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**LA THÈSE A ÉTÉ
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ON TESTING THE BLACK-SCHOLES
OPTION PRICING MODEL

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in the Department

of

Economics

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ABSTRACT

Empirical studies have found the Black-Scholes prices to deviate from the actual call prices in certain systematic ways. But there is pronounced disagreement regarding the sources and directions of this mispricing. The conclusions may critically depend upon the underlying testing procedures. The purpose of this dissertation is to show the inadequacies of the existing testing procedures and offer some alternative tests.

We propose a multiple regression model of Black-Scholes mispricing and derive an estimable regression equation. We find that one of the components in the expected response function will persist despite the assumption that the Black-Scholes model is valid. This persistent component is the bias of the Black-Scholes price calculated with an estimated volatility rate.

Three alternative estimators for the Black-Scholes model price are considered. In our Monte Carlo, none seem to show clear superiority over the simple formula estimator. We further note that the relationships of the bias with moneyness, time to maturity or variance rate are option-specific. This in turn implies that the coefficients in the regression equation will be option-specific. Thus, the coefficient estimates in a constant coefficient estimation framework, prevalent in current empirical options literature, may not reflect the marginal biases.

We are further concerned with the stochastic regressor problem arising from the use of estimated volatility rate as a

regressor and the effect of omitted variables. They are identified as two probable sources of sign reversal of coefficient estimates across studies.

As an alternative to handle dividend effect on mispricing, we propose a Chow Test for the reported systematic biases. Our test results indicate significant dividend inducement for the entire regression relationship of Black-Scholes mispricing. The volatility rate seems to play a dominant role behind this dividend inducement.

We undertake an indirect test of Black-Scholes validity by testing restrictions among the regression coefficients. These restrictions are available because they are implied by the bias in the Black-Scholes formula estimator. Unfortunately, our small sample and large sample test results are contradictory.

Finally, in order to combat option specificity, we conduct multivariate cubic spline regression to allow curvature in the relationship of Black-Scholes formula mispricing to individual factors. The spline predictions of these relationships are then visually compared with our Monte Carlo results, which have been generated on the assumption that the Black-Scholes model is valid. The comparisons do not show support for the Black-Scholes model.

DEDICATION

To my mother
Mrs. Mazeda Begum

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INTRODUCTION AND SUMMARY

Since the pioneering work of Black and Scholes(1973), there has been substantial research in extending and modifying their European call valuation model, along with the monte carlo and empirical testings. The American call valuation model developed by Roll(1977), Geske(1979a), and Whaley(1981) has inspired another surge of research to investigate the adequacy of the Black-Scholes European call valuation model or its modified versions in the context of American calls on dividend-paying stocks.

Black and Scholes(1973) assumed a known volatility rate in deriving their valuation formula. Since the true volatility rate is not known, researchers have commonly used some estimate of the volatility(or variance) rate in the formula, in order to investigate the Black-Scholes pricing empirically. The resulting price is an estimate of the price that would have been given by the formula with the true volatility rate. Since the difference between these two price magnitudes would assume a vital role in our discussions ahead, we would like to make clear the terms to be used while referring to them. The price given by the formula with the true volatility(or variance) rate would be referred to as the 'Black-Scholes model price', or simply the 'model price', when there is no chance of confusion. On the other hand, whenever any estimate of the true volatility (or variance) rate, rather than the true volatility(or variance) rate itself is used in the formula, we would refer to the resulting price

alternatively as the 'estimated model price', the 'estimated formula price', or simply the 'formula estimate'. Thus, when the authors of empirical studies make statements about the model's pricing, almost invariably those statements are in fact about the estimated formula's pricing'. In what follows, when we mention the results of various studies based on empirical data, they should be interpreted as for the formula estimate.

Black and Scholes(1972) undertook the first empirical study of their model. They found the model to overestimate (underestimate) the market option prices for options on high(low) historically estimated variance stocks. This has come to be known as the variance bias of the Black-Scholes pricing. Later, Black(1975) found that the model tended to overprice(underprice) deep-in-the-money(deep-out-of-the-money) options. This is referred to as the direct striking price bias. He also found options with short maturities(less than 90 days) to be underpriced by the model.

MacBeth and Merville(1979,1980) found that the direction of the striking price bias of the dividend-adjusted Black-Scholes prices is exactly opposite to what was reported by Black(1975). This bias came to be known as the inverse striking price bias. Since the data used for empirical purposes were data on unprotected American calls, the impact of the early exercise possibility on the probable mispricing and the systematic but

1 These comments apply, whether the European, the simple stock price adjustment, or the Pseudo-American version of the Black-Scholes is being used.

reportedly conflicting biases of Black-Scholes pricing came to be the subject matter of a number of studies, e.g., Whaley (1982), Sterk (1982, 1983), Gultekin, Rogalski, and Tinic (1982), Geske and Roll (1984, 1984a). Whaley (1982) compared two modified (for dividend) versions of the Black and Scholes model against the Roll-Geske-Whaley American valuation model and found that the latter model eliminates all the biases of the former, except for the variance bias as reported by Black and Scholes (1972). The striking price bias was not significant, and the time to expiration bias was in the same direction as found by Black (1975). Sterk (1982) compared the pseudo-American version of the Black and Scholes and the American call valuation model as developed by Roll (1977) and Geske (1979a). He found the latter model to reduce the striking price bias of the former. He also confirmed the direct striking price bias of the Black and Scholes model. Gultekin, Rogalski, and Tinic (1982) considered the Roll-Geske-Whaley model and found that its bias characteristics are identical to that of the Black-Scholes. They confirmed the direct striking price bias and the direct estimated variance bias. That both models have identical bias characteristics was also supported by Blomeyer and Klemkosky (1983).

Geske and Roll (1984) showed that the conflicting results on the direction of the striking price bias, in particular, could be explained by the early exercise probability of the sample options. Geske and Roll (1984a) reported that the striking price and the time to expiration biases are essentially

dividend-induced, while the estimated variance bias is measurement-error-in-variance-induced. They showed that the use of James-Stein estimator of the variance rates eliminates the estimated variance bias.

It may be said that the conclusions of empirical studies about the validity of the Black-Scholes model are far from being decisive, especially in view of the fact that the results of various studies are not directly comparable due to the varied nature of (1) the data used, (2) the technique of volatility rate estimation used, and (3) the approach to model validation and/or comparison adopted². Moreover, confusion arose from the difference in the types of statements used to report their findings. After a brief survey of the theoretical literature in the first chapter, we detail these complaints in chapter 2. For example, we will raise questions such as:

(a) What problems lie behind the use of ISD, as was used by MacBeth and Merville (1979) among many others?

(b) Were the reportedly deeper-away-from-the-money options of Black (1975) truly deeper-away-from-the-money?

(c) Is MacBeth and Merville (1979)'s result, based on the classification in-the-money and out-of-the-money, comparable to Black (1975)'s result which does not seem to be based on similar classification?

² In this thesis, we would not be concerned with the data differences.

(d) Can MacBeth and Merville (1979)'s, Whaley (1982)'s or Geske and Roll (1984a)'s regression coefficients be construed as the marginal biases or their unbiased estimates?

(e) Can we offer an alternative approach to address general problems such as testing the validity of the Black-Scholes model or more specific problems such as testing the dividend inducement of the reported empirical systematic biases of the Black-Scholes formula estimate?

In dealing with these type of questions, our discussion about Implied Standard Deviation (ISD) plays an important role. ISD is computed by equating the market price to the model price and solving for the only unknown--the volatility rate. That ISD is not the true volatility rate is indicated by the empirical finding of varying ISDs of options on the same stock. If it is an estimate of the true volatility rate only, it will be a biased estimate, even if the model is valid. This bias arises since the inverse function of the formula will be nonlinear in the estimation error (of the volatility rate) when the model is valid. But more importantly the bias of ISD as an estimate of the volatility rate would be related to the moneyness and time to maturity of the option, and the true volatility rate. Thus the technique of model validation which takes systematic tendencies of ISD as evidence against the model would be flawed ³.

³Butler and Schachter (1984) provide an interesting discussion of these issues. They also numerically analyze the bias of ISD as an estimator of the volatility rate and the biases in option prices generated by the use of ISD.

This also raises fundamental questions such as how does the market price options, and what can be considered as a valid model. Since the volatility rate would in general be unknown, we are not permitted to assume that the market price would be identically equal to the fair value of the option⁴. Nor we can assume that the market plugs-in some estimate of the volatility rate in a nonlinear formula such as the one of Black-Scholes, since that would admit the possibility of the market being systematically off the fair value of the option⁵. If we believe in the wisdom of the market, then we need to assume that the market prices options in such a way as to allow only zero-mean perturbations around the fair value of the option⁶. This perturbation can also accommodate the idiosyncratic behavior of traders.

Under this scenario, a model can be said to be 'valid' if it is the model of fair valuation. Thus, if the Black-Scholes model is valid, the only source of systematic deviation for the formula estimate from the market price would be its bias with respect to the model price.

⁴ If the set of assumptions under which a theoretical model of option valuation is developed, are correct, the model price would be the fair value of the option, and the model can be called the model of fair valuation. In general, this model will be unknown to a researcher.

⁵ To be discussed later, formula estimate would be a biased estimate of the model price.

⁶Whaley(1982) introduced this perturbation in estimating the volatility rate from a regression using ISDs; but the economic purpose of the error term was not clear.

As mentioned earlier, formula estimate has commonly been used in empirical studies as proxies for the model price. Much less commonly it has been recognized that the formula estimate is a biased estimate of the model price. The bias arises due to the statistical fact that the expected value of a nonlinear function of a random variable is not equal to the function evaluated at the expected value of the random variable. In our context, the Black-Scholes formula is the nonlinear function, and the random variable is the estimate of the volatility (or variance) rate⁷. Boyle and Ananthanarayanan (1977)'s controlled experiment indicates that the formula estimate would underestimate the model price for at and around-the-money options, and overestimate for deeper-away-from-the-money options. The implication for empirical testing is that the observed deviations of the formula estimates from the market prices may predominantly be the nonlinearity bias, and/or their systematic tendencies primarily induced by the nonlinearity bias. The nature of the nonlinearity bias has also important ramifications for the model validation techniques, the appropriateness of the statistical techniques used, and their effects on the reported results.

For example, the hedging technique of validating the Black-Scholes model utilizes the formula estimates to establish the hedge positions in securities. These positions would in fact

⁷ Though we cannot show it explicitly, we are assuming that biased estimate of the volatility (or variance) rate would not produce unbiased estimate of the model price.

be distorted due to the bias of the formula estimates , and so will be the result of testing excess hedge returns ⁸. Moreover, the hedging technique relies upon asset pricing model such as the CAPM, the latter itself being subject to empirical testing.

In addition to the ISD-based technique mentioned earlier, and the hedging technique, there is another technique of model validation that has been used rather extensively. This technique relies upon comparing the actual market prices with the formula estimates, and examining whether the deviations are systematically related to factors such as the moneyness and the time to maturity of the option, the volatility rate, et cetera. Three major statistical techniques of comparison have been used: (1)classify sample options into some broad categories such as in-the-money versus out-of-the-money, or short versus long maturity, and then examine the signs and magnitudes of group averages for dollar or percentage deviation of the formula estimates from the market prices; (2)plot dollar or percentage deviations against a factor, and see if any pattern emerges; (3)regress mispricing on factors such as moneyness and time to maturity of the option, the estimated volatility rate, estimate of early exercise probability, et cetera., and see if the slope coefficients are significant⁹

⁸Butler and Schachter(1984a), pp 21, mentions this by pointing out that the hedge ratios using estimated volatility rate will be biased.

⁹The two measures of mispricing used are dollar deviation of the formula estimate from the market price, or the percentage deviation. Our expositions would be in terms of dollar deviations.

As statistical techniques of relating the mispricing to the status of a factor, both grouping and plotting suffer from the problem of inadequate or no control for other relevant factors. Thus the primary source of under or overpricing of an in-the-money option may not be its moneyness status, even though under these two techniques it may appear to be so. Moreover, it appears that, when the Black-Scholes model is valid, the nonlinearity bias for options within a broad group such as in-the-money cannot be expected to have the same pattern. Thus the sample mixture of options can seriously affect the results. In general, this criticism would be valid irrespective of the validity of the Black-Scholes model. On the other hand, finer groupings would be rather arbitrary. For example, the nonlinearity bias of the formula indicates that the same option could be considered near-in-the-money (thus underpriced) or deeper-in-the-money (thus overpriced), depending upon the time to maturity and/or true volatility rate of the option. This contradictory needs of finer grouping and less arbitrary grouping would be called the 'dichotomous bias dilemma'. The word dichotomous bias arises from the fact that bias statements are often made attributing two different directions (signs) of mispricing to two different broad groups of options. Black(1975)'s direct striking price bias, and MacBeth and Merville(1979)'s inverse striking price bias can be cited as two examples. By their very nature, dichotomous bias statements and

9 (cont'd)

the grouping technique suit each other. But, as we have indicated they are rather inappropriate and misleading for judging the validity of the Black-Scholes model.

The regression technique is an improvement upon the grouping or plotting techniques, in the sense that it attempts to control for other factors when evaluating the relationship of mispricing to a particular factor. By the very nature of a regression model, it is not suitable for issuing a statement of the dichotomous bias type. The other category of bias statements, which we would call the functional bias type, are more coherent with the regression technique. Thus, if we say that the mispricing increases as the moneyness increases, it would be a functional bias statement, since it gives the reader a sense of functional relationship between mispricing and the level of moneyness. But notice that this statement does not allow us to say that the model underprices(overprices) in-the-money(out-of-the-money), since the sign of the mispricing would depend upon the total effect of all relevant factors. Unfortunately, there has been a tendency in the current literature to use the signs of the regression coefficients as the basis for dichotomous bias statements.

That the regression results and the grouping results are not comparable should be clear by now. Also, whether the regression coefficients in a constant coefficient regression model can be interpreted as the marginal biases is cast into doubt by the option-specificness of the nonlinearity bias alone.

It appears that given our market pricing scenario, the complications of the formula's nonlinearity bias, the limitations of the general model validation techniques and of the specific statistical techniques used, an appropriately defined regression model may be necessary to elicit a concise but comprehensive understanding about the problems involved. We hope to present such a model in chapter 3, which we shall call an 'alternative regression model'. We derive an estimable regression equation which clearly shows the various probable components of the formula estimate's deviation from the market price. The regression coefficients in this equation would generally be option-specific, and reflect model misspecification bias over and above the nonlinearity bias. The use of estimated volatility rate as a proxy for the true volatility rate gives rise to the stochastic regressor problem for the least squares coefficients, which may or may not be substantial, as is also the degree of heteroskedasticity.

The model shows the costs of simplification (such as assuming the regression coefficients constant across options) or incompleteness (such as omitting relevant factors, e.g., the early exercise probability), or lack of attention to econometric problems (such as stochastic regressor problem). The alternative regression model also shows how one can address general problems like the validity of the Black-Scholes model or more specific problems such as dividend inducement of the systematic relationships of mispricing with factors.

In our discussion above, we have seen that the nonlinearity bias of the formula may assume a vital rôle in empirical investigation of Black-Scholes pricing. The nature of this bias is examined in chapter 4.

On the basis of second order approximation to the nonlinearity bias, we shall try to provide an explanation for the pattern of striking price bias found by Boyle and Ananthanarayanan(1977). The systematic relationships of the nonlinearity bias with the moneyness and the time to maturity of the option, and the true variance rate will be given monte carlo investigation ¹⁰. In doing so, the option specificness and the marginal nature of the systematic relationships will be emphasized. We will also discuss how the sample mixture of the options can affect the conclusions about systematic tendency of the formula estimate.

We will also consider four alternative estimators and see if they are better than the formula estimator:¹¹ ¹² (1) pseudo estimator: if we plug-in estimated variance rate or volatility

¹⁰Butler and Schachter(1983a) used numerical integration technique to do a similar analysis with respect to a composite measure of moneyness and the variance. See our chapter 4 for more details.

¹¹ In this thesis, we would call one estimator 'better' than another, if the former has lower bias, and lower standard deviation or lower mean square error, unless otherwise stated explicitly. Another qualitative measure of goodness of an estimator would be the lack of any showing of systematic deviation.

¹²Butler and Schachter(1985) propose a technique to minimize mean squared error of estimating the Black-Scholes price ; the technique requires the knowledge of true volatility rate.

rate into a specific function of the true variance (or volatility) rate, the estimator should have zero bias, when the bias is considered in the form of a truncated Taylor series; (2) Butler-Schachter estimator: the estimator proposed by Butler and Schachter (1983a); (3) CC estimator: it is similar to Butler-Schachter estimator, but eliminates some of the latter's deficiencies; (4) potentially unbiased estimator: both Butler-Schachter and CC are approximately unbiased, but another estimator is suggested, which would be unbiased, subject to operational limitation.

We undertake a Monte Carlo for comparing the behaviors of the pseudo estimator and the formula estimator. Yet another monte carlo is undertaken to compare the performance of the Butler-Schachter and the CC estimator with that of the formula estimator. We do not find evidence of clear superiority of these estimators over the formula estimator, though the CC estimator seems to have slight edge.

Finally, we entertain some indirect tests of the Black-Scholes model, and dividend-induced systematic relationships. We continue to use the formula estimate, and utilize the structure of our alternative regression model.

In chapter 5, as a preliminary basis of our tests in later chapters, we start our empirical estimations by assuming a constant coefficient version of the regression model developed in chapter 3. Following Geske and Roll (1984a), we divide our total sample into two subsamples, one containing the options

with no dividend payment on the stock prior to maturity and the other with options having one dividend payment prior to maturity. We discuss the signs and statistical significance of the coefficients estimated for the two subsamples, and compare them with previous studies. We also discuss the stochastic regressor problem, and find it as a potential source of sign reversal of coefficients across samples.

As indicated earlier, recent studies have considered the lack of proper treatment for the early exercise possibility of unprotected American calls as the source of the systematic deviations of the formula estimate. The question asked under testing dividend inducement is: if we find the mispricing of formula estimate to be systematically related to the factors (such as moneyness and time to maturity of the option, the volatility rate on stock return), is it mainly because of the formula's not accounting for the early exercise feature of the option? Previous studies except Geske and Roll (1984a) and Whaley (1982) used techniques other than the multiple regression technique, and are subject to the criticisms we have made earlier¹³. Geske and Roll (1984a) ran two regressions, one for the total sample of options, and the other for the subsample of non-dividend-paying options. The significance of the coefficient of a factor in the total sample, but insignificance in the zero-dividend subsample was taken as evidence of

¹³Whaley (1982) did not present the estimated multiple regression equations

dividend-inducement of formula's systematic relationship with that factor. We shall comment on their sample mixture of dividend-paying and non-dividend-paying options, as it can seriously affect the significance of a coefficient in the two regressions. Moreover, they undertook no statistical test of difference in the coefficient estimates across the two equations.

In chapter 6, we propose a Chow test for testing dividend inducement. Three regressions are run, assuming constant coefficient; one each for the zero-dividend subsample, the dividend-paying subsample, and the total sample. The first two implies that we are not restricting the coefficients to be the same for zero-dividend and dividend-paying subsamples, but the last implies the restrictions. If there is no significant dividend inducement, the restrictions would not matter. Our test result indicate significant dividend inducement for the whole regression function. Tests of differences in the individual coefficients and pairs of coefficients indicate that the significant difference in the coefficient of the volatility rate may have caused the significant dividend inducement for the whole regression function. The coefficients of moneyness and the time to expiration do not seem to have any perceptible role in dividend inducement. These results are in contrast to those of Geske and Roll(1984a), or Whaley(1982).

In chapter 7, we present one of the indirect tests of the validity of the Black-Scholes model. We utilize the structure of

the second order approximation to the nonlinearity bias, and the fact that when the Black-Scholes model is valid, the expected response function in our regression model would consist of the nonlinearity bias alone. It appears that the approximation implies specific restrictions among the slope coefficients in our regression model. Thus an indirect test of the Black-Scholes model would be to test the implied restrictions among the coefficients. The merit of the proposed test lies in that we take advantage of the existence of the nonlinearity bias of the Black-Scholes formula estimate, rather than being victimized by it, as the existing tests have been. We perform the small sample F-test as well as the large sample Wald test. Unfortunately, the two test results are contradictory.

We noted earlier that the coefficients in a proper regression model would be option-specific, while our estimations in chapter 7 and elsewhere relied on constant coefficient estimation. In chapter 8, we undertake cubic spline regression to allow the coefficient of a factor to be nonlinear with respect to that factor alone.

Once the spline regression equation is estimated, we use the estimated equation to predict the path of the mispricing as only one of the factors is allowed to vary over its sample range. This predicted path is then compared with that of the nonlinearity bias alone, the latter being known from the Monte Carlo studies. If the former is extremely different from the latter, the validity of the Black-Scholes model would be in

doubt. For example, Boyle and Ananthanarayanan(1977)'s study and our results in chapter 4 show that the negative of the nonlinearity bias of the formula as a function of the moneyness would exhibit four phases of decrease, increase, decrease, and increase. If the functional relationship predicted by the spline regression shows just the reverse cycle, we would have enough reason to suspect the validity of the Black-Scholes model. Unfortunately, our results indicate evidence against the Black-Scholes model.

We also compare across the two subsamples, spline prediction of the relationship of mispricing to moneyness and variance rate ¹⁴. As found in chapter 6, dividend-related effect do not seem to induce noticeable departures in striking price bias of the Black-Scholes formula estimate. The variance rate bias, on the other hand, exhibits remarkable difference across the two subsamples. This strengthens what we found in chapter 6, i.e., the important role played by the volatility rate in dividend-induced mispricing.

¹⁴In chapter 4, we found the nonlinearity bias to have similar relation with time to maturity and variance rate. Also, the coefficient of time to maturity did not seem to have significant dividend inducement in our tests of chapter 6.

CHAPTER 1:
SURVEY OF THEORETICAL LITERATURE

A simple call option written on a stock is a contract promising to pay $\max[0, P(t)-X]$, if exercised at time t , where $P(t)$ is the stock price at time t , X is the striking price. For a European call, the time of exercise, if to be exercised at all, is contractually defined to be t^* , the time when the contract expires. On the other hand, an American call entitles the holder to exercise any time on or before t^* . An American call not having provisions of change so as to preserve its value in the event of cash distributions on the underlying stock prior to contractual maturity, is commonly referred to as Payout Unprotected American Call, hereafter UAC.

Both European and UAC will have nonnegative values, since they will be exercised only if the holder finds it profitable to do so. Unlike a forward or futures contract, exercise of an option is a privilege, not an obligation. But the holder of an UAC has an additional exercise in optimality, namely, that of whether to exercise prematurely, and if so, when. This early exercise feature of an UAC gives rise to considerable complications in its valuation when dividends are paid on the underlying stock.

Though substantial amount of research took place prior to 1973, the seminal article of Black and Scholes revolutionized the science of derivative asset pricing. It will hardly be an overemphasis if we call the European call pricing model of Black and Scholes(1973) as 'the premier model' of option pricing. Using the argument of the absence of arbitrage in a market

equilibrium , the valuation formula of Black-Scholes was derived as the solution to a linear parabolic partial differential equation, subject to time-independent boundary conditions¹⁵. The Black-Scholes valuation formula is given below:^{16 17}

$$CB(\theta) = P \phi(d_1) - XB(T) \phi(d_2) \dots\dots(1)$$

where the notations are as follows:

P the stock price¹⁸

X the striking price

T the contractual time to maturity

r the risk-free rate per unit of time

B(T) the price of a risk-free bond with time to maturity T and face value of \$1

σ the volatility of stock return per unit of time

θ the set of arguments (P,X,r,T, σ)¹⁹

ϕ the standard normal distribution function

¹⁵ Merton(1973) showed that the stock price need not be the equilibrium price.

¹⁶ Merton(1973) provides an alternative derivation of Black-Scholes valuation equation by using 3-security, instead of Black-Scholes's two-security hedge. Merton's is zero-equity, because proceeds from short selling and borrowing could be used to finance the long position. Black-Scholes's hedge necessitates net equity, though both Black-Scholes's and Merton's hedging portfolios are self-financing.

¹⁷ Hereafter, d_1 and d_2 will be understood as magnitudes, based on relevant stock price adjustment for dividends.

¹⁸ For future reference, the stock price net of escrowed dividends will be denoted by S. For an option with no dividend payments prior to maturity, S would be equal to P.

¹⁹ In the context of dividend adjustment, P will be replaced by S.

$$d_1 = [\ln(P/X) + (r + 0.5\sigma^2)T] / \sigma\sqrt{T} \text{ and}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The assumptions under which (1) was derived are:

1. There is no restriction on borrowing and short selling, with full proceeds available.

2. Borrowing and lending rates are equal.

3. There are no transaction costs or differential taxes.

4. Trading takes place continuously in time.

5. The short term risk-free rate is known and constant through time.

6. There are no dividend payments or other distributions during the life time of the option.

7. The option is European.

8. The sample path of the stock price is continuous²⁰.

Specifically, the stock price follows a Geometric Brownian Motion through time (i.e., follows a random walk in continuous time with a variance rate proportional to the square of the stock price).

Thus, the distribution of the stock price over a finite interval of time is lognormal.

9. The variance rate for the stock return is a known constant.

The first three assumptions constitute what has been termed as the "ideal conditions" in the market. The "ideal conditions", together with the assumptions 4 and 5, specify the capital

²⁰ Increments of the stock price exhibit local Markov property, i.e., in a short interval of time, the stock price can only change by a small amount.

market environment. Assumptions 8 and 9 specify the stock return distribution.

The plethora of research that followed Black and Scholes(1973), showed that the basic analysis of Black and Scholes is not affected in any significant way by the relaxation of the capital market assumptions²¹.

Merton(1976) extends Black-Scholes by considering a stock price process that combines the Gauss-Weiner and the Poisson. Even though completely riskless hedge cannot be formed under such a process, Merton derived a formula similar to the Black-Scholes's in spirit, by invoking the CAPM.

Cox and Ross(1976) considered several other processes, and also proposed the risk-neutral approach of option valuation which has become very popular as a conceptual and simulation tool. The risk-neutrality argument is based on the observation that if a riskless hedge can be formed and maintained, then the option valuation becomes preference-free, and thus could be undertaken as if the economy were risk-neutral²².

Jarrow and Rudd(1982) introduce the construct of deriving an approximate valuation for an arbitrary stochastic process. The approximate valuation is the sum of Black-Scholes valuation and adjustment terms involving the higher order moments of the

²¹ See Merton(1973) for stochastic interest rate, Ingersoll(1976) for differential taxes for capital gains versus dividend or interest, and Rubinstein(1976) for discrete trading.

²² For some simple types of processes, it is possible to form riskless hedge, and then the risk-neutrality approach can be applied to value the option.

underlying stock price process.

Geske(1979) derived a valuation equation for a call, viewing the call as a compound option(on the value of the firm), with the Black-Scholes valuation as a special case when the firm in question is unlevered. The important distinction is that Geske's formula includes leverage effects, and accommodates a varying variance rate for the stock return. Cox(1975) also considered a non-stationary variance rate in deriving the Constant Elasticity of Variance(CEV) diffusion formula. The CEV formula, though similar in appearance, is little more complicated than the Black-Scholes²³. Black-Scholes is, of course, contained as a special case of the CEV when the constant elasticity assumes the value of zero(i.e., the elasticity parameter is equal to 2).

The Displaced Diffusion Option Pricing model of Rubinstein(1983) considered the asset structure of the firm in addition to the leverage considered by Geske(1979)²⁴. The resulting valuation formula has the same structure as the Black-Scholes, except that both the striking price and the stock price to be used in valuation are displaced by amounts dependent

²³ Both Merton's jump-diffusion and Cox's CEV diffusion involves the evaluation of infinite sums, and for empirical purpose, estimation of additional parameters than the Black-Scholes.

²⁴ Rubinstein assumes that the firm holds a risky asset with lognormal return, and a riskless asset. Capital structure consists of riskless bonds and equity. Also the time and the size of dividend payment are assumed to be known.

on the asset structure and the capital structure²⁵.

All the models mentioned above assumes that either the option is European or that early exercise is not optimal if it is an UAC. Merton(1973) and Smith(1976) have shown that early exercise of an American call is not optimal, if there is no distribution on the stock prior to the contractual maturity of the option²⁶. In that case, an American call will have the same value as an European option.

For a proportional dividend policy $D(P,T)=\nu P$, $\nu>0$, Merton(1973) derived the value of a European warrant as:²⁷

$$CM(\theta;\nu)=\exp(-\nu T)P\phi(d_1)-XB(T)\phi(d_2) \dots(2)$$

Rubinstein(1976) derived the valuation formula for similar policy in discrete time.

Let us note the following:

(i) $CB(\theta)> CM(\theta;\nu)$ for $\nu >0$. Thus the Black-Scholes valuation will overestimate an European option with distributions on the stock prior to maturity of the option.

²⁵ The striking price adjustment depends on the dividend policy also.

²⁶ At each point in time prior to the contractual maturity of the option, the holder will compare the value on exercise($P-X$) with the value if held to maturity($CB(\theta)$). But $CB(\theta)$ is bounded below by $P-XB(T)$. Given $B(T)<1$, $[P-XB(T)]>[P-X]$. Thus, $\min[CB(\theta)]>[P-X]$. An option is worth more alive than dead. Intuitively, if not exercised early, the dollars for the exercise price could earn riskless rate over the remaining life of the option.

²⁷ The notation is similar to that used by Merton(1973), pp 171. It is assumed that the underlying stock pays dividends continuously at the rate ν . Smith(1976), pp 26, has similar exposition, but his valuation formula differs slightly from that reported in Merton(1973).

(ii) For large P , $\phi(d_1) \approx 1$, $\phi(d_2) \approx 1$, and thus $CM(\theta; \nu) \approx \exp(-\nu T)P - XB(T)$. Then $CM(\theta; \nu) < (P - X)$ for some large P . Hence, for large stock price, an American call could be worth more dead than alive.

Merton(1973) also considered a constant dividend policy. Though a closed form solution for $T < \infty$ was not given, a solution for the perpetual warrant was derived.

Geske(1978) derived an option valuation formula for the case of lognormally distributed dividend yield in a discrete time framework, using Rubinstein's technique of discounting uncertain income stream²⁸. The distinction that the stochastic dividend brings about is that the variance rate for the stock return is to be adjusted to accommodate the instantaneous variance of compound dividend yield and the covariance between the return on the stock and the dividend yield²⁹. Like (2), Geske's stochastic dividend valuation adjusts the Black-Scholes prices downward. But the early exercise feature of an UAC has not been dealt with, in these basically-European-call-oriented valuations.

²⁸ Rubinstein's technique allows discrete trading to occur in discrete time and does not require a riskless hedge. But restrictions on investor preference and probability distribution of asset returns are required. For stochastic dividend, riskless hedge cannot be formed, because, in general, the dividend cannot be expressed as a nonstochastic function of stock price and time.

²⁹ The variance rate for the stock return needs to be adjusted also in the case of stochastic risk-free rate, as shown by Merton(1973), pp 162-169.

Schwartz(1977) developed a numerical solution to the partial differential equation governing the value of an UAC, with known date and size of dividend payments on the underlying stock prior to contractual maturity of the option. Roll(1977) provided the first closed form solution for such valuation problem. Roll considered the case of an UAC, where the stock has a single dividend of known size D on a known date t_1 , prior to the expiration date t^* . It was also assumed that on the ex-dividend date, the stock price would go down by a known fraction of D^{30} .

Instead of directly solving the partial differential equation governing the price dynamics of the UAC, Roll resorts to the technique of valuation by duplication³¹. Roll duplicates the cashflow of an UAC by forming a portfolio of two European calls and an European compound option on one of the European calls.

Noting that an UAC is, in fact, a compound option, Geske(1979a) applies the Geske(1979)'s compound option formula to directly value an UAC. Whaley(1981) corrected some minor

³⁰ In what follows we will assume the value of the fraction to be equal to one, as has been assumed by the empirical studies employing Roll's model. Whaley(1982), of course, hinted that this simplifying assumption could be a probable source of the persistent variance bias, to be discussed later in this thesis.

³¹ Ross(1976) showed that arbitrary simple options are equivalent to a portfolio of call options. When the primitive assets are of limited liability nature, and the striking price of the options are nonnegative, Ross proved, in his Theorem 3, that all simple options can be thought of as portfolios of puts and calls.

mistakes in Roll(1977)'s and Geske(1979a)'s valuation formulae³². Whaley pointed out that the duplicating portfolio to replicate the cashflows of an UAC is not unique. Geske(1981) showed that there are simpler duplicating portfolios than the ones used by Roll(1977) and Whaley(1981), containing fewer securities, and resulting in more compact solutions. Though both Roll(1977) and Geske(1979a)'s corrected versions will result in the same value of an UAC, Geske(1981) argued that the use of direct solution could prove cost-economic. Hereafter, we will refer to the UAC valuation model developed by Roll(77), Geske(1979a), and Whaley(1981), as the RGW model.

The RGW price for a single dividend case is given by:

$$C(R) = S[\phi(b_1) + \phi_2(a_1, -b_1; -\sqrt{(T_1/T)})] - XB(T)[\{\phi(b_2)/B(T-T_1)\} + \phi_2(a_2, -b_2; -\sqrt{(T_1/T)})] + DB(T_1)\phi(b_2) \dots\dots\dots(3)$$

where

S is the stock price net of escrowed dividend

T₁ is the time to ex-dividend instant or date

S* is the critical level of the ex-dividend stock price above which the American call will be exercised

R denotes the set of arguments (S, X, r, T, σ, S*)

φ₂(a, b; c) is the bivariate normal distribution function with upper integral limits a and b, and correlation coefficient c

a₁ = [ln(S/X) + (r + 0.5σ²)T] / σ√T, a₂ = a₁ - σ√T

³² The corrections are for the striking price (S*, instead of S* + aD) of one of the duplicating options in Roll(1977), and the sign of the correlation coefficient (-√(T₁/T), rather than √(T₁/T), in our notation) in Geske(1979a).

$$b_1 = [\ln(S/S^*) + (r + 0.5\sigma^2)T_1] / \sigma\sqrt{T_1}, \quad b_2 = b_1 - \sigma\sqrt{T_1}$$

Following Geske(1979a), pp 377, and Whaley(1981), pp 45, and as mentioned in Sterk(1983), the following interpretation of the RGW valuation equation can be given.

A. On the ex-dividend date, $S=S(t_1)$, and $T_1=0$. Thus if $S>S^*$, $b_1=b_2=\infty$, $\phi_2(\cdot)$ s will be equal to zero, and $\phi(\cdot)$ s will be equal to one, and $C(R)$ becomes $S-X+D$, the exercise value if exercised prematurely. If $S<S^*$, $b_1=b_2=-\infty$, $\phi_2(a_1, -b_1; -\sqrt{(T_1/T)}) = \phi(a_1)$, $\phi_2(a_2, -b_2; -\sqrt{(T_1/T)}) = \phi(a_2)$, $\phi(b_2) = \phi(b_1) = 0$. Thus, $C(R)$ becomes $S\phi(a_1) - XB(t^*-t_1)\phi(a_2)$, the value of the option if kept alive.

B. Note that b_1 and b_2 are in fact d_1 and d_2 of the Black-Scholes valuation equation (1), with a time to maturity T_1 and striking price S^* . Thus, following Jarrow and Rudd(1983), $\phi(b_2)$ could be interpreted as the probability that the hypothetical European call will end up in-the-money at maturity, i.e., $S(t_1) > S^*$. Hence, $\phi(b_2)$ could be interpreted as the probability of early exercise³³.

C. Note that a_2 represents the d_2 of the Black-Scholes valuation equation (1) for a European option with time to expiration T , and thus $\phi(a_2)$ represents the probability of this hypothetical option to end up in-the-money at maturity. On the other hand, $\phi(-b_2) = 1 - \phi(b_2)$ represents the probability that the hypothetical option with time to maturity T_1 in part B above will not end up in-the-money, i.e., $p[S(t_1) < S^*]$. Thus

³³See Whaley(1982), footnote 17.

$\phi_2(a_2, -b_2; -\sqrt{(T_1/T)})$ seems to represent the joint probability that $S(t^*) > X$ and $S(t_1) < S^*$, i.e., the probability that the call is not exercised early and is in-the-money at contractual expiration.

D. In a risk-neutral economy, or if a riskless hedge can be formed with the call and the stock, the value of an UAC with dividend payment ahead could be shown as the sum of: ³⁴ ³⁵

(i) $B(T_1)[E\{S(t_1) | S(t_1) > S^*\} + D - X] p[S(t_1) > S^*]$, the present value of the conditional expected value of the exercise value when exercised just before the ex-dividend time.

(ii) $B(T)[E\{S(t^*) | S(t^*) > X\} - X] p[S(t^*) > X \text{ and } S(t_1) \leq S^*]$, the present value of the option's value unexercised just before the ex-dividend time.

To account for the early exercise possibility in the framework of Black and Scholes (1973), Black (1975) suggested an approximation currently known as the pseudo-American valuation. Since the pseudo-American valuation has been in popular use, and the early exercise possibility is a vital part of an UAC valuation, we will end our discussion of the theoretical research with a short digression on them.

Let the stock prices be represented by:

$$S(t) = P(t) - DB(t_1, -t) \text{ for } t < t_1$$

$$= P(t) \text{ for } t \geq t_1$$

³⁴Note that the probabilities in B and C are risk-neutral probabilities. Thanks to Professor Whaley for pointing this out.

³⁵See Whaley (1982), footnote 17.

Let Δt represent an instant of time. Jarrow and Rudd(1983) ,pp 49, proves that :

$$\lim_{\Delta t \rightarrow 0} P(t_1) + D = P(t_1 - \Delta t) \text{ with certainty.}$$

That is to say, at the ex-dividend time, the stock price would fall by the amount D with certainty, from the level prevailing at an arbitrarily close previous instant. This also means that the stock price cum-dividend at $t_1 - \Delta t$ will be : $P(t_1 - \Delta t) = P(t_1) + D$. Let us now see what would be the value of the option for $t < t_1$, for the two hypothetical cases: (i) certain early exercise at $t_1 - \Delta t$, and (ii) certain exercise at t_1 (i.e., no early exercise).

(i) Certain exercise at $t_1 - \Delta t$

The relevant stock price at $t < t_1$, would be $P(t) - DB(T_1)$, because value of the option corresponds to the risky component of the stock price. The effective exercise price at $t_1 - \Delta t$ will be:

$$X - DB(t_1 - t_1 + \Delta t) \approx X - D$$

The reason is that if the option is exercised at $t_1 - \Delta t$, D can be claimed at t_1 , thus reducing the effective exercise price by the amount D . The option value, then, is given by:

$$CB(\theta) = CB[\{P(t) - DB(T_1)\}, (X - D); \sigma, r, T_1]$$

We have T_1 as the time to maturity here, because of early exercise taking place at $t_1 - \Delta t$, the option's life effectively shrinks to that moment. It is to be mentioned here that Merton(1973) has shown that an American call, if exercised early

optimally, will be exercised only an instant before the ex-dividend time.

(ii) Certain exercise at t^* or no early exercise:

The relevant stock price and exercise prices are $P(t)-DB(T_1)$ and X respectively. The value of the option will, then, be given by :

$$CB(\theta) = CB[\{P(t)-DB(T_1)\}, X; \sigma, r, T]$$

But the option will be exercised to the best interest of the holder. Thus, if the probability of early exercise were a zero-one variable, American call should have the value:

$$C_t = \max[CB\{(P(t)-DB(T_1)), (X-D); \sigma, r, T_1\}, \\ CB\{(P(t)-DB(T_1)), X; \sigma, r, T\}]$$

The above is the pseudo-American valuation, originally suggested by Black (1975). Pseudo valuation will be a good approximation if the probability of early exercise (p) is close to either of the extreme values.

We will now examine when the early exercise will take place. Early exercise at $t_1 - \Delta t$ will take place if the option value on exercise at $t_1 - \Delta t$ exceeds the value when exercise is postponed till t^* . If exercised at $t_1 - \Delta t$, option's value at that moment will be : $P(t_1 - \Delta t) - X$. If it is not exercised at $t_1 - \Delta t$, it will be worth:

$$CB[\{P(t_1 - \Delta t) - DB(t_1 - t_1 + \Delta t)\}, X; \sigma, r, (t^* - (t_1 - \Delta t))]$$

$$=CB[\{P(t_1, -\Delta t) - D\}, X; \sigma, r, (t^* - t_1)]$$

$$=CB[P(t_1), X; \sigma, r, (t^* - t_1)]$$

So option will be exercised at $t_1 - \Delta t$, if $[P(t_1, -\Delta t) - D]$ or $P(t_1)$ is such that :

$$P(t_1, -\Delta t) - X > CB[P(t_1), X; \sigma, r, (t^* - t_1)]$$

$$\text{i.e., } P(t_1) + D - X > CB[P(t_1), X; \sigma, r, (t^* - t_1)]$$

Note that $S(t_1) = P(t_1)$.

We now have the condition of early exercise as:

$$S(t_1) + D - X > CB[S(t_1), X; \sigma, r, (t^* - t_1)]$$

If it is an equality, then the holder will be indifferent about whether to exercise prematurely or not. On the other hand, a less than sign will lead to postponement of exercise. The solution S^* to :

$$S^*(t_1) + D - X = CB[S^*(t_1), X; \sigma, r, (t^* - t_1)]$$

is the critical level of the ex-dividend stock price above which option will be exercised early optimally. We will refer to this price as simply S^* .

It has been shown in Jarrow and Rudd (1983) that the sufficient condition for no early exercise is:³⁶

$$X - XB(t^* - t_1) > D$$

$$\text{or, } X - D > XB(t^* - t_1)$$

In other words, it means that if the effective exercise price, when exercised an instant before the ex-dividend time is greater than the present value, at $t_1 - \Delta t$, of the cost of exercise at t^* , the option will not be exercised early.

³⁶This was first noted by Roll (1977).

Now, if we are at $t < t_1 - \Delta t$, given $S(t)$, we can find out the probability of $S(t_1) > S^*$, because S^* will be known in advance.

Using the property of Geometric Brownian Motion for S :

$$p = p(S(t_1) > S^* | S(t) = s(t)) = 1 - F(S^*/s(t); (t_1 - t))$$

where F is the distribution function.

The larger the size of the known dividend D , the cheaper the cost of exercise at $t_1 - \Delta t$, compared to the cost of exercise at t^* . Also, closer the ex-dividend date to the contractual maturity date, lower will be the ex-dividend option value, if not exercised at $t_1 - \Delta t$. In both of these cases, S^* becomes smaller, when the schedule $S(t_1) + D - X$ shifts to the left (for larger D) and the curve $C[S(t_1), (t^* - t_1), X]$ swings down (for smaller $t^* - t_1$) along with the schedule $S(t_1) - XB(T - T_1)$ shifting down, in a diagram like Figure 1, pp 253, of Roll (1977). The implication is that by monotonicity of F , p becomes larger.

Also notice that as the option goes in-the-money (or deeper-in-the-money), p would become larger.

CHAPTER 2:

REVIEW OF EMPIRICAL LITERATURE

The body of applied research in the area of call valuation since the early 70's has centred predominantly around the model of Black and Scholes(1973). The principal questions asked were:

Q1. Does the Black-Scholes European call valuation model, or some modified or extended version of it represent 'well' the actual market prices of traded call options ?

Q2. If the answer to Q1 is not a strong yes, then, (a) what is the nature of the weakness of the Black-Scholes model, and (b) what are the sources of weakness?

Q3. Does an alternative model perform better than the Black-Scholes model in a prespecified sense ?

Q1 can be interpreted as testing the validity of the Black-Scholes model against an unspecified alternative. Galai(1983) outlined the complications of testing model validity. In general, such tests represent joint tests of the market synchronization, option market efficiency, and the validity of the model.

Q2 is, in fact, an integral part of answering Q1. Q3 is a problem in model selection, nested or non-nested, as the case may be³⁷. In this thesis, we do not consider testing the Black-Scholes model against any specific alternative model, though it remains an important agenda for our future research.

³⁷In the existing studies, the models against which the Black-Scholes model has been tested, contain the latter as a special case. Thus, they belong to the nested case.

Our focus, here, would be on Q1 and Q2.³⁸

One would find the answers to these questions as diverse as the tools of research producing them. Although findings such as Black(1975)'s direct striking price bias and subsequently MacBeth and Merville (1979)'s inverse striking price bias have led to interesting debates, insufficient attention has been paid to issues such as the differences in the types of tools used, their limitations, their probable effects on the results reported, et cetera. Moreover, there has been a lack of care in the interpretation and comparison of results. It is with this critical eye that we shall review the empirical research in this chapter.

There are nine sections in this chapter. The major approaches to model validation are briefly reviewed in section 1. Though these approaches differ in significant ways, the studies based on them do share a common finding that the estimated Black-Scholes prices tend to exhibit systematic deviations from the market prices. The findings of these systematic deviations are summarised in section 2. Since these findings are mostly for UAC data, the treatment of dividend in these studies is a potentially important issue, and is briefly

³⁸ To be clear, option market efficiency will be a maintained hypothesis, not a hypothesis to be tested as is the case in many studies. We feel that the proper technique of investigating option market efficiency is the hedging technique first proposed by Black and Scholes(1972). But we do not pursue the question of efficiency, and thus our results about model validity may have been affected by any departure from the maintained hypothesis of market efficiency.

surveyed in section 3. A summary of the nature of empirical testing follows in section 4. The methodological and statistical problems of ISD and their possible resolution via regression models are discussed in section 5. In section 6, we examine some of the important regression-based studies.

In section 7, we classify bias statements into two broad categories: the dichotomous bias and the functional bias. It is surprising that the bias statements of these two differing types were indiscriminately used to compare empirical results by researchers. We discuss the problem associated with making dichotomous statements in section 8. Finally, section 9 reviews the problems and prospects of the functional bias approach³⁹.

³⁹ Approaches to validation of the Black-Scholes model were classified by Galai (1983), according to the techniques of validation. An alternative classification can be made based upon the type of bias statement or the tool of investigation used. As we would see later on in this chapter, the two approaches under this classification are dichotomous bias approach and functional bias approach.

SECTION 1

Approaches to model validation differ in at least two important dimensions⁴⁰:

(i) The estimator for the unknown variance or volatility rate.

(ii) Given the estimator, the technique of validation.

Examples of studies using different types of estimators are:

(i) Historical variance rate estimator:

Black-Scholes(1972), Boyle and Ananthanarayanan(1977), Galai(1977), Butler and Schachter(1983,1983a,1984a), Merton(1976a) ⁴¹.

(ii) Actual variance rate estimator: Black-Scholes(1972), Latane and Rendleman(1976), Bhattacharya(1980), Chiras and Manaster(1978) ⁴².

(iii) Implied Standard Deviation (ISD) estimator: Latane and Rendleman (1976), Schmalensee and Trippi(1978), Chiras and Manaster(1978), MacBeth and Merville(1979,1980), Whaley(1982),

⁴⁰The definition of moneyness may be considered as yet another dimension.

⁴¹Historical variance rate estimators across studies differ in the unit of time and the length of the period over which they are estimated.

⁴²Actual variance rate estimators imply estimation of volatilities from the stock return data over the life of the option, and as such may be appropriate only for ex-post research.

Emanuel and MacBeth(1982), Sterk(1982,1983) ⁴³.

Black(1975), Bhattacharya(1980), Geske and Roll(1984a) compared the stock price(or stock price adjusted for dividend) with the undiscounted exercise price. However, MacBeth and Merville(1979,1980), Merton(1976a), Whaley(1982), Sterk(1982, 1983), Butler and Schachter(1983,1983a), Jarrow and Rudd(1982) among others compared the stock price(or stock price adjusted for dividend) and the discounted value of the exercise price. The difference between these two definitions could be pronounced if the the time to expiration is substantial. In general, the latter definition will be biased towards finding long-maturity options to be in-the-money.

Four major techniques of validation can be identified:

(i) Monte Carlo studies: Merton(1976a), Boyle and Ananthanarayanan(1977), Butler and Schachter(1983,1983a), Jarrow and Rudd(1982), and to a limited extent Bhattacharya(1980), Roll(1977), Geske(1978). The technique relies on relaxing one or more of the assumptions of the Black-Scholes model, and evaluation of the success of the Black-Scholes prices in tracking the true prices.

(ii) Studies based on risk-adjusted return of hedge positions: Black(1972) first proposed the technique, and it was

⁴³Latane and Rendleman(1976) compare the performance of the historical and the actual estimators in predicting volatility. Whaley(1982), footnote 3, points out that even though Latane and Rendleman(1976), and Chiras and Manaster (1978) call this estimate the 'actual standard deviation of return', it is merely an estimate. Black and Scholes(1972) used both the historical and the actual estimator in two separate sets of tests.

later used by Galai(1977), Chiras and Manaster(1978), Phillips and Smith(1980), Blomeyer and Klemkosky(1983), Butler and Schachter(1984a) among others ⁴⁴. The idea is to establish positions in multiple securities(stocks and options) on the basis of comparison between the estimated model price and the actual market price, so as to make the overall position riskless, and rebalance the position at some prespecified interval of time. If the model is valid, the overall position should indeed be riskless(or no risk that would be priced by the market). Also, the model should be able to identify the temporarily under or overvalued options. In an efficient market, the temporary under or overvaluation relative to the fair value of the option would eventually get eliminated. Hence, the return on the riskless hedge should be higher than the risk-free rate.

