AN INVESTIGATION OF THE RELATIVE EFFECTS OF SPECIFICATION AND ESTIMATION ERRORS ON ESTIMATES OF OPTION PRICES GENERATED BY TWO COMPETING MODELS

by

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An Investigation of the Relative Effects of Specification and Estimation

Errors on Estimates of Option Prices Generated by Two Competing Models

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ABSTRACT

Since the seminal paper of Black-Scholes was published in 1973, a number of alternative option pricing models have been developed in the finance literature. A great deal of research has been done to validate these competing models empirically. A major outcome of the empirical tests is that there is no compelling evidence in favour of any single model including Black-Scholes. The lack of such evidence is, in part, due to interaction of biases arising from model misspecification and errors in estimation of the variance of stock returns. A valid test for competing models requires a simultaneous yet separate examination of these two sources of biases. However, no attempts have previously been made to carry out such a test. The objective of this thesis is to fill this vacuum.

We identify estimation and specification error biases of the Black-Scholes and Cox-Ross models by expanding the relevant formulas in a Taylor series. Both these sources of biases are then examined by using both analytical and Monte Carlo simulation techniques. First, estimation error biases of the two models are examined. Then, the effects of misspecification of the stock price process are investigated. Finally, the combined effects of errors in estimation and specification on estimates of option prices are analysed.

We test a number of hypotheses. The central hypothesis of the research is that the estimation error bias in the correctly specified model is sometimes large enough to make us pick the "wrong" model as "correct". Our results support this hypothesis, and indicate that there is a bias towards accepting the Cox-Ross model as correct, even if the Black-Scholes model is the true model for pricing the option. Importantly, in order to provide a valid test for competing option pricing models, the results suggest an adjustment of the models' estimated price for the effects of the second moment of the estimated variance. This adjustment gives a more accurate estimate of the option price and enables

us to identify the true model on the basis of the specification error bias of the models. Extension of our analysis to other versions of the Cox-Ross model is also discussed.

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CHAPTER I

INTRODUCTION

1.1. Prelude

Option pricing models have been the focus of considerable research in the finance literature over the past decade. Since the seminal paper of Black and Scholes (B-S) was published in 1973, a number of alternative models and a burgeoning number of empirical tests have been proposed and implemented. The debate over the choice of the "appropriate" model and the "correct" test has been a long and interesting one. The debate continues, in part, because there have been serious estimation and specification problems in the tests used to examine the appropriateness of various models, and in particular the Black-Scholes model, which has been the primary focus of these tests. These problems have manifested themselves in tests of option pricing models through systematic mispricing of the options considered. Consequently there is still considerable doubt regarding the empirical validity of the theoretical models proposed. Any choice between competing models is hampered by the interaction of errors arising from misspecification and mismeasurement. The choice is hampered because the error arising from mismeasurement makes it difficult to diagnose the specification error of the models. The examination of the individual sources of the errors thus becomes an important task if we hope to unravel the tangled threads of existing empirical work.

We shall see that the existing studies have either examined the two sources of error jointly without attempting to decompose the individual sources of error or examined only one source of error without simultaneously investigating the other. As noted above, these shortcomings are especially significant in tests which compare the performance of competing models. Since all tests of the models are joint tests of the two hypotheses of model specification and data accuracy, we may be more confident about our conclusions only when we simultaneously identify and examine the two sources of error. Simultaneous examination of the two sources of error is necessary not only because we are concerned about the "truth" or "falsity" of these hypotheses but also because it allows investors and researchers to better understand their significance with respect to the option pricing problem. Knowledge of the component sources of mispricing can be used to improve investors performance by reducing the risk of misidentifying over- and underpriced options. At the same time this knowledge aids researchers in their quest for a better validation test for various option pricing models.

In the following sections we review the literature on tests of the option pricing models and highlight the importance of this study. We include a theoretical discussion of the sources of bias revealed in the tests of the models. We discuss the problems associated with the empirical tests of options pricing models and delineate the objective of this thesis in attempting to deal with these problems. In particular we focus on the methodological and procedural problems involved in conventional tests and outline the basis for an alternative test which attempts to decompose and simultaneously examine the two sources of pricing bias mentioned above.

1.2. Review of Tests on Option Pricing Models

The pioneering Black-Scholes (1973) model of option pricing has been extended and modified in several ways in papers by Cox (1975), Cox-Ross (1976), Merton (1976), Roll (1977), Geske (1979, 1979a), Whaley (1981), Rubinstein (1983), Ritchey (1984), Bookstaber-McDonald (1985), Hull-White (1985), Johnson-Shanno (1985) and others. As a result several option valuation models have been developed as alternatives to the Black-Scholes model. A great deal of research has been done to validate the alternative models empirically. Empirical tests of these models have met with varying levels of success. A major outcome of the validation tests has been that the prices estimated by the various models have systematically differed from observed market prices of options. The pattern of mispricing has varied across tests between models and for different tests of a given model.

For example, Black (1975) and Finnerty (1978) reported that deep in-the-money (out-of-the-money) call options generally had B-S model prices that were greater (less) than market prices. MacBeth and Merville (1979), on the other hand, found exactly the opposite result with regard to the direction of the pricing bias when they analysed the standardized difference between the actual and B-S model prices. Rubinstein (1985) confirmed the MacBeth-Merville (1979) results, but also found that the biases found by Black (1975) recurred in some sampled periods. While Black (1975) explained the deviations of the model price from the actual price in terms of errors in the estimated variance of stock returns, Macbeth and Merville (1979), assuming market efficiency, attributed these deviations to the weaknesses of the model, especially its assumption of a constant variance of the stock's rate of return. Therefore, in another paper (1980), they compared the B-S model with the Cox-Ross (C-R) model of constant elasticity of variance (CEV) and claimed that "the stochastic process generating stock prices can best be considered, at least for the pricing of call options, as a constant elasticity of variance process" (1980, p. 299). However, in some cases, they also found that the CEV model mispriced options more than the Black-Scholes model (1980, p. 297).

For theoretical considerations, Thorp and Gelbaum (1980) also believed that the CEV is a better model than the Black-Scholes model. But they found only small deviations of the B-S model prices from the observed prices of options. However, they did find that the Black-Scholes model underpriced out-of-the-money options. Similar pricing bias for the Black-Scholes model was also found by MacBeth (1981) and Gultekin-Rogalski-Tinic (1982). While contrary to the findings of MacBeth and Merville (1979), this conforms to the conclusions drawn by Black (1975).

Emanuel and MacBeth (1982) also compared the pricing bias of the Black-Scholes and CEV models and found that both models worked poorly for options in or out of the money. They found similar pricing bias characterisics for the two models and concluded that the B-S model performed as well as the CEV model in predicting market prices of options.

Ball and Torous (1985), on the other hand, compared pricing bias characteristics of the B-S and Merton (1976) models. Even though they found that the Black-Scholes model mispriced options in one way or another, they could not diffferentiate the performance of the two models in identifying over- and underpriced options.

Blomeyer and Resnick (1982) and Barone-Adesi (1984) tested the B-S model against the Geske model of compound options . Their results showed that prices based on the Geske and Black-Scholes models overvalued (undervalued) out-of-the-money options and undervalued (overvalued) in the money options. However, in most cases the B-S model prices were found to be less biased (in absolute terms) than the Geske model prices. Similar pricing bias for the B-S model was also found by Blomeyer and Klemkosky (1983). They attributed these biases to the "no-dividends on the underlying stock" assumption of the Black-Scholes model. Therefore, they compared the performance of the B-S and the Roll (1977) option pricing (a model developed for American call options) models. They found identical pricing-bias characteristics for the two models and concluded that the B-S model is as good as the Roll model even for high-dividend-yield stocks.

Other studies such as those by Whaley (1982) and Sterk (1982) also compared the American model (Roll (1977), Geske (1979) and Whaley (1981)) against an ad-hoc dividend adjusted B-S model. Even though the dividend adjusted B-S model is obviously flawed, it has been difficult to establish the superiority of the RGW model empirically. While the proponents of the American model claimed that the model would eliminate the pricing bias of the Black-Scholes, Sterk (1983) showed empirically that the American model performs more poorly than the B-S when the probability of early exercise is less than 30 percent or over 70 percent.

While the authors mentioned above examined only the Black-Scholes model or the Black-Scholes and one other model, Rubinstein (1985) tested a number of alternative models [e.g. the constant-elasticity-of-variance model of Cox-Ross (C-R, 1976), the diffusion-jump process model of Merton (1976), the compound option model of Geske (1979) and the displaced-diffusion model of Rubinstein (1983)]. He observed systematic deviations of market prices from the B-S prices and attributed them to model misspecification. By examining the Implied Standard Deviations (ISDs, standard deviation implied in the market price of the option) across maturities and exercise prices, he also tried to distinguish which option pricing model would explain better the observed biases from Black-Scholes values. But he could not differentiate between the predictive ability of the models considered. He found the direction of the pricing bias to vary from model to model and from period to period.

Thus the empirical evidence is contradictory in terms of both the magnitude and direction of the pricing bias of the models. The only consensus is that systematic mispricing exists with respect to each model tested. In some cases, competing models were found to exhibit pricing biases identical to those of the Black-Scholes model, and, as noted above, the B-S model itself has provided different patterns of pricing biases in different studies. The bottom line is that no model has consistently performed better than the other models in explaining the maket prices of options. We believe the lack of such evidence can on priori reasons be attributed to the joint effects of the two sources of error — from misspecification and mismeasurement. This is because the estimation error bias in the correctly specified model might be large enough in some cases to make researchers pick the misspecified model as correct.

Besides the empirical tests discussed above, a number of simulation tests of the competing models have been conducted . The results once again leave the choice of the most appropriate model in doubt. Using simulated option prices, Merton (1976a), Madansky (1977), Boyle (1977), Boyle and Anathanarayanan (1977), MacBeth and Merville (1980), Beckers (1980), Bhattacharya (1980), Jarrow and Rudd (1983a, 1983b), Butler and Schachter (1983, 1984, 1986), Johnson-Shanno (1985), Bookstaber-McDonald (1985), Hull-White (1985) and others find systematic deviations of the estimated model prices from the true price of the option. Some of these studies assume that the true variance of the underlying stock returns is known and compare the performance of the Black-Scholes model with that of the competing models, while others examine the impact of the estimated variance on the B-S estimated price.

There is little consensus among these studies as regards to the bias pattern of the option valuation models. For example, Merton concludes that the B-S model underprices (overprices) in-the-money and out-of-the-money (at-the-money) options when the actual process of stock returns is assumed to be a mixture of jump and diffusion processes. Beckers (1980) and MacBeth and Merville (1980), on the other hand, find that the B-S model underprices (overprices) in-the-money (out-of-the-money) options when the underlying security prices are characterised by constant elasticity of variance (CEV) processes. Bookstaber-MacDonald (1985) use an option valuation model based on a "generalised beta of kind 2" (GB2) distribution and compare its prices with those of the Black-Scholes model. Their results suggest that the Black-Scholes model underprices (overprices) out-of-the (in-the) money options which is exactly the opposite to the findings of Beckers and MacBeth-Merville. Sometimes explanations for observed mispricing patterns are also found in terms of changing return volatilities (Johnson-Shanno (1985) and Hull-White (1985)) and the methods used to measure the amount of mispricing in options (French-Martin (1984)). It is also found that (MacBeth-Merville (1980) and Ritchey (1984), among others), at the money, the options are valued by the B-S model with very small error due to misspecification of stock price movements. In contrast, Bhattacharya (1980) finds that the B-S model misprices options only at-the-money while Madansky (1977) finds underpricing of options by 1-2% by the B-S model in all cases.

The other studies by Boyle and Ananthanarayanan (1977) and those by Butler and Schachter (1983 , 1983a) suggest that the bias in the B-S estimated option price arising from estimation error of the variance of the underlying stock may be significant. The B-S model is found to overprice in- and out-of-the-money options and underprice options at the money. It is also found that these biases decrease with an increase in the sample size used to estimate the variance rate (Boyle and Ananthanarayanan, 1977) as well as with an efficient method of estimating the sample variance (Boyle (1977), Parkinson (1980), Garman-Klass (1980) Butler and Schachter (1983a) , Geske-Roll (1984) and Ball-Torous (1984)). In their recent study, Butler and Schachter (1986) also find that the usual estimator (with an estimated variance inserted directly into the B-S formula) of the B-S model, as compared with an unbiased B-S option price estimator, overprices all options except those near the money.

1.3. Sources of Pricing Bias in Option Pricing Models and Importance of the Study : A Theoretical Discussion

It is apparent from the preceding discussion that the bias identified in the empirical and simulation literature can be attributed to several sources of error. We are going to focus on two sources of bias abstracting from the others.

(a) Errors in estimation of the variance of the underlying stock , and

(b) Misspecification of the process characterising the movement of the underlying stock price.

Estimation error biases arise because the true variance of the underlying asset is unknown and an estimate of the variance rate is used to generate the model price. As a result , the model formula option price becomes a function of the variance estimate used. Thorp (1976) points out that this bias will exist in option price estimates using an estimate of the variance rate in the model even if the variance estimate is unbiased . This would occur as a result of the nonlinearity of the model in the variance. This is because the unbiasedness property does not hold under a nonlinear transformation.

Specification error bias arises because the underlying 'true' but unknown stock-price generating process may be different from the stock price process selected by the investigator. Several simulation tests (e.g., Merton (1976a), Macbeth and Merville (1980) and Beckers (1980)) have attempted to examine this source of bias by comparing the estimates provided by alternative models with assumed "true" variance of the underlying stock.

The major problem associated with the tests discussed above is that the two sources of bias are examined either jointly without attempting to decompose them into the constituent sources of bias or examine only one source of bias without simultaneously investigating the other. The empirical tests conform to the former while the simulation tests fit the latter mold. In addition, there is a total lack of information from simulation tests about estimation error biases in models other than the Black-Scholes model. Consequently, simulation tests investigating the estimation error problem for competing models would provide much useful information.

Since the two sources of error may interact systematically and unless the two sources can be simultaneously yet separately identified, a proper basis for comparing the performance of the competing models in valuing option prices may not exist. As a result, it is an important task to decompose the total pricing-bias of the models into their estimation error and specification error components. Such decomposition will help determine how and to what extent the observed pricing-bias characteristics of the models are attributable to the individual sources of error, and therefore, will enable the analyst to isolate and remedy the bias so as to obtain better validation tests for the competing models.

It is also an important task to examine the sensitivity of the competing models with respect to the two sources of error. Examination of the sensitivity of option price estimates to errors in specification and estimation can help to indicate which source of bias we should be more concerned about over what range of parameter values. On the basis of such an analysis, the investigator can decide whether to investigate certain estimates more thoroughly before making a final judgement on the performance of the models. If the investigator does decide to investigate the direction of bias more intensively as a result of the sensitivity analysis, he may be able to improve the estimates of the option price and thereby reduce the risk of making a wrong decision about the ability of alternative option valuation models to explain the market price of options. This examination can allow an investor to better understand the significance of the two sources of error on the option valuation problem and therefore help improve his performance by providing additional information as regards to identifying over- and undervalued options.

Empirical tests for decomposing the two sources of bias present a major hurdle . The component parts of a joint hypothesis are not independently testable because of the underlying logic of all empirical tests. The form of any empirical test allows logic to be used in one of two permissible ways. Only when all the assumptions (i.e the joint configuration consisting of the underlying "core" hypothesis , procedural specifications, and conventions) of a theory being tested are true, will the conclusions be true. On the other hand, if the results of an empirical test turn out to be false, as they do in the case of all the empirical tests of the competing option pricing models, then it has to be that one of the assumptions ("hypotheses") constituting the "joint" empirical model is false. But there is no way of isolating the sub-hypothesis among the joint hypotheses which is causing the "poor" results. In the option pricing context, this implies that it is not possible to identify whether it is the estimation error or the specification error which is causing the anamolous results as long as we must rely exclusively on market data.

The only means of isolating the two sources of error therefore is through simulation. In such a study the investigator can simulate a known underlying stock-price generating process and then compare the performance of the competing models. The data required to validate the models can be generated by simulation and the impact of the underlying distribution of stock returns and of the estimated variance rate examined in an unbiased fashion. The distributions of stock prices underlying each of the competing models can be simulated by design to abstract from specification errors so as to isolate the effect of the estimation error. Similarly other stock price processes can be superimposed on the models and further isolation of the two sources of bias can be achieved.

1.4. Objective of the Study

In this thesis we investigate the relative effects of specification and measurement errors on option price estimates generated by competing option pricing models. As explained earlier, it is necessary to examine the estimation problem and the specification problem simultaneously to provide a proper basis for comparing the ability of various option pricing models to explain market prices of options. It is possible to identify the two sources of bias in option price estimates by expanding the relevant option pricing formula in a Taylor series. Although it is unlikely that the estimated model price will converge fully to its true value with just the first few terms of the series, it is assumed that the contributions of the higher order terms when taken together will be very small.[1] Estimation error biases will be related to the moments of the estimated parameter(s) and the derivatives of the formula. The derivatives show the sensitivity and bias transmission characteristics of the model to estimation error and interact systematically with the moments to produce biases in the estimate of the option price. Specification error biases will be related to the alternative option valuation formulas themselves. Both these sources of biases can be analysed using Monte Carlo simulation techniques.

To be more concrete, we examine biases from specification error arising from assumptions about the return distribution of the asset on which the option is traded. We

[1]It will be seen in the later discussion that the absolute values of the terms beyond the second term are very small, and that they are alternating in signs to their previous terms. Since the magnitudes of these biases are very small, the cumulative effects of the higher order terms will be negligible.

examine measurement error biases arising from estimation of the variance rate of the underlying asset. The Cox-Ross and Black-Scholes models are the subjects of our analysis. Our motivation to choose these two models rests on two major considerations:

(i) The two models require the same inputs which makes it easier to compare the effects of the two sources of error, especially the impact of the estimation error of the variance rate, on the option pricing models, and

(ii) The Cox-Ross model has features with respect to the stock price process which several studies (e.g., Beckers (1980), Black (1975)) have found to be more characteristic of actual stock price movement than the lognormal process of the Black-Scholes. As a result, the Cox-Ross model has been considered as an ideal candidate to study the likely magnitude of specification error bias that may be found by implementing the B-S model on actual prices of options and vice versa.

Because the two models make different assumptions about the constancy of the variance rate of the underlying asset and the resulting option pricing formulas differ in form, the pricing bias induced by estimation and specification errors should also differ.

First, we examine bias transmission characteristics of the models arising from the estimation problem of the variance rate. The analysis of estimation error biases of the B-S model is substantially a replication of other simulation studies (e.g., Boyle-Ananthanarayanan (1977) and Butler-Schachter (1983, 1986)). The analysis of biases for the Cox-Ross model is, however, new in this study. To the extent that the bias is related to the estimated variance, we examine the issue of whether a particular model, regardless of its specification, might perform more poorly as compared to the other one simply because the model is relatively more sensitive to errors in the variance estimate. This is important because a large bias in the model price might exist even with a more accurate estimate of the variance rate when the specific bias transmitted through the derivatives of the model is very large. Similarly, a large error in the variance estimate might not produce a large bias in the model price if the values of the derivatives are very small.

Second, we consider the effects of misspecification of the stock price process in order to examine whether a model based on different specifications of stock return distributions yields a significantly different option price as compared to the price which would prevail under the true process. This analysis is similar to the work done by MacBeth–Merville (1980). Because the response system of the model is different and disproportionate to the specification problem, and because we can never be sure that a given model is correctly specified, examination of this issue will improve the ability of practitioners and researchers to identify mispriced options, among other things.

Third, we compare the combined effects of measurement error and of process misspecification on the option price generated by alternative models. By so doing, we examine the issue of whether the option price is more sensitive to process specification or to the estimation problem of the variance rate. Consideration of this issue is important in order to understand the conflicting evidence contained in the literature as well as to provide a broader basis for evaluating the performance of competing option pricing models. This issue has not previously been examined.

Existing simulation tests which attempt to compare the performance of alternative option pricing models have not addressed many of the issues outlined above. For example, the studies which examine estimation error related problem have only been concerned with investigating the Black–Scholes model. The variance induced pricing bias characteristics of the competing models have not been examined. By specifically incorporating such an analysis into the simulation study, we can subsequently decompose the total bias of the models into their estimation and specification error components and simultaneously examine these two sources of pricing bias in order to provide a broader basis for comparing the ability of the two models to explain market prices of options.

A brief overview of the two models we shall be using is given in the next Chapter. In Chapter III a detailed analytical examination of the two sources of bias is undertaken. In Chapter IV the results of the Monte Carlo tests as regards to the estimation error of the variance rate are reported for the Black-Scholes model. The Cox-Ross model Monte Carlo results for the estimation error problem are reported in Chapter V. This Chapter also compares the bias transmission characteristics of the Cox-Ross and B-S models with respect to the estimation problem of the variance rate. Chapter VI compares the performance of the Black-Scholes and Cox-Ross models with an examination of the seriousness of specification and measurement error biases of the models. The combined effects of the two sources of error are also examined in this Chapter in order to understand the conflicting results that have been observed in the empirical literature. While Chapters I through VI examine the estimation and specification error biases of the Black-Scholes and Cox-Ross models, in Chapter VII we examine whether the results obtained in the preceding chapters can be generalized for other cases of the Cox-Ross model. In this chapter we also show how an ex-post test for the Black-Scholes and Cox-Ross models can be done by using simulated options and stock prices. Finally, Chapter VIII contains a summary and outlines some directions for further work in this area.

