascension. Moreover, to determine the sun's declination, the plumb line is placed over the known longitude or right ascension. The bead on the plumb line is placed over the point where the plumb line crosses the ecliptic. Then the plumb line is placed over the line of declination (without moving the bead) to provide the sun's declination. And vice versa, we may determine the longitude or the right ascension of

[20]Gunter, *On the sector*, p. 202.

the sun from a given solar declination. Care, however, must be taken to note whether the sun lies in the northern or southern celestial hemisphere, *i.e.*, whether the sun's declination is positive or negative.

To demonstrate, we cite the following by Gunter who says

> As if the place of the Sunne giuen be the fourth degree of [Gemini], the beade first set to this place, and then moued to the line of declination, shall there shew the declination of the Sunne at that time to be 21 gr. from the equator.[21]

Likewise he goes on to say

> As if the declination be 21 gr. the beade first set to this declination, and then moued to the ecliptique, shall there shew the fourth of [Gemini], the fourth of [Sagittarius], the 26 of [Cancer], and the 26 of [Capricorn] and which of these foure is the place of the Sunne, may appeare by the quarter of the yeare.[22]

4.11.3 On Using the Cursor

The cursor may be used to determine the solar meridional altitude. If we lay the plumb line across the day of the month in which we are interested, the sun's meridional altitude is provided by where the plumb line crosses the limbus. Moreover, the sun's place in the ecliptic may be determined when either the day of the month is known or the solar meridional altitude is known. In the subsection, *Finding the Solar Altitude*, the sun's altitude at noon could have been determined without doing an observation.

4.11.4 On Using the Seasonal Hour Lines

To use the seasonal hour lines, we need to take a sighting at noon or determine the sun's altitude at noon by some other means to position the bead over the arc corresponding to the hour of twelve. Once the bead is rectified, if a sighting is taken

[21] *Ibid.*, p. 202.
[22] *Ibid.*, p. 203.

at some other time of the day, the position of the bead indicates the time of this observation.

4.11.5 On Using the Equal Hour Lines

Those lines which are labelled $6, 7, 8, 9, 10, 11$, and 12 at the equator and those marked $1, 2, 3, 4, 5$, and 6 at the tropic represent the hour circles. The hour line which begins at 12 on the equator and terminates at 0 at the tropic near the middle of June on the cursor, represents the hour circle for noon when the sun lies in the northern celestial hemisphere. Those lines drawn from 11 to 1, 10 to 2, 9 to 3, *etc.*, are the hour circles for $11a.m./1p.m.$, $10a.m./2p.m.$, *etc.*, when the sun lies in the northern celestial hemisphere. The other collection of arcs emanating from $6, 7, 8, 9, 10, 11$, and 12 represents the same hour circles for the case when the sun lies in the southern celestial hemisphere.

If the day of the month or the solar meridional altitude is known, the position of the sun in the ecliptic can be found. We begin by laying the plumb line across the day of the month on the cursor (or across the limbus for the meridional height). Where the plumb line crosses the hour circle for noon, set the bead at that point of intersection. The bead has now been rectified (to use Gunter's term) for the given day of the month in which we are interested. (Now that the bead has been rectified, we can determine the sun's position in the ecliptic. Simply move the plumb line until the bead falls on the ecliptic. And vice versa, if the point in the ecliptic is known, we can determine the day of the month.)

At any other time of the day for which the bead has been rectified, the time may be determined by sighting the sun. The point where the bead lies indicates the time. It may lie between two consecutive hour circles, and, thus the hour may need to be interpolated. This also provides the observer with the altitude of the sun at that time. Likewise, if we know the solar altitude, the hour can be determined.

Moreover, we can determine when the sun meets the horizon in the morning or evening. Once the bead on the plumb line has been rectified, move the plumb line until the bead falls on the horizon. The hour can then be determined as it is described above. This also allows us to find the rising and setting sun's distance from the points due east or west on the horizon, and where the plumb line crosses the limbus gives the ascensional difference (useful for computing the length of daylight on a given day).

4.11.6 Using the Azimuth Circles

The remaining arcs which lie between the equator and the tropic are the azimuth circles. The left-most azimuth represents the meridian. That which is marked 10 is the azimuth that is 10^o from the meridian.

Again, we must rectify the bead on the plumb line for that day of the month in which we are interested. If we then move the plumb line so that the bead falls upon some azimuth, then the point where the plumb line crosses the limbus provides us with the zenith distance of the sun. And vice versa, if the zenith distance is known, then the azimuth can be determined. Moreover, we can determine the sun's azimuth for a given time since we can determine the solar altitude at that time; the zenith distance is just the complement of the solar altitude.

We shall not describe the uses of the shadow square at this point, since we have already decribed them earlier in this chapter.

This completes our examination of quadrant constructions and applications.

