# AN APPLICATION OF NON-STANDARD MODEL THEORETIC METHODS TO TOPOLOGICAL GROUPS AND INFINITE GALOIS THEORY

by

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# ABSTRACT

The purpose of this paper is to review some of the work done by Abraham Robinson in topological groups and infinite Galois Theory using ultrapowers as our method of obtaining non-standard models. Chapter One contains the basic logical foundations needed for the study of Non-Standard Analysis by the method of constructing ultrapowers.

In Chapter Two, we look at non-standard models of topological groups and give the characterizations of some standard properties in non-standard terms. We also investigate a non-standard property that has no direct standard counterpart. In Chapter Three, we analyze an infinite field extension of a given field $F$ and arrive at the correspondence between the subfields of our infinite field that are extensions of $F$ and the subgroups of the corresponding Galois group through the Krull topology by non-standard methods.

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# INTRODUCTION

In 1961, Abraham Robinson pointed out that the methods available in contemporary mathematical logic were sufficient to construct a theory of analysis with infinitely large and infinitely small numbers. The resulting theory is termed Non-Standard Analysis. To date, such topics as topology, real and complex analysis, field theory and class field theory have been investigated by non-standard methods.

The purpose of this paper is to review some of the work done by Abraham Robinson in topological groups and infinite Galois Theory using ultrapowers as our method of obtaining non-standard models. Chapter One contains the basic logical foundations needed for the study of Non-Standard Analysis by the method of constructing ultrapowers.

In Chapter Two, we look at non-standard models of topological groups and give the characterizations of some standard properties in non-standard terms. We also investigate a non-standard property that has no direct standard counterpart. In Chapter Three, we analyze an infinite field extension of a given field $F$ and arrive at the correspondence between the subfields of our infinite field extension that are extensions of $F$ and the subgroups of the corresponding Galois group through the Krull topology by non-standard methods.

We shall assume throughout the paper a basic knowledge of topology, group theory and finite Galois Theory. The Axiom of Choice is tacitly assumed throughout the paper as it is fundamental to the ultrapower construction.

# CHAPTER I

# FOUNDATIONS

The formal system in which we work consists on the one hand of a formal language $\mathcal{K}$ and on the other our notions of satisfiability and truth in $\mathcal{K}$. As we shall be restricting ourselves to the '$\in$' relation, we shall incorporate into our language $\mathcal{K}$ a simple theory of types.

Definition 1

We begin with the number 0.

i) 0 is a type.

ii) If $\tau_1, \ldots, \tau_n$ are types, so is $(\tau_1, \ldots, \tau_n)$.

Let $\mathbb{T}$ be the smallest set satisfying i and ii. $\mathbb{T}$ is called the set of types.

We make the following natural correspondence between the objects connected with a non-empty set $\mathbb{X}$ and $\mathbb{T}$: the elements of $\mathbb{X}$ are of type 0 in $\mathbb{X}$. If $\mathbb{Y} = \{y_1, \ldots, y_n, \ldots\}$ is a collection of objects of type $\tau$ in $\mathbb{X}$, then $\mathbb{Y}$ is of type $(\tau)$ in $\mathbb{X}$. If $x_1, \ldots, x_n$ are objects of type $\tau_1, \ldots, \tau_n$ in $\mathbb{X}$ respectively, then $\langle x_1, \ldots, x_n \rangle$ is of type $(\tau_1, \ldots, \tau_n)$ in $\mathbb{X}$. We shall only consider objects of a definite type in $\mathbb{X}$.

Our formal language $\mathcal{K}$ shall consist of the following symbols:

i) $,\ (\ )\ \neg\ \rightarrow\ =\ \{\ \}\ \langle\ \rangle$ (read comma, parentheses, negation, implication, equality, set brackets, pointed brackets)

ii) $x_1, x_2 \ldots$ (individual variables)

iii) $a_1, a_2 \ldots$ (individual constants)

iv) $\epsilon$ (epsilon relation)

v) $(\forall\ )$ (universal quantifier)

vi) for each $\tau \epsilon \mathbb{T}$, the symbol $\mathbb{T}_\tau(\ )$ (type predicate)

The number of constants in our language $\mathcal{K}$ shall vary with respect to any particular theory we wish to study within $\mathcal{K}$. The number of constants shall be large enough to put into one-to-one correspondence[1] with any desired ultrapower of a model of the theory. This will be made precise once ultrapowers are defined.

We shall call any finite sequence of symbols an expression. We do not wish to consider all possible expressions but just those expressions which are formed in a regular manner. To this end, we make the following definitions.

Definition 2

Individual variables and individual constants are terms. If $t_1, \ldots, t_n$ are terms, so is $\langle t_1, \ldots, t_n \rangle$.

There are no other terms.

Definition 3

i) If $t_1$ and $t_2$ are terms, then $t_1 \in t_2$ is an atomic formula.

ii) If $t_1$ and $t_2$ are terms, then $t_1 = t_2$ is an atomic formula.

iii) If $t$ is a term, then for any $\sigma \in \Pi$, $\Pi_\sigma(t)$ is an atomic formula.

There are no other atomic formulas.

The group of expressions we wish to consider is embodied in the following definition.

Definition 4

i) If $\mathcal{A}$ is an atomic formula, then $\mathcal{A}$ is a well-formed-formula (WFF).

ii) If $\mathcal{A}$ and $\mathcal{B}$ are WFF's, so are $(\neg\mathcal{A})$, $(\mathcal{A} \Rightarrow \mathcal{B})$, and $((\forall x)\mathcal{A})$ for any individual variable $x$.

There are no other WFF's.

We shall adopt the standard convention for the omission of parentheses (see [5]). In a WFF $\mathcal{A}$, the occurrence of a variable $x$ is bound iff it is either the variable of the quantifier $(\forall x)$ in the WFF $\mathcal{A}$ or is under the scope of the quantifier $(\forall x)$ in the WFF $\mathcal{A}$, whereby scope we mean that the WFF $\mathcal{B}$ is the scope of $(\forall x)$ in $((\forall x)\mathcal{B})$. Otherwise $x$ is said to be free.

Definition 5

A WFF in $\mathcal{K}$ is said to be a sentence iff every variable in the WFF is bound.

Definition 6

A mathematical structure in $\mathcal{K}$ is a non-empty set $A$ called the domain together with an assignment to each individual constant[2] in $\mathcal{K}$ an object of finite type in $A$. We denote the structure by $\mathfrak{A}$.

We are now in a position to discuss whether a WFF $\mathcal{A}$ in our language $\mathcal{K}$ is satisfiable or true in a given mathematical structure $\mathfrak{A}$. To this end, let $\Sigma(A)$ be the collection of denumerable sequences of objects of arbitary finite type in $A$, the domain of $\mathfrak{A}$. Let $A'$ be the collection of objects of arbitrary finite type in $A$. Let $T$ be the set of terms of $\mathcal{K}$. Let $s=(s_1,s_2,\dots)\in\Sigma(A)$. Define $\hat{s}: T \to A'$ dependent on $s$ by[3]

i) if $t$ is $x_i$, $\hat{s}(t)=s_i$ ;

ii) if $t$ is an individual constant, let $\hat{s}(t)$ be that member of $A'$ assigned to $t$.[4]

Definition 7

i) if an atomic formula is of the form $t_1 \in t_2$ , then $s \in \Sigma(A)$ satisfies $t_1 \in t_2$ iff $\hat{s}(t_1) \in \hat{s}(t_2)$;

ii) if an atomic formula is of the form $t_1 = t_2$, then $s \in \Sigma(A)$ satisfies $t_1 = t_2$ iff $\hat{s}(t_1) = \hat{s}(t_2)$;

iii) if an atomic formula is of the form $\mathbb{T}_\sigma(t)$. then $s \in \Sigma(A)$ satisfies $\mathbb{T}_\sigma(t)$ iff $\hat{s}(t)$ is of type $\sigma$ in $A$;

iv) for any WFF $\mathcal{Q}$ in $\mathcal{K}$, $s \in \Sigma(A)$ satisfies $\neg\mathcal{Q}$ iff $s$ does not satisfy $\mathcal{Q}$;

v) for any WFF's $\mathcal{Q}$ and $\mathcal{B}$ in $\mathcal{K}$, $s \in \Sigma(A)$ satisfies $\mathcal{Q} \Rightarrow \mathcal{B}$ iff $s$ satisfies $\mathcal{B}$ or $s$ does not satisfy $\mathcal{Q}$;

vi) for any WFF $\mathcal{Q}$ in $\mathcal{K}$, $s \in \Sigma(A)$ satisfies $(\forall x_i)\mathcal{Q}$ iff for each sequence $s'$ in $\Sigma(A)$ differing from $s$ in at most the $i^{th}$ place, $s'$ satisfies $\mathcal{Q}$.

Definition 8

A WFF is true in a mathematical structure $\mathfrak{A}$ iff every sequence in $\Sigma(A)$ satisfies it. A WFF is false in a mathematical structure $\mathfrak{A}$ iff no sequence in $\Sigma(A)$ satisfies it.

Definition 9

A mathematical structure is a model of a set of WFF's $\Gamma$ iff every WFF in $\Gamma$ is true in the given structure.

Our interest lies in various models of a mathematical theory. A mathematical theory consists of the formal language $\mathcal{K}$ together with a schema of logical axioms, axioms for our type theory, and any other axioms that are particular to the theory in question, e.g., group theory, field theory, etc. For all theories, we adopt the usual rules of inference and the standard definitions of proof in a theory.

Definition 10

A model of a theory $\mathcal{T}$ is a mathematical structure in which all axioms of $\mathcal{T}$ are true.

Definition 11

A WFF $\mathcal{a}$ in $\mathcal{K}$ is valid in a theory $\mathcal{T}$ iff [5] is true in all models of $\mathcal{T}$.

We have seen that every model of a mathematical theory $\mathcal{T}$ is a mathematical structure. Now given any mathematical structure $\mathfrak{A}$, $\mathfrak{A}$ is a model of the theory $\mathcal{T}_{\mathfrak{A}}$ whose axioms are just those statements which are true in $\mathfrak{A}$. Hence se[6] may freely interchange the terms model and mathematical structure as they are, in the above sense, equivalent.

We know that we may define abstract mathematical relations in terms of the '$\in$' relation alone.[7] Often our interests lie in investigating some of these relations within a given mathematical structure. Hence we may designate certain of the assigned constants in our structure as relations. Using the axiom of choice, we can well-order them in some manner, placing them in one-to-one correspondence with all of the ordinals less than some initial ordinal, say $\rho$. So we may now write our structure $\mathfrak{A}$ as $\mathfrak{A} = \langle A, R_1, \ldots, R_\alpha, \ldots \rangle$ for all $\alpha < \rho$. We call $\rho$ the order of $\mathfrak{A}$.

Given two structures $\mathfrak{A} = \langle A, R_1, \ldots, R_\alpha, \ldots \rangle$ and $\mathfrak{B} = \langle B, S_1, \ldots, S_\beta, \ldots \rangle$, we call $\mathfrak{A}$ and $\mathfrak{B}$ similar iff they have the same order, $\rho$, and for all $\gamma < \rho$, $R_\gamma$ and $S_\gamma$ are of the same type. $\mathfrak{A}$ is an extension of $\mathfrak{B}$ iff they are similar and $A \supseteq B$ and for $R_\gamma$ a relation of type $\sigma$ in $A$, if $R_\gamma$ is restricted to the objects of type $\sigma$ in $B$, denoted $R_\gamma|_B$, we have $R_\gamma|_B = S_\gamma$.

Now given a mathematical structure, we are interested in constructing non-standard models of the structure. A non-standard model of a structure is a generalized model--a model of the theory of the structure in which quantification is interpreted as limited to a particular subcollection of the objects of finite type, known as internal objects, such that the objects of any finite type in the original structure are members of the collection of internal objects, and such that any mathematical notion definable in the original structure is defined in the new model and that any mathematical statement true in the original structure is true in the new model with the quantification restricted to the internal objects only. There are various methods for obtaining non-standard models. The method we shall use is the ultrapower construction. We shall identify precisely those objects which are selected to be internal and give a relatively simple counterexample to show why all of the objects of finite type in a particular non-standard model can not be considered as internal.

For our construction, we need the purely set theoretic notion of a filter.

Definition 12

Let $I$ be a non-empty set. A filter $\mathbb{\Delta}$ on $I$ is a non-empty set of subsets of $I$ satisfying the following conditions:

i) $\phi \notin \mathbb{\Delta}$

ii) $\Delta \in \mathbb{\Delta}$ and $I \supseteq \Omega \supseteq \Delta$ implies $\Omega \in \mathbb{\Delta}$

iii) $\Delta_1$ and $\Delta_2$ in $\mathbb{\Delta}$ implies $\Delta_1 \cap \Delta_2 \in \mathbb{\Delta}$.

Theorem 1

Given any filter $\mathbb{\Delta}$ on $I$, there exists a

maximal filter $\Gamma$ on $I$ that contains $\boldsymbol{\Delta}$.
Such maximal filters are called ultrafilters.

Proof:

The proof is a straight forward application of Zorn's Lemma to the collection of all filters on $I$ that contain $\boldsymbol{\Delta}$ and is left to the reader.

$\square$

Perhaps one of the most useful of all theorems on ultrafilters is the following:

Theorem 2

Let $\boldsymbol{\Delta}$ be any ultrafilter on $I$. Then for any $\Delta \subseteq I$, either $\Delta \in \boldsymbol{\Delta}$ or $\complement\Delta = \{i \mid i \in I \text{ and } i \notin \Delta\} \in \boldsymbol{\Delta}$.

Proof:

Let $\Delta \subseteq I$ and suppose $\Delta \notin \boldsymbol{\Delta}$. Then $\Delta \neq I$ and furthermore no subset of $\Delta$ is in $\boldsymbol{\Delta}$ due to the closure under supersets. Hence any $B \in \boldsymbol{\Delta}$ is such that $B \cap \complement\Delta \neq \emptyset$. So let $\boldsymbol{\Delta}'$ be that collection of subsets of $I$ that are supersets of sets of the form $B \cap \complement\Delta$ for each $B \in \boldsymbol{\Delta}$. Straight forward checking shows that $\boldsymbol{\Delta}'$ is a filter on $I$. Clearly $\boldsymbol{\Delta}'$ contains each $B \in \boldsymbol{\Delta}$, hence $\boldsymbol{\Delta}' \supseteq \boldsymbol{\Delta}$. But $\boldsymbol{\Delta}$ is maximal, hence $\boldsymbol{\Delta} \supseteq \boldsymbol{\Delta}'$. Thus $\complement\Delta \in \boldsymbol{\Delta}$.

