DENSITIES AND SUMMABILITY

by

Chong Suh Chun

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Name: Chong Suh Chun

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Examining Committee:

Chairman: Prof. C. Villegas

Dr. A.R. Freedman
Senior Supervisor

Dr. J.J. Sember

Dr. B.S. Thomson

Dr. A. Mekler

Dr. Bruce Shawyer
Department of Mathematics and Statistics
Memorial University

Date Approved: April 3, 1986

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Author: ______________________

(signature)

CHONG SUH CHUN

(name)

April 9 1986

(date)

ABSTRACT

The concept of density of a set of positive integers is introduced along with some of the basic properties. A very general density is defined in terms of a sequence of non-negative regular matrices and two filters. It is shown that most of the known densities, i.e., matrix method densities, 0-1 densities, uniform density and some complete densities are subsumed under the general formulation.

The class of sets of upper density zero are called zeroclasses. Special zeroclasses are studied, in particular zeroclasses consisting of lacunary sets. Some surprising inclusions between some of these are proved.

An R-type summability method (RSM), S, is a regular linear functional on a real sequence space c_s such that $|c_s|^0$, the set of all sequences which are s-strongly summable to zero, is a solid subspace of c_s. It is shown that s is non-negative and continuous. A Bounded Consistency type theorem for the strong convergence fields of RSMs is proved. RSMs and non-negative regular summabilities are compared and interesting matrix methods are examined. Progress is made regarding the characterization of RSMs in terms of densities and zeroclasses.

DEDICATION

This thesis is dedicated to my father.

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CHAPTER 0

INTRODUCTION

In this thesis we shall discuss three aspects of sequence space theory. The first is general density theory, the second is the concept of zeroclasses consisting of lacunary sets and, lastly the theory of strong summability with respect to R-type summability methods. Most of our work extends that of Freedman and Sember [3], [4], [8].

In Chapter 1 we define a general set function $d = d_{\mathcal{M},\mathcal{G},\mathcal{F}}(A)$ where A is a subset of I, the set of positive integers. $\mathcal{M}$ is a certain type of sequence of infinite matrices and $\mathcal{G}$ and $\mathcal{F}$ are filters on I. It turns out that d is a density (see Definition 1.2) on I. By judicious choices of $\mathcal{M}$, $\mathcal{G}$ and $\mathcal{F}$, d can become identical with any of densities considered in the literature: e.g., matrix method densities, 0-1 densities, uniform density and complete densities.

Chapter II comprises combinatorial results concerning classes of lacunary sets. Such classes arise naturally in some sequence space and combinatorial studies. For example, Freedman [5] has shown that the class $\mathcal{L}$ of lacunary sets corresponds exactly to the set of 0-1 sequences in the space $bs + c$ where bs is the space of sequences with bounded partial sums and c is the space of all convergent sequences. More recently, Brown and Freedman [2], have shown that the famous conjecture of Erdös (that the set $A \subseteq I$ has arbitrarily long arithmetic progressions whenever $\Sigma\,\{\frac{1}{a} : a \in A\} = \infty$)

is true if and only if it is true for every lucunary set and that the conjecture does indeed hold when A is an L_1-lucunary set (see definition 2.1). In Chapter II we introduce many subclasses of lucunary sets, show which of them are full (see definition 2.4), [8] and relate them to one another by means of set theoretic inclusions of which some are surprising.

In Chapter III we carry on the investigation of R-type summability methods (RSMs) introduced by Freedman and Sember [3]. The connection between RSMs and densities is made clear through the use of an analytic definition based on the concept of zeroclass. Our efforts culminate with a somewhat surprising result which amounts to bounded consistency for RSMs on the associated strong summability field. The result does not require that the RSM be generated by a regular matrix and so is in a sense not comparable to the traditional bounded consistency theorem (BCT). On the other hand, since it applies only to the strong convergence field, it does not require the powerful analytic machinery for its proof as does the traditional BCT.

Finally, Chapter III attempts to add to the knowledge of RSMs that are generated by regular matrices. This, as the reader will see, is a difficult topic.

Many of the propositions are well known but we have not bothered to cite sources.

Our special notation will be introduced as needed. The notation used in set theoretic or sequence space discussions is all standard. A list of symbols and their definitions can be found in the Appendix.

CHAPTER I

GENERAL DENSITIES

In this Chapter, a general concept of density is defined. In particular, density is defined in terms of a sequence of non-negative regular matrices and two filters. Many of the standard densities will be subsumed under our definition. These include ordinary asymptotic density, uniform density, non-negative regular matrix densities and 0-1 densities defined by zeroclasses. [3], [9]

Definition 1.1. Two subsets, A and B of I are asymptotic if $A \Delta B$ is finite. ($A \Delta B$ means the symmetric difference of A and B). In this case we write $A \sim B$.

Definition 1.2. [3] A function $\delta: 2^I \to R$ is called a lower asymptotic density (or just a density) if the following five axioms hold:

(D1) for each $A \in 2^I$, $0 \le \delta(A) \le 1$;

(D2) if $A \sim B$, then $\delta(A) = \delta(B)$;

(D3) if $A \cap B = \emptyset$, then $\delta(A) + \delta(B) \le \delta(A \cup B)$;

(D4) for all A, B, $\delta(A) + \delta(B) \le 1 + \delta(A \cap B)$;

(D5) $\delta(I) = 1$.

Definition 1.3. If δ is a density, we define $\bar{\delta}: 2^I \to R$, the upper density associated with δ , by $\bar{\delta}(A) = 1 - \delta(A^c)$ where $A^c = I - A$.

At first, we will list some basic properties of δ and $\bar{\delta}$ omitting most of the proofs since the verifications involve only simple arguments and most appear in [3].

Proposition 1.4. Let δ be a lower asymptotic density and $\bar{\delta}$ its associated upper density. For $A,B \in 2^I$, we have

(i) $A \subset B \Rightarrow \delta(A) \leq \delta(B)$,

(ii) $A \subset B \Rightarrow \bar{\delta}(A) \leq \bar{\delta}(B)$,

(iii) $\delta(\emptyset) = \bar{\delta}(\emptyset) = 0$,

(iv) $A \cap B = \emptyset \Rightarrow \delta(A \cup B) \leq \delta(A) + \bar{\delta}(B)$,

(v) $A \sim B \Rightarrow \bar{\delta}(A) = \bar{\delta}(B)$,

(vi) For all A, $\delta(A) \leq \bar{\delta}(A)$,

(vii) For all A,B, $\bar{\delta}(A) + \bar{\delta}(B) \geq \bar{\delta}(A \cup B)$,

(viii) $A \cup B = I \Rightarrow \bar{\delta}(A) + \bar{\delta}(B) \geq 1 + \bar{\delta}(A \cap B)$.

Proof: (iv) Suppose that $A \cap B = \emptyset$. By (D3) $\delta(A \cup B) + \delta(B^c) \leq 1 + \delta((A \cup B) \cap B^c)$. On the other hand, $A \cap B = \emptyset$ implies $(A \cup B) \cap B^c = A$. Thus $\delta(A \cup B) + \delta(B^c) \leq 1 + \delta(A)$. Therefore we have $\delta(A \cup B) \leq \delta(A) + 1 - \delta(B^c) = \delta(A) + \bar{\delta}(B)$.

(viii) Suppose that $A \cup B = I$ and so we get $A^c \cap B^c = \emptyset$. By (D2) $\delta(A^c) + \delta(B^c) \leq \delta(A^c \cup B^c)$. Therefore $1 - \delta(A^c) + 1 - \delta(B^c) \geq 1 + 1 - \delta(A^c \cup B^c)$. Thus we get $\bar{\delta}(A) + \bar{\delta}(B) \geq 1 + \bar{\delta}(A \cap B)$.

Definition 1.5. Let δ and $\bar{\delta}$ be associated lower and upper densities. We define

$$\eta_\delta = \{A \subset I : \delta(A) = \bar{\delta}(A)\} ,$$

$$\eta_\delta^0 = \{A \subset I : \bar{\delta}(A) = 0\} .$$

We say that $A \subset I$ has natural density (resp. has natural density zero) with respect to δ in case $A \in \eta_\delta$ (resp. $A \in \eta_\delta^0$).

Note that $A \in \eta_\delta$ and $\bar{\delta}(A) = \delta(A) = 0$ if and only if $\bar{\delta}(A) = 0$.

The basic facts concerning η_δ and η_δ^0 are contained in the following proposition. We omit the proofs.

Proposition 1.6. For any $A,B \in 2^I$.

(i) $A \sim I \Rightarrow A \in \eta_\delta$,

(ii) $A \sim \phi \Rightarrow A \in \eta_\delta^0$,

(iii) $A \in \eta_\delta$ and $A \sim B \Rightarrow B \in \eta_\delta$.

Definition 1.7. A class $\mathcal{X}$ of subsets of I will be called a zeroclass [4] if the following conditions hold:

(i) A is finite $\Rightarrow A \in \mathcal{X}$,

(ii) $A,B \in X \Rightarrow A \cup B \in \mathcal{X}$,

(iii) $A \subset B \in \mathcal{X} \Rightarrow A \in \mathcal{X}$,

(iv) $I \notin \mathcal{X}$.

Note that a zeroclass is just a non principal ideal on 2^I .

Proposition 1.8. If $\delta : 2^I \to R$ is a lower asymptotic density then η_δ^0 is a zeroclass.

Proof: (i) If A is finite, then $A \sim \phi$ and so $\phi \in \eta_\delta^0$.
(ii) Let $A,B \in \eta_\delta^0$ so that $\bar{\delta}(A) = 0$ and $\bar{\delta}(B) = 0$. By Proposition 1.4. (vii) $\bar{\delta}(A \cup B) \le \bar{\delta}(A) + \bar{\delta}(B) = 0$. Therefore $\bar{\delta}(A \cup B) = 0$. Hence $A \cup B \in \eta_\delta^0$.
(iii) If $A \subset B \in \eta_\delta^0$, then $0 \le \bar{\delta}(A) \le \bar{\delta}(B) = 0$. Thus $\bar{\delta}(A) = 0$ and so $A \in \eta_\delta^0$.
(iv) Since $\bar{\delta}(I) = 1 - \bar{\delta}(\phi) = 1$, we have $I \notin \eta_\delta^0$.

<u>Definition 1.9</u>. A filter on a set X is a family F of subsets of X which has the following properties:

(i) $A,B \in F \Rightarrow A \cap B \in F$,

(ii) $A \subset B$ and $A \in F \Rightarrow B \in F$,

(iii) $\phi \notin F$.

<u>Definition 1.10</u>. Let $F_0 = \{A \in 2^I : A^c \text{ is finite}\}$. Then F_0 is called the Fréchet filter.

Remark: If X is a zeroclass then $F_X = \{A^c \mid A \in X\}$ is a filter finer than the Fréchet filter, i.e., $F_X \supset F_0$.

In order to introduce a particular method of constructing densities, we will first present several lemmas. In these lemmas, G and F will be filters on I.

<u>Lemma 1.11</u>: For any positive integer m, n let $P(m,n)$ and $Q(m,n)$ be corresponding statements such that for each m, n,

$$P(m,n) \Rightarrow Q(m,n) .$$

Then we have

(i) for each n

$$\{m: P(m,n) \text{ is true}\} \in \mathcal{G}$$

$$\Rightarrow \{m: Q(m,n) \text{ is true}\} \in \mathcal{G} .$$

(ii) $$\{n: \{m: P(m,n) \text{ is true}\} \in \mathcal{G}\} \in \mathcal{F}$$

$$\Rightarrow \{n: \{m: Q(m,n) \text{ is true}\} \in \mathcal{G}\} \in \mathcal{F} .$$

Proof: (i) For each n, by the hypothesis, $\{m: P(m,n) \text{ is true}\} \subset \{m: Q(m,n) \text{ is true}\}$. Since $\mathcal{G}$ is a filter, we have the result.

(ii) By (i), $\{n: \{m: P(m,n) \text{ is true}\} \in \mathcal{G}\} \subseteq \{n: \{m: Q(m,n) \text{ is true}\} \in \mathcal{G}\}$. Since $\mathcal{F}$ is a filter, we have the result.

Corollary 1.12. For any $m,n \in I$, let $S(m,n)$ and $T(m,n)$ be corresponding real numbers with $S(m,n) \leq T(m,n)$. Then for any $\alpha \in R$, we have

(i) for each $n \in I$,

$$\{m: \alpha \leq S(m,n)\} \in \mathcal{G} \Rightarrow \{m: \alpha \leq T(m,n)\} \in \mathcal{G} .$$

(ii) $$\{n: \{m: \alpha \leq S(m,n)\} \in \mathcal{G}\} \in \mathcal{F}$$

$$\Rightarrow \{n: \{m: \alpha \leq T(m,n)\} \in \mathcal{G}\} \in \mathcal{F} .$$

Proof: Take $P(m,n)$ to mean $\alpha \leq S(m,n)$ and $Q(m,n)$ to mean $\alpha \leq T(m,n)$. By Lemma 1.11 (i) and (ii) hold.

Lemma 1.13. For any $m,n \in I$, let $P(m,n)$, $Q(m,n)$ and $S(m,n)$ be corresponding statements. Suppose that for each $m,n \in I$,

$$(P(m,n) \text{ and } Q(m,n)) \Rightarrow S(m,n).$$

Then

(i) for each n, if $\{m: P(m,n) \text{ is true}\} \in \mathcal{G}$ and $\{m: Q(m,n) \text{ is true}\} \in \mathcal{G}$ then $\{m: S(m,n) \text{ is true}\} \in \mathcal{G}$.

(ii) if $\{n: \{m: P(m,n) \text{ is true}\} \in \mathcal{G}\} \in \mathcal{F}$ and $\{n: \{m: Q(m,n) \text{ is true}\} \in \mathcal{G}\} \in \mathcal{F}$ then $\{n: \{m: S(m,n) \text{ is true}\} \in \mathcal{G}\} \in \mathcal{F}$.

Proof: (i) For each fixed n, by the hypothesis,

$$\{m: P(m,n) \text{ is true}\} \cap \{m: Q(m,n) \text{ is true}\} \subset \{m: S(m,n) \text{ is true}\}. \quad (1)$$

Suppose that $\{m: P(m,n) \text{ is true}\} \in \mathcal{G}$ and $\{m: Q(m,n) \text{ is true}\} \in \mathcal{G}$. Since $\mathcal{G}$ is a filter $\{m: P(m,n) \text{ is ture}\} \cap \{m: Q(m,n) \text{ is true}\} \in \mathcal{G}$. By (1), $\{m: S(m,n) \text{ is true}\} \in \mathcal{G}$.

(ii) For each n, let

$$P_1(n) \equiv \{m: P(m,n) \text{ is true}\} \in \mathcal{G},$$

$$Q_1(n) \equiv \{m: Q(m,n) \text{ is true}\} \in \mathcal{G}, \text{ and}$$

$$S_1(n) \equiv \{m: S(m,n) \text{ is true}\} \in \mathcal{G}.$$

By the proof of (i), for each $n \in I$, $(P_1(n) \text{ and } Q_1(n)) \Rightarrow S_1(n)$. Since $\mathcal{F}$ is a filter, we can apply (i). Thus, we have

$$\{n: P_1(n) \text{ is true}\} \in \mathcal{F} \text{ and } \{n: Q_1(n) \text{ is true}\} \in \mathcal{F}$$

$$\Rightarrow \{n: S_1(n) \text{ is true}\} \in \mathcal{F}.$$

Hence the proof of (ii) is completed.

Corollary 1.14. For each $m,n \in I$, let $S(m,n)$ and $T(m,n)$ be corresponding real numbers and $\alpha \in R$. Then we have

(i) for each n, if $\{m: \alpha \le S(m,n)\} \in G$, and $\{m: \beta \le T(m,n)\} \in G$, then $\{m: \alpha + \beta \le S(m,n) + T(m,n)\} \in G$.

(ii) if $\{n: \{m: \alpha \le S(m,n)\} \in G\} \in F$ and $\{n: \{m: \beta \le T(m,n)\} \in G\} \in F$ then $\{n: \{m: \alpha + \beta \le S(m,n) + T(m,n)\} \in G\} \in F$.

Proof: Let $P(m,n) \equiv$ "$\alpha \le S(m,n)$", $Q(m,n) \equiv$ "$\beta \le T(m,n)$" and $S(m,n) \equiv$ "$\alpha + \beta \le S(m,n) + T(m,n)$". By the lemma 1.13, we get the results (i) and (ii).

Lemma 1.15. Let F be a filter on I which is finer than the Fréchet filter, F_0. Then for any $A \in 2^I$ and for any $N \in I$, $A \in F \Leftrightarrow A \cap J_N^c \in F$, where $J_N = \{1,2,3,\dots,N\}$.

Proof: Since F is finer than F_0, for any $N \in I$, $J_N^c \in F$. Suppose that $A \in F$. Since any intersection of two members of a filter is also a member, $A \cap J_N^c \in F$. Conversely suppose that $A \cap J_N^c \in F$. Any superset of a member of a filter is also a member. Thus $A \in F$.

Lemma 1.16. For any $n,m \in I$, let $S(m,n)$ and $T(m,n)$ be corresponding real numbers with the property that there exists $N \in I$ such that $n > N \Rightarrow S(m,n) \le T(m,n)$ for all m. Suppose that F is a filter finer than F_0. Then

$$\sup\{\alpha: \{n: \{m: \alpha \le S(m,n)\} \in G\} \in F\}$$

$$\le \sup\{\alpha: \{n: \{m: \alpha \le T(m,n)\} \in G\} \in F\}.$$

Proof: Since for any $n > N$ and for any $m \in I$, $S(m,n) \le T(m,n)$.

By Corollary 1.12(i), for any $n > N$ and for any real number α, we have

$$\{m\colon \alpha \le S(m,n)\} \in \mathcal{G} \Rightarrow \{m\colon \alpha \le T(m,n)\} \in \mathcal{G} .$$

Thus we have

$$\{n\colon \{m\colon \alpha \le S(m,n)\} \in \mathcal{G}\} \cap J_N^{\,c}$$

$$\subset \{n\colon \{m\colon \alpha \le T(m,n)\} \in \mathcal{G}\} \cap J_N^{\,c} .$$

Since $\mathcal{F}$ is a filter,

$$\begin{aligned} &\{n\colon \{m\colon \alpha \le S(m,n)\} \in \mathcal{G}\} \cap J_N^{\,c} \in \mathcal{F} \\ &\Rightarrow \{n\colon \{m\colon \alpha \le T(m,n)\} \in \mathcal{G}\} \cap J_N^{\,c} \in \mathcal{F} . \end{aligned} \tag{A}$$

By Lemma 1.15, since $\mathcal{F}$ is finer than $\mathcal{F}_0$, (A) is logically equivalent to

$$\{n\colon \{m\colon \alpha \le S(m,n)\} \in \mathcal{G}\} \in \mathcal{F}$$

$$\Rightarrow \{n\colon \{m\colon \alpha \le T(m,n)\} \in \mathcal{G}\} \in \mathcal{F} .$$

Thus we have,

$$\{\alpha\colon \{n\colon \{m\colon \alpha \le S(m,n)\} \in \mathcal{G}\} \in \mathcal{F}\}$$

$$\subset \{\alpha\colon \{n\colon \{m\colon \alpha \le T(m,n)\} \in \mathcal{G}\} \in \mathcal{F}\} .$$

Hence

$$\sup\{\alpha\colon \{n\colon \{m\colon \alpha \le S(m,n)\} \in \mathcal{G}\} \in \mathcal{F}\}$$

$$\le \sup\{\alpha\colon \{n\colon \{m\colon \alpha \le T(m,n)\} \in \mathcal{G}\} \in \mathcal{F}\} .$$

Lemma 1.17. Let $\mathcal{G}$ and $\mathcal{F}$ be filters and $t \in \mathbb{R}$ then we have

(i) $\sup\{\alpha : \{n : \alpha \le t\} \in \mathcal{F}\} = t$.

(ii) $\sup\{\alpha : \{n : \{m : \alpha \le t\} \in \mathcal{G}\} \in \mathcal{F}\} = t$.

Proof: (i) For each n , we have

$$\{n : \alpha \le t\} = \begin{cases} I & \text{if } \alpha \le t \\ \phi & \text{if } \alpha > t . \end{cases}$$

Thus $\{\alpha : \{n : \alpha \le t\} \in \mathcal{F}\} = \{\alpha : \alpha \le t\}$.

Hence $\sup\{\alpha : \{n : \alpha \le t\} \in \mathcal{F}\} = t$.

(ii) If $\beta \le t$, then $\{m : \beta \le t\} = I \in \mathcal{G}$ and so , $\{n : \{m : \alpha \le t\} \in \mathcal{G}\} = I \in \mathcal{F}$. If $\beta > t$, then $\{m : \beta \le t\} = \phi \notin \mathcal{G}$ and so, $\{n : \{m : \beta \le t\} \in \mathcal{G}\} = \phi \notin \mathcal{F}$. Therefore $\{\alpha : \alpha \le t\} = \{\alpha : \{n : \{m : \alpha \le t\} \in \mathcal{G}\} \in \mathcal{F}\}$. Thus we have the result (ii).

Definition 1.18. Let $x = (x_n) \in \omega$ (the space of all real sequences) and let $A = (a_{nk})$, $n,k = 1,2,3,\ldots$ be an infinite real matrix. Then the product Ax denote the sequence (y_i), if it exists, where $y_i = \sum_{j=1}^{\infty} a_{ij}x_j$. We denote $(Ax)_i = y_i = \sum_{j=1}^{\infty} a_{ij}x_j$. We also define $c_A = \{x \in \omega : Ax \in c\}$. In Chapter 3, we will write C_A for c_A.

Definition 1.19. An infinite matrix A is called regular if $c \subset c_A$ and for any $x \in c$, $\lim_i x_i = \lim_i (Ax)_i$.

Let us state the well known Silverman-Toeplitz Theorem [6] without proving it. Furthermore

<u>Proposition 1.20.</u> An infinite matrix A is regular if and only if

(1) $\sup_i \sum_j |a_{ij}| < \infty$,

(2) $\lim_i a_{ij} = 0$ for $j = 1,2,3,\ldots$,

(3) $\lim_i \sum_j a_{ij} = 1$.

<u>Definition 1.21.</u> A matrix $A = (a_{ij})$ is called nonnegative if $a_{ij} \geq 0$ for any $i,j = 1,2,3,\ldots$.

Note that, for any matrix A , c_A is a linear subspace of ω . For any $x,y \in c_A$ and $\alpha \in R$, $A(x+y) = Ax + Ay$ and $A(\alpha x) = \alpha Ax$.

The following proposition introduces our general method of constructing densities.

<u>Proposition 1.22.</u> Let $M = \{M_m\}$ be a sequence of nonnegative regular matrices. Let us denote $M_m = (a^m_{ik})$. Suppose that the following uniformity conditions hold.

