ON PATH DECOMPOSITIONS OF COMPLETE GRAPHS

by

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On Path Decompositions of Complete Graphs

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ABSTRACT

Given an undirected (directed) graph, we say G has a path decomposition if the edge-set (arc-set) E of G can be partitioned, into disjoint subsets $E_1, \ldots, E_r$ such that each of the subgraphs induced by E_i is a path.

In this thesis, we will look at two path decomposition problems on complete graphs.

Problem I: Path number problem

Given any complete directed graph G, what is the minimum number of paths in any path decomposition of G?

In other words, we are interested in the minimum value of r as described above; this value is called the *path number* of G. An expository account of results on path numbers of tournaments is given. In addition, a new result is given in which the path number of a Walecki tournament is determined.

Problem II: Path arboreal problem

Given any complete undirected graph K_n, and a sequence of natural numbers $(m_1, \ldots, m_r)$ such that $m_i \le n-1$ for $i=1, \ldots, r$ and $\sum_{i=1}^{r} m_i = \binom{n}{2}$. Is there a path decomposition of K_n, such that E_i contains exactly m_i edges for $i=1, \ldots, r$?

If the answer to this problem is yes for any sequence $(m_1, \ldots, m_r)$ such that $m_i \le n-1$ for $i=1, \ldots, r$ and $\sum_{i=1}^{r} m_i = \binom{n}{2}$,

then we call G *path arboreal*. An exposition of the present status of the attempt to prove that K_n is path arboreal is given as well as some original work that extends the results in the literature.

DEDICATION

To my parents.

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INTRODUCTION

The topic of graph decompositions is probably one of the most widely considered topics in the field of graph theory. Hundreds and hundreds of papers and books have been published on different aspects of this problem. Chung and Graham [7] give a survey on many decomposition problems — what has been done and what has yet to be done, together with a list of more than one hundred references. They also formulated the general graph decomposition problems as follows:

> Given a graph G and a family of graphs H, we say G has an H-decomposition if the edge-set E of G can be partitioned into disjoint subsets $E_1, E_2, \ldots, E_r$ such that each of the subgraphs induced by E_i is isomorphic to a member of H.

By allowing G and H to be directed, we get the analogue for directed graphs. The more commonly known problems involve using families of complete graphs, cycles or paths for H. In this thesis, we will restrict ourselves to the latter case, that is, H consists of paths only. Furthermore, we are mostly interested in H-decompositions of complete simple graphs (directed or undirected), that is, complete graphs with neither self-adjacent vertices nor multiple edges. Henceforth the term "graph" refers to a simple graph unless otherwise specified and the term "digraph" refers to a simple directed graph. Other terminology that will be used quite frequently in this thesis includes "circuits" for cycles in digraphs and "paths" for simple paths (containing no cycles) in graphs or digraphs. The following notations and definitions will also be used here :

NOTATION 0.1 : The set of vertices is always labeled by $Z_n = \{0, 1, \ldots, n-1\}$.

NOTATION 0.2 : An arc going from vertex u to vertex v in a digraph is denoted by (u,v).

NOTATION 0.3 : An edge joining vertex u and vertex v in a graph is denoted by $\langle u,v \rangle$.

NOTATION 0.4 : The in-degree and out-degree of a vertex in a digraph is denoted by id(v) and od(v), respectively.

NOTATION 0.5 : $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

NOTATION 0.6 : $\lceil x \rceil$ denotes the smallest integer larger than or equal to x.

NOTATION 0.7 : The directed path $\{(x_0,x_1), (x_1,x_2), \ldots, (x_{n-1}, x_n)\}$ is briefly denoted by $(x_0,x_1, \ldots, x_n)$ and the undirected path $\{\langle x_0,x_1 \rangle, \langle x_1,x_2 \rangle, \ldots, \langle x_{n-1},x_n \rangle\}$ is similarly denoted by $\langle x_0,x_1, \ldots, x_n \rangle$.

DEFINITION 0.1 : The *length* of an arc (u,v) denoted by $l(u,v)$ is defined as v-u (mod n) where n is the number of vertices in the digraph.

DEFINITION 0.2 : The *length* of an edge <u,v> denoted by $l\langle u,v\rangle$ is defined as min {v-u,u-v} (mod n) where n is the number of vertices in the graph.

DEFINITION 0.3 : For any two vertices u, v in a digraph, we say u *dominates* v if (u,v) is an arc.

DEFINITION 0.4 : The degree deg(v) of v in a digraph is defined as od(v)+id(v).

DEFINITION 0.5 : For any vertex v in a digraph, we define $\mu(v)$ to be max {id(v),od(v)}.

DEFINITION 0.6 : An *asymmetric digraph* is a digraph such that (v,w) is an arc implies (w,v) is not an arc.

DEFINITION 0.7 : A *tournament* of order n, denoted by T_n, is a complete asymmetric simple digraph on n vertices.

DEFINITION 0.8 : A *regular tournament*, denoted by RT_n, is a tournament such that for any vertex v in RT_n, od(v)=id(v).

DEFINITION 0.9 : A *near-regular tournament*, denoted by NT_n, is a tournament such that for any vertex v in NT_n, |od(v)-id(v)|=1.