The studies in this category, of course, differ in the choice of securities to combine, the time and prices at which to establish the hedge position, and the length of time over which the position to be held.

This approach requires unbiased estimates of Black-Scholes model prices to identify under or over-valued options in order to establish the hedge position⁴⁵. To give an example of the probable error involved in using the formula estimate, let us assume that a near-out-of-the-money option is trading at \$1.00,

⁴⁴Phillips and Smith(1980) provides a good survey of the use of this technique.

⁴⁵Butler and Schachter(1984a) discusses this problem.

while the formula estimate is \$0.80 . For such an option, the formula estimate would, on the average, underestimate the Black-Scholes model price. If the latter exceeds \$1.00, given that the Black-Scholes model is valid, the formula estimate would indicate the wrong hedge position, on the average.

Moreover, since the discretely adjusted hedges are not completely riskless, users of this approach tend to depend on asset pricing models such as the CAPM to estimate risk-adjusted excess return on the hedge. Given that the asset pricing models themselves are subject to controversy and need to be validated empirically, the hedging approach adds yet another hypothesis in the already crowded pool of joint hypotheses.

(iii) Studies based on the behavior of ISD: Latane and Rendleman(1976), MacBeth and Merville(1979), Schmalensee and Trippi(1978), Geske and Roll(1984) belong to this category. The idea is that , assuming synchronous and efficient market, if the model is valid, then the ISDs should not have any systematic relationship with the features of the option(moneyness, time to expiration).

If one claims that ISD is the true volatility rate, then the finding that ISDs of options on the same stock, traded at about the same time period, differ, cannot be adequately explained. If it is only an estimate of the true volatility rate, then the implicit assumption is that the market plugs-in an estimate of volatility rate in an exact pricing function such as the Black-Scholes formula. But then one is accepting the fact

that the market will be systematically off the fair value of the option, since the formula estimates will have systematic biases. This scenario is difficult to agree upon. In fact, the problem could lie in the assumption of an exact pricing function generating the actual option prices. We intend to discuss these problems in more details in section 5 of this chapter.

(iv) Studies based on comparison of the estimated model prices and the actual market prices of options: MacBeth and Merville(1979,1980), Geske and Roll(1984a), Black (1975), Whaley(1982), Sterk(1982,1983), Emanuel and MacBeth(1982), Gultekin,Rogalski,and Tinic(1982), Blomeyer and Klemkosky(1983), Butler and Schachter(1984a) belong to this category ⁴⁶. This technique dwells on estimating the volatility rate, calculating the model price using the the estimated volatility rate, taking the difference of the actual price and the estimated model price(dollar difference or percentage),group the options and compare statistics on mispricing across groups, or plot mispricing against the variables of interest, or regress on the variables of interest. If the model is valid, the deviation of market price from the model price would be random⁴⁷.

The grouping technique faces the dilemma that if the classification is inadequate, the sample mixture of options would affect the conclusions severely and finer classification

⁴⁶Note that some studies appear in more than one category, since the authors considered more than one approaches to validation.

⁴⁷But the deviation of formula estimate from the model price is not random.

would entail arbitrariness⁴⁸. Plotting or regressing the mispricing against individual factors lack the control for other factors needed to examine the impact of one factor. Although multiple regression technique may be an appropriate vehicle of investigation, unless properly laid out, simple interpretation of regression results could be misleading. For example, in chapter 7, we show that the regression coefficients do not necessarily indicate the directions of marginal biases, nor can their significance be construed as evidence against a pricing model, in our case, the Black-Scholes.

SECTION 2

Though the techniques mentioned above differ in significant ways, they have at least two things in common. The answer to Q1 is almost invariably not a strong yes, and the finding that when the Black-Scholes prices deviate from the true (in Monte Carlo) or actual (in empirical) market prices, the deviations tend to exhibit some systematic pattern. The former of these two items is less than surprising, but the latter deserves at least a brief discussion.

Using over-the-counter options market data, Black and Scholes (1972) first reported that the model overpriced (underpriced) options on high (low) estimated variance stocks.

⁴⁸A detailed discussion of this dilemma is provided in section 8 of this chapter.

The problem was attributed to the errors-in-variable problem arising from the measurement error in the variance rate on stock return. Black(1975) found that the model systematically overpriced(underpriced) deep-in-the-money(deep-out-of-the-money) options and underpriced options with less than 90 days to maturity. Examining some probable sources of these biases, Black ultimately concluded that "...we don't know why some kinds of options are consistently overpriced according to the formula and others are consistently underpriced".

Merton(1976) provided the qualitative results and Merton(1976a) tested the robustness of the Black-Scholes model postulating the jump-diffusion process to be the proper return generating process, and found that the Black-Scholes prices tended to underprice both deep-in-the-money and deep-out-of-the-money options, and overprice around-the-money options. The mispricing is expected to magnify for shorter maturity options, since for longer maturity, mispricing decreases as the distributions tend to converge to each other⁴⁹. Also Merton(1976),pp.140-141, states that the qualitative results correspond to practitioners' claim that deep-in , deep-out, short maturity options are underpriced, and marginally in-the-money and longer maturity options are overpriced by the Black-Scholes.

Boyle and Ananthanarayanan(1977) considered the bias of the Black-Scholes price with an estimated variance rate, against the

⁴⁹See Merton(1976),pp.140, and Merton(1976a) ,pp., 345.

Black-Scholes price with the true variance rate. Using numerical integration to compute the bias, they found that the formula estimate underprices at and around-the-money options and overprices deeper-away-from-the-money options⁵⁰. They also reported that the size of these biases is small even when the sample size for estimating the variance rate is as low as 15.

In a simulation study, Bhattacharya(1980) concluded that the formula bias is significant for at-the-money options with less than one day to expiration. His other findings are:(a)near-in-the-money and near-out-of-the-money options are underpriced;(b)the bias decreases as the time to expiration increases;(c)no systematic bias with respect to the variance estimator.

Galai(1977) and, Chiras and Manaster(1978) found some evidence of market inefficiency using the Black and Scholes model to establish riskless hedge (according to the Black and Scholes(1972) hedging technique). Bookstaber(1981), using the same data as Chiras and Manaster(1978), suggested that the observed inefficiency could be due to the nonsimultaneity of the data used. Phillips and Smith(1980) suggested that the transaction cost would eliminate any inefficiency observed by Galai(1977) and, Chiras and Manaster(1978). As a result, the joint tests of the validity and the market efficiency seem to remain indecisive.

⁵⁰Butler and Schachter(1983,1983a) confirmed this result.

MacBeth and Merville(1979) considered the data for CBOE options on six stocks over the year of 1976. They observed that:

(a)the Black-Scholes underprices in-the-money options and overprices out-of-the-money options, irrespective of the time to maturity; (b)the time to expiration bias is similar to Black(1975).

MacBeth and Merville's finding about the striking price bias is diametrically opposed to the finding of Black(1975). This was attributed to the nonstationarity of the variance rate overlooked in Black(1975).

MacBeth and Merville(1980) compared Cox's CEV and the Black-Scholes, by doing simulations and also empirical testing using the same data as in their 1979 paper. Their simulation confirmed their earlier finding of the inverse striking price bias of the Black-Scholes, due primarily to its constant variance rate assumption. On the empirical side, they found the CEV doing consistently better than the Black-Scholes.

Emanuel and MacBeth(1982) enlarged the sample of MacBeth and Merville(1979,1980) by including the observations from 1978. They questioned the validity of the constancy of the CEV parameters as assumed by MacBeth and Merville(1980), and concluded that: (a)the superior performance of the CEV wanes as the interval of prediction(of the variance rate) increases; (b)depending on the parameter values of the CEV, the Black-Scholes will consistently either overprice (underprice) or underprice(overprice) in-the-money(out-of-the-money) options.

The studies and the results cited in this section are but a fraction of the growing empirical literature related to the Black-Scholes. In the next section we shall survey studies with the dividend-induced Black-Scholes biases. However, we have already encountered several factors to which the mispricing could be systematically related. For example, Merton(1976,1976a) relate the mispricing to moneyness and time to maturity, Boyle and Ananthanarayanan(1977) relates to moneyness, Black(1975) and, MacBeth and Merville(1979) relate to both moneyness and time to maturity. Most interestingly, the sources of bias vary among the investigators. For examples, the striking price or the moneyness bias is traced to the misspecification of the stock price process by Merton(1976,1976a) and MacBeth and Merville(1979,1980), but to the nonlinearity of the Black-Scholes formula with respect to the variance(or volatility) rate by Boyle and Ananthanarayanan(1977). Even more interesting is the fact that the directions of the systematic relationships to a factor, in this case, the moneyness, found by different authors (Black(1975) and, MacBeth and Merville(1979)) are exactly opposite. Emanuel and MacBeth(1981) tried to explain this by means of nonstationarity of the CEV elasticity parameter, while Sterk(1982) and, Geske and Roll(1984) couch their explanations in terms of dividend-inducement.

The questions we shall pose are somewhat diagnostic, and to certain extent, prescriptive:

(a) Were the reportedly deeper-away-from-the-money options of Black(1975) truly deeper-away-from-the-money?

(b) Is MacBeth and Merville(1979)'s result, based on the classification in-the-money and out-of-the-money, comparable to Black(1975)'s result which does not seem to be based on similar classification?

(c) Can MacBeth and Merville(1979)'s regression coefficients be construed as the marginal biases or their unbiased estimates?

(d) What problems lie behind the use of ISD, as was used by MacBeth and Merville(1979) among many others?

(e) Can we offer an alternative approach to address general problems such as testing the validity of the Black-Scholes model or more specific problems such as testing the dividend-induced biases?

SECTION 3

We have discussed earlier in chapter 1 the effect of dividend payment on the underlying stock prior to contractual maturity of an UAC. Neither the simple dividend adjustment nor the pseudo-American version of the Black-Scholes appear viable relative to the Roll-Geske-Whaley model in capturing the early exercise phenomenon of an UAC ⁵¹. Since the publication of Roll(1977), there has been growing awareness of the impact of the early exercise possibility on the adequacy of simpler

⁵¹We are assuming here, the truth of the lognormal diffusion of the dividend-adjusted stock price.

(compared to Roll's) modified versions of the Black-Scholes , and the nature of observed mispricing. Attempts to estimate Roll-Geske-Whaley model prices with empirical data are to be found in Whaley(1982), Blomeyer and Klemkosky (1983), Sterk(1983), Gultekin,Rogalski,and Tinic(1982), and to a limited extent in MacBeth and Merville (1980), Geske and Roll(1984). Geske and Roll(1984) also discussed the probable implications of improper treatment of dividend in the earlier studies.

Merton(1976)'s Monte Carlo did not consider dividend payment. Latane and Rendleman(1976) used the original Black-Scholes model directly without any adjustment for dividends. Galai(1977), MacBeth and Merville(1979,1980), Blomeyer and Klemkosky(1983), Geske and Roll(1984a) used the simple dividend adjustment of the stock price approach. Chiras and Manaster(1978) transformed the discrete dividend payments into a constant continuous yield. Schmalensee and Trippi(1978) applied the Black-Scholes without any dividend adjustment to compute ISDs, but they tried to concentrate on low dividend-yield stocks. Sterk(1982,1983) used the pseudo-American version of the Black-Scholes to compare the Black-Scholes against the Roll-Geske-Whaley model. Whaley(1982) computed for both of the modified versions(simple dividend adjustment, and pseudo-American) of the Black-Scholes in comparing Black-Scholes pricing against Roll-Geske-Whaley pricing.

Simple adjustment and Merton's continuous dividend yield modification imply that the probability of early exercise is

essentially zero. On the other hand, pseudo-American valuation implies either certain early exercise or no early exercise. But as the probability of early exercise can take any value in the closed interval of 0 to 1, the results about the pricing error and its relationship to the features of the option will be affected to varying extents.

Whaley(1982) found that when the estimated Roll-Geske-Whaley model price is used, the striking price bias and the time to expiration bias get almost eliminated, and the variance bias is reduced, but not eliminated. Sterk(1982) confirmed Black(1975)'s finding that the Black-Scholes underprices (overprices) out-of-the-money(in-the-money) options, and found the Roll-Geske model to reduce the striking price bias⁵². Sterk's finding is in conflict with MacBeth and Merville(1979,1980)'s findings, and he attributed this to the larger number of firms used in his study(his 63 versus MacBeth and Merville's 6). Sterk's other finding was that 27 of the 182 options used in the study showed strong probability of early exercise, which could have offset the bias found by MacBeth and Merville.

But Sterk(1982) did not show the extent of better performance by the Roll-Geske-Whaley model over the pseudo-American version of the Black-Scholes. Sterk(1983) attempted to accomplish this task by using the same data as in

⁵² Sterk(1982) used the version of American valuation model prior to the corrections of Whaley(1981), while Sterk(1983) used the corrected version.

Sterk(1982), and developing a dividend measure which would incorporate the determinants of early exercise(dividend size, moneyness, time interval between the ex-dividend time and the time of contractual expiration)⁵³. As expected, he found that the Roll-Geske-Whaley model performs significantly better than the pseudo version of the Black-Scholes as dividend size increases. For small dividends, the performance of the Roll-Geske-Whaley model is not significantly better ⁵⁴. For probability of early exercise between 0.3 and 0.68(the dividend measure between 2.5 and 4), the Roll-Geske-Whaley model performs significantly better than the pseudo-American version of the Black-Scholes.

Blomeyer and Klemkosky(1983) compared both the simple stock price adjustment version of the Black-Scholes and the Roll-Geske-Whaley model prices(estimated with weighted ISD, or WISD) with the actual market prices, using transaction data for CBOE options on 18 stocks over the period 1977-1978 ⁵⁵. Examining the graphical relationship between the two measures

⁵³ The rationale for this composite measure as advanced by Sterk is that it is difficult to separate out the effects of the various determinants. According to Sterk's dividend measure, an option with average dividend, average value for the degree of moneyness, and average time interval between the ex-dividend time and the time of contractual expiration, will have a measure 1.

⁵⁴The Roll-Geske-Whaley model performs significantly better for dividends exceeding one dollar.

⁵⁵WISD from the Black-Scholes was used in estimating both Black-Scholes and Roll-Geske-Whaley model prices, thus biasing estimated Roll-Geske-Whaley model prices upward.

(degree of moneyness $\left[\frac{S}{XB(T)} - 1\right]$, and relative prediction error $\left[\frac{C(\text{market}) - C(\text{estimated model})}{C(\text{estimated model})}\right]$) suggested by MacBeth and Merville(1979), Blomeyer and Klemkosky concluded: the two models have identical pricing bias characteristics; both models tended to underprice out-of-the-money options and price fairly well the at-the-money and in-the-money options. The out-of-the-money bias is in conformity with Black(1975) and Merton(1976), but contradicting MacBeth and Merville(1979).

Blomeyer and Klemkosky also performed ex-post hedging tests with the two models. Both models succeeded in identifying under-valued and over-valued calls, and the ex-post performance between the two models was not statistically significant.

Thus Blomeyer and Klemkosky concluded that the observed mispricing of the Black-Scholes is not dividend-induced, a conclusion which is in conformity with Gultekin, Rogalski, and Tinic(1982), but somewhat in conflict with Whaley(1982) and Sterk(1982, 1983).

Geske and Roll(1984) discussed why the earlier studies failed to detect the expected dominance of the Roll-Geske-Whaley model. They found an inverse relationship between ISDs solved from the Black-Scholes and the striking price in their Monte Carlo, as was also suggested by MacBeth and Merville(1979) , for the no-early-exercise case⁵⁶. But the relationship reversed for

⁵⁶ MacBeth and Merville's finding was for simple adjustment version of the Black-Scholes, which essentially implies zero probability of early exercise.

the certain early exercise case. Geske and Roll therefore suggested that, depending on whether the probability of early exercise is close to zero or one, one may find that the striking price bias reverses.

Geske and Roll examined the data on dividend yield of S & P 500 composite and the yield to maturity of short term bonds over the period 1976-1978, which covered the period considered by MacBeth and Merville. The inverse bias was detected when the average dividend yield was low, and thus the probability of early exercise was probably low.

Geske and Roll also pointed out that the assumption of correctness of at-the-money pricing by the Black-Scholes is untenable, since the Black-Scholes(pseudo-American version) will underprice all options on dividend-paying stocks irrespective of their moneyness.

MacBeth and Merville(1979) excluded the options whenever significant probability of early exercise was detected, thus biasing their sample towards no early exercise options, and ultimately finding the reverse striking price bias.

Geske and Roll forced the Black-Scholes price to equal to the Roll-Geske-Whaley model price for at-the-money. But previous tests did not have this forced equality. Thus all Black-Scholes prices would be lower than Roll-Geske-Whaley model prices in those studies, but the in-the-money options will be most underpriced because of their higher probability of early exercise.

Geske and Roll also suggested that dividend uncertainty may lead to option premia decreasing in volatility rather than the usual increasing pattern. Thus, the Roll-Geske-Whaley model will not be able to eliminate the variance bias.

SECTION 4

We give below a summary of the nature of the empirical works for which we have given a brief description.

A. Use of ISD (with different weighting schemes by different authors) has been more popular than the historical variance rate estimator.

B. Almost invariably, model prices were estimated by plugging in the estimate of the variance rate in the valuation formula.

C. With the exception of Black and Scholes (1972), data used for empirical testing were data for UACs.

D. The tools used for reaching conclusions were:

(i) Scatter diagrams: See, for example, MacBeth and Merville (1979, 80), Geske and Roll (1984), Sterk (1982).

(ii) Ordinary Least Squares (OLS) regression:

(a) To estimate ISD: MacBeth and Merville (1979), Sterk (1982, 1983), Whaley (1982), Emanuel and MacBeth (1982).

(b) To estimate CEV elasticity parameter: MacBeth and Merville (1980).

(c) To compare performance of ISD (or WISD) versus historical variance rate estimator: Latane and Rendleman (1976), Chiras and Manaster (1978), Schmalensee and Trippi (1978)⁵⁷.

(d) To examine systematic risk of hedge positions: Black and Scholes (1972), Galai (1977), Blomeyer and Klemkosky (1983), Whaley (1982).

(e) To relate the mispricing of a model to variables of interest: Whaley (1982), Gultekin, Rogalski, and Tinic (1982), Geske and Roll (1984a), MacBeth and Merville (1979).

(iii) Simple comparison of means of the variables of interest: MacBeth and Merville (1979, 1980), stratified according to the degree of moneyness and the time to expiration, Blomeyer and Klemkosky (1983), stratified according to the dividend yield.

(iv) Mispricing in the form of forecast error: Emanuel and MacBeth (1982), root mean forecast error for options stratified according to the degree of moneyness and time to expiration.

(v) Nonparametric test: Sign test in Sterk (1983).

SECTION 5⁵⁸

Black and Scholes (1972) suggested that if the historical estimate of stock return volatility over the life of the option were known in advance, and used in the Black-Scholes formula to

⁵⁷ Schmalensee and Trippi (1978) used also Cochrane-Orcutt -Generalized Least Squares combination in the presence of autocorrelation.

⁵⁸ Part of our discussion in this section follows closely that given by Butler and Schachter (1984).

predict the actual option premium, the Black-Scholes would perform much better⁵⁹ ⁶⁰. Ceteris paribus, the performance of the model depends on the goodness of the estimate of stock return volatility. However, if the return distribution is non-stationary, then the historical estimator of volatility rate will be a biased estimator. It is also argued that the historical estimator does not incorporate information other than the series of past returns, which may be deemed relevant by the market in pricing the option. Latane and Rendleman(1976)'s ISD measure is designed to circumvent both of these alleged deficiencies of the historical estimator. ISDs are numerically solved out of an equation that equates the actual market price of an option to its model value, in our case, the Black-Scholes model price. Cox and Rubinstein(1985) interprets ISD as 'market's estimate' of volatility.

Let us start our discussion of ISD by exploring the investor behavior implicitly assumed. Based on the information set available at the time of trading, which may include the historical series on return, an investor would form an ex-ante estimate of the volatility rate. Investors, using these ex-ante estimates of volatility rates, would come up with estimates about the fair value of the option, probably using some familiar

⁵⁹ This ex-post variance rate is sometimes referred to as the 'actual variance rate' in the literature.

⁶⁰ Better in the sense of identifying the undervalued and the overvalued calls, smaller serial correlation, and lower variance of hedge portfolio's excess return).

model of fair valuation. In the market place, investor interactions then lead to a market price. One can ask, is ISD, solved from the market price formed in the above-mentioned way, the true volatility rate?

Given the stock price process, the true volatility rate could be a function of the stock price, time, and other parameters such as the elasticity parameter for the CEV diffusion process. It can be a constant as it is assumed by the Black-Scholes model.

If the stochastic process and its parameters were known, everyone would know the volatility rate, and also the model of fair valuation. Thus barring for idiosyncratic behavior of investors, the ISD should be the true volatility rate. Then, if the true process is constant volatility rate lognormal diffusion process, and other assumptions of the Black-Scholes model are valid, the ISD solved from the Black-Scholes formula should be the true volatility rate. And ISDs solved from different options on the same stock should not differ.

In general, the true stock price process and its parameters will not be known to either the investors or the researchers. So investors will form estimates of volatility rate on the basis of the information set available and their individual expectations. Then they will use some model of fair valuation which is deemed appropriate. Again, the interactions of the investors lead to a market price. But now, if we (researchers) use a particular model to solve for ISD, that ISD would merely be an estimate of the

true volatility rate of the true stock price process.

To simplify matters, let us assume that all investors agree about the model of fair valuation, even though the agreed upon model could be an inappropriate model. The investors could thus differ in their estimates of the fair value of the option, to the extent that they may differ in their estimates of the volatility rate or other parameters determining it. As researchers, if we use the same model as the investors did, the solved ISD would reflect market's estimate of volatility. What we are assuming in addition, is that the 'market's mind' plugs-in the forecasted volatility rate in the pricing function of the model.

If our purpose is to search for a model of fair valuation, the investigation should rather be directed towards identifying the true stock price process, given that we know other relevant factors, such as dividend payments, tax treatment, et cetera. On the other hand, our interest may be the investigation of whether a particular model is able to describe the observed market prices. Let this model be the Black-Scholes model.

Researchers, in the past, put ISD into two types of uses: (a) examining the behavior of ISDs as a direct test of the Black-Scholes validity; (b) using ISDs of options on a stock (on a given date or of a common maturity) to form an estimate of the volatility rate, which in turn is plugged into the Black-Scholes formula, the resulting formula estimate to be compared with the actual market price. If the market used the Black-Scholes

formula and plugged-in the market's forecast of the volatility rate, the ISDs of options on the same stock would exhibit:

(a) constancy, if the market used the same estimate of volatility rate for all options on the same stock; or

(b) systematic relationships to features of option, if the market changes its estimate of volatility rate in certain ways, depending on these features; or

(c) erratic variations, if the 'market's mind' chose estimates of volatility rate randomly for different options on the same stock.

Given that the market deemed the Black-Scholes model as the model of fair valuation, patterns (b) and (c) are rather untenable, since the Black-Scholes model uses a constant volatility rate for all options on the same stock. Cox and Rubinstein(1985) advanced two explanations of differing ISDs of options on the same stock:

(i) when the volatility rate is changing over time, options on the same stock, but differing in maturity, may not be expected to yield the same ISD;

(ii) if the calls are UACs, options with the same contractual life, but differing in moneyness, may have different effective lives, and thus ISDs would be different when the contractual life is used to solve for ISD.

But these explanations can be consistent with only the assumption that the market knowingly used the Black-Scholes model in situations where it is not the fair model of valuation.

This assumption would appear unacceptable.

There may be another explanation, namely, that the market simply used some other model in valuing the options. In that case, the Black-Scholes is misspecified and would cause ISD to exhibit systematic tendencies.

So far, we have assumed that the market plugged-in its forecast of the volatility rate in the pricing function given by a theoretical model of fair valuation. But the existing models of call valuation, including the Black-Scholes model, are all nonlinear in the volatility rate itself or the parameters determining it. Hence, even plugging-in unbiased estimate of the volatility rate or its parameters in the model's pricing function would only produce biased estimate of the model price. Thus to entertain the idea that the market plugged-in the estimated parameter(s) in a nonlinear pricing function would be equivalent to admitting the possibility of the market being systematically off the price given by the model which it thinks is the model of fair valuation. To avoid such possibility, one can assume that the market would use an unbiased estimator of the model price deemed to be the fair value.

Butler and Schachter(1983a) proposed an estimator for the Black-Scholes model price which they claim to be approximately unbiased. If it is truly so, it would not exhibit any systematic deviation from the Black-Scholes model price, while the formula estimator would. If the market considers the Black-Scholes model price to be the fair value, it may be thought that the

Butler-Schachter estimator was used by the market in an effort to establish the fair value on the average. Under such circumstances, the ISDs harvested from the (Black-Scholes) formula would exhibit systematic tendencies. The reason is that the market used an unbiased estimator of the model price, and the formula produces biased estimate of the latter.

If the market uses an estimator such as Butler and Schachter's, then the latter must show superiority over the usual formula estimator in that it always has lower bias, lower standard deviation, and negligible systematic tendency. However, our Monte Carlo study fails to establish such superiority for the two alternative estimators considered in chapter 4, one being the Butler-Schachter estimator.

In light of the above discussion and if we believe that an efficient market cannot be systematically off the fair value of an option, we need to assume that the market comes fairly close to knowing the fair value. From the viewpoint of a researcher, the market price could be taken to represent the fair value of the option, allowing for a random error. Does this mean that, the ISDs from the Black-Scholes formula would show no systematic variation? That the answer is 'no' can be visualized from an example ⁶¹. Let the market price be:

$$y = \ln(\sigma)(S/X) + \epsilon$$

where $\ln(\sigma)(S/X)$ is the fair value, and $E(\epsilon) = 0$.

⁶¹The example is for the purpose of exposition only. It should not be considered as a valid option pricing function.

Now, we have:

$$ISD = \exp(yX/S), \text{ and } E(ISD) = \sigma E[\exp(\epsilon X/S)].$$

The bias of ISD would be $\sigma[E\{\exp(\epsilon X/S)\}-1]$. It will, in general, not be equal to zero, and more importantly, would depend on the moneyness S/X and σ . Though, the Black-Scholes formula is much more complicated than our example, the qualitative result would be similar.

Thus, the ISDs solved from the Black-Scholes formula are expected to exhibit systematic tendencies even if the Black-Scholes model is the model of fair valuation. Therefore, examining the behavior of ISDs in order to judge the validity of the Black-Scholes model would be inappropriate. As for the other use of ISDs, namely, that of combining them to form an estimate of the volatility rate, plugging-in this estimate into the formula, and comparing this formula estimate with the market price, could be even less appropriate, since, in fact, a biased estimate of the volatility rate would be used in the formula.

In addition to the abovementioned problems, the methods of using ISDs of options on a stock in forming an estimate of the volatility rate, have come under criticisms. Practitioners of ISD have applied weighting functions to the ISDs of options on the same stock to arrive at a final estimate of the volatility rate of the stock. As indicated by Chiras and Manaster(1978), pp 216, Latane and Rendleman(1976)'s weighted average is not truly

a weighted average ⁶². It is also biased towards zero, the bias being directly related to the number of options used in weighting. Also, Schmalensee and Trippi(1978) ,pp132, pointed out that Latane and Rendleman(1976)'s weighting tends to attach little weight to short (time to)maturity options, and deep-in-the-money or deep-out-of-the-money options.

Trippi(1977) and, Schmalensee and Trippi(1978) used equal weighting for all options(the latter paper weeds out short maturity, far in-the-money or out-of-the-money options) on a stock. Latane and Rendleman(1976) argued that such an average would be unreasonable , given that the sensitivity of the option prices differ depending on other characteristics of the options.

Chiras and Manaster(1978) notes that Latane and Rendleman(1976)'s weighting by partials (of option price with respect to the volatility rate) ignores the size of investment(level of option price) effect, and suggest the weighting by elasticity instead ⁶³.

MacBeth and Merville(1979) recalls that Black(1975)'s result implied that the Black-Scholes formula will approximately correctly price an at-the-money option with time to maturity greater than 90 days. They suggest the regression of ISDs on the degree of moneyness and taking the estimated intercept as the estimate of the stock's volatility rate. Two objections have

⁶²Weights do not sum to unity.

⁶³For more criticisms of weighting, see Butler and Schachter (1983).

been advanced against this technique: (a) Butler and Schachter(1983a): ISD's bias is not linearly related to the degree of moneyness. (b) Blomeyer and Klemkosky(1983): According to the assumed linear relationship, out-of-the-money options would have lower ISD than the intercept would indicate. Thus the use of the intercept estimate as the estimate of the volatility rate would bias the out-of-the-money prices upward, increasing the extent of overpricing (by the Black-Scholes formula) for out-of-the-money options.

In addition to the above, the intercept estimator of MacBeth and Merville would suffer from the limited availability of the type of options to be used.

Whaley(1982) proposes a nonlinear optimization procedure to estimate the volatility rate from the ISDs of options on a stock and the first partials of option premia with respect to the volatility rate. He also controls for time to maturity by estimating one volatility rate for each time to maturity (on each day, for each stock). Whaley's procedure overcomes many of the problems associated with ISD, but still is subject to the following criticisms: (a) a model is assumed valid in its entirety, while the historical estimator requires only the validity of the specification for the stock price process; (b) econometric problems of estimation have been overlooked ⁶⁴.

⁶⁴The lack of orthogonality of regression errors to the regressor is an example.

Moreover, Butler and Schachter(1983a) found that the use of the Black-Scholes formula with an estimated variance rate may lead to serious mispricing for an at-the-money option. Thus estimating the volatility rate assuming the at-the-money correctness of the Black-Scholes formula pricing is questionable.

SECTION 6

It is surprising that regression has been used as a primary vehicle of investigation only in Whaley(1982), Gultekin,Rogalski and Tinic(1982), and Geske and Roll(1984a), with MacBeth and Merville(1979)'s use of regression much less emphatic. The apparent lack of popularity of this powerful, yet simple, technique can probably be traced to the following.

With the exception of Geske and Roll(1984a), all studies undertaken to date are based on pooling of time series and cross-section data. The otherwise simple technique of least squares regression becomes little more complicated in such cases. In particular, when the time series on an option is expected to have serial correlation, correcting for serial correlation in addition to the pooling problem may become economically and computationally burdensome. Moreover, one will reasonably raise the issue of nonstationarity(at least of the stock return variance rate) of inputs to the regression problem.

In light of the above, what then will be the rationale for the use of regression in addressing the basic problem of

investigating the validity of a valuation model, its strength against a competing model, and the nature and sources of mispricing, if any, by a model. To provide a rationale, we need to notice the following.

In the empirical studies, attempts are made to relate the observed mispricing to various features of the option, e.g., the degree of moneyness, time to expiration, volatility, probability of early exercise or its determinants, et cetera. Though the results are conflicting some times, existence of systematic relationships between observed mispricing by a model and the state of one or more factors have been evidenced. For example:

(i) Black(1975) reported that the Black-Scholes underprices (overprices) deep-out-of-the-money (deep-in-the-money) options.

(ii) MacBeth and Merville(1979) reported that the Black-Scholes overprices (underprices) out-of-the-money(in-the-money) options.

(iii) Geske and Roll(1984) reported that depending on the probability of early exercise, either of the above two cases may occur.

(iv) MacBeth and Merville(1979) also reported that the extent of mispricing for a specific degree of moneyness depends on the time to maturity⁶⁵.

It is these latter two types of evidences which bear the seed of rationale for the use of regression as a primary vehicle

⁶⁵See MacBeth and Merville(1979), pp.1183.

of investigation. What is suggested is that to assert a relationship between one of the factors and the observed mispricing, we need to control for the others, which the (multiple) regression is designed for. So far in the literature, controls in the form of control for a single factor has been more popular. For example, see Galai(1977), Emanuel and MacBeth(1982), Chiras and Manaster(1978) attempting to control the dividend impact, though the soundness of its implementation is in doubt⁶⁶. If multiple explanatory factors covary with each other and all of them are relevant, then the lack of control for one of the factors will lead to biased results about the impacts of all others ⁶⁷.

We have another compelling reason to resort to the multiple regression technique. Studies to date based their conclusions on considering the formula estimates, while disregarding the fact that the the biases observed can essentially be the formula's nonlinearity bias (with respect to the model). Another possibility is that the observed biases are due to model misspecification over and above the nonlinearity bias of the formula. If the model is valid, the misspecification error would vanish. As indicated earlier, the model price then can be thought as the price of the option, which will on the average, prevail in the market.

⁶⁶MacBeth and Merville(1979) institute controls for two factors through multiple regression.

⁶⁷See Kennedy(1980) for a lucid discussion about this issue. Kennedy introduces the use of Venn diagrams in this context.

These considerations lead one to believe that the difference between the estimated model value and the actual market price may contain several components:

- (1) model misspecification;
- (2) bias of the estimated formula price with respect to the true model price;
- (3) random (zero-mean) disturbance.

Regression technique will allow us an opportunity to investigate these components and lead us to devise a test of model validity.

We begin by reviewing some important regression studies by Whaley (1982), Gultekin, Rogalski, and Tinic (1982), and Geske and Roll (1984a).

Whaley (1982) compared the pricing bias of (i) Black-Scholes valuation with simple dividend adjustment, (ii) pseudo-American valuation, and (iii) Roll-Geske-Whaley (RGW) valuation. Data used were the weekly closing prices of all call options (with single dividend prior to contractual maturity) on 91 stocks over the period January 17, 1975 through February 3, 1978. The factors to which the pricing biases were related through simple regressions are volatility rate, degree of moneyness, time to contractual expiration, probability of early exercise, and dividend yield on the underlying stock. The measure of mispricing used is the pricing error of the estimated model value relative to

itself ⁶⁸.

Whaley performed 160 cross-sectional simple regressions for six regressors (and thus six equations), and Student's t-test on the time average of coefficients⁶⁹. The test results of Whaley(1982) may be summarized as:

(1) Contrary to previous evidences, striking price bias for none of the three valuation models is statistically significant. But the coefficient is smaller for the RGW model.

(2) The bias with respect to the probability of early exercise is significant for both versions of Black-Scholes, the RGW completely eliminating this bias, as expected. The pseudo-American version does better than the simple stock price adjustment version.

(3) Statistically significant inverse relationship between the time to expiration and the relative prediction error of the simple stock price adjustment version exists. Coefficient for the pseudo-American version is smaller, and the coefficient for the RGW model, still smaller and statistically insignificant(at 5% significance level).

⁶⁸The rationale for using the relative prediction error was advanced as : it will reduce the influence of hetroskedasticity. Geske and Roll(1984a) argued back that relative prediction error is very sensitive to whether the option is in- or out-of-the-money, since out-of-the-money model prices are usually very low.

⁶⁹The technique is somewhat similar to the Fama-MacBeth procedure used in the empirical testing of the Sharpe-Lintner-Mossin Capital Asset Pricing Model.

(4) For both versions (simple dividend adjustment and pseudo-American) of the Black-Scholes model, the coefficient of the dividend yield is positive and statistically significant. For the RGW model, the coefficient is not significant.

(5) The variance bias is direct and significant for all the three models. Thus the RGW model succeeds in eliminating all the biases except for the variance bias, indicating it may not be a dividend-related problem at all.

Whaley also reported that multiple regressions using various combinations of regressors did not change the above simple regression results in any significant way. But the multiple regression results were not presented in the paper.

We shall now turn to Gultekin, Rogalski, and Tinic (1982) who used multiple regression to investigate the pricing bias of the RGW model. Weekly data on 36 call options (25 CBOE and 11 AMEX) during 1975-1976 was used. In total, there were 1296 pooled time series cross-sectional observations. The measure of mispricing used was the dollar difference (or the absolute dollar difference in some of the regressions), and the measure of moneyness was $P-X$ (or a dummy variable assuming the value of 1 for $P < X$ in some of the regressions). The regressors were the measure of moneyness, the time to expiration, the historical estimate of the volatilities ⁷⁰. The test results of Gultekin, Rogalski, and Tinic (1982) may be summarized as:

⁷⁰Historical estimates were estimated from the most recent 6 months' observations.

(1) RGW tends to underestimate(overestimate) the market price for out-of-the-money(in-the-money) options⁷¹.

(2) Irrespective of the moneyness of the option, as the time to expiration increases, the dollar deviations tend to decrease⁷². This finding is in conflict with Whaley(1982)'s finding, but similar to Geske and Roll(1984a)'s.

(3) In conformity with Whaley(1982), the RGW overprices(underprices) options on high(low) estimated variance stocks⁷³.

What is noticeable in the results of Gultekin, Rogalski, and Tinic(1982) is that the RGW could not eliminate the striking price and the time to expiration biases as Whaley(1982) reported.

Regressions using the absolute value of the absolute prediction error as the regressand indicate that the prediction errors are larger for in-the-money options and options on high estimated variance stocks.

In comparing the results of Whaley(1982) and Gultekin, Rogalski, and Tinic(1982), we should bear in mind that their measures of mispricing are different.

⁷¹This reporting is on the basis of conditional frequency distribution of average dollar deviations and multiple regression. See Gultekin, Rogalski, and Tinic(1982), pp.64-65.

⁷²This result is based on multiple regression. See Gultekin, Rogalski, and Tinic(1982), pp.65.

⁷³This reporting is based on rank correlation of dollar deviations and estimated stock return volatilities, and multiple regression. See Gultekin, Rogalski, and Tinic(1982), pp.63 and 65.

Geske and Roll(1984a) used⁷⁴ transaction data of 667 CBOE options on 85 stocks, the date of transaction being randomly chosen as August 24, 1976. Historical variance rates were estimated using the 6 months' data prior to the chosen date. Geske and Roll(1984a)'s study is free from any time series related problem, except for the nonstationarity of the variance rate. They ran two cross-sectional regressions, one for the whole sample, and the other for the subsample of 119 options(on 28 stocks) which did not have any dividend payments prior to contractual maturity. The regression equation looks like:

$$C-CB(\theta) = \beta_0 + \beta_1 \ln(S/X) + \beta_2 T + \beta_3 \hat{\sigma}$$

Comparing the statistical significance of coefficients in the two regressions, Geske and Roll(1984a) concluded that the striking price bias and the maturity bias are essentially dividend-induced biases⁷⁴. While the variance bias does not appear to be dividend-related, Geske and Roll conjectured that the variance bias is a measurement error problem, and could be redressed if James-Stein estimator, instead of the historical estimator, is used for estimating the volatilities. James-Stein estimator for the simultaneous estimation of several variance rates pulls upward (downward) the smaller(larger) historically estimated variance rates, and thus alleviates the errors-in-variable problem, originally mentioned by

⁷⁴The variance bias and the striking price bias are in conformity with Whaley(1982) and Gultekin, Rogalski, and Tinic(1982), but the maturity bias is in conflict with Whaley(1982).

Black-Scholes(1972). Using the James-Stein estimator for the variance rates, Geske and Roll(1984a) found that the variance bias got eliminated ⁷⁵.

We level the same criticisms as applied to Gultekin,Rogalski, and Tinic(1982). We plan to examine the results of Geske and Roll(1984a) more closely in a later section.

Some notable deficiencies of the existing regression studies can be summarized as:

(1) Linear regression models have been postulated and estimated, without justification.

(2)The existing option pricing models including the Black-Scholes model are nonlinear in the volatility rate or parameters determining it. Thus, if we replace the true parameters even by their unbiased estimates, the resulting formula estimate will be a biased estimate of the formula price with true parameter values, the latter price being referred to as the model price. No attention has been paid to this nonlinearity bias of the valuation formulae with respect to the volatility rate, and its probable impact on the estimate of

⁷⁵ By adjusting the variance rates towards the grand average, essentially the variability of the volatility regressor is drastically reduced. This leads to a larger standard error of the volatility coefficient in the regression, and thus to smaller t-value for the least squares estimate of the coefficient. Thus the elimination of the variance bias by using the James-Stein estimator of variance rates could just be a regression artifact without economically viability.

coefficients ⁷⁶.

(3) Estimated volatility rate has been used as a regressor without ever noticing the probable impacts on the least squares coefficient estimates. The least squares estimates of coefficients will be biased and asymptotically biased, though these biases may be negligible under certain conditions ⁷⁷.

(4) Care has not been taken towards examining the nature of the disturbance term in the regression⁷⁸. The disturbance may have economically meaningful components affecting the estimates of regression coefficients, as our discussion in the next chapter and in chapter 5 indicates. We also find the errors to be heteroskedastic, in general⁷⁹.

(5) Coefficients across estimated equations were compared without an individual or a joint test for statistical significance of the difference between them. We intend to undertake such test(s) in chapter 6, in the context of investigating the dividend inducement of the systematic biases of estimated Black-Scholes prices.

⁷⁶In chapter 4, the problem of nonlinearity bias will be discussed extensively.

⁷⁷A discussion of this stochastic regressor problem is forthcoming in chapter 5.

⁷⁸ Bhattacharya (1980) recognised the problem of heteroskedasticity in Galai (1977)'s regression of excess hedge returns on market returns, while Schmalensee and Trippi (1978) attempted to deal with serial correlation in a different context.

⁷⁹ Whether the problem is such that it warrants the use of more complicated techniques such as the Estimated Generalized Least Squares, is a debatable question and we leave it for future research.

SECTION 7

So far in our discussion, we did not differentiate between the types of statements used by authors to indicate their findings about systematic relationships of model mispricing or formula estimate's mispricing to various factors. To the best of our knowledge, and to our surprise, this seems to have gone unnoticed in the literature. Consider the following statements:

A. "...the extent to which the B-S model price exceeds the market price decreases as the expiration date approaches." (MacBeth and Merville(1979),pp.1184)

B. "Options that are way out of the money tend to be overpriced, and options that are way into the money tend to be underpriced." (Black(1975),pp.64).

Statement A gives one the flavor of a continuous functional relationship between mispricing and a factor(time to maturity here), and we will refer to such statements as of the functional bias type. Statement B, on the other hand, reveals no more than the reversal of the sign of mispricing for two particular groups of options, and such statements would be referred to as of the type dichotomous bias.

That adequate differentiation is not made in comparing results of different authors breed as much uneasiness as the following types of attempts:

a. To base statements of one type on techniques best suited for the other type. For example, based on the mean dollar deviations of broad categories of options, MacBeth and

Merville(1979),pp.1182, states: "...as τ approaches zero so does y ...," where τ is the time to expiration and y is the dollar deviation.

b. To state unqualified equivalence of the two types of biases. For example, consider Whaley(1982),pp.48, "...there exists a significantly negative relationship ...(i.e., it underprices short-lived options and overprices long-lived options)... ." Also, consider Gultekin,Rogalski and Tinic(1982),pp 65, "...since $(P_S - P_E)$ would be negative for out-of-the-money options, the formula tends to overestimate their values." ⁸⁰ That such equivalences do not necessarily follow, can be seen from a hypothetical sample of options in figure 2.1 . For this sample, functional time to maturity bias would be negative, but the sign reversal of a dichotomous bias statement is not present(since both short and long maturity options would be on the average overpriced).

It appears that broad classification technique may be more appropriate for dichotomous bias type statements, and regression technique would be more suitable for functional bias type statements. On the basis of the technique used, some of the important findings can be categorised as follows: ⁸¹

Striking Price Bias

⁸⁰The latter statement is on the basis of the sign of regression coefficient estimate.

⁸¹Some authors used both classification and regression techniques, and/or issued both types of statements.

Black(1975) dichotomous

MacBeth and Merville(1979) dichotomous and functional

Whaley(1982) functional

Sterk(1982) dichotomous

Gultekin,Rogalski, and Tinic(1982) dichotomous and
functional

Geske and Roll(1984a) functional

Time to Maturity Bias

Black(1975) dichotomous

MacBeth and Merville(1979) dichotomous and functional

Whaley(1982) functional

Gultekin,Rogalski, and Tinic(1982) functional

Geske and Roll(1984a) functional

Estimated Variance or Volatility Rate Bias

Black and Scholes(1972) dichotomous

Whaley(1982) functional

Gultekin,Rogalski, and Tinic(1982) dichotomous and
functional

Geske and Roll(1984a) functional

SECTION 8

MacBeth and Merville(1979),pp.1185, emphasized that their finding of inverse (both dichotomous and functional) striking price bias is exactly opposite to that of Black(1975)'s direct(dichotomous) striking price bias. Geske and Roll(1984) tried to explain this reversal in terms of the effect of early

exercise possibility of unprotected American calls. But let us now see how the dichotomous striking price bias could be affected by mere variation of the sample mixture of options.

Let us assume that the Black-Scholes model is valid. As reported by Boyle and Ananthanarayanan(1977), and Butler and Schachter(1983,1983a), and to be discussed in our chapter 4, at-the-money and around-the-money options will be underpriced by the formula with an estimated volatility rate, and the deeper-away-from-the-money options will be overpriced. Suppose that the sample is such that the proportion of near-out-of-the-money (near-in-the-money) among the out-of-the-money (in-the-money) options is much larger(smaller) than 0.5. We would then expect the direct dichotomous striking price bias of Black(1975) to emerge. If the term larger(smaller) is replaced by smaller(larger), we would end up with the inverse dichotomous striking price bias similar to MacBeth and Merville(1979).

On the other hand, if, for example, Merton(1976)'s jump-diffusion model is valid, the misspecification error of the Black-Scholes model alone would reverse the dichotomous biases in our example. It is not, of course, clear whether and to what extent the nonlinearity bias of the formula would actually neutralize the misspecification bias.

We now turn to another problem associated especially with the dichotomous bias type statements, viz., that of the arbitrariness of the dichotomy.

In our Monte Carlo study in chapter 4, we find that the dichotomous striking price bias of Boyle and Ananthanarayanan (1977) holds irrespective of the levels of time to maturity and/or true variance rate. At first, this robustness might appear to nullify our criticism of the lack of control for other factors when relating the mispricing to any one factor. But a more careful consideration would reveal that the problem lies in the arbitrariness of the dichotomy deeper-away-from-the-money versus around-the-money. The results of Boyle and Ananthanarayanan(1977), Butler and Schachter(1983,1983a) indicate the existence of two points of moneyness, one in the in-the-money and the other in the out-of-the-money range, at which the expected mispricing would be zero, when the Black-Scholes model model is valid. Merton(1976) have also indicated the existence of two such points for the pure model misspecification bias of the Black-Scholes model with respect to the jump-diffusion model. The definition of closeness conformable to the dichotomy of underpricing versus overpricing, found by the above authors, appear to be the one which would term as closer-to-the-money (or around-the-money) those options having moneyness between the two points of zero bias. Then, at least, given the validity of the Black-Scholes model, whether an option is around-the-money(thus underpriced on the average) or deeper-away-from-the-money (thus overpriced on the average) would depend on the levels of time to maturity and/or the true variance rate, since these factors affect the boundaries of

closeness. Thus an option with 0.6 as the ratio of stock price to the present value of the striking price, may appear deep-out-of-the-money, though, according to the aforementioned definition of closeness, it could be near-out-of-the-money, if the time to maturity and/or true variance rate is rather high. If this option is actually found to be underpriced by the estimated Black-Scholes formula relative to the market price, one not caring for the above definition of closeness may be inclined to say that deep-out-of-the-money option is underpriced by the Black-Scholes model. If the Black-Scholes model is valid, a near-out-of-the-money option will, in fact, be underpriced by the formula estimate.

We cannot also rule out the possibility that Merton's jump-diffusion model is valid. The zeros of pure model misspecification bias of the Black-Scholes model could be such that a truly deep-out-of-the-money option is, in fact, being underpriced with the model misspecification error overcoming the nonlinearity bias of the Black-Scholes formula estimate.

The above discussion sheds doubt on the interpretation of Black(1975)'s widely discussed finding that "...way out of the money tend to be overpriced, and ...way into the money tend to be underpriced."⁸² Unless one knows the true variance rates of the associated stocks, one cannot really say whether the overpriced(underpriced) options were truly way out of(way into)

⁸²To recall, Black's overpricing is our underpricing, and vice versa.

the money. It then, appears that the striking price bias reversal reported by MacBeth and Merville(1979,1980) may not be a reversal at all, for at least two reasons: (a)Black's dichotomy is way out versus way in, while MacBeth and Merville's is out versus in; (b)it is not clear whether Black's way out(in) was truly way out(in).

The technique of grouping into broad classes of options and then comparing the mispricing across groups seems to be in a methodological dilemma. The impact of the sample mixture of options indicates the necessity of finer grouping. The arbitrariness of dichotomy, on the other hand, seems to frustrate such an attempt. Functional bias approach, supported by an appropriate regression model has, at least, the potential of breaking out of such a dilemma.

SECTION 9

Under the functional bias approach, one tries to associate changes in mispricing with changes in some factors such as the degree of moneyness, time to maturity, stock volatility rate , etc. Thus the functional bias, being marginal in nature, necessitates simultaneous control of other factors when the impact of variation in any one factor is being considered. As we have indicated earlier, multiple regression technique is suitable for this purpose. But in using multiple regression technique, more care is called for than just regressing a measure of mispricing on several factors, as the most of the

existing users in empirical options literature have done.

First of all, if the intention behind using the multiple regression technique by authors such as MacBeth and Merville(1979) or Geske and Roll(1984a), were to investigate the signs and magnitudes of marginal biases(with respect to different factors), let us check whether they did so investigate.