(i) For any $\varepsilon > 0$, there exists N such that $n > N$ implies

$$1 - \varepsilon \leq \sum_{k=1}^{\infty} a^m_{nk} \leq 1 + \varepsilon \quad \text{for all } m .$$

i.e., $\lim_n \sum_{k=1}^{\infty} a^m_{nk} = 1$ uniformly in m .

(ii) For any $\varepsilon > 0$ and for any $s \in I$, there exists N such that $n > N$ implies $a^m_{n1} + a^m_{n2} + \ldots + a^m_{ns} < \varepsilon$, for all m.

Suppose further that $\mathcal{F}$ is a filter finer than $\mathcal{F}_0$ and $\mathcal{G}$ is any filter. Let $d_{M,\mathcal{G},\mathcal{F}}(A) = \sup\{\alpha\colon \{n\colon \{m\colon \alpha \leq (M_m\chi_A)_n\} \in \mathcal{G}\} \in \mathcal{F}\}$, where $\chi_A(n)$ ($= 1$ if $n \in A$, $= 0$ otherwise) is the characteristic sequence of A. Then $d_{M,\mathcal{G},\mathcal{F}}$ is a lower asymptotic density.

Proof: In this proof, we denote $d_{M,\mathcal{G},\mathcal{F}}(A)$ by $d(A)$. For any $A \in 2^I$, $(M_m\chi_A)_n = \Sigma_{j=1}^{\infty} a^m_{nj}\chi_A(j) \leq \Sigma_{j=1}^{\infty} a^m_{nj}$. Since M_m is a non-negative regular matrix $\Sigma_{j=1}^{\infty} a^m_{nj}$ converges, thus $(M_m\chi_A)_n = \Sigma_{j=1}^{\infty} a^m_{nj}\chi_A(j)$ is a real number. Hence our definition of $d(A)$ is well defined.

Suppose that $A \subset B$. Then $\chi_A(j) \leq \chi_B(j)$ for all j. For each m,n, $(M_m\chi_A)_n = \sum_{n=1}^{\infty} a^m_{nj}\chi_A(j) \leq \sum_{j=1}^{\infty} a^m_{nj}\chi_B(j) = (M_m\chi_B)_n$. By lemma 1.6 we have

$$\sup\{\alpha\colon \{n\colon \{m\colon \alpha \leq (M_m\chi_A)_n\} \in \mathcal{G}\} \in \mathcal{F}\}$$

$$\leq \sup\{\alpha\colon \{n\colon \{m\colon \alpha \leq (M_m\chi_B)_n\} \in \mathcal{G}\} \in \mathcal{F}\}.$$

Hence $d(A) \leq d(B)$.

Since $\chi_\phi = 0$, we have, for any m,n, $(M_m\chi_\phi)_n = 0$. Thus by Lemma 1.17, with $t = 0$, $d(\phi) = \sup\{\alpha\colon \{n\colon \{m\colon \alpha \leq (M_m\chi_\phi)_n\} \in \mathcal{G}\} \in \mathcal{F}\} = 0$.

Next we will show $d(I) = 1$. Since $\chi_I = (1,1,1,\ldots) = e$, $(M_m\chi_I)_n = \sum_{j=1}^{\infty} a^m_{nj}$. By the assumption (i) there exists N such that

$n > N$ implies $1 - \varepsilon \le \sum_{j=1}^{\infty} a_{nj}^{m} \le 1 + \varepsilon$ for all m .

Let $S(m,n) = 1 - \varepsilon$, $T(m,m) = (M_m \chi_I)_n$ and $V(m,n) = 1 + \varepsilon$. By Lemma 1.16

$$\sup\{\alpha: \{n: \{m: \alpha \le 1 - \varepsilon\} \in G\} \in F\}$$

$$\le \sup\{\alpha: \{n: \{m: \alpha \le (M_m \chi_I)_n\} \in G\} \in F\}$$

$$\le \sup\{\alpha; \{n: \{m: \alpha \le 1 + \varepsilon\} \in G\} \in F\} .$$

By Lemma 1.17 and the definition of d , $1 - \varepsilon \le d(I) \le 1 + \varepsilon$. Since ε is arbitrary $d(I) = 1$. Let $A \in 2^I$, since $\phi \subset A \subset I$ we have, $0 = d(\phi) \le d(A) \le d(I) = 1$.

Let $A \in 2^I$, and L be a positive integer. Then $M_m \chi_{(A \cup J_L)} \le M_m(\chi_A + \chi_{J_L}) = M_m\chi_A + M_m\chi_{J_L}$. For any $\varepsilon > 0$, by the assumption (ii) of the proposition there exists a positive integer N such that $n > N \Rightarrow (M_m\chi_{J_L})_n = \sum_{j=1}^{L} a_{nj}^{m} < \varepsilon$ for all m . Let $T(m,n) = (M_m\chi_A)_n + \varepsilon$, then by Lemma 1.16, we have

$$\sup\{\alpha: \{n: \{m: \alpha \le (M_m\chi_{(A \cup J_L)})_n\} \in G\} \in F\}$$

$$\le \sup\{\alpha: \{n: \{m: \alpha \le (M_m\chi_A)_n + \varepsilon\} \in G\} \in F\}$$

$$= \sup\{\alpha: \{n: \{m: \alpha - \varepsilon \le (M_m\chi_A)_n \in G\} \in F\}$$

$$= \sup\{\beta + \varepsilon: \{n: \{m: \beta \le (M_m\chi_A)_n\} \in G\} \in F\}$$

$$= \varepsilon + \sup\{\beta: \{n: \{m: \beta \le (M_m\chi_A)_n\} \in G\} \in F\} .$$

Hence $d(A \cup J_L) \le \varepsilon + d(A)$. Since $\varepsilon > 0$ is arbitrary $d(A \cup J_L) \le d(A)$. Also $d(A) \le d(A \cup J_L)$ whence $d(A) = d(A \cup J_L)$.

If $A \sim B$, then $A \cup J_L = B \cup J_L$ for some $L \in I$. Hence $d(A) = d(A \cup J_L) = d(B \cup J_L) = d(B)$.

Next we prove that:

$$A \cap B = \phi \Rightarrow d(A) + d(B) \le d(A \cup B).$$

By the definition of d, for any $\varepsilon > 0$, there exist real numbers α and β such that

$$d(A) - \varepsilon < \alpha, \quad d(A) - \varepsilon < \beta,$$

$$\{n: \{m: \alpha \le (M_m \chi_A)_n\} \in G\} \in F$$

and

$$\{n: \{m: \beta \le (M_m \chi_B)_n\} \in G\} \in F.$$

Since $A \cap B = \phi$, we have $\chi_{A \cup B} = \chi_A + \chi_B$ and so $M_m \chi_{(A \cup B)} = M_m(\chi_A + \chi_B) = M_m \chi_A + M_m \chi_B$. Therefore $(M_m \chi_A)_n + (M_m \chi_B)_n = (M_m \chi_{(A \cup B)})_n$

By Corollary 1.14

$$\{n: \{m: \alpha + \beta \le (M_m \chi_A) + (M_m \chi_B)_n\} \in G\} \in F$$

equivalently,

$$\{n: \{m: \alpha + \beta \le (M_m \chi_{(A \cup B)})_n\} \in G\} \in F.$$

Therefore $\alpha + \beta \le d(A \cup B)$ so we have $d(A) - \varepsilon + d(B) - \varepsilon < \alpha + \beta \le d(A \cup B)$. Hence $d(A) + d(B) \le d(A \cup B) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary $d(A) + d(B) \le d(A \cup B)$.

Finally we want to show that $d(A) + d(B) \le 1 + d(A \cap B)$.

For any $\varepsilon > 0$, there exist α and β such that

$d(A) - \varepsilon \le \alpha$, $d(B) - \varepsilon \le \beta$,

$\{n\colon \{m\colon \alpha \le (M_m\chi_A)_n\} \in G\} \in F$ and

$\{n\colon \{m\colon \beta \le (M_m\chi_B)_n\} \in G\} \in F$.

By Corollary 1.14,

$$\{n\colon \{m\colon \alpha + \beta \le (M_m\chi_A)_n + (M_m\chi_B)_n\} \in G\} \in F .$$

On the other hand $\chi_A + \chi_B = \chi_{A \cup B} + \chi_{A \cap B} \le \chi_I + \chi_{A \cap B}$. So that $(M_m\chi_A)_n + (M_m\chi_B)_n = (M_m\chi_A + M_m\chi_B)_n = (M_m(\chi_A + \chi_B))_n \le M_m(\chi_I + \chi_{(A \cap B)})_n =$ $(M_m\chi_I)_n + (M_m\chi_{(A \cap B)})_n$. By Corollary 1.12(ii), we have

$$\{n\colon \{m\colon \alpha + \beta \le (M_m\chi_I)_n + (M_m\chi_{(A \cap B)})_n\} \in G\} \in F .$$

By the hypothesis (i) for any $\varepsilon > 0$, there exists $N \in I$ such that $n > N$ implies

$$(M_m\chi_I)_n = \sum_{j=1}^{\infty} a_{nj}^m < 1 + \varepsilon .$$

By Lemma 1.16, we have

$$\alpha + \beta \le \sup\{\delta\colon \{n\colon \{m\colon \delta \le (M_m\chi_I)_n + (M_m\chi_{(A \cap B)})_n\} \in G\} \in F\}$$

$$\le \sup\{\delta\colon \{n\colon \{m\colon \delta \le 1 + \varepsilon + (M_m\chi_{(A \cap B)})_n\} \in G\} \in F\}$$

$$= \sup\{\delta\colon \{n\colon \{m\colon \delta - 1 + \varepsilon \le (M_m\chi_{(A \cap B)})_n\} \in G\} \in F\}$$

$$= \sup\{\gamma + 1 + \varepsilon\colon \{n\colon \{m\colon \gamma \le (M_m\chi_{(A \cap B)})_n\} \in G\} \in F\}$$

$$= 1 + \varepsilon + \sup\{\gamma\colon \{n\colon \{m\colon \gamma \le (M_m\chi_{(A \cap B)})_n\} \in G\} \in F\}$$

$$= 1 + \varepsilon + d(A \cap B) .$$

Hence $d(A) - \varepsilon + d(B) - \varepsilon < \alpha + \beta \le 1 + \varepsilon + d(A \cap B)$. Since $\varepsilon > 0$ is arbitrary we get $d(A) + d(B) \le 1 + d(A \cap B)$.

<u>Proposition 1.23.</u> Let M be a nonnegative regular matrix and $M = \{M_m\}$ where $M_m = M$ for each $m = 1,2,3,\ldots$. Let $G = \{A \in 2^I\colon 1 \in A\}$ and F be a filter finer than F_0 , then

$$d_{M,G,F}(A) = \sup\{\alpha\colon \{n\colon \alpha \le (M\chi_A)_n\} \in F\}$$

for each $A \in 2^I$. In this case, we write

$$d_{M,F} = \sup\{\alpha\colon \{n\colon \alpha \le (M\chi_A)_n\} \in F\} .$$

Proof: Clearly conditions (i) and (ii) of Proposition 1.22 are satisfied. Hence $d_{M,G,F}$ is a density. Since for each $A \in 2^I$,

$$\{m: \alpha \leq (M_m\chi_A)_n\} \in G \Leftrightarrow 1 \in \{m: \alpha \leq (M_m\chi_A)_n\}$$

$$\Leftrightarrow \alpha \leq (M\chi_A)_n .$$

Hence $$d_{M,G,F}(A) = \sup\{\alpha: \{n: \{m: \alpha \leq (M_m\chi_A)_n\} \in G\} \in F\}$$

$$= \sup\{\alpha: \{n: \alpha \leq (M\chi_A)_n\} \in F\}.$$

<u>Example 1.24</u>. Let F_0 be the Fréchet filter and M a non-negative regular matrix, then $d_{M,F_0}(A) = \liminf_n (M\chi_A)_n$. In this case we write $d_M'(A) = \liminf_n (M\chi_A)_n$, which is called a matrix method density. ([9] Definition 3.5).

Proof: Let

$$r = d_{M,F_0}(A) = \sup\{\alpha: \{n: \alpha \leq (M\chi_A)_n\} \in F_0\}.$$

Then for any $\varepsilon > 0$, there exists α such that

$$r - \varepsilon < \alpha \text{ and } \{n: \alpha \leq (M\chi_A)_n\} \in F_0 .$$

Since F_0 is the Fréchet filter, there exists N such that $n > N$ implies $\alpha \leq (M\chi_A)_n$. Hence

$$r - \varepsilon < \alpha \leq \liminf_n (M\chi_A)_n$$

and

$$r \le \liminf_n (M\chi_A)_n .$$

Also

$$\{n\colon r + \varepsilon \le (M\chi_A)_n\} \notin \mathcal{F}_0 ,$$

and since $\mathcal{F}_0$ is the Fréchet filter, for infinitely many n, $r + \varepsilon > (M\chi_A)_n$. Hence

$$r + \varepsilon \ge \liminf_n (M\chi_A)_n .$$

Therefore we have

$$r = \liminf_n (M\chi_A)_n .$$

Example 1.25. Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \ldots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \ldots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

be the Cesáro matrix. Then

$$\begin{aligned} d_M(A) &= \liminf_n (M\chi_A)_n \\ &= \liminf_n \Sigma_{i=1}^{n} a_{ni}\chi_A(i) \\ &= \liminf_n \frac{\chi_A(1) + \chi_A(2) + \ldots + \chi_A(n)}{n} \\ &= \liminf_n \frac{A(n)}{n} \end{aligned}$$

where $A(n)$ is the cardinality of the set $A \cap \{1,2,\ldots,n\}$. We write $d(A) = \liminf_n \frac{A(n)}{n}$, and say it is the ordinary asymptotic density ([9], [3]).

Example 1.26. Let M be the Cesáro matrix and let

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ . & . & . & . & . & \end{bmatrix}$$

then $MN^{m-1} = (a^m_{ni})$, where

$$a^m_{ni} = \begin{cases} \frac{1}{n} & \text{if } m \le i \le m+n-1 \\ 0 & \text{otherwise .} \end{cases}$$

Let $M_m = MN^{m-1}$, $\mathcal{G} = \{I\}$ and $\mathcal{F}_0$ be the Frechet filter. Then $d_{M,\mathcal{G},\mathcal{F}_0}(A) = \liminf_n \min \frac{A[m+1,m+n]}{n}$ where $A[m+1,m+n]$ is the cardinality of the set $A \cap \{m+1,m+2,\ldots,m+n\}$. In this case we write $u(A) = \liminf_n \min_{m \ge 0} \frac{A[m+1,m+n]}{n}$ and say that it is the uniform density [9].

Proof: By the definition of $d_{M,\mathcal{G},\mathcal{F}_0}(A)$

$$d_{M,G,F}(A) = \sup\{\alpha\colon \{n\colon \{m\colon \alpha \le (M_m\chi_A)_n\} \in G\} \in F_0\}$$

$$= \sup\{\alpha\colon \{n\colon \{m\colon \alpha \le (M_m\chi_A)_n\} = I\} \in F_0\}$$

$$= \sup\{\alpha\colon \{m\colon \alpha \le (M_m\chi_A)_n \text{ , for all } m \in I\} \in F_0\}$$

$$= \sup\{\alpha\colon \{n\colon \alpha \le \inf_{m \ge 1} (M_m\chi_A)_n\} \in F_0\}.$$

By the same method of proof as in the previous example

$$d_{M,G,F} = \liminf_{n} \inf_{m \ge 1} (M_m\chi_A)_n$$

$$= \liminf_{n} \inf_{m \ge 1} \sum_{i=1}^{\infty} a_{ni}^{m}\chi_A(i)$$

$$= \liminf_{n} \inf_{m \ge 1} \sum_{i=m}^{m+n-1} \frac{1}{n}\chi_A(i)$$

$$= \liminf_{n} \inf_{m \ge 1} \frac{A[m,m+n-1]}{n} .$$

Since for $m \ge 1$, $\frac{A[m,m+n-1]}{n} \in \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{m}\}$,

$$\inf_{m \ge 1} \frac{A[m,m+n-1]}{n} = \min_{m \ge 1} \frac{A[m,m+n-1]}{n}$$

$$= \min_{m \ge 0} \frac{A[m+1,m+n]}{n} .$$

Therefore we have

$$d_{M,G,F}(A) = \liminf_{n} \min_{m \geq 0} \frac{A[m+1,m+n]}{n} .$$

The last two examples show that two of the important densities which have fundamental differences (see [9]) are subsumed under our general density in Proposition 1.22.

Example 1.27. Let X be a zeroclass and $F = \{I - A \mid A \in X\}$. Note that F is finer than F_0. Let M_m be the identity matrix for all m. Let $G = \{I\}$. Then

$$d_{M,G,F}(A) = \begin{cases} 1 & \text{if } A^c \in X \\ 0 & \text{otherwise} . \end{cases}$$

We write $d_{M,G,F}(A) = d_X(A)$. (If X_0 is the set of all finite sets of I, then d_{X_0} is called the discrete density.)

Proof. Since $(M_m \chi_A)_n = \chi_A(n)$, we have

$$\{m\colon \alpha \leq (M_m \chi_A)_n\} = \begin{cases} I & \text{if } \alpha \leq \chi_A(n) \\ \phi & \text{if } \alpha > \chi_A(n) . \end{cases}$$

Thus $\{n\colon \{m\colon \alpha \leq (M_m \chi_A)_n\} \in G\} = \{n\colon \alpha \leq \chi_A(n)\}$. Therefore we have $d_{M,G,F}(A) = \sup\{\alpha\colon \{n\colon \alpha \le \chi_A(n)\} \in F\}$. Since

$$\{n : \alpha \le \chi_A(n)\} = \begin{cases} \phi & \text{if } 1 < \alpha \\ A & \text{if } 0 < \alpha \le 1 \\ I & \text{if } \alpha \le 0 , \end{cases}$$

it follows that

$$\{\alpha\colon \{n\colon \alpha \le \chi_A(n)\} \in F\} = \begin{cases} \{\alpha\colon \alpha \le 1\} & \text{if } A \in F \\ \{\alpha\colon \alpha \le 0\} & \text{if } A \notin F . \end{cases}$$

Hence

$$d_{M,G,F}(A) = \begin{cases} 1 & \text{if } A \in F \\ 0 & \text{if } A \notin F \end{cases}$$

$$= \begin{cases} 1 & \text{if } I - A \in X \\ 0 & \text{otherwise} . \end{cases}$$

Now we want to express $\bar{d}_{M,G,F}(A)$ by a formula similar to the definition of d .

Proposition 1.28. Suppose that M, G and F are defined as in Proposition 1.22. Then we have

$$\bar{d}_{M,G,F}(A) = \inf\{\alpha\colon \{n\colon \{m\colon \alpha \ge (M_m\chi_A)_n\} \in G\} \in F\} .$$

Proof: Consider $d_{M,G,F}(A^c)$ first.

$$d_{M,G,F}(A^c) = \sup\{\alpha: \{n: \{m: \alpha \le (M_m\chi_{A^c})_n\} \in G\} \in F\}$$

$$= \sup\{\alpha: \{n: \{m: \alpha \le (M_m(\chi_I - \chi_A))_n\} \in G\} \in F\}$$

$$= \sup\{\alpha: \{n: \{m: \alpha \le (M_m\chi_I)_n - (M_m\chi_A)_n\} \in G\} \in F\}$$

By the condition (ii) in Proposition 1.22 and Lemma 1.16, it follows that for any $\varepsilon > 0$,

$$\sup\{\alpha: \{n: \{m: \alpha \le 1 - \varepsilon - (M_m\chi_A)_n\} \in G\} \in F\}$$

$$\le \sup\{\alpha: \{n: \{m: \alpha \le (M_m\chi_I)_n - (M_m\chi_A)_n\} \in G\} \in F\}$$

$$\le \sup\{\alpha: \{n: \{m: \alpha \le 1 + \varepsilon - (M_m\chi_A)_n\} \in G\} \in F\} .$$

Thus

$$1 - \varepsilon + \sup\{\alpha: \{n: \{m: \alpha \le -(M_m\chi_A)_n\} \in G\} \in F\}$$

$$\le d_{M,G,F}(A^c)$$

$$\le 1 + \varepsilon + \sup\{\alpha: \{n: \{m: \alpha \le -(M_m\chi_A)_n\} \in G\} \in F\} .$$

Hence

$$1 - \varepsilon - \inf\{\alpha: \{n: \{m: \alpha \geq (M_m\chi_A)_n\} \in G\} \in F\}$$

$$\leq d_{M,G,F}(A^c)$$

$$\leq 1 + \varepsilon - \inf\{\alpha: \{n: \{m: \alpha \geq (M_m\chi_A)_n\} \in G\} \in F\} .$$

Since $\varepsilon > 0$ is arbitrary,

$$d_{M,G,F}(A^c) = 1 - \inf\{\alpha: \{n: \{m: \alpha \geq (M_m\chi_A)_n\} \in G\} \in F\} .$$

Consequently,

$$\overline{d}_{M,G,F}(A) = \inf\{\alpha: \{n: \{m: \alpha \geq (M_m\chi_A)_n\} \in G\} \in F\}.$$

<u>Lemma 1.29</u>. For any $m,n \in I$, let $S(m,n)$ be the corresponding real number. Let G, F be filters. Then we have

$$\sup\{\alpha: \{n: \{m: \alpha \leq S(m,n)\} \in G\} \in F\}$$

$$= \sup\{\alpha: \{n: \{m: \alpha < S(m,n)\} \in G\} \in F\} .$$

Proof: For any $\alpha \in R$, $\alpha < S(m,n) \Rightarrow \alpha \leq S(m,n)$. By Lemma 1.11(ii),

$$\{n: \{m: \alpha < S(m,n)\} \in G\} \in F \Rightarrow \{n: \{m: \alpha \leq S(m,n)\} \in G\} \in F .$$

Therefore we have

$$\{\alpha: \{n: \{m: \alpha < S(m,n)\} \in G\} \in F\} \subset \{\alpha: \{n: \{m: \alpha \leq S(m,n)\} \in G\} \in F\}.$$

Thus

$$\sup\{\alpha\colon \{n\colon \{m\colon \alpha < S(m,n)\} \in G\} \in F\}$$

$$\le \sup\{\alpha\colon \{n\colon \{m\colon \alpha \le S(m,n)\} \in G\} \in F\} .$$

Conversely, let $\varepsilon > 0$ be fixed, then for any $\alpha \in R$, it follows that $\alpha \le S(m,n) \Rightarrow \alpha - \varepsilon < S(m,n)$. By Lemma 1.11 (ii), we have

$$\sup\{\alpha\colon \{n\colon \{m\colon \alpha \le S(m,n)\} \in G\} \in F\}$$

$$\le \sup\{\alpha\colon \{n\colon \{m\colon \alpha - \varepsilon < S(m,n)\} \in G\} \in F\}$$

$$= \sup\{\beta + \varepsilon\colon \{n\colon \{m\colon \beta < S(m,n)\} \in G\} \in F\}$$

$$= \varepsilon + \sup\{\beta\colon \{n\colon \{m\colon \beta < S(m,n)\} \in G\} \in F\} .$$

Since ε is arbitrary,

$$\sup\{\alpha\colon \{n\colon \{m\colon \alpha \le S(m,n)\} \in G\} \in F\}$$

$$\le \sup\{\alpha\colon \{n\colon \{m\colon \alpha < S(m,n)\} \in G\} \in F\}.$$

Thus we have the result.