It can be easily seen from the degrees that regular tournaments must have odd order and near-regular tournaments must have even order.

Now, let us consider the directed analogue of the general problem with $\mathcal{H}$ being the family of all directed paths and G a digraph. Clearly, an $\mathcal{H}$-decomposition exists since each arc by itself is a directed path. What we want to obtain here is the minimum size of such partitions, where the size of an $\mathcal{H}$-decomposition is defined as the number of subsets $E_1, E_2, \ldots, E_r$ (as defined in the general decomposition problem) that the arc-set of G is decomposed into. In other words, it is the number r. This number is called the *path number* of G and that is why we call this problem the "path number problem". The idea of finding minimum path decompositions of simple graphs is due to Erdös [9] and The term "path number" was first introduced by Harary [11].

DEFINITION 0.10 : The *path number* of a graph G, denoted by pn(G), is the minimum number of edge-disjoint paths in G whose union is G.

Replacing the words graph by digraph and edge by arc, we get the definition for path number of digraphs.

DEFINITION 0.10': The *path number* of a digraph G, denoted by pn(G), is the minimum number of arc-disjoint paths in G whose union is G.

The problem of determining the path numbers of digraphs was first attempted by Alspach and Pullman [2] in 1974. They established lower and upper bounds for path numbers of asymmetric digraphs and conjectured that the same upper bound holds for all digraphs. This conjecture was solved by O'Brien [16] in 1975. Later in 1976 [3], Alspach, Pullman and Mason showed that $pn(T_n)$ satisfies the inequality

$$\lfloor (n+1)/2 \rfloor \le pn(T_n) \le \lfloor n^2/4 \rfloor$$

and they also showed which numbers in that interval are indeed path numbers of some tournaments. In Chapter 1 we will look at how the lower and upper bounds were derived, some properties for the path numbers of tournaments and compute the path number for some special tournaments. Also in Chapter 1 is a brief discussion of a conjecture which is closely related to the path number problem. It turns out that solving the path number problem for near-regular tournaments is equivalent to solving the following famous conjecture.

<u>CONJECTURE 1 (KELLY [15, p.7])</u>

The arc set of a regular tournament of odd order n can be decomposed into $(n-1)/2$ arc-disjoint Hamilton circuits.

This is the same as saying there exists a C_n-decomposition (that is $H=\{C_n\}$) for every regular tournament of odd order n. We shall see later how the two problems relate to each other, and discuss some of the results on Conjecture 1. Alspach [1] confirmed that Kelly's Conjecture holds for regular tournaments of odd order at most nine. We shall exhibit all these tournaments with corresponding decompositions.

Having seen what happened in the directed case, we turn our attention to the undirected case. The path number problem for undirected graphs was first examined by Lovász [13]. He showed that for any simple graph G with u odd vertices and $g \geq 1$ even vertices, $pn(G) \leq u/2 + g - 1$. This bound was later improved by Donald [8] to $pn(G) \leq \lfloor 3n/4 \rfloor$. As for complete graphs, the result is well known (see Stanton, Cowan and James [20]). For odd order complete graphs we have $pn(K_{2m+1}) = m+1$ whereas for the even case we have $pn(K_{2m}) = m$. In Chapter 2 we shall look at a slightly different version of the path decomposition problem. Consider K_n, the undirected complete graph of order n, and let $m_1, m_2, \ldots, m_r$ be positive integers such that $m_i \leq n-1$ for $i=1, \ldots, r$ and $\sum_{i=1}^{r} m_i = \binom{n}{2}$. If, given any such sequence $(m_1, m_2, \ldots, m_r)$, we can decompose K_n into edge-disjoint paths of lengths $m_1, m_2, \ldots, m_r$, we call K_n *path arboreal*. The problem we are going to investigate was first asked by Slater [19]: Is K_n path arboreal for all positive n? If not, for which n is K_n path arboreal? Tarsi [21] also asked a similar question for multigraphs. He showed that for any integer λ, n, if $m_1 = m_2 = \ldots = m_{r-1}$ and $m_r \le m_{r-1} \le n-1$ and $\sum_{i=1}^{r} m_i = \lambda \cdot \binom{n}{2}$, then λK_n, the complete graph with λ edges joining every pair of vertices, can be partitioned into paths of lengths $m_1, m_2, \ldots, m_r$. In this chapter, we shall look at some results on this problem for odd n. They will include how to partition K_n into edge-disjoint paths of lengths $m_1, m_2, \ldots, m_r$ (i) if all $m_i \le n-2$ or (ii) if $m_1 = m_2 = \ldots = m_k = n-1$ and there exists $I \subseteq \{k+1, \ldots, r\}$ such that $k \le \sum_{i \in I} m_i \le \max\{n-k-1, m+k\}$. The remaining cases (iii) when $m_i \ge n-k$ for some $i \ge k+1$ and (iv) for all even n are still open.

CHAPTER 1 : PATH NUMBERS OF TOURNAMENTS

This chapter is divided into three sections : Section I discusses the problem that is closely related to the path number problem, namely Kelly's Conjecture. We consider some of the known results on this problem and how these two problems are related. Section II is a list of all regular tournaments of orders 3, 5, 7 and 9. It is known that Kelly's Conjecture holds for all these tournaments. We will give a circuit decomposition for each one of them. Section III is a survey of results on the path numbers of tournaments.

