TWO ESSAYS IN ARBITRAGE PRICING ANALYSIS

by

Andrew M.K. CHEUNG

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Name: Andrew Mun Keung Cheung

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Examining Committee:

Chairman: Dr. Stephen T. Easton

Dr. Dr. Robert Jones
Senior Supervisor

Dr. Avi Bick
Supervisor

Dr. Geoffrey Poitras
Supervisor

Dr. Terence Heaps
Internal/External Examiner

Dr. Joseph Ostroy, Professor
University of California
External Examiner

Date Approved: Dec 9 1994

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Author: ______________________

(signature)

____

(name)

Dec 9/94

(date)

ABSTRACT

The two essays in this thesis explore some aspects of the fundamental theorem of arbitrage pricing in modern finance. The focus in the first essay is on the existence of a state price functional under the presumption of no arbitrage opportunity in the financial market. Two cases are developed. In case one, we tackle the existence problem in the tradition of making no preference assumption. Here a "multiple-version" of Hahn Banach theorem is applied at the cost of introducing a less used continuity concept. The payoff of that approach allows us to remove some strong assumptions made in existing models. In case two, we strengthen the 'viability' of a price system by incorporating a recently improvised preference relations from general equilibrium theory. A continuous price function is derived and used to obtain the familiar Black-Scholes pricing density.

In the second essay, effort is made to extract some implications by modeling an arbitrage free term structure. First, it is shown that this yield curve model enables one to price interest rate related contingent claims such as a bond option which is similar in spirit to the Black-Scholes approach to equity option pricing. A second result is that we derive a random variable that relates the pair of risk-adjusted probabilities obtained from the two closely related yield curve models. The existence of such a random variable throws light on characterizing futures and forward bond prices. Finally the two yield curve models are blended to validate one version of expectations hypothesis in continuous time.

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Dedication

To the memory of my god-parents Mr. Antonio Francisco Carmo and Mrs. Eulalia Carmo.

CONTENTS

A FUNCTIONAL ANALYSIS
OF
ARBITRAGE CONTINGENT CLAIMS PRICING

CHAPTER 1. INTRODUCTION TO THE FIRST ESSAY

Modern financial theory places considerable emphasis on the presumption of no arbitrage opportunity in pricing financial securities. Arbitrage opportunities represent riskless plans for profit without any initial investment. The absence of such opportunities is a necessary requirement in any meaningful asset pricing model.

By arbitrage valuation is meant that suitably defined financial assets are identified with their 'rational' prices as long as profitable arbitrage is precluded. The existing finance literature (see for instance Ingersoll, 1987, ch.2, p.57; Dothan, 1990, ch.2, p.24) establishes an operational principle under 'the absence of arbitrage': namely, one is able to construct a set of arbitrage free *linear state prices* from a subset of observed asset values. The standard tool used to derive these state prices is Farkas lemma, or the theorem of the alternative, familiar from linear programming.

The theorem of the alternative is well established in a linear algebraic framework (Mangasarian, 1969) but it often disguises the existence problem in a more general setting. One can develop a deeper insight by formalizing the notion of a 'linear state spaces'. This entails interpreting the space of asset payoffs as a linear state space which embeds the idea of different states of the world. By modeling the state space as a vector space, an element can be interpreted as the payoff of a contingent consumption claim. A particularly useful connection between the linear state space and its dual space of linear functionals is then obtained by the following observation.

The presumption of no profitable arbitrage opportunity in the vector subspace of marketed securities implies an empty intersection between the subset with arbitrage opportunity and the set of strictly positive future payoff. Given this condition, a basic *Separating Hyperplane Theorem* (Takayama, 1984, p.32) states there exists a closed hyperplane which is related to a continuous linear functional. This functional can be interpreted as the value for the contingent claims.

Defining a state space for arbitrage valuation shares a similar spirit to the *Second Welfare theorem* in general equilibrium analysis. According to this theorem, if agents' preference are defined on a nonempty convex subset of a linear commodity space, then a Pareto optimal allocation can be found and is associated with a continuous linear functional (Lucas and Stokey, 1988, p.424). This functional can be interpreted as competitive market clearing prices.

There remains a relation between arbitrage theory and general equilibrium analysis richer than the mathematical fact that both are founded on Separating Hyperplanes. A brief historical review of this connection is developed in section 1. Identifying this linkage between the two theories at the outset has the advantage of identifying some variables in arbitrage theory with their counterparts in general equilibrium theory.

The canonical arbitrage model is presented in section 2 using the concept of convex cones and the dual cones. Cones are elementary geometric objects and present a compelling visualization of the arbitrage problem in a finite dimensional Euclidean space. Convex cones reappear in later development of the arbitrage valuation problem in more abstract linear spaces. After the linear state prices are derived, they are represented in three equivalent formulations articulated by the fundamental theorem of arbitrage valuation. Finally, in section 3 we discuss the existence problem in an infinite dimensional linear space. We end the present chapter by pointing out where the next two chapters are heading.

1. An overview of the relation between the arbitrage theory and the general equilibrium model

The fundamental theorems of welfare economics state conditions for a competitive equilibrium allocation to be a Pareto optimal allocation and vice versa. This equivalence between competitive and optimal allocations originated in the seminal paper by Arrow and Debreu (1954). That classic analysis presents in an axiomatic framework the properties of an economy with a finite number of agents and commodities.

Uncertainty enters into the Arrow-Debreu model via an elaboration of a two-period economy ($t = 0,1$). There are ℓ different goods available for trade in the two periods and S different states of the world at $t = 1$. Define $p \in \mathbb{R}_{+}^{\ell(S+1)}$ as a price vector of the $\ell(S+1)$ number of goods. To close the model, Arrow-Debreu assume:

(i) every agent knows which state obtains at $t = 1$ when it occurs;
(ii) there is a complete trust that contingent promises will be honored;
(iii) every agent knows p; and
(iv) exchange is costless.

These four assumptions form a basis leading to the proof of the existence of an equilibrium price and a set of corresponding resource allocations in a simple exchange economy. One of the remarkable features of the equilibrium results is that it reduces a two-period model to a static setting. In particular, trading only takes place at t = 0 in which each agent faces one budget constraint.

Within the same uncertainty setting, Debreu (1959, ch.5,7) proves that the competitive equilibrium price vector gives rise to Pareto efficient allocations. This is the first theorem of welfare economics. The separating hyperplane theorem from convex analysis is the key to the demonstration of the Second Welfare theorem.

While the Arrow-Debreu equilibrium is epitomized by its simplicity and elegance, it suffers from a lack of subsequent market transactions after t = 0. This contradicts observed reality. Radner (1968, 1972), maintaining most of the setup of Arrow-Debreu, introduces the concept of a *sequence economy*. A sequence economy is one that allows trading at every date.

The cost of organizing a large number of markets for complete insurance at t = 0 is the usual justification for introducing sequential markets; but such introduction rapidly complicates the original Arrow-Debreu model. Because of the opening of future spot markets, agents must be assumed to form future spot price expectations. The possibility of information asymmetry and potential for moral hazard all lead to a vast literature on transactions cost and market incompleteness.

For pricing assets in a financial market setting, one can still redeem the relevance of most of the static general equilibrium result by focusing on a "stripped down" version of the sequential market model. This is achieved by assuming perfect foresight expectations on the part of the agents. That is, for $s \in S$, there is a price vector $p(s) \in \mathbb{R}_+^{\ell}$ expected with certainty for $t = 1$ and is in fact faced by all agents at $t = 1$. This simplification in the expectation mechanism, coupled with assuming a contingent futures spot market for a good, say good 1, almost leads to the same Arrow-Debreu result except agents face two constraints in their choice problem.

Define the payoff of a futures contract in terms of the S different spot prices of good 1 in a diagonal return matrix. Finally assume that return matrix has a full rank. The Pareto efficient allocation in this simple sequence economy can be shown to replicate the allocation attained by the Arrow-Debreu economy (Laffont, 1989, ch.6). Of course, by the Second Welfare theorem such efficient allocation is also an equilibrium allocation.

Much earlier than Radner's formulation of an incomplete market model there existed an interesting result due to Arrow (1953). One of the main insight from Arrow's model is that we can use a trading mechanism (a security market) to reproduce the static Arrow-Debreu state prices. A fundamental contribution of Arrow's paper is the clever use of the arbitrage concept. Unlike Radner's setting that relies on a futures good market at $t = 0$, Arrow considers securities market at $t = 0$ that allows agents to trade wealth across future states. The following is a brief account of this model which serves as an inspiration for the modern theory of finance.

2. The canonical arbitrage model

Let n be the total number of securities traded at $t = 0$ and m the total number of states at $t = 1$. (Notation here follows the modern literature on arbitrage pricing with m states of the world and n traded securities.) A marketed security, say j, yields a vector of state return denoted by

$$d_j^T = [d_{j1},\ldots,d_{jm}] \qquad j = 1,\ldots,n$$

where 'T' denote the transpose of a column vector. The return is denominated in a numeraire unit of account called money.

Given that m can be much larger than n, investors' interest is in the securities' future payoff which are captured by the m×n return matrix denoted by D. As in Radner's expectation mechanism, the investor is assumed to have perfect foresight regarding D. Finally, the current prices of the securities are given by the vector

$$p^T = [p_1,\ldots,p_n].$$

The problem is reduced to finding a relationship between p and D characterizing the absence of arbitrage opportunities.

An arbitrage opportunity is a portfolio of the n assets denoted by the vector of quantities held

$$\theta^T = [\theta_1,\ldots,\theta_n]$$

with two properties. First, θ does not cost anything at $t = 0$. Second, θ has a nonnegative payoff at $t = 1$ with a positive payoff in at least one state. Formally, the two statements can be expressed as

$$(1) \qquad p^T\theta = 0, \quad D\theta \geq 0 \quad \text{and} \quad D\theta \neq 0.$$

Arrow and others argue that a necessary condition for any meaningful relation between the price vector p and the matrix D to exist is that one cannot find any θ that satisfies (1). Loosely put, to rule out arbitrage situation like (1) for a given D, the price vector p must adjust until no θ can be found that satisfies the definition for arbitrage. It should be emphasized that no serious adjustment process is provided for p. The word 'adjust' in the previous statement merely conveys the existence of a set of state price functionals once arbitrage is ruled out.

Technically this entails finding a solution to the dual problem to (1). This dual problem involves finding a set of positive state prices, one for each state of the world, so that the vector p and the matrix D are linearly related. The Farkas lemma and the theorem of the alternative are the reigning methods of deriving the arbitrage free price functional in a discrete state space model. Here, we retreat to a less used yet more graphical concept known as convex cones and their dual cones for derivation. (See Gale, 1961).

The theorem of the alternative and the theorems of convex cones are similar ways of solving system of linear inequalities. However the latter method has the advantage of offering geometric intuition in finite Euclidean spaces. Moreover, while infinite dimensional Farkas lemma is not well known, the insights from finite convex cones analysis can be extended to the infinite dimensional linear spaces. The representation of cones in a finite setting therefore provides some intuition for the general case.

In what follows, λ is a real number.

Definition. A subset S in a vector space L is said to be <u>convex</u> if

$$\lambda x+(1-\lambda)y \in S \qquad \text{whenever} \quad x,y \in S \quad \text{and} \quad 0 \le \lambda \le 1.$$

Geometric objects in $\mathbb{R}^n$ such as a linear subspace, a line, a halfspace and a hyperplane are examples of convex sets.

Definition. Convex cones are a class of convex set, having the property that

$$\lambda x \in S \qquad \forall\ \lambda \ge 0 \quad \text{if } x \in S.$$

Important examples of convex cones are $\mathbb{R}^n$ and all linear subspaces. Moreover if H is a hyperplane through the origin, H is a convex cone. The difference between a halfline and a line is given by

$$\{x \mid x = \lambda y,\ \ \lambda \ge 0\} \qquad \text{and} \qquad \{x \mid x = \lambda y,\ \ \forall\ \lambda \in \mathbb{R}\}$$

for any vector y; thus a halfline satisfies the defining property of a convex cone. Halfspaces are also convex cones. This brings us to a useful correspondence between linear homogeneous inequalities and convex cones. To introduce this correspondence requires a concept of a finite cone.

Definition. A set C is a finite cone if every element in C is expressed as a linear combination of a finite number of vectors. Alternatively

(i) C is a finite cone if there exists a finite number of vectors v^i such that

$$x = \Sigma_i \lambda_i v^i, \qquad \lambda_i \ge 0 \quad \forall\ x \in C$$

(ii) C is a finite cone if there is a finite number of halflines (v^i) such that

$$C = \sum_i v^i$$
$$= \{x | x = \sum_i y_i, \; y_i \in v^i\}.$$

The advantage of introducing a finite cone is that one can use it to represent the solution to a set of linear equations:

$$C = \{x | x = Au, \quad u \geq 0\} \qquad \text{for A is a } m \times n \text{ matrix.}$$

As defined earlier absence of arbitrage is equivalent to placing some restrictions on a set of homogeneous inequalities. Since these inequalities are now identified a set of finite cones, the arbitrage restriction is reflected on "the other side of the same coin", that is the dual cone.

Definition. If C is a convex cone, the set

$$C^* = \{y | y^T x \leq 0, \quad \forall \, x \in C\} \qquad \text{is the dual cone of C.}$$

In geometric terms, the dual of a convex cone is the set of vectors making a nonacute angle with the vectors of the original cone.

The two fundamental duality theorems about finite cones are stated below (see also Gale, (1960)).

Theorem 1. If C is a finite cone, then C* is a finite cone.

Proof. Given that C is a finite cone, we can write $C = \Sigma_i (v^i)$. Then C* is given by:

$$C^* = \{y \mid yv^i \le 0\}$$

which is a finite cone. If C is expressed in the form $C = \{x \mid x = Au,\ u \ge 0\}$, then $C^* = \{y \mid yA \le 0\}$.

□

Theorem 2. $(C^*)^* = C$.

Proof. For notational convenience, write $(C^*)^*$ as C^{**}. For all $z \in C^{**}$, we have $yz \le 0$ if $y \in C^*$. But if $y \in C^*$, we have $yx \le 0$, for all $x \in C$. Now C^* is a finite cone, so its dual C^{**} is a finite cone. Thus we have $C \subset C^{**}$. If $C^{**} \subset C$, we are done.

Suppose $C^{**} \not\subset C$, then since C, C* are both finite cone and $C \subset C^{**}$, we have

$$C^{**} = C + \Sigma_j (b^j)$$

where (b^j) are halflines not in C. Take dual again and define the resulting cone by C^{****}. This second dual is related to C^{**} in the same way as C^{**} is related to C, that is

$$C^{****} = C^{**} + \Sigma_j (c^j) = C + \Sigma_j (b^j) + \Sigma_j (c^j)$$

where c^j are halflines not in C^{**}.

Taking duals repeatedly in this way, we add new halflines to C^{2n} (where n is the number of times double duals have been taken) at each round. Since C^{2n} is obtained by continually taking duals of finite cone, it must be a finite cone itself. But letting $n \to \infty$ C^{2n} is not a finite cone which is a contradiction. Thus the hypothesis that $C^{**} \neq C$ cannot be true. Hence the desired result follows.

□

Application of finite convex cones and their duals to the arbitrage valuation boils down to showing the following result.

Proposition 1: There is no portfolio θ that satisfies $p^T\theta \le 0$ and $D\theta \ge 0$ if and only if there exists a $m\times 1$ vector $q > 0$ such that we have $p = D^Tq$.

Proof. Necessity. Denote

$$\bar{D} = \begin{bmatrix} D \\ -p^T \end{bmatrix},$$

and by the definition of an arbitrage portfolio θ, we have $\bar{D}\theta > 0$. Given that there exists a $m\times 1$ vector $q > 0$ such that $p = D^Tq$, we claim that the existence of an arbitrage opportunity creates a contradiction. Let θ be an arbitrage portfolio. Postmultiply θ to the transpose of $p = D^Tq$ gives

$$q^TD\theta = p^T\theta;$$

and rearranging yields

$$\begin{aligned} 0 &= (q^TD-p^T)\theta \\ &= [q^T \ \ 1]\begin{bmatrix} D \\ -p^T \end{bmatrix}\theta \\ &= [q^T \ \ 1]\bar{D}\theta. \end{aligned}$$

Since $\bar{D}\theta > 0$ and the q vector is positive, the above equalities lead to an immediate contradiction.

Sufficiency. Absence of arbitrage opportunity implies

$$\{\theta | \theta^T p \leq 0\} \cap \{\theta | D\theta \geq 0\} = \{0\}.$$

Now rewrite $D\theta \geq 0$ as $-D\theta \leq 0$ and consider the convex cone

$$A = \{\theta | \bar{D}\theta \leq 0, \quad \theta \text{ unrestricted}\}, \qquad \text{where } \bar{D} = \begin{bmatrix} -D \\ p^T \end{bmatrix}$$

is a $(m+1)\times n$ matrix.

By the above fundamental theorem of duality for convex cones, we have the dual cone denoted by A* such that

$$A^* = \{q^* | \bar{D}^T q^* = b, \quad q \geq 0\}.$$

Setting b = 0, a $n\times 1$ null row vector, the above implies a set of n hyperplanes through the origin. Since $q^* \geq 0$, let the m+1-th element be the row sum and explicitly consider the set of n equalities in A* as follows:

$$\begin{bmatrix} -d_{11} & \cdots & -d_{1m} & p_1 \\ & \cdots\cdots & & \\ & \cdots\cdots & & \\ -d_{n1} & \cdots & -d_{nm} & p_n \end{bmatrix} \begin{bmatrix} q_1^* \\ \cdot \\ \cdot \\ q_m^* \\ q_{m+1}^* \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

Expand the LHS to yield

$$(-d_{11}q_1^*)+\ldots+(-d_{1m}q_m^*)+p_1q_{m+1}^* = 0$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$(-d_{n1}q_1^*)+\ldots+(-d_{nm}q_m^*)+p_nq_{m+1}^* = 0.$$

Rearrange the above:

$$p_1q_{m+1}^* = d_{11}q_1^*+\ldots+d_{1m}q_m^*$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$p_nq_{m+1}^* = d_{n1}q_1^*+\ldots+d_{nm}q_m^*.$$

Let $q_i = \dfrac{q_i^*}{q_{m+1}^*}$, $i = 1,2,\ldots,m$, we have

$$p_1 = d_{11}q_1+\ldots+d_{1m}q_m$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$p_n = d_{n1}q_1+\ldots+d_{nm}q_m,$$

or in matrix notation $p = D^Tq$. This completes the proof.

□

The $(m+1)\times 1$ vector of q^* embodies a useful market interpretation. Given there is no arbitrage opportunities in trading the n marketed securities, one can imagine there exists simultaneously a market for m state securities. Each of the first m elements of the q^* vector, say q_i, then represents the cost of obtaining one unit of numeraire at $t = 1$ if state i occurs and nothing otherwise. Viewed in this fashion, the existence of these m states securities traded at $t = 0$ allows one to fully insure one unit of numeraire good regardless which states occurs by buying one of each m securities at $t = 0$. The cost of this portfolio is

$\sum_{i=1}^{m} q_i^*$ which is the m+1-th element.

As a package, one can interpret the absence of arbitrage as equivalent to the existence of a m+1 state securities market, the last security being the riskless asset. q_i can be treated as the normalized state security price (relative to the price of the riskless asset). In the light of these state prices, three equivalent representation of the security price functionals are readily available.

<u>First</u>, since by construction summing over all q_i gives one, q_i inherits a basic property of a probability measure and can be called the risk-neutral probability denoted by $\mathbb{Q}_i$. (Note that implicit in the designation for q_{m+1}^* to be $\Sigma_i q_i^*$ is the presumption that the implied interest rate is zero. In a more general case let $q_{m+1}^* = 1$, so that $\Sigma_i^m q_i = \frac{1}{1+r}$.) Therefore the pricing equation can be written as

$$p_j = \sum_{i=1}^{m} q_i d_{ij} = \sum_{i=1}^{m} \mathbb{Q}_i d_{ij} = E_{\mathbb{Q}}(d_i), \qquad \text{where } q_i \equiv \mathbb{Q}_i.$$

<u>Second</u>, one can enrich the state space setting by adding probability assessments of different states occurring. Denote the investor's subjective probability of state i by $\mathbb{P}_i$. The pricing equation can be expressed as

$$p_j = \sum_{i=1}^{m} \mathbb{P}_i \left(\frac{q_i}{\mathbb{P}_i}\right) d_{ij} = \sum_{i=1}^{m} \mathbb{P}_i \lambda_i d_{ij}, \qquad \text{where} \quad \lambda_i \equiv \frac{q_i}{\mathbb{P}_i}$$

is price per unit of probability of state i occurrence. Intuitively, one can view λ_i as the risk premium per unit of payoff in state i.

<u>Third</u>, since $q_i \equiv \mathbb{Q}_i$, the pricing equation can further be rewritten as

$$p_j = \sum_{i=1}^{m} \mathbb{P}_i \left(\frac{\mathbb{Q}_i}{\mathbb{P}_i}\right) d_{ij}$$

where $\frac{\mathbb{Q}_i}{\mathbb{P}_i}$ is called the Radon Nikodym derivative. In some general equilibrium models, this variable is identified with the marginal utility of an infinitely lived representative agent (Cox, Ingersoll and Ross, theorem 4, 1985a).

In a simple linear state-space model, the three equivalent representations of p_j in the absence of arbitrage opportunity constitute the *fundamental theorem of arbitrage asset pricing* (Ross and Dybvig, 1987). A special case worth stressing is where n = m, and the D matrix is nonsingular. The vector q from the proposition is then the unique arbitrage free state price functional.

The resulting security market is said to be *complete* in the following sense. Any other payoff that is spanned by the D matrix can be priced uniquely by q: As Arrow (1953) implicitly points out, it is via securities trading at t = 0 and contingent spot market trading between the numeraire good and the other goods at t = 1 that one can replicate the Arrow-Debreu static budget constraint and simultaneously economize on the use of the contingent claims market.

3. Discussion and direction of the thesis

Two aspects of the linear state price functional q require emphasis. First, the existence proof of q does not require any preference specification of the agents. Except making the crucial assumption that there is no arbitrage opportunity, the entire derivation is due to the geometry of the finite Euclidean space. In order for the linear functional to carry economic meaning, it suffices to attach to q the mildest presumption that agents prefer more wealth to less. This implies the arbitrage free price functional is consistent with risk neutral or risk-averse preferences (the latter being a standard assumption in many finance models such as the Capital Asset Pricing Model). The definite merit of this result is that it removes the modeling and estimation of an unobserved preference parameter.

A more important characterization of the price functional is that it is a continuous linear functional. This aspect is often subsumed when the underlying state space is a finite dimensional vector space. In this case continuity of the linear functional is exemplified by the standard Euclidean norm topology. Needless to say, continuity is a useful requirement from any price functional and only in this way can any arbitrary (contingent payoff) bundle in the state price be unambiguously valued. However, in an infinite dimensional vector space, which is the prevailing setup for many finance models, issues regarding the continuity of a linear price functional rapidly turn complicated.

In an infinite dimensional state space setting, one is confronted with a vast number of linear topologies. While some of these topologies are simply natural generalization of the finite Euclidean topology, unfortunately these norm topologies are too strong to induce a continuous price functional. It follows that merely making appeal to the absence of arbitrage is far from necessary and sufficient to yield a meaningful valuation result.

In the next chapter we endeavor to look for a more robust existence result in the sense that we are motivated to use a specific class of linear topology for the linear space. The nature of the research in that direction is inevitably technical but fortunately in functional analysis there are well developed results suitable for our analytical setting. It will be shown that the existence of a continuous price functional is founded on the powerful Hahn-Banach theorem. Most of the topological considerations are embedded in the statements of the Hahn-Banach theorem.

The plan of the next chapter is as follows. We begin to look for a version of the general Hahn-Banach theorem which allows us to derive a continuous linear price functional. Then we identify some existing arbitrage valuation models as consistent with the general result we present in that chapter. Because of the wide range of potential applications in pricing, the topological approach that retains the preference-free property in the general setting is deemed promising.

The <u>second</u> aspect of the state price functional q is the concern about its role as a shadow price. Granted that the absence of arbitrage opportunity plus a linear topology are sufficient for the existence of q, there is no simple guide as to which topology to choose. As noted by Kreps (1982), in order to obtain a sound economic interpretation, one needs to endogenize any given price in the model. In a simple state space model with exchange only, the obvious fundamental related to the shadow price is the preference relation assumed for agents.

While enriching the arbitrage model can be achieved by incorporating a preference relation, this preference approach interestingly presents an alternative solution to some of the topological difficulties raised in chapter two. The idea is that assuming a continuous preference relation implies the model builder has input a topology compatible with the linear space topology. This is then sufficient to permit the Hahn Banach theorem to yield a continuous price functional. The economic reasoning behind this is quite familiar. The arbitrage free security prices in the market model that can be extended to the entire state space is defined to be a viable price system if agents can find a solution to their optimization problem.

The topological approach to valuation by making a set of assumptions about the preference relation is a useful device. Along this line of modeling and with a marginal effort, one can even treat the shadow prices as prices in a Walrasian equilibrium. One of the advantages in constructing an arbitrage equilibrium in this way is that one can skip over a full description of demand and supply and market clearing. Indeed this approach is very similar in spirit to the idea of the second theorem of welfare economics.

The arbitrage equilibrium model based on standard of assumptions about preference relations were first developed in two influential papers by Harrison and Kreps (1979) and Kreps (1981). Recent theoretical advance in general equilibrium analysis suggest that there is room to improve these earlier models. In chapter 3, two examples illustrate that in some linear spaces where all the preference assumptions are satisfied, one is still unable to derive a nontrivial continuous price functional. We are then led to adopt a stronger notion of viability. As an application, this modification is then combined with a stochastic setting to derive the well known Black-Scholes state price density function.

CHAPTER 2. A RECONSIDERATION OF ARBITRAGE VALUATION

The goal of this chapter is to generalize the theory of asset valuation by arbitrage from a finite dimensional Euclidean setting to an infinite dimensional vector space. A vector space of infinite dimension can be thought of as a space of functions. In finance and economics, in which uncertainty is involved, function spaces are usually identified as state spaces with elements called random variables. Among all functions spaces, the normed linear spaces play an important role in this kind of stochastic analysis primarily because most of their defining characteristics can be matched with the concepts from finite dimensional Euclidean spaces. For instance, a norm can be treated as a generalization of Euclidean distance.

A Banach space is a complete normed vector space. Linear functionals defined on a Banach space form a dual space of functionals. For any analysis that involves optimization, Banach spaces and a subset of their duals are functionally connected. This means any element in a linear space can be associated with a continuous linear functional in its dual space. In the arbitrage valuation theory, these continuous functionals are naturally interpreted as implicit state prices.

Generalization of analysis to infinite dimensional spaces is not a straightforward exercise. Normed linear spaces do not in general have the desirable properties found in finite Euclidean spaces. For instance, in the last chapter convex analysis is employed to derive the extended price functional in a standard setting with m states and n securities. That approach, and many variants, to finding prices in the dual space are based on the single most important Hahn Banach theorem in functional analysis.

In its entirety, the Hahn Banach theorem is composed of the separation form and the extension form. The separation part of the theorem stipulates that provided with two disjoint convex sets, at least one of which has a nonempty interior, one can find a hyperplane slipping between the two sets. The extension part of the theorem states that provided a linear functional in a subspace is dominated by a convex functional, one can find a continuous extension of the subspace linear functional to the entire linear space.

In spite of its usefulness in the arbitrage valuation problem and in optimization theory, application of the Hahn Banach theorem raises many difficulties. The present chapter focuses on two problems that arise mainly in finding a separating hyperplane. First, separation requires one of the convex sets to have a nonempty interior; unfortunately most infinite dimensional normed linear spaces fail to have this topological property. Second, if a linear subspace is closed, then a linear functional defined on the subspace is continuous. However, closedness of linear subspace is not guaranteed in infinite dimensional function spaces.

Both of the above problems reveal that application of the Hahn Banach theorem depends crucially on the topological structure of the linear space. The lack of nonempty interior in normed linear spaces causes us to search for other weaker topologies compatible with the linear space. A class of topological vector spaces known as locally convex spaces is introduced. It will be shown that locally convex spaces include most of the useful function spaces adopted in economics and finance.

In addition, associated with locally convex spaces is a wide class of weak topologies that are sufficient to satisfy the Hahn Banach theorem. This is indicated by the Mackey-Arens theorem which can be used to assert the existence of a weak topology for any pairing of L_p spaces. Two applications will be demonstrated to illustrate the relevance of this theorem.

Aside from the mathematical desirata, modeling arbitrage valuation in a locally convex space allows us to rediscover a number of features familiar from the arbitrage analysis in the finite dimensional setting. Similar to the finite setting with regard to investor's characteristics, the general setting specifies nothing other than that more wealth is better. This similarity of analysis by arbitrage between finite and infinite dimensional state spaces thus confirms its theoretical advantage that it is primarily a preference-free methodology.

The plan of this chapter is as follows. Section 1 and 2 recall some important facts for analyzing linear spaces. These two sections also serve to introduce notation and preliminary results that motivate two complementary formulations of the "Panglossian" functional. In section 3 we first deliver the "imprecise" Hahn Banach theorem. Crucial to this section is a device called the Minkowski functional that is used to prove the existence of an extended linear functional.

However, the full-blown version of extension of a linear function from the subspace to the entire linear space ultimately depends on the possibility to separate two nonempty convex sets by a hyperplane. When the normed linear space is used, the nonexistence problem enters the picture since most of these spaces do not have subsets containing a nonempty norm interior. The exact nature of the problem is demonstrated in Section 4.

In section 5, we consider the weak topology as a substitute for the strong norm topology. Then the Mackey Aren theorem is introduced. In the presence of this important topological result, we are able to derive a weaker version of the Hahn Banach theorem and later apply this theorem to the market model introduced by Ross (1978). After the general existence theorem for the market model is derived, we use the result to reconsider two existing arbitrage pricing models that used L_p spaces as the commodity spaces.

In the first model, which uses L_2 as its commodity space, it is shown that some strong assumptions can be removed if the functional analysis result developed here is adopted. In the second model, which uses L as its commodity space, the separation of two convex subsets in L_∞ is satisfied but the existence of an unambiguous continuous linear functional in the norm dual is still problematic since the dual space of L_∞ consists of uninterpretable elements. The duality theorem developed in this chapter combined with a result from Bewley (1972) is shown to resolve the problem. Finally, we discuss some further implications of using weak topology in the arbitrage valuation.

