THE USE OF THE CALCULATOR AS AN INSTRUCTIONAL MODE FOR TEACHING LIMITS

by

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THE USE OF THE CALCULATOR AS AN INSTRUCTIONAL MODE FOR TEACHING LIMITS

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ABSTRACT

This thesis reported on the development and implementation of a supplement on the topic of limits in a first calculus course at the university level. The supplement was designed so that the calculator was an integral part of the instructional mode. The intent of the supplement was to promote students' understanding of the concept of the 'limit' of a function. The primary purpose of this thesis was to investigate whether or not students' understanding of the concept of limit was enhanced by using calculators as an integral component of the instructional mode.

The supplement provided guidelines about the use of a calculator (i.e., the "calculator method") in determining limits of functions and in verifying the answers thus obtained by employing standard problem solving techniques mentioned in the course.

In reviewing the literature prior to the preparation of the supplement, the claims, counterclaims and suggestions made vis-a-vis using a calculator were examined, so that the construction of the supplement was informed by this review.

The supplement included a questionnaire that sought to identify which of the claims, issues and suggestions as determined by the literature review were of concern to calculus students. More precisely, the questionnaire invited students' comments about the following four broad questions:

1. Did the use of the calculator enhance students' understanding of the concept of the limit of a function?

2. Did the students consider the calculator to be an effective learning device?
3. Did the students favor the use of the calculator as an integral part of the calculus curriculum?
4. Did the students judge the supplement to be a useful instructional guide?

After a pilot run in the Fall of 1981, a revised supplement was implemented with Mathematics Department students at Simon Fraser University in the Spring of 1982. Their responses were analyzed. Based on that analysis, four main conclusions were drawn:

1. The use of the calculator aided in students' understanding of limits.
2. The students considered the calculator to be an effective learning device.
3. Though a clear-cut conclusion regarding the integration of the use of the calculator in the calculus curriculum could not be made, the students evidently favored the use of both methods in the class as well as in the examination in dealing with limits.
4. The students judged the supplement to be a good guide for the study of limits. They felt that it gave a clear and concise overview on the subject.

DEDICATION

To my brothers and sisters

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C H A P T E R I

INTRODUCTION

1. The Importance and Validity of the Study.

In our present complex society, calculators and computers are among the most significant and ubiquitous technological advances of our time. They are having an ever-increasing interaction with all spheres of our daily existence. They send us bills, run our telephone systems, keep our records, generate, administer and score tests. Their importance is being felt in all phases of our lives including home, school, supermarket, banking, business, industry, transportation, aviation, food production, government communications, administration, sciences, education as well as leisure.

Due to their low-cost, increasing availability and portability in comparison with computers, calculators became accessible to more people within an extraordinarily short span of time. As a consequence, educators, school administrators and parents became concerned about the calculator's potential impact on education - particularly, on mathematics education.

This concern was indicated by numerous articles about calculators* which appeared in various journals and the general press as well as by many hours of discussions about calculators by

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*Henceforth, the word calculator refers to any calculator, including programmable calculators.

interested groups such as the National Council of Teachers of Mathematics.

2. An Overview of the Use of the Calculator in Classrooms.

Like many other technological devices, the advent of calculators was marked by both enthusiasm and hesitation. There was a diversity of opinions as to their introduction, use and effects in the school setting among educators, school administrators, teachers, parents and others working in this field throughout the world. Their use in classrooms became a highly controversial issue. Points of view regarding their introduction in classrooms ranged from the positive to negative extremes. In an analysis done by Shumway (cited in Suydam, 1976) the extremes and a more central position were depicted as follows:

1. From first grade on, hand-held calculators should be made readily available to all children, for all school work. (p. 19)
2. Restrict calculator use to checking answers only; or to certain days of the week; or to the upper grades (10-12); or restrict capabilities of the calculator by making electronic changes or masking so that capacities surpassing the curriculum are not within students' reach and paper-and-pencil algorithms are still necessary. (p. 23)
3. Classroom use of hand-held calculators for mathematics should be banned. (p. 21)

Since these controversies were outcomes of varied beliefs and opinions concerning objectives and priorities of school mathematics instruction among different sectors of society, it is essential as well as pertinent to study various developments in mathematical education over roughly the past two decades.

A. Evolution within Mathematical Education from 1960 to Date .

The 1960s was a period of crises in mathematical education. As a consequence, emphasis was placed on the "why" of mathematics (i.e., on understanding the structure and principles of mathematics) (Butts (cited in Pikaart et al., 1980, p. 113); NAEP, Jan. 1975, p. xiii). Mathematics curriculum and instruction underwent a revolution. The need to revise mathematics programmes was realized by the public (NCTM Agenda, 1980, p. i).

According to the National Advisory Committee on Mathematical Education (NACOME, 1975) attempts to improve school mathematics curricula were spurred by public and professional debates in the mid 1950s. Curricular innovations included new mathematical topics, new organization of mathematical programmes and new grade placements of traditional topics as well. Focus of these "new math" content innovations was on powerful but abstract structuring concepts and processes such as set theoretic concepts, algebraic field properties (commutativity, associativity and distributivity) and number bases. It was believed that set theoretic concepts would provide another means for explaining and drilling basic concepts and skills of mathematics. (pp. 1, 3, 15-16)

The rationale and design for the so-called "new math" curriculum were devised at a number of conferences, both American and international, held in the late 1950s and early 1960s. Though initial innovations were geared toward high school mathematics programmes for college preparatory students, modifications had to be made also in elementary and junior high school mathematics programmes. (NACOME, 1975, pp. 1, 3)

Some of the events that guided the above curricular innovations were the following (NACOME, 1975):

1. The 1959 publication of the report of the College Entrance Examination Board (CEEB), by the Commission on Mathematics, which suggested reform in secondary school mathematics curriculum on these grounds: (a) change in pure mathematics structure because of vast developments of new concepts and methods, (b) application of both classical and new ideas to biological, social and management sciences, and (c) satisfaction of need for mathematically developed scientific manpower. The Commission Report suggested inclusion of topics from logic, modern algebra, probability and statistics in the new content but the main emphasis was on efficient reorganization and treatment of traditional topics. (NACOME, 1975, pp. 1-2)
2. The 1963 Cambridge Conference approval of school mathematics reforms - Goals for School Mathematics, grades K-12, that proposed acceleration and enrichment of traditional curricula

and influenced all succeeding curriculum research and development, especially at the elementary school level. (NACOME, 1975, p. 2)

3. The 1960 publication of Jerome Bruner's The Process of Education that led mathematics teachers to stress conceptual understanding of mathematical methods and had a strong impact on decisions about goals for mathematics instruction K-8. Bruner's hypothesis was "any subject can be taught effectively in some intellectually honest form to any child at any stage of development". (NACOME, 1975, p. 3)
4. Influence of Piaget's theories that directed math educators to consider meticulously what children could learn at various stages of development which, in turn, guided formulation of new curricula for elementary mathematics in which the "intellectually honest form" of teaching was usually a concrete model of mathematical ideas. The aim was to de-emphasize rote learning. (NACOME, 1975, pp. 3-4)

In November 1975, the NACOME, appointed by the Conference Board of the Mathematical Sciences and funded by the National Science Foundation (NSF), released a report Overview and Analysis of School Mathematics, Grades K-12 to exhibit the national impact of the "new math" goals on the United States school curricula. The year 1972-73 was chosen as a benchmark. A quick glance of the impact is presented below:

1. At the high school level, implementation of the recommendations of Commission on Mathematics for CEEB was immediate. School

Mathematics Study Group (SMSG) texts stressed treatment of inequalities along with equations, structure and proof in algebra, integration of plane and solid geometry with coordinate methods, integration of algebra and trigonometry and a course in elementary functions for twelfth grade. (p. 5)

2. At the junior high school level, curriculum change was evidenced by the use of new textbooks. Texts following the SMSG and UMMaP (University of Maryland Mathematics Project) experimental materials contained concepts and language of sets, both algebraic and informal properties of number systems, non-standard numeration systems and number theory. In addition, several nationally standardized tests were developed and used due to the rising movement relating to "accountability" for educational programmes. (pp. 9-10)
3. At the elementary school level, curricular changes were gradual, but substantial. Incorporation of geometry, probability and statistics, functions, graphs, equations, inequalities, and algebraic properties of number systems in curricula transmuted "arithmetic" into "mathematics". (pp. 10-11)

Curriculum guides reflecting the above innovations were developed throughout the 1960s. Although (NAEP, Jan. 1975) the mathematics being taught was not specifically "new" or "modern", the fear that traditional topics such as computation might not receive due importance in schools prevailed. (p. xiii)

Furthermore, the NACOME (1975) related that some critics of the "new math" reform complained: the new content (set theory, Boolean algebra, topology, symbolic logic and abstract algebra) was not appropriate for school curricula, was deductively structured and was too formal (since it put too much emphasis on symbolism and terminology). It ignored interaction of abstract ideas with applications (i.e., was too theoretical) and was not suitable for average and low ability students because it did not satisfy their requirements for basic mathematical literacy. Proposals were made for interdisciplinary and career oriented curricula as well as for return to skill oriented curricula. (pp. ix, 14, 24)

Commenting on developments in the 1970s, Carl (NAEP, 1979), one of the interpreters of the data on changes in mathematical achievement between 1973 and 1978, recorded that

> The early 1970s introduced an era of experimentation in teaching approaches. New approaches - open classrooms, team teaching, performance contracting, individualized instruction and alternative schools - were instituted with lofty expectations, though many of them were left unfulfilled. This, coupled with concern for decline in achievement test scores, provoked the slogan that schools go "back-to-the-basics" and focus on the fundamentals of reading, writing and mathematics. Another issue - "accountability" of schools - gathered momentum. Furthermore, to ensure that high school graduates possessed so-called minimal abilities, "minimal competency" requirements were installed in some places. (p. 24)

> The mid 1970s was marked by the "back-to-the-basics" movement which stressed computational skills (i.e., ability to perform the four basic operations - addition, subtraction, multiplication and division) and knowledge of facts and definitions. Moreover, textbooks focusing on basic mathematics were in widespread use during this period. (p. 24)

According to the NACOME (1975), the label "new math" refers to the vague phenomenon or the diversified series of developments that occurred in school mathematics between 1955 and 1975 (p. 22). It had a general push and trend but sprang from many roots, evolved and assumed many phases (p. 21). The crux of the logistics of that era was to generate alternative innovative programmes, and not just a unique approved model (p. xii). In spite of its several accomplishments, it had many unachieved goals, and problems that ignited criticism among educators, parents and politicians (p. 147). These developments (NAEP, 1979) led to considerable public debate and left many people uncertain as to what the schools were and what they should have taught, and anxious about what had happened to the students in the era 1955-1975 (p. xi).

It should be noted that the "mathematics curriculum" reform or the "new math" reform of 1955-1975 was a worldwide movement to up-date the content and teaching of school mathematics. It had its counterparts in countries other than the United States - Canada, Japan, the Soviet Union, Germany, Denmark, Belgium, France, and Great Britain. It was also subject to professional and public criticism in many of those countries. (NACOME, 1975, pp. x-xi)

The advent of calculators added another aspect to the uncertainty regarding curricular decision making for it challenged the traditional preference given to arithmetic skill development in grades K-8. Furthermore, availability of computers led to profound questions about arrangement and emphasis on topics in algebra, geometry and calculus (NACOME, 1975, pp. ix-x). The NACOME envisioned that the use of calculators would lead to restructuring of elementary school mathematics curriculum (by earlier introduction and more stress on decimal fractions, with postponement and reduction of stress on common fractions), exposition of new and significant mathematical concepts for low achieving students, and nullification of existing standards of mathematical achievement (pp. 41-42). The NACOME recommended that beginning no later than the end of the eighth grade, every mathematics student should be provided with a calculator in every mathematics class and allowed its use in all mathematical work including tests (p. 138). Furthermore, it recommended research regarding the uses of calculators and computers in curriculum at all grade levels as well as revision of curriculum in view of calculator and computer advances (pp. 144-145).

As mentioned earlier, during the 1970s, more emphasis was given to the acquisition of meaningful skills than to curriculum content because of the "back-to-the-basics" and "minimum competency" issues. Nonetheless, there was an equal concern that too much emphasis on low-level skills might prevent understanding of applying

those skills in problem situations. (Meiring (cited in Pikaart et al., 1980, p. 6))

Two national studies of mathematics achievement within the American population were conducted on 9, 13 and 17 year olds by the National Assessment of Educational Progress (NAEP), funded by the National Institute of Education, during the 1972-73 and 1977-78 school years respectively. On the basis of the NAEP findings, panelists made a number of recommendations. Those that pertain to problem solving are outlined below (NAEP, 1979):

1. Expansion of the definition of "basic" in mathematics so as to include emphasis on problem solving ability.
2. Modification of textbooks to include more variety of problem solving tasks.
3. More emphasis on teaching problem solving in schools than on exercising mastery of skills.
4. Performance on tests not to be the sole criterion for evaluating the effectiveness of mathematics programmes.
5. Equal importance to be given to both ability to analyze a problem situation and its correct solution. (p. 27)

The recommendations were made following significant average declines shown by all three age groups on the application items of mathematics assessments between 1973 and 1978. Mathematical application includes the use of mathematical knowledge, understanding and skills in the solution of problems. Problem solving requires more than just computations. It requires the ability to choose the right procedures, facts, understandings, interpretations as well as

the ability to apply required processes in the right order. (NAEP, 1979, p. 12)

The fact, that problem solving needs much more than computation, was revealed by the hand calculator portion of the Second Mathematics Assessment of the NAEP, on comparisons of performances of all age groups on problem solving with and without the use of a calculator. (Carpenter et al., 1981, pp. 127-128)

In their position paper on basic mathematical skills, the National Council of Supervisors of Mathematics (NCSM) (cited in Pikaart et al., 1980, p. 28) commented that the narrow definition of basic skills, which regarded mathematical competence as equivalent to computational ability, was a result of: declining scores on college entrance examinations and standardized achievement tests, reactions to the NAEP reports, soaring costs of education, accountability issues, shift in emphasis from curriculum content to instructional modes and alternatives, growing need to provide remedial and compensatory programmes, and publicity by the media of all these issues.

During their 1976 Annual Meeting in Atlanta, Georgia, the need for a unified position on basic mathematical skills was expressed by more than one hundred participating members. Consequently, the NCSM established a task force to formulate a position on basic mathematical skills. Reasons cited for expanding the definition of basic skills included calculator and computer availability, and changing needs of the existing technological society such as - daily use of skills of estimating, problem solving, interpreting data,

organizing data, measuring,predicting and application of mathematics to everyday situations. (p. 28)

According to the NCSM (cited in Pikaart et al., 1980), basic mathematical skills fall under ten vital areas. The first area is problem solving. They state that

> Learning to solve problems is the principal reason for studying mathematics. (p. 29)

The above discussion reveals that the 1970s was a period of unrest for mathematics education. Some of the high-lights of this era were: back-to-the-basics movement, accountability of schools, minimum competency issues, achievement test results, calculator and computer developments, and curriculum changes.

As we enter into the 1980s , attention becomes focused on the area of problem solving. In 1979, the National Council of Teachers of Mathematics (NCTM Agenda, 1980) funded by the NSF, conducted a survey of the beliefs of both professionals and non-professionals about school mathematics. This project was known as Priorities in School Mathematics (PRISM). After giving serious consideration to the results of this survey, in April 1980, the NCTM released An Agenda for Action: Recommendations for School Mathematics of the 1980s. The agenda is meant for a decade of action and consists of eight recommendations. The first recommendation stresses problem solving. The recommended actions include, among other things, organization of mathematics programmes of the 1980s around problem solving that exploit the use of various mathematical concepts and techniques. The third recommendation proposes the use of calculator and computer in mathematics programmes at all grade levels. It is

suggested that these devices be exploited in problem solving. Furthermore, the NCTM believes that computational skills are still required, emphasizes development of imaginative materials and stresses teacher education. (pp. i, 1-2, 8, 13, 25-26)

The above discussion demonstrates a brief history of the evolution within mathematical education from 1960 to date. As a consequence of this evolution, the issue of the use of calculators in classrooms displayed several facets. The following three seem to be most prominent:

1. Should calculators be used in classrooms?
2. When should calculators be used in classrooms?
3. How should calculators be used in classrooms?

These facets will be explained in the next few pages.

B. Evolution of the Use of Calculators in Classrooms.

As indicated earlier, educators, teachers, parents and lay persons differed in opinions concerning introduction of calculators into classrooms. Most parents seemed to be resisting the introduction of calculators into classrooms, particularly in the lower grades. They felt that the use of the calculator would threaten the acquisition of computational skills and cause a decline in students' ability to perform the paper-and-pencil algorithms. (Recall that in the early 1970s, need for return to skill oriented curricula was felt, and "back-to-the-basics" movement was the major force of the mid 1970s). They also held that children would not develop abstract thinking and that they would solve problems by guessing and

not by thinking. Furthermore, they would become so dependent on the calculator that they would not be able to solve mathematics problems without them. The idea of using calculators in classrooms seemed to be shocking even to many professional educators.

On the other hand, some educators mildly supported the use of the calculator. They were of the view that the calculator should not be used until after the students have, to some degree, mastered the basic operations. Those on the positive extreme believed that calculators were instructional tools for computational skill development and problem solving activities. They seemed to be convinced that calculators could provide another method for helping children to think, create and learn mathematics. Another view supported introduction of calculators only with appropriate changes in the content of curriculum.

Thus, concerns of parents, teachers, educators and school administrators seemed to revolve around all the three issues listed before, though the central point of discussion was whether or not calculators should be used in classrooms.

However, passage of time alleviated, to some extent, hesitency about introduction of calculators in classrooms. Their wider accessibility, dropping cost and use at home made some educators think that calculators could not be ignored because they existed in this real world. This, in turn, led them to recommend their use in classrooms and focus their attention on 'when' and 'how' to make the best use of calculators to reinforce mathematical skills and

ideas. For example, in September 1974, the NCTM adopted the following position statement (Bell, esty, Payne & Suydam, 1977):

> Because of reduction in its cost, the minicalculator is becoming increasingly available to students at all levels. It is for mathematics teachers to realize its latent contribution as a valuable teaching aid. It should be used in the classroom in imaginative ways for reinforcement of learning and motivation of the learner as he becomes adept in mathematics. (p. 224)

Bell et al. (1977) stated that teachers, parents and administrators were concerned with questions related to both immediate and long-range futures. For example:

> How can I use calculators in my class tomorrow? Should I allow my child use of the family's calculator for homework? Should I buy a classroom set of calculators for my primary group? What kind should I buy? Should I wait a few years so that the prices drop while mathematical capabilities of calculators increase? Will calculator use prepare my child better to deal with real life problems or not? How will the calculator fit into the total mathematics programme of my students? (pp. 224-225)

The NACOME (1975) insisted strongly on reducing the emphasis on computational skills because of the widespread accessibility of calculating aids (p. 42). On the other hand, approximation, interpretation of numerical data and estimation were recommended for emphasis (pp. 37, 42). In addition, the NACOME suggested provision of calculators for those students who had not acquired arithmetic proficiency even by the end of the eighth grade (pp. 41-42). The

Committee envisioned a few changes in school programmes that were mentioned before. Furthermore, the Committee suggested some questions to be researched:

1. When and how to introduce calculator use so that it does not prohibit development of students' understanding on computational skills and algorithms.
2. Will calculator use facilitate or hinder memory of basic facts?
3. Which mathematics procedures require step-by-step paper-and-pencil calculations for complete understanding and retention?
4. What kind of machine logic and display are required for satisfactory uses of school?
5. What curricular materials are needed to exploit classroom use of calculators?
6. How does the use of calculators affect instructional emphasis, curriculum organization, and student learning styles in secondary mathematics subjects? (pp. 42-43)

The Committee expressed the opinion that calculators allowed students to feel the power of mathematics and free time for teachers to stress conceptual aspects of the subject (p. 43). Finally, the Committee made the major recommendation that beginning no later than the end of the eighth grade, each mathematics student should be provided with a calculator in each mathematics class and be allowed to use it during all of his or her mathematical work including tests (p. 138). Other suggestions included need of research on the uses of calculating aids in curriculum at all levels and their relationship

to instructional objectives; development of instructional materials at all levels for calculators; curricular revision or reorganization in view of the emergence of calculators and computers (pp. 144-145).

The National Institute of Education (NIE) sponsored a Conference on Basic Mathematical Skills and Learning in October 1975 in Euclid, Ohio. Each of the thirty-three participants presented a position paper on the following issues (Bell et al., 1977):

1. What, in fact, are basic mathematical skills and learning?
2. What problems keep a child from acquiring basic mathematical skills and what role should the NIE play in dealing with these problems? (p. 227)

Some of the topics discussed in the position papers included:

1. The impact of calculators on the curriculum.
2. Calculators as a vehicle for re-investigating the place and emphasis of various topics in the mathematics curriculum.
3. The use of the calculator as an aid in early counting skills.
4. The amount and types of paper-and-pencil procedures needed in and out of school. (p. 228)

Furthermore, they realized the need to re-examine curriculum structures and priorities at the secondary school level because of the powers of calculators and computers to perform traditional computations. They felt that this influenced the definition of basic skills. (p. 228)

Suydam (cited in Bell et al., 1977) conducted research for the NSF about the range of beliefs and reactions about calculators. In addition to a literature search, questionnaires were sent to

teachers and other school personnel, state supervisors of mathematics, mathematics educators in colleges and universities, and textbook publishers. The questionnaires sought arguments for and against the use of calculators as well as answers to such questions:

How should calculators be used?

What uses are important at different levels?

What modifications should be made in the curriculum? (pp. 229-230)

Furthermore (Bell et al., 1977), to obtain additional arguments, articles in educational and non-educational journals and in news-papers, conference reports, curriculum materials, position papers, and other documents were surveyed. In order to attain information on current and future sales and development, calculator manufacturers were surveyed. (p. 230)

For the Interim Report, Shumway (cited in Suydam, 1976, p. 19) developed a section in which he expanded on arguments in favor of and against using calculators. Based on Suydam's (1976) analysis of the reasons given by educators and others, as well as on Shumway's analysis, arguments against the use of calculators included:

1. The use of calculators could replace development of computational skills. Students would lose motivation to learn basic facts and algorithms. In addition, children's ability to perform the paper-and-pencil algorithms would be impaired. These algorithms are still necessary because calculators can never be everywhere. Since the primary objectives of mathematics teaching (at least in grades K-9) are that children learn the basic facts and paper-and-pencil algorithms for addition,

subtraction, multiplication and division , the use of calculators would demolish the fundamental structure of elementary school mathematics. This was the most common argument given by opponents, which included most parents and other members of the lay public. [Recall that "back-to-the-basics" movement was a major force in mathematical education during the 1970s].

2. Children would create a wrong impression about mathematics, that mathematics is nothing else but pushing buttons on a black box and involves only computations performed without any thinking. Moreover, they would think that in mathematics, emphasis is on product rather than on process--structure is not important.
3. Children would not develop mathematical (abstract) thinking because they would become dependent on calculators and would stop making use of their brains. Some children and teachers would misuse calculators by taking advantage of them for every simple calculation.
4. Calculators are unsuitable for slow learners, because they would destroy their motivation to learn basic skills. Moreover, they cause decline in children's ability to detect errors.
5. There is lack of research on calculator effects.
6. Calculator's cost makes it inaccessible to every child.
7. Calculators lead to maintenance and security problems. Batteries are unreliable. They lose their charge and wear out. (pp. 18, 21-23)

The above views represent the negative extreme. Views representing the positive extreme as consolidated from Shumway (cited in Suydam, 1976) and Suydam (1976) are the following:

1. Calculators assist in computation. They reduce drudgery of tedious calculations, and increase speed and accuracy. Their calculational power facilitates understanding and concept development. Concepts such as properties of functions (simple, logarithmic, exponential, trigonometric), exponents, compounding continuous interest, limits and number theory can become more interesting. Moreover, calculators reinforce basic facts and concepts with immediate feedback and hence they reduce memorization work. As far as the usual paper-and-pencil algorithms for basic operations are concerned, they will no longer be required, because the calculator is the best calculational algorithm. In addition, extensive drill-and-practice exercises will not be necessary because most children would learn basic operations to make estimations and to save time.
2. Calculators facilitate intuitive number sense because children can have early experience with numbers of all sizes with increased frequency. This would increase the power of mathematics used by the common man inconceivably.
3. Calculators aid in solving problems that are more realistic because of their calculational power.
4. Calculators motivate by stimulating curiosity, positive attitudes and independence. Consequently, they encourage

discovery, exploration, creativity,estimation, approximation and verification. Thus they help in exploring,understanding and learning algorithmic processes.

5. Calculators motivate low achievers by removing the frustration and fear of being unable to perform necessary calculations.
6. The use of calculators permits early introduction of, as well as postponement of, some topics. For instance, decimals and scientific notations can be introduced in first grade (because children encounter them more frequently on calculators) and algorithms of fractions can be delayed until algebra. Furthermore, de-emphasis of paper-and-pencil algorithms allows introduction of new topics in mathematics curriculum. Moreover, increased speed and accuracy allow more time to teach concepts and principles of mathematics in depth.
7. Since calculators exist in this real world, they will have to be recognized. Their cost will not prohibit their widespread use because it is falling. (pp. 17, 19-21)

Some more moderate positions as reported by Shumway (cited in Suydam, 1976) are as follows:

1. Restrict the use of calculators to grades 10-12, so that students have already mastered the basic facts and paper-and-pencil algorithms.
2. Restrict the use of calculators to particular days only or to checking answers only, so that students still have to learn basic facts and paper-and-pencil algorithms.