Section I : Motivation

It is easy to see that every regular tournament of odd order can be decomposed into arc-disjoint circuits since every vertex has in-degree equal to out-degree. However, a further restriction that all circuits have to be hamiltonian, proves to be a much more difficult problem. P. Kelly conjectured this decomposition problem in the early 1960's [15, p.7] and so far little is known about this conjecture other than a few special cases. One of these special cases is the construction of a class of regular tournaments that satisfy Kelly's Conjecture. One way to achieve this is by partitioning the edges of an undirected complete graph into Hamilton cycles and then orienting each of these cycles in one of two ways. The following construction, known as "Walecki's construction", for partitioning the edges of K_n into Hamilton cycles was found by Walecki and introduced by Lucas [14] in 1891. The proof given here is due to Berge [5].

LEMMA 1.1.1 : Every complete graph K_n of order $n=2m+1$ can be decomposed into m edge-disjoint Hamilton cycles.

PROOF : Let $C_0=\langle 0,1,2,n-1,3,n-2, \ldots ,m,m+2,m+1,0\rangle$ as shown in Figure 1.1. This is clearly a Hamilton cycle. Now define

$$C_i=\langle 0,1+i,2+i,n-1+i,3+i, \ldots ,m+i,m+2+i,m+1+i,0\rangle$$

for $i=1, \ldots ,m-1$ modulo $n-1$ (notice that $n-1$ is used in place of 0 when performing modulo $n-1$ arithmetic). Then all C_i's are again Hamilton cycles, because each C_i is just a rotation of C_0 about the vertex 0. To show that they are pairwise edge-disjoint notice that every $\langle u,v\rangle$ in C_0, with $u,v\neq 0$ has $u+v\equiv 2$ or 3 (mod $n-1$), so every edge $\langle u,v\rangle$ in C_1 with $u,v\neq 0$ must have $u+v\equiv 4$ or 5 (mod $n-1$) and every edge $\langle u,v\rangle$ in C_i with $u,v\neq 0$ must have $u+v\equiv 2+2i$ or $3+2i$ (mod $n-1$). Furthermore, for edges incident with 0 we have $\langle 0,i+1\rangle$ and $\langle m+1+i,0\rangle$ in C_i where $i=0, \ldots ,m-1$. Hence every edge $\langle u,v\rangle$ lies in exactly one cycle C_i.

∎

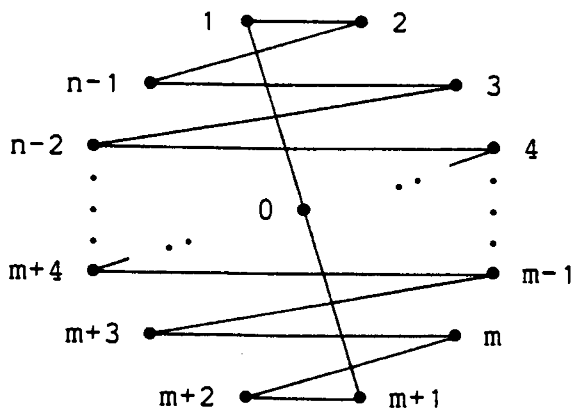


FIGURE 1.1

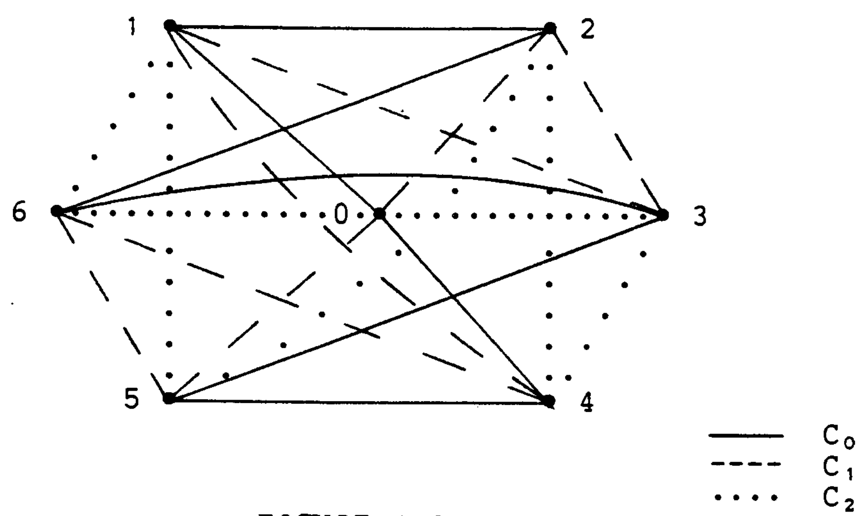


FIGURE 1.2

Figure 1.2 gives an example of Walecki's construction for n=7. With this lemma, we can now construct a special class of tournaments that obviously satisfy Kelly's conjecture by giving an orientation to each of the cycles C_i arising in the proof of Lemma 1.1.1.