Chapter 5

Conclusions

Scholarship in the history and philosophy of science and technology which examines developments in scientific instruments and their relationship to science is a relatively recent development within the world of academia. The history of scientific instruments like the quadrant is slowly becoming a field of great interest by researchers who are interested in how the instruments were constructed, who constructed them, how the instruments affected developments in science and vice versa, and texts which described their use and construction. We have considered many of these areas in this study; however, we have barely begun to see the entire picture of the quadrant's developments in the Latin West during the early Renaissance.

We have drawn upon many works by other scholars and available texts and manuscripts to put together this study. The primary texts have helped us to understand the construction and some applications of the quadrant; however, we have had to rely upon a lot of secondary material to gain a glimpse into approximately 800 years of history in astronomy, mathematics, navigation, surveying and scientific instruments. Undertaking such a study is quite daunting, and it is clear that historians of science and technology must work together with each other to uncover a fairly accurate interpretation of instruments like the quadrant and their relationship to science.

While we have relied upon the work of other scholars, this work is not a mere derivative. Based upon our knowledge of today's available literature, scholars have not previously attempted to view the developments of the quadrant within the same parameters which we have set for this thesis. That is, there has been no visible attempt to construct the interlocking pieces of several different puzzles which examine the influence from various sciences, other areas of technology and the educational atmosphere within each of these disciplines. Still, many pieces of the puzzle are missing, for this is hardly a definitive work. However, we believe it presents a plausible framework within which we and other scholars can continue to look at the developments of the quadrant in this period of history and its role within scientific developments and the ideas of scientists.

Chapter 6

Glossary

azimuth The angular measurement of an arc along an observer's celestial horizon measured from the southern most point on the horizon to another point on the horizon where a vertical circle at a right angle to the horizon passes through some celestial body. It can also refer to the angle measured at one's zenith contained by the prime meridian and another celestial circle which passes through the zenith and some celestial body.

azimuth circles/lines These are great circles on the celestial sphere which pass through an observer's zenith point. Since they are great circles through the pole of the horizon they must meet the horizon at right angles and pass through the nadir (the point on the celestial sphere directly opposite the observer's zenith point). We will refer to those lines which represent projections of these circles on the quadrant as azimuth lines. They are also called azimuth circles.

celestial equator The great circle on the celestial sphere which is co-planar with the Earth's equator and is perpendicular to the Earth's axis of rotation.

circles of declination These are celestial circles which are parallel to the celestial equator. In the context of this thesis, by and large they represent the apparent path of the sun across the sky over one solar day.

cursor That part of the quadrant which represents the sun's declination at noon throughout a given year. It is constructed on the quadrant so as to represent the sun's solar altitude at noon throughout a given year.

declination An angular measurement or the measurement of an arc of a great circle through the pole of the celestial sphere as measured between the celestial equator and some celestial body.

ecliptic The great circle upon the celestial sphere which marks the apparent path of the sun in one solar year as seen from the earth which results from the earth's orbit around the sun.

equinoctial The apparent circle on the celestial sphere which coincides with the sun's apparent daily path when the sun lies in either of the equinoxes.

equinoxes The two points on the celestial sphere where the equator intersects with the ecliptic. These points are identified on the ecliptic as the first point of the zodiacal signs of Aries and Libra (*i.e.*, the first days of Spring and Fall).

hour lines	On the quadrant these represent the hour circles on the sphere, which are great circles through both celestial poles and serve to identify a given part of a day as measured by equal hours. The hour angles are the angles these circles make with the meridian.
limbus	The 90° graduated arc of the quadrant.
longitude	The measurement of an arc of the ecliptic measured from the first point of Aries.
meridian	A great circle through the poles of the equator and the observer's zenith.
obliquity of the ecliptic	The angle at which the ecliptic meets the celestial equator.
prime meridian	The great circle on the celestial sphere which passes through the pole and an observer's zenith. It necessarily passes though the most northern and southern points on the horizon.
prime vertical	The great circle which passes through the poles of the prime meridian and an observer's zenith point. It necessarily passes through the most easterly and westerly points on the horizon.
right ascension	The measurement of the arc of the celestial equator measured from the Spring equinox to the point where an hour circle passes through some celestial body.
scale of declination	A scale on Gunter's quadrant which represents the sun's declination.

shadow square A scale on the quadrant which ultimately represent the tangent and cotangent trigonometric functions. It was used in surveying for determining the length of a side of some right angle triangle which is similar to some other given right angle triangle.

solstices Correspond to those points on the celestial sphere when the sun has the greatest declination north and south of the celestial equator. These points are identified on the ecliptic as the first points of Cancer and Capricorn respectively, or the first day of Summer and Winter.

Tropic of Cancer The circle on the celestial sphere which is parallel to the celestial equator and tangent to the ecliptic at Cancer. It marks the northern limit of the sun during the year.

Tropic of Capricorn The circle on the celestial sphere which is parallel to the celestial equator and tangent to the ecliptic at Capricorn. It marks the southern limit of the sun during the year.

zenith The point directly over an observer. It is the pole of the observer's own horizon.

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