1. Geometry of the vector space

The essence of the Hahn Banach theorem lies in its irresistible geometric intuition: given certain conditions are satisfied, two nonempty convex subsets of a linear space can be separated by a closed hyperplane. To motivate this important result requires some basic definitions and properties of vector spaces. A *vector space* is a set L along with two algebraic operations on the elements of L: addition and multiplication by a scalar. The elements of L are referred to as *vectors.* By convention, there exists a unique vector 0 in L referred to as the *zero vector* or *origin* of L.

As usual in economics, one interprets a vector as a commodity bundle with elements representing everything that an economic agent consumes. In analysis with uncertainty, a vector can be a contingent commodity bundle. The vector space most frequently used in economics and finance is Euclidean space, denoted as $\mathbb{R}^n$. Most of properties of vector spaces, however, carry over to spaces other than $\mathbb{R}^n$.

Letting $S = \{v^i \in L | i \in I\}$ be any collection of vectors indexed by the set I (of nonnegative integers), the *linear combination* is defined as

$$\Sigma_{i \in I} \alpha_i v^i \in L \qquad \text{for } \alpha_i \in \mathbb{R}$$

provided that only a finite number of α_i are not equal to zero. For a set $S \subset L$, consider the set of all possible linear combination of vectors in S. The *span* of S is given by

$$sp(S) = \{x \in L | x = \Sigma_i \alpha_i v^i,\ \alpha_i \in \mathbb{R},\ v^i \in S\}$$

for which a finite number of scalars α_i are nonzero. If $S \subset L$, then sp(S) is a *subspace* of L. If sp(S) does not coincide with L, it is called a *proper subspace*. An example of a proper subspace is a one dimensional line through the origin of $\mathbb{R}^2$.

A collection of vectors $S \subset L$ is called *linearly independent* if

$$v^i \notin sp(S)-\{v^i\} \qquad \forall\ v^i \in S.$$

That is, no vector in S can be expressed as a linear combination of the remaining vectors in S. Consider S as a subset of L. If S *spans* all of L, i.e., sp(S) = L, and if elements of S are linearly independent, then S is called a *basis* in L. The number of elements in a basis is called *cardinality* (which is a term allowing for sets with infinite number of elements). A vector space having a finite basis is called *finite dimensional.* All other vector spaces are said to be *infinite dimensional.*

Let L be a linear space and L′ be a subspace of L. Then two elements $x,y \in L$ are said to belong to the same *class* generated by L′ if $x-y \in L'$. The set of all such classes form a *quotient space* denoted by L-L′. The dimension of the quotient space is called the *codimension* of L′ in L. Elements from L and L′ are related by the following:

Lemma 1: Let L′ be a subspace of a linear space L. Then L′ has finite codimension n if and only if there are linear independent elements $x_1,\ldots,x_n$ in L such that every element $x \in L$ has a unique representation

$$x = \alpha_1 x_1 + \ldots + \alpha_n x_n + y$$

where $\alpha_1,\ldots,\alpha_n$ are nonzero scalars and $y \in L'$.

The proof of of this result is in Kolmogorov and Fomin (1972, p.112). Given that M is a nonempty proper subspace of L, the *translation* of the subspace is called a *linear variety* (also called *affine subspace, flat*, or *linear manifold*). It is written as

$$V = x_0 + M \qquad \text{for } x_0 \notin M.$$

2. Linear functional and hyperplanes

A *linear functional* on a vector space L is a mapping $p:L \to \mathbb{R}$ which obeys

(a) $p(x+x') = p(x)+p(x') \quad \forall\ x,x' \in L$ and

(b) $p(\alpha x) = \alpha p(x) \quad \forall\ x \in L$ and $\forall\ \alpha \in \mathbb{R}$.

A functional that satisfies (b) is called *homogeneous*. The set $L^{\sim}$ of linear functionals on L is called the *dual space* of L and is itself a vector space. If $L = \mathbb{R}^n$, the dual space $L^{\sim}$ is again $\mathbb{R}^n$ and the linear functional is given by the *scalar product:*

$$p\cdot x \equiv \sum_{i=1}^{n} p_i x_i$$

where p and x are elements of $\mathbb{R}^n$.

Consider the linear functional p defined on a linear space L. Then the set M_p of all elements $x \in L$ such that $p(x) = 0$ is called the *kernel* of p:

$$\begin{aligned} \ker(p) &\equiv M_p \\ &= \{x \in L \mid p(x) = 0\}. \end{aligned}$$

Note that M_p is a subspace of L since for $x,y \in M_p$ implies

$$\begin{aligned} p(ax+\beta y) &= \alpha p(x)+\beta p(y) \\ &= 0. \end{aligned}$$

Two cases arise from the definition of a kernel. If $p = 0$, then $\ker(p) = L$. If $p \neq 0$, then $\ker(p)$ is one dimension less than

L, and the resulting kernel is called a *hyperplane.* A further generalization is obtained by the translation of the kernel:

$$\ker(p)+x_0 = \{x \in L \mid p(x) = p(x_0)\}.$$

A translated subspace is called an *affine subspace*; and if $p \neq 0$, the resulting affine subspace is called an *affine hyperplane.* Note that L and M_p have the following relationship.

Lemma 2: Let x_0 be any fixed element of $L-M_p$. Then every element of $x \in L$ has a unique representation of the form

$$x = \alpha x_0+y \qquad \text{where } y \in M_p.$$

Proof. By hypothesis $x_0 \neq 0$ and $p(x_0) \neq 0$. Take $p(x_0) = 1$, otherwise renormalize x_0 by $\dfrac{x_0}{p(x_0)}$ so that $p\left(\dfrac{x_0}{p(x_0)}\right) = 1$. Given any $x \in L$, let

$$y = x-\alpha x_0 \qquad \text{where } \alpha = p(x).$$

We claim that $y \in M_p$, because

$$\begin{aligned} p(y) &= p(x-\alpha x_0) \\ &= p(x)-\alpha p(x_0) \\ &= p(x)-p(x) \\ &= 0. \end{aligned}$$

Therefore $x = \alpha x_0+y$.

To prove uniqueness of such representation of x, assume to the contrary, there exists another representation

$$x = \alpha' x_0 + y' \qquad y' \in M_p.$$

Taking difference of the two distinct representations yields

$$(\alpha - \alpha')x_0 = y - y'$$

implying that $x_0 = \dfrac{y-y'}{\alpha-\alpha'}$ which belongs to M_p (since $y-y' \in M_p$). This contradicts $x_0 \notin M_p$.

□

The one-to-one correspondence between hyperplane and linear functionals is given by the following theorem.

Theorem 1: Given a linear space L, let p be a nontrivial linear functional on L. Then the set $M = \{x \mid p(x) = 1\}$ is a hyperplane M' parallel to the kernel M_p of the functional. Conversely, let

$$M' = L' + x_0 \qquad \text{for } x_0 \notin L'$$

be any set parallel to a subspace $L' \subset L$ of codimension 1. Then there exists a unique linear functional p on L such that $M' = \{x \mid p(x) = 1\}$.

Proof. For a given p, choose x_0 such that $p(x_0) = 1$. The above lemma 2 states that every element $x \in M'$ can be represented as

$$x = x_0 + y \qquad \text{for } y \in M_p.$$

Conversely, given $M' = L' + x_0$ (for $x_0 \notin L'$) it follows from lemma 1 that every vector $x \in L$ can be uniquely represented as

$$x = \alpha x_0 + y \qquad \text{for } y \in L'$$

The desired linear functional is obtained by setting $p(x) = \alpha$. We claim that p is unique. To see this, consider another linear functional q such that $q(x) \equiv 1$ for $x \in M'$ and $q(y) = 0$. Then, $q(\alpha x_0 + y) = \alpha = p(\alpha x_0 + y)$.

□

The above theorem of correspondence between a hyperplane and a linear functional provides some analytical convenience. Any result that yields the former can allow one to conclude the existence of the latter. However, the theorem does not say anything about the boundedness and continuity of the linear functional given the existence of a hyperplane. Continuity of a linear functional is an enormously useful feature in economic and finance models. With suitable interpretation of the linear functional as a price vector in the arbitrage state space model, for instance, continuity of the linear functional implies that claims on every (infinitesimal) state of the world are given positive values.

A price functional is discontinuous when it is not bounded (Luenberger, p.105 1969). Both concepts require a precise notion of openness defined on the linear space. A relevant topological concept that motivates the continuity of price functional is the denseness of the hyperplane in L.

Definition: A subset A of a topological space T is *dense* if its closure is the entire T.

To apply the above definition to analyze a vector space L, a topology must be introduced on L. Then T can be viewed as a subset of L and A is the subspace represented by the hyperplane. Intuitively, denseness of the hyperplane A in T means that there are sequences in the subspace that converge to any element of T. Since the entire linear space is unbounded, naturally the associated price functional is unbounded and hence discontinuous. To rule out such pathological situation, one need the following requirement for the hyperplane.

Definition: A subset A of a topological space T is *nowhere dense* if its closure has empty interior.

Again the abstract topological space T in the above definition can be viewed as a subset of the linear vector space L. Then the interior corresponds to the strictly positive orthant. Therefore to yield a non-trivial hyperplane requires that no sequence from the subspace "enter" into the positive orthant. A formal restatement of this intuition is the following:

Lemma 3: Let L be a linear space. If p is continuous, then ker(p) is closed and nowhere dense in L.

The proof of this result is delayed as that involves more topological concepts that are developed later.

3. Valuation by Hahn Banach extension theorem

As indicated in the previous subsection, the dual space of a linear space L is itself a large vector space of linear functionals, some of which are discontinuous. Our interest is restricted to finding the set of bounded continuous linear functionals so that all conceivable contingent claims can be unambiguously valued. (By valued is meant that the linear functional is positive.)

The classic Hahn Banach theorem states conditions for the existence of continuous linear functionals extended from the subspace to the entire linear space L. As mentioned at the beginning of the chapter, the theorem is divided into a portion that deals with the separation of nonempty convex subsets and the remaining portion deals with the extension of linear functionals from the linear subspace to the whole space. In a general linear space, the topological consideration largely shows up in the separation part of the theorem. In particular it requires at least one of the convex sets separated to have a nonempty interior.

If the linear space is a finite dimensional Euclidean space, the Hahn Banach theorem is usually presented in an algebraic form (see Nakaido, 1968, p.26) called the theorem of supporting hyperplane. In this elementary version, the topological requirement is often satisfied by the Euclidean topology. It is a basic fact that all subsets in $\mathbb{R}^n$ have interior given by open balls.

Of interest here is the separation theorem in infinite dimensional linear space and different definitions of the topology on such spaces yield different versions of the separation theorem. The strategy at the moment is to present the Hahn Banach theorem in an imprecise form without explicitly identifying a specific topology. Doing this has the advantage of examining first the extension part of the theorem and then checking out its implications for arbitrage pricing. The crucial concept at this stage of the problem development is that of a convex functional whose characteristics are described by the following definitions.

Definition: A functional ρ defined on a linear space L is called a *convex functional* if it obeys

(i) $\rho(x) \geq 0 \qquad \forall\ x \in L$ (nonnegativity)

(ii) $\rho(\alpha x) = |\alpha| \cdot \rho(x) \qquad \forall\ x \in L$ and $\forall\ \alpha \geq 0$

(iii) $\rho(x+y) \leq \rho(x)+\rho(y) \qquad \forall\ x,y \in L.$

As properties (i) - (iii) are basic criterion for a distance measure, ρ can be interpreted as a measure of distance for elements in L.

Definition: A set $C \subset L$ is called *convex* if $x,y \in L$, $0 \leq t \leq 1$ implies $tx+(1-t)y \in C$. Furthermore, C is called

(i) *balanced* (or *circled*) if $x \in C$, and $|t| = 1$ implies $tx \in C$;

(ii) *absorbing* (or *absorbent*) if $\bigcup_{t>0} tC = L$.

Holmes (1975) calls a set C that satisfies the above characteristics a *convex body*.

Definition: The interior of a convex body denoted by I(C) is the set of all points $x \in C$ with the following property: Given any $y \in L$, there exists a number $\varepsilon > 0$ such that

$$x+\mu y \in C \qquad \text{if } |\mu| < \varepsilon.$$

Note that in defining the "encompassing" concept of an interior, no topology is mentioned.

Definition: Let C be a convex body whose interior contains the point 0. The functional

$$\rho_C(x) = \inf\{r \mid \frac{x}{r} \in C,\ r > 0\}$$

is called the *Minkowski functional* of C.

The connection between a convex functional and a convex set is stated below.

Theorem 1: If ρ is a convex functional on a linear space L and K is any positive number, then the set $C = \{x \mid \rho(x) \le K\}$ is convex. If $\rho(x) < \infty$, for all $x \in L$, then C is a convex body with interior

$$I(C) = \{x \mid \rho(x) < K\}.$$

Conversely, given a convex body C with 0 in its interior, $\rho_C(x)$ is a finite convex functional and $C = \{x \mid \rho_C(x) \le 1\}$.

Proof. If $x,y \in C$, $\lambda_1,\lambda_2 \ge 0$, $\lambda_1+\lambda_2 = 1$, then

$$\rho(\lambda_1 x+\lambda_2 y) \le \lambda_1\rho(x)+\lambda_2\rho(y) \le K$$

which shows that C is a convex set. By hypothesis, $\rho(x)$ is finite. Let $\rho(x) < K$, $\mu > 0$, $y \in L$. Then

$$\rho(x\pm\mu y) \le \rho(x)+\mu\rho(\pm y).$$

If $\rho(-y) = \rho(y) = 0$, then $x\pm\mu y \in C$ for all μ. If at least one of the numbers $\rho(y)$, $\rho(-y)$ is nonzero, then $x\pm\mu y \in C$ provided

$$\mu < \frac{K-\rho(x)}{\max(\rho(y),\rho(-y))}.$$

Conversely, given any $x \in L$, pick a sufficiently large r so that $\frac{x}{r} \in C$. Then $\rho_C(x)$ is nonnegative and finite. Clearly, $\rho_C(0) = 0$. To check the homogeneity of ρ_C, if $\alpha > 0$, then

$$\begin{aligned}
\rho_C(\alpha) &= \inf\{r > 0 | \frac{\alpha x}{r} \in C\} \\
&= \inf\{\alpha r' > 0 | \frac{x}{r'} \in C\} \\
&= \alpha \cdot \inf\{r' > 0 | \frac{x}{r'} \in C\} \\
&= \alpha \cdot \rho_C(x).
\end{aligned}$$

To check convexity of ρ_C, consider $\varepsilon > 0$ and any $x_1, x_2 \in L$, choose r_i $(i = 1,2)$ so that

$$\rho_C(x_i) < r_i < \rho_C(x_i)+\varepsilon.$$

Then $\frac{x_i}{r_i} \in C$. If $r = r_1+r_2$, then

$$\frac{x_1+x_2}{r} = \frac{r_1}{rr_1}x_1+\frac{r_2}{rr_2}x_2$$

belongs to the segment with end points $\frac{x_1}{r_1}$ and $\frac{x_2}{r_2}$. Since C is convex, this segment and hence the point $\frac{x_1+x_2}{r}$ belongs to C. It follows that

$$\rho_C(x_1+x_2) = r = r_1+r_2 < \rho_C(x_1)+\rho_C(x_2)+2\varepsilon.$$

Since ε is arbitrary, we can conclude that

$$\rho_C(x_1+x_2) \le \rho_C(x_1)+\rho_C(x_2)$$

which proves that ρ_C is convex.

□

Note that the Minkowski functional $\rho(x)$ defines a measure of distance from the origin to x with respect to the convex body. The finiteness of $\rho_C(x)$ is precisely the prerequisite to use the Hahn Banach theorem. In its extension form, the theorem allows the extension of a bounded linear functional from a subspace of L to bounded continuous linear functional defined on the entire space. To prove the Hahn Banach extension theorem, a simplifying assumption about L is needed, namely it is a *separable* space, (that is, containing a countable dense subsets).

Hahn Banach extension theorem. Let L be a linear space and $\rho(x)$ be a finite convex functional on L. Suppose f is a linear functional defined on a subspace M of L satisfying

$$f(m) \le \rho(m) \qquad \forall\ m \in M.$$

Then there is an extension F of f from M to L such that $F(x) \le \rho(x)$ on L.

Proof. Suppose y is a point in L but not in M. Consider all elements of the subspace denoted by $[M+y]$. Then $x \in [M+y]$ has a unique representation

$$x = m+\alpha y, \qquad \text{where } m \in M \text{ and } \alpha \text{ is a real scalar.}$$

An extension g of f from M to $[M+y]$ has the form

$$g(x) = f(m)+\alpha g(y).$$

Hence the extension is specified by prescribing the constant $g(y)$. It must be shown that this constant can be picked so that

$$g(x) \leq \rho(x) \qquad \text{on } [M+y].$$

Let $m, m_2 \in M$, we have

$$\begin{aligned} f(m_1)+f(m_2) &= f(m_1+m_2) \\ &\leq \rho(m_1+m_2) \\ &\leq \rho(m_1-y)+\rho(m_2+y). \end{aligned}$$

Rearranging the above yields

$$f(m_1)-\rho(m_1-y) \leq \rho(m_2+y)-f(m_2).$$

By hypothesis, f is dominated by ρ which is finite and m_1 and m_2 are arbitrary; therefore let

$$c'' = \sup_{m \in M} [f(m)-\rho(m-y)]; \qquad c' = \inf_{m \in M} [\rho(m+y)-f(m)]$$

or we have $c'' \leq c'$.

Hence we can find a real constant c such that the following holds:

$$c'' \leq c \leq c'.$$

Replace $g(y)$ by c so that

$$g(x) = f(m)+\alpha c.$$

If $\alpha > 0$, then

$$
\begin{aligned}
\alpha c+f(m) &= \alpha[c+f(\tfrac{m}{\alpha})] \\
&\le \alpha[\rho(\tfrac{m}{\alpha}+y)-f(\tfrac{m}{\alpha})+f(\tfrac{m}{\alpha})] \\
&= \alpha\rho(\tfrac{m}{\alpha}+y) \\
&= \rho(m+\alpha y).
\end{aligned}
$$

If $\alpha = -\beta < 0$, then

$$
\begin{aligned}
-\beta c+f(m) &= \beta[-c+f(\tfrac{m}{\beta})] \\
&\le \beta[\rho(\tfrac{m}{\beta}-y)-f(\tfrac{m}{\beta})+f(\tfrac{m}{\beta})] \\
&= \beta\rho(\tfrac{m}{\beta}-y) \\
&= \rho(m-\beta y).
\end{aligned}
$$

Thus $g(m+\alpha y) \le \rho(m+\alpha y)$ $\forall$ α and g is an extension of f from M to $[M+y]$, then to $[[M+y_1]+y_2]$ and so on.

Finally, g (which is continuous since ρ is continuous in the metric space defined by ρ) can be extended by continuity from the dense subspace S to the entire linear space L. To see this, suppose $x \in L$, then there is a sequence $\{s_n\}$ of vectors in S converging to x. Define $F(x) = \lim_{n\to\infty} g(s_n)$. F is linear and

$$F(x) \leftarrow g(s_n) \le \rho(s_n) \rightarrow \rho(x)$$

and so $F(x) \le \rho(x)$ on L.

□

To sum up, the Hahn Banach theorem relates the linear subspace and its dual by a continuous linear price functional on L. The first rigorous application of the Hahn-Banach extension theorem to financial asset pricing problem is in Ross (1978, appendix). In Ross setting, there are a finite number of marketed securities in a linear subspace characterized by the absence of arbitrage opportunities. However, Ross acknowledges that the state space of returns is an infinite dimensional linear space and "a version" of the Hahn Banach theorem is required to generate a continuous price functional.

We choose to interpret the Ross' (unproven) result in term of the Hahn Banach extension theorem presented above. Despite its 'vague' topological treatment, the extension form does convey some good intuition. That is, on a linear subspace of L there is a linear functional with some "viable" economic properties, this functional can be "carried over" to the entire linear space according to the theorem.

The above Hahn Banach theorem is derived under the presumption that the separation part of the theorem is satisfied by some unidentified convex functional. Any explicit consideration of the separation aspect of the theorem gradually reveals some analytical difficulties. First, if L is modeled as a normed linear space using an L_p-norm, there is a lack of interior in the positive cone of such L_p spaces. Since separation is only assured if at least one of the sets separated has nonempty interior, this poses a problem of existence of a separating hyperplane if the L_p-norm is used.

Second, unlike finite dimensional Euclidean space, linear subspaces in an infinite dimensional linear spaces are not necessarily closed. This means that merely having a linear functional defined over a linear subspace does not automatically lead to continuity of that functional.

Third, the working of the Hahn-Banach extension theorem hinges on the linear space being separable. This separability property is unfortunately not available in the space of essentially bounded functions, i.e. L_∞. Each of these problems are examined in the rest of the chapter.

4. Valuation in normed linear spaces

A normed linear space is a class of functions space that combines the characteristics of a vector space and a metric space; the former embeds only the algebraic operations whereas the latter deals with the notion of distance between any two elements. This combination is captured by a norm. Formally, a norm in a linear space is a real-valued function defined by $\|\cdot\|: L \to \mathbb{R}$. For all x,y $\in$ L, and $\alpha \in \mathbb{R}$, $\|\cdot\|$ obeys the following axioms:

(i) $\|x\| \geq 0$;
(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$;
(iii) $\|x+y\| \leq \|x\|+\|y\|$.

The finite dimensional $\mathbb{R}^n$ is a classic example of a normed linear space. A Banach space is a complete normed vector space where all Cauchy sequences converge.

An important family of normed linear spaces is called the L_p space (ℓ_p space if the elements are real valued sequences). In addition to obeying the above properties, a L_p space can be further induced by a measure space and in this case it is denoted as $L_p(\Omega,\mathcal{F},\mu)$ where the triple represent more primitive objects.

For instance, under uncertainly, Ω represents different states of the world, $\mathcal{F}$ is a σ-algebra of subsets and μ is a measure over all these subsets. (The interaction between a measure space and L_p spaces are discussed in Bartle, 1966). For $\mu(\Omega) = 1$, the measure is called a *probability measure* which is customarily denoted by $\mathbb{P}$. If μ is a counting measure, L_p is reduced to a sequence space denoted by ℓ_p. In the analysis to follow, $(\Omega,\mathcal{F},\mu)$ is understood as the underlying measure space and will be omitted whenever appropriate to simplify notation.

The *norm* of an element x in ℓ_p spaces is given by

$$\|x\|_p = \Big(\sum_{t=1}^{\infty} |x_t|^p\Big)^{1/p}, \qquad \text{for } 1 \le p < \infty \qquad \text{and}$$
$$\|x\|_\infty = \sup_t |x_t| \qquad p = \infty.$$

Define the space $\mathcal{L}_p[a,b]$, for $p \ge 1$, consisting of those mappings x from the interval [a,b] to $\mathbb{R}$ such that $|x|^p$ is *Lebesgue integrable*. The norm for $x \in \mathcal{L}_p$ is given by

$$\|x\|_p = \Big(\int_a^b |x_t|^p dt\Big)^{1/p}$$

where the expression inside the bracket is a *Lebesgue integral* and $t \in [a,b]$.

Note that $\|x\|_p = 0$ does not imply $x = 0$ since x may be nonzero on a set of measure zero. Taken into account of this possibility, we consider a family of related normed linear spaces of *equivalence classes* of measurable functions. A standard notation for this class of function space is given by $L_p(\Omega,\mathcal{F},\mathbb{P})$. Two functions are said to be $\mathbb{P}$-equivalent if they are equal $\mathbb{P}$-almost everywhere. Elements of L_p are normed by

$$\|x\|_p = \left(\int_\Omega |x(\omega)|^p d\mathbb{P}\right)^{1/p} \qquad 1 \le p < \infty.$$

The sup norm on L_∞ is given by

$$\begin{aligned} \|x\|_\infty &= \inf\{S(N) \mid N \in \mathcal{F},\ \mathbb{P}(N) = 0\} \\ &\equiv \text{essential supremum } |x(\omega)|, \end{aligned}$$

where $S(N) = \sup\{|x(\omega| \mid \omega \notin N\}$. An element of L_∞ is called an *essentially bounded* measurable functions.

In finance theory, elements of L_p spaces are interpreted as random variables. The norms of these elements are merely transformations of the various moments of these random variables. The algebraic dual of L_p space is denoted by $L_p^\sim$, which is a space of linear functionals over L_p. Of significance is the subspace of the algebraic dual consisting of bounded continuous linear functionals. Let L be a normed linear space. The space of bounded linear functionals on L are called *norm dual* of L and is denoted by L^* (also corresponding to the space of continuous functions on L). An element $f \in L^*$ is normed by

$$\begin{aligned} \|f\| &= \inf_M \{M \mid f(x) \le M \cdot \|x\|,\ \forall\ x \in L\} \\ &= \sup_{\|x\|=1} |f(x)|. \end{aligned}$$

One of the important properties about L^* is that it is also a Banach space (Luenberger, 1969, p.106). For L_p, $1 \le p < \infty$, define the conjugate index of p as $q = \dfrac{p}{p-1}$ so that $\dfrac{1}{p}+\dfrac{1}{q} = 1$. The consequent L^* is then L_q with one exception. The exception is $p = \infty$ as the norm dual of L_∞ is larger than L_1.

The economic interpretations of elements for L_p and L_q are that the former is a space of state contingent payoff while the latter represents a linear space of price functionals for these contingent claims. As Banach spaces are vector spaces, they are typically characterized by two algebraic operations, namely addition of vectors and multiplication of any given vector by a scalar. These two operations have interpretable counterparts in the price-taking assumption of a security market model.

Linearity of the functional in L_q over elements in L_p implies the value of two separate commodities is the same as the values of two commodities added together. In a security market characterized by the absence of arbitrage opportunity, this linearity property of the price functional is then called value additivity.

Although L_p spaces provide a natural setting for contingent claims analysis, one of the crucial argument for applying the Separation theorem is missing: for infinite dimensional L_p spaces, the positive orthants have empty interiors. To demonstrate this important fact, consider first the definition of the L_p norm interior.

Definition: Let $\mathcal{P}$ be a subset of a normed linear space L. The point $p \in \mathcal{P}$ is said to be an *interior point* of $\mathcal{P}$ if there is an $\varepsilon > 0$ such that all vectors x satisfying $\|x-p\| < \varepsilon$ are also elements of $\mathcal{P}$. The collection of all interior points $\mathcal{P}$ is called the *interior* of $\mathcal{P}$.

Lemma 1: **(i)** The positive orthant of ℓ_∞ has a nonempty interior. **(ii)** The positive orthant of ℓ_p for $1 \le p < \infty$ has a empty interior.

Proof. **(i)** Recall the ℓ_∞ norm is $\|x\| = \sup_t |x_t|$. Denote $\mathcal{P}$ as the set of all x with nonnegative coordinates. Take any point x′ in $\mathcal{P}$ which is bounded from zero i.e., $|x_t| > m$ for all t. Then x′ is an interior point. To see this, since x′ is bounded away from zero, one can find an ε-neighborhood around x′ such that any element p in this neighborhood has distance from x′ measured by $\|x'-p\| < \varepsilon$. Hence x′ is an interior point.

(ii) Consider ℓ_p for p = 2. Its norm is $\|x\|_2 = (\sum_{t=1}^{\infty} |x_t|^2)^{1/2}$. Given any $\varepsilon > 0$, denote x as an arbitrary element of the nonnegative orthant. Since $\|x\|_2 < \infty$ there exists N such that $\forall$ n $\geq N$, $x_n \leq \frac{\varepsilon}{2}$. Define z with

$$z_n = x_n \quad \text{for } n \neq N, \qquad z_n = x_n - \frac{\varepsilon}{2} \leq 0 \quad \text{for } n = N.$$

Thus $z_N < 0$ and z is not in the nonnegative orthant of ℓ_2 but is in ℓ_2. Also,

$$\|x-z\| = |(x_N-z_N)^2|^{1/2} = \frac{\varepsilon}{2} < \varepsilon.$$

Since ε and x are arbitrary, this shows that the nonnegative orthant of ℓ_2 has an empty interior.

□

The implication of lemma 1 is that the Separating Hyperplane theorem, which stipulates one of the convex subsets to have a nonempty interior, fails to apply to the L_p spaces. This is so since the discussion from section 2 illustrates that without a norm interior the hyperplane can be dense in L_p^+ and the resulting linear functional is discontinuous. If one insists to use L_p norm as a measure of openness in L_p spaces, the absence of interior points in these spaces seriously hinders the use of Hahn Banach theorem. Furthermore, the theoretical forces of arbitrage pricing which hinges on the existence of a continuous state price functional is heavily discounted.

To appreciate the source of nonexistence problem, it is useful to recapitulate the pricing analysis where existence is not a problem. This occurs in a finite dimensional Euclidean space where an interior point in $\mathbb{R}_+^n$ is guaranteed (Debreu, 1959, p.14). Harrison and Pliska (1981) explicitly consider an economy with finite number of terminal states. Contingent claims payoffs are defined on $\mathbb{R}_+^n$ and these payoffs can be replicated by marketed securities with payoff defined on a subspace of $\mathbb{R}^n$.

The no-arbitrage restriction in this finite setting can then be translated as a requirement that the subspace has empty intersection with the positive orthant except at the origin (Harrison and Pliska, 1981, theorem 2.7). Therefore this provides a necessary condition that satisfies the Separation theorem, and the existence argument can go through. The required separation however fails in infinite L_p spaces ($1 \leq p < \infty$) since the nonempty interior for L_p^+ is missing. It follows that the subspace and L_p^+ are not disjoined, and one is unable to push the existence argument through this case.

As noted in the above lemma, of all L_p spaces, only the positive orthant of L_∞ contains a nonempty interior which suggests separating hyperplane theorem can be applied. Unfortunately, the use of L_∞ as a state space setting for asset valuation leads to another dilemma. The norm dual of L_∞ is larger than L_1 and containing functionals that have no economic meaning. This observation is first pointed out by Radner (1967), extensively developed by Bewley (1972) and recently emphasized by Back and Pliska (1991).

5. Topological vector space approach to valuation

Granted that the Hahn Banach theorem is the pivotal step in obtaining a continuous price functional, the absence of L_p-norm interior becomes a stumbling block to extending linear price functions from the subspace to the entire state space. As the vector space is a natural setting for modeling price-taking behavior, (rather than abandoning the linear framework) a better way to tackle the problem is to look for other definitions of interior in general linear spaces.

Mathematically, this entails introducing a topology weaker than the L_p norm topology to the linear space. The study of general topology is a vast subject in the mathematics literature. General references that are constantly adhered in working out the relevant materials below are from Royden (1968), Berge (1963) and Robertson and Robertson (1973).