DEFINITION 1.1.1 : A *Walecki tournament* of order $n=2m+1$ is a tournament whose vertices are labeled $\{0,1, \ldots ,n-1\}$, which has a symbol set $S=\{s_0,s_1, \ldots ,s_{m-1}\}$ such that $s_i \in \{-1,+1\}$, and which is obtained by orienting the complete undirected graph of order n in the following fashion. If $s_i=-1$, then orient the i^{th} Walecki cycle C_i so that C_i is a circuit and $(i+1,0)$ is an arc. If $s_i=+1$, then orient C_i so that $(0,i+1)$ is an arc. The resulting tournament is regular and is denoted by $LT_n(S)$. For $n=2m$ we construct $LT_{n+1}(S)$ first and then remove vertex 0 and all its incident arcs. This results in an arc-disjoint union of Hamilton paths which

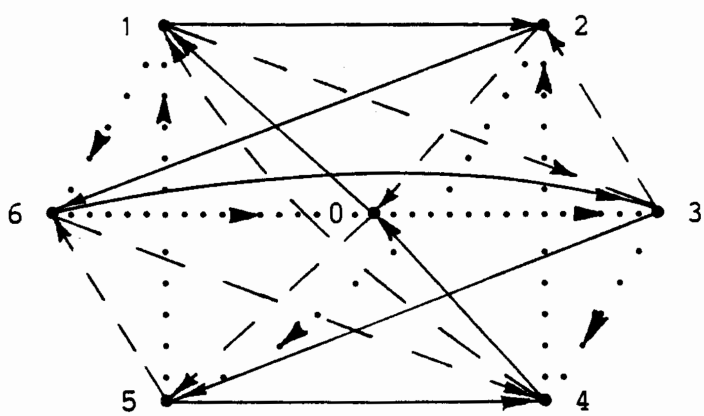


FIGURE 1.3

is a near-regular tournament of order 2m, also denoted by $LT_n(S)$.

The tournament shown in Figure 1.3 is an $LT_7(+1,-1,+1)$. A second class of tournaments that satisfy Kelly's conjecture are the circulant tournaments which can be defined as follows.

DEFINITION 1.1.2 : A *circulant tournament* is a tournament of odd order n, with vertices labeled $\{0,1, \ldots ,n-1\}$, and a symbol set $S \subset \{1,2, \ldots ,n-1\}$ such that $|S|=(n-1)/2$ and for all $i,j \in S$, we have $i+j \neq n$. Then each vertex i dominates vertex $i+j \pmod{n}$ for every $j \in S$. The resulting tournament is a regular tournament of order n and is denoted by $CT_n(S)$.

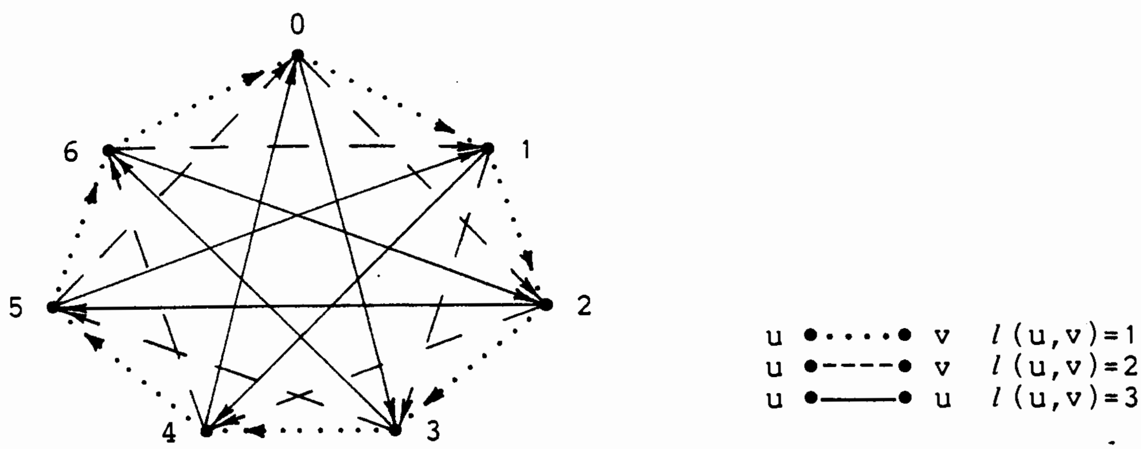


FIGURE 1.4

Figure 1.4 gives us a $CT_7(1,2,3)$. Unlike Walecki tournaments, not every circulant tournament has been proven to have the nice property that it is an arc-disjoint union of Hamilton circuits. Instead we have the following theorem.

THEOREM 1.1.2 : Every circulant tournament of prime order $p \geq 3$ can be decomposed into $(p-1)/2$ Hamilton circuits.

PROOF : Suppose $p=2m+1$ and CT_p has symbol set $S=\{s_0,s_1, \ldots ,s_{m-1}\}$ then $(i,i+j)$ is an arc if and only if $j \in S$. Let $C_i=\{(x,x+s_i) \mid x=0,1, \ldots ,p-1, s_i \in S\}$. Clearly, there are p arcs in C_i. Each vertex x in C_i has $od(x)=id(x)=1$ because $(x,x+s_i)$ and $(x-s_i,x)$ are both in C_i. Therefore, C_i is a union of vertex-disjoint circuits. Suppose we have $(x,x+s_i, \ldots ,x+ks_i)$ where $k \leq p$ and $x+ks_i \equiv x \pmod{p}$; that is, $ks_i \equiv 0 \pmod{p}$. Since $k \leq p$ and $s_i<p$, in order for $ks_i \equiv 0 \pmod{p}$ we must have $k=p$. Hence C_i is a Hamilton circuit; furthermore, all C_i's are disjoint because all s_i's are distinct. Thus $\{C_0,C_1, \ldots ,C_{m-1}\}$ forms a Hamilton circuit decomposition of $CT_p(S)$.