$\square$

Definition 13

An ultrafilter is called principal iff it consists of all supersets of $\{i_0\}$, for some $i_0 \in I$. Otherwise the ultrafilter is called non-principal.

Now let $\{\mathfrak{A}_i \mid i \in I\}$ be a non-empty family of similar structures, say $\mathfrak{A}_i = \langle A_i, R_1^i, \ldots, R_\alpha^i, \ldots\rangle$. Let $\Delta$ be a filter on $I$.

Definition 14[8]

The reduced direct product of the family of mathematical structures $\{\mathfrak{A}_i \mid i \in I\}$ relative to $\Delta$ is

$$\prod_{i\in I} \mathfrak{A}_i/_\Delta = \langle \prod_{i\in I} A_i/_\Delta, R_1, \ldots, R_\alpha, \ldots\rangle$$

where the domain is the set of equivalence classes $f/_\Delta$ where $f$ is a function on $I$ with $f(i) \in A_i$ and where the equivalence relation is $\equiv_\Delta$ defined by $f \equiv_\Delta g$ iff $\{i \mid f(i) = g(i)\} \in \Delta$. We say that $R_\alpha = \prod_{i\in I} R_\alpha^i/_\Delta$ is of type $\sigma$ in $\prod_{i\in I} A_i/_\Delta$ iff $\{i \mid R_\alpha^i$ is of type $\sigma$ in $A_i\} \in \Delta$ (whereby $\prod_{i\in I}(\langle t_1, \ldots, t_n\rangle_i)/_\Delta$ we mean $\langle \prod_{i\in I} t_{1_i}/_\Delta, \ldots, \prod_{i\in I} t_{n_i}/_\Delta\rangle$). We define $\prod_{i\in I} t_i/_\Delta \in \prod_{i\in I} t_i'/_\Delta$ iff $\{i \mid t_i \in t_i'\} \in \Delta$.

If $\Delta$ is an ultrafilter, we call the reduced direct product an ultraproduct. If, in addition, $\mathfrak{A}_i = \mathfrak{A}$ for each $i \in I$, we call $\prod_{i\in I} \mathfrak{A}_i/_\Delta = \mathfrak{A}^I/_\Delta$ an ultrapower.

It is straight forward to show that $\mathfrak{A}^I/_\Delta \cong^9 \mathfrak{A}$ if $\Delta$ is principal or if $I$ is finite or if $\mathfrak{A}$ is finite. However, if $I$ and $\mathfrak{A}$ are at least countably infinite and if $\Delta$ is non-principal, then $\mathfrak{A}^I/_\Delta$ has cardinality at least $2^{\aleph_0}$. For further details, see [11].

In any case, there is a natural embedding of $\mathfrak{A}$ in $\mathfrak{A}^I/_\Delta$, namely for $a \in A$, let $\varphi(a)$ denote the equivalence class to which the function $f$ on $I$ with $f(i) = a$, for all $i \in I$, belongs. By a standard point in $\mathfrak{A}^I/_\Delta = {}^*\mathfrak{A}$, we mean the point $\varphi(a)$ for some $a \in A$. We generally denote the standard element $\varphi(a)$ in ${}^*\mathfrak{A}$ by $a$ and arbitrary elements of ${}^*\mathfrak{A}$ (standard or not) by ${}^*a$. For any element ${}^*a \in {}^*A = A^I/_\Delta$, ${}^*a$ is the equivalence class of some functions $\alpha : I \longrightarrow A$, and we may often represent ${}^*a$ by ${}^*a = (\alpha(i))_{i \in I}/_\Delta$ for some representative function $\alpha$. Similar comments hold for standard objects and arbitrary objects of arbitrary finite type in ${}^*\mathfrak{A}$. We note in particular that for $\{S_i \mid i \in I\}$ a family of sets of type $(\sigma)$ in $A$, we can embed in ${}^*\mathfrak{A}$ the ultraproduct of the family of sets, $\prod_{i \in I} S_i/_\Delta$, as the set of ${}^*s$ such that ${}^*s = (s(i))_{i \in I}/_\Delta$, where $s(i) \in S_i$ for each $i \in I$.

Definition 15[10]

A set ${}^*B$ of objects of type $\sigma$ in $\mathfrak{A}^I/_\Delta$ is internal iff there exists a family $\{S_i \mid i \in I\}$ of sets of type $(\sigma)$ in $A$ such that ${}^*B = \prod_{i \in I} S_i/_\Delta$."[11]

We now state the fundamental theorem of ultraproducts due originally to Łos. As a notational convenience, for a WFF $\mathcal{Q}$ in $\mathcal{K}$ we write $\mathfrak{A} \models \mathcal{Q}$ iff $\mathcal{Q}$ is true in $\mathfrak{A}$.

Theorem 3

Let $\{\mathfrak{A}_i \mid i \in I\}$ be a non-empty collection of similar mathematical structures, and let $\Delta$ be an ultrafilter on $I$. Let ${}^*s = ({}^*s_1, \ldots)$ be any denumerable sequence of objects[12] of arbitrary finite type in $\prod_{i \in I} A_i/_\Delta$. Then for $\mathcal{Q}({}^*s)$ a WFF in $\prod_{i \in I} \mathfrak{A}_i/_\Delta$,

$$\prod_{i \in I} \mathfrak{A}_i/_\Delta \models \mathcal{Q}({}^*s) \text{ iff } \{i \mid \mathfrak{A}_i \models \mathcal{Q}((s_1(i), s_2(i), \ldots))\} \in \Delta$$

for ${}^*s_j = (s_j(i))_{i \in I}/_\Delta$.[13]

Proof:

The proof is by induction on the length of[14] and is similar to that given in [11] with the following additions:[15]

Case 1:

Suppose $\mathcal{Q}$ is of the form $t_1 \in t_2$.

$${}^*t_1 \in {}^*t_2 \text{ iff } (t_1(i))_{i \in I}/_\Delta \in (t_2(i))_{i \in I}/_\Delta$$

$$\text{iff } \{i \mid t_1(i) \in t_2(i)\} \in \Delta \quad \text{by definition 14.}$$

Case 2:

Suppose $\mathcal{Q}$ is of the form $t_1 = t_2$.

Then again by definition 14 we see that

$${}^*t_1 = {}^*t_2 \text{ iff } \{i \mid t_1(i) = t_2(i)\} \in \Delta.$$

Case 3:

Suppose $\mathfrak{a}$ is of the form $\ulcorner \mathbb{I}_\sigma(t)$.

Then $\ulcorner \mathbb{I}_\sigma({}^*t)$ iff ${}^*t$ is of type $\sigma$ in $\prod_{i\in I} A_i/\Delta$

iff $\{i \mid t(i)$ is of type $\sigma$ in $A_i\} \in \Delta$

iff $\{i \mid \ulcorner \mathbb{I}_\sigma(t(i))$ is true in $A_i\} \in \Delta$.

□

It can now be seen that if $\mathfrak{A}$ and $I$ are at least countably infinite and if $\Delta$ is a non-principal ultrafilter on $I$, then $\mathfrak{A}^I/\Delta$ is a non-standard model of $\mathfrak{A}$. There are, in general, many types of non-standard models of a given structure $\mathfrak{A}$. We wish to consider those that have desirable properties. One of these properties deals with the notion of concurrency.

Definition 16

Let $\mathfrak{A}$ be a mathematical structure, $R_\rho$[16] a binary relation of certain type in $\mathfrak{A}$. $a$ is in the domain of $R_\rho$ iff there exists an object $b$ in $\mathfrak{A}$ such that $\langle a,b\rangle \in R_\rho$. We say that $R_\rho$ is concurrent in $\mathfrak{A}$ (or finitely satisfiable in $\mathfrak{A}$) if for each finite set of objects $a_1,\ldots,a_n$ in the domain of $R_\rho$, there exists a $b$ in $\mathfrak{A}$ such that $\langle a_1,b\rangle \in R_\rho, \ldots, \langle a_n,b\rangle \in R_\rho$.

Definition 17

A non-standard model ${}^*\mathfrak{A}$ of a structure $\mathfrak{A}$ is an enlargement of $\mathfrak{A}$ iff for every concurrent relation $R_\beta$ in $\mathfrak{A}$, there exists an object ${}^*b$ in ${}^*\mathfrak{A}$ such that $\langle a, {}^*b\rangle \in R_\beta$ in ${}^*\mathfrak{A}$ for all standard objects $a$ in the domain of $R_\beta$ in ${}^*\mathfrak{A}$.

One can naturally ask when ultrapowers are enlargements. To that end, we introduce the notion of an ultrafilter on a set $I$ being adequate. The original definition appears in [1], but we shall use a slightly more general definition as put forth by W. A. J. Luxemburg.

Definition 18

Let $\kappa$ be an infinite cardinal. A filter $\Delta$ on $I$ is called $\kappa$-adequate iff for every family $\mathcal{B}$ of subsets of $\kappa$ with the finite intersection property, there is a mapping $f: I \rightarrow \kappa$ such that the filter generated by all supersets of sets of the form $f(\Delta)$, for each $\Delta \in \Delta$, contains $\mathcal{B}$.

It can be shown that if $I$ has cardinality greater than or equal to $2^\kappa$, then $\kappa$-adequate ultrafilters exist on $I$ for some prechosen cardinal $\kappa$ (see [1] or [2]). The following theorem gives a sufficient condition for an ultrapower to be an enlargement.

Theorem 4

If $\Delta$ is a non-principal ultrafilter on $I$ which is $\kappa$-adequate for $\kappa \geq \mathrm{card}(A')$ where $A'$ is the set of all objects of all finite types in $\mathfrak{A}$, then $\mathfrak{A}^I/_\Delta$ is an enlargement of $\mathfrak{A}$.

Proof:

Let $R_\beta$ be any concurrent relation in $\mathfrak{A}$. Then for each $a$ in the domain of $R_\beta$, let $F_a = \{b \mid \langle a,b\rangle \in R_\beta\}$. Thus the family $\mathcal{F} = \{F_a \mid a \text{ IN THE DOMAIN OF } R_\beta\}$ is a non-empty set of non-empty sets which have the finite intersection property as $R_\beta$ is concurrent. Hence, $\Delta$ being $\kappa$-adequate for $\kappa \geq \mathrm{card}(A')$, it follows that there exists a map $f: I \to A'$ such that for every $a$ in the domain of $R_\beta$, there exists a subset $\Delta_a \in \Delta$ such that $f(\Delta_a) \subseteq F_a$. That is, for every $a$ in the domain of $R_\beta$ we have $\{i \mid \langle a, f(i)\rangle \in R_\beta\} \supseteq \Delta_a \in \Delta$. Therefore the object $^*f = (f(i))_{i\in I}/_\Delta$ is such that $\langle a, {}^*f\rangle \in R_\beta$ in $^*\mathfrak{A} = \mathfrak{A}^I/_\Delta$ for all standard objects $a$ in the domain of $R_\beta$ in $^*\mathfrak{A}$ as seen from theorem 3. That is, $^*\mathfrak{A}$ is an enlargement of $\mathfrak{A}$.

$\square$

We have mentioned that we can not make all objects of arbitrary finite type in ${}^*A$ internal. To see this, let $\mathfrak{A}$ be the structure consisting of the domain $A$ the set of positive natural numbers, along with the usual relations of order, addition and subtraction. Let $I$ be any infinite index set, and $\Delta$ any non-principal ultrafilter on $I$. Then ${}^*A = A^I/_\Delta$ is the set of positive natural numbers, both finite and "infinite"-- that is, for any ${}^*\rho \in {}^*A - A$, ${}^*\rho > a$ for all standard $a \in A$ [for ${}^*\rho = (\rho(i))_{i\in I}/_\Delta$, either $\{i \mid \rho(i) > a\}$ or $\{i \mid \rho(i) \leq a\}$ is in $\Delta$ for any $a \in A$. But $\{i \mid \rho(i) \leq a\} \in \Delta$ implies $\{i \mid \rho(i) = b\} \in \Delta$ for some $b < a$[17] in $A$ due to the nature of ultrafilters and that there are only finitely many elements less than any fixed $a \in A$. That is, ${}^*\rho \notin {}^*A - A$. If, for each $a \in A$ we have that $\{i \mid \rho(i) > a\} \in \Delta$, then ${}^*\rho > a$ for all standard $a \in A$[18]. That is, ${}^*\rho \in {}^*A - A$.]

The following statement is true in $\mathfrak{A}$: every non-empty subset of $A$ has a least element. In ${}^*\mathfrak{A}$ it reads: every non-empty internal subset of ${}^*A$ has a least element. Now if we allow all objects of arbitrary finite type in ${}^*A$ as internal, then ${}^*A - A$ is internal. Hence it has a least element, ${}^*\rho = (\rho(i))_{i\in I}/_\Delta$ say. Then $({}^*\rho) - 1 = (\rho(i) - 1)_{i\in I}/_\Delta$ is less than ${}^*\rho$, hence $({}^*\rho) - 1$ is in $A$. But then $({}^*\rho) - 1$ is standard, so $({}^*\rho) - 1 + 1 = {}^*\rho$ is standard also as $A$ has no maximal element. Thus ${}^*\rho \in A$ and ${}^*\rho \in {}^*A - A$, a contradiction. So we can not allow all objects of arbitrary finite type in $^*\Delta$ to be internal.

Now consider the language $\mathcal{K}$. If we wish to study a theory $\mathcal{T}$ that has a standard model whose set of arbitrary finite type has cardinality $\alpha$, then we shall require $\mathcal{K}$ to have $\alpha^{(2^\alpha)}$ constants so that we may construct within $\mathcal{K}$ any adequate ultrapower of $\mathcal{T}$ that is desired. This may, in general, be many more constants than we need, but it assures us of being able to construct within $\mathcal{K}$ both standard and non-standard models of $\mathcal{T}$.