CHAPTER II

A BRIEF OVERVIEW OF TWO OPTION VALUATION MODELS

In this chapter we present an overview of the Black-Scholes and Cox-Ross option pricing models and show their differences with respect to specification of the stock price process. These models are used in subsequent chapters in order to examine, both analytically and numerically, systematic biases in the estimated option price arising from errors in estimation and specification.

The Black-Scholes (1973) option pricing model is based on the following assumptions.[1]

1. All individuals can borrow and lend without restrictions at the instantaneous riskless rate of interest, r, and that rate is constant over the life of the option, T.
2. The capital market is perfect in that there are no transaction costs and no differential taxes.
3. The stock price movement through time is characterized by a lognormal diffusion process:

$$dX = \mu \cdot X \cdot dt + \sqrt{v} \cdot X \cdot dZ \tag{1}$$

where $\mu =$ the instantaneous expected return on the stock, $X =$ the stock price, $v =$ the instantaneous variance of stock returns, assumed to be constant through the life of the option, $dt =$ a small increment in time, and $dZ =$ a Gauss-Weiner process with zero mean and unit variance.

4. The stock pays no dividends, D, during the option's time to expiration.

[1]The assumptions are taken from Whaley (1982).

The Black-Scholes valuation formula is given by,

$$C_{BS} = X \cdot N(d1) - E e^{-rT} \cdot N(d2) \tag{2}$$

where

$$d1 = [\ln (X/E) + (r + .5v)T] / \sqrt{vT} \ , \ d2 = d1 - \sqrt{vT} \ ,$$

r = the instantaneous riskless rate of interest, E = the exercise price, T = the time to maturity , and $N(di)$ is the univariate standard normal cumulative distribution function with upper integral limit di.

Assumption (3) of the Black-Scholes analysis has two important characteristics, namely, (a) the variance of stock returns is constant, $Var(dX/X) = v$, and (b) over a very small interval in time the size of change in the stock price is also very small. [2] Cox (1975) and Cox-Ross (1976) question the validity of characteristic (a) and derive an option valuation formula for a "Constant Elasticity of Variance" (CEV) diffusion process:

$$dX = \mu \cdot X \cdot dt + \delta \cdot X^{\rho} \cdot dZ \tag{3}$$

where ρ = the elasticity parameter ($0 \leq \rho < 1$) and $\delta > 0$, a constant. [3] The implication of the CEV process is that the instantaneous variance of stock returns, given by the following equation, is a decreasing function of the stock price, X, for $\rho < 1$.

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[2]Cox and Rubinstein (1983, p.10) discuss these characteristics. For detailed explanation see Cox-Rubinstein in Option Pricing : Theory and Applications, Brenner, M. (ed.) Lexington Books, Toronto.

[3]Several other authors also question the validity of the characteristics of assumption (3) of the Black-Scholes model and derive alternative valuation formulas. The discussion of these models is outside the scope of this study, however. For detailed discussion see Cox-Rubinstein (1983).

$$\mathrm{Var}(dX/X) = \delta^2 \cdot X^{2\rho-2} \tag{3a}$$

One special case of the CEV process is the Absolute process. That is when $\rho = 0$, the CEV process reduces to the Absolute process

$$dX = \mu \cdot X \cdot dt + \delta \cdot dz \tag{3b}$$

and the variance of stock returns becomes

$$\mathrm{Var}(dX/X) = \delta^2/X^2 \tag{3c}$$

Cox-Ross (1976) discuss the absolute process as a limiting case of pure Markov jump processes and derive an option valuation formula for the absolute process. The Cox-Ross formula is given by :

$$C_{CR} = [X - E \cdot e^{-rT}]\ N(h1) + [X + E \cdot e^{-rT}]\, N(h2) + S[n(h1) + n(h2)] \tag{4}$$

where

$S = \sqrt{[(\delta^2 - \delta^2 \cdot e^{-2rT})/2r]}$, $\delta^2 = v \cdot X^2$,

$h1 = (X - E \cdot e^{-rT})/S$, $h2 = (-X - E \cdot e^{-rT})/S$,

and n (.) and N (.) are respectively the density and cumulative density functions of the standard normal random variable.

A justification for the absolute process may be found in the empirical work of MacBeth-Merville (1980). From the study of six stocks, they infer that the elasticity parameter, ρ, is usually less than 1, and the parameter values range from .5 to -2. Importantly, three out of their six stocks appeared to follow the absolute diffusion process (i.e. $\rho = 0$).

Like the Absolute model, the Black-Scholes model is special case of the CEV model. That is when $\rho = 1$, the diffusion process is lognormal. The difference between the models is that the Black-Scholes model assumes a constant variance of stock returns, whereas the Cox-Ross model allows the variance to change with the stock price. To make this distinction more concrete, we follow MacBeth-Merville (1980, p. 287) and derive a relationship between the variances of the two models. Specifically, we write an expression for the variance input of the Cox-Ross model in terms of a constant variance of stock returns (Black-Scholes variance). In other words, given the variance of stock returns, v, we obtain the value of δ^2 from equation (3c). That is $\delta^2 = v \cdot X^2$. This relationship implies that the variance input to the Cox-Ross model is equal to the Black-Scholes variance "v" times stock price squared. It is this difference between the Black-Scholes and Cox-Ross models that we consider in order to examine the effects of misspecification of the stock price process on the option valuation problem.

We have chosen to distinguish the two models in the above manner for two reasons: First, in this way we are able to obtain and compare Black-Scholes and Cox-Ross model prices based on the same instantaneous variance, v, and, therefore, we can examine the effects of different diffusion processes on the option valuation problem using the same variance. Second, we can use the same estimates of the variance in the Black-Scholes and Cox-Ross models, and, therefore, can compare their estimation error biases in an unbiased fashion.

In this chapter we have presented the Black-Scholes and Cox-Ross option pricing models and shown that the models differ in the assumptions about the variance of the underlying asset. It has also been shown that, even though the two models differ in specifications of the diffusion process of the stock price, we can use the same instantaneous variance of stock returns in order to examine specification and estimation error biases of the two models. In the following chapter we use these models [i.e equations (2), (4)] in order to provide a detailed analytical examination of biases in the estimated option price which arise from errors in estimation of the variance rate and from misspecification of the model.

CHAPTER III

ANALYTICAL EXAMINATION OF ESTIMATION AND SPECIFICATION ERROR BIASES

In Chapter II we provided an overview of the Black–Scholes and the Cox–Ross option pricing models. In this chapter we provide an analytical examination of systematic biases in their estimated option prices arising from errors in estimation and specification. To the extent that we are able to do so, the major issues we discussed in the introductory chapter are also examined analytically here.

We assume throughout that the market is efficient. Thus, empirically, researchers would compare a model generated option price with the reported option price. If the model used is correct and the true parameters are known, then it must follow that the model price equals the reported option price. Of course this is the ideal case, and would not be found in practice, even under our assumption of an efficient market. Systematic deviations of model prices from market prices of options certainly would occur. Such would be the case when an estimate of a parameter is used in the valuation formula, or when the model is misspecified . For example, when an estimate of the variance rate is used to generate the option price, systematic mispricing of options by the model is found due to errors in the sample variance. Systematic pricing biases also arise from specification error when (among other possibilities) investigators use a model based on incorrect assumptions about the return distribution of the underlying asset.

To begin with, assume that the option in the market is priced by some ideal ("true") model, say $C_M(v, X, E, r, T)$, where the option price is a non-linear function of the parameters of the model (denoted hereafter by $C_M(v)$; the parameters other than v are suppressed for expositional convenience). We define the reported option price by

$C_R(v)$. By assumption for the ideal model, $C_R(v) = C_M(v)$, where v is the true variance of the stock's rate of return.

We assume, temporarily, that the formula for the "ideal" model is known to the investigator. Suppose, however, the true variance of stock returns is unknown. Such is certainly the case in actual empirical work. The investigator, as a result, must use an estimate of the variance in the formula to generate the option price. Because the formula is non-linear in the variance rate, the variance estimate produces systematic biases in the model estimated price. This is true even if the variance estimate is unbiased [Thorp (1976), Boyle and Ananthanarayanan (1977), Butler and Schachter (1983, 1984, 1986)]. This is because the unbiasedness property is not preserved under a non-linear transformation. Following Butler and Schachter (1986), the bias can be shown by expanding the estimated model value, $C_M(\hat{v})$, around v by fourth order Taylor Series expansion.[1]

$$C_M(\hat{v}) = C_M(v) + C'_M(v)\cdot(\hat{v}-v) + (1/2)\cdot C''_M(v)\cdot(\hat{v}-v)^2 + (1/6)\cdot C'''_M(v)\cdot(\hat{v}-v)^3$$

$$+ (1/24)\cdot C''''_M(v)\cdot(\hat{v}-v)^4 + R \tag{5}$$

The expectation of (5) is,

$$E[C_M(\hat{v})] = C_M(v) + (1/2)\cdot C''_M(v)\cdot E(\hat{v}-v)^2 + (1/6)\cdot C'''_M(v)\cdot E(\hat{v}-v)^3$$

$$+ (1/24)\cdot C''''_M(v)\cdot E(\hat{v}-v)^4. \tag{6}$$

where

$C''_M(v)$, $C'''_M(v)$, and $C''''_M(v)$ are respectively the second, third and fourth derivatives of the model evaluated at v, $\hat{v}$ = an estimate of the variance rate, E = the expected

[1]An investigation of higher order derivatives might be necessary if the remainder term seems to be very high.

value operator and R = the remainder term, assuming $E(R) \approx 0$.

The bias { i.e $E[C_M(v)] - C_M(v)$ } in the option price estimates generated by the model is given by the second term onwards in the right hand side (RHS) of (6). These terms specifically indicate that the higher order moments of the estimated variance interact systematically with the derivatives of the model with respect to the variance rate to produce estimation error biases in the estimated option price. The derivatives show biases specific to the model and transmit the error due to the moments of the estimated variance to the model where the moments themselves represent estimation error in the variance rate.[2]

Because the bias in equation (6) is shown to be the sum of the products of the moments and the derivatives of the model, the bias pattern in the estimated option price will depend on the signs and magnitudes of the last three terms of the right hand side of (6). Therefore, the analysis not only lends support to the notion that an (unbiased) estimate of the variance produces biases in a model estimated price but also suggests that, in general, the estimation error biases will have different signs for different parameter values.

Let us now assume that the variance of stock returns is known, but the true model is not.[3] Specification error biases in the estimated option price will be observed if the model used to generate the option price is based on incorrect assumptions about the underlying return generating process of the stock . The impact of this source of specification error can be illustrated by using Taylor series expansion in a manner similar

[2] We assume that the four moments of the estimated variance are sufficient to capture the effects of estimation error.

[3] We are able to assume this because the analysis of our results is based on a controlled experiment.

to that employed above. Following this procedure we show specification error biases as a decomposition of the total bias arising from both estimation and specification errors. As an illustration, consider two option pricing models, $C_M(v)$ and $C_{MS}(v)$. As before, $C_M(v)$ represents the "true"(assumed) model. Let $C_{MS}(v)$ denote a "misspecified" model which differs from the "true" model. In other words, while $C_M(v)$ reflects the "actual" return–generating process, $C_{MS}(v)$ reflects some return–generating process other than the "actual". By assumption, $C_M(v) \neq C_{MS}(v)$. We define the difference between the prices of these two models by

$$h(v) = C_M(v) - C_{MS}(v) , \tag{7}$$

where v is the true variance of the stock returns.

The option price generated by the "true" model with an estimate of the variance is, again, represented by $C_M(\tilde{v})$. Similarly, $C_{MS}(\tilde{v})$ represents the option price generated by the "misspecified" model using an estimate of the variance. Now define the function

$$h(\tilde{v}) = C_M(\tilde{v}) - C_{MS}(\tilde{v}). \tag{7a}$$

We expand $h(\tilde{v})$ around v by a fourth order Taylor Series expansion

$$h(\tilde{v}) = h(v) + h'(v)(\tilde{v}-v) + (1/2)h''(v)(\tilde{v}-v)^2 + (1/6)h'''(v)(\tilde{v}-v)^3$$

$$+ (1/24)h''''(v)(\tilde{v}-v)^4 + R \tag{8}$$

where $h'(v)$, $h''(v)$, $h'''(v)$ and $h''''(v)$ represent the first, second, third and fourth derivatives of $h(v)$. Taking the expectation of (8), we have

$$E[h(\tilde{v})] = h(v) + (1/2)h''(v)\cdot E(\tilde{v}-v)^2 + (1/6)h'''(v)\cdot E(\tilde{v}-v)^3 + (1/24)h''''(v)\cdot E(\tilde{v}-v)^4. \tag{9}$$

$E(R) \simeq 0$ by assumption.

By substituting (7) and (7a) in (9), we obtain

$$E[C_M(\hat{v})] - E[C_{MS}(\hat{v})] = [C_M(v) - C_{MS}(v)] + (1/2)[C''_M(v) - C''_{MS}(v)]E(\hat{v}-v)^2$$

$$+ (1/6)[C'''_M(v) - C'''_{MS}(v)]E(\hat{v}-v)^3 + (1/24)[C''''_M(v) - C''''_{MS}(v)]E(\hat{v}-v)^4. \qquad (10)$$

The first term on the LHS shows the average option price estimated by the "true" model with an estimator of the variance. Similarly, the second term represents the average option price generated by the "misspecified" model. However, the extant differences between the two model prices do not tell us whether such discrepencies should be attributable to the process misspecification or to the estimation error.

We add and subtract $C_M(v)$ from the LHS of (10). Rearranging terms then gives

$$E\{[C_M(\hat{v})] - [C_M(v)]\} - E\{[C_{MS}(\hat{v})] - [C_M(v)]\} = [C_M(v) - C_{MS}(v)]$$

$$- [(1/2)C''_{MS}(v)E(\hat{v}-v)^2 + (1/6)C'''_{MS}E(\hat{v}-v)^3 + (1/24)C''''_{MS}E(\hat{v}-v)^4]$$

$$+ [(1/2)C''_M(v)E(\hat{v}-v)^2 + (1/6)C'''_M(v)E(\hat{v}-v)^3 + (1/24)C''''_M(v)E(\hat{v}-v)^4]. \qquad (11)$$

Since $C_M(v)$ reflects the option price obtained from the "true" model using the "true" variance rate , the first term on the LHS gives estimation error biases of the true model. Similarly, the second term on the LHS gives the bias in the misspecified model arising from the combined effects of estimation and specification errors. The difference between these two terms indicates whether the estimation error bias in the "true" model is sufficiently large to cause the "misspecified" model to perform better than the "true" model when predicting market prices of options. In other words, when we compare the performance of the "true" and "misspecified" models in explaining market prices of options, the terms on the LHS of equation (11) indicate whether the estimation error bias in the "true" model is large enough to cause us to pick the misspecified model as correct. In fact, there is no reason why this could not be so, in some cases. What we would like to be able to determine is the relevance of such cases for empirical work.

Subtracting (11) from (6) gives

$$E\{[C_{MS}(\hat{v})]-[C_M(v)]\} = [C_{MS}(v)-C_M(v)] + [(1/2)C''_{MS}(v)E(\hat{v}-v)^2 + (1/6)C'''_{MS}(v)E(\hat{v}-v)^3$$

$$+ (1/24)C''''_{MS}(v)E(\hat{v}-v)^4]. \tag{12}$$

The first term on the RHS is the misspecification bias and the other terms together are the estimation error bias.

This decomposition of the total bias into its constituent components (equation (12)) makes it possible to separately examine the estimation and specification error biases of the model under study. Specification error biases can be examined by comparing the prices generated by the $C_{MS}(v)$ and $C_M(v)$ models using the "true" variance, v, and the estimation error bias can be examined by analysing the products of the moments of the sample variance and the derivatives of the $C_{MS}(v)$ model. In general, the direction and magnitude of the two sources of biases would be different. A comparison of these two biases, however, will indicate whether the model estimated option price is relatively more sensitive to process misspecification or to estimation errors. The combined effects of specification and estimation errors will shed light on the pricing-bias characteristics of the model which may lead to a better understanding of the contradictory findings of the existing empirical literature.

Ideally we would like to conduct a completely analytical examination of these biases. Unfortunately analytical results are not easily obtained. A complete analysis requires a numerical investigation as well. First, however, we present those insights which are available through the use of analytical techniques. First we look at the individual sources of biases in the Black-Scholes and Cox-Ross models. Then we examine the combined effects of the estimation and specification biases in the two models.

A. Biases due to Estimation Error

We start with the Black-Scholes (B-S) model and assume, temporarily, that the option is priced by the B-S model. By assumption, $C_{BS}(v) = C_R(v)$. We assume the true variance is not known , hence the bias arising from estimation error in the variance can be shown by substituting $C_{BS}(v)$ for C_M in equation (6) :-

$$E[C_{BS}(\hat{v})] - C_{BS}(v) = C'_{BS}(v) \cdot E(\hat{v}-v) + (1/2)C''_{BS}(v) \cdot E(\hat{v}-v)^2$$

$$+ (1/6)C'''_{BS}(v) \cdot E(\hat{v}-v)^3 + (1/24)C''''_{BS}(v) \cdot E(\hat{v}-v)^4. \qquad (13)$$

Following Thorp (1976) and Butler and Schachter (1983, p. 8), we assume that the sample variance in the Black-Scholes model is gamma distributed.[4] The moments of the estimated variance are always positive. This implies that the signs of the biases in the estimated model price arising from the higher order moments of the sample variance will depend on the signs of the derivatives of the model. Thus, a close look at the derivatives is called for.

[4]Boyle and Ananthanarayanan (1977) assume the sample variance to be Chi-square distributed. Note that the chi-square density is a particular case of the gamma density.

The derivatives of the Black-Scholes model obtained from equation (2) are

$$C'_{BS} = (1/2)X \cdot N'(d1) \cdot \sqrt{(T/v)}$$

$$= (1/2)[(X/\sqrt{2\pi})\, e^{-(\{\ln(X/E)+(r+.5v)T\}^2/2vT)} \sqrt{(T/v)}] > 0. \tag{14}$$

where

$N'(d1)=(\ 1/\sqrt{2\pi})e^{-d1^2/2}$, the first derivative of the univariate standard normal cumulative distribution function with respect to the variance v.

As is well known, we see that the first derivative of the model with respect to the variance rate is always positive.

The second derivative of the model is,

$$C''_{BS} = (1/2v)C'_{BS}[\{(\ln(X/E)+rT)^2/vT\} - (vT/4) - 1] \gtrless 0. \tag{15}$$

The sign of the derivative is indeterminate and depends on the values of the inputs of the model. We can be more clear about the nature of the bias arising from the second moment of the estimated variance if we look at $[\{(\ln(X/E)+rT)^2/vT\} - (vT/4) - 1]$ of equation (15) more closely.

Consider the case when $[\{(\ln(X/E)+rT)^2/vT\} - (vT/4) - 1] = 0$. And, therefore, we find

$C''_{BS} \gtrless 0$ as

$$\{\ln(X/Ee^{-rT}\}^2 \gtrless ((vT/2)-1)^2 - 1 . \tag{15a}$$

It appears from (15a) that, to the extent that the bias is related to the second moment of the estimated variance, for low values of $\{\ln(X/Ee^{-rT}\}^2$ the Black-Scholes model will tend to underprice options while for its high values the reverse would be true. Naturally, this statement depends on the fact that the parameters, v, r, T, X and E, assume only positive values in equation (15a). For given values of v, r and T, we can now determine the type of option which would tend to be under or over-priced by the Black-Scholes model.

First, consider an option which is at the money. That is when $X = Ee^{-rT}$, the LHS of (15a) is equal to zero which is always smaller than $((vT/2) + 1)^2 - 1$. Therefore it follows that, to the extent the bias is related to the second moment of the estimated variance, the Black-Scholes model will always underprice options at the money. However, as the stock price changes (i.e., when the options move away from at the money), ceteris paribus, $(\ln(X/Ee^{-rT}))^2$ increases monotonically and eventually becomes greater than $((vT/2)+1)^2-1$. This indicates that deep-in, in- and out-of-the money options will be overpriced by the Black-Scholes model. With the changes in the values of T, r, and v, the sign of the derivative may change also. Therefore, the analysis seems to suggest that, depending on the values of the parameters of the model, the variance of the sample variance will induce different bias patterns into the model price.

The third and fourth derivatives of the model are :

$$C'''_{BS} = C''_{BS}[\{(\ln(X/Ee^{-rT})^2/2v^2T\}-(T/8)-(1/2v)]$$

$$- [\{(\ln(X/Ee^{-rT})^2/Tv^3\} - (1/2v^2]C''_{BS} \gtrless 0. \qquad (16)$$

and

$$C''''_{BS} = C'''_{BS}[\{(\ln(X/Ee^{-rT})^2/2Tv^2\}-(T/8)-1/2v] + C''_{BS}\{1/v^2 -(\ln(X/Ee^{-rT})^2/Tv^3\}$$

$$- C'_{BS}\{1/v^3-3(\ln(X/Ee^{-rT})^2/Tv^4\} \gtrless 0. \tag{17}$$

In order to determine the signs of the third and fourth derivatives, we, again, consider an option which is at the money (i.e when $X = Ee^{-rT}$). Equations (16) and (17) respectivly then become

$$C'''_{BS} = - C''_{BS}\{(T/v)+(1/2v)\} + C'_{BS}(1/v^2) > 0. \tag{17a}$$

and

$$C''''_{BS} = - C'''_{BS}\{(T/8)+(1/2v)\} + C''_{BS}\ (1/v^2) - C'_{BS}(1/v^3)<0. \tag{17b}$$

Since C'_{BS} is always positive and since C''_{BS} is negative when the option is at the money (shown earlier), the sign of the third derivative is positive. Similarly, we can show that the sign of the fourth derivative is negative at the money. Therefore, the bias in the model estimated price arising from the third and fourth moments of the estimated variance for options at the money will respectively be positive and negative. However, the signs of the bias arising from the third and fourth moments may change when the options are in, deep-in and out-of-the money. This is because the second derivative of the model which is positive for these options makes the sign of the third derivative indeterminate. For the same reason, the sign of the fourth derivative is also indeterminate. Therefore, the analysis suggests that the biases induced by the third and fourth moments will also depend on the parameter values of the model.