Remark 1.30. By Lemma 1.29, we have

$$\underline{d}_{M,G,F}(A) = \sup\{\alpha\colon \{n\colon \{m\colon \alpha < (M_m\chi_A)_n\} \in G\} \in F\} .$$

$$\bar{d}_{M,G,F}(A) = \inf\{\alpha\colon \{n\colon \{m\colon \alpha > (M_m\chi_A)_n\} \in G\} \in F\} .$$

Definition 1.31. Let $d: 2^I \to R$ be a lower asymptotic density, we say that d is complete if, for any $A \in 2^I$, $d(A) = \bar{d}(A)$. That is, every set in I has natural density with respect to d or $2^I = \eta_d$.

Definition 1.32. An ultrafilter F on a set X is a filter such that there is no filter on X which is strictly finer than F.

Proposition 1.33. Let M, G, F and $d_{M,G,F}$ be defined as in Proposition 1.12. If G, F are ultrafilters then $d_{M,G,F}$ is a complete density.

Proof. For $A \in 2^I$, let

$$K = \{\alpha\colon \{n\colon \{m\colon \alpha \le (M_m\chi_A)_n\} \in G\} \in F\}.$$

By Lemma 1.11, if $\alpha_1 \in K$ and $\alpha_2 \le \alpha_1$, then $\alpha_2 \in K$. Since K is bounded above by $d_{M,G,F}(A) = r \le 1$, it follows that $K = \{x \in R\colon x < r\}$ or $K = \{x \in R\colon x \le r\}$. Thus $K^c = \{x \in R\colon x \ge r\}$ or $K^c = \{x \in R\colon x > r\}$. Hence

$$\begin{aligned} r &= \inf K^c \\ &= \inf\{\alpha\colon \{n\colon \{m\colon \alpha \le (M_m\chi_A)_n\} \in G\} \in F\}^c \\ &= \inf\{\alpha\colon \{n\colon \{m\colon \alpha \le (M_m\chi_A)_n\} \in G\} \notin F\}. \end{aligned}$$

Since F is an ultrafilter, for any $D \in 2^I$, $D \notin F \Leftrightarrow D^c \in F$. Thus we have

$$\begin{aligned} r &= \inf\{\alpha\colon \{n\colon \{m\colon \alpha \le (M_m\chi_A)_n\} \in G\}^c \in F\} \\ &= \inf\{\alpha\colon \{n\colon \{m\colon \alpha \le (M_m\chi_A)_n\} \notin G\} \in F\}. \end{aligned}$$

G is also an ultrafilter, so that we have

$$r = \inf\{\alpha: \{n: \{m: \alpha \le (M_m \chi_A)_n\}^c \in G\} \in F\}$$

$$= \inf\{\alpha: \{n: \{m: \alpha > (M_m \chi_A)_n\} \in G\} \in F\}$$

By Remark 2.30, $r = \bar{d}_{M,G,F}(A)$, and so $d_{M,G,F}(A) = \bar{d}_{M,G,F}(A)$.

Corollary 1.34. (1) If M is regular matrix and F is an ultrafilter then $d_{M,F}(A) = \sup\{\alpha: \{n: \alpha \le (M\chi_A)_n\} \in F\}$ is complete.

(2) Let X be a zeroclass, $G = \{I - A: A \in X\}$. If G is ultrafilter, then

$$d_X(A) = \begin{cases} 1 & \text{if } I - A \in X \\ 0 & \text{otherwise} \end{cases}$$

is complete.

Proof: By Proposition 1.23, it is obvious that (1) is true. By example 1.27 it is obvious that (2) is true.

Example 1.35. Let M_m be defined as in Example 1.27. Let $G = \{I\}$, and let F be any filter finer than the Fréchet filter. Then $d_{M,G,F}$ is not complete.

Proof: By Example 1.27 we have

$$d_{M,G,F}(A) = \sup\{\alpha: \{n: \alpha \le \min_{m \ge 0} \frac{A[m+1,m+n]}{n}\} \in F\},$$

$$\bar{d}_{M,\mathcal{G},\mathcal{F}}(A) = \inf\{\alpha\colon \{n\colon \alpha \geq \max_{m \geq 0} \frac{A[m+1,m+n]}{n}\} \in \mathcal{F}\} .$$

Let $A = \bigcup [2^{2n}, 2^{2n+1}]$. Since for each n,

$$\min_{m \geq 0} \frac{A[m+1,m+n]}{n} = 0 \quad \text{and} \quad \max_{m \geq 0} \frac{A[m+1,m+n]}{n} = 1 ,$$

it follows that $d_{M,\mathcal{G},\mathcal{F}}(A) = 0 < 1 = \bar{d}_{M,\mathcal{G},\mathcal{F}}(A)$. Hence d is not complete. Since $\mathcal{F}$ may be taken to be an ultrafilter, this example shows that, in general we do not get a complete density if $\mathcal{G}$ is not also an ultrafilter.

CHAPTER II

LACUNARY SETS

As mentioned in the Introduction, the family of lacunary sets arises naturally in sequence space and combinatorial studies. In this Chapter we introduce several natural subclasses of the lacunary sets, show their inter-relationships and consider their "fullness".

First we introduce several types of lacunary sets.

Definition 2.1. Let the elements of a set $A = \{a_i\} \in 2^I$ be represented by an increasing sequence (a_n) and let (d_n) be the difference sequence, that is, $d_n = a_{n+1} - a_n$. Then we define the following:

(1) $L = \{A \in 2^I : \lim_n d_n = \infty\} \cup \chi_0$, where χ_0 is the class of all finite subsets of I,

(2) $L_1 = \{A \in L \mid d_n \leq d_{n+1}$, for each $n\} \cup \chi_0$,

(3) $L_2 = (L_1 \cap \{A \in 2^I \mid \Sigma_{a \in A} \frac{1}{a} = \infty\}) \cup \chi_0$,

(4) $L_3 = \{A \in L \mid d_n < d_{n+1}$, for each $n\} \cup \chi_0$,

(5) $L_{M_i} = \{A \in L \mid d_m \leq d_n + i$, for $m \leq n\} \cup \chi_0$.

A set $A \in 2^I$ is called a lacunary (resp. L_T lacunary) set if A is a finite union of members of L (resp. L_T). Note also that $L_{M_0} = L_1$.

<u>Definition 2.2</u>. Suppose that $\mathcal{A}$ is a family of a sets. Let $\hat{\mathcal{A}} = \{B \mid B \subseteq A \text{ , for some } A \in \mathcal{A}\}$, the hereditary closure of $\mathcal{A}$, and let $[\mathcal{A}]$ be the family of all finite unions of all members of $\mathcal{A}$.

We prove a simple proposition.

<u>Proposition 2.3</u>. For any family $\mathcal{A}$ of sets, $[\hat{\mathcal{A}}] = \widehat{[\mathcal{A}]}$.

Proof: For any $B \in [\hat{\mathcal{A}}]$, $B = B_1 \cup B_2 \cup \ldots \cup B_n$ for some $B_i \in \hat{\mathcal{A}}$, $i = 1,2,\ldots,n$.

For each $i \in \{1,2,3,\ldots,n\}$, there exists $A_i \in \mathcal{A}$ such that $A_i \supset B_i$. Then $B = B_1 \cup B_2 \cup \ldots \cup B_n \subset A_1 \cup A_2 \cup \ldots \cup A_n$. Since $A_1 \cup A_2 \cup \ldots \cup A_n \in [\mathcal{A}]$, $B \in \widehat{[\mathcal{A}]}$. Hence $[\hat{\mathcal{A}}] \subseteq \widehat{[\mathcal{A}]}$.

Conversely suppose that $B \in \widehat{[\mathcal{A}]}$, then $B \subset A$ where $A \in [\mathcal{A}]$. Then $A = A_1 \cup A_2 \cup \ldots \cup A_n$ where $A_i \in \mathcal{A}$. Let $B_i = B \cap A_i$, for $i \in \{1,2,\ldots,n\}$, then $B_i \subset A_i$ and $B_i \in \hat{\mathcal{A}}$ for $i \in \{1,2,\ldots,n\}$. Hence $B = B_1 \cup B_2 \cup \ldots \cup B_n \in [\hat{\mathcal{A}}]$. Therefore $\widehat{[\mathcal{A}]} \subset [\hat{\mathcal{A}}]$.

Now we proceed to investigate the fullness of the various classes of lacunary sets just defined. We begin with a definition:

<u>Definition 2.4</u>. A class $\Phi \subset 2^I$ is full if,

(a) $\cup\{S : S \in \Phi\} = I$ (covering);

(b) $S \in \Phi$ whenever $S \subseteq T \in \Phi$ for some T (hereditary); and

(c) if (t_k) is a sequence of real numbers and $\Sigma_{k \in S} |t_k| < \infty$ for each $S \in \Phi$, then $\Sigma_{k=1}^{\infty} |t_k| < \infty$.

<u>Proposition 2.5.</u> If a class $A \subset 2^I$ is full and $[A] \neq 2^I$, then $[A]$ is a full zeroclass (see definition 1.7.).

Proof: Suppose that A is a full class, for each $a \in I$, since $\cup A = I$, there exists $S \in A$ such that $a \in S$. Thus $\{a\} \subset S$. By the hereditary property of full classes, $\{a\} \in A$, since A contains any singleton, $[A]$ contains all finite sets. Clearly $[A]$ is closed under finite unions. By the hereditary property of full classes, we have $A = \hat{A}$. Thus $[A] = [\hat{A}] = \widehat{[A]}$. Hence $[A]$ also has hereditary property. By the hypothesis, $[A] \neq 2^I$ implies $I \notin [A]$. Hence $[A]$ is a zeroclass.

<u>Proposition 2.6.</u> L, $\hat{L}_1$, $\hat{L}_2$ are full classes. (Note that $L = \hat{L}$).

Proof: Since $\hat{L}_2 \subset \hat{L}_1 \subset L$ and L, $\hat{L}_1$ and $\hat{L}_3$ are hereditary, $\hat{L}_2$ full would imply that L and $\hat{L}_1$ are also full. Hence we prove only that $\hat{L}_2$ is full. That $\hat{L}_2$ is covering is obvious. We show $\hat{L}_2$ has property (c) of definition 2.4. Let (t_k) be a sequence of real numbers for which $\sum_{k=1}^{\infty} |t_k| = \infty$. For each $n \in I$, there exists $b_n \in I$ such that $\sum_{k=1}^{\infty} |t_{(b_n+2^n k)}| = \infty$.

We will construct two sequences $(M_n)_{n \geq 2}$, and $(N_n)_{n \geq 1}$ in I with the following properties:

$$N_n < M_{n+1} < N_{n+1} \qquad (n \geq 1) \qquad (1)$$

$$M_n \equiv b_n \bmod 2^n \qquad (n \geq 2) \qquad (2)$$

$$N_n \equiv M_n \bmod 2^n \qquad (n \geq 2) \qquad (3)$$

$$M_{n+1} \equiv N_n \mod(2^n+1) \qquad (n \geq 1) \qquad (4)$$

$$M_n > b_n \qquad (n \geq 2) \qquad (5)$$

$$\sum_{a \in B^{2^n}[M_n,N_n]} |t_a| > 1 \qquad (n \geq 2) \qquad (6)$$

$$\sum_{a \in B^{2^n}[M_n,N_n]} \frac{1}{a} > 1 \qquad (n \geq 2) \qquad (7)$$

where $$B^s_{[a,b]} = \{a, a+s, a+2s, \ldots, a+[\frac{b-a}{s}]s\} .$$

Take $N_1 = b_1$ and suppose that we have constructed two sequences $(M_n)_{n=2}^{m-1}$ and $(N_n)_{n=1}^{m-1}$ such that (1) and (4) are true for $n = 1,2,\ldots,m-2$ and further (2), (3), (5), (6), (7) are true for $n = 2,3,4,\ldots,m-1$. Since 2^m and 2^m+1 are relatively prime, by the Chinese remainder theorem there exists $x_0 \in I$ such that

$$\begin{aligned} x_0 &\equiv b_m \mod 2^m \\ x_0 &\equiv N_{m-1} \mod(2^m + 1). \end{aligned} \qquad (*)$$

As long as $x \equiv x_0 \mod 2^m(2^m + 1)$, x is also a solution of the system (*). Therefore we can take $M_m \in I$ such that

$$M_m \equiv b_m \mod 2^m ,$$

$$M_m \equiv N_{m-1} \mod (2^m + 1),$$

$$M_m > b_m \quad \text{and} \quad M_m > N_{m-1} .$$

Since $\Sigma_{k=1}^{\infty} |t_{(b_m + 2^m k)}| = \infty$ and $M_m = b_m \bmod 2^m$ we have $\Sigma_{k=1}^{\infty} |t_{(M_m + 2^m k)}| = \infty$. By the integral test, $\Sigma_{k=1}^{\infty} \frac{1}{M_m + 2^m k} = \infty$.

Now we can take $N_m \in I$ such that

$$N_m \equiv M_m \bmod 2^m ,$$

$$\sum_{a \in B^{2^m}_{[M_m, N_m]}} |t_a| > 1$$

and

$$\sum_{a \in B^{2^m}_{[M_m, N_m]}} \frac{1}{a} > 1 .$$

This completes inductive definitions of (M_m) and (N_n). Let

$$A = \bigcup_{k=1}^{\infty} (B^{2^k+1}_{[N_k, M_{k+1}]} \cup B^{2^{(k+1)}}_{[M_{k+1}, N_{k+1}]}) .$$

Clearly $A \in L_2$ and $\Sigma_{a \in A} |t_a| = \infty$.

Proposition 2.7. The class $\hat{L}_3$ is not full.

Proof: For the real sequence $(\frac{1}{k})_{k=1}^{\infty}$, $\Sigma_{k=1}^{\infty} \frac{1}{k} = \infty$. For any $A \in \hat{L}_3$, then there exists $B \in L_3$ such that $A \subset B$. Suppose that

B is expressed as a sequence (b_n), and let $d_n = b_{n+1} - b_n$ for $n = 1,2,\ldots$. For $n \geq 2$

$$b_n = b_1 + d_1 + d_2 + \ldots + d_{n-1} > 1 + 2 + \ldots + n-1 = \frac{n(n-1)}{2} .$$

So that we have

$$\sum_{a\in A} \frac{1}{a} \leq \sum_{b\in B} \frac{1}{b} \leq \frac{1}{b_1} + \sum_{n=2}^{\infty} \frac{2}{n(n-1)} < \infty .$$

Hence $\hat{L}_3$ is not full.

We next show that if we perform the hereditary closure on L_{M_1} we get all the lacunary sets.

<u>Proposition 2.8</u>. $\hat{L}_{M_1} = L$.

Proof: Let $A = \{a_i\} \in L$. Let $N_0 = 1$. Since $\lim_n d_n = \infty$, for any $k \geq 1$, there exists $N_k > N_{k-1}$ such that $d_n > k^2$ whenever $n > N_k$. For each n with $N_k \leq n \leq N_{k+1}$,

$$d_n = q_n k + r_n \quad , \quad \text{where} \quad 0 \leq r_n < k .$$

So that $q_n k = d_n - r_n > k^2 - k = (k-1)k$. Hence $q_n > k-1$ and $d_n = (q_n - r_n)k + (k+1)r_n$. Let $\alpha_n = (\alpha_{n1},\alpha_{n2},\ldots,\alpha_{nq_n})$ be the finite sequence $(k,k,\ldots,k,k+1,\ldots,k+1)$ such that there are $(q_n - r_n)$ many k's in the first part and r_n many $(k+1)$'s in the second part. Let

$$\begin{aligned}(e_m) &= (\alpha_1,\alpha_2,\alpha_3,\ldots)\\ &= (\alpha_{11},\ldots,\alpha_{1q_1},\alpha_{21},\alpha_{22},\ldots,\alpha_{2q_2},\ldots,\alpha_{n1},\ldots,\alpha_{nq_n},\ldots) .\end{aligned}$$

Note that $e_i \le e_j + 1$ for $i \le j$ but, if $N_k \le n_1 \le n_2 \le N_{k+1}$ then $e_i = e_j - 1$ when e_i is the last term of α_{n_1} and e_j is the first term of α_{n_2+1}. Also, clearly $d_n = \alpha_{n1} + \alpha_{n2} + \ldots + \alpha_{nq_n}$. Letting $b_1 = a_1$ and $b_m = a_1 + e_1 + e_2 + \ldots + e_{m-1}$ where $m > 1$, we have $B = \{b_m : m \in I\} \in L_{M_1}$ (and in general $B \notin L_{M_0}$).

For any $a_n \in A$,

$$a_n = a_1 + d_1 + d_2 + \ldots + d_{n-1}$$

$$= a_1 + \sum_{i=1}^{q_1} \alpha_{1i} + \sum_{i=1}^{q_2} \alpha_{2i} + \ldots + \sum_{i=1}^{q_n} \alpha_{ni}$$

$$= b_m \in B$$

where $m = 1 + q_1 + q_2 + \ldots + q_n$. Hence $A \subset B$. Therefore $L \subset \hat{L}_{M_1}$.

Conversely since $L = \hat{L}$ and $L_{M_1} \subset L$ we have $\hat{L}_{M_1} \subset \hat{L} = L$.

Note that for $i \ge 2$, $L_{M_1} \subset L_{M_i} \subset L$, thus we conclude that for any $i \ge 1$, $\hat{L}_{M_i} = L$.

We now proceed to show that $\hat{L}_1 = \hat{L}_{M_0} \subsetneq L$. Actually, this result is included in the stronger Proposition 2.28 below. We present it here, however, in order to see the method of proof which is different. Let's introduce some definitions and lemmas before we prove $\hat{L}_1 \subsetneq L$.

<u>Definition 2.9</u>. (1) Let $a, x_1, x_2, \ldots, x_n$ be positive integers

with $a = x_1 + x_2 + \ldots + x_n$ and $x_1 \le x_2 \le \ldots \le x_n$. Then $(x_1, x_2, \ldots, x_n)$ is called a <u>partition</u> of a and n is called the <u>length</u> of the partition $(x_1, x_2, \ldots, x_n)$.

(2) Let $(a_1, a_2, \ldots, a_n)$ be any finite sequence of positive integers and let $(y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, y_{22}, \ldots, y_{2k_2}, \ldots, y_{n1}, \ldots, y_{nk_n})$ be a nondecreasing sequence such that $(y_{i1}, y_{i2}, \ldots, y_{ik_i})$ is a partition of a_i. Then $(y_{11}, y_{12}, \ldots, y_{1k_1}, \ldots, y_{n1}, \ldots, y_{nk_n})$ is called a (monotone) partition of the sequence $(a_1, a_2, \ldots, a_n)$.

<u>Definition 2.10</u>. Let $(x_n)_{n=1}^{\infty}$ be a sequence. Then the finite subsequence $(x_s, x_{s+1}, \ldots, x_t)$ is called a <u>part</u> of the sequence $(x_n)_{n=1}^{\infty}$.

<u>Lemma 2.11</u>. Let $(a_1, a_2, \ldots, a_n)$ be a strictly decreasing sequence of prime numbers with $n \ge a_n$. Then there does not exist a partition of $(a_1, a_2, \ldots, a_n)$ such that its first term is larger than 1.

Proof: Suppose that there exists a partition $(y_{11}, y_{12}, \ldots, y_{1k_1}, \ldots, y_{n1}, \ldots, y_{nk_n})$ of $(a_1, a_2, \ldots, a_n)$ with $y_{11} > 1$. Assume that $k_i = 1$ for some $i < n$. Since $y_{(i+1)1}$ is a member of the partition of a_{i+1}, we have $y_{(i+1)1} \le a_{i+1}$. Since $(y_{11}, \ldots, y_{1k_1}, \ldots, y_{n1}, \ldots, y_{nk_n})$ is nondecreasing, $a_i = y_{i1} \le y_{(i+1)1} \le a_{i+1}$. This contradicts that $(a_1, a_2, \ldots, a_n)$ is a strictly decreasing sequence. Therefore, for any $i = 1, 2, \ldots, n-1$, $k_i > 1$.

Clearly $y_{i1} < y_{ik_i}$ for each $i = 1,2,\ldots,n-1$, since otherwise, $a_i = y_{i1}+y_{i2}+\ldots+y_{ik_i} = k_i y_{i1}$, which contradicts that a_i is a prime number. Hence $y_{i1} < y_{ik_i} \le y_{(i+1)1}$ for $i = 1,2,\ldots,n-1$. Therefore $1 < y_{11} < y_{21} < \ldots < y_{n1} \le a_n$ and thus $n < y_{n1} \le a_n$, which contradicts the hypothesis $a_n \le n$.

<u>Proposition 2.12</u>. $\hat{L}_1 \subsetneq L$.

Proof: Let $p_1 < p_2 < \ldots < p_n < \ldots$ be the sequence of prime numbers. Let $\{a_n\}$ be a sequence of natural numbers whose difference sequence $(d_n = a_{n+1} - a_n)$ is given by

$$\{d_n\} = \{\underbrace{p_{k_1},\ldots,p_2,p_1}_{k_1},\underbrace{p_{k_2},\ldots,p_3,p_2}_{k_2-2+1},\ldots,\underbrace{p_{k_\ell},p_{(k_\ell-1)},\ldots,p_\ell}_{k_\ell-\ell+1},\ldots\}$$

where $k_\ell-\ell+1 > p_\ell$, for each $\ell \ge 1$.

By the construction $\lim_n d_n = \infty$. Thus $A \in L$. We want to show that $A \notin \hat{L}_1$.

Suppose that $A \in \hat{L}_1$ and thus $A \subset B$ for some $B \in L_1$. Let $\{e_m\}$ be the difference sequence of $B = \{b_m\}$ so that $e_i \le e_{i+1}$ for all i. Since $A \subset B$, for any n, $a_n = b_{f(n)}$ for some function f on I and $d_n = a_{n+1} - a_n = b_{f(n+1)} - b_{f(n)} = (b_1 + e_1 + \ldots + e_{f(n+1)}) - (b_1 + e_1 + \ldots + e_{f(n)}) = e_{f(n)+1} + e_{f(n)+2} + \ldots + e_{f(n+1)}$. So that $(e_{f(n)+1}, e_{f(n)+2}, \ldots, e_{f(n+1)})$ is a partition of d_n. Since

$\lim_m e_m = \infty$, $e_{f(n)} > 1$ for some n . Suppose that $(p_{k_\ell}, p_{k_\ell-1}, \ldots, p_{\ell+1}, p_\ell)$ is the part of the sequence $\{d_n\}$ such that $(p_{k_\ell}, p_{k_\ell-1}, \ldots, p_{\ell+1}, p_\ell) = (d_m, d_{m+1}, \ldots, d_{m+(k_\ell-\ell)})$, and $e_{f(m)} > 1$. Then $(p_{k_\ell}, p_{k_\ell-1}, \ldots, p_{\ell+1}, p_\ell) = (d_m, d_{m+1}, \ldots, d_{m+(k_\ell-\ell)})$ is partitioned into $(e_{f(m)+1}, \ldots, e_{f(m+1)}, \ldots, e_{f(m+k_\ell-\ell)+1}, \ldots, e_{f(m+k_\ell-\ell+1)})$ which is a contradiction by previous lemma. Therefore $A \notin \hat{L}_1$.