■

Figure 1.4 also shows a decomposition of $CT_7(1,2,3)$ into Hamilton circuits. Apart from these two results, not much is known about Kelly's Conjecture, although Häggkvist claims to have proven that for large enough n Kelly's Conjecture is true. A related theorem was obtained by Kotzig [12] in 1969.

THEOREM 1.1.3 : The arcs of every regular tournament of order n can be partitioned into $(n-1)/2$ sets of size n, each of which is a vertex-disjoint union of circuits.

This theorem can also be viewed as an immediate consequence of Hall's theorem of 1935 [10]. Consider the following. Given any regular tournament RT_n, construct a bipartite graph G with bipartition (X,Y) where $X=Y=V(RT_n)$, the vertex set of RT_n. Then for any $u \in X$, $v \in Y$, $\langle u,v \rangle$ is an edge in G if and only if (u,v) is an arc in RT_n. Thus G is an $(n-1)/2$-regular bipartite graph. Now by a corollary of Hall's theorem (see Bondy and Murty [6, p.73]), G is 1-factorable and each 1-factor represents a union of vertex-disjoint circuits in RT_n. Thus RT_n can be decomposed into $(n-1)/2$ unions of vertex-disjoint circuits, each union having n arcs.

Now let us turn our attention back to Kelly's Conjecture. In an effort to solve Kelly's Conjecture, the path number problem on regular tournaments was developed. To see the connection between the two problems, we need the following lemmas.

LEMMA 1.1.4 : If $pn(NT_n)=n/2$ for even n, then every path in the minimum partition is hamiltonian.

PROOF: Each directed path has length at most n-1 and thus the maximum number of arcs covered by n/2 paths is $(n-1)\cdot n/2 = \binom{n}{2}$. If one of the n/2 paths in a minimum path decomposition of NT_n has length less than n-1, then the total number of arcs covered is less than $\binom{n}{2}$. Therefore, the path number equal to n/2 implies that every path in the minimal path partition of NT_n is hamiltonian.

■

LEMMA 1.1.5 : Every regular tournament can be obtained by inserting a vertex into some near-regular tournament and every near-regular tournament can be obtained by removing one vertex from some regular tournament.

PROOF : By definition, a near-regular tournament is an even-order tournament NT_{2m} with $|od(v)-id(v)|=1$ for all v. Let V^+ denote the set $\{v|v\epsilon NT_{2m}$ and $od(v)-id(v)=+1\}$ and V^- denote the set $\{v|v\epsilon NT_{2m}$ and $od(v)-id(v)=-1\}$, then the union of V^+ and V^- is the vertex-set of NT_{2m} and their intersection is the empty set. Since the sum of the out-degrees equals the sum of the in-degrees in any tournament, we have $|V^+|=|V^-|=m$. Now introduce a vertex w, with an arc joining w to every $v \epsilon V^+$ and an arc joining to w from every $v \epsilon V^-$. This increases $od(v)$ by 1 for every $v \epsilon V^-$ and $id(v)$ by 1 for every $v \epsilon V^+$. Also $od(w)=id(w)=|V^+|=|V^-|=m$. Thus $NT_{2m} \cup \{w\}$ is a regular tournament, call it RT_{2m+1}. In other words, NT_{2m} can be obtained from RT_{2m+1} by deleting w.

Similarly, given any regular tournament RT_{2m+1}, we can delete any one vertex w to form $RT_{2m+1} \setminus \{w\}$. Notice that every vertex v in $RT_{2m+1} \setminus \{w\}$ has either $od(v)-id(v)=1$ or $od(v)-id(v)=-1$ because there is exactly one arc joining w to or from every v. Upon the removal of w, the degree of v drops by 1 (either in or out-degree). Thus $RT_{2m+1} \setminus \{w\}$ is a near-regular tournament NT_{2m}, which implies RT_{2m+1} can be constructed by inserting w and all its adjacent arcs into NT_{2m}.

■

Using these facts we can now show that solving Kelly's Conjecture is equivalent to solving the path number problem for near-regular tournaments of even order. We shall state it as a theorem.

THEOREM 1.1.6 : Kelly's Conjecture holds if and only if $pn(NT_n)$ equals $n/2$ for all even n.