Because the three derivatives may assume different signs depending on the parameter values considered, and because the values of the products of the moments and the derivatives of the model are not known, the overall bias induced into the model price is therefore analytically indeterminate. Further analysis of estimation error biases requires a numerical approach.

Boyle and Ananthanarayanan (1977), and Butler and Schachter (1983), among others, have examined the bias in the Black-Scholes model arising from the estimation problem of the variance rate. They mentioned that the distribution of the estimated variance is important in analysing the estimation error bias of the model. But, they studied the bias without decomposing the estimation error bias into its moment components. By examining the contribution of the moments of the estimated variance to the total bias of the model, we have a better understanding of the pricing-bias characteristics of the Black-Scholes model. This is because, as noted above, the bias of the model is a function of the moments of the estimated variance. We take this analysis further in the numerical work to follow.

Following the same line of reason as was used to examine the Black-Scholes model, the bias in the estimated Cox-Ross model price can be shown by the following equation, assuming $C_{CR}(v) = C_R(v)$.

$$E[C_{CR}(\hat{v})] - C_{CR}(v) = (1/2)C''_{CR}(v)E(\hat{v}-v)^2 + (1/6)C'''_{CR}(v)E(\hat{v}-v)^3$$

$$+ (1/24)C''''_{CR}(v)E(\hat{v}-v)^4. \tag{18}$$

As in the Black-Scholes model, estimation error biases in the Cox-Ross model arise from the interaction of the derivatives of the model with the moments of the estimator of the variance rate. The derivatives of the Cox-Ross model obtained from equation (4) are below.

$$C_{CR} = [-(X-Ee^{-rT})n(h1)(h1/2v)] - [(X+Ee^{-rT})n(h2)(h2/2v)]$$

$$+ \; S \cdot M + \{n(h1)-n(h2)\}\{Q/4rS\} > 0 \,. \qquad (19)$$

where

$$S = \sqrt{\{(V \cdot X^2 - V \cdot X^2 \cdot e^{-2rT})/2r\}}.$$

$$M = [\{n(h1)(h1^2/2v)\} - \{n(h2)(h2^2/2v)\}]$$

and

$$Q = X^2 - X^2 e^{-2rT}.$$

The second derivative of the model is,

$$C''_{CR} = [X-Ee^{-rT}][(3h1-h1^3)/4v^2]n(h1) + [X+Ee^{-rT}][(3h2-h2^3)/4v^2]n(h2)$$

$$+ \; S \cdot W + (2QM/4rS) - \{n(h1) - n(h2)\}\{Q^2/16r^2S^3\} \gtrless 0. \qquad (20)$$

where

$$W = \{(h1^4-4h1^2)n(h1)/4v^2\} - \{(h2^4 - 4h2^2)n(h2)/4v^2\}.$$

The third and fourth derivatives are:

$$C'''_{CR} = [X-Ee^{-rT}][\{(20h1^3-2h1^5-30h1)/16v^3\}n(h1)].$$

$$+ \; [X+Ee^{-rT}][\{(20h2^3-2h2^5-30h2)/16v^3\}n(h2)] + (3WQ/4rS)$$

$$+ \ [(SY-3MQ^2)/16r^2S^3] \ + \ 3[n(h1)-n(h2)][Q^3/64r^3S^3] \gtrless 0. \qquad (21)$$

where

$$Y = [(2h1^6-24h1^4+48h1^2)/16V^3]n(h1) - [(2h2^6-24h2^4+48h2^2)/16v^3]n(h2).$$

And

$$\tilde{C}''''_{CR} = (X-Ee^{-rT})A_1 + (X+Ee^{-rT})A_2 - (6Q^2W/16r^2S^3) + (4QA_3/4rS)$$

$$+ \ S.A_4 + (12Q^3M/64r^3S^3) - 15[n(h1)-n(h2)][Q^4/256r^4S^7] \gtrless 0 . \qquad (22)$$

where

$$A_1 = [(336h1^5-1680h1^3-16h1^7+1680h1)/256v^4]n(h1),$$

$$A_2 = [(336h2^5-1680h1^3-16h1^7+1680h1)/256v^4]n(h2),$$

$$A_3 = [(2h1^6-24h1^4+48h1^2)/16V^3]n(h1) - [(2h2^6-24h2^4+48h2^2)/16v^3]n(h2),$$

$$A_4 = [(16h1^8-384h1^6+2304h1^4-3072h1^2)/256v^4]n(h1)$$

$$- [(16h2^8-384h2^6+2304h2^4-3072h2^2)/256v^4]n(h2).$$

Attempts to look at the behaviour of the derivatives of the Cox-Ross model proved analytically intractable even for options at the money. The only conclusion we can draw is that the signs of the derivatives of the Cox-Ross model depend on the parameter values of the model.

Since the signs of the derivatives of the Black-Scholes and Cox-Ross models depend on the parameter values of the models, and since the moments of the estimated variance are not known, we are not in a position to compare the estimation error biases of the two models analytically. For that we must examine them numerically.

However, to get a feel for the relative effects, of the moments of the sample variance on the two models, we consider the following data: X = 50, E = 50, v = .025, r =.015 and T = 1. The values of the second derivatives of the Cox-Ross and Black-Scholes models are −1235.32 and −1239.21 respectively. This implies that, to the extent the bias is related to the second moment of the estimated variance , both the Black-Scholes and Cox-Ross models will tend to underprice options which are near the money (in this case $(X/Ee^{-rT} = 1.0151)$) . The magnitude of the bias in the two models will also be similar. Since the curvature of a function is usually examined using the the second derivative , the above values seem to suggest that the two models have almost identical inflection points for options at the money. Therefore, we suspect that the models will tend to produce similar biases for options at the money. However, with the change of the value of the stock price, ceteris paribus, the bias pattern may change. For example, when the stock price changes from 50 to 60 or 40, C'_{CR} becomes 236.68 and 396.58 respectively. The corresponding values for C'_{BS} are 345.93 and 341.11. The implication of these findings is that, even though both the models tend to overprice in and out-of-the money options, the Black-Scholes model, as compared to the Cox-Ross model, will produce greater biases for options in the money and vice versa for options out-of-the-money. Examination of the higher order terms does not yield much in the way of additional insight, so we will simply restrict our further examination of the estimation error biases to numerical work.

B. Biases due to Specification Error

We again begin with Black-Scholes.

Suppose, however, that the stock price movement is actually characterised by the Absolute process and the option, as a result, is priced by the Cox-Ross model. In this case, by assumption, the reported (market) option price is equal to the Cox-Ross model price, i.e $C_R(v) = C_{CR}(v)$. However, when we use the Black-Scholes formula to price the option, the bias arising from misspecification of the model can be shown by a process analogous to that used in deriving equation (12). We assume that the variance of stock returns is known and denote the bias by ,

$$BSSE = C_{BS}(v) - C_{CR}(v) \tag{23}$$

which gives, in the Black-Scholes case, the following specification error bias (the difference between equation (2) and equation (4)).

$$C_{BS}(v)-C_{CR}(v) = - X[N(h2)+N(h1)-N(d1)] + E \cdot e^{-rT}[N(h1)-N(h2)-N(d2)]$$
$$- S[n(h1)+n(h2)]. \tag{24}$$

The bias in the B-S option price due to misspecification of the stock price process will depend on the relative strength of the three terms on the RHS of equation (24). The biases might be positive, zero or negative depending on the values of the parameters used in generating the option price.

To determine under what conditions underpricing and overpricing may occur, we made attempts to look at the second derivative of (24) with respect to the stock price but, we could not analytically determine the specification error bias of the model.

Therefore , we have followed an alternate route. That is, we set the RHS of equation (24) equal to zero and rearrange terms to find

$$[C_{BS}(v) - C_{CR}(v) \gtrless 0$$

as

$$[X/Ee^{-rT}] \ [N(h1)+N(h2)-N(d1)+\{n(h1)+n(h2)\}\{(1/2r)\sqrt{(v-ve^{-2rT})}\}]$$

$$+ \ [N(h2)+N(d2)-N(h1)] \gtrless 0. \qquad (24a)$$

To determine the type of option which will be over- or underpriced, we consider the the data, E=50, r=.015, v=.025, and T=1, and find the value of the stock price, (i.e., X=49.85) at which the specification bias is equal to zero. This implies that, when the option is in the neighbourhood of at the money (in this case, (X/Ee^{-rT}) = 1.01206), the Black-Scholes and Cox-Ross models will give exactly the same price for the option. As a result, when the stock price is different from 49.85 (i.e., when the options are in- or out-of the money), ceteris paribus, the B-S model will either overprice or underprice options. For example, when X = 40 (i.e., when the option is out-of the money), the LHS of equation (24a) is equal to .1129 which is greater than zero, implying that the B-S model will overprice out of the money options. Similarly, we can show that the B-S model will underprice options when the stock price is greater than 49.85. That is, the B-S model will underprice in-the-money options. Thus, for high values of the stock price, X, ceteris paribus, the Black-Scholes model will underprice the "true" option price while for low values of X the reverse would be true. One possible explanation for this mispricing behaviour can be provided by looking at the relationship between the variances of the two models.

Recall from chapter II that the variance input to the Cox-Ross model is equal to the Black-Scholes variance times stock price squared (a scale parameter). This implies that each time the stock price changes, the Cox-Ross model uses a different variance input (i.e the Black-Scholes variance times the scale parameter) to price the option. Therefore, the variance used in the Cox-Ross model to price out-of (in-) the-money options is smaller (larger) than the variance used in pricing options at the money. Since the Black-Scholes model does not allow the variance to change with the stock price, and since the option price is positively related to the variance, the price generated by the Black-Scholes model for out-of (in-) the-money options will, in general, be greater (smaller) than the "true" price of the option. As a result, the specification error bias (BSSE) calculated from equation (24) will be positive, zero or negative depending on whether the option is out, at, or in the money. This result is supported by the numerical results which follow.

By switching the roles of the Cox-Ross model and the Black-Scholes model in the foregoing discussion, we can obtain exactly the opposite specification error bias for the Cox-Ross model. This is -BSSE in equation (24) above.

C. Biases due to Errors in Estimation and Specification

In the previous two sub-sections, we illustrated the biases in the Black-Scholes and Cox-Ross models arising independently from errors in estimation and specification. We now examine the biases in terms of their combined effects. We assume that the reported option price is denoted by, $C_{CR}(v)$. The bias in the Black-Scholes estimated price can be shown similarly to equation (12) [changing M, MS to CR, BS] by ,

$$E[C_{BS}(\tilde{v})] - C_{CR}(v) = [C_{BS}(v) - C_{CR}(v)] + [(1/2)C''_{BS}(v)E(\tilde{v}-v)^2$$

$$+ (1/6)C'''_{BS}(v)E(\tilde{v}-v)^3 + (1/24)C''''_{BS}(v)E(\tilde{v}-v)^4]. \tag{25}$$

The overall bias in the Black–Scholes model price depends on the contributions of the two sources of bias discussed above. Since both the specification and estimation error biases depend on the parameter values considered, the resulting bias pattern in the model price is analytically indeterminate. This is true even when the options are at the money. This is because the estimation error bias in the B–S model is indeterminate.[5] Possibly because of this, there exists conflicting evidence as to the magnitude or even direction of the biases in the tests of the model (e.g., Black (1975), MacBeth and Merville (1979), Boyle and Ananthanarayanan (1977), Butler and Schachter (1983) and Rubinstein (1985), among others). We hope to find out if this is a reasonable explanation.

When we compare the ability of the Black–Scholes and Cox–Ross models to explain the reported option price (reported option price is assumed to be generated by the Cox–Ross model), we usually compare whether $E[C_{BS}(\tilde{v})] - C_{CR}(v)$ is equal to, greater than, or smaller than $E[C_{CR}(\tilde{v})] - C_{CR}(v)$. Because the estimation problems associated with attempts to validate the models are related to both estimation and specification errors, and because the two sources of error have different effects on the two models, conflicting evidence as regards the performance of the models can repeatedly be observed in validation tests of the models. It is possible that the models will perform equally well, even when the Black–Scholes model is not the correct model. This would be the case when the estimation error biases of the Cox–Ross model are equal to the sum of the specification and estimation error biases of the Black–Scholes model. Our Monte Carlo

[5]If we restrict our analysis to the second derivative we can sign the bias for options at the money.

results, in a later chapter, seem to suggest that the Black-Scholes model performs as well as the Cox-Ross model when the option is in the neighbourhood of at the money. In this situation it is difficult, if not impossible, to compare the performance of the models without separately but simultaneously examining the two sources of bias. The implication of this examination is that: once we identify the sources underlying the pricing bias of the models, it would be useful to know which source of bias we should be more concerned about over what range of parameter values. And accordingly, measures can be undertaken to remedy the bias so as to provide a better validation test for the competing option valuation models. For example, to the extent the bias is related to the estimation problem, improved methods for estimating the variance rate can be devised and one model can be said to perform more poorly than the other when the model is merely more sensitive to estimation error of the sample variance. A theoretically better model can be used in pricing options if the majority of the bias arise from misspecification of the process of the underlying stock price.

In this chapter we have analytically examined systematic biases of the Black-Scholes and Cox-Ross models resulting from errors in estimation and specification . We have seen that both the B-S and Cox-Ross models may produce positive , zero or negative biases in the estimated option price due to the estimation problem of the variance rate. We have also seen that the bias due to misspecification of the process might be positive, zero, or negative depending on whether the option is out, at, or in the money. However, the analytical discussion of these two sources of biases has not provided enough information to come to any definite conclusions about the extent and magnitude of biases in the Black-Scholes and Cox-Ross models. For that, in the following chapters, we restrict our examination of the two sources of bias to Monte-Carlo simulation tests.

CHAPTER IV

MONTE-CARLO TESTS FOR ESTIMATION ERROR BIASES IN THE BLACK-SCHOLES MODEL

In Chapter III we provided an analytical examination of biases arising from errors in estimation and specification in the Black-Scholes and Cox-Ross option pricing models. In this chapter we examine the pattern and magnitude of estimation error biases of the Black-Scholes model in a quantitative fashion. We examine these biases in terms of the first four moments of the estimated variance. This is done by employing Monte-Carlo simulation techniques. The contributions of the moments to the overall bias of the Cox-Ross model are examined in the following chapter. We also employ Monte-Carlo simulation techniques in a later chapter in order to examine biases arising from errors in both estimation and specification in the Black-Scholes and Cox-Ross models. Section A of this chapter provides a brief discussion of the potential usefulness of the Monte Carlo study. In section B we provide a discussion of Monte Carlo simulation techniques which were used. Section C reports the results of Monte Carlo tests of estimation error biases for the Black-Scholes model.

A. The Potential Usefulness of Monte Carlo Study

Previous studies to examine estimation error biases in option pricing models have focussed on the Black-Scholes model. Boyle and Anathanarayanan (1977) examined the impact of variance estimates on the Black-Scholes option pricing model. They investigated the bias in the Black-Scholes formula by comparing the expected value (obtained by using the probability density function of the estimated variance) of the Black-Scholes estimated option price with the true price of the option (obtained by directly inserting the true variance of stock returns into the Black-Scholes formula). The argument in Boyle-Ananthanarayanan runs as follows: If $f(\bar{v})$ represents the probability density function (pdf) of the sample variance, $\bar{v}$, and $g(\bar{v})$ is the option valuation function which is random in $\bar{v}$, then the expected option value is given by

$$E[g(\bar{v})] = \int g(\bar{v})f(\bar{v})d\bar{v}. \tag{26}$$

The bias is then calculated as $E\,[g(\bar{v})] - g(v)$, where $g(v)$ is the true price of the option calculated by plugging the true variance, v, into the function g. Boyle and Ananthanarayanan use numerical integration techniques to calculate the extent of the bias. They do not provide a breakdown of the sources of estimation error bias in terms of the moment components of the sample variance. Similarly, Butler and Schachter (1983) use the numerical integration method to estimate the magnitude and direction of the bias in the Black-Scholes formula without decomposing the estimation error bias into its moment components. Thus, as noted earlier, a useful extension of their work would be to examine the contribution of the higher order moments of the estimated variance to the total estimation error bias of the model. In last chapter we have already (analytically) examined the contribution of the moments to the overall bias of the model. Below we continue to examine the moment components of the bias by Monte Carlo methods.

Boyle (1977) uses a Monte Carlo approach to obtain a numerical solution to the option pricing problem without attempting to look at the bias issue. His analysis is able to use this simpler method, as compared to the numerical integration method used by Boyle-Ananthanarayanan and Butler-Schachter, to get an unbiased estimate of the expected value of the option price. Boyle has established the validity of the Monte Carlo method in his analysis of obtaining estimates of option values on dividend paying stocks. This Monte Carlo approach is useful in our context in order to examine the related bias problem discussed above.

(i) The aggregate estimation error bias in option pricing models can be decomposed into the moment components of the variance estimate. By doing so the proportion of bias induced by each of the components can be examined, and a better understanding of the pricing-bias of the models can be obtained.

(ii) Since the variance of stock returns is the only stochastic input variable in the option pricing formula and since the stochastic variance estimates can be generated independently of the particular model specification, the estimation error bias and its components can be isolated from the specification bias. As a result, the estimation error biases of the competing option pricing models (e.g., B-S and C-R) can be compared.

(iii) The use of Monte Carlo procedures provides a convenient means to examine the bias issue as compared to the numerical integration technique hitherto used. The Monte Carlo method is easy to understand and easy to implement.

However, the method is not without drawbacks. It provides only statistical estimates rather than exact results. It is also a slow and costly way to study a problem.

B. Monte Carlo Simulation Techniques

Monte Carlo simulation techniques perform sampling experiments on the data and model of a system. This technique involves the construction of the sampling distribution of an estimator when mathematical techniques are inadequate to establish the underlying nature of the true distribution. The essence of this technique is that an analogue of the relevant situation is created, and one simulates the relevant process to generate a body of data from which the desired solution can be obtained.[1] Typically, in a Monte Carlo study, the response (dependent) variable is expressed as a function of stochastic input variables and an average estimate of the response variable is obtained by the Monte-Carlo sample mean method. [2] This estimate is then compared with the known true value of the variable of interest in order to obtain an estimate of the bias.

The Sample-Mean Monte Carlo Algorithm can be characterised by the following steps:

(i) A sequence $\hat{v}_i$ (where $i = 1, 2, \ldots, K$) of K random numbers is generated.

(ii) Given a known functional form, the response variable is computed as a function of the input variables generated in step (i) above, i.e compute $g(\hat{v}_i)$.

(iii) The sample mean is computed by $g = (1/K) \Sigma g(\hat{v}_i)$.

In the context of the option pricing problem discussed here, $\hat{v}_i$, the estimated variance of stock returns, represents the only stochastic input variable in the option pricing formula. As a result, the expected value of the option price generated by any particular option pricing model (i.e B-S or C-R) can be estimated by the sample mean method. The moments of the generated variance estimates can be obtained so that the contributions of each of these moments to the total estimation error biases of the model can be established.

[1]An excellent exposition of the Monte Carlo method can be found in Hammersley and Handcomb (1964).

[2]There is another Monte Carlo technique for computing the integral (26). The technique is called "the hit or miss Monte Carlo method.". This technique is based on the geometrical interpretation of an integral as an area. Nevertheless, the sample mean method is more efficient than the hit or miss method. That's why we have used the sample mean method in this study.

As noted earlier, estimation error in the sample variance induces bias in Black–Scholes option price estimates. Of course the estimation error derives from the distribution of the variance estimate. Therefore, in order to examine estimation error induced biases in option price estimates, we generate K different estimates of the variance, $\hat{v}$, each representing a sample of size N. The sampling distribution is assumed to be the gamma density function such that $E(\hat{v}) = v$, the true variance, and $Var(\hat{v}) = 2v/N$.[3] These simulated variance estimates are then plugged into the B–S formula in order to obtain K different estimates of the option price. Since researchers do not know the true variance of stock returns and use an estimate of the variance in the formula in order to obtain an estimated option price, these K different option price estimates represent the option price that can be obtained by researchers who undertake K independent experiments to obtain the B–S model price. Given that the B–S model is correctly specified, one might at first expect that the average of these K estimated option prices should represent the true (market) price of the option. But Boyle–Anathanarayanan and Butler–Schachter have shown that the expected option price is not equal to the true value of the option, as noted in chapter III. As a result, the bias that the B–S model on average produces, due to estimation of the variance rate, is given by the difference of the average price of the option from the price that would prevail were the true variance, v, used. For our purposes, the average price of the option generated by the model is found by the Monte Carlo Sample mean method outlined above.

We also vary the size of N and generate different sets of K variance estimates and then reestimate the model prices in order to examine the effects of sample size.

[3]Our choice of the gamma density function in generating the variance estimates is based upon the analyses of Thorp (1976) and Butler and Schachter (1986), among others. They argue that the variance is estimated as the mean of the squared percentage changes in the stock price. As Black–Scholes (1973) assume that the percentage change in the stock price is distributed N(0,v), the variance estimate is therefore gamma distributed. For a detailed discussion of this point see Butler and Schachter (1986).