Next let's prove that $\hat{L}_2 \subsetneqq \hat{L}_1$. First we will prove some lemmas.

<u>Lemma 2.13</u>. Let $p > 2$ be a prime number and let $(a_1, a_2, \ldots, a_p)$ be the sequence with $a_i = p$, for all $i = 1,2,\ldots,p$. Let $(y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, \ldots, y_{2k_2}, \ldots, y_{p1} y_{p2}, \ldots, y_{pk_p})$ be a monotone partition of $(a_1, a_2, \ldots, a_p)$ with $y_{11} > 1$. Then $k_p = 1$ and $y_{p1} = p$.

<u>Proof:</u> Suppose that $k_p > 1$. Then $y_{pk_p} < p$ and, since p is a prime, $y_{p1} < y_{pk_p}$. It follows that $k_i > 1$ for all $i < p$ since if $k_i = 1$, then $y_{i1} = p > y_{pk_p}$, which is a contradiction. Furthermore, $y_{i1} < y_{ik_i}$ since $a_i = p$ is a prime. Therefore $1 < y_{11} < y_{21} < \ldots < y_{p1} < p$ which a contradiction.

Lemma 2.14.

$$\lim_{n\to\infty} \frac{p_{n+1}^2}{p_1^2 + p_2^2 + \ldots + p_n^2} = 0$$

where, of course, p_n is the n-th prime number.

Proof: By the prime number theorem

$$\lim_{n} \frac{p_n}{n \ \ell n \ n} = 1 .$$

So that $\lim_{n} \frac{p_n}{p_{n+1}} = 1$. It follows, for any $k \in I$, that $\lim_{n} \frac{p_n}{p_{n+k}} = 1.$

Hence

$$\overline{\lim_{n\to\infty}} \frac{p_{n+1}^2}{p_1^2 + p_2^2 + \ldots + p_n^2}$$

$$= \overline{\lim_{n\to\infty}} \frac{p_{n-r}^2}{p_1^2 + p_2^2 + \ldots + p_n^2}$$

$$\leq \overline{\lim_{n\to\infty}} \frac{p_{n-k}^2}{(k+1)\ p_{n-k}^2} = \frac{1}{k+1}$$

since k is arbitrary the lemma follows.

Lemma 2.15.

(1) $$\sum_{m=1}^{\infty} \frac{1}{p_m} \ \ell n \ \frac{2(p_1^2 + p_2^2 + \ldots + p_m^2)}{2(p_1^2 + p_2^2 + \ldots + p_{m-1}^2) + p_m^2} < \infty \ ;$$

(2) $$\sum_{m=1}^{\infty} \frac{1}{p_m} \ln \frac{2(p_1^2 + p_2^2 + \ldots + p_m^2) + p_{m+1}^2}{2(p_1^2 + p_2^2 + \ldots + p_m^2)} < \infty \quad .$$

Proof: The proof of (1) and (2) are similar. We prove (2) only.

We know $\lim_{x \to 0^+} (\ln(1+x))/x = 1$. Now using this and lemma 2.14, we have

$$S = \sum_{m=1}^{\infty} \frac{1}{p_m} \ln \frac{2(p_1^2 + p_2^2 + \ldots + p_m^2) + p_{m+1}^2}{2(p_1^2 + p_2^2 + \ldots + p_m^2)}$$

$$= \sum_{m=1}^{\infty} \frac{1}{p_m} \ln \left(1 + \frac{p_{m+1}^2}{2(p_1^2 + p_2^2 + \ldots + p_m^2)}\right)$$

$$= O\left(\sum_{m=1}^{\infty} \frac{1}{p_m} \frac{p_{m+1}^2}{2(p_1^2 + p_2^2 + \ldots + p_m^2)}\right) .$$

Since $\lim_{m \to \infty} p_{m+1}/p_m = 1$

$$S = O\left(\sum_{m=1}^{\infty} \frac{p_m}{p_1^2 + p_2^2 + \ldots + p_m^2}\right)$$

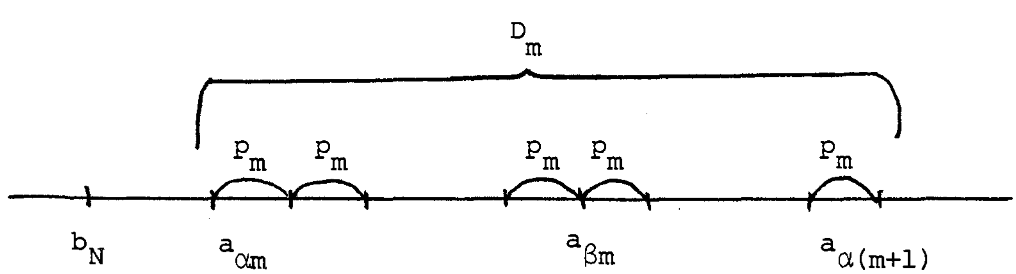
$$= O\left(\sum_{m=10}^{\infty} \frac{m \ln m}{1^2 + 1^2 + \ldots + m^2}\right)$$

$$= O\left(\sum_{m=10}^{\infty} \frac{\ln m}{m^2}\right)$$

Since $\sum_{m=10}^{\infty} (\ln m)/m^2$ converges, S is finite.

<u>Proposition 2.16</u>. $\hat{L}_2 \subsetneq \hat{L}_1$.

<u>Proof</u>: We construct $A \in \hat{L}_1$ such that $A \notin \hat{L}_2$. Let $\{p_1 < p_2 < \dots < p_m < \dots\}$ be the set of all prime numbers. Let $D_m = (p_m, p_m, \dots, p_m)$ be the sequence of $2p_m$ repetitions of p_m . Let $\{d_n\} = (D_1, \dots, D_m, \dots)$ and finally let $A = \{a_m\} \subset I$ be the sequence such that $a_n = a_1 + d_1 + \dots + d_{n-1}$ for $n \geq 2$ where $a_1 = 1$. Clearly $A \in L_1 \subset \hat{L}_1$. Suppose that $A \in \hat{L}_2$ so that $A \subset B = \{b_u\}$, where $B \in L_2$. Let $\{e_u\}$ be the difference sequence of B . Since $B \in L$, there exists N such that for any $k \geq N$, $e_k > 1$. Let us consider D_m , the m-th part of $\{d_n\}$ such that $b_N \leq a_{\alpha m}$. Consider the following diagram:



Clearly $\alpha m = 1 + 2\,(p_1 + p_2 + \dots + p_{m-1})$ and $\beta m = \alpha m + p_m$. Since $A \subset B = \{b_u\}$, some part of $\{e_u\}$ is a partition of the m-th part D_m of $\{d_n\}$. (See the proof of Lemma 2.11). Suppose that $(e_s, e_{s+1}, \dots, e_t)$ is the partition of D_m . Then $b_N \leq a_{\alpha m} = b_s$. Thus $N \leq s$ and $e_s > 1$. By Lemma 2.13, if

$$a_{\beta m} \leq b_u < b_{u+1} < a_{\alpha(m+1)} \quad \text{then} \quad e_u = p_m \,. \tag{3}$$

Since $\{e_u\}$ is a nondecreasing sequence if

$$a_{\alpha(m+1)} \leq b_u \leq a_{\beta(m+1)} \text{ then } p_m \leq e_u \leq p_{m+1} . \tag{4}$$

Let

$$B_m = \{x \in B: a_{\alpha m} < x \leq a_{\beta m}\}$$

$$B_m^* = \{x \in B: a_{\beta m} < x \leq a_{\alpha(m+1)}\}$$

for $m = 1,2,3,\ldots$.

Then from (3), we have

$$\sum_{b \in B_m^*} \frac{1}{b} = \sum_{k=1}^{p_m} \frac{1}{a_{\beta m} + kp_m}$$

$$< \int_0^{p_m} \frac{1}{a_{\beta m} + xp_m}\, dx$$

$$= \frac{1}{p_m} \ell n \frac{a_{\alpha(m+1)}}{a_{\beta m}} . \tag{5}$$

From (4) we have

$$\sum_{b \in B_{m+1}} \frac{1}{b} \leq \sum_{k=1}^{p_m} \frac{1}{a_{\alpha(m+1)} + kp_m}$$

$$< \int_0^{p_m} \frac{1}{a_{\alpha(m+1)} + xp_m}\, dx$$

$$= \frac{1}{p_m} \ell n \frac{a_{\beta(m+1)}}{a_{\alpha(m+1)}} . \tag{6}$$

Now we have

$$\sum_{b\in B} \frac{1}{b} = \sum_{b\in S} \frac{1}{b} + \sum_{b\in T} \frac{1}{b}$$

where $S = \bigcup_{m=1}^{\infty} B_m$ and $T = \bigcup_{m=1}^{\infty} B_m^*$. Let

$$S_1 = \bigcup_{a_{\alpha m} \geq b_N} B_{m+1} \,, \quad T_1 = \bigcup_{a_{\alpha m} \geq b_N} B_m^* \,.$$

Then $S - S_1$ and $T - T_1$ are finite. Thus for some real M

$$\sum_{b\in B} \frac{1}{b} = M + \sum_{b\in S_1} \frac{1}{b} + \sum_{b\in T_1} \frac{1}{b}$$

$$= M + \sum_{\alpha_m \geq N} \sum_{b\in B_{m+1}} \frac{1}{b} + \sum_{\alpha_m \geq N} \sum_{b\in B_m^*} \frac{1}{b} \,.$$

By (5) and (6)

$$\sum_{b\in B} \frac{1}{b} \leq M + \sum_{\alpha_m \geq b_N} \frac{1}{p_m} \ell n \frac{a_{\alpha(m+1)}}{a_{\beta m}} + \sum_{\alpha_m \geq b_N} \frac{1}{p_m} \ell n \frac{a_{\beta(m+1)}}{a_{\alpha(m+1)}}$$

$$\leq M + \sum_{m=1}^{\infty} \frac{1}{p_m} \ell n \frac{a_{\alpha(m+1)}}{a_{\beta m}} + \sum_{m=1}^{\infty} \frac{1}{p_m} \ell n \frac{a_{\beta(m+1)}}{a_{\alpha(m+1)}} \,.$$

By the construction of $a_{\alpha m}, a_{\beta m}$, we have

$$a_{\alpha m} = 1 + 2\,(p_1^2 + p_2^2 + \dots + p_{m-1}^2) \,,$$

$$a_{\beta m} = 1 + 2\,(p_1^2 + p_2^2 + \dots + p_{m-1}^2) + p_m^2 \,.$$

Therefore by Lemma 2.15, we have

$$\sum_{b\in B} \frac{1}{b} \le M + \sum_{m=1}^{\infty} \frac{1}{p_m} \ln \frac{1 + 2(p_1^2 + p_2^2 + \ldots + p_m^2)}{1 + 2(p_1^2 + p_2^2 + \ldots + p_{m-1}^2) + p_m^2}$$

$$+ \sum_{m=1}^{\infty} \frac{1}{p_m} \ln \frac{1 + 2(p_1^2 + p_2^2 + \ldots + p_m^2) + p_{m+1}^2}{1 + 2(p_1^2 + p_2^2 + \ldots + p_m^2)} < \infty .$$

This contradicts $B \in L_2$.

<u>Proposition 2.17</u>. $[\hat{L}_3] \subsetneqq [\hat{L}_1]$.

<u>Proof</u>: Obviously $[\hat{L}_3] \subset [\hat{L}_1]$. Let $\mathcal{D} = \{A \subset I: \sum_{a\in A} \frac{1}{a} < \infty\}$. Then obviously $\mathcal{D} = [\hat{\mathcal{D}}]$. Since $L_3 \subset \mathcal{D}$ (see the proof of Proposition 2.7), it follows that $[\hat{L}_3] \subset [\hat{\mathcal{D}}] = \mathcal{D}$. Since L_2 is full (Proposition 2.6) there exists $X \in L_1$ with $\sum_{x\in X} \frac{1}{x} = \infty$. Therefore $X \notin \mathcal{D}$ and so $X \notin [\hat{L}_3]$. Thus $[\hat{L}_3] \subsetneqq [\hat{L}_1]$.

The following lemma will be used to prove that $[L_{M_i}] \subsetneqq [L_{M_j}]$ for $0 \le i < j$. (Compare the remark following Proposition 2.7).

<u>Lemma 2.18</u>. Suppose that d, k, s, t, u, v, i and j are integers such that $d > k^2 + k$, $0 \le i < j < k$, $0 \le s \le k$, $1 \le v \le k$ and $1 \le t \le k$, then

(1) $v(d+j) \le t(d+j)+i \Rightarrow v \le t$ and $v(d+j) \le t(d+j)$,

(2) $vd \le td+i \Rightarrow v \le t$ and $vd \le td$,

(3) $v(d+j) \le s(d+j)+td+i \Rightarrow v < s+t$ and $v(d+j) < s(d+j)+td$,

(4) $v(d+j)+sd \le td+i \Rightarrow v+s < t$ and $v(d+j)+sd < td$,

(5) $vd \le sd+t(d+j)+i \Rightarrow v \le s+t$ and $vd < sd+t(d+j)$,

(6) $vd+s(d+j) \le t(d+j)+i \Rightarrow v+s \le t$ and $vd+s(d+j) < t(d+j)$.

Proof: The proofs of (1) and (2) are similar so we prove (1) only: If $v > t$ then $v(d+j) \ge t(d+j)+d+j > t(d+j)+i$ contrary to hypothesis, thus $v \le t$ and clearly (1) holds.

The proofs of (3) and (4) are similar, so we prove (3) only: If $v \le s$ then the conclusion clearly holds. Assume $v > s$ then $t > 0$. If $v \ge s+t$ then $v(d+j) \ge s(d+j)+t(d+j) > s(d+j)+td+i$ contrary to hypothesis. Hence $v < s+t$ and

$$\begin{aligned} v(d+j) &\le s(d+j)+t(d+j)-(d+j) \\ &= s(d+j)+td+(t-1)j-d \\ &< s(d+j)+td . \quad (\text{Since } (t-1)j - d < 0). \end{aligned}$$

The proofs of (5) and (6) are similar and so we prove (5) only. Since $vd \le sd+t(d+j)+i$ is equivalent to $-i-tj \le (s+t-v)d$, we have $-d < -k-k^2 < -i-tj \le (s+t-v)d$. Thus we get $-1 < (s+t-v)$ or, equivalently, $v \le s+t$.

If $vd \geq sd+t(d+j)$ then we have $(v-s)d \geq t(d+j)$. Since $1 \leq t$, $1 \leq d$ and $1 \leq j$ we have $v-s > t$ which is a contradiction. Therefore (5) holds.

<u>Proposition 2.19</u>. If $0 \leq i < j$, then $[L_{M_i}] \subsetneq [L_{M_j}]$.

<u>Proof</u>: Let

$L_m = (m^3+j, m^3+j, \ldots, m^3+j)$, m repetitions of m^3+j,

$R_m = (m^3, m^3, \ldots, m^3)$, m repetitions of m^3,

and

$B_m = (L_m, R_m, L_m, R_m, \ldots, L_m, R_m)$, m repetitions of L_m, R_m.

Let $\{d_n\} = (B_1, B_2, \ldots, B_m, \ldots)$. Let $A = \{a_n | n \in I\}$ such that $a_n = 1 + d_1 + \ldots + d_{n-1}$ for $n \geq 1$. Let

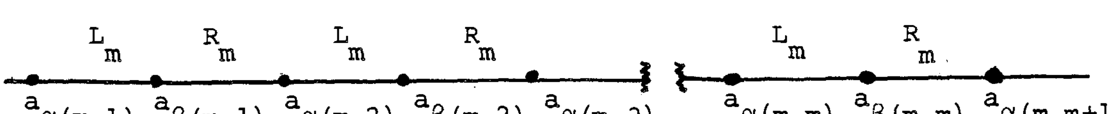
$$[a_m, a_n] = \{a_r \in A \mid m \leq r \leq n\}$$

and

$$(a_m, a_n) = \{a_r \in A \mid m < r < n\} = [a_{m+1}, a_{n-1}] .$$

Clearly $A \in L_{M_j}$.

Let us consider the m-th part of A corresponding to B_m (diagram is actually illustrated below).



Here $\alpha(m,t) = 1 + 2(1^2 + 2^2 + \ldots + (m-1)^2) + 2(t-1)m$ for $1 \le m$ and $1 \le t \le m+1$ and $\beta(m,t) = \alpha(m,t) + m$. Note that $\alpha(m+1,1) = \alpha(m,m+1)$. Let

$$A_{Lmt} = [a_{\alpha(m,t)}, a_{\beta(m,t)}], \quad A^0_{Lmt} = (a_{\alpha(m,t)}, a_{\beta(m,t)})$$

$$A_{Rmt} = [a_{\beta(m,t)}, a_{\alpha(m,t+1)}], \quad A^0_{Rmt} = (a_{\beta(m,t)}, a_{\alpha(m,t+1)}) .$$

Suppose that $X = \{x_q\} \in L_{M_i}$ and $X \subset A$. We want to show that if $m^3 > m^2 + m$ and $j < m$ then

$$|X \cap A_{Rmm}| \le 2 . \qquad (*)$$

Let $\{y_q\}$ be the difference sequence of $\{x_q\}$ and f be the function on I such that $x_q = a_{f(q)}$. Then $f(s+1) - f(s)$ equals the number of d_n's in the sum $y_s = d_{f(s)} + d_{f(s)+1} + \ldots + d_{f(s+1)-1}$. At first we will consider the following six cases which will be used in proving $|X \cap A_{Rmm}| \le 2$.

(1) If $a_{\alpha(m,t)} \le x_q < x_{q+1} < x_{q+2} \le a_{\beta(m,t)}$, i.e., three consecutive elements of X are in A_{Lmt} , then since $X \in L_{M_i}$ and $y_q \le y_{q+1} + i$, we have

$$x_{q+1} - x_q \le x_{q+2} - x_{q+1} + i ,$$

$$a_{f(q+1)} - a_{f(q)} \le a_{f(q+2)} - a_{f(q+1)} + i ,$$

$$(f(q+1) - f(q))(d+j) \le (f(q+2) - f(q+1))(d+j) + i, \text{ where } d = m^3 .$$

By the previous lemma, case (1), we conclude that

$f(q+1) - f(q) \leq f(q+2) - f(q+1)$ and $y_q \leq y_{q+1}$.

(2) Similarly, if $x_q < x_{q+1} < x_{q+2}$ are in the interval A_{Rmt} , then we apply the previous lemma case (2) and we get

$f(q+1) - f(q) \leq f(q+2)- f(q+1)$ and $y_q \leq y_{q+1}$.

(3) If $a_{\alpha(m,t)} \leq x_q < x_{q+1} \leq a_{\beta(m,t)} < x_{q+2} \leq a_{\alpha(m,t+1)}$ that is, x_q, x_{q+1} are in A_{Lmt} and x_{q+1} is in A_{Rmt} , then, since $x_{q+1} - x_q \leq x_{q+2} - x_{q+1} + i$, it follows that

$$a_{f(q+1)} - a_{f(q)} \leq a_{f(q+2)} - a_{f(q+1)} + i$$

$$= a_{\beta(m,t)} - a_{f(q+1)} + q_{f(q+2)} - a_{\beta(m,t)} + i ,$$

which is equivalent to

$$(f(q+1) - f(q))(d+j) \leq (\beta(m,t) - f(q+1))(d+j) + (f(q+2) - \beta(m,t))d + i.$$

Now we apply the previous lemma case (3) and we get

$$f(q+1) - f(q) < f(q+2) - f(q+1) \text{ and so } y_q < y_{q+1} .$$

(4) Similarly if

$$a_{\alpha(m,t)} \leq x_q < a_{\beta(m,t)} \leq x_{q+1} < x_q \leq a_{\alpha(m,t+1)} ,$$

that is, x_q is in A_{Lmt} and x_{q+1} , x_{q+2} are in A_{Rmt} , then we can apply the previous lemma case (4) and get

$$f(q+1) - f(q) < f(q+2) - f(q+1) \text{ and } y_q < y_{q+1} .$$

(5) If

$$a_{\beta(m,t)} \leq x_q < x_{q+1} \leq a_{\alpha(m,t+1)} < x_{q+2} \leq a_{\beta(m,t-1)}$$

where $t \leq m$, that is, x_q and x_{q+1} are in A_{Rmt} and x_{q+2} is in $A_{Lm(t+1)}$, then we have

$$a_{f(q+1)} - a_{f(q)} \leq a_{f(q+2)} - a_{f(q+1)} + i$$

$$= a_{\alpha(m,t+1)} - a_{f(q+1)} + a_{f(q+2)} - a_{\alpha(m,t+1)} + i$$

equivalently

$$(f(q+1) - f(q))d \leq (\alpha(m,t+1) - f(q+1))d + (f(q+2) - \alpha(m,t+1))(d+j) + i.$$

We apply the previous lemma case (5) and we get

$$f(q+1) - f(q) \leq f(q+2) - f(q+1) \text{ and } y_q < y_{q+1} .$$

(6) Again, if

$$a_{\beta(m,t)} \leq x_q < a_{\alpha(m,t+1)} \leq x_{q+1} < x_{q+2} \leq a_{\beta(m,t+1)}$$

then we can apply the previous lemma case (6) and obtain

$$f(q+1) - f(q) \leq f(q+2) - f(q+1) \text{ and } y_q < y_{q+1} .$$

Now assume that $|X \cap A_{Rmm}| \geq 3$ and so there exist three consecutive elements x_w, x_{w+1}, x_{w+2} of X in A_{Rmm}. By the case (2)

$$f(w+1) - f(w) \leq f(w+2) - f(w+1)$$

and so

$$2(f(w+1) - f(w)) \leq f(w+1) - f(w) + f(w+2) - f(w+1)$$

$$= f(w+2) - f(w) \leq m .$$

Thus $f(w+1) - f(w) \leq \frac{1}{2}m$ and $y_w = (f(w+1) - f(w))d \leq \frac{1}{2}md = \frac{1}{2}m^4$, the half length of A_{Rmm}.

We claim: for any $u < w$ and $x_u \geq a_{\alpha(m,1)}$, we have $y_u \leq y_w$.