Implicit in the results of Boyle and Ananthanarayanan (1977), Merton(1976a), and explicitly discussed in our chapter 3, the marginal bias (or mispricing, which is the negative of bias) with respect to a factor is a function(possibly nonlinear) of the levels of all relevant factors including the factor in question. This implies that coefficients estimated in a constant coefficient estimation, can hardly be interpreted as estimates of marginal biases. What is required for the said purpose is fully option-specific estimation of an appropriate nonlinear regression equation. In addition, the econometric environment of estimation has to be evaluated.

Note that fully option-specific estimation of an appropriate nonlinear regression equation would be free from both the problems of sample mixture of options and arbitrariness of dichotomy, as faced by the technique of grouping and comparison of mispricing statistics across groups.

We derive an option-specific nonlinear regression equation in the next chapter. Given the enormous complexities of fully option-specific estimation, one may want to limit the objective

to estimation allowing for polynomial approximation to the specificness of a coefficient with respect to the level of the corresponding factor only. We make a limited attempt of this nature in chapter 8.

As indicated earlier in this chapter, and to be taken up in chapter 5, a frustrating and rather unavoidable stochastic regressor problem arises whenever stock volatility rate is included as a regressor, since the estimates of volatility rates are subject to measurement error. In addition, there exists the problem of heteroskedastic error term⁸³. Thus, the prospect for acceptably accurate and sound investigation of marginal biases does not appear promising. However, there remain two areas of investigation that appear less troublesome. They are:

(i) Testing the dividend inducement of the alleged systematic mispricing by the estimated Black-Scholes formula. This we do in chapter 6.

(ii) Testing the validity of the Black-Scholes model against unspecified alternative through either testing restrictions on regression coefficients, or through allowing limited degree of option specificness in coefficients and then comparing predicted functional relation of mispricing to a factor with that implied by the validity of the Black-Scholes model. We shall pursue these in chapters 7 and 8.

⁸³The form of heteroskedasticity looks intractably complex even under second order Taylor series approximation to the nonlinearity bias of the formula.

CHAPTER 3:

AN ALTERNATIVE REGRESSION MODEL

In the previous chapter, while discussing the deficiencies of the ISD approach, we argued that from the viewpoint of a researcher, the market price of an option could be taken to represent its fair value, allowing for a zero-mean random error. The need for an appropriately defined multiple regression model was emphasized in order to undertake empirical investigation of whether the Black-Scholes model price represent the market price on the average, and hence the fair value of the option. The underlying rationale is that if it does so, the deviation of the model's price from the market price should not be systematically related to factors such as moneyness, time to maturity, or volatility rate. But the nonlinearity bias of the formula estimate poses a practical problem, namely, its use as a proxy for the Black-Scholes model price induces systematic relationships, which are independent of any model misspecification error. Moreover, the systematic relationships induced by the nonlinearity bias and any probable model misspecification error would in general be option-specific.

In our empirical survey, we have seen that the existing regression studies attempting to validate the Black-Scholes model empirically seem uninterested in these issues. The purpose of this chapter is therefore to present an alternative regression model which would embody explicit concern for these issues. It is also intended to help illuminate the limitations of the existing studies and the econometric environment of estimation. Though our exposition is in terms of validating the

Black-Scholes model, the basic framework can be applied to the validating other option pricing models as well.

Let us assume that the actual option prices are generated according to the model:

$$C_j = f(Z_j^T) + u_j \dots\dots\dots(3.1)$$

where

C_j : the actual market price for the j -th sample option

Z_j^T : row vector of nonstochastic treatment variables' observations which may or may not include unity as an element

$u_j \sim N(0, \sigma^2)$.

Notice two things:

(i) We are assuming that the model of fair valuation is the same for all options, whether written on the same stock or not.

(ii) The disturbance term which may incorporate the idiosyncratic behavior of market participants is not option-specific or stock-specific. And it does perturb the deterministic part in no systematic way.

Let us denote the Black-Scholes inputs for the j -th option as θ_j^T . The following locational shift of (3.1) would be useful:

$$C_j - CB(\theta_j^T) = f(Z_j^T) - CB(\theta_j^T) + u_j \dots\dots\dots(3.2)$$

In this form, the response variate is the dislocation of the Black-Scholes model price from the market price, and the

expected response conditional on $Z_j^T \cup \theta_j^T$ is $f(Z_j^T) - CB(\theta_j^T)$.

For arbitrary realisations of $Z_j^T \cup \theta_j^T$, the expected response in (3.2) would be zero, if Black-Scholes is the true model of fair valuation, i.e., if (i) Z_j^T and θ_j^T are of the same dimension and their difference is a null vector, and (ii) CB is identical function as f.

Note that if a researcher's interest is model selection vis-a-vis the Black-Scholes, $f(Z_j^T)$ could be thought as the competing deterministic model function (rather than the assumed true deterministic model function). Then $Z_j^T \cap \theta_j^T = \theta_j^T$ and $Z_j^T \cap \theta_j^T \neq Z_j^T$, where \cap denotes intersection, would imply nesting of the Black-Scholes model under the competing model. An example of such a $f(Z_j^T)$ would be the jump-diffusion model of Merton or Cox's CEV. If $Z_j^T \cap \theta_j^T \neq \theta_j^T$ and $Z_j^T \cap \theta_j^T \neq Z_j^T$, then we have a case of non-nested model selection, when f is treated as a competing model function. An example would be the pure jump model as f.

In this thesis, we would be dealing with only the case of Black-Scholes being nested (or not). And the competing model is unspecified. In other words, our framework is that of testing the Black-Scholes against an unspecified alternative, with the assumption that the alternative is not nested in the Black-Scholes model.

For developing a convenient regression model, we need to hypothesize a simple form for $f(Z_j^T) - CB(\theta_j^T) = g(Z_j^T \cup \theta_j^T)$. Sometimes, a nonlinear expected response function could be

conveniently expressed or approximated (by suitable normalisation and/or transformations) as the dot product of the gradient vector with respect to the treatment variables and the vector of treatment variables. An example would be the expected response function $E(Y) = X_1^a X_2^\beta$. For simplicity, we would assume that the case for $g(\cdot)$ is alike.

We can express (3.2) as :

$$C_j - CB(\theta^T)_j = B_j^T \Phi_j + u_j \dots \dots \dots (3.3)$$

where

$B_j^T \Phi_j$ is the assumed replacement for $g(\cdot)$

B_j^T is the row vector of suitable arguments chosen from $Z_j^T U_j \theta^T$ and covering $Z_j^T U_j \theta^T$.

Φ_j is the conformable (to B_j^T) column vector of coefficients.

Notice that the coefficient vector is subscripted. In general, the coefficients would depend on the realisation of treatment variables, whenever dealing with the Black-Scholes

⁸⁴Arguments, if chosen suitably, would make the interpretation of empirical results more tractable. For example, because r would be the same for all options (at least for the same maturity options), and option prices have been observed to be extremely insensitive to r , (see Cox and Rubinstein (1985), pp. 217), we might decide to drop r as an argument. Also, instead of dealing with S and X as separate arguments, S/X or $\ln(S/X)$ would make more sense in terms of interpretation.

model⁸⁵.

It is now evident from (3.3) that if Black-Scholes is the model of fair valuation, then for arbitrary realisation of B^T_j , conditional mean response ($B^T_j \Phi_j$) would be zero, only if Φ_j is a null vector. Thus, if B^T_j were known and nonstochastic in repeated sampling sense, a test of the validity of the Black-Scholes model would be the test whether $\Phi_j = 0$, a null vector.

There are two immediate problems we encounter.

First, the column dimension of B^T_j is unknown⁸⁶. Let us do the hypothetical partition $B^T_j = \{\theta^T_j, \delta^T_j\}$ and $\Phi_j = \{\Phi_{Bj}^T, \Phi_{\delta j}^T\}$, the latter partition being conformable to the former. To be consistent with our earlier assumption of unspecified alternative to the Black-Scholes model, it is only reasonable to use θ^T_j in the regression.

Second, both the response variate [$C_j - CB(\theta^T_j)$] and the treatment variable σ_j (included in θ^T_j as an argument) are observed with error. We are assuming, all other variables are measured without error.

⁸⁵See our discussion of nonlinearity bias in chapter 4.

⁸⁶Thus the dimension of parameter space is unknown.

Corresponding to the true regression model (3.3), the following would be the model with observed regressand:

$$C_j - CB(\theta^T_j) - CB(\hat{\theta}^T_j) = \theta^T_j \Phi_j - CB(\hat{\theta}^T_j) + v_j$$

$$\text{or, } C_j - CB(\hat{\theta}^T_j) = \theta^T_j \Phi_j + [CB(\theta^T_j) - CB(\hat{\theta}^T_j)] + v_j \dots\dots\dots(3.4)$$

where $v_j = u_j + \delta^T_j \Phi_j$.

In terms of equation (3.4), we can identify four different sources to explain why the estimated Black-Scholes formula price would differ from the actual market price:

- (a) u_j : The irreducible noise common to all stochastic models of market price generation.
- (b) $CB(\theta^T_j) - CB(\hat{\theta}^T_j)$: The error in estimation of the model price resulting from the volatility being measured with error.
- (c) $\theta^T_j \Phi_j$: The error of functional misspecification in the Black-Scholes model, even though the arguments are the same as in the unknown market's model of fair valuation.
- (d) $\delta^T_j \Phi_j$: The error of model truncation in the Black-Scholes model, or incompleteness in arguments of the Black-Scholes model.

If Black-Scholes is the true model of fair valuation, $\Phi_{\delta j} = 0$, a null vector, then (3.4) boils down to:

$$C_j - CB(\hat{\theta}^T_j) = CB(\theta^T_j) - CB(\hat{\theta}^T_j) + u_j \dots\dots\dots(3.5)$$

If we do the Taylor series expansion of $CB(\hat{\theta}_j^T)$ around $\hat{\sigma}_j = \sigma_j$, truncate after the second-order term for simplicity and tractability, then after some manipulations, we can arrive at the following : ⁸⁷

$$CB(\hat{\theta}_j^T) - CB(\theta_j^T) = E[CB(\hat{\theta}_j^T) - CB(\theta_j^T)] + \tilde{\epsilon}_j \dots \dots \dots (3.6)$$

where

$$\tilde{\epsilon}_j = 0.5 [\partial^2 CB(\theta_j^T) / \partial (\hat{\sigma}_j^2) | \hat{\sigma}_j = \sigma_j] \{ \text{Var}(\hat{\sigma}_j) - (\hat{\sigma}_j - \sigma_j)^2 \} - [\partial CB(\theta_j^T) / \partial (\hat{\sigma}_j) | \hat{\sigma}_j = \sigma_j] (\hat{\sigma}_j - \sigma_j)$$

and $\hat{\sigma}_j$ is an unbiased estimate of σ_j . ⁸⁸

Note that $E(\tilde{\epsilon}_j) = 0$, and $E[CB(\hat{\theta}_j^T) - CB(\theta_j^T)]$ is the negative of the bias of the estimated Black-Scholes formula price with respect to the Black-Scholes model price.

If we substitute (3.6) in (3.4), and at the risk of oversimplification, express $E[CB(\hat{\theta}_j^T) - CB(\theta_j^T)]$ as $\theta_j^T \beta_j$, then

⁸⁷The truncation is not essential to the derivation of the regression equation. If not truncated, the error term in (3.6) would be the sum of series of terms similar to $\tilde{\epsilon}_j$.

⁸⁸Sample standard deviation s is not an unbiased estimator of the population standard deviation. The unbiased estimator $\hat{\sigma}$ is equal to $s / [1 - \{1/4(n-1)\} + \{1/32(n-1)^2\}]$. Both s and $\hat{\sigma}$ are distributed as a constant times a χ variable with $n-1$ degrees of freedom, but differing in the constant. Here n denotes the number of stock return observations from which the historical variance rate estimate is calculated. The difference in the central moments of s and $\hat{\sigma}$ are negligible for large n .

the following equation emerges:

$$C_j - CB(\theta^T)_j = \theta^T_j \Psi_j + \epsilon_j \quad \dots\dots\dots(3.7)$$

where $\Psi_j = \Phi_j + \beta_j$

and $\epsilon_j = v_j + \tilde{\epsilon}_j$

If we denote the coefficients of σ as ϕ_j and β_j , then the final estimable form looks like:

$$C_j - CB(\theta^T)_j = \theta^T_j \Psi_j + \eta_j \quad \dots\dots\dots(3.8)$$

Where $\eta_j = \epsilon_j - (\phi_j + \beta_j) (\hat{\sigma}_j - \sigma_j)$

The error in the regression equation deserves some economic and econometric discussion. The j-th error term is:

$$\eta_j = u_j + \delta_j^T \Phi_j + \{ \partial CB(\theta^T)_j / \partial (\hat{\sigma}_j) |_{\hat{\sigma}_j = \sigma_j} \} [\{ d_1, d_2 / 2\sigma_j \}$$

$$\{ \text{var}(\hat{\sigma}_j) - (\hat{\sigma}_j - \sigma_j)^2 \} (\hat{\sigma}_j - \sigma_j)] - \psi_j (\hat{\sigma}_j - \sigma_j)$$

where d_1 and d_2 are as defined in chapter 1.

Clearly there are four components of the disturbance term from four different sources.

(i) u_j : The irreducible noise.

(ii) $\delta_j^T \Phi_{\delta j}$: The error from the omission of the relevant variables. As we have indicated earlier, when the Black-Scholes is complete in arguments, $\Phi_{\delta j} = 0$, a null vector.⁸⁹ Thus, this component will vanish when the Black-Scholes is complete in regressors.

(iii) $-\psi_j (\hat{\sigma}_j - \sigma_j)$: The error emerging from the measurement error in the regressor σ_j .

(iv) $\eta_j - u_j - \delta_j^T \Phi_{\delta j} + \psi_j (\hat{\sigma}_j - \sigma_j)$: This component is emitted by the nonlinearity bias of the Black-Scholes formula estimated with respect to the Black-Scholes model. Note that this error emerged when we expressed the difference between the estimated Black-Scholes formula price and the Black-Scholes model price in terms of the expected value of the difference (negative of the formula bias) and the sampling error in the difference.

Notice that if $\Phi_{\delta j} \neq 0$, then the effect of this component will partially (when δ_j^T and θ_j^T has non-zero covariance) get incorporated in the intercept term, when it is included. And thus the remaining of the composite error will have mean zero.

For the purpose of testing validity of the Black-Scholes model, the following observations may prove useful:

⁸⁹We are equating the arguments with the explanatory variables. Though in regression terminology, regressor is differentiated from explanatory variable, we would be less strict here and use these terms interchangeably, unless explicit differentiation is made.

(1) Even if the Black-Scholes is the true model of fair valuation, we might find the estimated coefficients to be significant. This would happen if β_j is non-null. Given the sensitivity of the Black-Scholes formula's bias with respect to the volatility, the coefficient of the volatility may come out significant even in a zero-dividend sample, where the Black-Scholes is least suspect.

(2) If the Black-Scholes is complete in arguments, but only functionally wrong, thus $\Phi = 0$, we may find the joint significance of the slope coefficients, not of the intercept if we include it.

(3) If the Black-Scholes is not the proper model, both functionally and in arguments, we may find both joint significance of the slope coefficients and significance of the intercept. ⁹⁰

The econometric environment of estimation is characterized by the following issues:

(i) The regression model is nonstationary across observations, since at least the nonlinearity bias of the Black-Scholes formula estimate is option-specific.

(ii) The design matrix would be stochastic in repeated sampling sense, and, hence the stochastic regressor problem emerges.

⁹⁰When we say functionally, we mean functionally for the common regressors. It is possible that whenever incomplete in regressors, also wrong functionally for the common regressors.

(iii) The composite error term η_j is heteroskedastic for two reasons:

(a) $\text{var}(\hat{\sigma}_j)$ may differ across options on different stocks;

(b) $\psi_j = \phi_j + \beta_j$ would vary across options.

The effects of these econometric problems are well-established, the cures are not, especially when the variance-covariance matrix of the disturbance vector is not known. In chapter 8, we partially address the first problem, and in chapter 5, the effects of the second problem in our context are discussed. The third problem is left for future research.

CHAPTER 4:
ON THE NONLINEARITY BIAS OF
THE BLACK-SCHOLES FORMULA ESTIMATE

In the foregoing chapter, we have seen that one of the components of the observed deviation between the market price of an option and the Black-Scholes price computed by inserting an estimate of the variance (or volatility) rate in the formula, is the nonlinearity bias of the formula estimate with respect to the Black-Scholes model price. In our regression model, the expected response function have, in fact, the negative of the nonlinearity bias as one of its components. For ease of exposition but at the risk of confusion, we will refer to the nonlinearity bias as the formula bias or formula mispricing(negative of the formula bias) interchangeably.

That even the use of an unbiased estimator for the variance rate leads to biased estimate of the model price, has been pointed out by Ingersoll(1976). This problem has been pursued in detail by Boyle and Ananthanarayanan(1977), Butler and Schachter(1983,1983a). This formula bias may assume significance as an issue, if one attempts to validate the Black-Scholes model empirically, and/or investigate the sources of observed mispricing by the formula estimate. In particular, as indicated in the foregoing chapter, it is possible to conclude that the Black-Scholes model misprices options in certain systematic ways, even though the Black-Scholes model price does represent the fair value of the option, the observed mispricing being induced by the nonlinearity bias of the formula.

Given the potential importance of the formula bias in empirical investigation of the Black-Scholes pricing, it seems

appropriate to examine carefully the nature of the formula bias, its implications for the empirical studies, and explore the existence of unbiased or approximately unbiased estimators of the Black-Scholes model price, and their properties, if they exist.

The remainder of this chapter is divided into eight sections. The Taylor series representation of the nonlinearity bias of the formula estimate is discussed in section 1. In section 2, we introduce the pseudo estimator, the bias of which would be identically equal to zero when the bias is approximated only up to a specific order of approximation. With a Monte Carlo, we compare the behavior of the pseudo estimator with that of the formula estimator, as the variance rate and the sample size varies.

Both the formula estimate and the pseudo estimate are formed by replacing the true variance rate with the estimated variance rate in the respective estimating functions. Butler and Schachter (1983a) proposed an estimator which first approximates the Black-Scholes price by a pair of Taylor series, which are linear in the powers of the variance rate, and then these powers are replaced with their unbiased estimates, rather than with the powers of the estimated variance rate. We review this estimator in section 3.

A legitimate concern with respect to such attempts as Butler and Schachter (1983a), is the validity of series representation of the Black-Scholes model price, in general, and

the validity of Taylor series representation around arbitrarily chosen points, in particular. These issues are our subject matter in section 4.

In section 5, we propose an approximately unbiased estimator which retains the spirit of the estimator proposed by Butler and Schachter(1983a), but tries to improve upon it in terms of logical consistency and validity of approximation, and sample size requirement. We undertake a separate Monte Carlo study to compare the performance of these two estimators with the formula estimator. In section 6, we consider an estimator, for which the parameters of a normal distribution function are to be estimated, the point of evaluation being known. This contrasts with the standard problem of considering estimators with known standard normal distribution function, but unknown point(s) at which to evaluate the function. The new estimator has the potential to be unbiased, rather than just approximately unbiased. But the information requirement to form an estimate in practical situations seems to limit its usefulness, at least, at this stage of our research.

Our Monte Carlo results did not find any of the alternative estimators to have clear superiority over the computationally simple formula estimator. So, in section 7, we explore further the nature of of the nonlinearity bias of the formula estimate. An explanation for the striking price bias found by Boyle and Ananthanarayanan (1977) is provided on the basis of second order approximation to the nonlinearity bias. Using Monte Carlo

results, we review the systematic relationships of the nonlinearity bias with moneyness and time to maturity of the option, and the true variance rate on the underlying stock's return. The marginal nature of these relationships are emphasized, along with the implications for the dichotomous bias and the functional bias approaches to model (Black-Scholes) validation.

The final section summarizes some of the results of this chapter.

SECTION 1

The nonlinearity bias of the formula arises when $E[g(x)] \neq g[E(x)]$, i.e., if the function g of the random variable x , is nonlinear in x .⁹¹ In our case, the unbiased estimate \hat{V} of the true variance rate V could be taken as x , and the Black-Scholes functional form CB as g . To see the nonlinearity bias more clearly, let us expand $CB(\hat{V})$ around the point $\hat{V}=V$,

⁹¹We are, of course, assuming the existence of the relevant expectations. Also, note that $\text{Plim}[g(x)] = g[\text{Plim } x]$, if the probability limit exist. Thus, a biased estimator can be unbiased in the sense of probability limit, or roughly speaking, asymptotically unbiased.

⁹²In terms of our previous notation, $CB(V)$ is equivalent to $CB(\theta^T)$, and $CB(\hat{V})$ is equivalent to $CB(\hat{\theta}^T)$. Since the only difference between $\hat{\theta}^T$ and θ^T is the replacement of σ with the random variable $\hat{\sigma}$, for the purpose of this chapter, we will suppress the other arguments than the volatility rate. Further, for the sake of comparability with the existing studies in this context, we will use the variance rate V as the argument, rather than the volatility rate.

using Taylor series expansion, and then take the expectation: ⁹²

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$$E[CB(\hat{V})] = CB(V) + [\{ \partial^2 CB(\hat{V}) / \partial (\hat{V}^2) | \hat{V}=V \} \mu_2(\hat{V}) / 2!] +$$

$$[\{ \partial^3 CB(\hat{V}) / \partial (\hat{V}^3) | \hat{V}=V \} \mu_3(\hat{V}) / 3!] + \dots + [\{ \partial^r CB(\hat{V}) / \partial \hat{V}^r | \hat{V}=V \} \{ \mu_r(\hat{V}) / r! \}] + \dots$$

$$+ [\{ \partial^n CB(\hat{V}) / \partial \hat{V}^n | \hat{V}=V \} \{ \mu_n(\hat{V}) / n! \}] +$$

$$[\{ \partial^{n+1} CB(\hat{V}) / \partial \hat{V}^{n+1} | \hat{V}=V \} \{ \mu_{n+1}(\hat{V}) / (n+1)! \}] + \dots (4.1)$$

where $\mu_r(\hat{V})$ is the r-th central moment of \hat{V} .

From (4.1), the formula bias is:

$$B(V) = E[CB(\hat{V}) - CB(V)] = \sum_{r=1}^{\infty} [\{ \partial^r CB(\hat{V}) / \partial \hat{V}^r | \hat{V}=V \} \{ \mu_r(\hat{V}) / r! \}] +$$

$$\sum_{r=n+1}^{\infty} [\{ \partial^r CB(\hat{V}) / \partial \hat{V}^r | \hat{V}=V \} \{ \mu_r(\hat{V}) / r! \}] \dots (4.2)$$

It appears from (4.2) that if we knew $B(V)$, then we could find an unbiased estimator $[CB(\hat{V}) - B(V)]$. If we could find an unbiased estimate of $B(V)$, subtracting that from $CB(\hat{V})$ would have yielded unbiased estimate of $CB(V)$. But it seems that we cannot form unbiased estimate of $B(V)$ for two reasons:

⁹³A somewhat similar exposition of the nonlinearity bias is given in Butler and Schachter (1983a).

(a) Due to the existence of non-zero derivative of any order for $CB(\hat{V})$, we are unable to form unbiased estimates of $\{\partial^r CB(\hat{V})/\partial \hat{V}^r | \hat{V}=V\}$. We can, of course, estimate $\mu(\hat{V})$ without bias, for all finite r . But even if we had unbiased estimate of $\{\partial^r CB(\hat{V})/\partial \hat{V}^r | \hat{V}=V\}$, the product of its unbiased estimate and that of $\mu(\hat{V})$ would not be an unbiased estimate of the product.

(b) In practice, it is not possible to account for all the terms in the infinite series. If we are prepared to consider the bias in truncated form, i.e., consider finite number of terms in the Taylor series, we have two alternatives. We can try to choose an n such that either $\sum_{r=0}^n [\{\partial^r CB(\hat{V})/\partial \hat{V}^r | \hat{V}=V\} \{\mu(\hat{V})/r!\}]$ or $\sum_{r=n+1}^{\infty} [\{\partial^r CB(\hat{V})/\partial \hat{V}^r | \hat{V}=V\} \{\mu(\hat{V})/r!\}]$ is rather small.

If one is successful in choosing an n such that $\sum_{r=n+1}^{\infty} [\{\partial^r CB(\hat{V})/\partial \hat{V}^r | \hat{V}=V\} \{\mu(\hat{V})/r!\}] \approx 0$, that would imply $B(V) = \sum_{r=0}^n [\{\partial^r CB(\hat{V})/\partial \hat{V}^r | \hat{V}=V\} \{\mu(\hat{V})/r!\}]$ to be not approximately zero if $B(V)$ is not approximately zero. In general, $B(V)$ would not be approximately zero or equal to zero. But there exists functional form $CS(V)$ for which $B(V) = \sum_{r=0}^n [\{\partial^r CS(\hat{V})/\partial \hat{V}^r | \hat{V}=V\} \{\mu(\hat{V})/r!\}]$ is identically equal to zero. If $CS(V)$ were to approximate the behavior of $CB(V)$ as V varies, $CS(\hat{V})$ could be thought as a pseudo estimator of $CB(V)$, for which the bias would be identically equal to zero when only the first n terms are considered in an expression similar to (4.2). As we search for alternative estimators, it would be interesting to investigate the nature and the behavior of pseudo estimator as compared to the formula estimator, especially when the variance rate V or

the sample size (from which the variance rate is estimated) N varies.

SECTION 2

In our discussion above, we indicated that there exists functional form in V , for which, insertion of \hat{V} would lead to zero bias when only the first n terms in the Taylor series are considered. This assertion follows from the fact that the functional form mentioned satisfies the equation

$B(V)=0$, or, in other words, it is a solution to the n -th order linear differential equation in V with variable coefficients:

$$\sum_{j=1}^n [p_{n-j} V^j D^j] W = 0 \quad \dots(4.3)$$

where

W is the dependent variable

D is the differential operator d/dV

p_{n-j} is equal to $\{V/(N-1)\}^j E[\chi^2_{N-1} - (N-1)]^j$

(4.3) is a special case of Legendre Linear Equation.⁹⁴ In order to solve (4.3), we can write it in the following form, using the transformation $Z=\ln(V)$, and the operator $D = d/dZ$:

$$\sum_{j=1}^n [p_{n-j} \{ \prod_{\ell=0}^{j-1} (D - \ell) \}] W = 0 \quad \dots\dots\dots(4.3a)$$

⁹⁴Legendre Linear Equation is:

$$[p_0 (aX+b) D^n + p_1 (aX+b) D^{n-1} + \dots + p_r (aX+b) D^{n-r} + \dots + p_{n-1} (aX+b) D + p_n] Y = 0,$$

where Y and X are dependent and independent variables respectively, and D denotes d/dY .

Zeros (including multiplicities) of the characteristic polynomial $g(D) = \sum_{j=0}^n p_{n-j} \{ \prod_{l=0}^{j-1} (D - l) \} = 0$, will determine the nature of the solution to (4.3a) or (4.3), if it exists. This solution was referred to as $CS(V)$ in the foregoing section.

Lemma 4.1: The solution to (4.3) exists, and it is unique.

Proof: The existence and uniqueness of solution theorem for general n -th order linear differential equation requires that the function $D^n Y = G(Y, X, D^{n-1} Y, \dots, D Y)$ exists, G be continuous with respect to all $n+1$ arguments, and be at least once differentiable with respect to the last n arguments. Assuming $W(V)$ to belong to the same class of differentiable functions as $CB(V)$, this theorem is satisfied for $Y=W(V)$, $X=V$, $D=D$. Hence, solution to (4.3a) exists, and is unique. By substitution for Z , the same follows for (4.3).

Two things to be noticed about (4.3). First, $V=0$ is a (in fact, the only, except infinity) singular (of irregular type) point of (4.3). Second, as we have assumed, when the estimate inserted for V is an unbiased estimate of V , the solution to the relevant differential equation would lack the property of boundedness, when considered in the semi-infinite interval of V .

Lemma 4.2: If \hat{V} is such that $E(\hat{V})=V$, $CS(V)$ is unstable with respect to V .

Proof: Whenever $E(\hat{V})=V$, at least one of the zeros of the characteristic polynomial $g(D)$ is unity. Hence, the solution would contain a segment $A \cdot V^z$, where A is a constant. This would explode as V tends to infinity.

Regarding singularity, since 0 is an irregular singular point, we cannot specify the value of the solution or its derivatives at that point. Thus, the solution would be unstable as V tends to 0.

To address the above problems, one can choose a lower bound ξ , arbitrarily close to 0, and an upper bound η , which, for most practical purposes, could be set to 1.

Solving (4.3) analytically for lower order n , and numerically for higher orders, the general solution can be derived. To form the pseudo estimator for the Black-Scholes model price, restrictions on the solution can be imposed by suitable choice of the constants in the general solution.

One of the boundary conditions can be used to incorporate the moneyness of the option. It is to be noted that the Black-Scholes model price approaches differing lower limits depending on the degree of moneyness and definition of moneyness. For the purpose of this section, we would be referring to $\ln(S/X)$ as the indicator of moneyness. For at-the-money ($\ln(S/X)=0$), and in-the-money ($\ln(S/X)>0$), the lower limit is $S-X\exp(-rT)$. For out-of-the-money ($\ln(S/X)<0$), the lower limit would be $S-X\exp(-rT)$ or 0, depending on whether $\ln(S/X)+rT >0$, or $\ln(S/X)+rT \leq 0$.⁹⁵ These limits are taken as V approaches 0. To avoid the singularity at $V=0$, one would set \lim

⁹⁵If we adhere to $\ln(g)=\ln(S/X)+rT$ as the indicator of moneyness, the lower limit would be 0 for at-the-money ($\ln(g)=0$), and out-of-the-money ($\ln(g)<0$), and $S-X\exp(-rT)$ for in-the-money ($\ln(g)>0$).

CS (V) as V^{-0} , equal to the appropriate lower limit of the Black-Scholes model price as V^{-0} , and thus try to incorporate some option specificness in the pseudo estimator.

A second boundary condition can be used to set $\lim_{\eta} CS (V)$ as $V^{-\eta}$, equal to S, the upper limit of the Black-Scholes model price, as $V^{-\infty}$.

To avoid under or overidentification of constants, one would need another $n-2$ equations in the n unknown constants. A definite candidate for one of the equations would be to set $CS (V) = CB(V)$, when the variance rate is known. The rest of the conditions can be supplied by choosing $n-3$ arbitrary values of V , V_i 's, and setting $CS (V_i) = CB(V_i)$. When the variance rate of the option at hand is not known, we would have to choose $n-2$ arbitrary values of V . In choosing the arbitrary V_i 's, one would, of course, see that they do not exceed η or fall short of ξ ⁹⁶.

Before we go for the Monte Carlo study undertaken to illustrate the behavior of the pseudo estimator as compared to the usual formula estimator, let us make a somewhat counterintuitive comment about the pseudo estimator. As the sample size N increases, the pseudo estimator may exhibit instability. This is due to the fact that as N increases, p_0 in (4.3) may tend to 0, for a given V , and thus may have similar

⁹⁶The above boundary conditions do not ensure nonnegativity for $CS (V)$, neither do they ensure that it be increasing in V .

Further ingenuity may be required to impose such desirable restrictions.

singularity effect as V tending towards 0.

Since the purpose the Monte Carlo undertaken is primarily expositional, we have chosen the value of n to be 3 only. The solution to $B(V)=0$ is:

s3

$$CS_2(V) = c_1 + c_2 V + c_3 V^a$$

where $a = 2.75 - 0.75 N$.

The lower and upper limits of V were chosen as those values of the estimated variance rate which cutoff 2.5% probability area on either tail of its distribution. For example, for $V=0.025$ and $N=15$, these limits are 0.0100535 and 0.0466071 respectively.

To solve for the constants, we have used the two limiting boundary conditions, and the equation $CS_3(V)=CB(V)$.

For a given degree of moneyness, time to expiration, riskless rate, variance rate, and sample size, 500 sample variance rates were generated with the given V and N as parameters. These sample variance rates were, in turn, used to compute 500 different values of the formula estimator and the pseudo estimator. For both the estimators, the mean of the 500 estimates was subtracted from the Black-Scholes model price $CB(V)$ to arrive at a measure of mispricing, which, in fact, is the negative of the bias.

The results reported in Table 4.1 seem to support the expected behavior for the estimators. Regardless of the situation, the mispricing of the formula estimator decreases in

absolute value, as the variance rate decreases. On the other hand, with a few exceptions, the absolute value of mispricing of the pseudo estimator increases as the variance rate decreases. When we come to variation of the sample size, again, regardless of the situation, the level of mispricing by the formula estimator decreases as the sample size increases.⁹⁷ As for the pseudo estimator, the level of mispricing increases with increasing sample size. Though not reported here, we observed that the standard deviation of the formula estimates is lower than that of the pseudo estimates.

In conclusion, it is less likely to uniformly improve upon the simple formula estimator by following the strategy of finding some alternative functional form in V in which a sample variance is inserted. In particular, for options on low variance stocks, the formula estimator seems to have performed relatively well. Moreover, whenever the use of larger sample size is not deemed inappropriate, the formula estimator appears viable.

SECTION 3

The discussion in the preceding section was focused upon insertion of the sample variance rate into functional forms nonlinear in the variance rate, and the bias considered was based upon Taylor series expansion of the estimate (of Black-Scholes model price) around the true variance rate. Butler

⁹⁷This is in conformity with Boyle and Ananthanarayanan (1977), Butler and Schachter (1983a).

and Schachter(1983a) proposed an estimator, which attempts to redress the nonlinearity bias problem from a different methodological perspective. Their approach differs in two important ways: first, Taylor series expansion is used to approximate the Black-Scholes model price, the approximating series being linear in the powers(positive and/or negative) of the true variance (or volatility) rate; second, the price estimate is formed by replacing the powers of the variance(or volatility) rate with their unbiased estimates instead of the powers of the sample variance(or volatility) rate. Use of the powers of the sample variance(or volatility) rate would have been equivalent to straight forward insertion of the sample variance(or volatility) rate in the estimating function, as is the case for both the formula estimate and the pseudo estimate.

Since each term in the approximating series can individually be estimated without bias, the Butler-Schachter estimate would be an unbiased estimate of the approximating series. But it would be only an approximately unbiased estimate of the Black-Scholes model price, the remaining bias being the approximation error of the finite order expansion.

To outline the Butler-Schachter estimator, we first note that , to form unbiased estimate of the Black-Scholes model price $CB(V)$, it is sufficient to form unbiased estimates of the cumulative standard normal probabilities $\phi(d_1)$ and $\phi(d_2)$, where,

to recollect, $d_1 = [\ln(S/X) + rT + 0.5\sigma^2 T] / \sigma\sqrt{T}$, and $d_2 = d_1 - \sigma\sqrt{T}$.⁹⁸ Rather than approximating $CB(V)$ by a single expansion, Butler and Schachter approximates $\phi(d_i)$, $i=1,2$ individually by Taylor series around $d_i = 0$:

$$\phi(d_i) \approx \phi(0) + \phi'(0)d_i + \phi''(0)\{d_i^2/2!\} + \dots + \phi^{(n)}(0)\{d_i^n/n!\} \dots (4.4)$$

where $\phi^{(r)}(0)$ denotes the r -th derivative of the standard normal distribution function, evaluated at the centre 0.

Using the features of the standard normal distribution, and after some manipulations, (4) can be written as:

$$\phi(d_i) \approx 0.5 + f(0)\{d_i - \{d_i^3/6\} + \{d_i^5/120 - \dots\} \dots (4.4a)$$

where $f(0)$ is the standard normal density evaluated at 0, and the m -th term in the bracketed expression is:

$$(-1)^{m-1} d_i^{2m-1} \{(1/2) \cdot (1/4) \dots (1/(2m-2)) \cdot (1/(2m-1))\}$$

Substituting for d_i 's and collecting terms in the like powers of \sqrt{V} lead to an expression, in which, each term is a constant multiplied by some positive or negative exponents of \sqrt{V} .

⁹⁸The following exposition of Butler-Schachter estimator is taken from Butler and Schachter (1983a).

Now,

$$E[s^r] = (2/k)^{r/2} \left[\frac{\Gamma\{(k+r)/2\}}{\Gamma(k/2)} \right] (\sqrt{V})^r,$$

where $K=N-1$ and s is the sample standard deviation or volatility rate.

So, unbiased estimate of $(\sqrt{V})^r$ would be:

$$s^r \left(\frac{k}{2} \right)^{r/2} \left[\frac{\Gamma(k/2)}{\Gamma\{(k+r)/2\}} \right]$$

Replacing for the r -th power of \sqrt{V} from the above expression leads to the Butler-Schachter estimate of the Black-Scholes model price.

Butler and Schachter reported numerical integration results for different combinations of the true variance, sample size, and moneyness ($g=S/(X \exp(-rT))$). The main observations made in their paper are:

B1. The bias of the Butler-Schachter estimator is considerably smaller than that of the formula estimator.

B2. Biases for the Butler-Schachter estimator are largest, when the variance is small, and the option is not at-the-money.

B3. (a) Except for out-of-the-money-and-small-variance case, the mean square error of the Butler-Schachter estimator is consistently higher than that of the formula estimator by one per cent or less.

(b) The mean square error is highest for at-the-money and increases with the variance.

In addition to the above observations, we notice the following:

M1. For at-the-money options, larger bias is associated with higher variance. For not-at-the-money, no such discernible pattern appears.

M2. Negative biases are more frequent for not-at-the-money with lower variance and/or small sample size.

M3. For at-the-money, the bias increases with sample size.

Let us put forward our first comment about the Butler-Schachter estimator in the form of a lemma.

Lemma 4.3: The expansion of $\phi(d_1)$ and $\phi(d_2)$ both around a common value (e.g., zero) is not logically consistent.

Proof: Let us assume that both the expansions are around zero. Then the expansion of $\phi(d_1)$ around $d_1=0$ implies a similar expansion (as function of V) around $V = -2\ln(g)/T$. On the other hand, the expansion of $\phi(d_2)$ around $d_2=0$ implies expansion around $V=2\ln(g)/T$. Only for an at-the-money option, both expansions are implicitly around the common value 0 of V . Otherwise, expansions of $\phi(d_1)$ and $\phi(d_2)$ around the common value 0 of d implies approximating the same option value at two different points of V at the same time.

Corollary 4.1: If $\phi(d_1)$ is expanded around $d_1=0$, $\phi(d_2)$ is to be expanded around $d_2=-\sqrt{VT}$, for the sake of logical consistency.

Proof: For $d_2=-\sqrt{VT}$, the implicit value of V is $-2\ln(g)/T$ which is the same as implied by $d_1=0$. Thus, the expansions would

preserve logical consistency.

Corollary 4.2: Expansions of both $\phi(d_1)$ and $\phi(d_2)$ around 0 implies, for small enough VT , expansion around an at-the-money option.

Proof: $d_1=0$ implies $\ln(g)=-0.5VT$ and $d_2=0$ implies $\ln(g)=0.5VT$, both of which would be approximately zero for small enough VT . Thus the expansion would be approximately around an at-the-money option's value.

According to Corollary 4.2, when the variance is small, the approximation error may tend to be larger for away-from-the-money option, since the expansions used would imply approximation at about an at-the-money option's value. In addition, according to Lemma 4.3, only for an at-the-money option, we would have logically consistent expansions. Thus, we may expect the result as mentioned in B1.

In the proof of Lemma 4.3, we have mentioned that for an at-the-money option, expansions of $\phi(d_i)$, $i=1,2$ around $d_i=0$ would imply logically consistent expansion around $V=0$ (or $\sqrt{V}=0$). Thus, when the Butler-Schachter estimator approximates the values of larger variance at-the-money options, the expansion point ($V=0$ or $\sqrt{V}=0$) would be at larger distances from the true variance rate. The approximations thus may tend to be poorer for larger variance, though not necessarily so. Our observation M1 is in conformity with this explanation.

There is another limitation of the Butler-Schachter estimator. For higher order expansions, away-from-the-money

options would necessitate larger sample sizes. This is due to the fact that for higher order negative powers of \sqrt{V} , the Gamma functions would otherwise encounter inadmissible argument values. ⁹⁹.

SECTION 4

The essence of the Butler-Schachter approach in redressing the problem of the nonlinearity bias of the formula lies in the alternative representation of the Black-Scholes model price by a Taylor series. Strictly speaking, the authors used Taylor series to approximate the two cumulative normal distribution functions individually, rather than the model price directly. A legitimate concern with respect to such attempts is the validity of the series representation of the model price in general and the validity of Taylor series representation around arbitrarily chosen points in particular. The following exposition is intended to shed some light on these issues.

For simplicity, let us assume that $S=T=1$. Then it can be shown that the Black-Scholes model price $CB(V)$ satisfies the following second order differential equation in V :

$$2V^2\{\partial^2 CB(V)/\partial V^2\} + \{\partial CB(V)/\partial V\} [2a - 2bV^2 + V] = 0 \dots\dots\dots(4.6)$$

where $a = -\{\ln(g)\}^2/2$, and $b = -1/8$.

⁹⁹This was pointed out by Dr. Pao Cheng and Dr. John Heaney.

Let us consider the following power series solution to

(4.6):

$$CB(V) = \sum_{j=0}^{\infty} c_j v^j \quad \dots(4.7)$$

where $v = V - V_0$, V_0 being an arbitrary point in the acceptable domain of $CB(V)$.

The coefficients c_j can be solved from:

$$c_0 = CB(V_0)$$

$$c_1 = \{ \partial CB(V) / \partial V |_{V=V_0} \}$$

$$c_{j+2} = -[(j+1)z_j / q_j] c_{j+1} - [\{2j(j-1) + j(1-2bV_0)\} / q_j] c_j + [2b(j-1) / q_j] c_{j-1}, \quad j=0,1,2,\dots$$

where

$$q_j = 2V_0^2(j+1)(j+2) \text{ and } z_j = 2a + V_0(1+4j-2bV_0).$$

This series solution is in fact a Taylor series representation of the Black-Scholes model price around $V = V_0$. But, for the hypothesized series solution to exist, the series should be convergent. In addition, the suggested solutions for c_j 's which led to Taylor series representation, require V_0 to be an ordinary point of (4.6).

The series solution would converge for all values of V such that $|V - V_0| < R$, where R is the radius of convergence, i.e., the

distance from the point V_0 to the nearest point of singularity for (4.6). It appears that $V=0$ is the nearest point of singularity for finite V .¹⁰⁰ Thus the hypothesized series solution would converge for $0 < V < 2V_0$.

Given that $V=0$ is a singular point, Taylor series representation of $CB(V)$ around $V=0$ is not valid. If $V=0$ were a singular point of regular type, then (4.6) would have had a Frobenius type series solution of the form:

$$CB(V) = \sum_{j=0}^{\infty} c_j V^{j+w}$$

where the series would converge for all V in the radius of convergence.

Unfortunately, $V=0$ fails to qualify as a regular singular point. If we want to represent $CB(V)$ around $V=0$, the proper expansion would be a Laurent expansion, since $V=0$ qualifies to be a point of isolated singularity.¹⁰¹

The Butler-Schachter approach expands $\phi(d_1)$ around $d_1=0$, implying, in essence, expansion around $V=-2\ln(g)/T$, and $\phi(d_2)$ around $d_2=0$ implying expansion around $V=2\ln(g)/T$. Considered in terms of \sqrt{V} , their expansions would be valid and also logically

¹⁰⁰An exception would be the case where the Black-Scholes model is considered as a function of \sqrt{V} , and the option is at-the-money. In this situation, $\sqrt{V}=0$ would not be a singular point for the corresponding differential equation in \sqrt{V} . But for not-at-the-money options, $\sqrt{V}=0$ would be a singular point.

¹⁰¹Thanks are due to Dr. John Heaney for pointing this out.

consistent for an at-the-money option.¹⁰² For an out-of-the-money option, their expansion for $\phi(d_1)$ would converge if the assumed true variance rate is in the range $0 < V < -4\ln(g)/T$. But for $\phi(d_2)$, then, V would be outside the radius of convergence, viz., 0 to $4\ln(g)/T$. Similar explanation applies to in-the-money option. Thus, for not-at-the-money options, the implied series representation of the Black-Scholes model price by the Butler-Schachter estimator would not be valid.

From our discussion in this section, it appears that a Taylor series representation of the Black-Scholes model price around a non-zero value of the variance rate would be valid, if the true variance rate lies in the radius of convergence.

SECTION 5

The distinguishing feature of the Butler-Schachter approach as discussed earlier is the series approximation of the Black-Scholes model price, which is linear in the powers of the variance (volatility) rate and amenable to unbiased estimation. But the specific type of approximation used by the authors led to problems of logical consistency, validity of the series representation, and convergence of the approximating series. Moreover, a practical limitation is imposed by the requirement of large sample size (for stock returns) to undertake higher

¹⁰² The problem of logical consistency can be visualized in two alternative forms: (i) for a given g , different V 's are implied, when $g \neq 1$; (ii) for a given V , different g 's are implied. The former case was discussed earlier in this chapter. We would continue to interpret in terms of this case.

order expansions for not-at-the-money options. A very simple approach that retains the spirit of the Butler-Schachter approach, but attempts to alleviate the aforementioned problems is delineated below.

Let us do the Taylor series expansion of $CB(V)$ around an arbitrary point $V=V_0$, $V_0 \neq 0$:

$$\begin{aligned}
 CB(V) = & CB(V_0) + \left\{ \frac{\partial CB(V)}{\partial V} \Big|_{V=V_0} \right\} (V - V_0) + \\
 & \left[\left\{ \frac{\partial^2 CB(V)}{\partial V^2} \Big|_{V=V_0} \right\} \frac{(V - V_0)^2}{2!} \right] + \dots + \\
 & \left[\left\{ \frac{\partial^n CB(V)}{\partial V^n} \Big|_{V=V_0} \right\} \frac{(V - V_0)^n}{n!} \right] + \dots \dots \dots (4.8)
 \end{aligned}$$

Substituting for $(V - V_0)^r$ by its binomial expansion leads to a series which is linear in the positive powers of V , irrespective of the moneyness of the option. Truncating the series after the term involving the n -th derivative, and replacing the powers of V by their unbiased estimates produces an unbiased estimate of the truncated expansion. The truncation (the finite order expansion) error can be considered to be the bias of the estimator.

Notice that we are expanding the model price directly, around the chosen value V_0 , thus avoiding the problem of logical consistency between the two individual expansions of the Butler-Schachter approach. Since we are choosing a non-zero V_0 , we also escape the problem of singularity. Moreover, only

positive powers of V are involved in contrast to both positive and negative powers of V for not-at-the-money option under the Butler-Schachter approach. Thus the estimator proposed here, alternatively referred to as the CC estimator, can be used for all types of options without being limited by the requirement of larger sample size.

A limitation of the CC estimator is that, since the true variance rate is unknown, we cannot guarantee that the convergence criterion $0 < V < 2V_0$ is satisfied. Given the positivity of the true variance rate V , choice of a larger V_0 may practically reduce the possibility of the lack of convergence. But we do not mean to say that the convergence problem is eliminated.

A computational problem that may arise for the CC estimator is the evaluation of higher order derivatives of the Black-Scholes formula with respect to the variance rate. The expressions for these derivatives are much more complicated than the derivatives of the standard normal density functions only, required under the Butler-Schachter approach. Towards this end, we present, in Appendix 4.1, a simple algorithm to compute the higher order derivatives of the Black-Scholes formula with respect to the variance rate ¹⁰³.

Now that we have two approximately unbiased estimators of the Black-Scholes model price, the Butler-Schachter

¹⁰³A similar algorithm applies to the computation of derivatives with respect to the volatility rate.

estimator(BTS) and the the CC estimator, it would be interesting to see how they compare in performance. In addition, we may ask the question: 'Is the performance of any of the approximately unbiased estimators compared to that of the formula estimator is such that one would forsake the extreme simplicity of the latter?' We have undertaken a Monte Carlo study with these issues in mind ¹⁰⁴. For a given mean rate of return(assumed to be zero), a variance rate, and a sample size, we generated 500 samples to yield the same number of sample variance rates. These sample variance rates were used to generate as many estimates of BTS, CC, and the formula estimator, for a given option ¹⁰⁵. On the basis of those estimates, we computed the mean mispricing(negative of the bias), the mean percentage (taken out of the Black-Scholes model price) error, the variance, and the mean square error, for each estimator.

In our computations, we truncated the series after the term containing the 31st derivative of the Black-Scholes formula with respect to the variance rate. Thus for the BTS estimator, the highest order of derivative for the standard normal density is 30. For the CC estimator, V was taken to be 50% higher than the

¹⁰⁴Both Boyle and Ananthanarayanan(1977), and Butler and Schachter(1983a) results are based on calculation of bias by numerical integration. Thus, their results are not directly comparable to our Monte Carlo results.

¹⁰⁵For the sake of comparability with the existing studies, we would use in this section, $S/Xexp(-rT)$ as the indicator of moneyness. The contracts are to buy one share with current price $S=\$1.0$.

true variance rate ¹⁰⁶

In total, we considered 40 different combinations of moneyness(g), variance rate(V), and sample size(N). Since the order of expansion for the BTS and the CC estimator was 31, we could compute for the BTS in only 16 of the cases. The results are presented in Tables 4.2 to 4.4 .