This is because the sample size used to estimate the variance rate affects the value as well as the distribution of the sample variance. [4] It does not affect the derivatives of the model. Nevertheless, the separate examination of the model's mispricing due to variations in the sample size has important implications as regards the conflicting evidence contained in the literature. Because the sample sizes used in different studies are different, and because the variance estimates may contain different magnitudes of errors across studies, the pricing bias revealed in various tests should also differ.

Estimation error biases are then examined in a second manner , specifically, in terms of the moments of the generated variance and the derivatives of the model. The signs and magnitudes of the derivatives are evaluated for different parameter values to provide a numerical examination of the sensitivity of the model to the moments of the estimated variance. The values of the derivatives of the model are then multiplied by the corresponding moments of the estimated variance in order to find the bias induced by each of the moments in the estimated model price. The biases arising from each of the moments are then compared to each other and to the overall bias of the model in order to examine how, and to what extent, the biases in model estimated price are attributable to the various moments of the sample variance.

C. Results of Monte-Carlo Tests of Estimation Error Biases for Black-Scholes Model

Following the procedures outlined above , we generate 50 different estimates of the variance rate from the gamma density function by using the subroutine "GGAMR" from the International Mathematical and Statistical Library (IMSL). We insert each of these variance estimates into the B-S formula (equation (2)) and get 50 different estimates of

[4] We expect the sample variance to change when we change the sample size.

the option price.[3] The average option price that the formula gives is then calculated by the sample mean method. The estimation error bias in the B-S model estimated price is obtained as the dollar excess of the average option price over the "true" option price. The "true" option price is assumed to be generated by the Black-Scholes formula and is obtained by inserting the true variance , v, into the formula. We assume throughout that the estimator of the variance rate is unbiased (i.e $E(\hat{v}) = v$).

Table-1 through Table-11 report the results of Monte-Carlo simulations for the B-S model for a variety of stock prices (40, 50, 60, 80), times to option's expiration (1, 3, 5, 7, 9, 15)[4], and sample sizes (60, 100, 140) used in estimating the variance of stock returns. The true variance rate, v, is assumed to be .025 per quarter.

1. Pattern and Magnitude of Biases in the B-S Model

Table-1 reports the "true" price of the option generated by the Black-Scholes model, and Table-2 provides the estimated Black-Scholes option prices with unbiased variance estimates. It should be noted that the sample mean, $\bar{\hat{v}}$, of the variance used in estimating the Black-Scholes option price is different for different sample sizes. For N=60, it is .02482513, whereas $\bar{\hat{v}}$ = .024949 and .02499679 when N= 100 and 140 respectively. Because of this, the estimated option prices in Table-2 decrease as the sample size, N, decreases.

[3]The CDFs contained in the B-S formula were estimated by using a Fortran program obtained from Butler and Schachter (1986).

[4]Note that the options traded in an organised exchange have maturity periods maximum of 3 quarters. Consideration of options with more than three quarters stems from the fact that the options traded in the over-the-counter market may have infinite times to expiration. Besides, the results for the long maturity options may be consistent with the results for different types of options when the types of options (such as, at-, in, deep-in, and out-of the money options) are defined in terms of X/Ee^{-rT}.

Table–3 shows the biases (Table–2 minus Table–1) found in the Black–Scholes model due to estimation of the variance rate. These biases are of different signs and of different orders of magnitude. Further, the bias pattern is found to change with a change in the stock price, the option's time to maturity, and the sample size. To facilitate an understanding of the bias pattern of the model, we first examine the bias of the model for a particular sample size and then explain the effects of the sample size on these biases of the model.

Table–3 indicates that for N=140 the Black–Scholes model overprices (with a few exceptions for long maturity options) deep in the money options (X=80). The overpricing is also observed for options out (X=40) and in the money (X=60) with one quarter to maturity, but ~~otherwise the~~ model underprices options. The absolute bias is greatest for options at the money.* These findings are consistent with those of Boyle and Ananthanarayanan (1977), and Butler and Schachter (1983), but they contradict the findings of other simulation studies, notably Merton (1976), Madansky (1977), MacBeth and Merville (1980), Beckers (1980), Bhattacharya (1980), Hull–White (1985), and Bookstabler–McDonald (1985). For example, MacBeth–Merville report that the model overprices options out-of-the-money (consistent with our findings) but underprices options in the money (contradicts our results). In addition, we observe that the model tends to underprice options, especially for X=40, and X=60, as the option's time to maturity increases. This implies that the variance estimate does not always force the model to overprice options in and out of the money as found by Boyle and Anathanarayanan (1977), among others. This difference in our results is due to the fact that Boyle and Ananthanarayanan (1977) considered an option which had only one quarter to maturity.

*For our purposes, being at the money means $X=E$ so that when $X=Ee^{-rT}$ (considered in Chapter III) we are slightly in the money. This definition however does not change the results of our analysis.

Table-4 lists the percentage biases of the model price (i.e the dollar difference between the estimated price and the reported price as a fraction of the reported price). Again for N=140, the table shows that the percentage bias is greatest for out of the money options, and that the biases, in general, decrease as the option's time to expiration increases except for options in the money (X=60). This indicates that the estimation error of the variance rate forces the model to increasingly misprice the option in the money when the maturity period of the option, ceteris paribus, is increased. Table-4 also shows that the magnitude of the percentage bias in the Black-Scholes model is negatively related to the stock price, given the option's expiration date. These are new insights into the mispricing behaviour of the model, which are not found in the existing simulation studies.

Table-3 shows that with the decrease of the sample size in estimating the variance rate, the bias in the model price increases in all cases except for in- and out-of-the-money options of one quarter to maturity and for deep-in-the- money options of long maturity. In addition , the biases for short maturity (T=1) in- and out-of-the-money options (X=40, 60) become negative with N=60, which implies that the increased deviation of the estimated parameter from the true value of the variance rate will change the bias pattern of the model. In particular, we observe that (when N=60) the model underprices all options except those which are deep in the money (X=80) with 1, 3, and 5 quarters to maturity . The implication of these results is that depending on the accuracy of the estimated variance (which here is being related to the sample size used), the pricing bias of the model may differ across studies. For example, the results we have found for N=60 conform with the most commonly observed bias of the Black-Scholes formula. That is, the model tends to underprice at- and out-of-the-money options while overpricing deep-in-the-money-options (e.g., see Black (1973), Finnerty

(1978), Thorp-Gelbaum (1980), Gultekin-Rogalski (1982), among others). But the results contradict the findings of other studies, notably MacBeth-Merville (1979), Sterk (1982, 1983), and Rubinstein (1985). Thus the results suggest that the conflicting evidence about the model's validity observed in the empirical literature can, in part, be attributable to the sample size used in estimating the sample variance.

Further, we observe from Table-4 that when N=60 and T=1, the percentage bias is greatest for options at the money, and it increases as the sample size decreases. However, the percentage bias for out of the money options decreases as N decreases. This observation is not surprising because the bias in the estimated price for options out-of-the-money decreases as the size of N decreases. The results thus imply that increasing the sample size used in estimating the variance rate will not always reduce the bias in the estimated model price , especially for out of the money options.

In short, for the range of parameter values considered in this study, we observe multi-directional biases in the Black-Scholes model arising from the problem of the estimation of the variance rate. The biases are also found to be of different orders of magnitude, even with an unbiased estimator of the variance rate. The results thus support the hypothesis that even if the B-S model is correctly specified , the variance estimate induces different patterns of systematic biases into the model estimated option price, depending on the model's other parameters. As a result, when researchers use an unbiased estimate of the variance rate , the B-S model will produce a biased estimate of the option price . This will , sometimes, lead the researcher to commit a "Type I" error by rejecting the null hypothesis when in fact it is true, i.e., the B-S model is the true model to price options. For investors, committing a "Type I" error is particularly serious when alternative models are recommended for pricing of options or when attempts are being made to buy or sell options on the basis of the information obtained from the estimated option price of the Black–Scholes formula.

Further, our results indicate that the sample size used in estimating the variance may change the bias pattern of the model estimated price. Since different studies use different sample sizes in estimating the variance (e.g., Finnerty used 8 years of weekly observations, Geske–Roll used 180 daily observations, Rubinstein used about 350 daily observations, and so on) , and since the error in the estimated variance differs across studies, the bias pattern in the Black–Scholes estimated option price should also differ across various tests of the model.

2. Biases and the Effects of the Moments of the Estimated Variance Rate

We have seen above that our results are consistent with those of Butler and Schachter (1983) and Boyle and Anathanarayanan (1977). We provide some new insights into the model's mispricing by examining the biases of the model in terms of the effects of the underlying distribution i.e., the moments of the estimated variance .

Table–5 reports biases arising from the second moment of the estimated variance in the Black–Scholes estimated option price. These biases are obtained by multiplying the second derivative of the model with the second moment of the variance estimate. The table shows that the bias arising from the second moment of the estimated variance in the Black–Scholes estimated price is largest and negative for the option around the exercise price, and it is positive for deep in the money options. Positive bias is also transmitted to the model price for out and in the money options with one quarter to maturity.

Interestingly, the biases observed in the Black–Scholes model (in Table–3) are of the same signs as the biases reported in Table–5 (a few exceptions are found when N=60).

For example, overpricing of out-of-the-money options by the Black-Scholes model is congruous with the positive bias induced by the second moment of the estimated variance, and so on. This is an insight that explains the biases in the Black-Scholes model found in Butler and Schachter (1983, 1984) and Boyle and Ananthanarayanan (1977). That is, the Black-Scholes model tends to overprice out-of-the-money options while it underprices in-and at-the-money options. This can be seen more clearly by examining the biases reported in Table-6. Table-6 reports the net bias in the estimated option price after we adjust the original bias (from Table-3) for the bias induced by the second moment of the sample variance. That is, we deduct the product of the second moment of the estimated variance and the second derivative of the model (Table-5) from biases reported in Table-3. Table-6 shows an uni-directional (negative) bias in the estimated option price for all cases. This clearly indicates that the multi-directional biases in the model' estimated price found in Butler-Schachter and Boyle-Anathanarayanan are due to the effects of the second moment of the variance estimate. The total remaining biases of the model arising from the higher order moments of the sample variance are uniformly negative, and the magnitude of these biases, in general, is very small.

Table-7 reports biases in the model estimated price arising from the third moment of the estimated variance. These biases are calculated by the product of the third derivative of the model and the third moment of the estimated variance. Similarly, the bias in the model estimated price induced by the fourth moment is obtained by the product of the fourth derivative of the model and the fourth moment of the variance estimate (Table-8). It is seen from Table-8 that the fourth moment induces biases which are of the same signs (except for in-the-money options of all maturity and for short maturity deep-in-the-money options) as the biases induced by the second moment (Table-5). But the bias pattern induced by the third moment (Table-7) is opposite (with a few exceptions) to that which is due to the second and fourth moments.[1] Therefore, the results suggest that the various moments of the estimated variance induce different patterns of biases into the model estimated price and that the total bias in the estimated option price depends on the relative sizes of biases induced by the various moments of the estimated variance.

Comparing Tables 5, 7, and 8, it is seen that the magnitude of the bias arising from the moments of the sample variance decreases (with a few exceptions) as the order of the moments increases. The second moment induces greater biases in the estimated option price as compared to the other two moments, and the biases induced by the third and fourth moments together are smaller than those of the second moment. The results seem to suggest that majority of the bias in the model estimated price arise from the second moment of the estimated variance and, therefore, a more accurate estimate of the model price can be obtained by concentrating on reducing the bias arising from the second moment of the variance estimate. This is possible by reducing the variance of the estimated variance. This is what most researchers have been doing. Existing studies suggest that the variance of the variance estimate can be reduced either by increasing the sample size used in estimating the variance rate, provided the variance is constant over that period, or by using some variance reduction techniques suggested by Geske-Roll (1984), Parkinson(1980), Boyle (1977) and Boyle-Ananthanarayanan (1977), among others. Since the variance of stock returns may not remain constant over longer periods, and since, for small samples, the variance reduction technique, such as the Bayes-Stein method used by Geske-Roll (1984), itself gives a biased estimator of the variance rate (e.g., see Butler-Schachter (1984) p.6), an easier approach would be to adjust the estimated price of

[1]Even though the third and fourth moments individually induces biases of different signs, we should not be surprised with the results reported in Table-6. This is because the bias reported in Table-6 is the outcome of the combined effects of the higher order, including third and fourth, moments of the sample variance.

the option for the bias induced by the second moment of the estimated variance. Of couse, this adjustment procedure will work better when the variance estimate used in the formula is unbiased. Even though this adjustment procedure is ad-hoc and requires repeated samples of stock prices to estimate the variance , it will provide us with a more accurate estimate of the option price and therefore will enable us to provide a better validation test for the model. This may also help investors improve their performance by reducing the probability of misidentifying over-or underpriced options.

The importance of the above adjustment for the effects of the variance of the estimated variance can be seen more clearly by examining the proportions of the bias induced by the three moments to the overall bias of the model. These biases are reported in Tables 9, 10 and 11. It is seen in Table-9 that the second moment induces almost all of the biases of the model. In some cases, the proportion of biases induced by the second moment is greater than 100%. This result is not surprising because the overall bias of the model consists of biases induced by all the moments of the estimated variance and some of the higher order moments (e.g., the third moment in our analysis) have induced negative biases into the model estimated price. The third and fourth moments individually do not induce more than 10% of the overall biases of the model. Even though the percentage biases reported in Tables 10 and 11 have changed with the sample size, the second moment is still found to induce much of the bias in the model estimated price. In a few cases (where we observed positive biases in the model estimated price with N=140 and these biases became negative with the decrease of the sample size (see Table-3)), the percentage biases induced by the three moments have increased, otherwise they have uniformly decreased with the decrease of the sample size. The rationale for this result can be found in the following analysis.

The majority of the generated variance estimates have come from the lefthand side of the gamma density function for which the expected value of the variance estimates have decreased with the decrease in sample size. This has caused the mean value of the B-S estimated prices to decrease. As a result, the positive (negative) bias observed in the model price with $N=140$ has decreased (increased) with the decrease of N. On the other hand, the magnitudes of the individual moments of the variance estimate have increased as the size of N has decreased. This has increased the biases induced by each of the moments. As a result, the percentage biases induced by the moments have increased in these cases. In other cases the percentage biases have decreased, probably because all the other moments (not considered here) have added more biases to the model estimated option price.

To conclude, we can say that the majority of the pricing bias of the model is due to the second moment of the estimated variance. It has also been found that the second moment induces multi-directional biases into the estimated price of the B-S model. That is, even when investigators use an unbiased estimator of the variance rate, the results suggest that the B-S model would yield biases of different signs and of different orders of magnitude. These results support the hypothesis that the nonlinearity of the Black-Scholes model in the estimated variance rate causes the model to misprice options in different directions. Bulter-Schachter (1983) and Boyle-Anathanarayan (1977) have also found similar implications by examining the estimation error biases of the model. We extend their studies by examining the estimation error biases of the model in terms of the moments of the estimated variance. We find that the variance of the variance estimate induces the majority of the bias of the model and these biases are subject to change with the change in the size of sample used in estimating the variance rate. Since the error in the variance estimate depends on the sample size used, and since different studies use different sample sizes in estimating the variance, contradictory results as regards to the bias pattern of the model can be observed among various tests of the model. For example, our results with N=60 (Table-3) seem to lend support to the most widely observed biases of the model. That is, the model tends to underprice at- and out-of-the-money options while it overprices deep-in-the-money options. But, for other values of N the mispricing behaviour of the model has been found to be different. Since any number of bias patterns can be obtained by the selection of different samples, it can be expected that even if the Black-Scholes model is the true model for pricing the option, the empirical validation of the model would be a difficult task, unless the model estimated price is approximated by taking into account the effects of the moments of the estimated variance. Our Monte Carlo test results suggest that the adjustment of the model estimated price for the effect of the second moment of estimated variance will reduce the bias of the model by a significant amount. Even though this adjustment procedure is ad-hoc, this would provide investigators with a more accurate estimate of the option price and would enable them to provide a better validation test for the model. This adjustment will eliminate the multidirectional bias pattern of the estimated price (as observed in Table-6) and therefore would help investors improve their performance by reducing the risk of misidentifying over- or underpriced options.

In this chapter we have reported the bias of the Black-Scholes model arising from the underlying distribution of the estimated variance rate. It is found that the model produces multi-directional systematic biases in option price estimates due to the estimation problem of the variance rate. These biases are also found to be of different orders of magnitude. Further, we have found that the multi-directional bias in the model estimated price arises from the second moment of the estimated variance. It is also found that the contradictory results of the model's performance in explaining market prices of options can repeatedly be observed due to the degree of accuracy of the estimated variance. Even though we do not know the true variance of the underlying asset, the results suggest that researchers should adjust the estimated model price for the effects of the variance of the estimated variance, even when they use an unbiased estimator of the variance rate in generating the model price.

We turn next to an examination of the effects of the estimation error of the sample variance on the Cox-Ross model. Estimation error biases in the Cox-Ross model estimates are examined in a manner similar to the above in the next chapter. The results for the two models will then be compared.

Specification error biases of the models are examined by comparing the prices generated by the competing models with the true variance inserted into the formulas. For the Cox-Ross model, as mentioned in chapter II, the variance rate is $\delta^2 = v \cdot X^2$, whereas v is the variance used for the Black-Scholes model.

Finally, the relative effects of estimation and specification errors on option price estimates generated by the B-S and C-R models are examined. It is investigated whether the estimated option price is relatively more sensitive to the specification error or to the estimation error, and whether the combined effects of the two sources of error provide some insights in order to understand the conflicting evidence that has repeatedly been observed in the empirical studies.

TABLE -1

Theoretical Option Prices Generated by the Black-Scholes Model

Exercise price=50, r=0.015, V=0.025(per quarter)
X=Stock price, T=Time to maturity of the option.*

T / X	1	3	5	7	9	15
40	0.30787	1.849	3.385	4.809	6.134	9.6496
50	3.513	6.497	8.719	10.606	12.285	16.548
60	11.18	13.862	16.132	18.121	19.913	24.498
80	30.747	32.396	34.18	35.94	37.63	42.222

* V is assumed to be the true variance of stock returns.

TABLE - 2

Estimated Black-Scholes Option Prices
with unbiased variance estimates

Exercise price=50, r=0.015, v=0.025 per quarter,
X=Stock price, T=Option's time to expiration.*

X	N \ T	1	3	5	7	9	15
40	140	0.309285	1.845146	3.376241	4.797353	6.119795	9.630898
	100	0.308775	1.839878	3.367678	4.786427	6.107065	9.61472
	60	0.307286	1.826531	3.346133	4.758995	6.075166	9.574243
50	140	3.507695	6.487656	8.706718	10.59219	12.27011	16.53098
	100	3.502521	6.478974	8.695884	10.57978	12.25648	16.51505
	60	3.489535	6.457207	8.668691	10.54862	12.22223	16.47502
60	140	11.18148	13.8595	16.12643	18.11385	19.9045	24.48717
	100	11.18061	13.85464	16.11906	18.10475	19.89406	24.47421
	60	11.17816	13.84219	16.1004	18.08167	19.86766	24.4415
80	140	30.74785	32.40053	34.18602	35.94763	37.63763	42.22371
	100	30.74803	32.40147	34.186	35.9463	37.63501	42.21806
	60	30.74849	32.40338	34.18536	35.9422	37.62761	42.20317

* V is assumed to be the true variance of stock returns.
N = Sample observations

TABLE - 3

Biases observed in the Black-Scholes model
with unbiased variance estimates

Bias= Estimated option price-True option price

Exercise price=50, r=0.015, v=0.025 per quarter,
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	0.001418	-0.004333	-0.008717	-.011868	-0.014265	-.018798
	100	0.000909	-0.009602	-0.01728	-.022795	-.026995	-.034976
	60	-.000581	-0.022948	-.038824	-.050226	-.058894	-.075454
50	140	-.005714	-.009585	-.011955	-.01366	-.01495	-.017349
	100	-.010889	-.018267	-.02279	-.026071	-.02858	-.033279
	60	-.023874	-.040034	-.049983	-.05723	-.06283	-.073303
60	140	.001341	-.00279	-.005508	-.007401	-.008789	-.011353
	100	.00047	-.00765	-.012878	-.016495	-.019226	-.024307
	60	-.00198	-.020101	-.03154	-.039581	-.045624	-.057022
80	140	.000473	.004929	.005844	.005295	.004303	.001724
	100	.000656	.005875	.005829	.003967	.001678	-.003922
	60	.001114	.007782	.005188	-.000137	-.005722	-.018814

* V is assumed to be the true variance of stock returns.
N = sample observations.

TABLE - 4

Percentage biases observed in the Black-Scholes Model with unbiased variance estimates

APB = (|Bias of Table 3|X100)/True option price

Exercise price=50, r=0.015, V=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	.460723	.234309	.257509	.246786	.232555	.194803
	100	.295229	.519151	.510413	.473979	.440079	.362457
	60	.188688	1.2408	1.14696	1.04437	.960117	.781929
50	140	.162646	.14753	.137122	.128801	.121691	.10484
	100	.309928	.281145	.261392	.245813	.232638	.201104
	60	.679619	.616174	.573287	.539608	.511434	.442965
60	140	.011993	.02013	.034146	.040839	.044137	.04634
	100	.004205	.055188	.079832	.091024	.096549	.099219
	60	.017708	.145002	.195512	.218425	.229112	.232757
80	140	.001538	.015214	.017098	.014731	.011434	.004084
	100	.002134	.018134	.017053	.011038	.00446	.009288
	60	.003623	.024022	.015178	.000382	.015205	.04456

* V is assumed to be the true variance of stock returns.
N= Sample observations.