Proof of claim: Since $X \in L_{M_i}$, we have $y_u \leq y_w + i$. Thus let $y_u = t(d+j) + vd$ and $y_w = qd$. Then we have $t(d+j) + vd \leq qd+i$.

If $t > 0$ (resp. $t = 0$), then we apply the previous lemma case (4) (resp. case (2)) and so we get $y_u \leq y_w$.

By the above claim we conclude that for any $u \leq w$ and $x_u \geq a_{\alpha(m,1)}$ we have $y_u \leq y_w \leq \frac{1}{2}m^4 = \frac{1}{2}$ length of $A_{Rmt} \leq \frac{1}{2}$ length of A_{Lmt} for $t = 1,2,\ldots,m+1$.

Hence, for any t, A_{Rmt} and A_{Lmt} each contain at least two elements of X.

Therefore we conclude that:

By case (3) and (4), if $x_q \in A^0_{Lmt}$ and $x_{q+2} \in A^0_{Rmt}$ then $f(q+1) - f(q) < f(q+2) - f(q+1)$.

By case (5) and (6), if $x_q \in A^0_{Rmt}$ and $x_{q+2} \in A^0_{Lm(t+1)}$ then $f(q+1) - f(q) \leq f(q+2) - f(q+1)$.

By case (1) and (2) if $x_q, x_{q+1}, x_{q+2} \in A_{Lmt}$ or $x_q, x_{q+1}, x_{q+2} \in A_{Rmt}$, then $f(q+1) - f(q) \leq f(q+2) - f(q+1)$.

Now if we let x_{s_q} be the element of X such that $x_{s_q} \in A^0_{Lmq}$ and $x_{s_{q+2}} \in A^0_{Rmq}$ for $q = 1,2,\ldots,m$. Then we have for $q = 1,2,\ldots,m-1$,

$$f(s_q+1) - f(s_q) < f(s_{q+1} + 1) - f(s_{q+1}) .$$

Therefore we get the sequence of inequalities

$$1 \leq f(s_1+1) - f(s_1) < f(s_2+1) - f(s_2) < \ldots < f(s_m+1) - f(s_m)$$

$$\leq f(w+1) - f(w) \leq \frac{1}{2} m .$$

Since there are $m-1$ inequalities, we get a contradiction. Therefore $|X \cap A_{Rmm}| \geq 2$.

Finally we show that $A \notin [L_{M_i}]$. Suppose that $A \in [L_{M_i}]$ and so $A = X_1 \cup X_2 \cup \ldots \cup X_n$ where $X_s \in L_{M_i}$. Since $A_{Rmm} = \bigcup_{i=1}^{n} (A_{Rmm} \cap X_i)$, if we take $m^3 > m^2 + m$ and $m > j$, we have

$$m = |A_{Rmm}| \leq \sum_{i=1}^{n} |A_{Rmm} \cap X_i| \leq 2n .$$

Since m can be arbitrarily large, we get a contradiction. Hence $A \notin [L_{M_i}]$.

Corollary 2.20. For all $i \geq 0$, $[L_{M_i}] \subsetneq [L]$. In particular $[L_1] \subsetneq [L]$.

Proof: Obviously, by Proposition 2.19, $[L_{M_i}] \subsetneq [L_{M_{i+1}}] \subset [L]$.

Proposition 2.21. $[L_2] \subsetneq [L_1]$.

Proof: Obviously we know $[L_2] \subset [L_1]$. We want to show that $A = \{n^2\} \in L_1$ but $A \notin [L_2]$.

Suppose that $A \in [L_2]$ then there exists an infinite $X \in L_2$ such that $X \subset A$. Then we have $\infty = \sum_{x \in X} (1/x) \leq \sum (1/n^2) < \infty$, which is a contradiction.

Proposition 2.22. Let $G = [\hat{L}_1]$, then $\hat{L}_1 \subsetneq G$.

Proof: Let $A = \{n^2\}$ and $B = \{n^2+1\}$. Then $A \cup B \in [L_1] \subset [\hat{L}_1] = G$. But $A \cup B \notin L = \hat{L} \supset \hat{L}_1$. Therefore $\hat{L}_1 \subsetneq G$.

We will also prove that $[L_1] \subsetneq G$. First, let's define some terms and prove a lemma.

Definition 2.23. Let $\{a_n\} = A \subset I$ be a sequence and $(a_s, a_{s+1}, \ldots, a_{s+r})$ be a part of $\{a_n\}$.

If $(d_s, d_{s+1}, \ldots, d_{s+r-1})$ is strictly decreasing sequence, where $d_i = a_{i+1} - a_i$, then we say that $(a_s, a_{s+1}, \ldots, a_{s+r})$ is a descending wave of length $r+1$ in A. Let $d_t = a_{t+1} - a_t$ be called the decreasing steps of the wave.

<u>Lemma 2.24.</u> There exists a function $f(n)$ such that, if $A_1, A_2, \ldots, A_n \in L_1$ and X is any descending wave in $A_1 \cup A_2 \cup \ldots \cup A_n$, then the length of X is less than or equal to $f(n)$.

<u>Proof:</u> We take $f(1) = 2$.

Suppose there exists $f(n-1)$ such that for any $A_1, A_2, \ldots,$ and A_{n-1} in L_1, and any descending wave X in $A_1 \cup A_2 \cup \ldots \cup A_{n-1}$, the length of $X \leq f(n-1)$.

Let $A = A_1 \cup A_2 \cup \ldots \cup A_{n-1}$ and $B = A_n$ where $A_1, A_2, \ldots, A_n \in L_1$, let

$$W_u = \{a \in A \mid b_u < a < b_{u+1}\},$$

$$V_u = \{c \in A \cup B \mid b_u \leq c \leq b_{u+1}\}.$$

(1) Suppose that X is a descending wave in $A \cup B$ and $V_u \subset X$ and $V_{u+1} \subset X$, then we wish to prove that $|W_u| < |W_{u+1}|$.

Let $e_1 > e_2 > \ldots > e_{q+1} > c_1 > c_2 > \ldots > c_{p+1}$ be the decreasing steps of $V_u \cup V_{u+1}$. Since $B \in L_1$,

$(q+1)e_{q+1} \leq e_1 + e_2 + \ldots + e_{q+1} = b_{u+1} - b_u \leq b_{u+2} - b_{u+1} =$ $c_1 + c_2 + \ldots + c_{p+1} \leq (p+1)c_1 < (p+1)e_{q+1}$. Therefore $q+1 < p+1$ and so $q < p$ where, clearly $q = |W_u|$ and $p = |W_{u+1}|$.

(2) If $X \subset A \cup B$ is a descending wave then we show $|X \cap B| \leq f(n-1) + 2$.

Suppose otherwise so that $|X \cap B| > f(n-1)+2$. Let $\{b_r, b_{r+1}, \ldots, b_s\} \subset X \cap B$, where $s = r + f(n-1) + 2$. Then $V_k \subset X$ for all $r \le k \le s-1$. By (1), $0 \le |W_r| < |W_{r+1}| < \ldots < |W_{s-1}|$, thus we have $|W_{s-1}| = |W_{r+f(n-1)+1}| \ge f(n-1)+1$ which is a contradiction since W_{s-1} is a descending wave in A .

Finally let X be a descending wave of $A \cup B$. Then by (2) we have $|X \cap B| \le f(n-1)+2$. Write $X \cap B = \{b_r, b_{r+1}, \ldots, b_s\}$. Then $X \subseteq H \cup V_r \cup \ldots \cup V_{s-1} \cup J$ where H and J are the (possibly empty) descending waves in $A \cap X$ which come before b_r and after b_{s-1} . Thus $|X| \le |H| + |J| + \sum_{i=r}^{s-1} |V_i| \le (f(n-1)+3)(f(n-1)+2)$ and so we can set $f(n) = (f(n-1)+2)(f(n-1)+3)$.

<u>Proposition 2.25</u>. Let $G = [\hat{L}_1]$, then $[L_1] \subsetneq G$.

<u>Proof</u>: Clearly $[L_1] \subset G$, let $B_n = (n^2, (n-1)n, (n-2)n, \ldots, 2n, n)$ and $\{d_n\} = \{B_1, B_2, \ldots, B_q, \ldots\} = (1,4,2,9,6,3,16,12,8,4,\ldots)$. Let $a_n = 1 + d_1 + \ldots + d_{n-1}$. Let $W_m = (m,m,\ldots,m)$, with $m(m+1)/2$ repetitions of m and $\{y_m\} = (w_1, w_2, \ldots, w_p, \ldots) = (1,2,2,2,3,3,3,3,3,3,4,4,\ldots)$. Let $x_m = 1 + y_1 + y_2 + \ldots + y_{m-1}$. Then $\{x_m\} \in L_1$ and $\{a_n\} \subset \{x_m\}$. Thus $\{a_n\} \in \hat{L}_1 \subset [\hat{L}_1]$. Since $\{a_n\}$ contains arbitrary long descending waves, by the previous lemma, $\{a_n\} \notin [L_1]$. Thus $[L_1] \subsetneq G$.

We have seen (Proposition 2.12) that $\hat{L}_1 \subsetneq L$. It follows that $\hat{L}_1 \neq [L]$. Proposition 2.25 shows that $[L_1] \subsetneq G$ and it follows that $[L_1] \subsetneq [L]$. These together, however, do not imply that $[\hat{L}_1] \subsetneq [L]$. This strict inclusion is the last goal of the present chapter. We first prove some useful lemmas.

<u>Lemma 2.26</u>. Suppose that H_1, G and B are given real numbers. Then the following two methods of defining $H_1, H_2, \ldots, H_k$ and $M_1, M_2, \ldots, M_k$ are equivalent,

(1) $M_t = G(H_t + B)$ and $H_{t+1} = H_t + M_t$ for $t = 1,2,\ldots,k$.

(2) $M_1 = G(H_1 + B)$, $M_t = (1+G)^{t-1} M_1$ and $H_{t+1} = H_t + M_t$
for $t = 1,2,\ldots,k$.

<u>Proof</u>: ((1) ⇒ (2)). Since $M_t = G(H_t + B)$, we get $M_1 = G(H_1 + B)$ and, for $t \geq 2$,

$$M_t = G(H_t + B) = G(H_{t-1} + M_{t-1} + B)$$

$$= G(M_{t-1} + H_{t-1} + B) = G(M_{t-1} + (1/G)M_{t-1})$$

$$= (G+1)M_{t-1} .$$

Thus $M_t = (G+1)^{t-1}M_1$.

((2) ⇒ (1)). Now

$$H_t - H_1 = (H_t - H_{t-1}) + (H_{t-1} - H_{t-2}) + \ldots + (H_2 - H_1)$$

$$= M_{t-1} + M_{t-2} + \ldots + M_1$$

$$= (1+G)^{t-2}M_1 + (1+G)^{t-3}M_1 + \ldots + M_1$$

$$= \frac{(1+G)^{t-1}-1}{G} M_1$$

$$= ((1+G)^{t-1}-1)(H_1 + B)$$

$$= (1+G)^{t-1}(H_1 + B) - (H_1 + B) .$$

Therefore we have $H_t + B = (1+G)^{t-1}(H_1 + B) = \frac{1}{G} M_t$ which proves the lemma.

Lemma 2.27. Let x, u, and v be positive integers. Suppose that:

$$x + (x-1) + \ldots + (x-u+1) = d_1 + d_2 + \ldots + d_\alpha ,$$

$$(x-u) + (x-u-1) + \ldots + (x-u-v+1) = d_{\alpha+1} + \ldots + d_{\alpha+\beta} .$$

$$d_1 \le d_2 \le \ldots \le d_{\alpha+\beta} \quad \text{and} \quad \frac{1}{2} uv(u+v) < d_1 .$$

Then we have $d_1 < d_{\alpha+\beta}$.

Proof: Suppose that $d_1 = d_2 = \ldots = d_{\alpha+\beta} = d$. Then we get

$$\frac{u(2x - u+1)}{2} = \alpha d$$

and

$$\frac{v(2x - 2u - v+1)}{2} = \beta d .$$

It follows that

$$ux - \frac{1}{2}u(u-1) = \alpha d$$

$$vx - \frac{1}{2}v(2u + v-1) = \beta d$$

and

$$uvx - \frac{1}{2}uv(u-1) = \alpha vd$$

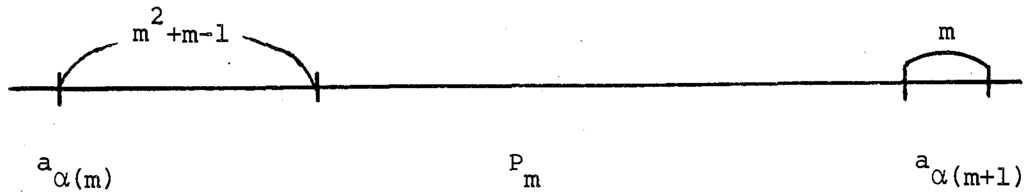
$$uvx - \frac{1}{2}uv(2u + v-1) = \beta ud .$$

By subtracting the second equation from the first, we get the equation $\frac{1}{2}uv(u+v) = (\alpha v - \beta u)d$. We conclude that $d \mid \frac{1}{2}uv(u+v)$ so that $d \le \frac{1}{2}uv(u+v)$. This is a contradiction to the hypothesis.

<u>Proposition 2.28</u>. $[\hat{L}_1] \subsetneq [L]$.

<u>Proof</u>: Let $D_m = (m^2+m-1, m^2+m-2, \ldots, m)$ and $(d_n) = (D_1, D_2, \ldots, D_m, \ldots)$. Let $A = \{a_n\} \in 2^I$ with $a_1 = 1$, $a_n = a_1+d_1+\ldots+d_{n-1}$. Then obviously $A \in L$. We claim that $A \notin [\hat{L}_1]$. Assume that $A \in [\hat{L}_1]$ so that $A \subset A_1 \cup A_2 \cup \ldots \cup A_k$ where $A_i \in L_1$ for $i = 1,2,\ldots,k$. Let us denote $A_i = \{a_n^i\}$, $d_n^i = a_{n+1}^i - a_n^i$ for $i = 1,2,\ldots,k$. Since A_i are lacunary sets, there exists N such that $n \ge N$ implies $d_n^i > (3k)^3$ for all $i = 1,2,\ldots,k$. Take $a^* = \max(a_N^1, a_N^2, \ldots, a_N^k)$.

Consider the part P_m of A corresponding to D_m as indicated in the diagram below,



Let $\alpha(m)$ be the function such that $a_{\alpha(m)}$ is the initial element of P_m. Then we compute

$$\alpha(m) = 1^2 + 2^2 + \ldots + (m-1)^2 + 1 = \frac{1}{6}(m-1)m(2m-1)+1 .$$

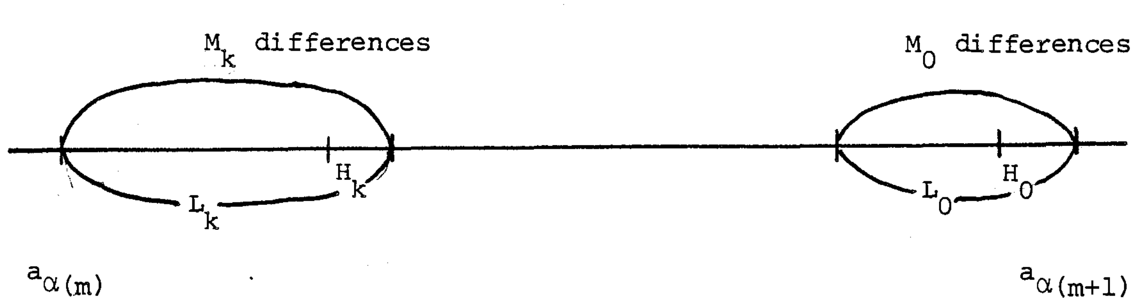
Take m such that $a^* \le a_{\alpha(m)}$. Let $M_0 = 3k$, $B = \frac{1}{2}(3k-1)$, $G = 9k^3$, $M_1 = G(m+3k+B)$ and $M_t = (1+G)^{t-1}M$, for $t = 2,3,\ldots,k$. Then we have

$$M_k + M_{k-1} + \ldots + M_1 + M_0 = \frac{1}{G}((1+G)^k - 1))M_1 + M_0$$

$$= ((1+G)^k - 1)(m+3k+B) + 3k .$$

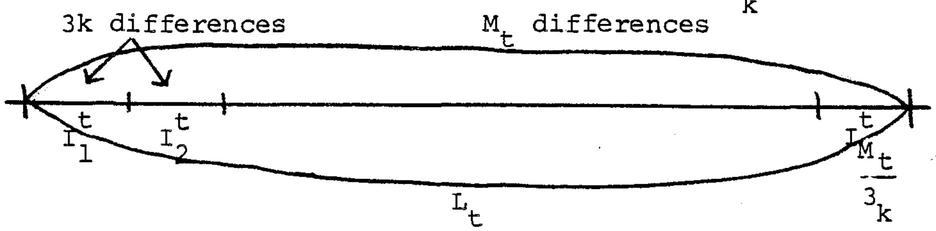
Since $M_k + \ldots + M_0 = ((1+G)^k - 1)(m+3k+B) + 3k$ is a polynomial of m of degree 1, we can further take m such that $m^2 \ge M_k + M_{k-1} + \ldots + M_0$. We will partition part of P_m backwards from $a_{\alpha(m+1)}$ so that we get intervals $L_0, L_1, \ldots, L_k$ from right to left where the number of differences of A in the interval L_t is M_t for $t = 0,1,2,\ldots,k$.

Let H_t be the smallest d_n in the interval L_t at the right hand end of L_t.



Then clearly $H_{t+1} = H_t + M_t$, $t \geq 0$. Since $H_1 = m+3k$, $M_1 = G(H_1+B)$, $B = \frac{1}{2}(3k-1)$, and $M_t = (1+G)^{(t-1)}M_1$, we apply lemma 2.26 and we get $M_t = G(H_t+B)$ for $t = 1,2,\ldots,k$.

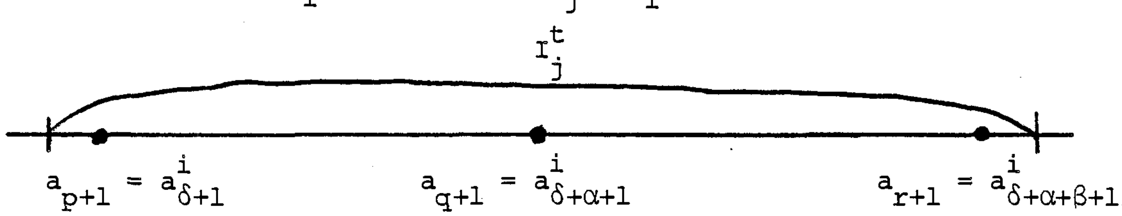
At first we partition L_t by $3k$ differences in A from left to right and we get intervals $I_1^t, I_2^t, \ldots, I^t_{\frac{M_t}{3k}}$ of A.



For each I_j^t, the number of elements of A in the interval is $3k+1$. Since $I_j^t \subset A_1 \cup A_2 \cup \ldots \cup A_k$, we get

$$3k+1 = |I_j^t| = |(I_j^t \cap A_1) \cup (I_j^t \cap A_2) \cup \ldots \cup (I_j^t \cap A_k)|$$
$$\leq |I_j^t \cap A_1| + |I_j^t \cap A_2| + \ldots + |I_j^t \cap A_k| .$$

Thus there exists A_i such that $|I_j^t \cap A_i| \geq 3$.



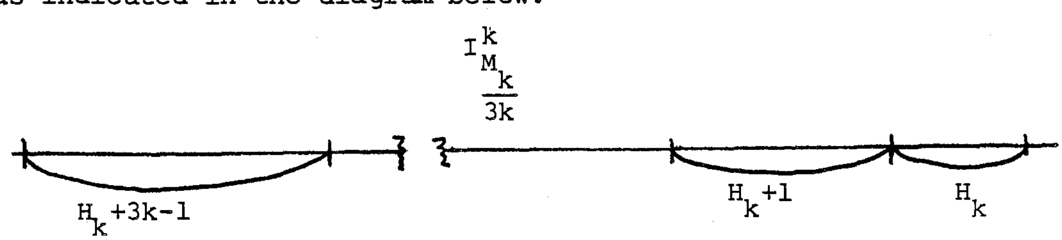
Let $a_{p+1} = a^i_{\delta+1}$, $a_{q+1} = a^i_{\delta+\alpha+1}$, and $a_{r+1} = a^i_{\delta+\alpha+\beta+1}$ be in $I_j^t \cap A_i$. Then we get the equations

$$x + (x-1) + \ldots + (x-u) = d^i_\delta + d^i_{\delta+1} + \ldots + d^i_{\delta+\alpha-1}$$

$$(x-u-1) + (x-u-2) + \ldots + (x-u-v) = d^i_{\delta+\alpha} + \ldots + d^i_{\delta+\alpha+\beta}$$

where $x = d_p$ $u = q-p$ and $v = r-q$. Of course $d^i_s \leq d^i_{s+1}$ and recall $d^i_\delta > (3k)^3$, so we can apply lemma 2.27 and get $d^i_\delta < d^i_{\delta+\alpha+\beta}$. Therefore we conclude that, for any I^t_j , there exists A_i such that d^i_n increases in I^t_j .

At first for the interval L_k , there are $M_k/3k$ I^k_j's. Thus there are at least $\frac{M_k}{3k}$ increases in the d^i_n's among $A_1, A_2, \ldots,$ and A_k . Thus there exists A_i such that there are at least $\frac{M_k}{3k^2}$ increases of d^i_n's in $A_i \cap [\min L_k, \max L_k]$. Let $d^i_{n_k}$ be the largest difference of A_i in the interval $[\min L_k, \max L_k]$, then $d^i_{n_k} > \frac{M_k}{3k^2}$. On the other hand $M_k/G = M_k/9k^3 = (H_k+B)$ and so $M_k/3k^2 = 3k(H_k+B) = 3k(2H_k+3k-1)/2$ is the length of the interval $I^k_{\frac{M_k}{3k}}$ as indicated in the diagram below.



Since D_m is a decreasing sequence, length of $I^k_{\frac{M_k}{3k}}$ > length of I^t_j whenever $k > t$. Therefore we get $d^i_{n_k}$ > length of I^t_j whenever $k > t$.