3. Restrict potentialities of calculators by making electronic changes or masking so that capacities surpassing the curriculum are not within students' reach and paper-and-pencil algorithms are still essential.
4. Introduce calculators only with appropriate changes to the content of curriculum.

In summary, Shumway suggested the following rational resolution of these concerns:

> Examine prevailing and future needs of the society for basic facts and paper-and-pencil algorithms. If no such skills are required, de-emphasize them and introduce widespread use of calculators. If such skills are needed, then examine if calculators can be introduced in classrooms in such a way that these skills can still be developed. (pp. 23-25)

In the above report, a summary of suggestions for research included (Bell et al., 1977):

1. When and how should calculators be introduced?
2. What are effective ways to learn basic facts, computational skills, problem solving and other mathematical concepts?
3. Is there a need for paper-and-pencil algorithms?
4. What are efficient procedures for calculators?
5. What are long-range effects of using calculators?
6. Which kinds of calculators are best for the classroom? (p. 232)

Appendices to the above report contained position papers prepared by interested educators. In their position paper, Immerzeel, Ockenga and Tarr wrote that to get rid of future shock, imaginative

software must be developed. The examples that they gave for using the calculator usually involved topics from existing curricula. (pp. 233-234)

The Calculator Information Centre at Columbus, Ohio, has published a series of information bulletins, reference bulletins and state-of-the-art reviews on calculators. The following paragraphs summarize the portions on background and types of uses being made of calculators in schools from the first review, released in April 1978 (Suydam) .

The price of calculators had reduced to one-tenth of what it was four years earlier. Their status had risen from an item of luxury to an item of necessity. Marketing figures revealed a sale of more than 80 million calculators in the United States, including sales to individual parents and schools.

The degree of acceptance of the calculator varied with school level as follows. At the college level, the calculator was easily recognized as a tool in mathematics, engineering,science and some other courses for all students from remedial to advanced. The approval was high at the secondary school level as well. The calculator was considered as a time saving device, so that the time saved on hand calculation could be spent on the development of mathematical ideas and more interesting content. Though their use was widespread, it was not incorporated into instruction by every secondary school mathematics teacher. The major issue - whether calculators should be used on tests - seemed to be disappearing because teachers were using tests where calculators did not affect the goals being tested.

From the junior high school level downward, reluctance about calculator use increased, particularly, in classes for low achievers because many teachers still believed in the mastery of computational facts and procedures by students before they used calculators. Nonetheless, some teachers felt that the low achievers be allowed the use of calculators so that they could learn some real mathematics instead of struggling with basic facts and algorithms. In the elementary school, the calculator use was more at the intermediate level than at the primary level because parents as well as teachers believed that children should have acquired mastery of basic facts and algorithms before they proceeded to use calculators. The "back-to-the-basics" bandwagon also accounted for the suppression of calculator use at the elementary school level, because in order to satisfy the demand of parents and school boards for a more "traditional" kind of arithmetic programme, teachers focused their attention on computation work. In lieu of exploring effective uses of calculators, drill-and-practice materials were developed.

Furthermore, uses for the calculator at the elementary school level included checking paper-and-pencil computations, games, calculation and exploratory activities. The secondary school level emphasized calculation, recreations and games, exploration and use of calculator - specific materials. (pp. 10-13)

The next few paragraphs record some of the highlights of the second state-of-the-art review published in May 1979 (Suydam).

Slowly but surely calculators were being incorporated into the teaching process at all levels. 'Should calculators be used in

classrooms' was still an issue for some parents and teachers but people were getting more used to this teaching aid. The calculator was being recognized as an instructional tool. Its use increased with grade level. (pp. 2-3)

The NCTM released a new position statement suggesting other ways the calculator could assist in teaching. In addition, the Council encouraged the use of the calculator in the classroom as an instructional aid and a computational tool. [The first position statement appeared in the NCTM Newsletter in December 1974]. (p. 2)

Moreover, many school districts recognized competency with calculators as one of the minimal competencies required for graduation. In addition, about 100 studies were conducted in the previous four or five years to assess the effects of calculator use. The goal of most of those studies was to determine whether or not calculator use harmed mathematical achievement of students. With the exception of a few, all the studies revealed that students who used calculators for mathematics instruction (but not on tests) achieved at least as high or higher scores than students not using calculators for instruction. (Note that this indicates that reduction in time spent on paper-and-pencil work did not appear to hurt the achievement of those who used calculators). Hence it was for teachers to teach children as to 'when' and 'how' to use calculators. (pp. 4, 8)

Finally, though a large number of studies revealed that the use of a calculator did not appear to have detrimental effects on the mathematical achievement of those who used calculators, several of them did not record scrupulously how the use of the calculator was

made by students or teachers. Usually, the calculator was used in the way it was deemed fit by the student or teacher, that is, for checking paper-and-pencil computations or for activities that indicate nothing more than confirming that the calculator is a calculating device. (pp. 4-5)

A summary of the third annual review prepared in August 1980 by Suydam follows.

The availability of calculators was fading as an issue. Their cost was not fluctuating any more. Battery life was prolonged. Decrease in their size and weight increased their portability. Awareness to their potential instructional applications kept on growing. This fact was evident from the third recommendation of the NCTM Agenda for Action: Recommendations for School Mathematics of the 1980s, released in April 1980. According to this recommendation,

> Mathematics programs must take full advantage of the power of calculators and computers at all grade levels. (p. 1)

The rationale for this understanding included: In addition to gaining familiarity with the part played by calculators and computers in society, most students must know how to use them, especially in problem solving. Furthermore, in view of the availability of these computing aids, the computational skills required by every citizen need to be re-examined. A major part of early schooling should deal with the study of number concepts and skills without a calculator, although, the calculator should be made available when tedious computations become more important than the educational value of the procedure. (pp. 1-2)

Perhaps the addition of this cautionary note helped calm down the fears of those who objected to the introduction of calculators in lower grades.

The actions recommended by the NCTM to achieve this objective included:

Calculators and computers should be made available to all students throughout their school mathematics programme. Their use should be made a part of the core mathematics curriculum. Further, they ought to be used in imaginative ways so as to discover, explore and develop mathematical concepts and not just for checking computations or for drill-and-practice. Curriculum materials that incorporate their use in various and imaginative ways should be developed and made available as well. Moreover, software should fit the goals of the programme. Goals and developmental sequence should not be twisted to fit the available software and technology. (p. 2)

Furthermore, in order to ensure the maximum advantage for students from the use of calculators and computers both at school and home, teachers and administrators are required to interact with parents. Finally, to deal with the needs of teachers, not only should colleges offer courses concerning instructional applications of calculator uses for preservice as well as in-service teachers, but also certification standards should demand such preparation. Professional organizations should disseminate information in every possible way. (p. 2)

As far as the effectiveness of the use of the calculator in teaching-learning process was concerned, both data from studies and

evidence from teachers' practical experience indicated that the use of the calculator aided in the teaching of a number of mathematical ideas (pp. 3-4). However, when beliefs and attitudes were surveyed, it was evident that many people did not take notice of the research evidence in relation to achievement and learning. Concepts regarding the uses and importance of the calculator in mathematics curriculum depended on the type of the group surveyed. The PRISM survey of preferences and priorities, conducted in 1979, by the National Council of Teachers of Mathematics, devoted almost 20% of its items to discover how educators at all levels (from primary through college), parents, and school board members felt about the uses of the calculator. Strongest support was received from supervisors and teacher educators (85% and 74% respectively); acceptance by teachers at all levels was lower (support averaged 50%); and very little support was given by parents and school board members to increased emphasis and to uses of calculators other than checking answers. Thus, the increased use of calculators was supported far more by educators than by lay persons and checking answers turned out to be a noncontroversial use of the calculator. (pp. 3-4)

Materials integrating the use of the calculator in order to teach mathematical concepts were scarce. Further, materials emphasizing only games were decreasing,while those supplementing the on-going instruction were increasing. The existing published articles involved work with operations, functions, exponents, polynomials, square roots, and problem solving. The following two collections of materials for teachers became available:

1. A collection of articles from the Arithmetic Teacher and the Mathematics Teacher (Burt, 1979).
2. A categorized listing of references on calculators (Suydam, June 1979). (Suydam, 1980, p. 7)

Nonetheless, materials which developed mathematical ideas were still needed.

Though there was very little support for using calculators in classrooms, this situation appeared to change with the acceptance of the calculator by people in their own as well as their children's lives. The existing concern was when should the calculator be used in teaching basic facts and algorithms? The fear that paper-and-pencil computational skills will be destroyed and achievement scores will deteriorate, still prevailed, even though research revealed that the use of the calculator did not lower computational skill accomplishment. Hence parents needed to be assured that calculator usage could enhance the understanding of several mathematical concepts and thus advance mathematical attainment. (p. 7)

Finally, we come to the fourth annual state-of-the-art review prepared in August 1981 by Suydam. A report of this review follows.

The use of the calculator in the classrooms was not normally a matter of dispute as it used to be in the mid 1970s. Instead, the approval of the calculator as a tool was increasing. Reasons for this change in people's attitude could be:

1. Addition of the cautionary note in the third recommendation of the NCTM Agenda that "a significant portion of instruction

in the early grades must be devoted to the direct acquisition of number concepts and skills without the use of calculators".

2. Research evidence that the use of the calculator does not affect computational skill achievement adversely.

3. Defusing of the issue due to passage of time. (p. 1)

Furthermore, it was apparent that 'should calculators be used' was no longer an issue on the educational front. (p. 2)

The NAEP findings on the Second Mathematics Assessment revealed that students showed better performance on routine computations when the calculator was used. However, they did poorly on all non-routine computational exercises as well as on exercises evaluating concepts and understanding. No improvement was shown when the calculator was used. Moreover, the data indicated that problem solving needed more than computational skills. Many studies conducted in the United States since January 1980 aimed at determining the effect of calculator usage on problem solving. This interest in problem solving was aroused by the first recommendation of the NCTM Agenda. According to this recommendation, "Problem solving must be the focus of school mathematics in the 1980s". (pp. 2-3)

The findings of these studies included:

1. The use of the calculator assists problem solving provided the problems are within the limits of students' paper-and-pencil computational ability.

2. When using a calculator, students are less afraid to handle hard problems and employ widely varying problem solving techniques.

3. There is not much difference in the number of problems done with or without a calculator.
4. The scores on problem solving are possibly not influenced by the use of the calculator. (p. 3)

Materials containing activities exploiting the use of the calculator on topics in the existing curriculum were being published. These activities were more frequent for the middle and secondary grades than for the primary grades. In addition, the number of materials for the programmable calculator was on the increase. The greatest frustration was that materials which integrated the use of the calculator throughout the curriculum were deficient. Perhaps microcomputers were diverting people's attention from developing such materials. Though the use of the calculator could alter both methodology and curriculum content,materials reflecting such changes were very rare and slow in appearing. (p. 5)

At the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics held in 1976, the participants had thought that new calculator-oriented materials would be developed within five years. That five-year interim period was over but the interim materials had not arrived because only meagre attention was given to that task. (pp. 5-6)

By way of summary, the above discussion reveals that the issues--'<u>should</u> calculators be used in classrooms?' and '<u>when</u> should calculators be used in classrooms?'--have almost disappeared. The present concern of interested groups revolves around the third issue--'<u>how</u> should calculators be used in classrooms?', so as to have

the best results concerning development and reinforcement of mathematical skills and concepts.

3. Scope of the Study.

Calculators have become commonplace due to their capacity to perform speedy and accurate calculations. People are increasingly accepting their existence in their own lives as well as in their children's lives. The very fact that they exist all around us makes them difficult to ignore. Now in schools children are having calculators to use. Calculators are being recognized as useful teaching and learning tools. They are being integrated into the school mathematics programmes at all grade levels. Their uses are being carefully explored. Students are becoming growingly familiar with their roles in society and are trying them in various ways.

Several interested individuals and groups are studying effects of calculator use. NACOME's (1975) concerns included: how to exploit calculator use so that it does not hinder students' understanding and skills in arithmetic operations and procedures (p. 42). Moreover, different ways of teaching mathematics by the use of this new tool are being investigated and required materials are being developed. Research (Suydam, 1978) has shown that the calculator can be used to improve the growth of mathematical concepts and skills, and thus promote mathematical achievement (p. 20). Parents and teachers are becoming aware of these findings.

In the overview section of this chapter, it was discussed that the presence of calculators can influence instructional emphasis,

curriculum organization , student learning styles, teaching patterns and strategies, and content of curriculum. Consequently, some topics become superfluous while some become accessible as well as easier. For example (NACOME, 1975), the calculator can be a boon for low achievers who fail to acquire functional levels of arithmetic computations even upto the end of the eighth grade. The calculator can handle their arithmetic needs of daily existence. In addition, topics such as probability, statistics, functions, graphs and coordinate geometry become easier and accessible for them. Furthermore, calculator use can lead to redesigning of elementary school mathematics curriculum by delaying and de-emphasizing fractions but emphasizing and introducing decimals, negative integers, exponents, square roots, scientific notation and large numbers earlier in the curriculum, because students encounter them while experimenting with the calculator. Thus, calculator availability definitely challenges traditional instructional priorities. (pp. 41-42)

In addition (NACOME, 1975), in view of calculator availability, less emphasis needs to be placed on purely mechanical aspects of arithmetic, while more emphasis is needed on crucial aspects of problem solving process, and more real problems with messy calculations can be treated (p. 42). Obviously, computational skills require less emphasis.

The NCTM (NCTM Agenda, 1980) expressed a need to reassess computational skills required by every citizen because they thought some of them would become more important, whereas others would become

less important or redundant due to the availability of computing aids (p. 8).

While efforts are being made (NCTM Agenda, 1980) to integrate the use of the calculator into the core mathematics curriculum and to develop calculator-oriented curricular materials (p. 9), there is still scarcity of such materials. Those that are available (Suydam, 1981) do not pay special regard to change in content and method of instruction (p. 5).

Thus, it would seem clear that if calculators are to be used successfully as an instructional device, then curricula need to be developed which make the use of calculators an integral part of the instructional mode. Moreover, teachers and parents need to be made aware of how the calculator can be a positive addition to the sets of tools children may use to learn and understand mathematics, and that calculators will not detract from student learning.

This thesis is concerned primarily with the development and testing of a calculator-based unit of work. Special consideration is given to methodology and content. The unit was designed for a topic in a first calculus course at the university level. Since calculus is also one of the advanced level secondary school mathematics courses, this unit is directly related to the school mathematics instruction from which most of the above information on calculator use is derived.

The topic of 'limits' constitutes the subject matter of the unit of work. The principal purpose of the unit was to enhance students' understanding of the concept of the 'limit' of a function.

The unit was designed so that the calculator was an integral part of the instructional mode. Guidelines were provided as to the use of a calculator in investigating limits of functions. Students were required to use calculators to carry out various numerical calculations (i.e., to use the "calculator method") in order to ascertain and approximate limits. In addition, the standard problem solving techniques mentioned in their text were also displayed with the goal of enabling students to verify their results.

Observe that the contents of the unit include some limits which require L'Hospital's Rule as the standard technique. This rule is introduced only after students are familiar with the notion of the derivative (because L'Hospital's Rule depends on derivatives). The notion of derivatives rests completely on the notion of the limit. Due to this reason, such limits must be delayed until the introduction of L'Hospital's Rule. Thus, the author attempts to display that the use of the calculator permits early introduction of some topics in the curriculum, and thus influences curriculum organization.

At the end of the unit, there is a questionnaire which is aimed at investigating, among other questions, whether or not the unit of work and the use of the calculator enhanced students' understanding of the concept of the limit of a function. A detailed description of the entire unit of work (including the questionnaire) is supplied in Chapter III.

4. Statement of the Problem.

This thesis reported on the development and testing of a supplement on the topic of limits in a first calculus course at the university level. The supplement was designed so that the calculator was an integral part of the instructional mode. The intent of the supplement was to promote students' understanding of the concept of the 'limit' of a function. The primary purpose of this thesis was to investigate whether or not students' understanding of the concept of limit was enhanced by using calculators as an integral component of the instructional mode. The other concerns were about the use of the calculator as an effective learning device, integration of the use of the calculator in the calculus curriculum, and quality of the supplement as a whole.

More precisely, the study was designed to seek answers to the following four broad questions:

1. Did the use of the calculator enhance students' understanding of the concept of the limit of a function?
2. Did the students consider the calculator to be an effective learning device?
3. Did the students favor the use of the calculator as an integral part of the calculus curriculum?
4. Did the students judge the supplement to be a useful instructional guide?

5. Closing Remarks.

The present chapter has presented an introduction to and a description of the nature of the study. The next chapter is composed of a review of the literature that encompasses claims, counterclaims, suggestions, research studies and instructional materials concerning calculator usage. Here the brief but dynamic growth of the use of calculators in classrooms described above will be fleshed out.

C H A P T E R II

REVIEW OF THE LITERATURE

1. Preliminary Remarks.

This chapter reviews the literature relevant to the study. Not only does it expand the rationale behind the problem under consideration, but it also lends insight into curriculum considerations concerning the calculator. More explicitly, this chapter records various claims, counterclaims and research studies regarding the use of the calculator. In addition, it reviews some suggestions for how and in which mathematical topics calculators might be used in order to obtain the best results pertaining to the development and reinforcement of desired mathematical skills and concepts. Furthermore, the chapter presents a glimpse of some of the activities conceived by various mathematics educators to exploit the use of the calculator as an effective instructional tool.

2. Counterclaims Regarding the Use of Calculators.

Implementing educational change (Gross, Giacquinta & Bernstein, 1971) is a complex process. This process includes three requirements: (a) locating or developing a new idea, (b) obtaining funds needed to execute it, and (c) convincing the staff of the educational value of the innovation. Most school administrators are of the opinion that if the initiation phase of the process is handled properly, innovations can be successfully implemented.

Teachers may have extremely positive feelings towards a proposed change or may meet frustrations or difficulties in carrying them out. Most innovations require significant alterations in the existing mode of behaviour. Switching to new ways may need considerable time. This may also involve stressful periods. But such periods are likely to constitute forward steps toward the implementation of an innovation. (pp. 208-209)

In order to develop new and better techniques and to implement their use, educators need to be aware of: the process of change, techniques that can be used at various steps in the process, and methods to obtain feedback on the effect of the change during the experimental phase and even after it has become an integral part of the system. To minimize the resistance to planned change affecting the educational community, educators must work toward (Eiben & Milliren, 1976):-

1. Improving the interpersonal competencies of teachers, administrative staff and students.
2. Bringing about a change in the educators' priorities so that humanistic concerns are given preference.
3. Enhancing understanding among all members of the school community so as to reduce tension and anxiety.
4. Giving assistance to the educational community in resolving conflicts and improving communication.
5. Selling the process of change to all members of the school and community organization to the extent where all commit to devote time and energy in planned change. (p. 110)

In connection with calculators, Bell (1976) suggested finding solutions to the problems of philosophy,curriculum and methodology, design, and school management of calculators. (p. 502)

As far as the techological innovations in education are concerned, they can be approached in a number of ways ranging between two extremes. The negative extreme considers the new technology to be an unnecessary and undesirable intrusion into the classroom, while the positive extreme regards it as the long-awaited solution to a myriad of difficulties which removes all obstructions and clears the route for effective and efficient teaching and learning.

A few years ago, a new technological innovation - the calculator - appeared in the market place and, by 1973, was available in considerable numbers (Bell, 1977, p. 7). The tininess of electronic components enabled the calculator industry to flourish like a crop of mushrooms. Furthermore, the declining costs and growing variety of calculators made them available to more and more people to the extent that they became one of the staples in almost every home.

This tool represented (Bell, 1977) at least one order of magnitude improvement in speed and at least two orders of magnitude decline in cost, size and weight over other mechanical devices which possess similar or lesser calculating capabilities (p. 7). This rendered other calculating devices--slide rules and tables--obsolete. As Willson (1978) remarked, "It is clear that logarithms and slide rules as means of calculation are already museum-pieces (except in some examinations!)" (p. 55).

Since the use of calculators by students increased dramatically, school administrators, and mathematics teachers were faced with decisions regarding the use of calculators in schools. Institutions responded to this phenomenon in a variety of ways. Some institutions prohibited their use in classrooms, while others encouraged it or left it to the discretions of the students. Reporting on the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics, Bell (1977) stated that the participants arrived very early at the following working conclusion:

> The increasing availability of hand-held calculators in the students' world at home and at school, forces educators to take a hard look to see what course the schools should take. (p. 7)

This conclusion was supported (Bell, 1977) by the fact that the order of magnitude changes in technology generally bring about fundamental changes in society. (p. 7)

Educators generally view (Roberts, 1980) with skepticism the effect of innovations which take place in schools or are found by children outside schools. The calculator received the same fate, because, many educators thought that teaching machines encouraged rote memory and not creative thinking. The calculator was rejected by many educators who viewed them as toys. (p. 71)

In its initial stages, the wave of this movement was small but all of a sudden the waters of this deluge swelled. More educators began to realize the imminent changes that this recent innovation could cause in programs. Slogans--pro and con--like the following were in the air: "As far as pocket calculators are concerned, I feel it would be

better if students had brains in their heads before they put them in their pockets" (Engel, Education Digest, 1976, p. 48); "If you have to ban the calculator to teach a mathematics course, then what you're teaching is trivial" (Fey, Education Digest, 1976, p. 48).

Bell (1978) listed some people's concerns. They are expressed in phrases like: "People are already too dependent on machines". "Hard things shouldn't be too easy". "It rewards sloth and ignorance to give a calculator to someone who hasn't learned to calculate without one". Also to some people, use of calculators in schools means "pampering", "frills", "waste of taxpayers' money" and other such code words concerning standard moral concerns about schools. (p. 406)

Besides technological conservatism, the reasons for peoples' initial aversion to the introduction of calculators in classrooms included the perception that the use of the calculator would ruin the basic, mainstream mathematics of the elementary school curriculum, hinder mathematical thinking, and decrease motivation to learn paper-and-pencil algorithms, as discussed in the last chapter. According to Shumway (1976), public debate concerning the use of the calculator for teaching mathematics seemed most controversial at the elementary school level (pp. 571-572). Bell (1977) reported:

> These small, portable, and inexpensive machines have the potential for replacing the paper-and-pencil calculations that have been the major (and often the sole) component of elementary school arithmetic. (p. 7)

The capability of replacing a large part of already existing curriculum made the issue both pressing and controversial (Bell, 1977). It

created enough dissatisfaction among some influential sectors of the public and guided proposals to "ban the calculator" in schools. At the same time, warnings originated from within mathematics education telling people to use calculators in early grades "with care, if at all". Another striking factor was the feeling expressed by industry consultants that microelectronic technology was changing at an astonishing rate. As a result, very soon, the four-function calculator could be replaced by one with many more functions, scientific calculators could be replaced by programmable calculators, which could further be replaced by medium-sized computers. (p. 7)

The misuse of the calculator also accounted for people's disapproval toward introduction of calculators in classrooms. Johnson (1978) described the abuses of the calculator as follows:

> Calculator books and magazines are in abundance, and much of what appears in these supplementary materials represents merely play activity, or worse forces the use of the machine with little attention to the goals of school mathematics. (p. 50)

Further, Johnson classified the most common abuses into four categories:

1. Calculations with awkward numbers with no apparent purpose.
2. Games and puzzles with no mathematical objectives.
3. Mystical button pushing such as making words by turning a calculator upside down.
4. Checking answers.