TABLE - 5

Biases arising from the second moment of the estimated variance in the Black-Scholes model price

Exercise price=50, r=0.015, V=0.025 (per quarter)
X=Stock price, T=Times to option's expiration.*

T		1	3	5	7	9	15
X	N						
	140	.001551	-.004149	-.008526	-.011672	-.014047	-.018531
40	100	.00218	-.005832	-.011985	-.016408	-.01975	-.026049
	60	.003659	-.009789	-.020115	-.02754	-.03314	-.04372
	140	-.005634	-.009416	-.011728	-.013388	-.01465	-.016976
50	100	-.00792	-.013236	-.016487	-.01882	-.02058	-.023863
	60	-.01329	-.022215	-.02767	-.03159	-.034554	-.04005
	140	.001573	-.002463	-.005727	-.006973	-.008337	-.010833
60	100	.002211	-.003462	-.007207	-.009802	-.01172	-.015228
	60	.003711	-.005811	-.01209	-.016451	-.01967	-.025558
	140	.000546	.005229	.006307	.005859	.005004	.002445
80	100	.000768	.007351	.008865	.008236	.007034	.003437
	60	.001289	.012338	.014879	.013822	.011805	.005769

* v is assumed to be the true variance of stock returns.

TABLE - 6

Biases in the Black-Scholes model after adjustment for the bias induced by the variance of variance estimates

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	-.000133	-.000185	-.000191	-.000196	-.000218	-.000268
	100	-.001271	-.00377	-.005295	-.006388	-.00725	-.008927
	60	-.00424	-.013316	-.018709	-.022689	-.025755	-.031735
50	140	-.000081	-.00017	-.000228	-.000273	-.000304	-.000381
	100	-.00297	-.005031	-.006304	-.007251	-.007992	-.00943
	60	-.010508	-.01782	-.022313	-.025643	-.028276	-.033264
60	140	-.000233	-.000328	-.000381	-.000443	-.000458	-.000534
	100	-.001741	-.004189	-.005676	-.006699	-.007507	-.009094
	60	-.005691	-.01429	-.019455	-.023132	-.025955	-.031479
80	140	-.000076	-.000305	-.000473	-.000565	-.000702	-.000732
	100	-.000122	-.00148	-.003036	-.004272	-.005356	-.00737
	60	-.000183	-.004562	-.009705	-.013962	-.017532	-.024597

* v is assumed to be the true variance of stock returns.

TABLE - 7

Biases arising from the third moment of variance estimates in the Black-Scholes formula price

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	-.000104	5.1156E-5	.000224	.00035	.000443	.000601
	100	-.000174	8.5569E-5	.000275	.000586	.000741	.001005
	60	-.000384	.000188	.000826	.001291	.001631	.002211
50	140	.000185	.000303	.000368	.00041	.000438	.000467
	100	.00031	.000506	.000616	.000686	.000732	.000781
	60	.000683	.001115	.001356	.001511	.001611	.00172
60	140	-.000124	-5.362E-5	1.5041E-5	5.952E-5	8.8163E-5	.000121
	100	-.000208	-8.968E-5	2.516E-5	.0001	.000147	.000202
	60	-.000458	-.000197	5.539E-5	.000219	.000325	.000445
80	140	2.521E-5	-.000134	-.000296	-.000372	-.000407	-.000435
	100	4.264E-5	-.000224	-.000496	-.000622	-.00068	-.000727
	60	9.283E-5	-.0000493	-.001091	-.00137	-.001498	-.0016

* v is assumed to be the true variance of stock returns.

TABLE 8

Biases arising from the fourth moment of variance estimates in the Black-Scholes formula price

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	4.717E-5	9.2451E-5	-7.343E-6	-.000136	-.000182	-2.59E-5
	100	9.436E-5	1.8496E-5	-1.469E-5	-.000293	-.000365	-5.12E-5
	60	.000273	5.348967	-4.249E-5	-.00079	-.001056	-.000148
50	140	-7.87E-5	-.000126	-.00015	-.000163	-.00017	-.000166
	100	-.000158	-.000252	-.0003	-.000327	-.0034	-.000332
	60	-.000456	-.00073	-.000869	-.000945	-.000983	-.000959
60	140	6.124E-5	6.5741E-5	4.6367E-5	3.412E-5	2.7786E-5	2.995E-5
	100	.000123	.000132	9.276E-5	6.825E-5	5.559E-5	5.99E-5
	60	.000354	.00038	.000268	.000197	.000161	.000173
80	140	-5.39E-6	2.1689E-6	9.4518E-5	.000157	.000196	.000252
	100	-1.08E-5	4.339E-6	.000189	.000315	.000393	.000505
	60	-3.12E-5	1.255E-5	.000547	.00091	.001135	.00146

* v is assumed to be the true variance of stock returns.

TABLE -9

Proportion of biases due to the second, third and fourth moments of variance estimates to the overall biases of the Black-Scholes model (N=140)

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T / X		1	3	5	7	9	15
40	2nd	109.34	95.7437	97.8138	98.3455	98.4692	98.5784
	3rd	7.3478	1.1805	2.5732	2.9531	3.1056	3.1946
	4th	3.3253	.2133	.0842	1.1498	1.279	.1361
50	2nd	98.5977	98.2314	98.1	98.0	97.9686	97.846
	3rd	3.2437	3.1585	3.08805	3.003	2.9266	2.6927
	4th	1.3786	1.3156	1.2556	1.1959	1.1364	.9558
60	2nd	117.3	88.27	93.0767	94.2219	94.861	95.4232
	3rd	9.2652	1.9214	.2731	.8042	1.003	1.0634
	4th	4.5675	.2356	.8418	.4611	.3161	.2638
80	2nd	115.49	106.11	107.915	110.649	116.285	141.818
	3rd	5.3288	2.714	5.0681	7.0269	9.4513	25.2057
	4th	1.1405	.044	1.617	2.97	4.56	14.643

* v is assumed to be the true variance of stock returns.

TABLE -10

Proportion of biases due to the second, third and fourth moments of variance estimates to the overall biases of the Black-Scholes model (N=100)

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

X \ T		1	3	5	7	9	15
40	2nd	239.86	60.7443	69.36	71.98	73.1468	74.47
	3rd	19.18	.8912	2.1713	2.5719	2.7451	2.872
	4th	10.38	.1926	.085	1.1977	1.3522	.1464
50	2nd	72.736	72.46	72.3416	72.1903	72.0384	71.7048
	3rd	2.8474	2.7724	2.7031	2.6321	2.5607	2.3481
	4th	1.4474	1.3811	1.3178	1.2536	1.1893	.9968
60	2nd	470.257	45.258	55.96	59.4248	60.959	62.648
	3rd	44.1997	1.1723	1.954	.6035	.767	.8307
	4th	26.06	1.719	.7203	.4137	.2891	.2465
80	2nd	117.009	125.134	152.096	207.589	419.063	87.65
	3rd	6.4261	3.8087	8.4997	15.6872	40.529	18.5383
	4th	1.6449	.0739	3.2441	7.931	23.3925	12.881

* v is assumed to be the true variance of stock returns.

TABLE -11

Proportion of biases due to the second, third and fourth moments of variance estimates to the overall biases of the Black-Scholes model (N=60)

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T / X		1	3	5	7	9	15
40	2nd	629.875	42.6556	51.81	54.8273	56.27	57.94
	3rd	66.07	.08209	2.127	2.5697	2.7701	2.9309
	4th	46.97	.02331	.01094	1.572	1.7924	.1962
50	2nd	55.678	55.488	55.3588	55.1933	54.99	54.6359
	3rd	2.859	2.7849	2.7134	2.6397	2.5644	2.3469
	4th	1.9091	1.8225	1.7376	1.6515	1.5645	1.3088
60	2nd	187.427	28.91	38.3518	41.56	43.11	44.82
	3rd	23.1082	.9823	.1756	.5537	.7116	.7796
	4th	17.8974	1.8923	.8506	.04986	.03524	.03039
80	2nd	115.676	158.542	286.8	1000.0	206.309	30.66
	3rd	8.3334	6.3299	21.024	997.71	26.1735	8.50
	4th	2.8021	.0161	10.541	662.596	19.84	7.7643

* v is assumed to be the true variance of stock returns.

CHAPTER V

MONTE-CARLO TESTS FOR ESTIMATION ERROR BIASES IN THE COX-ROSS MODEL

In Chapter IV we examined the pattern and magnitude of the estimation error biases of the Black-Scholes model using Monte Carlo simulation techniques. A similar examination of the estimation error biases of the Cox-Ross model is provided in this Chapter. Estimation error biases of the Cox-Ross model are then compared with those of the Black-Scholes model. The relative effects of the second, third and fourth moments of the estimated variance on option price estimates generated by the two models are also compared.

A. Estimation Error Biases in the Cox-Ross Model

In Chapter III we analytically examined the estimation error biases of the Cox-Ross model. It was shown there that the C-R estimated option prices are biased even when they are obtained by using an unbiased estimate of the variance of stock returns in the C-R formula. [1] We discussed that the bias in the C-R estimated option price will be of different signs and of different orders of magnitude, depending on the bias induced by each of the moments of the estimated variance. We could not unambigously determine the extent of the bias in the model estimated price because the values of the products of the moments of the estimated variance and the derivatives of the model were unknown analytically. Therefore, in this chapter we employ Monte Carlo simulation techniques to calculate the extent of the estimation error bias of the Cox-Ross model.

[1]This is because the C-R formula is nonlinear in the variance, and the unbiasedness property does not hold under a nonlinear transformation.

Table–12 through Table–22 report the results of Monte–Carlo simulations for the Cox–Ross model for the same set of stock prices (40, 50, 60, 80), times to option's expiration (1, 3, 5, 7, 9, 15) and sample sizes (60, 100, 140) considered for the Black–Scholes model in Chapter IV. We use equivalent inputs in both the Cox–Ross and Black–Scholes models in order to ensure a consistent comparison of the estimation error biases of the two models. Since the B–S and C–R models differ in functional form, we expect that the estimation error biases of the two models should also differ.

The results reported in Tables 12 through 22 are obtained in a manner similar to that adopted for the Black–Scholes model except that here we assume that the market price of the option is generated by the Cox–Ross model. In other words, the Cox–Ross model is the "true" (correctly specified) model for pricing of options. However, as before, we assume that the true variance of stock returns, v, is .025 per quarter.

1. Pattern and Magnitude of Biases in the Cox- Ross Model

Table–12 reports the market price of the option generated by the Cox–Ross model. These prices are obtained by directly inserting the true value of the variance of stock returns, v, into the C–R formula. Table–13 provides the average C–R option prices with unbiased variance estimates. The unbiased variance estimates used in generating these prices are the same as those used in generating the B–S model estimated price (Table 2 of Chapter IV). [2] It is seen in Table–13 that the C–R estimated option prices decrease (except for short maturity (T=1) deep–in–the–money option) as the sample size decreases. This is because the sample mean of the variance used in estimating the option price decreases with the decrease of the size of N. For example, as noted in chapter IV,

[2] We use the same variance estimates in order to facilitate a consistent comparison of estimation error biases of the B–S and C–R models. This comparison is done in a later section of this chapter.

when $N=60$ it is .02482513, whereas $\bar{v}$ = .024949 and .02499679 when $N=$ 100 and 140 respectively.

Table–14 gives the estimation error biases in the Cox–Ross model which are obtained as the dollar excess of the C–R estimated option price (Table 13) over the market price of the option (Table–12).[3] The biases are found to be of different signs and of different magnitudes. The biases also change with changes in the stock price, the option's time to maturity, and the sample size. Specifically, when $N=140$ the Cox–Ross model overprices (except for long maturity) deep in–the–money options ($X=80$). The overpricing is also observed for options out– and in– the– money with one quarter to maturity, otherwise the model underprices options . The absolute pricing bias is greatest for options at the money.[4]

Table–15 lists the percentage biases in the Cox–Ross model price (biases in proportion to market prices of options) and shows that the percentage bias is greatest for out– of–the money options. This is because the model prices are typically low for these options. It is seen in Table 15 that for $N=140$ the percentage biases decrease with the increase of the option's time to expiration in all cases except for options in the money ($X=60$). This indicates that the estimation error of the variance rate forces the model to increasingly misprice in the money options when the maturity period of the option, ceteris paribus, is increased. Table–15 also shows, again for $N=140$, that the magnitude of the percentage bias in the Cox–Ross model is negatively related to the stock price, given the option's expiration date. This implies that the bias in the Cox–Ross model estimated price

[3]We call it the market price of the option because here we assume that the option is priced by the Cox–Ross model.

[4]As explained in f.n. 6 of Chapter IV, being at the money means $X=E$ so that when $X=Ee^{-rT}$ (considered in Chapter III) we are slightly in the money. This definition however does not change the results of our analysis.

will always increase (given our parameter sets) with a decrease in the price of the stock, given the exercise price.

However, with the decrease of the sample size in estimating the variance rate, the bias in the model estimated price increases (Table–14) in all cases except for options out of the money with one quarter to maturity and deep-in-the money options with $T=5$. The direction of the bias in the estimated price has also changed with changes in the sample size used. For example, the bias in the model price is negative for the short maturity ($T=1$) in-the-money options ($X=60$) with $N=100$ and 60. This implies that the increased deviation of the estimated variance from the true value of the variance will change the bias pattern of the model. Because of this, it is possible that conflicting evidence as regards to the bias pattern of the model can be observed across studies which use different sample sizes in estimating the variance of stock returns. For example, Thorp–Gelbaum (1980) found the Cox–Ross model to underprice (overprice) at- and out of the (in the) money option whereas MacBeth–Merville (1980) found the model to overprice (underprice) out-of-the (in- and at-the) money options.

Further, we observe from Table–15 that the percentage bias in the estimated price for out of the money options (with $T=1$) decreases as the size of the sample used decreases. But the opposite result holds for other options (with a few exceptions for $X=80$) considered in our simulations. This indicates that the use of a larger sample size in estimating the variance will not always reduce the bias in the estimated model price , especially for options out of the money. This result for the out-of-the-money option is not surprising because the total bias in the estimated price (Table–14) for this option decreases as the size of N decreases.

In short, for the range of parameter values considered in our simulation tests we observe multi-directional biases in the Cox-Ross model arising from the estimation problem of the variance of the underlying stock. The biases are also observed to be of many orders of magnitude, even with an unbiased estimator of the variance rate. The results thus support our analytical discussion (in chapter III) that even if the Cox-Ross model is the true model to price the option, the variance estimate induces different patterns of biases in the model estimated option price.

Even though it is mentioned elsewhere (e.g Butler and Schachter (1984)) that the estimation problem associated with the unknown but estimated variance of the underlying asset is shared by any option pricing model (which are non-linear in the variance), the examination of the effects of the estimation error of the variance on the Cox-Ross model estimated price is lacking in the existing literature. The results of this chapter thus provide useful insights about the model's mispricing of options that might be observed in any validation tests of the model. That is, the model will tend to overprice deep-in-the-money options and short maturity in- and out-of-the money options. Otherwise, the model will tend to underprice options. Of course the pattern of the bias explained above may change depending on the accuracy of the sample variance used in any tests of the model. This pattern of bias is (dis)similar to the pattern of bias in B-S. We return to this point shortly.

We now examine estimation error biases of the Cox-Ross model in terms of the effects of the moments of the estimated variance.

2. Biases and the Effects of the Moments of the Estimated Variance

Table-16 reports biases arising from the second moment of the estimated variance in the Cox-Ross estimated option price. These biases are obtained by multiplying the second derivative of the C-R formula by the second moment of the estimated variance. The table shows that the bias induced by the second moment of the estimated variance is largest and negative for options at the money, and it is positive for most options deep in the money (X=80). The biases are also positive for in- and out-of-the-money options of one quarter to maturity.

Interestingly, the biases observed in the C-R estimated option price (in Table-14) have mostly the same signs as the biases reported in Table-16. For example, overpricing of out-of-the-money options by the Cox-Ross model is congruous with the positive bias induced by the second moment of the estimated variance, and so on. Therefore, it is possible that the multidirectional biases in the model estimated option price arise due to the second moment of the estimated variance. In order to examine this possibility, we closely look at Table-17 which reports the net bias in the estimated option price after we adjust the original bias (Table-14) for the bias induced by the second moment of the sample variance. That is, we subtract the product of the second moment of the estimated variance and the second derivative of the model from biases reported in Table-14. Table-17 shows an uni-directional (negative) bias in the estimated option price for all cases. This indicates that the multi-directional stock specific and maturity-period specific biases of the model are due to the effects of the second moment of the variance estimate. The remaining biases of the model arising from the higher order moments of the sample variance when taken together are uniformly negative, and the magnitude of the biases are quite small. Even though we shall see in the following paragraph that the third and fourth moments individually induces different patterns of biases in the estimated model price, we should not be surprised with the results of Table-17. This is because the negative bias reported in Table-17 is the outcome of the combined effects of the higher order, including the third and fourth moments, of the estimated variance.

Table-18 reports biases in the model estimated price arising from the third moment of the estimated variance. These biases are obtained by taking the product of the third derivative of the model and the third moment of the estimated variance. Similarly, the bias arising from the fourth moment in the estimated model price is obtained by the product of the fourth moment and the fourth derivative of the model (Table-19). It is seen from from Table 19 that the fourth moment induces biases in the model estimated price which have mostly the same signs as the biases induced by the second moment (Table-16). A few exceptions are found for long maturity in- and deep-in-the money options. The sign of the bias induced by the third moment (Table-18) is opposite to that which is due to the second and fourth moments. Therefore, the results suggest that the various moments of the sample variance induce different patterns of biases into the model estimated price and that the total bias in the estimated option price depends on the relative sizes of biases induced by the various moments of the estimated variance.

Comparing Tables 16, 18, and 19, it is seen that the second moment of the sample variance induces greater biases in the estimated option price as compared to the other two moments, and that the biases induced by the third and fourth moments together are smaller than those of the second moment. The results thus suggest that majority of the bias in the model estimated option price arise from the second moment of the estimated variance, and therefore, an accurate estimate of the option price can be obtained by concentrating on reducing the bias arising from the second moment of the variance estimate. As we suggested in Chapter IV, an easier approach to do this would be to adjust the estimated price of the option for the bias induced by the second moment of the estimated variance. This adjustment will reduce the bias of the estimated price by a significant amount and would enable researchers to provide a better validation test for the model.

The importance of the above adjustment for the bias induced by the second moment can be seen more clearly by examining the proportions of the bias induced by the three moments to the overall bias of the model. These biases are reported in Tables 20, 21, and 22. It is seen in Table-20 that the second moment induces almost all of the biases of the model (a few exceptions are found for X=80). In some cases, the proportion of biases induced by the second moment is greater than 100%. This result is not surprising because, as we explained in section IV for the Black-Scholes model, the overall bias in the estimated option price consists of biases induced by all the moments of the sample variance and some of the moments induce negative biases into the model estimated price. The third and fourth moments individually or jointly do not induce significant amount of biases in the C-R estimated price. Even though the percentage biases reported in Tables 21 and 22 have changed with the sample size, the second moment is still found to induce much of the bias in the model estimated price. However, in a few cases the fourth moment has induced a significant amount of bias in the estimated option price. In addition, the percentage biases induced by the three moments have increased for some cases, otherwise they have decreased with the decrease of the sample size. The rationale for these results can be found in the following analysis.

The majority of the generated variance estimates have come from the lefthand side of the gamma distribution for which the expected value of the variance estimates have decreased with the decrease of the sample size. This has caused the mean value of the C-R estimated prices to decrease. As a result the positive (negative) bias observed in the model price with N=140 has decreased (increased) with the decrease of N. On the other hand, the magnitudes of the individual moments of the variance estimate has increased as the size of the sample has increased. This has increased the biases induced by each of the moments. As a result, the percentage biases induced by the moments have increased in these cases. In other cases the percentage biases have decreased, probably because all the other moments (not considered here) have added more bias to the model estimated opion price.

To conclude, we can say that the majority of the pricing bias of the model is due to the second moment of the estimated variance. It has also been found that the second moment induces multidirectional biases into the estimated price of the Cox-Ross model. That is, even when investors use an unbiased estimator of the variance rate, the results suggest that the Cox-Ross model would yield biases of different signs and of different orders of magnitudes. These results support our hypothesis that the non-linearity of the C-R model in the estimated variance rate causes the model to misprice options in directions. We also find that the mispricing behaviour of the model is subject to change with the change in the size of the sample used in estimating the the variance rate. Since the error in the variance estimate depends on the sample size used, and since different studies use different sample sizes in estimating the variance , contradictory results as regards to the bias pattern of the model can be observed among various tests of the model. For example, our results with N=60 (Table-14) seem to lend support, in part, to the empirically observed biases by MacBeth-Merville (1980). That is the model tends to overprice (underprice) the short maturity out-of-the-money (at- and in-the-money) options. But, for other values of N the mispricing behaviour of the model has been found to be different. Since any number of bias patterns can obtained by the selection of different samples, it can be expected that even if the Cox-Ross model is the true model for pricing the option, the empirical validation of the model would be a difficult task, unless the model estimated price is approximated by taking into account the effects of the moments of the estimated variance. Our results suggest that the adjustment of the model estimated price for the effects of the second moment of the estimated variance will reduce the bias of the model by a significant amount. Even though this adjustment procedure is ad-hoc, this would provide investors with a more accurate estimate of the option price and would enable them to provide a better validation test for the model. This adjustment will eliminate the multidirectional bias pattern of the estimated option price (as observed in Table-17) and therefore would help investors improve their performance by reducing the risk of misidentifying over- or underpriced options.