In general, the ranking of the absolute magnitudes is the same whether we consider the mispricing or the percentage error. Also, variance gives identical ranking as the mean square error. Except for the cases of $g=0.8$, all rankings are the same for the two variance rates considered.

When we consider the absolute magnitude of mispricing in the 16 common cases, both BTS and CC improves upon the formula estimator, but CC improves most. CC is lowest in 12 cases, BTS in 3(all 3 at-the-money) cases, and the formula estimator in the lone case of relatively deeper-in-the-money($g=1.2$) option with higher variance rate($V=0.04$) for the stock. But in only 3(all 3 at-the-money lower sample size) out of the 13 cases, where BTS improves upon the formula, the difference in the absolute value of the percentage error(dp) is greater than 1%. Considering the common 16 cases only, in 5 out of the 15 cases where CC improves

¹⁰⁶These choices were to some extent constrained by the capability of the software used. But the knowledge of the true variance rate V is not required to choose V . Any value in the range of convergence would do. For better comparisons across cases, we have chosen the V 's to be the same percentage distance away from V 's. 0

upon the formula, dp is greater than 1% ¹⁰⁷.

When we compare between BTS and CC, in 2 out of 13 cases where CC performs better (in terms magnitude of mispricing), the percentage error gain (i.e., dp) is more than 1%. BTS performs better in 3 (all 3 at-the-money) cases, but in none of these, the percentage error gain is more than 1%.

If we consider performance in terms of variance or the mean square error of the estimators, formula outperforms both BTS and CC convincingly. Out of the 16 common cases, BTS is lowest in 4 (all 4 at-the-money higher sample sizes), CC is lowest in 3 (2 relatively deeper-in-the-money, 1 relatively deeper-out-of-the-money, and all largest sample size), and the formula in 9 cases ¹⁰⁸. Between BTS and CC, CC performs better in 10 out of the 16 cases.

Overall, BTS seems to have some advantage over the other two, for at-the-money option. But at-the-money options are least frequently traded options. Majority of the traded options are around-the-money. For such options, CC enjoys advantage over formula in terms of mispricing magnitude, but the advantage reverses in terms of the variance or mean square error.

It is to be noted that our Monte Carlo results for the mispricing (in particular, its magnitudes) of BTS relative to that of formula are very different from the results of Butler

¹⁰⁷ Among all 40 cases, CC improves upon formula in 38 cases, $dp > 1\%$ in 18 of these cases (mostly lower sample sizes).

¹⁰⁸ Out of the all 40 cases, formula wins over CC in 29 cases.

and Schachter(1983a). The results are not directly comparable due to the nature of the two studies and the difference in the sample sizes used.

Considering the directions of mispricing, the formula, as found by Boyle and Ananthanarayanan(1977), underprices at-the-money and near-the-money options, and overprices relatively deeper-away-from-the-money options. Our Monte Carlo results are in conformity with this pattern. We find BTS to underprice at-the-money options, and overprice not-at-the-money options ¹⁰⁹. For CC, no such patterns seem to emerge.

The variance and mean square error of both BTS(at-the-money only) and formula, and the mispricing of the latter, appear to decrease in magnitude as the sample size increases. Variance and mean square error of CC seem to follow the same pattern, but the mispricing of CC does not exhibit any such pattern, as is also the case with BTS.

For BTS, mean square error seems to be larger for at-the-money options, and does seem to increase with variance rate ¹¹⁰. Same is the case for formula and CC.

From the above discussion, it appears that CC may have slight (not clear) advantage in terms of lack of systematic pattern in its mispricing, and lower magnitudes of mispricing

¹⁰⁹Butler and Schachter(1983a)'s results also indicate underpricing of at-the-money option by BTS, and the cases of overpricing are all not-at-the-money. See our observation M2 in section 3.

¹¹⁰See observation B3(b) in section 3.

for the widely traded around-the-money variety of options. But the formula estimator performs best in terms of variability and mean square error, and shows promise of advantage for larger sample size situations. Our results do not indicate a case of practical advantage for the BTS estimator. In summary, we do not find any of the alternative approximately unbiased estimators to be superior to the biased formula estimator ¹¹¹. SECTION 6

In the context of the estimators we have considered so far, the standard normal distribution function was known, but the points at which this function is to be evaluated were not known. By suitable manipulation, we can transform the problem into one, where the value at which the distribution function of a normal (as opposed to standard normal) variable is to be evaluated is known. But now, mean and variance of the normal distribution function are to be estimated.

More specifically, the following can be shown:

$$\phi(d_1) = \text{Prob}[Y_1 < \ln(g)], \text{ and}$$

$$\phi(d_2) = \text{Prob}[Y_2 < \ln(g)]$$

where $Y_1 = Z\sqrt{VT} - VT/2$, and

$$Y_2 = Z\sqrt{VT} + VT/2,$$

and Z is a standard normal variable.

It is sufficient to estimate $\phi(d_i)$, $i=1,2$, without bias for unbiased estimation of $CB(V)$. Healey (1956) and Guenther (1971)

¹¹¹As indicated in the introductory chapter, superiority is established if the estimator has always lower magnitude of bias, and lower variance and/or lower mean square error. It should also show no systematic mispricing.

among others have proposed small sample unbiased estimators for the cumulative normal distribution function's value at a specific point. These estimators require the estimation of the sample mean and variance of the normal variable. In our context, if we could find two normally distributed variables similar to Y_1 and Y_2 , for which samples are available, we could form unbiased estimates of $\phi(d_1)$ and $\phi(d_2)$, and hence unbiased estimate of $CB(V)$.

Lognormal distribution for stock price at the end of interval T implies:

$$\ln\left(\frac{S_{t+T}}{S_t}\right) - \alpha T = -(VT/2) + Z\sqrt{VT}$$

where α is the geometric mean stock return per unit of time.

The variable on the left hand side, is normally distributed with mean $-VT/2$ and variance VT , as is Y_1 . And negative of the left hand side would be distributed normally with mean $VT/2$ and variance VT , as is Y_2 . The $\ln(S_{t+T}/S_t)$ part of these variables are observable. In addition, if we knew α , there would have been no problem in forming the unbiased estimate of $CB(V)$. If we replace α with its unbiased estimate, the means of the constructed variables would remain unaffected, but the variance would no longer be VT . This creates a problem for further research. It also remains to be seen whether the use of unbiased estimate of α or even the use of the riskless rate as a proxy could improve upon the approximately unbiased estimators considered earlier in this chapter. It is ironic as well as unfortunate that, although the potentially unbiased estimator

does not suffer from the arbitrariness of series truncation or the point of expansion, and is unbiased rather than approximately unbiased, it necessitates the estimation of the geometric mean rate of return, the absence of which in the Black-Scholes model endears all researchers.

SECTION 7

Our purpose in this section is to investigate the nature of the nonlinearity bias of the formula beyond what has been accomplished by Boyle and Ananthanarayanan(1977) and, Butler and Schachter(1983,1983a). There are at least three reasons why this investigation is called for:

(a) Neither of the above studies explain why the formula estimator underestimates the Black-Scholes model price for at and around-the-money options, and overestimates for deeper-away-from-the-money options.

(b) The systematic relationships, if any, with time to maturity and variance rate did not receive adequate attention. Also, the marginal nature of systematic relationships were not discussed.

(c) The implications of these systematic relationships for commonly applied techniques of Black-Scholes validation were not clearly brought out.

We will touch these issues in succession.

Boyle and Ananthanarayanan(1977) first reported that the formula with the sample variance rate tends to underprice at and

around-the-money options, and overprice deeper-away-from-the money options ¹¹². But no explanation was provided regarding this pattern of mispricing. Butler and Schachter(1983,1983a) confirmed this pattern, and in Butler and Schachter(1983a),pp. 5, the following explanation was advanced:

"The pattern of the biases examined by Boyle/Ananthanarayanan (1977) in their Table 2 results from the behavior of $\partial^2 \phi(d_i) / \partial (s^2)^2$, $i=1,2$, over the range $-\infty < d_i < \infty$ (see Figure 1). They found that, as the stock price, and hence d_1 and d_2 , rises, a small positive bias reaches a maximum, becomes a large negative bias, and then becomes positive again."

The following points need to be made about the foregoing:

A. Butler and Schachter(1983a) missed one of the four phases in the changing pattern of bias. We think it is the 4th phase, where the dollar bias, after becoming positive in the 3rd phase, begins to decline again. On the other hand, Boyle and Ananthanarayanan(1977), considering the percentage error, missed the 1st phase, about which Butler and Schachter(1983a) says that "...a small positive bias reaches a maximum...". ¹¹³

B. Though the bias of the formula estimator would be affected by higher than second order derivatives of the standard

¹¹²The implicit indicator of moneyness seems to be the comparison of stock price and striking price. For Butler and Schachter(1983,1983a), the indicator of moneyness is explicitly mentioned as the ratio of stock price to the discounted value of the striking price.

¹¹³When we plotted the dollar deviations of Boyle and Ananthanarayanan(1977), this phase reappeared.

normal distribution function, for practical purposes, it may suffice to consider the second order bias only, since the type of options for which the bias pattern is missed by the second-order derivative, are relatively infrequently traded. But it remains to be noted that even the second order bias of the formula estimator is the result of the mixing of two different-valued (in general) second order derivatives of the standard normal distribution function. Thus, it is not clear, whether one should merely look at the relationship of $\partial^2 \phi(d) / \partial d^2$ to d , as was considered in Butler and Schachter (1983a).

We will try to provide an explanation on the basis of second order Taylor series approximation to the nonlinearity bias of the formula, which incorporates the combined effect of the two different-valued second order derivatives of the standard normal distribution function.

The second order approximation to the nonlinearity bias can be written as: ¹¹⁴

$$E = H d_1 d_2 \exp(-0.5d_1^2)$$

where

$$H = \{ \text{variance}(\sqrt{V}) / 2\sqrt{V} / (2\pi) \} S_V / T$$

The sign of this bias depends on whether d_1 and d_2 are of the same or opposite signs. Formula will overprice if they are

¹¹⁴This approximation is taken treating the formula as function of the volatility rate, since later in this thesis we will be using the volatility rate as a regressor following the existing regression studies such as Whaley (1982), Geske and Roll (1984a). In particular, in chapter 7, we shall use this approximation to form an indirect test of the validity of the Black-Scholes model.

of the same signs, and underprice if they are of opposite signs. Given the positivity of \sqrt{VT} , d_1 and d_2 will be of opposite signs whenever d_1 is in the range $0 < d_1 < \sqrt{VT}$, i.e., $|\ln(S/X) + rT| < 0.5\sqrt{VT}$. Beyond this range, they will be of the same sign. If moneyness is measured by $g = S/X \exp(-rT)$, this implies that out-of-the-money options which have moneyness in the range $-0.5\sqrt{VT} < \ln(g) < 0$, will be underpriced ¹¹⁵. Similarly, in-the-money options with $\ln(g)$ lying between 0 and $0.5\sqrt{VT}$ will be underpriced ¹¹⁶. Figure 4.1 clearly shows these ranges.

For an at-the-money option, d_1 and d_2 will be of opposite signs, and thus the formula will underprice.

The range of moneyness over which underpricing takes place has width \sqrt{VT} , independent of the indicator of moneyness ¹¹⁷. Butler and Schachter (1983a) found, on the basis of simulation results for the total nonlinearity bias, that the region of overpricing (underpricing) shrinks (broadens), as the variance of the stock return increases. Our exposition of the second order bias alone predicts the same. Though a strong enough prediction cannot be made, it seems likely that options with high

¹¹⁵If $\ln(S/X)$ is the indicator of moneyness, the range of underpricing for out-of-the-money ($\ln(S/X) < 0$) options would be $-(0.5\sqrt{VT} + rT) < \ln(S/X) < 0$, if $r < 0.5V$, and $-(0.5\sqrt{VT} + rT) < \ln(S/X) < 0.5\sqrt{VT} - rT$, if $r > 0.5V$.

¹¹⁶When $\ln(S/X)$ is the indicator of moneyness, the underpricing range for in-the-money ($\ln(S/X) > 0$) options is $0 < \ln(S/X) < 0.5\sqrt{VT} - rT$, if $r < 0.5V$. For $r > 0.5V$, no in-the-money option would be underpriced.

¹¹⁷The indicator $\ln(S/X)$ shifts the boundaries of this width to the left compared to the indicator $\ln(g)$, by the amount rT .

(true)variance rate and/or long time to maturity would tend to be underpriced more often than not, if the Black-Scholes model is the model of fair valuation. The estimated variance bias of Black and Scholes(1972) and and time to maturity bias of Black(1975) are in contrast to this.

Though the extent of bias may differ, the studies mentioned above found the sign of bias to be positive for both deep-in-the-money and deep-out-of-the-money, negative for near-in-the-money and near-out-of-the-money options. This similarity in the direction of bias for in-the-money and out-of-the-money options is clearly indicated by our finding that the range of underpricing or overpricing is dictated by the absolute value of the moneyness measure $\ln(g)$.

It appears that the second order bias of the formula captures some important regularities of the total nonlinearity bias of the formula, if not the latter's mirror image.

Our next issue is the systematic relationships of the total nonlinearity bias with the three important features (alternatively referred to as factors) of the option, i.e., the moneyness, the time to maturity, and the variance rate on the underlying stock's return. Boyle and Ananthanarayanan(1977) provided a graphic view of the relationship between percentage mispricing(negative of the statistical percentage bias) and the stock price(for a fixed striking price, moneyness), controlling

for other factors. The relationships with time to maturity or the variance rate were not considered explicitly. Butler and Schachter(1983a) produced level surfaces indicating the combinations of moneyness $g(=S/X\exp(-rT))$ and variance VT that would lead to a given amount of dollar bias. But this does not reveal the individual relationships of bias with V or T , or even moneyness $\ln(S/X)$ which is independent of T . Moreover, none of these studies consider the marginal nature of the relationships.

We have undertaken a Monte Carlo study to investigate the nature of the nonlinearity bias with emphasis on the abovementioned issues. To separate out the effects as much as possible, we would consider $\ln(S/X)$ as the indicator of moneyness here. Also, since it has become almost customary to graph the mispricing(negative of the statistical bias), rather than the statistical bias, against a factor, we will do the same. Hence, it may be convenient hereafter to refer to the mispricing as bias, keeping well in mind their distinction. To clarify, for example, if we say that the marginal variance rate bias is positive, what we truly mean is that for marginal increase in the variance rate, the statistical nonlinearity bias of the formula decreases.

The design of the Monte Carlo is as follows. For a given variance rate, we generated 500 sample variances using its distribution and assuming the sample size to be 10. ¹¹⁸

¹¹⁸The nature of the relationships do not change substantially for larger sample sizes, the magnitudes become smaller.

Following Boyle and Ananthanarayanan(1977), we used 0.015/quarter as the riskless rate, and 50 as the striking price¹¹⁹. For a given option, 500 formula estimates are computed using the sample variances. These estimates are subtracted from the Black-Scholes model price(the price with the assumed true variance rate). Averaging the deviations lead to our measure of bias or mispricing.

Figures 4.2 to 4.4 confirms(with Monte Carlo) the already established nature of moneyness bias. At-the-money($S=50$) and, near-the-money($S=45, S=55$) options are underpriced, and deeper-away-from-the-money($S=30, S=80$) options are overpriced. As predicted by the second order bias, the range of underpricing widens for larger V or T . Note also the problem of dichotomous bias arising from the arbitrariness of naming the moneyness, mentioned earlier in chapter 2. Options with the same degree of moneyness could be underpriced or overpriced depending on V and/or T .

The option-specificness of marginal moneyness bias is also illustrated in these figures¹²⁰. Though both near-in-the-money and near-out-of-the-money options are underpriced, the marginal moneyness bias is negative for the former, and positive for the latter. This can be seen from Boyle and Ananthanarayanan(1977)'s

¹¹⁹The variance rate is also on per quarter basis, and time to maturity is measured in quarters.

¹²⁰We use the term 'moneyness bias', rather than the usual 'striking price bias', since we feel that the former brings out more directly the connotation of the comparison between the stock price and the striking price.

diagram also. The deeper-away-from-the-money options may have both positive and negative marginal moneyness bias.

Figures 4.5 to 4.7 illustrates the nature of time to maturity bias. If we define T less than 1 quarter as short maturity as most existing studies have done, it can be seen that these options can both be underpriced or overpriced, depending upon the moneyness and the variance rate. For example, let us consider the option with $T=0.9$ in Figure 4.5. If the stock price is 55, the option is overpriced; if, on the other hand, the stock price is 45, it is underpriced. In Figure 4.6, we see that when variance rate is 0.025 (rather than 0.01), the option with $T=0.9$ and $S=55$ is underpriced (rather than overpriced). This example highlights the problem of the usual dichotomy--short maturity versus long maturity, and the directions of bias attached to them.

For at-the-money options, the marginal time to maturity bias is always positive. This is also evident from the comparison of peaks in Figures 4.2 to 4.4, for a given variance rate.

As for not-at-the-money options, the range of underpricing versus overpricing again comes into play. For $S=45$ and $S=55$, except for very short maturity low variance rate options, the marginal time to maturity bias is positive. This exception can be understood going back to Figure 4.2, and exploiting the fact that higher time to maturity shifts the curves (for moneyness bias) in similar way as higher variance rate. When the time to

maturity or the variance rate decreases, as shown earlier, the range of underpricing around the at-the-money option shrinks. Thus the stock prices 45 and 55, with the striking price fixed at 50, would move out of the underpricing range for low enough T and V , and be positioned deep in the deeper-away-from-the-money range. In that range, the higher maturity curve would be situated below the relatively lower maturity curve, thus indicating negative marginal time to maturity bias.

Similarly, we find in Figures 4.5 to 4.7 that except for very long maturity-high variance rate options, the marginal time to maturity bias for $S=30$ and $S=80$ is negative. We can understand the exception by recollecting that when the time to maturity or the variance rate increases, the range of underpricing widens around the at-the-money option, and then imagining that the $S=30$ and $S=80$ points will be positioned in the underpricing range for long maturity-high variance rate options. In this range, marginally higher time to maturity curve would be situated higher, resulting in positive time to maturity bias.

Figures 4.8 to 4.10 produces results for the systematic relationship with the variance rate which are largely similar to those for the time to maturity. For the range of parameter variation considered, here we could not detect the exception in the direction of marginal bias for $S=45$ and $S=55$.

Now that we have considered the nature of systematic relationships of the nonlinearity bias with the factors, both in the dichotomous and the functional sense, we are in a position to mention the implications for the validation techniques. Since the consequences for the dichotomous bias approach is hoped to be clear from our discussion above, the following is offered in the context of functional bias approach.

Consider the effect of incomplete control for other factors when considering the systematic relationship to a factor. Let us suppose that we have two options represented by the points c_1 and c_2 in Figure 4.4. Both these options have the same time to maturity, but c_1 is relatively deeper-out-of-the-money, and has a lower variance rate. If we do not control for the variance rate difference, and try to relate mispricing functionally to moneyness, we would end up with the prediction that higher degree of moneyness leads to higher mispricing, if, in fact, the Black-Scholes model is valid¹²¹. That this prediction would be misleading can be seen by considering the point c_3 . This is an option with the same variance rate as c_1 , and the same degree of moneyness as c_2 . By controlling the variance rate, we would roughly move from c_1 to c_3 , not to c_2 . This indicates that our prediction of positive relationship with moneyness was, in fact, induced by the difference in the variance rate, not by the difference in moneyness.

¹²¹When Black-Scholes model is valid, the only source of systematic mispricing is nonlinearity bias, for the formula estimate.

The previous example also indicates the need for option-specific estimation in order to investigate marginal bias. If we would have regressed mispricing on moneyness using the two options c_1 and c_2 , differing only in moneyness, we would get a coefficient close to zero, when the Black-Scholes model is valid. But the true marginal bias at c_1 is negative, and at c_2 , close to zero. To identify such differences in marginal bias, fully option-specific estimation is required.

Next we consider the effect of sample mixture of options. Suppose that we have two samples, the first consisting of a_1 and a_2 , and the second consisting of a_1 and a_3 , the points shown in Figure 4.4. In both samples, the sample options differ only in moneyness. If we plot mispricing against moneyness, or perform corresponding linear regression, we would predict direct relationship in the first sample, and inverse relationship in the second sample. Again, this anomaly could have been avoided by option-specific estimation.

The above examples were in the context of investigating moneyness bias, but their essence applies to the investigation of time to maturity and variance rate biases as well. It is also clear that the findings of systematic relationships even when the Black-Scholes model is valid, are as likely as the errors involved in prediction due to the limitations of the techniques of investigation used. In particular, a sound investigation of the marginal biases looms rather difficult, and may even be infeasible.

SECTION 8

Let us summarize our findings in this chapter. The Black-Scholes formula with an estimated variance (or volatility) rate produces biased estimates of the Black-Scholes model price. Three alternative estimators for the Black-Scholes model price were considered, one of these previously proposed by Butler and Schachter (1983a). Our Monte Carlo results do not indicate superiority for any of these estimators over the conventional formula estimator.

The nature of the formula estimator's bias was explored in detail. We found that the sign of this bias would depend on whether the indicator of moneyness ($\ln(S/X) + rT$) is greater or less than half of the variance (VT) in absolute value. The Monte Carlo results confirmed the striking price bias of Boyle and Ananthanarayanan (1977), and the time to expiration and variance rate biases seemed to depend on the level of moneyness. The option-specificness of the marginal nature of these biases require fully option-specific estimation, and may have important bearings upon the value of results based on alternative procedures.

APPENDIX 4.1

A SIMPLE METHOD OF COMPUTING HIGHER ORDER DERIVATIVES OF
BLACK-SCHOLES WITH RESPECT TO THE VARIANCE RATE

Let us define the operator D as d/dV , and denote the r -th derivative of the Black-Scholes model price $CB(V)$ as $D^r C$. Then it can be shown that:

$$D^1 C = a_1 \left(\frac{1}{V} \right)^V b_1 \left(\frac{1}{\sqrt{V}} \right) h \dots (1)$$

$$D^2 C = [D^1 C] K(V) \dots (2)$$

where

$$a_1 = \exp\{-0.5(\ln(g))^2\}$$

$$b_1 = \exp(-1/8)$$

$$h = 0.5/\sqrt{2\pi g}, \text{ and } g = S/[X \exp(-rT)] .$$

$$a = \ln(a_1), \quad b = \ln(b_1) \text{ and}$$

$$K(V) = b - (a/V^2) - (1/2V) \dots (2A)$$

At this point, let us recall Leibniz's Formula:

$$D^n (uv) = D^n (u)v + C_1 D^{n-1} (u) D^1 (v) + C_2 D^{n-2} (u) D^2 (v) + \dots$$

$$+ C_r D^{n-r} (u) D^r (v) + \dots + u D^n (v) \dots (3)$$

where $u = u(V)$ and $v = v(V)$.

In our case, $u=D^1C$ and $v=K(V)$.

Now, the n -th derivative of uv corresponds to $(n+2)$ -th derivative of $CB(V)$. So, for $n=1,2,\dots$,

$$\begin{aligned}
 D^n(uv) &= D^{n+2}C \\
 &= (D^{n+1}C)K + C_1(D^nC)(D^1K) + C_2(D^{n-1}C)(D^2K) + \dots \\
 &+ \dots + C_r(D^{n-r+1}C)(D^rK) + \dots + (D^1C)(D^nK) \dots (4)
 \end{aligned}$$

Thus the steps of computing higher order derivatives will be:

Step 1: Compute the quantities a_1, a, b_1, b , and h for a given moneyness ($g=S/X\exp(-rT)$).

Step 2: For a given V , compute D^1C from (1).

Step 3: Compute $K(V)$ from (2A).

Step 4: Compute D^2C from (2), utilizing the values of D^1C and $K(V)$ from steps 2 and 3.

Step 5: For the highest order of derivative for $CB(V)$ being J , compute the followings:

$$D^r K(V) = (-1)^{r+1} r! \left[\frac{a(r+1)}{V} + 0.5 \right] / (V^{r+1}) \dots (5)$$

for $r=1,2,\dots,(J-2)$.

Step 6: Compute $D^3C = (D^2C)K(V) + (D^1C)(D^1K)$, by utilizing the values from steps 2 through 5.

Step 7: Compute $D^4C = (D^3C)K(V) + {}^2C_1 (D^2C)(D^1K) + {}^2C_2 (D^1C)(D^2K)$ utilizing the values from steps 2 through 6.

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In general, there will be $(J-1)$ terms for the J -th derivative of $CB(V)$.

CHAPTER 5:
CONSTANT COEFFICIENT REGRESSION RESULTS

In chapter 2, we emphasized the need of an appropriate regression model for empirical investigation of various aspects of Black-Scholes pricing. The limitations of the existing regression studies were also examined there. A major shortcoming seemed to be the lack of understanding about the linear regression results, over and above the inattentiveness to the effects of probable econometric problems. Much of this may be attributed to the absence of effort in defining a regression model with features special to the problem of empirical investigation of Black-Scholes pricing. In chapter 3, we presented such a model, and derived an estimable regression equation. This regression model shows that, in general, the regression coefficients are option-specific and do not necessarily reflect the marginal biases. But the existing regression results are produced exclusively by constant coefficient estimation, which may or may not be a reasonable approximation. Moreover, the stochastic regressor problem arising from the use of estimated volatility rate as a regressor has been overlooked so far.

In this chapter we report further constant coefficient regression results in continuity with previous studies. But our purpose is different here. First, to produce preliminary results which would be used as inputs and/or basis for comparison with the results in the forthcoming chapters. Second, to point out some of the effects associated with the existing procedures by replicating them.

The remainder of the chapter is organised as follows. The data sources are mentioned in section 1. Sample information about the explanatory variables of regression is provided in section 2. Section 3 deals with the nature of the Black-Scholes formula estimates' deviations from the actual market prices¹²². The results of broad classification technique, plotting mispricing against individual factors, and simple regressions are presented here. In section 4, constant coefficient multiple regression results are discussed, keeping in view the regression model offered in chapter 3. Section 5 is on the problem of stochastic regressor. Finally, in section 6, the findings of this chapter are summarized.

SECTION 1

Following the strategy of sampling of Geske and Roll(1984a), we selected the date February 05, 1981 randomly. Then a sample of 383 call options written on 54 stocks traded on the chosen date in the CBOE was drawn randomly from the data base installed at Simon Fraser University¹²³. The stock prices and the option prices are daily closing prices¹²⁴

¹²² We have used the simple stock price adjustment version of the Black-Scholes model.

¹²³ The data tapes were obtained from the Interactive Data Corporation, Options History Service

¹²⁴ See Cox and Rubinstein(1985), pp 341, for the problems associated with using closing prices. Some of these problems are: (i) often the stock and the options close at different times, the former being an inadequate approximation for the contemporaneous stock price; (ii) it is difficult to discern whether the closing option is at the bid, at the ask, or in

. The maximum number of options for a single stock is 17 and the minimum number is 1¹²⁵. Out of the 383 options, 101 options (on 30 different stocks) had no dividend payment prior to their contractual maturity, 149 had a single dividend, and 133 had two or more. We screened out the 133 options with two or more dividends, thus retaining a total sample of 250 options.

The volatility rates (daily) for the stock returns have been estimated from the daily return data of the Center for Research in Security Prices data base, over the period of 180 days prior to February 05, 1981. Unbiased estimates of the volatility rates were formed by adjusting the sample standard deviations of percentage stock returns. Given the expositions in Merton (1973a), pp 871-873, and, Jarrow and Rudd (1983), pp 90-91, we expect this to be a good approximation to the estimation of the volatility rate. Cox and Rubinstein (1985), pp 257-258, suggests the sample standard deviation of logarithms of one plus the stock return as the estimator, while Butler and Schachter (1983a) proposes the sample average of squared percentage changes. The latter authors assume zero drift for the stock return¹²⁶

¹²⁴ (cont'd) between, thus giving rise to an interval of uncertainty which may be a significant proportion of the option price; (iii) closing quotations do not provide information about the depth of the market. Also, see Patell and Wolfson (1979), pp 135-136, for the biases in ISD estimation using closing prices.

¹²⁵ Only 3 out of the 54 stocks have 1 option each.

¹²⁶ Yet other estimators are available in the literature, which may be termed 'non-classical'. Parkinson (1980) proposed an estimator based upon the high and low prices, which was modified by Garman and Klass (1980) to include opening and closing prices

We have used a risk-free rate of 14.83% per year for all options ¹²⁷. This is the 1981's second quarter's average discount rate on new issues of three-month U.S. Treasury Bills, taken from International Financial Statistics, June, 1982, volume 35(6). Information from the Wall Street Journal shows that on February 4, 1981, the new 13 week issue was trading at an average return of 14.85%. It appears that our rate can be considered as a good proxy ^{128 129}.

SECTION 2

In this section, sample information about moneyness, time to maturity, estimated volatility rates, and dividend-related

¹²⁶(cont'd) as well, and to adjust for the number of trades. The estimator of Beckers(1983) combines the traditional close-to-close estimator and Parkinson's estimator in a stock-specific manner. Becker goes on to include the ISDs to form better predictions of close-to-close stock price variability.

¹²⁷We transformed it to daily rate.

¹²⁸The choice of proxy for the riskless rate differs across studies. For example, Schmalensee and Trippi(1978) used daily rate equivalent to the preceding Monday's auction rate on 13-week U.S. Treasury Bill, while Blomeyer and Klemkosky(1983) used the mean of bid-asked quotations on the day before the transactions observation date for a Treasury Bill with similar maturity as the option. Whaley(1982) interpolated the effective yields of the two Treasury Bills whose maturities closely preceded and exceeded the time to maturity of the option.

¹²⁹It has been observed by Cox and Rubinstein(1985), pp 217, that even more than doubling the risk-free rate leads to only about 8% increase in Black-Scholes value. MacBeth and Merville(1979), pp 1174, observes that their results would have remained virtually unchanged if a single risk-free rate were used instead of maturity-specific rates.

information are provided¹³⁰.

Zero-dividend Subsample

According to both indicators of moneyness $\ln(S/X)$ and $\ln(g)=\ln(S/X)+rT$, this subsample of 101 options seem to be in-the-money on the average. But according to $\ln(S/X)$, we have more out-of-the-money than in-the-money options; while, $\ln(g)$ says we have more in-the-money than out-of-the-money options. This is probably due to the fact that marginally out-of-the-money options, according to $\ln(S/X)$, came out of the out-of-the-money range when rT was added according to $\ln(g)$. But due to the right-skewness of the distribution within the subsample, the in-the-moneyness of in-the-money options is possibly not very deep.

On the time to expiration side, the options seem to be short maturity (less than 90 days) on the average. The right-skewed distribution indicates the abundance of less-than-average short maturity options.

Estimated volatility rates have symmetric distribution. But the average estimated volatility rate in this subsample is slightly higher than that of the single-dividend subsample.

Single-dividend Subsample

According to both indicators of moneyness, this subsample of 149 options seem to be near-out-of-the-money on the average. Though both indicators of moneyness indicate greater number of

¹³⁰See also Tables 5.1 and 5.1A.

out-of-the-money options than in-the-money, as expected the $\ln(g)$ indicator gives us a significantly smaller proportion of out-of-the-money options compared to $\ln(S/X)$. This is, in part, coming from the average longer maturity nature of options in this subsample. The left-skewed distribution for time to maturity says that there are more options with above-average time to maturity than with below-average time to maturity.

The distribution of estimated volatility rates is approximately the same as that for the zero-dividend subsample, but with slightly lower mean.

Total Sample

The mingling of the effects of the differences in the degree of moneyness and time to expiration of the two subsamples led to contradictory characterization of moneyness for the total sample, according to the two indicators of moneyness. Average near-in-the-moneyness is indicated by $\ln(g)$, while $\ln(S/X)$ says that the total sample is near-out-of-the-money on the average. The right-skewness of the distribution, of course, shows the preponderance of near-out-of-the-money options.

In spite of the push-up by longer maturity options in the single-dividend subsample, the total sample remained short maturity on the average. This may be due to the abundance of very short maturity options in the zero-dividend subsample¹³¹.

¹³¹ Since dividends are usually paid on quarterly basis, we would expect the zero-dividend options to be of short maturity. Thanks to Professor Whaley for pointing this out to me.

The distribution of estimated volatility rates is similar to the two subsamples.

It is to be noted that the average characterizations are meant for general overview, and not to be related to the empirical results to follow. The average moneyness characterization, in particular, is very arbitrary, in view of our discussion in chapter 2 and our results about the nonlinearity bias in chapter 4.

Dividend-related Information

The average size of the single dividend in the single-dividend subsample is about 41 cents, and the dividend as a proportion of the stock price averaged to about 1.02%, varying from 0.01% to 2.5%. When we escrowed dividend, the mean of the stock price adjusted for dividend was only 40 cents lower than the mean of the unadjusted stock prices.

Comparing the time to ex-dividend day and, the lag between the ex-dividend day and the day of contractual maturity, we find that the ex-dividend day is, on the average, closer to current date (February 05, 1981) than to the maturity date ¹³². This, of course, is expected to have dampening effect on the probability of early exercise.

¹³²The average time to ex-dividend day is 44.5033, and the average lag between the ex-dividend day and the maturity date is 52.302.

The necessary condition for early exercise is:¹³³
 $D - X\{1 - B(T - T_1)\} > 0$. We checked for this condition in our single-dividend subsample. The mean value for the left-hand side expression came out as -0.559942, indicating probably a lack of strong early exercise possibility in the subsample. There are only 32 options where it may be optimal to exercise early. We may speculate about probable reasons for this: (i) the options were on the average out-of-the-money; (ii) the ex-dividend date was on the average closer to the current date than to the maturity date; and (iii) the size of the dividend did seem to be low on the average.

SECTION 3

Information about mispricing by the Black-Scholes formula estimates is provided in this section¹³⁴.

Absolute Prediction Error versus Relative Prediction Error

In both the zero-dividend and the single-dividend subsamples, the dollar difference of formula estimate from the market price, referred to as absolute prediction error says that

¹³³ Jarrow and Rudd (1983) provides the sufficient condition for no early exercise. This implies the necessary condition for early exercise.

¹³⁴ We did not find any unusual market behavior around the time of our observation, except for the general market uncertainty about President Reagan's economic policies.

the options are, on the average, overpriced by the model¹³⁵. On the other hand, the percentage (taken out of formula estimate) difference, alternatively called the relative prediction error, indicates average underpricing¹³⁶.

For the zero-dividend subsample, we observe the following from Table 5.2:

(a) there are more out-of-the-money options than in-the-money options¹³⁷;

(b) the number of overpriced options is almost double the number of underpriced options;

(c) more out-of-the-money options are overpriced than in-the-money options;

(d) the overpricing of out-of-the-money options is substantially higher than that of the in-the-money options.

Thus, the absolute prediction errors are expected to be negative (meaning overpricing) on the average. These, when weighted by the inverse of the formula estimates, the highly overestimated out-of-the-money option values are expected to

¹³⁵ The mean absolute prediction error is -0.229993 for the zero-dividend subsample and -0.293016 for the single-dividend subsample. The mean relative prediction errors are 2.38457 and 0.062257 respectively.

¹³⁶ Black (1975), pp.41, observed that there are times when most traded options seem underpriced (overpriced). He advanced two explanations: (a) market's estimates of volatilities to be generally lower (higher) than the estimates used in the formula; (b) factors unrelated to the Black-Scholes model may be affecting option prices.

¹³⁷ The indicator of moneyness we would use here, is, $\ln(S/X)$, unless otherwise mentioned explicitly.

lead towards positive value for the average relative prediction error.

A similar explanation applies to the single-dividend case. In addition, we note that a greater proportion of overpriced out-of-the-money options and their larger mean absolute prediction error lead to to a smaller (relative to the zero-dividend subsample) mean relative prediction error for the single-dividend subsample.

The above discussion indicates that a probable source of conflicting empirical results could be the difference in the measures of formula mispricing adopted.

Formula Mispricing and Categorisation of Options

As we have mentioned in 2, one of the tools of reaching conclusion about formula pricing, used in the existing empirical studies, has been to do broad stratification of the sample options, and then compare the strata means of the measure of mispricing. Our grouping in Table 5.2 is not exactly similar. But if we compare the mean mispricing (dollar deviation) of our groups, we would notice the following:

in both the subsamples, options that are relatively deeper-away-from-the-money (closer-to-the-money), of relatively shorter (longer) time to maturity, and with relatively lower (higher) estimated volatility rate on the underlying stock's return, tend to be underpriced (overpriced).

To be more careful, this observation does not enable us to relate formula mispricing systematically to any one of the individual factors. For example, it would not be proper to say that shorter(longer) time to maturity options are underpriced(overpriced), since we are unable to say, at least at this stage, whether the underpricing -overpricing is due to the difference in any one of the factors alone, or some combination of their differences. As was pointed out in chapter 2, this is a major shortcoming of the broad classification technique., and hence of the existing results based on this technique.

We also note that neither Black(1975)'s nor MacBeth and Merville(1979)'s dichotomous striking price bias is supported by our subsamples. Both in-the-money and out-of-the-money options are, on the average, overpriced, rather than one group being being underpriced, while the other being overpriced. As can be seen from Table 5.2, dichotomous bias pattern can be seriously affected by the sample mixture of options. For example, had the zero-dividend subsample been comprised of only the groups of ZUO(ZOO) and ZOI(ZUI), MacBeth and Merville(Black) would have been supported by the grouping technique¹³⁸.

Plotting or Regressing Formula Mispricing

Against a Single Factor

¹³⁸ We are not being strict here about Black's grouping deeper-in-the-money versus deeper-out-of-the-money, rather than in-the-money versus out-of-the-money, the latter used by MacBeth and Merville.

When we plot formula mispricing against moneyness, no discernible pattern in the relationship emerges in any of the subsamples. For time to maturity and estimated volatility rate, there seems to appear inverse relationships, these relationships being less strong in the case of the zero-dividend subsample.

The simple regression results reported in Table 5.3 bear support to the above visual observations. Slope coefficient for moneyness is not statistically significant (at 5% significance level) in any of the two subsamples. Those for time to maturity and estimated volatility rate are negative and significant in both the subsamples. As we may like to recall, Whaley (1982)'s simple regression results for the formula estimate are similar ¹³⁹.

We should keep in mind that although visual observations or simple regression results do not really help establish conjectures about marginal biases, they prompt us to explore with more appropriate techniques.

Formula Mispricing and Black-Scholes Validity

As established in chapter 4, a familiar result about the nonlinearity bias of the formula is that the formula estimate tends to underestimate (overestimate) the model price for

¹³⁹ Whaley considered both simple stock price adjustment and Pseudo-American versions of the Black-Scholes, and the measure of mispricing was relative prediction error. We are considering only the simple stock price adjustment version of the Black-Scholes, and our measure of mispricing is absolute prediction error.

closer-to-the-money and at-the-money(deeper-away-from-the-money) options. Though an observed(empirically) pattern similar to this does not necessarily validate the Black-Scholes model, one nearly opposite to the above pattern would raise reasonable doubts about the validity of the Black-Scholes model.

In both of our subsamples, the mispricing pattern is nearly opposite to that of the nonlinearity Bias of the formula. This can be seen from Table 5.2 and Figures 5.1 and 5.4 . Relatively closer-to-the-money(away-from-the-money) options tend to be overpriced(underpriced). In addition, the two at-the-money options in the zero-dividend subsample are overpriced by the formula estimate. Though there is the possibility of sampling error, these patterns(all in the opposite direction to the nonlinearity bias) would be too much of a chance fluctuation.

Thus, at this elementary level of investigation at least, we do not find support for the Black-Scholes model. This conclusion of ours is based upon analysis similar to the dichotomous bias approach, the soundness of which is very much in question. In chapter 2, the functional bias approach and the multiple regression analysis was suggested as a relatively better package. We now turn to our multiple regression results.

SECTION 4

Constant Coefficient Multiple Regression Results

The multiple regression model we presented in chapter 3, yielded the estimable regression equation:

$$C_j - CB(\theta_j^T) = \theta_j^T \psi_j + \eta_j$$

As we remember, this estimable version followed the specification of the volatility rate as a regressor in θ_j^T , and using estimate of the volatility rate as a proxy for the true volatility rate. For comparability with the existing results, and also for the ease of interpretation, we are now suggesting the moneyness ($\ln(S_j/X_j) = m_j$) and the time to maturity (T_j) of the option as the two other regressors in θ_j^T .¹⁴⁰ Adding an intercept, we can write the regression equation as:

$$C_j - CB(\theta_j^T) = a_j + \psi_{1j} m_j + \psi_{2j} T_j + \psi_{3j} \hat{\sigma}_j + \eta_j \dots \dots \dots (5.1)$$

The coefficients in (5.1) are subscripted by observation to denote option-specificness of marginal biases. In chapter 4, we have seen that the nonlinearity bias of the formula alone leads to option-specific marginal biases. This means that the coefficients in (5.1) are functions of at least the included variables.¹⁴¹ Note, in addition, the coefficients are not the

¹⁴⁰To recall, we are using simple stock price adjustment version of the Black-Scholes. The formula is the same as that for the European call version, but the stock price is adjusted for the escrowed dividends, if any, and the volatility rate is from the stochastic process of the adjusted stock price.

¹⁴¹ When the Black-Scholes model is not valid, the coefficients may be functions of some additional variables not included here. An example would be the variable--the lag between the ex-dividend day and the maturity day, if the Roll-Geske-Whaley model is valid. We remind the reader that we are not attempting to test options market efficiency, we are rather accepting it as a maintained hypothesis.

option-specific marginal biases¹⁴². If there were no nonlinearity bias of the formula, then the validity of the Black-Scholes model would have implied zero values for the coefficients. The indifference to the nonlinearity bias may have led the users of regression to interpret the coefficients as (marginal)biases.

If we assume that the coefficients are constant across observations, considerable simplification in estimation is achieved. Constant coefficient estimation implies: for each factor, we are approximating the function relating mispricing to the factor by a straight line, the slope of which is invariant to the level of any of the factors, at least over the range of sample variation. For example, let us consider the coefficient of moneyness in the zero-dividend subsample. Suppose that the Black-Scholes model is valid. Then, Figures 4.2 to 4.4 tell us that the marginal moneyness bias would depend, both in terms of sign and magnitude, on the levels of moneyness, time to maturity, and true volatility rate. Thus, if it were possible to write the nonlinearity bias in the form of $\theta^T \beta_j$, the coefficient Ψ_j in (5.1), which is equal to β_j in this case, would be a function of the variables mentioned¹⁴³.

¹⁴² In chapter 7, our exposition on the basis of second order approximation to the nonlinearity bias shows that when the Black-Scholes model is valid, the coefficients would be functions of the marginal biases.

¹⁴³ In chapter 7, we show that the second order approximation to the nonlinearity bias can be written in the form $\theta^T \beta_j$.

Now suppose that Merton's jump diffusion model is valid, and the model misspecification error of Black-Scholes overwhelmingly outweighs the nonlinearity bias of the Black-Scholes formula estimate. Merton(1976a)'s Figure 1, pp 341, indicate that the marginal moneyness bias would depend on the level of moneyness. Though figures were not drawn for varying levels of time to maturity, total volatility rate, jump frequency, or the relative contribution of the jump component to the total volatility rate, his tables bear indirect support that the curve in Figure 1 would be shifting due to these variations. Thus, the coefficient Ψ_{1j} in (5.1), which would now be equal to ϕ_{1j} , will be a function of the included variables plus some extra variables.

In both cases of our example, the assumption of a constant slope coefficient would be reasonable only if the sample options have extreme similarity in terms of the variables in question. A practical problem to identify such a sample is the unknown nature of the true volatility rate or parameters determining it¹⁴⁴.

Unless a truly homogeneous sample is available, one should be careful in interpreting the estimated constant coefficients as the marginal biases, though they may appear to be so. With this thought in mind, we have taken advantage of the simplicity of constant coefficient estimation in chapters 5 to 7. But we

¹⁴⁴The design of a homogeneous sample is an item on our agenda for future research.

have tried to avoid unqualified interpretation of the coefficient estimates as marginal biases.

The Ordinary Least Squares(OLS) regression results for (5.1) are presented in Table 5.4, both for the zero-dividend subsample and the single-dividend subsample. As we consider the statistical significance of individual coefficients at 5% significance level, we do not find the coefficient of the moneyness regressor to be significant in any of the subsamples. Whaley(1982) and, Geske and Roll(1984a) found similar result. Geske and Roll(1984a) pointed out that the (so-called)striking price bias is more pronounced in comparing options on the same stock. But we would have reservation about interpreting this insignificance as the insignificance of the marginal striking price or moneyness bias, though our exposition in chapter 7 indicates that this may be the case under certain circumstances¹⁴⁵.

The coefficient of time to maturity is significant in both subsamples. We noted that this regressor has high variability which may have contributed to the significance.

The coefficient of volatility rate has been found to be significant in the single-dividend subsample, but not in the zero-dividend subsample. The latter part of this result is in contrast to all previous studies, where the coefficient was

¹⁴⁵ On the basis of second order approximation to the nonlinearity bias, the coefficient of moneyness is a ratio of two option-specific magnitudes, the marginal moneyness bias being in the numerator. Thus smallness of this marginal bias may also imply smallness of the coefficient.

found to be persistently significant.

Similar to the preceding result, the intercept gained statistical significance from the zero-dividend subsample to the single-dividend subsample. It is to be mentioned that both Whaley(1982) and, Geske and Roll(1984a)'s regression results had significant intercepts. As we may recall from chapter 3, the intercept term would partially capture the effect of any relevant variable excluded from the regression. In this case, some early-exercise-related variable might have caused the result¹⁴⁶.

The F-tests of the significance of regressions indicate rejection in both the subsamples of the hypothesis that all the slope coefficients in a regression are jointly zero.

In the light of our observations about testing model validity in chapter 3, the above results seem to lend support to the view that: (a) the Black-Scholes model may be complete in regressors, but functionally wrong in the non-dividend-paying case; (b) the Black-Scholes model may both be incomplete in regressors, and functionally wrong (in the included regressors) for the dividend-paying case. But we would like to emphasize that our observations in chapter 3 were based upon option-specific regression, while the present results are from constant coefficient regression.

¹⁴⁶ There may also be an statistical explanation in that the sheer magnitude of the coefficient of volatility rate in the single-dividend subsample is responsible for the significance of the intercept there.

Regarding the signs of the coefficients, if we may refer to the forthcoming expositions in chapter 7, one is more likely to be misled in interpreting the signs of the estimated constant coefficients as the directions of marginal biases than in equating the magnitudes (zero versus non-zero). However, an interesting feature of our result regarding the signs, which deserves some discussion, is that the coefficients of moneyness and volatility rate changed their signs from the zero-dividend subsample to the single-dividend subsample¹⁴⁷. In the zero-dividend subsample, the sign of the coefficient of moneyness seem to support MacBeth and Merville(1979)'s finding, while in the single-dividend subsample, Whaley(1982) and, Geske and Roll(1984a) are supported. For the coefficient of volatility rate, the less-debated negative sign appears in the single-dividend subsample, while the zero-dividend subsample produces the first(to our knowledge) finding of positive sign.

Let us see if we can advance some suggestion as to the cause of such reversal on the basis of our regression model in chapter 3.

Consider the hypothetical case that the jump-diffusion model of Merton provides the fair value(CM) of a non-dividend-paying option . In the dividend-paying case, some appropriate model, corresponding to the jump-diffusion process, would provide the fair value(CMD). In the zero-dividend case, the

¹⁴⁷ It may not be quite appropriate to consider sign reversal of coefficients which tested to be not significantly different from zero. Thanks are due to Francis Boabang for pointing this out.

coefficient vector $\Psi^{(0)}$ would have two parts, one part $\Phi^{(0)}$ coming from the difference CM-CB, and the other $\beta^{(0)}$ arising from the nonlinearity bias of the Black-Scholes formula. In the single-dividend case, $\Phi^{(1)}$ can be thought to have two parts, $\Phi^{(1)}(a)$ due to CMD-CM, and $\Phi^{(1)}(b)$ from CM-CB. Thus, including $\beta^{(1)}$ from the nonlinearity bias of the Black-Scholes formula, $\Psi^{(1)}$ would have three components. Unless the options in the two cases are very different (except for dividend), we would expect $\Phi^{(0)} \approx \Phi^{(1)}(b)$, and $\beta^{(0)} \approx \beta^{(1)}$. But $\Phi^{(1)}(a)$, the difference in pure model misspecification error of Merton's in terms of θ^T , may cause the signs of some or all elements in $\Psi^{(1)}$ to be different from the signs of the corresponding elements in $\Psi^{(0)}$.¹⁴⁸

Note also that part of the model misspecification error may come in the form of omitted variable effect, $\delta^T \Phi_j$. Ordinarily, we would imagine that this effect get embodied in the intercept estimate, thus leaving the slope estimates unaffected. But we should not overlook the possibility that the omitted variable(s) is(are) significantly correlated with some or all of the included regressors. Under such circumstances, the OLS estimates of the included regressors would be biased. The signs and

¹⁴⁸ Thus, for example, the coefficient for moneyness would have three components in each subsample. Let the component coming from CM-CB be equal across the subsamples and same be the case for the component coming from nonlinearity bias. But the remaining component coming from CMD-CM in the single-dividend subsample, which is zero in the zero-dividend subsample, will lead to different coefficient values across the subsamples and may even lead to different signs.

magnitudes of these biases would depend upon that of the correlation (between the included and the excluded) and the coefficients of the excluded variables (if they were included). Thus, the phenomenon of omitted variable may contribute to sign reversal of included regressors' coefficient estimates in two ways: (i) variables omitted in the two cases are different; (ii) the same omitted variables, but having different coefficients in the two cases.