Shumway (cited in Suydam, 1976), who quoted as well as summarized arguments proposed for and against the use of the hand calculators in school mathematics (see Chapter I), maintained that the opponents' argument is essentially the following:

> The principal objectives of mathematics instruction (at least in grades K-9) are that children learn the basic facts and the paper-and-pencil algorithms. Such learning will not occur if calculators are made available in schools. (p. 23)

On the other hand, the proponents' argument is essentially:

> The hand-held calculator is the tool used in society today for calculations. Schools are 'burying their heads in sand' if hand-held calculators are not recognized and used as the calculational tool that they are. (p. 23)

Furthermore, as discussed in the previous chapter, Shumway (cited in Suydam, 1976) reported that the extreme of the point of view against using calculators was represented by the statement:

> Hand-held calculators should be banned from classroom use for mathematics. (p. 21)

The extreme of the point of view in favor of using calculators was represented by the statement:

> Hand-held calculators as sophisticated as the so-called 'scientific calculator' should be made readily available to all children, for all school work, from first grade on. (p. 19)

Roberts (1980) reported that most of the concerns about the effects of the use of the calculator emerged from educational institutions, and from parents, teachers and principals of elementary school children. Moreover, the opponents contended that the use of the calculator might damage the growth of children's mathematical abilities. (p. 72)

As an illustration, McKinney (1974), a professor of mathematics, advanced the following arguments:

> If what we're talking about is reducing tedious calculations, then perhaps minicalculators can be an aid, but teaching a student to push buttons won't help him if what he needs is more instruction in actual addition, subtraction, multiplication and division - I can't think of any reason why a fourth or fifth grader should even see one, after all, that's when we're trying to teach basic arithmetic. (p. 13)

In an article which appeared in the News Exchange, it was mentioned that the 1978 NAEP results showed that the students were weak in mathematics problem solving. Causes for this situation were given by a few educators. O'Brien ("Issue", 1980) contended that it was due to the use of a calculator:

> Is it any wonder that students do poorly? Perhaps more distressing are the data concerning the use of a calculator. Of 13 year olds, 85 percent never use a calculator in math classes (94 percent said "never" for science classes), yet 73 percent report the use of a calculator in their home activities. (p. 5)

In connection with an experiment conducted in a town in Pennsylvania on seventh-grade students,a questionnaire was sent to their parents and 60% response was received. Sample comments from the parents were (Rudnick & Krulik, 1976):

> "Let's go back to teaching the basics, not teach our children to be dependent upon a machine."
>
> "Stop experimenting with our kids; you have already lost one generation to modern math."
>
> "No way our kids should use the machines. Teach them basics."
>
> "It's a good idea! But what will teachers do with the time left over?"
>
> "It's all right to introduce the calculator in the higher grades, after the students learn their basic skills."
>
> "The calculators are too easily stolen."
>
> "Under no circumstances should the tax-payers' money be spent on this." (p. 655)

The above statements reveal the strong indignation and fear of some of the parents regarding the use of the calculator in the classroom.

They also indicate the degree to which the introduction of calculators in schools has not been a well-managed innovation. This innovation certainly has not followed the suggestions made by Eiben & Milliren (see page 39) as to the factors one has to be cognizant of when introducing changes. But despite the fact that there was much resistance initially to the use of calculators in classrooms, the proponents did win out, or so it seems. The arguments they presented to buttress their case are examined next.

3. Claims and Suggestions Regarding the Use of Calculators.

This section deals with both claims and suggestions in favor of the use of calculators in classrooms. Because many of the articles reviewed contained both claims and suggestions, it was not feasible to divide these into two separate sections for review.

There is an abundance of articles in various journals and the general press as well as research papers that proclaim that the use of calculators in classrooms has positive instructional value. Several arguments have been put forth in favor of the use of calculators in classrooms. Suggestions have been made as to the ways in which the use of a calculator may facilitate and promote student learning so that the calculator may be a positive addition to the set of tools children may use to acquire and enhance mathematical skills and ideas.

The National Council of Teachers of Mathematics evinced a keen interest in encouraging the use of the calculator as a valuable instructional aid in the classroom. In September 1974, the NCTM Board of Directors adopted a position statement which appeared in the NCTM Newsletter in December 1974 (cited in Shumway, 1976):

> With the decrease in cost of the minicalculator, its accessibility to students at all levels is increasing rapidly. Mathematics teachers should recognize the potential contribution of this calculator as a valuable instructional aid. In the classroom, the minicalculator should be used in imaginative ways to reinforce learning and to motivate the learner as he becomes proficient in mathematics. (p. 572)

Furthermore, the NCTM Instructional Affairs Committee (1976) identified nine justifications, with sample problems, for suggesting the potential use of the hand-held calculator in the mathematics classroom. According to the Committee, the minicalculator can be used as follows:

1. To encourage students to be inquisitive and creative as they experiment with mathematical ideas.
2. To assist the individual to become a wiser consumer.
3. To reinforce the learning of basic number facts and properties in addition, subtraction, multiplication and division.
4. To develop the understanding of computational algorithms by repeated operations.
5. To serve as a flexible "answer-key" to verify the results of computation.
6. To be like a resource tool that promotes student independence in problem solving.

7. To solve problems that previously have been too time-consuming or impractical to be done with paper and pencil.
8. To formulate generalizations from patterns of numbers that are displayed.
9. To decrease the time needed to solve difficult computations. (pp. 72-74)

Pollak (1977) commented that in many instances, new teaching techniques and devices received either "missionary enthusiasm" or "uncompromising disdain" by mathematicians. But he advocated that instead of opting for either of the above routes, we should take a deeper look at the problems we have in school mathematics instruction, where a calculator might be an aid. Furthermore, Pollak suggested a few topics where the use of the hand-held calculator might provide pedagogic advantages. These topics were functions, inverse functions, iteration methods for solving simultaneous linear equations, probability, statistics and linear programming. (pp. 293-295)

As stated in Chapter I, Shumway (cited in Suydam, 1976) reported a few moderate positions concerning the above issue. While some people view these approaches as unworkable efforts to "have your cake and eat it too", the others consider them as "democratic compromises" to the attainment of the best solution to the issue (p. 23). Furthermore, Shumway suggested the following rational approach to the above issue: namely, examine prevailing and future needs of the society. If no such skills are needed, de-emphasize them and introduce calculators. If such skills are needed, then examine if calculators

can be introduced in classrooms in such a way that these skills can still be developed. (p. 24)

Bell (1976) proposed orderly investigation of some questions concerning student reactions to calculators, some pedagogical issues, and classroom management problems.

Sloyer (1980) contended:

> We should stop asking,"Should we allow our students to use calculators?" and instead ask, "How can we teach more and better mathematics with the use of a calculator?" (p. 617)

Hawthorne (1973) claimed that calculators could be of great help to the elementary school children because they allow "immediate verification, which is an important motivational factor" (p. 671). Moreover, "far more significantly, hand-held calculators can eliminate tedious, unnecessary calculations that consume precious time and destroy interest" (p. 672). However, Hawthorne thought that calculators should not be introduced until after a child has developed some number sense and familiarity with the basic operations of arithmetic. For otherwise, the calculator would be just like a "black box" that serves as an answer-key without any indication to the way the answers were obtained. He affirmed that teachers should stress <u>how</u> an answer is obtained. (p. 672)

Machlowitz (1976) recommended the use of a calculator in the general mathematics classrooms. She claimed:

> More significantly, the calculator can also present dramatic, attractive, and speedy opportunities for discovery, demonstration, and reinforcement in the general mathematics classroom with even the lowest-ability student. In fact, the slower student, usually resistant to deductive logic and abstractions, yet too sophisticated for arrays of ducks and portions of pies, can profit most. (p. 104)

Machlowitz also remarked that immediate surge of interest and participation on the part of student due to the presence of the calculator in the classroom increases the value of the calculator as an instructional tool that can assist a teacher in introducing concepts. (p. 106)

Bruni and Silverman (1976) maintained that the calculator should be used in the elementary school classroom because it provides

> Instant motivation! The most "reluctant" learner is anxious to have a chance to use the calculator. (p. 494)

Schnur and Lang (1976) related that the advocates of the calculator describe it

> As an essential implement in the newest mathematics (Higgins 1974) - as a motivating device (Mastbaum 1969) - as a means toward immediate reinforcement of results, a significant learning strategy (Lewis 1974)". (p. 559)

In her discussion about potential values of using a calculator in the classroom, Denman (1974) remarked that the use of the calculator has potential values in the areas of intrinsic motivation and reinforcement as a checking device. (p. 56)

Referring to the use of a programmable calculator, DuRapau and Bernard (1979) commented:

> Proper use can aid in developing intuitive understandings for some rather sophisticated mathematical concepts and can be a useful motivational tool. (p. 424)

Olson (1979) presented the following three examples to illustrate the use of a calculator in determining patterns as well as stimulating and testing conjectures:

Example 1. Does the following generate a pattern?

$$5^2 - 5 = 4^2 + 4$$

$$7^2 - 7 = 6^2 + 6 .$$

Example 2. Is it always true that a product of two sums of two squares of whole numbers is a sum of two squares of whole numbers? That is, given whole numbers a,b,c,d do whole numbers A, B exist such that

$$(a^2+b^2)(c^2+d^2) = A^2+B^2 \quad ?$$

Example 3. Does the following generate a pattern?

$$(20+25)^2 = \_\_\_$$

$$(30+25)^2 = \_\_\_$$

(pp. 288-289)

Maor (1976) felt that the use of a calculator can impart new insight, interest and fun to the teaching of many topics in mathematics. It was stated:

> The stimulation of exploring with these calculators apparently has a psychological effect on people's attitude toward numbers and arithmetic. (p. 471)

Hopkins (1976) proposed fullest possible use of calculators at all grade levels. He suggested that calculators be accepted as inevitable and ways to make their best-possible use be studied. He argued:

> It must be granted that it is a better instrument. It is faster, more accurate, and in the long run, cheaper. (p. 658)

Gawronski and Coblentz (1976) related that Etlinger characterized two different views on the use of the calculator -- functional and pedagogical. In the functional use, the calculator is regarded as a device like an eraser or classroom desk; whereas in the

pedagogical use, the calculator is viewed as a textbook, flashcards, or a manipulative device. They went on to argue that the type of use would depend on the ability of the students. More able students can use a calculator as a tool (functional use) with skills they have already mastered. The calculator might also assist the less able students in comprehending mathematical concepts (pedagogical use). They contended:

> If handled properly, the calculators will help a student develop a better understanding of the algorithms. (p. 511)

Furthermore, they asserted:

> The calculator has the potential for becoming a viable instructional tool for problem-solving activities and computational skill development. There are some researchable questions to be examined as well as curriculum uses to be identified. At the present we are convinced that calculators can provide another strategy for helping children to think, create and learn mathematics. (p. 512)

Usiskin (1978) is against the belief that calculators are a crutch. He maintained that the "crutch premise" consists in letting students use a calculator for arithmetic problems that can be done by hand. A crutch (bad!) can be used as a tool (good!) in many situations. Therefore, many value judgements depend on the type of label which is perceived as applicable. (p. 412)

Ockenga (1976) suggested explorations of calculator uses in junior high grade topics such as exponents, percents, solving equations, and common fractions. He also claimed,

> These amazing little devices can also become powerful instructional tools in the classroom. (p. 519)

Henry (1977) provided techniques for finding prime numbers and solving trigonometric equations and polynomial equations using a calculator. In the author's view:

> The hand-held calculator thus has the capability to become an invaluable tool for providing the best possible mathematical education for each of our students. (p. 591)

In Goodson's (1978) opinion, calculators have a real place in the junior high classroom. Moreover,

> They go right along with the overhead projector, movie projector, and all the other teaching aids. The real question is where and how to use them to the best advantage. (p. 20)

Johnsonbaugh (1976) displayed how calculators and computers can be integrated with theory, instead of being viewed as mere supplementary devices. It was also shown that these machines afford an opportunity to introduce topics which have previously been regarded as advanced. The author believed that by observing how the machine interacts with the problem, a student may gain insight into the theoretical aspects of the problem.

Webb (1976) discussed educational advantages of calculators as follows. Calculators can (a) provide more time for deeper study of certain topics; (b) provide encouragement and incentive by allowing checking of calculation techniques which still have not been learnt; and (c) allow investigating of number patterns and sequences. The author suggested exploration of possible advantages and exchange of investigations. At the same time, the author warned about the dangers of misuse of these devices.

Johnson (1978) mentioned some of the abuses of calculators (see Section 2 of the present chapter) and also suggested the following activities:

1. Calculations in working with formula for combinations, solutions of trigonometric equations.
2. Pattern generation and pattern search.
3. Exploration for concept demonstration, concept-reinforcement, problem solving, situations involving formulas which are both motivating and interesting.
4. Applications— consumer and social.
5. New/Renewed content including key topics such as estimation, errors, algorithms and iteration, and mathematical modelling.

Stolovich (1976) in his article "A Pocket Calculator Never Loses Patience", enumerated several needs of handicapped learners, and mentioned that teachers used calculators with their handicapped pupils as a tool for guided discovery, drill practice and motivation. The author referred to specially designed calculators for mentally retarded children, on which structured drill in basic computations can be provided.

In Gibb's (1975) view, possible uses of calculators in the classroom include checking answers, debugging problems, checking knowledge of basic facts in the four computation areas, assessing insight, making the calculator speak, forming patterns and solving problems.

Stultz (1975) maintained that in the classroom, the calculator may be used in: (a) counting in preschool, kindergarten, and first grade; (b) motivating students by permitting them to make up their own problems; (c) checking answers and debugging problems; (d) teaching place value; (e) immediate reinforcement; (f) changing fractions to decimals; (g) enforcing correct order in chain operation; (h) number approximations, truncation errors, and rounding off numbers; and (i) evaluation of formulas.

Teitelbaum (1978) demonstrated the use of a calculator in teaching of verbal problem solving skills. In the words of the author,

> Real life will demand a much greater use of verbal problem-solving ability than knowledge of basic computational processes. (p. 19)

Immerzeel (1976b) claimed, "Even one calculator in a classroom can be helpful" (p. 230). It can make the teaching work easier as well as more interesting because it becomes a portable answer-key which is better than the usual answer-key. When checking papers, it helps in detecting the type of error a student is making. It also aids in preparing worksheets or designing other activities for students. He proclaimed, "So far, I have not observed the development of any dependency on the calculator" (p. 231).

Immerzeel (1976a) also believed that concepts could be built through student experiences with the calculator. But he felt that the instruction in concept development was still essential. He remarked that the back-to-the-basics movement changed into back to concept-development and understanding (p. 50). Furthermore,

> The calculator is a portable, hand-held, math lab providing a source of experiences with numbers in a fast, efficient manner. (p. 50)

Moreover, in Immerzeel's (1976a) opinion, the calculator assists in developing better problem-solving skills because the problems can now be related to the real world experiences. (pp. 51, 148)

Duea and Ockenga (1982) suggested that "students should be encouraged to be authors, to create problems as well as to solve them (p. 50)". They should not depend on textbooks and teachers. Every day life situations provide ample opportunities for practicing problem solving skills. For example, collection of personal information (such as number of heart beats) using realistic data. Furthermore, problem solving requires methods of gathering, organizing, and interpreting information, drawing and testing inferences from data and communicating results. Calculator should be used as a tool for problem solving. It can aid only in computational work. Consequently, its speed, accuracy and efficiency help diminish dislikes to problem-solving.

The claims and suggestions made above and drawn from a broad spectrum of expert opinion seem to have won the great calculator debate, though it is difficult to see why these claims should be viewed any more positively than the counterclaims described earlier. Perhaps it was because the calculator suddenly appeared everywhere - or so it seemed - and rather than fight the inevitable the opponents of calculator use in schools simply gave in. Regardless,

there very soon appeared many articles describing how calculators should or could be used constructively in classrooms. The next section presents in tabular form some of these suggested activities.

4. Activities Exploiting the Use of Calculators.

This section affords a quick glance at some of the activities designed by various mathematics educators for use with calculators. The table below characterizes these activities by author, type and grade level. In general, the classification is determined by the descriptors on an ERIC search. In some cases, the author(s) of the article makes mention of the level where the activity should take place. Where this was not explicit, the present investigator inferred the grade level(s) at which the activity would seem most appropriate.

TABLE I

A CATEGORIZED LISTING OF

REVIEWED CALCULATOR-BASED ACTIVITIES

AUTHOR/YEAR[1]	TYPE[2]	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL[3]
Boyle (1976)	G	Calculator charades, that is, recreational games involving computations done on a calculator so as to produce results which can be read as words or phrases, when the display is inverted. For example, the calculation of $\{(16.599)^2 - (29.59)\}$ is 91851345; and when inverted, it displays SHEISBIG.	S

[1] Arranged chronologically beginning with articles in 1976, and, within any one year, alphabetically.

[2] C = concept formation, P = problem solving, G = games and recreation, M = miscellaneous.

[3] E = elementary, El = lower elementary, E2 = upper elementary, M = middle grades, S = secondary, P = post-secondary.

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
Bruni and Silverman (1976)	CP	Basic arithmetic operations, analyzing algorithms, mental calculation and estimation, solving word problems, getting acquainted with the calculator.	E
Guthrie and Wiles (1976)	GP	A calculator tournament to give practice in problem solving.	EMS
Immerzeel (1976a)	CP	Computations with decimals and decimal fractions, activities involving mental arithmetic and estimation skills, solving of problems related to the real world experiences.	E
Immerzeel (1976b)	CG	Use in the interest centre of the class, calculation of answers by one student during class discussions.	E
Johnsonbaugh (1976)	C	Representation of numbers by a machine, sum of an infinite series, and a convergent test (i.e., a positive term series converges if its partial sums are bounded). The calculator used was programmable.	SP
Judd (1976)	GC	Seven games - Nim, Wipeout, Before, After, Solitaire, Target K and The Big One - to develop concepts.	E
Maor (1976)	C	(i) Verification of the trigonometric identity $\sin^2 A + \cos^2 A = 1$ for various values of A. (ii) Calculation of numerical values of functions such as $y = ax^2 + bx + c$, $y = \frac{2x+1}{2x-1}$, $\sin(ax+b)$. (iii) Choosing any positive number, then pressing the square-root key any number of times so that the result approaches 1. This implies that $\lim_{n \to \infty} \sqrt[n]{a} = 1$. (iv) Testing of Wallis's product $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots$	S

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
Ockenga (1976)	CPG	Solving real-world problems, computing an approximation for pi, measurement and geometry, functions, making interpretations of data, game of "Check" which involved basic fact combinations, estimating and then checking answers of division exercises or multiplication exercises.	M
Willson (1976)	C	(i) Use of the square-root key in determining cube roots, nth roots, logarithms, inverse trigonometric functions and other trigonometrical functions using cos θ. (ii) Use of the square key in determining cos θ, e^x, a^b .	S
Bolduc (1977)	C	Using a calculator and ideas from geometry, determining of value of π correct to five decimal places.	S
Henry (1977)	CP	Instructional techniques to (i) find prime numbers, (ii) solve trigonometric equations and (iii) solve polynomial equations.	EMS
Russell (1977)	C	Outlined some activities to enhance students' knowledge of place value and of basic arithmetic operations.	MS
Billings and Moursund (Sep.-Oct. 1978)	MCG	Getting started (testing various keys), checking answers concerning four basic operations with whole numbers, order of operations, exponents, operations with powers, chaining (for example, 8+9-5 = ...). Suggested games included: (i) Familiarity with letters obtained by turning the video display upside down (calculator digit 5 gives, upside-down, letter S). (ii) Playing word games, for example, "What did the cannibal cook say when asked if supper was ready? To find out: Find the product of 6 and 4759. Add 17. Double the result."	S

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
		(iii) Displaying numbers with symmetries, for example: "What is the smallest two-digit number you can display with point symmetry?"	
Billings and Moursund (Nov.-Dec. 1978)	P	(i) Real world problems that involve numbers or geometry. (ii) Problems to give practice in identifying parts of a problem (givens, restrictions and goals) and then solving them. (iii) Problems to give practice in four steps in problem solving -- (a) understand the problem, (b) devise a plan, (c) carry out the plan, (d) look back -- and then solve the problem. (iv) Problem solving by guessing. (v) Problems involving calculations to be done by using some or all of the five methods of calculation - mental arithmetic, math tables, pencil-and-paper, calculators and computers.	S
Goodson (1978)	C	Factoring, prime factoring, determining square roots and percentages. Included a worksheet displaying the method to find a square root.	M
Hiatt (1978)	CP	Determining the area of a circle by finding points on the circle and then applying the formula for the area of a polygon.	S
Hobbs and Burris (1978)	C	Provision of an algorithm for generating as many digits as desired in the decimal representation of a rational number .	E
Jurgensen (1978)	CP	Measurement of the sides of right triangles to discover the Pythagorean property.	S

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
Keller (1978)	C	Discussed the construction of tables of numbers as a learning activity which involves reciprocals, factors, formulas, patterns, discovery, prime numbers and other types of numbers. The author felt that every student - including the slowest one - can contribute to this activity.	S
Lappan and Winter (1978)	GP	Bingo game, which emphasizes quick calculations and problem solving.	E2
Miller and Hazekamp (1978)	CP	Enclosed three worksheets which contained the following activities: (i) graphing $y = x^2$ on $[0,1]$; (ii) graphing $y = \frac{1}{x}$ on $[1,2]$; (iii) graphing $x^2 + y^2 = 1$ on $[0,1]$. Calculator is used in computing values.	MS
Morgan and Warnock (1978)	M	(i) Pointed out a few problems that arise when using a calculator to explain derivatives; (ii) gave some examples of numerical differentiation techniques.	SP
Snover and Spikell (1978)	CP	Solution of $153 = x^3 + y^3 + z^3$ using a programmable calculator.	S
Willson (1978)	C	Showed that repeatedly taking the square root of numbers with a calculator produces numbers possessing properties similar to logarithmic properties, which can be used to define the logarithm function.	S
Blume (1979)	PC	Problems on population growth and inflation which may serve as a basic structure for a unit to replace much of the usual algebra unit dealing with exponential functions and logarithms.	S

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
DuRapau and Bernard (1979)	GC	Three games which make use of a programmable pocket calculator, so that the calculator facilitates the learning of the function concept.	S
Jamski (1979a)	C	The relationship between simple interest and compound interest was explored to develop the required formula inductively and intuitively.	MS
Olson (1979)	C	Three examples to determine patterns as well as stimulate and test conjectures.	S
Russakoff (1979)	CP	Solution of a class of problems $x^a + y^a = 5^a$ for different values of a .	SP
Snover and Spikell (1979b)	P	Solutions of several problems requiring advanced techniques using programmable calculators.	S
Snover and Spikell (1979a)	CP	Solutions to difficult equations with numerical techniques.	S
Toth (1979)	C	Prime factorization of a number. Provided several examples and a worksheet.	E2
Wagner (1979)	C	Discovering of cyclic patterns that appear in repeating decimals of the families of primes P from 7 through 97 (that is, family of P means all of the proper fractions with the denominator P). Presentation of a table which demonstrated the number of cycles, length of cycles, a listing of cycles and the sum of the digits of a cycle, in case of each prime family.	S

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
Waits and Schultz (1979)	PC	Solutions of four problems using a calculator and an iterative method for computing solutions to equations: (i) determining the zeros of $f(x) = x^2-x-3$; (ii) determining a zero of $f(x) = x^9-x^3-1$; (iii) determining the solution of $3^x + x = 0$; and (iv) a word problem.	S
Woodward and Hamel (1979)	CP	Determination of the rule of 72. That is, if money is invested at r% compounded annually, it will take approximately $\frac{72}{r}$ years to double the amount. If interest is compounded semi-annually or instantaneously, the doubling period will be approximately $\frac{70}{r}$ years or $\frac{69.3}{r}$ years, respectively. The use of these lessons was made to investigate problems including population, inflation and energy reserves.	S
Bone (1980)	CP	Design of activity united to: find some patterns of sine and cosine functions, draw graphs and round numbers to the nearest hundredth. Provided worksheets.	S
Dorn and Councilman (1980)	C	Use of the calculator square-root function to introduce the notions of limit of a sequence, monotone function, and step function. Suggested activities included discussion of: for any $x > 0$, $\lim_{n \to \infty} x^{2^{\frac{1}{n}}} = 1$, by repeated application of the square-root key.	S

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
Fearnley-Sander (1980)	PC	Discussion of a calculator solution of the equation $x^x = 3$, correct to three decimal places. The author felt that an algorithmic approach is more meaningful to a child than formalism because it emphasizes concrete facts about numbers, which the latter leaves out.	S
Maor (1980)	MCP	Use of the calculator exchange key to interchange the numerator and denominator of a function, generating the Fibonacci sequence and geometric series, and finding square roots.	S
Sloyer (1980)	C	Introduction to the sum of a geometric series.	S
Stover (1980)	CP	Evaluation of some functions, such as square roots, $\log_b x$, b^x, cos x, $\cos^{-1} x$.	S
Wheatley (1980)	CP	Presentation of four activities - (i) estimating sums and addendands; (ii) problem solving; (iii) application; and (iv) developing the concept of decimal.	M
Adkins (1981)	C	A procedure to find the greatest common factor (GCF) of two numbers. The method involves subtraction instead of division. Once the GCF is known, a calculator can be used to find the LCM (least common multiple) also.	MS
Bestgen (1981)	M	Devised a scheme to introduce children to calculators. Devised a sequence of activities for teachers.	El

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
Cheung (1981)	CP	Approach to equation-solving using the trial-and-error strategy. The author recommended inclusion of this approach in the secondary school mathematics program as a way of moving to equations harder than the linear and quadratic ones traditionally studied.	S
Kluepfel (1981)	PMC	Designed seven problems to depict the importance and use of the log key. The author stated that memorizing of two-decimal-place common logarithms of 2, 3 and 7 helps in approximation. Moreover, logarithms are useful in computing large factorials and for mental calculations of powers, roots, and products because a logarithm is an inverse operation to raising to a power. These problems involve respectively the ideas of interest, brightness of stars, ear's response to sound, musical scale, atmospheric pressure and height above sea level, and the game of Master Mind.	S
Seber (1981)	CP	Presented Gauss-Siedel method of successive approximations for solving a system of linear equations. The author recommended inclusion of this method in an algebra one course due to reasons including for example, (i) reinforcement of geometry of a system of linear equations and (ii) provision of background for sequential processes and the idea of a convergent sequence, by the presentation.	SP
Weaver (1981)	C	Illustrated ways in which the use of calculator facilitates consideration of selected unary operations. Emphasized unary operations which are suitable for exploration and investigation for pre-algebra level students.	M

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
Wiebe (1981)	CPM	Use of a calculator to develop understanding of the meaning of square roots, and operations on whole and rational numbers by employing three techniques: (a) use of the "counting" function; (b) directed, repeated estimation; and (c) entering the problem in meaningful pieces.	M
Woodward (1981)	MC	Presented six activities to exploit the use of the constant arithmetic feature of the calculator.	E
Aichele (1982)	CP	Presented three examples which involve geometric constructions. These constructions employ a compass, a centimeter ruler as a straightedge, and a hand-held calculator. In the first example, lengths of sides and lengths of medians of a triangle are given. The problem is to construct a sequence of triangles such that the lengths of the sides of each "new" triangle are the lengths of the medians of the triangle immediately preceding it. An analysis of the sequence of areas leads to the conjecture area Δ (N+1) $\doteq$ .75 area Δ (N), for N = 1,2,... . The other two examples deal with the concept of inradius and altitudes of a triangle. According to the author, "Employing mathematical tools to support the understanding and discovery of mathematical principles is clearly a desire of mathematics educators" (p. 707).	S
Bestgen, Stuart and Taylor (1982)	PC	"Hit the target". This activity involves two digit addition and subtraction, and incorporates the use of the calculator as an aid to problem solving within a trial-and-error context. Make-believe darts are shot at the target, so as to hit a "bullseye". For example, to reach the target (i.e.,	E1

AUTHOR/YEAR	TYPE	SUGGESTED CALCULATOR-BASED ACTIVITY	GRADE LEVEL
		the desired number 45), students seek to find combinations of the numbers on the dart (i.e., 3 and 8), in order to make an addition or subtraction problem. Calculators are used to find whether bullseye has been hit (i.e., the answer 45 has been arrived). The authors felt that a calculator is very necessary for the successful completion of this activity for it provides immediate feedback to students' guesses and hence inspires them to continue their trials.	
Goodman (1982)	PC	Estimation of factors and divisors. Checking of estimation using a calculator. The goal was to enable students to learn problem solving while they worked to improve their estimation skills. As an illustration: 975 ÷ $\square$ so that the quotient falls within the range (10, 20).	MS
Lappan, Phillips and Winter (1982)	C	A sequence of activities to explore powers of numbers and patterns of numbers within the powers. They guide solution to the following problem: "With only one guess, can you give a number to place in the box so that the problem can be correctly completed? $\square^7 = \_\_\_\_\_^2$ Each dash on the right of the equal sign stands for one digit". (p. 42) The authors suggested use of the table of units digits for solving the above problem.	M

The above table has depicted the activities suggested for calculator use by mathematics educators. While efforts are being made

to meet the requirements of the NCTM Agenda for Action (1980), much work still remains to be done. In particular, these activities are of a "stand alone" variety, i.e., they do not involve any systematic or comprehensive revision of a significant portion of the mathematics curriculum. This is not to say, however, that major curricular revisions have not been suggested. Indeed, the next section outlines some of the suggestions for basic curriculum revision which have been suggested.