B. Comparison of Estimated Error Biases of the Cox-Ross and Black-Scholes Models

In this section we compare and contrast the estimation error biases of the Black-Scholes and Cox-Ross models. Since the two models differ in functional form, we expect that estimation error biases of the models should also differ. First, we compare the total estimation error biases of the B-S and C-R models reported respectively in Table-3 of chapter IV and in Table-14 of this chapter. We then compare the effects of the moments of the estimated variance on the option price generated by the two models.

The results reported in Table-3 and Table-14 suggest that both the Black-Scholes and Cox-Ross models result in similar bias patterns in the estimated price of the option. That is, both the B-S and C-R models produce multi-directional biases in the estimated option price and these biases are of same signs for the two models when they are viewed across stock prices, given maturity periods, or across maturity periods, given stock prices. In other words, we find that both the Black-Scholes and the Cox-Ross models have similar pricing bias characteristics with respect to estimation error of the variance of stock returns.

Nevertheless, the magnitude of biases in the estimated price of the two models are different except for options at the money (i.e when $X=E$). For example, the Cox-Ross model produces greater biases for options out- and deep-in- the- money , whereas the bias produced by the Black-Scholes model is greater for options in the money at least when $N=140$. However, with the change of sample observations from 140 to 100 or 60, it is observed that the Black-Scholes model produces smaller (greater) biases for options in the money (out of the money). Tables 3 and 14 also show that when the option is at the money, the estimation error in the variance estimate induces the same bias for the two models, even though the two models differ in functional form. This result occurs because the two models are equally sensitive as regards to the effects of each of the three moments of the estimated variance (Table-9 vs. Table-20, Table-10 vs. Table-21, and Table-11 vs. Table-22). For example, the second, third and fourth moments of the estimated variance respectively induce 98, 3, and 1 percent of the total estimation error bias of each of the models when the option is at the money (Table-9 vs. Table-20). For other options considered, the proportion of the bias induced by these three moments to the overall bias of each of the models is different across the models.

To conclude, we can say that both the Black-Scholes and Cox-Ross models produce multidirectional biases in the estimated option price and these biases are of same signs for the two models when they are viewed across stock prices of given maturity periods or across maturity periods for given stock prices. However, the magnitude of the biases in the estimated price of the two models are different except for options at the money. For example, the Cox-Ross (B-S) model produces greater biases for out and deep-in-the (in) money options. The major implications of these results are:

(i) The Cox–Ross model , regardless of its specification, might perform poorly in predicting market prices of out– and deep–in–the money options as compared to the B–S model simply because the Cox–Ross model is relatively more sensitive to errors in the variance estimate for these types of options. Similarly, the B–S might perform poorly in predicting the price of the in–the–money option as compared to the Cox–Ross model.

(ii) In both models, the majority of the pricing bias is due to the second moment of the estimated variance and these biases are very large as compared those induced by the higher order moments. To provide a better comparison of the ability of the models in predicting market prices of options, our results suggest an adjustment of the models' estimated prices for the effects of the second moment of the sample variance. This will help reduce biases of the two models by a considerable amount and thereby allow researchers to provide a better validation test for the competing models.

(iii) Finally, and most important, our results suggest that as long as the options are at the money the estimation error bias of the two models might provide the same misinformation about over–or underpriced options.

However, it is important to note that the above results are only based upon the effects of the estimation error of the variance. Since the two models differ in specification of the underlying process of the stock price, it is also of interest to examine specification error biases in option price estimates generated by the two models. This is because the specification error may confuse the interpretation of the evidence of the estimation error and that might lead researchers to validate the "wrong" model as correct. Therefore, the examination of specification error biases and their comparison to the estimation error biases of the two models are provided in the following chapter.

TABLE - 12

Theoretical Option Prices Generated by the Cox-Ross Model

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T= Time to maturity of the option.*

T / X	1	3	5	7	9	15
40	.1949649	1.459059	2.829634	4.145098	5.39519	8.805538
50	3.516669	6.514113	8.754669	10.66496	12.37043	16.72328
60	11.33978	14.27437	16.70585	18.82101	20.71853	25.5448
80	30.77386	32.82386	35.04062	37.16156	39.14848	44.35401

* v is assumed to be the true instantaneous variance of stock returns.

TABLE - 13

Estimated Cox-Ross Option Prices
with Unbiased Variance Estimates

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T= Option's time to expiration.*

X	N \ T	1	3	5	7	9	15
40	140	0.196655	1.456078	2.822534	4.135027	5.382905	8.789028
	100	0.196556	1.451866	32.815186	4.125404	5.371557	8.774277
	60	0.196104	1.441150	2.796661	4.101215	5.343041	8.737299
50	140	3.510981	6.504609	8.742886	10.65158	12.35584	16.70576
	100	3.505805	6.495927	8.732021	10.63912	12.34213	16.68932
	60	3.492813	6.474115	8.704736	10.60782	12.30768	16.64799
60	140	11.34060	14.27032	16.69885	18.81195	20.70786	25.52969
	100	11.33918	14.26434	16.69002	18.80115	20.6955	25.51366
	60	11.33531	14.24898	16.66769	18.77379	20.66415	25.47328
80	140	30.776	32.8297	35.0448	37.16333	39.14767	44.34457
	100	30.77666	32.82954	35.04185	37.15773	39.13968	44.33055
	60	30.77814	32.82846	35.03366	37.14288	39.11885	44.29491

* v is assumed to be the true variance of stock returns.
N = Sample obsevations

TABLE - 14

Biases Observed in the Cox-Ross Model
with Unbiased Variance Estimates

Bias=Estimated option price - True option price

Exercise price=50, r=0.015, V=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	0.00169	-0.002981	-0.0071	-.01007	-0.012285	-.01651
	100	0.001591	-0.007193	-0.014448	-.019693	-.023633	-.031261
	60	0.001139	-0.017913	-0.032972	-.04388	-.052149	-.068239
50	140	-.005688	-.009505	-.011784	-.01338	-.01459	-.017517
	100	-.010863	-.018187	-.022648	-.02584	-.0283	-.033966
	60	-.023855	-.039998	-.049933	-.05714	-.06275	-.075287
60	140	.00082	-.00405	-.007004	-.009064	-.010681	-.015106
	100	-.00047	-.00765	-.012878	-.016495	-.019226	-.024307
	60	-.00447	-.025391	-.038162	-.047226	-.054382	-.071518
80	140	.002136	.005844	.004181	.00177	-.000809	-.00943
	100	.002792	.005692	.001236	-.00383	-.008804	-.023453
	60	.004272	.004608	-.006958	-.018677	-.029633	-.059097

* v is assumed to be the true variance of stock returns.
N = sample observations.

TABLE - 15

Percentage Biases Observed in the Cox-Ross Model with Unbiased Variance Estimates

APB= (|Bias of Table 14|X100)/True option price

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

X	N \ T	1	3	5	7	9	15
40	140	.866991	.204323	.25092	.242957	.227707	.187496
	100	.816089	.492962	.510602	.4751	.438038	.35502
	60	.584261	1.2277	1.16525	1.05879	.96658	.774958
50	140	.161736	.145903	.134598	.125458	.117945	.104747
	100	.308909	.279187	.258695	.242287	.228774	.203106
	60	.678347	.614021	.570363	.535776	.507257	.450192
60	140	.007233	.028374	.041924	.048157	.051554	.059136
	100	.005295	.070545	.094809	.105557	.111208	.121916
	60	.039418	.177876	.228436	.250921	.262481	.279971
80	140	.006942	.017804	.011932	.004763	.002066	.021261
	100	.009074	.01734	.003527	.010306	.02249	.052876
	60	.013883	.014039	.019857	.050258	.075693	.1332[illegible]

* v is assumed to be the true variance of stock returns.
N= Sample observations.

TABLE -16

Biases arising from the second moment of the estimated variance in the Black-Scholes model price

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Times to option's expiration.*

X	N \ T	1	3	5	7	9	15
40	140	.001803	-.002802	-.006913	-.009877	-.012072	-.01621
	100	.002535	-.003938	-.009718	-.013885	-.01697	-.022787
	60	.004254	-.00661	-.01631	-.023303	-.028481	-.038244
50	140	-.005617	-.009326	-.011541	-.013091	-.014256	-.017047
	100	-.00789	-.01311	-.016224	-.018402	-.02004	-.023964
	60	-.01325	-.022004	-.027229	-.030884	-.033634	-.040219
60	140	.001076	-.003702	-.006613	-.008606	-.010158	-.014491
	100	.001513	-.005203	-.009297	-.012098	-.01428	-.02037
	60	.002539	-.008733	-.015603	-.020304	-.023966	-.034188
80	140	.00227	.006325	.004806	.002515	1.8903E-5	-.008548
	100	.00319	.008891	.006756	.003536	2.657E-5	-.012015
	60	.005355	.014922	.011339	.005934	4.451E-5	-.020166

* v is assumed to be the true variance of stock returns.

TABLE - 17

Biases in the Cox-Ross Model after Adjustment
for the Bias induced by the Variance of Variance Estimates

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	-.000113	-.00018	-.000187	-.000194	-.000214	-.0003
	100	-.000944	-.003255	-.00473	-.005809	-.006663	-.008475
	60	-.003115	-.011303	-.016663	-.020586	-.023668	-.029996
50	140	-.000072	-.000178	-.000243	-.00029	-.000335	-.000473
	100	-.002969	-.005076	-.006425	-.007439	-.008261	-.01001
	60	-.010605	-.017995	-.022705	-.026257	-.029116	-.035067
60	140	-.000257	-.000349	-.000397	-.000458	-.000534	-.000626
	100	-.002113	-.004867	-.006546	-.007782	-.008774	-.010788
	60	-.00702	-.016658	-.022560	-.026922	-.030426	-.037338
80	140	-.000137	-.000488	-.000626	-.000748	-.000839	-.000885
	100	-.000412	-.003204	-.005524	-.00737	-.008835	-.011444
	60	-.001083	-.010315	-.01830	-.024612	-.029678	-.03894

* v is assumed to be the true variance of stock returns.

TABLE - 18

Biases arising from the third moment of variance estimates in the Cox-Ross formula price

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Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
	140	-9.06E-5	6.6248E-6	.00017	.000293	.000383	.000507
40	100	-.000151	1.1081E-5	.000284	.000491	.000641	.000848
	60	-.000338	2.4396E-5	.000626	.00108	.00141	.001867
	140	.000185	.000302	.000366	.000406	.000427	.000401
50	100	.00031	.000505	.000612	.000679	.000715	.000671
	60	.000682	.001111	.001348	.001495	.001574	.001478
	140	-.000123	-1.225E-5	7.0478E-5	.000121	.000143	.000115
60	100	-.000206	-2.048E-5	.000118	.000202	.000239	.000192
	60	-.000453	-4.510E-5	.00026	.000444	.000526	.000422
	140	1.501E-5	-.000304	-.000392	-.000408	-.00042	-.000421
80	100	2.510E-5	-.000508	-.000656	-.000682	-.000703	-.000702
	60	5.527E-5	-.001118	-.001445	-.001501	-.001549	-.001546

* v is assumed to be the true variance of stock returns.

TABLE - 19

Biases arising from the fourth moment of variance estimates in the Cox-Ross formula price

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	.000357	4.545E-5	-4.722E-5	-.000115	-.000202	-.001254
	100	.000715	9.0934E-5	-9.447E-5	-.000229	-.000404	-.002508
	60	.002066	.000263	-.000273	-.000663	-.001167	-.00725
50	140	-7.87E-5	-.000126	-.00015	-.000201	-.000427	-.00254
	100	-.000157	-.000252	-.000301	-.000403	-.000853	-.005082
	60	-.000455	-.000728	-.000871	-.001165	-.002468	-.014697
60	140	2.308E-5	8.5551E-5	3.6832E-5	-.000137	-.000703	-.00383
	100	.000462	.000171	7.368E-5	-.000273	-.001406	-.00767
	60	.001336	.000495	.000213	-.00079	-.004066	-.022142
80	140	.001986	.001295	.000777	-8.42E-5	-.001606	-.005975
	100	.003973	.00259	.001554	-.000169	-.003213	-.011954
	60	.011489	.00749	.004494	-.000488	-.009291	-.03457

* v is assumed to be the true variance of stock returns.

TABLE -20

Proportion of Biases of the Cox-Ross Model arising from the Second, Third and Fourth Moments of the Estimated Variance (N=140)

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

X	T	1	3	5	7	9	15
40	2nd	106.34	93.97	97.37	98.07	98.26	98.18
40	3rd	5.357	.22222	2.3937	2.91	3.1165	3.0709
40	4th	21.1297	1.5247	.6651	1.137	1.6425	7.594
50	2nd	98.75	98.13	98.94	97.83	97.71	97.32
50	3rd	3.254	3.1742	3.1065	3.034	2.9286	2.2905
50	4th	1.3839	1.3238	1.277	1.5052	2.9235	14.50
60	2nd	131.2	91.38	94.436	94.95	95.15	95.92
60	3rd	15.012	.3024	1.0063	1.33	1.3361	.7588
60	4th	2.815	2.112	.5258	1.5069	6.5802	25.334
80	2nd	106.25	108.22	114.95	142.11	2.33	90.64
80	3rd	.7026	5.1958	9.3831	23.0236	52.005	4.45
80	4th	92.95	22.15	18.57	4.76	198.5	63.36

* v is assumed to be the true variance of stock returns.

TABLE -21

Proportion of Biases of the Cox-Ross Model
arising from the Second, Third and Fourth Moments
of the Estimated Variance (N=100)

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T / X		1	3	5	7	9	15
40	2nd	159.305	54.7543	67.26	70.504	71.806	72.89
	3rd	9.52	.1541	1.9677	2.49	2.7099	2.7129
	4th	44.91	1.264	.6539	1.1639	1.7082	8.024
50	2nd	72.6795	72.088	71.635	71.215	70.813	70.553
	3rd	2.85506	2.7748	2.7036	2.6283	2.52255	1.9759
	4th	1.449	1.384	1.329	1.559	3.0154	14.9618
60	2nd	252.172	51.67	58.69	60.89	61.97	65.407
	3rd	34.3339	.2035	.7443	1.015	1.036	.6156
	4th	76.999	1.6998	.4652	1.3754	6.1027	24.5841
80	2nd	114.268	156.21	546.63	92.32	.3018	51.23
	3rd	.8991	8.924	53.09	17.7985	7.9897	2.9947
	4th	142.277	45.507	125.717	4.403	36.492	50.972

* v is assumed to be the true variance of stock returns.

TABLE -22

Proportion of Biases of the Cox-Ross Model
arising from the Second, Third and Fourth Moments
of the Estimated Variance (N=60)

Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T / X		1	3	5	7	9	15
40	2nd	373.454	36.899	49.466	53.096	54.615	56.044
	3rd	29.276	.1362	1.8982	2.4613	2.7037	2.7361
	4th	181.408	1.468	.828	1.5103	2.2388	10.6305
50	2nd	55.54	55.01	54.531	54.05	53.6	53.42
	3rd	2.8578	2.7777	2.699	2.6167	2.5076	1.9626
	4th	1.909	1.82	1.7437	2.03	3.933	19.52
60	2nd	56.59	34.39	40.88	42.99	44.069	47.803
	3rd	10.1439	.1776	.6801	.94	.9664	.5902
	4th	29.88	1.949	.5584	1.673	7.4775	30.4594
80	2nd	125.341	323.81	162.96	31.77	.1505	34.12
	3rd	1.293	24.266	20.762	8.0353	5.22	2.6164
	4th	268.91	162.54	64.58	2.6114	31.35	58.499

* v is assumed to be the true variance of stock returns.

CHAPTER VI

MONTE CARLO TESTS FOR BIASES DUE TO ERRORS IN SPECIFICATION AND ESTIMATION

In Chapters IV and V respectively we reported Monte-Carlo tests results for the estimation error biases of the Black-Scholes and Cox-Ross models. Specification error biases for the Black-Scholes and Cox-Ross models, and the combined effects of errors in specification and estimation on option price estimates generated by these two models are examined in this chapter. We examine the seriousness of the above two sources of error on the model estimated option price in order to find out whether the estimation error bias in the correctly specified model is sufficiently large to cause the misspecified model to be selected as correct. Consideration of this issue is important in order to understand the conflicting evidence contained in the empirical literature as well as to provide a broader basis for evaluating the performance of the Black-Scholes and Cox-Ross option pricing models to explain the market price of options.

A. Specification and Estimation Error Biases in the Black-Scholes Model

We assume, temporarily, that the stock price movement is characterised by the Absolute process and the option, as a result, is priced by the Cox-Ross model. In this case, by assumption, the true option price is equal to the Cox-Ross model price (obtained by inserting the true value of the variance of stock returns in the Cox-Ross formula). Thus, when researchers use the Black-Scholes formula with an estimate of the variance rate to generate the option price, systematic biases in the B-S estimated option price would arise from the combined effects of model misspecification and the error in the estimated variance.

Table 23 reports biases in the Black-Scholes estimated option price arising from the combined effects of the above two sources of errors. These biases are obtained as the dollar excess of B-S estimated option prices (Table-2) over the true price of options (Table-12). It is seen in Table-23 that the B-S model overprices out-of-the- money options (of all maturities) and it underprices all other options. The biases in the B-S estimated option price are least when the options are at the money (X=50). Interestingly, the above findings conform with the biases empirically observed by MacBeth-Merville (1979, 1980) and Rubinstein (1985). That is, the Black-Scholes model overprices (underprices) out-of-the (in- and deep-in-the) money options. But, they contradict the most widely observed biases of the B-S model contained in the empirical literature. As noted in Chapter IV, the most widely observed bias of the B-S formula is a tendency to underprice (overprice) out-of-the (in-the) money options (e.g., see Black (1975), Finnerty (1978), Thorp-Gelbaum (1980), and Gultekin-Rogalski (1982), among others).

It is important to note that, even though Rubinstein (1985) found biases opposite to those reported by Black (1975) and others, he also reported that the biases found by Black recurred in some sample periods. Within the framework of the present study, this reversal in the bias pattern of the B-S model may be explained by a change in the parameter value of the CEV process over different time periods. For example, when the stock price movement is characterised by the lognormal process (i.e., when the B-S model contains only the estimation error bias), the B-S formula will tend to misprice options in a manner reported in Black (1975) and other studies. This point has already been dicussed in Chapter IV where we examined the estimation error biases of the B-S model. However, when the stock price movement follows the absolute process (i.e., the B-S model contains both specification and estimation error bias), the B-S formula will tend to misprice options in a manner reported in MacBeth-Merville (1979, 1980). As a result, the

B–S model with only estimation error corresponds more with most empirical results than the B–S with errors in specification and estimation.

In order to understand the significance of the two sources of biases in the B–S estimated price , we now separately examine the specification and estimation error biases of the model. Table–24 reports the specification error bias in the B–S estimated option price. These biases are obtained by using equation (24) of Chapter III. That is, the difference between option prices generated by the B–S and C–R models (with the true variance of stock returns inserted into the respective formula) gives the specification error bias of the B–S model. Table– 24 indicates that the B–S model overprices (underprices) out– of– the–money (in and at–the–money) options due to misspecification of the underlying process of the stock price. These findings are consistent with the analytical results discussed in Chapter III as well as with the findings of other simulation studies, notably MacBeth–Merville (1980), Beckers (1980),and Ritchey (1985).

Further, we observe that the signs of the biases reported in Table–24 , in most cases, are different from those of the estimation error biases of the model (Table–3 of Chapter IV). For example, when N=140 the estimation error bias is positive for options deep in the money whereas the specification error bias is negative for these options, and so on. The magnitudes of biases arising independently from errors in estimation (Table–3) and specification (Table–24) are also quite different except for at the money options. Comparing Tables 3 and 24, it is seen that the specification error, as compared to the estimation error, induces greater biases into the B–S estimated price for all options, except for the option at the money. Because of this, the biases reported in Table–23 are of the same signs as the specification error biases of the model (Table–24). For example, underpricing of deep–in–the–money options by the Black–Scholes model (Table–23) is congruous with the negative bias induced the specification error (Table–24).

Interestingly, at the money, the options are valued by the Black-Scholes model with small errors due to misspecification of stock price movements, and that the magnitude of this specification error bias is smaller than the estimation error bias of the model. The results thus suggest that when the options are at the money, the B-S estimated option price is relatively insensitive with respect to the specification problem, and that the majority of the biases in the B-S estimated price for options at the money are due to errors in the estimated variance. This can be seen more clearly by comparing Tables 25 and 26. Tables 25 and 26 respectively report the percentage biases in the B-S estimated option price arising from errors in specifiaction and estimation. It is seen that for options at the money, the percentage bias in the B-S estimated price due to errors in the estimated variance is at least two times greater than that of the specification error (this is even true when we compare Tables 4 and 25). For other options, the specification error induces greater percentage biases into the B-S option price. The highest percentage specification error bias of the model is found for options out-of-the money. Therefore, we can say that the model estimated option price is very sensitive to the specification problem when the options are out-of-the money, and that the estimation problem is more serious when the options are at the money.

Since the two sources of errors induce different patterns of biases in the B-S estimated option price, and since the specification error, in most cases, changes the interpretation of the evidence of the estimation error, it is possible that the estimation error bias in the correctly specified model is sufficiently large in some cases to cause the misspecified model to be selected as correct. In order to examine this possibility, we now compare the ability of the B-S and C-R models to explain the true price of the option. Specifically, we compare the estimation error biases of the Cox-Ross model (Table 14) with the overall bias of the Black-Scholes model reported in Table-23. As before, we assume that the option in the market is priced by the Cox-Ross model.