Our ultimate goal is to incorporate a class of topological vector spaces called the locally convex space (LCS) into the valuation analysis. As will be shown shortly, LCS includes some features akin to L_p spaces. Its advantage over other linear topological spaces lies in its ability to square up some problems that arise in applying the Hahn Banach separation theorem in infinite dimensional L_p spaces. In particular, we show that there exists a whole spectrum of locally convex weak topologies by the Mackey-Aren theorem. Each of these topologies presents a meaningful topological interior satisfying the requirement for deriving a closed separating hyperplane. The existence of such a wide variety of topologies then places arbitrage pricing in the general linear spaces on robust ground.

An additional benefit of using a locally convex linear topological space is that these spaces embody a lot of structures that are expressible in terms of convex cones and dual cones. The duality of convex cones has already shown its immensely useful geometric insights given in our derivation of the state price functional in chapter one. Even though presenting geometry is nearly impossible in an infinite dimensional scenario, the basic idea of separation theorem between the finite state space and the infinite state space is not too remotely disconnected.

Let X be a nonempty set. A collection τ of subsets of X is said to be a *topology* on X if the following holds:

(i) The empty set $\emptyset$ and the set X itself belongs to τ.

(ii) If τ_1 and τ_2 are members of τ, then the intersection $\tau_1 \cap \tau_2$ belongs to τ.

(iii) If $\{\tau_\lambda\}_{\lambda \in \Lambda}$ is an arbitrary collection of members of τ, then the union $\cup_{\lambda \in \Lambda}$ belongs to τ.

The pair (X,τ) is called a *topological space* and the members are called the *open sets* in X. Complements of open sets are called the *closed sets.*

A given set X can have more than one topology. Comparison of alternative topologies τ and τ' on a set X can be attained by set inclusion. If $\tau < \tau'$, so that every open set under τ is an open set under τ', then τ is said to be *coarser* then τ'. Equivalently τ' is *finer* than τ in the sense that the former contains more open sets.

The most frequently employed topological concepts are the neighborhood base and the Hausdorff topology. A *neighborhood* of a point x is an open set containing x. Denote $\mathcal{U}(x)$ as the collection of all neighborhoods of x. An important class of open sets that separate elements in X are defined by a Hausdorff topology. Formally X is a *Hausdorff space* and τ is a *Hausdorff topology* if for two arbitrary distinct points, $x,y \in X$, there exists neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

A subcollection $\mathcal{U}^*(x)$ of $\mathcal{U}(x)$ is called a *fundamental neighborhood system* of x if it satisfies the following

for any $U \in \mathcal{U}(x)$*, there exists* $V \in \mathcal{U}^*(x)$ *such that* $V \subset U$.

X is said to satisfy the *first axiom of countability* if for each $x \in X$, there exists a fundamental neighborhood system of x which has countably many members.

A family of open sets in X is called an *open base* for X if every open set can be expressed as a union of members of this family. X is said to satisfy the *second axiom of countability* if there exists an open base for X which has countably many members. X with a countable open base is *separable.*

The primary reason to consider different topological space is that one can introduce weaker topologies than the norm induced topology for a normed space and its dual space of linear functionals. Formally,

Definition: A topological vector space is a linear space L with a topology such that

(i) the single valued mapping f of $L \times L$ into L given by $f(x,y) = x+y$ is continuous; in other words, for each neighborhood $V(x_0+y_0)$, there exists neighborhoods $U_1(x_0)$ and $U_2(y_0)$ so that

$$x \in U_1(x_0),\ y \in U_2(y_0) \quad \text{implies} \quad x+y \in V(x_0+y_0).$$

(ii) the single valued mapping g of $\mathbb{R}\times L$ into L given by $g(\lambda,x) = \lambda x$ is continuous; in other words, for each neighborhood $V(\lambda_0,x_0)$, there exists a number η and a neighborhood $U(x_0)$ such that

$$|\lambda-\lambda_0| \le \eta,\ x \in U(x_0) \quad \text{implies} \quad \lambda x \in U(\lambda_0,x_0).$$

Behind the above definition is the following intuition: any topology τ which makes both algebraic operations f and g continuous is called a linear topology. τ is *translation invariant* in the sense that a subset $G \subset L$ is open if and only if the translate $x+G$ is open for every $x \in L$. It conveys the idea that one can characterize a linear topology in L in terms of a basis at any point in L. More precisely, if a convenient choice of a local base at 0 for L is made, then a local base at x is defined by translation

$$\beta_x = \{B_x \subset L \,|\, B_x = x+B_0, \quad B_0 \subset \beta_0\}.$$

Two important examples of a linear topology are given respectively. First, a normed space L is a topological vector space and the open balls induced by its norm

$$\tau_S = \{x \in L \,|\, \|x\| < \varepsilon\} \qquad \text{for } x \in L \quad \text{and} \quad \varepsilon > 0.$$

form a local base. τ_S can then be called a linear topology (or sometimes strong topology).

Second, let L be a normed space and L′ be its dual formed by a set of continuous linear functionals on L. Let Φ be a finite subset of L′. Given $\varepsilon > 0$, define

$$N_\varepsilon^\Phi = \{x | x \in L, \ |f(x)| \leq 1 \quad \forall f \in \Phi\}.$$

One can verify as Φ and ε vary, the sets of the form

$$N_\varepsilon^\Phi(x) = x+N_\varepsilon^\Phi$$

give rise to a fundamental base of neighborhoods for a topology in L, called the weak topology of L, denoted by τ_W. L together with the weak topology is a *topological vector space*.

An important class of topological vector spaces is called the *locally convex spaces*. In this case, every open set containing zero contains a convex open set containing 0. We shall begin to verify the two previously looked at topological vector spaces as locally convex spaces and then consider more general cases.

Lemma 1. A normed space L with its strong topology τ_S is a locally convex space.

Proof. The fundamental base of neighborhood is given by the form

$$B_\varepsilon(0) = \{x | \, \|x\| \leq \varepsilon\} \qquad \text{for } \varepsilon > 0.$$

Now, consider two points x,y such that

$$x \in B_\varepsilon(0), \quad y \in B_\varepsilon(0) \qquad \text{implies} \quad \|x\| \leq \varepsilon, \quad \|y\| \leq \varepsilon.$$

The convex combination of x and y is normed by

$$\|\lambda x+(1-\lambda)y\| \le \lambda\varepsilon+(1-\lambda)\varepsilon$$
$$= \varepsilon \qquad \text{where } 0 < \lambda < 1.$$

Hence $\lambda x+(1-\lambda)y \in B_\varepsilon(0)$ which verifies the neighborhood $B_\varepsilon(0)$ is therefore convex.

□

Lemma 2. A normed space L with its weak topology τ_W is a locally convex space.

Proof. Consider the fundamental base of neighborhoods:

$$N_\varepsilon^\Phi = \{x \,|\, |f(x)| \le \varepsilon, \quad \forall\, f \in \Phi\}$$

where Φ is a finite subset of the dual L' and $\varepsilon > 0$. The set N_ε^Φ is convex since it is the intersection of closed halfspaces.

□

A generalization of the previous two results is possible by introducing the concept of a seminorm.

Definition: A seminorm on a vector space L is a real-valued map $\rho: L \to [0,\infty)$ such that

$$\rho(x+y) \le \rho(x)+\rho(y) \quad \forall\, x,y \in L \quad \text{and}$$

$$\rho(ax) = |a|\cdot\rho(x) \quad \forall\, a \in \mathbb{R} \text{ and } x \in L.$$

A seminorm is a norm if $\rho(x) = 0$ implies $x = 0$.

Definition: A linear topology is locally convex if it contains a basis whose elements are open convex sets containing zero. The resulting topological vector space is called a locally convex space.

The connection between seminorms and locally convex spaces is given by the following theorem.

Theorem 1: To each seminorm ρ on a vector space L, there is a coarsest topology τ on L compatible with the algebraic structure. Under τ, L is a locally convex space.

Ignoring the proof (which is given in Robertson and Robertson (1973, p.15)), the statement of the theorem points out clearly that in a locally convex topological vector space, the topology is given by a family of seminorms. In proving the Hahn Banach extension theorem earlier, the Minkowski functional is introduced. The defining properties of the convex Minkowski functional constitutes a useful example of a seminorm.

Unraveled in this fashion, Hahn Banach theorem is a topological statement since for a continuous linear functional, one is able to uncover a linear topology for the given vector space. It follows that a seminorm induced topology can be substituted for the strong norm topology in the event that the latter fails to have an interior necessary for establishing a separating hyperplane.

Rather than presenting the correspondence between a seminorm and a topology τ as stated above, we shall use this result as the next stepping stone to motivate a more encompassing theorem, known as the Mackey-Aren theorem. The latter result identifies all seminorm induced locally convex topologies that are sufficient to derive a continuous linear functional in the dual space of L. Some definitions are in order.

Definition: A dual system $\langle L,L'\rangle$ is a pair of vector spaces L and L' together with a bilinear function $(x,x') \to \langle x,x'\rangle$ from $L \times L'$ into $\mathbb{R}$ satisfying two properties.

(i) if $\langle x,x'\rangle = 0 \quad \forall\, x' \in L'$ then $x = 0$, and

(ii) if $\langle x,x'\rangle = 0 \quad \forall\, x \in L$ then $x' = 0$.

Definition: A locally convex topology τ on L is said to be compatible with the dual system $\langle L,L'\rangle$ whenever $(L,\tau)' = L'$ holds. Equivalently τ is a compatible topology whenever there exists a linear functional $f:L \to \mathbb{R}$ belonging to the topological dual of (L,τ) if and only if there exists exactly one $x' \in L'$ such that

$$f(x) = \langle x,x'\rangle \qquad \text{holds for each } x \in L.$$

Two locally convex topologies that satisfy the above definitions for dual pair are the weak topology and the Mackey topology.

Definition: Let (L,L') be a dual pair. To each $x' \in L'$ corresponds a seminorm ρ on L given by

$$\rho_W(x) = |\langle x,x'\rangle|.$$

The coarsest topology on L making this seminorm continuous is the weak topology on L' and is denoted by $\sigma(L,L')$.

Earlier on it is shown that the collection of the sets $\{x | \rho(x) < \varepsilon\}$ forms a neighborhood base around zero and these bases topologize the vector space L. It is of interest to inquire whether there exists other seminorms topologizing L in a similar fashion. The next two definitions and the lemma immediately after makes one step towards addressing this inquiry.

Definition: For each $\sigma(L',L)$-compact convex subset C of L', consider the seminorm on L given by

$$\rho_C(x) = \sup_{x' \in C} |\langle x,x'\rangle| .$$

Definition: Let (L,L') be a dual pair. The Mackey topology on L denoted by $\xi(L,L')$ is the topology of uniform convergence on $\sigma(L',L)$-compact convex subsets of L'. That is

$$x_\alpha \xrightarrow[\xi(L,L')]{} x \qquad \text{if and only if} \qquad x'(x_\alpha) \longrightarrow x'(x)$$

uniformly as x' runs through any fixed $\sigma(L',L)$-compact convex subset of L'.

Lemma 3: $\xi(L,L')$ is a dual topology.

Proof. The $\{\rho_C\}$ as C varies over all $\sigma(L',L)$-compact, absolutely convex sets of L' generating the $\xi(L,L')$-topology. Consider $C \subset L' \subset L^\sim$ where $L^\sim$ is the algebraic dual of L. Since the restriction of $\sigma(L^\sim,L)$ to L' is $\sigma(L',L)$, C is $\sigma(L^\sim,L)$-compact and so $\sigma(L^\sim,L)$-closed in $L^\sim$. From the bipolar theorem (introduced in the appendix), $(C^o)^o_{L^\sim} = C$. But the polar of the convex sets are given by $C^o = \{x \mid |\rho_C(x)| \le 1\}$. The family of C^o

$$\{C^o \mid C \text{ is convex, balanced } \sigma(L',L)\text{-compact subset of } L'\}$$

forms a neighborhood base at $0 \in L'$ for the Mackey topology $\xi(L,L')$. Therefore

$$L_{\xi}^{\sim} = \cup_C (C^o)^o_{L^{\sim}} = \cup_C C = L'.$$

□

As stated at the beginning of this section, to search for a robust aspect of arbitrage valuation in infinite dimensional linear spaces is equivalent to look for a general result that can establish the existence of a separating hyperplane. The following fundamental duality theorem meets this objective.

Mackey-Aren theorem: Let (L,L') be a dual pair. A locally convex topology τ on L is a dual topology if and only if

$$\sigma(L,L') \subseteq \tau \subseteq \xi(L,L').$$

Proof. See the appendix.

□

A crucial message of the Mackey-Aren theorem states that there exists a spectrum of linear topologies ranged from the weak topology to the Mackey topologies such that L under τ is precisely L'. All these topologies are linear, Hausdorff and locally convex. Furthermore, to every τ corresponds a continuous and finite seminorm (convex functional) so that the prerequisites for applying the extension and the separation forms of Hahn-Banach theorem are implied by these inclusive topologies. This is so since the Mackey-Aren theorem has established a well-defined τ-interior for one of the disjoint convex subsets of L. Consequently, a nontrivial continuous linear functional is warranted to exist in the topological dual L' The next section illustrates how this topological result fits into an arbitrage valuation framework.

6. Arbitrage valuation in a locally convex space

Duality pairing via the locally convex spaces appears sparsely in economics literature. Two classical papers by Debreu (1954) and Bewley (1972) respectively make implicit and explicit appeal to Mackey-Aren theorem to extend the welfare theorem of Walrasian equilibrium to infinite dimensional vector spaces. More recently, Magill (1981) also exploits locally convex spaces to study infinite horizon programs in growth theory.

As noted earlier, the application of the Hahn Banach theorem to finance and asset pricing is found in a terse analysis by Ross' (1978, appendix). Ross' paper is motivated by the prevailing use of a Brownian motion in financial valuation theory concerning options pricing. A Brownian motion is a continuous-time stochastic process that satisfies the defining property of an element in an infinite dimensional linear space. More precisely, both the time and state on which the Brownian motion defined fall into a continuum.

Ross introduces an abstract linear space and a subspace of marketed securities. His problem is therefore reduced to finding a closed hyperplane that separates the linear subspaces and the positive orthant. A fundamental assumption in that development is the absence of arbitrage opportunity in the subspace of marketed securities, which then implies the existence of a linear functional defined over that subspace. This part of Ross' argument overlaps the finite state space model developed in chapter 1.

The departure of the two models begins when Ross assumes a topological interior for the positive orthant. This assumption however hardly leads to a direct derivation of a continuous price functional defined for the entire space for two reasons. <u>First</u>, the subspaces of a general topological space are not automatically closed. This forces us to consider the closure argument and the separating hyperplane theorem has to be applied in a roundabout fashion. Note that, if the marketed subspace is assumed to be closed, the exercise is enormously simplified. In this case, the subspace is the desired closed hyperplane.

A <u>second</u> problem arises from the fact that the extension part of the Hahn Banach theorem requires the separability of the underlying linear space. This poses some difficulty when the linear space is L_∞ which is inseparable (Aliprantis and Burkinshaw, 1981, p.212). The rest of this section endeavors to resolve these two problems using results from linear topological spaces developed in the last section.

Let X be a topological vector space. A *convex cone* C is a convex subset of X such that

$$x \in C \quad \text{implies} \quad \lambda x \in C \qquad \text{for any scalar } \lambda > 0.$$

let A and B be convex cones in X.

Definition: A continuous linear functional $f{:}X \to \mathbb{R}$ separates A from B if

$$f(x) \geq f(0) \quad \forall\ x \in A, \qquad f(x) \leq f(0) \quad \forall\ x \in B.$$

The function f strictly separates A from B if

$$f(x) > f(0) \quad \forall\ x \in A, \qquad f(x) \leq f(0) \quad \forall\ x \in B.$$

The following basic result records a relationship between a linear functional and linear subspaces of X.

Lemma 1: Let f be a nonzero linear functional on X. Then the hyperplane $H = \{x | f(x) = c\}$ is closed for every c if and only if f is continuous.

Proof. It suffices to show the argument by letting X be a normed space. Suppose f is continuous. Let $\{x_n\}$ be a sequence from H convergent to $x \in X$. Then $c = f(x_n) \to f$ and thus $x \in H$ and H is closed. Conversely, assume that $M = \{x | f(x) = 0\}$ is closed. Let $x = x_0 + M$ and suppose $x_n \to x$ in X. Then

$$x_n = \alpha_n x_0 + m_n, \qquad x = \alpha x_0 + m.$$

Let d denote the distance of x_0 from M, we have

$$|\alpha_n - \alpha| d \leq \|x_n - x\| \to 0$$

and hence $\alpha_n \to \alpha$. Also

$$\begin{aligned} f(x_n) &= \alpha_n f(x_0) + f(m_n) \\ &= \alpha_n f(x_0) \qquad \text{(since } f(m_n) = 0) \\ &\longrightarrow \alpha f(x_0) = f(x). \end{aligned}$$

Thus f is continuous on X.

□

The above lemma can be directly applied to the market model. If M is defined as a linear subspace, then if M is closed, any linear price functional from M can be extended to X. The next results streamline the nice property about linear subspaces of finite dimensional X.

Theorem 1 (a) Every finite dimensional subspace of a linear topological space is closed. **(b)** Every linear functional on a finite-dimensional linear topological space is continuous.

Proof. See Day (1973, p.15).

□

It follows from theorem 1 that in a finite security market model, absence of arbitrage opportunity is sufficient to derive a continuous linear price functional defined on the entire X.

In a general infinite dimensional topological vector space, linear subspaces are not necessarily closed. Consider the space ℓ_2 with infinite sequences. Let Y be the subspace of ℓ_2 containing vectors that possesses only a finite number of nonzero components, i.e.

$$y = (a_1, a_2, \ldots, a_k, 0, \ldots).$$

To show that Y is not closed, consider the sequence $\{y_n\}$ in Y defined by $y_1 = (1,0,..)$, $y_2 = (1,\frac{1}{2},0,\ldots)$, $y_3 = (1,\frac{1}{2},\frac{1}{3},0,..)$, ... We claim that the limit of this sequence is the vector $(1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{k},\ldots)$. First, observe that $\|x\| = \Sigma_{n=1}^{\infty}(\frac{1}{n^2}) < \infty$, hence $x \in \ell_2$. Next,

$$\begin{aligned}\|x-y^n\| &= \|(0,\ldots,0,\frac{1}{n+1},\frac{1}{n+2},\ldots)\| \\ &= (\sum_{k=n+1}^{\infty} \frac{1}{k^2})^{1/2}\end{aligned}$$

tends to zero as $n \to \infty$. However all components of y are not zero, showing the limit of the sequence $\{y^k\}$ is not in Y. It follows that Y is not a closed subspace.

In the light of this daunting example, the development of the next general separation theorem relies on using closures of linear subspaces. Denote the closures of convex sets A and B by $\bar{A}$ and $\bar{B}$.

Proposition 1: Suppose A and B are nonempty, disjoint convex cones in a locally convex topological vector space X. Then there exists a nonzero, continuous linear functional

$$F:X \to \mathbb{R} \quad \text{separating A from B} \qquad \text{if } \overline{(B-A)} \neq A.$$

Proof. Consider two nonempty disjoint convex sets $\tilde{A}$ and $\tilde{B}$. Assume zero is an interior point of either one of them. Otherwise by translation,

$$A = \tilde{A}-x_0 = \{z | z = x-x_0, \ x \in \tilde{A}\},$$

$$B = \tilde{B}-x_0 = \{z | z = y-x_0, \ y \in \tilde{B}\}.$$

By hypothesis, $\overline{(B-A)} \neq A$, there exists $v_0 \notin \overline{(B-A)}$. W.l.o.g. let v_0 belong to A. Therefore the set

$$K = B-A+v_0$$

is convex with 0 as its interior and $v_0 \notin K$. Since X is a locally convex space, there is a convex neighborhood of v_0, $N(v_0)$ such that

$$N(v_0) \cap \overline{(B-A)} = \emptyset.$$

Let C be the convex cone generated by $N(v_0)$ such that

$$C = \{x \in X | x = \lambda y \quad \text{for some } \lambda > 0 \ \text{ and } \ y \in N(v_0)\}.$$

The rest of the proof is to construct a linear subspace and define a linear functional in that subspace. To this end, let E be a flat subset of C which contains no interior point of K and let C_0 be the smallest linear subspace of C containing E. Then E is a hyperplane in C_0 with $E = \{x | f(x) = 1\}$. Denote the Minkowski seminorm of K by ρ_K. Since E contains no point of interior of K, we have

$$f(x) = 1 \le \rho_K(x) \qquad \forall\ x \in E.$$

By homogeneity,

$$f(tx) \le \rho_K(tx) \qquad \text{if } x \in E \ \text{ and } \ t > 0;$$
$$f(tx) \le 0 \le \rho_K(tx) \qquad \text{if } t \le 0.$$

Hence f is dominated by ρ_K. By Hahn Banach extension theorem proven in section 3, there is an extension F of f from E to X with $F(x) \le \rho_K(x)$. Let

$$H = \{x | F(x) = 1\}.$$

Continuity of F comes from the continuity of the semi-norm ρ_K. This implies H is closed.

□

The next corollary shows that the hyperplane H separates the sets A and B.

Corollary: Let A and B be disjoint convex sets as above. Then there is a closed hyperplane H separating A and B.

Proof. From the above separation, F is continuous so that

$$F(x) \le 0 \qquad \text{for } x \in K' = B-A.$$

This implies

$$\text{for } x_1 \in B \text{ and } x_2 \in A, \qquad F(x_1) \le F(x_2).$$

We can therefore find a real number c such that

$$\sup_{x_1 \in B} F(x) \le c \le \inf_{x_2 \in A} F(x).$$

The separating hyperplane is identified to be

$$H = \{x | F(x) = c\}.$$

□

6.1. *Interpretation of the separation theorem*

Two aspects of the above separation theorem require comment; one is technical whereas the other concerns the economic interpretation. Throughout the above proof we incorporate an idea very similar in spirit to the well known theorem of minimum norm in Hilbert space (see Luenberger p.118). In a general linear topological space, the convex set K with zero as its interior point has its Minkowski functional $\rho_K(x) = \inf\{r | \frac{x}{r} \in K,\ r > 0\}$ defines a kind of distance from the origin.

If distance is given by an L_p norm one can then identify $\rho_K(x)$ as a distance measure for a unit sphere. However since K is arbitrary, especially including convex sets that have no L_p norm interior, $\rho_K(x)$ represents a weaker but more robust notion of distance from the origin. This robustness of $\rho_K(x)$ is reflected by the fact that implied by $\rho_K(x)$ is a family of locally convex topology ranged from weak topology to Mackey topology. The role of Mackey-Aren theorem is to transform the earlier imprecise extension theorem into a precise one with an identifiable family of locally convex topologies.

Next, a subtle economic reasoning of the separation theorem is crucially embodied in the restriction $(B-\overline{A}) \neq A$. In terms of the market model with a linear subspace M representing portfolio of traded securities, both B and A are subsets of M. On the one hand, the set B consists of elements that are portfolio combination such that current cost is nonpositive whereas the future payoff is nonnegative. Therefore B is the feasible subset of M that has the arbitrage opportunity. On the other hand, the set A represent subset of M that are portfolio of securities with positive payoff and command positive initial cost. The objective of applying the Hahn Banach theorem is to separate the set with arbitrage opportunities from the set that is free of arbitrage profits by a linear price functional.

Note that the standard presumption of absence of arbitrage opportunity is not sufficient enough for extension in a general setting. This is due to the earlier mentioned phenomenon that in an infinite dimensional vector space setting, linear subspaces are not closed. Kreps (1981) has characterized an approximate arbitrage opportunity called the free lunch in the following manner.

Definition: A free lunch is a sequence $\{(m_n, x_n)\}$ in $M \times X_+$ satisfying

(i) $m_n \geq x_n$;

(ii) x_n converges to some nonzero $k \in X_+$, and

(iii) $\liminf f(m_n) \leq 0$.

The closure of the set of arbitrage portfolio represented by $\overline{(B-A)}$ capture the essence of Kreps' notion of free lunch. To obtain a meaningful separation theorem for valuation, it is therefore necessary to rule out such asymptotic arbitrage opportunity. This is given by $\overline{(B-A)} \neq A$ in the theorem. Define $v_0 \in M$ as a sure payoff in the market model with a value of one. Then the absence of free lunch can be equivalently stated as

$$N(v_0) \cap \overline{(B-A)} = \emptyset,$$

where $N(v_0)$ represents a convex neighborhood of v_0.

A stronger condition is often invoked to substitute $\overline{(B-A)} \neq A$, namely, the marketed subspace M is closed. In this case, absence of arbitrage opportunity in M is equivalent to having M as the closed hyperplane. Any linear function defined on M is continuous and can be extended to the entire linear space. Translated into economic language, the assumption of M being closed is equivalent to assuming that the security market as being complete. This M is effectively reduced to be a linear span of the space X.

6.2. Application of the separation to valuation

Two valuation models are reviewed in this subsection. The objective here is to consider how the general separation theorem developed above can fit into these existing models. The first model is developed by Hansen and Richard (1987). While it is a generic market model, Hansen and Richard's framework are strongly colored by two features. The linear space of payoff is modeled by an infinite dimensional space $X = L_2(\Omega,\mathcal{F},P)$ with the mean-square norm given by

$$\|x\| = (\int_{\Omega} x(\omega)^2 dP)^{1/2}.$$

Hansen and Richard assume the subspace M with marketed securities has no arbitrage opportunities. Furthermore, M is assumed to be a closed subspace in the sense that for any sequence $\{m_n\}$ in M such that $m_n \to m$, it follows that $m \in M$. The resulting linear functional extended from the subspace M to X is essentially an application of a separating hyperplane theorem (see Duffie, 1992, p.227).

As mentioned above, assuming M is closed is equivalent to making the strong assumption that the security market is complete. Following the more general approach developed here, $X = L_2(\Omega,\mathcal{F},P)$ is treated as a topological vector space. The topology is induced by the open neighborhood of seminorm convex functionals. More specifically, let τ be a locally convex topology that is compatible with the dual system $\langle L_2, L_2\rangle$. That is $(L_2,\tau)' = L_2'$ where L_2' denote the topological dual of L_2. Suppose M is the marketed subspace on which is defined a linear price functional f. Let A and B be subsets of M and $\overline{(B-A)} \neq A$. Then by the general separation theorem, we obtain a linear extended price functional F defined on X.

A second application of valuation in locally convex space is to pricing of contingent claims in a space of bounded functions, L_∞. Of all the L_p spaces, L_∞ is the only one that has a nonempty norm interior; hence applying Hahn Banach separation theorem does not seem to pose any problem in this case. Unfortunately valuation in L_∞ is confounded by the fact that the norm dual of L_∞ is not L_1, but a space of bounded additive linear set functions. From the classic theorem of Yosida and Hewitt (1952), the linear functional is decomposed into a countably additive component and the finitely additive component. While the countably additive set functional is an element of L_1, the finitely additive set function has very little economic interpretation.

In a general equilibrium setting where the commodity space is chosen to be L_∞, Bewley (1972) introduces Mackey topology into his model and under that topology, the topological dual is L_1. More precisely, treating L_∞ as a locally convex space and by the Mackey Aren theorem discussed earlier, there exists a locally convex topology such that (L_∞, L_1) forms a dual pair. That topology is the Mackey topology. The same method applies to arbitrage valuation but in this case one needs to incorporate a mild assumption that investor's preference relation is upper semicontinuous with respect to the Mackey topology. All that said, we shall illustrate here how existence of a price functional can be resolved, retrieving most of the insights from Bewley (1972).

Denote the linear space by $L = L_\infty(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi(L,L')$ be the linear Mackey topology defined on L. As before, M (not the same M as denoted in $\xi(L,L')$) is the linear subspace where a finite number of securities are traded. Investors' preference is given by $\succsim$.

Definition: $\succsim$ is said to be convex and upper-semicontinuous in the sense that for each $x \in L_+^\infty$, $\{y \in L_\infty^+ | y \succsim x\}$ is convex, and $\{y$

$\in L_\infty^+ | y \succsim x\}$ is a closed subset of L_∞ in the τ_M.

Absence of arbitrage leads to the existence of a nonzero continuous linear function Ψ on L_∞ since the upper contour preference set is separated from the budget set by a closed hyperplane. The rest of the problem becomes exclusively an analysis of Ψ. For any $x \in L_\infty^+$, $\Psi(x)$ has the representation (Yosida and Hewitt, 1952, theorem 2.3)

$$\Psi(x) = \int X d\Psi_c + \int X d\Psi_p$$

where Ψ_c is a countably additive measure and Ψ_p is a nonnegative purely finitely additive measure. Also by Yosida and Hewitt (1952, Theorem 1.22), there exists sequence of measurable events $\mathcal{F}_i$, such that

$$\lim_{i\to\infty} \mathbb{P}(\mathcal{F}_i) = 0, \quad \lim_{i\to\infty} \Psi_c(\mathcal{F}_i) = 0 \quad \text{and} \quad \Psi_p(\mathcal{F}_i^{\,c}) = 0.$$

The interpretable content of $\mathcal{F}_i$ is that it is a distantly remote event with very low likelihood of occurrence and therefore is assigned a value insignificantly different from zero by Ψ_c. Its almost singular value is derived mainly from the finitely additive measure Ψ_p. The next result (the proof of which mimics that offered in Bewley, 1972) argues that if $\succsim$ is Mackey upper-semicontinuous, $\Psi_p = 0$.

Theorem. If $\succsim_i$ is Mackey upper semicontinuous, Ψ is a countably additive set function on Ω.

Proof. If Ψ is not countably additive on Ω, there exists an increasing sequence of sets $\mathcal{F}_k$, $\mathcal{F}_k \subset \Omega$, such that

$$\Psi(\mathcal{F}_k) < \Psi(\bigcup_{k=1}^{\infty} \mathcal{F}_k)-\varepsilon \qquad \forall\ k,\ 0 < \varepsilon < 1.$$

Let $E_k = \mathcal{F}_k \cup (\Omega \backslash \cup_k \mathcal{F}_k)$ (where "\" represents set subtraction) and $\Psi(\Omega) = 1$. Then $\cup_k E_k = \Omega$ and $\Psi(E_k) = 1-\varepsilon$ for all k.

Next, consider the measurable functions

$$x = \hat{x}+\varepsilon\chi_{\Omega} \qquad \text{and} \qquad x_k = x-2\chi_{\Omega\backslash E_k}.$$

We claim that $x \succ \hat{x}$ for sufficiently large k. This is so since by Alaoglus's theorem (Dunford and Schwartz, 1958, p.468), subsets of L_1 are Mackey compact under Mackey topology; hence

$$x_k \succ \hat{x} \quad \text{for large enough } k, \qquad \text{therefore} \quad \Psi(x_k) > \Psi(\hat{x}).$$

But

$$\begin{aligned}\Psi(x_k)-\Psi(\hat{x}) &= \varepsilon\Psi(\Omega)-2\Psi(\Omega\backslash E_k)\\ &= \varepsilon-2+2-2\varepsilon\\ &= -\varepsilon\\ &< 0.\end{aligned}$$

This yields a contradiction.