5. Curriculum Considerations Concerning the Calculator.

Several mathematics educators have envisioned considerable changes in curriculum due to the classroom use of calculators. Proposals were made suggesting necessary alterations in the existing curricula. Workshops were conducted. A number of conferences were also organized by government agencies, educational organizations and community groups to discuss the impact of this novel innovation on school mathematics curriculum. Highlighting these statements is the description that follows.

Bell (cited in Hopkins, 1978, p. 33) exclaimed:

> Finally, I have become convinced during just this past year that the widespread availability of cheap electronic calculators will have profound effects and must move us very soon to re-evaluate many of our current practices in the teaching of school mathematics.

Bell (1976) envisioned fundamentally new curriculum development initiatives that would lead to enrichment of elementary school curriculum with increased use of concrete materials, calculators and problem solving material which is meaningful and interesting so as to involve students in experiments with numbers.

Moreover, in the context of secondary school mathematics education, Bell (1978) envisioned unrestricted use of calculators, dealing with real problems with real data in college preparatory courses, and establishment of interaction between science and mathematics courses.

Gibb (1975) perceived (a) earlier appearance of the study of rational numbers expressed as decimals in curriculum; (b) change in

how we teach computation; (c) greater emphasis on estimating and error checking skills; and (d) greater emphasis on problem solving. (p. 44)

In Immerzeel's (1976a) view, "The part of the curriculum where the calculator has been most effective has been in developing the students' problem-solving skills" (p. 51). Also, the calculator helps students in solving more problems in less time than with pencil-and-paper. Furthermore, now the problems can be related to real world experiences (p. 148).

Gawronski and Coblentz (1976) foresaw (a) availability of more time to emphasize problem solving skills when the calculator is used to eliminate drudgery of tedious and unnecessary calculations; (b) the idea of a basic skill may require a reexamination; (c) the curriculum may contract in some directions but it will definitely expand in the problem solving direction; (d) the calculator is neither going to be a substitute for basic skills of mathematics nor is it going to produce a generation of machine-dependent learners.

Bitter (1977) envisioned much broader application of the calculator to curriculum, that is, use of the calculator for inter-disciplinary projects and in curriculum areas other than mathematics, such as those which have economics, consumer mathematics, business mathematics and shop. He perceived advantages of calculator use (a) in making projections and estimates concerning population and production statistics, (b) in attacking and successfully solving those problems which would have been ignored otherwise, (c) in emphasizing and integrating estimation strategy and consequently, (d) in developing concepts employing an experimentation and

discovery process, even before having a thorough understanding of algorithmic processes. Furthermore, since calculator use facilitates understanding of decimals, they could be introduced at younger ages. Some educators are recommending introduction of decimals, instead of fractions, in the middle grades, because metric computation needs comprehension of decimals. Therefore, powers of ten could be emphasized earlier. This would prepare children to deal with decimals.

In connection with application and planning, Bitter was of the view that effective use of the calculator inculcates innovative thinking, inductive and deductive reasoning as well as generalizing skills, which can be interpreted into the content areas by using the calculator in various situations. Moreover, calculator use facilitates number sense, which can be applied to everyday situations. Thus, calculator has special importance in survival math - an area which involves a recognition and knowledge of practical information and skills needed to survive in a changing world.

Quadling (1975) was concerned about goals of curriculum and testing practices. For example, can a marking scheme be thought of in which some points are allotted for correct computations, if some students have made use of an electronic calculator while the others have not? Would it become impossible to set certain desirable questions in the exam due to the availability of these machines? Should there be no questions in the exam in which any kind of calculator could be of help?

Pollak (1977), while rethinking the content and teaching of secondary mathematics in the calculator era, advocated that the curriculum should be based on two partial orderings - one of which is essentially supplied by the discipline and the other by society (p. 295). Since the use of the hand-held calculator might provide pedagogic advantages with some topics (see section on claims and suggestions), both the content and societal orderings are going to be effected, to a large extent, by these machines. Thus, the availability of calculators may reorder the sequence of topics in curriculum, cause some of the topics to disappear from the curriculum and allow early appearance of some topics which could not be approached in the past due to some pedagogic difficulties.

Wheatley (1980) presented a proposal that computationally oriented curriculum may be shifted to a conceptually oriented curriculum using the calculator as an instructional vehicle and the teaching of complex computations in the elementary school be eliminated.

As reported in Chapter I, the NACOME (1975) viewed availability of calculators as a challenge to traditional instructional priorities and envisioned restructuring of elementary school mathematics curriculum (p. 41). Furthermore, in its report Overview and Analysis of School Mathematics: Grades K-12, the Committee listed a few questions to be investigated through research. These questions included, among others, effect of calculator availability on instructional emphasis, curriculum organization, and student learning styles in advanced level secondary mathematics subjects such as algebra,

geometry, trigonometry and calculus (p. 43). In addition, according to the Committee, one of the areas of urgent concern was - "Instructional materials at all levels in: the use of calculators, applications and modelling,statistics and the general ability to collect, organize, interpret,and understand quantitative information, combinatorial mathematics, and metric system measurement"(p. 145). The Committee also recommended curricular revision or reorganization in view of the increasing significance of calculators and computers (p. 145).

The calculator and curriculum related issues were also treated (Bell et al., 1977) in the Conference on Basic Mathematical Skills and Learning held in Euclid, Ohio, in October 1975, sponsored by the NIE (see also Chapter I). The final report of this Conference consisted of two volumes - one containing the thirty-three position papers submitted by each of the thirty-three participants, and the other containing a description of the background and organization of the Conference, four working-group reports, and an essay by the Conference co-chairmen. In the essay, the co-chairmen warned against putting any restrictions on calculator usage so that important research might not be blocked. They stressed the need for good curriculum materials to:

> Support and extend conceptual understanding of mathematics and to facilitate the application of arithmetical techniques to the solution of real life problems. (p. 229)

The group working on curriculum development and implementation recommended studies of:

1. Alternative sequences for elementary instruction in arithmetic.

2. Uses of the calculator as an aid and stimulus for arithmetic instruction.
3. The impact of calculator availability on problem-solving instruction.
4. The relative importance of various familiar fraction concepts in an environment of calculators (to include an investigation of curriculum topics in later courses such as algebra. (p. 228)

Furthermore, a need to re-examine curriculum structures and priorities at the secondary level was noted because of the increasing potentialities of calculators and computers. It was stated that this clearly affected the definition of basic skills. (p. 228)

One professional group - the NCSM - took an active part in studying school mathematics curriculum concerns (see Chapter I). During the 1976 Annual Meeting in Atlanta, Georgia, the NCSM held a special session to discuss the Euclid Conference Report. More than one hundred participating members mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. As a result, the definition of basic skills was expanded so as to enclose ten vital areas (cited in Pikaart et al., 1980, p. 29):

1. Problem solving.
2. Applying mathematics to everyday situations.
3. Alertness to the reasonableness of results.
4. Estimation and approximation.
5. Appropriate computational skills.

6. Geometry.
7. Measurement.
8. Reading, interpreting,and constructing tables, charts and graphs.
9. Using mathematics to predict.
10. Computer literacy.

In March 1975, the NSF funded an investigation (Bell et al., 1977, p. 229) involving a critical analysis of the role of calculator with the intent of studying the impact of the calculator on the pre-college mathematics curriculum. The study was undertaken by Suydam (see Chapter I). The report prepared by Suydam was entitled Electronic Hand Calculators: The Implications for Pre-College Education. Appendices to the report present an annotated list of references, the complete set of responses from questionnaires and from position papers by various educators.

In their position paper, Usiskin and Bell did not support the idea of merely incorporating the calculator into the existing curricula: "It is thus our belief that the insertion of calculators into K-6 classrooms using [mostly] existing curricula is fraught with peril" (Bell et al., 1977, p. 234). They supported an alternative curriculum and supplied an appraisal for restructuring the curriculum. (p. 234)

Weaver, in his paper, distinguished among three types of curricula - (a) calculator-assisted, (b) calculator-modulated and (c) calculator-based and remarked that "research should not be unmindful of such differential roles" (Bell et al., 1977, pp. 234-235).

The major recommendations of the above report included (Suydam, 1976, p. 46) a complete analysis of mathematics and other curricula of elementary and secondary schools in order to find (a) how optimal use of calculators could be integrated with existing curricula and (b) how existing curricula could be revised/redeveloped so as to incorporate optimal use of calculators. In addition, systematic research concerning development of curricula should be undertaken.

A conference was sponsored (Werner, 1980) by the NIE/NSF on the Uses of Hand-Held Calculators in Education in Arlington, Virginia, from June 26-30, 1976. One of the working groups stated that because of the arrival of calculators, new initiatives in the school mathematics curriculum are not only desirable, but are also imperative (p. 29). It was concluded by the Conference that the education community and mathematics educators, in particular,

> ...must lead in delineating curriculum applications of hand-held calculators. It must not default, allowing manufacturers and publishers to make most of the crucial decisions. (p. 29)

According to Werner (1980), the most powerful document to provide direction for future research was prepared by participants who attended the above Conference held in Arlington. The title of the written document was: Report of the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics. It was available in 1977. As a result of the discussion, twenty two recommendations were formulated. They belong to the following broad areas of concern:

1. Development of an information base.
2. Curriculum development for the immediate future.
3. Curriculum development for the long-range future.
4. Research and evaluation.
5. Teacher education.
6. Dissemination. (pp. 33-36)

Curriculum development for the immediate future included development of (Werner):

1. Materials to exploit the calculator as a teaching tool at every point in the curriculum.
2. Curriculum materials for K-12 to teach estimation, approximation, significant digits, order of magnitude calculations and similar ideas.
3. Curriculum materials to teach problem solving strategies more effectively so as to build students' confidence in their ability to solve problems.
4. Curriculum materials for topics that are not now taught but which become feasible with the use of calculators. (p. 34)

Curriculum development for the long-range included development of (Werner):

1. Full-scale alternatives to the K-6 elementary school mathematics program that use calculators wherever appropriate and broaden the range of mathematical ideas.
2. New courses for secondary school students including consumer-industrial and data-oriented statistics courses, alternatives

for junior high school students, and for students not electing standard college preparatory mathematics courses. (pp. 34-35)

In Chapter I, it was discussed that the third recommendation of the NCTM Agenda for Action: Recommendations for School Mathematics of the 1980s, emphasized full advantage of the powers of calculators and computers at all grade levels by mathematics programs. Recommended actions included provision of calculators and computers by schools for use in elementary as well as secondary school mathematics classrooms, integration of the use of these devices into the core mathematics curriculum, and development and dissemination of curriculum materials that integrate and require the use of these tools in diverse and imaginative ways. A note of warning was added for the developers of software that the use of conventional material and techniques newly translated to the medium of these electronic tools will not be enough. The Guideline provided for choosing software was: software should fit the goals or objectives of the program and not vice versa.

Finally, in a recent state-of-the-art review, Suydam (1982, p. 9) reported that many researchers have analyzed the curriculum in order to determine the topics in which the use of the calculator can be most effective. As far as the secondary school level is concerned, many mathematics textbooks integrate the use of calculators. But materials at the elementary school level are generally supplementary and most of them emphasize computations

instead of teaching mathematical ideas. Few of these focus on coordinated use of manipulative materials, whereas the research evidence indicates that this is essential to the development of mathematical ideas. [Note that this report became available after the study was conducted].

The present section has reviewed concerns and recommendations of some of the interested mathematics educators and curriculum developers in the light of increasing availability and the use of the calculator.

Although the section has reviewed a large number of suggestions and recommendations mostly from professional educators, regarding the impact of the calculator on curricula, it appears that to date, little or nothing has been done about them. The difficulties regarding change to the educational system mentioned at the beginning of Section 2 seem to be applicable here. Of course, suggestions and recommendations should be based on the solid groundwork of scientific studies.

The subsequent section reviews research studies conducted to assess the effects of the use of this tool on mathematics achievement and attitudes.

6. Research on Calculator Effects.

Chapter I recorded the positions which educators hold in concern with the use of calculators in pre-college education. While opponents contend that the use of calculators might have detrimental impacts on the growth of children's mathematical abilities, proponents affirm that it might facilitate and promote mathematical learning. Several research studies have been undertaken with the goal of assessing calculator effects on factors such as computations, achievement and attitudes. This section reviews such studies.

Cech (1972) conducted an experimental study with ninth grade, low-achieving mathematics students. The following three hypotheses were tested. (a) The use of calculators in the instructional program improves students' attitude toward the study of mathematics. (b) The use of calculators in the instructional program improves students' computational skills. (c) Students can compute better with calculators than without calculators. The general mathematics students,in a high school in Illinois, were distributed into 4 groups - two experimental and two control. Each group was given seven weeks of instruction concerning addition, subtraction, multiplication and division of whole numbers by two teachers. Each teacher had one control group and one experimental group. Lesson plans were developed. They were used for both groups by both teachers. Each experimental group received four days of additional instruction time dealing with the operation of the calculator. The students in an experimental group were told to check

their work by using calculators, while those in a control group were told only to check their work. The analysis, which was a t-test on mean differences, supported the third hypothesis only; namely, that students can compute better with calculators than without them.

O'Loughlin (1976) reported a study conducted in 1973 at SUCC (SUNY, at Cortland, Cortland, N.Y.) to investigate the effect of the incorporation of an electronic programmable calculator in a first course in calculus. The control group was taught topics covered in beginning calculus using the traditional lecture method. The treatment group was taught an experimental course, the content of which consisted of topics covered in the control group plus additional topics in limits of functions, applications of the derivative and numerical integration. The method of instruction for this group was the lecture method augmented by the use of the calculator as a teaching aid. A series of tests were constructed to measure achievement regarding a few topics.

The data from the analysis of variance indicated no significant difference in achievement with respect to limits of functions, continuity and local extrema of functions. The data did indicate that the treatment group revealed significantly higher student achievement in observing the inter-relationships between a function and its first two derivatives, in solving verbal problems involving derivatives, and in interpreting the definition of the definite integral. The investigators commented that it was confirmed that the incorporation of the calculator into the first

calculus course allows inclusion of topics usually omitted or reserved for subsequent calculus courses and that it does not detract from efficiency on the usual topics covered.

Sosebee and Walsh (1975) were interested in assessing the impact of the use of calculators on introductory chemistry grades on in-class examinations. A comparison of student scores indicated that the students who used calculators scored higher on every test than those who did not. The differences in math scores were not significant.

Schnur and Lang (1976) conducted a study with sixty youngsters in Muscatine, Iowa for four weeks. Results indicated significantly more improvement in whole number computational ability of groups using calculators than those not using calculators. The sex of a student did not influence calculator usage.

Sullivan (1976), one of the co-directors of classroom trials of hand-held calculators in 1973-1974, reported that in a trial conducted with two sixth-grade classes in New York, it was evident that calculators encouraged children to explore many topics not usually studied intensively in sixth grade, such as probability, exponents, sequences, prime numbers, palindromes, negative numbers, division by zero, divisibility and permutations. In addition, topics from the regular program - averages, rounding numbers, numeration, factoring, and the fundamental operations - were supported very well by calculators.

Hopkins (1978) conducted a study with ninth-grade general mathematics students in order to investigate the effects on

achievement and attitude resulting from the use of a calculator-based curriculum and a classroom set of hand-held calculators. Six teachers participated in the study, each having one class in the calculator treatment and one class in the non-calculator treatment. Both treatment groups were given units of instruction on estimation, computation, and problem solving involving the four arithmetic operations on whole numbers. The students in a calculator group used a classroom set of hand-held calculators in instruction, while those in a non-calculator group used paper-and-pencil only. Both groups were given pretests and posttests in mathematics achievement and attitude. Half of each group used a hand-held calculator as an aid in the achievement posttest, whereas the other half of each group used paper-and-pencil only. Analysis of covariance was used to analyze the resulting data. The findings of this study indicated that the use of the calculator-based curriculum did not significantly affect student achievement in computation or student attitude toward mathematics but did have a positive effect on student achievement in problem solving.

Shin (1978) gave an account of a 14-question survey on the attitude of form 4 school children of Hong Kong towards the use of calculators, particularly in schools. The results of the survey revealed that the children liked the calculator better than a slide rule or mathematical table. They considered calculator to be a useful aid and favored its use in public examinations. Further, Shin discussed the use of mathematical tables as opposed to calculators. Shin is of the view that educationally, calculators

are better than tables because some ideas can be conveyed better with calculators than with tables, for example, the idea of a limit and convergence. Shin claimed that

> The calculator is not just a computational tool, good as it is at that. It is a useful teaching aid for the exposition of certain topics, and its use makes it possible to teach others that are otherwise difficult or impossible to teach. (p. 40)

Creswell and Vaughn (1979) reported a study the subjects of which were ninth-grade, Fundamentals of Mathematics, students. (The achievement of these students is at least two years below grade level). Students in the experimental group used the calculator on both the post-test and the retention test. Neither the experimental group nor the comparison group made use of the calculator on the pretest. It was concluded that these students could achieve at a higher level when using hand-held calculator and a specially designed curriculum.

Jamski (1979b) conducted a study on 162 seventh-graders from University Middle School in Bloomington, Indiana. The purpose of the study was to investigate the effect on achievement, of learning conversion algorithms among fractions, decimals and percents with the hand calculator. The students were divided into six classes, all taught by the same teacher. Three classes were randomly assigned to the hand calculator treatment (C), the rest becoming the control group (NC). A pretest was used to partition students into three ability levels (Hi, Med, Lo). Each member of C was allowed the use of a calculator during both instruction and posttest , whereas those in NC used only paper-and-pencil. Four weeks of instruction was given. The instructional materials were basically from the text

in use, supplemented by exercises given on the blackboard. All C and NC classes were given the same materials and the same instruction from the same teacher. Two-way ANOVAs were employed to analyze the data. In relation to achievement, the calculator appeared likely to be successful with some topics, such as the fraction-decimal conversion, but not with others. Furthermore, if achievement was to be the only criterion, segregation of calculator use to specific ability group did not seem to be justified, because there was no evidence in this respect to support either a claim of significant interaction between student ability and calculator use or significant bridging of ability gaps.

Szetela (1979) used calculators as a tool in teaching trigonometric ratios. The investigator hypothesized that the students whose instruction was calculator based (CBI's) would perform better than those who were not allowed to use calculators (NUC's). Special lessons and materials were developed for calculators. The instruction was given by two teachers to 131 grade 9 and 10 students, of low to average ability level, for 18 days. Two quizzes, plus a 12 item five-point attitude scale and a 20 item achievement test led to the conclusion that CBI's performed at least as well as NUC's.

Bitter (1980) conducted a study the subjects of which were primary, middle and upper grade teachers. It was concluded that teacher attitudes toward the classroom use of calculators can be improved through inservice education organized around familiarizing teachers with the calculator and its classroom applications. (p. 326)

A study was conducted by West (1980) to compare the effectiveness of two drill strategies--paper-and-pencil and electronic calculator--in facilitating the learning of basic multiplication facts. The finding of the study was that paper-and-pencil drill was more effective than the calculator drill. In the investigator's view, perhaps a longer time period for treatment would have shown considerable differences for the calculator group. Also, the study created more questions than it answered. However, one tentative conclusion drawn from the study was that the teachers who use only the paper-and-pencil strategy for drill should not feel their students are being seriously handicapped in learning multiplication facts.

Wheatley (1980) conducted a study with two groups of sixth-grade students of above-average ability. The intent of the study was to compare the problem solving processes of students using calculators with those of students not using calculators. Both groups received six weeks of training in problem solving. Initially, each group had 23 students. Both groups studied a unit on adding, subtracting, multiplying and dividing decimal fractions with emphasis on application. Verbal problems involved decimal fractions. At the conclusion of the training period, each student was interviewed. Each student was given five problems to solve. During the interviews, calculators were supplied to each student in the calculator group only. The students were asked to think aloud as they solved the problems. These interviews were tape-recorded and transcribed. The processes used by the students were identified by these transcripts. The checklist

coding system developed by Days was used in this study. Computational processes used by two treatment groups were compared. Also computational errors, production scores, and time-on-task were analyzed. In addition, the total number of processes used were compared using t tests of differences between means of the calculator and noncalculator groups. It was revealed that the calculator group used a total of 152 facilitative processes, against 104 for the noncalculator group. The greatest differences were recorded on these items - 'has bright ideas', 'estimates', 'uses unexpressed equations', 'checks conditions', 'retraces steps'. The experimenter felt that this suggested that calculators stimulate students to think about approaches to problems.

Behr and Wheeler (1981) conducted a clinical study with 30 kindergarten and first-grade children with the goal of examining whether or not these children could be taught, with minimal instruction, to use and perceive successive punches of the counter button ("=" key) of a hand-held calculator as a means for carrying out counting activities concerning addition and subtraction. It was concluded by the authors that the use of a calculator to develop counting behaviours might facilitate a child's acquisition of addition and subtraction concepts.

Hector and Frandsen (1981) compared three methods for teaching fractions which used (a) conventional algorithms, (b) conventional algorithms and calculators, and (c) calculator based algorithms, respectively. 72 community college students were the subjects of this study. Their scores were compared on three measures - (i) fraction computation, (ii) fraction understanding

and (iii) attitude toward mathematics. All groups received considerable pretest to posttest gain. Fraction computation scores made maximum contribution to the increase in scores. In this experiment, calculators were not used in pretesting. They were used in posttesting only by the last group which used calculator based algorithms. The authors claimed:

> Thus, the calculator algorithms can serve as an effective alternative instructional strategy where computational skill is a goal of instruction. (p. 354)

Shumway, White, Wheatley, Reys, Coburn and Schoen (1981) conducted a study the subjects of which were teachers and their classes. 56 classes Grades 2-6 were selected from 5 mid-western states and randomly assigned to calculator and no calculator treatments. The treatments were in effect for 18 weeks. The goal of the study was to investigate the effect that availability of calculators to students; and availability of calculator-related curriculum resources, inservice workshops on the use of calculators for teachers; and researchers' interactions with teachers as consultants had on children's attitudes and achievement in mathematics. Results revealed no measurable detrimental effects for calculator use (that is, no development of debilitations because of calculator use for instruction, when tested without calculators). In addition, significant gains on basic facts and achievement tests (taken without the use of calculators) were made irrespective of calculator use during instruction. It was apparent that children enjoyed calculators and the use of calculators increased children's computational ability with little instruction.