Comparing Tables 14 and 23, it is seen that the absolute values of the biases in the B-S estimated option price are, in all cases, greater than the biases in the Cox-Ross estimated option price. This result indicates that if the actual stock price movement follows the 'absolute process , it is unlikely that the Black-Scholes model will perform better than the Cox-Ross model in explaining the market price of options. Therefore, we can say that the above result does not support our hypothesis that the estimation error bias in the correctly specified model is so large as to cause the misspecified model to be selected as correct. Thus, the choice between the Black-Scholes and Cox-Ross models is not hampered by the interaction of errors in estimation and specification.

However, by switching the roles of the Cox-Ross model and the Black-Scholes model in the following discussion (i.e., by assuming that the B-S model is correctly specified and the Cox-Ross model is misspecified), a different picture emerges.

B. Specification and Estimation Error Biases in the Cox-Ross Model

Specification error biases for the Cox-Ross model are examined by assuming that the B-S model generates the market price of options (that is, the actual stock price movement follows the lognormal process). The biases in this case are nothing other than the negative of specification error biases of the Black-Scholes model (i.e., negative of BSSE in equation 24 of chapter III) . The specification error biases of the Cox-Ross model are reported in Table-27. It is seen in Table-27 that the Cox-Ross model underprices (overprices) out-of-the-money (at-, in-, and deep-in-the) options. Further, we observe that the signs of the specification error biases of the Cox-Ross model are different (except for out-of and most deep-in-the-money options) from those of the estimation error bias of the model (Table–14). For example, the specification error biases are positive for options at the money whereas the estimation error biases are negative for these options. The magnitudes of biases arising independently from errors in specification (Table–27) and estimation (Table–14) are also different from each other. Comparing Tables 14 and 27, it is seen that the specification error biases are greater than the estimation error bias when the options are in, deep–in, and out–of the money. At the money, the options are valued by the Cox–Ross model with small (positive) errors due to misspecification of stock price movements, and that the magnitude of this specification error bias is smaller than the estimation error bias of the model (estimation error biases are negative in this case). This indicates that as long as the options are at the money, the bias arising from the joint effects of errors in estimation and specification in the Cox–Ross model will be smaller than the estimation error bias of the C–R model. This can be seen more clearly in Table–28, where we report the combined effects of errors in specification and estimation on option price estimates generated by the Cox–Ross model.

The biases reported in Table–28 arise when an estimate of the variance is used in the Cox–Ross formula to generate the option price. These biases are obtained as the dollar difference of Cox–Ross estimated option prices over the true prices of options. As noted above, the true price of the option is generated by the B–S formula (with the true value of the variance inserted into the B–S formula).

It is seen in Table–28 that the Cox–Ross model overprices (underprices) in– and deep–in (out–of and some at–) the money money options. The biases in the C–R estimated option price are least when the options are at the money. The above findings conform with the biases empirically observed by Thorp and Gelbaum (1980). That is, the Cox–Ross model overprices in– and deep–in–the money options and it underprices at– and out–of– the money options. But, they contradict the findings of MacBeth–Merville (1980).

MacBeth–Merville found that the Cox–Ross model overpriced (underpriced) out–of–the (in– and deep–in–the) money options.

It is important to note that, even though MacBeth–Merville (1980) found the Cox–Ross model to misprice options in the above manner, they concluded that the Cox–Ross model, as compared to the B–S model, best describes the market price of the option. In some cases, however, they found that the Cox–Ross model mispriced options more than the B–S model . Therefore, the question arises: whether the estimation error bias in the B–S model is sufficiently large in many cases to make MacBeth–Merville pick the Cox–Ross model as correct. We shall see in the following discussion that this is certainly the case when the options are at the money.

In order to examine whether the estimation error bias in the B–S model is large enough to cause the Cox–Ross model to be selected as correct, we compare the overall bias of the Cox–Ross model (Table–28) with the estimation error bias of the Black–Scholes model (Table–3). Comparing Tables 3 and 28 , it is seen that the Cox–Ross model misprices out–of, in–, and deep–in the money options more than the Black–Scholes model (the signs of the biases across the models are different though). However, for options, at the money, the estimation error bias of the B–S model is greater than the overall bias of the C–R model (Table–28). The results thus imply that as long as the options are at the money (most of the options used in empirical tests and traded in the market are at the money), the Cox–Ross model may perform better than the Black–Scholes, even if the Cox–Ross is mispecified. It is possible that this could explain MacBeth–Merville's (1980) findings that the Cox–Ross model performed better than the Black–Scholes model. The implication of the above analysis is that there is a bias towards accepting the Cox–Ross model even when the B–S model is the true model to price options.

Importantly, the above results support our hypothesis that the estimation error bias in the correctly specified model is large enough in some cases to cause the misspecified model to be selected as correct. Since the estimation error biases in the correctly specified model may differ across studies (depending on data set considered and sample size used) and since specification error biases differ across models (depending on the deviation of the assumed process from the true process of stock returns), it is expected that even if the Black-Scholes model is the true model for pricing the option, the empirical validation of the model would be a difficult task, unless we undertake necessary measures to remedy the estimation error biases of the model. Our Monte Carlo results (in Chapters IV and V) suggest that the adjustment of the model estimated price for the effects of the second moment of the estimated variance will reduce the bias of the model by a significant amount and therefore would help researchers provide a better validation test for the B-S and Cox-Ross models. This is because the remaining biases of the models will then exist only due to misspecification of the true underlying process of the stock price. Knowledge of the specification error bias can be used to identify the true model for pricing the option.

In this chapter we have reported the bias of the Black-Scholes and Cox-Ross option pricing models arising from errors in estimation and specification. We have examined the seriousness of the two sources of error on the model estimated option price in order to find out whether the estimation error bias in the correctly specified model is sufficiently large in some cases to cause the misspecified model to be selected as correct . It is found that the specification error bias is more serious for both the models when the options are in, out-of and deep-in the money. At the money, the options are priced by the models with small errors due to misspecification of stock price movements, and that the magnitude of this specification error bias is smaller than the estimation error biases of the models. These results thus suggest that majority of the biases in the model estimated price for options at the money are due to errors in the estimated variance. We have examined whether the estimation error bias in the correctly specified model is large in any case to make researchers pick the misspecified as correct. It has been found that, when the B-S is the true model, the estimation error bias of the B-S model (for options at the money) are greater than the overall bias of the Cox-Ross model, implying that even if the B-S model is the true model for pricing the option, researchers might empirically validate the Cox-Ross model as correct (this is because most of the options traded in the market are at the money). In order to circumvent this model selection problem, our results suggest an adjustment of the model estimated price for the effects of the second moment of the estimated variance. This adjustment gives a more accurate estimate of the option price generated by the relevent model and enables us to identify the true model on the basis of specification error biases of the models.

To put the results in another way, we say that when the B-S model is the true model of option pricing, the estimation error bias of the B-S model corresponds to most widely observed biases of the model. However, we find a bias towards accepting the Cox-Ross model, even if the B-S model is the true model. The bias in favour of the Cox-Ross model is due to the joint effects of errors in estimation and specification on the Cox-Ross estimated option price. The results thus imply that even if the stock price movement is characteristised by a lognormal process , the Cox-Ross model (as against the B-S model) may be selected as the correct model, unless we undertake necessary measures to remedy the estimation error biases of the models. This is because the estimation error bias makes it difficult to diagnose the specification error bias of the Cox-Ross model. Therefore, the present study suggests that the elimination of the variance induced biases is essential in order to provide a proper basis for comparing the performance of the B-S and C-R models to explain the market price of options.

TABLE -23

Biases in the Black-Scholes Estimated Price
due to Errors in Specification and Estimation.*

Bias=Estimated B-S prices-Cox-Ross prices.
Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Times to option's expiration.**

T		1	3	5	7	9	15
X	N						
	140	.[illegible]432	.386087	.546607	.652255	.724605	.82536
40	100	.113811	.380819	.538044	.641329	.711875	.809182
	60	.112321	.367473	.5165	.613895	.679976	.768704
	140	-.008973	-.026458	-.047951	-.07277	-.10032	-.192307
50	100	-.014148	-.035139	-.058785	-.08518	-.11395	-.208237
	60	-.027133	-.056907	-.085979	-.11634	-.1482	-.24826
	140	-.158299	-.41487	-.579422	-.707169	-.814041	-1.05763
60	100	-.15917	-.41973	-.586792	-.716263	-.824478	-1.07059
	60	-.16162	-.43218	-.605453	-.739349	-.850876	-1.1033
	140	-.026016	-.423325	-.854599	-1.21393	-1.51085	-2.13029
80	100	-.025833	-.422379	-.854614	-1.21526	-1.51347	-2.13594
	60	-.025375	-.420471	-.855255	-1.21936	-1.52087	-2.15083

* The Cox-Ross model is assumed to the correctly specified model to generate market prices of options.
** v is the variance of stock returns.

TABLE -24

Biases in the Black-Scholes Estimated Price due to Misspecification of the Stock Price process. *

Exercise price=50, r=0.015, V=0.025(per quarter)
X=Stock price, T=Times to option's expiration.**

T / X	1	3	5	7	9	15
40	.11290	.390421	.555324	.664124	.73887	.844158
50	-.003259	-.016872	-.035995	-.05911	-.08537	-.174957
60	-.15964	-.41208	-.573914	-.699768	-.805252	-1.04628
80	-.026489	-.428253	-.860443	-1.21922	-1.51515	-2.13202

* Absolute diffusion process is assumed to be the true process.
** V is assumed to be true variance of stock returns.

TABLE -25

% Biases in the Black-Scholes Estimated price due to Misspecification of the Stock Price Process. *

%Bias=(|Bias in Table 24|*100)/True option price.
Exercise price=50, r=0.015, V=0.025(per quarter)
X=Stock price, T=Times to option's expiration.**

T / X	1	3	5	7	9	15
40	57.9087	26.7584	19.6253	16.0219	13.695	9.58667
50	.092665	.259013	.411158	.554242	.690114	1.04619
60	1.40779	2.88685	3.4354	3.71801	3.88663	4.09586
80	.086077	1.3047	2.4556	3.2808	3.87027	4.80682

* Absolute diffusion process is assumed to be the true process.
** V_r is assumed to be true variance of stock returns.

TABLE -26

% Biases in the Black-Scholes Estimated Price due to Estimation error in the Variance Estimate*

Exercise price=50, r=0.015, V=0.025(per quarter)
X=Stock price, T=Times to option's expiration.**

T		1	3	5	7	9	15
X	N						
40	140	.727522	.297006	.308046	.286326	.264403	.213478
	100	.466192	.658068	.610666	.54992	.500348	.397208
	60	.297955	1.5728	1.37205	1.2117	1.0916	.85689
50	140	.162495	.147148	.136559	.128087	.120851	.103743
	100	.309641	.280417	.260318	.244451	.231032	.199
	60	.678889	.614578	.57093	.536617	.507904	.43833
60	140	.011824	.019549	.032973	.03932	.042421	.044442
	100	.004146	.053595	.077089	.08764	.092796	.095155
	60	.017459	.0140816	.188796	.210304	.220208	.223224
80	140	.001537	.015015	.016678	.014248	.010991	.003887
	100	.002132	.017897	.016635	.010676	.004287	.008841
	60	.00362	.023708	.014806	.00037	.014616	.042418

* The Cox-Ross model generates true prices of options.
** v is the true variance of stock returns.

TABLE -27

Biases in the Cox-Ross Estimated Price due
to Misspecification of the Stock Price process. *

Exercise price=50, r=0.015, v=0.025(per quarter)
X=Stock price, T=Times to option's expiration.**

T / X	1	3	5	7	9	15
40	-.11290	-.390421	-.555324	-.664124	-.73887	-.844158
50	.003259	.016872	.035995	.05911	.08537	.174957
60	.15964	.41208	.573914	.699768	.805252	1.04628
80	.026489	.428253	.860443	1.21922	1.51515	2.13202

* Lognormal process is assumed to be the true process.
** v is the true variance of stock returns.

TABLE -28

Biases in the Cox-Ross Estimated Price due to Errors in Specification and Estimation.*

Bias=Estimated C-R prices - BLack-Scholes prices.
Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Times to option's expiration.**

T		1	3	5	7	9	15
X	N						
40	140	-.111211	-.393402	-.562424	-.614194	-.751155	-.860668
	100	-.11131	-.397614	-.569772	-.683817	-.762503	-.87542
	60	-.111763	-.408334	-.588296	-.708012	-.791018	-.912397
50	140	-.002429	.007368	.024212	.04573	.07078	.15744
	100	-.00705	-.0001314	.013348	.03327	.05707	.140991
	60	-.020597	-.023126	-.013938	.001969	.02262	.09967
60	140	.16046	.40803	.56691	.690704	.794571	1.03117
	100	.15904	.42201	.5580752	.679901	.782211	1.01514
	60	.15517	.386689	.535751	.652542	.75087	.974762
80	140	.028625	.434097	.864624	1.22099	1.51434	2.12259
	100	.029282	.433695	.861679	1.21539	1.50635	2.10857
	60	.030762	.432861	.853485	1.20055	1.48552	2.07292

* The Black-Scholes model is assumed to the correctly specified model to generate market prices of options.
** v is the variance of stock returns.

CHAPTER VII

SOME EXTENSIONS

In the preceding chapters we examined the estimation and specification error biases of the Black-Scholes model and Absolute version of the Cox-Ross model in order to find out whether the option mispricing by the correctly specified model is large enough to make us validate the "wrong" model as "correct". In this chapter two additional questions are examined. In the first section we examine whether our results can be generalized for other cases of the Cox-Ross model. The Cox-Ross model encompasses the CEV family of processes which depend on the value of the elasticity parameter of the variance. In the second part we describe an ex-post test for the Black-Scholes and Cox-Ross models using simulated options and stock prices. We show how estimation error and specification error impact on tests of this sort.

A. Other Versions of the Cox-Ross Model

Cox (1975) and Cox-Ross (1976) derived the following option pricing formula for a CEV Process (given by equation (3) in Chapter II).

$$C_{CEV} = X \sum_{0}^{\infty} g(\lambda X^{-\theta},\ n+1)\ G(\lambda(Ee^{-rT})^{-\theta},\ n+1-(1/\theta))$$

$$-Ee^{-rT}\sum_{0}^{\infty} g(\lambda X^{-\theta},\ n+1-(1/\theta))G(\lambda(Ee^{-rT})^{-\theta},\ n+1). \qquad (27)$$

where $\theta = 2\rho - 2$, ρ = the elasticity parameter,

$\lambda = 2r/\ (\delta^2\ \theta(e^{\theta rT} -1))$, $\delta^2 = VX^{(2-2\rho)}$.

$g(Z,n) = e^{-Z} Z^{n-1} / \Gamma(n)$ is the gamma density function,

$\Gamma(n) = \int e^{-\upsilon} \upsilon^{n-1}$ is the gamma function and

$G(w,n) = \int g(Z,n)\, dZ$, the complementary gamma distribution function.

The Cox–Ross model we considered was the case when $\rho = 0$. That is when $\rho = 0$, the CEV model reduces to the Absolute (Cox–Ross) model. We chose the Absolute model for two reasons:

(i) Since the CEV process can be used to fit a wide range of processes (including the lognormal), it is impossible to make any comparison between the CEV and Black–Scholes models without restricting the parameter value, ρ, to some extent.

(ii) Most important, a justification for the Absolute model is found in the empirical work of MacBeth–Merville (1980). They found that the Absolute model best fits the empirical data as compared to other models considered.

When $\rho = 1$, the diffusion process is lognormal. As a result, by considering the Absolute ($\rho = 0$) and Black–Scholes ($\rho = 1$) models in our analysis, we examined two extreme cases of the CEV model. Other cases of the CEV model could have been used by restricting ρ between 0 and 1.[1] We did not consider these intermediate cases in our previous discussion. In this section we explicitly consider these cases in order to examine

[1]For values of ρ greater than 1, the CEV process could also be analyzed. Emanuel and MacBeth (1982), however, explain that for mathematical reasons (i.e., for explosive nature of the stochastic process) and because of different boundary behaviour (the density function has a flat upper tail), the analysis is not identical. The analysis is not identical because the option price does not tend to zero as the exercise price tends to infinity (this is a result of the flat tail of the density function). This implies that no matter how large the exercise price, there is some probability that the stock price will exceed the exercise price, and that an option with zero probability of being exercised will have (non zero) positive price. In contrast, for $\rho < 1$ the option will expire worthless. Furthermore, the formula with $\rho > 1$ is different from the formula for $\rho < 1$. Therefore, we may not be able to generalize our analysis for these cases.

whether the other versions of the model would also result in similar kind of biases due to errors in estimation and specification. We found in our preceding analysis that both the Black-Scholes and Absolute models have similar pricing bias characteristics with respect to estimation error in the variance. We used two different functional forms (equations (2) and (4) in Chapter II) for the two models. It could have been possible instead to use the CEV formula (equation (27)) choosing the ρ values equal to 0 and 1 in equation (27). The results, of course, would have been the same.

The question arises whether the estimation error biases for other cases ($0<\rho<1$) of the CEV process would have the same signs as the biases of the Black-Scholes and Absolute models. In order to provide a thorough analysis of this question, it would be necessary to examine the derivatives of the the CEV formula with respect to the variance for different values of the elasticity parameter ρ. That is, first we have to obtain the derivatives of the general CEV model with respect to the variance and then to examine whether the signs of the derivatives change with changes in the elasticity parameter. This examination will shed light on the effects of the moments of the estimated variance on the estimated option price of the CEV model, and will allow us to determine the estimation error biases of different versions of the CEV model (the analysis would be similar to that provided for the B-S and Absolute models in Chapter III). Unfortunately, we could not obtain the derivatives of equation (27) with respect to the variance rate. Failing that, the estimation error biases for another special case of the CEV model has been examined. If the estimation error biases of this model are found to be similar to those of the Absolute and Black-Scholes models, we may be able to say that the other cases of the CEV would also produce similar kinds of biases with respect to the estimation problem of the variance rate.

The model we consider here is the case when $\rho = .5$. When $\rho = .5$, the CEV model becomes the Square-Root model. The Square-Root model is another popular version of the CEV model. In addition, Beckers (1980) has found the square root process to be more characteristic of the actual stock price movement than the lognormal process assumed by Black-Scholes.

The square root model is given by:

$$C_{SR}(w) = A/B, \tag{28}$$

where $A = 1 + h(h-1)((W+2y)/(w+y)^2) - h(h-1)(2-h)(1-3h)((w+ +2y)^2/2(w+4)^4) - ((k/(w+y))^h,$

$B = [(2h^2((w+2y)/(w+y)^2)(1-(1-h)(1-3h)((w+2y)/(w+y)^2)]^{1/2},$

$h(w) = 1-(2/3)(w+y)(w+3y)(w+2y)^{-2},$

$y = 4rX/(VX(1-e^{-rT}))$, $K = 4rE/(VX(e^{rT}-1))$, and w is a parameter which takes on the values 0 or 4.

Table 29 reports the estimation error biases of the Square-Root model for the same set of stock prices (40, 50, 60, 80), quarters to option's expiration (1, 3, 5, 7, 9, 15) and sample sizes (60, 100, 140) considered for the Black-Scholes and Absolute models. These biases are the difference between the estimated and the theoretical option prices generated by the Square-Root model. The estimated option prices are the average Square-Root model option prices with unbiased variance estimates. The variance estimates used in generating these prices are the same as those used in the Black-Scholes and Absolute models.

It is seen in Table 29 that the estimation error biases of the Square-Root model vary in sign and magnitude. Specifically, when $N=140$ the Square-Root model overprices deep-in-the money options (X=80). A few exceptions are seen for long maturity options. The overpricing is also observed for options out- and in-the- money with one quarter to maturity, otherwise the model underprices options. These biases have also changed with changes in the stock price , the option's time to maturity, and the sample size. Importantly, the biases of the Square-Root model are of the same signs as the biases of the Black-Scholes (Table-3) and Absolute (Table-14) models when they are viewed across stock prices, given maturity periods, or across maturity periods, given stock prices. For example, at the money, the options of all maturities are underpriced by the three models. Since these three models are three special cases of the CEV model, and since we have found the models to produce biases of the same signs, we may generalize the results by saying that the estimation error biases for other cases of the CEV model will be of the same signs as the biases of the three models considered.

Nevertheless, the magnitude of biases in the estimated price of the different versions of the CEV model are different except for options at the money. This can clearly be seen by comparing the magnitudes of biases of the Black-Scholes (Table-3), Absolute (Table 14) and Square-Root (Table-29) models. It is seen from these tables that the absolute values of the biases for the Square-Root model are between the magnitudes of the biases of the Black-Scholes and Absolute models. This result is not surprising, rather, this is what we expected. This is because the Square-Root model is an intermediate case of the two extreme versions of the CEV model. This result thus suggests that the magnitudes of biases for other cases of the CEV model will depend on the ρ value to be considered in equation (27). That is, if we choose the ρ value close to zero (one), the magnitudes of the biases for this case would be closer to those of the Absolute (Black-Scholes) model.

We turn next to examine the specification error biases of the Square-Root model. We noted in Chapter II that the variance input to the CEV model is equal to the B-S variance times $X^{2-2\rho}$. This implies that, for a given value of ρ, each time the stock price changes, the CEV model uses a different variance input to price the option. Therefore, the variance input to the CEV model to price out-of-the (in-the) money options is smaller (greater) than the variance input used in pricing options at the money. Since the Black-Scholes model does not allow the variance to change with the stock price, and since the option price is positively related to the variance rate, the price generated by the Black-Scholes model for out-of-the (in-the) money options will be greater (smaller) than the price generated by the CEV model. As a result, the specification eror biases for the CEV model, given that the Black-Scholes is the true model, will be negative , zero, or positive depending on whether the option is out, at, or in the money. In order to find out whether this would be the case we examine the specification error biases of the Square-Root model. These biases are then compared with those of the Absolute model.