□

7. Conclusion

This chapter has developed a self-contained functional analysis of the arbitrage pricing model. A weak existence result for the linear extended price functional can be established by means of defining a Minkowski convex functional. However for squaring up the nonexistence problem in L_p spaces due to lack of norm interior, we are motivated to explore the linear topological spaces. The key to the existence of an extended linear price functional is based on the duality theorem in a locally convex linear topological space.

In a loose sense, the present analysis is an anologue to an infinite dimensional Farkas-Lemma, a result not known to the author. Such analogy aside, the present analysis has its own merit for it interprets a general commodity space as a space of function which then incidentally the customarily used Euclidean $\mathbb{R}^n$ as a special case. This in turn stresses the role played by Hahn Banach theorem in terms of relating a function space and its dual.

In this chapter absence of arbitrage can be identified as a economic force that induces the Hahn Banach theorem. However other economic presumption can also be made to invoke the same theorem. To head off a bit more in that direction, note that it is a paradigm in finance that investor maximizes their expected utility While a shadow price functional is derived in the present setting that says nothing much about specific investor's preference, it is natural to wonder whether one can tightly relate some familiar preference characteristics such as marginal rate of substitution to the linear price functional. That possibility is investigated in the next chapter. A more challenging objective in the next chapter, however, is to present an alternative solution approach, which again relies on applying the Hahn Banach theorem, to the pricing of contingent claims by absence of arbitrage in an infinite dimensional setting.

Appendix: Proof of Mackey Aren theorem

The discussion of Mackey Aren theorem is found in a number of advanced functional analysis texts, for instance, Robertson and Robertson (1973), Choquet (1969) and Narici/Bockenstein (1985). The material here follows from the more easily assessable proof of Reed and Simon (1980). One of the crucial concepts that derives the Mackey Aren theorem is that of polar sets.

Definition: Let $\langle L, L' \rangle$ be a dual pair and $A \subset L$. The polar of A, denoted by A°, is given by

$$\{f \in L' \mid |f(e)| \leq 1, \quad \forall\, e \in A\}.$$

An equivalent notation for A° is $A^{\circ}_{L'}$.

Some basic facts about A° are

(a) A° is convex, balanced and $\sigma(L,L')$ closed.

(b) If $A \subset B$, then $B^{\circ} \subset A^{\circ}$.

(c) If $\lambda \neq 0$, $(\lambda A)^{\circ} = |\lambda|^{-1} A^{\circ}$.

(d) $(\cup_{\alpha} A_{\alpha})^{\circ} = \cap_{\alpha} A_{\alpha}^{\circ}$.

Lemma A.1. (The bipolar theorem). Let L and L′ be a dual pair. Then using $\sigma(L,L')$-topology on L, we have

$$L^{\circ\circ} = \overline{\mathrm{ach}(L)}$$

where ach(L), the absolutely convex hull of L, is the smallest balanced convex set containing L. That is

$$\mathrm{ach}(L) = \{ \sum_{n=1}^{N} \alpha_n x_n \,|\, x_1, \ldots, x_n \in L, \ \sum_{n=1}^{N} |\alpha_n| = 1, \quad N = 1,2,..\}$$

and the closure is in the $\sigma(L,L')$ topology.

Proof. Let $L_C = \overline{\mathrm{ach}(L)}$. Clearly $L \subset L^{\circ\circ}$ and since $(L^\circ)^\circ$ is convex, balance, and $\sigma(L,L')$-closed, $L_C \subset (L^\circ)^\circ$. On the other hand, if $x \notin L_C$, we can find $f \in L'$ with $f(e) \le 1$ for $e \in L_C$ and $f(x) > 1$. Since L_C is balanced, $\sup_{e \in L_C} |f(e)| \le 1$, so $f \in L^\circ$. But then

$$|f(x)| > 1 \qquad \text{implies} \qquad x \notin L^{\circ\circ}.$$

$\square$

Lemma A.2. The Mackey topology is a dual topology.

Proof. This is done in the text.

$\square$

Lemma A.3. Let $U \subset L$ be a balance, convex neighborhood of 0 in some $\langle L,L' \rangle$ dual topology. Then $U_{L'}{}^\circ$ is a $\sigma(L',L)$-compact set in L'.

Proof. This is a restatement of the Banach-Alaoglu theorem.

$\square$

Lemma A.4. Every dual topology is weaker than the Mackey topology.

Proof. Let ρ be a seminorm on E in some given dual topology. We will shown that $\rho = \rho_C$ for some $\sigma(L,L')$-compact, convex subset, C, in L'. Let $U = \{x \mid |\rho(x)| \le 1\}$. Then U is balanced, convex and $\sigma(L,L')$-closed. Thus $(U^\circ)^\circ = U$ by the double polar theorem. Let $C = U^\circ \subset L'$. By Lemma A.3, C is $\sigma(L',L)$-compact and it is convex. By definition $(U^\circ)^\circ = \{x \mid |\rho_C(x)| \le 1\} = U$, so $\rho_C = \rho$.

□

Proof of Mackey Aren theorem: Since $\sigma(L,L')$ and $\xi(L,L')$ topologies are dual topologies (Lemma A.2) any τ in between is also a dual topology. By definition, $\sigma(L,L')$ is the weakest possible dual topology and by Lemma A.4, $\xi(L,L')$ is the strongest possible dual topology.

□

CHAPTER 3. VALUATION BY VIABILITY OF PRICE SYSTEM

This chapter probes deeper with the issue of price extension from a subspace of random variables to the entire space of contingent payoffs. While the analysis is still based on the topological method in the sense that extension of a continuous price functional is equivalent to finding a closed hyperplane the treatment of the existence problem here differs from the preference free approach in the last chapter. As discussed in the previous chapter, the general existence problem can be handled without referring to preference characteristics. In that framework, the correspondence between the absence of arbitrage and the existence of a continuous linear functional is confirmed since the pricing problem is then reduced to a reformulation of the Hahn Banach theorem in a locally convex space.

The analysis of arbitrage pricing problem is more far-reaching than merely motivating the existence of a price functional, however. Intuition suggests that pricing in economics should be ultimately related to optimization and equilibrium. In a simple finite setting with a linear state space, the equivalence among the absence of arbitrage, the optimal solution to an investor's portfolio choice problem, the existence of a linear price functional, the use of a risk neutral probability for asset pricing and the representation of the price functional by the marginal utility of an average agent can be shown by the fundamental theorem of arbitrage valuation (Dybvig and Ross, 1987 and Back and Pliska, 1991).

In the finance literature, the formalization of a price extension that takes into account of preference continuity with respect to a topology is due to an influential analysis by Harrison and Kreps (1979). One of the important results from these authors is a theorem about correspondence between the existence of a price functional and a risk-adjusted probability measure. This leads to further insights developed by Kreps (1982) and Duffie and Huang (1985) in terms of an interesting connection between static economic equilibrium and multiperiod dynamic economic equilibrium under uncertainty in a Walrasian model. To sum up briefly, these results are mainly consolidations of Arrow's insight (1953) about the role of security market in an optimal allocation of risk.

While the static-dynamic correspondence is an important theoretical achievement, a more fundamental contribution of Harrison and Kreps' paper is its application of the separating hyperplane argument to financial asset pricing problem. In particular they generalize the earlier arbitrage options pricing theories from Black and Scholes (1973) to Cox and Ross (1976) by developing a mathematical economics approach to these finance models. Implicit in these earlier models is an assertion about the existence of a continuous state price functional that, upon a probabilistic transformation, can be used to value random payoffs defined on an abstract infinite dimensional vector space.

Harrison and Kreps observe that the extension form and separation form of the general Hahn-Banach theorem yielding such price functional can be combined as a problem of finding a geometric separating hyperplane in their model. The idea is to assume absence of arbitrage opportunity in a linear subspace of marketed securities and then deduce a closed hyperplane that separates the subspace from the positive orthant.

As the analysis from the last chapter can testify, this separation is hardly straightforward in an infinite dimensional setting. The difficultly arises since, on the one hand, the L_p spaces are traditionally used to model state spaces for stochastic finance models. On the other hand these spaces in general suffer from a lack of norm interior in L_p^+ which is a basic requirement for the separation theorem to work.

Harrison and Kreps' attack on the problem is to invent a concept called the viability of the security price system. A price system is viable when it meets two criteria. First, no arbitrage opportunities in the marketed subspace implies the values of all portfolio combination of assets can be represented by a linear functional in that subspace. Second, given the subspace of securities and the price functionals, agents with a prespecified preference are able to solve their portfolio choice problem. The solution of the agent's optimization implies the linear price functional from the subspace can then be extended to the entire state space.

One of the sufficient conditions, as pointed out by Harrison and Kreps, satisfying the definition of viability is that agent's preference is representable by an expected utility functional. In the usual finite dimensional state space, the solution to the maximization of expected utility is both necessary and sufficient for the existence of a continuous state price functional as demonstrated by Rubinstein (1974). Via the solution to the expected utility maximization, the extended state price functional can be interpreted as the familiar Lagrange multiplier (Back, 1991 appendix).

Expected utility representation of preference is unduly restrictive since it calls for the existence of infinite moments of a random variable. Moreover the assumptions for preference to satisfy are subject to strong criticisms (Kasui and Schmeidler, 1991). Recent development by Duffie and Skiadas (1994) looks into two extensive classes of functional representations of preference. The first class, originally discussed by Constantinides (1989), is called the habit-formation preference. The second class, motivated by Duffie and Epstein (1992), is called the differential utility. These modifications lead to the more general non-expected utility functional that is developed to tackle the "equity premium puzzle" (Prescott and Mehra, 1985). The relation of these general utility functions to the extended state price is collectively expressed as the utility gradient approach to asset pricing (Duffie and Skiadas, 1994).

A more subtle interpretation of Harrison and Kreps notion of viability than stipulating preference to be representable by expected utility can also be offered. This view is more in line with the usual general equilibrium modeling. A preference relation is assumed to be transitive, convex, increasing and continuous with respect to a topology denoted by τ. The last topological assumption about preference is then combined with the linear price function from the marketed subspace to induce the theorem of separating hyperplane.

Note that the preference continuity especially plays a productive role for the existence of a closed hyperplane in the case where the linear space does not have an open interior in its positive orthant. One of the advantages of this approach over the utility gradient approach is that it neither asks for any specific functional form such as a quadratic utility function nor requires differentiability assumption.

However, there is a danger associated with the topological interpretation of viability. As an analogy to the discrete state space theory, given a τ continuous preference one would like to conclude that in an infinite dimensional function space the state price functional is represented by a continuous marginal rate of substitution function. This is unfortunately not always the case and two examples are used in this chapter to illustrate the possible source of the existence problem. We are therefore motivated to 'strengthen' the restrictions on preference so that the resulting marginal rate of substitution can be representable as a continuous price functional.

In an independent path-breaking paper on the general equilibrium analysis Mas-Colell (1986a) introduces an instrumental concept known as uniform proper preference. Other advances on general equilibrium problems utilizing the same concept is found in Richard and Zame (1986), Mas-Colell (1986b) and Aliprantis, Brown and Burkinshaw (1987). A relaxation for uniform properness to pointwise properness is found in Yannelis and Zame (1986), and Araujo and Monteiro (1989).

Primarily developed to deal with a Walrasian general equilibrium problem in an infinite dimensional commodity space, Mas-Colell's notion of uniform proper preference turns out to be an ideal candidate to handle the above arbitrage pricing problem in general state spaces as well. It will be shown below that a useful feature of bringing uniform preference into the model is that it leads to a bounded marginal rate of substitution, sufficient for the existence of a continuous price functional. A price system in which preference is uniformly proper is also consonant with Harrison and Kreps notion of viable price system.

Incorporating some of the tools from general equilibrium analysis for the arbitrage pricing has an additional payoff as it illuminates an underlying methodological issue. It brings closer the linkage between the arbitrage theory and the Walrasian equilibrium theory so that the two perfect foresight information models can be treated as complementary to each other. There is an on-going tradition in finance that for the purpose of valuing derivative securities one can derive the continuous pricing functional in the dual space without explicitly identifying the underlying equilibrium allocations. However, that same setting can be enriched if one is concerned with the issues regarding Pareto optimality of the model parameters since the same valuation framework can be readily expanded for such purpose.

This chapter unfolds as follows. Section 1 reexamines the original idea of pricing by viability. In this context, the continuity of preference relation plays an important role in deriving the separating hyperplane. Merely having preference continuity in an infinite dimensional setting does not necessarily lead to a price extension. In section 2 two examples are recollected from the general equilibrium literature to illustrate this unfortunate pathology. This motivates the introduction of the uniform proper preference due to Mas-Colell in section 3. The mathematical significance of uniform proper preference is that it can be well coordinated with most of the commonly used linear spaces in finance and most importantly it induces a nontrivial separating hyperplane.

In section 4, the canonical market model is retrieved and some basic feature of the market model can be derived quite independently of the preference characteristics. However, incorporation of the uniform proper preference is the main key to the existence of a continuous linear price functional in this model. In this market model, we define the state space of payoff as a topological vector lattice. Some of the characteristics of vector lattices are collected in the appendix. Finally, by further specializing the commodity space to be a Banach lattice, the Black-Scholes state price density is rediscovered in section 5.

1. On the traditional notion of price by viability

Part of the thesis from the last chapter unravels the fact that arbitrage pricing in a general linear space is a topological problem. This is so since the separation part of the Hahn Banach theorem entails one of the pair of disjoint convex sets to have a nonempty interior. Among most of the commonly used L_p spaces in finance, the above topological requirement presents difficulty for obtaining an extended linear price functional as the interior of positive orthant of these spaces is proven to be empty.

In their seminal paper, Harrison and Kreps (1979) and Kreps (1981) introduce a Separating Hyperplane argument by invoking an assumption about continuous preference defined on the positive orthant of L_p spaces. Attached to this methodology is a presumption that one can associate a security market model with a general equilibrium model. The resulting continuous price functional is also dubbed the arbitrage equilibrium price functional. Incidentally, the same connection between absence of arbitrage and the equilibrium of the security market is also foreshadowed in the original Black-Scholes paper (1973).

There are two principal components to the Harrison and Kreps pricing argument. First, a finite number of marketed securities are traded in a subspace of a given linear commodity space L. However, separation of the subspace that embodies arbitrage opportunity from the positive orthant is not possible if L is modeled by a L_p space as the interior under the L_p norm is empty. This problem is removed by regarding the commodity space as a topological space where investor's preference is specified. As a consequence of preference continuity, one is able to recreate a topological interior.

More specifically, the space is endowed with a Hausdorff, metrizable topology τ that is compatible with the L_p norm topology. An axiomatic specification of preference can then be introduced. Namely, a preference relation denoted by $\succsim$ is assumed to be

(i) reflexive, transitive and complete;

(ii) convex: the set $\{y \in L | y \succsim x\}$ is convex for every $x \in L$;

(iii) continuous: $\succsim$ is both upper- and lower-semicontinuous.

Upper-semicontinuous $\succsim$ implies the set $\{x \in L | x \succsim y\}$ are τ-closed for all $y \in L$; lower-semicontinuous $\succsim$ implies the set $\{x \in L | y \succsim x\}$ is τ-closed for all $y \in L$.

The next result follows immediately from the above characterization of $\succsim$. Let L be a topological space with a topology τ.

Theorem 1: For a preference relation $\succsim$ defined on L, the following are equivalent.

(a) The preference $\succsim$ is continuous.

(b) The preference $\succsim$ is closed in $L \times L$.

(c) If $x \succ y$ holds in L, then there exists disjoint neighborhoods U_x and U_y of x and y respectively such that $a \in U_x$ and $b \in U_y$ implying $a \succ b$.

Proof. (a) $\Rightarrow$ (c). Let $x \succ y$. We have two cases.

<u>Case I</u>: There exists some $z \in L$ such that $x \succ z \succ y$. In this case, the two neighborhoods

$$U_x = \{a \in L | a \succ z\} \quad \text{and} \quad U_y = \{b \in L | z \succ b\}$$

satisfying the desired properties.

<u>Case</u> <u>II</u>: There is no $z \in L$ satisfying $x \succ z \succ y$. In this case, take

$$U_x = \{a \in L \mid a \succ y\} \quad \text{and} \quad U_y = \{b \in L \mid x \succ b\}.$$

(c) $\Rightarrow$ (b). Let $\{(x_\alpha, y_\alpha)\}$ be a net of $\succsim$ satisfying $(x_\alpha, y_\alpha) \to (x,y)$ in $L \times L$. If $y \succ x$ holds, then there exists two neighborhoods U_x and U_y of x and y respectively, such that

$$a \in U_x \quad \text{and} \quad b \in U_y \qquad \text{imply} \quad b \succ a.$$

In particular, for all sufficiently large α, we must have $y_\alpha \succ x_\alpha$, contradiction. Hence $x \succsim y$ holds and so (x,y) belongs to $\succsim$. That is, $\succsim$ is a closed subset $L \times L$.

(b) $\Rightarrow$ (a). Let $\{y_\alpha\}$ be a net of $\{y \in L \mid y \succsim x\}$ satisfying $y_\alpha \to z$ in L. Then the net $\{(y_\alpha, x)\}$ of $\succsim$ satisfies $(y_\alpha, x) \to (z,x)$ in $L \times L$, we see that $(z,x) \in \succsim$. Thus $z \succsim x$ holds, proving that the set $\{y \in L \mid y \succsim x\}$ is a closed set.

In a similar fashion, we can show that the set $\{y \in L \mid x \succsim y\}$ is a closed set for each $x \in L$ and the proof is complete.

$\square$

The above result is an important building block to the application of the Separating Hyperplane theorem. Harrison and Kreps motivates the separation argument by introducing the concept of a viable price system. In words, a price system with a set of marketed securities is viable if agents with the above preference characteristics are able to form an optimal portfolio of securities. As an example if preferences are representable by a smooth expected utility function, viability is readily captured by the familiar first order condition of utility maximization.

More generally, the feasible sets of portfolio of securities and the set of preference relation are convex. In addition, the continuity of preference relation has induced a nonempty open neighborhood by the above theorem 1. Given this scenario, the Separating Hyperplane theorem can then be appealed to yield a linear functional which by continuity of $\succsim$ can be extended to the entire L. This is the basic logic behind the viability proposition of Harrison and Kreps (1979, theorem 1 p.386). In the next subsection, it is shown that because of the important role born by the preference relation, some further restrictions on $\succsim$ will be needed to ensure viability of the price system is a sufficient condition to generate a continuous price functional.

2. The insufficiency of preference continuity for valuation

As discussed in the last chapter, the derivation of the linear state price functional can be deduced without any need for preference characterization. However, with preference incorporated in deriving the arbitrage price functional does have a conceptual advantage. Given that a linear topology is chosen for the linear state space of payoff, the latter approach implies there exists investors' preference that is continuous with respect to the choice of the topology. It follows that the resulting price functional can be represented by the marginal rate of substitution of the investor. Further, such marginal utility representation of the state price functional can be suitably readjusted to yield a "risk-neutral" probability measure for valuing contingent claims.

Pushing the above reasoning one step down, it is tempting to offer the following conclusion. A continuous linear price functional implies the marginal rate of substitution is a continuous function; conversely a continuous preference relation likely yields a marginal rate of substitution that is a continuous linear functional in the dual (price) space. While the first implication may be valid by definition, the reverse implication can be found on a shaky ground if the linear space does not have any clear-cut interior. The following two examples illustrate the need for additional topological characterization for preference other than continuity.

Example 1 (Jones 1984)

Consider the commodity space $L = L' = \ell_2$ which is a space of square summable (infinite) sequence. L is endowed with the weak topology, that is $\tau = \sigma(\ell_2, \ell_2)$. There is only one consumer in this economy and his utility function is given by

$$U(x) = \sum_{t=1}^{\infty} u(x(t),t), \qquad \text{where} \quad u(x,t) = \frac{1}{t^2}(1-e^{-t^2x})$$

is called the felicity function. U(x) is continuous with respect to τ, which captures a good economic intuition. Brown and Lewis (1981) has shown that continuity of preference relation with respect to the weak topology $\sigma(L,L')$ is equivalent to assuming patience on the part of the economic agents in some intertemporal models.

Introduce the endowment bundle as $\omega(t) = \frac{1}{t^2}$. Cox, Ingersoll and Ross (1985) demonstrate that an equilibrium price is obtained if the single agent is induced to optimally choose his own endowment in the economy. Yet this resulting price is not continuous. To see the problem, note that the only price that clears the market is found by setting it equal to marginal utility evaluated at endowment. That is

$$\begin{aligned} p(t) &= u'(\omega(t),t) \\ &= e^{-1}. \end{aligned}$$

But the condition $[\sum_{t=1}^{\infty} (u'(\cdot),t)^2]^{1/2} < \infty$ is not satisfied since the above sequence is not square summable. Therefore, the resulting p(t) represented by a marginal utility function is linear but not continuous.

□

Example 2 (Mas-Colell, 1986)

The commodity space is $L = ca(K)$, where $K = Z_+ \cap \{\infty\}$ is the compactification of positive integers. This is a linear space of countably signed additive measures with the bounded variation norms. For $x \in L$ and $i \in K$, let $x_i = x(\{i\})$ and define a felicity function $u_i : [0,\infty) \to [0,\infty)$ by

$$u_i(t) = \begin{cases} 2^i t & \text{for } t \leq \frac{1}{2^{2i}} \\ \frac{1}{2^i} - \frac{1}{2^{2i}} + t & \text{for } t > \frac{1}{2^{2i}}. \end{cases}$$

The preference relation on L_+ is represented by a concave utility function $U(x) = \sum_{i=1}^{\infty} u_i(x_i)$ where $U(x)$ is continuous for the weak convergence for measure (i.e. weak* continuous). Introduce an endowment by

$$\omega_i = \frac{1}{2^{2i+1}} \quad \text{for } i < \infty \qquad \text{and} \qquad \omega_\infty = 1.$$

Within the relevant range where the endowment lies, the marginal utility is given by $u_i' = 2^i$. The infinite sum of the above sequence of marginal utilities is unbounded. The only value for the given endowment bundle in this one person economy is zero. To see this, let p be a nonzero positive linear functional. For any $x \geq 0$,

$$\omega + x \succsim \omega \qquad \text{hence} \qquad p \cdot x \geq 0.$$

For $i \in K$, define $p_i = p \cdot e_i$

$$\text{where} \quad e(\{j\}) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Assume $p \cdot \omega > 0$ and by equating the marginal rate of substitution to relative prices, we have

$$\frac{p_i}{p_1} = \frac{2^i}{2} = 2^{i-1} \qquad i < \infty.$$

Next create a nonnegative bundle as follows. Define $z \in L_+$ by $z_i = \frac{1}{p_i}$; and $z^n \in L_+$ by

$$z_i^n = \begin{cases} z_i & \text{if } i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$z-z^n \geq 0 \quad \text{implying} \quad p\cdot z^n \leq p\cdot z \qquad \forall\ n.$$

However,

$$\begin{aligned} p\cdot z^n &= \sum_{i=1}^{n} p_i z_i \\ &= \sum_{i=1}^{n} p_i \cdot \frac{1}{p_i} \\ &= n. \end{aligned}$$

For a sufficiently large n, $p\cdot z^n >> p\cdot z$. This is a contradiction, which can only be avoided when $p = 0$, i.e. $p\cdot\omega = 0$.

□

In the previous examples 1 and 2, preferences are representable by an increasing concave utility function. More specifically the utility function in the first example is differentiable in addition to being continuous whereas in the second example it is only continuous. However none of these continuous preference generates a nontrivial continuous linear price functional, since their corresponding marginal utilities are unbounded.

In principle, prices are measured by marginal utility. Given the underlying commodity spaces for these two examples are infinite dimensional linear spaces, the implication is that imposing continuity on preference alone does not place enough restriction on the resulting marginal utility to yield a continuous price functional. One is tempted to conjecture that in the dual valuation space, the set of continuous price functional is contained in larger et of functionals representable by marginal rate of substitution. The next section establishes more substance to this conjecture.

3. Uniform proper preference

In two seminal papers Mas-Colell (1986a, 1986b) introduces the concept of uniform proper preference to tackle existence problems for a wide class of general equilibrium models. These models share a number of similar characteristics. The underlying commodity spaces are infinite dimensional linear spaces, including the L_p spaces and ca(K) which is the space of countable additive signed measures on a compact metric space K. Moreover, all these linear spaces can be ordered so that they can be treated as vector lattices (also called Riesz spaces). An important generalization of vector lattices gives rise to the topological vector lattices.

In finance, the space of contingent payoff consists of elements that are random variables. As discussed in the previous chapter, these random variables with suitably defined norm are merely elements of L_p spaces. It is shown in the appendix that normed L_p spaces induce an important class of topological vector lattices known as Banach lattices. Given this environment modeled by lattices, uniform proper preference defines an open cone in the positive orthant. Two consequences of the induced openness from uniform properness will be derived in this section which serves as preliminaries to invoke the separation theorem in the next section.

Let L be a Riesz space and define τ to be a linear topology on L. Also, let $\succsim$ be a preference relation defined on L^+. That is, denote the better than or indifferent set by

$$P(x) = \{y \in L^+ | y \succsim x\}.$$

The following definitions of $\succsim$ are due to Mas-Colell (1986a).

Definition: The preference relation $\succsim$ is τ-proper at some point $x \in L^+$ if there exists some $v > 0$ and some τ-neighborhood V of zero such that

$$x-\alpha v+z \succsim x \quad \text{in } L^+ \quad \text{with} \quad \alpha > 0 \quad \text{implies} \quad z \notin \alpha V.$$

Definition: The preference relation $\succsim$ is uniformly τ-proper if there exists some $v > 0$ and some neighborhood V of zero such that for any arbitrary $x \in L^+$ satisfying

$$x-\alpha v+z \succsim x \quad \text{in } L^+ \quad \text{with} \quad \alpha > 0 \quad \text{implies} \quad z \notin \alpha V.$$

The requirement of the point $v > 0$ to exist in the above definitions may not be clearly justified in most economic models as noted by Yannelis and Zame (1986). However it is common in finance to assume the existence of a riskless asset relative to other risky assets in the state space. One can therefore interpret the point v as the return of a riskless asset. An immediate consequence of the definition of uniform proper preference is the following.

Theorem 1: Let τ be a locally convex topology on a Riesz space L and let $\succsim$ be a preference on L^+. Then $\succsim$ is uniformly τ-proper if and only if there exists a nonempty τ-open convex cone Γ such that

(a) $\Gamma \cap (-L^+) \neq \emptyset$, and

(b) $(x-\Gamma) \cap P(x) = \emptyset \quad \forall \; x \in L^+$.

Proof. Let $\succsim$ be uniformly τ-proper and suppose $v \geq 0$ be a vector of uniform properness corresponding to some open, convex, τ-neighborhood V of zero. We construct the τ-open convex cone as follows:

$$\Gamma = \{w \in L \mid \exists \; \alpha > 0 \text{ and } y \in V \quad \text{with} \quad w = \alpha(y-v)\}.$$

Since $-v \in \Gamma$, $\Gamma \cap (-L^+) \neq \emptyset$.

By the method of contradiction, assume that $(x-\Gamma) \cap P(x) \neq \emptyset$. Let $z \in (x-\Gamma) \cap P(x)$ and write

$$z = x+\alpha(y-v) = x-\alpha v+\alpha y \succsim x.$$

By uniform τ-properness of $\succsim$, $\alpha y \notin \alpha V$, which implies $y \notin V$. This is impossible since $y \in V$ and $y \notin V$ cannot hold simultaneously.

Conversely, let a non-empty τ-open convex cone that satisfies (a) and (b). Consider $w \in \Gamma \cap (-L^+)$ and some τ-neighborhood V of zero with $w+\Gamma \subseteq \Gamma$. Define $v = -w > 0$ and let

$$x-\alpha v+z \succsim x \quad \text{in } L^+ \quad \text{with } \alpha > 0.$$

Suppose $z \in \alpha V$, then $z = \alpha y$, for some $y \in V$ and so

$$\begin{aligned} x-\alpha v+z &= x+\alpha(y-v) \\ &= x+\alpha(w+y) \end{aligned}$$

is an element of $(x+\Gamma) \cap P(x)$. This violates the hypothesis that (b) holds. We therefore conclude that

$$x-\alpha v+z \succsim x \quad \text{in } L^+ \quad \text{with} \quad \alpha > 0 \quad \text{implies} \quad z \notin \alpha V.$$

□

The intuition behind theorem 1 can be expressed as follows. Proper preference has induced a τ-open convex cone at a given point $x \in L^+$ and restricted the Γ-cone to have an empty intersection with the better than set of x. This is part (b) of the theorem. "Uniformity" ensures that such property holds for every $x \in L^+$. That Γ is a τ-open convex cone forms a key argument to apply the separation Theorem subsequently. Part (a) of the theorem captures the property that the τ-neighborhood V is a topological base around the origin and V spans Γ.

The definition of a uniform proper preference also induces the following property regarding marginal rate of substitution. Let L^+ be a norm lattice. Then

$$x-\alpha v+z \succsim x \quad \text{implies} \quad \|z\| \geq \alpha\varepsilon.$$

This reflects the idea that the vector v is so much valued and will not be given up unless the compensating bundle z is of a certain size measured by the norm. A more familiar characterization of this aspect of uniform proper preference is that the marginal rate of substitution is bounded.

To make the above argument precise, we adopt a modified argument from Zame (1987, p.1087) who shows that a uniform proper production set leads to a bounded marginal rate of technical substitution. Let $\succsim$ arise from a continuously differentiable monotone utility function u and let $D_x u(y)$ denote the directional derivative of u at y in the direction x, so that

$$D_x u(y) = \lim_{t \to 0} \frac{1}{t}\left(u(y-tx)-u(y)\right).$$

Denote $v^* = \dfrac{v}{\|v\|}$ and $z^* = \dfrac{z}{\|z\|}$ as per unit of the commodity bundle v and z measured by their respective norms. The mean value theorem implies that

$$u(x-\alpha x+z) = u(x)+D_{-\alpha v+z}u(h) \qquad \text{where} \qquad h \in (x,x-\alpha v+z).$$

Since u is continuously differentiable, $D_x u(y)$ is linear in x. Therefore,

$$D_{-\alpha v+z} = -\alpha\|v\|D_{v^*}u(h)+\|z\|D_{z^*}u(h).$$

Trade will occur whenever

$$u(x-\alpha v+z)-u(x) > 0$$

$$\text{which implies} \qquad \frac{D_{v^*}u(h)}{D_{z^*}u(h)} < \frac{\|z\|}{\alpha\|v\|}.$$

In the above development, $\dfrac{D_{v^*}u(h)}{D_{z^*}u(h)}$ is the marginal rate of substitution between bundle v^* and z^*. Thus uniform properness preference has the implication that marginal rate of substitution is bounded by the quantity $\dfrac{\|z\|}{\alpha\|v\|}$. The next stage of the analysis is to incorporate this important preference feature into a security market model to derive a price extension.