Gimmestad (1982) conducted a study with nineteen Calculus II students at the Department of Mathematical and Computer Sciences in Michigan Technological University, Houghton, Michigan. The students were randomly assigned to calculator (n = 9) and non-calculator groups (n = 10). Each student was given 24 Advanced Placement calculus problems to solve. The students were asked to think aloud as they worked through the problems. Their interviews were videotaped, coded and analyzed for reasoning process as well as outcome. The results revealed that the use of calculators affected the testing of basic facts and reasoning processes at the individual problem level but basic concepts and principles remained unaffected. A follow-up teaching experiment indicated that if students had greater calculator expertise, the content validity of Advanced Placement calculus problems could be more seriously affected.

Szetela (1982) executed two studies. The main study involved 187 students of Grades 3, 5, 7 and 8 from different schools in Richmond, B.C. for a period of 8 to 12 weeks. In each grade, both the Calculator group (C) and the Noncalculator group (N) were taught problem solving by the same teacher. A parallel study (that is, supplementary study) was conducted simultaneously. In this study all of 116 students of Grades 5, 6 and 7 made use of calculators for problem solving. The aim of this study was to compare the use of calculators on a posttest of problem solving with that of paper-and-pencil only, on such a posttest, after all groups had used calculators during an instruction period of 8 weeks.

Posttesting with and without the use of calculators revealed few differences on the number of problems attempted and the number with correct operations. Furthermore, when calculators were used on posttests, in 7 out of 12 comparisons, C groups attained significantly more right answers to problems than did N groups. In addition, on paper-and-pencil tests of computation and problem solving, C groups performed at least as well as N groups.

Bartalo (1983) conducted a project for ten days with elementary school students. Structured lessons were designed to help students focus on the process involved in solving common problems. The aim of the project was to make students think and talk about what they were doing. The students were encouraged to think aloud to help their class fellows understand. The author felt that the educators should learn how to make use of calculators in order to help students learn better and to help students learn how to solve common everyday problems (that is, practical situations which we face as citizens and consumers). Furthermore, in the author's opinion, the use of the calculator can definitely help elementary students to become better problem solvers because by having a calculator available, children could concentrate on the solution process and not merely on the needed computation.

In summary, the above research findings indicate:

1. The use of the calculator can effect computational benefits.
2. Students may achieve at a higher level when using calculators with some topics.

3. When using calculators, students may use more approaches to problems.
4. Calculator expertise may affect content validity of some topics.
5. Teacher attitudes toward the classroom use of calculators may be improved through inservice education.

However, in his recent review of 34 articles, Roberts (1980), while noting the generally positive outcomes of the various studies, did identify some serious research difficulties. In summarizing his review of the articles' "by-effects", Roberts noted that

1. Computational - Computational benefits occurred when students using calculators during a treatment could do routine computations (and not solutions to word problems) more accurately and/or rapidly than those not using calculators during the treatment. Such advantages took place whether or not students were permitted the use of a calculator on the posttest. 30 studies examined computational skills, out of which 19 (63%) reported positive findings in case of the E (Experimental) groups. None of the studies, where the E group was better than the C (Control) group, revealed an overall difference in performance. As far as allowing the E-group to use the calculator on the posttest, 11 (58%) allowed this while 6 (32%) did not, and the remaining 2 do not give any clear indication to this effect. Thus the data suggest real computational benefits due to the use of the calculator. In addition, the data seem to support the hypothesis that using calculators during instruction benefits

routine calculations and that the benefit is most pronounced when students continue to use calculators while actually performing the test computations. This hypothesis is supported strongly by the results of a series of investigations conducted by Roberts and his colleagues. They used three criterion performance measures - number correct, time to work problems, and efficiency. Systematic increase in the sophistication level of the calculation mode led to large increases in performance.

2. Conceptual - The empirical data do not support the proposition that the use of the calculator can have an impact on mathematical concept formation. In fact, this hypothesis has not been properly tested because few studies made a real attempt to integrate the use of the calculator into the curriculum. This would have illustrated how concept learning could be facilitated by calculator usage. Of the 16 studies investigating concepts, only 4 (25%) indicated superiority of the E group over the C group on tests that could be considered to emphasize concepts. Acquiring of concepts is a more complex task. Therefore, in order to bring about conceptual benefits, careful attempts should be made to fully integrate calculator use into mathematics instruction. For instance, merely showing students how to operate the calculator is not enough when it is to be used in mathematics problem solving situations. Similarly, if the goal is to facilitate concept acquisition using a calculator, then more efficient and/or effective ways of using the calculator to solve the problem must be demonstrated.

One reason for not noticing the conceptual advantages more often was that the learning settings in which these studies were conducted did not usually focus on concept-formation skills. In only two of the four positive findings, calculators were allowed on the posttest. Thus, the concept-formation advantages of calculator usage will remain an unresolved issue until the calculator is used as a strategy for problem solving.

3. Attitudinal - Of the 20 studies examining attitudes, only 7 reported results in favor of the E group. Four of these seven studies were in the Roberts series. These series placed emphasis on students' immediate reactions regarding their feelings about themselves and about the problems they had just completed. Three reasons can be cited for disappointing results on attitudinal criteria - (i) the measures used were too oriented-toward general traits, (ii) short time frame for most studies, and (iii) disallowing the use of the calculator to the E group on the posttest. Thus the evidence appears to support the proposition that calculators influence immediate and specific attitudinal perceptions but there is no evidence to support more general and lasting changes.

Furthermore, Roberts identified research difficulties as follows:

1. Assignment of Students to Groups. Assignment of students to E and C groups has been the most serious design strategy problem of the calculator research. Only in 12 of the 34 studies, students were assigned at random. The usual procedure was to assign classes at

random. Incorrect use of ANCOVA was made to correct design inadequacies. This could have produced effects or prohibited demonstration of effects.

2. Contamination of Treatment with Control Groups. The contamination of the treatment by the control group would have very likely occurred due to two reasons - (a) availability of calculators to C students in the home or at other locations, (b) extent of communication between E and C students about the experiment in progress. The first type of contamination is uncontrollable, while the second type could have been controlled by randomly assigning E and C conditions to multiple school and/or multiple grade levels in the same school. Thus, the possibility of the contamination of the second kind could have confounded the results on the conceptually oriented tests and the attitudinal measures, even when the calculators were not available to C students on criterion tests.

3. Control of the Teacher Variable. The calculator impacts have been either increased or concealed because of the inconsistencies in the implementation of E and C routines. In some cases, the researcher or the same teacher handled both E and C conditions, while in others, one teacher taught E and another teacher taught C. If the same teacher taught both groups, but was not very enthusiastic about calculator use, this could have entered into the daily instruction and reduced the actual effects. If the teacher teaching the E group was more enthusiastic about calculator use, extra but inappropriate help might have been given to the E students.

4. Use of Calculators on Criterion Tests. One of the most peculiar features of the calculator research is banning the use of calculators on the posttests. 47% investigations did not allow calculator use on posttests. The logic behind this practice is that the real question to be explored is whether the use of the calculator will harm students' performance on calculations to be done by paper-and-pencil methods. This is a negative orientation. Instead of investigating possible positive impacts, the emphasis is on showing the lack of negative impacts. It appears more realistic to assume that the calculator may have more positive effects than negative effects. If this approach is adopted, then allowing calculator use on the criterion test (posttest) seems more useful research strategy.

In the light of Roberts' review and the questions he raised, further studies are required in which the methodological concerns raised by him are dealt with systematically. Until this is done, the studies reviewed here cannot be considered to be conclusive. This conclusion is supported by Suydam (May 1979) who has written that it is not clear from many of the studies as to how the calculator was used: "Often the calculator is used as the teacher or student sees fit" (pp. 4-5).

In view of the entire above discussion, it would seem justifiable to infer that no conclusive evidence was found regarding the conceptual benefits of calculator use. Nonetheless, there were some indications that adequately integrated calculator use into the instructional process may reveal conceptual advantages. The present study was prompted by those indications.

7. Closing Remarks.

The review of the literature undertaken here clearly indicates that the central question regarding the use of calculators in classrooms is no longer whether or not they should be used. Rather, the question of central impact is how and where these instructional tools will be used.

Thus, instructional materials and curricula need to be developed so as to exploit the best use of calculators. Though interested groups and persons are trying to describe how calculators can be used constructively in classrooms, the activities described in Section 4 do not involve any systematic revisions of mathematics curricula. There seems to be a dearth of specially designed materials relevant to specific courses.

Some suggestions and recommendations have been made for basic curriculum revisions but very little has been done so far to effect these changes. Of course, recommendations should be based on valid empirical research, however, the issue is so vast that no single study can provide definitive answers.

With respect to the present study, the research studies completed which used calculators in the teaching of calculus, provided mixed evidence. One study (O'Loughlin, 1976) indicated that the calculator was a useful tool, while another study (Gimmestad, 1982) implied that calculator use may not effect the understanding of basic concepts and principles. Because of this mixed reaction, the present study attempted to clarify the above results by using the calculator as a vital instructional tool.

While claims were made that the use of calculators would enhance concept formation, Roberts' (1980) review, for example, suggested that this claim has not been critically tested, but he did indicate that conceptual benefits may accrue if calculators are used as an integral part of the instructional process.

Again, claims were made that student attitudes and motivation will be improved by allowing the use of calculators. Roberts' (1980) results question the validity of these claims. Consequently, the present study sought data from students as to the effectiveness of calculators as motivating devices.

These results with respect to the use of the calculator in a calculus course, attitudes, and concept formation led to the present study.

Though it was impossible in the setting of the study to completely revamp the calculus course, the supplement did develop a mini-unit in which calculators were used as an integral part of the instructional process. Consequently, the study, like the activities reviewed in Section 4, represents another limited attempt to use calculators for instruction. This study, however, is designed specifically to treat the central concept of any beginning calculus course, namely, limits. In doing so, data were sought as to the effectiveness of calculators as motivating devices, and as aids in the development of understanding of the limit concept.

The next chapter describes in detail the study which was undertaken.

C H A P T E R III

DESIGN OF THE STUDY

1. Limit - The Central Notion of Calculus.

This study was concerned with a calculator-based unit of work on a topic in calculus. Special consideration was given to methodology and content. Calculus is one of the mathematics courses offered at both the secondary school and the university levels. Moreover, calculus is a branch of mathematics indispensable to modern science and technology.

It is interesting to observe that some of the ideas of calculus were first developed by ancient Greek mathematicians - Eudoxus, Euclid and Archimedes. In the seventeenth century, ideas of calculus were given a new life by the pioneering work of Fermat, Descartes, Barrow, Newton and Leibniz. Indeed, Newton and Leibniz are regarded as the inventors of calculus because they contributed to its development and applications far more than their predecessors. However, the names of the Bernoulli family, Euler and Lagrange also need to be mentioned as inventive intellects in this field during the eighteenth century. Finally, the structure of calculus was completed in the nineteenth century, when some of the basic ideas of calculus were made really precise and clear by Cauchy, Dedekind and Weierstrass.*

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*See for example, Boyer, C.B., The History of the Calculus and Its Conceptual Development. New York: Dover Publications, Inc., 1949.

The notion of the 'limit' is fundamental to calculus. The two main foci of calculus - derivatives and integrals - rest essentially on this notion. One is frequently confronted with the limit of some expression because most concepts in calculus involve limits. "Calculus consists of a body of theorems and techniques which enable one to calculate various types of limits and to use the limit concept to solve certain problems", maintain Bartle and Tulcea (1970, p. 108).

Thus the subject matter of the unit of work dealt only with limits. Guidelines were provided regarding the use of a calculator in investigating limits of functions. Besides demonstrating the use of the "calculator method", various standard problem solving techniques were exhibited so as to enable students to verify their results. It was hoped that the calculator method would help them in obtaining a better "feeling" for as well as a clearer understanding of the limit concept due to the fact that the use of the calculator enables them to "see" what actually happens to the 'value of a function $f(x)$ as x approaches a'. It was assumed that students had made some previous attempts at understanding the concept of the limit. Since the unit of work was designed to enhance students' understanding of the concept of the 'limit' of a function and its contents were in accordance with those of the Math 157 - a first calculus course for social science students at the university level - the unit of work was entitled 'A Supplement on Limits for Mathematics 157' (see Appendix).

It is worth mentioning here that limits which require a device called L'Hospital's Rule (and are usually delayed till the introduction of this rule), were also included in the supplement because the calculator method can handle these limits as well.

However, the supplement did not address the problem of round-off errors which sometimes lead to spurious results. The author did not feel it appropriate to discuss this in the supplement as very few of the readers at the time the supplement was presented to them would have been in a position to understand a careful explanation. Nonetheless, the students were told that the calculator method only indicates what the limit might be, whereas more analytical methods are required for proving that a particular number is the correct limit.

2. Purpose of the Study.

As stated in Chapter I, the primary purpose of this study was: to investigate whether or not students' understanding of the concept of limit was enhanced by using calculators as an integral component of the instructional mode. In particular, the study was aimed at finding answers to the following four broad questions:

1. Did the use of the calculator enhance students' understanding of the concept of the limit of a function?
2. Did the students consider the calculator to be an effective learning device?
3. Did the students favor the use of the calculator as an integral part of the calculus curriculum?

4. Did the students judge the supplement to be a useful instructional guide?

3. Research Procedures and Data Collection Techniques.

A. The Pilot Study.

In the Fall of 1981, the supplement was developed and then tested on a sample of Mathematics Department students at Simon Fraser University, with the goal of reporting the conclusions and implications of the pilot study to the Mathematics Department.

The supplement was administered during November, 1981. Section IV of the supplement consisted of a questionnaire that contained nineteen questions pertaining to the following four areas:

1. Effect of the experiment on development of students' understanding of the limit concept.
2. Use of the calculator as an effective learning device.
3. Integration of the use of the calculator in the calculus curriculum.
4. Quality of the supplement as a whole.

After working through the unit, students were asked to return the attached questionnaire. Due to a conflict with exam period, the response rate was quite low. The number of questionnaires distributed was 200, of which 21 were returned.

An analysis of the data from the pilot study led the investigator to make some alterations in the supplement. For example, question three of the questionnaire, "Do you think a calculator

makes you think for yourself?", needed to be placed after question seven, because it seemed that the question was misinterpreted by the respondents. Moreover, answers to question sixteen indicated that it might be necessary to introduce more challenging material into the supplement. However, almost 80% of the respondents indicated that subject matter presentation was good or excellent. Almost three-quarters felt that quality of the supplement as a whole was above average. In addition, more than half felt that contribution of the supplement to their understanding of limits was good or excellent.

Suggestions by the respondents for improvement of the supplement included:

1. Introduction of limits as x becomes infinite.
2. Display of more examples involving exponents, logarithms and complex expressions.
3. Provision of an answer-key to Section III of the supplement.

In addition, student responses indicated that the experiment was not a waste of time. Their comments included: It was a helpful review. It clarified their "foggy" areas of concern and thus strengthened the limit concept.

As a brief summary of the results of this pilot study, the following five conclusions were drawn:

1. The use of a calculator enhanced students' understanding of the concept of the limit of a function.
2. The students considered the calculator to be an effective learning device.

3. A teacher should use both the calculator method as well as the problem solving techniques in teaching limits.
4. The supplement was a good guide for the study of limits.
5. The supplement should be expanded so as to contain limits as x becomes infinite.

Though the number of responses on which these conclusions were drawn was small, the responses were nonetheless extremely useful in making constructive revisions to the supplement.

B. The Main Study.

Since the unit of work was well-received by students (though their number was small) and some members of the Mathematics Department, an expanded and revised version of the supplement (including limits of functions when the variable x becomes infinite) was developed in 1982 Spring Semester. It was administered in February, 1982. After working through the unit, students were again asked to return the attached questionnaire. The number of questionnaires distributed was 408, of which 99 were returned. The response rate for the study rose, therefore, to 24.26% from the 10% response rate for the pilot study. The next section presents a detailed account of the expanded and revised version of the supplement, a copy of which is reproduced in the Appendix.

4. Discussion of the Supplement.

The supplement was designed so that the calculator was an integral part of the instructional mode. Students were required to

use calculators to do various numerical calculations (i.e., to use the "calculator method") in order to ascertain and approximate limits. In addition, the problem solving techniques that are mentioned in the course text (Haeussler & Paul, 1980) were displayed so that students could verify the answers thus obtained. In particular, the intent of the supplement was to promote students' understanding of the concept of the limit of a function, i.e., limit of a function $f(x)$ as x approaches a and as x becomes infinite.

The supplement included a description of the "calculator method" as follows:

> Two situations can arise:
>
> (a) When x approaches some (finite) real number a (see Sections I and II of the supplement).
>
> (b) When x becomes positively (or negatively) infinite (see Section III of the supplement).
>
> In situation (a), this method consists in taking values of x closer and closer to a (but not equal to a), writing values of x and corresponding values of $f(x)$ in the form of a table, and then ascertaining from the table whether, as x approaches a, the values of $f(x)$ appear to get arbitrarily close to one particular real number ℓ or to more than one real number or become unbounded (i.e., very large and positive or very large (in absolute value) and negative or both). The limit exists in the first case only.
>
> In situation (b), when x becomes positively infinite (or negatively infinite), values of x are taken to be positive (or negative) and they become bigger and bigger in absolute value. The rest of the procedure is the same as for situation (a). (Appendix, p. 2)
>
> The supplement was organized into five sections as follows:

Section I defined 'limit of a function $f(x)$ as x approaches a real number a' , stated important properties of limits and demonstrated the use of the "calculator method" as well as the "problem solving techniques". Furthermore, this section exhibited various possibilities in relation to the limit of a function and its value at a point. In particular, Section I dealt only with limits of functions which are defined by the same formula for $x < a$ and $x > a$ and formed a basis for the subsequent sections.

Section II studied limits of functions that are defined by different formulae for $x < a$ and $x > a$, respectively. In particular, this section studied one-sided limits and the (ordinary) limit in terms of the one-sided limits.

Section III discussed limits of functions when the variable x becomes infinite, i.e., limits at infinity (or minus infinity). In other words, limits of functions when x becomes positively large (or negatively large) or, when x increases (decreases) without bound through positive (negative) values. In fact, Section III was an extension of the limit concept of Sections I and II. (Note that the pilot version of the supplement did not contain this section).

Section IV involved students in the use of calculators in investigating limits.

Finally, Section V invited students' comments regarding use of the calculator as an effective learning device, integration of the use of the calculator in the calculus curriculum, quality of the supplement as a whole, and effectiveness of the experiment on

development of students' understanding of the limit concept, in particular.

It was strongly suggested that the students would actually calculate some of the limits of Sections I, II and III using their calculators before proceeding to Section IV. This would help them become familiar with the calculator method.

It was taken for granted that the students were familiar with the notion of a function and the algebra of functions. Furthermore, some familiarity with the concept of the limit and various problem solving techniques was also presumed. More precisely, it was assumed that the students had made some previous attempts at understanding the concept of the limit through lectures and homework assignments.

The students were asked to use their own calculators because the type of calculator was not of particular importance as long as it had the capacity to calculate a few functions that were specified in the supplement, and could calculate up to at least eight significant digits to obtain a better approximation to the limit. Moreover, no instruction was given to the students on how to operate a calculator because of the widely varying types of calculators. Of course, the calculator manual always contains all the necessary information.

Sometimes it is difficult to determine the existence or non-existence of limits without the aid of special problem solving techniques. However, the supplement revealed that a calculator is often an aid in ascertaining the existence of limits as well as in approximating the values of those that do exist. Furthermore, there are certain limits that need L'Hospital's Rule as a technique. This

rule is introduced only after students are familiar with the notion of the derivative (because L'Hospital's Rule depends on derivatives), which rests essentially on the notion of the limit. With the integration of calculators as an integral part of the instructional mode, students can be introduced to such limits even before the introduction of L'Hospital's Rule (viz., Examples 14-16 on pp. 22-24 and Examples 25-26 on pp. 39-40 of the supplement).

Section V of the supplement, which consisted of a questionnaire, needs special mention.

5. The Questionnaire.

The questionnaire (see pages 47-49 of Appendix) was comprised of nineteen questions (items) pertaining to the following four areas:

1. Effect of the experiment on development of students' understanding of the limit concept (Q. 1, 2).
2. Use of the calculator as an effective learning device (Q. 3-7).
3. Integration of the use of the calculator in the calculus curriculum (Q. 8-13).
4. Quality of the supplement as a whole (Q. 2, 14-19).

In addition, the items were mainly of two types:

A. Open-ended questions or free-response items (Q. 1, 13(b), 14, 15, 19).

B. Scaled items (the remaining questions).

Scaled items (that used scaled response mode) fell under the following three classes:

(i) Those utilizing a two-point scale (two choices for each question) as the response mode (see items 3-7, 12).

(ii) Those utilizing a three-point scale as the response mode (see items 8, 9).

(iii) Those utilizing a five-point scale as the response mode (see items 2, 10, 11, 13(a), 16-18).

6. Closing Remarks.

This chapter has included a description of limit as the central notion of calculus, purpose of the study, the research procedures and data collection techniques used, a description of the essential aspects of the supplement, and an overview of the questionnaire used in the study. The next chapter reports the results of the study.

C H A P T E R IV

ANALYSIS OF THE DATA

AND

DISCUSSION OF THE RESULTS

1. Preliminary Remarks.

This chapter includes the analysis of the questionnaire data. Basically two types of items were used in the questionnaire:

A. Open-ended questions or free-response items (Q. 1, 13(b), 14, 15, 19).

B. Scaled items (the remaining questions), which fell under three classes: two-point, three-point and five-point scaled items.

Analysis of the data from free-response items is done in the form of discussion followed by a table and that of the data from scaled items is exhibited in the form of tables.

There were 408 questionnaires and supplements distributed to students in February,1982. By the end of March,1982, 99 questionnaires had been returned. The rate of return was 24.26% which was much lower than the rate anticipated but nonetheless, it was felt that sufficient responses were received to allow for an empirical and statistical analysis to be done.

2. Analysis of the Data.

A. Free-Response Items.

With the exception of one response, all of the student responses to statement 1 indicated that the experiment was not a waste of time. Out of 99 questionnaires which were returned, the enumeration of student responses to this statement 1 was: positive responses = 80, negative responses = 1, answers making no sense = 1, no answer made at all = 17. Therefore, the number of no responses was 18. Consequently, the corresponding percentages are: positive responses = 81%, negative responses = 1%, no responses = 18% (rounding the percentages to the nearest integer). Observe that "no response" means either no answer is made at all, or the answer makes no sense.

The students were of the opinion that the supplement aided their understanding of limits. Their comments included that the experiment was a great idea, worthwhile, beneficial, excellent, and of insurmountable help. It was also commented that the supplement was an excellent booklet both for those who do not understand limits, and for those who used it as a review. It was a useful review because it went over theories in the text and allowed students to practice them, provided more and better practice questions, was more thorough and steps were more simplified than those in the text. Thus, it clarified their "blurred" areas of concern and strengthened the limit concept.

Given these results, it was surprising that one student stated that the supplement confused previous understanding. Perhaps

the student did not like the use of the calculator as was evident from this student's negative responses to statements 3-7.

Regarding naming of some topics besides limits, where a teacher should use a calculator (statement 13(b)), 87% made no response. The remaining 13% mentioned application formulas, logarithms, continuous interest, differentiation and economics related functions.

As far as statement 14 (about things they especially liked about the supplement) was concerned, by and large, the respondents felt the supplement was well-organized, readable, intelligible, thorough and researchable. They commented that it gave a very thorough and good discussion on limits, and was a methodical aid to study and solving of limits. In particular, comments were made concerning language, explanations, subject matter, examples and organization of the supplement. These will be dealt with in turn.

It was mentioned that the language of the supplement was clear, simple, and easy to comprehend. In addition, students especially liked explicit statements of definitions, properties and theorems; clear and concise instructions; simple, basic, clear, succinct, thorough, careful, good, well-thought-out and step-by-step explanations.

Furthermore, the respondents indicated that the subject matter was presented very well. It was commented that the supplement provided other sources of information to enhance students' knowledge of limits. Demonstration of the calculator method was applauded due to the fact that this method presented a different approach from the text and consequently, provided a different perspective on

viewing limits. Inclusion of the standard techniques was also appreciated because it enabled them to check their answers as well as review theories in the text.

In addition, regarding examples in the supplement, respondents' comments included: the supplement contains useful, very clear, concise, straightforward, excellent, and a wide variety of sample questions that give plenty of information and clear-up many doubts concerning limits. Moreover, they stated that step-by-step analysis of each example with simple, detailed and full explanations was done; thus enabling students to solve problems themselves. Also, each step was clear and easy to follow. Some students even felt that the explanation of the subject was much more simple and better in the supplement than that in the course text.

As stated before, the respondents believed that the supplement had a very well-thought-out and careful organization. Various reasons cited in support of the above claim were the following: progression of the entire supplement in a logical and easily understandable manner, division of the supplement into sections, listing of properties straight-out, reinforcement of definitions and properties of limits, summarizing of properties in one place, referring back to the property numbers, explicit statements of theorems and rules, clear and good use of examples and definitions, spelling out of each step, the underlining of key words, grouping of problems in the back of the supplement as well as provision of their answers, answers near the questions and hence allowing students to check correctness of their work, good type spacing as well as absence of typographical errors.