# CHAPTER II

# TOPOLOGICAL GROUPS

Definition 19

The 4-tuple $(G, \cdot, {}^{-1}, \tau)$ is a topological group iff

i) $(G, \cdot, {}^{-1})$ forms a group

ii) $\tau$ is a topology on $G$

iii) ${}^{-1}: G \to G$ is a continuous function with respect to the topology $\tau$ on $G$

iv) $\cdot: G \times G \to G$ is a continuous function on $G \times G$ with the product topology on $G \times G$.

Let $(G, \cdot, {}^{-1}, \tau)$ be a topological group. We would like to analyze $(G, \cdot, {}^{-1}, \tau)$ using non-standard methods. Hence, it will prove beneficial to see what kind of topology $\tau$ induces on $G^I/\Delta$ (which shall henceforth be written ${}^*G$), for $I$ an infinite set, $\Delta$ a non-principal ultrafilter on $I$ which may be assumed to be adequate for a fixed infinite cardinal $\kappa$ if necessary. We shall always indicate when such an assumption is needed.

Lemma 1

$${}^*\tau_Q = \{ \prod_{i \in I} T_i/\Delta \mid T_i \in \tau \text{ FOR EACH } i \in I \}$$

forms a base for a topology on ${}^*G$ which we shall call the Quasi-$\tau$-topology (or Q-topology) on ${}^*G$. (The terminology is due essentially to A. Robinson).

Proof:

A base for a topology on $G$ is a collection $\tau'$ of subsets of $G$ that satisfy:

i) $\phi \in \tau'$, $G \in \tau'$

ii) $T_1, T_2 \in \tau' \Rightarrow T_1 \cap T_2 \in \tau'$.

The resulting topology consists of all possible unions of sets in the base.

i) $\phi \in {}^*\tau_Q$ as $\phi \in \tau$ and $\phi = \phi^I/\Delta \in {}^*\tau_Q$. ${}^*G \in {}^*\tau_Q$ as $G \in \tau$ and thus ${}^*G = G^I/\Delta \in {}^*\tau_Q$.

ii) Let ${}^*T_1, {}^*T_2 \in {}^*\tau_Q$. Then ${}^*T_1 = \prod_{i \in I} T_{1,i}/\Delta$ and ${}^*T_2 = \prod_{i \in I} T_{2,i}/\Delta$ for $T_{1,i} \in \tau$ and $T_{2,i} \in \tau$ for each $i \in I$. Now ${}^*(T_1 \cap T_2) = \prod_{i \in I}(T_{1,i} \cap T_{2,i})/\Delta$ $\in {}^*\tau_Q$ as $T_{1,i} \cap T_{2,i} \in \tau$ for each $i \in I$.

Thus ${}^*x = (x_i)_{i \in I}/\Delta \in {}^*(T_1 \cap T_2)$

iff $\{i \mid x_i \in T_{1,i} \cap T_{2,i}\} \in \Delta$

iff $\{i \mid x_i \in T_{1,i}\} \cap \{i \mid x_i \in T_{2,i}\} \in \Delta$

iff $\{i \mid x_i \in T_{1,i}\} \in \Delta$ AND $\{i \mid x_i \in T_{2,i}\} \in \Delta$

iff ${}^*x \in {}^*T_1$ AND ${}^*x \in {}^*T_2$

iff ${}^*x \in {}^*T_1 \cap {}^*T_2$. Hence ${}^*T_1 \cap {}^*T_2 \in {}^*\tau_Q$.

Therefore ${}^*\tau_Q$ is a base for a topology on ${}^*G$.

$\square$

Theorem 5

The group operations[20] on ${}^*G$ are continuous with respect to the Q-topology on ${}^*G$.

Proof:

Let ${}^*a = (a_i)_{i\in I}/\Delta$, ${}^*b = (b_i)_{i\in I}/\Delta$, ${}^*c = (c_i)_{i\in I}/\Delta$ be elements of ${}^*G$, such that ${}^*a = {}^*b\,{}^*c$. Let ${}^*T = T^I/\Delta$ be a neighborhood of ${}^*a$, for ${}^*T \in {}^*\tau_Q$.[21] Then $\{i \mid T_i \in \tau\} = I \in \Delta$, $\{i \mid a_i \in T_i\} \in \Delta$ and $\{i \mid a_i = b_i c_i\} \in \Delta$. So $J = \{i \mid a_i = b_i c_i \in T_i \in \tau\} \in \Delta$.

Now for each $i \in J$, $T_i$ is an open neighborhood of $a_i = b_i c_i$, so by the continuity of multiplication in $(G, \cdot, {}^{-1}, \tau)$, there exist open sets $U_i$ and $V_i$ in $\tau$ such that $b_i \in U_i$, $c_i \in V_i$ and $U_i V_i \subseteq T_i$. For $i \in I - J$, let $U_i = V_i = G$. Then

$J = \{i \mid a_i = b_i c_i \in T_i \supseteq U_i V_i$ AND $b_i \in U_i \in \tau$ AND $c_i \in V_i \in \tau\} \in \Delta$.

Hence, for ${}^*U = \prod_{i\in I} U_i/\Delta$, ${}^*V = \prod_{i\in I} V_i/\Delta$, we have by theorem 3 that ${}^*a = {}^*b\,{}^*c \in {}^*T \supseteq {}^*U\,{}^*V$ and ${}^*b \in {}^*U \in {}^*\tau_Q$, ${}^*c \in {}^*V \in {}^*\tau_Q$ and ${}^*T \in {}^*\tau_Q$. That is, multiplication in ${}^*G$ is continuous with respect to the Q-topology on ${}^*G$.

The proof that the inverse operation in ${}^*G$ is continuous with respect to the Q-topology is similar to that of multiplication and is left to the reader.

□

Hence, when we wish to analyze a topological group $(G, \cdot, ^{-1}, \tau)$ by non-standard methods, we shall consider $({}^*G, \cdot, ^{-1}, {}^*\tau)$ where ${}^*\tau$ is the Q-topology on ${}^*G$, generated by the topological base ${}^*\tau_Q$.

Definition 20

Let ${}^*a \in {}^*G$. We define the $\tau$-monad of ${}^*a$, $\mu_\tau({}^*a)$, by $\mu_\tau({}^*a) = \cap\{T^I/_\Delta \mid T \in \tau$ and ${}^*a \in T^I/_\Delta\}$.

Theorem 6

Let $a$ be a standard point of ${}^*G$. Then $\mu_\tau(a^{-1}) = (\mu_\tau(a))^{-1}$.

Proof:

For any open neighborhood $V$ of $a^{-1}$, there exists an open neighborhood $U$ of $a$ such that $a^{-1} \in U^{-1} \subseteq V$. Thus ${}^*V = V^I/_\Delta \supseteq \mu_\tau(a^{-1})$ and ${}^*U = U^I/_\Delta$ is such that ${}^*U \supseteq \mu_\tau(a)$, and ${}^*U^{-1} \subseteq {}^*V$. Therefore $(\mu_\tau(a))^{-1} \subseteq {}^*U^{-1} \subseteq {}^*V$. As $V$ was an arbitrary open neighborhood, $(\mu_\tau(a))^{-1} \subseteq \cap\{V^I/_\Delta \mid V \text{ is an open neighborhood of } a^{-1}\} = \mu_\tau(a^{-1})$.

Similarly, $(\mu_\tau(a^{-1}))^{-1} \subseteq \mu_\tau((a^{-1})^{-1}) = \mu_\tau(a)$, hence $\mu_\tau(a^{-1}) \subseteq (\mu_\tau(a))^{-1}$. Hence $\mu_\tau(a^{-1}) = (\mu_\tau(a))^{-1}$.

□

Theorem 7

Let $a, b$ be any two standard points of ${}^*G$ .
Then $\mu_\tau(a)\mu_\tau(b) = \mu_\tau(ab)$ .

Proof:

Let ${}^*c \in \mu_\tau(a)$ , ${}^*d \in \mu_\tau(b)$ . We wish to show that for any open neighborhood of $ab$ , $W$ say, in $\tau$, ${}^*c\,{}^*d \in {}^*W = W^I/_\Delta \in {}^*\tau_\omega$ . But for any open neighborhood $W$ of $ab$, there exist open neighborhoods $U, V$ of $a, b$ respectively, such that $UV \subseteq W$ by the continuity of multiplication in $G$. Hence for ${}^*U = U^I/_\Delta$ , ${}^*V = V^I/_\Delta$ , we have ${}^*U\,{}^*V \subseteq {}^*W$ . Now ${}^*U \supseteq \mu_\tau(a)$ , ${}^*V \supseteq \mu_\tau(b)$ , hence ${}^*c \in {}^*U$, ${}^*d \in {}^*V$ , so ${}^*c\,{}^*d \in {}^*W$. As $W$ was an arbitrary neighborhood of $ab$, ${}^*c\,{}^*d \in \bigcap\{{}^*W = W^I/_\Delta \mid W$ IS AN OPEN NEIGHBORHOOD OF $ab\}$ . That is, ${}^*c\,{}^*d \in \mu_\tau(ab)$ . Hence $\mu_\tau(a)\mu_\tau(b) \subseteq \mu_\tau(ab)$ .

Now let ${}^*c \in \mu_\tau(ab)$. For ${}^*d^{-1} \in \mu_\tau(a^{-1})$ , ${}^*d^{-1}\,{}^*c \in \mu_\tau(a^{-1})\mu_\tau(ab)$ . By the above, we see that ${}^*d^{-1}\,{}^*c \in \mu_\tau(a^{-1}ab) = \mu_\tau(b)$ . Hence ${}^*c = {}^*d({}^*d^{-1}\,{}^*c) \in \mu_\tau(a)\mu_\tau(b)$ . Thus $\mu_\tau(ab) \subseteq \mu_\tau(a)\mu_\tau(b)$ . So $\mu_\tau(ab) = \mu_\tau(a)\mu_\tau(b)$.

$\square$

Lemma 2

$\mathcal{T}$ is open in $\mathcal{G}$ iff for all $p \in \mathcal{T}$, $\mu_\tau(p) \subseteq \mathcal{T}^I/_\Delta$, provided $\Delta$ is adequate for $2^{\mathrm{CARD}(\mathcal{G})}$.

Proof:

Assume $\mathcal{T}$ is open in $\mathcal{G}$. As $\mathcal{T}$ is an open neighborhood of all its points, clearly $\mu_\tau(p) \subseteq \mathcal{T}^I/_\Delta$ for all $p \in \mathcal{T}$.

So, suppose for all $p \in \mathcal{T}$, $\mu_\tau(p) \subseteq \mathcal{T}^I/_\Delta$, and that $\Delta$ is adequate for $2^{\mathrm{CARD}(\mathcal{G})}$. The relation $R_x(A,B)$ which holds between two subsets of $\mathcal{G}$ iff $A$ is an open neighborhood of $x$ and $B$ is an open neighborhood of $x$ and $A \supseteq B$, is concurrent on $\mathcal{G}$ as for any finite collection of open neighborhoods of a point $x$, $A_1, \ldots, A_n$ say, $\bigcap_{i=1}^{n} A_i$ is an open neighborhood of $x$ and $R_x(A_i, \bigcap_{j=1}^{n} A_j)$ holds in $\mathcal{G}$ for all $i = 1, \ldots, n$. Hence, as $\Delta$ is adequate for $2^{\mathrm{CARD}(\mathcal{G})}$, we know that for any collection of open neighborhoods of the point $x$ in $\mathcal{G}$, $\{A_j \mid j \in J\}$ say, there exists by the methods of theorem 4 a function $f: I \to J$ such that $\{i \mid x \in A_{f(i)} \subseteq A_j\} \in \Delta$ for every $j \in J$. Thus $\prod_{i \in I} A_{f(i)}/_\Delta = {}^*A$ is such that $x \in {}^*A \subseteq A_j^I/_\Delta$, for every $j \in J$. Hence this holds if $\{A_j \mid j \in J\}$ is the collection of all open neighborhoods of $x$ in $\mathcal{G}$. That is,

$c \in {}^*A \subseteq \mu_\tau(c)$. Therefore $\mu_\tau(c)$ is a neighborhood of $c$ in ${}^*G$.

Hence, as $T^I/_\Delta \supseteq \mu_\tau(c)$ for all $c \in T$, $T^I/_\Delta$ is a neighborhood of $c$ in ${}^*G$. Thus $T^I/_\Delta$ must contain a basic open set, $\prod_{i \in I} T_i/_\Delta$ say, where $\{i \mid c \in T_i \in \tau\} \in \Delta$. Hence $\{i \mid T \supseteq T_i \supseteq \{c\} \text{ AND } T_i \in \tau\} \in \Delta$. That is, $T$ is a neighborhood of $c$ in $G$, for all $c \in T$. Thus $T$ is an open set.

□

Henceforth we shall assume that $\Delta$ is adequate for $2^{\text{CARD}(G)}$.

Theorem 8

Let $a \in G$, $W \subseteq G$. $W$ is open iff $Wa$ is open.

Proof:

If $W$ open implies $Wa$ is open then $Wa$ open implies $Wa(a^{-1}) = W$ is open. Hence it suffices to show that $W$ open implies $Wa$ is open.

Let $W$ be open in $G$. Let $b \in W$. Then $\mu_\tau(b) \subseteq {}^*W = W^I/_\Delta$. Let $c \in Wa$. Then $c = da$ for some $d \in W$. Thus $(\mu_\tau(d))a \subseteq \mu_\tau(d)\mu_\tau(a) = \mu_\tau(da) = \mu_\tau(c)$ by theorem 7. If ${}^*e \in \mu_\tau(c)$, then ${}^*f = {}^*e(a^{-1})$ is such that ${}^*f \in \mu_\tau(c)\mu_\tau(a^{-1}) = \mu_\tau(ca^{-1}) = \mu_\tau(d)$. Thus ${}^*e = {}^*ea^{-1}a = {}^*fa \in \mu_\tau(d)a$. That is, $\mu_\tau(c) \subseteq (\mu_\tau(d))a$.

So $\mu_\tau(c) = (\mu_\tau(d))a$ . Hence, as $(\mu_\tau(d))a \subseteq {}^*Wa$,[22] $\mu_\tau(c) \subseteq {}^*Wa$ .