3. The canonical market model

The present analysis retains most of the elements from the Harrison and Kreps framework (1979). As the current focus is on examining the concept of viability of a price system and its extension, most of the continuous time details of their model regarding information flows and dynamic trading strategies are stripped away for simplicity. While these details are crucial ingredients for a model of valuation under uncertainty, suitable extension of the current simple formulation can retrieve these continuous time insights.

For instance, a given linear commodity space can be induced by an underlying measure space $(\Omega,\mathcal{F},\mathbb{P})$ and $\mathcal{F}$ can be further partitioned into a family of increasing sub-sigma-algebra. Similarly, admissible trading strategies can be defined in a linear subspace with securities payoff that can be identified as square integrable random variables (see Harrison and Pliska, (1981) and Duffie and Huang (1985)) in order to avoid nontrivial continuous time arbitrage strategies.

Formally, let L be a vector lattice. Denote a subset of L by X which is given a locally convex, linear Hausdorff topology τ. A topology is Hausdorff if for any two elements x,y of a set X, there exists open neighborhoods for x,y which are denoted U_x and U_y and which are disjoint. Note that τ is compatible with the algebraic and a lattice structure of X. This means both the addition and scalar multiplication as well as the two order operations inf(x,y) and sup(x,y) are continuous functions with respect to τ. The resulting commodity space is therefore a topological vector lattice. Our focus is placed exclusively on a class of topological vector lattice called Banach lattice. Some relevant properties of a Banach lattice are reported in the appendix.

Economic activity only occurs at the two extremes of the time interval $[0,T]$. There is only one single good available for consumption. An element of X are interpreted as a state contingent commodity bundle. Agents in the economy are represented by their preferences for terminal consumptions. Each agent's preference is denoted by $\succsim$ and is assumed to satisfy the following conditions:

(i) continuous in τ: for all $x \in X$, the sets $\{x' \in X | x \succsim x'\}$ and $\{x' \in X | x' \succsim x\}$ are closed in τ;

(ii) convex: $x, x' \succsim x''$ and $\lambda \in [0,1]$ imply $\lambda x + (1-\lambda) x' \succsim x''$;

(iii) strictly monotonic: let $k \in X_+$ and $k \neq 0$, then $x+k \succ x \quad \forall \quad x \in X$;

(iv) uniformly proper: there exists some $v > 0$ and some neighborhood V of zero such that for any arbitrary $x \in X_+$ satisfying $x - \alpha v + z \succsim x$ in X_+ with $\alpha > 0$, we have $z \notin \alpha V$.

In a finite state space setting, conditions (i) - (iii) are sufficient for the existence of a continuous state price functional.

Agents are allowed to have terminal endowments $\bar{x} \in X$ but to simplify the setup, preferences on net trade bundle are instead defined as follows:

$$x \succsim y \quad \text{if} \quad x + \bar{x} \succsim_* y + \bar{x} \qquad \forall \quad x, y \in X$$

when x and y are net trades. In this way, $\succsim$ represents preference on net trades that is derivable from the more primitive preference given by $\succsim_*$.

Denote a subspace of X by M which represents the subset of terminal space of all attainable commodity bundles. Elements of M are denoted by m that can be obtained by a combination of existing marketed commodity bundles. More specifically there is a basis of bundles denoted by M_0 that spans elements in M. M_0 is called the marketed subspace where tradings do not incur any transactions costs.

In the parlance of Harrison and Kreps, M_0 is a subspace consisting of n+1 marketed long-lived securities indexed by $j = 0,1,\ldots,n$. Each of these securities is characterized by its terminal payoff denoted by d_j. One can interpret $d_j(\omega)$ as number of units of the single good entitled to the owner of one share of security j if state ω occurs. Also assume that security zero promises its owner one unit of consumption good regardless which states of the world occur at $t = 1$.

The initial value of each of the long-lived securities is defined by a functional $S_j : M_0 \rightarrow \mathbb{R}$. In vector notation, $S = (S_0, S_1, \ldots, S_n)^T$, where "T" stands for the transpose of a row vector. A trading strategy is (n+1)-dimensional vector denoted by $\theta = (\theta_0, \theta_1, \ldots, \theta_n)^T$. One can interpret θ_j as the number of shares of the j-th security held by an investor. Elements in M can be attained by agents via the initial tradings of portfolios of marketable securities.

Definition: A consumption plan $m \in M$ is attained if there exists a trading strategy θ such that $m = \theta^T d$, where $d = (d_0, \ldots, d_n)^T$.

All that developed so far is a set up for a two period model with infinite number of terminal states of the world. Agents are assumed to agree on the possibility of each state although their probability assessments of the states occurrence may vary. To expand the setting to a continuous time framework would entail a number of specifications such as defining S as a vector of stochastic processes. Furthermore, information flow in the model would have to be suitably restricted in order to motivate a reasonable class of dynamic trading strategy.

These developments are important in their own right especially for modeling multiperiod asset valuation (see Duffie 1988 and Dothan 1900). The present focus is however a more modest treatment of a price extension by viability in a two period model. An immediate issue to confront at the moment is to define a reasonable value for claims in M. The standard procedure is to impose the absence of arbitrage trading opportunities in M_0.

Definition: An arbitrage opportunity is a trading strategy θ such that $\theta^T S \leq 0$ and $\theta^T d \geq 0$ with $\theta^T d > 0$ for some states.

In words, an arbitrage opportunity is a trading strategy that gives rise to a nonnegative consumption plan with zero initial cost. The implication of the existence of an arbitrage opportunity is that an agent who prefers more to less will not find a solution to his portfolio problem. On the other hand, the absence of arbitrage opportunity allows us to assign a particular functional form for values of all attainable claims in M. Define $\pi : M \to \mathbb{R}$ as the value for any attainable $m \in M$.

Proposition 1: Given there is no arbitrage opportunity in M_0. Then π is a unique linear functional on M.

Proof. To show that π is unique, we use the method of contradiction. Assume $\pi(m)$ is nonunique for some $m \in M$. Consider π' and π'' and let $\pi' > \pi''$ such that

$$m = \theta'^{T}d \qquad \text{with initial cost} \qquad \pi' = \theta'^{T}S$$

$$m = \theta''^{T}d \qquad \text{with initial cost} \qquad \pi'' = \theta''^{T}S.$$

Next define a claim $\hat{m} \in M$ as follows:

$$\hat{m} = \left(\frac{\pi'-\pi''}{S_0}\right)d_0+(\theta''-\theta')^{T}d.$$

$\hat{m}$ has a strictly positive value regardless of the terminal states of the world. The initial cost of $\hat{m}$ is given by

$$\begin{aligned}\pi(\hat{m}) &= \left(\frac{\pi'-\pi''}{S_0}\right)\pi(d_0)+(\theta''-\theta')\pi(d) \\ &= (\pi'-\pi'')+(\theta''-\theta')^{T}S \\ &= 0.\end{aligned}$$

This violates the assumptions of no arbitrage opportunity and we conclude that $\pi(m)$ is unique.

To show that π is linear, consider $m_1, m_2 \in M$ and $\tilde{m}$ is given by the formula

$$\tilde{m} = \lambda_1 m_1+\lambda_2 m_2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Assume that $\pi(\tilde{m}) \neq \lambda_1\pi(m_1)+\lambda_2\pi(m_2)$. Then uniqueness of $\pi(\tilde{m})$ is violated and it contradicts the assumption of no arbitrage opportunity. Therefore,

$$\pi(\lambda_1 m_1+\lambda_2 m_2) = \lambda_1\pi(m_1)+\lambda_2\pi(m_2)$$

which proves the linearity of π.

□

The first intuition behind linearity of π is compelling. Given a reasonable price system. It is impossible to yield the same terminal bundle by repackaging two different portfolio of basis bundles with different initial values. The second intuition about linearity of π is that as a consequence of no arbitrage opportunity, the terminal bundles in M is forced to be independent of trading strategies.

The following two definitions are an embodiment of a viable price system introduced in Harrison and Kreps.

Definition: The pair (M,π) is supported if there exists some $\succsim$ and $m^* \in M$ such that

$$\pi(m^*) \leq 0 \quad \text{and} \quad m^* \succsim m \quad \forall\, m \in M \qquad \text{so that} \qquad \pi(m) \leq 0.$$

That is an agent with $\succsim$ can always find a solution to his portfolio optimization problem in the marketed subspace. Such preference $\succsim$ is said to support (M,π). Denote Ψ to be the set of τ continuous and L_+ strictly positive linear functionals in L.

Definition: The pair (M,π) has extension property for (L,τ) if π can be extended to all $x \in L$.

Proposition 2: The pair (M,π) is supported by preference $\succsim$ if and only if it has the extension property.

Proof. Two cases are considered. Case **(i)** The topology τ is generated by L_p-norm. Define the better than set

$$B(x) = \{x \in X_+ | x \succ 0\}.$$

The set B is convex since $\succ$ is a convex preference. Consider a point $\hat{x}$. By hypothesis, $\succ$ is proper at $\hat{x}$. This implies (from theorem 1 of the last subsection) the existence of an open cone. Also from theorem 1, we conclude that both

$$\hat{x}-\Gamma(\hat{x}) \quad \text{and} \quad B(\hat{x})$$

are disjoint implying $\hat{x}-\Gamma(\hat{x}) \cap B(\hat{x}) = \emptyset$.

Furthermore, $\hat{x}-\Gamma(\hat{x})$ is convex with a nonempty interior. Accordingly, the Separation theorem (Holmes, 1975, p.63) can be applied to yield a hyperplane passing through $\hat{x}$. Let the linear functional associated with the hyperplane be denoted by ψ. Uniform properness then implies that the hyperplane is defined on any arbitrary $x \in X_+$. Since L is a Banach lattice, a result form the appendix (theorem A.3) shows that ψ is a continuous linear functional.

To verify ψ is consistent with (M,π), two things need to be shown. First, pick $m_0 \in M$ such that $m_0 \succ 0$. Since $\succ$ supports (M,π), $\pi(m_0) > 0$. It must be shown that $\psi(m_0) > 0$ as well. To see this, let $x \in X_+$ such that $\psi(x) > 0$. By continuity of preference, there must exist $\lambda \in \mathbb{R}$ so that $m-\lambda x \succ 0$. Therefore,

$$\psi(m_0-\lambda x) \geq 0.$$

Linearity of ψ implies

$$\psi(m_0) \geq \lambda\psi(x) > 0$$

leading to the conclusion that $\psi(m_0) > 0$. Finally, since $\psi(m_0) > 0$ and $\pi(m_0) > 0$, ψ can be normalized to yield $\psi(m_0) = \pi(m_0)$.

Next choose any $m \in M$ and let λ be such that

$$\pi(m)+\lambda\pi(m_0) = \pi(m+\lambda m_0) = 0.$$

By linearity of π, both $m+\lambda m_0$ and $-m-\lambda m_0$ are both in M implying

$$\psi(m+\lambda m_0) \le 0 \quad \text{and} \quad \psi(-m-\lambda m_0) \le 0.$$

Therefore, $\psi(m+\lambda m_0) = 0$. It follows

$$\begin{aligned}\psi(m) &= -\lambda\psi(m_0) \\ &= -\lambda\pi(m_0) \\ &= \pi(m).\end{aligned}$$

Thus, we have shown that ψ extends π.

Case **(ii)** The topology τ is a semi-norm generated weak topology. Then the upper contour set B(x) has a τ-open interior. In this case, both B(x) and $x-\Gamma(x)$ are convex, disjoint and having nonempty interior. Therefore, the separation theorem again applies.

□

When (M,π) has an extension to (X,τ), the resulting security market is called "viable", a term first employed by Harrison and Kreps. It must be emphasized that viability in the current context has a stronger meaning. This is so because extension of price functional as shown in the above proof relies mainly on the additional topological property defined by proper preference. When the upper contour set has empty interior, properness induces an open cone leading to the separating hyperplane. In the case that upper contour set has a τ-interior, the role of properness is again reinforced.

However, in both cases, the resulting linear functional is ensured to be bounded. This is a defining property of proper preference, a provision not found in Harrison and Kreps. The possibility that a hyperplane exists and yet the resulting functional being discontinuous is ruled out.

4. Derivation of the Black-Scholes state price density function

This section applies the above linear functional to the famous Black-Scholes economy and deduce the state price density process. Two specializations have to be taken into account for this economy. First, Black and Scholes (1973) model a dynamic economy which involves a description of the market securities as stochastic processes. In principle a full fledged dynamic information model will be entailed to describe a general stochastic security price process. However for a constant coefficient price model like Black-Scholes, the analysis can be dramatically simplified since the underlying uncertainty is easily seen to be generated by a Brownian motion process. It follows that the derivation of the state price process is reduced to applying a few mathematical properties associated with a Brownian motion.

Second, since a Brownian motion is a square integrable random variable, the Banach lattice used for the above modified Harrison and Kreps economy is specialized to be a Hilbert lattice. The main result of this section is to exploit the representation of a linear functional on Hilbert lattice by an expectation of the inner product of two random variables in the Hilbert space.

In the Black-Scholes economy, only two securities are traded. One of them is risky stochastic process but does not pay dividends on the time interval [0,T] and the other is a riskless process. More specifically the former is a traded security price process S(t) with a stochastic representation given by:

$$S(t) = \exp\left((\mu-\frac{1}{2}\sigma^2)t+\sigma w(t)\right) \qquad t \in [0,T]$$

where μ and σ are two strictly positive constants. $\{w(t)\}$ is a standard Brownian motion that starts at zero at $t = 0$ with probability one (w.p.1). The riskless security does not pay dividends on [0,T] and has a price process B(t) with a deterministic representation given by:

$$B(t) = \exp(rt) \qquad t \in [0,T].$$

An investor in this economy is interested in trading in the two securities to achieve a desired random wealth a time T. It is assumed that terminal random variables have finite second moments. Traders' preference satisfy the properties discussed in the previous section. The vector $(B(t),S(t))$ is restricted to be a viable price system. This means that w.p.1 it is impossible for any trader to obtain a strictly positive terminal wealth with an initial portfolio strategy that has nonpositive cost. Equivalently by proposition 2 of the last section a viable price system has an extended continuous price functional ψ defined over the entire space of terminal random wealths.

We emphasize that there are two kinds of arbitrage opportunities in models that allow dynamic tradings. The first type is what has been discussed so far and is ruled out by the existence of a price extension. The second kind of arbitrage opportunity can occur even in the absence of the first kind.

Harrison and Kreps (1979, p.403) illustrate the second kind by the doubling strategy which can be removed by admitting only simple trading strategies in the model. Formally, a trading strategy is said to be *simple* if it is bounded and if it only changes its values a finite number of times in a given time interval.

Given the existence of the price functional ψ and a simplified stochastic security model, the derivation of the Black-Scholes state price density will be obtained in two stages. The next subsection retrieves some brief but essential details about Ito's calculus. These details will then be used as ingredients for obtaining a specific formula for a state price density.

3.4.1 *A quick summary of Ito's integral related to Black-Scholes economy*

Investors are assumed to observe the realization of the two securities process over $t \in [0,T]$ and these realizations are accumulated to form the information set to the traders at time t. As the riskless asset is deterministic, traders know its future value at t. On the other hand, traders infer the future values of w(s) at time t, s > t indirectly from observing S(s).

Formally the information set is given by a filtration $\mathbb{F} = \{\mathcal{F}_t | t \in [0,T]\}$. In this case as information is generated by w(t), the filtration is therefore denoted as $\mathbb{F}^W = \{\mathcal{F}_t^W | t \in [0,T]\}$.

Definition: A stochastic process x_t on a filtered probability space $\{\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P}\}$ is adapted to $\mathbb{F}$ if and only if x_t is measurable on $\mathcal{F}_t$. (In our case, S(t) is adapted to $\mathcal{F}_t^W$.)

Definition: An adapted process x_t on $\{\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P}\}$ is called a martingale on [0,T] if and only if

(a) for each $0 \le t \le$, $E_{\mathbb{P}}(|x_t|) < \infty$,

(b) for each $0 \le u \le t \le T$, w.p.1. $E_{\mathbb{P}}(x_t | \mathcal{F}_u) = x_u$.

It can be shown that a Brownian motion process is a $\mathbb{P}$-martingale on $\mathcal{F}_t^{w}$ (Breiman, 1968). The two sample path properties of a Brownian motion process that are useful for later purpose are the optional quadratic variation and predictable quadratic variation processes. The former describes the limit of the sum of squared changes of w(t) while the latter describes the limit of the sum of conditional expected squared changes of w(t).

Definition: Let x_t be a stochastic process on $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P})$ and time interval [0,t]. Corresponding to dyadic partitions of [0,t], **(i)** consider sequences of sums of squared changes of x_t.

$$S_m(x)(t,\omega) = x^2(0,\omega) + \sum_{j=1}^{\infty} [x(t_{m,j} \wedge t, \omega) - x(t_{m,j-1} \wedge t, \omega)]^2.$$

It there exists a stochastic process, denoted by $\{[x,x]_t\}$ such that for every $0 \le t \le T$ and every $\varepsilon > 0$,

$$\lim_{m \to \infty} \mathbb{P}\{\sup_{0 \le u \le t} |S_m(x)(u,\omega) - [x,x](u,\omega)| \ge \varepsilon\} = 0,$$

then we say $[x,x]$ is the optional quadratic variation of x_t. **(ii)** Consider consequences of sums of conditional expected squared changes, $S_m(x)(t,\omega)$ of x_t

$$S_m(x)(t,\omega) = x^2(0) + \sum_{j=1}^{\infty} E_{\mathbb{P}}\{[x(t_{m,j} \wedge t) - x(t_{m,j-1} \wedge t)]^2 | \mathcal{F}_{t_{m,j-1} \wedge t}\}(\omega).$$

If there exists a stochastic process, denoted $\{\langle x,x \rangle_t\}$ such that, for every $0 \le t \le T$ and every $\varepsilon > 0$,

$$\lim_{m\to\infty} \mathbb{P}\{\sup_{0\le u\le t} |S_m(x)(u,\omega) - \langle x,x\rangle(u,\omega)| \ge \varepsilon\} = 0,$$

then we say that $\langle x,x\rangle$ is the predictable quadratic variation process of x.

For a Brownian motion process, it can be shown that

$$[w,w]_t = t \qquad \text{and} \qquad \langle w,w\rangle_t = t.$$

An additional characteristics of $[x,x]_t$ and $\langle x,x\rangle_t$ is that both are increasing processes. In differential form they are expressed as

$$d[x,x]_t = (dx_t)^2, \qquad d\langle x,x\rangle_t = E_{\mathbb{P}}((dx_t)^2 | \mathcal{F}_t).$$

The mathematical development of Ito's integral is built on the above sample properties of a Brownian motion process. In finance and economics literature, the Ito's integral is defined to reflect that it has a martingale property. This particular route to define an Ito's integral can be motivated by the following existence theorem.

Theorem 1: Suppose $\alpha(t)$ is a stochastic process on the interval $[0,T]$, adapted to $\mathcal{F}_t^w$ measurable, and such that, w.p.1. $\int_0^T \alpha(t)^2 dt < \infty$. Then there exists a sequence of adapted, measurable, simple stochastic process $\{\alpha_{mt}\}$ such that w.p.1.

$$\int_0^T \alpha_{mt}^2 dt < \infty, \qquad \lim_{m\to\infty} \int_0^T (\alpha_t - \alpha_{mt})^2 dt = 0$$

and w.p.1. the sequence of integrals $\int_0^t \alpha_{ms} dw_s$ converges uniformly on the interval [0,T]. Furthermore, the quantity $\lim_{m\to\infty} \int_0^t \alpha_{ms} dw_s$ does not depend on the choice of approximating sequence of adapted, measurable simple processes $\{\alpha_{mt}\}$ such that $\int_0^T \alpha_{mt}^2 dt < \infty$.

Definition: For a stochastic process $\{\alpha_t\}$ in the previous theorem,

$$\int_0^t \alpha_s dw_s = \lim_{m\to\infty} \int_0^t \alpha_{ms} dw_s.$$

The left hand side is called the Ito's integral of the process $\{\alpha_t\}$.

In precise term (Chung and Williams, 1990 chapter 2), Ito's integral is an isometry. Loosely this means the integral is a transform of the process α_t by the Brownian motion process $\{w_t\}$. Two important properties of Ito's integral are recorded below.

Lemma 1: If the adapted measurable process $\{\alpha_{1t}\}$ and $\{\alpha_{2t}\}$ are such that $\int_0^T \alpha_{1t}^2 dt < \infty$ and $\int_0^T \alpha_{2t}^2 dt < \infty$, then consider x_t and y_t defined by $x_t \equiv \int_0^t \alpha_{1s} dw_s$, $y_t \equiv \int_0^t \alpha_{2s} dw_s$. Then we have

$$\text{(i)} \quad [x,x]_t = \langle x,x\rangle_t = \int_0^t \alpha_s^2 ds$$

$$\text{(ii)} \quad [x,y]_t = \langle x,y\rangle_t = \int_0^t \alpha_{1s}\alpha_{2s} ds.$$

Part (ii) of the above lemma gives the *optional* and *predictable quadratic covariation* of two Ito integrals. In terms of increments this can be written as $\alpha_{1t}\alpha_{2t}dt$. Associated with Ito's integral is an important representation result according to H. Kunita and S. Watanabe (1967).

Theorem (Kunita-Watanabe). If $\{x_t\}$ is a square integrable martingale on the filtration $\mathcal{F}_t^{w}$, then there exists an adapted measurable process $\{\alpha_t\}$ such that $E_{\mathbb{P}}(\int_0^T \alpha_t^2 dt) < \infty$ and

$$x_t = x_0 + \int_0^t \alpha_s dw_s, \qquad \forall\ 0 \le t \le T.$$

This result, also called martingale representation can be heuristically explained as follows. The right side of the equality can be viewed as the resulting application of Ito lemma to $f(t,w_t) = \exp(w_t - \frac{1}{2}t)$ with the fact that the coefficients of the time differential and the quadratic variation cancels each other out before integration from 0 to t. One can also extract a familiar interpretation from this representation theorem. That is in a space of square integrable martingales, the Brownian motion can be treated as an infinite dimensional basis and spans other martingales in $t \in [0,T]$.

3.4.2 Black-Scholes state price as an Ito integral

The extended linear functional ψ from section 3 itself has very little applicable value unless it can be transformed into a tractable form ready for asset pricing. This is implied by the construction of an equivalent martingale measure. The fundamental Riesz Representation theorem is a vehicle through which the change of measure can be subsequently performed. This theorem allows a real-valued linear functional to be expressed as an inner product of a random terminal wealth and its random state prices. Prior to stating that result, it must be shown that any square integrable random wealth is a $\mathbb{P}$-martingale.

The first defining property of a martingale is easily satisfied by a square integrable random variable since square integrability implies absolute integrability $\forall\ t \in [0,T]$. The second property of a $\mathbb{P}$-martingale is obtained by the law of iterated expectation. Define $x_T \equiv E_{\mathbb{P}}(x_T | \mathcal{F}_T)$. Then

$$\begin{aligned} E_{\mathbb{P}}(x_T | \mathcal{F}_t) &= E_{\mathbb{P}}(E_{\mathbb{P}}(x_T | \mathcal{F}_T) | \mathcal{F}_t) \\ &= x_t, \end{aligned}$$

since the filtration formed by a Brownian motion is increasing, $\mathcal{F}_t \subset \mathcal{F}_s$, $s > t$.

Theorem 3 (Riesz representation). Given that ψ is a continuous linear functional on $x \in L^p(\mathbb{P})$ with $p \in [1,\infty)$. Then there exists a unique $z \in L^q(\mathbb{P})$ such that

$$\psi(x) = \int_\Omega x(\omega) z(\omega) d\mathbb{P} \qquad \forall \quad x \in L^p(\mathbb{P}).$$

Moreover, if ψ is positive, then $z \geq 0$ a.s. and if ψ is strictly positive, then $z > 0$ a.s.

Proposition 1. Suppose the price system is viable in the Black-Scholes economy and assume all positive terminal random wealth are square integrable, i.e., $x \in L^p(\mathbb{P})$, $p = 2$. Then the time zero value of x is given by

$$\psi(x) = E_{\mathbb{P}}\left(\int_0^T \alpha_t d[w,x]_t\right).$$

Furthermore, there exists a strictly positive random variable z_T given by $z_T = \int_0^T \alpha_t dw_t$.

Proof. By martingale representation theorem, any square integrable random variable can be written as $x_T = x_0 + \int_0^T \eta_t dw_t$ where η_t is a square integrable predictable process. Since the price system is viable, time zero value of x_T is given by $\psi(x_T)$.

Next by Riesz representation theorem, there exists z_T such that the linear functional can be expressed as expectation of scalar product of x_T and z_T. That is $\psi(x) = E_{\mathbb{P}}(x_T z_T)$. We argue that z_T has a stochastic integral representation. To see this, note that the RHS of the above representation is a Stieltjes integral. On the other hand Protter (1990, p.75) shows that the optional quadratic covariation process $[w,x]$ has finite variation on compact interval. Given that x_T is a square integrable martingale, it is necessary that z_T has a stochastic integral representation

$$z_T = \int_0^T \alpha_t dw_t,$$

where α_t is an adapted process and is square integrable. That is

$$\begin{aligned} E_{\mathbb{P}}(x_T z_T) &= E_{\mathbb{P}}\left(\int_0^T \eta_t dw_t \int_0^T \alpha_t dw_t\right) \\ &= E_{\mathbb{P}}\left(\int_0^T \alpha_t \eta_t dw_t{}^2\right) \\ &= E_{\mathbb{P}}\left(\int_0^T \alpha_t d\left(\int_0^T \eta_t dt\right)\right) \end{aligned}$$

$$= E_{\mathbb{P}}\left(\int_0^T \alpha_t d(dw_s \int_0^t \eta_s dw_s)\right)$$

$$= E_{\mathbb{P}}\left(\int_0^T \alpha_t d[w,x]_t\right).$$

□

The Ito integral representation for z_T can be interpreted as the state price process for the strictly positive random variable x_T. Since $\psi(x_T)$ is a nontrivial positive linear functional, it follows that z_T is strictly positive. Let $z = \ln \tilde{z}$. By Ito's lemma

$$\tilde{z}(t) = \exp\left(\int_0^T \alpha_t dw_t + \frac{1}{2}\int_0^T \alpha_t^2 dt\right).$$

The next result shows that the adapted process $\tilde{z}(t)$ has the familiar form of the market price of risk in the Black-Scholes economy. A definition is in order.

Definition: Let two measures $\mathbb{P}$ and $\mathbb{Q}$ define on the measurable space $(\Omega,\mathcal{F})$. The measure $\mathbb{Q}$ is said to be absolutely continuous with respect to $\mathbb{P}$ and is denoted by $\mathbb{Q} << \mathbb{P}$ such that

$$\mathbb{P}(B) = 0 \quad \text{implies} \quad \mathbb{Q}(B) = 0 \qquad \forall \quad B \in \mathcal{F}.$$

Theorem 4 (Girsanov). Let $\{w_t\}$ be a Brownian motion process on the probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. Let $\{\alpha_t\}$ be measurable process adapted to the natural filtration $\{\mathcal{F}_t^{\,w}\}$ such that

$$E_{\mathbb{P}}\left(\exp(\theta\int_0^T \alpha_t^2 dt)\right) < \infty \qquad \text{for some } \theta > 1.$$

Furthermore, let $\mathbb{Q}$ be a probability measure on $(\Omega,\mathbb{F})$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^T \alpha_t dw_t + \frac{1}{2}\int_0^T \alpha_t^2 dt\right).$$

Denote $\tilde{z} = E_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathcal{F}_t^{\,w}\right)$. Then $w_t^* = w_t - \int_0^t \frac{d\langle w, \tilde{z}\rangle_s}{\tilde{z}_s}$ is a Brownian motion process on the filtered probability space $(\Omega, \mathbb{F}, \mathbb{Q}, \{\mathcal{F}_t\})$.

Proposition 2: Suppose the two price processes $(B(s), S(s))$ form a viable price system in the Black-Scholes model. Then there exists a measure $\mathbb{Q}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \tilde{z}(T)$$
$$= \exp\left(\int_0^T \alpha_t dw_t + \frac{1}{2}\int_0^T \alpha_t^2 dt\right),$$

where $\alpha_t \equiv \frac{-(\mu - r)}{\sigma}$. Moreover the discounted security price process $\frac{S(t)}{B(t)}$ is a $\mathbb{Q}$-martingale.

Proof. Given that $\tilde{z}(T) = \exp\left(\int_0^T \alpha_t dw_t + \frac{1}{2}\int_0^T \alpha_t^2 dt\right)$, where $\alpha_t = \frac{-(\mu - r)}{\sigma}$, it must be shown that $\mathbb{Q}$ is a probability measure and the discounted stock price process is a $\mathbb{Q}$-martingale. The first claim can be verified by directly integrating $\tilde{z}(T)$ with respect to the density function of $\mathbb{P}$-Brownian motion. This yields

$$E_{\mathbb{P}}(\tilde{z}(T)) = \int_\Omega \tilde{z}(T) d\mathbb{P}$$
$$= 1.$$

Let $Q(A) \equiv \int_A \tilde{z}(T)d\mathbb{P}$ for $A \in \mathcal{F}$. By Ito's lemma, the discounted price process is given by

$$d\left(\frac{S(t)}{B(t)}\right) = (\mu-r)\frac{S(t)}{B(t)}dt+\sigma\frac{S(t)}{B(t)}dw_t.$$

One the other hand, $d\tilde{z}_t = \frac{-(\mu-r)}{\sigma}\tilde{z}dw_t$. Now the predictable quadratic covariation of a Brownian motion and $\tilde{z}$ is given by

$$d\langle w,\tilde{z}\rangle_t = \frac{-(\mu-r)}{\alpha}\tilde{z}_t dt.$$

From Girsanov theorem, the process w_t^* given by

$$\begin{aligned} w_t^* &= w_t-\int_0^t \frac{d\langle w,\tilde{z}\rangle_s}{\tilde{z}_s}ds \\ &= w_t-\int_0^t\left(\frac{-(\mu-r)}{\sigma}\tilde{z}_s\right)\frac{1}{\tilde{z}_s}ds \\ &= w_t+\frac{(\mu-r)}{\sigma}t \end{aligned}$$

is a $\mathbb{Q}$-Brownian motion. Substituting w_t^* into the discounted price process gives

$$d\left(\frac{S(t)}{B(t)}\right) = \sigma\left(\frac{S(t)}{B(t)}\right)dw_t^*,$$

where $\frac{S(t)}{B(t)}$ satisfies $E_{\mathbb{Q}}\left(\int_0^T \left(\frac{B(t)}{S(t)}\right)^2 ds\right) < \infty$. Hence $\frac{S(t)}{B(t)}$ is a $\mathbb{Q}$-martingale process.