In the light of the above data, it would seem justifiable to conclude that the respondents felt that the supplement was a good guide especially for those who did not understand limits because it gave a clear and concise overview of the subject. 14% gave no answer to this statement 14.

On the other hand (statement 15), respondents who felt that the organization of the supplement could be improved, focused on such things as making properties stand out more (typographically), simplifying calculations, giving further explanation of L'Hospital's Rule, and making the supplement shorter. This statement 15 (concerning the things they disliked about the supplement) educed a 55% response rate.

It is worth remarking here that as far as L'Hospital's Rule was concerned, it was not possible to give its full explanation within this supplement due to the reason cited on page 24 of the supplement. The intent of the author was to demonstrate how the use of the calculator allows early introduction of the limits that require L'Hospital's Rule. These limits may otherwise need to be delayed till the students have learned this rule, which is taught necessarily after the acquisition of basic limits and derivatives.

Finally, in response to statement 19 (about suggestions for improvement), the students indicated that they would like to have: inclusion of more and harder problems and further explanation of L'Hospital's Rule. Some students suggested abridgement of the supplement. However, the author feels that an abridged version may

not convey the whole information as effectively as the original version and consequently, may fail to produce the desired results.

A few of the respondents agreed with the author that the use of the calculator should be introduced after students have got some understanding of the particular concept.

It was indeed heartening to note that one of the respondents attributed his/her attainment of 100% score on the mid-term to this supplement. In addition, many others made positive remarks indicating that there was no need of further improvement because the supplement already fulfilled their needs.

Table II (p. 116 below) summarizes the above analysis. The table displays page number (of the supplement), question number, description of the question, area to which the question belongs and responses to each question as a percentage of total number of questionnaires returned. Furthermore, in computing percentages, all figures are rounded to the closest integer. Recall that the category "No Response" means either no answer is made at all, or the answer makes no sense.

TABLE II

FREE-RESPONSE ITEMS

PAGE	QUESTION NUMBER	QUESTION	AREA	RESPONSE (%)		NO RESPONSE (%)
				Positive (%)	Negative (%)	
47	1	Was this experiment a waste of time or did it give you a better understanding of "limits"?	Concept development	81	1	18
48	13(b)	Can you name some topics (besides limits in the teaching of which your teacher should use a calculator)?	Curriculum	13		87
48	14	Name a few things that you especially like about this supplement.	Supplement	86		14
48	15	Name a few things that you especially dislike about this supplement.	Supplement	55		45
49	19	Give a few suggestions for the improvement of this supplement.	Supplement	43		57

B. Scaled Items.

The analysis is exhibited in Tables III, IV and V (see pp. 118-120 below). In addition to page number (of the supplement), question number, description of the question and area to which the question belongs, each table displays total responses to each category (choice) of a particular item as a percentage of total number of questionnaires received. As an illustration, consider statement number 2 on page 47 of the supplement which appears in Table V. Obviously, the total number of responses to this statement under all categories (including No Response) was 99 [Recall that the number of questionnaires returned was 99]. The numbers of responses to various categories were: Excellent = 9, Good = 62, Average = 24, Below Average = 3, Poor = 0, No Response = 1. Hence the category "Good" received a 62.63% response, which is rounded to 63%.

Observe that in computing the percentages, all figures are rounded to the closest integer so that the sum of the numbers under all categories corresponding to a statement may not always be 100. Furthermore, the category "No Response" means either no choice is made at all; or a choice, different from those already given, is made; or the response makes no sense. Tables III, IV and V deal with two-point, three-point and five-point scaled items respectively. In addition, each table is followed by conclusions based on the analysis of the data.

TABLE III

TWO-POINT SCALED ITEMS

PAGE	QUESTION NUMBER	QUESTION	AREA	CATEGORY (%)		
				Yes	No	No Response
47	3	Do you think a calculator is a motivating device?	Calculator use	60	37	3
47	4	Do you think a calculator enhances your independence in problem solving?	"	58	38	4
47	5	Do you think a calculator can provide another method for helping you to think, create and learn mathematics?	"	65	31	4
47	6	Does a calculator help you gain more confidence?	"	70	30	0
47	7	Do you think a calculator makes you think for yourself?	"	29	69	2
48	12	Do you think that the Math 157 curriculum should include more calculator problems?	Curriculum	38	56	6

Conclusions -- From Table III, it is concluded that roughly 60% responses indicated that a calculator is a motivating device (Q. 3 = 60%), enhances indepedence (Q. 4 = 58%), provides another method (Q.5 = 65%) and helps gain confidence (Q. 6 = 70%). In view of these results, it is strange that Q. 7, guided only 29% of responses, that a calculator makes them think for themselves. This question caused problems in pilot study. It might have been wise to rephrase it.

Moreover, only 38% felt the need to include more calculator problems in the Math 157 Curriculum (Q. 12 = 38%). Perhaps the remaining 62% thought that the use of calculators was too time-consuming.

TABLE IV

THREE-POINT SCALED ITEMS

PAGE	QUESTION NUMBER	QUESTION	AREA	CATEGORY (%)			
				All the Time	Some of the Time	Never	No Response
47	8	Did you ever use a calculator in problems on limits?	Curriculum	2	60	38	0
48	9*	Do you think that your teacher should use		(a)	(b)	(c)	No Response
		(a) only calculator method (b) only problem-solving techniques (c) both in the class in teaching limits?	Curriculum	0	28	72	0

*In question 9, category (c), some students have suggested more emphasis on problem-solving techniques.

Conclusions--In view of the above table, it is concluded that roughly 60% responses indicated that in problems on limits, they used calculators some of the time (Q. 8 = 60%) and that both methods should be used by a teacher in the class (Q. 9 = 72%).

TABLE V

FIVE-POINT SCALED ITEMS

PAGE	QUESTION NUMBER	QUESTION	AREA	CATEGORY (%)					
				Excellent	Good	Average	Below Average	Poor	No Response
47	2	How would you rate the contribution of the supplement to your understanding of limits?	Concept development/ Supplement	9	63	24	3	0	1
47	17	How would you rate the way the subject matter in this supplement was presented?	Supplement	22	65	11	1	0	1
				Agree Strongly	Agree	Neutral Opinion	Disagree	Disagree Strongly	No Response
48	10	Do you think students should be allowed the calculator method in problems on limits in the examination?	Curriculum	26	35	26	8	4	0
48	11	Do you think there should be a question in the exam exploiting the use of calculator alone?	Curriculum	1	10	34	31	20	3
48	13(a)	Do you think your teacher should use a calculator in teaching other chapters of Math 157 besides limits?	Curriculum	2	24	43	17	8	5
				Highly Challenging	Challenging	Average	Little Challenging	No Challenge	No Response
48	16	How challenging did you find this supplement?	Supplement	2	53	37	4	1	3
				Far Above Average	Somewhat Above Average	Average	Somewhat Below Average	Far Below Average	No Response
49	18	Rate the quality of this supplement.	Supplement	17	63	17	0	0	3

Conclusions--Please see following page.

Conclusions (Table V).

From Table V, it is concluded that almost three-quarters felt that the contribution of the supplement to their understanding of limits was good or excellent (Q. 2 Excellent or Good = 72%). More than 85% responses indicated that the subject matter presentation was good or excellent (Q. 17 Excellent or Good = 87%). Precisely 80% expressed that the quality of the supplement was above average (Q. 18 Far above average = 17%, somewhat above average = 63%). Answers to question 16 indicated that the level of the material in the supplement was about right--neither too challenging nor lacking challenge (Q. 16 Highly Challenging or Challenging = 55%, Average = 37%).

Furthermore, about three-fifths favored allowing of the calculator method in problems on limits in the exam (Q. 10 = 61% Agree or Agree Strongly). As far as the question concerning the use of a calculator by the teacher in teaching topics other than limits was concerned, ignoring the Neutral and No Response categories, responses were roughly evenly split (Q. 13(a) Agree or Agree Strongly = 26%, Disagree or Disagree Strongly = 25%). Finally, responses were generally negative regarding exam question exploiting the use of calculator alone (Q. 11 = 88% Neutral or below). Perhaps the remaining students thought that the use of the calculator consumes too much time; or perhaps that it would add to, rather than simplify, an already heavy course.

3. Discussion of the Results.

The following is a discussion of the major results of this study, based on the preceding analysis of data. The study was conducted in an attempt to answer the four broad questions stated in Chapter I. This section presents each of those questions followed by a response to it on the basis of the results obtained.

The first question was--did the use of the calculator enhance students' understanding of the concept of the limit of a function?

81% of the responses to statement 1 indicated that the experiment was not a waste of time. Indeed, it enhanced their understanding of limits (only 1% gave a negative response. The remaining 18% gave no response). In addition, almost three-quarters (72%) felt that the contribution of the supplement to their understanding of limits was good or excellent (Q. 2). Hence, it would seem reasonable to conclude that the use of the calculator significantly enhanced students' understanding of the concept of the limit of a function.

The second broad question was--did the students consider the calculator to be an effective learning device?

Roughly 60% responses indicated that a calculator is a motivating device (Q. 3 = 60%), enhances independence (Q. 4 = 58%), provides another method (Q. 5 = 65%) and helped students gain confidence (Q. 6 = 70%). However, only 29% thought that a calculator makes them think for themselves. Perhaps this question

needed rewording (Q. 7). Hence, it can be inferred that the students considered the calculator to be an effective learning device.

The third broad question was--did the students favor the use of the calculator as an integral part of the calculus curriculum?

Almost two-thirds of the responses indicated that in problems on limits,both methods should be used by a teacher in the class (Q. 9 = 72%), the calculator method should be allowed in the examination (Q. 10 = 61% Agree or Agree Strongly) and the respondents used calculators some of the time (Q. 8 = 60%). Furthermore, only 38% felt the need to include more calculator problems in the Math 157 curriculum (Q. 12). Responses were predominately negative regarding exam questions exploiting the use of calculator alone (Q. 11 = 88% Neutral or below). Perhaps the remaining students thought that the use of the calculator was too time-consuming; or perhaps that it would add to, rather than simplify, an already heavy course. Moreover, ignoring the Neutral and No Response categories, responses were roughly evenly split concerning the use of a calculator by the teacher in teaching topics other than limits (Q. 13(a) Agree or Agree Strongly = 26%, Disagree or Disagree Strongly = 25%). 13% suggested topics other than limits, where a teacher should use a calculator. The topics mentioned were application formulas, logarithms, continuous interest, differentiation and economics related functions (Q. 13(b)).

In the light of the above data, it can be inferred that, though a clear-cut conclusion regarding the integration of the use

of the calculator in the calculus curriculum could not be made, the students evidently favored the use of both methods in the class as well as in the examination in dealing with limits.

The fourth question of interest to the investigator was--did the students judge the supplement to be a useful instructional guide?

72% stated that the contribution of the supplement to their understanding of limits was good or excellent (Q. 2). Furthermore, 86% of the respondents felt that the supplement was well-organized, readable, intelligible, thorough or researchable (Q. 14 No Response = 14%). Answers to statement 16 indicated that the level of the material in the supplement was about right--neither too challenging nor lacking challenge (Q. 16 = Highly Challenging or Challenging = 55%). Precisely 80% of the respondents indicated that the quality of the supplement was above average (Q. 18 Far above average = 17%, Somewhat above average = 63%) and 87% stated that the subject matter presentation was good or excellent (Q. 17). A few students did make some suggestions for improvements to the supplement including the introduction of more and harder problems (Q. 19). However, most students felt that there was no need for further improvement because the supplement already fulfilled their requirements.

In the light of the above data, it would seem justifiable to conclude that the students judged the supplement to be a good guide for the study of limits. They felt that it gave a clear and concise overview on the subject.

The above discussion implies that the calculator method did enable students to "see" what actually happened when a function approached a limit and thus, gave them a better "feeling" for as well as a clearer understanding of the notion of the limit as hoped. Consequently, the supplement and hence the use of the calculator definitely enhanced students' understanding of the concept of the limit of a function.

4. Closing Remarks.

The present chapter has presented the analysis of the data from the returned questionnaires and treated the results of the study at length. Though the supplement was very well-received by the students, the response rate was much lower than expected, in spite of tremendous efforts on the part of the professor teaching the course and the teaching assistants. Perhaps this could be due to the fact that the large number of students, not having English as their first language, were unprepared to answer the open-ended questions due to perceived language difficulties. This could have possibly prevented them from returning their questionnaires. Perhaps the questionnaire to be used in the replication of this study should present such questions differently. The next chapter contains conclusions and implications of the study.

C H A P T E R V

SUMMARY, CONCLUSIONS AND IMPLICATIONS

1. Summary.

The world is presently undergoing an explosion of electronic technology, the two major components of which are the calculator and the computer. The availability, affordability and portability of calculators made them accessible to more and more people in a remarkably short period of time. These tiny marvels, almost unknown in the early 1970s, are now omnipresent. In almost every home, they are as common a staple as the television, the transistor radio or the cassette tape recorder. Not only do they prevail in most homes or offices, but they also can be found in many pockets or handbags like combs, credit cards and other essential items.

The proliferation, wide variety, easy accessibility and widespread use of calculators in society caused a minor revolution in mathematics education as reported in Chapter I. Students brought calculators to class and used them for doing their homework. Teachers needed immediate direction as to how to deal with this phenomena. As a consequence, the impact of the use of the calculator on mathematics education became one of the most widely discussed topics in every meeting where mathematics educators gathered. Meetings were held at the State, Provincial, Regional and National levels to discuss the impact of calculator usage on the classroom environment. Their use in the classroom was a widely

discussed and sometimes a controversial issue among educators, parents and other community members. Articles -- pro and con -- appeared in several journals, research papers, magazines and the general press. Inspite of the apparent controversy, student use of calculators within the classroom increased due to several factors including those mentioned above as well as the endorsement of their use by teacher educators, teacher organizations and textbook publishers.

The proclamation made by the Board of Directors of the NCTM in 1974 supported the use of this device in the classroom to reinforce learning and to motivate learners as they become proficient in mathematics. Several sessions, workshops and debates were conducted at all levels. Research proposals were made and studies were undertaken. It was felt that the calculator had the potential for reshaping computationally oriented mathematics curriculum. Since the narrow definition of basic skills equated mathematical competence with computational ability, in the mid 1970s, mathematics textbooks emphasized the basics of computational skills and knowledge of facts. Due to reasons including the advent of calculators and computers, the NACOME insisted on putting less emphasis on computational skills. The need for redefining basic mathematical skills was felt by members attending the NCSM 1976 Annual Meeting held in Atlanta, Georgia. The basic mathematical skills, redefined by the NCSM included, among others, problem solving and appropriate computational skills. Furthermore, the NCTM felt that the computational skills needed by every citizen required a reexamination because of the availability of computing aids. Moreover, the NCTM recommended integration of the

use of electronic tools into the core mathematics curriculum. The Council further suggested that curriculum materials which integrate and require the use of the calculator and computer in different and imaginative ways should be developed and made available. While efforts are being made to augment mathematics instruction with the use of the calculator, and curriculum materials are being developed to achieve this objective, few of these materials use the calculator to teach mathematical ideas.

The purpose of the present study was to develop a calculator-based unit of work--supplement on limits--to enhance the understanding of the concept of the limit. The subjects of the study were students in a first year calculus course at the university level. The unit is directly related to the school mathematics instruction because calculus is one of the advanced level mathematics courses at the secondary school level. A pilot test of the supplement was conducted in the Fall of 1981. The extended and revised supplement was tested in the Spring of 1982. There was a questionnaire attached to the supplement. The students were requested to return the questionnaire after working through the unit.

The questionnaire sought answers to the following four broad questions:

1. Did the use of the calculator enhance students' understanding of the concept of the limit of a function?
2. Did the students consider the calculator to be an effective learning device?

3. Did the students favor the use of the calculator as an integral part of the calculus curriculum?
4. Did the students judge the supplement to be a useful instructional guide?

The responses were analyzed and interpreted.

2. Conclusions.

As a result of the findings of this study, the following four major conclusions were drawn.

1. The use of the calculator aided in students' understanding of limits.
2. The students considered the calculator to be an effective learning device.
3. Unfortunately, a clear-cut conclusion regarding the integration of the use of the calculator in the calculus curriculum could not be made based on the data. However, the results clearly indicated that the students would like their teacher to use both the calculator method and the problem solving techniques in the class in teaching limits. In addition, they would like the use of the calculator method to be allowed in problems on limits in the examinations.
4. The students judged the supplement to be a good guide for the study of limits. They felt that it gave a clear and concise overview on the subject.

3. Limitations of the Study.

The limitations of the study resided with the questionnaire, namely:

1. The number of questionnaires returned was much lower than anticipated. It may be that the students who responded were not of a heterogeneous level of ability but formed a group of a homogeneous level of ability.

2. 20% of the questions were open-ended. A large number of students did not have English as their first language. This could have caused them to ignore open-ended questions or prevented them from returning their questionnaires altogether.

4. Suggestions for Future Research.

This study was designed to investigate the effectiveness of the use of the calculator in fostering student understanding of the concept of the limit of a function. Since the number of questionnaires returned was not as large as expected, the results of this study do not afford definitive answers to all of the questions posed in the study. These questions are still open for investigation. However, important results were arrived at, which warrant further research. The following are a few suggestions for research ensuing from the present study:

1. Replicate this study with the same target population but rephrase the open-ended questions with a view to minimizing the effect of limitation #2.

2. Conduct this study with different levels of students, for example--(a) first level scientific calculus courses, or (b) Grade 12 calculus students, or (c) a combination of both--with (i) the same questionnaire, or (ii) the questionnaire appropriately modified.

3. Replicate the study but provide for an experimental and control group, i.e., divide students into two groups such that one group uses the supplement and the other does not. Compare their performance on a test which measures their comprehension of the limit concept and on which they are not allowed the use of a calculator.

4. Replicate this study, but after students have worked through the supplement, give them a test on limits. To obtain the maximum response, let the score on this test be counted towards students' final grade and have the questionnaire returned along with this test. Assess students' performance on this test. This data could then be used to determine whether or not there is a correlation between student performance and positive response to the supplement. It would be hoped, of course, that there would be no correlation between achievement and response to the supplement, because no correlation would imply that the supplement was beneficial for all students regardless of achievement level.

5. Extend the instructional material so as to include either (a) verbal problems, or (b) trigonometric functions, or (c) both. Use it in conjunction with any of the other suggestions regarding future research.

6. Repeat the basic design of this study but extend the instructional material so as to include continuity and/or derivatives.

5. Implications for Education.

The use of the calculator as an integral part of the instructional mode can be beneficial to students. Materials can be developed to use this tool effectively. By simplifying or speeding up complex calculations, the student can acquire a better feeling for and a deeper insight into the theoretical aspects of the problem. As a result, the student can have a clearer and better understanding of some mathematical principles and concepts.

Since calculators are becoming a part of children's lives, more interesting and imaginative materials need to be prepared on how best to use calculators. This implies that math educators and math teachers must possess the mathematical competence necessary to use the calculator as a teaching aid.

As the use of the calculator can have a positive instructional value, it can motivate learners. Moreover, children can be more eager to do and confident about mathematics when calculators are available, because they have no fear of being bogged down by tedious calculations. Even low achievers can generate new enthusiasm for mathematics because they can be better prepared to deal comfortably with drudgery of calculations and large numbers. The use of the calculator can also provide immediate verification, which is an important motivational factor.

Furthermore, the calculator can be used to encourage students to be inquisitive and creative. Students can be inspired to create their own problems while they are practicing mathematical

ideas. Ideas can be explored further using various kinds and sizes of numbers as well as different techniques. Thus, mathematical concepts can be learned in a more interesting manner and in more detail.

The calculator method can provide a new and different approach from the text and consequently, a different perspective on viewing the subject matter. Open exploration and new problems can be offered to students because of the calculating power which the calculator provides. The use of the calculator can impart new insight, interest and fun to the teaching of many mathematical concepts. It can also allow early consideration of some topics. This implies that a teacher should use the calculator as an instructional tool in class.

Good instructional materials can be developed for use with the calculator so as to meet different needs of students. Because of the growing utility of this device, materials need to be designed in order best to teach mathematics. In today's world, mathematical competence is essential to every individual's meaningful and productive existence. The hand-held calculator is the most practical machine used in society today. Almost all extensive calculations are done with the use of this miniature marvel. Mathematics educators and curriculum developers need to play their parts in developing and disseminating good and imaginative instructional materials so as to ensure best advantage of the modern calculator technology.

6. Concluding Remarks.

Calculators are now acceptable in the classroom. While on-the-job and in-the-home use of calculators is almost standard, schools have not yet fully incorporated their use into the mathematics curriculum. The NCTM recommends that mathematics programs take "full advantage" of the power of the calculator at all grade levels. The research evidence reveals that the use of the calculator does not harm achievement. On the contrary, when calculators are used, the achievement is as high or higher than when they are ignored. Furthermore, with the use of the calculator, some mathematics content can be taught better. Hence, all persons involved in the preparation of new math materials and curricula need to take careful note of the above statements. Finally, all interested persons and groups need to join in a massive cooperative endeavor toward better mathematics education for all students. As Emerson wrote:

> The true test of civilization is not the census, nor the size of cities, nor the crops - no, but the kind of man the country turns out.

APPENDIX

A

SUPPLEMENT

ON

LIMITS

FOR

MATHEMATICS 157

Prepared by

Pushpa K. Jain

This supplement is designed to enhance your understanding of the concept of the 'limit' of a function, i.e., 'limit of a function $f(x)$ as x approaches a' (and 'as x becomes infinite'). In addition, the supplement provides guidelines regarding the use of a calculator in investigating limits of functions and in verifying the answers thus obtained by employing problem-solving techniques that are mentioned in the Math 157 text.

The supplement is organized into five sections. Section I states the definition of the limit and outlines various important properties of limits. In addition, it contains examples of limits that are determined using a calculator as well as various problem-solving techniques. Section II discusses one-sided limits and the (ordinary) limit in terms of one-sided limits. Section III is an extension of the limit concept of Sections I and II. It studies limits at infinity, i.e., limits of functions when x becomes positively infinite or negatively infinite. Section IV involves you in adventures (concerning limits) with your calculators. Finally, Section V invites your comments about your adventures and about the supplement as a whole.

To be able to participate in this experiment, you need a familiarity with the concept of a function and the algebra of functions. It is assumed that you have made some previous attempts at understanding the concept of the 'limit'. The type of calculator you use is not of particular importance, but

it should have the capacity to calculate squares and other powers, square roots and other roots, reciprocals, exponential functions and logarithmic functions. Furthermore, to obtain better accuracy, it would be desirable to have a calculator that calculates to at least eight significant digits.

Sometimes it is difficult to prove the existence or non-existence of limits without the aid of special problem-solving techniques. However, a calculator can be an aid in ascertaining the existence of limits as well as in approximating the values of those that do exist.

What is the "calculator method" which may help you understand limits? Two situations can arise:

(a) When x approaches some (finite) real number a (see Sections I and II).

(b) When x becomes positively (or negatively) infinite (see Section III).

In situation (a), this method consists in taking values of x closer and closer to a (but not equal to a), writing values of x and corresponding values of $f(x)$ in the form of a table, and then ascertaining from the table whether, as x approaches a, the values of $f(x)$ <u>appear</u> to get arbitrarily close to <u>one</u> particular real number ℓ or to <u>more than one</u> real number or become <u>unbounded</u> (i.e., very large and positive or very large (in absolute value) and negative or both). The limit <u>exists</u> in the <u>first</u> case only.

In situation (b), when x becomes positively infinite (or negatively infinite), values of x are taken to be positive (or negative) and they become bigger and bigger in <u>absolute value</u>. The rest of the procedure is the same as for situation (a).

Thus, it is hoped that the calculator method will help you in obtaining a better "feeling" for as well as a clearer understanding of the limit concept.

Before proceeding to Section IV, you should ACTUALLY CALCULATE some of the limits in Sections I, II and III using your calculator. This will help you become familiar with the calculator method.

SECTION I

We begin this section with the definition of the limit.

Definition 1. Let $f(x)$ be a real-valued function which is defined for all values of x close to a real number a (except possibly at a). We say that the "limit of $f(x)$, as x approaches a, is ℓ" (where ℓ is a real number) and write

$$\lim_{x \to a} f(x) = \ell ,$$

if $f(x)$ is as close to the number ℓ as we please for all x sufficiently close to the number a, but not equal to a.

Note that in the above definition, we are not concerned with what happens to $f(x)$ when x equals a, but only with what happens to $f(x)$ when x is close to a, on either side of a (i.e., for values of x such that x is near a and $x < a$ or $x > a$ respectively).

Remark. Some authors allow ℓ (in the definition of the limit) to be $+\infty$ or $-\infty$. Here $+\infty$ and $-\infty$ are not real numbers, but convenient notations. That is, $\lim_{x \to a} f(x) = +\infty$ (or $= -\infty$) is a way of stating that $f(x)$ increases without bound (or decreases without bound) as x approaches a. However, we emphasize that the limit does not exist in all cases except when ℓ is a (finite) real number and satisfies the definition. For example, $f(x) = \frac{1}{x^4}$ increases without bound as x approaches 0. We may write that $\lim_{x \to 0} \frac{1}{x^4} = +\infty$, but we shall say that this limit does not exist (see Examples 2, 3 and 10 for instance).