Comparison with Simple Regression Results

It may also be interesting to consider the omitted variable effect in the context of simple regression results such as ours or Whaley (1982)'s.

There are two sign reversals in our case, as we go from multiple regression to simple regression results. In the zero-dividend subsample, the multiple regression coefficient of volatility is 1.60941, whereas the simple regression coefficient is -15.2183. In the single-dividend subsample, the coefficient of moneyness change from -0.000703 in the multiple regression to 0.202008 in the simple regression.

Following Goldberger (1964), pp. 194, we decomposed each of these multiple regression coefficients into the corresponding simple regression coefficient and another term incorporating the omitted variable effect. In the case of volatility rate's coefficient, strong positive covariance of estimated volatility rate with time to expiration, in association with negativity of

time to expiration's multiple regression coefficient produced the sign reversal. In the case of the coefficient of moneyness, large negative multiple regression coefficient of volatility rate coupled with negative covariance between moneyness and estimated volatility led to the sign reversal.

Our results above raise doubt about the robustness of the simple regression results claimed by Whaley(1982), pp.48. ¹⁴⁹ MacBeth and Merville(1979)'s finding of positive coefficient for the measure of moreyness in their volatility rate-excluded two variable regressions, is also cast in doubt by our results. It appears that the omitted variable effect may have contributed to the conflicting results about the direction of the so-called striking price bias.

SECTION 5

In the foregoing section, we noted that the coefficients of moneyness and volatility rate changed their signs from the zero-dividend subsample to the single-dividend subsample. We advanced two probable reasons, (i) the difference in the model misspecification error of the Black-Scholes, and (ii) the difference in the omitted variable effect across the two subsamples. In this section, the stochastic regressor problem is identified as yet another source of sign reversal. The stochastic regressor problem arises due to the fact that the

¹⁴⁹Whaley reports that his simple regression results remain virtually unaffected when various combinations of the regressors are used instead.

regressor volatility rate is measured with error, when we use estimated volatility rate in its place.

To begin with, the stochastic regressor problem is to be separated from a similar problem, namely, the errors-in-variable problem. The latter problem has been discussed extensively in the context of grouping of portfolios for empirical testing of the Capital Asset Pricing Model, and was first mentioned by Black and Scholes(1972) in the context of testing the Black-Scholes option valuation model.

Errors-in-variable Problem of Black and Scholes(1972)

On the basis of historically estimated variances, Black and Scholes ranked the stocks from the minimum to the maximum estimated variance, and assigned options on the 25 percent of the stocks with the lowest estimated variances to the first portfolio, the options on the next 25 percent of the stocks to the second portfolio, and so on. Then buying the options at model prices estimated with the estimated variance rates, and adjusting the hedge return of portfolios for the market risk, Black and Scholes found that the two portfolios with the lower(higher) estimated variances gave positive(negative) excess hedge returns. Thus, they concluded that the model(estimated) overpriced options on high variance(estimated) stocks, and underpriced options on low variance(estimated) stocks. In this last sentence, the parenthesized words are ours, and we think they are important. They attributed this phenomenon to the

errors-in-variable problem.

Let us now outline the errors-in-variable problem ¹⁵⁰.

If the true variance rate of a stock is low, then if the estimated variance rate for the stock is also low (among the set of estimated variance rates), it will duly be assigned to a low variance category. But due to measurement error, if the estimated variance rate comes out high, it will be unduly placed in the higher variance rate category.

If the true variance rate of a stock is high, then if the estimated variance rate is high, it will duly be categorised as high variance. But due to measurement error, if it is estimated low, it will erroneously be placed in a lower variance category.

Thus Black and Scholes's portfolios of lower (higher) 50 percent estimated variances will probably contain options on stocks with the true variance rate being high (low).¹⁵¹ For these options, the estimated model price will be lower (higher) than the fair price or the model price. For the duly assigned options, expected excess hedge return, when transaction at model price, being zero, the erroneously assigned options will lead to positive (negative) excess hedge return for the respective portfolio. The estimated model price thus would seem to underprice (overprice) options on low (high) estimated variance stocks. But the note of caution is that, due to measurement

¹⁵⁰For this part, I have greatly benefitted from Prof. Barry Schachter's ECON 817 lectures.

¹⁵¹The errors would be severe for the extreme groups.

error, true low(high) variance stocks are expected to have positive(negative) measurement errors more often. Thus if the model is valid, options on low(high) variance stocks will be overpriced(underpriced) by the model, relative to the market.

Stochastic Regressor Problem

The errors-in-variable problem discussed above arises in the context of the hedging-technique-cum-dichotomous-bias approach to the validation of the Black-Scholes model. The stochastic regressor problem, on the other hand, arises in the context of the regression-technique-cum-functional-bias approach to validation.

The regression model we presented in chapter 3 shows that one of the regressors, the volatility rate, can only be measured with error. In other words, given that the true volatility rates are not known, we can only use their estimates, which are subject to measurement error. Thus the applied regressor, the estimated volatility rate, would be correlated with the error term in the estimable regression equation. Under such circumstances, it is a well-established econometric result that the OLS estimates of the regression coefficients would be biased and asymptotically biased. Under the simplifying assumption that the measurement errors in volatility estimates are identically distributed across stocks, and that the moments of the measurement errors of order greater than two are approximately

zero, we derived an expression for the asymptotic bias of the OLS estimates ¹⁵².

Asymptotic bias vector:

$$[0 \ 0 \ H] \Sigma^{-1}$$

where

$$H = \sigma_j^{2*} EA \left[\left\{ \frac{d_1}{d_2} \right\} \left\{ \frac{\partial CB(\theta^T)}{\partial(\hat{\sigma}_j)} \right\} \Big|_{\hat{\sigma}_j = \sigma_j} \right] - \sigma_j^{2**} EA \left\{ \frac{\partial CB(\theta^T)}{\partial(\hat{\sigma}_j)} \right\} \Big|_{\hat{\sigma}_j = \sigma_j} - \psi_3 \sigma_j^{2**} \Big],$$

σ_j^{2*} the population variance of the regressor σ_j

σ_j^{2**} the assumed common variance of the measurement errors

ψ_3 the assumed option-non-specific coefficient of volatility in our estimated regressions ¹⁵³

EA is the asymptotic expectation operator

and Σ the population dispersion matrix of the regressors, assumed to exist and nonsingular.

As we can see, typical of the stochastic regressor problems, the asymptotic bias contains the unknown parameter ψ_3 .

The direction of the asymptotic bias in the OLS coefficients will depend on the sign of the covariance of the composite error with the volatility estimate, and the magnitudes and signs of the covariances of the regressors among themselves.

¹⁵² These simplifying assumptions are much less restrictive than they might appear, when the sample size from which the volatilities are estimated is large.

¹⁵³ In option-specific estimation, ψ_3 can be thought as the population first central moment of the ψ_3 's.

The covariance of estimated volatility and the composite error term will depend on the relative strengths of the components in the latter term. If the sampling error in the nonlinearity bias of the formula is the major component in the composite disturbance, the first two terms in the covariance expression H will dominate. In that situation, if d_{1j} and d_{2j} tend to be of the same sign, and half of their product be less than one, then a negative covariance will be expected. And the extent will depend upon $\{\partial CB(\theta^T) / \partial(\hat{\sigma}_j) | \hat{\sigma}_j = \sigma_j\}$'s and (d_{1j}, d_{2j}) 's.

By now, it is clear that if the asymptotic bias problem is severe, this may lead to sign reversal of coefficients across different samples. Let us now examine the sign reversal in our case.

In both the subsamples, d_{1j} and d_{2j} tended to have same sign, and $(d_{1j}, d_{2j} / 2)$ seemed to be less than 1, on average. We would thus expect negative covariance, larger for the single dividend subsample, for two reasons:

- (i) mean $\{\partial CB(\theta^T) / \partial(\hat{\sigma}_j) | \hat{\sigma}_j = \sigma_j\}$ higher;
- (ii) mean $(d_{1j}, d_{2j} / 2)$ lower.

The sign pattern of the elements in the last column of the inverse of the estimated variance-covariance matrix of regressors is:

[+ - +]^T

Thus, in both the subsamples, we would expect underestimation for the coefficients of moneyness and volatility rate, and overestimation for the coefficient of time to

expiration, but more so for the single-dividend subsample. The observed sign reversal from the zero-dividend subsample to the single-dividend subsample is in conformity with this expectation. Bear in mind, however: (i) our expectation is on the basis of some simplifying assumptions, and sample estimates of the relevant magnitudes in the asymptotic bias expression; (ii) more than one probable sources of sign reversal might have interacted.

The existing regression studies using the estimated volatility rate as a regressor also suffer from the stochastic regressor problem similar to ours. But the problem was not identified as such. The question remains: what can be done about this problem and whether the problem is severe at all.

To avoid the stochastic regressor problem, a natural alternative that may come into consideration is that of dropping the volatility rate as a regressor, as was the case in MacBeth and Merville(1979)¹⁵⁴. We may think that this would improve (lower bias) the coefficient estimates for the remaining regressors. McCallum(1972) and Wickens(1972) showed that dropping the errored variable would lead to higher asymptotic biases for the remaining coefficients. Aigner(1974)'s results, based on mean square error, also broadly supports the use of the errored variable. Thus, if the problem of stochastic regressor is serious, MacBeth and Merville(1979)'s and Whaley(1982)'s

¹⁵⁴ MacBeth and Merville, of course, do not talk about the stochastic regressor problem.

results would be more severely affected than Geske and Roll(1984a)'s or ours.

The other alternative is to use some alternative estimation procedure than OLS. Two of the well known procedures are the Instrumental Variables and the Maximum Likelihood estimation. Under the former procedure, any instrument correlated with the volatility rate, but uncorrelated with the error term in the regression equation, would lead to consistent estimates for all the coefficients. It would be difficult to find such an instrument given the nature of the composite error term in our regression model. Even if we could find one, the Instrumental Variables Estimators do not have the minimum asymptotic variance. Moreover, given our objective of validating the Black-Scholes model, use of instrument for the volatility rate may bias the results against it.

Maximum Likelihood Estimation, on the other hand, leads to consistent and efficient estimates. But as shown in Judge, et al,(1980), the procedure breaks down without information in addition to that provided by the sample. Unfortunately, the information requirements considered there will be unknown in our context.

We may, then, wonder whether the problem of stochastic regressor in our context is serious at all. There is no clear answer to this. If we overlook the sampling error of the nonlinearity bias, or assume it to be relatively a negligible component in the composite error term, some comments can be

made. The complex covariance expression H simplifies to $-\psi_3 \sigma^{2**}$ ¹⁵⁵.

The asymptotic bias of the coefficient for volatility rate would be small if the cross-stock variability of the volatility rate is relatively larger than the variability of the measurement errors ¹⁵⁶. Our situation does not look very different from this.

The asymptotic bias for the other coefficients will be small if that for the coefficient of volatility rate is small, and in addition, the probability limits of the coefficients of regression of volatility rate on these regressors are small.

Admittedly, the stochastic regressor problem will persist, for which we neither find a satisfactory solution, nor a strong a priori basis to gauge its severity. For the rest of this thesis, we shall overlook this problem.

SECTION 6

To summarise our findings in this chapter, neither Black(1975)'s nor MacBeth and Merville(1979)'s dichotomous striking price bias is supported by our zero-dividend and single-dividend subsamples. Contrary to the prediction of the nonlinearity bias of the formula, we find relatively

¹⁵⁵Note that the previous predictions about the directions of asymptotic biases in our subsamples would remain good, if $\psi_3 > 0$.

¹⁵⁶A smaller multiple correlation coefficient of volatility rate with the other regressors will also help reduce the asymptotic bias.

closer-to-the-money(away-from-the-money) options to be overpriced(underpriced) by the formula estimates. But the soundness of the technique producing this result is questionable.

We then moved to constant coefficient multiple regression, and found the coefficients of moneyness and volatility rate to change direction from the zero-dividend subsample to the single-dividend subsample. Three probable sources, all in the light of the regression model presented in chapter 3, of this sign reversal were discussed, since such reversals prompted substantial research, e.g., MacBeth and Merville(1979,1980), Emanuel and MacBeth(1981), Sterk(1982), Whaley(1982), Geske and Roll(1984). These are: (i)the difference in the model misspecification error of the Black-Scholes, in terms of the included regressors, and (ii)the difference in the omitted variable effect across the subsamples; and (iii)the asymptotic bias of the OLS estimators arising from the use of estimated volatility rates as proxy for the unknown true volatility rates.

The estimation results in this chapter did not provide any substantial evidence about the validity of the Black-Scholes model or the dividend inducement of the systematic deviations of Black-Scholes formula estimates. In the next chapter, we utilize the multiple regression results of this chapter to test dividend inducement.

CHAPTER 6:
TESTING DIVIDEND-INDUCED SYSTEMATIC BIASES OF
BLACK-SCHOLES FORMULA ESTIMATE

The European call valuation model of Black and Scholes(1973) does not take into account the early exercise possibility of Unprotected American Calls(UAC) which are the most widely traded options. Black(1975) proposed the pseudo-American valuation of UAC which uses the Black-Scholes European formula. But this valuation constrains the early exercise probability to a zero-one variable. Schwartz(1977) proposed a numerical valuation procedure. Later, in other papers, Roll(1977), Geske(1979a), and Whaley(1981) developed a closed form solution ¹⁵⁷.

On the empirical front, with possibly a few exceptions such as Black and Scholes(1972), researchers continued to examine the Black-Scholes pricing using UAC data. A common finding is that the Black-Scholes formula (European, Simple Stock Price Adjustment, or pseudoAmerican version) estimates tend to deviate from the actual market prices in certain systematic ways. Three of the most popular factors to which the deviations have been systematically related to, are: moneyness (or striking price) and time to maturity of the option, and the volatility rate on the underlying stock's return. Since the data used for empirical tests were mostly on UACs, a legitimate concern is whether the findings of these systematic relations are essentially

dividend-induced. If they are, the use of a model such as the

¹⁵⁷These valuations assume lognormal diffusion process for the stock price adjusted for escrowed dividends. No closed form solution has yet been developed for the two other well known stochastic processes--the constant elasticity of variance and the jump-diffusion.

one developed by Roll, Geske, and Whaley should be able to eliminate the systematic tendencies of the Black-Scholes.

Whaley(1982) found the Roll-Geske-Whaley model to eliminate all the systematic tendencies of Black-Scholes except for the one related to the volatility rate. Sterk(1982) reported that the Roll-Geske model reduces the striking price bias¹⁵⁸. Sterk(1983) found that the improvement of the Roll-Geske-Whaley model is economically significant for the range 0.3 to 0.7 of the early exercise probability, and size of the single dividend more than a dollar. In two other studies, Blomeyer and Klemkosky(1983), and Gultekin, Rogalski, and Tinic(1982) concluded that the bias characteristics of the Roll-Geske-Whaley model is identical to that of the Black-Scholes. Later, Geske and Roll(1984a) suggested that the systematic deviations of the Black-Scholes with respect to moneyness and time to maturity are essentially dividend-induced, while the one related to the volatility rate is a measurement error problem.

The empirical studies are to be evaluated keeping in view the limitations of the testing procedures used. The interpretation of the test results is equally important. For example, we need to know what is being interpreted as systematic relationship and dividend inducement of such relationship. In this chapter, we draw on our discussions in the previous chapters to throw light on these aspects and offer an

¹⁵⁸ Sterk(1982) used the version of American valuation model prior to the corrections of Whaley(1981), while Sterk(1983) used the corrected version.

alternative test of dividend inducement.

Section 1 is a brief review of the studies mentioned above. In section 2, we propose an alternative test of dividend inducement. The test results are presented in the section 3. Section 4 concludes the chapter.

SECTION 1

Whaley(1982), using ISD, computed formula estimates for the simple stock price adjustment and the pseudo-American versions of the Black-Scholes, and also for the Roll-Geske-Whaley model. He then ran simple regressions of percentage deviation (from the market price) on moneyness(g), time to maturity, estimated volatility(from ISDs) rate, early exercise probability estimate, et cetera¹⁵⁹. The statistical significance of the simple regression coefficients was interpreted as the existence of systematic relationships, and decrease(or elimination) in the statistical significance, from the Black-Scholes to the Roll-Geske-Whaley, as significant dividend inducement.

Sterk(1982) plotted both the percentage and dollar deviations of pseudo-American and Roll-Geske estimates against moneyness(g) ¹⁶⁰. He found that the latter, compared to the former, have lower number of underpriced out-of-the-money and

¹⁵⁹ Whaley(1982) performed multiple regressions also, but did not report the estimated equations. He mentioned that no essential difference followed from the use of multiple regression.

¹⁶⁰ Sterk used MacBeth and Merville(1979)'s procedure to estimate volatility rate from the ISDs.

overpriced in-the-money options. Since this reduces Black(1975)'s direct striking price bias, this was taken to mean that the the bias was essentially dividend-induced. The Roll-Geske estimates were also found to have lower average percentage deviation and average dollar deviation.

Blomeyer and Klemkosky(1983) plotted the percentage deviations of the simple stock price adjustment version of the Black-Scholes , and the Roll-Geske-Whaley, against moneyness(g). They found that both underprice out-of-the-money and price fairly well at and in-the-money options. They also compared ex-post mean holding period return of hedges based upon the estimated prices of the two models, and did not find statistically significant difference. In addition, Blomeyer and Klemkosky stratified the options into three portfolios based upon the dividend yield of underlying stocks. For none of these portfolios, the two models yielded significantly different mean hedge return.

Gultekin, Rogalski, and Tinic(1982) performed multiple regression of the dollar deviations of the Roll-Geske-Whaley estimates on moneyness(S-X), time to maturity, and estimated(from past series of return) volatility rate. The coefficients were found significant. The negative coefficients for moneyness and estimated volatility rate were interpreted as similar to Black(1975)'s striking price and, Black and Scholes(1972)'s variance bias of the Black-Scholes formula estimates. Thus, follows the implication that these biases are

not dividend-related¹⁶¹.

All the studies above are based on pooling of time series and cross-section data¹⁶². Geske and Roll(1984a) used cross-section data only, thus avoiding the econometric problems related to pooling and time series. They ran two regressions of the dollar deviations of the Black-Scholes formula(simple stock price adjustment version) estimates on moneyness($\ln(S/X)$), time to maturity, and estimated(from historical series of returns) volatility rate. One regression was for the whole sample of options with dividend payments as well as no dividend payments on the stock prior to maturity, and the other for the subsample of options with no dividend payments prior to maturity. The coefficients for time to maturity and estimated volatility rate were significant for the whole sample, but not significant for the zero-dividend subsample. The coefficient for moneyness was not significant in any of the regressions, but the t-statistic declined in the zero-dividend subsample. These results were interpreted as evidence of significant dividend inducement for the striking price and the time to maturity biases, but not for the variance bias.

When we have a critical look at the studies mentioned above, we see that Whaley's suffer from absence of statistical test of difference in coefficients, and pooling and time

¹⁶¹ The authors do not make such a statement though.

¹⁶²See Judge et al(1980),pp 325-358, for an extensive discussion of pooling problems.

series-related problems¹⁶³. Sterk's suffer from insufficient control for variables, dichotomous bias dilemma, and absence of statistical test of difference in mean errors. Blomeyer and Klemkosky's is subject to inadequate control for variables (when plotting), and the problems associated with the hedging approach to model validation. Gultekin et al does not exactly compare the Black-Scholes with the Roll-Geske-Whaley, and may have pooling and time-series related problems. Also, they do not take into account that when the Black-Scholes model is not valid in the non-dividend paying case, the Roll-Geske-Whaley is not also valid in the dividend-paying case¹⁶⁴.

Geske and Roll's probably came close to a more reliable test of dividend inducement. But the sample mixture of dividend-paying versus non-dividend-paying options in the total sample may seriously affect their results through affecting the statistical significance of coefficients. Moreover, no statistical test of difference in the coefficient values across the equations was undertaken. In the following section, we propose an alternative test which uses the basic idea of Geske and Roll, but attempts to eliminate its deficiencies.

¹⁶³The reader is referred to chapter 2 for detailed discussion of most of the shortcomings to be mentioned here.

¹⁶⁴ The Roll-Geske-Whaley model differs from the Black-Scholes model in that the former assumes that the underlying stock pays dividend prior to maturity of the American call. In the absence of dividend payments, Black-Scholes gives the same value for an American as for a European. Thus, if the Black-Scholes does not provide the fair value of an American call with no dividends, we would not expect the Roll-Geske-Whaley model to provide the fair value were there any dividends prior to maturity.

SECTION 2

The test of dividend inducement we propose in this section is based upon the regression model presented in chapter 3, and our discussion of the omitted variable effect in chapter 5.

In our regression model, the expected response function embodies the (negative of) nonlinearity bias of the Black-Scholes formula estimate, and any model misspecification error in terms of the included regressors¹⁶⁵. The coefficients of the regressors would have two components corresponding to these two broad sources of systematic mispricing.

Now consider two options, one with a single dividend payment of known size prior to maturity, and the other with no such payment.

Consider the hypothetical case that the jump-diffusion model of Merton provides the fair value (CM) of the zero-dividend option. In the dividend-paying case, some appropriate model, corresponding to the jump-diffusion process, would provide the fair value (CMD). In the zero-dividend case, the coefficient vector $\Psi^{(0)}$ would have two parts, one part $\Phi^{(0)}$ coming from the difference $CM - CB$, and the other $\beta^{(0)}$ arising from the nonlinearity bias of the Black-Scholes formula. In the single-dividend case, $\Phi^{(1)}$ can be thought to have two parts, $\Phi^{(1)}(a)$ due to $CMD - CM$, and $\Phi^{(1)}(b)$ from $CM - CB$.

¹⁶⁵These regressors are the usual ones--moneyness($\ln(S/X)$), time to maturity, and estimated (from historical return data) volatility rate.

Thus, including $\beta_j^{(1)}$ from the nonlinearity bias of the Black-Scholes formula, $\Psi_j^{(1)}$ would have three components. Unless the options in the two cases are very different (except for dividend), we would expect $\Phi_j^{(0)} \approx \Phi_j^{(1)}(b)$, and $\beta_j^{(0)} \approx \beta_j^{(1)}$. But $\Phi_j^{(1)}(a)$, the difference in pure model misspecification error of Merton's in terms of θ_j^T , could still distinguish the coefficients for the two options. If dividend payment does make a significant difference in terms of the included regressors, some or all of the coefficients should be (statistically) significantly different across the two options.

Now suppose that the Black-Scholes model provides the fair value (CB) of the zero-dividend option. The Roll-Geske-Whaley model provides correspondingly the fair value (CR) of the single-dividend option. In the zero-dividend case, the coefficient vector $\Psi_j^{(0)}$ would have just one part $\beta_j^{(0)}$ arising from the nonlinearity bias of the Black-Scholes formula. In the single-dividend case, $\Psi_j^{(1)}$ will have two parts, $\Phi_j^{(1)}$ due to CR-CB, and $\beta_j^{(1)}$ from the nonlinearity bias of the Black-Scholes formula. Unless the options in the two cases are very different (except for dividend), we would expect $\beta_j^{(0)} \approx \beta_j^{(1)}$. But $\Phi_j^{(1)}$, the pure model misspecification error of the Black-Scholes model in terms of θ_j^T in the dividend-paying case, would still distinguish the coefficients for the two options. If dividend payment does make a significant difference in terms of the included regressors, again, some or all of the coefficient estimates should be

(statistically) significantly different across the two options.

We have seen that irrespective of the Black-Scholes validity in the zero-dividend case, dividend inducement would distinguish the coefficients of the dividend-paying case from those of the non-dividend-paying case. In empirical testing, if the dividend inducement is significant, we should find statistically significant difference across the two cases for some or all of the coefficients, no matter whether the Black-Scholes model is valid or not in the zero-dividend case.

In our example, the coefficients or parts thereof were subscripted to denote option-specificness of the coefficients. To estimate the coefficients, we would need samples of zero-dividend and single-dividend options. If we want to maintain the option-specificness, we would have too many differences to test with too few observations ¹⁶⁶. Thus, as in chapter 5, we would make the simplifying assumption of constant coefficient (within the sample for any case) estimation. Then, there will be at most 4 (including the intercept) differences (or equality) to be tested jointly, across the samples of zero-dividend and single dividend options.

Under the assumed constant coefficient estimation, the difference (or equality) of the coefficients across the two cases can be tested by undertaking a Chow-test of switching regime. Fortunate for us, the switching point is known. If we have a

¹⁶⁶The problem, in fact, is also to estimate too many parameters with too few observations.

total sample of single-dividend and zero-dividend options, and arrange the sample such that the single-dividend subsample follows the zero-dividend subsample, the point where the single-dividend subsample starts can be considered as the switching point, in analogy to time series data.

In total, three regressions are to be run, one each for the two subsamples, and the third for the total sample to test the joint dividend inducement for all the parameters (including the intercept, if included) ¹⁶⁷. The error sum of squares from the first two regressions would sum to the unrestricted error sum of squares (SSEU), and that of the last regression to be treated as the restricted error sum of squares (SSER) ¹⁶⁸. Then, under the null hypothesis of the restrictions being true, the following ratio will be central F-distributed with K and N-2K degrees of freedom for the numerator and the denominator respectively: ¹⁶⁹

$$\left[\frac{SSER - SSEU}{K} \right] / \left[\frac{SSEU}{(N - 2K)} \right] \dots\dots\dots (6.1)$$

where K is the number of parameters estimated in each

¹⁶⁷The parameters here refer to the ones to be estimated from regression.

¹⁶⁸SSER would be greater than SSEU, because of the additional constraint imposed upon the minimization problem.

¹⁶⁹ The Chow-test assumes that the variance of the error term is the same across regimes. Toyoda (1974) and, Schmidt and Sickles (1977) found that the nominal level of significance would differ from the true level of significance (probability of rejecting the null hypothesis when it is true) under unequal variances. To cope with such circumstances, Jayatissa (1977) offered an alternative central F-test and Watt (1979) proposed an asymptotic central chi-square test, while Honda (1982) compared the two tests with Monte Carlo. Given Honda's results, Watt's test may be preferred for our case. Thanks to Professor p. Kennedy for exposing me to this literature

regression, and N is the size of the total sample.

The test above is for the dividend inducement of the whole regression relationship (including the intercept). But we may be interested in the dividend inducement of a subset of the parameters. For this purpose, first observe that the SSEU above could have alternatively been computed from the following regression equation run over the total sample: ¹⁷⁰

$$C_j - CB(\theta^T)_j = a_j + D_j + \psi_{1j} m_j + \pi_{1j} D_{1j} + \psi_{2j} T_j + \pi_{2j} D_{2j} + \psi_{3j} \hat{\theta}_j + \pi_{3j} D_{3j} + \eta_j \dots (6.2)$$

where

$D_j = D_{1j} = D_{2j} = D_{3j} = 0$ for all observations of the zero-dividend subsample and

for observations of the single-dividend subsample

$$D_j = 1$$

$$D_{1j} = m_j$$

$$D_{2j} = T_j$$

$$D_{3j} = \hat{\theta}_j$$

Note that (6.2) is, in fact, the following two equations written together: ¹⁷¹

zero-dividend subsample

$$C_j - CB(\theta^T)_j = a^{(0)}_j + \psi_1^{(0)} m_j + \psi_2^{(0)} T_j + \psi_3^{(0)} \hat{\theta}_j + \eta_j$$

¹⁷⁰See Maddala(1977), pp.136-137,194-201.

¹⁷¹In (6.2), $\pi_i = \psi_i^{(1)} - \psi_i^{(0)}$.

single-dividend subsample

$$C_j - CB(\theta^T)_j = a^{(1)}_j + \psi_1^{(1)} m_j + \psi_2^{(1)} T_j + \psi_3^{(1)} \hat{\sigma}_j + \eta_j$$

The dividend inducement of a subset of parameters can be tested by treating (6.2) as the unrestricted version, and deleting corresponding dummy variables in (6.2) as the restricted version.¹⁷² Using the error sum of squares from these two versions, the relevant F-statistic can be computed from the general form of (6.1):

$$[(SSER - SSEU)/J] / [SSEU / (N - 2K)] \dots\dots\dots (6.1A)$$

where J is the number of restrictions, here the number of parameters for which we are testing dividend inducement.

This ratio will be distributed central F with J and N-2K degrees of freedom for the numerator and the denominator respectively, when the restrictions are true.

To give an example, suppose that we want to test joint dividend inducement for the intercept and the coefficient of moneyness only. The restrictions are represented in the null hypothesis of F-test:

$$H_0 :$$

$$a^{(0)} = a^{(1)}, \psi_1^{(0)} = \psi_1^{(1)}$$

The alternative hypothesis, of course, is that none of the four (including the above two) parameters need remain the same across the two subsamples.

¹⁷²See Skvarcius and Cromer (1971).

Two regressions are to be run over the total sample for this test. (6.2) would provide the unrestricted error sum of squares, and the regression deleting D_j and D_{j_1} from (6.2) would provide the restricted error sum of squares. These error sum of squares can be used to compute the ratio in (6.1A), which will be distributed central F with 2 and N-8 degrees of freedom.

Test of dividend inducement for a single parameter only, can be undertaken merely by a central t-test of the corresponding dummy variable's coefficient estimated from (6.2)¹⁷³. Alternatively, we can follow the procedure of the previous paragraph¹⁷⁴.

SECTION 3

Since we have four parameters to estimate for a subsample, there would be 14 subsets of the parameters, in addition to the set of all four parameters, for which test of dividend inducement can be undertaken. The unrestricted error sum of squares in all these tests would be the same. It can be calculated by adding the error sum of squares from the two multiple regressions of chapter 5, or alternatively from running (6.2)¹⁷⁵

¹⁷³See Gujarati(1970).

¹⁷⁴The resulting ratio in (6.1A) would be approximately equal to the square of the t-statistic in (6.2).

¹⁷⁵Due to computational rounding in regression results, we found the latter to be 0.002% higher than the former. To calculate the F-ratios, the larger magnitude was used. Since this would relatively lower the F-ratios, a significant F-ratio may indicate even stronger evidence against the null hypothesis. Our

Table 6.1 summarizes the test results¹⁷⁶. The constant coefficient regression relationship (including the intercept) for the deviation of Black-Scholes formula estimate from the market price as a whole seems to be significantly affected by the early exercise possibility of UAC's. This significant dividend inducement occurs in spite of the lack of strong early exercise possibility we found in chapter 5.

The coefficients of moneyness and time to maturity do not show significant dividend inducement, either individually or as a pair. The coefficient of volatility rate, on the other hand, exhibit strong dividend-induced effects both by itself and in combination with the coefficients of moneyness or time to maturity. This was indicated by the large difference between the estimates of the coefficient of volatility rate in the two multiple regressions of chapter 5.

The intercept shows dividend-induced effects similar to the coefficient of volatility rate. As we have mentioned in chapter 5, these effects may reflect those through the coefficient of volatility rate and/or part of the effects of relevant omitted variable(s), such as the time to ex-dividend date (given the time to expiration).

¹⁷⁵ (cont'd) test results are, however, unaffected by the choice between the two alternative magnitudes.

¹⁷⁶ To see whether the assumption of equal error variances across the two subsamples affects our test results, we undertook Watt (1979)'s chi-square test for our tests 1 and 2. We found similar results as reported in Table 6.1

It appears that the principal source of the significant dividend inducement for the entire regression relationship may be attributed to the coefficient of the volatility rate alone. This apparent importance of the volatility rate could be genuine in the sense that it is 'the' critical determinant of early exercise possibility. Or, the importance could be an econometric illusion. As mentioned in chapter 5, the use of estimated volatility rate as a proxy for the true volatility rate regressor leads to biased OLS coefficient estimates in both small and large samples. The sign as well as the magnitude of this bias for a coefficient estimate will depend upon the nature of sample options. If it is the coefficient of volatility rate (and thus affecting the intercept estimate too) which is affected most, it is more likely to reject the hypothesis of equality of a set of parameters when the volatility rate's coefficient is included in the set than otherwise. Thus, the volatility rate may appear to play an important role in the dividend inducement of the regression relationship, while in fact the stochastic regressor problem is responsible for this. However, our discussion in chapter 5 also indicates that the stochastic regressor problem may not be severe at all.

SECTION 4

To summarize, using the Chow test of switching regime, we have found that the constant coefficient regression relationship for the mispricing of the Black-Scholes formula estimate is

significantly affected by the early exercise possibility feature of UAC's. Previous conclusions that the striking price and the time to maturity biases are essentially dividend-induced are contradicted, while the volatility rate is found to be an important ingredient behind the dividend inducement of the mispricing by the Black-Scholes formula estimate.

Our test results are subject to the limitations of constant coefficient estimation. Further evidence of dividend inducement is to follow in chapter 8, where we allow limited degree of option-specificness in estimation. But the test of dividend inducement here was rather indifferent to the validity of the Black-Scholes model in the zero-dividend case, where the assumed known dividend plays no role. In the next chapter, we offer an indirect test of the validity of the Black-Scholes model in the zero-dividend case.

CHAPTER 7:

AN INDIRECT TEST OF BLACK-SCHOLES

BY TESTING RESTRICTIONS AMONG REGRESSION COEFFICIENTS

Whether the European call valuation model of Black and Scholes(1973) or some modified version can be considered to be a valid model representing the actual market prices of traded calls has been a key concern in recent empirical studies ¹⁷⁷. Galai(1983) surveys the studies in this area and also outlines the complications of testing model validity. An important problem is that such tests are, in general, joint tests of market synchronization, option market efficiency, and the validity of the model in question, in our case the Black-Scholes model¹⁷⁸ Further complications arise, since an essential ingredient of the Black-Scholes model- the volatility rate of the underlying stock's return is not known and has to be estimated.

Since the Black-Scholes formula is nonlinear with respect to the volatility rate, even if an unbiased estimate of the volatility rate is used, the resultant formula estimate would be a biased estimate of the model price(the formula price with the true volatility rate). In chapter 4, we have studied the systematic nature of this bias in detail.

The implication of the systematic nonlinearity bias of the formula estimate for testing the validity of the Black-Scholes

¹⁷⁷A popular modified version is the pseudo-American call valuation originally proposed by Black(1975). This model seeks to accommodate the early exercise possibility of unprotected American calls.

¹⁷⁸ It is to be noted that the hypotheses other than the validity of the Black-Scholes model would be accepted as maintained hypotheses in our testing and thus are not being tested.

model is that such tests will be biased towards finding evidence against the model, if the formula estimate is used as a proxy for the Black-Scholes model price. Given that the practitioners have almost invariably used some estimate of the volatility rate (or the variance rate) in the Black-Scholes European formula or the pseudo-American formula, the existing tests of the Black-Scholes model are themselves biased.

The purpose of this chapter is to undertake a test of Black-Scholes validity by taking advantage of the nonlinearity bias of the formula estimate rather than being victimized by it as the existing studies have been. For analytical tractability, we consider only up to second order term in the Taylor series of the nonlinearity bias ¹⁷⁹. Note that the bias implies restrictions among coefficients in our regression model. Since the nonlinearity bias alone would be the expected response function in our regression model when the Black-Scholes model is valid, an indirect test of the Black-Scholes model is provided by the test of implied restrictions among the coefficients.

In section 1, we describe our test of the Black-Scholes model validity. The test results are presented in section 2. And, some concluding thoughts follow in section 3.

¹⁷⁹Let $CB(\sigma)$ be the Black-Scholes model price, where σ is the true volatility rate. If we expand $CB(\hat{\sigma})$ around $\hat{\sigma}=\sigma$, and take expectation on both sides, then the nonlinearity bias would be:

$$E[CB(\hat{\sigma})]-CB(\sigma) = \sum_{r=1}^{\infty} \frac{\partial^r CB(\hat{\sigma})}{\partial \hat{\sigma}^r} \Big|_{\hat{\sigma}=\sigma} E(\hat{\sigma}-\sigma)^r / r! ,$$
where the summation runs over 1 to ∞ .

SECTION 1

If the Black-Scholes model is the unknown true model of fair valuation, the following equation emerges in our regression model:

$$C_j - CB(\theta_j^T) = \theta_j^T \beta_j + \epsilon_j \dots\dots\dots(7.1)$$

For the sake of convenience and continuity with previous research, we have chosen the moneyness $m(=\ln(S/X))$, time to maturity T , and volatility rate σ as the regressors in θ . For simplicity if we drop the subscript j , but keep in mind its existence, (7.1) can be written as :

$$C - CB(\theta^T) = [(\partial E / \partial m)m / R] + [(\partial E / \partial T) T / R] + [(\partial E / \partial \sigma)\sigma / R] + \epsilon \dots(7.2)$$

where E stands for the bias of the formula estimate in a second order Taylor series expansion, and

$$R = w_1 m + w_2 T + w_3 \sigma,$$

with

$$w_1 = [2h / (\sigma^2 T d_1 d_2)] - d_1 / (\sigma \sqrt{T})$$

$$w_2 = (1/2T) + [(8rTh - \sigma^4 T^2) / \{T(4h^2 - \sigma^4 T^2)\}] - [(2h + \sigma^2 T)(4rT - 2h + \sigma^2 T) / 8(\sigma T)^2]$$

$$w_3 = [d_1 d_2 - 1 - \{\sigma^2 T / (d_1 d_2)\}] / \sigma$$

$$h = \ln(S/X) + rT = \ln(g)$$

and to recall

$$d_1 = (h + 0.5\sigma^2 T) / (\sigma \sqrt{T})$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

where

S is the stock price

x is the striking price and
 r is the risk-free rate

In terms of our regression model,
 the coefficient of m is $(\partial E/\partial m)/R = \beta_1$
 the coefficient of T is $(\partial E/\partial T)/R = \beta_2$
 the coefficient of σ is $(\partial E/\partial \sigma)/R = \beta_3$

It appears that the coefficients of the regressors commonly used are less likely to be the marginal biases. They are more likely to be weights for the linear combination of m, T, and σ to represent the total (here the second order) bias. To see this, note that :

$$R = \{(\partial E/\partial m)m/E\} + \{(\partial E/\partial T)T/E\} + \{\partial E/\partial \sigma\} \sigma/E$$

That is, R is the sum of bias elasticities, and thus the coefficients are marginal biases relative to the sum of bias elasticities. In a linear regression of $C - CB(\theta^T)$ on m , T , and $\hat{\sigma}$, we would be assuming these ratios to be constant. And, of course, we would be estimating the assumed constant ratios, not the marginal biases. But it surely would be misleading to assume the signs and magnitudes of the estimated coefficients to be the same as of the marginal biases¹⁸⁰. This important aspect has hitherto been overlooked

¹⁸⁰ In chapter 5, we have mentioned that smallness of the coefficients may reflect smallness of the marginal biases.

and cannot be overemphasized.

Now notice that the marginal biases are:

$$(\partial E / \partial m) = E w_1, \quad (\partial E / \partial T) = E w_2, \quad (\partial E / \partial \sigma) = E w_3$$

Since they all have E in common, we can deduce that:

$$(\partial E / \partial m) / (w_1 R) = (\partial E / \partial T) / (w_2 R) = (\partial E / \partial \sigma) / (w_3 R)$$

$$\text{or, } (\beta_1 / w_1) = (\beta_2 / w_2) = (\beta_3 / w_3)$$

If the Black-Scholes model is valid, and the second order approximation is good enough, the coefficients of the regression model should satisfy the above restrictions for all observations. Note that for any observation, only two of the restrictions are independent.

One of the three probable sets of restrictions would be:

$$\beta_{2j} = \beta_{1j} w_{2j} / w_{1j}$$

$$\beta_{3j} = \beta_{1j} w_{3j} / w_{1j}$$

In this fully option-specific form, we would have too few(N) observations to test too many(2N) restrictions. In the context of constant coefficient estimation, a compromise would be:

$$\beta_2 = \beta_1 w_{21} \text{ and } \beta_3 = \beta_1 w_{31}$$

where w_{21} and w_{31} are the population central moments of (w_{2j} / w_{1j}) and (w_{3j} / w_{1j}) respectively, and can be estimated by

their sample means. In this form, we would have only two restrictions to test.

SECTION 2

For options with dividend payments prior to maturity, the Black-Scholes model is not acceptable as the model of fair valuation on theoretical grounds. The test of Black-Scholes validity is to be undertaken in the context of options with no dividend payments prior to maturity. Hence, in our tests below, we have used only the zero-dividend subsample.

The first problem we face in testing is that the w_{ij} 's are not known. The use of sample estimates for w_{ij} 's lead to a situation of nonlinear stochastic restrictions¹⁸. For example, one such set of restrictions would be:

$$(\beta_2/\beta_1) - (ws_{21} + v_{21}) = 0$$

$$(\beta_3/\beta_1) - (ws_{31} + v_{31}) = 0$$

where we have decomposed the w_{ij} 's into their sample means ws_{ij} 's and measurement errors v_{ij} 's.

The greatest simplification is achieved by pretending that the ws_{ij} 's are the respective population central moments, that is, overlooking the stochastic nature of the restrictions. In that case, we would have the following set of linear restrictions:

¹⁸ See Appendix 7.1 for a brief discussion of testing restrictions in a linear regression model.

$$\beta_2 - w_{2,1}\beta_1 = 0$$

$$\beta_3 - w_{3,1}\beta_1 = 0$$

These restrictions, under the assumption of multivariate normality for disturbance vector, can be tested by undertaking F-test. Under the null hypothesis of restrictions being true, the following ratio would be central F-distributed with 2 and N-4 degrees of freedom:

$$[(SSER-SSEU)/2] / (SSEU/N-4)$$

where SSER and SSEU are restricted and unrestricted error sum of squares respectively.

The results of F-test, both with and without the inclusion of intercept, are presented in Table 7.1. At 5% significance level, the restrictions are being rejected for all the cases.

If the stochastic nature of the restrictions is not assumed away, the restrictions would be nonlinear in the parameters estimated which now include w_{ij} 's. Thus, we would have to rely on some large sample test. In addition, we need to make some assumption about the nature of v_{ij} 's¹⁸².

The choice between Wald test, Lagrange Multiplier test, and likelihood ratio test depends mainly upon the ease of estimation of the restricted and/or the unrestricted regression equation. Since, in our case, it is easier to estimate the unrestricted version of the regression, we would be using the Wald test.

¹⁸²In the test to follow, we have assumed that the observational errors of w_{ij} 's have zero covariance with the regression disturbances.

Under the null hypothesis of the restrictions being true, the Wald statistic is asymptotically chi-square distributed with J degrees of freedom, where J is the number of independent restrictions. The critical value of chi-square with $J=2$ at 5% significance level is 5.991. If the sample value of Wald statistic exceeds 5.991, the evidence would be against the null hypothesis of the restrictions being true.

The test results of Wald test, both with and without the inclusion of intercept, are presented in Table 7.2 . In all of the cases, we are unable to reject the null hypothesis of the restrictions being true.

It appears that the results of the F-test and the Wald test are contradictory. As discussed in Appendix 7.1, the additional variability introduced by the stochastic nature of restrictions in the Wald test may have caused this. On a priori basis, it is rather difficult to conclude superiority of one test over the other in our context.

SECTION 3

Our test results indicate mixed evidence regarding the acceptability of the Black-Scholes as a reasonable model describing the market prices of options without any dividend payment prior to maturity. Our testing procedure was radically different from the existing procedures of testing the Black-Scholes validity. We used the nonlinearity bias of the commonly used Black-Scholes formula estimate to the advantage of testing, rather than being victimized by it as the existing

studies have been. It is to be mentioned that our results are subject to the effect of the approximations we have used. In particular, the assumption of constant coefficient may have important bearing for the test results. In the next chapter, we would relax this assumption through the use of multivariate cubic spline regression and offer another indirect test of Black-Scholes validity.

APPENDIX 7.1

SHORT DIGRESSION ON TESTING RESTRICTIONS IN A LINEAR REGRESSION MODEL

In what follows, we briefly describe some key econometric results about testing equality restrictions among parameters in a classical normal linear regression model. The references are Goldberger(1964), Judge et al(1980), and White(1984). We shall refer to them as GB, JG and WT respectively.

The regression model we are considering is:

$$Y = X\beta + e$$

where

Y is $N \times 1$ column vector of responses

X is $N \times K$ design matrix

β is $K \times 1$ column vector of response coefficients

e is $N \times 1$ column vector of disturbances and

e has multivariate normal distribution with mean vector 0 and variance-covariance matrix $\sigma^2 I$.

Four types of restrictions among the elements of β may arise:

Case 1: Linear non-stochastic restrictions. Such restrictions are conventionally expressed as $R\beta=r$, where R is a $J \times K$ matrix of rank $J < K$, and r is a $J \times 1$ column vector of known elements. ¹⁸³ An alternative expression for the restrictions is: $g(\beta)=0$, where $g(\beta)=R\beta-r$. g is written as a

¹⁸³Rank(R)= J means we have J independent restrictions.

function of β only, since r is known and fixed.

Example: $\beta_1 + \beta_2 = 1$ in $Y = X_1\beta_1 + X_2\beta_2 + e$.¹⁸⁴

Case 2: Linear stochastic restrictions. JG represents them as $r = R\beta + v$ or, alternatively $E(r) = R\beta$, where r is now an observable random vector and v is an unobservable normally distributed random vector with 0 and $\sigma^2\Omega$ as the mean vector and the covariance matrix respectively.¹⁸⁵ In this case, an alternative expression for the set of restrictions is $q(\theta) = 0$, where θ contains $E(r)$ as a subvector in addition to β , and $q(\theta) = R\beta - E(r)$. The restrictions are written as functions of all the unknown parameters, rather than as functions of β only.

Example: $0.5 = \beta_1 + v_1$, and $0.5 = \beta_2 + v_2$. This example is given by JG, pp 72-73.

Case 3: Nonlinear non-stochastic restrictions. These restrictions are conventionally expressed as $g(\beta) = 0$, where $g(\beta) = h(\beta) - r$. The elements in $h(\beta)$ are nonlinear in the elements of β , and r is known and fixed as in case 1.

The example given by WT, pp 76, is: $\beta_3 - \beta_1\beta_2 = 0$ in $Y = X_1\beta_1 + X_2\beta_2 + X_3\beta_3 + e$. Here, r is assumed to be zero.

¹⁸⁴ Here, $R = [1 \ 1]$ and $r = 1$.

¹⁸⁵ See JG, pp 72-76. Note that r is normally distributed random vector with mean $R\beta$ and covariance matrix $\sigma^2\Omega$.

Case 4: Nonlinear stochastic restrictions. Such restrictions are the mixture of cases 2 and 3, and can be expressed as $h(\beta)=E(r)$, or alternatively as $q(\theta)=0$, where $q(\theta)=h(\beta)-E(r)$.

Example: $\beta_3 - \beta_1 \beta_2 - (0.5 + v) = 0$, where 0.5 is the value of r observed with error.

Of all the above cases, case 1 is the easiest to test. We can minimize the error of sum of squares without restrictions and with restrictions. The restricted error sum of squares (SSER) would be higher than the unrestricted (SSEU). But, if the restrictions are true, it would not be significantly higher, since the data would have already embodied the restrictions. It can be shown that:

$$SSER - SSEU = \hat{g}^T G^{-1} \hat{g}$$

where

$$\hat{g} = R\hat{\beta}_u - r$$

$\hat{\beta}_u$ is the unrestricted OLS estimate of β

$$\text{and } G = [R(X^T X)^{-1} R^T]$$

Note that the covariance matrix for \hat{g} is $\sigma^2 G$, and therefore, $(SSER - SSEU) / \sigma^2$ will be central χ^2 distributed with J degrees of freedom, if the restrictions are true. ¹⁸⁶

¹⁸⁶To see this, let Σ denote the covariance matrix $\sigma^2 G$. Then $(SSER - SSEU) / \sigma^2$ can be expressed as $(P\hat{g})^T (P\hat{g})$, where $P^T P = \Sigma^{-1}$. According to theorem 5.9 of GB, the random vector $P\hat{g}$ is a standard normal vector. Thus, theorem 5.21 of GB shows that $(SSER - SSEU) / \sigma^2$ would be distributed central χ^2 with J degrees of freedom, since I is idempotent of rank J .

J

The SSEU is of course equal to $\epsilon^T M \epsilon$, where $M = [I - X(X^T X)^{-1} X^T]$ is an idempotent with rank $(N-K)$. SSEU will be distributed central $\sigma^2 \chi^2$ with $N-K$ degrees of freedom, according to theorem 5.22 of GB. Then, $SSEU/\sigma^2$ will be distributed central χ^2 with $N-K$ degrees of freedom. Thus, if the restrictions are true, the following ratio will be central F distributed with J and $N-K$ degrees of freedom:¹⁸⁷

$$[(SSER - SSEU)/J] / [SSEU/(N-K)] \dots (1)$$

Using this statistic, a conventional F-test can be undertaken.

In addition to the F-test, three other tests are commonly used. These are Wald test, Lagrange multiplier test and Likelihood ratio test. They rely on asymptotic normality property, and in all cases, the test statistic is asymptotically central χ^2 distributed if the restrictions are true. A discussion of all these tests and their relative merits is beyond the scope of this appendix. In our context, it is convenient to compute the Wald statistic. Given our expositions above, it would also be easier to describe the Wald test.