Without loss of generality let us assume that $A_i = A_1$. Then for any $t < k$, $|A_1 \cap I_j^t| \le 1$. For I_j^{k-1} with $1 \le j \le \frac{M_{k-1}}{3k}$, there exists as before A_i such that $|I_j^{k-1} \cap A_i| \ge 3$ and clearly $A_i \ne A_1$. Thus we know that there are at leat $M_k/3k$ increases of the d_n^i's among $A_2, \ldots,$ and A_k in $[\min L_{k-1}, \max L_{k-1}]$. So there exists $A_i \ne A_1$ such that there are at least $M_{k-1}/3k^2$ increases of d_n^i's in the interval $[\min L_{k-1}, \max L_{k-1}]$. Without loss of generality we assume that $A_i = A_2$. Then $d^2_{n_{(k-1)}} > M_{k-1}/3k^2$ where $d^2_{n_{(k-1)}}$ is the largest difference of A_2 in the interval $[\min L_{k-1}, \max L_{k-1}]$. Since $M_{k-1}/3k^2 = 3k(M_{k-1}/G) = 3k\ (H_{k-1} + B) = 3k(2H_{k-1} + 3k-1)/2$ is the length of the last interval $I^{k-1}_{\frac{M_{k-1}}{3k}}$. Thus A_2 cannot appear more than once in I_j^k's where $h < k-1$. Thus there exists A_3 such that there are at least $M_{k-3}/3k^2$ increases in the differences of A_3 in the interval $[\min L_{k-3}, \max L_{k-3}]$.

By repeating this process k times, for all $d^i_{n_i}$ are larger than the length of $I^1_{\frac{M_k}{3k}}$. Thus $|A_i \cap I_0| \le 1$, for $i = 1,2,\ldots,k$. Hence

$$|I_0 \cap (A_1 \cup A_2 \cup \ldots \cup A_k)|$$

$$\le |I_0 \cap A_1| + |I_0 \cap A_2| + \ldots + |I_0 \cap A_k|$$

$$\le k < 3k+1 = |I_0| .$$

Therefore I_0 is not covered by $A_1 \cup A_2 \cup \ldots \cup A_k$, which is a contradiction. This completes the proof.

Summary for Chapter II.

In Chapter II, we have shown that:

(1) L, $\hat{L}_1$, $\hat{L}_2$ and $\hat{L}_{M_i}$ (for $i \geq 1$) are full. (Propositions 2.6 and 2.8).

(2) $\hat{L}_{M_i} = L$ (for $i \geq 1$). (Proposition 2.8).

(3) $\hat{L}_{M_0} = \hat{L}_1 \subsetneqq \hat{L}$ and $\hat{L}_2 \subsetneqq \hat{L}_1$. (Propositions 2.12 and 2.16).

(4) $[\hat{L}_3] \subsetneqq [\hat{L}_1]$ $[L_1] \subsetneqq [L]$ $[L_{M_i}] \subsetneqq [L_{M_j}]$ if $0 \leq i < j$, and $[L_2] \subsetneqq [L_1]$. (Propositions 2.17, 2.20, 2.19 and 2.21).

(5) $[L_1] \subsetneqq [\hat{L}_1]$ and $\hat{L}_1 \subsetneqq [\hat{L}_1]$ (Propositions 2.25 and 2.22).

(6) $[\hat{L}_1] \subsetneqq [L]$. (Proposition 2.28).

Freedman has found the relation:

$A \in [L]$ if and only if $\chi_A \in bs + c_0$, where

$bs = \{x \in w : \sup_n |\sum_{i=1}^{n} x_i| < \infty\}$ and c is the space of convergent sequences.

We have tried to find a sequence space V_1 with $A \in [\hat{L}_1]$ if and only if $\chi_A \in V_1$. This is especially important because, as we have seen, $[\hat{L}_1]$ turns out to be unequal to $[L]$. In particular we wanted to construct a V_1 which can be defined by an analytic expression as the space bs is defined by $\sup_n |\sum_{i=1}^{n} x_i| < \infty$, or bs + c is defined by $\sup_n \limsup_m |\sum_{i=m+1}^{m+n} (x_i - r)| < \infty$ (5). The existence of such an analytic formulation for V_1 is still an open question. However, for any zero class Z, if we take

$$V = \{x \in w : \text{ for any } \alpha > 0, \{i \in I : |x_i| > \alpha\} \in Z\}$$

then we have $A \in X$ if and only if $\chi_A \in V^0$ (Proposition 3.29). This is the main motivation of our study in Chapter III.

CHAPTER III

R-TYPE SUMMABILITY METHODS

The concept of an R-type summability method (RSM) is introduced and studied to some depth. Each RSM is defined on a subspace of ω , the space of all real sequences. We topologize ω with the topology induced by uniform convergence (this is somewhat unorthodox). It turns out that an RSM is regular, non-negative and continuous with respect to this topology.

We will build on the results of Freedman and Sember [3] and ultimately obtain a bounded consistency type theorem for RSMs on their strong summability fields. When the RSM is induced by a regular matrix, our result is implied by the standard Bounded Consistency Theorem [7] although our proof does not require the same degree of depth. There are however RSMs which are not generated by any matrix. (See Proposition 3.46).

The thesis finishes with an attempt to understand those RSMs which are generated by regular matrices.

Most of the notation employed in this chapter can be found in the appendix.

Definition 3.1. Let C_s be a subspace of ω . Let $S: C_s \to R$ be a linear functional. We let

$$c_s^0 = \{x \in c_s : S(x) = 0\} ;$$

$$|c_s|^0 = \{x \in \omega : |x| \in c_s^0\} ;$$

$$|c_s| = \{x \in \omega : \exists r \in R \text{ such that } x-r \in |c_s|^0\}.$$

We say that S is a summability method, c_s is the convergence field associated with S, $|c_s|$ is the strong convergence field associated with S.

Remark: Since S is a linear functional c_s^0 is the kernel of S and so c_s^0 is a linear space.

We introduce several types of summability methods.

Definition 3.2. Let $S: c_s \to R$ be a summability method. We say that

(1) S is regular, in case $c \subset c_s$ and for any $x \in c$, $S(x) = \lim x$.

(2) S is nonnegative if $x \in c_s$, $x \geq 0$ (i.e., $x_i \geq 0$ for each i) then $S(x) \geq 0$.

(3) S is an R-type summability method (RSM) in case $m|c_s|^0 = |c_s|^0$ and S is regular.

Note: For matrix methods "regular" has the usual meaning but "nonnegative" is somewhat different here. Definition (1) and (3) are from [3].

Proposition 3.3. For any summability method $S: c_s \to R$, the condition $m|c_s|^0 \subset c_s$ is equivalent to the condition $x \in |c_s|^0$ and $|y| \leq |x|$ implies $y \in c_s$.

Proof. Assume that $m|c_s|^0 \subset c_s$. Suppose that $x \in |c_s|^0$ and

and $|y| \leq |x|$. Let $z \in \omega$ with

$$z_i = \begin{cases} 1 & \text{if } x_i = 0 , \\ \dfrac{y_i}{x_i} & \text{if } x_i \neq 0 . \end{cases}$$

Then $|z| \leq e = (1,1,\ldots)$ and $y = zx$. By the assumption $m|C_s|^0 \subset C_s$, $y \in C_s$.

Assume that $x \in |C_s|^0$ and $|y| \leq |x|$ implies $y \in C_s$. Suppose $a \in m$, $x \in |C_s|^0$. Let $|a_i| \leq M$ for each $i = 1,2,\ldots$. Then $S(|Mx|) = MS(|x|) = M0 = 0$. Therefore $Mx \in |C_s|^0$. Since $|ax| \leq |Mx|$ and $Mx \in |C_s|^0$, by the assumption, $ax \in C_s$.

<u>Proposition 3.4.</u> Let $S: C_s \to R$ be a regular summability method. Then S is an RSM if and only if

$$x \in |C_s|^0 \text{ and } |y| \leq |x| \text{ implies } y \in |C_s|^0 .$$

<u>Proof:</u> Assume that $m|C_s|^0 = |C_s|^0$. Suppose that $x \in |C_s|^0$ and $|y| \leq |x|$. Let $z \in \omega$ with

$$z_i = \begin{cases} 1 & \text{if } x_i = 0 , \\ \dfrac{y_i}{x_i} & \text{if } x_i \neq 1 . \end{cases}$$

As in the proof of the previous proposition $z \in m$ and $y = zx$. By the assumption $y \in |C_s|^0$.

Assume that, if $x \in |C_s|^0$ and $|y| \leq x$, then $y \in |C_s|^0$. Since $|C_s|^0 \subset m|C_s|^0$, it is enough to show $m|C_s|^0 \subset |C_s|^0$. The proof is similar to that of the previous proposition and is omitted.

Proposition 3.5. ([3], Proposition 4.8). If $S: C_s \to R$ is an RSM, then

(1) $|C_s|^0 \subset C_s^0$,

(2) $m|C_s|^0 \subset C_s^0$,

(3) $|C_s| \subset C_s$.

Proof: We omit the proof since readers can find the proof in the reference.

Proposition 3.6. ([3]. Proposition 4.9). If S is an RSM, then $|C_s|$ and $|C_s|^0$ are subspaces of C_s and C_s^0, respectively. Furthermore, $c \subset |C_s|$ and $c_0 \subset |C_s|^0$.

Proof: We omit the proof.

Now we proceed to investigate further properties of an RSM.

Proposition 3.7. If S is an RSM then S is nonnegative.

Proof: Suppose that $x \in C_s$ and $x \geq 0$. We want to show that $S(x) \geq 0$.

Assume that $S(x) = -r$, where $r > 0$. Then $|x+r| = x+r$ and $S(x+r) = S(x) + S(re) = S(x)+r = -r+r = 0$. Hence $x+r \in |C_s|^0$. By proposition 3.4, and since $re \le x+r$, $re \in |C_s|^0$. Thus $S(re) = 0$, which contradicts that S is regular. Thus $S(x) \ge 0$.

Proposition 3.8. If $S: C_s \to R$ is a nonnegative summability method then for any x, y in C_s, $x \le y$ implies $S(x) \le S(y)$.

Proof: This is a standard result.

Proposition 3.9. If $S: C_s \to R$ is a nonnegative and regular summability method then $S: C_s \to R$ is a continuous function where C_s is given the topology of uniform convergence (write T_∞ for this topology).

Proof: For $x \in m$, we denote $\|x\|_\infty = \sup_n |x_n|$. It is sufficient to show that, for any $\varepsilon > 0$, there exist $\delta > 0$ such that $|S(x) - S(y)| \le \varepsilon$ whenever $x,y \in C_s$, $x-y \in m$ and $\|x-y\|_\infty < \delta$.

Take $\delta = \varepsilon$. If $x,y \in C_s$, $x-y \in m$ and $\|x-y\|_\infty < \delta$, then $-\delta e \le x-y \le \delta e$. Thus, by Proposition 3.8, $S(-\delta e) \le S(x-y) \le S(\delta e)$. Since S is regular and linear, $-\delta \le S(x) - S(y) \le \delta$.

Corollary 3.10. If $S: C_s \to R$ is an RSM then S is continuous, where C_s has the topology T_∞.

Proof: If $S: C_s \to R$ is an RSM then S is a nonnegative and regular by Proposition 3.7. It follows that, by Proposition 3.9, S is continuous.

Proposition 3.11. If $S: C_s \to R$ is an RSM then for any $x \in C_s$,

$$\liminf_n x_n \le S(x) \le \limsup_n x_n .$$

<u>Proof</u>: Let $x \in C_s$. If $\liminf_n x_n = -\infty$ then obviously $\liminf_n x_n \le S(x)$. Suppose that $\liminf_n x_n > -\infty$. Let $x \in C_s$ and $y_n = \inf_{k \ge n} x_k \in R$. Then $y = (y_1, y_2, \ldots, y_n, \ldots) \le x$ and $\lim y = \liminf_n x_n$. Consider the eventually constant sequences z^n, where

$$z_i^n = \begin{cases} y_i & \text{if } i \le n , \\ y_n & \text{if } i > n . \end{cases}$$

Then $z^n \in c$ and $\lim z^n = y_n$ and $z^n \le x$. Since S is an RSM, $z^n \in c \subset C_s$ and $S(z^n) = \lim z^n = y_n$. Since S preserves order by Proposition 3.8, $y_n = S(z^n) \le S(x)$. Therefore $\liminf_n x_n = \lim y \le S(x)$. Finally, $\liminf_n (-x_n) \le S(-x)$. Thus $S(x) \le \limsup_n x_n$.

<u>Corollary 3.12</u>. If $S\colon C_s \to R$ is an RSM then $C_s \ne \omega$ (the space of all sequences).

<u>Proof</u>: By the previous proposition $\liminf_n x_n \le S(x) < \infty$ and $\limsup_n x_n > -\infty$. Hence no sequence which diverges to ∞ or $-\infty$ can be in C_s.

<u>Proposition 3.13</u>. Suppose that $S\colon C_s \to R$ is a regular summability method, $m|C_s|^0 \subset C_s$ and $m \not\subset C_s$. Then S is an RSM.

Proof: By Proposition 3.4, it is sufficient to show, $x \in |C_s|^0$ and $|y| \le |x|$ implies $y \in |C_s|^0$. Suppose that $x \in |C_s|^0$ and $|y| \le |x|$. By Proposition 3.3, $|y| \in C_s$.

Case (1). Assume that $S(|y|) = -r < 0$. Then $|y| + r > 0$ and $S(|y| + r) = S(|y|) + S(re) = -r + r = 0$. By the definition of $|C_s|^0$, $|y| + r \in |C_s|^0$.

Let $z \in m$ with $\|z\|_\infty = b$. Then, for each i,

$$\frac{|z_i|}{|z_i| + r} \le \frac{1}{r} |z_i| \le \frac{b}{r} .$$

Thus

$$\frac{z}{|y| + r} \in m \quad \text{and so}$$

$$z = \frac{z}{|y| + r} (|y| + r) \in m|C_s|^0 \subset C_s$$

by the hypothesis. Therefore $m \subset C_s$, which contradicts the hypothesis. Hence $S(|y|) \ge 0$.

Case (2). Assume that $S(|y|) = r > 0$. Take $w = |x| - |y|$. Then $|w| = w \le |x|$. By Proposition 3.3 and the condition $m|C_s|^0 \subset C_s$, $|w| \in C_s$. Therefore $S(|w|) = S(|x| - |y|) = S(|x|) - S(|y|) = 0 - r < 0$. We can apply the argument of Case (1) to w and get $S(|w|) \ge 0$, a contradiction. Thus $S(|y|) = 0$ and $y \in |C_s|^0$.

Example 3.14. Let B be a Hamel basis for c and $B \cup D$ be a Hamel basis for ω where $B \cap D = \phi$. For any $x \in \omega$, we can express uniquely

$$x = \sum_{b \in B}^{\cdot} \alpha_b b + \sum_{d \in D}^{\cdot} \beta_d d$$

where α_b, β_d are all zero except for finitely many. Let us write $x_B = \sum_{b \in B} \alpha_b b$, $x_D = \sum_{d \in D}^{\cdot} \beta_d d$. We define linear functional $S: \omega \to R$ such that $S(x) = \lim x_B$ for any $x \in \omega$. Then $S: C_s = \omega \to R$ is a regular summability method such that $m|C_s|^0 \subset C_s$, $m \subset C_s$ but S is not an RSM .

Proof: Obviously $m|C_s|^0 \subset C_s$ and $m \subset C_s$. By Corollary 3.12, since $C_s = \omega$, S cannot be an RSM .

If $S: C_s \to R$ is a regular summability method and C_s is small, that is, $m \not\subset C_s$, then we can replace the condition $m|C_s|^0 = |C_s|^0$ to $m|C_s|^0 \subset C_s$ for S being an RSM. For example, AC, ω_δ where δ is ordinary density (see Definition 3.22), V_χ when χ is not an ultra zeroclass (see Definition 3.27 and Proposition 3.45).

We have studied some properties of RSMs . Next we will illustrate some examples which show the difference between nonnegative regular summabilities and RSMs .

Example 3.15. Let $A \in 2^I$ be an infinite set with $\bar{\delta}(A) = 0$, where δ is the ordinary density. Let $C_s = c \oplus < \chi_A >$ where $< \chi_A >$

denotes the linear subspace of ω spanned by χ_A . Let $S: C_s \to R$ be defined by $S(x+t) = \lim x$, where $x \in c$ and $t \in \langle \chi_A \rangle$. Then S is regular and nonnegative, C_s is a closed subset of (ω, T_∞) but $m|C_s|^0 \notin C_s$.

Proof: S is nonnegative and regular immediately from the definition. Next we want to show: $c \oplus \langle \chi_A \rangle$ is a closed subset of (ω, T_∞) .

Suppose that $x^n \in c$ and $x^n + t_n\chi_A \to z \in \omega$ with respect to T_∞ , that is, $\|x^n + t_n\chi_A - z\|_\infty \to 0$ as $n \to \infty$. Suppose that $\lim x^n = r_n$ for each n . First let us show $\{r_n\}$ is a Cauchy sequence. For any $\varepsilon > 0$, there exists $N \in I$ such that if $n,m > N$ then $\|x^n + t_n\chi_A - (x^m + t_m\chi_A)\|_\infty < \varepsilon$. Then

$$|r_n - r_m| = |r_n - x_i^n + x_i^n - x_i^m + x_i^m - r_m|$$

$$\leq |x_i^n - r_n| + |x_i^n - x_i^m| + |x_i^m - r_m|.$$

Since I-A is infinite, for each $i \in I-A$, we get

$$|r_n - r_m| \leq |x_i^n - r_n| + |x_i^n - t_n\chi_A(i) - (x_i^m - t_m\chi_A(i))| + |x_i^m - r_m|$$

$$\leq |x_i^n - r_n| + |x_i^m - r_m| + \|(x^n - t_n\chi_A) - (x^m - t_m\chi_A)\|_\infty$$

$$\leq |x_i^n - r_n| + |x_i^m - r_m| + \varepsilon .$$

Thus, since $x_i^n \to r_n$ and $x_i^m \to r_m$, we have

$$|r_n - r_m| \leq \inf_{i \in I-A} (|x_i^n - r_n| + |x_i^m - r_m|) + \varepsilon$$

$$= 0 + \varepsilon .$$

Therefore $\{r_n\}$ is a Cauchy sequence. Further we know that for any $\varepsilon > 0$ there exists $N \in I$ such that $n,m > N$ implies $\|(x^n - t_n\chi_A) - (x^m - t_m\chi_A)\|_\infty < \varepsilon$ and $|r_n - r_m| < \varepsilon$.

Suppose $n,m > N$ and $i \in A$, then we have

$$\begin{aligned} |t_n - t_m| &= |t_n\chi_A(i) - t_m\chi_A(i)| \\ &= |t_n\chi_A(i) - x_i^n + x_i^n - r_n + r_n - r_m + r_m - x_i^m + x_i^m - t_m\chi_A(i)| \\ &\leq |t_n\chi_A(i) - x_i^n + x_i^m - t_m\chi_A(i)| + |x_i^n - r_n| + |r_n - r_m| + |r_m - x_i^m| \\ &\leq \|(x^n - t_n\chi_A) - (x^m - t_m\chi_A)\|_\infty + |r_m - r_n| + |x_i^n - r_n| + |x_i^m - r_m| \\ &\leq 2\varepsilon + |x_i^n - r_n| + |x_i^m - r_m| . \end{aligned}$$

Since $x_i^m \to r_m$, $x_i^n \to r_n$ and A is given to be infinite, we have $\inf_{i \in A} (|x_i^n - r_n| + |x_i^m - r_m|) = 0$. Therefore $|t_n - t_m| \leq 2\varepsilon$. Hence $\{t_n\}$ is a Cauchy sequence. Let $\lim_n t_n = t$. Then $t_n\chi_A$ uniformly converges to $t\chi_A$, and so $x_n = (x_n - t_n\chi_A) + t_n\chi_A$ uniformly converges to $z - t\chi_A$. Since c is a closed subset of (ω, T_∞), it follows that $z - t\chi_A \in c$ and so $z \in c \oplus < \chi_A >$. Consequently $c \oplus < \chi_A >$ is a closed subset of (ω, T_∞).

Finally we prove that $m|C_s|^0 \not\subset C_s$. Let $B \subset A$ such that B and $A - B$ are infinite. Then $\chi_B \in m|C_s|^0$ since $\chi_A \in |C_s|^0$. We show that $\chi_B \notin C_s$.

Suppose otherwise and assume that $\chi_B = x + r\chi_A$ where $x \in c$ and $r \in R$. We get

$$0 = \lim_{\substack{i\to\infty \\ i\notin A}} \chi_B(i) = \lim_{\substack{i\to\infty \\ i\notin A}} (x_i + r\chi_A(i)) = \lim_{\substack{i\to\infty \\ i\notin A}} x_i = \lim x .$$

Again,

$$0 = \lim_{\substack{i\to\infty \\ i\in A-B}} \chi_B(i) = \lim_{\substack{i\to\infty \\ i\notin A-B}} (x_i + r\chi_A(i)) = \lim_{\substack{i\to\infty \\ i\in A-B}} r\chi_A(i) = r .$$

Consequently $\chi_B = x + 0\chi_A = x$. But $\chi_B \notin c$, a contradiction.

By this example we can declare the following proposition.

<u>Proposition 3.17.</u> S being regular and nonnegative does not imply $m|C_s|^0 \subset C_s$.

<u>Example 3.18.</u> Suppose that $f: m \to R$ and $g: m \to R$ are continuous regular summabilities from (m, T_∞) into R. e.g., f and g can be "Banach limits" [4]. Let us define $h: m \to R$ such that

$$h(x) = 2f(x_1, x_3, \ldots, x_{2n+1}, \ldots) - g(x_2, x_4, x_6, \ldots, x_{2n}, \ldots) .$$

Then h is continuous and regular and $m|C_s|^0 \subset C_s$ but h is not nonnegative.

Proof: Given x, let $y = (x_1, x_3, x_5, \ldots)$ and $z = (x_2, x_4, x_6, \ldots)$. Then we have

$$|h(x)| = |2f(y) - g(z)|$$

$$\leq 2|f(y)| + |g(z)|$$

$$\leq 2\|f\|\|y\|_\infty + \|g\|\|z\|_\infty$$

$$\leq (2\|f\| + \|g\|)\|x\|_\infty .$$

Thus h is bounded, equivalently h is continuous.

For any $x \in c$, since f and g are regular,

$$h(x) = 2f(x_1, x_3, \ldots, x_{2n+1}, \ldots) - g(x_2, x_4, \ldots, x_{2n}, \ldots)$$

$$= 2 \lim_n x_{2n+1} - \lim_n x_{2n} = \lim x .$$

Thus h is regular. But $h(0,1,0,1,\ldots) = 2f(0,0,0,\ldots) - g(1,1,1,\ldots) = 2.0 - 1 = -1$. Thus h is not nonnegative.

By the above example we conclude that:

Proposition 3.19. A summability method $S: C_S \to R$ being regular, continuous and satisfying $m|C_S|^0 \subset C_S$ does not imply that S is nonnegative (compare example 3.14).

Proposition 3.20. If $S: C_s \to R$ is regular and nonnegative and $m|C_s|^0 \subset C_s$ then S is an RSM. (Compare Proposition 3.4.).

Proof: Suppose that $x \in |C_s|^0$ and $|y| \le |x|$. Since $m|C_s|^0 \subset C_s$ and $|y| \in m|C_s|^0$, we get $|y| \in C_s$. Since S is nonnegative, $0 \le S(|y|) \le S(|x|) = 0$, it follows that $S(|y|) = 0$, equivalently $y \in |C_s|^0$. By Proposition 3.4. S is an RSM.