Furthermore, if $f(x)$ is not defined for all values of x close to a (on either side of a), we say the limit does not exist. For example, $f(x) = \sqrt{x-2}$ is not defined for $x < 2$. Therefore, $\lim_{x \to 2} f(x)$ does not exist.

On the other hand, $\sqrt{x}$ is defined for all x sufficiently close to .00001 on both sides of .00001 even though $\sqrt{x}$ is not defined for -.000001 (or any other negative number). The reason being that this number -.000001 is "fairly" close to .00001 but not "sufficiently" close. In this case, all the positive numbers between 0 and .00001 are "sufficiently" close to .00001.

Thus, in general, you will have to check whether the function is defined for all values of x sufficiently close to a (except possibly at a) before testing for the limit.

Here are a few important properties of limits.

Let $f(x) = c$, where c is a real number.

(i.e., $f(x)$ is a "constant function"). Then

1. $\lim_{x \to a} f(x) = c$ (for any a) .

Let $f(x) = x^n$, where n is a positive integer. Then

2. $\lim_{x \to a} x^n = a^n$.

If both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then the functions $f(x) + g(x)$, $f(x) - g(x)$ and $g(x) \cdot f(x)$ have a limit. Furthermore, properties 3-5 below hold.

3. $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$;

4. $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$;

5. $\lim_{x \to a} (g(x) \cdot f(x)) = (\lim_{x \to a} g(x)) \cdot (\lim_{x \to a} f(x))$.

If both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and $\frac{f(x)}{g(x)}$ is defined for all x close to a , then

6. $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.

Let $\lim_{x \to a} f(x)$ exist. Then the function $cf(x)$ has a limit and

7. $\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$, where c is a real number.

Let $f(x)$ be a polynomial function of degree n. Then

8. $\lim_{x \to a} f(x) = f(a)$.

Let $\lim_{x \to a} f(x)$ exist. If the functions $\sqrt[n]{f(x)}$ and $[f(x)]^n$ are defined for all x close to a (where n is a fixed real number), then properties 9 and 10 below hold.

9. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$.

10. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$.

It is worth remarking that properties 3 - 6 hold only if <u>both</u> $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. In addition, property 6 holds only if $\lim_{x \to a} g(x) \neq 0$. Furthermore, properties 3 and 5 can be extended to the limit of a sum and product of a <u>finite</u> number of functions.

It is interesting to note that in case of a polynomial function, the limit is obtained by <u>direct substitution</u> of a for x (refer to prop. 8). Also note that prop. 7 is a special case of prop. 5, when $g(x)$ is a constant function.

Let's try out this "calculator method" on a few examples.

Example 1. Consider the constant function $f(x) = 3$. Investigate $\lim_{x \to 2} f(x)$.

Solution. The given function can be written as $f(x) = 3 + 0 \cdot x$. Observe that $f(x)$ is defined for all values of x close to 2. [In fact, $f(x)$ is defined for all reals x]. After choosing the values of x as shown, we obtain the following table.

for $x < 2$	$f(x) = 3 + 0 \cdot x$	for $x > 2$	$f(x) = 3 + 0 \cdot x$
2 - .1	3	2 + .1	3
2 - .01	3	2 + .01	3
2 - .001	3	2 + .001	3
2 - .0001	3	2 + .0001	3

Since $f(x)$ assumes only one value 3, no matter how close x is to 2, it follows that

$$\lim_{x \to 2} f(x) = 3 .$$

Remark. The above is an example of property 1. It shows that the limit of a constant function is equal to value of the function itself, no matter what real number a may be.

Example 2. Let $f(x) = \frac{1}{x^2}$, $x \neq 0$. Find $\lim_{x \to 0} f(x)$.

Solution. Obviously, $f(x)$ is defined for all values of x close to 0 (on either side of 0, but not at 0). Now let us see how the values of $f(x)$ change as x approaches zero (from either side) using a calculator.

for $x < 0$	$f(x) = \frac{1}{x^2}$	for $x > 0$	$f(x) = \frac{1}{x^2}$
-.3	11.111111	.3	11.111111
-.2	25	.2	25
-.1	100	.1	100
-.01	10,000	.01	10,000
-.001	1,000,000	.001	1,000,000

The above table indicates that $f(x)$ increases without bound as x approaches 0, from <u>either</u> side. Consequently, $f(x)$ becomes unbounded when x is sufficiently close to 0 and thus, $\lim_{x \to 0} \frac{1}{x^2}$ does not exist. However, as we mentioned earlier, we may <u>write</u> $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

<u>Example 3.</u> Let $F(x) = \frac{x+1}{x-1}$. Investigate $\lim_{x \to 1} F(x)$.

<u>Solution.</u> Here both numerator and demominator are polynomial functions defined for all x near 1 (on either side of 1, including 1). But $\lim_{x \to 1} (x-1) = 0$ (property 8). Therefore, the quotient rule (property 6) does not apply to this function. Let us apply our calculator method to determine the behaviour of $F(x)$, for x close to 1 .

for $x < 1$	$F(x) = \frac{x+1}{x-1}$	for $x > 1$	$F(x) = \frac{x+1}{x-1}$
1 - .3 = .7	-5.6667	1 + .3 = 1.3	7.6667
1 - .2 = .8	-9	1 + .2 = 1.2	11
1 - .1 = .9	-19	1 + .1 = 1.1	21
1 - .01 = .99	-199	1 + .01 = 1.01	201
1 - .001 = .999	-1,999	1 + .001 = 1.001	2,001
1 - .0001 = .9999	-19,999	1 + .0001 = 1.0001	20,001

Assuming that the above pattern of values continues as x approaches 1, $F(x)$ gets negatively large when x approaches 1 from the left, and positively large when x approaches 1 from the right. Hence the limit does not exist.

Other technique. Limits in Examples 2 and 3 can be seen to not exist by the following well-known theorem.

Theorem 1. Let $\lim_{x \to a} f(x) = \ell \neq 0$ and $\lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{does not exist.}$$

Example 4. Given $f(x) = x^2 + x - 2$. Evaluate $\lim_{x \to 1.2} f(x)$.

Solution. Since the given function is a polynomial function, it is defined for all x close to 1.2. Further, using a calculator we obtain the following table.

for $x < 1.2$	$f(x) = x^2+x-2$	for $x > 1.2$	$f(x) = x^2+x-2$
$1.2 - .2 = 1.0$	0	$1.2 + .2 = 1.4$	1.36
$1.2 - .1 = 1.1$	.31	$1.2 + .1 = 1.3$	.99
$1.2 - .01 = 1.19$	.6061	$1.2 + .01 = 1.21$	.6741
$1.2 - .001 = 1.199$	Calculate	$1.2 + .001 = 1.201$	Calculate
$1.2 - .0001 = 1.1999$	these	$1.2 + .0001 = 1.2001$	these
$1.2 - .00001 = 1.19999$	yourself	$1.2 + .00001 = 1.20001$	yourself
$1.2 - .000001 = 1.199999$		$1.2 + .000001 = 1.200001$	

From the table we can conclude that

$$\lim_{x \to 1.2} f(x) = .64 \quad . \tag{1}$$

By direct substitution, we have

$$f(1.2) = (1.2)^2 + (1.2) - 2 = .64 \ . \qquad (2)$$

In view of (1) and (2) ,

$$\lim_{x \to 1.2} f(x) = f(1.2) \ .$$

Remark. The above is an example of property 8 which states that the limit of a polynomial function is obtained by direct substitution of a for x in $f(x)$. However, for other functions, this is not always the case as is illustrated by the next three examples.

Example 5. Given $f(x) = \frac{x^2-1}{x+1}$. Investigate $\lim_{x \to -1} f(x)$.

Solution. Evidently, $f(x)$ is defined for all x close to -1 (on either side of -1, except at -1). Substituting -1 for x , we get

$$f(-1) = \frac{0}{0} \ ,$$

which is not defined, but the limit of $f(x)$ still exists! Calculate some of the values of $f(x)$ as x approaches -1 .

for $x < -1$	$f(x) = \frac{x^2-1}{x+1}$	for $x > -1$	$f(x) = \frac{x^2-1}{x+1}$
$-1 - .1$	-2.1	$-1 + .1$	-1.9
$-1 - .01$	-2.01	$-1 + .01$	-1.99
$-1 - .001$	Calculate	$-1 + .001$	Calculate
$-1 - .0001$	these	$-1 + .0001$	these
$-1 - .00001$	yourself	$-1 + .00001$	yourself
$-1 - .000001$		$-1 + .000001$	

The table indicates that, as x approaches -1 from either side, $f(x)$ approaches $\square$. That is,

$$\lim_{x \to -1} f(x) = \square .$$

Other technique. We do not want to have to use a calculator every time we wish to find a limit. There are quicker ways as illustrated below. [However, "seeing" what actually happens to the limit as $x \to a$ by using a calculator may help you to get a "feel" for what a limit is.] The quick solution to the above problem is as follows: express $f(x)$ in a different form by cancelling the common factor $(x + 1)$ from the numerator and denominator. [This is allowed since $x \neq -1$, whence $x + 1 \neq 0$]. Then take the limit. The entire process can be exhibited thus:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2-1}{x+1} \qquad (\tfrac{0}{0} \text{ form})$$

$$= \lim_{x \to -1} \frac{\cancel{(x+1)}\,(x-1)}{\cancel{(x+1)}}$$

$$= \lim_{x \to -1} (x-1) = -1-1 = -2 ,$$

where the limit in the last step is obtained by using property 8.

Example 6. Given $f(x) = \dfrac{3x}{\sqrt{1+x}-1}$. Determine $\lim_{x \to 0} f(x)$.

Solution. The function is defined for all x close to 0 (except at 0). On substituting 0 for x in $f(x)$, we obtain

$$f(0) = \frac{0}{0} ,$$

which is not defined. But the limit exists! The following table exhibits some of the calculated values of $f(x)$ as x gets closer and closer to 0.

for $x < 0$	$f(x) = \frac{3x}{\sqrt{1+x}-1}$	for $x > 0$	$f(x) = \frac{3x}{\sqrt{1+x}-1}$
-.1	5.8460499	.1	6.1464265
-.01	5.9849623	.01	6.0149627
-.001	5.9984992	.001	6.0014992
-.0001	5.999844	.0001	6.000144
-.00001	6	.00001	6
-.000001	6	.000001	6
-.0000001	6	.0000001	6

It appears from the table that the limit exists and has the value 6.

Other technique. Though $f(0) = \frac{0}{0}$, the method of the last example does not apply here, because the numerator and denominator have no common factor. The usual technique in this case is: express $f(x)$ in a different form by "rationalizing" the denominator [by multiplying both numerator and denominator by $(\sqrt{1+x}+1)$], simplify, and then take the limit. This can be displayed as follows:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{3x}{\sqrt{1+x}-1} \qquad (\tfrac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{3x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}$$

$$= \lim_{x \to 0} \frac{3x(\sqrt{1+x}+1)}{(1+x)-1}$$

$$= \lim_{x \to 0} \frac{3x(\sqrt{1+x}+1)}{x}$$

$$= \lim_{x \to 0} 3(\sqrt{1+x}+1)$$

$$= 3(1+1) = 6 \ ,$$

where the limit in the last step is obtained by substitution as a result of properties 7, 1, 3, 9. Thus both methods yield the same limit.

Remark. 1. Note that $\sqrt{1+x} + 1 \neq 0$ for x close to 0 but not equal to 0 so that $\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$ is defined (and equals 1).

2. Also, note that sometimes we rationalize the numerator instead of the denominator or both, and then proceed to the limit. See the following.

Example 7. Investigate $\lim_{x \to 16} f(x)$, where $f(x) = \frac{\sqrt{x}-4}{x-16}$.

Solution. The given function is defined for all x close to 16 (except at 16). Moreover, $f(16) = \frac{0}{0}$, which is not defined. Using the calculator method, we obtain the following table.

for $x < 16$	$f(x) = \frac{\sqrt{x}-4}{x-16}$	for $x > 16$	$f(x) = \frac{\sqrt{x}-4}{x-16}$
16-.1	.1251959	16+.1	.1248053
16-.01	.1250195	16+.01	.1249805
16-.001	.125002	16+.001	.124998
16-.0001	.125	16+.0001	.125
16-.00001	.125	16+.00001	.125
16-.000001	.125	16+.000001	.125

It appears that the limit exists and has the value .125.

Other technique.

$$\lim_{x \to 16} f(x) = \lim_{x \to 16} \frac{\sqrt{x}-4}{x-16} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 16} \frac{(\sqrt{x}-4)(\sqrt{x}+4)}{(x-16)(\sqrt{x}+4)}$$

$$= \lim_{x \to 16} \frac{\cancel{(x-16)}}{\cancel{(x-16)}(\sqrt{x}+4)}$$

$$= \lim_{x \to 16} \frac{1}{\sqrt{x}+4}$$

$$= \frac{1}{\sqrt{16}+4} = \frac{1}{4+4} = \frac{1}{8} = .125 \text{ .}$$

Remark. Observe that $\sqrt{x}+4 \neq 0$ for x close to 16 but not equal to 16 so that $\frac{\sqrt{x}+4}{\sqrt{x}+4}$ is defined .

The following example demonstrates that, in general, limit of a function as x approaches a is not related to its value at $x = a$.

Example 8. Examine limits of the following functions as x approaches 2 .

(a) $f(x) = x$.

(b) $f(x) = x$, $x \neq 2$.

(c) $f(x) = \begin{cases} x, & x \neq 2 \\ 1, & x = 2 \end{cases}$.

Solution. (a) Since $f(x)$ is a polynomial function, in view of property 8,

$$\lim_{x \to 2} f(x) = f(2)$$
$$= 2 \quad ,$$

whence, limit = value at the point.

(b) The function is not defined at $x = 2$, though it is defined for all other x close to 2 (on either side of 2). For the purpose of the limit, the value of f at the point 2 is immaterial. We are only concerned with values of x that are not equal to 2 . Thus, the behaviour of this function is the same as that of the function in part (a), as x gets closer and closer to 2 . Hence

$$\lim_{x \to 2} f(x) = 2 \ ,$$

though value of f is not defined at $x = 2$.

(c) The function is defined for all x close to 2 (including 2). Further, this function differs from the function in part (a) at one point only, namely, at $x = 2$ which causes no difference to the limit. Therefore, reasoning as in part (b), we have

$$\lim_{x \to 2} f(x) = 2 \ ,$$

though value of f is 1 at the point $x = 2$.

Thus, we have the following situations: -

(a) Value of the function defined at a , and value = limit.

(b) Value not defined at a , but the limit still exists.

(c) Value defined at a , but value $\neq$ limit.

SECTION II

In the last section we discussed limits of functions that were defined by the same formula for $x < a$ and $x > a$. In this section we are concerned with limits of functions that are defined by different formulae to the left and right of a. Before we consider some examples, we state a few definitions.

Definition 2. Let $f(x)$ be defined for all x close to a but less than a. We say that ℓ is the left limit (or left-handed limit) of $f(x)$ if $f(x)$ approaches ℓ as x approaches a from the left (i.e., x takes values close to a but less than a), and write

$$\lim_{x \to a^-} f(x) = \ell .$$

Definition 3. Let $f(x)$ be defined for all x close to a but greater than a. We say that m is the right limit (or right-handed limit) of $f(x)$ if $f(x)$ approaches m as x approaches a from the right (i.e., x takes values close to a but greater than a), and write

$$\lim_{x \to a^+} f(x) = m .$$

Both the left and the right limits are called one-sided limits.

In Section I, we stated that "limit of $f(x)$, as x approaches a, is ℓ" means $f(x)$ is as close to the number ℓ as we please for all values of x sufficiently close to a on either side of a. This means the limit must be the same whether x approaches a from the left or from the right. Thus, the limit (ordinary limit) will exist if both one-sided limits exist and are equal.

Hence, if $f(x)$ is defined by different formulae to the left and right of a, then in order to determine the limit of $f(x)$ as x approaches a, we should determine both left and right limits and examine their equality.

Furthermore, it may happen that

(a) one one-sided limit does not exist,

(b) both one-sided limits do not exist,

(c) both one-sided limits exist, but are not equal,

(d) both one-sided limits exist and are equal.

The (ordinary) limit exists only in the last case.

Remark. We remark that all the properties 1-10 of Section I apply to one-sided limits as well.

The following examples illustrate the above mentioned cases.

Example 9. Given $f(x) = \sqrt{x-2}$. Investigate $f(x)$ for both left and right limits as x approaches 2.

Solution. Here $f(x)$ is not defined for values of $x < 2$. On the other hand, $f(x)$ is defined for all $x > 2$. After choosing the values of x as shown, we obtain the following table.

for $x < 2$	$f(x) = \sqrt{x-2}$	for $x > 2$	$f(x) = \sqrt{x-2}$
2-.3	ERROR	2+.3	.5477
2-.2	"	2+.2	.4472
2-.1	"	2+.1	.3162
2-.01	"	2+.01	.1
2-.001	"	2+.001	.0316
2-.0001	"	2+.0001	.01
2-.00001	"	2+.00001	.0032
2-.000001	"	2+.000001	.001
2-.0000001	"	2+.0000001	.0003

The errors arise because the square root of a negative number is not defined. Thus,

$$\lim_{x \to 2^-} f(x) \quad \text{does not exist.}$$

Also, it appears from the table that

$$\lim_{x \to 2^+} f(x) = 0 .$$

Other technique. Since $f(x)$ is not defined for $x < 2$, $\lim_{x \to 2^-} f(x)$ does not exist. On the other hand, since $f(x)$ is defined for all $x > 2$, in view of property 9, $\lim_{x \to 2^+} f(x) = \sqrt{\lim_{x \to 2^+} (x-2)} = \sqrt{2-2} = 0$.

Example 10. Given $f(x) = \frac{1}{x}$, $x \neq 0$. Investigate $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$.

Solution. Clearly, $f(x)$ is defined for all x close to 0 on either side of 0 (except at 0).

for $x < 0$	$f(x) = \frac{1}{x}$	for $x > 0$	$f(x) = \frac{1}{x}$
-.01	-100	.01	100
-.001	-1,000	.001	1,000
-.0001	Calculate	.0001	Calculate
-.00001	these	.00001	these
-.000001	yourself	.000001	yourself
-.0000001		.0000001	

From the table what happens to $f(x)$ as x approaches 0 from the left, and as x approaches 0 from the right? What can you conclude about the left and right limits?

Example 11. Examine the following function for left and right limits as x approaches 0.

$$f(x) = \begin{cases} \dfrac{|x|}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases} .$$

Solution. Evidently, $f(x)$ is defined for all x close to 0. In using your calculator, recall that $|d| = \sqrt{d^2}$, for any real number d (in case your calculator does not have the absolute value key).

for $x < 0$	$f(x) = \dfrac{\lvert x \rvert}{x}$	for $x > 0$	$f(x) = \dfrac{\lvert x \rvert}{x}$
-.01	-1	.01	1
-.001	-1	.001	1
-.0001	Calculate	.0001	Calculate
-.00001	these	.00001	these
-.000001	yourself	.000001	yourself
-.0000001		.0000001	

The table indicates that $\lim_{x \to 0^-} f(x) =$ ☐ and $\lim_{x \to 0^+} f(x) =$ ☐.

If you use another technique, do you get the same results?

Other technique. By virtue of the definition of $|x|$, $f(x)$ can be written as

$$f(x) = \begin{cases} \dfrac{-x}{x} = -1 & , \quad x < 0 \\ 0 & , \quad x = 0 \\ \dfrac{x}{x} = 1 & , \quad x > 0 . \end{cases}$$

Therefore, by property 1, $\lim_{x \to 0^-} f(x) = -1$ and $\lim_{x \to 0^+} f(x) = 1$.

Example 12. Ascertain $\lim_{x \to 1} \dfrac{x + |x-1| - 1}{|x-1|}$.

Solution. Let $f(x) = \dfrac{x + |x-1| - 1}{|x-1|}$. Then $f(x)$ is defined for all values of x close to 1 (except at $x = 1$). Keeping in mind $|d| = \sqrt{d^2}$, some of the calculated values of $f(x)$ are demonstrated below.

for $x < 1$	$f(x)$	for $x > 1$	$f(x)$
$1-.1 = .9$	0	1.1	2
$1-.01 = .99$	0	1.01	2
$1-.001 = .999$	0	1.001	2
$1-.0001 = .9999$	0	1.0001	2
$1-.00001 = .99999$	0	1.00001	2

It appears that $\lim_{x \to 1^-} f(x) = 0$ and $\lim_{x \to 1^+} f(x) = 2$. Since the left limit is not equal to the right limit, $\lim_{x \to 1} f(x)$ does not exist.

Other technique. For $x < 1$, $x - 1 < 0$. Therefore, $|x-1| = -(x-1)$. Consequently, $f(x) = \dfrac{x - (x-1) - 1}{-(x-1)} = 0$. Hence

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0 . \qquad \text{(Property 1)}$$

On the other hand, for $x > 1$, $x-1 > 0$. Therefore, $|x-1| = x-1$. Consequently, $f(x) = \dfrac{x + (x-1) - 1}{x-1} = \dfrac{2x-2}{x-1} = \dfrac{2(x-1)}{x-1} = 2$. Hence

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2) = 2 .$$

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, it follows that $\lim_{x \to 1} f(x)$ does not exist.

Example 13. Examine $\lim_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} \sqrt{x} , & 0 \leq x < 1 \\ 1 , & x = 1 \\ x , & x > 1 . \end{cases}$$

Solution. Clearly, $f(x)$ is defined for all x close to 1.

for $x < 1$	$f(x) = \sqrt{x}$	for $x > 1$	$f(x) = x$
1-.3 = .7	.8367	1+.3	1.3
1-.2 = .8	.8944	1+.2	1.2
1-.1 = .9	.9486833	1+.1	1.1
1-.01 = .99	.9949874	1+.01	1.01
1-.001 = .999	Calculate	1+.001	Calculate
1-.0001 = .9999	these	1+.0001	these
1-.00001 = .99999	yourself	1+.00001	yourself
1-.000001 = .999999		1+.000001	

Hence the table you constructed indicates that $\lim_{x \to 1^-} f(x) =$ ☐ and

$\lim_{x \to 1^+} f(x) =$ ☐ . Thus, the limit exists and has the value ☐ .

Other technique. By property 9, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{x} = \sqrt{\lim_{x \to 1^-} x} = 1.$

Also by property 8, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x = 1$. Hence $\lim_{x \to 1} f(x) = 1$.

We conclude this section with a few examples that demonstrate how the use of a calculator simplifies complicated limits.

Example 14. Investigate $f(x) = x \ln x$ for left and right limits as x approaches 0 .

Solution. $f(x)$ is not defined for values of $x < 0$ (since $\ln x$ is not defined for $x < 0$). Therefore, $\lim_{x \to 0^-} f(x)$ does not exist. On the other hand, $f(x)$ is defined for all $x > 0$.

for $x < 0$	$f(x) = x \ln x$	for $x > 0$	$f(x) = x \ln x$
-.5	ERROR	.5	-.3465736
-.2	"	.2	-.3218876
-.1	"	.1	-.2302585
-.01	"	.01	-.0460517
-.001	"	.001	-.0069078
-.0001	"	.0001	-.000921
-.00001	"	.00001	-.0001151
-.000001	"	.000001	-.0000138

The errors arise because $\ln x$ is not defined for $x < 0$. The table indicates that $\lim_{x \to 0^-} x \ln x$ does not exist but $\lim_{x \to 0^+} x \ln x = 0$.

Other technique. $f(x) = x \ln x = x/(1/\ln x)$, which has the form $\frac{0}{0}$ as x approaches 0 from the right, but none of the methods of Section I apply here because there is no common factor in the numerator and denominator nor can we rationalize numerator or denominator. The other technique that is employed here is a device called L'Hospital's Rule. (In this case, the rule will be easier to apply if we write $f(x) = (\ln x)/(1/x)$.) However, the calculator method still works here!

Example 15. Investigate $\lim_{x \to 0} f(x)$, where $f(x) = \frac{3^x - 1}{x}$.

Solution. Evidently, $f(x)$ is defined for all x close to 0 (except at 0). Using a calculator, we obtain the following table.

for $x < 0$	$f(x) = \frac{3^x - 1}{x}$	for $x > 0$	$f(x) = \frac{3^x - 1}{x}$
-.01	1.0925996	.01	1.1046692
-.001	1.098009	.001	1.099216
-.0001	1.098552	.0001	1.098673
-.00001	1.09861	.00001	1.09862
-.000001	1.0986	.000001	1.0986

It seems that the limit exists and has the approximate value 1.0986, which we note is approximately $\ell n\ 3$. Hence we guess that

$$\lim_{x \to 0} \frac{3^x - 1}{x} = \ell n\ 3 .$$

Other technique. As x approaches 0, $f(x)$ has the form $\frac{0}{0}$, which can be handled using L'Hospital's Rule.

Example 16. Investigate $\lim_{x \to 0^+} f(x)$, where $f(x) = x^x$.

Solution. Obviously, $f(x)$ is defined for all $x > 0$ and close to 0.

for $x > 0$	$f(x) = x^x$
.5	.7071068
.2	.7247797
.1	.7943282
.01	Calculate
.001	these
.0001	yourself
.00001	
.000001	
.0000001	
.00000001	
.000000001	

What do you conclude from the table with respect to $\lim_{x \to 0^+} x^x$?