By lemma 2, as $c \in Wa$ was arbitrary, $Wa$ is open.

$\square$

Definition 21

A topological space $(G, \tau)$ is said to be Hausdorff (or $T_2$) iff for any two points $p, q \in G$ such that $p \neq q$, there exist two open sets $T_1, T_2 \in \tau$ such that $p \in T_1$, $q \in T_2$ and $T_1 \cap T_2 = \emptyset$.

Theorem 9

A topological group $(G, \cdot, ^{-1}, \tau)$ is Hausdorff iff for any two standard points $p, q \in {}^*G$ such that $p \neq q$, $\mu_\tau(p) \cap \mu_\tau(q) = \emptyset$.

Proof:

If $(G, \cdot, ^{-1}, \tau)$ is Hausdorff (more precisely, if $(G, \tau)$ is Hausdorff) then for any two distinct points $p, q \in G$, there exist open sets $T_1, T_2$ such that $p \in T_1$, $q \in T_2$ and $T_1 \cap T_2 = \emptyset$. Hence $p \in T_1{}^{I}/_{\Delta}$, $q \in T_2{}^{I}/_{\Delta}$, and $T_1{}^{I}/_{\Delta} \cap T_2{}^{I}/_{\Delta} = (T_1 \cap T_2)^{I}/_{\Delta} = \emptyset$ as seen in the proof of lemma 1. As $\mu_\tau(p) \subseteq T_1{}^{I}/_{\Delta}$ and $\mu_\tau(q) \subseteq T_2{}^{I}/_{\Delta}$, $\mu_\tau(p) \cap \mu_\tau(q) = \emptyset$.

Let $\mu_\tau(p) \cap \mu_\tau(q) = \phi$ for all standard $p, q \in {}^*G$ such that $p \neq q$. Then, as seen in the proof of lemma 2, there exist open sets in ${}^*\tau_Q$, ${}^*T$ and ${}^*O$ say, such that ${}^*T \subseteq \mu_\tau(p)$ and $p \in {}^*T$, and ${}^*O \subseteq \mu_\tau(q)$ and $q \in {}^*O$. Hence ${}^*T \cap {}^*O = \phi$. For ${}^*T = \prod_{i \in I} T_i/\Delta$, ${}^*O = \prod_{i \in I} O_i/\Delta$, we necessarily have $\{i \mid p \in T_i \text{ AND } q \in O_i \text{ AND } T_i \cap O_i = \phi \text{ AND } T_i \in \tau \text{ AND } O_i \in \tau\} \in \Delta$. That is, there exist non-empty disjoint open sets about $p$ and $q$ respectively, hence $(G, \cdot, {}^{-1}, \tau)$ is Hausdorff.

□

Theorem 10

Let ${}^*c, {}^*d \in \mu_\tau(a)$ for some standard point $a \in {}^*G$. Then ${}^*c\, {}^*d^{-1} \in \mu_\tau(e)$ for $e$ the identity of the group $G$.

Proof:

${}^*d \in \mu_\tau(a)$ implies ${}^*d^{-1} \in (\mu_\tau(a))^{-1} = \mu_\tau(a^{-1})$ by theorem 6. Hence ${}^*c\, {}^*d^{-1} \in \mu_\tau(a)\mu_\tau(a^{-1}) = \mu_\tau(aa^{-1}) = \mu_\tau(e)$ by theorem 7.

□

Theorem 11

$\mu_\tau(e)$ is a subgroup of ${}^*G$, for $e$ the identity of the group $G$.

Proof:

Clearly $e \in \mu_\tau(e)$. Let ${}^*a, {}^*b \in \mu_\tau(e)$. Then ${}^*a{}^*b \in \mu_\tau(e)\mu_\tau(e) = \mu_\tau(ee) = \mu_\tau(e)$ by theorem 7. Let ${}^*a \in \mu_\tau(e)$. ${}^*a^{-1} \in (\mu_\tau(e))^{-1} = \mu_\tau(e^{-1}) = \mu_\tau(e)$ by theorem 6. Hence $\mu_\tau(e)$ is a subgroup of ${}^*G$.

□

Definition 22

Let ${}^*a \in {}^*G$. ${}^*a$ is said to be near-standard in ${}^*G$ iff there exists a standard point $b \in {}^*G$ such that ${}^*a \in \mu_\tau(b)$.

Theorem 12

The near-standard points of ${}^*G$, denoted $ns({}^*G)$, form a subgroup of ${}^*G$.

Proof:

If ${}^*c \in \mu_\tau(a)$, ${}^*d \in \mu_\tau(b)$ for $a, b$ standard points in ${}^*G$, then we have ${}^*c{}^*d \in \mu_\tau(a)\mu_\tau(b) = \mu_\tau(ab)$ by theorem 7. Hence ${}^*c{}^*d \in ns({}^*G)$. If ${}^*b \in \mu_\tau(a)$, for $a$ a standard point of ${}^*G$, then ${}^*b^{-1} \in (\mu_\tau(a))^{-1} = \mu_\tau(a^{-1})$ by theorem 6. Thus ${}^*b^{-1} \in ns({}^*G)$. Hence $ns({}^*G)$ is a subgroup of ${}^*G$.

□

Theorem 13

$\mu_\tau(e)$ is a normal subgroup of $ns({}^*G)$.

Proof:

Clearly $\mu_\tau(e) \subseteq ns(^*G)$. Actually, by the argument of theorem 11, $\mu_\tau(e)$ is a subgroup of $ns(^*G)$.

Let $^*a \in \mu_\tau(e)$, $^*g \in ns(^*G)$. Then there exists a standard point $b \in {}^*G$ such that $^*g \in \mu_\tau(b)$. Thus $^*g\,^*a\,^*g^{-1} \in \mu_\tau(b)\mu_\tau(e)(\mu_\tau(b))^{-1} = \mu_\tau(b)\mu_\tau(e)\mu_\tau(b^{-1}) = \mu_\tau(b)\mu_\tau(eb^{-1}) = \mu_\tau(b)\mu_\tau(b^{-1}) = \mu_\tau(bb^{-1}) = \mu_\tau(e)$ by theorem 6 and theorem 7. That is, $\mu_\tau(e)$ is a normal subgroup of $ns(^*G)$.

$\square$

Utilizing theorems 9, 10, 11, and 12, we have:

Theorem 14

Let $(G, \cdot, {}^{-1}, \tau)$ be a Hausdorff space. Then $ns(^*G)/\mu_\tau(e) \cong G$ ..

Proof:

Let $p \in G$. Then $p\mu_\tau(e) \subseteq \mu_\tau(p)\mu_\tau(e) = \mu_\tau(pe) = \mu_\tau(p)$ but $p\mu_\tau(e) = \{p\,^*a \mid {}^*a \in \mu_\tau(e)\} \supseteq \{p(p^{-1}\,^*q) \mid {}^*q \in \mu_\tau(p)\} = \{(pp^{-1})\,^*q \mid {}^*q \in \mu_\tau(p)\} = \mu_\tau(p)$. Thus $p\mu_\tau(e) = \mu_\tau(p)$, so the cosets of $\mu_\tau(e)$ in $ns(^*G)$ are precisely the monads of standard points in $^*G$. Thus any element of $ns(^*G)/\mu_\tau(e)$ looks like $\mu_\tau(p)/\mu_\tau(e) = p\mu_\tau(e)/\mu_\tau(e)$ for $p$ a standard point of $^*G$.

Now define $\psi: ns(^*G)/\mu_\tau(e) \to G$ by $\psi(\mu_\tau(p)/\mu_\tau(e)) = p$ for $p$ a standard point in $^*G$. Let $\psi(\mu_\tau(p)/\mu_\tau(e)) \neq \psi(\mu_\tau(q)/\mu_\tau(e))$. Then $p \neq q$. So by theorem 9, $\mu_\tau(p) \cap \mu_\tau(q) = \phi$, so $\mu_\tau(p)/\mu_\tau(e) \neq \mu_\tau(q)/\mu_\tau(e)$. Thus $\psi$ is well defined. Let $p = q$. Then $\mu_\tau(p) = \mu_\tau(q)$, hence $\mu_\tau(p)/\mu_\tau(e) = \mu_\tau(q)/\mu_\tau(e)$. Thus $\psi$ is 1:1.

$\psi$ is onto as any point in $G$ forms a distinct monad in $^*G$ by theorem 9. Now $\psi((\mu_\tau(p)/\mu_\tau(e))(\mu_\tau(q)/\mu_\tau(e))) = \psi(\mu_\tau(p)\mu_\tau(q)/\mu_\tau(e))$ $\psi(\mu_\tau(pq)/\mu_\tau(e)) = pq = \psi(\mu_\tau(p)/\mu_\tau(e))\psi(\mu_\tau(q)/\mu_\tau(e))$. Hence $\psi$ is an isomorphism and thus $ns(^*G)/\mu_\tau(e) \cong G$.

□

Before we look at something in $^*G$ which has no direct counterpart in $G$, we present the characterization of compactness by means of monads, which will prove useful in chapter 3.

Definition 23

A topological space $(G, \mathcal{T})$ is compact iff for any non-empty family of open sets $\{O_i \mid i \in I\}$ such that $\bigcup_{i \in I} O_i = G$, there exists a finite subfamily $\{O_i \mid i \in J\}$, $J \subseteq I$, $J$ finite, such that $\bigcup_{i \in J} O_i = G$.

Theorem 15

A topological space $(G,\tau)$ is compact iff all points of $^*G$ are near-standard.

Proof:

Let $(G,\tau)$ be compact. Suppose there exists a point $^*p = (p_i)_{i\in I}/\Delta$ such that $^*p \notin ns(^*G)$. Then for any standard point $q \in {}^*G$, $^*p \notin \mu_\tau(q)$. Hence there exists an open neighborhood $U_q$ of $q$ in $G$ such that $^*p \notin (U_q)^I/\Delta$. Now $G \supseteq \bigcup_{q\in G} U_q \supseteq \bigcup_{q\in G} \{q\} = G$. Hence $\{U_q \mid q \in G\}$ is such that $\bigcup_{q\in G} U_q = G$.

Thus there exists a finite subfamily, $U_1,\ldots,U_n$ say, such that $\bigcup_{j=1}^{n} U_j = G$. So $(\bigcup_{j=1}^{n} U_j)^I/\Delta = {}^*G$. That is, $\{i \mid p_i \in \bigcup_{j=1}^{n} U_j\} \in \Delta$. Thus $\{i \mid p_i \in U_j\} \in \Delta$ for some $j=1,2,\ldots,n$, otherwise $\{i \mid p_i \notin U_j\} \in \Delta$ for $j=1,2,\ldots,n$, hence $\bigcap_{j=1}^{n} \{i \mid p_i \notin U_j\} \in \Delta$. But this is impossible as $\bigcap_{j=1}^{n} \{i \mid p_i \notin U_j\} = \{i \mid p_i \notin \bigcup_{j=1}^{n} U_j\} = \complement\{i \mid p_i \in \bigcup_{j=1}^{n} U_j\}$, and by theorem 2 and that $\{i \mid p_i \in \bigcup_{j=1}^{n} U_j\} \in \Delta$, $\bigcap_{j=1}^{n} \{i \mid p_i \notin U_j\} \notin \Delta$.

That is, for some $j=1,2,\ldots,n$, $\{i \mid p_i \in U_j\} \in \Delta$. Hence $^*p \in (U_j)^I/\Delta$, which is a contradiction. Therefore $^*p$ must be a near-standard point of $^*G$.

Suppose $(G,\tau)$ is not compact. Then for some family of open sets $\{O_j \mid j \in J\}$ such that $\bigcup_{j\in J} O_j = G$ , for any finite $J' \subseteq J$, $\bigcup_{j\in J'} O_j \neq G$ . Consider the relation $R(A,a)$ which holds iff $A \in \{O_j \mid j \in J\}$ and $a \notin A$ . Then by our hypothesis, $R$ is concurrent. Hence, as $\Delta$ is adequate for $2^{\mathrm{CARD}(G)}$ , there exists an element $^*k \in {}^*G$ such that $^*k \notin (O_j)^I/_\Delta$ for all $O_j \in \{O_j \mid j \in J\}$ . That is, as for any $g \in G$, $\mu_\tau(g) \subseteq (O_j)^I/_\Delta$ for a suitable $j \in J$ , $^*k$ is not in $\mu_\tau(g)$ for any standard $g \in {}^*G$. That is, $^*k$ is not near-standard.

□

Let $^*H$ be any subgroup of $^*G$ . $^*H$ may or may not be internal. We know that $^*H$ is closed under finite products. Let $\mathbf{N}$ be the non-negative natural numbers. $^*\mathbf{N} = \mathbf{N}^I/_\Delta$ is a non-standard model of the non-negative natural numbers. We speak of elements in $^*\mathbf{N}$ but not in $\mathbf{N}$ as star-finite elements. We wish to consider the collection of all star-finite products of length $^*\omega \in {}^*\mathbf{N} - \mathbf{N}$ in $^*H$ , that is, $^*H^{(^*\omega)}$ . To do this we need the definition of an internal sequence of length $^*n \in {}^*\mathbf{N}$, of elements of $^*H$ .

Definition 24

Let $^*n \in {}^*\mathbf{N}$ . $(^*a_1, \ldots, ^*a_{^*n})$ is an internal sequence of elements of $^*H$ iff it can be obtained from a doubly indexed array

$$\begin{array}{llll} a_{11} & a_{12} & \cdots & a_{1m_1} \\ a_{21} & a_{22} & \cdots & a_{2m_2} \\ \vdots & & & \ddots \\ a_{i1} & a_{i2} & \cdots & a_{im_i} \\ \vdots & & & \ddots \end{array}$$

such that ${}^*n = (m_i)_{i\in I}/_\Delta$ and for any $k: I \longrightarrow I$ such that $k(i) \leq m_i$, we have ${}^*a_{*k} = (a_{i,k(i)})_{i\in I}/_\Delta$ a member of ${}^*H$. If these conditions hold, then $({}^*a_1, \ldots, {}^*a_{*n}) = \prod_{i\in I} K_i/_\Delta$ where $K_i = (a_{i1}, \ldots, a_{im_i})$.[23]

Consider the functions $\pi_n: G^n \longrightarrow G$ by

$$\pi_1((a)) = a$$

$$\pi_2((a,b)) = ab$$

$$\vdots$$

$$\pi_n((a_1, \ldots, a_n)) = a_1 \cdot \ldots \cdot a_n$$

$$\vdots$$

Then $\pi_n$ is defined for each $n \in \mathbf{N}$. Then for ${}^*n = (n_i)_{i\in I}/_\Delta$, ${}^*\pi_{*n}: {}^*G^{*n} \longrightarrow {}^*G$ is defined by ${}^*\pi_{*n} = \prod_{i\in I}(\pi_{n_i})/_\Delta$, and, for any sequence of length ${}^*n$, ${}^*\pi_{*n}(({}^*a_1, \ldots, {}^*a_{*n})) = {}^*a_1 \cdot \ldots \cdot {}^*a_{*n}$. Hence we can now multiply together the members of any sequence of star-finite length.