PROOF : If $pn(NT_n)=n/2$ for all even n, then given any regular tournament RT_{2m+1}, by Lemma 1.1.5, RT_{2m+1}.. can be constructed from some NT_{2m} by inserting one vertex w and all its incident arcs. Also, by Lemma 1.1.4 the m paths that partition the arcs of NT_{2m} are hamiltonian. Now let us define V^+ and V^-, as in the proof of Lemma 1.1.5, for NT_{2m} and observe that in any path partition of NT_{2m} every vertex $v \epsilon V^+$ must be the initial vertex of at least one path and every vertex $v \epsilon V^-$ must be the terminal vertex of at least one path. In particular, if there are only m paths in the partition, then every $v \epsilon V^+$ is the initial vertex of exactly one path and every $v \epsilon V^-$ is the terminal vertex of exactly one path. To each path in the minimal path partition, we assign a pair of arcs $\{(u,w),(w,v)\}$ where v, u are the initial and terminal vertices of that path, respectively. Since all initial and terminal vertices are distinct, every arc adjacent to w belongs to exactly one pair. Now each Hamilton path in the minimal partition of NT_{2m} together with its corresponding pair of arcs forms a Hamilton circuit in RT_{2m+1}. This gives us a Hamilton circuit decomposition of RT_{2m+1}.

Conversely, if every RT_n can be decomposed into $(n-1)/2$ Hamilton circuits, then given any near-regular tournament NT_{2m}, by Lemma 1.1.5, there exists a regular tournament RT_{2m+1} such that the removal of a vertex w and all its incident arcs yields NT_{2m}. By assumption RT_{2m+1} can be decomposed into Hamilton circuits $C_0, C_1, \ldots, C_{m-1}$, and then removing w and all its incident arcs from these circuits we get m paths $C_0 \setminus \{(u_0,w),(w,v_0)\}$, $C_1 \setminus \{(u_1,w),(w,v_1)\}$, ... , $C_{m-1} \setminus \{(u_{m-1},w),(w,v_{m-1})\}$ which form a partition of the arc set of NT_{2m}. Hence $pn(NT_{2m})=m$.

∎

This marks the beginning of the path number problem for tournaments. Before we start investigating results on path numbers, let us first examine some regular tournaments of small orders. The following section is devoted to that purpose.

Section II : Some regular tournaments

It is already known that Kelly's Conjecture holds for tournaments of odd order through nine (unpublished work by B. Alspach [1]). In this section we will see how these tournaments can be decomposed into circuits and paths. But first we need a little help in identifying small order tournaments.

DEFINITION 1.2.1 : Let (u,v) be an arc of a tournament T_n. Then $\tau(u,v)$ is defined to be the number of 3-circuits of T_n containing (u,v).

DEFINITION 1.2.2 : For each regular tournament RT_n of order $n=2m+1$, we define a τ-vector $(a_1, a_2, \ldots, a_m)$ where a_i denotes the number of arcs (u,v) in RT_n with $\tau(u,v)=i$.

DEFINITION 1.2.3 : The *score vector* of a tournament T_n is the ordered n-tuple $(s_1, s_2, \ldots, s_n)$ where $s_i=od(v_i)$, and is called the *score* of vertex v_i.

First of all, every τ-vector must satisfy $\sum_{i=1}^{m} a_i = \binom{n}{2}$ since every arc must lie on at least one 3-circuit and no more than m 3-circuits. This is because for every arc (u,v) in RT_n, the out-degree of v in $RT_n \setminus \{(u,v)\}$ is m and the out-degree of u in $RT_n \setminus \{(u,v)\}$ is m-1. Furthermore, a result found by Kendall and Babington (1940), Szele (1943) and Clark (1964) (see Moon [15, p.9]) shows that there are exactly $\binom{n}{3} - \sum_{i=1}^{n} s_i(s_i-1)/2$ 3-circuits in a tournament of order n with score vector $(s_1, s_2, \ldots, s_n)$. For regular tournaments we have $s_1=s_2= \ldots =s_n =m=(n-1)/2$, and thus the total number of 3-circuits in RT_n is

$$\binom{n}{3} - \sum_{i=1}^{n} \binom{m}{2},$$

which is equal to $(n^3-n)/24$. We then have $\sum_{i=1}^{m} ia_i=(n^3-n)/8$. Since every isomorphism of a tournament preserves its circuit-structure, 3-circuits are mapped onto 3-circuits and we have the following lemma.

LEMMA 1.2.1 : Let RT'_n and RT''_n be two regular tournaments of order $n=2m+1$ with corresponding τ-vectors $(a'_1, a'_2, \ldots, a'_m)$ and $(a''_1, a''_2, \ldots, a''_m)$. If RT'_n and RT''_n are isomorphic tournaments then

$$(a'_1, a'_2, \ldots, a'_m) = (a''_1, a''_2, \ldots, a''_m).$$

This lemma gives us a quick way of identifying non-isomorphic tournaments for the lower order cases. To compute the τ-vector of a tournament T_n all we need is its tournament matrix M defined as follows.

DEFINITION 1.2.4 : The *tournament matrix* M of a tournament T_n is an n by n 0-1 matrix such that

$$(M)_{ij} = \begin{cases} 1 & \text{if } v_i \text{ dominates } v_j \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be easily seen that the (i,j)-entry of M^t (t is any positive integer, and multiplication is performed using Boolean arithmetic) is 1 if and only if there exists a directed walk of length t from v_i to v_j in T_n (see Moon [15, p.34]). We can find $\tau(v_i,v_j)$ by determining the (j,i)-entry in M^2 (by ordinary arithmetic). By collecting these numbers for all arcs, we get the τ-vector for RT_n. This would be a good tool for identification of non-isomorphic tournaments if it had not been for the fact that as n increases we have more flexibility in putting $(n^3-n)/24$ 3-circuits together and this ruins the uniqueness of the τ-vectors as we shall see in the order 9 case. There are two pairs of non-isomorphic regular tournament of order 9 having the same τ-vector. The regular tournaments in Figures 1.13 and 1.22 both have τ-vector (0,18,18,0), but the first one contains three 3-circuits composed of arcs with τ equal to 3, whereas the second one has six of this kind of 3-circuit. The regular tournaments in Figures 1.15 and 1.16 both have τ-vector (3,15,15,3), but the first one contains three 3-circuits which are composed solely of arcs with τ equal to 3, whereas the second one has five of them. Nevertheless, we will use this technique to identify tournaments of small order.