Other technique. As x approaches 0 from the right, $f(x)$ has the form 0^0, which is evidently different from all those discussed earlier. Again, the other technique that is used here is L'Hospital's Rule.

Remark. As has been mentioned above, the other technique that is employed to find the last three limits is the L'Hospital's Rule. This rule is not introduced till students have learned derivatives, but with the integration of the calculator as an integral part of the instructional mode, such limits can be introduced even before the introduction of L'Hospital's Rule.

SECTION III

This section studies limits at infinity (or minus infinity) i.e., limits of functions when x becomes positively large (or negatively large) or, in other words, when x increases (decreases) without bound through positive (negative) values.

Recall that infinity is not a real number. But the symbol "$x \to \infty$" merely indicates that x increases without bound through positive values. On the other hand, "$x \to -\infty$" indicates that x decreases without bound through negative values.

A few definitions are needed here.

Definition 4. Let $f(x)$ be a real-valued function. We say that the "limit of $f(x)$, as x increases without bound through positive values, is ℓ" (where ℓ is a real number) and write

$$\lim_{x \to \infty} f(x) = \ell ,$$

if $f(x)$ is as close to the number ℓ as we please, when x is sufficiently large.

Note that in the above definition we do not state "for all x sufficiently close to infinity but not equal to infinity" because infinity is not a number of any kind. A real number can never be "close" to infinity.

Definition 5. Let $f(x)$ be a real-valued function. We say that the "limit of $f(x)$, as x decreases without bound through negative values, is m" (where m is a real number) and write

$$\lim_{x \to -\infty} f(x) = m ,$$

if $f(x)$ is as close to the number m as we please, when x is sufficiently large (in absolute value) and negative.

In the above definition, note the omission of "for all x sufficiently close to minus infinity but not equal to minus infinity".

Definition 6. Let $f(x)$ be a real-valued function. We say that "$f(x)$ increases without bound as x increases without bound" and write

$$\lim_{x \to \infty} f(x) = \infty ,$$

if $f(x)$ becomes arbitrarily large when x becomes sufficiently large.

Remark. Similar definitions can be stated for

$$\lim_{x \to \infty} f(x) = -\infty , \quad \lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty .$$

Note that in these definitions, limit of $f(x)$ does not exist. (Recall that we say the limit exists only when the limit is a finite real number).

Remark. Properties 1-10 of Section I remain valid if $x \to a$ is replaced by $x \to \infty$ or $x \to -\infty$.

The question now is: how can the calculator method be used in determining limits of this section? There can be two possibilities for x :

(i) x increases without bound through positive values;

(ii) x decreases without bound through negative values.

In case (i), choose values of x so that they are positive and become bigger and bigger. Write the corresponding values of $f(x)$. Make a table as in Sections I and II. Then ascertain from the table whether the values of $f(x)$ appear to get arbitrarily close to one particular real number, or to more than one real number, or become positively infinite or negatively infinite. The limit exists in the first case only.

In case (ii), choose values of x so that they are negative and become bigger and bigger (in absolute value). Then make the corresponding table and ascertain limit of $f(x)$ as in case (i).

Let us ascertain a few limits using the calculator method as well as other techniques.

Example 17. Find $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$, where $f(x) = \frac{1}{x}$.

Solution. Using a calculator, we obtain the following table.

for $x \to -\infty$	$f(x) = \frac{1}{x}$	for $x \to \infty$	$f(x) = \frac{1}{x}$
-1,000	-.001	1,000	.001
-10,000	-.0001	10,000	.0001
-100,000	-.00001	100,000	.00001
-1,000,000	-.000001	1,000,000	.000001
-10,000,000	-.0000001	10,000,000	.0000001

It appears that $\lim_{x \to -\infty} \frac{1}{x} = 0$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Remark. The above example demonstrates that as x decreases (or increases) without bound through negative (or positive) values, $\frac{1}{x}$ approaches 0 through negative (or positive) values.

As a consequence of property 10, we state a very useful theorem.

Theorem 2. If p is a positive real number, then $\lim_{x \to \infty} \frac{1}{x^p}$ is zero.

Our next venture is to investigate the limit of a quotient of two polynomials as the variable x becomes infinite.

Example 18. Find $\lim_{x \to \infty} f(x)$, where $f(x) = \frac{x^2+x-1}{x^3+2x+2}$.

Solution. Using the calculator method, we get

for $x \to \infty$	$f(x) = \frac{x^2+x-1}{x^3+2x+2}$
1,000	.001001
10,000	.0001
100,000	.00001
1,000,000	.000001
10,000,000	.0000001

The table indicates that as x increases without bound, $f(x)$ approaches 0 through positive values. Hence $\lim_{x \to \infty} f(x) = 0$.

Other technique. Obviously, both numerator and denominator become infinite as $x \to \infty$ (i.e., $f(x)$ has the form $\frac{\infty}{\infty}$), so from this nothing can be said about the behaviour of $f(x)$. However, we can draw some conclusions about the limit of $f(x)$, by expressing $f(x)$ in a

different form. The usual technique is: divide both numerator and denominator by the largest power of x that occurs either in the numerator or in the denominator (here, it is x^3) and then proceed to the limit. Note that we are concerned only with large values of x so that we can assume $x \neq 0$. The entire process can be exhibited as follows:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2+x-1}{x^3+2x+2} = \lim_{x \to \infty} \frac{\frac{x^2+x-1}{x^3}}{\frac{x^3+2x+2}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3}}{1 + \frac{2}{x^2} + \frac{2}{x^3}}$$

$$= \frac{\lim_{x \to \infty} (\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3})}{\lim_{x \to \infty} (1 + \frac{2}{x^2} + \frac{2}{x^3})}$$

$$= \frac{\lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2} - \lim_{x \to \infty} \frac{1}{x^3}}{\lim_{x \to \infty} 1 + 2 \cdot \lim_{x \to \infty} \frac{1}{x^2} + 2 \cdot \lim_{x \to \infty} \frac{1}{x^3}}$$

$$= \frac{0 + 0 - 0}{1 + 2(0) + 2(0)} = \frac{0}{1} = 0 .$$

Remark. Note that $\lim_{x \to \infty}$ (denominator) $= \lim_{x \to \infty} (1 + \frac{2}{x^2} + \frac{2}{x^3}) = 1 \neq 0$, so that we can apply the quotient rule (property 6) for limits at infinity.

Example 19. Investigate $\lim_{x \to -\infty} f(x)$, where $f(x) = \frac{x-1}{2x^2}$.

Solution. The calculator method gives the following table.

for $x \to -\infty$	$f(x) = \frac{x-1}{2x^2}$
-100	-.00505
-1,000	-.0005005
-10,000	-.00005
-100,000	-.000005
-1,000,000	-.0000005

From the table, we can conclude that $\lim_{x \to -\infty} f(x) = 0$.

Other technique. Since $\lim_{x \to -\infty} (x-1) = -\infty$ and $\lim_{x \to -\infty} 2x^2 = \infty$, $f(x)$ is of the form $\frac{\infty}{\infty}$. Using the technique mentioned in the previous example,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x-1}{2x^2} = \lim_{x \to -\infty} \frac{\frac{x-1}{x^2}}{\frac{2x^2}{x^2}}$$

$$= \lim_{x \to -\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{2}$$

$$= \frac{\lim\limits_{x \to -\infty} \frac{1}{x} - \frac{1}{x^2}}{\lim\limits_{x \to -\infty} 2}$$

$$= \frac{\lim\limits_{x \to -\infty} \frac{1}{x} - \lim\limits_{x \to -\infty} \frac{1}{x^2}}{\lim\limits_{x \to -\infty} 2}$$

$$= \frac{0 - 0}{2} = \frac{0}{2} = 0 .$$

Example 20. Investigate $\lim\limits_{x \to -\infty} f(x)$, where $f(x) = \frac{2x^2}{x}$ ..

Solution. Calculating a few values of $f(x)$, we get:

for $x \to -\infty$	$f(x) = \frac{2x^2}{x}$
-100	-200
-1,000	-2,000
-10,000	-20,000
-100,000	-200,000
-1,000,000	-2,000,000

It appears that $\lim\limits_{x \to -\infty} f(x) = -\infty$, that is, no limit exists.

Other technique. Here $f(x)$ can be considered as a quotient of two polynomials. Moreover, as $x \to -\infty$, $f(x)$ has the form $\frac{\infty}{\infty}$. Therefore, employing the usual technique,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{2x^2}{x} = \lim_{x \to -\infty} \frac{\frac{2x^2}{x^2}}{\frac{x}{x^2}}$$

$$= \lim_{x \to -\infty} \frac{2}{\frac{1}{x}} = \frac{\lim_{x \to -\infty} 2}{\lim_{x \to -\infty} \frac{1}{x}} .$$

Since the denominator approaches 0 and the numerator approaches a non-zero limit, the limit does not exist.

Alternate technique. Since $x \to -\infty$, we can assume $x \neq 0$. Hence

$$\lim_{x \to -\infty} \frac{2x^2}{x} = \lim_{x \to -\infty} 2x = -\infty ,$$

so that the limit does not exist (for the limit in the last step, see problem 23, Section IV).

Example 21. Examine $\lim_{x \to -\infty} f(x)$, where $f(x) = \frac{5x^3 + 2}{3x^3 + x}$.

Solution. Some of the calculated values of $f(x)$ are shown on the next page.

for $x \to -\infty$	$f(x) = \dfrac{5x^3 + 2}{3x^3 + x}$
-123	1.6666296
-1234	1.6666663
-12345	1.6666667
-123456	1.6666667
-1234567	1.6666667

From the table, it appears that the limit exists and has the value 1.6666667. That is, the limit is probably $1\frac{2}{3} = 5/3$.

Other technique. Since $f(x)$ is a quotient of two polynomials and has the form $\frac{\infty}{\infty}$ as $x \to -\infty$, applying the usual technique,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{5x^3 + 2}{3x^3 + x}$$

$$= \lim_{x \to -\infty} \frac{\dfrac{5x^3 + 2}{x^3}}{\dfrac{3x^3 + x}{x^3}}$$

$$= \lim_{x \to -\infty} \frac{5 + \dfrac{2}{x^3}}{3 + \dfrac{1}{x^2}}$$

$$= \frac{\lim\limits_{x \to -\infty} \left(5 + \dfrac{2}{x^3}\right)}{\lim\limits_{x \to -\infty} \left(3 + \dfrac{1}{x^2}\right)}$$

$$= \frac{5 + 0}{3 + 0} = \frac{5}{3} .$$

Thus, both methods give the same limit.

Remark. The technique of dividing both numerator and denominator by the highest power of x that occurs either in the numerator or in the denominator, is used only in cases when $x \to \infty$ or $x \to -\infty$. When x approaches a (finite) real number, we employ techniques mentioned in Sections I and II of this supplement.

Example 22. Determine $\lim_{x \to \infty} f(x)$, where $f(x) = x - \frac{x^2+1}{x-1}$.

Solution.

for $x \to \infty$	$f(x) = x - \frac{x^2+1}{x-1}$
100	-1.020202
1,000	-1.002002
10,000	-1.0002
100,000	-1.00002
1,000,000	-1.00000

In view of the above table, the limit exists and has the value -1.

Other technique. As $x \to \infty$, $f(x)$ has the form $\infty - \infty$. A standard technique which is used to handle this form is to take the least common denominator first and then take the limit. This can be displayed as follows:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x - \frac{x^2+1}{x-1}$$

$$= \lim_{x \to \infty} \frac{x(x-1)}{x-1} - \frac{x^2+1}{x-1}$$

$$= \lim_{x \to \infty} \frac{x(x-1) - (x^2+1)}{x-1}$$

$$= \lim_{x \to \infty} \frac{\cancel{x^2} - x - \cancel{x^2} - 1}{x-1}$$

$$= \lim_{x \to \infty} \frac{-x-1}{x-1}$$

$$= \lim_{x \to \infty} \frac{\frac{-x-1}{x}}{\frac{x-1}{x}}$$

$$= \lim_{x \to \infty} \frac{-1 - \frac{1}{x}}{1 - \frac{1}{x}}$$

$$= \frac{\lim_{x \to \infty} -1 - \frac{1}{x}}{\lim_{x \to \infty} 1 - \frac{1}{x}}$$

$$= \frac{-1 - 0}{1 - 0} = -1 .$$

Example 23. Investigate $\lim_{x \to \infty} f(x)$, where $f(x) = \dfrac{\sqrt{x^4+5x^2}-2x}{4x+1}$.

Solution. Some of the calculated values of $f(x)$ are as shown below.

for $x \to \infty$	$f(x) = \dfrac{\sqrt{x^4+5x^2}-2x}{4x+1}$
100	24.943865
1,000	249.93814
10,000	2499.9376
100,000	24999.938
1,000,000	249999.94

It appears from the table that $f(x)$ becomes arbitrarily large when x becomes sufficiently large. Thus, no limit exists.

Other technique. Here $f(x)$ is not a quotient of two polynomials (because the numerator is not a polynomial). However, since $f(x)$ has the form $\frac{\infty}{\infty}$ as $x \to \infty$, a method similar to that in case $f(x)$ is a quotient of two polynomials, can be applied to this case as well. The largest power of x occurs in the numerator and it is $\sqrt{x^4}$ $(= x^2)$. Thus, we divide the numerator and denominator by x^2 (This can be done, assuming $x \neq 0$. Note that we are concerned only with large values of x). Consequently,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sqrt{x^4+5x^2}-2x}{4x+1}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\frac{x^4+5x^2-2x}{x^2}}}{\frac{4x+1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\frac{x^4+5x^2-2x}{x^4}}}{\frac{4x+1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1 + \frac{5}{x^2} - \frac{2}{x^3}}}{\frac{4}{x} + \frac{1}{x^2}}$$

$$= \frac{\lim_{x \to \infty} \sqrt{1 + \frac{5}{x^2} - \frac{2}{x^3}}}{\lim_{x \to \infty} \frac{4}{x} + \frac{1}{x^2}}$$

Since the denominator approaches 0 and the numerator approaches a non-zero limit, the limit does not exist.

Example 24. Ascertain $\lim_{x \to \infty} f(x)$, where $f(x) = \frac{\sqrt[3]{x^3+x+1}}{2x-1}$.

Solution. A few calculated values of $f(x)$ are shown below.

for $x \to \infty$	$f(x) = \dfrac{\sqrt[3]{x^3+x+1}}{2x-1}$
100	.5025295
1,000	.5002503
10,000	.500025
100,000	.5000025
1,000,000	.5000003
10,000,000	.5
100,000,000	.5

Evidently, the limit exists and has the value .5 .

Other technique. Here again, $f(x)$ has the form $\frac{\infty}{\infty}$ as $x \to \infty$, though $f(x)$ is not a quotient of two polynomials. Proceeding as in the last example, we divide both numerator and denominator by x. Thus,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sqrt[3]{x^3+x+1}}{2x-1}$$

$$= \lim_{x \to \infty} \frac{\frac{\sqrt[3]{x^3+x+1}}{x}}{\frac{2x-1}{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt[3]{\frac{x^3+x+1}{x^3}}}{\frac{2x-1}{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt[3]{1 + \frac{1}{x^2} + \frac{1}{x^3}}}{2 - \frac{1}{x}}$$

$$= \frac{\lim\limits_{x \to \infty} \sqrt[3]{1 + \frac{1}{x^2} + \frac{1}{x^3}}}{\lim\limits_{x \to \infty} \left(2 - \frac{1}{x}\right)}$$

$$= \frac{\sqrt[3]{1 + 0 + 0}}{2 - 0} = \frac{1}{2} = .5 .$$

We end this section with a couple of examples that reveal how the calculator method simplifies limits of exponential functions.

Example 25. Examine $\lim\limits_{x \to \infty} f(x)$, where $f(x) = \frac{x}{e^x}$.

Solution. Since $\lim\limits_{x \to \infty} x = \infty$ and $\lim\limits_{x \to \infty} e^x = \infty$, $f(x)$ has the form $\frac{\infty}{\infty}$ as $x \to \infty$ [Recall that $e = 2.7182818$, which is obviously greater than 1]. As the denominator is an exponential function, none of the techniques of this section apply here. But our calculator method still works!

for $x \to \infty$	$f(x) = \frac{x}{e^x}$
10	.000454
50	$9.6437492 \times 10^{-21}$
100	3.720076×10^{-42}
200	$2.7677931 \times 10^{-85}$
225	$4.3243824 \times 10^{-96}$
230	$2.9784958 \times 10^{-98}$

On many calculators (those without "scientific notation") the last five values of $f(x)$ in the table may be 0. The table indicates that the limit exists and has the value 0.

Other technique. An application of L'Hospital's Rule.

Example 26. Investigate $\lim_{x \to \infty} f(x)$, where $f(x) = (1 + \frac{1}{x})^x$.

Solution. Employing the calculator method:

for $x \to \infty$	$f(x) = (1 + \frac{1}{x})^x$
25	2.6658363
365	2.7145675
6824	2.7180826
10568	2.718154
1234567	2.7182798

It seems that the limit exists and has the approximate value 2.718, which is approximately e. Hence we guess that

$$\lim_{x \to \infty} (1 + \frac{1}{x})^x = e .$$

Other technique. $f(x)$ has the form 1^∞ as $x \to \infty$. This form is evidently different from all those discussed so far. The standard technique which can handle this limit is again L'Hospital's Rule (which is not available until after you study derivatives).

Remark. In view of the last two examples and examples 14-16 of Section II, it can be concluded that the use of a calculator definitely simplifies calculation of many complicated limits. Furthermore, in view of the entire supplement it can be said that, although the standard problem-solving techniques provide quick solutions to problems on limits, the calculator method enables you to acquire a better feeling for the limit concept.

SECTION IV

Using your calculator, find the limit indicated or state that it does not exist. You may wish to check your results using standard techniques if known.

Answers

1. $\lim_{x \to 2.3} f(x)$, where $f(x) = 5^{\frac{3}{4}}$. ($5^{\frac{3}{4}}$)

2. $\lim_{x \to 1} f(x)$, where $f(x) = \sqrt[3]{x^5}$. (1)

3. $\lim_{x \to 0} f(x)$, where $f(x) = \frac{x^4}{e^x}$. (0)

4. $\lim_{x \to 1} f(x)$, where $f(x) = x^3 + 1 - \frac{x+2}{x-1}$. (does not exist)

5. $\lim_{x \to -2} f(x)$, where $f(x) = \sqrt{\frac{4x-1}{x+1}}$. (3)

6. $\lim_{x \to 1} f(x)$, where $f(x) = \frac{x-1}{x^2-1}$. ($\frac{1}{2}$)

7. $\lim_{x \to 1} f(x)$, where $f(x) = \frac{\sqrt{2-x}-1}{\sqrt{x}-1}$. (-1)

(Hint: To use a standard technique, rationalize both numerator and denominator.)

8. $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$, where $f(x) = |x|$. (1)

Answers

9. $\lim_{x \to -1^-} f(x)$, where $f(x) = \frac{1}{x+1}$. ($-\infty$)

10. $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1} f(x)$, where $f(x) = \frac{x+1}{x^2-1}$. ($-\infty$, ∞, does not exist)

11. $\lim_{x \to \frac{1}{3}} f(x)$, where $f(x) = \frac{1}{(3x-1)^5}$. (does not exist)

12. $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$ and $\lim_{x \to 3} f(x)$, where $f(x) = \sqrt{3-x} + x^2 - 1$. (8, does not exist, does not exist)

13. $\lim_{x \to 0^-} f(x)$, $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0} f(x)$, where $f(x) = x\,|x|$. (0,0,0)

14. $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$ and $\lim_{x \to 3} f(x)$, where

$$f(x) = \begin{cases} \frac{2|x-3|}{x-3} , & x \neq 3 \\ 7 , & x = 3 . \end{cases}$$

(-2,2, does not exist)

Answers

15. $\lim_{x \to 5^-} f(x)$, $\lim_{x \to 5^+} f(x)$ and $\lim_{x \to 5} f(x)$, where $f(x) = \dfrac{x - |x-5|}{x-5}$, $x \neq 5$. ($-\infty$, ∞ , does not exist)

16. (a) $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} x^2 , & x < 0 \\ 5 , & x = 0 \\ -2x , & x > 0 \end{cases}$. (0)

(b) Is the limit = value at 0 ? (no)

17. (a) $\lim_{x \to -1} f(x)$, where $f(x) = \begin{cases} |x|+1, & x < -1 \\ 2 , & x = -1 \\ x+3 , & x > -1 \end{cases}$. (2)

(b) Is the limit = value at -1 ? (yes)

18. $\lim_{x \to 0^+} f(x)$, where $f(x) = x^{3x}$. (1)

19. $\lim_{x \to 0^+} f(x)$, where $f(x) = \left(\dfrac{1+x}{1-x}\right)^{1/x}$. ($7.389 = e^2$)

20. $\lim_{x \to 1} f(x)$, where $f(x) = \dfrac{\ell n\, x}{x-1}$. (1)

Answers

21. $\lim_{x \to 0} f(x)$, where $f(x) = \frac{e^x - 1}{x}$. (1)

22. $\lim_{x \to \infty} 1.99$. (1.99)

23. $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$, where $f(x) = 2x$. $(-\infty , \infty)$

24. $\lim_{x \to \infty} f(x)$, where $f(x) = \frac{1}{x^{1.7}}$. (0)

25. $\lim_{x \to \infty} f(x)$, where $f(x) = \frac{1}{2x^3} - \frac{5x}{x^2 - 1}$. (0)

26. $\lim_{x \to -\infty} f(x)$, where $f(x) = \frac{x^2}{x-1} + \frac{2}{x}$. $(-\infty)$

27. $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$, where

$$f(x) = \begin{cases} 5x , & x \leq 1 \\ -1 , & x > 1 \end{cases} .$$ $(-\infty , -1)$

28. $\lim_{x \to \infty} f(x)$, where $f(x) = \left(1 + \frac{1}{x}\right)^{\frac{1}{x}}$. (1)

Answers

29. $\lim_{x \to -\infty} f(x)$, where $f(x) = \frac{x}{\sqrt{x^2+1}}$. (-1)

(Hint: To use a standard technique, note that for

$x < 0$, $\sqrt{x^2} = -x$.)

30. $\lim_{x \to 0} f(x)$, where $f(x) = \frac{\sin x}{x}$. (1)

(This problem can be handled if your calculator has the sin function.)

SECTION V

Please answer all the questions honestly and frankly. If you require more room than is provided, please use the back of the page and/or attach an additional page. Your identity will be kept secret. A summary of the answers by the students as well as the conclusions and implications of this pilot study will be compiled for the Mathematics Department.

1. Was this experiment a waste of time or did it give you a better understanding of "limits"?
2. How would you rate the contribution of the supplement to your understanding of limits? (Check (✓) the response that corresponds most closely with your opinion.)

   EXCELLENT ☐ GOOD ☐ AVERAGE ☐ BELOW AVERAGE ☐ POOR ☐
3. Do you think a calculator is a motivating device?

   YES ☐ NO ☐
4. Do you think a calculator enhances your independence in problem solving?

   YES ☐ NO ☐
5. Do you think a calculator can provide another method for helping you to think, create and learn mathematics?

   YES ☐ NO ☐
6. Does a calculator help you gain more confidence?

   YES ☐ NO ☐
7. Do you think a calculator makes you think for yourself?

   YES ☐ NO ☐
8. Did you ever use a calculator in problems on limits?

   ALL THE TIME ☐ SOME OF THE TIME ☐ NEVER ☐

9. Do you think that your teacher should use

   (a) only calculator method _____

   (b) only problem-solving techniques _____

   (c) both _____

   in the class in teaching limits (check (✓) one)?

10. Do you think students should be allowed the calculator method in problems on limits in the examination?

AGREE STRONGLY ☐ AGREE ☐ NEUTRAL OPINION ☐ DISAGREE ☐ DISAGREE STRONGLY ☐

11. Do you think there should be a question in the exam exploiting the use of calculator alone?

AGREE STRONGLY ☐ AGREE ☐ NEUTRAL OPINION ☐ DISAGREE ☐ DISAGREE STRONGLY ☐

12. Do you think that the Math 157 curriculum should include more calculator problems?

YES ☐ NO ☐

13. (a) Do you think your teacher should use a calculator in teaching other chapters of Math 157 besides limits?

AGREE STRONGLY ☐ AGREE ☐ NEUTRAL OPINION ☐ DISAGREE ☐ DISAGREE STRONGLY ☐

   (b) Can you name some such topics?

14. Name a few things that you especially like about this supplement.

15. Name a few things that you especially dislike about this supplement.

16. How challenging did you find this supplement?

HIGHLY CHALLENGING ☐ CHALLENGING ☐ AVERAGE ☐ LITTLE CHALLENGING ☐ NO CHALLENGE ☐

17. How would you rate the way the subject matter in this supplement was presented?

EXCELLENT ☐ GOOD ☐ AVERAGE ☐ BELOW AVERAGE ☐ POOR ☐

18. Rate the quality of this supplement. Use this scale:

(a) Far above average.

(b) Somewhat above average.

(c) Average

(d) Somewhat below average.

(e) Far below average.

19. Give a few suggestions for the improvement of this supplement.

Thank you for taking time to answer these questions.

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