Now let ${}^*\omega \in {}^*\mathbf{N}\text{-}\mathbf{N}$. Let $\mathbf{N}({}^*\omega) = \{{}^*n \mid {}^*n \in {}^*\mathbf{N}$ AND THERE EXISTS $m \in \mathbf{N}$ SUCH THAT ${}^*n \leq m\,{}^*\omega\}$. It is straight forward to show that $\mathbf{N}({}^*\omega)$ is an initial segment of ${}^*\mathbf{N}$.

Definition 25

Let ${}^*H$ be any subgroup of ${}^*G$ . Define ${}^*H^{({}^*\omega)}$ by ${}^*H^{({}^*\omega)} = \{ {}^*a_1 \cdot \ldots \cdot {}^*a_{*n} \mid {}^*a_{*i} \in {}^*H$ AND $({}^*a_1, \ldots, {}^*a_{*n})$ IS AN INTERNAL SEQUENCE OF ELEMENTS OF ${}^*H$ AND ${}^*n \in N({}^*\omega)$, AND ${}^*a_1 \cdot \ldots \cdot {}^*a_{*n} = {}^*\pi_{*n}(({}^*a_1, \ldots, {}^*a_{*n}))\}$. Define ${}^*H^{\infty} = \bigcup_{{}^*\omega \in {}^*N} {}^*H^{({}^*\omega)}$ .

Theorem 16

${}^*H^{({}^*\omega)}$ is a subgroup of ${}^*G$ .

Proof:

The identity ${}^*e$ of ${}^*G$ is in ${}^*H^{({}^*\omega)}$ as ${}^*e$ is in ${}^*H$ and $({}^*e)$ is an internal sequence (of length 1 in $N({}^*\omega)$), the product of whose members is ${}^*e$ .

Now suppose $({}^*a_1, \ldots, {}^*a_{*n})$ is an internal sequence of length ${}^*n$ of elements of ${}^*H$ , for ${}^*n \in N({}^*\omega)$. Then, as seen from definition 24, $({}^*a_1, \ldots, {}^*a_{*n}) = \prod_{i \in I} K_i/\Delta$ for $\{K_i \mid i \in I\}$ a family of sequences of elements of $G$ ( $K_i$ is the sequence formed by the $i^{th}$ row in our doubly indexed array).

Let $\{K_i^{-1} \mid i \in I\}$ be the family of sequences of elements of $G$ such that if $K_i = (a_{i1}, \ldots, a_{im_i})$, then $K_i^{-1} = (a_{i1}^{-1}, \ldots, a_{im_i}^{-1})$ . Then $\prod_{i \in I} K_i^{-1}/\Delta$ is an internal sequence of length ${}^*n$ of elements of ${}^*H$. In fact, $\prod_{i\in I} K_i^{-1}/_{\Delta} = ({}^*a_1^{-1}, \ldots, {}^*a_{*n}^{-1})$ , hence $({}^*a_1^{-1}, \ldots, {}^*a_{*n}^{-1})$ is an internal sequence of length ${}^*n$ of elements of ${}^*H$. We need, however, $({}^*a_{*n}^{-1}, \ldots, {}^*a_1^{-1})$ an internal sequence of length ${}^*n$ of elements of ${}^*H$.

Let us define a sequence of functions $(\xi_n)_{n\in \mathbf{N}}$, by

$$\xi_1 : G \longrightarrow G \quad \text{by} \quad \xi_1((a)) = (a)$$
$$\xi_2 : G^2 \longrightarrow G^2 \quad \text{by} \quad \xi_2((a,b)) = (b,a)$$
$$\vdots$$
$$\xi_n : G^n \longrightarrow G^n \quad \text{by} \quad \xi_n((a_1,\ldots,a_n)) = (a_n,\ldots,a_1)$$
$$\vdots$$

So $(\xi_n)_{n\in\mathbf{N}}$[24] defines a sequence of functions in $G$, hence for ${}^*n = (n_i)_{i\in I}/_{\Delta}$, and ${}^*\xi_{*n} = \prod_{i\in I}(\xi_{n_i})/_{\Delta}$, $({}^*\xi_{*n})_{*n\in{}^*\mathbf{N}}$ is a sequence of internal functions in ${}^*G$. Now for ${}^*\xi_{*n}$, ${}^*\xi_{*n}(({}^*a_1^{-1}, \ldots, {}^*a_{*n}^{-1})) = ({}^*a_{*n}^{-1}, \ldots, {}^*a_1^{-1})$ and as ${}^*\xi_{*n}$ and $({}^*a_1^{-1}, \ldots, {}^*a_{*n}^{-1})$ are internal, so is $({}^*a_{*n}^{-1}, \ldots, {}^*a_1^{-1})$. It is clearly of length ${}^*n$ and is constructed of elements of ${}^*H$.

Thus we have ${}^*a_1 \cdot \ldots \cdot {}^*a_{*n} \in {}^*H^{(*\omega)}$, ${}^*a_{*n}^{-1} \cdot \ldots \cdot {}^*a_1^{-1} \in {}^*H^{(*\omega)}$, and if ${}^*H^{(*\omega)}$ is closed under multiplication, we have ${}^*a_1 \cdot \ldots \cdot {}^*a_{*n} \cdot {}^*a_{*n}^{-1} \cdot \ldots \cdot {}^*a_1^{-1} = {}^*e$ in ${}^*H^{(*\omega)}$. That is, ${}^*H^{(*\omega)}$ is closed under inversion.

It remains to show that ${}^*H^{(*\omega)}$ is closed under multiplication and that multiplication in ${}^*H^{({}^*\omega)}$ is associative. So let ${}^*a_1\cdot\ldots\cdot{}^*a_{*n}$, ${}^*b_1\cdot\ldots\cdot{}^*b_{*m}$ be elements of ${}^*H^{({}^*\omega)}$. Then there exist $k_1, k_2 \in \mathbb{N}$ such that ${}^*n \le k_1{}^*\omega$, ${}^*m \le k_2{}^*\omega$. Hence, ${}^*n + {}^*m \le (k_1+k_2){}^*\omega$. Thus ${}^*n+{}^*m \in \mathbb{N}({}^*\omega)$. It is now straight forward to show that ${}^*a_1\cdot\ldots\cdot{}^*a_{*n}\cdot{}^*b_1\cdot\ldots\cdot{}^*b_{*m} \in {}^*H^{({}^*\omega)}$, and that the multiplication in ${}^*H^{({}^*\omega)}$ is associative.

Therefore ${}^*H^{({}^*\omega)}$ is a subgroup of ${}^*G$.

□

Theorem 17

${}^*H^{\infty}$ is a subgroup of ${}^*G$.

Proof:

The proof is similar to that of theorem 16 and is left to the reader.

□

Theorem 18

Let ${}^*H$ be a normal subgroup of ${}^*G$. Then ${}^*H^{({}^*\omega)}$ and ${}^*H^{\infty}$ are normal subgroups of ${}^*G$.

Proof:

Let $({}^*h_1\cdot\ldots\cdot{}^*h_{*n}) \in {}^*H^{({}^*\omega)}$. Then $({}^*h_1,\ldots,{}^*h_{*n})$ is an internal sequence of elements of ${}^*H$ and ${}^*n \in \mathbb{N}({}^*\omega)$. As ${}^*H$ is normal, for any ${}^*g \in {}^*G$, $({}^*g{}^*h_1{}^*g^{-1}, \ldots, {}^*g{}^*h_{*n}{}^*g^{-1})$ is an internal sequence in ${}^*H$. Thus ${}^*g{}^*a_1{}^*g^{-1}\cdot{}^*g{}^*a_2{}^*g^{-1}\cdot\ldots\cdot{}^*g{}^*a_{*n}{}^*g^{-1} =$

${}^*g({}^*a_1 \cdot \ldots \cdot {}^*a_{*n}){}^*g^{-1} \in {}^*H^{(*\omega)}$ . That is, ${}^*H^{(*\omega)}$ is a normal subgroup of ${}^*G$ .

Let ${}^*h \in {}^*H^\infty$ . Then ${}^*h \in {}^*H^{(*\omega)}$ for some ${}^*\omega \in {}^*N$. Thus for any ${}^*g \in {}^*G$, ${}^*g\,{}^*h\,{}^*g^{-1} \in {}^*H^{(*\omega)}$ by the above. Thus ${}^*g\,{}^*h\,{}^*g^{-1} \in {}^*H^\infty$ . Hence, ${}^*H^\infty$ is a normal subgroup of ${}^*G$ .

□

The connected component of the identity, $G(e)$ , is the largest connected subset of $G$ containing $e$ . That is, the largest subset of $G$ containing $e$ that can not be decomposed into $A, B$ such that $A \neq \emptyset \neq B$ , $A \cap B = \emptyset$ and $A, B$ are both closed[25] subsets of $G$ . It is straight forward to show that $G(e)$ is a normal subgroup of $G$ . We now give a non-standard proof of a standard theorem.

Theorem 19

If $U$ is an open neighborhood of $e$ and $U \subseteq G(e)$ then $\bigcup_{n=1}^{\infty} U^n = G(e)$.

Proof:

Let $G' = \bigcup_{n=1}^{\infty} U^n$ . $G'$ is open as it is the union of open sets in $G$ . To show it is closed, let $a \in G$ be such that $a$ is in the closure of $\bigcup_{n=1}^{\infty} U^n$. Then any neighborhood $V$ of $a$ is such that $V \cap G' \neq \emptyset$ . Thus for ${}^*V = V^I/_\Delta$ , ${}^*G' = (G')^I/_\Delta$ ,

${}^*V \cap {}^*G' \neq \emptyset$ , for $V$ any neighborhood of $a$ in $G$ . Thus $\mu_\tau(a) \cap {}^*G' \neq \emptyset$[26]. Let ${}^*b \in {}^*U^{({}^*n)} = (U^I/_{\Delta})^{({}^*n)}$ . ${}^*b \in \mu_\tau(a)$ implies

${}^*ba^{-1} \in \mu_\tau(a)\mu_\tau(a^{-1}) = \mu_\tau(aa^{-1}) = \mu_\tau(e)$ . Thus $({}^*ba^{-1})^{-1} = a^{-1^{-1}}\,{}^*b^{-1} = a\,{}^*b^{-1} \in \mu_\tau(e)$ .

But $\mu_\tau(e) \subseteq {}^*U$ , thus $a\,{}^*b^{-1} \in {}^*U$. Hence ${}^*b \in {}^*U^{({}^*n)}$, so $a = a\,{}^*b^{-1}\,{}^*b \in {}^*U\,{}^*U^{({}^*n)} = {}^*U^{({}^*n+1)}$ . Hence $a \in {}^*G'$. So $a \in G'$ and hence $G'$ is closed.

$G(e) \supseteq G'$ , hence $G(e) - G'$ is open[28] and $G(e)$ is both open[29] and closed. Also, $G(e) - G'$ is closed as $G'$ is open. Thus $G(e) = G'$ for otherwise $G(e)$ is not connected, a contradiction.

$\square$

# CHAPTER III

# INFINITE GALOIS THEORY

Let $F$ be a commutative field. Let $\Phi$ be an infinite, normal, seperable extension of $F$. We know that if $\Phi$ is a finite extension of $F$, then there is a 1:1 correspondence between the extensions of $F$ that are subfields of $\Phi$ and the subgroups of the group of automorphisms on[30] that leave $F$ invariant. If $\Phi$ is an infinite extension of $F$, this correspondence fails to be 1:1. So let us analyze by non-standard methods the relationship between extensions of $F$ that are subfields of $\Phi$ and the group $G$ of automorphisms of $\Phi$ that leave $F$ invariant. We shall do this with the aid of an infinite index set $I$ and a non-principal ultrafilter $\Delta$ on $I$. We shall assume that $\Delta$ is adequate for $2^{\mathrm{CARD}(G)}$ (as noted in chapter 2). We shall also require one further degree of adequacy of $\Delta$ which will be noted accordingly. We shall denote $\Phi^I/\Delta$ as ${}^*\Phi$ and $F^I/\Delta$ as ${}^*F$ throughout this chapter.

Lemma 1

Let $G$ be the Galois group of $\Phi/F$, that is, the group of all automorphisms of $\Phi$ that leave $F$ invariant. Then ${}^*G = G^I/\Delta$ is the Galois group of internal automorphisms of ${}^*\Phi$ that leave ${}^*F$ invariant. ${}^*G$ is a subgroup of the Galois group of ${}^*\Phi/{}^*F$ , ${}^*G^{\square}$ say.

Proof:

As ${}^*G$ is internal, it consists of internal automorphisms, all of which are automorphisms of ${}^*\Phi$ and leave ${}^*F$ invariant as $\{i \mid \sigma_i \Phi = \Phi$ AND $\sigma_i$ LEAVES $F$ INVARIANT$\} = I \in \Delta$ for ${}^*\sigma = (\sigma_i)_{i \in I/\Delta} \in {}^*G$ .