We have shown that an RSM is nonnegative and continuous under the topology T_∞. Next we will find some relation between densities and RSMs Freedman and Sember have found a connection between RSMs and densities. ([3], Proposition 4.10). We will extend this result so that we obtain a "Bounded Consistency Theorem" on strong convergence fields.

Definition 3.21. Let $x \in \omega$ and $r \in R$ and $A \in 2^I$ with $I-A$ infinite. We write $x \underset{(A)}{\longrightarrow} r$ in case for each $\varepsilon > 0$ there exists $N > 0$ such that $|x_n - r| < \varepsilon$ whenever $n \ge N$, $n \notin A$.

Definition 3.22. For any density δ, let

$$\omega_\delta = \{x \in \omega: \exists r \in R \text{ and } A \subset I \text{ with } \bar{\delta}(A) = 0 \text{ and } x \underset{(A)}{\longrightarrow} r\}.$$

For any zeroclass $\mathcal{X}$, let $\omega_{\mathcal{X}} = \{x \in \omega: \exists r \in R$ and $A \subset I$ with $x \in \mathcal{X}$ and $x \underset{(A)}{\longrightarrow} r\}$. We call ω_δ the set of (δ-) nearly convergent sequences. We call $\omega_{\mathcal{X}}$ the set of ($\mathcal{X}$-) nearly convergent sequences.

Proposition 3.23. For any density δ, $\eta_\delta^0 = \{A \in \omega \mid \bar{\delta}(A) = 0\}$ is a zeroclass (definition 1.7.).

Proof: Let $A \in 2^I$ be finite. Then $\bar{\delta}(A) = \bar{\delta}(\phi) = 0$ (by Proposition 1.1, V). Thus $A \in \eta_\delta^0$. Suppose that A and B are in η_δ^0. Then $\bar{\delta}(A \cup B) \le \bar{\delta}(A) + \bar{\delta}(B) = 0 + 0 = 0$ and $A \cup B \in \eta_\delta^0$. Let $A \subset B$ and $B \in \eta_\delta^0$. Then $\bar{\delta}(A) \le \bar{\delta}(B) = 0$ and $A \in \eta_\delta^0$. Finally $\bar{\delta}(I) = 1$ so that $I \notin \eta_\delta^0$.

Proposition 3.24. For any zeroclass X let

$$d_X(A) = \begin{cases} 1 & \text{if } I-A \in X \\ 0 & \text{otherwise.} \end{cases}$$

Then d_X is a density with $\eta_{d_X}^0 = X$.

Proof: By example 1.27 d_X is a density. For any $A \in 2^I$, $A \in X$ if and only if $I - (I-A) \in X$ if and only if $d_X(I-A) = 1$ if and only if we $\bar{d}_X(A) = 0$ if and only if $A \in \eta_{d_X}^0$. Therefore $X = \eta_{dX}^0$.

Remark: For any density δ, η_δ^0 is a zeroclass. And for any zeroclass X, there exists a density δ such that $\eta_\delta^0 = X$. Therefore we do not have to distinguish between ω_δ and ω_X. ω_δ may be considered as $\omega_{\eta_\delta^0}$ and ω_X may be considered as ω_{d_X}.

Proposition 3.25. For any zeroclass X, ω_X is a linear space of sequences with $c \subset \omega_X$. ([3], Proposition 4.3.).

Proof: If $x \in c$ then $x \xrightarrow[(\phi)]{} r$ for some $r \in R$. Since $\phi \in X$, $x \in \omega_X$.

Let x and y be in ω_X and let r_1, r_2, A, B be such that $A, B \in X$ and $x \xrightarrow[(A)]{} r_1$ and $y \xrightarrow[(B)]{} r_2$, $A \cup B \in X$ and $x + y \xrightarrow[(A \cup B)]{} r_1 + r_2$. So that $x+y \in \omega_X$. Suppose that $k \in R$, then $kx \xrightarrow[(A)]{} kr_1$. Therefore $kx \in \omega_X$. Hence ω_X is a linear space.

Definition 3.26. [3]. A density δ (resp. zeroclass X) and RSM are related if, for each $A \in 2^I$

$$\bar{\delta}(A) = 0 \text{ (resp. } A \in X) \Leftrightarrow \chi_A \in |c_s|^0 .$$

Now, let us introduce a new technique using the zeroclass concept which will pave the way to the bounded consistency theorem on the strong convergence fields.

Definition 3.27. For any zeroclass X, we denote

$$v_X^0 = \{x \in \omega: \text{ For any } \alpha > 0 \ \{i: \alpha \dot{<} |x_i|\} \in X\}$$

$$v_X = \{x \in \omega: \exists r \in R, \ x - r \in v_X^0\} .$$

Proposition 3.28. For any zeroclass X,

(1) V_X is a linear space of sequences.

(2) V_X^0 is a subspace of V_X.

Proof: Suppose that $x \in V_X$ and $y \in V_X$ and $r_1, r_2 \in R$ with $x - r_1 \in V_X^0$, $y - r_2 \in V_X^0$. Since, for any i,

$$|x_i + y_i - (r_1 + r_2)| \le |x_i - r_1| + |y_i - r_2|,$$

for any $\alpha > 0$,

$$|x_i - r_1| \le \frac{\alpha}{2} \text{ and } |y_i - y_2| \le \frac{\alpha}{2} \Rightarrow |x_i + y_i - (r_1 + r_2)| \le \alpha .$$

Thus

$$\{i: |x_i + y_i - (r_1 + r_2)| > \alpha\} \subseteq \{i: |x_i - r_1| > \tfrac{\alpha}{2}\} \cup \{i: |y_i - y_2| > \tfrac{\alpha}{2}\}.$$

By the definition of V_X^0 and the properties of zeroclasses $\{i: |x_i - r_1| > \frac{\alpha}{2}\} \cup \{i: |y_i - r_2| > \frac{\alpha}{2}\} \in X$. Thus $\{i: \alpha < |(x_i + y_i) - (r_1 + r_2)|\} \in X$. Consequently it follows that $x+y \in V_X$. If $k \in R$, then for any $\alpha > 0$,

$$\{i: \alpha < |kx_i - kr_1|\} = \begin{cases} \phi & \text{if } k = 0 \\ \{i: \frac{\alpha}{|k|} < |x_i - r_1|\} & \text{if } k \ne 0 \end{cases} .$$

Therefore for any $\alpha > 0$, $\{i: \alpha < |kx_i - kr_1|\} \in X$, which implies

$kx \in V_X$. Hence V_X is a linear space of sequences.

(2) In the proof of (1) we can put $r_1 = r_2 = 0$. The other steps are all the same. Hence V_X^0 is a subspace of V_X.

<u>Proposition 3.29</u>. For any zeroclass X, let $T_X\colon V_X \to R$ be the function from V_X to R defined by

$$T_X(x) = r \Leftrightarrow x-r \in V_X^0 .$$

Then T_X is an RSM with domain $C_{T_X} = V_X$ and $|C_{T_X}| = C_{T_X} = V_X$ and $|C_{T_X}|^0 = C_{T_X}^0 = V_X^0$. Further, T_X is related with the zeroclass X.

<u>Proof</u>: In this proof we will write T_X as T for convenience.

We first prove T is well defined.

Suppose that $x-r_1 \in V_X^0$ and $x-r_2 \in V_X^0$. Since V_X^0 is a linear space $(x-r_1) - (x-r_2) = (r_2-r_1)e \in V_X^0$. From the facts

$$\{i\colon \alpha < |(r_1-r_2)e_i|\} = \begin{cases} \phi & \text{if } \alpha \geq |r_2-r_1| \\ I & \text{if } \alpha < |r_2-r_1| \end{cases}$$

and $\{i\colon \alpha < |(r_2-r_1)e_i|\} \in X$, for any $\alpha > 0$, it follows that $r_1 = r_2$. Thus T is well defined.

Suppose that $x,y \in V_X$ and $T(x) = r_1$ and $T(y) = r_2$.

Then $x-r_1 \in V_X^0$ and $y-r_2 \in V_X^0$. Since V_X^0 is a linear space, $x+y - (r_1+r_2) \in V_X^0$ and $kx - kr_1 \in V_X^0$ for any real number k. Therefore $T(x+y) = T(x) + T(y)$ and $T(kx) = kT(x)$. Hence T is a linear functional.

We have

$$
\begin{aligned}
C_T^0 &= \{x \in \omega: \ T(x) = 0\} \\
&= \{x \in \omega: x \in V_X^0\} = V_X^0 \\
&= \{x \in \omega: \text{for any } \ \alpha > 0, \{i: \alpha < |x_i|\} \in X\} \\
&= \{x \in \omega: |x| \in V_X^0\} \\
&= \{x \in \omega: |x| \in C_T^0\} \\
&= |C_T|^0 .
\end{aligned}
$$

Therefore

$$C_T^0 = V_X^0 = |C_T|^0 .$$

Furthermore,

$$
\begin{aligned}
C_T &= V_X \\
&= \{x \in \omega: \exists r \in R \ \ x-r \in V_X^0\} \\
&= \{x \in \omega: \exists r \in R \ \ x-r \in |C_T^0|\} \\
&= |C_T|.
\end{aligned}
$$

Thus

$$C_T = V_X = |C_T| .$$

We now use Proposition 3.4 to show that T is an RSM.

Suppose that $x \in |C_T|^0$ and $|y| \leq |x|$. For any i, $|y_i| \leq |x_i|$. Thus $\alpha < |y_i| \Rightarrow \alpha < |x_i|$. Hence for any $\alpha > 0$, $\{i: \alpha < |y_i|\} \subset \{i: \alpha < |x_i|\}$. On the other hand $\{i: \alpha < |x_i|\} \in X$ for any $\alpha > 0$. Hence, for any $\alpha > 0$, $\{i: \alpha < |y_i|\} \in X$. Consequently, we have $y \in |C_T|^0$. Hence T is an RSM.

For any $A \in 2^I$,

$$\{i: \alpha < \chi_A(i)\} = \begin{cases} A & \text{if } 0 < \alpha < 1 \\ \phi & \text{if } 1 \leq \alpha . \end{cases}$$

Hence $\chi_A \in V_X^0 = |C_T|^0 \Leftrightarrow A \in X$. Thus T and X are related.

<u>Proposition 3.30</u>. For any zeroclass X, V_X is closed with respect to the topological space (ω, T_∞).

<u>Proof</u>: Suppose that $x \in \bar{V}_X$ and choose $\{x^n\} \subset V_X$ such that $\|x^n - x\|_\infty = \sup_{i \geq 1} |x_i^n - x_i| < \frac{1}{n}$ $(n \geq 1)$. Suppose that $T_X(x^n) = T(X) = r_n$. Since $\{x^n\}$ converges to x, we have, for any $\varepsilon > 0$, that there exists $N \in I$ such that $n,m \geq N \Rightarrow \|x^n - x^m\|_\infty < \varepsilon$. We want to show that

$n,m \geq N \Rightarrow |r_n - r_m| \leq \varepsilon$. Suppose that $n,m \geq N$. For each $i \in I$

$$|r_n - r_m| \leq |r_n - x_i^n| + |x_i^n - x_i^m| + |x_i^m - r_m|$$

$$< |r_n - x_i^n| + \varepsilon + |x_i^m - r_m| \ .$$

Clearly

$$I = \{i: |r_n - r_m| - \varepsilon < |r_n - x_i^n| + |x_i^m - r_m|\}$$

$$\subset \{i: \frac{|r_n - r_m| - \varepsilon}{2} < |r_n - x_i^n|\} \cup \{i: \frac{|r_n - r_m| - \varepsilon}{2} < |x_i^m - r_m|\}.$$

If

$$|r_n - r_m| > \varepsilon ,$$

then

$$\{i: \frac{|r_n - r_m| - \varepsilon}{2} < |x_i^n - r_n|\} \in X$$

and

$$\{i: \frac{|r_n - r_m| - \varepsilon}{2} < |x_i^m - r_m|\} \in X$$

and it follows that $I \in X$, which is a contradiction. Therefore $|r_n - r_m| \leq \varepsilon$. Consequently it follows that $\{r_m\}$ is a Cauchy sequence of real numbers. Let $\lim_n r_n = r \in R$.

Now we claim that $x \in V_X$. For any $\alpha > 0$ there exists $N \in I$ such that $n > N \Rightarrow \|x - x^n\|_\infty < \frac{\alpha}{3}$ and $|r_n - r| < \frac{\alpha}{3}$. For any $i \in I$,

$$|x_i - r| \le |x_i - x_i^n| + |x_i^n - r_n| + |x_n - r|$$

$$< \frac{2\alpha}{3} + |x_i^n - r_n| \ .$$

So that

$$\{i: \alpha < |x_i - r|\} \subset \{i: \frac{\alpha}{3} < |x_i^n - r_m|\} \ .$$

Since $T(x^n) = r_n$,

$$\{i: \frac{\alpha}{3} < |x_i^n - r_n|\} \in X \ .$$

Therefore $\{i: \alpha < |x_i - r|\} \in X$. Hence $x \in V_X$, and so, it follows that $\bar{V}_X \subset V_X$.

<u>Proposition 3.31</u>. For any zeroclass X , V_X^0 is a closed subset of (ω, T_∞).

<u>Proof</u>: $T_X: (V_X, T_\infty) \to R$ is an RSM and so it is continuous. Thus $T_X^{-1}(0) = V_X^0$ is a closed subset of (V_X, T_∞). Since, by previous proposition, V_X is a closed subset of (ω, T_∞), V_X^0 is a closed subset of (ω, T_∞) .

<u>Proposition 3.32</u>. For any zeroclass X , $\bar{\omega}_X = V_X$ (where $\bar{\omega}_X$ denotes the closure of ω_X with respect to the topology T_∞).

<u>Proof</u>: Suppose that $x \in \omega_X$ and $r \in R$ and $A \in X$ with $x \xrightarrow[(A)]{} r$. Then by the definition of $x \xrightarrow[(A)]{} r$, we have, for any $\alpha > 0$ there exists $N \in I$ such that $\{i: \alpha < |x_i - r|\} \subset A \cup \{1,2,\ldots,N\}$. Since $A \in X$ and $\{1,2,\ldots,N\} \in X$, we have $A \cup \{1,2,\ldots,N\} \in X$. Thus for any $\alpha > 0$,$\{i: \alpha < |x_i - r|\} \in X$. So that $x \in V_X$. Therefore $\omega_X \subset V_X$. Since V_X is closed $\bar{\omega}_X \subset V_X$.

Suppose that $x \in V_X$ and $T(x) = r$. For each n , let $\{i: \frac{1}{n} < |x_i - r|\} = A_n$. Then $A_n \in X$. Let us define $x^n \in \omega$ by

$$x_i^n = \begin{cases} r & \text{if } i \in I - A_n \\ x_i & \text{if } i \in A_n \text{ .} \end{cases}$$

Obviously, $x^n \xrightarrow[(A_n)]{} r$ and $A_n \in X$, thus $x^n \in \omega_X$. Since

$$|x_i^n - x_i| = \begin{cases} |r - x_i| & \text{if } i \in I - A_n \\ 0 & \text{if } i \in A_n \text{ ,} \end{cases}$$

We get $\|x^n - x\|_\infty \le \frac{1}{n}$. So that it follows that $x \in \bar{\omega}_X$. Hence $\bar{\omega}_X = V_X$.

Proposition 3.33. For any zeroclass $\mathcal{X}$, $\overline{\omega^0_{\mathcal{X}}} = v^0_{\mathcal{X}}$ where

$$\omega^0_{\mathcal{X}} = \{x \in \omega : \exists A \in \mathcal{X} \quad x \xrightarrow[(A)]{} 0\} .$$

Proof: In the proof of the previous proposition, we change $\omega_{\mathcal{X}}$ to $\omega^0_{\mathcal{X}}$, $v_{\mathcal{X}}$ to $v^0_{\mathcal{X}}$ and r to 0. Then we get the proof.

Proposition 3.34. ([3], Proposition 4.10). If $\mathcal{X}$ and S are related zeroclass and RSM then

(1) $\omega^0_{\mathcal{X}} \cap m \subset |c_s|^0 \subset v^0_{\mathcal{X}}$.

(2) $\omega_{\mathcal{X}} \cap m \subset |c_s| \subset v_{\mathcal{X}}$,

(3) S and $T_{\mathcal{X}}$ have some value on $|c_s|$, that is,

$$S|_{|c_s|} = T_{\mathcal{X}}|_{|c_s|} .$$

Proof: (1) Let $x \in \omega^0_{\mathcal{X}} \cap m$. Then there exist a set $A \in 2^I$ such that $A \in \mathcal{X}$ and $x \xrightarrow[(A)]{} 0$. Since $A \in \mathcal{X}$, we have $\chi_A \in |c_s|^0$. Writing $x = x\chi_A + x\chi_{(I-A)}$ and noting that $x \in m$, we have, by the definition of RSM, $x\chi_A \in |c_s|^0$. Further $x\chi_{I-A} \in c_0 \subset |c_s|^0$ by Proposition 3.6, it follows that $x \in |c_s|^0$.

Next we consider any $x \in |c_s|^0$. Then for any $\alpha > 0$ $\alpha\chi\{i: \alpha < |x_i|\} \le |x| \in |c_s|^0$. By Proposition 3.4, $\alpha\chi\{i: \alpha < |x_i|\} \in |c_s|^0$. Thus $\chi\{i: \alpha < |x_i|\} \in |c_s|^0$, equivalently $\{i: \alpha < |x_i|\} \in \mathcal{X}$. Therefore $x \in v^0_{\mathcal{X}}$.

(2) Suppose that $x \in \omega_X \cap m$, then there exist $r \in R$ and $A \in X$ such that $x \underset{(A)}{\longrightarrow} r$. Since $x-r \in \omega_X^0 \cap m$ and by (1), $x-r \in |C_s|^0$. Thus we get $x \in |C_s|$. Hence $\omega_X \cap m \subset |C_s|$.

Suppose that $x \in |C_s|$ and $r \in R$ such that $x-r \in |C_s|^0$. Then, by (1), $x-r \in V_X^0$. Therefore $x \in V_X$. Thus $|C_s| \subset V_X$.

(3) Suppose that $x \in |C_s|$ and $r \in R$ with $x-r \in |C_s|^0$. By Proposition 3.5 (1), $x-r \in C_s^0$, so that $S(x-r) = 0$ and $S(x) = r$.

On the other hand $x-r \in |C_s|^0 \subset V_X^0$ by (1). Therefore $T_X(x) = r$. Hence, we get $S(x) = r = T_X(x)$.

<u>Proposition 3.35.</u> If X_1 and X_2 are zeroclasses with $X_1 \subset X_2$. Then we have

(1) $V_{X_1}^0 \subset V_{X_2}^0$,

(2) $V_{X_1} \subset V_{X_2}$,

(3) $T_{X_2}\big|_{V_{X_1}} = T_{X_1}$.

<u>Proof:</u> (1) Suppose that $x \in V_{X_1}^0$. Then for any $\alpha > 0$, $\{i: \alpha < |x_i|\} \in X_1 \subset X_2$. Therefore for any $\alpha > 0$, $\{i: \alpha < |x_i|\} \in X_2$, equivalently $x \in V_{X_2}^0$. Hence $V_{X_1}^0 \subset V_{X_2}^0$.

(2) and (3). For any $x \in V_{X_1}$, let $T_{X_1}(x) = r$. Then we have

$x-r \in V^0_{X_1} \subset V^0_{X_2}$. Thus $x-r \in V^0_{X_2}$ and so $T_{X_2}(x) = r = T_{X_1}(x)$ and $x \in V_{X_2}$.

Proposition 3.36. (The bounded consistency theorem on strong convergent fields). Let $S_1: C_{s_1} \to R$ be an RSM related with a zeroclass X_1 and $S_2: C_{s_2} \to R$ be an RSM related with a zeroclass X_2 . Suppose that $X_1 \subset X_2$ and $C_{s_1} \cap m \subset C_{s_2}$. Then:

(1) $|C_{s_1}|^0 \cap m \subset |C_{s_2}|^0 \cap m$,

(2) $|C_{s_1}| \cap m \subset |C_{s_2}| \cap m$,

(3) $S_1\Big|_{(|C_{s_1}| \cap m)} = S_2\Big|_{(|C_{s_1}| \cap m)}$.

Proof: If $x \in |C_{s_1}|^0 \cap m$. Then $|x| \in C_{s_1} \cap m$, $S_1(|x|) = 0$, $|x| \in V^0_{X_1}$ (by Proposition 3.34) and $T_{X_1}(|x|) = 0$. Since by hypothesis, $C_{s_1} \cap m \subset C_{s_2}$, we also have $|x| \in C_{s_2}$ so that $S_2(|x|)$ is defined. By Propositions 3.34 and 3.35, $\omega_{X_2} \cap m \subset |C_{s_2}| \cap m \subset \bar{\omega}_{X_2} \cap m$ and $|x| \in V^0_{X_2} \cap m \subset \bar{\omega}_{X_2} \cap m$. Thus we can find a sequence $\{x^n\}$ in $|C_{s_2}| \cap m$ such that $x^n \to |x|$ in the sense of T_∞ . Since S_2 is a RSM and by Proposition 3.9, S_2 is continuous. Thus $S_2(x^n) \to S_2(|x|)$. Since we know that $x^n \in V_{X_2}$, by Proposition 3.34, $T_{X_2}(x^n) = S_2(x^n)$. On the other hand $|x| \in V^0_{X_1} \subset V^0_{X_2}$ and so by Proposition 3.35, $0 = T_{X_1}(|x|) = T_{X_2}(|x|)$. Hence we have

$$0 = T_{X_2}(|x|) = \lim_n T_{X_2}(x^n) = \lim_n S_2(x^n) = S_2(|x|).$$

So that $x \in |C_{s_2}|^0$. Consequently, we have

$$|C_{s_1}|^0 \cap m \subset |C_{s_2}|^0 \cap m.$$

(2) If $x \in |C_{s_1}| \cap m$, then there exists $r \in R$ with $x-r \in |C_{s_1}|^0 \cap m$. By (1) $x-r \in |C_{s_2}|^0 \cap m$. Thus $x \in |C_{s_2}| \cap m$.

(3) Let $x \in |C_{s_1}| \cap m$ and $r \in R$ with $x-r \in |C_{s_1}| \cap m$. Then by Proposition 3.34, $x-r \in V^0_{X_1}$ and $T_{X_1}(x) = r$. Again by Proposition 3.35, $x-r \in V^0_{X_2}$ and $T_{X_2}(x) = r$. By Proposition 3.34, $S_1(x) = T_{X_1}(x) = r = T_{X_2}(x) = S_2(x)$.

Hence we have $S_1\big|_{|C_{s_1}| \cap m} = S_2\big|_{|C_{s_1}| \cap m}$.