NOTATION 1.2.1 : If a regular tournament can be decomposed into arc-disjoint Hamilton circuits, then we call it HC-decomposable and the corresponding decomposition an HC-decomposition.

There is one RT_3, one RT_5, three RT_7's and fifteen RT_9's. Figures 1.5 to 1.24 form a list of all of them together with their types (if known), HC-decompositions and τ-vectors. The number on each arc (u,v) is the value of $\tau(u,v)$. A minimum path decomposition can also be found by removing the underlined arc from each Hamilton circuit of the HC-decomposition. These arcs together form a path, hence giving us a partition of the arc-set of RT_n into $(n+1)/2$ paths. We will see why this is minimum in Section III.

Before we leave this section, we shall take a brief look at another application of the τ-vector. First, we need one more definition.

DEFINITION 1.2.5 : A tournament T_n is *doubly-regular* if for all pairs of vertices, the numbers of vertices dominated by both vertices of each pair are the same.

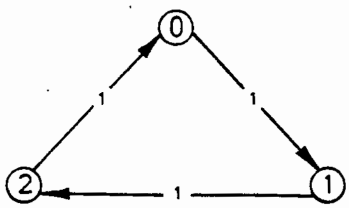
A necessary condition for a tournament to be doubly-regular is that it has to be regular as shown by Reid and Beineke [18]. They also characterized doubly-regular tournaments by the following theorem.

THEOREM 1.2.2 : The following statements are equivalent for any non-transitive tournament of order $n \geq 5$:

i) T_n is doubly-regular

ii) Every arc of T_n lies on the same number of cyclic triples

iii) Every $(n-2)$-subtournament has the same score vector



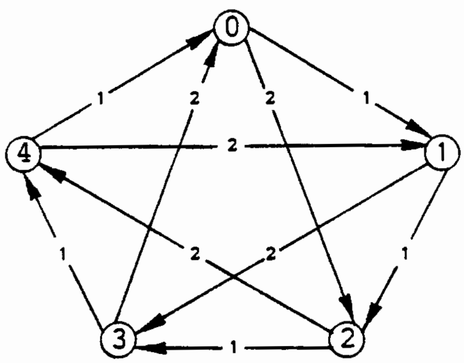
HC-decomposition:
$(\underline{0,1},2,0)$

τ-vector:
(5)

TYPE: $CT_3(1)$

Isomorphic to:
$CT_3(2)$, $LT_3(+1)$, $LT_3(-1)$.

FIGURE 1.5



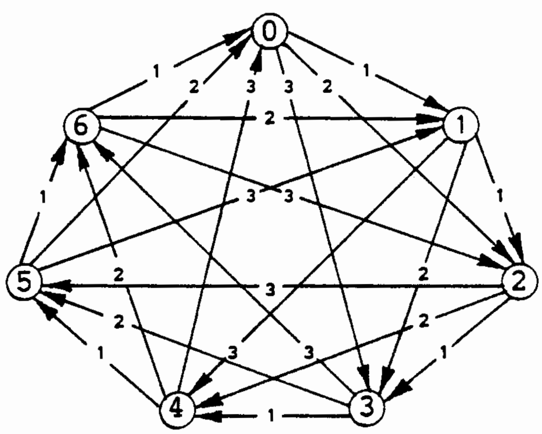
HC-decomposition:
$(\underline{0,1},2,3,4,0)$
$(0,2,4,\underline{1,3},0)$

τ-vector:
(5,5)

TYPE: $CT_5(1,2)$

Isomorphic to:
$CT_5(1,3)$, $CT_5(4,2)$,
$CT_5(4,3)$,
$LT_5(+1,+1)$, $LT_5(+1,-1)$,
$LT_5(-1,+1)$, $LT_5(-1,-1)$.

FIGURE 1.6



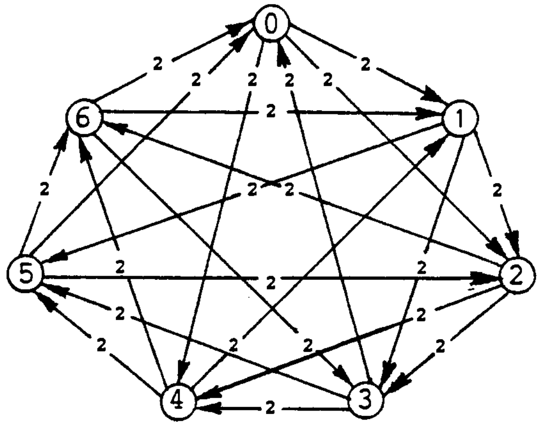
HC-decomposition:
$(\underline{0,1},2,3,4,5,6,0)$
$(0,2,4,6,\underline{1,3},5,0)$
$(0,\underline{3,6},2,5,1,4,0)$

τ-vector:
(7,7,7)

TYPE: $CT_7(1,2,3)$

Isomorphic to:
$CT_7(6,5,4)$, $CT_7(1,5,3)$,
$CT_7(6,2,4)$, $CT_7(1,5,4)$,
$CT_7(6,2,3)$.