Let ${}^*\sigma$ be any internal automorphism of ${}^*\Phi$ that leaves ${}^*F$ invariant. Then ${}^*\sigma = (\sigma_i)_{i \in I/\Delta}$ and $\{i \mid \sigma_i \Phi = \Phi$ AND $\sigma_i$ LEAVES $F$ INVARIANT$\} = J \in \Delta$ . For $i \notin J$, let $\alpha_i$ be the identity automorphism. Then ${}^*\sigma = (\sigma_i)_{i \in I/\Delta} = (\gamma_i)_{i \in I/\Delta}$ for $\gamma_i = \sigma_i$ if $i \in J$, $\gamma_i = \alpha_i$ if $i \notin J$. But $\gamma_i \in G$ for all $i \in I$ , hence ${}^*\sigma \in {}^*G$ . Hence ${}^*G$ consists of all automorphisms on ${}^*\Phi$ that leave ${}^*F$ invariant and that are internal.

${}^*G$ is clearly a subgroup of ${}^*G^{\square}$ as the composition and inversion of internal automorphisms are internal.

□

Let $\Xi$ index the finite, normal, algebraic extensions $K_\xi$ of $F$ that are subfields of $\Phi$. For each $\xi \in \Xi$, let $\Gamma_\xi = \{\gamma \mid \gamma \in \Xi$ AND $K_\gamma \supseteq K_\xi\}$ . Let $\mathbb{\Gamma} = \{\Gamma_\xi \mid \xi \in \Xi\}$ .

Theorem 20

$\Gamma$ is closed under the taking of finite intersections.

Proof:

Let $\Gamma_\alpha, \Gamma_\beta \in \Gamma$. Then $K_\alpha = F(a_1, \ldots, a_n)$ say, and $K_\beta = F(b_1, \ldots, b_m)$ say. Let $K_\tau = F(a_1, \ldots, a_n, b_1, \ldots, b_m)$. Then $K_\tau \supseteq K_\alpha$, $K_\tau \supseteq K_\beta$, thus for any $\eta \in \Gamma_\tau$, $K_\eta \supseteq K_\alpha$ and $K_\eta \supseteq K_\beta$. Hence $\eta \in \Gamma_\alpha$ and $\eta \in \Gamma_\beta$. Therefore $\Gamma_\tau \subseteq \Gamma_\alpha \cap \Gamma_\beta$. Let $x \in \Gamma_\alpha \cap \Gamma_\beta$. Then $K_x \supseteq F(a_1, \ldots, a_n)$ and $K_x \supseteq F(b_1, \ldots, b_m)$. Hence $K_x \supseteq F(a_1, \ldots, a_n, b_1, \ldots, b_m)$. Thus $x \in \Gamma_\tau$. That is, $\Gamma_\alpha \cap \Gamma_\beta = \Gamma_\tau$.

Assume for our induction hypothesis that any $k$-elements of $\Gamma$ have their intersection equal to a set in $\Gamma$.

Let $\Gamma_{\alpha_1}, \ldots, \Gamma_{\alpha_k}, \Gamma_{\alpha_{k+1}}$ be $k+1$ elements of $\Gamma$. Then $\Gamma_{\alpha_1} \cap \cdots \cap \Gamma_{\alpha_k} = \Gamma_\rho$ for some $\rho \in \Xi$, by the induction hypothesis. Hence $\Gamma_{\alpha_1} \cap \cdots \cap \Gamma_{\alpha_k} \cap \Gamma_{\alpha_{k+1}} = \Gamma_\rho \cap \Gamma_{\alpha_{k+1}} = \Gamma_\xi$ for some $\xi \in \Xi$ by the above. That is, the intersection of $n$-elements of $\Gamma$ is again an element of $\Gamma$. So $\Gamma$ is closed under finite intersections.

$\square$

Thus $\Gamma$ generates a filter on $\Xi$, by taking all supersets of elements of $\Gamma$. Call it $\hat{\Gamma}$. If $\boldsymbol{\Delta}$ is adequate for $\mathrm{CARD}(\Xi)$, there exists a function $g : I \to \Xi$ such that the filter generated by sets of the form $g(\Delta)$ for each $\Delta \in \boldsymbol{\Delta}$, contains $\hat{\Gamma}$. Henceforth we shall assume that $\boldsymbol{\Delta}$ is adequate for $\mathrm{CARD}(\Xi)$.

Now let ${}^*K = \prod_{i\in I} K_{g(i)} / \boldsymbol{\Delta}$. Then as $\{i \mid K_{g(i)}$ IS A FINITE, NORMAL, ALGEBRAIC EXTENSION OF $F$ AND SUBFIELD OF $\Phi\} = g^{-1}(\Xi) \in \boldsymbol{\Delta}$ by the construction of $g$, we have that ${}^*K$ is a star-finite (in the sense mentioned in chapter 2), normal, algebraic extension of ${}^*F$ and subfield of ${}^*\Phi$. Also, as $\{i \mid K_{g(i)} \supseteq K_\beta\} \supseteq g^{-1}(\Gamma_\beta) \in \boldsymbol{\Delta}$, we have that ${}^*K \supseteq K_\beta$, where $K_\beta$ in ${}^*\Phi$ means the standard points $\sigma \in {}^*\Phi$ such that $\sigma \in K_\beta$. As $\Phi = \bigcup_{\beta \in \Xi} K_\beta$, we have ${}^*K \supseteq \Phi$.

Now as, for each $i \in I$, $K_{g(i)}$ is a finite, normal, algebraic extension of $F$ and subfield of $\Phi$, there exists a corresponding Galois group $H_{g(i)}$ of automorphisms of $K_{g(i)}$ that leave $F$ invariant. By the nature of $g$, we see that ${}^*H = \prod_{i\in I} H_{g(i)} / \boldsymbol{\Delta}$ is an internal group of automorphisms of ${}^*K$ that leave ${}^*F$ invariant. By an argument similar to lemma 3, ${}^*H$ is the group of all internal automorphisms of ${}^*K$ that leave ${}^*F$ invariant. We can carry through in much the same manner the usual 1:1 Galois correspondence between the internal subfields of ${}^*K$ which extend ${}^*F$ and the internal subgroups of ${}^*H$.

Theorem 21

Let ${}^*\sigma \in {}^*H$. Then ${}^*\sigma$ , when restricted to $\Phi$ , is an automorphism of $\Phi$ .

Proof:

Since ${}^*\sigma \in {}^*H$, ${}^*\sigma$ is internal. So let ${}^*\sigma = \prod_{i\in I} \sigma_i/_\Delta$ . Now for each $i \in I$, $\sigma_i$ is an automorphism of $K_{g(i)}$ , and by a fundamental theorem of Galois theory, $\sigma_i$ is the restriction of an element of $G$ to $K_{g(i)}$ , say $\varphi_i$. Then ${}^*\sigma$ is the restriction of ${}^*\varphi = \prod_{i\in I} \varphi_i/_\Delta$ to ${}^*K$ , where ${}^*\varphi \in {}^*G$.

As $\Phi = \bigcup_{a\in\Phi-F} F(a)$ , we have ${}^*\sigma\, \Phi = {}^*\sigma \left( \bigcup_{a\in\Phi-F} F(a) \right) = \bigcup_{a\in\Phi-F} F({}^*\sigma a)$ as ${}^*\sigma$ leaves ${}^*F$, and hence $F$, invariant.

Now $a \in \Phi$ satisfies some minimal polynomial $f(x) \in F[x]$, that is, $f(a) = 0$. Thus as ${}^*\sigma(x^p) = ({}^*\sigma x)^p$ and ${}^*\sigma b = b$ for all $b \in F$, ${}^*\sigma$ can only map $a$ to one of the $k$-roots of $f(x) \in F[x]$ , say $a_1, \ldots, a_k$ in $\Phi$. Hence $\{i \mid \sigma_i a = a_1 \text{ or } \sigma_i a = a_2 \text{ or } \cdots \text{ or } \sigma_i a = a_k\} \in \Delta$ . But, similar to the argument in theorem 15, we see that $\{i \mid \sigma_i a = a_j\} \in \Delta$ for some $j \in \{1, 2, \ldots, k\}$ . That is, ${}^*\sigma a = a_j \in \Phi$ .

Thus ${}^*\sigma\Phi = \bigcup_{a\in\Phi-F} F({}^*\sigma a) \subseteq \Phi$ .

Similarly, ${}^*\sigma^{-1}\Phi \subseteq \Phi$ , so ${}^*\sigma\Phi = \Phi$. Thus any automorphism ${}^*\sigma$ of ${}^*K$ that leaves ${}^*F$ invariant is, when restricted to $\Phi$, an automorphism of $\Phi$ that necessarily leaves $F$ invariant.

□

Definition 26

For any ${}^*\sigma \in {}^*H$, let ${}^\circ({}^*\sigma) = {}^*\sigma|_\Phi$.

Then evidently ${}^\circ({}^*\sigma) \in G$ for all ${}^*\sigma \in {}^*H$ by theorem 21.

Now let $\Theta$ be a subfield of $\Phi$ and extension of $F$. Then ${}^*\Theta = \Theta^I/\Delta$ is an internal subfield of ${}^*\Phi$ and extension of ${}^*F$. Let ${}^*\Theta_K = {}^*\Theta \cap {}^*K$. This is again an internal subfield of ${}^*\Phi$ and extension of ${}^*F$ and, in fact, a subfield of ${}^*K$. As we have the Galois correspondence between the internal subfields of ${}^*K$ that are extensions of ${}^*F$ and the internal subgroups of ${}^*H$, let ${}^*H_\Theta$ be the subgroup corresponding to ${}^*\Theta_K$. Define ${}^\circ({}^*H_\Theta)$ by ${}^\circ({}^*H_\Theta) = \{T \mid T = {}^\circ({}^*\sigma) \text{ for some } {}^*\sigma \in {}^*H_\Theta\}$. Then ${}^\circ({}^*H_\Theta) \subseteq G$. In fact, as ${}^*H_\Theta$ is a subgroup of ${}^*G$, ${}^\circ({}^*H_\Theta)$ is a subgroup of $G$.

Lemma 4

$\Theta$ is the set of invariants of $\Phi$ under ${}^\circ({}^*H_\Theta)$.

Proof:

$\Theta \subseteq {}^*\Theta_K$, thus as ${}^*\sigma \in {}^*H_\Theta$ leaves ${}^*\Theta_K$ invariant, ${}^*\sigma$ leaves $\Theta$ invariant. Hence ${}^\circ({}^*\sigma)$ leaves $\Theta$ invariant.

Suppose $a \in \Phi - \Theta$. Then $a \notin \Theta$. Hence $a \notin {}^*\Theta$, thus $a \notin {}^*\Theta_K$. Therefore there exists a ${}^*\sigma \in {}^*H_\Theta$ such that ${}^*\sigma a \neq a$. $a \in \Phi$, hence ${}^\circ({}^*\sigma)a \neq a$. Thus $\Theta$ is precisely the set of invariants of $\Phi$ under ${}^\circ({}^*H_\Theta)$.

□

Lemma 5

Let $\sigma$ be an automorphism of $\Phi$ leaving $F$ invariant. Then for $D$ a finite, normal, algebraic extension of $F$ and subfield of $\Phi$, $\sigma(D) = D$.

Proof:

Suppose $D = F(a_1, \ldots, a_n)$. As $a_i$ satisfies a minimal polynomial $f_i(x)$ in $F[x]$, and since $D$ is normal, $\sigma a_i$ must be a root of $f_i(x)$, say $b_i$, which is necessarily in $D$ due to the normality of $D$. That is, $\sigma D \subseteq D$. Similarly $\sigma^{-1} D \subseteq D$. Thus $\sigma D = D$.

□

Corollary 5.1

For ${}^*\sigma$ any internal automorphism of ${}^*\Phi$ leaving ${}^*F$ invariant, if ${}^*D$ is an internal, star-finite, normal, algebraic extension of ${}^*F$ and subfield of ${}^*\Phi$ , then ${}^*\sigma\,{}^*D = {}^*D$ .

Proof:

This is a direct consequence of theorem 3 and lemma 5.

□

Lemma 6

${}^\circ({}^*H_\Theta)$ is precisely the set of automorphisms of $G$ leaving the elements of $\Theta$ invariant.

Proof:

Suppose $\sigma \in G$ is such that $\sigma$ leaves the elements of $\Theta$ invariant. Then the standard element $\sigma \in {}^*G$ leaves the elements of ${}^*\Theta$ invariant. Thus $\sigma|_{{}^*K}$ leaves the elements of ${}^*\Theta_K$ invariant. Thus by corollary 5.1, and the fact that ${}^*K$ is a star-finite, normal, algebraic extension of ${}^*F$ and subfield of ${}^*\Phi$ , $\sigma|_{{}^*K}$ is an automorphism of ${}^*K$ . Hence, as $\sigma|_{{}^*K}$ is internal, $\sigma|_{{}^*K}$ is in ${}^*H_\Theta$ . Hence ${}^\circ(\sigma|_{{}^*K}) = \sigma \in {}^\circ({}^*H_\Theta)$ which is a subset of $G$ .

Hence, using lemma 4, we have that ${}^\circ({}^*H_\Theta)$ is precisely the set of automorphisms of $G$

leaving the elements of $\Theta$ invariant.

$\square$

Hence we have a map $\nu$ from the subfields of $\Phi$ that are extensions of $\Gamma$ into the set of subgroups of $G$. We wish to characterize the range of $\nu$.

Definition 27

For any ${}^*\sigma \in {}^*G$, let ${}^\circ({}^*\sigma) = {}^\circ({}^*\sigma|_{{}^*K})$.

For any ${}^*\Sigma \subseteq {}^*G$, let ${}^\circ({}^*\Sigma) = \{\tau \mid \tau = {}^\circ({}^*\sigma)$ FOR SOME ${}^*\sigma \in {}^*\Sigma\}$.

Theorem 22

A subgroup $J$ of $G$ is in the image[31] of $\nu$ iff ${}^\circ({}^*J) = J$ for ${}^*J = J^I/_\Delta$.