Table 30 reports the specification error biases of the Square-Root model. These biases are obtained as the difference between option prices generated by the Square-Root and Black-Scholes models with the true variance inserted in the respective formulas. It is seen in Table 30 that the Square-Root model underprices (overprices) out-of-the-money (all other) options. Importantly, these biases are of the same signs as the biases of the Absolute model (Table 27). Therefore, this result support our assertion that the CEV model will underprice (overprice) out-of-the-money (all other) options when the Black-Scholes model is the true model for pricing the option.

Nevertheless, the magnitudes of the biases are different for the two models. Comparing Tables 27 and 30 we find that the specification error biases of the

Square-Root model are smaller than those of the Absolute model. This result therefore suggest that the greater is the value of ρ the smaller would be the specification error bias of the CEV model. This is because the higher the value of the elasticity parameter the smaller would be the variance input to the CEV model. Since the option price is positively related to the variance, and since the variance input to the absolute model (VX^2) is greater than that of the the Square-Root model (VX), the magnitudes of the specification error biases of the Square-Root model are smaller than those of the Absolute model.

Now we compare the overall biases of the Square-Root model (Table-31) with the estimation error biases of the B-S model (Table-3) in order to examine whether the estimation error bias in the B-S model is large enough to cause the Square-Root model to be selected as correct. Comparing Tables 3 and 31, it is seen that the Square-Root model misprices out-of, in-, and deep-in-the money options more than the Black-Scholes model. However, for options at the money , the estimation error biases of the B-S model are greater than the sum of the estimation and specification error biases of Square-Root model. The results thus indicate that as long as the options are at the money (most of the options used in empirical tests and traded in the market are at the money), the Square-Root model may appear to perform better than the Black-Scholes model even if the Square-Root model is the misspecified model. Since the same result was also found for the Absolute model, this suggests that there is a bias towards accepting the CEV model even when the B-S model is the true model of option valuation.

Up to this point we have examined the estimation and specification error biases of the Black-Scholes and Cox-Ross models in order to examine whether the estimation error bias in the correctly specified model is large enough to cause us to choose the wrong model as correct. As mentioned above, we have found that there is a bias towards validating the Cox-Ross model even when the Black-Scholes is the true model for pricing the option. This ex-ante test for the comparative performance of the two models has been done by comparing the deviations of the models' estimated option prices (obtained by using an estimate of the variance) from the true prices of options. The question remains whether the Cox-Ross model would also demonstrate superior performance in an ex-post test based on the hedged portfolios. This is a test where the information on prices at which options and stocks can be bought or sold is given in advance. The purpose of the following section is to show how an ex-post test for the Black-Scholes and Cox-Ross models can be done by using simulated options and stock prices.

B. Expost Tests Based on the Hedged Portfolios

Black and Scholes (1972) developed a technique to measure the ex-post performance of the Black-Scholes model. This technique measures the ex-post risk adjusted return to option positions, and has been used by Blomeyer-Klemkosky (1983) in order to compare the ex-post performance of the Black-Scholes and Roll models. They argue that if a model identifies the equilibrium option pricing process, the model could be used to identify options that are over or underpriced in the market. Since the market forces cause the mispriced option to return to its equilibrium price, a trading strategy using the pricing model can be employed to develop an appropriate short or long position in the mispriced option with a resultant positive mean holding-period return. They argue that if one model can consistently develop higher mean holding-period returns, then that model would demonstrate superior ex-post performance.

Blomeyer and Klemkosky calculate the excess return on a hedge (established independently by the Black–Scholes and Roll models) held from time t until t+1 for an initially underpriced option (the model price is greater than the market price) by:

$$R_H = C_{t+1}^{A} - C_t^{A} - (\partial C_t^{M}/\partial X)(X_{t+1} - X_t)$$

$$- (C_t^{A} - (\partial C_t^{M}/\partial X)X)\ r\Delta t \qquad (29)$$

C_{t+1}^{A} and C_t^{A} = the option prices in the market at time t+1 and t respectively.

X_{t+1} and X_t = the stock prices at time t+1 and t respectively.

$\partial C_t^{M}/\partial X$ = the hedge ratio for the model used to establish a hedge position.

r = the risk-free interest rate, and Δt = the holding period of the option position. Note that, for an initially overvalued call option the excess return is simply $-R_H$.

It is implicit in the Blomeyer–Klemkosky analysis that the hedge is correctly established. However, if the Black–Scholes model is correct (assuming that the true variance is known), then the hedge established by the Roll model must be incorrect (this is because the Roll model is the misspecified model). This implies that the excess mean holding-period returns from an incorrectly established hedge is no longer the true excess returns. This is because the hedge ratio (the first derivative of the model with respect to X) is biased due to using the misspecified model, and this biased hedge does not give the riskless return on the model investment. Therefore, even if the Roll model is found to generate higher excess returns than the Black–Scholes, we can not claim that the Roll model has demonstrated superior ex-post performance. The only conclusion we can make is that the higher returns for the Roll model are due to higher risks associated with the hedge established by the wrong model. To put this in another way, we can say that the difference between the excess returns generated by the two models is due to misspecification of the Roll model (assuming that the true variance of stock returns has been used in establishing the hedge). [2] Therefore, the excess mean-holding-period returns can be considered as a measure of option mispricing by the Roll model when the Black-Scholes model is the true model of option pricing. This makes sense because if the market prices are the B-S model prices (estimated with the true variance), the excess returns from the hedge position established by Black-Scholes model (with the true variance) will be equal to zero. For the Roll model the excess returns should be different from zero.

Following the above line of argument, we show below how the ex-post risk-adjusted returns to a hedge position can be used to examine the comparative performance of the Black-Scholes and Cox-Ross models.

To begin with, assume that the option in the market is priced by the Black-Scholes model. By assumption, the market price is equal to the B-S model price (obtained by inserting the true variance in the formula). Suppose, however, that the true variance of stock returns is not known to an investor and therefore, he obtains the B-S model price with an estimate of the variance. If he finds that the market price is higher (lower) than the estimated model price, it would not imply that the option is overpriced (underpriced) in the market, rather, it would imply that the model has underpriced (overpriced) options due to estimation error in the variance. Suppose, however, that the

[2] Note that if the true variance is unknown, and if an estimate of the variance is used in establishing the hedge, the hedge ratio will biased for the two models. In that case, the excess return generated by the Cox-Ross model would be due to the combined effects of errors in estimation and specification, and for the Black-Scholes model, the excess return would result from errors in estimation (even if the B-S model is the true model for pricing the option). We discuss this point in details in the rest of this chapter.

investor is not aware of this problem. He acts as if the option is mispriced in the market and employs a trading strategy using the estimated model price. If the model price is higher (lower) than the market price, he takes a long (short) position in the option with a corresponding short (long) position in the stock. Suppose that he maintains the hedge position (the hedge is established at the model price) until the next option transaction at which it is closed at the new transaction prices and calculates the mean-holding-period excess returns using equation (29). If the excess return thus calculated is found to be positive, it does not mean that the model has correctly identified over or underpriced options, rather, the excess return has resulted from the incorrectly established hedge based on the estimated model price. This is because, if the hedge was established at the theoretical (true) option 'price of the model, the excess return would be equal to zero. Thus, we can say that the calculated excess return for the Black-Scholes model is an outcome of errors in the estimated variance, and that the excess return provides a measure of option mispricing by the Black-Scholes model on an ex-post basis. The closer the excess return to zero the better would be the performance of the Black-Scholes model when explaining the market prices of options.

Similarly, the excess return from the hedge position established at the Cox-Ross model price (with an estimate of the variance) would provide a measure of option mispricing by the Cox-Ross model. This mispricing of options by the Cox-Ross model would be due to errors in estimation and specification, if the Black-Scholes model is the true model for pricing the option. If the Cox-Ross model generates returns smaller than the Black-Scholes model, then we can say that the Cox-Ross model has demonstrated superior ex-post performance even if the Cox-Ross model is misspecified.

Since our ex-ante tests results suggest that the estimation error biases in the Black-Scholes model are sometimes large enough to cause the Cox-Ross model to be selected as correct, we shall expect that the excess returns generated from the Cox-Ross model will sometimes be smaller than that of the Black-Scholes model should the Cox-Ross model demonstrate superior ex-post performance. We show below how an ex-post test for the Black-Scholes and Cox-Ross models can be done by using simulated options and stock prices.

To begin with, assume that the stock price movement is characterized by a lognormal process (equation 1 in Chapter II). In other words, the Black-Scholes model is the true model for pricing the option. The option price in the market at time t will then be equal to the theoretical Black-Scholes model price, say C_t^A, obtained by inserting among other parameters, an arbitrarily selected stock price , X_t, and the true variance of stock returns in the formula. Assume that this theoretical model price is the first transaction price of the day, t, for a particular option. Now we calculate the model price of the option using an estimate of the variance and the stock price X_t, and compare it to the market price. If the model price is lower (higher) than the market price, we take a short (long) position in the option (at the market price) with a corresponding long (short) position in the stock. The investment in the hedge is however determined by the estimated model price, which is given by :

$$V_{H1} = C_t^A - (\partial C^M_t/\partial X)X, \tag{30}$$

if the market price is lower than the estimated model price, and $-V_{H1}$, if the market price is higher than the estimated model price. The hedge position is then maintained until time t+1 at which it is closed out at the new transaction prices. We assume that the stock price at time t+1 is X_{t-1}, which can be generated based on the lognormal diffusion process. This is because, as noted above, the stock price follows a lognormal process.

Following Boyle (1976), we can obtain

$$X_{t+1} = X_t e^{(r-(V/2)+VZ)}, \tag{31}$$

where Z is the weiner process with zero mean and unit variance. The market price of option for time t+1 can then obtained by inserting the true variance and X_{t+1} in the Black-Scholes formula. The excess return on the hedge held from time t to t+1 for the option overpriced (underpriced) by the model is then calculated by R_H ($-R_H$). The above procedure is repeated across all transactions on all options written on a particular stock, and the mean-holding-period returns are found for the Black-Scholes model. If the mean-holding- period returns are positive, we will conclude that the Black-Scholes model underpriced options relative to the market. This underpricing of options by the B-S model is due to estimation error in the variance.

Following the above procedures, the mean-holding-period excess returns for the Cox-Ross model can also be obtained. The only difference in this case is that the investment in the hedge (equation 30) will be determined by the Cox-Ross estimated hedge ratio (the first derivative of the model with respect to X). Since the hedge ratios for the two models are different, the risks associated with the hedge positions for the two models should be different. Therefore, the excess returns generated from the hedge positions for the two models will be different. If the Cox-Ross model is found to develop lower excess returns, we would conclude that the Cox-Ross model has shown superior performance on an ex-post basis.

In this chapter we have examined the estimation and specification error biases of the Square-Root model in order to find out whether the results obtained in the preceding chapters can be generalized for the CEV model. We have found that the Square-Root model produces biases similar to those of the Black-Scholes and Absolute models. Importantly, it is seen that there is a bias towards validating the Square-Root model even when the Black-Scholes model is the true model of option pricing. We have discussed the generality of this result for other cases of the Cox-Ross model and found that the estimation error bias in the Black-Scholes model is sometimes large enough to cause the Cox-Ross model to be selected as correct, even if the Cox-Ross model is misspecified. In this chapter we have also discussed how an ex-post test for the Black-Scholes and Cox-Ross models can be done by using simulated options and stock prices. We have shown analytically that the closer is the mean-holding-period returns to zero the better would be the performance of the models on an ex-post basis.

TABLE - 29

Biases Observed in the Square Root Model
with Unbiased Variance Estimates

Bias=Estimated option price - True option price

Exercise price=50, r=0.015, V=0.025 (per quarter)
X=Stock price, T=Option's time to expiration.*

T		1	3	5	7	9	15
X	N						
40	140	0.00157	-0.00354	-0.00771	-.010641	-0.012848	-.017213
	100	0.001288	-0.008247	-0.015601	-.020822	-.024746	-.03264
	60	0.000359	-0.020195	-0.035473	-.046428	-.054811	-.071422
50	140	-.005709	-.009472	-.011722	-.013249	-.01388	-.017059
	100	-.010881	-.018148	-.022548	-.02564	-.02753	-.033279
	60	-.023866	-.039902	-.043221	-.056829	-.06185	-.07402
60	140	.0005	-.003361	-.006119	-.008087	-.009308	-.012756
	100	-.000565	-.00877	-.01416	-.018036	-.020691	-.027115
	60	-.00376	-.02264	-.034622	-.043243	-.04953	-.0634
80	140	.00119	.005722	.006622	.003983	-.002106	-.002304
	100	.001602	.006348	.005402	.00702	-.002899	-.011368
	60	.002533	.007263	-.001587	-.008026	-.016205	-.034882

* v is assumed to be the true variance of stock returns.
N = sample observations.

TABLE -30

Biases in the Square Root Model Estimated Price due to Misspecification of the Stock Price process. *

Exercise price=50, r=0.015, v=0.025(per quarter)
X=Stock price, T=Times to option's expiration.**

T / X	1	3	5	7	9	15
40	-.061164	-.205362	-.287868	-.33735	-.36632	-.385917
50	.001221	.009281	.023824	.04285	.06394	.129669
60	.076651	.20191	.285553	.352356	.408188	.530075
80	.007431	.170593	.36293	.527557	.661667	.932022

* Lognormal process is assumed to be the true process.
** v is the true variance of stock returns.

TABLE -31

Biases in the Square Root Model Estimated Price due to Errors in Specification and Estimation.*

Bias=Estimated C-R prices - BLack-Scholes prices.
Exercise price=50, r=0.015, v=0.025 (per quarter)
X=Stock price, T=Times to option's expiration.**

X \ T	N	1	3	5	7	9	15
40	140	-.059588	-.208902	-.295583	-.348092	-.379168	-.40313
	100	-.059876	-.21361	-.303469	-.358273	-.391066	-.418556
	60	-.060805	-.225557	-.323346	-.383879	-.42113	-.457338
50	140	-.004488	.000191	.012102	.0296	.05006	.11261
	100	-.009661	-.008867	.001277	.01721	.03641	.09639
	60	-.022645	-.030621	-.019397	.01398	.00209	.055649
60	140	.07715	.198549	.279434	.344269	.398888	.517319
	100	.071	.19314	.271393	.33432	.357497	.50216
	60	.07289	.17927	.250931	.30113	.358658	.466675
80	140	.008621	.176315	.369553	.53154	.663713	.929718
	100	.009033	.176941	.368332	.528259	.658768	.920654
	60	.009964	.177856	.364516	.519531	.645462	.897141

* The Black-Scholes model is assumed to the correctly specified model to generate market prices of options.
** v is the variance of stock returns.

CHAPTER VIII

SUMMARY AND CONCLUSIONS

Option pricing models have been the focus of considerable research in the finance literature over the past decade. Since the seminal paper of Black–Scholes was published in 1973, a number of alternative option pricing models have been developed. A great deal of research has been done to validate these competing models empirically. A major outcome of the empirical tests is that there is no compelling evidence in favour of any single model including the Black–Scholes. We recognise that the lack of such evidence is due to interaction of biases arising from model misspecification and errors in estimation of the variance of stock returns, and that a valid test for competing models requires a simultaneous yet separate examination of these two sources of biases. However, no attempts have previously been made to carry out such a test. The objective of this thesis is to fill this vacuum.

We have identified and examined the estimation and specification error biases of the Black–Scholes and Cox–Ross models by expanding the relevant formulas in a Taylor series. Estimation error biases have been related to the moments of the estimated variance and the derivatives of the models with respect to the variance of stock returns. Specification error biases have been related to the alternative option valuation formulas themselves. Both these sources of biases were then analysed using both analytical and Monte Carlo simulation techniques.

First, we have analysed pricing bias characteristics of the models arising from the estimation problem of the variance of stock returns. Existing studies which examine estimation error related problem have only been concerned with the B–S model (e.g., Boyle–Ananthanarayanan (1977) and Butler–Schachter (1983, 1986)). The variance induced pricing bias characteristics of the Cox-Ross model have not been examined. By incorporating such an analysis in the present study, we have examined the issue of whether a particular model, regardless of its specification, will perform more poorly than the other one simply because the model is relatively more sensitive to errors in the variance estimate. The results reveal that to the extent that the bias in the estimated option price is related to estimation error in the sample variance, the Cox-Ross model will produce greater biases for out-of-the and deep-in-the money options as compared to the B-S model simply because the Cox-Ross model is relatively more sensitive to errors in the estimated variance for these types of options. Similarly, the bias produced by the B-S model is greater when the option is in-the money. Importantly, we have found that when the options are at the money, the estimation error in the sample variance induces the same bias for the two models , even though the models differ in functional form. The major implication of this result is that as long as the option is at the money , the estimation error bias of the two models provides the same information about the mispriced options.

Further, we have found that in both models the majority of the estimation error bias is due to the second moment of the estimated variance. To provide a better comparison of the ability of the models in predicting market prices of options, the results thus suggest an adjustment of the models' estimated prices for the effects of the second moment of the sample variance.

We have considered the effects of misspecification of the stock price process on the estimated option price. The object of this consideration has been to examine whether a model based on different specifications of stock return distributions yields a significantly different option price as compared to the price which would prevail under the true process. It has been found that when the options are near the money, the bias arising from misspecification of the stock price process is virtually nil. In other words, both the B-S and C-R models with the true variance of stock returns give the same option price estimates when the options are near the money. Otherwise, the option prices generated by the two models are different. That is, specification error biases have been found to be serious when the options are in, deep-in, and out-of the money. These results, however, have been found to be consistent with the findings of MacBeth-Merville (1980) and Beckers (1980).

Finally, the combined effects of estimation error and of process misspecification on the option price generated by the B-S and C-R models have been examined. In particular, we have examined the seriousness of the above two sources of bias on the model estimated option price in order to find out whether the estimation error bias in the correctly specified model is large enough in some cases to make us pick the "misspecified" model as correct . It has been found that the specification error bias, as compared to the estimation error bias, is more serious for both the models when the options are in, deep-in, and out-of-the money. However, the majority of the biases in the estimated price for options at the money are due to errors in the estimated variance. We have examined whether the estimation error bias in the correctly specified model is large enough to cause researchers to validate the misspecified model as correct. It has been found that, when the B-S model is the true model, the estimation error bias of the B-S model for options at the money are greater than the overall bias of the Cox-Ross model, implying that even if the B-S model is the true model for pricing the option, there is a bias towards accepting the Cox-Ross as correct (this is because most of the options used in empirical tests and traded in the market are at the money). Similar results have also been found for another case of the CEV model. Then, we have discussed how estimation error and specification error affect the tests of the Black-Scholes and Cox-Ross models on an ex-post basis. It has been shown there that if the excess returns on the hedge position established by the Cox-Ross model is smaller than those of the Black-Scholes, the Cox-Ross model would demonstrate superior ex-post performance even when the Black-Scholes model is the true model for pricing the option.

In order to circumvent this model selection problem, the results of this study suggest an adjustment of the models' estimated prices for the effects of the second moment of the estimated variance. This adjustment reduces (eliminates) the estimation error bias of the models by a significant amount and enables us to identify the true model on the basis of specification error biases of the models. This adjustment, also, helps improve investors' performance by reducing the risk of misidentifying under or overpriced options.

However, the adjustment of the model estimates for the effects of the second moment of the estimated variance deals with only one of many data problems which exist in any attempt to validate alternative models. The other well-known data problems such as the bid-ask spread and nonsynchronous data should also be dealt with in order to provide a better validation test of the models. Nevertheless, measures to remedy the bias arising from estimation error in the sample variance should be undertaken in order to provide a better validation test for the models.

Several studies have used ISDs (an estimate of the variance rate implied in market prices of options) to mitigate the concern about the bias arising from the estimation problem of the variance. Unfortunately, ISDs could not solve this problem. This is because (i) the ISD is still an estimate and not the true standard deviation which is impossible to infer except by chance (Butler-Schachter (1983a)), and (ii) the ISD can not be equal to the market's estimated standard deviation, and it is in fact a biased estimate of the true standard deviation (cf. Butler-Schachter (1983a)). In fact, the ISD estimates have been found to produce, in some cases, greater biases in the Black-Scholes model than the historical estimates (Butlter-Schachter, 1985). As a result, whether the historical estimates of the variance or the ISDs are used in testing the comparative performance of various option pricing models, some measures to remedy the bias arising from estimation error of the variance should be necessary in order to provide a better validation test of the models.

The present work may be extended in several ways. One immediate extension of the work would be to consider several other option pricing models under the banner of our analysis. A direct test of the models would also be a worthwhile task, if we want to look at the implications of our simulation results for validating alternative option pricing models. Since we have compared the theoretical and estimated model prices of option, and since empirical works deal with the market and the model estimated option prices, the results of the empirical tests should be interpreated with proper caution. This is because the other sources of error may cause the models to misprice options in a different way.

Further, we note that we have investigated the effects of estimation error of a constant variance of stock returns on the model estimated option price. Hence, a parallel problem dealing with estimation error of a changing variance of stock returns (Gennotte (1985), Scott (1986), Melville-Pieptea (1985)) along with the effects of errors in specification can be explored. The analysis of this problem is expected to be more complicated than that presented here.

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