Conversely, given $\frac{S(t)}{B(t)}$ is a $\mathbb{Q}$-martingale, it must be shown that the state price process take the form stated. Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$. By Radon Nikodym theorem (Bartle 1966, p.85), $\frac{d\mathbb{Q}}{d\mathbb{P}} = \tilde{z}$ is a $\mathbb{P}$-square integrable random variable. Martingale representation theorem implies that $\tilde{z}_t = 1+\int_0^t \beta_s dw_t$. Note also that by Bayes rule, $\frac{S(t)}{B(t)}$ is a $\mathbb{Q}$-martingale if and only if $\tilde{z}_t\frac{S(t)}{B(t)}$ is a $\mathbb{P}$-martingale. This implies

$$\begin{aligned} d(\tilde{z}_t\frac{S(t)}{B(t)}) &= \tilde{z}_t d(\frac{S(t)}{B(t)})+(\frac{S(t)}{B(t)})d\tilde{z}_t+d[\tilde{z},\frac{S(t)}{B(t)}]_t \\ &= ((\mu-r)\tilde{z}_t+\sigma\beta_t)\frac{S(t)}{B(t)}dt+(\sigma\tilde{z}_t+\beta_t)\frac{S(t)}{B(t)}dw_t. \end{aligned}$$

By observation the "drift" term of the above is zero since $z_t\frac{S(t)}{B(t)}$ is a $\mathbb{P}$-martingale. It follows that $\beta_t = \frac{-(\mu-r)}{\sigma}\tilde{z}_t$. Therefore

$$d\tilde{z}_t = \frac{-(\mu-r)}{\sigma}\tilde{z}_t dw_t \qquad \text{and}$$

$$\tilde{z}(T) = \exp(\int_0^T \frac{-(\mu-r)}{\sigma}dw_t+\frac{1}{2}\int_0^T (\frac{\mu-r}{\sigma})^2 dt).$$

The proof of necessity is completed.

□

The formula for z(T) in the above proposition defines a state price density. As a result of modeling the commodity space as a Hilbert lattice, a closed form state price density is derived in a modified Harrison and Kreps framework which is further rigged to be the Black Scholes economy. Relaxing the whole exercise to other Banach lattice in principle will retain the spirit of the above result. A subtle feature of the Hilbert lattice will be lost nevertheless if the analysis is extended to other norm lattices. The uniqueness of the equivalence martingale measure is not preserved in other lattices partially because the martingale representation theorem does not hold in these other spaces.

The nonuniqueness problem and the resulting incompleteness of securities market is further emphasized by Harrison and Pliska (1983) and Duffie and Huang (1985). The problem has not been resolved since yet although recent work by Aase (1988) and He and Pearson (1992) show some promising progress.

5. Conclusion

The previous sections have derived a linear price functional by means of an arbitrage partial equilibrium approach. It differs from Harrison and Kreps formulation in that it imposes strong restriction on the preference of the investor, namely the preference satisfies the uniform proper condition. The resulting price functional has the property that it can be represented by a bounded marginal rate of substitution in the dual valuation space. An investor in this economy is able to attain an optimal terminal wealth given a strongly viable price system.

Furthermore with presence of a continuous state price function and specialization of the terminal random variables to be elements of Hilbert lattices, a formula for the state price density process is obtained. It follows that we have obtained the risk-neutral martingale probability measure. This thesis therefore represents one formulation of the fundamental asset pricing theorem popularized by Dybvig and Ross (1987) and Back and Pliska (1991). There remains a few issues that are not explored thoroughly in the above research program.

The topological vector lattice is a useful mathematical structure that can be exploited in a richer analysis than is presented here. In the current partial equilibrium model, the state price functional is exogenously taken but as expressed cogently by Kreps (1982), it is the responsibility of a good economist to endogenize basic data like prices in an economic model. In other words, it should be possible to push forth the result here to obtain a representation of the state price functional as a general equilibrium price functional.

Kreps's proposal can be approached on two fronts. Cox, Ingersoll and Ross (1985) have formed a fully dynamic model with marketed securities and production and the state variables are represented by diffusion processes. Relying on the usual market clearing and rational expectations assumptions, these authors are able to derive the marginal utility of a representative individual as the static Arrow-Debreu general equilibrium state price functional in their theorem 4. (Other general equilibrium formulations include Huang (1987) and Richard and Sundaresan (1981)).

On the flip side of this dual economic equilibrium formulation, the existence of Arrow-Radner dynamic equilibrium can be taken for granted initially. Then one can carry out the static Arrow-Debreu equilibrium analysis where the mathematical property of a Riesz space can manifest its full strength. In particular, Aliprantis-Brown-Burkinshaw (1987,1989) have introduced rich mechanics of vector lattice in analyzing a general Walrasian model. How that general model can be narrowed down to incorporate a subspace of marketed securities remain an interesting research topic.

A less obvious aspect of incorporating uniform proper preference in defining a viable price system must be unraveled in this conclusion. While that preference specification has delivered a desirable property that the price functional is bounded, it also rules out unfortunately some utility function (logarithmic utility function in particular) which are widely adhered to in many finance models. The popularity of log-utility is understandable for it is one of the few examples that has a closed form solution to a stochastic dynamic control problem via solving a highly nonlinear Bellman partial differential equation. It remains to explore how uniform proper preference can be relaxed so that this important utility function can become admissible to a strengthened viable price.

Appendix

In this appendix, some properties of vector lattices are defined since these properties are occasionally employed in the text. A relation $\geq$ on a non-empty set X is said to be an *order relation* if it satisfies

(i) $x \geq x$ holds $\forall\, x \in X$;

(ii) $x \geq y$ and $y \geq x$ implies $x = y$;

(iii) $x \geq y$ and $y \geq z$ implies $x \geq z$.

The resulting X is an order set. A *lattice* is an ordered set X such that $\sup(x,y)$ and $\inf(x,y)$ exists for each pair $x,y \in X$. In notations

$$x \vee y \equiv \sup(x,y) \qquad \text{and} \qquad x \wedge y \equiv \inf(x,y).$$

A *partially* ordered vector space X is called a *Riesz space* or a vector lattice whenever for any two elements x and y of X both $x \vee y$ and $x \wedge y$ exist. The set

$$X^{+} = \{x \in X \mid x \geq 0\}$$

is called the *positive cone* of X. For each $x \in X$, the "parts" of x can be expressed in terms of "v" operator, namely

$$x^{+} = x \vee 0; \qquad x^{-} = (-x) \vee 0; \qquad |x| = x \vee (-x).$$

Intuitively the above equalities represent the positive, negative and absolute values of x respectively.

The two results below are readily verifiable.

Lemma A.1. **(a)** $x^{+} \wedge x^{-} = 0$; **(b)** $x = x^{+} - x^{-}$; **(c)** $|x| = x^{+} + x^{-}$.

Lemma A.2. Let x,y,z be elements of a vector lattice. Then the following inequalities hold:

(a) $|x+y| \leq |x|+|y|$

(b) $||x|-|y|| \leq |x-y|$

(c) $|x^{+}-y^{-}| \leq |x-y|$

(d) $|x \vee z - y \vee z| \leq |x-y|$

(e) $|x \wedge z - y \wedge z| \leq |x-y|$.

Similar to a topological vector space that generalizes the normed vector space, a topological Riesz space is defined by a linear topology τ consistent with the algebraic and lattice structures. In particular, if τ is induced by a norm $\|\cdot\|$ on a vector space X, a *norm lattice* is resulted. That is

$$|x| \leq |y| \text{ in } X \qquad \text{implies} \qquad \|x\| \leq \|y\|.$$

When a norm lattice X is complete, then X is referred to as a

Banach lattice.

As an important example of the norm lattice on X, recall the linear space induced by a measurable space is denoted as $L^p(\Omega,\mathcal{F},\mathbb{P})$ and is normed by

$$\|x\|_p = \left(\int_\Omega |x|^p d\mathbb{P}\right)^{1/p} \qquad \text{for } x \in L^p.$$

One typically treat x as a random variable. A special case of an element defined on the positive cone is the lognormal random variable.

Proposition A.1: L^p is a vector lattice.

Proof. First, from lemma 1, $x = x^+ - x^-$ and it can be shown that x^+ and x^- are nonnegative random variables and belong to L^p. Moreover, for any pair of random variables in L^p, say x and y, we have

$$x \vee y = (x-y)^+ + y \qquad \text{and} \qquad x \wedge y = y-(y-x)^+.$$

We therefore conclude that the space of random variables are normed lattices.

□

The well known Riesz-Fischer theorem for $L^p(\Omega,\mathcal{F},\mathbb{P})$ also applies to this norm lattice.

Theorem A.1: If $1 \le p < \infty$, then $L^p(\Omega,\mathcal{F},\mathbb{P})$ is a Banach lattice.

Proof. It suffices to show that $L^p(\Omega,\mathcal{F},\mathbb{P})$ is a Banach space. This is a standard result shown in Bartle (1966).

□

Theorem A.2: $L^{\infty}(\Omega,\mathcal{F},\mathbb{P})$ is a Banach lattice.

Proof. It suffices to show that $L^{\infty}(\Omega,\mathcal{F},\mathbb{P})$ is a Banach space. This is a standard result shown in Bartle (1966).

□

The above two theorems together constitute a formal definition for a Banach lattice.

Definition: A lattice is said to be a Banach lattice if it is a Banach space and the lattice operations are continuous in the norm. That is, if $\{x_n\}$ converges in the norm to x in the space, then $\{x_n^+\}$ also converges in theorem to x^+ which is an element of the lattice.

Another useful fact regarding linear functional on Banach lattices is as follows.

Theorem A.3: A positive linear functional on a Banach lattice is continuous in the norm. If the norm is given by $L^p(\Omega,\mathcal{F},\mathbb{P})$, then a positive linear functional on L^p is L^p-norm continuous.

Proof: See Schaefer (1974, theorem II.53, p.84).

□

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IMPLICATIONS OF ARBITRAGE APPROACH TO BOND OPTIONS PRICING

CHAPTER 4. INTRODUCTION TO THE SECOND ESSAY

1. Early literature review on bond options pricing

Historically, the valuation of a European call option on a pure discount bond can be linked to the original equity option pricing model. In this development, Merton's insight (1973) should be credited for he extends the Black-Scholes model by incorporating a stochastic interest rate. Given a specific interest rate process, Merton is able to generate a bond option solution that maintains much of the original flavor of the Black-Scholes formula.

In order to distinguish from the modern treatment of bond option valuation adopted in this thesis, Merton's approach will be referred to as the spot rate approach. Other papers that employ similar techniques for bond pricing include Vasicek (1977), Richard (1978), Dothan (1978) and Brennan and Schwartz (1979). In this class of models, a bond option formula is obtained in a two-step procedure.

First, an equilibrium bond pricing problem is solved with a risk premium parameter introduced to represent the compensation to investors for random changes in the instantaneous spot rates (usually the only state variable in these models). Then the conditional expectation of the bond option payoff (which also incorporates the risk premium function) is computed. Alternatively the same bond option solution is obtained by solving a second order parabolic partial differential equation.

Aside from the technical treatments, there is a tyrannsaurausic difference between taking a risk adjusted conditional expectation and solving a partial differential equation in finding a solution to a bond option. Cheng (1991) has shown the potential trouble from exogenously specifying a bond price process and then deriving a partial differential equation via a simple hedging argument. In some extreme cases, the resulting partial differential equation is nothing but an 'empty mathematical shell'. However, with 'careful' selection of the market price of risk function, both solution approaches satisfy the necessary and sufficient conditions for pricing bond option by absence of arbitrage (Cheung, 1992). In fact, the logical connection between these two solution approaches can be shown by the Feynman-Kac formula (Duffie, 1992).

A variant and in-depth treatment of this spot rate approach is to construct a general equilibrium model so that the preference parameter can be directly derived from market clearing conditions instead of arbitrarily determined in some partial equilibrium models. A leading example of the general equilibrium approach to solving a bond option pricing problem is developed by Cox, Ingersoll and Ross (1985a). (CIR is now a customarily used pseudonym for these authors' names.)

As a theoretical advantage, specifying a dynamic general equilibrium formulation for option pricing provides consistency between a viable interest rate process and the equilibrium interest rate. Thus, in a variety of contexts, Cox, Ingersoll and Ross (1981a,b, 1982) propose an arbitrage free square root interest rate process and then separately (1985b) illustrate how the coefficients of that process are all derivable from a typical economy with the preference of an infinitely lived individual and with carefully specified production. Standard market clearing plus the rational expectations assumptions are the keys to close the CIR model. In other words, one can claim that all the parameters in the square root interest rate process are the embodiment of the essential optimal conditions that characterize a

Walrasian competitive equilibrium.

Derivation of the bond option formula according to the spot rate approach suffers from two interrelated drawbacks. Although the solution shares a similar structure to the Black-Scholes equation, it differs from the latter formula in one crucial aspect. Whereas the Black-Scholes takes the currently observed stock price as given, the current bond price in the bond option case has to be obtained from an equilibrium model. This implies in principle the applicant of the model would have to find an estimate of the market price of risk function. Such a risk premium is not needed in applying the stock option model.

A second unsatisfactory aspect of the traditional bond option formula arises from its insufficient use of currently observed information. Unlike the Black-Scholes formula, the currently available bond prices are not incorporated in the bond option. If one were to view currently observed prices as conveying relevant information about future states of the world, then an efficient pricing formula should embody this information as part of its elements.

The two aforementioned drawbacks have rendered the spot rate approach to bond option pricing undesirable. In particular, the information aspect of the model cannot match the insight offered by the Black-Scholes case. Recent researchers have taken seriously these drawbacks and started reformulation of the bond option model in a manner closer in spirit to the Black-Scholes' methodology.

In a discrete time framework, Ho and Lee (1986) have exogenously specified a dynamic fluctuation of the yield curve according to a binomial process. Placing restrictions on the yield curve movement via an appeal to the absence of arbitrage opportunity, these authors are able to derive a set of martingale probabilities which they then use to price a bond option. This approach has the beauty that it takes the initially observed term structure as input data to the option pricing problem. An important assumption of the Ho and Lee model is that there are always enough zero coupon bonds traded to span the yield curve for a given time interval.

Heath Jarrow and Morton (1992) (hereafter denoted as HJM) advocate an approach similar to the Ho and Lee model. Instead of building a discrete time model, HJM construct a stylized scenario with continuous trading. A crucial assumption in these authors' models is that at every instant there exists a continuum of discount bonds to span the yield curve. The exogenous stochastic process that governs the evolution of the term structure is identified as the forward rate process. Choosing a savings account as the numeraire and expressing the bond price function relative to this numeraire, HJM work out the necessary and sufficient conditions for the relative bond and option prices to be martingale processes.

Merely for the purpose of pricing a bond option, we argue in this thesis that HJM's methodology can be simplified. This simplification is inspired by an idea from Bick (1987). One of the insights in Bick's analysis is to transform the payoff of a call option with a positive exercise price to a payoff with zero exercise price. This is achieved by introducing a theoretical asset called ZEPO (zero exercise price option). Pricing a call option on the usual terminal equity value net of exercise price can be shown to be equivalent to pricing a call option on a ZEPO.

The interesting feature of a ZEPO asset is that when it is combined with different number of discount bonds in a dynamic trading strategy one can exactly replicate the payoff from a forward price contract. Equivalently, specifying a dynamic trading strategy of forward contracts alone is sufficient to produce the payoff of a ZEPO. An extra arrangement with the latter strategy is required to produce the standard payoff of the European call option. That is, one needs to determine an initial borrowing to replicate the exercise price of the option at maturity.

The key to understand the equivalence of the standard approach to solving a general call option pricing problem and Bick's proposal is that in the latter approach, one needs to specify as numeraire the discount bond with the same maturity as the option, and then express the payoff of the forward contract in terms of this numeraire. This subtlety in Bick's approach makes it especially relevant to the pricing of a bond option. The following paragraphs provide a synopsis of this thesis that attempts to relate the use of two different numeraires to price a bond option: one from the saving account in HJM model and the other from the discount bond in Bick's model.

Instead of denominating the terminal bond options payoff in units of the savings account, one picks as the numeraire an initially traded discount bond having the same maturity as the option. Next, one chooses the current number of initial forward contracts for discount bonds. This effectively creates the deterministic exercise price of the option. The remaining business is to specify a dynamic strategy for trading forward contracts in order to produce the random bond price at maturity (which plays the role of the ZEPO asset in the terminology of Bick's framework). Because of this last requirement, one needs to introduce a stochastic process to model the forward bond price movements prior to the planned maturity. Consequently, this formulation using forward contracts allows one to produce the same payoff function as that from the direct HJM model.

The above description of replicating the ZEPO payoff via a forward price process has likened the bond option pricing problem to the original Black-Scholes version of an equity-option payoff replicated by a stock trading process. This analogy allows us to appeal to the standard arbitrage analysis of Harrison and Pliska (1981). According to a fundamental result of Harrison and Pliska, the absence of arbitrage opportunity restricts the forward bond price process to be a martingale. One of the principal theorems in the next chapter is to derive the necessary and sufficient conditions for a forward bond price process to be a martingale under a risk-adjusted probability measure.

It is worth emphasizing that the valuation problem here is based on a transformation of pricing a bond option on a stochastic term structure into pricing of the same option on a forward bond price process. Note that the maturity matching between the option and a discount bond is strongly facilitated by the assumption of a complete bond market. This is the same assumption made by the HJM model in terms of a fully spanned term structure. The other focus of the thesis extends well beyond the valuation of a bond option. With this objective in mind, the results developed here are not to be considered as theoretically competitive but rather complementary to HJM's results. Given the existence of both the forward equivalent martingale measure and the risk neutral measure, a number of existing results about a stochastic term structure will be re-examined in the next chapter.

First, a basic intuition suggests that the value of the option should reman unchanged even though there is a change of numeraire in the price system. This invariance principle will be formalized by a necessary condition for the existence of a random variable called the Radon Nikodym derivative. The sole function of this random variable is to preserve the 'fair game' characterization of the option as a result of a martingale to martingale transformation.

Next, the difference between the forward price and futures prices can be explored once again in the presence of the Radon Nikodym derivative. The fact that in general the two prices differ is thoroughly presented by Cox, Ingersoll and Ross (1981). Here the difference between the prices is phrased in terms of an implausibility proposition.

Finally, by carefully blending the forward equivalent martingale measure with the risk neutral measure, we are able to recover a version of the traditional *expectations hypothesis*. This last result makes the reformulation of the bond option valuation particularly rewarding since some earlier influential studies by Cox, Ingersoll and Ross (1981) have expressed concern about the validity of the expectations hypothesis in a continuous time setting.

The rest of this chapter is to present a brief review of the rigorous model of Heath, Jarrow and Morton (1992). The insights and notations of this model will then be used to compare and contrast with the results developed in the next chapter.

2. Review of Heath Jarrow Morton model

The starting point of the HJM model is to exogenously specify a stochastic movement of the implied forward rates. The probability space is described by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Here Ω is the underlying state space, $\mathcal{F}$ is the σ-algebra representing measurable events and $\mathbb{F} = \{\mathcal{F}_t \mid t \in [0,\tau]\}$ is a family of sub-σ-algebra of $\mathcal{F}$ satisfying the usual conditions (Duffie, 1992, appendix C).

Lastly, $\mathbb{P}$ is a probability measure.

HJM's paper assumes that uncertainty is generated by multiple Brownian motions. The present review only assumes a one dimensional Brownian motion adapted to $\mathcal{F}_t$ in order to highlight the important issue at hand.

Consider an economy with continuous trading in an interval $[0,\tau]$ for a given $\tau > 0$. It is assumed that a continuum of default free zero coupon (discount) bonds trade with various maturities denoted by $T \in [0,\tau]$. This presumption guarantees the term structure is dynamically spanned. Define $P(t,T)$ as time t price of a T maturity discount bond for $\forall\ T \in [0,\tau]$ and $t \in [0,\tau]$, $t \leq T$. Bond prices are required to satisfy the following properties

(i) $P(T,T) = 1 \qquad \forall\ T \in [0,\tau]$

(ii) $P(t,T) > 0 \qquad \forall\ T \in [0,\tau]$ and $t \in [0,\tau]$.

As a note, implicit in the HJM economy is a complicated mathematical framework which has 'double infinity'. The state space is an infinite dimension because of the introduction of a Brownian notion. The assumption of a dynamically spanned term structure implies an infinite number of bond assets traded in this economy. This latter aspect of the model therefore necessitates more boundedness restriction on the bond price process parameters below.

A yield curve describes the relationship between spot rates (yields to maturity) and a spectrum of maturities for a given set of discount bonds at a single point in time. This relationship is also called term structure of interest rates. While the fluctuation of the yield curve can be captured by specifying either the bond prices dynamics for all maturities $T \in [0,\tau]$ or a process for the forward rates, HJM has chosen the latter because of its stationarity property. The crucial idea however is that once the forward rate process is specified, the stochastic processes for bond prices of various maturities are also determined.

Following the argument of HJM , the continuous stochastic movements of the forward rates process is modeled by the Ito processes. Let the instantaneous forward rates for date T viewed from date t be f(t,T). Bond prices and forward rates are connected by the following basic relationship:

$$(1) \qquad f(t,T) = \frac{-\partial}{\partial T}\ln\big(P(t,T)\big) \qquad \forall\ T \in [0,\tau], \qquad t \in [0,T].$$

The evolution of the forward rates is given by

$$(2) \qquad f(t,T)-f(0,T) = \int_0^t \mu_f(v,T,\omega)dv+\int_0^t \sigma_f(v,T,\omega)dB(v)$$

where f(0,T) is a set of nonrandom initial forward rates, $\forall\ T \in [0,\tau]$ and B(v) is a one dimensional Brownian motion process with the standard properties (see Friedman, 1975).

The following regularity conditions are imposed on the drift and volatility of the forward rate process. The drift $\mu_f:\{(t,s)|0\le t\le s\le T\}\times\Omega \rightarrow \mathbb{R}$ is jointly measurable from $\mathcal{B}\{(t,s)|0\le t\le s\le T\}\times\mathbb{F} \rightarrow \mathcal{B}$, adapted, with

$$\int_0^T |\mu_f(t,T,\omega)|dt < +\infty \qquad \mathbb{P}\text{-a.e.}$$

Here $\mathcal{B}(\cdot)$ is the Borel σ-algebra restricted to $[0,\tau]$. The volatility $\sigma:\{(t,s)|0\le t\le s\le T\}\times\Omega \rightarrow \mathbb{R}$ is jointly measurable from

$\mathcal{B}\{(t,s)|0\le t\le s\le T\}\times\mathbb{F} \to \mathcal{B}$, adapted and satisfies

$$\int_0^T \sigma_f^2(t,T,\omega)dt < +\infty \qquad \mathbb{P}\text{-a.e.}$$

In differential form, the fluctuation of forward rates is described by

(3) $\quad df(t,T) = \mu_f(t,T,\omega)dt+\sigma_f(t,T,\omega)dB(t).$

Note that the spot rate at time t, r(t), is defined by the instantaneous forward rate at time t, namely

$$r(t) = f(t,t) \qquad \forall\ t \in [0,\tau].$$

It follows that by satisfying the regularity conditions on the forward rates process the spot rate process can be defined by

(4) $\quad r(t)-f(0,t) = \int_0^t \mu_f(v,t,\omega)dv+\int_0^t \sigma_f(v,t,\omega)dB(v) \qquad t \in [0,T].$

Note that f(0,t) is the initially observed forward rates (at t = 0).

In addition to discount bonds, there exists a saving account traded in this economy. Define the saving account process as

$$Z^0(t,\omega) = \exp\left(\int_0^t r(u)du\right) \qquad \forall\ t \in [0,\tau].$$

The interpretation of the saving account process is quite straightforward. An investor with an initial one dollar can invest into this saving account and let it grow instantaneously at the stochastic spot rate. The time t value of rolling over the dollar is given by $Z^0(t,\omega)$. Note that $Z^0(t,\omega)$ satisfies the regularity conditions since r(t) is transformed by an exponential function.

Given the forward rates process, there is a functional relationship between the bond return process and the forward rates process. Define the instantaneous return on the discount bond by

$$(5) \qquad \frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)dt - \sigma_P(t,T)dB(t)$$

where the dependence of all the variables on ω is suppressed for notational ease. From (1) and Ito's lemma,

$$(6) \qquad \begin{aligned} df(t,T) &= \frac{-\partial}{\partial T} d(\ln[P(t,T)]) \\ &= \frac{-\partial}{\partial T}([\mu_P(t,T) - \frac{1}{2}\sigma_P^2(t,T)]dt - \sigma_P(t,T)dB(t)). \end{aligned}$$

The partial differential operator (w.r.t. T) can be loosely treated as a linear operator on the variable T inside the bond price process P(t,T). A more rigorous description of this partial differential operator is found in HJM (lemma A.1. appendix, p.96, 1992).

Matching volatility terms from equations (2) and (6) results in

$$(7) \qquad \sigma_f(t,T) = \frac{\partial}{\partial T}\sigma_P(t,T) \qquad \text{which implies}$$

$$(8) \qquad \sigma_P(t,T) = \int_t^T \sigma_f(t,u)du.$$

Also, matching the drift of the forward rates gives

$$\mu_f(t,T) = \frac{-\partial}{\partial T}\mu_P(t,T)+\frac{\partial}{\partial T}\left(\frac{\sigma_P^{\,2}(t,T)}{2}\right)$$

which upon rearranging leads to

$$(9) \qquad \frac{\partial}{\partial T}\mu_P(t,T) = -\mu_f(t,T)+\sigma_P(t,T)\sigma_f(t,T).$$

Note that the second term on the right is obtained by chain rule of differentiation.

Up to this point, all developments are primarily definitional. Theoretical substance can now be introduced to the model. The necessary condition for absence of arbitrage is stated by the following classic condition:

$$(10) \qquad \mu_P(t,T)-r(t) = \lambda(t)\sigma_P(t,T)$$

where $r(t)$ and $\lambda(t)$ (the market price of risk) are common parameters to all bonds of various maturities and hence independent of T for $T \in [0,\tau]$. An original justification for λ to be independent of T is developed by Merton (1973) and Vasicek (1977) in a one stochastic interest rate model for bond pricing. (10) is couched in a highly interpretable form namely: the excess expected return on holding a risky discount bond is measured by its total risk times per unit risk price.

The derivation of equation (10) for a simple one state variable is given by Vasicek (1977); and for the more general multiple state variables is given by Cox, Ingersoll and Ross (1981). Differentiation of (10) with respect to T gives

$$(11) \qquad \frac{\partial}{\partial T}\mu_P(t,T) = \lambda(t)\frac{\partial}{\partial T}\sigma_P(t,T)$$

$$= \lambda(t)\sigma_f(t,T).$$

Substituting this result into (9) produces one of the main results in Heath, Jarrow, Morton (c.f. HJM Proposition 3, p.86, 1992):

$$\text{(12)} \qquad \begin{aligned} \mu_f(t,T) &= \sigma_P(t,T)\sigma_f(t,T)-\lambda(t)\sigma_f(t,T) \\ &= \sigma_f(t,T)[\sigma_P(t,T)-\lambda(t)]. \end{aligned}$$

Equation (12) represents an arbitrage restriction on the drift of the forward rates process. Note also the market price of risk function, for $t \rightarrow T$, becomes

$$\text{(13)} \qquad \lambda(t) = \frac{-\mu_f(t)}{\sigma_f(t)}$$

since $\sigma_P(t,T)\big|_{T=t} = 0$ (and $\mu_f(t,t)$ and $\sigma_f(t,t)$ are now simplified as $\mu_f(t)$ and $\sigma_f(t)$). That is a T-maturity bond has no volatility at $T = t$ by definition.

For contingent claims to be priced by arbitrage, HJM proceeds to show in their theorems 1 to 3 (HJM p.84-86, 1992) that there exists a risk neutral measure $\mathbb{Q}^*$ such that bond prices relative to a saving account,

$$Z^1(t,\omega) = \frac{P(t,T)}{Z^0(t,\omega)},$$

is a $\mathbb{Q}^*$-martingale process. By Girsanov theorem (Duffie, 1992, appendix D), they introduce

$$\text{(14)} \qquad dB(t)^* = dB(t)-\lambda(t)dt$$

where $B(t)^*$ is a Brownian motion measurable with respect to probability $\mathbb{Q}^*$.

The theoretical break-through of the HJM model lies in its ability to eliminate the market price of risk in contingent claims valuation. To see this, substitute (12) and (14) into equation (4) for the spot rate process:

$$
\begin{aligned}
(15) \qquad r(t) &= f(0,t)+\int_0^t \mu_f(v,t)dv+\int_0^t \sigma_f(v,t)dB(v) \\
&= f(0,t)+\int_0^t [\sigma_P(v,t)\sigma_f(v,t)-\lambda(v)\sigma_f(v,t)]dv \\
&\quad +\int_0^t \sigma_f(v,t)[dB(v)^*+\lambda(v)dv] \\
&= f(0,t)+\int_0^t \sigma_P(v,t)\sigma_f(v,t)dv+\int_0^t \sigma_f(v,t)dB(v)^*.
\end{aligned}
$$

Both the market price of risk parameter $\lambda(t)$ and the forward rate drift $\mu_f(\cdot)$ vanish in the last equality. In this reduced form, the spot rate process depends on the initially observed forward rates as well as on the volatility of the term structure which consists of $\sigma_P(\cdot)$ and $\sigma_f(\cdot)$.