Proof:

Suppose $J$ is in the range of $\nu$. Then $J = {}^\circ({}^*H_\Theta)$ for some field $\Theta$, $\Gamma \subseteq \Theta \subseteq \Phi$. Thus for any $\sigma \in J$, $\sigma$ leaves $\Theta$ invariant. Hence ${}^*\sigma \in {}^*J$ leaves ${}^*\Theta = \Theta^I/_\Delta$ invariant, for ${}^*J = J^I/_\Delta$. Also, as any ${}^*\tau \in {}^*G$ is internal, say ${}^*\tau = \prod_{i \in I} \tau_i/_\Delta$, then if ${}^*\tau$ leaves ${}^*\Theta$ invariant, we must have $\{i \mid \tau_i \in G$ AND $\tau_i$ LEAVES THE ELEMENTS OF $\Theta$ INVARIANT$\} = U_1 \in \Delta$. That is, $\{i \mid \tau_i \in J\} = U_1 \in \Delta$, so ${}^*\tau \in {}^*J$.[32]

Now ${}^\circ({}^*J)$ is such that ${}^\circ({}^*\sigma) \in {}^\circ({}^*J)$

leaves $\Theta$ invariant. Hence ${}^\circ({}^*\mathcal{J}) \subseteq \mathcal{J}$ , as $\mathcal{J} = {}^\circ({}^*H_\Theta)$ . Clearly $\mathcal{J} \subseteq {}^*\mathcal{J}$ by the natural embedding, thus $\mathcal{J} \subseteq {}^\circ({}^*\mathcal{J})$ . Hence $\mathcal{J} = {}^\circ({}^*\mathcal{J})$ .

Suppose ${}^\circ({}^*\mathcal{J}) = \mathcal{J}$. Let ${}^*\mathcal{J}_K$ be the group of automorphisms of ${}^*\mathcal{J}$ restricted to ${}^*K$ . By corollary 5.1, any element of ${}^*\mathcal{J}_K$ is an automorphism of ${}^*K$ .

Now ${}^*\mathcal{J}_K = \{{}^*\sigma|_{{}^*K} \mid {}^*\sigma \in {}^*\mathcal{J}\} = \prod_{i \in I} \mathcal{J}_{g(i)} / \Delta$ for $\mathcal{J}_{g(i)} = \{\sigma_i|_{K_{g(i)}} \mid \sigma_i \in \mathcal{J}\}$ . Hence ${}^*\mathcal{J}_K$ is internal.

So let ${}^*\Lambda$ be its corresponding subfield under the "usual" Galois correspondence. Then ${}^*\sigma \in {}^*\mathcal{J}_K$ leaves ${}^*\Lambda$ invariant. Thus $\Theta = {}^*\Lambda \cap \Phi$ is left invariant by the elements of ${}^*\mathcal{J}_K$ , hence by the elements of ${}^\circ({}^*\mathcal{J}_K) = {}^\circ({}^*\mathcal{J}) = \mathcal{J}$ .

If $a \notin \Theta$, $a \in \Phi$ , then $a \in {}^*K - {}^*\Lambda$. Thus there exists ${}^*\sigma \in {}^*\mathcal{J}_K$ such that ${}^*\sigma a \neq a$. Hence ${}^\circ({}^*\sigma)a \neq a$ , and ${}^\circ({}^*\sigma) \in \mathcal{J}$ . Therefore $\Theta$ is the set[33] of invariants under all members of $\mathcal{J}$. Thus ${}^*\Theta = \Theta^I/\Delta$ is the set of all invariants under all members of ${}^*\mathcal{J}$. Therefore ${}^*\Theta_K$ is the set of all invariants under all members of ${}^*\mathcal{J}_K$ . That is, ${}^*\Theta_K = {}^*\Lambda$ .

Thus ${}^*\mathcal{J}_K = {}^*H_\Theta$, hence $\mathcal{J} = {}^\circ({}^*\mathcal{J}) = {}^\circ({}^*H_\Theta)$ . That is, $\nu(\Theta) = \mathcal{J}$ .

$\square$

Using the above result, we may prove the following standard theorem:

Theorem 23

$\mathcal{J} = \nu(\Theta)$ is a normal subgroup of $\mathcal{G}$ iff $\Theta$ is a normal extension of $\mathcal{F}$.

Proof:

If $\mathcal{F} \subseteq \Theta \subseteq \Phi$ for $\Theta$ a normal extension of $\mathcal{F}$, then ${}^*\Theta = \Theta^I/\Delta$ is a normal extension of ${}^*\mathcal{F}$. Hence ${}^*\Theta_K$ is a normal extension of ${}^*\mathcal{F}$. Conversely, if ${}^*\Theta_K$ is a normal extension of ${}^*\mathcal{F}$ then ${}^*\Theta_K \cap \Phi$ is a normal extension of $\mathcal{F}$. That is, $\Theta$ is a normal extension of $\mathcal{F}$ iff ${}^*\Theta_K$ is a normal extension of ${}^*\mathcal{F}$.

If $\mathcal{J}$ is a normal subgroup of $\mathcal{G}$, ${}^*\mathcal{J}$ is a normal subgroup of ${}^*\mathcal{G}$, thus ${}^*\sigma\, {}^*\mathcal{J}\, {}^*\sigma^{-1} \subseteq {}^*\mathcal{J}$ for all ${}^*\sigma \in {}^*\mathcal{G}$. Thus by corollary 5.1, $({}^*\sigma|_{{}^*K})({}^*\mathcal{J}_K)({}^*\sigma^{-1}|_{{}^*K}) \subseteq {}^*\mathcal{J}_K$ for all ${}^*\sigma \in {}^*\mathcal{G}$. Thus ${}^*\mathcal{J}_K$ is a normal subgroup of ${}^*\mathcal{G}$. In fact, it is normal in the Galois group ${}^*\mathcal{H}$ of ${}^*K$ over ${}^*\mathcal{F}$.[34] Conversely, if ${}^*\mathcal{J}_K$ is a normal subgroup of ${}^*\mathcal{H}$, then ${}^\circ({}^*\mathcal{J}) = \mathcal{J}$ is normal in $\mathcal{G}$ by the previous theorem and the assumption that $\nu(\Theta) = \mathcal{J}$. Thus $\mathcal{J}$ is normal in $\mathcal{G}$ iff ${}^*\mathcal{J}_K$ is a normal subgroup of ${}^*\mathcal{H}$.

Now $^*J_K$ corresponds to $^*\Theta_K$ iff[35] $J$ corresponds to $\Theta$.[36] So suppose $\Theta$ is a normal extension of $F$. Then $^*\Theta_K$ being normal corresponds to $^*J_K$ under the "usual" Galois correspondence and the fundamental theorem of Galois theory tells us that $^*J_K$ must be normal in $^*H$. Hence $J$ is normal in $G$. Similarly, if $J$ is normal in $G$, $\Theta$ must be a normal extension of $F$.

□

We now state the existence theorem for the Krull topology on $G$.

Theorem 24

There is a topology $\tau$ on $G$ which is compatible with the group structure of $G$ and which has $\mathcal{N} = \{J \mid J$ IS A SUBGROUP OF $G$ AND THE SUBFIELD CORRESPONDING TO $J$ IS A FINITE EXTENSION OF $F\}$[37] as a fundamental system of neighborhoods of the identity.

Proof:

The proof is a standard one and may be found in [6].

□

Now consider the following statement: if $\mathcal{J}$ is open in the Krull topology on $G$ and if $\sigma \in \mathcal{J}$ then there exist $a_1, \ldots, a_n \in \Phi$ such that for all $\tau \in G$, if $\sigma a_1 = \tau a_1, \ldots, \sigma a_n = \tau a_n$, then $\tau \in \mathcal{J}$.

If this statement is false, then for any $a_1, \ldots, a_n \in \Phi$ there exists a $\tau \in G$ such that $\sigma a_1 = \tau a_1, \ldots, \sigma a_n = \tau a_n$ and $\tau \notin \mathcal{J}$.[38] Then the following relation, $R_\sigma(a, \gamma)$, defined by $(a, \gamma) \in R_\sigma$ iff $a \in \Phi$ and $\gamma \notin \mathcal{J}$ and $\sigma a = \gamma a$, is concurrent. Then by theorem 4, we see that for $\Omega = \{\Omega_x \mid x \in \mathrm{Dom}\, R_\sigma \text{ AND } \Omega_x = \{\gamma \mid (x, \gamma) \in R_\sigma\}\}$, if $\Delta$ is adequate for $\kappa = \mathrm{CARD}(G)$, then there exists a function $h: I \longrightarrow G$ such that for any $\Omega_x \in \Omega$, there exists $\Delta_x \in \Delta$ such that $h(\Delta_x) \subseteq \Omega_x$. That is, $\{i \mid (x, h(i)) \in R_\sigma\} \in \Delta$. So ${}^*h = (h(i))_{i \in I}/_\Delta$ is such that ${}^*h a = \sigma a$ for all $a \in \Phi$, and ${}^*h \notin \mathcal{J}^I/_\Delta$. Thus ${}^\circ({}^*h) = \sigma \notin \mathcal{J}$, a contradiction, provided $\Delta$ is adequate for $\kappa = \mathrm{CARD}(G)$. But in chapter 2 we assumed $\Delta$ adequate for $\kappa = 2^{\mathrm{CARD}(G)}$, hence it must be adequate for $\mathrm{CARD}(G)$ also.

Theorem 25

For any ${}^*\sigma \in {}^*G$, ${}^*\sigma \in \not\mu_\tau({}^\circ({}^*\sigma))$.

Proof:

Let ${}^\circ({}^*\sigma) = \tau$. Then for every $a \in \Phi$, ${}^*\sigma a = \tau a$. Thus for every $a \in \Phi$, $\{i \mid \sigma_i a = \tau a\} \in \Delta$. Hence for any finite collection of elements of $\Phi$,

$a_1, \ldots, a_n$ say, $\{i \mid \sigma_i a_1 = \tau a_1, \ldots, \sigma_i a_n = \tau a_n\} = \bigcap_{j=1}^{n} \{i \mid \sigma_i a_j = \tau a_j\} \in \Delta$ as ultrafilters are closed under finite intersections.

Thus, by the observation made above, we see that for $J$ any open neighborhood of $\tau$, $\{i \mid \sigma_i \in J\} \in \Delta$, hence ${}^*\sigma \in J^I/_\Delta$. As $J$ was arbitrary, we necessarily have ${}^*\sigma \in \mu_\tau(\tau)$.

□

Corollary 25.1

$(G, \tau)$ is compact.

Proof:

This is a direct consequence of theorem 15 and theorem 25.

□

Theorem 26

$(G, \tau)$ is Hausdorff.

Proof:

Due to the continuity of multiplication, it suffices to show that $\bigcap_{J \in \eta} J = \{e\}$.

Let $\sigma \in \bigcap_{J \in \eta} J$. Then $\sigma$ leaves each element of each finite extension of $F$ and subfield of $\Phi$ invariant. As $\Phi = \bigcup_{a \in \Phi - F} F(a)$, we see that $\sigma$ is necessarily the identity.

□

Theorem 27

$\mathcal{J}$ is closed in the Krull topology iff $°({}^*\mathcal{J}) = \mathcal{J}$.

Proof:

Let $°({}^*\mathcal{J}) = \mathcal{J}$. $\sigma$ in the closure of $\mathcal{J}$ implies $\mu_\tau(\sigma) \cap {}^*\mathcal{J} \neq \phi$, for otherwise $\mu_\tau(\sigma)$ is contained in the complement of ${}^*\mathcal{J}$. But in this case, as $\sigma \in {}^*V \subseteq \mu_\tau(\sigma)$ for some ${}^*V$ open in the Q-topology on ${}^*G$ as seen in lemma 2, we see that ${}^*\mathcal{J}$ is contained in the complement of ${}^*V$ which is closed. Thus $\sigma$ is not in the closure of ${}^*\mathcal{J}$, hence not in the closure of $\mathcal{J}$.

Hence there exists ${}^*\xi \in {}^*\mathcal{J} \cap \mu_\tau(\sigma)$. Hence, as ${}^*\xi \in \mu_\tau(°({}^*\xi))$ by theorem 25, and that $\mu_\tau(\alpha) \cap \mu_\tau(\beta) = \phi$ if $\alpha \neq \beta$ by theorem 26 and theorem 9, we see that $°({}^*\xi) = \sigma$. But $°({}^*\mathcal{J}) = \mathcal{J}$, $°({}^*\xi) = \sigma$, hence $\sigma \in \mathcal{J}$. Thus $\mathcal{J}$ is closed in the Krull topology on $G$.

Let $\mathcal{J}$ be closed in the Krull topology on $G$. Now $\mathcal{J} \subseteq °({}^*\mathcal{J})$ as each element of $\mathcal{J}$ is a standard element of ${}^*\mathcal{J}$ under the natural embedding. Let $°({}^*\pi) \in °({}^*\mathcal{J})$. Then ${}^*\pi \in {}^*\mathcal{J}$. So $\mu_\tau(°({}^*\pi)) \cap {}^*\mathcal{J} \neq \phi$ as ${}^*\pi \in \mu_\tau(°({}^*\pi))$ by theorem 25. So $°({}^*\pi)$ is not in the complement of $\mathcal{J}$ as it is open, hence by lemma 2,

$$\mu_\tau(\sigma) \subseteq {}^*\Gamma_{\mathcal{J}} = \Gamma_{{}^*\mathcal{J}} \text{ for any } \sigma \in \Gamma_{\mathcal{J}} .$$

So ${}^\circ({}^*\pi) \in \mathcal{J}$ . That is, ${}^\circ({}^*\mathcal{J}) = \mathcal{J}$ .

$\square$

Also, $(G, \tau)$ is totally disconnected, for if $\sigma \notin H \in \eta$, then $\sigma H$ is an open set containing $\sigma$ and $H \cap \sigma H = \phi$. Hence $H$ is closed and, utilizing the fact that $(G, \tau)$ is Hausdorff, $(G, \tau)$ is totally disconnected.

## FOOTNOTES

1. This should read: "... large enough so that some subset of the constants may be put into one-to-one correspondence with ..." .

2. Change "... to each individual constant in ..." to " ... to some of the individual constants in...".

3. Note that $\hat{S}$ is a partial function from $T_c$ to $A'$.

4. Add the following clause to the definition of $\hat{S}$ :
"iii) if $t = \langle t_1, \dots, t_n \rangle$ and $\hat{S}$ has been defined
for $t_1, \dots, t_n$, then $\hat{S}(t) = (\hat{S}(t_1), \dots, \hat{S}(t_n))$ ."