<u>Corollary 3.37.</u> Let $S_1: C_{s_1} \to R$ and $S_2: C_{s_2} \to R$ be RSMs defined on the same domain $C_s = C_{s_1} = C_{s_2}$ and related with the same zeroclass X. Then $|C_{s_1}| \cap m = |C_{s_2}| \cap m$ and

$$S_1\big|_{(|C_{s_1}| \cap m)} = S_2\big|_{(|C_{s_2}| \cap m)}.$$

<u>Proof</u>: By the previous proposition $|C_{s_1}| \cap m \subset |C_{s_2}| \cap m$ and $|C_{s_1}| \cap m \supset |C_{s_2}| \cap m$. Thus $|C_{s_1}| \cap m = |C_{s_2}| \cap m$ and $S_1\big|_{(|C_{s_1}| \cap m)} = S_2\big|_{(|C_{s_2}| \cap m)}$.

Remark: Let $\mathcal{F}$ be the collection of all RSMs which are related with zeroclass $\mathcal{X}$. Then $T_{\mathcal{X}}$ is a member of $\mathcal{F}$ and for any RSM $S: C_s \to R$ in $\mathcal{F}$, S is $T_{\mathcal{X}}$ on the bounded strong convergence field associated with S.

Next, we will study RSMs induced from matrices. Also, we will show that in the matrix case Proposition 3.36 is a special case of the standard Bounded Consistency Theorem on regular matrices. We will also show that Proposition 3.36 is not subsummed under the standard Bounded Consistency Theorem on regular matrices.

<u>Definition 3.38.</u> Let A be a regular matrix. Let $f_A: C_A \to R$ be a function defined by $f_A(x) = \lim_i (Ax)_i$ for any $x \in C_A$. Note that C_A is a linear space of sequences and f_A is a linear functional.

<u>Definition 3.39.</u> A matrix A is called an RSM if $f_A: C_A \to R$ is an RSM.

<u>Proposition 3.40.</u> Suppose that A is nonnegative (i.e., $A_{nk} \geq 0$ for all $n,k = 1,2,3,\ldots$) regular matrix. Then $f_A: C_A \to R$ is an RSM.

<u>Proof</u>: Let us write $C_{f_A} = C_A$, $|C_{f_A}|^0 = |C_A|^0$ and $|C_{f_A}| = |C_A|$. (see definition 3.1.). Clearly f_A is regular. Now suppose that $x \in |C_A|^0$ and $|y| \leq |x|$. Since A is nonnegative

$$0 \leq (A|y|)_n = \sum_{k=1}^{\infty} a_{nk}|y_k| \leq \sum_{k=1}^{\infty} a_{nk}|x_k| = (A|x|)_n.$$

Thus

$$0 \le \overline{\lim_n} \, (A|y|)_n \le \lim_n \, (A|x|)_n = f_A(|x|) = 0$$

Hence

$$\lim_n \, (A|y|)_n = 0 \quad \text{and} \quad |y| \in |c_A|^0 .$$

By Proposition 3.4, f_A is an RSM.

Let us state the Bounded Consistency Theorem (BCT) without proof. [7].

<u>Proposition 3.41</u>. If A and B are regular matrices such that $c_A \cap m \subset c_B$ then $f_A(x) = f_B(x)$ for all $x \in c_A \cap m$.

Remark: The BCT is very general and so this theorem implies Proposition 3.36 when the RSMs S_1 and S_2 are induced by matrices.

<u>Corollary 3.42</u>. Suppose that A and B are RSM matrices and f_A and f_B are corresponding RSMs. If $c_A \cap m \subset c_B$ then

(1) $|c_A|^0 \cap m \subset |c_B|^0 \cap m$,

(2) $|c_A| \cap m \subset |c_B| \cap m$,

(3) $f_A\big|_{(|c_A| \cap m)} = f_B\big|_{(|c_B| \cap m)}$.

<u>Proof</u>: By the ordinary BCT, $f_A\big|_{(c_A \cap m)} = f_B\big|_{(c_A \cap m)}$. Hence (3) is true. Let $x \in |c_A|^0 \cap m$, then $|x| \in c_A \cap m$ and

$f_A(|x|) = 0$. By the hypothesis $C_A \cap m \subsetneq C_B$ and by the BCT, $|x| \in C_B \cap m$ and $f_B(|x|) = 0$. Therefore $x \in |C_B|^0 \cap m$. Hence (1) is true.

(2) Follows from (1) as before.

Next we want to show that our Proposition 3.36 is meaningful. In other words Proposition 3.36 is not deduced from the BCT for the non-matrix case. For this purpose, we will construct an RSM $S: C_s \to R$ such that for any RSM matrix A, $|C_s| \cap m \supsetneq |C_A| \cap m$.

At first let us introduce a definition and some propositions.

Definition 3.43. An ultra zeroclass on I is a zeroclass $\mathcal{X}$ such that there is no zeroclass on I which is strictly finer than $\mathcal{X}$. (In other words, a maximal element in the ordered family of all zero-classes on I). This is the same as maximal ideal in 2^I [1].

Proposition 3.44. Let $\mathcal{X}$ be an ultra zeroclass on I. Then for any $A \in 2^I$, $A \in \mathcal{X}$ or $I-A \in \mathcal{X}$.

Proof: Let $\mathcal{F} = \{A \in 2^I;\ I-A \in \mathcal{X}\}$. Then $\mathcal{F}$ is an ultrafilter. Thus for any $A \in 2^I$, $A \in \mathcal{F}$ or $I-A \in \mathcal{F}$ (see [1], Chapter I, §6.4, Proposition 5). Hence for any $A \in 2^I$, $A \in \mathcal{X}$ or $I-A \in \mathcal{X}$.

Proposition 3.45. $\mathcal{X}$ is an ultra zeroclass if and only if $m \subset V_{\mathcal{X}}$.

Proof: Suppose that $\mathcal{X}$ is an ultra zeroclass. Then for any $A \in 2^I$, $A \in \mathcal{X}$ or $I-A \in \mathcal{X}$, if and only if $\chi_A \in V_{\mathcal{X}}$ or $\chi_{(I-A)} \in V_{\mathcal{X}}$ if and only if $\chi_A \in V_{\mathcal{X}}$ or $-\chi_A + 1 \in V_{\mathcal{X}}$. Thus, since $V_{\mathcal{X}}$ is a linear space which contains c , $-\chi_A + 1 \in V_X$ is equivalent to $\chi_A \in V_X$. Therefore, for any $A \in 2^I$, $\chi_A \in V_X$. Thus $\{\chi_A : A \in 2^I\} \subset V_X$. Since V_X is a linear space of sequence, $m_0 \subset V_X$. Since V_X is closed in (ω, T_∞), $\bar{m}_0 = m \subset V_X$ (see [10], p. 24, 15 Example).

Suppose that X is not an ultra zeroclass, then there exists an $A \in 2^I$ such that $A \notin X$ and $I-A \notin X$.

Suppose that $\chi_A \in V_X$. Then there is an $r \in R$ such that $\{i: \alpha < |\chi_A(i) - r|\} \in X$ for any $\alpha > 0$.

If $r = 1$, then $\{i: \frac{1}{2} < |\chi_A(i) - 1|\} = I-A \notin X$.

If $r = 0$, then $\{i: \frac{1}{2} < |\chi_A(i) - 0|\} = A \notin X$.

If $r \neq 0$ and $r \neq 1$, take $0 < \alpha_0 < \min \{|r|, \{r-1|\}$.

Then $\{i: \alpha_0 < |\chi_A(i) - r|\} = I \notin X$. And so we have a contradiction.

Hence $\chi_A \notin V_X$ since $\chi_A \in m$ we have $m \not\subset V_X$.

<u>Proposition 3.46</u>. If X is an ultra zeroclass then there does not exist an RSM matrix A such that $|V_X| \cap m = C_A \cap m$.

<u>Proof</u>: Since X is an ultra zeroclass $m \subset V_X$, and so $|V_X| \cap m = V_X \cap m = m$. On the other hand, for any regular matrix A , $C_A \cap m \subsetneq m$ (see [6], p. 187 Theorem 14 (The Steinhaus Theorem)).

Therefore for any RSM matrix A , $|C_A| \cap m \subset C_A \cap m \subsetneq m = |V_X| \cap m$.

Corollary 3.47. Let X be an ultra zeroclass and let $c_s = v_X$ and let $S: c_s \to R$ be T_X. Then for any RSM matrix A,

$$|c_A| \cap m \subsetneq |c_s| \cap m .$$

Proof: By the previous proposition

$$|c_s| \cap m = |v_X| \cap m = m \supsetneq c_A \cap m \supset |c_A| \cap m .$$

Hence our BCT (Proposition 3.36) can be applied in cases unapproached by matrix methods.

In the matrix cases, there are many interesting examples which are not RSMs. For the rest of this chapter we will study regular matrices vis-a-vis RSMs. At first let us define a regular matrix which is not an RSM.

Example 3.48. Let A be given by

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{16} & \cdots \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} & \cdots \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \cdots \end{bmatrix} .$$

By the Silverman-Toeplitz Theorem, we can easily see that A is regular. Let $x = (1,2,1,2,1,2,\ldots)$. Then $Ax = 0$. Hence $x \in |c_A|^0$. But if we take $y = (1,1,1,2,1,1,1,2,\ldots)$ then $Ay = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \ldots)$ and

$y \notin c_A$. Thus $m|c_A|^0 \notin c_A$.

Also, if we take $x = (0,1,0,1,0,1,\ldots)$ then $Ax = (-1,-1,-1,\ldots)$. Therefore f_A is not nonnegative. This example is generalized in Proposition 3.50. The previous example is also interesting in view of the following.

<u>Proposition 3.49.</u> If A is a regular matrix then $f_A\colon (c_A, T_\infty) \to R$ is continuous.

We omit the proof.

If A is a nonnegative regular matrix then A is an RSM. Thus being a nonnegative regular matrix is a sufficient condition of being an RSM. But it is hard to find nice necessary conditions for a matrix to be an RSM. The following proposition is an attempt to find necessary conditions for being an RSM.

<u>Proposition 3.50.</u> Let A be a regular matrix. Suppose that for each column of A, all members of that column are either nonnegative or nonpositive and $\lim_n \sum_{k=1}^{\infty} a_{nk}^- = r$ and $r > 0$ where $a^+ = \max(a,0)$ and $a^- = \max(-a,0)$. Then f_A cannot be nonnegative.

<u>Proof</u>: Since A is regular and $\sum_{k=1}^{\infty} |a_{nk}| = M < \infty$, the series $\sum_{k=1}^{\infty} a_{nk}^-$ converges for each $n \geq 1$.

Now let $x \in \omega$ such that

$$x_k = \begin{cases} 0 & \text{if } k\text{-th column of } A \text{ is nonnegative or all zero} \\ 1 & \text{if } k\text{-th column of } A \text{ is nonpositive.} \end{cases}$$

Then

$$f_A(x) = \lim_n (Ax)_n$$

$$= \lim_n \left(\sum_{k=1}^{\infty} a_{nk}^{+} x_k - \sum_{k=1}^{\infty} a_{nk}^{-} x_k \right)$$

$$= - \lim_n \sum_{k=1}^{\infty} a_{nk}^{-}$$

$$= -r < 0 \text{ .}$$

Therefore f_A is not nonnegative. Hence f_A cannot be an RSM.

Remark: Thus an RSM matrix cannot have the property stated in Proposition 3.50.

Example 3.51. Let a matrix A be given by:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & -1 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & -1 & 0 & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & -1 & 0 & 0 & \dots \end{bmatrix} .$$

Then A is an RSM.

Proof: We can easily see that A is regular. Suppose that $x \in |c_A|^0$ and so

$$\frac{|x_1| + |x_2| + \ldots + |x_n|}{n} \to 0$$

and

$$2|x_n| - |x_{n+1}| \to 0 .$$

Then we show $|x_n| \to 0$. Assume that $|x_n| \not\to 0$ so that there exists $\varepsilon > 0$ such that for infinitely many n, $|x_n| > \varepsilon$. Since $2|x_n| - |x_{n+1}| \to 0$, there exists N such that $n > N$ implies $2|x_n| - |x_{n+1}| < \frac{\varepsilon}{2}$. Take n_0 such that $|x_{n_0}| > \varepsilon$ and $n_0 > N$. Then we have $2\varepsilon < 2|x_{n_0}| < \frac{\varepsilon}{2} + |x_{n_0+1}|$. Therefore $\frac{3\varepsilon}{2} < |x_{n_0+1}|$. If $k > n_0$ and $|x_k| > \varepsilon$ then $2\varepsilon < 2|x_k| < \frac{\varepsilon}{2} + |x_{k+1}|$. Thus $\frac{3\varepsilon}{2} < |x_{k+1}|$. Therefore by induction for any $n \geq n_0$, $\varepsilon \leq |x_n|$. Thus we have

$$\liminf_{n\to\infty} \frac{|x_1| + |x_2| + \ldots + |x_n|}{n} \geq \varepsilon ,$$

which is a contradiction.

Hence $\lim_n |x_n| = 0$ and $|c_s|^0 = c_0$. Since $x \in c_0$ and $|y| \leq |x|$ implies $\lim_n y_n = 0$ we have that A is an RSM.

Example 3.52. Let A be given by

$$a_{ij} = \begin{cases} \frac{1}{i} & \text{if } j \text{ is odd and } 1 \le j \le 2i+1 \\ 0 & \text{if } j \text{ is even and } 2 \le j \le 2i \\ -\frac{1}{i} & \text{if } j = 2i+2 \\ 0 & \text{if } j > 2i+2 \end{cases}$$

that is

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & \cdots \end{bmatrix}$$

By the Silverman-Toepliz Theorem, we can easily see that A is regular.

Let $x \in \omega$ be given by

$$x_i = \begin{cases} j & \text{if } i = 2j + 2 \text{ and } j \in I \\ 0 & \text{otherwise.} \end{cases}$$

that is, $x = (0,0,0,1,0,2,0,3,0,4,0,\ldots)$. Then $(Ax)_n = -1$ for all n. Hence $\lim_n (Ax)_n = -1$ and $x \in C_A$ and $f_A(x) = -1$. Since $x \ge 0$ and $f_A(x) = -1$, f_A is not nonnegative so that f_A is not an RSM.

Next let us take $x \in \omega$ by

$$x_i = \begin{cases} 1 & \text{if } i \text{ is odd} \\ j & \text{if } i = 2j+2 \text{ and } j \in I \cup \{0\}, \end{cases}$$

that is, $x = (1,1,1,2,1,3,1,4,1,5,1,6,\ldots)$. Take $y \in \omega$ by

$$y_i = \begin{cases} 2j+1 & \text{if } i = 4j+2 \text{ and } j \in I \\ 1 & \text{otherwise,} \end{cases}$$

that is $y = (1,1,1,1,1,3,1,1,1,5,1,1,1,7,\ldots)$. We know that $(Ax)_n = 0$ for each $n \in I$ so that $x \in |C_A|^0$. Since

$$(Ay)_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}, \end{cases}$$

it follows that $y \notin C_A$. Obviously $|y| \leq |x|$. Therefore $m|C_A|^0 \not\subset C_A$.

The previous example is important in the sense that there exists a regular matrix A which is essentially nonnegative and still not an RSM. Essentially, nonnegative matrices were studied by Sonnenschein [9] because of the following proposition.

<u>Proposition 3.53</u>. Suppose that A is a regular and essentially nonnegative matrix. Let $d_A(S) = \liminf_n (A\chi_S)_n = \liminf_n \sum_{k=1}^{\infty} a_{nk}\chi_S(k)$ for any $S \in 2^I$. Then d_A is an asymptotic density.

Proof: (See [9], Page 26, Theorem 3.3).

Definition 3.54. A matrix A is said to be essentially non-negative if

$$\lim_n \sum_k a^-_{nk} = 0 .$$

While we have shown in example 3.52 there exists an essentially non-negative regular matrix A such that $f_A: C_A \to R$ is not an RSM, if A is an essentially nonnegative regular matrix, then the restriction of f_A to $C_A \cap m$ is an RSM. To prove that, at first, let us prove two lemmas.

Definition 3.55. Let A be a matrix, then we denote

$$A^+ = (a^+_{ni}) \quad \text{and} \quad A^- = (a^-_{ni}) .$$

Lemma 3.56. Let A be an essentially nonnegative regular matrix. Then we have

(1) $C_A \cap m = C_{A^+} \cap m$,

(2) $f_A\big|_{(C_A \cap m)} = f_{A^+}\big|_{(C_A \cap m)}$.

Proof: (1) For any $x \in m$, $x \in C_A$ if and only if $Ax \in c$. Since A is a regular matrix, if $Ax \in c$ then A^+x and A^-x exist and further $Ax = A^+x - A^-x$. Since A is essentially nonnegative and $x \in m$, we have

$$|(A^{-}x)_i| = |\sum_{k=1}^{\infty} a^{-}_{ik}x_k| \le \|x\|_\infty \sum_{k=1}^{\infty} a_{ik} \to 0 .$$

Thus we get $A^{-}x \in c_0$, and so $Ax \in c$ if and only if $A^{+}x \in c$. Hence $c_A \cap m = c_{A^+} \cap m$.

(2) For any $x \in c_A \cap m$, $f_A(x) = \lim_i (Ax)_i = \lim_i ((A^{+}x)_i - (A^{-}x)_i) = \lim_i (A^{+}x)_i - \lim_i (A^{-}x)_i = \lim_i (A^{+}x)_i = f_{A^+}(x)$. Therefore

$$f_A\Big|_{(c_A \cap m)} = f_{A^+}\Big|_{(c_A \cap m)}.$$

<u>Lemma 3.57.</u> Let $S: c_s \to R$ be an RSM and let $T = S|_{(c_s \cap m)}: c_s \cap m \to R$ and $c_T = c_s \cap m$. Then

(1) $|c_T|^0 = |c_s|^0 \cap m$

(2) $T: c_T \to R$ is also an RSM.

<u>Proof:</u> (1) For any $x \in \omega$.

$$\begin{aligned} x \in |c_T|^0 &\Leftrightarrow |x| \in c_T^0 \\ &\Leftrightarrow |x| \in c_T \text{ and } T(|x|) = 0 \\ &\Leftrightarrow |x| \in c_s \cap m \text{ and } S(|x|) = 0 \\ &\Leftrightarrow x \in |c_s|^0 \text{ and } x \in m \\ &\Leftrightarrow x \in |c_s|^0 \cap m . \end{aligned}$$

Thus $|c_T|^0 = |c_s|^0 \cap m$.

(2) Let $x \in |C_T|^0$ and $|y| \le |x|$. Then $x \in |C_S|^0 \cap m$ and $|y| \le |x|$. Since S is an RSM, $|y| \in |C_S|^0$. Since $x \in m$ and $|y| \le |x|$, it follows that $y \in m$. Thus, by (1) $y \in |C_S|^0 \cap m = |C_T|^0$. Therefore T is an RSM.

Proposition 3.58. Let A be an essentially nonnegative regular matrix. Let $S = f_A\big|_{(C_A \cap m)} : (C_A \cap m) \to R$ be the restriction of f_A to $C_A \cap m$. Then S is an RSM.

Proof: Since A^+ is a nonnegative regular matrix $f_{A^+}: C_{A^+} \to R$ is an RSM. By the previous lemma $f_{A^+}\big|_{(C_{A^+} \cap m)} : (C_{A^+} \cap m) \to R$ is also an RSM. By Lemma 3.57, $f_{A^+}\big|_{(C_{A^+} \cap m)} = f_A\big|_{(C_A \cap m)}$. Thus S is an RSM.

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APPENDIX

List of Notations

R is the set of real numbers.

I is the set of positive integers.

2^X is the power set of a given set X.

If $a,b \in R$, then $\max(a,b)$ (resp. $\min(a,b)$) is the maximum (resp. minimum) of the set $\{a,b\}$.

If $S \subset R$, $\sup S$ (resp. $\inf S$) is the supremum (resp. infimum) of S.

If $a \in R$, $a^+ = \max(a,0)$, $a^- = \max(-a,0)$.

ω is the set of all real sequences.

If $x \in \omega$, then (x_i) or $(x_1, x_2, \ldots)$ denote x.

$\lim_i x_i$ or $\lim x$ denote the limit of a real sequence x.

$m = \{x \in \omega: \sup_k |x_k| < \infty\}$.

$c = \{x \in \omega: \lim x \text{ exists}\}$.

$c_0 = \{x \in \omega: \lim x = 0\}$.

e, e^n are the sequences given by $e_k = 1$ for all k and $e^n_k = 0$ for $k \neq n$, $e^n_n = 1$.

If $A \in 2^I$, χ_A is the characteristic sequence of A, that is $(\chi_A)_n = 1$ if $n \in A$, $(\chi_A)_n = 0$ if $n \notin A$.

m_0 is the linear span of $\{\chi_A: A \in 2^I\}$.

$bs = \{x \in \omega: \sup_n |\sum_{k=1}^{n} x_k| < \infty\}$.

$xy = (x_i y_i)$ is the co-ordinate wise product of two sequences x and y.

For $x \in \omega$, we let $|x| = (|x_i|)$.

For $x \in \omega$ and $r \in R$, we write $x+r = (x_i+r)$ and $rx = (rx_i)$.

For $A,B \subset \omega$, we write

$$AB = \{xy \mid x \in A \text{ and } y \in B\},$$

$$A+B = \{x+y \mid x \in A \text{ and } y \in B\}.$$

For $x,y \in \omega$, $x \le y$ means $x_i \le y_i$ for each i.

$M = (a_{nk})$ denotes infinite matrices.

If $M = (a_{nk})$ is an infinite matrix and (x_i) is any sequence, the product Mx denotes the sequence (y_i), if it exists, where $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$. We also define $c_M = \{x \in \omega: Mx \in c\}$. In Chapter 3, we write C_M for c_M.

$\bar{0} \in \omega$ denotes zero sequence $(0,0,0,\ldots)$.

Zero matrice $A = (a_{ni})$, where $a_{ni} = 0$ for all i, j is denoted by 0.

$f: S \to T$ denotes a function from a set S into a set T.

If $f: S \to T$ is a function and $U \subset S$ then $f|_U: U \to T$ is the restriction of f into U.

$a,b \in I$, $\quad [a,b] = \{x \in I: a \le x \le b\}$,

$\quad (a,b) = \{x \in I: a < x < b\}$,

$|A|$ is the cardinal number of a given set A .

For two sets A and B , we let $A \Delta B = (A-B) \cup (B-A)$ be the symmetric difference of A and B .

For two sets A and B , $A \sim B$ means $A \Delta B$ is finite.

For $A \in 2^I$, we let $A^c = I-A$ be the complement of A .

$F_0 = \{A \in 2^I : A^c \text{ is finite}\}$ is the Fréchet filter.

$J_N = \{1,2,3,\ldots,N\}$ when $N \in I$.

$\chi_0 = \{A \in 2^I : A \text{ is finite}\}$.