¹⁸⁷See theorem 5.7 of GB, and notice that under the truth of restrictions, $(SSER - SSEU)$ can be alternatively expressed as:

$\epsilon^T [X(X^T X)^{-1} R^T G^{-1} R(X^T X)^{-1} X^T] \epsilon$, where the bracketed matrix is an idempotent of rank J . Then, observe that the product of this matrix and M is a null matrix.

The underlying idea of Wald test can be described as follows. Due to sampling variation, the unrestricted estimate of β would satisfy the restrictions only on the average, even if the restrictions are true. Thus in our case 1, if the restriction $g(\beta)=0$ is true, then \hat{g} would be on the average equal to zero, given that $\hat{\beta}$ is an unbiased estimate of β . If we know the distribution of \hat{g}_u , we can test how much deviation from 0 can be allowed as mere sampling variation, before we reject the hypothesis of restrictions being true.

For the convenience of exposition, let us first write down the expression for Wald Statistic(W) in the more general context of case 3: ¹⁸⁸

$$g_u(\hat{\beta}_u)' [g_u(X'X)^{-1}g_u']^{-1} g_u(\hat{\beta}_u) / \hat{\sigma}^2 \dots (2)$$

where

$$g_u = [\{ \partial g(\beta) / \partial \beta^T \} |_{\beta = \hat{\beta}_u}]$$

$$\text{and } \hat{\sigma}^2 = \text{SSE}_u / (N - K)$$

¹⁸⁸For a more general expression, see WT, pp 71, 77.

WT used the mean value theorem in deriving the Wald statistic. But more ordinarily, we can imagine the following linearization underlying (2):

$$g(\beta_u) \approx g(\beta) + g_2(\beta_u - \beta) \dots (3)$$

where $g_2 = \{ \{ \partial g(\beta_u) / \partial \beta_u^T \} | \beta_u = \beta \}$ and

wherefrom the covariance of $g(\beta_u)$ is:¹⁸⁹

$$\Gamma = \sigma^2 g_2 (X^T X)^{-1} g_2^T \dots (4)$$

Consistent estimate of Γ can be formed from:

$$\hat{\sigma}^2 g_1 (X^T X)^{-1} g_1^T \dots (5)$$

This estimate has been used in (2). Note from WT, pp 77, that $\{L \cdot g(\beta_u)\}$, where $L^T L = \Gamma^{-1}$, will be asymptotically distributed as a standard normal vector. Same would be the case when Γ is replaced by its consistent estimate. Thus, W would be asymptotically χ^2 distributed.

For linear nonstochastic restrictions (case 1), the W statistic is equal to J times the F statistic in (1). This can be seen by noting that in this case, $g_1 = g_2 = R$.

¹⁸⁹ Under the null hypothesis of restrictions being true, $g(\beta) = 0$.

Let us now consider the situation of stochastic linear restrictions (case 2). The restrictions in all cases involve the parameter vector β , which is not known. The testing of restrictions is carried on using estimate of β , and the latter's sampling variation is the only source of variability for the random vector \hat{g} and $g(\beta)$ in cases 1 and 3. In the case of stochastic restrictions, additional variability is introduced by the observational error in r . Thus, in devising a test statistic, the variability of r as well as its possible covariance with the regression disturbance ϵ are to be taken into account. If it is assumed that the latter covariances are zero, then Theil's compatibility statistic can be applied in case 2: ¹⁹⁰

$$u_1 = \hat{q} [G + \Omega]^{-1} \hat{q} / J \hat{\sigma}^2 \dots (1A)$$

where

$$\hat{q} = R\hat{\beta} - r.$$

Under the null hypothesis of restrictions being true, u_1 is central F distributed with J and N-K degrees of freedom. Note that the only difference between the F in (1) and u_1 is that the covariance matrix of estimate for the restrictions in the latter case has the additional component $\sigma^2 \Omega$. This would tend to decrease the value of test statistic and thus reduce the possibility of rejecting the restrictions merely because of the

¹⁹⁰See JG, pp 76.

errors in observing $E(r)$.

In the case of nonlinear stochastic restrictions, the linearisation similar to (3) would now involve derivative of $q(\theta)$ with respect to θ^T :

$$q(\theta) \approx q(\theta) + g_3(\theta - \theta) \dots (3A)$$

where $g_3 = \{ \partial q(\theta) / \partial \theta^T \mid \theta = \theta \}$.

The covariance of $q(\theta)$ is:

$$\Gamma_1 = \sigma^2 g_3 T g_3^T \dots (4A)$$

where T is a block-diagonal matrix with $(X^T X)^{-1}$ and Ω as the two non-null blocks.¹⁹¹

Replacing Γ_1 by its consistent estimate Γ leads to the asymptotically χ^2 distributed statistic:¹⁹²

$$q(\theta) [g_4 (X^T X)^{-1} g_4^T + \Omega]^{-1} q(\theta) / \hat{\sigma}^2 \dots (2A)$$

where

$g_4 = \{ \partial q(\theta) / \partial \beta^T \mid \theta = \theta \}$ and

$\hat{\Omega}$ is a consistent estimate of Ω .

Note again that the stochastic nature of restrictions lowers the value of the test statistic by introducing more variability into the observed (or estimated) restrictions.

¹⁹¹ It can be shown that:

$$\Gamma_1 = \sigma^2 [\underbrace{q}_{\beta} (X^T X)^{-1} \underbrace{q^T}_{\beta} + \Omega \dots (4B)], \text{ where } \underbrace{q}_{\beta} = \{ \partial q(\theta) / \partial \beta^T \mid \theta = \theta \}.$$

¹⁹² We are assuming that $q(\theta)$ is a multivariate normal vector.

The expositions above show that the test results may depend upon how we view the restrictions. In a large sample Wald test, we may unduly overemphasize the stochastic nature and/or impose wrong covariance structure. On the other hand, in a small sample F-test, we overlook the additional variability of stochastic restrictions and thus would tend to reject the restrictions more often.

CHAPTER 8:
MULTIVARIATE SPLINE REGRESSION AND
AN INDIRECT TEST OF BLACK-SCHOLES VALIDITY

The empirical studies trying to validate the Black-Scholes model, have largely overlooked the role of the nonlinearity bias in inducing the observed systematic deviations of the Black-Scholes formula estimates. We have also argued in various parts of this thesis that the existing studies are plagued with host of problems. They range from the fundamental deficiency of the ISD approach (to model validation), to tools-of-analysis related problems such as the dilemma of dichotomous bias studies or the inappropriate estimation and/or interpretation of the results of functional bias studies.

The estimable regression equation we derived in chapter 3 clearly showed that if the Black-Scholes model is valid, the only component in the expected response function is the nonlinearity bias of the formula estimate. And thus, the systematic tendency of the formula estimates should be that induced by its nonlinearity bias. When, on the other hand, the Black-Scholes model is not valid, the model misspecification error is expected to generate tendency which may reinforce or counteract those induced by the nonlinearity bias alone. Observed systematic tendencies radically different from that induced by the nonlinearity bias would thus create reasonable doubts about the validity of the Black-Scholes model. That this fact can be used to test the validity of the Black-Scholes model have gone unnoticed in the literature.

In chapter 7, we used the structure of second order approximation to the nonlinearity bias, and offered an indirect

test of the validity of the Black-Scholes model. The approximation imposes parameter restrictions in our regression model. We tested these restrictions in the zero-dividend subsample, where the Black-Scholes is least suspect. But an important limitation of this test was the fact that we assumed the slope coefficients to be constant across options, while they are most likely to be option-specific as indicated by the nature of nonlinearity bias and portrayed in our regression model.

In this chapter, we will try to approximate the option-specificness by the use of multivariate cubic spline regression technique. More specifically, we would allow the coefficient of a factor to be nonlinear in that factor alone. Thus the nonlinearity would be non-interactive. Since the cubic splines are very flexible functional forms, we hope to achieve good approximations to the option-specificness of the coefficients.

Using the estimated equation, we would trace the path of predicted mispricing (by the formula estimate) as any one of the factors varies over its sample range, the other factors fixed at prespecified levels. Then, for a similar combination of parameters, the predicted path will be compared with that of the nonlinearity bias (monte carlo) found in chapter 4. If the Black-Scholes model is valid, we should not observe radical differences.

We would consider the variation of moneyness and variance rate only in generating spline predictions. As found in chapter

4, the relations of time to maturity and variance rate to the nonlinearity bias are similar. Moreover, the volatility (square root of variance) rate was found to be the dominant source of significant dividend inducement in chapter 6. Thus, comparison of the spline predictions of the relation to the variance rate across the two subsamples may provide further evidence in this respect.

After briefly reviewing the spline approach in section 1, the estimation results would be presented in section 2. Some concluding comments follow in section 3.

SECTION 1

In its most common usage, a spline is a mechanical device used by draftsmen much like a French Curve to draw smooth curves. The device consists of a flexible rod with weights attached to make the curve go through specific points. A class of functions commonly used in the approximation theory are referred to as spline functions, because their properties are very similar to those of the draftsmen's spline.

If we consider an interval $[a, b]$ and its partition $a = x_0 < x_1 < \dots < x_k = b$, then a function $S(x)$ is called a spline function, when it satisfies the following properties:

(i) On each subinterval $(x_{i-1} < x < x_i)$, $i = 1, 2, \dots, k$, $S(x)$ coincides with a polynomial of degree less than or equal to t , a given integer.

(ii) The r -th derivative of $S(x)$ is continuous for $r=0,1,\dots,t-1$, except when $t=0$.

It is apparent that a spline function is a step function for $t=0$, and a piecewise linear function for $t=1$. The most widely used spline functions are cubic splines, corresponding to $t=3$. Cubic splines were shown to achieve improvement upon piecewise cubic Lagrange and piecewise cubic Hermite interpolates for a given function, both in terms of smoothness and bound on approximation error¹⁹³.

The cubic spline interpolate $S(x)$ to a function $f(x)$ satisfies:

$$(i) S(x_i) = f(x_i) = y_i, \quad 0 \leq i \leq K \quad (\text{pure interpolatory constraint})$$

$$(ii) \left(\frac{\partial^r S(x)}{\partial x^r} \right) \Big|_{x=x_i} = \left(\frac{\partial^r f(x)}{\partial x^r} \right) \Big|_{x=x_i}, \quad 0 \leq i \leq K; \quad r=1,2,\dots$$

(smoothness constraint)

The x_i 's are called knots, since at these points the spline's values are tied down by constraints, or alternatively join points, since the cubics over adjacent subintervals are connected at these points.

Poirier(1973) showed that the above continuity conditions of cubic spline interpolation boil down to $k-1$ equations in the $k+1$ unknowns, the second derivatives of the spline at the knots, alternatively referred to as the moments¹⁹⁴. Two end conditions

¹⁹³ See Prenter(1975), pp., 78, 84.

¹⁹⁴ $k-1$ equations come from the equality of the one-sided limits of the spline's first derivatives at the $k-1$ intermediate knots. See Poirier's appendix.

at the extreme knots were imposed to eliminate this deficiency. Poirier also showed that, for any vector X (of dimension n , greater than $k+1$) of abscissa values, the corresponding vector of spline interpolants $S(X)$ can be expressed as a linear function of y_i 's, the ordinate values at the $k+1$ knots. Thus, for known y_i 's, we can write:

$$S(X) = W y$$

where W is $n \times (k+1)$ transformed data matrix and y is the $(k+1) \times 1$ vector of y_i 's.

In the context of unknown y_i , Poirier considered the statistical model:

$$Y = S(X) + \epsilon$$

where

Y is the $n \times 1$ vector of dependent variable observations and $E(\epsilon) = 0$, and $E(\epsilon\epsilon^T) = \sigma^2 I$, 0 being an $n \times 1$ null vector, and I is an identity matrix of dimension n .

The least squares estimator of Y is the BLUE of Y , and, given the normality of ϵ , standard statistical tests can be applied.

The above model was named Cubic Spline Regression Model (CSRM) by Poirier. In this model, the expected response function was approximated by the cubic spline, and the unknown expected values of the dependent variable at the known knots were the parameters to be estimated. Using these estimates, one can, of course, write down the estimated spline as the set of k cubic polynomials over the subintervals.

Buse and Lim(1977) considered direct estimation of the coefficients of the k polynomials:

$$S_i(x) = a_i + b_i x + c_i x^2 + d_i x^3, \\ x_{i-1} \leq x \leq x_i, \quad i=1,2,\dots,k$$

Writing the observations on x as a block diagonal matrix, all the coefficients can be jointly estimated subject to the continuity conditions and the end point conditions of Poirier. Buse and Lim showed that their restricted least squares estimates are identical to Poirier's indirect estimates. They also mentioned that the arbitrary end point conditions are not necessary for restricted least squares estimation, but may improve efficiency of the estimates.

Both of the above papers assumed that the join points are known. Gallant and Fuller(1973) considered the same regression model in the context of unknown join points. They introduced some dummy variables and used the continuity restrictions to write down the regression equation similar to Buse and Lim, in a compact reparameterized form. This reparameterized form is, of course, nonlinear in the parameters(to be estimated). The authors suggested a complicated(relative to conventional least squares) iterative least squares algorithm for the estimation of the nonlinear equation.

It was until Suits, Mason and Chan(1978), the expositions on spline estimation remained mathematically formidable, and thus this powerful technique was relatively obscure to

practitioners of economics ¹⁹⁵. With simple graphs, they showed the relative advantages of spline regression over dummy variable, piecewise linear, or polynomial regression. The greatest advantage is that it is not necessary to prespecify a functional form for the expected response function ¹⁹⁶.

Suits, et al used the structure of Gallant and Fuller(1973), but the assumption of known join points led to a reparameterized version linear in the parameters(to be estimated). Also, following Barth et al, they used the displacements $(x-x_i)$'s, rather than the x_i 's. Their spline estimation finally boils down to a conventional multiple regression of the dependent variable on a set of composite variables. The latter variables are formed from: (i) the data on the explanatory variable, (ii) the knots, (iii) the widths of the intervals between knots, and (iv) $(k-1)$ dummy variables. Since we would be following Suits et al in our estimation, let us write down their equation 7 in slightly different notation:

$$Y = a_1 + b_1(x-x_0) + c_1(x-x_0)^2 + d_1(x-x_0)^3 + (d_2-d_1)(x-x_1)^3 D_1 + (d_3-d_2)(x-x_2)^3 D_2$$

where x_0, x_1, x_2 are the first three of the four knots considered ¹⁹⁷

¹⁹⁵One of the few exceptions is Barth, Kraft, and Kraft(1976).

¹⁹⁶When the knots are not known, the piecewise linear regression can prove fatal.

¹⁹⁷Though more than three intervals can be used, it requires greater number composite variables, and in an expected response function of several explanatory variables, that can be cumbersome.

$D_i = 1$, iff $x \geq x_i$, and $D_i = 0$ otherwise

The parameter subscripts refer to the subinterval polynomials they belong to. Estimates for the rest of the parameters of these polynomials can be recovered from the continuity restrictions, using the ones in the above equation.

Generally we would have more than one explanatory variable in the expected response function of the regression. In that case, we would just have to add more composite variables similar to the above ones to allow curvature with respect to the additional explanatory variables. The splines for different explanatory variables need not be of the same order, though the use of cubics all through may provide more flexibility.

Finally, it is noted by Suits et al, that the standard statistical tests can be applied to spline estimation results.

SECTION 2

We have noted in chapter 3 that the expected response function in our regression model comprises of the nonlinearity bias of the Black-Scholes formula estimate and probable model misspecification error of the Black-Scholes model. In chapter 7, we have seen that even only the second order approximation to the nonlinearity bias gives rise to an expected response function in terms of the three explanatory variables, which is much too complicated for description by any simple function. The assumption of constant coefficient there led to considerable simplification in estimation.

Now we would use three noninteractive additive cubic splines (with respect to the three explanatory variables) to approximate the unknown but foreseeably complex expected response function in our regression model ¹⁹⁸. This would not still fully capture the option specificness of the coefficients in our regression model of chapter 3. But by cubic spline regression, we would implicitly allow the coefficient of an explanatory variable to be nonlinear with respect to that variable. We hope that this may be a good enough approximation.

We have followed the spline regression construct of Suits et al. Thus, for each of the three splines, we have four knots. The minimum and maximum of the sample values for the explanatory variables were chosen as the terminal knots. The intermediate knots were chosen so as to allow evenness in distribution across the subintervals, and sufficient observations in each subinterval. During estimation, we, of course, varied the knots around the final choices, and found no significant departure in the key estimation results such as the error sum of squares.

The estimated spline regression equations for the zero-dividend subsample and the single-dividend subsample are presented in table 8.1. The regressions are significant at 5% significance level. Our purpose in this chapter was defined as to use spline regression for predicting possibly nonlinear functional relationships of Black-Scholes formula estimates'

¹⁹⁸There has been some work on incorporating interaction in bivariate spline approximation and estimation. See Poirier(1973) for these references.

mispricing with respect to the explanatory variables, and then make visual comparison with that of the nonlinearity bias (from monte carlo in chapter 4). Hence, we would skip the computation and discussion of the subinterval polynomials, which can be provided to interested readers.

Figures 8.1 to 8.6 pictures the spline predictions of the moneyness bias in the two subsamples. The curves in these figures look almost upside down of the curves for nonlinearity bias in figures 4.2 to 4.4. There does not, of course, seem to be any perceptible difference in the curvature across the two subsamples. Thus the indication is that the Black-Scholes model is not valid in the zero-dividend subsample, and that there is no significant dividend inducement in the moneyness bias. Though it is not warranted, we cannot resist saying that the spline prediction of moneyness bias is very similar to Merton(1976a)'s model misspecification error diagram, and Ball and Torous(1985)'s empirical counterpart of this diagram.

The variance bias predicted by the spline regressions in figures 8.7 and 8.8 is also at contrast to the figures 4.8 to 4.10 corresponding to the nonlinearity bias. The contrasts here are not as prominent as in the case of the moneyness bias. But the sharp difference in the variance bias across the two subsamples is rather noticeable. This produces further evidence about the varinace(or volatility) rate's dominant role in the significant dividend inducement of the entire regression relationship. The strong negative tendency of the variance bias

in the single-dividend subsample is in conformity with the large negative coefficient for volatility rate in the constant coefficient results of chapter 5.

SECTION 3

Overall, our indirect test via visual comparison in this chapter produces evidence against the validity of the Black-Scholes model in the zero-dividend case. We also find support for our constant coefficient result in chapter 6 that the volatility rate played a dominant role in the dividend inducement of the entire regression relationship.

Our results in this chapter are subject to the noninteractive option-specificity we have allowed in estimation. Also, the assumption of known join points may have affected the results.

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TABLE 4.1

COMPARISON OF THE MISPRICING OF PSEUDO ESTIMATOR
AND FORMULA ESTIMATOR FOR THE BLACK-SCHOLES PRICE

Mispricing is the negative of statistical bias.
Striking price of 50 and riskless rate of 0.015/quarter
were used following Boyle and Ananthanarayanan(1977).
Time to maturity(T) is measured in quarter and V stands for
quarterly variance rate. N denotes the sample size from which
the variance rate is estimated and S is the current stock price.

FORMULA ESTIMATOR

| N | T | S | V=0.01 | V=0.025 | V=0.045 |
|----|-----|----|---------------|---------------|-----------|
| 10 | 0.5 | 30 | -8.754103E-09 | -0.000106 | -0.002329 |
| | | 50 | 0.041279 | 0.065966 | 0.088798 |
| | | 80 | -2.441400E-07 | -0.000319 | -0.005405 |
| | 1.0 | 30 | -2.611765E-05 | -0.003819 | -0.018059 |
| | | 50 | 0.057256 | 0.092415 | 0.124853 |
| | | 80 | -5.969238E-05 | -0.007113 | -0.031188 |
| | 1.5 | 30 | -0.000441 | -0.013151 | -0.030953 |
| | | 50 | 0.068755 | 0.112110 | 0.152042 |
| | | 80 | -0.000663 | -0.020685 | -0.050307 |
| 15 | 0.5 | 30 | -6.204268E-10 | -5.469823E-05 | -0.000157 |
| | | 50 | 0.028822 | 0.046064 | 0.062003 |
| | | 80 | -6.103511E-08 | -0.000175 | -0.003748 |
| | 1.0 | 30 | -1.148765E-05 | -0.002667 | -0.013637 |
| | | 50 | 0.039991 | 0.064541 | 0.087197 |
| | | 80 | -2.752685E-05 | -0.005014 | -0.023621 |
| | 1.5 | 30 | -0.000262 | -0.009730 | -0.023865 |
| | | 50 | 0.048035 | 0.078304 | 0.106212 |
| | | 80 | -0.000392 | -0.015397 | -0.038679 |
| 30 | 0.5 | 30 | 0.000000 | -1.980407E-05 | -0.000788 |
| | | 50 | 0.007899 | 0.012703 | 0.171270 |
| | | 80 | -2.136300E-07 | -6.842040E-05 | -0.001931 |
| | 1.0 | 30 | -3.361905E-06 | -0.001387 | -0.007824 |
| | | 50 | 0.010913 | 0.017785 | 0.024137 |
| | | 80 | -1.022339E-05 | -0.002632 | -0.013633 |
| | 1.5 | 30 | -0.000113 | -0.005469 | -0.014693 |
| | | 50 | 0.013039 | 0.021554 | 0.029472 |
| | | 80 | -0.000169 | -0.008632 | -0.023794 |

PSEUDO ESTIMATOR

| | | | | | |
|----|-----|----|-----------|-----------|-----------|
| 10 | 0.5 | 30 | -0.741289 | -0.741312 | -0.741257 |
| | | 50 | -1.125320 | -1.058410 | -0.996164 |
| | | 80 | -1.226230 | -1.226250 | -1.226090 |
| | 1.0 | 30 | -0.741289 | -0.741187 | -0.739116 |
| | | 50 | -1.082950 | -0.089147 | -0.901781 |
| | | 80 | -1.217070 | -1.216830 | -1.212930 |
| | 1.5 | 30 | -0.741296 | -0.740166 | -0.731834 |
| | | 50 | -1.051680 | -0.937712 | -0.831511 |
| | | 80 | -1.207960 | -1.206250 | -1.193140 |
| 15 | 0.5 | 30 | -0.835758 | -0.835781 | -0.835738 |
| | | 50 | -1.283060 | -1.217110 | -1.155790 |
| | | 80 | -1.382520 | -1.382520 | -1.382410 |
| | 1.0 | 30 | -0.835757 | -0.835650 | -0.833622 |
| | | 50 | -1.240040 | -1.147560 | -1.061480 |
| | | 80 | -1.372190 | -1.371960 | -1.368130 |
| | 1.5 | 30 | -0.835765 | -0.834650 | -0.826461 |
| | | 50 | -1.207900 | -1.095590 | -0.000950 |
| | | 80 | -1.361940 | -1.360240 | -1.347330 |
| 30 | 0.5 | 30 | -0.973390 | -0.973445 | -0.973374 |
| | | 50 | -1.531810 | -1.479880 | -1.431470 |
| | | 80 | -1.610180 | -1.610260 | -1.610100 |
| | 1.0 | 30 | -0.973390 | -0.973345 | -0.971707 |
| | | 50 | -1.493990 | -1.421140 | -1.353240 |
| | | 80 | -1.598160 | -1.598040 | -1.594960 |
| | 1.5 | 30 | -0.973397 | -0.972560 | -0.966051 |
| | | 50 | -1.464800 | -1.376330 | -1.293790 |
| | | 80 | -1.586230 | -1.584950 | -1.574710 |

TABLE 4.2

PERFORMANCE OF FORMULA ESTIMATOR FOR THE BLACK-SCHOLES PRICE

The symbol g is the ratio of the stock price to the present value of the striking price.

V is the true variance rate; time to maturity and stock price are assumed to be equal to 1, and N is the sample size from which V is estimated.

The numbers in a row are mean mispricing($\times 1000$), mean percentage error, variance($\times 10000$) and mean square error($\times 10000$) respectively.

| $V=0.03$ | | | | |
|----------------------------|------------|------------|-----------|-----------|
| <u>$g=0.80$</u> | | | | |
| <u>$N=35$</u> | -0.2427514 | -2.6929514 | 0.1676188 | 0.1882081 |
| $N=31$ | -0.2520018 | -2.7955707 | 0.1956182 | 0.1962532 |
| $N=15$ | -0.4437155 | -4.9223387 | 0.3873705 | 0.3893394 |
| $N=11$ | -0.6891022 | -7.6445253 | 0.5963708 | 0.6011194 |
| <u>$g=0.95$</u> | | | | |
| <u>$N=35$</u> | 0.3176129 | 0.6674272 | 0.6906412 | 0.6916500 |
| $N=31$ | 0.4182462 | 0.8788967 | 0.7952151 | 0.7969643 |
| $N=15$ | 1.4660706 | 3.0807802 | 1.5784583 | 1.5999519 |
| $N=11$ | 1.6817069 | 3.5339153 | 2.1142948 | 2.1425752 |
| <u>$g=1.00$</u> | | | | |
| <u>$N=35$</u> | 0.3717014 | 0.5385996 | 0.7192473 | 0.7206289 |
| $N=31$ | 0.4814856 | 0.6976780 | 0.8279887 | 0.8303070 |
| $N=15$ | 1.6200778 | 2.3475108 | 1.6607279 | 1.6869743 |
| $N=11$ | 1.8813720 | 2.7261290 | 2.2217960 | 2.2571915 |

g=1.05

| | | | | |
|------|-----------|-----------|-----------|-----------|
| N=35 | 0.3076896 | 0.3279331 | 0.6303466 | 0.6312934 |
| N=31 | 0.4044612 | 0.4310715 | 0.7257749 | 0.7274108 |
| N=15 | 1.4115896 | 1.5044609 | 1.4419760 | 1.4619019 |
| N=11 | 1.6212168 | 1.7278798 | 1.9312179 | 1.9575014 |

g=1.20

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.1139159 | -0.0638031 | 0.1927383 | 0.1928680 |
| N=31 | -0.1022985 | -0.0572963 | 0.2234309 | 0.2235355 |
| N=15 | -0.1450963 | -0.0812669 | 0.4389998 | 0.4392103 |
| N=11 | -0.2824841 | -0.1582164 | 0.6527238 | 0.6535217 |

V=0.04

g=0.80

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.1978470 | -1.3346248 | 0.3354644 | 0.3358558 |
| N=31 | -0.2562891 | -1.7288604 | 0.3388653 | 0.3395222 |
| N=15 | -0.3215196 | -2.1688884 | 0.8331181 | 0.8341518 |
| N=11 | -0.0042801 | -0.0288722 | 1.0177396 | 1.0177398 |

g=0.95

| | | | | |
|------|-----------|-----------|-----------|-----------|
| N=35 | 0.3857000 | 0.6638506 | 0.9398362 | 0.9413239 |
| N=31 | 0.7356461 | 1.2661628 | 1.0526356 | 1.0580474 |
| N=15 | 1.0942295 | 1.8833413 | 2.0376738 | 2.0196471 |
| N=11 | 1.8821047 | 3.2393988 | 2.7467117 | 2.7821349 |

g=1.0

| | | | | |
|-------------|-----------|-----------|-----------|-----------|
| <u>N=35</u> | 0.4302459 | 0.5401319 | 0.9565785 | 0.9584296 |
| N=31 | 0.5570824 | 0.6993628 | 1.1011644 | 1.1042678 |
| N=15 | 1.8718538 | 2.3499306 | 2.2087229 | 2.2437612 |
| N=11 | 2.1744215 | 0.0272977 | 2.9545326 | 3.0018137 |

g=1.05

| | | | | |
|-------------|-----------|-----------|-----------|-----------|
| <u>N=35</u> | 0.3717858 | 0.3579586 | 0.8559209 | 0.8573031 |
| N=31 | 0.7062284 | 0.6799628 | 0.9587188 | 0.9637064 |
| N=15 | 1.0527393 | 1.0135865 | 1.8562953 | 1.8673779 |
| N=11 | 1.8085845 | 1.7413207 | 2.5034353 | 2.5361451 |

g=1.20

| | | | | |
|-------------|------------|------------|-----------|-----------|
| <u>N=35</u> | -0.0502059 | -0.0272029 | 0.3386162 | 0.3386415 |
| N=31 | -0.1001262 | -0.0542511 | 0.4146760 | 0.4147762 |
| N=15 | -0.2355243 | -0.1276134 | 0.7149487 | 0.7155034 |
| N=11 | -0.4516799 | -0.2447320 | 0.9869581 | 0.9889982 |

TABLE 4.3

PERFORMANCE OF BUTLER-SCHACHTER ESTIMATOR FOR
THE BLACK-SCHOLES PRICE

The symbol g denotes the ratio of the stock price to the present value of the exercise price.

V is the true variance rate, time to maturity and stock price are assumed to be equal to 1, and N is the sample size from which V is estimated.

The numbers in a row are mean mispricing($\times 1000$), mean percentage error, variance($\times 10000$) and mean square error($\times 10000$) respectively.

The Taylor series expansions of the cumulative normal distribution functions were truncated after the term involving the 31st derivative.

| $V=0.03$ | | | | |
|---------------------|------------|------------|-----------|-----------|
| $\overline{G=0.80}$ | | | | |
| $\overline{N=35}$ | -0.1807991 | -2.0056870 | 0.1743087 | 0.1746356 |
| $\overline{G=0.95}$ | | | | |
| $\overline{N=35}$ | -0.2258770 | -0.4746549 | 0.7082651 | 0.7087753 |
| $\overline{G=1.00}$ | | | | |
| $\overline{N=35}$ | 0.1365863 | 0.1979150 | 0.6575194 | 0.6577059 |
| $N=31$ | 0.0106656 | 0.0154545 | 0.7102703 | 0.7102714 |
| $N=15$ | 0.3161313 | 0.4580778 | 1.7341208 | 1.7351202 |
| $N=11$ | 0.0911227 | 0.1320379 | 2.3541830 | 2.3542660 |
| $\overline{G=1.05}$ | | | | |
| $\overline{N=35}$ | -0.2152029 | -0.2293615 | 0.6462582 | 0.6467213 |
| $\overline{G=1.20}$ | | | | |
| $\overline{N=35}$ | -0.1636570 | -0.0916626 | 0.2009645 | 0.2012323 |

V=0.04

G=0.80

N=35 -0.2628432 -1.7730726 0.3514824 0.3521733

G=0.95

N=35 -0.2958485 -0.5092019 0.9656459 0.9665211

G=1.00

N=35 0.1247157 0.1565685 0.8766820 0.8768374

N=31 0.2337891 0.2934996 0.9393952 0.9399418

N=15 0.3319917 0.4167833 2.3121275 2.3132297

N=11 0.0721875 0.0906244 3.1388611 3.1389132

G=1.05

N=35 -0.2817248 -0.2712470 0.8792380 0.8800317

G=1.20

N=35 -0.2299517 -0.1245940 0.3539068 0.3544356

TABLE 4.4

PERFORMANCE OF CC ESTIMATOR FOR
THE BLACK-SCHOLES PRICE

The symbol g is the ratio of the stock price to the present value of the striking price.

V is the true variance rate, time to maturity and stock price are assumed to be equal to 1, and N is the sample size from which V is estimated.

The numbers in a row are mean mispricing(X 1000), mean percentage error, variance(X 10000) and mean square error(X 10000) respectively.

A 32-term Taylor series expansion of the Black-Scholes model price around the arbitrarily chosen point of $1.5V$ was used.

| V=0.03 | | | | |
|---------------|------------|------------|-----------|-----------|
| <hr/> | | | | |
| <u>g=0.80</u> | | | | |
| N=35 | -0.0815318 | -0.9044715 | 0.1719397 | 0.1720061 |
| N=31 | -0.0767263 | -0.8511610 | 0.2034179 | 0.2034768 |
| N=15 | 0.1704859 | 1.8912814 | 0.3973426 | 0.3976333 |
| N=11 | -0.1581061 | 1.7539465 | 0.5613893 | 0.5616393 |
| <hr/> | | | | |
| <u>g=0.95</u> | | | | |
| N=35 | -0.2220565 | -0.4666258 | 0.6957164 | 0.6962095 |
| N=31 | -0.1903877 | -0.4000775 | 0.8084436 | 0.8088060 |
| N=15 | 0.3214269 | 0.6754411 | 1.6152515 | 1.6162846 |
| N=11 | 0.1319883 | 0.2773580 | 2.1861742 | 2.1863485 |
| <hr/> | | | | |
| <u>g=1.00</u> | | | | |
| N=35 | -0.0951202 | -0.1378303 | 0.7019970 | 0.7020874 |
| N=31 | -0.0245757 | -0.0356104 | 0.7953749 | 0.7953809 |
| N=15 | 0.3301224 | 4.7835100 | 1.7299251 | 1.7310149 |
| N=11 | 0.0617085 | 0.0894163 | 2.4007224 | 2.4007605 |

g=1.05

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.2129004 | -0.2269074 | 0.6317456 | 0.6321989 |
| N=31 | -0.1802518 | -0.1921108 | 0.7325740 | 0.7328989 |
| N=15 | 0.2970226 | 0.3165641 | 1.4731588 | 1.4740410 |
| N=11 | 0.1113794 | 0.1187072 | 1.9969524 | 1.9970765 |

g=1.20

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.0347012 | -0.0194358 | 0.1886853 | 0.1886973 |
| N=31 | 0.0024305 | 0.0013613 | 0.2133721 | 0.2133721 |
| N=15 | 0.1603146 | 0.0897905 | 0.4571467 | 0.4574037 |
| N=11 | 0.0759024 | 0.0425121 | 0.6758054 | 0.6758630 |

V=0.04

g=0.80

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.1177599 | -0.7943813 | 0.3229171 | 0.3230558 |
| N=31 | -0.0766070 | -0.5167732 | 0.3673446 | 0.3674033 |
| N=15 | 0.1461615 | 0.9859715 | 0.7812130 | 0.7814266 |
| N=11 | 0.0544888 | 0.3675686 | 1.1495169 | 1.1495466 |

g=0.95

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.1757010 | -0.3024095 | 0.9521773 | 0.9524860 |
| N=31 | -0.1296213 | -0.2230989 | 1.0998270 | 1.0999950 |
| N=15 | 0.4321024 | 0.7437170 | 2.2403947 | 2.2422618 |
| N=11 | 0.1912969 | 0.3292525 | 3.0427831 | 3.0431490 |

g=1.0

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.1621391 | -0.2035502 | 0.9230851 | 0.9233479 |
| N=31 | -0.0818940 | -0.1028101 | 1.0461069 | 1.0461739 |
| N=15 | 0.3269544 | 0.4104601 | 2.2723086 | 2.2733776 |
| N=11 | 0.0207923 | 0.0261027 | 3.1545902 | 3.1545945 |

g=1.05

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.1674993 | -0.1612699 | 0.8635489 | 0.8638294 |
| N=31 | -0.1210242 | -0.1165232 | 0.9957343 | 0.9958807 |
| N=15 | 0.4030697 | 0.3880792 | 2.0394339 | 2.0410586 |
| N=11 | 0.1684902 | 0.1622239 | 2.7740093 | 2.7742932 |

g=1.20

| | | | | |
|------|------------|------------|-----------|-----------|
| N=35 | -0.1581696 | -0.0857005 | 0.3257955 | 0.3260457 |
| N=31 | -0.1130641 | -0.0612612 | 0.3696524 | 0.3697802 |
| N=15 | 0.1354131 | 0.0733705 | 0.7691110 | 0.7692943 |
| N=11 | 0.0131593 | 0.0071301 | 1.1209038 | 1.1209055 |

TABLE 5.1

SUMMARY INFORMATION ON THE SAMPLE

| | | ZERO-DIVIDEND SUBSAMPLE | SINGLE DIVIDEND SUBSAMPLE | TOTAL SAMPLE |
|------------------------------------|---------|----------------------------|------------------------------|-----------------|
| ln(S/X) +rT | mean | 0.033654 | -0.012376 | 0.00622 |
| | median | 0.022897 | -0.009338 | 0.001703 |
| | maximum | 0.541091 | 0.458952 | 0.541091 |
| | minimum | -0.401377 | -0.493288 | -0.493288 |
| ln(S/X) | mean | 0.013220 | -0.049046 | -0.023886 |
| | median | -0.002503 | -0.044979 | -0.028510 |
| | maximum | 0.535030 | 0.442285 | 0.535030 |
| | minimum | -0.418044 | -0.520562 | -0.520562 |
| T(days) | mean | 53.91090 | 96.80540 | 79.47600 |
| | median | 44.00000 | 100.00000 | 72.00000 |
| | maximum | 254.00000 | 163.00000 | 254.00000 |
| | minimum | 16.00000 | 16.00000 | 16.00000 |
| Unbiased Estimate of Volatility | | | | |
| Rate/day | mean | 0.021709 | 0.020436 | 0.020951 |
| | median | 0.022392 | 0.019314 | 0.019314 |
| | maximum | 0.033423 | 0.033573 | 0.033573 |
| | minimum | 0.011418 | 0.011015 | 0.011015 |
| Cases of ln(S/X) | >0 | 48 | 56 | 104 |
| | =0 | 2 | 0 | 2 |
| | <0 | 51 | 93 | 144 |
| Cases of g | >0 | 56 | 70 | 126 |
| | =0 | 0 | 0 | 0 |
| | <0 | 45 | 79 | 124 |

TABLE 5.1A

FREQUENCY DISTRIBUTION OF MONEYNES($\ln(S/X)$), TIME TO MATURITY (DAYS) AND UNBIASED ESTIMATE OF DAILY VOLATILITY RATE FOR THE ZERO-DIVIDEND AND THE SINGLE DIVIDEND SUBSAMPLES

| | | ZERO-DIVIDEND SUBSAMPLE | SINGLE-DIVIDEND SUBSAMPLE |
|--|-----------|----------------------------|------------------------------|
| MONEYNES($\ln(S/X)$) | | | |
| -0.520562 ... | -0.309444 | 4 | 11 |
| -0.309444 ... | -0.098325 | 22 | 47 |
| -0.098325 ... | 0.112793 | 45 | 66 |
| 0.112793 ... | 0.323910 | 26 | 22 |
| 0.323910 ... | 0.535030 | 4 | 3 |
| TIME TO MATURITY(Days) | | | |
| 16 days | | 35 | 9 |
| 44 days | | 27 | 15 |
| 72 days | | 30 | 36 |
| 100 days | | 0 | 45 |
| 135 days | | 3 | 23 |
| 163 days and over | | 6 | 21 |
| UNBIASED ESTIMATE OF DAILY VOLATILITY RATE | | | |
| 0.011015 ... | 0.015527 | 18 | 24 |
| 0.015527 ... | 0.020038 | 28 | 63 |
| 0.020038 ... | 0.024550 | 21 | 30 |
| 0.024550 ... | 0.029061 | 26 | 18 |
| 0.029061 ... | 0.033573 | 8 | 14 |

TABLE 5.2

INFORMATION ON FORMULA PRICING IN THE ZERO-DIVIDEND
AND THE SINGLE-DIVIDEND SUBSAMPLES

| ----- OVERPRICED OPTIONS ----- | ZERO-DIVIDEND | SINGLE-DIVIDEND |
|---|---------------|-----------------|
| Number of options | 67 | 99 |
| Mean overpricing | -0.4089987 | -0.542326 |
| ln(S/X) | 0.0054388 | -0.052691 |
| T | 62.179105 | 100.283000 |
| Estimated volatility rate | 0.022509 | 0.021598 |
| Overpriced in-the-money (ln(S/X)>0) options ----- | GROUP ZOI | GROUP SOI |
| Number of options | 30 | 33 |
| Mean overpricing | -0.370346 | -0.516902 |
| ln(S/X) | 0.144376 | 0.122707 |
| T | 61.033300 | 93.424200 |
| Estimated volatility rate | 0.022371 | 0.022049 |
| Overpriced at-the-money (ln(S/X)=0) options ----- | GROUP GOA | |
| Number of options | 2 | |
| Overpricing | -0.092918 and | -0.352585 |
| T | 72 and | 44 |
| Estimated volatility rate | 0.017773 and | 0.018219 |
| Overpriced out-of-the-money (ln(S/X)<0) options ----- | GROUP ZOO | GROUP SOO |
| Number of options | 35 | 66 |
| Mean overpricing | -0.452771 | -0.555037 |
| ln(S/X) | -0.113339 | -0.140391 |
| T | 63.400000 | 103.712000 |
| Estimated volatility rate | 0.022886 | 0.021373 |

 UNDERPRICED OPTIONS

| | | |
|---------------------------|-----------|-----------|
| Number of options | 34 | 50 |
| Mean underpricing | 0.1227536 | 0.200617 |
| ln(S/X) | 0.0285897 | -0.041828 |
| T | 37.617647 | 89.920000 |
| Estimated volatility rate | 0.0202484 | 0.018136 |

Underpriced in-the-money
 (ln(S/X)>0) options

GROUP ZUI

GROUP SUI

| | | |
|---------------------------|-----------|-----------|
| Number of options | 18 | 23 |
| Mean underpricing | 0.175067 | 0.291098 |
| ln(S/X) | 0.220534 | 0.127096 |
| T | 45.944400 | 93.304300 |
| Estimated volatility rate | 0.019603 | 0.017577 |

Underpriced out-of-the-money
 (ln(S/X)<0) options

GROUP ZUO

GROUP SUO

| | | |
|---------------------------|-----------|-----------|
| Number of options | 16 | 27 |
| Mean underpricing | 0.063901 | 0.123540 |
| ln(S/X) | -0.187349 | -0.185726 |
| T | 28.250000 | 87.037000 |
| Estimated volatility rate | 0.020726 | 0.018613 |

TABLE 5.3: SIMPLE REGRESSION RESULTS FOR THE
ZERO-DIVIDEND AND THE SINGLE-DIVIDEND SUBSAMPLES**

zero-dividend subsample

| | Constant | Slope | F(1,99) |
|---------------------------|--------------------------|--------------------------|---------|
| Regressor:ln(S/X) | -0.231886 (-5.594060) | 0.143127 (0.652033) | 0.425 |
| Regressor:T | 0.030441 (0.547676) | -0.004831 (-6.074020) | 36.894 |
| Regressor: $\hat{\sigma}$ | 0.100380 (0.588410) | -15.21830 (-1.993960) | 3.976 |

single-dividend subsample

| | Constant | Slope | F(1,147) |
|---------------------------|--------------------------|--------------------------|----------|
| Regressor:ln(S/X) | -0.283108 (-6.457880) | 0.202008 (0.832085) | 0.692 |
| Regressor:T | -0.004336 (-0.041048) | -0.002982 (-2.966230) | 8.799 |
| Regressor: $\hat{\sigma}$ | 0.515668 (3.478420) | -39.57080 (-5.647030) | 31.889 |

**The terms in parenthesis are t-statistics.

TABLE 5.4

MULTIPLE REGRESSION RESULTS FOR
THE ZERO-DIVIDEND AND THE SINGLE-DIVIDEND SUBSAMPLES**

| | Zero-dividend | Single-dividend |
|-------------------------------|------------------------|-------------------------|
| R ² | 0.27755 | 0.2316 |
| Error Sum of Squares | 12.405 | 30.101 |
| F(3,97) | 12.422 | |
| F(3,145) | | 14.568 |
| Constant | -0.00200 (-0.01311) | 0.78926 (4.68013) |
| Coefficient of ln(S/X) | 0.17267 (0.90009) | -0.000703 (-0.00324) |
| Coefficient of T | -0.00492 (-5.64883) | -0.0029 (-3.16259) |
| Coefficient of $\hat{\sigma}$ | 1.60941 (0.21868) | -39.2290 (-5.70881) |

**The terms in parenthesis are t-statistics.

TABLE 6.1

TEST RESULTS FOR DIVIDEND INDUCEMENT

Restrictions under Null Hypothesis(H_0)

| | |
|---------|--|
| Test 1 | $a^{(0)} = a^{(1)}, \psi_1^{(0)} = \psi_1^{(1)}, \psi_2^{(0)} = \psi_2^{(1)}, \psi_3^{(0)} = \psi_3^{(1)}$ |
| Test 2 | $\psi_1^{(0)} = \psi_1^{(1)}, \psi_2^{(0)} = \psi_2^{(1)}, \psi_3^{(0)} = \psi_3^{(1)}$ |
| Test 3 | $\psi_1^{(0)} = \psi_1^{(1)}, \psi_2^{(0)} = \psi_2^{(1)}$ |
| Test 4 | $\psi_1^{(0)} = \psi_1^{(1)}, \psi_3^{(0)} = \psi_3^{(1)}$ |
| Test 5 | $\psi_2^{(0)} = \psi_2^{(1)}, \psi_3^{(0)} = \psi_3^{(1)}$ |
| Test 6 | $\psi_1^{(0)} = \psi_1^{(1)}$ |
| Test 7 | $\psi_2^{(0)} = \psi_2^{(1)}$ |
| Test 8 | $\psi_3^{(0)} = \psi_3^{(1)}$ |
| Test 9 | $a^{(0)} = a^{(1)}$ |
| Test 3A | $a^{(0)} = a^{(1)}, \psi_1^{(0)} = \psi_1^{(1)}, \psi_2^{(0)} = \psi_2^{(1)}$ |
| Test 4A | $a^{(0)} = a^{(1)}, \psi_1^{(0)} = \psi_1^{(1)}, \psi_3^{(0)} = \psi_3^{(1)}$ |
| Test 5A | $a^{(0)} = a^{(1)}, \psi_2^{(0)} = \psi_2^{(1)}, \psi_3^{(0)} = \psi_3^{(1)}$ |
| Test 6A | $a^{(0)} = a^{(1)}, \psi_1^{(0)} = \psi_1^{(1)}$ |
| Test 7A | $a^{(0)} = a^{(1)}, \psi_2^{(0)} = \psi_2^{(1)}$ |
| Test 8A | $a^{(0)} = a^{(1)}, \psi_3^{(0)} = \psi_3^{(1)}$ |

| Test | J | SSER | F | F(J,242) | R/UTR |
|------|---|--------|---------|----------|-------|
| 1 | 4 | 45.228 | 3.8743 | 2.37 | R |
| 2 | 3 | 45.130 | 4.9798 | 2.60 | R |
| 3 | 2 | 42.968 | 1.3152 | 3.00 | UTR |
| 4 | 2 | 45.067 | 7.2903 | 3.00 | R |
| 5 | 2 | 45.130 | 7.4696 | 3.00 | R |
| 6 | 1 | 42.564 | 0.3302 | 3.84 | UTR |
| 7 | 1 | 42.914 | 2.3228 | 3.84 | UTR |
| 8 | 1 | 45.067 | 14.5806 | 3.84 | R |
| 9 | 1 | 44.471 | 11.1874 | 3.84 | R |
| 3A | 3 | 45.220 | 5.1506 | 2.60 | R |
| 4A | 3 | 45.068 | 4.8621 | 2.60 | R |
| 5A | 3 | 45.228 | 5.1657 | 2.60 | R |
| 6A | 2 | 44.475 | 5.6051 | 3.00 | R |
| 7A | 2 | 45.219 | 7.7514 | 3.00 | R |
| 8A | 2 | 45.068 | 7.2931 | 3.00 | R |

** F(J,242) is the value of F, for 2 and ∞ degrees of freedom with 5% right-tail probability.
 ** SSER is the restricted error sum of squares. In all cases, the unrestricted error sum of squares is 42.506.
 ** J denotes number of restrictions tested.
 ** R denotes rejection of null hypothesis and UTR denotes that we are unable to reject null hypothesis.