In the light of equations (14) and (15), contingent claims valuation can be carried out according to the *standard procedure* spelled out clearly in Harrison and Pliska (1981). First, since $P(T,T) = 1 \quad \forall\ T \in [0,\tau]$, the sufficient condition for absence of arbitrage implies

$$\frac{P(t,T)}{Z^0(t)} = E_{\mathbb{Q}^*}\left[\frac{P(T,T)}{Z^0(T)} \,\middle|\, \mathcal{F}_t\right].$$

Rearranging the above together with the definitions of $Z^0(t)$ and $Z^0(T)$ gives

$$P(t,T) = E_{Q*}[\exp(-\int_t^T r(s)ds)] \qquad t \in [0,T].$$

Second, the terminal payoff of a European bond option with expiration date t is given by

$$C(t) = \mathrm{Max}(0, P(t,T)-K).$$

We have therefore rederived the following

Proposition 1 (Heath, Jarrow, Morton (1992)). Given an arbitrage free forward rates process, the initial value of a European bond option (which expires at T) is given by

$$C(t_0) = E_{Q*}[\exp(-\int_{t_0}^{t} r(u)du)\cdot C(t)] \qquad t_0 < t < T.$$

Two aspects of the above formula need emphasis. First, the right hand side of the above equality does not involve the variable $\lambda(t)$. It should be pointed out that the price of risk parameter is indirectly reflected in the risk-adjusted probability Q*. This is the preference free property of HJM valuation approach. Second, the present value of the bond option requires the joint distribution between the stochastic exponential function and the terminal option payoff at time t. This presents cumbersome computation for a closed form solution even if the forward rate process is a Guassian random walk.

3. Conclusion

To sum up, the HJM approach to bond option valuation has a clear advantage over the spot rate approach. Their major contribution is primarily in terms of deriving an arbitrage free restriction on the forward rates drift. Combining this constraint with the sufficient condition for absence of arbitrage results in the elimination of the market price of risk function.

In this regard, the contingent claim valuation problem is simplified considerably. Merely specifying a particular forward rates process and applying standard procedure will lead to a closed form option solution that involves only initially observed data as well as volatility parameters. These nice properties will reappear in the next chapter in a slightly different model which is also targeted for pricing a bond option with a stochastic term structure.

CHAPTER 5. ARBITRAGE APPROACH TO BOND OPTION PRICING AND ITS IMPLICATIONS

The thesis of this chapter shares the same spirit with a basic tenet in the general equilibrium analysis. In the Walrasian equilibrium price system, only a change in relative price can have real effects in the economy. On the other hand a change in the numeraire used in the price system cannot lead to any reallocation of resources. In finance, one would expect the same principle to hold in a viable price system that precludes all free lunches. That is a change in the numeraire should not change the fundamental state prices and similarly arbitrage free prices of contingent claims should be independent of the unit of account.

In this chapter we adopt the preference-free approach to the continuous time bond option pricing problem which is advocated by Heath Jarrow Morton. Instead of using a savings account on which the bond price function is denominated, a discount bond is chosen as the numeraire which has the same maturity date as the European option written on an underlying discount bond with a more distant maturity date than the numeraire. This has the effect of converting the terminal value of a bond option to be a function of the prevailing forward price which must be identical to the underlying discount bond at the delivery date. Given this observation, our bond option valuation problem begins by specifying a forward price process and then employs a dynamic forward strategy to replicate the terminal bond option payoff.

The necessary and sufficient conditions for absence of arbitrage opportunities in trading forward contracts allows us to derive a probability measure equivalent to the investor's subjective probability measure. This equivalent measure will be called the forward equivalent martingale measure. The value of a contingent claim (with the same maturity date as the European bond option) relative to the value of the numeraire discount bond is a martingale under the forward equivalent martingale measure.

One of the major themes of this chapter is to show that pricing a bond option on a forward bond price process produces the same present dollar value for the option originally priced on a stochastic term structure. The switch of numeraire, however, does change the appearance of some price processes. The bond option is transformed from a martingale under the risk-neutral measure (via the HJM approach) to be a martingale under the forward equivalent martingale. A principal advantage of our approach is the resulting simplification of computing the present value of the bond option.

This approach via specifying a forward price process in valuing a bond option has been first pointed out by Merton (1973) and later analyzed by Jamshidian (1987). Targeting for different purposes, these earlier approaches do not explicitly use the assumption of a dynamically spanned term structure which plays a crucial role in the results derived below. Also the solution of this early literature is derived by solving a partial differential equation. Here, the bond option is priced by necessary and sufficient conditions of absence of arbitrage.

One can therefore argue that one of the principal payoffs of the present approach over the HJM approach is that computing the arbitrage free bond option prices is cloning the procedure used for evaluating the Black-Scholes equity option. In addition, a by-product of the forward bond price approach is that it motivates the existence of a Radon Nikodym derivative. With the aid of this state price density function, a number of existing theories related to stochastic term structure can be analyzed from a different perspective.

Section 1 develops the technical aspects of the forward equivalent martingale measure and their interpretations. The valuation of a bond option with respect to this forward equivalent martingale measure is presented in section 2. Section 3 examines the consequence of adopting a different numeraire in contingent claims pricing. Here a neutrality principle is introduced and discussed. The result from section 3 provides another chance to look at the difference between the futures and forward prices. This is illustrated in section 4.

In section 5, both the risk neutral measure and the forward equivalent martingale measure are combined so that the unbiased expectation hypothesis can be seen in a new light. We are able to show that the expectations hypothesis is basically an arbitrage statement. Section 6 concludes this chapter with a suggestion for further research.

1. The forward equivalent martingale measure

Let $G(t,t^*,T)$ be the forward bond price at time t defined by a forward contract that entitles the holder to buy a T-maturity discount bond at the delivery date t^*. This implies

$$(1) \qquad G(t,t^*,T) \equiv \frac{P(t,T)}{P(t,t^*)}.$$

The above definition has the following important meaning. Although a forward price can be contracted explicitly at time t and effective at time t^* (for $t^* > t$), equation (1) states that the forward contract can be replicated by the currently available time t discount bonds.

Consider at time t a portfolio of going long one unit of T maturity discount bond and simultaneously going short $\frac{P(t,T)}{P(t,t^*)}$ number of t* maturity bonds. The initial time t cost of this portfolio is

$$P(t,T)-(\frac{P(t,T)}{P(t,t^*)})P(t,t^*) = 0.$$

At time t*, the portfolio has an obligation to deliver $\frac{P(t,T)}{P(t,t^*)}$ dollars, and at time T one dollar will be received. Consequently, the payoff of this portfolio duplicates the payoff of a forward contract of T-maturity discount bond.

Turning the definition in (1) around, the payoff at t* of the T maturity bond denoted by P(t,t*,T) can be replicated by a combination of forward contracts and borrowing. This implies that any derivative asset's terminal payoff that is a function of P(t,t*,T) can be attained by forming a dynamic portfolio of forward contracts. An interesting characteristic of this approach is that no additional borrowing or lending is required prior to t* since it does not cost anything to enter into forward contracts.

Our first objective is to specify the stochastic behavior of the forward bond price process. Given the definition of a forward bond price, its dynamics evolution can be obtained from the following result where dB is a standard Brownian motion process.

Lemma 1. The forward price process is given by

$$(2) \qquad \frac{dG(t,t^*,T)}{G(t,t^*,T)} = \mu_G(t,t^*,T)dt+\sigma_G(t,t^*,T)dB(t)$$

where

(3a) $\mu_G = \mu_P(t,T)-\mu_P(t,t^*)-\sigma_P(t,t^*)\sigma_G(t)$,

(3b) $\sigma_G = \sigma_P(t,T)-\sigma_P(t,t^*) \qquad t < t^* < T.$

Proof. Note that both P(t,T) and P(t,t*) have the following dynamics:

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)dt-\sigma_P(t,T)dB(t),$$

$$\frac{dP(t,t^*)}{P(t,t^*)} = \mu_P(t,t^*)dt-\sigma_P(t,t^*)dB(t).$$

Then apply Ito's lemma to (1) and simplify to obtain the dynamic evolution for the forward price, namely

$$\begin{aligned}\frac{dG}{G} &= \mu_P(t,T)dt+\sigma_P(t,T)dB(t)-\mu_P(t,t^*)-\sigma_P(t,t^*)dB(t)\\ &\quad+\sigma_P^2(t,t^*)dt-\sigma_P(t,t^*)\sigma_P(t,T)dt\\ &= [\mu_P(t,T)-\mu_P(t,t^*)-\sigma_P(t,t^*)(\sigma_P(t,T)-\sigma_P(t,t^*))]dt\\ &\quad+[\sigma_P(t,T)-\sigma_P(t,t^*)]dB(t)\end{aligned}$$

which yields the desired result.

□

A well functioning financial market with zero transaction cost can be characterized by the absence of arbitrage opportunities. An arbitrage opportunity is defined to be a trading strategy with zero initial cost and a nonnegative future payoff with probability one. In terms of the forward bond prices process, no financial free lunch means that it is impossible to form a riskless arbitrage portfolio by exploiting these forward price processes. This in turn leads to a set of restrictions on the forward price process parameters as demonstrated in the following lemma.

Lemma 2. If there is no arbitrage opportunity in the forward price process, then

$$\frac{\mu_G(t,t^*,T)}{\sigma_T(t,t^*,T)} = \lambda(t,t^*).$$

Proof. The value of a forward contract, denoted by g, at the initiation date t is zero and the embedded forward price is $G(t,t^*)$ where t^* is the delivery date. For any later date u such that $t < u < t^*$, $g(u) = \big(G(u,t^*)-G(t,t^*)\big)P(u,t^*)$ which can be established by an arbitrage argument.

Now at t, choose any two dates $T_1,T_2 > t^*$. Next from a portfolio of newly initiated forward contracts with delivery dates at t^*. In particular long θ_1 number of t^*-maturity forward contracts that deliver the underlying discount bond maturity at T_2; and simultaneously short θ_2 number of t^*-maturity forward contracts that delivers the underlying discount bond maturity at T_1. Denote the current value of the portfolio by $V(t)$ so that

$$V(t) = \theta_1 g\big(t,G(t,t^*);T_2\big)-\theta_2 g\big(t,G(t,t^*);T_1\big).$$

By construction, $g(\cdot;T_1) = g(\cdot;T_2) = 0$. After an instant, Δt, the value of the portfolio is given by

$$\begin{aligned} V(t+\Delta t) &= \theta_1 g\big(t+\Delta,G(t,t^*);T_2\big)-\theta_2 g\big(t+\Delta,G(t,t^*);T_2\big) \\ &= \theta_1\big[\big(G(t+\Delta,t^*;T_2)-G(t,t^*;T_2)\big)P(t+\Delta,t^*)\big] \\ &\quad -\theta_2\big[\big(G(t+\Delta,t^*;T_1)-G(t,t^*;T_1)\big)P(t+\Delta,t^*)\big]. \end{aligned}$$

Let $\Delta V = V(t+\Delta t)-V(t)$, and let $\Delta \to 0$

$$dV = \theta_1 dG(t,t^*,T_2)P(t,t^*)-\theta_2 dG(t,t^*,T_1)P(t,t^*).$$

Choose $\theta_1 = \dfrac{\sigma_G(t,t^*,T_1)}{G(t,t^*,T_2)}$ and $\theta_2 = \dfrac{\sigma_G(t,t^*;T_2)}{G(t,t^*;T_1)}$. This then implies

$$dV = [\sigma_G(t,t^*;T_1)\frac{dG(t,t^*;T_2)}{G(t,t^*;T_2)}-\sigma_G(t,t^*;T_2)\frac{dG(t,t^*;T_1)}{G(t,t^*;T_1)}]P(t,t^*)$$

One cannot express the above in percentage change since $V(t) = 0$. However the above can be simplified by substituting in the respective forward price dynamics:

$$\begin{aligned}
dV = {} & [\sigma_G(t,t^*;T_1)\mu_G(t,t^*;T_2)dt \\
& +\sigma_G(t,t^*;T_1)\sigma_G(t,t^*;T_2)dB \\
& -\sigma_G(t,t^*;T_2)\mu_G(t,t^*;T_1)dt \\
& -\sigma_G(t,t^*;T_2)\sigma_G(t,t^*;T_1)dB]\cdot P(t,t^*) \\
= {} & [\sigma_G(t,t^*;T_1)\mu_G(t,t^*;T_2)dt \\
& -\sigma_G(t,t^*;T_2)\mu_G(t,t^*;T_1)dt]\cdot P(t,t^*)
\end{aligned}$$

Since $P(t,t^*) > 0$ by construction and the initial cost of the portfolio is equal to zero, to rule out riskless arbitrage, the terms inside $[\cdots]$ after the second equality must be zero. That is

$$\sigma_G(t,t^*;T_1)\mu_G(t,t^*;T_2) = \sigma_G(t,t^*;T_2)\mu_G(t,t^*;T_1)$$

which implies

$$\frac{\mu_G(t,t^*;T_1)}{\sigma_G(t,t^*;T_1)} \overset{e}{=} \frac{\mu_G(t,t^*;T_2)}{\sigma_G(t,t^*;T_2)}.$$

As T_1, T_2 are arbitrary, absence of arbitrage opportunity in trading forward contracts implies the ratio of the drift to volatility functions of the forward price process is independent of maturities of underlying bonds. Therefore we can define

$$\frac{\mu_G(t,t^*)}{\sigma_G(t,t^*)} = \lambda(t,t^*).$$

$\square$

The above result has a nice interpretation. Given a future date t*, any two different maturity bonds ($T_1, T_2 > t^*$) purchased at t* will bear the same source of risk that comes from dB(t). The usual assumption of no default applies at maturity which consequently does not command any premium. As discussed earlier, the forward price process is used to replicate the discount bond process: at t*, G(t*,t*,T) must converge to P(t*,T). The combination of these two observations allows us to rationalize $\lambda(t,t^*)$ as the ratio of $\mu_G(\cdot)$ and $\sigma_G(\cdot)$ in the above theorem. Moreover, the ratio is independent of the maturities of the constituent bonds.

Except for the missing opportunity cost, r(t), $\lambda(t,t^*)$ plays a similar role to the classical necessary condition found in Vasicek (1977) for valuing a pure discount bond

$$\lambda(t) = \frac{\mu_P(t,T)-r(t)}{\sigma_P(t,T)}$$

The variable $\lambda(t)$ in Vasicek's model is called the market price of risk that arises from the fluctuation of the Brownian motion process. While it can be specified to be a function of r(t), it is independent of any arbitrary maturity T. Because of its resemblance, $\lambda(t,t^*)$ will be called hereafter the forward market price of risk. Note that the missing $r(s)$, $s \in [t,t^*]$ in the expression for $\lambda(t,t^*)$ is understandable since the holding of the bond asset is not effective until t^*.

The link between $\lambda(t)$ and $\lambda(t,t^*)$ can be established by the following:

Theorem 1. **(i)** $\lambda(t,t^*) = \lambda(t)-\sigma_P(t,t^*)$,

(ii) $\lambda(t,t^*)\big|_{t^*=t} = \lambda(t)$.

Proof. Part (ii) follows trivially from (i) since $\sigma_P(t,t) = 0$ for a bond that matures at t^*. To verify (i), use the definition

$$
\begin{aligned}
\lambda(t,t^*) &= \frac{\mu_G(t,t^*)}{\sigma_G(t,t^*)} \\
&= \frac{\mu_P(t,T)-\mu_P(t,t^*)-\sigma_P(t,t^*)\sigma_G(t,t^*)}{\sigma_G(t,t^*)} \\
&= \frac{\mu_P(t,T)-\mu_P(t,t^*)}{\sigma_G(t,t^*)} - \sigma_P(t,t^*) \\
&= \frac{(r(t)+\lambda(t)\sigma_P(t,T))-(r(t)+\lambda(t)\sigma_P(t,t^*))}{\sigma_G(t,t^*)} - \sigma_P(t,t^*) \\
&= \frac{\lambda(t)(\sigma_P(t,T)-\sigma_P(t,t^*))}{\sigma_G(t,t^*)} - \sigma_P(t,t^*) \\
&= \lambda(t)-\sigma_P(t,t^*)
\end{aligned}
$$

since $\sigma_G(t,t^*) = \sigma_P(t,T)-\sigma_P(t,t^*)$.

□

Theorem 1 states that the forward market price of risk and the usual market price of risk for holding a risk bond asset differs by $\sigma_P(t,t^*)$ prior to t^*. Provided that both $\lambda(t)$ and $\lambda(t,t^*)$ are positive values (since bond prices and forward prices are randomly fluctuating) part (i) implies that risk premium from holding the bond asset is higher than the premium from entering into a dynamic portfolio of forward prices contracts; that is

$$\lambda(t)-\lambda(t,t^*) = \sigma_P(t,t^*).$$

The rationale for this difference comes from the recognition that with the case of a bond price strategy, the asset is physically held and rebalanced at each instant. On the contrary, the forward price is not a traded asset, the risk exposure with the forward contract strategy is lowered but not entirely eliminated as the forward price is ultimately used to replicate the terminal random $P(t^*,T)$.

Part (ii) states that at the expiry date of the forward contract, the classic risk premium is identical to the forward market price of risk. This is so since the long position of the forward contract has an obligation to purchase a T-maturity bond at t^*. That is the time when the bearing of T-maturity risky bonds begins.

Provided that $\sigma_P(t,t^*)$ obeys a set of regularity condition, $\lambda(t,t^*)$ will inherit the properties of $\lambda(t)$. The following assumption is therefore adopted.

Assumption. $\sigma_P(t,t^*,\omega)$ is adapted with respect to $\mathcal{F}_t$, jointly measurable and uniformly bounded on $\{(t,v)|0 \le t \le v \le t^*\}\times\Omega$.

Proposition 1. Define

$$K(t) = \int_0^t \mu_G(v,t^*)dv+\int_0^t \sigma_G(v,t^*)dB(v).$$

Then $\lambda(t,t^*):\Omega\times[0,\tau] \to \mathbb{R}$ satisfies

(i) $\mu_G(t,t^*)+\sigma_G(t,t^*)\lambda(t,t^*) = 0 \quad \ell\times\mathbb{P}\text{-a.e.}$

where ℓ is a Lebesgue measure

(ii) $\int_0^t \lambda(v,t^*)^2 dv < \infty \quad \mathbb{P}\text{-a.e.}$

(iii) $E_{\mathbb{P}}\{\exp(\int_0^t \lambda(v,t^*)dB(v)-\frac{1}{2}\int_0^t \lambda(v,t^*)^2 dv)\} = 1$

(iv) $E_{\mathbb{P}}\{\exp[\int_0^t (\sigma_G(v,t^*)+\lambda(v,t^*))dB(v)$

$-\frac{1}{2}\int_0^t (\sigma_G(v,t^*)+\lambda(v,t^*))^2 dv]\} = 1$

if and only if there exists a probability measure $\tilde{Q}$ such that

(a) $\dfrac{d\tilde{Q}}{d\mathbb{P}} = \exp(-\int_0^t \lambda(v,t^*)dB(v)-\frac{1}{2}\int_0^t \lambda(v,t^*)^2 dv)$

(b) $\tilde{B}(t) = B(t)-\int_0^t \lambda(v,t^*)dv$

is a Brownian motion on $\{\Omega,\mathcal{F},\mathbb{F},\tilde{Q}\}$

(c) $dK(t,t^*) = \sigma_G(t,t^*)d\tilde{B}(t)$

(d) the forward bond price is a $\tilde{\mathbb{Q}}$-martingale process.

Proof. Given (i), (ii) and (iii), Girsanov's theorem (Elliot, 1982, ch.13) implies (a) and (b). Substitution of (b) into the definition of the process $K(t,t^*)$ gives (c). By construction

$$dG(t,t^*) = G(t,t^*)dK(t,t^*).$$

Elliot (theorem 13.5, (1982)) shows that there is a unique solution to the above stochastic differential equation, namely

$$G(t,t^*) = G(0,t^*)\exp[K(t,t^*)-\frac{1}{2}\int_0^t \sigma_G(v,t^*)^2dv].$$

Since the exponential function is strictly positive, the above shows that $G(t,t^*)$ is a $\tilde{\mathbb{Q}}$-supermartingale. $G(t,t^*)$ is a $\tilde{\mathbb{Q}}$-martingale only if $E_{\tilde{\mathbb{Q}}}(G(t,t^*)) = G(0,t^*)$, $\forall\ t \in [0,\tau]$. Therefore it has to be shown that

$$E_{\mathbb{P}}[\exp(\int_0^t \mu_G(v,t^*)dv+\int_0^t \sigma_G(v,t^*)dB(v)-\frac{1}{2}\int_0^t \sigma_G(v,t^*)dv)\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}] = 1.$$

Substituting $\mu_G(t,t^*) = \lambda(t,t^*)\sigma_G(t,t^*)$ and $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}$ into the above and simplifying yields

$$\begin{aligned}
&E_{\mathbb{P}}[\exp(\int_0^t \lambda(v,t^*)\sigma_G(t,t^*)dv+\int_0^t \sigma_G(v,t^*)dB(v)-\frac{1}{2}\int_0^t \sigma_G(v,t^*)dv \\
&\qquad -\int_0^t \lambda(v,t^*)dB(v)-\frac{1}{2}\int_0^t \lambda(v,t^*)^2dv)] \\
&= E_{\mathbb{P}}[\exp[\int_0^t (\sigma_G(v,t^*)+\lambda(v,t^*))dB(v)-\frac{1}{2}\int_0^t (\sigma_G(v,t^*)+\lambda(v,t^*))^2dv]].
\end{aligned}$$

Now by (iv), part (d) is obtained.

Conversely, given (a), (b), (c) and (d), (ii) and (iii) follows because of Radon-Nikodym theorem (Bartle, theorem 8.9, (1966)). Substituting (b) into the definition of $K(t,t^*)$ gives

$$dK(t,t^*) = \mu_G(t,t^*)dt+\lambda(t,t^*)\sigma_G(t,t^*)dt+\sigma_G(t,t^*)d\tilde{B}(t).$$

Given (c), it follows that (i) holds a.e. $\tilde{\mathbb{Q}}$. Finally, from (d) $G(t,t^*)$ is a $\tilde{\mathbb{Q}}$-martingale implying that (iv) holds.

□

Proposition 1 has transformed the forward bond price to be a martingale process with respect to the forward equivalent martingale measure. In contrast to the HJM model which places a nontrivial restriction on the drift of the forward rate process, here the forward bond price is restricted to have a zero drift.

Harrison and Kreps (1979, theorem 2) have shown that a viable price system is a martingale after a suitable normalization. Now the forward bond price at maturity must be identical to the spot price of a discount bond to avoid obvious arbitrage at the settlement date. That is,

$$G(t^*,t^*,T) = P(t^*,T) \qquad t^* < T.$$

Furthermore, the forward price process can be replicated by managing a dynamic portfolio of two discount bonds with their respective values $P(t,t^*)$ and $P(t,T)$. This is implied by the definition of a forward price

$$G(t,t^*,T) = \frac{P(t,T)}{P(t,t^*)}.$$

Therefore the forward bond price is a discounted price process in the sense that the T-maturity bond is discounted by t^*-maturity bond which is selected to be the numeraire. While this reasoning suggests why the arbitrage free forward price as a martingale goes back to the insights of Cox and Ross (1976), the actual transformation is performed by the application of the Girsanov theorem as shown in the above proof.

2. Pricing of bond option by the forward equivalent martingale measure

The forward prices restrictions derived from the last section significantly simplify the evaluation of the bond option. To see this, define a terminal payoff of a European (discount) bond option as follows

$$C(t^*,T:K) = \mathrm{Max}[0,P(t^*,T)-K]$$

where $t^* \equiv$ expiration date of the option
$K \equiv$ exercise price

and $t^* < T$.

In the original HJM formulation, the present value of the above payoff is evaluated by taking the conditional expectation with respect to the risk neutral martingale measure, i.e.

$$C(t_0,t^*,T;K) = E_{Q^*}\{\exp(-\int_{t_0}^{t^*} r(u)du)\mathrm{Max}[0,P(t^*,T)-K]|\mathcal{F}_{t_0}\}$$

for $t_0 < t^* < T$. This formula requires the knowledge of the joint distribution of the discount factor and $P(t^*,T)$ before the expectation can be taken. The computation turns cumbersome rapidly if both $\exp(\cdot)$ and $P(t^*,T)$ are complicated functions of the stochastic interest rates.

On the contrary, using the forward equivalent martingale approach can avoid such complication. Rewrite the terminal payoff as

$$\begin{aligned}\frac{C(t^*,T;K)}{P(t^*,t^*)} &= \mathrm{Max}[0,\frac{P(t^*,T)}{P(t^*,t^*)} - \frac{K}{P(t^*,t^*)}] \\ &= \mathrm{Max}[0,G(t^*,t^*,T)-K]\end{aligned}$$

since $P(t^*,t^*) = 1$. Consequently expressing the terminal call payoff in terms of a t^*-maturity discount bond has transformed the payoff to be a function of the forward price at the expiration date.

Given the above interpretation of the call option payoff, its present value can be determined by taking conditional expectations of the terminal payoff. This is given by

$$\frac{C(t_0,t^*,T;K)}{P(t_0,t^*)} = E_{\tilde{Q}}\{\mathrm{Max}[0,G(t^*,T)-K]|\mathcal{F}_{t_0}\} \qquad \text{or}$$

$$C(t_0,t^*,T:K) = P(t_0,t^*)E_{\tilde{Q}}\{\mathrm{Max}[0,G(t^*,T)-K]|\mathcal{F}_{t_0}\}.$$

The only information required to compute the current call value is the univariate distributional property of the forward price at the option's expiration date.

Since the arbitrage free forward price is a driftless martingale process with respect to the forward equivalent martingale measure, the forward price volatility structure σ_G fully determines the bond option value. In several special cases where σ_G is a deterministic function, the solution of the bond option resembles the Black-Scholes formula.

Proposition 1. If $\sigma_G(t,t^*,T)$ is nonstochastic, then

$$C(t_0,t^*,t;K) = P(t_0,T)N(\Phi+\frac{1}{2}\tilde{\sigma})-P(t_0,t^*)K\cdot N(\Phi-\frac{1}{2}\tilde{\sigma})$$

where $$\tilde{\sigma}^2 = \int_{t_0}^{t^*}\sigma_G(t,t^*,T)^2dt, \qquad \Phi = \frac{\ln[\dfrac{P(t_0,T)}{P(t_0,t^*)\cdot K}]}{\tilde{\sigma}}$$

and $N(\cdot)$ is the cumulative normal distribution function.

Proof. Given that $\frac{dG(t)}{G(t)} = \sigma_G(t,t^*,T)d\tilde{B}(t)$ and $\sigma_G(t,t^*,T)$ is nonrandom, the forward price process is a simple stochastic differential equation with solution given by

$$\ln\left(\frac{G(t^*)}{G(t_0)}\right) = \int_{t_0}^{t^*} \sigma_G(t,t^*,T)d\tilde{B}(t).$$

Let $t_0 = 0$ for ease of notation. Note that since $d\tilde{B}(t)$ is a Gaussian random variable, the right hand side loosely represents a linear combination of Gaussian random variables. Denote

$$\int_0^{t^*} \sigma_G(t,t^*,T)d\tilde{B}(t) = \sigma^* z(t^*)$$

where $z(t^*)$ is a normal random variable with zero mean and unit variance and

$$\sigma^{*2} = \int_0^{t^*} \sigma_G(t,t^*,T)^2 dt.$$

The solution of the bond option can now be computed as follows:

$$\text{(P.1)} \qquad \begin{aligned} \frac{C(0,t^*,T;K)}{P(0,t^*)} &= E_{\tilde{Q}}\{\mathrm{Max}[0,G(t^*,T)-K]\} \\ &= (1-\tilde{Q})\cdot 0+\tilde{Q}(E_{\tilde{Q}}[G(t^*,T)\,|\,G(t^*,T) > K]-K) \\ &= \tilde{Q}\cdot E_{\tilde{Q}}[G(t^*,T)\,|\,G(t^*,T) > K]-\tilde{Q}\cdot K \end{aligned}$$

where $\tilde{Q} \equiv \mathrm{Prob}(G(t^*) > K)$.

The first term can be simplified as follows:

(P.2) $\tilde{Q}\cdot E_{\tilde{Q}}[G(t^*,T)\,|\,G(t^*,T) > K]$

$$= \int_K^{\infty} G(0)\exp(\sigma^* z)\exp(\frac{-1}{2}z^2)\frac{1}{\sqrt{2\pi}}dz$$

$$= G(0)\int_K^{\infty} \exp(\sigma^* z-\frac{1}{2}z^2)\frac{1}{\sqrt{2\pi}}dz$$

$$= G(0)\int_K^{\infty} \exp(\frac{\sigma^* z}{2}-[\frac{\sigma^{*2}}{(\sqrt{2})^2}-\frac{2\sigma^* z}{\sqrt{2}\sqrt{2}}+\frac{z^2}{(\sqrt{2})^2}])\frac{1}{\sqrt{2\pi}}dz$$

$$= G(0)\int_K^{\infty} \exp(\frac{\sigma^{*2}}{2})\exp[\frac{-(\sigma^*-z)^2}{2}]\frac{1}{\sqrt{2\pi}}dz.$$

Deflate $G(t^*)$ by $\exp(\frac{\sigma^{*2}}{2})$ so that the last equality can be turned into a cumulative normal distribution. Note that

$$\frac{G(t^*)}{\exp(\frac{\sigma^{*2}}{2})} > K \quad \text{implies} \quad G(t^*) > \exp(\frac{\sigma^{*2}}{2})K$$

so that

$$\ln(G(0))+\sigma^* z > \frac{\sigma^{*2}}{2}\ln K$$

$$\Rightarrow \qquad \sigma^* z > \frac{\sigma^{*2}}{2}-\ln(\frac{G(0)}{K})$$

$$\Rightarrow \qquad z > \frac{-\ln(\frac{G(0)}{K})+\frac{\sigma^{*2}}{2}}{\sigma^*}$$

$$\Rightarrow \qquad -z < \frac{\ln(\frac{G(0)}{K})-\frac{\sigma^{*2}}{2}}{\sigma^*}$$

$$\Rightarrow \qquad \sigma^*-z < \frac{\ln(\frac{G(0)}{K})}{\sigma^*}-\frac{\sigma^*}{2}+\sigma^* = \frac{\ln(\frac{G(0)}{K})}{\sigma^*}+\frac{1}{2}\sigma^*.$$

Define $\Phi = \dfrac{\ln(\frac{G(0)}{K})}{\sigma^*}+\dfrac{1}{2}\sigma^*$ and $y = \sigma^*-z$ so that

$$\int_K^{\infty}\exp(\frac{\sigma^{*2}}{2})\exp[\frac{-(\sigma^*-z)^2}{2}]\frac{1}{\sqrt{2\pi}}dz = \int_{-\infty}^{\Phi}\exp(\frac{-y^2}{2})\frac{1}{\sqrt{2\pi}}dy.$$

Therefore (P.2) can be rewritten as

$$\text{(P.3)} \qquad \tilde{Q}\cdot E_{\tilde{Q}}[G(t^*,T)\,|\,G(t^*,T) > K] = G(0)\int_{-\infty}^{\Phi}\exp(\frac{-y^2}{2})\frac{1}{\sqrt{2\pi}}dy$$

$$= G(0)\cdot N(\Phi+\frac{1}{2}\sigma^*).$$

Similar manipulation is applied to $\tilde{Q}\cdot K$ with $G(t^*)$ being deflated by $\exp(\frac{\sigma^{*2}}{2})$. Hence

$$\tilde{Q} = \mathrm{Prob}(\ln[\frac{G(t^*)}{\exp(\frac{\sigma^{*2}}{2})}] > \ln K)$$

$$= \mathrm{Prob}(\ln G(t^*) > \ln K+\frac{1}{2}\sigma^{*2})$$

$$= \mathrm{Prob}(\ln G(0)+\sigma^* z > \ln K+\frac{1}{2}\sigma^{*2})$$

$$= \mathrm{Prob}(z > \frac{\ln[\frac{G(0)}{2}]}{\sigma^*}-\frac{1}{2}\sigma^*)$$

$$= \mathrm{Prob}(z < \Phi-\frac{1}{2}\sigma^*).$$

This yields

$$(P.4) \qquad \tilde{Q} = K\cdot N(\Phi-\frac{1}{2}\sigma^*).$$

Substituting (P.3) and (P.4) into (P.2) gives

$$\begin{aligned} C(0,t^*,T;K) &= P(0,t^*)G(0)N(\Phi+\frac{1}{2}\sigma^*)+P(0,t^*)\cdot K\cdot N(\Phi-\frac{1}{2}\sigma^*) \\ &= P(0,T)\cdot N(\Phi+\frac{1}{2}\sigma^*)+P(0,t^*)\cdot K\cdot(\Phi-\frac{1}{2}\sigma^*) \end{aligned}$$

because $G(0) = \dfrac{P(0,T)}{P(0,t^*)}$.