5. Change "... iff is true..." to "... iff $\mathfrak{A}$ is true..."

6. Change "se" to "we" .

7. Delete this paragraph (beginning with "We know that...") and insert in its place the following paragraph:

"Using the axiom of choice we can well-order the set $C$ of constants of $\mathcal{K}$ placing them in one-to-one correspondence with all of the ordinals less than some initial ordinal $\kappa$ . We shall agree to consider only structures in which the constants given an assignment from an initial segment of $C$ . So we may now write our structure $\mathfrak{A}$ as $\mathfrak{A} = \langle A, R_1, \dots, R_\alpha, \dots \rangle$, where $\alpha$ runs through some initial segment of ordinals, of order type $\rho$ say, and where $R_\alpha$ is the object of finite type in $A$ which is assigned to the $\alpha^{\text{th}}$ constant. We call $\rho$ the order of $\mathfrak{A}$ .

8. Delete definition 14. In its place insert the following introductory paragraphs and revised definition :

"We define $\prod_{i\in I} A_i/_\Delta$ to be the set of equivalence classes $f/_\Delta$ where $f$ is a function on $I$ with $f(i) \in A_i$ and where the equivalence relation is $\equiv_\Delta$ defined by $f \equiv_\Delta g$ iff $\{i \mid f(i) = g(i)\} \in \Delta$ .

"We now proceed to define by induction on $\sigma$ the object $\prod_{i\in I} S_i/_\Delta$ where for all $i \in I$ , $S_i$ is of type $\sigma$ in $A_i$ . Further we shall show that $\prod_{i\in I} S_i/_\Delta$ has type $\sigma$ . When $\sigma = 0$ , define $\prod_{i\in I} S_i/_\Delta$ to be $f/_\Delta$ where $f(i) = S_i$ for all $i \in I$ . In this case, $\prod_{i\in I} S_i/_\Delta$ is certainly of type $0$ in $\prod_{i\in I} A_i/_\Delta$ .

"Suppose now that $\prod_{i\in I} R_i/_\Delta$ has been defined of type $\tau$ in $\prod_{i\in I} A_i/_\Delta$ whenever $R_i$ is of type $\tau$ for all $i \in I$ . Then if $S_i$ is of type $\sigma = (\tau)$ in $A_i$ for all $i \in I$, we define

$$\prod_{i\in I} S_i/_\Delta = \Big\{ \prod_{i\in I} R_i/_\Delta \,\Big|\, R_i \in S_i \text{ FOR ALL } i \in I \Big\}$$

which is clearly of type $(\tau)$ in $\prod_{i\in I} A_i/_\Delta$ .

"Suppose now that $\sigma = (\tau_1, \ldots, \tau_n)$ and that for each $j$ in $1 \le j \le n$ , $\prod_{i\in I} R_i^j/_\Delta$ has been defined of type $\tau_j$ in $\prod_{i\in I} A_i/_\Delta$ whenever $R_i^j$ is of type $\tau_j$ in $A_i$ for all $i \in I$ . Then we define

$\prod_{i\in I} S_i/_{\Delta} = \langle \prod_{i\in I} R_i^1/_{\Delta}, \ldots, \prod_{i\in I} R_i^n/_{\Delta} \rangle$
where $S_i = \langle R_i^1, \ldots, R_i^n \rangle$ is of type $\sigma$ in $A_i$ for each $i \in I$. Then clearly $\prod_{i\in I} S_i/_{\Delta}$ is of type $\sigma$ in $\prod_{i\in I} A_i/_{\Delta}$.

Definition 14

The reduced direct product of the family of mathematical structures $\{\mathfrak{A}_i \mid i \in I\}$ relative to $\Delta$ is

$$\prod_{i\in I} \mathfrak{A}_i/_{\Delta} = \langle \prod_{i\in I} A_i/_{\Delta}, \mathbb{R}_1, \ldots, \mathbb{R}_\alpha, \ldots \rangle$$

where $\mathbb{R}_\alpha = \prod_{i\in I} R_\alpha^i/_{\Delta}$.

If $\Delta$ is an ultrafilter, we call the reduced direct product an ultraproduct. If, in addition, $\mathfrak{A}_i = \mathfrak{A}$ for each $i \in I$, we call $\prod_{i\in I} \mathfrak{A}_i/_{\Delta} = \mathfrak{A}^I/_{\Delta}$ an ultrapower.

"The reader may now easily prove by a straight forward induction argument that $\prod_{i\in I} S_i^1/_{\Delta} = \prod_{i\in I} S_i^2/_{\Delta}$ iff $\{i \mid S_i^1 = S_i^2\} \in \Delta$."

9. $\cong$ is the usual isomorphism notation.

10. Definition 15 should read: "An object ${}^*B$ of type $\sigma$ in $A^I/_{\Delta}$ is internal iff there exists a family $\{S_i \mid i \in I\}$ of objects of type $\sigma$ in $A$ such that ${}^*B = \prod_{i\in I} S_i/_{\Delta}$."

11. Note that in the definition of $\mathfrak{A}^I/_{\Delta}$, all of the $\mathbb{R}_\alpha$'s are internal objects.

12. This should read: "Let ${}^*S = ({}^*S_1, \cdots)$ be any denumerable sequence of internal objects of arbitrary ..."

13. Change "... for $^*S_j = (S_j(i))_{i\in I}/_{\Delta}$ ." to "... where for each positive integer $j$ , $S_j$ is any function on $I$ such that $^*S_j = (S_j(i))_{i\in I}/_{\Delta}$ , and where $\models$ on the left is interpreted in terms of quantification restricted to internal objects."

14. Change "... on the length of ..." to "... on the length of $\mathcal{Q}$ ..."

15. The proof of theorem 13 should be amended as to be consistent with the $\hat{S}$ notation. For example, consider case 3 (on page 15). Let $t$ be a term and let $^*S_t$ be that internal object assigned to $t$ by $\hat{^*S}$ . Then we have: $\prod_{i\in I}\mathfrak{A}_i/_{\Delta} \models \Pi_\sigma(t)(^*s)$

    iff $\hat{^*S}(t)$ is of type $\sigma$ in $\prod_{i\in I}\mathfrak{A}_i/_{\Delta}$

    iff $^*S_t$ is of type $\sigma$ in $\prod_{i\in I}\mathfrak{A}_i/_{\Delta}$

    iff $S_t(i)$ is of type $\sigma$ in $\mathfrak{A}_i$ for each $i\in I$ , where $^*S_t = (S_t(i))_{i\in I}/_{\Delta}$

    iff $\{i \mid \mathfrak{A}_i \models \Pi_\sigma(S_t(i))\} = I \in \Delta$

    iff $\{i \mid \mathfrak{A}_i \models \Pi_\sigma(t)(S_1(i), S_2(i), \ldots)\} \in \Delta$.

16. $R_\beta$ is the object to which the $\beta^{TH}$ constant is mapped.

17. For "$b<\lambda$" read "$b\leq\lambda$".

18. For "$a\in A$" read "$a\in{}^*A$".

19. For "$\tau$ is a topology on" read "$\tau$ is a topology on $G$".

20. Note that the group operations in $^*G$ come from $G$ by

way of theorem 3 and appropriate $R_\alpha$'s.

21. For "... ${}^*T = T^I/\Delta$ be a neighborhood of ${}^*a$, for ${}^*T \in {}^*\mathcal{T}_Q$." read "... ${}^*T = \prod_{i \in I} T_i/\Delta \in {}^*\mathcal{T}_Q$ be a basic open neighborhood of ${}^*a$ in ${}^*G$."

22. Note that $({}^*W)a = {}^*(Wa)$ by theorem 3.

23. Note that we can define a finite sequence of elements of $G$ as being a function (i.e. a binary relation of the obvious type) defined on an initial segment of $N$ with values in $G$. So finite sequences have type $((0,0))$. Let $S$ represent the set of all finite sequences of elements in $G$. $S$ is of type $(((0,0)))$. Hence an internal sequence of elements of ${}^*G$ is an element of ${}^*S = S^I/\Delta$. Note that this gives us all sequences of star-finite length in ${}^*G$.

24. For $(\xi_n)_{n \in N}$, as well as for $(\pi_n)_{n \in N}$, we can define functions $\pi : S \to G$, $\xi : S \to S$ and $\ell : S \to N$ as functions corresponding to the product of all elements of a sequence, reversal of a sequence and the length of a sequence respectively. They are of type $((((0,0)),((0,0))))$, $((((0,0)),0))$ and $((((0,0)),0))$ respectively. Hence we may speak of their counterparts in ${}^*G$ by means of theorem 3. For example, ${}^*\xi_{*n}$ is the restriction of ${}^*\xi : {}^*S \to {}^*S$ to the elements ${}^*\sigma \in {}^*S$ such that ${}^*\ell({}^*\sigma) = {}^*n$.

25. For "... closed ..." read "... relatively closed..."

26. Note that $\mu_{\tau}(a) \cap {}^*G \neq \phi$ comes from the fact that the relation $R_a(V, U)$ defined by $V$ is a neighborhood of $a$ and $U$ is a neighborhood of $a$ and $V \supseteq U$ is concurrent. Hence as $\Delta$ is adequate for $2^{\mathrm{CARD}(G)}$, we see by the methods of theorem 4 that if $\{U_j \mid j \in J\}$ is the set of all open neighborhoods of $a$, we know that there is a function $f$ from $I$ to $J$ such that $\{i \mid U_{f(i)} \subseteq V\} \in \Delta$ for any neighborhood $V$ of $a$. Hence $\mu_{\tau}(a) \supseteq {}^*U = \prod_{i \in I} U_{f(i)}/\Delta$ and as $\{i \mid U_{f(i)} \cap G' \neq \phi\} = I \in \Delta$ as $a$ is in the closure of $G'$, we have ${}^*U \cap {}^*G' \neq \phi$. Hence $\mu_{\tau}(a) \cap {}^*G' \neq \phi$ .

27. Note that we define $U^n$ by means of a function $F$ from pairs of the form $\langle U, n\rangle$ to the powerset of $G$ such that $F(\langle U, n\rangle) = U^n = \{u_1 \cdot \ldots \cdot u_n \mid u_i \in U \text{ for } 1 \leq i \leq n\}$ . Hence we may consider the corresponding function ${}^*F$ in ${}^*G$ which maps pairs of the form $\langle {}^*U, {}^*n\rangle$ into the powerset of ${}^*G$ for ${}^*U$ an internal subset of ${}^*G$ such that ${}^*F(\langle {}^*U, {}^*n\rangle) = {}^*U^{({}^*n)}$ .

28. For "... open ..." read "... relatively open ..."

29. For "... open ..." read "... relatively open ..."

30. For "... automorphisms on ..." read "... automorphisms on $\Phi$ ..."

31. For "... image ..." read "... range ..."

32. Note that the two lines beginning with "Also, as any ..." and "That is, ..." are unnecessary for the proof of theorem 22 and may be deleted.

33. For "... is the set ..." read "... is precisely the set ..."

34. Delete "Thus ${}^*\mathcal{J}_K$ is a normal subgroup of ${}^*G$. In fact, it is normal in the Galois group ${}^*H$ of ${}^*K$ over ${}^*F$." Insert the following: "${}^*\mathcal{J}_K$ is thus a normal subgroup of ${}^*H$ the internal Galois group of ${}^*K$ over ${}^*F$.

35. For "... iff ..." read "... if ..."

36. This correspondence comes from the assumption that $\nu(\Theta) = \mathcal{J}$ and that ${}^*\mathcal{J}_K = {}^*H_\Theta$ by theorem 3 and corollary 5.1.

37. The definition of $\eta$ should read as follows:
"$\eta = \{\mathcal{J} \mid \mathcal{J}$ IS A SUBGROUP OF $G$ AND $\mathcal{J} = G(K/F)$ FOR SOME FIELD $K$ THAT IS A SUBFIELD OF $\Phi$ AND FINITE EXTENSION OF $F\}$."

38. Delete this entire paragraph (lines 5-18, page 52) and insert in its place the following paragraph:

"If $\mathcal{J}$ is open, then $\mathcal{J} = \bigcup \tau_i \mathcal{J}_i'$ for an appropriate collection of $\mathcal{J}_i' \in \eta$ and $\tau_i \in G$. Thus any $\sigma \in \mathcal{J}$ is such that $\sigma = \tau_i \sigma'$ for an appropriate $\sigma'$ in some $\mathcal{J}_i'$. Now $\mathcal{J}_i'$ consists of all automorphisms leaving some finite extension of $F$

invariant, say $F(a_1,\dots,a_n)$. Then if $\sigma'' a_i = \sigma a_i$ for $i=1,\dots,n$, then $\tau_i^{-1}\sigma'' a_i = \tau_i^{-1}\sigma a_i = \tau_i^{-1}\tau_i\sigma' a_i = a_i$, hence $\tau_i^{-1}\sigma'' \in \mathfrak{J}_i'$. Thus $\tau_i\tau_i^{-1}\sigma'' = \sigma'' \in \mathfrak{J}$. So the statement is true."

39. Delete this entire paragraph (lines -1 through -7 on page 54 and lines 1,2 on page 55) and insert the following paragraph:

"Let $\mathfrak{J}$ be closed in the Krull topology on $\mathcal{G}$. If $\sigma \in {}^{\circ}({}^*\mathfrak{J})$ but $\sigma \notin \mathfrak{J}$ then as $\mathfrak{J}$ is closed, there exists an open set $V$ such that $\sigma \in V \subseteq \complement\mathfrak{J}$. Thus the standard element $\sigma \in {}^*\mathcal{G}$ is such that $\sigma \in \mu_{\tau}(\sigma) \subseteq V^I/\!\Delta \subseteq (\complement\mathfrak{J})^I/\!\Delta = {}^*\complement\mathfrak{J}$, so $\mu_{\tau}(\sigma) \subseteq {}^*\complement\mathfrak{J} = \complement{}^*\mathfrak{J}$ by theorem 3. Now $\sigma = {}^{\circ}({}^*\tau)$ for some ${}^*\tau \in {}^*\mathfrak{J}$. So we have that ${}^*\tau \in \mu_{\tau}(\sigma)$ by theorem 25. Thus $\mu_{\tau}(\sigma) \cap {}^*\mathfrak{J} \neq \emptyset$, a contradiction so ${}^{\circ}({}^*\mathfrak{J}) = \mathfrak{J}$.

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