TABLE 7.1

RESULTS OF F-TEST FOR BLACK-SCHOLES VALIDITY

Test Restrictions under null hypothesis

1 $\beta_2 - ms_{21}\beta_1 = 0, \beta_3 - ms_{31}\beta_1 = 0$

2 $\beta_1 - ms_{12}\beta_2 = 0, \beta_3 - ms_{32}\beta_2 = 0$

3 $\beta_1 - ms_{13}\beta_3 = 0, \beta_2 - ms_{23}\beta_3 = 0$

| Test | F(2,97) (with constant) | R/UTR | F(2,98) (without constant) | R/UTR |
|------|----------------------------|-------|-------------------------------|-------|
| 1 | 15.6467 | R | 37.5445 | R |
| 2 | 18.2857 | R | 29.8741 | R |
| 3 | 11.7135 | R | 12.3039 | R |

** F(2,60) with 5% right-tail probability is 3.15 and
F(2,120) with 5% right-tail probability is 3.07.

** R denotes rejection of null hypothesis and UTR denotes that
we are unable to reject null hypothesis.

TABLE 7.2

RESULTS OF WALD TEST FOR BLACK-SCHOLES VALIDITY

Test Restrictions under null hypothesis

1 $(\beta_2/\beta_1) - m_{21} = 0, (\beta_3/\beta_1) - m_{31} = 0$

2 $(\beta_1/\beta_2) - m_{12} = 0, (\beta_3/\beta_2) - m_{32} = 0$

3 $(\beta_1/\beta_3) - m_{13} = 0, (\beta_2/\beta_3) - m_{23} = 0$

| Test | W(2) (with constant) | R/UTR | W(2) (without constant) | R/UTR |
|------|-------------------------|-------|----------------------------|-------|
| 1 | 0.8387 | UTR | 0.7350 | UTR |
| 2 | 0.9048 | UTR | 0.9035 | UTR |
| 3 | 0.4244 | UTR | 0.1198 | UTR |

** χ^2_2 with 5% right-tail probability is 5.99147.

** R denotes rejection of null hypothesis and UTR denotes that we are unable to reject null hypothesis.

TABLE 8.1: SPLINE REGRESSION RESULTS FOR
THE ZERO-DIVIDEND AND THE SINGLE-DIVIDEND SUBSAMPLES

Regression Equation: ¹

$$\begin{aligned}
 C-CB(\hat{\sigma}) = & a_1 + b_1 (m-m_0) + c_1 (m-m_0)^2 + d_1 (m-m_0)^3 + \\
 & (d_2 - d_1) (m-m_1)^3 D_1 + (d_3 - d_2) (m-m_2)^3 D_2 + \\
 & b_1 (T-T_0) + c_1 (T-T_0)^2 + d_1 (T-T_0)^3 + \\
 & (d_2 - d_1) (T-T_1)^3 D_1 + (d_3 - d_2) (T-T_2)^3 D_2 + \\
 & b_1 (\hat{\sigma}-\hat{\sigma}_0) + c_1 (\hat{\sigma}-\hat{\sigma}_0)^2 + d_1 (\hat{\sigma}-\hat{\sigma}_0)^3 + \\
 & (d_2 - d_1) (\hat{\sigma}-\hat{\sigma}_1)^3 D_1 + (d_3 - d_2) (\hat{\sigma}-\hat{\sigma}_2)^3 D_2
 \end{aligned}$$

Zero-dividend Single-dividend

| | | |
|---|-------------------------|-------------------------|
| R ² | 0.4356 | 0.4339 |
| F(15,85) | 4.3740 | |
| F(15,133) | | 6.795 |
| a ₁ | 0.37093 (0.86638) | 0.79078 (1.82333) |
| b ₁ M | 2.66899 (0.54526) | -0.60125 (-0.13273) |
| c ₁ M | -26.35960 (-1.03693) | -13.47250 (-0.79637) |
| d ₁ M | 46.52390 (1.21584) | 24.01680 (1.26298) |
| (d ₂ - d ₁) M M | -61.81710 (-1.27439) | -47.11380 (-1.61277) |

¹The coefficients, knots and dummy variables of the splines are subscripted by the variable name.

| | | |
|----------------------------------|----------------------------|----------------------------|
| $(d_3 - d_2)$ M M | 15.34860 (0.56402) | 39.13480 (0.80983) |
| b_1 T | -0.22003 (-1.80704) | -0.00917 (-0.53228) |
| c_1 T | 0.01307 (1.81318) | 1.53709E-04 (0.28404) |
| d_1 T | -1.89584E-04 (-1.83548) | -5.91044E-07 (-0.13211) |
| $(d_2 - d_1)$ T T | 2.49617E-04 (1.85917) | -2.35739E-06 (-0.23295) |
| $(d_3 - d_2)$ T T | -6.19842E-05 (-1.93970) | 3.49201E-06 (0.40866) |
| b_1 σ | -197.61500 (-0.94567) | 200.9800 (1.28335) |
| c_1 σ | 43504.9000 (1.04103) | -61757.000 (-1.68455) |
| d_1 σ | -2.71384E+06 (-1.06224) | 4.21069E+06 (1.77766) |
| $(d_2 - d_1)$ $\sigma \sigma$ | 3.25912E+06 (0.97503) | -7.83935E+06 (-1.92965) |
| $(d_3 - d_2)$ $\sigma \sigma$ | 4.86329E+05 (0.21663) | 4.64327E+06 (1.86295) |
| m_0 | -0.418044 | -0.520562 |
| m_1 | -0.150000 | -0.150000 |
| m_2 | 0.150000 | 0.150000 |
| T_0 | 16 | 16 |
| T_1 | 46 | 75 |
| T_2 | 71 | 100 |
| $\hat{\sigma}_0$ | 0.011418 | 0.011015 |
| $\hat{\sigma}_1$ | 0.018000 | 0.018000 |

$\hat{\sigma}_2$

0.025000

0.022000

** Figures in parenthesis are t-statistics.

FIGURE 2.1

PLOTTING OF BLACK-SCHOLES FORMULA ESTIMATE'S DEVIATION FROM MARKET PRICE AGAINST TIME TO MATURITY FOR A HYPOTHETICAL SAMPLE OF OPTIONS

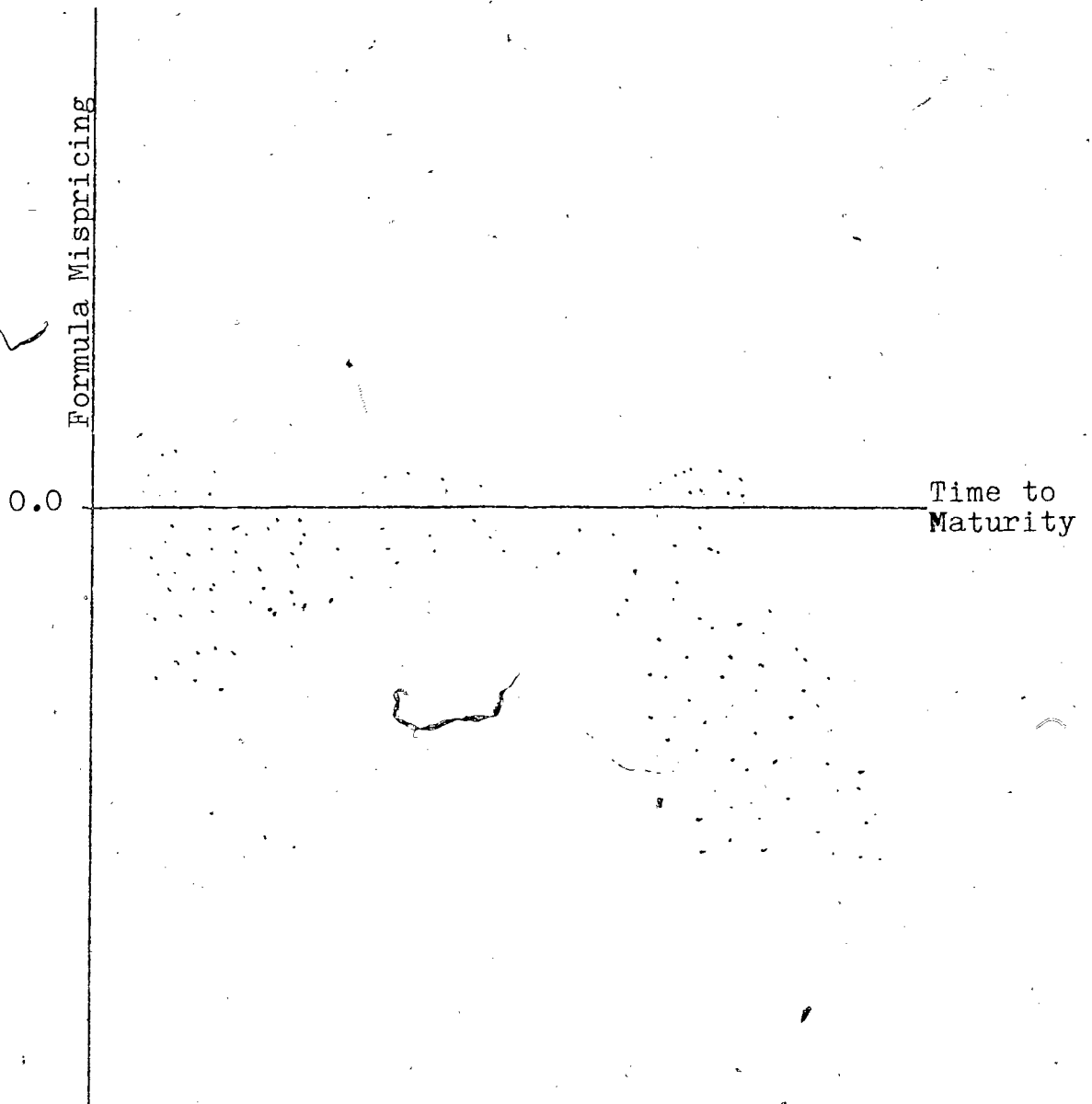
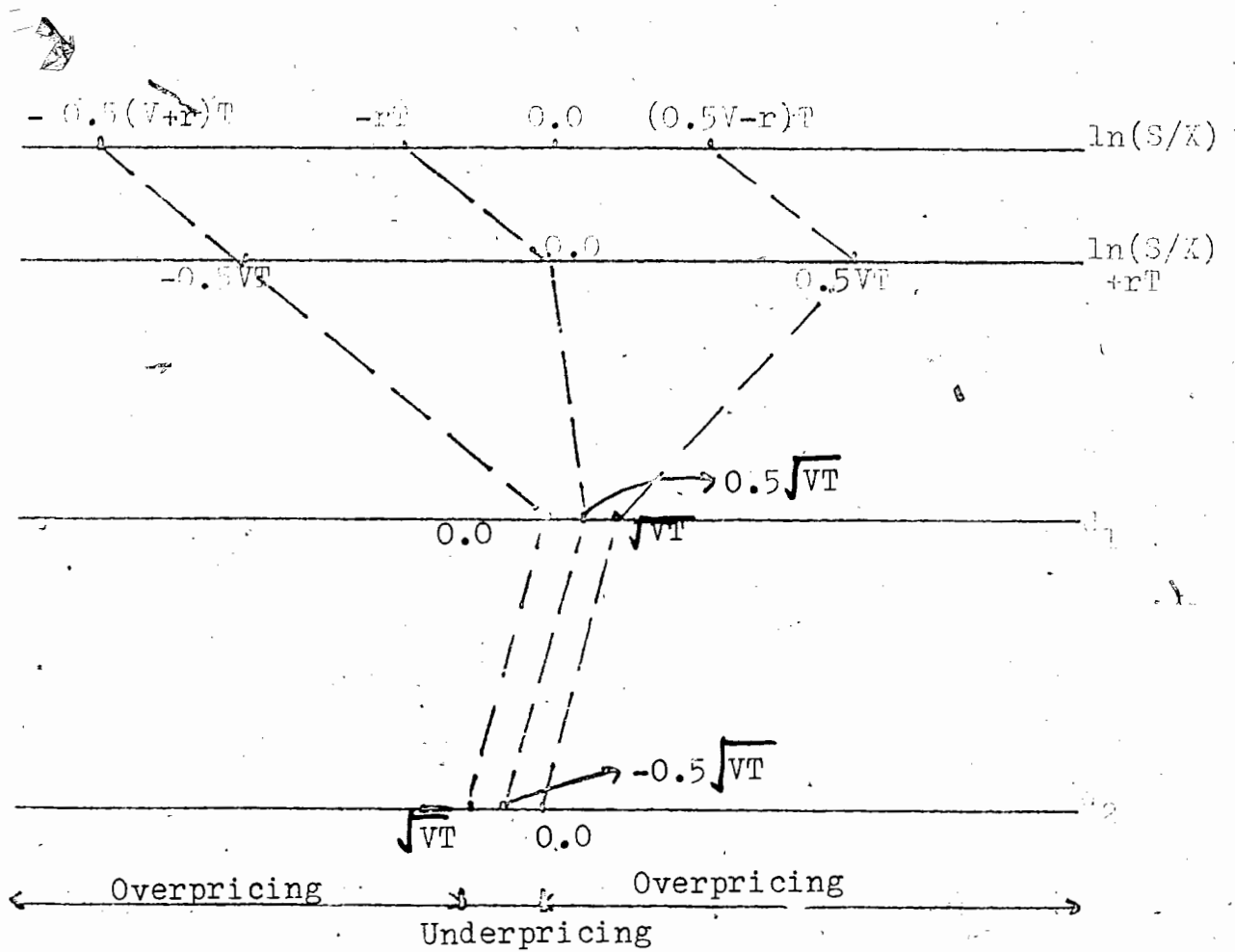


FIGURE 4.1

SECOND ORDER APPROXIMATION TO NONLINEARITY BIAS OF THE
BLACK-SCHOLES FORMULA ESTIMATE AND THE RANGE OF UNDER OR
OVERPRICING



The figure is drawn here for r less than $0.5V$.

FIGURE 4.2

FORMULA MISPRICING AS FUNCTION OF MONEYNES, STRIKING PRICE= 50,
TIME TO MATURITY=0.5 QUARTER, RISKLESS RATE=0.015/QUARTER

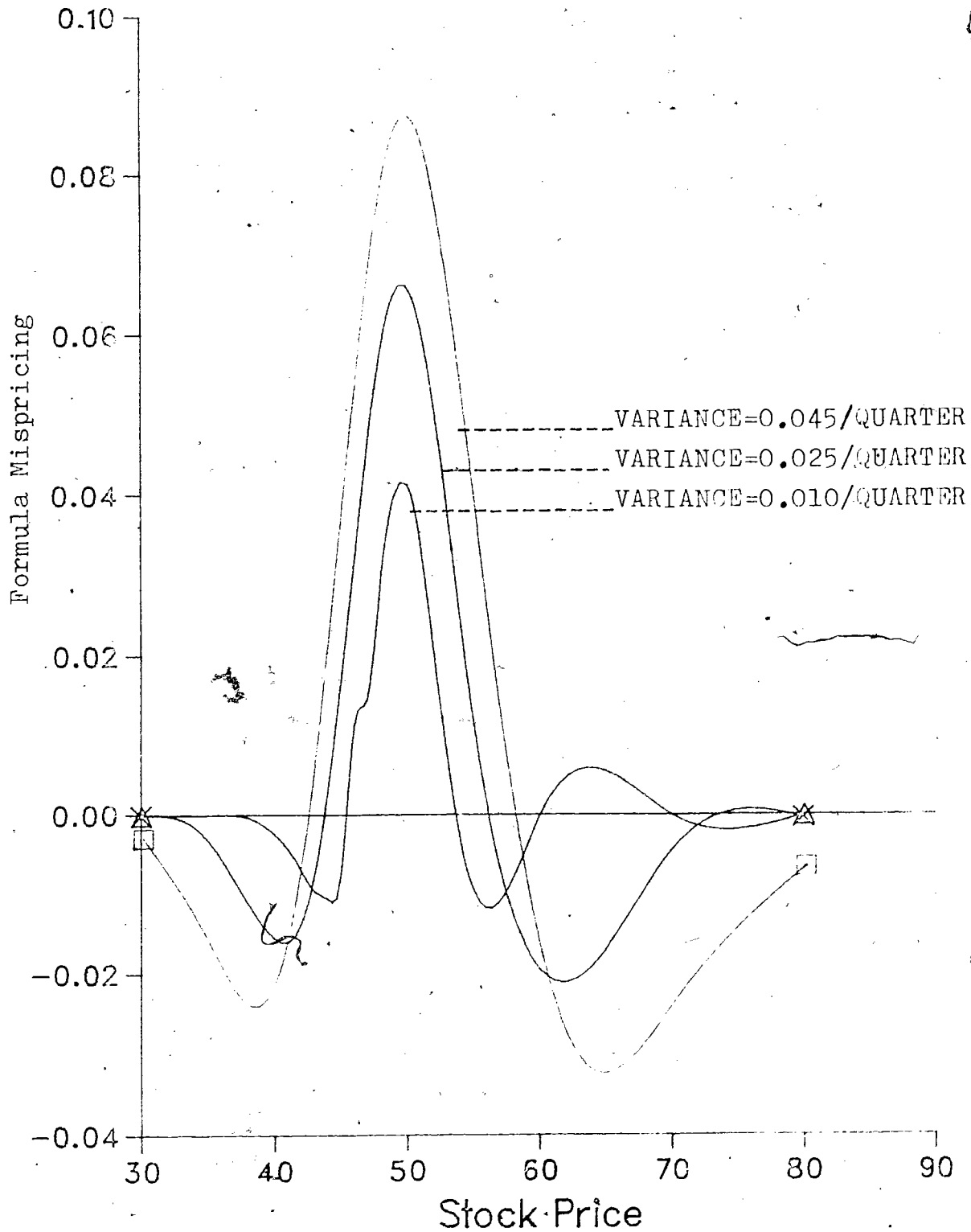


FIGURE 4.3

FORMULA MISPRICING AS FUNCTION OF MONEYNES, STRIKING PRICE= 50,
TIME TO MATURITY=1.0 QUARTER, RISKLESS RATE=0.015/QUARTER

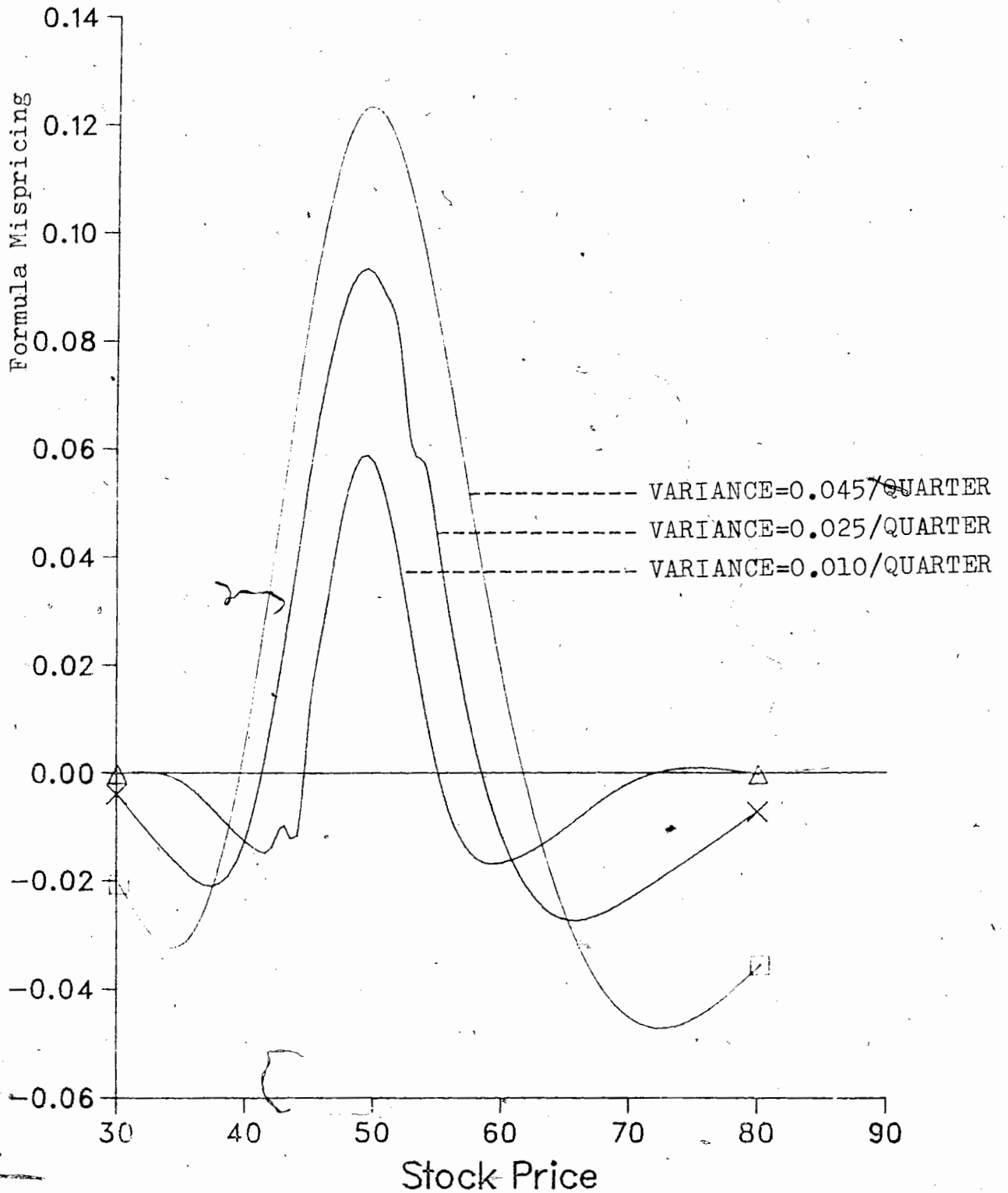


FIGURE 4.4

FORMULA MISPRICING AS FUNCTION OF MONEYNES, STRIKING PRICE= 50,
TIME TO MATURITY=1.5 QUARTER, RISKLESS RATE=0.015/QUARTER

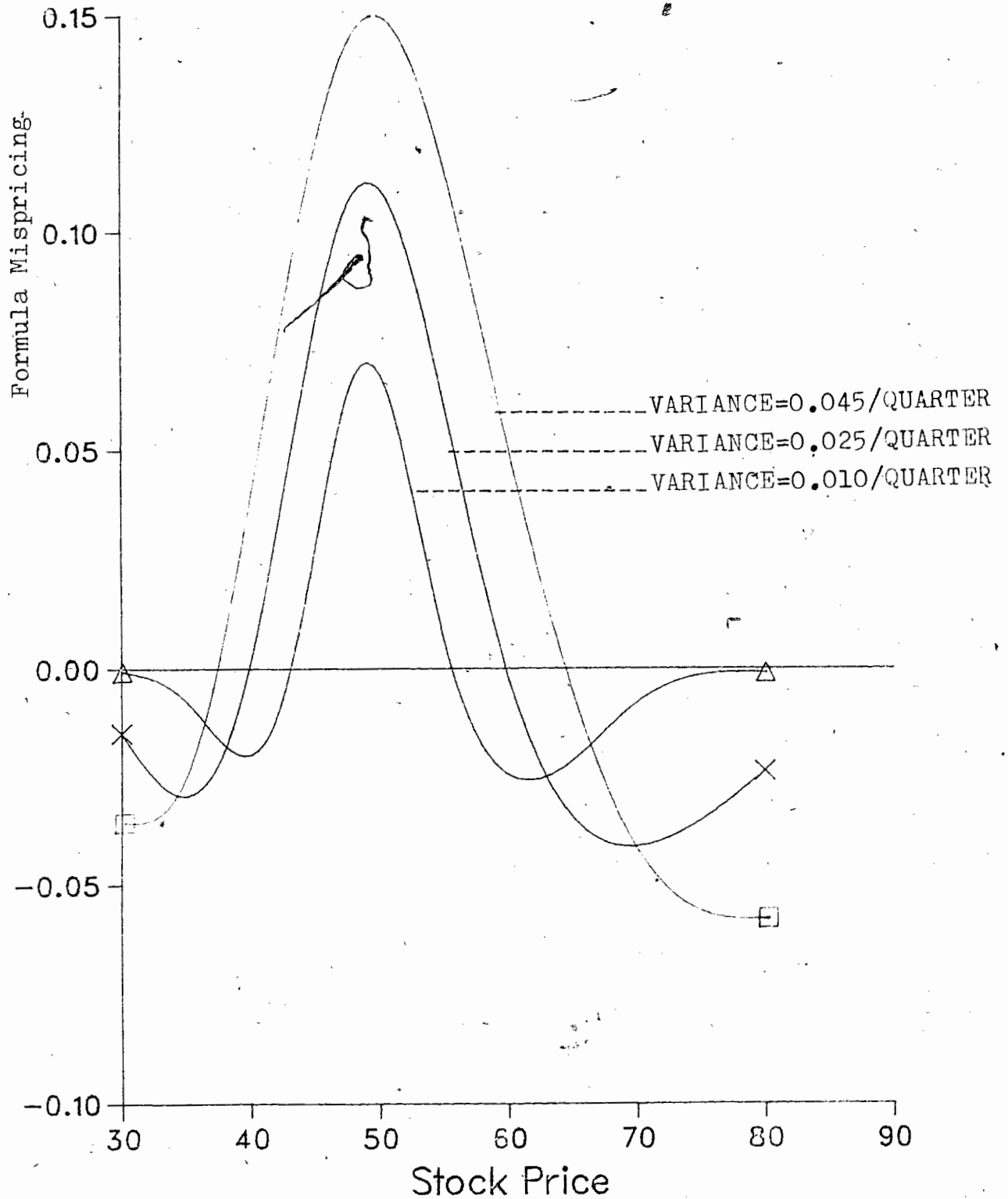


FIGURE 4.5

FORMULA MISPRICING AS FUNCTION OF TIME TO MATURITY, STRIKING PRICE=50, VARIANCE=0.010/QUARTER, RISKLESS RATE=0.015/QUARTER

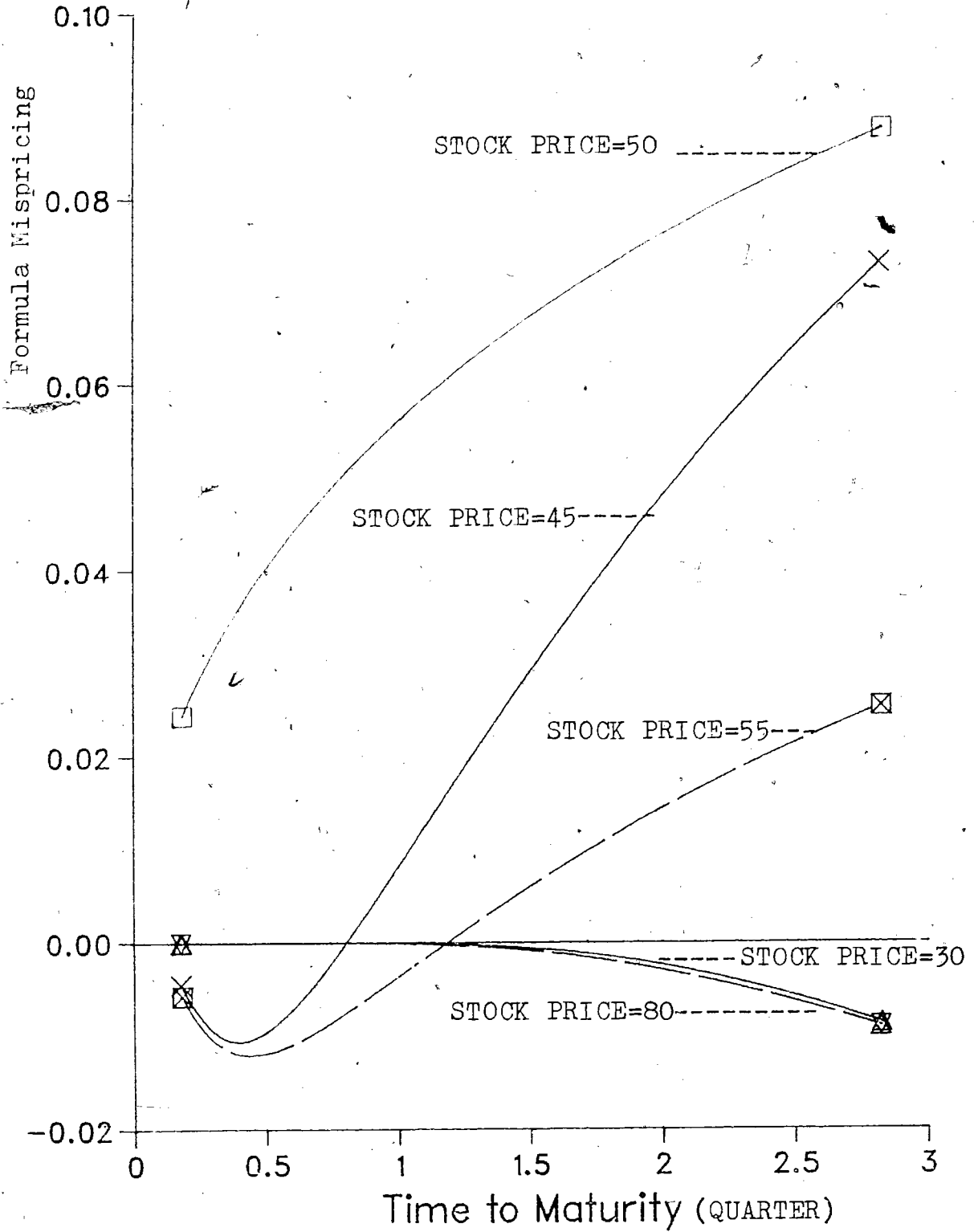


FIGURE 4.6

FORMULA MISPRICING AS FUNCTION OF TIME TO MATURITY, STRIKING PRICE=50, VARIANCE=0.025/QUARTER, RISKLESS RATE=0.015/QUARTER

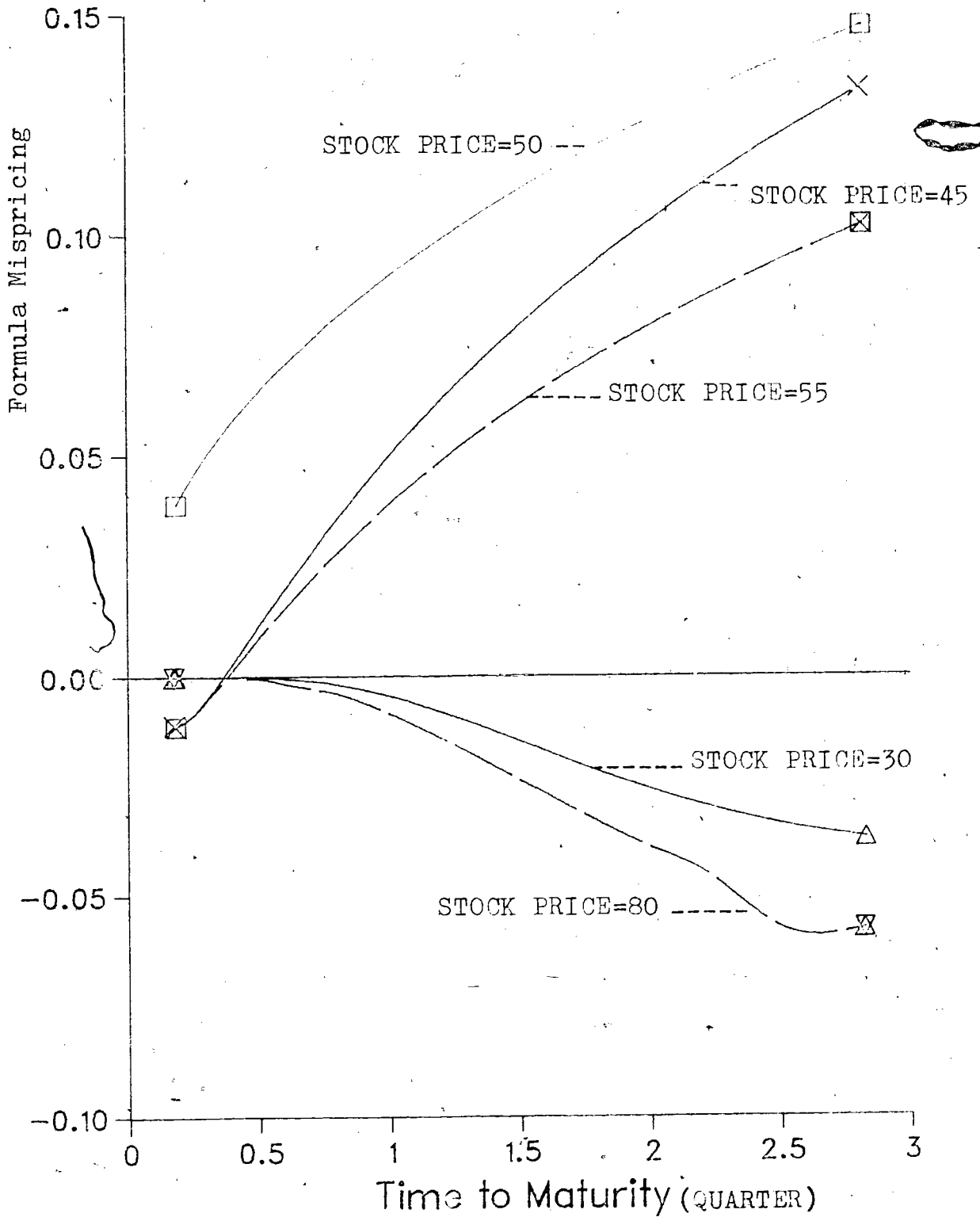


FIGURE 4.7

FORMULA MISPRICING AS FUNCTION OF TIME TO MATURITY, STRIKING PRICE=50, VARIANCE=0.045/QUARTER, RISKLESS RATE=0.015/QUARTER

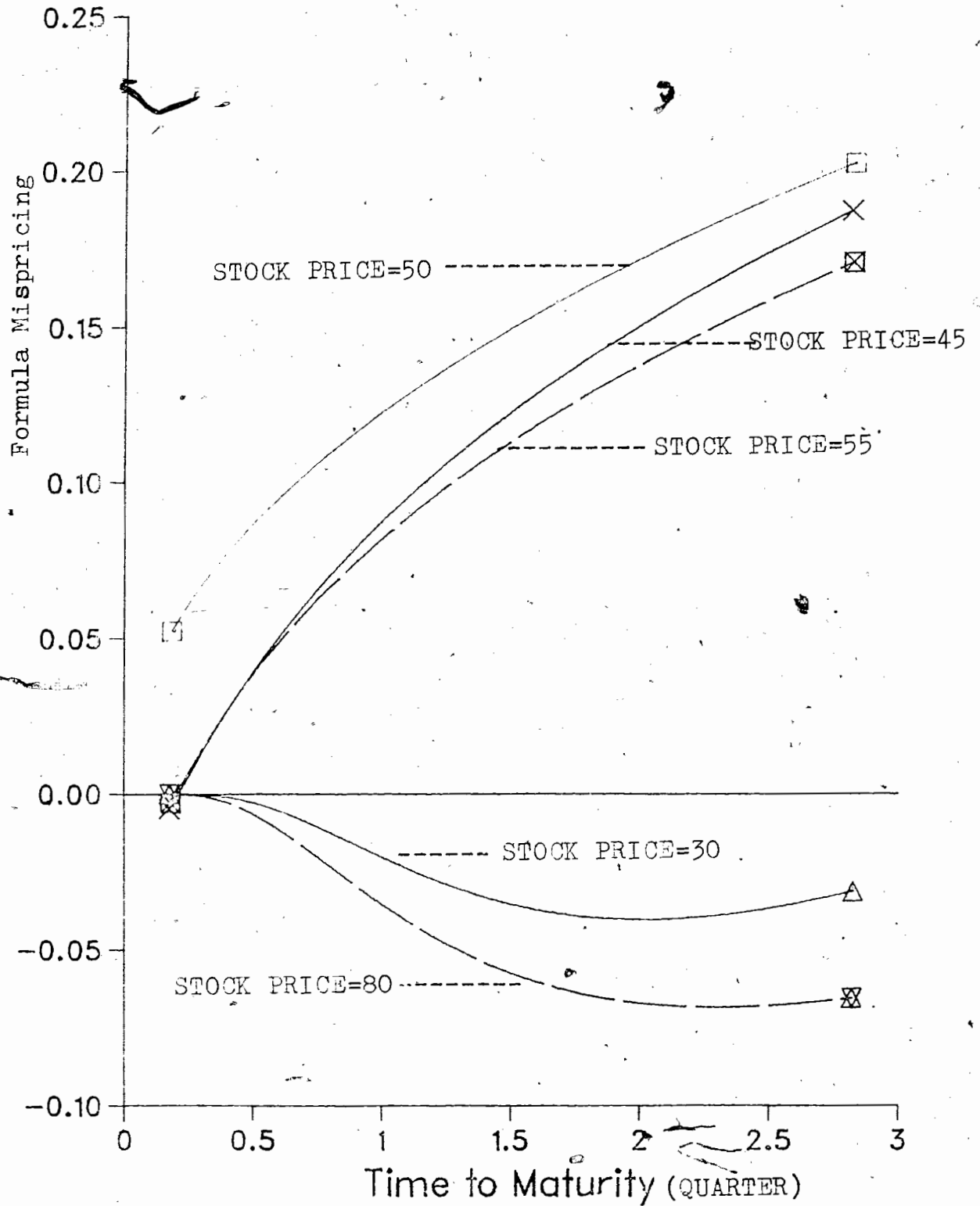


FIGURE 4.8

FORMULA MISPRICING AS FUNCTION OF VARIANCE RATE, STRIKING
PRICE=50; TIME TO MATURITY=0.5 QUARTER, RISKLESS
RATE=0.015/QUARTER

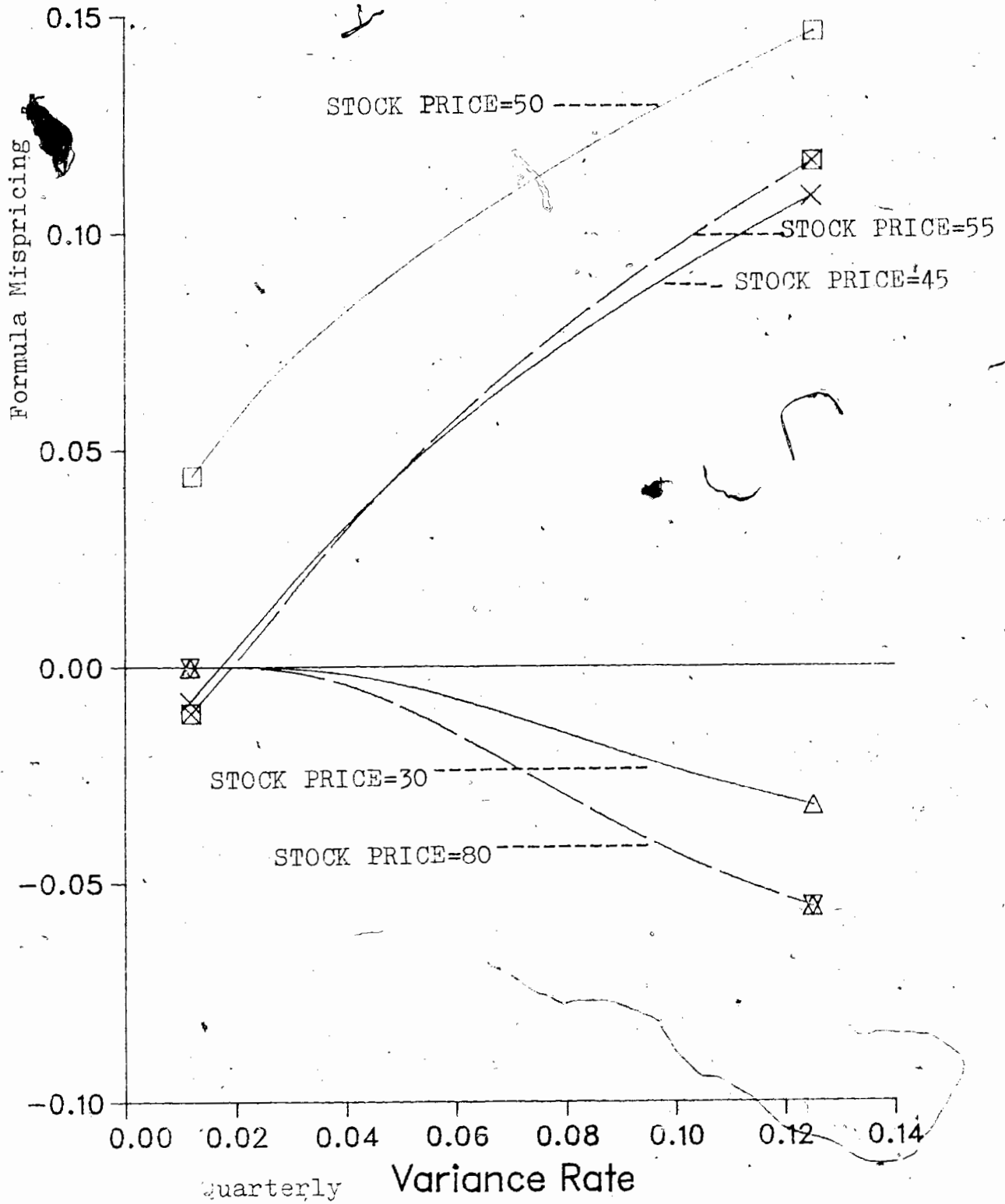


FIGURE 4.9

FORMULA MISPRICING AS FUNCTION OF VARIANCE RATE, STRIKING
PRICE=50, TIME TO MATURITY=1.0 QUARTER, RISKLESS
RATE=0.015/QUARTER

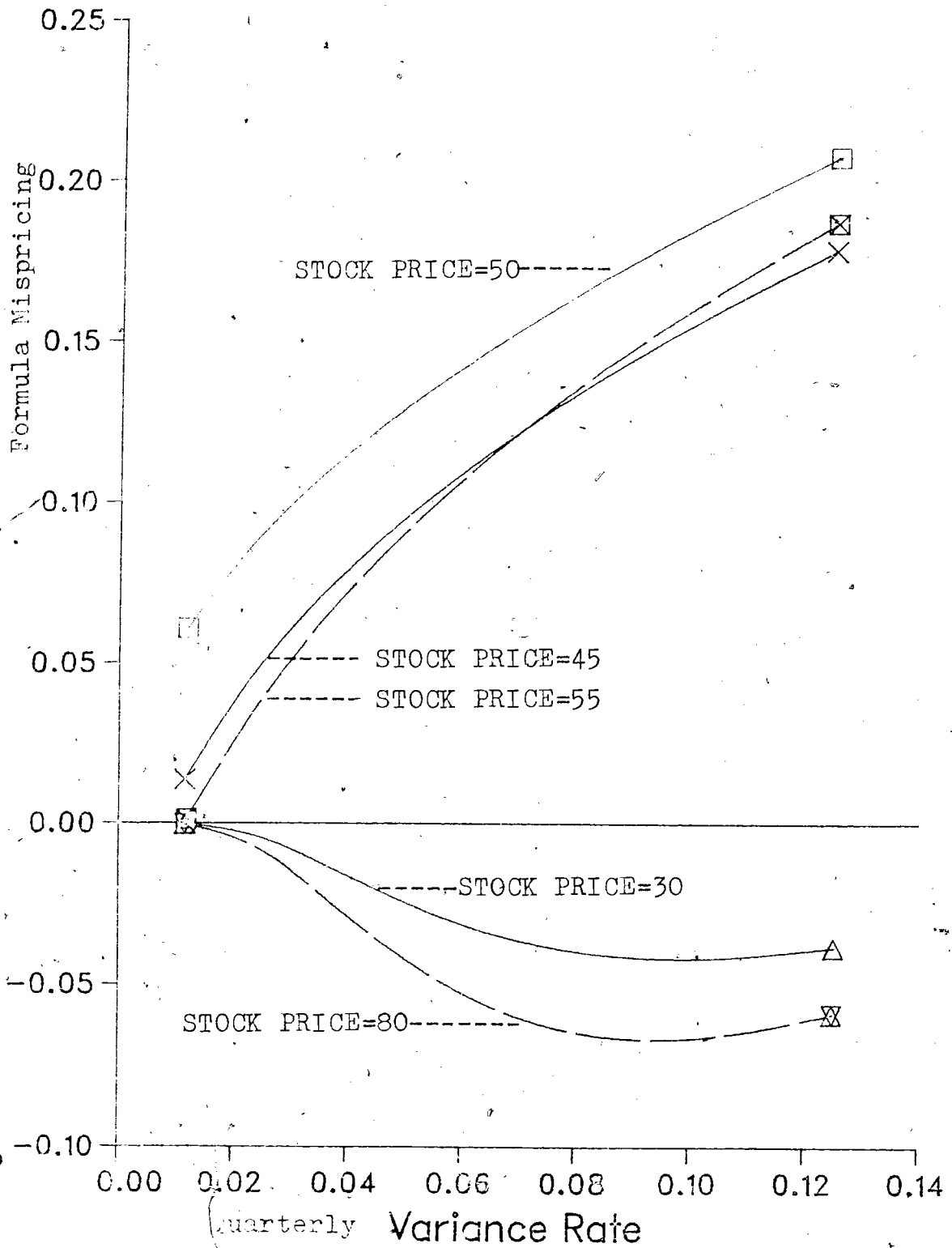


FIGURE 4.10

FORMULA MISPRICING AS FUNCTION OF VARIANCE RATE, STRIKING PRICE=50, TIME TO MATURITY=0.5 QUARTER, RISKLESS RATE=0.015/QUARTER