□

The above bond option formula is similar to the Black-Scholes equity option closed form solution in the sense that it takes the initially observed term structure as an input. Note that $P(t_0,T)$ is the discount bond that matures at T while $P(t_0,t^*)$ represents the bond that has the same maturity as the underlying option's expiration. This maturity matching does not create a problem since the entire term structure is spanned by assumption.

A second similarity between the Black-Scholes case and the present formula is found regarding the role played by $P(t_0,t^*)$. Whereas Black-Scholes model treat the riskless bond as a numeraire, here $P(t_0,t^*)$ is taken as a numeraire so that any contingent claim's terminal payoff expressed in terms of $P(t_0,t^*)$ is a martingale process with respect to the forward equivalent martingale measure.

Lastly as a reflection of Black-Scholes model, the only estimable parameter in the bond option formula is the volatility of the forward price process. However the two analyses diverge at this point. While Black-Scholes formula has a constant stock price volatility, the bond option formula is a function of the forward price volatility. This variable in turn is related to the bond yield volatilities according to lemma 1 in the last section:

$$\sigma_G(t,t^*,T) = \sigma_P(t,T)-\sigma_P(t,t^*) \qquad t < t^* < T.$$

Therefore the condition in proposition 1 will be satisfied if the bond yield volatilities are nonstochastic. A particularly convenient two factor term structure model can be used to meet the deterministic volatility requirement. This is chosen primarily to illustrate the simplicity of the present approach. The model of interest is expressed as:

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)dt-\sigma_{1,P}(t,T)dB_1(t)-\sigma_{2,P}(t,T)dB_2(t)$$

$\forall$ t,T $\in [0,\tau]$, where

$$\sigma_{1,P} = \bar{\sigma}_1 \cdot (T-t), \qquad \sigma_{2,P} = \frac{\bar{\sigma}_2}{\kappa}(1-e^{-\kappa(T-t)}),$$

κ = mean reverting parameter,

$\bar{\sigma}_1$ and $\bar{\sigma}_2$ are constants and where dB_1 and dB_2 are two uncorrelated Brownian motions.

Both volatility specifications have rich intuitions in that as t approaches maturity T in the limit, bond price uncertainty vanishes entirely. This aspect display the convenience for specifying a term structure movement since having constrained the term structure to be arbitrage free will automatically impose constraint on a discount bond price process as well. Working the other way around by means of imposing an absence of arbitrage constraint on a bond price process need not necessarily produce a simultaneous constraint on the term structure movement. Cheng

(1991) has shown that modeling a 'viable' bond price movement by a Brownian bridge process can still lead to arbitrage in the model.

Returning to the description of the two factor model, the first factor has a relatively straightforward interpretation, namely the random influence of dB_1 on bond returns is the same for all maturities. On the contrary, dB_2 has a larger influence on yield with short maturity than distant maturity. As T enlarges, the influence of dB_2 dwindles and the bond yield gets pulled towards a mean value by the mean reverting parameter κ.

According to proposition 1, the drift of the bond yield plays no role in determining the bond option value. Thus to complete the computation, substitute the specifications of $\sigma_{1,P}(\cdot)$ and $\sigma_{2,P}(\cdot)$ into $\tilde{\sigma}^2$. This is performed in the following

Lemma 2. Given the bond yield volatility $\sigma_{1,P}(\cdot)$, $\sigma_{2,P}(\cdot)$, the corresponding forward prices volatility is computed as

$$
\begin{aligned}
&\left(\int_{t_0}^{t^*} \sigma_G(t,t^*,T)^2 dt\right)^{1/2} \\
&= \left(\int_{t_0}^{t^*} \sigma_{G,1}(t,t^*)^2 dt\right)^{1/2} + \left(\int_{t_0}^{t^*} \sigma_{G,2}(t,t^*)^2 dt\right)^{1/2} \\
&= \bar{\sigma}_1 (T-t^*)\sqrt{(t^*-t_0)} + \frac{\bar{\sigma}_2}{2}\left(1-e^{-\kappa(T-t^*)}\right)\left(\frac{1-e^{-2\kappa(t^*-t_0)}}{2\kappa}\right)^{1/2}
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_{G,1}(t,t^*) &= \sigma_{P,1}(t,T)-\sigma_{P,1}(t,t^*) \\
\sigma_{G,2}(t,t^*) &= \sigma_{P,2}(t,T)-\sigma_{P,2}(t,t^*).
\end{aligned}
$$

Corollary. Given the bond yield volatilities as above, the corresponding bond option formula is given by

$$C(t_0,t,T;K) = P(t_0,)N(\Phi+\frac{1}{2}\tilde{\sigma})-P(t_0,t^*)K\cdot N(\Phi-\frac{1}{2}\tilde{\sigma})$$

where Φ and $N(\cdot)$ are given in proposition 1 and

$$\tilde{\sigma} = (\int_{t_0}^{t^*}\sigma_G(t,t^*,T)dt)^{1/2}$$

$$= \bar{\sigma}_1(T-t^*)\sqrt{(t^*-t_0)}+\frac{\bar{\sigma}_2}{\kappa}(1-e^{-\kappa(T-t^*)})(\frac{1-e^{-2\kappa(t^*-t_0)}}{2\kappa})^{1/2}.$$

The above corollary has been obtained as a one factor model by Heath, Jarrow, Morton (1992) and a two factor model by Jarrow and Brenner (1990). It is characterized by the full use of initially observed term structure as input parameters. Unlike the spot rate approach which entails a market premium function, there is no need for estimating such preference parameter in the option formula. Furthermore, instead of computing a cumbersome joint distribution of the terminal payoff as in HJM, the approach here requires merely simple integration once the nonstochastic volatility assumption is adopted.

3. Comparison between risk neutral measure and forward equivalent martingale measure

Granted that the forward equivalent martingale measure has produced an arbitrage free bond option value, it is natural to question whether it is by accident that this value is identical to that calculated from the risk neutral measure via the HJM model. Intuition suggests that these two values cannot differ. This is so since in a viable price system that is free of arbitrage opportunity, the 'real' cash flow should be determined independent of the numeraire chosen. In other words, the riskiness of the terminal payoff has already been reflected by the shadow state price, whereas any chosen numeraire only plays the role of a scaling factor so that the state price density can be transformed to be a risk adjusted probability measure.

The following proposition formalizes the above intuition. Before stating this useful result, denote the terminal payoff of a contingent claim (that may pay continuous dividend) by $c(\tau,T)$ where $\tau < T$. Furthermore, let

$$\tilde{c}(s) = \frac{c(s,\tau,T)}{P(s,\tau)} \quad \text{and} \quad c^*(s) = \frac{c(s,\tau,T)}{\exp(\int_t^\tau r(s)ds)}$$

and let $\mathfrak{z}(\tau) = \dfrac{1}{P(t,\tau)\exp(\int_t^\tau r(s)ds)}$.

Proposition 1. $E_{\tilde{Q}}(\int_0^\tau \tilde{c}(s)ds) = E_{Q^*}(\int_0^\tau c^*(s)ds)$.

Proof. From the left hand side,

$$E_{\tilde{Q}}(\int_0^\tau \tilde{c}(s)ds) = \int_\Omega \int_0^\tau \frac{c(s,\tau,T)}{P(s,\tau)} ds\, d\tilde{Q}$$

$$= \int_\Omega \int_0^\tau \frac{c(s,\tau,T)}{P(s,\tau)} ds \frac{d\tilde{Q}}{dQ^*} dQ^*$$

$$= \int_\Omega \int_0^\tau \frac{c(s,\tau,T)}{P(s,\tau)\exp(\int_0^\tau r(s)ds)\exp(-\int_0^\tau r(s)ds)} ds \frac{d\tilde{Q}}{dQ^*} dQ^*$$

$$= \int_\Omega \int_0^\tau \frac{\dfrac{c(s,\tau,T)}{\exp(-\int_0^\tau r(s)ds)}}{P(s,\tau)\exp(\int_0^\tau r(s)ds)} \, ds \, \mathcal{Z}(\tau) \, dQ^*$$

$$= \int_\Omega \int_0^\tau \frac{c^*(s)\mathcal{Z}(\tau)}{\mathcal{Z}(s)} \, ds \, dQ^*$$

$$= \int_0^\tau \int_\Omega \frac{c^*(s)\mathcal{Z}(\tau)}{\mathcal{Z}(s)} \, dQ^* \, ds$$

$$= \int_0^\tau \int_\Omega \int_{\mathcal{F}_s} \frac{c^*(s)\mathcal{Z}(\tau)}{\mathcal{Z}(s)} \, dQ^*(\mathcal{F}_T) dQ^* ds$$

$$= \int_0^\tau \int_\Omega \frac{c^*(s)\mathcal{Z}(s)}{\mathcal{Z}(s)} \, dQ^* \, ds$$

$$= \int_\Omega \int_0^\tau c^*(s) \, ds \, dQ^*$$

$$= E_{Q^*}(\int_0^\tau c^*(s)ds).$$

□

Throughout the proof of the above proposition, we have assumed the conditions for Fubini theorem is satisfied. Therefore interchanging integrals (applied twice) is justified. This proposition manifests the 'numeraire invariance principle' for it illustrates the irrelevance of the particular choice of numeraire in computing the arbitrage free contingent claim value.

The invariance principle is not a surprising result since an arbitrage free viable price system shares a feature familiar from a Walrasian general equilibrium models. That is that a change in numeraire does not cause a reallocation of resources in the economy. Only changes in relative price can trigger real economic changes. The necessary condition that upholds the equality in the above proposition is the existence of the R-N derivative $\frac{d\tilde{Q}}{dQ^*} = \mathcal{Z}(\tau)$.

An alternative interpretation of the invariance principle is that in an arbitrage free viable system "martingale to martingale" transformations should be permissible. With this interpretation, $\gamma(\tau)$ is merely performing a *risk reshuffling* function, but in doing so guaranteeing the fair game feature of the model is preserved. One must be careful of not using the term risk transformation to qualify $\gamma(\tau)$, for in that context, as explained by Cox and Ross (1976) and Harrison and Kreps (1979), the usual role played by the Radon-Nikodym derivative is drift removal for viable price processes.

If it is just a matter of interchanging martingale measures, the risk transformation may just be discarded as an esoteric exercise. However, it is an immensely important transformation for the last section is a testimony of the analytical convenience of valuing a bond option with respect to $\tilde{Q}$ rather than Q^*. The cumbersome joint conditional expectation of the terminal options payoff and the stochastic terms structure under Q^* has suddenly become a matter of finding a simplified univariate conditional expectation of the terminal option payoff with respect to $\tilde{Q}$. Thus the use of a forward equivalent martingale measure has recovered the attractive Cox and Ross approach of obtaining Black Scholes formula.

A slightly different way of comparing $\tilde{Q}$ and Q^* can be accomplished by re-examining the concept of futures and forward prices given a stochastic term structure. In their famous paper, Cox, Ingersoll and Ross (1981) have couched the analysis of the difference between the futures and forward prices in terms of applying the fundamental arbitrage principle. One of the important insights of these authors is to express the futures and forward prices as values of traded assets in the absence of arbitrage opportunities. The determination of futures and forward prices are then reduced to the determination of the rational values of these assets even though the futures and forwards and not themselves asset prices.

Specializing CIR's general result to the present context with a stochastic term structure, the futures bond price can be viewed as the present value of a terminal discount bond price $P(\tau,T)$ times a saving account that accumulates interest from present until time τ. From the arbitrage-free analysis of HJM, such a payoff with the saving account chosen as numeraire immediately implies that the futures price is necessarily a Q^*-martingale process. On the other hand, the forward bond price can be viewed as the present value of the terminal payoff given by $\frac{P(\tau,T)}{P(\tau,\tau)}$.

The last section has already justified that the forward bond price is a $\tilde{Q}$ martingale process. Here it is useful to focus on Q^* to present an alterative to CIR's characterization of the difference between futures and forward price. Using CIR's notation momentarily, define $H(t)$ and $G(t)$ as futures bond prices and forward bond prices respectively.

In a fairly general setup, CIR demonstrate the necessary condition that a contingent claim F must obey. That is in a continuous time and continuous state economy, the valuation equation for F is a fundamental partial differential equation (their equation (43))

$$\frac{1}{2}\Sigma_{i,j}\mathrm{cov}(X_i,X)F_{X_iX_j}+\Sigma_i(\mu_i-\phi_i)F_{X_i}+F_t-r(X,t)F+\delta(X,t) = 0$$

where subscripts on F defines partial derivatives and X is a vector containing all variables necessary to describe the current state of the economy. Also,

$\mu_i \equiv$ the local mean of the changes in X_i

$cov(X_i, X_j) \equiv$ the local covariance of changes in X_i with the changes in X_j

$r(X_i, t) \equiv$ the spot interest rate

$\delta(X,t) \equiv$ the continuous payment flow received by the claim

$\phi_i \equiv$ factor risk premium associated with X_i.

The above necessary condition combined with sufficient condition for valuation by arbitrage is now applied to streamline the essential difference between a future price and a forward price.

Proposition 2: In the absence of arbitrage opportunity, the futures bond price is a Q*-martingale process. The forward price is a Q*-martingale if and only if the R-N derivative $\frac{d\tilde{Q}}{dQ^*} = 1$ almost everywhere.

Proof. By proposition 2 of CIR, futures bond prices is the current value of an asset that has a terminal value given by

$$P(\tau,T)\exp\left(\int_t^{\tau} r(u)du\right) = H(\tau).$$

Also by proposition 7 of the same paper, these authors have shown that

$$\delta(X,t) = r(X,t)H(X,t).$$

Substituting these results into the fundamental partial differential equation and with appropriate relabelling $H \equiv F$, we have their equation (44):

$$\frac{1}{2}\Sigma_{ij}\operatorname{cov}(X_i X_j)H_{X_i X_j}+\Sigma_i(\mu_i-\phi_i)H_{X_i}+H_t = 0$$

subject to terminal condition $H(X,\tau) = P(\tau,T)\exp(\int_t^\tau r(u)du)$.

By the sufficient condition of pricing by arbitrage, there exists an equivalent martingale measure Q* such that $H(X,\tau)$ relative to the saving account given by $\dfrac{H(X,\tau)}{\exp(\int_t^\tau r(u)du)}$ is a Q*-martingale process. This yields

$$H(X,t) = E_{Q^*}(P(\tau,T)|\mathcal{F}_t)$$

which is a specialized version of CIR equation (46).

Next by proposition 1 of CIR, the forward bond price is the current value of an asset that has terminal payoff $G(\tau) = \dfrac{P(\tau,T)}{P(t,\tau)}$. This $G(\tau)$ can only be a Q*-martingale if the conditional expectation of $\dfrac{G(\tau)}{\exp(\int_t^\tau r(u)du)}$ yielding

$$\begin{aligned} G(t) &= E_{Q^*}\left(\frac{G(\tau)}{\exp(\int_t^\tau r(u)du)}\right) \\ &= E_{Q^*}(P(\tau,T)|\mathcal{F}_t). \end{aligned}$$

The second equality can hold only if $\dfrac{dQ}{dQ^*} \equiv P(t,\tau)\exp(\int_t^\tau r(u)du) = 1$.

□

The first part of the above proposition has verified an intuitive aspect of a futures price. Namely, in a viable price system H(t) is simply the risk adjusted predictor of the terminal random spot price. The second part of the proposition states that the forward price can also be expressed as a Q*-martingale, and therefore equivalent to futures price if the stringent condition $\frac{d\tilde{Q}}{dQ^*} = 1$ is satisfied. However, this effectively reduces $\frac{dQ}{dQ^*}$ to be a deterministic constant one almost everywhere, contradicting the property that $\frac{d\tilde{Q}}{dQ^*}$ is a Q*-measurable random variable.

Cox, Ingersoll and Ross express the same concern about the seeming contradiction if the equivalence between the futures and forward prices is maintained. They explain this implausible equivalence between the two prices as the equivalence of the strategy of "going long" with "rolling over" strategy.

Within the argument developed above, we can observe that for the equivalence of the two strategies to hold,

$$P(t,\tau) = \exp\left(-\int_t^{\tau} r(u)du\right)$$

is required almost everywhere. This leads to an extremely stringent requirement in a stochastic interest rate context. It constrains the geometric rate of return from a sure deposit to be the average of a sequence of stochastic instantaneous returns. Such an implausibility is the intuition that motivates the concern from Cox, Ingersoll and Ross about the equivalence of forward and futures prices.

4. A comment on the reexamination of the Expectations Hypothesis

This section reviews the validity of the traditional Expectations Hypothesis of the term structure of interest rates. The Hypothesis has a long history in financial economics and can be traced back to the writings of Irving Fisher (1896) and Hicks (1939). Despite a number of possible formulations, the original Expectations Hypothesis attains its popularity by the assertion that the implied forward rates are the unbiased predictor of the random future spot rates. The status of this version, however, has been shaken by the modern term structure literature (notably led by the paper of Cox, Ingersoll and Ross (1981c).

Cox, Ingersoll and Ross' attack on the above Hypothesis rests fundamentally on the Jensen inequality. In this regard, their analysis is a prime example of the spot rate approach to the determination of an equilibrium term structure. In particular, given a stochastic specification of the instantaneous spot rates as

$$dr = \alpha(r,t)dt+\sigma(r,t)dB(t),$$

one can use a simple hedging argument to obtain the classic no arbitrage condition, namely

$$\frac{\mu_P(t,T)-r(t)}{\sigma_P(t,T)} = \lambda(r,t)$$

where $\mu_P(\cdot)$ and $\sigma_P(\cdot)$ are the drift and volatility of the bond price process and $\lambda(t)$ is the equilibrium market risk premium.

Applying Ito's lemma to the coefficients μ_P and σ_P of a T-maturity discount bond can turn the no-arbitrage condition into a partial differential equation

$$rP = \frac{1}{2}\sigma(r,t)^2P_{rr}+[\alpha(r,t)-\lambda(r,t)\sigma(r,t)]P_r+P_t$$

subject to the boundary condition that P(T,T) = 1. The solution to the above fundamental valuation equation is proven by Cox, Ingersoll and Ross using a result from Friedman (Theorem 5.2, 1975)

$$P(r,t,T) = E_{\mathbb{P}}[\exp(-\int_t^T r(s)ds-\frac{1}{2}\int_t^T \lambda(s)^2 ds+\int_t^T \lambda(s)dB(s))|\mathcal{F}_t].$$

Note that the expression

$$\exp(-\int_t^T \lambda(s)dB(s)-\frac{1}{2}\int_t^T \lambda(s)^2 ds)$$

is the Radon Nikodym derivative $\frac{dQ^*}{d\mathbb{P}}$ which can be interpreted as the equilibrium state price density function.

It can be further shown that the Expectations Hypothesis does not hold in this framework. To see this, denote

$$A(T) \equiv -\int_t^T r(s)ds-\frac{1}{2}\int_t^T \lambda(s)^2 ds+\int_t^T \lambda(s)dB(s).$$

In this model, the forward rate is derived from the equilibrium bond price solution via the following relationship:

$$f(t,T) = \frac{-P_T(r,t,T)}{P(r,t,T)},$$

where subscript denotes partial derivative. By Ito's lemma

$$P_T(r,t,T) = -E_{\mathbb{P}}\left(r(T)e^{A(T)}\right).$$

To test the consistency of Expectations Hypothesis with this model, one merely need to check if $E_{\mathbb{P}}(r(T))$ is equal to $f(t,T)$. That is

$$E_{\mathbb{P}}(r(T)) \gtrless \frac{E_{\mathbb{P}}\left(r(T)e^{A(T)}\right)}{E_{\mathbb{P}}\left(e^{A(T)}\right)} \qquad \text{or}$$

$$E_{\mathbb{P}}(r(T))E_{\mathbb{P}}\left(e^{A(T)}\right) \gtrless E_{\mathbb{P}}\left(r(T)e^{A(T)}\right).$$

Since $r(T)$ and $e^{A(T)}$ are likely correlated, the RHS is larger than the left hand side by Jensen inequality. (More precisely, the inequality should be called the Holder's inequality which arises because of the nonzero covariance.) Note that even if λ = 0, the two sides of the above expression still remain unequal. This result therefore also falsifies a tendency to infer that Expectations Hypothesis may become valid in a risk neutral world.

Next, we approach the Expectations Hypothesis from the perspective of the pure arbitrage analysis developed here. The notable difference comes from taking the bond yield as given instead of being derived from the equilibrium. In this regard, the forward rates process is an exogenous stochastic process:

$$df(t,T) = \mu(t,T)dt+\sigma_f(t,T)dB(t).$$

In integral form,

$$f(t,T)-f(0,T) = \int_0^t \mu_f(v,T)dv+\int_0^t \sigma_f(v,T)dB(v).$$

The spot rate is obtained by having $T \to t$ so that $r(t) = f(t,t)$ or

$$r(t)-f(0,t) = \int_0^t \mu_f(v,t)dv+\int_0^t \sigma_f(v,t)dB(v).$$

Applying the arbitrage free condition to the forward rate process via the HJM model yields

$$dB^*(t) = dB(t)-\lambda(t)dt$$
$$\mu_f(t,T) = \sigma_f(t,T)[\sigma_P(t,T)-\lambda(t)] \qquad \forall\ t,T \in [0,\tau]$$

where $dB^*(t)$ is a Brownian motion with respect to the risk neutral probability measure Q^* and the second equation is the forward rate drift restriction. Substituting $dB^*(t)$ and $\mu_f(\cdot)$ into the spot rate yields

$$\begin{aligned} r(t) &= f(0,t)+\int_0^t \sigma_f(v,t)[\sigma_P(v,t)-\lambda(v)]dv \\ &\quad +\int_0^t \sigma_f(v,t)[dB^*(v)+\lambda(v)dv] \\ &= f(0,t)+\int_0^t \sigma_f(v,t)\sigma_P(v,t)dv+\int_0^t \sigma_f v,t)dB^*(v). \end{aligned}$$

Note that first integral from the second equality is not zero and the future spot rate is not a driftless Q^*-martingale.

On the other hand, applying the arbitrage free conditions of the forward price process converts the future spot rate process to be a $\tilde{Q}$ martingale. This is shown in the next proposition:

Proposition 1. Given that the forward price is a $\tilde{Q}$-martingale process, then the future instantaneous spot rate process $r(t)$, where $0 < t^* < t < T$, is also a forward equivalent martingale process with respect to $\tilde{Q}$.

Proof. From theorem 1 and proposition 1 of section 1,

$$d\tilde{B}(v) = dB(v)-\lambda(v,t^*,t)dt$$
$$\lambda(v,t^*,t) = \lambda(t)-\sigma_P(v,t^*,t).$$

Because the forward rates restriction holds for all times, for $t^* < t$

$$\mu_f(v,t^*,t) = \sigma_f(v,t^*,t)[\sigma_P(v,t^*,t)-\lambda(t)]$$
$$= -\sigma_f(v,t^*,t)[\lambda(t)-\sigma_P(v,t^*,t)].$$

Then

$$\frac{\mu_f(t^*,t)}{\sigma_f(t^*,t)} = -\lambda(t^*,t).$$

Substituting these expressions into the spot rate process yields

$$\begin{aligned} r(t) &= f(0,t)+\int_0^t \sigma_f(v,t^*,t)\frac{\mu_f(v,t^*,t)}{\sigma_f(v,t^*,t)}dv \\ &\quad +\int_0^t \sigma_f(v,t^*,t)[d\tilde{B}(v)+\lambda(t^*,v)dv] \\ &= f(0,t)-\int_0^t \sigma_f(v,t^*,t)\lambda(t,v)dv+\int_0^t \sigma_f(v,t^*,t)d\tilde{B}(t) \\ &\quad +\int_0^t \sigma_f(v,t^*,t)\lambda(t^*,v)dv \\ &= f(0,t)+\int_0^t \sigma_f(v,t^*,t)d\tilde{B}(v). \end{aligned}$$

Finally taking conditional expectations with respect to forward equivalent martingale measure yields

$$E_{\tilde{Q}}[r(t)|\mathcal{F}_0] = f(0,t).$$

□

To sum up, two approaches have been used to examine the validity of a version of the Expectations Hypothesis under uncertainty. On the one hand, this Hypothesis is inconsistent with the prediction from the equilibrium spot rates approach. The problem is mainly caused by the Jensen's inequality. The investor's preference plays no role in causing such inconsistency.

On the other hand, deriving the instantaneous spot rate process from the exogenously specified forward rates process recovers the Expectations Hypothesis. This is achieved by applying the arbitrage free conditions from the forward price process to the spot rate so that it becomes a forward equivalent martingale process. The spot rate process however is not a martingale with respect to the risk neutral measure.

Early analysis of Cox, Ingersoll and Ross (1981) have pointed out that a seemingly special case for unbiased Expectations Hypothesis would be a scenario of full certainty. The catch of their remark is that trivial risk neutrality alone is not sufficient to produce the unbiasedness of forward rate as a prediction of future spot rate. In the above analysis the bond market is assumed to be dynamically complete and the term structure is fully spanned by existing discount bonds. One of the fundamental insights from an Arrow-Debreu complete securities market setting is that it effectively reduces the economy to full certainty.

Therefore this comparison of a stochastic economy with an equilibrium world of perfect certainty allows us to recover a classic version of Expectations Hypothesis. Of course, this rationalization of the unbiased expectations hypothesis in a complete market is just heuristic. The main result is primarily driven by the absence of arbitrage opportunities which transforms the spot rate process to be forward equivalent martingale process. It suffices to conclude that the unbiased Expectations Hypothesis is a statement of absence of arbitrage.

6. Summary and conclusion

This chapter presents the arbitrage free approach to valuation of a bond option and its implications. The approach is based on the simple presumption that there is a completely spanned term structure. It is still a relatively fresh methodology and likely promising more interesting results than the few presented above. Therefore rather than conclusively closing the topic, it is perhaps more useful to streamline further the idiosyncrasy of this approach.

Our modern treatment of valuing interest rate related contingent claims manifests a fundamental guiding principle in finance. That is by observing a set of traded assets prices, one is able to extend these prices to value other derivative securities by an appeal to the absence of arbitrage opportunity. The payoffs to this approach is quite far-reaching and so it is worth to reiterate some of them here. The arbitrage free methodology stress the preference free advantage of pricing financial assets. This advantage spans both the theoretical and empirical aspects of the topic. The empirical convenience of the preference free feature is quite obvious. One is freed of the nagging chore of estimating the market price of risk function in this case. The only set of parameters left for estimation are those embedded in the volatility functions of the term structure.

On the theoretical side, the principle of parsimonious parameterization is almost always the advisable approach to asset pricing. The advance of options pricing as a major finance paradigm since Black-Scholes' contribution is primarily founded on risk neutral pricing. The only parameter in that model requiring specification is the volatility of the equity process.

Cox, Ingersoll and Ross (1985b) argue that it may be inappropriate to remove the preference parameter from bond options pricing problem when the underlying state variable is the non-traded interest rate. The fact that the interest rate is a fundamental economy wide variable forms a basic motivation for these authors to use an equilibrium approach to the valuation problem. Indeed, the equilibrium approach to pricing bond options is more driven by the need to endogenize the value of a pure discount bond.

The priority interestingly works in reverse if the objective is primarily to value an interest rate related derivative security. Once the dramatic assumption of a complete market is adapted, the power of Black-Scholes and Harrison and Kreps absence of arbitrage methodology reveals itself immediately by the removal of the drift of the price process. Thus as shown above, pricing bond options by the absence of profitable arbitrage entails merely the specification of a volatility function for the forward price process.

Furthermore, a by-product of this option pricing problem is the preference free pricing of the unit discount bond can also be solved as well. First, one can express the terminal payoff of the discount bond relative to a chosen numeraire. The conditional expectation of this numerated payoff with respect to the corresponding equivalent martingale measure must be the initially observed term structure. This is first pointed out in Ho and

Lee's discrete time model (1986). The validity of this observation is promoted in further Hull and White (1992) and Jamshidian (1987).

Given the above credit for supporting the preference free methodology to bond options pricing, it is fair to square up some of the remaining unresolved problems with this approach. As mentioned above, the arbitrage approach adopts a reverse priority to the equilibrium spot rate approach. Initially specifying a forward rate process, one is led to a random spot rate that is highly non-Markovian. The path-independence feature of a Markov interest rate process is an appealing feature for much analysis. Needless to say this is one of the main reasons motivating the early spot rate literature.

With an arsenal of mathematical tools in the stochastic calculus literature, one would be less surprised that the non-Markovian part of the problem will soon be resolved. At this point, we conjecture the technique used will be a time change Brownian motion. The creative part of the problem, however, is to provide a sound justification for employing any relevant mathematical tool. After all, Black Scholes' contribution is not about introducing PDE to economics and finance but rather foreshadowing the concept of risk-neutral pricing.

Similarly, Harrison and Kreps should not be merely credited for first applying the mathematical martingale representation theorem but also the dynamic spanning concept and the deep justification for the continuous tradings. There is definitely a distinction between the mathematics of passion and the passion for mathematics.

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