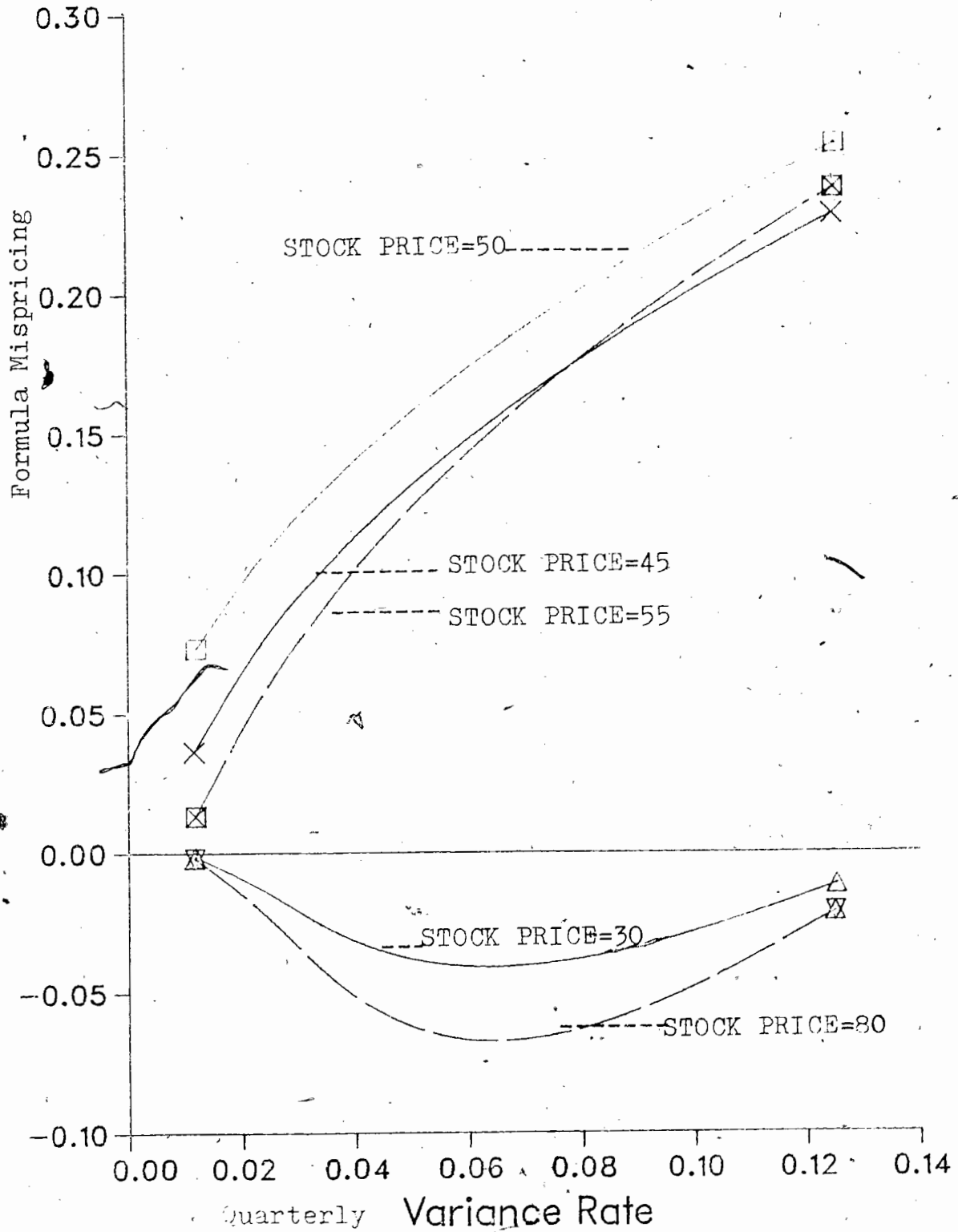


FIGURE 5.1

FORMULA MISPRICING AND MONEYNESS, ZERO-DIVIDEND SUBSAMPLE

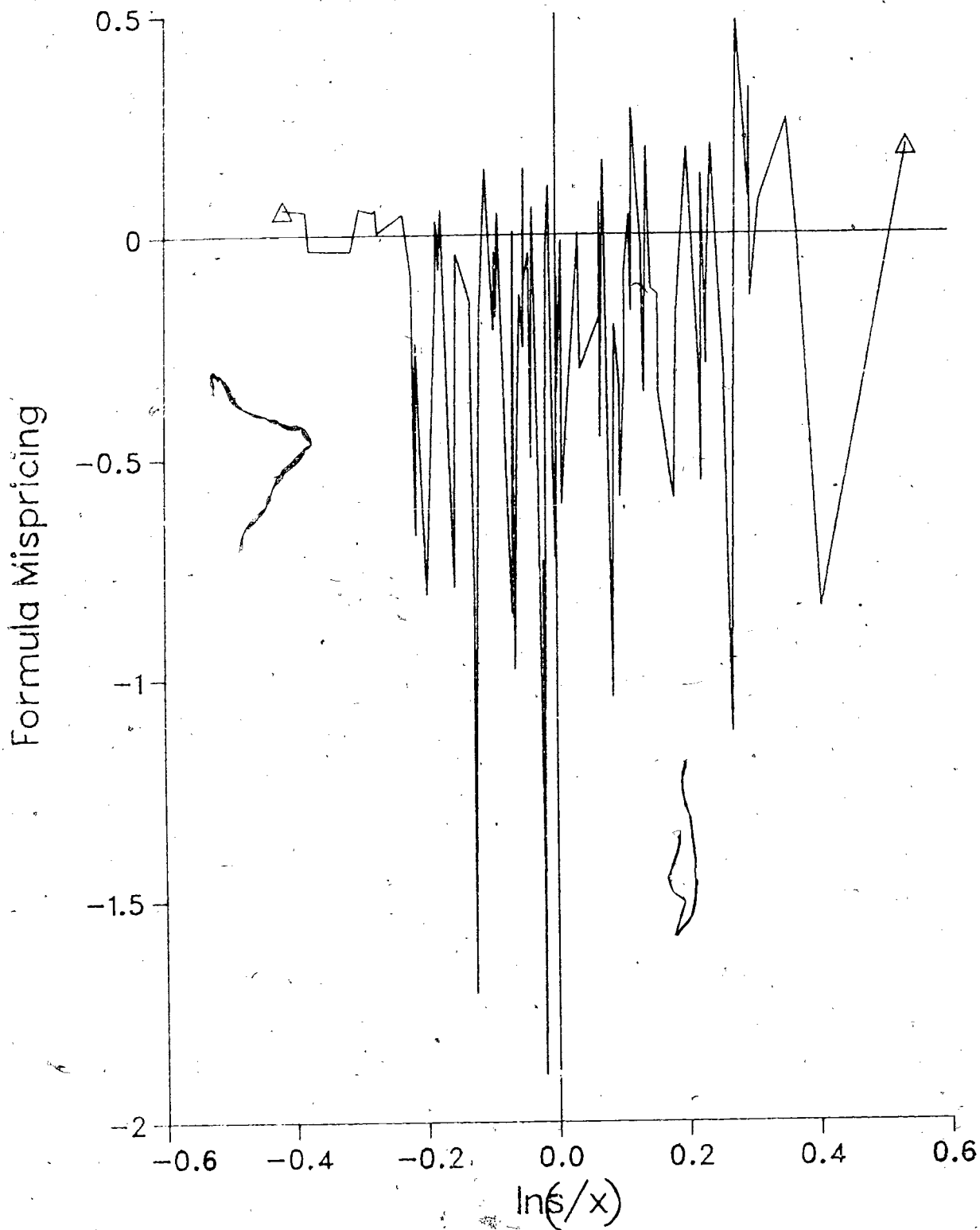


FIGURE 5.2

FORMULA MISPRICING AND TIME TO MATURITY, ZERO-DIVIDEND SUBSAMPLE

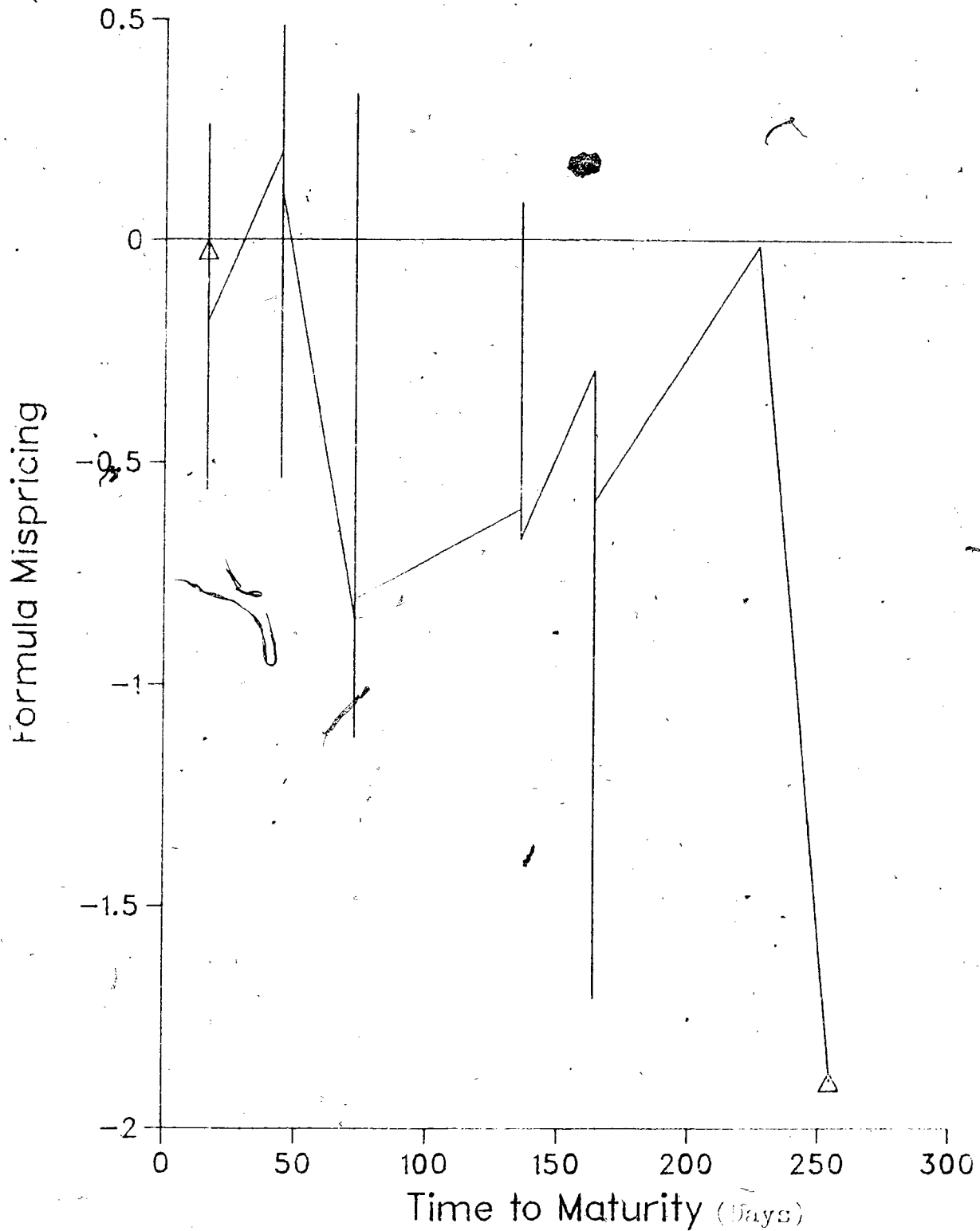


FIGURE 5.3

FORMULA MISPRICING AND ESTIMATED VOLATILITY RATE, ZERO-DIVIDEND
SUBSAMPLE

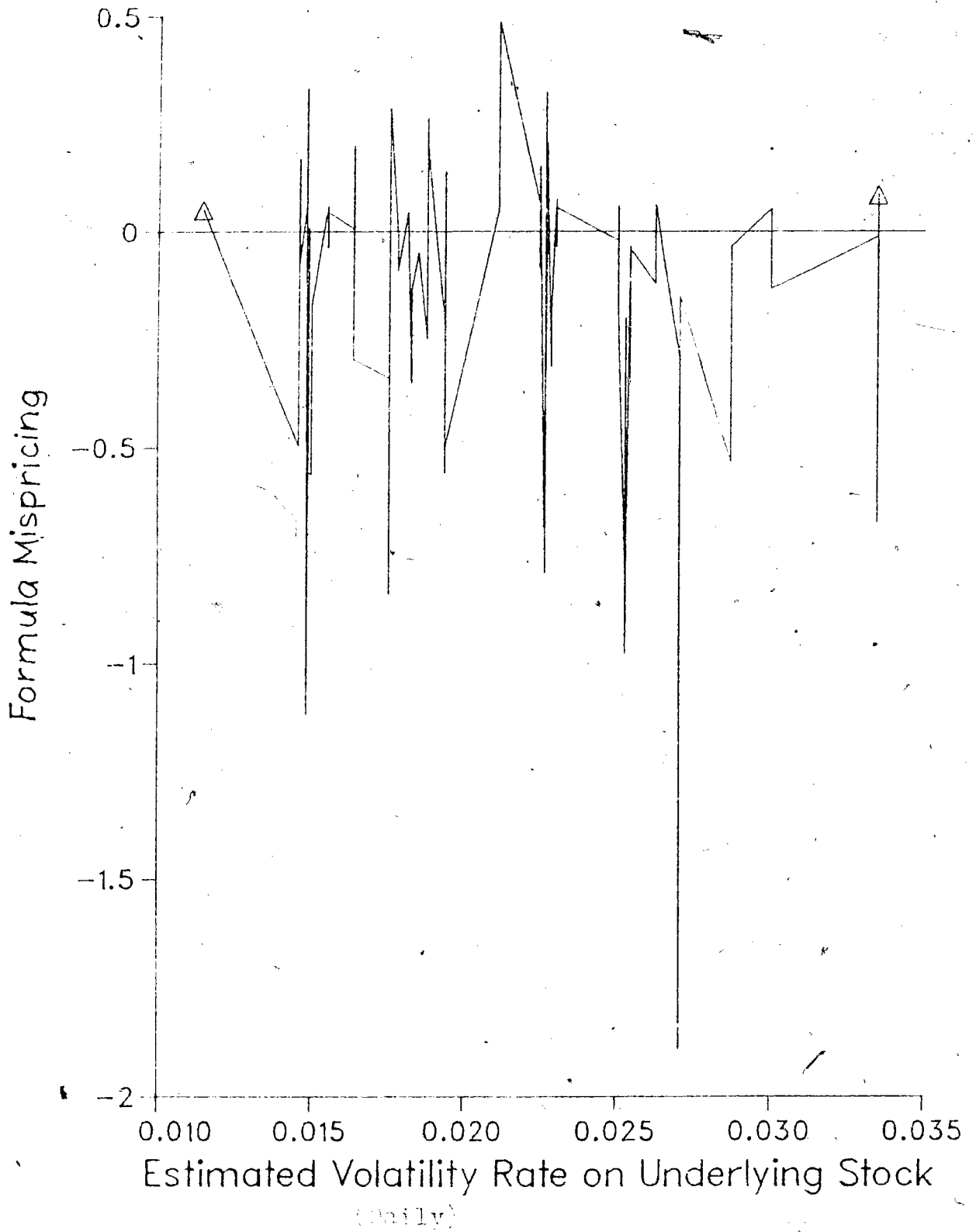


FIGURE 5.4

FORMULA MISPRICING AND MONEYNES, SINGLE-DIVIDEND SUBSAMPLE

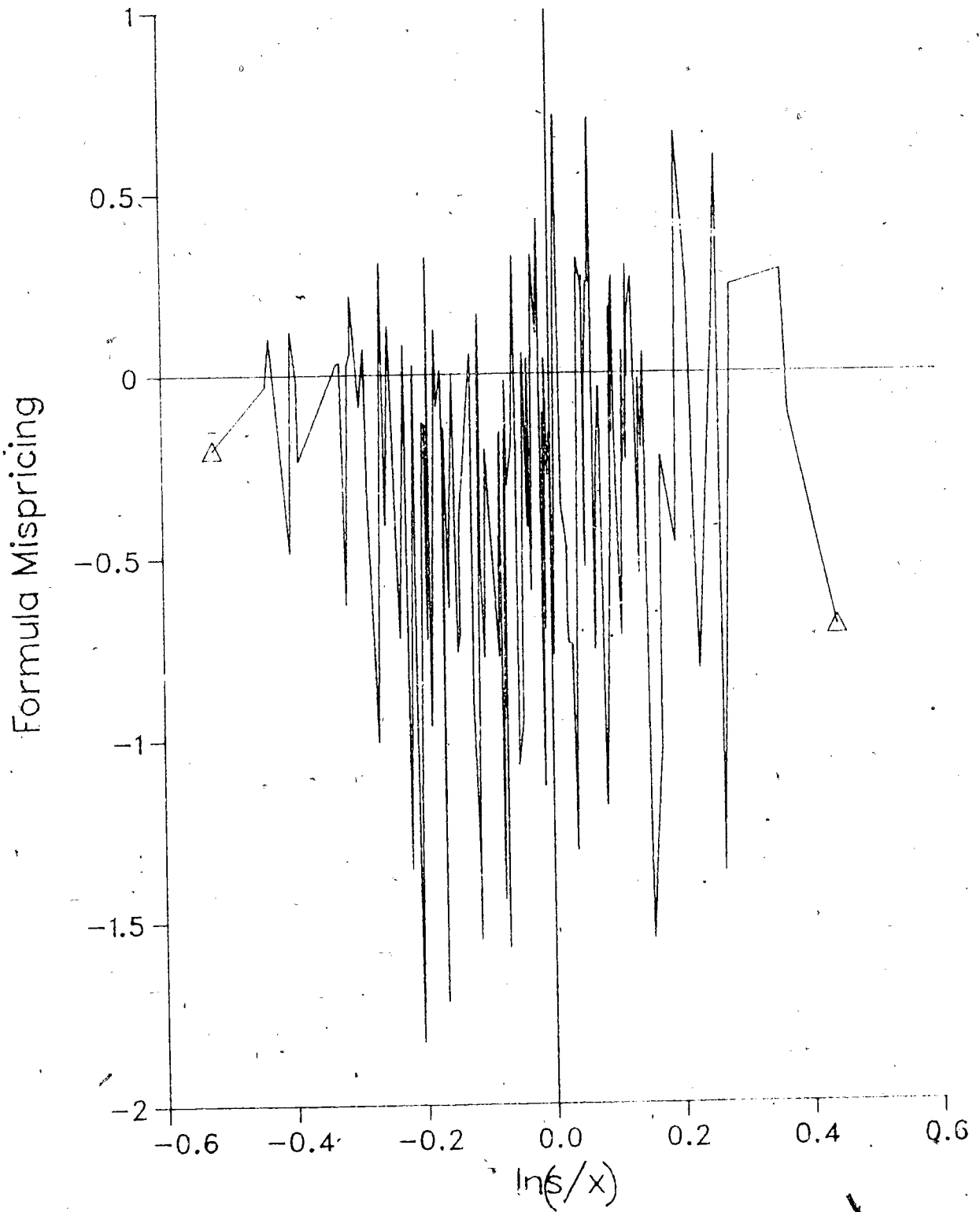


FIGURE 5.5

FORMULA MISPRICING AND TIME TO MATURITY, SINGLE-DIVIDEND
SUBSAMPLE

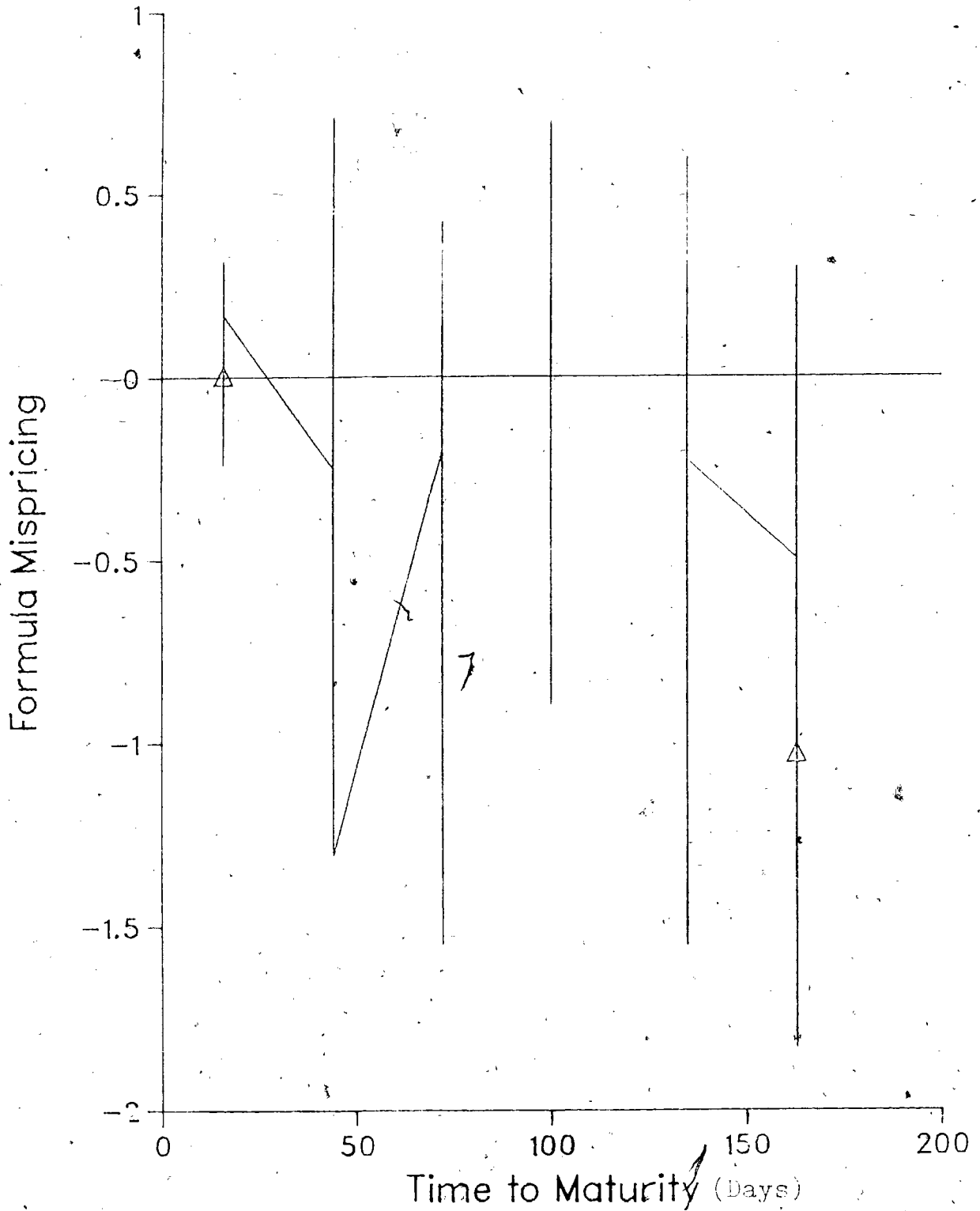


FIGURE 5.6

FORMULA MISPRICING AND ESTIMATED VOLATILITY RATE,
SINGLE-DIVIDEND SUBSAMPLE

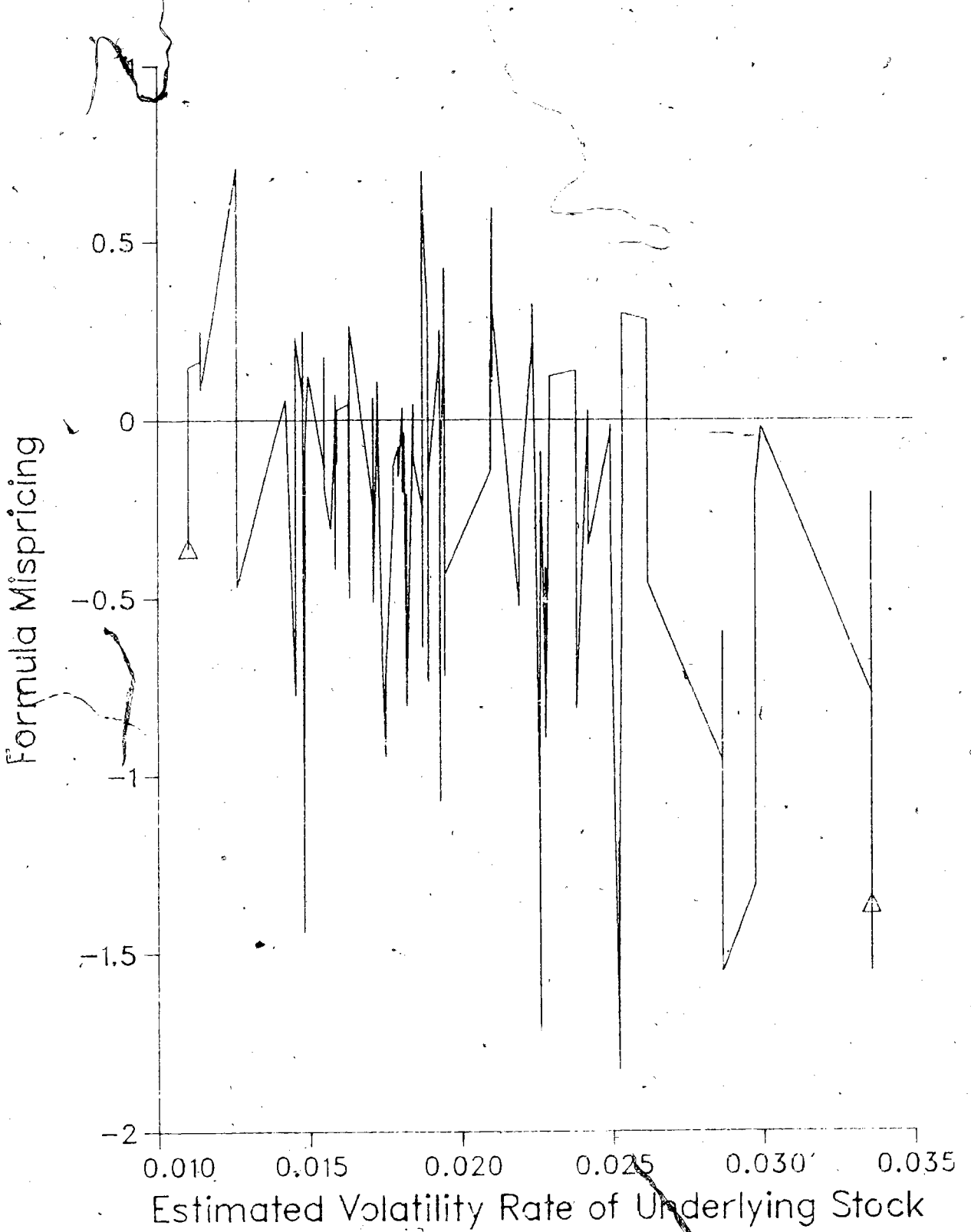


FIGURE 8.1

SPLINE PREDICTION OF MONEYNES BIAS IN THE ZERO-DIVIDEND
SUBSAMPLE FOR QUARTERLY VARIANCE OF 0.010

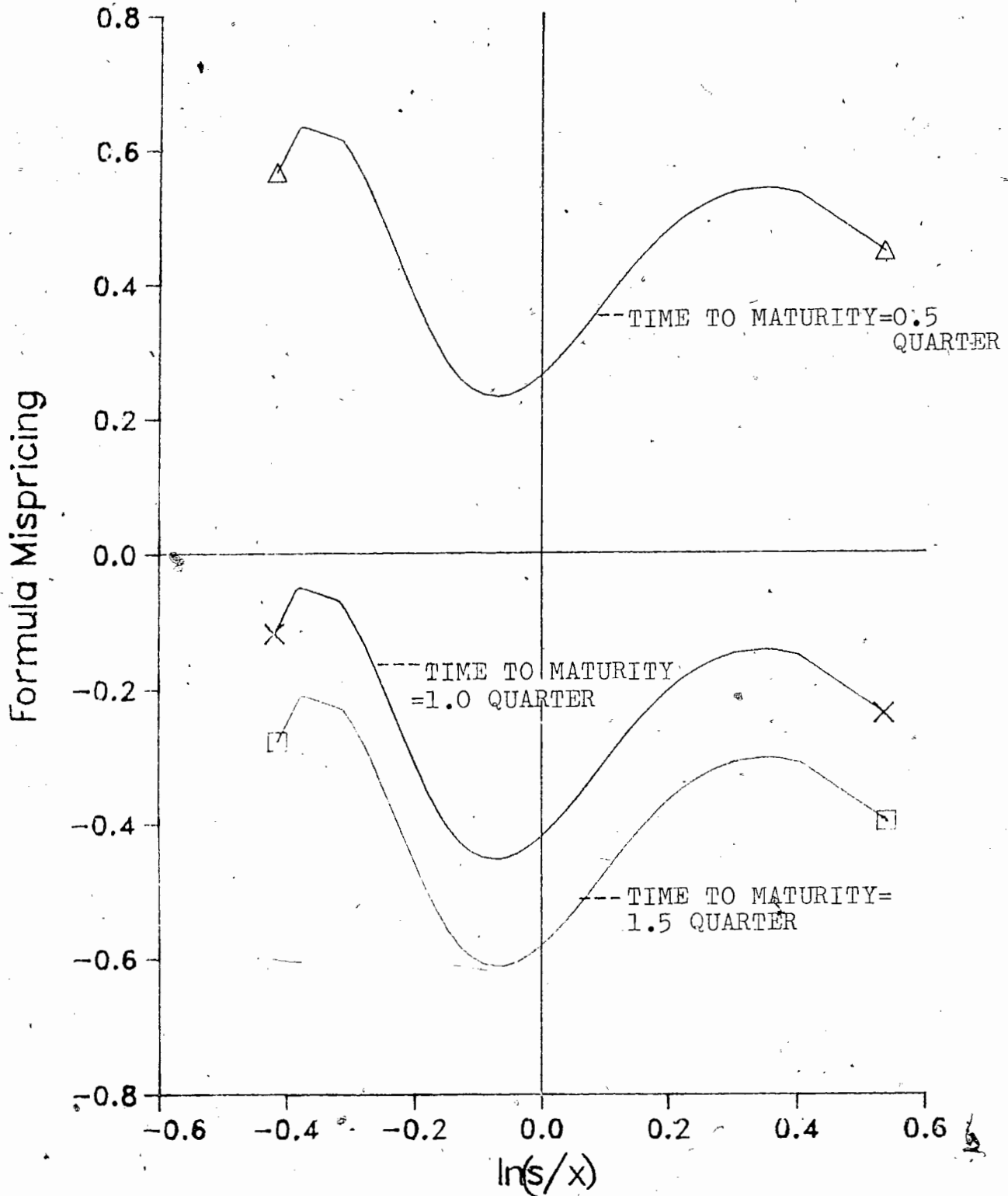


FIGURE 8.2

SPLINE PREDICTION OF MONEYNES BIAS IN THE ZERO-DIVIDEND.
SUBSAMPLE FOR QUARTERLY VARIANCE OF 0.025

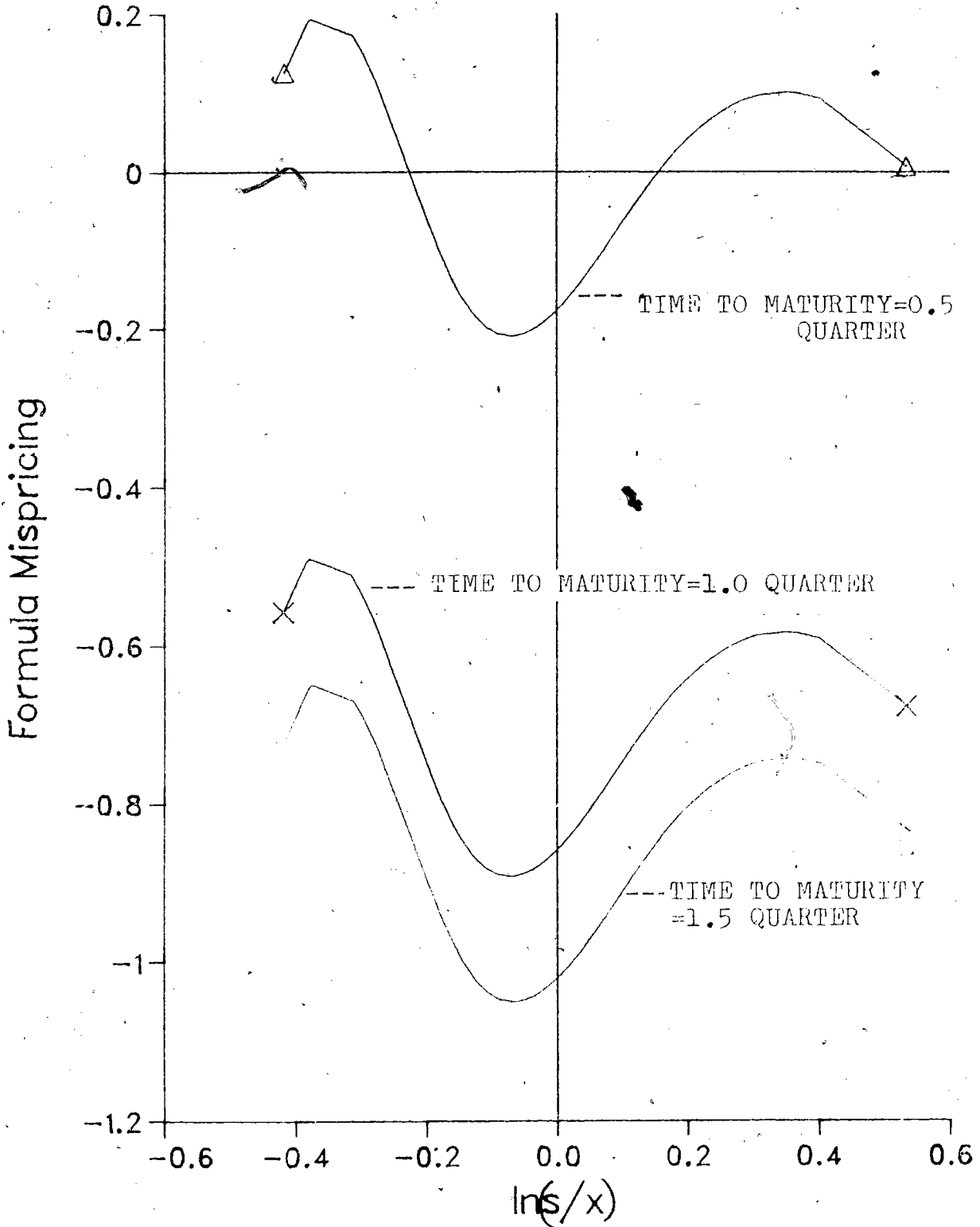


FIGURE 8.3

SPLINE PREDICTION OF MONEYNES BIAS IN THE ZERO-DIVIDEND
SUBSAMPLE FOR QUARTERLY VARIANCE OF 0.045

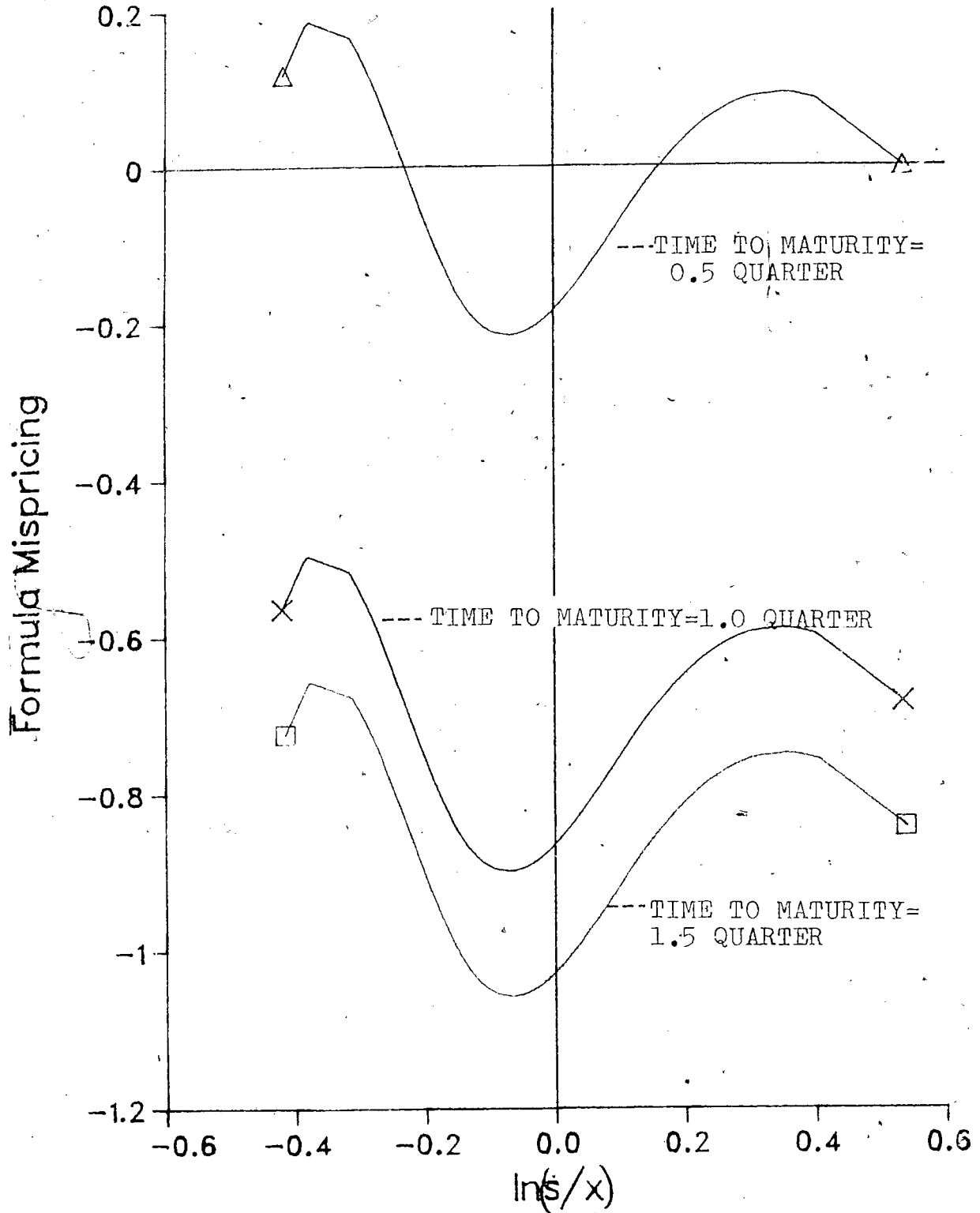


FIGURE 8.4

SPLINE PREDICTION OF MONEYNES BIAS IN THE SINGLE-DIVIDEND

SUBSAMPLE FOR QUARTERLY VARIANCE OF 0.010

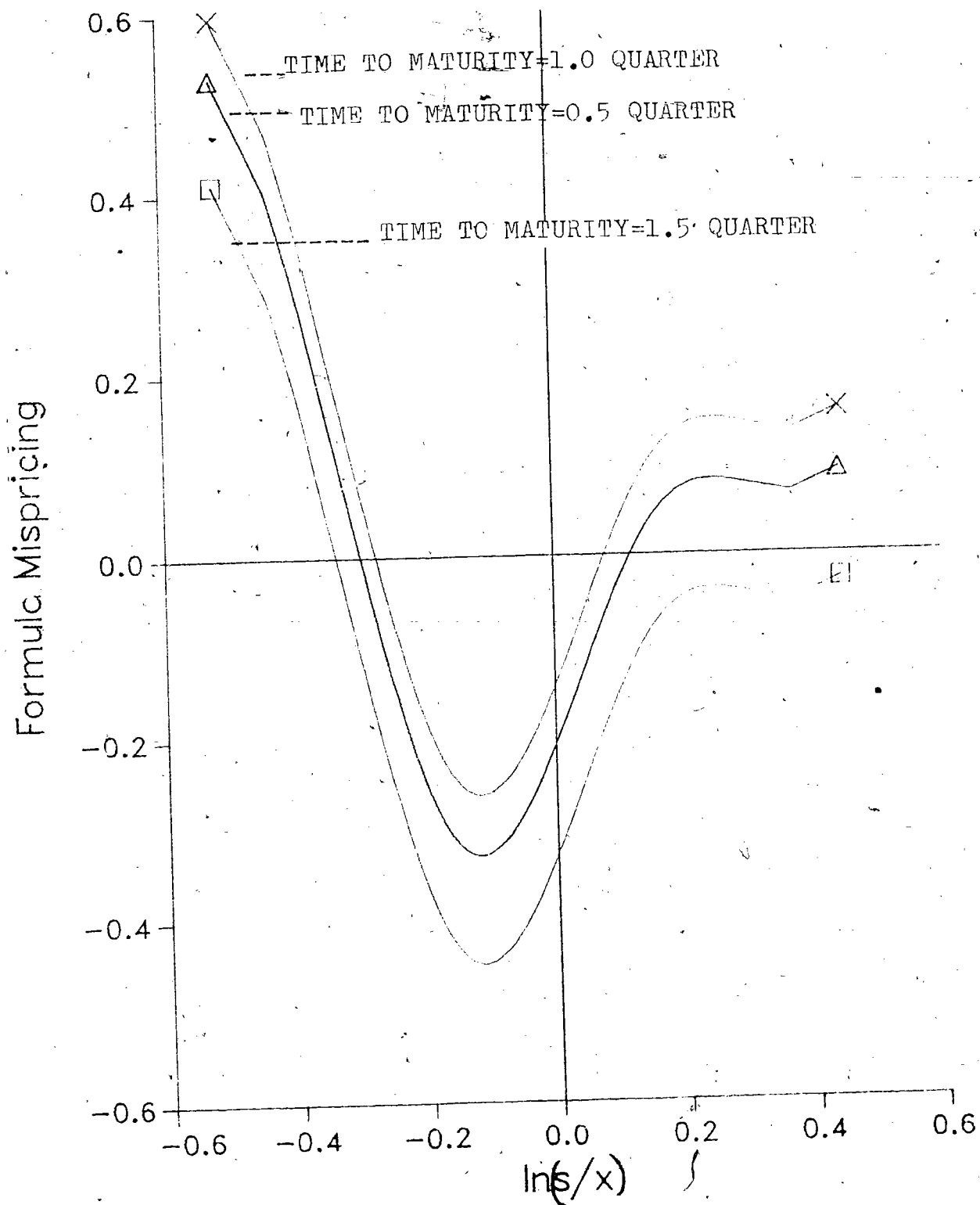


FIGURE 8.5

SPLINE PREDICTION OF MONEYNES BIAS IN THE SINGLE-DIVIDEND
SUBSAMPLE FOR QUARTERLY VARIANCE OF 0.025

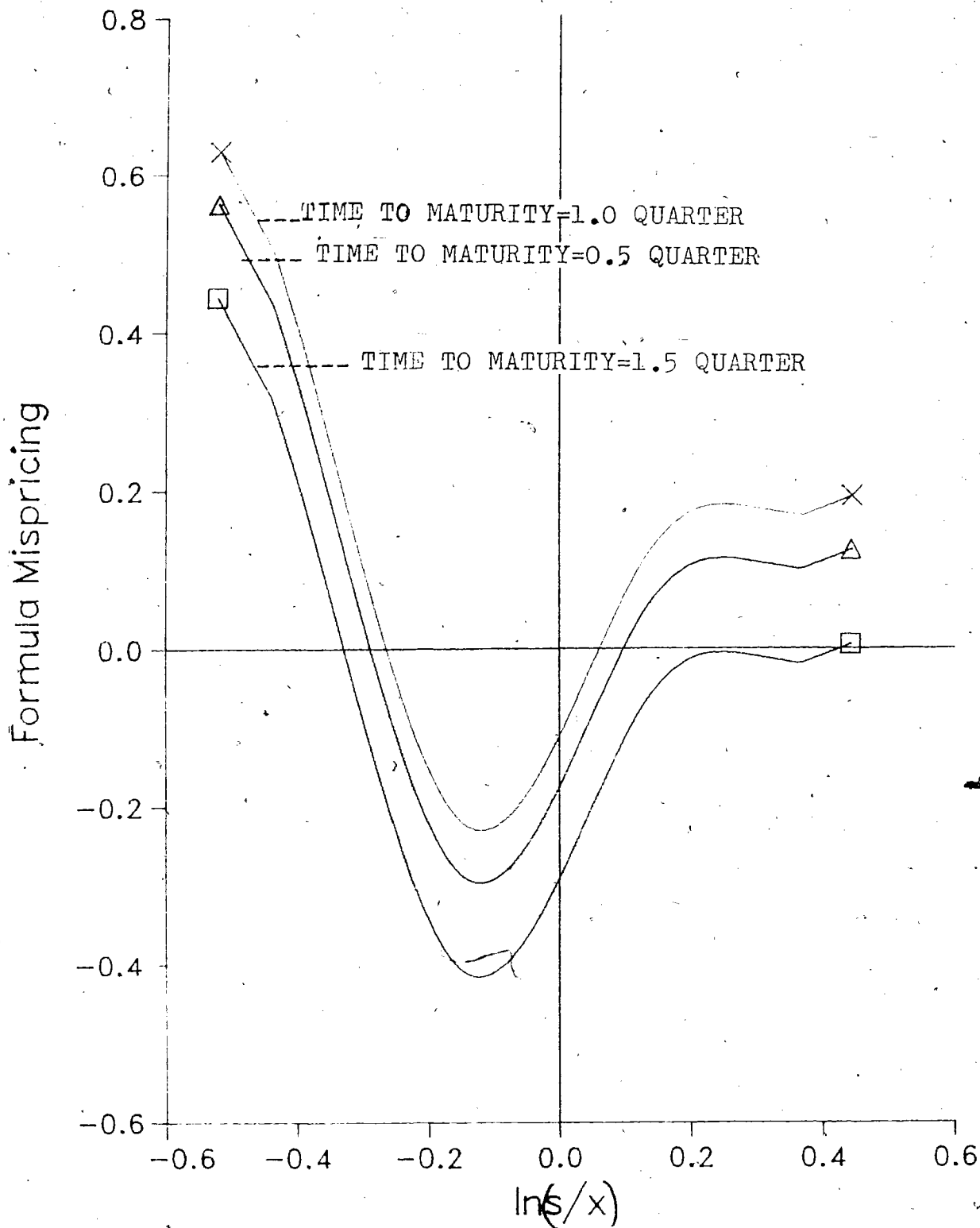


FIGURE 8.6

SPLINE PREDICTION OF MONEYNES BIAS IN THE SINGLE-DIVIDEND
SUBSAMPLE FOR QUARTERLY VARIANCE OF 0.045

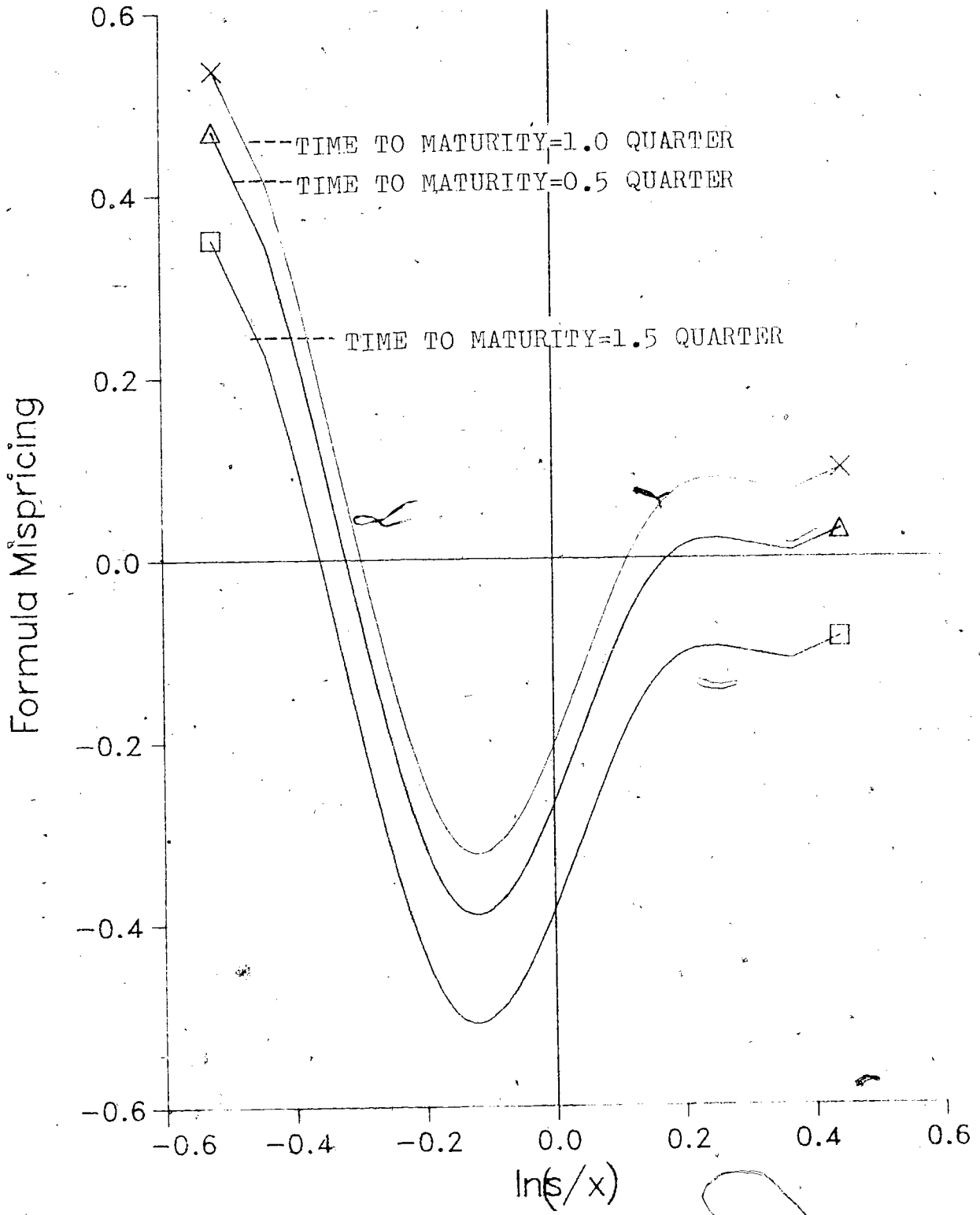


FIGURE 8.7

SPLINE PREDICTION OF VARIANCE BIAS IN THE ZERO-DIVIDEND AND THE SINGLE-DIVIDEND SUBSAMPLES FOR IN-THE-MONEY OPTIONS

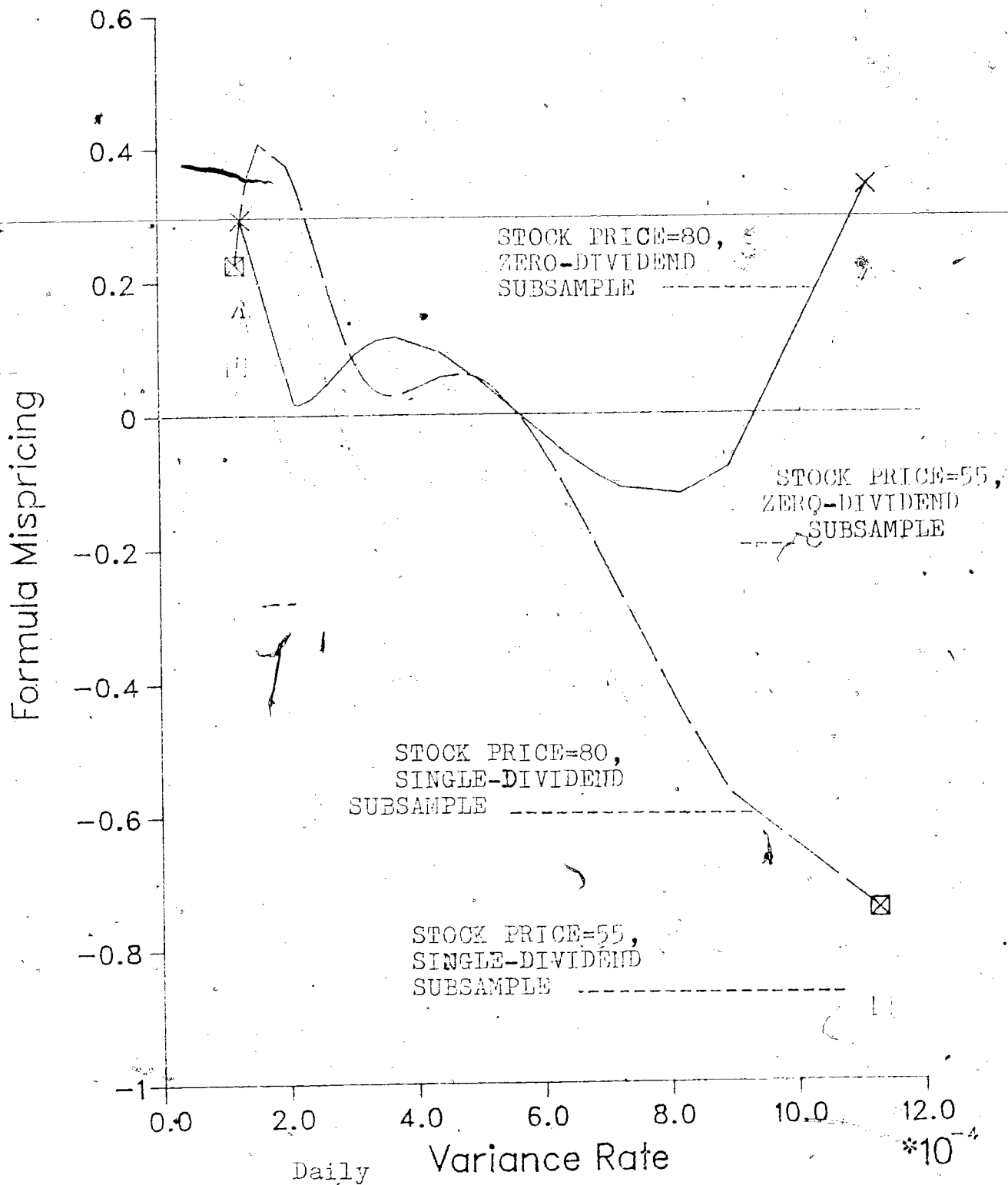


FIGURE 8.8

SPLINE PREDICTION OF VARIANCE BIAS IN THE ZERO-DIVIDEND AND THE SINGLE-DIVIDEND SUBSAMPLES FOR OUT-OF-THE-MONEY OPTIONS