FIGURE 1.7



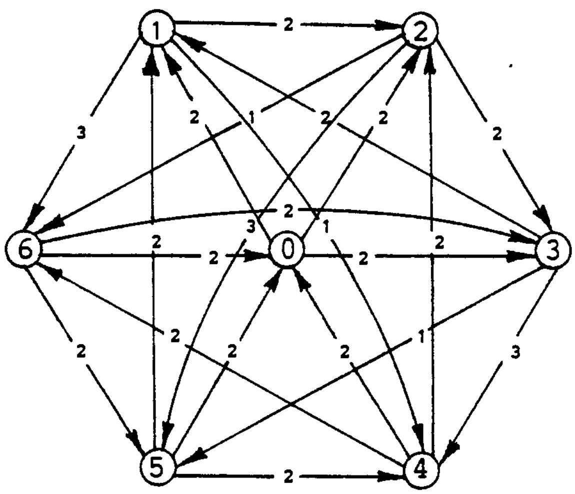
HC-decomposition:
(0,1,2,4,5,6,3,0)
(0,2,3,4,6,1,5,0)
(0,4,1,3,5,2,6,0)

τ-vector:
(0,21,0)

TYPE:$CT_7(1,2,4)$

Isomorphic to:
$CT_7(6,5,3)$,$LT_7(+1,-1,+1)$,
$LT_7(-1,+1,-1)$.

FIGURE 1.8



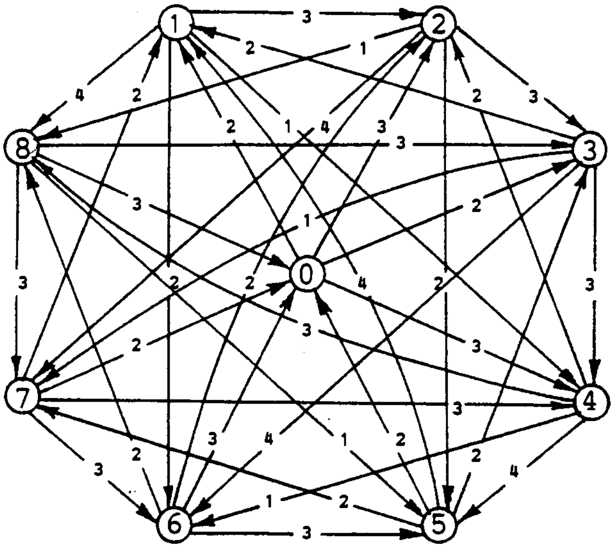
HC-decomposition:
(0,1,2,6,3,5,4,0)
(0,2,3,1,4,6,5,0)
(0,3,4,2,5,1,6,0)

τ-vector:
(3,15,3)

TYPE:$LT_7(+1,+1,+1)$

Isomorphic to:
$LT_7(-,+,+)$,$LT_7(-,-,+)$,
$LT_7(-,-,-)$,$LT_7(+,-,-)$,
$LT_7(+,+,-)$.

FIGURE 1.9



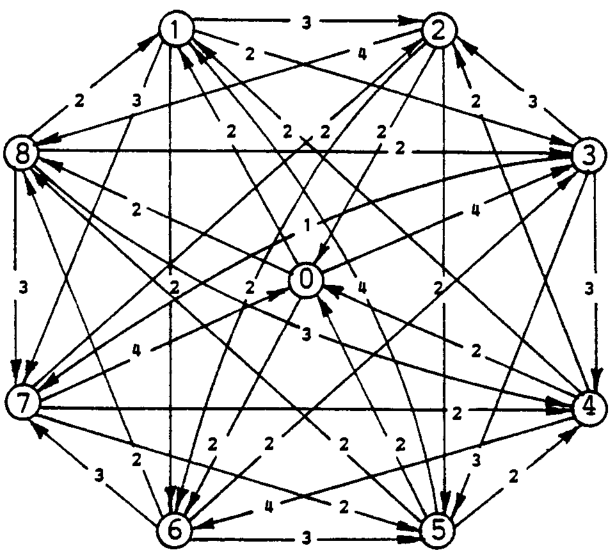
HC-decomposition:
(0,1,2,8,3,7,4,6,5,0)
(0,2,3,1,4,8,5,7,6,0)
(0,3,4,2,5,1,6,8,7,0)
(0,4,5,3,6,2,7,1,8,0)

τ-vector:
(5,13,13,5)

TYPE:LT_9(+1,+1,+1,+1)

Isomorphic to:
LT_9(−,+,+,+),LT_9(−,−,+,+),
LT_9(−,−,−,+),LT_9(−,−,−,−),
LT_9(+,−,−,−),LT_9(+,+,−,−),
LT_9(+,+,+,−).

FIGURE 1.10



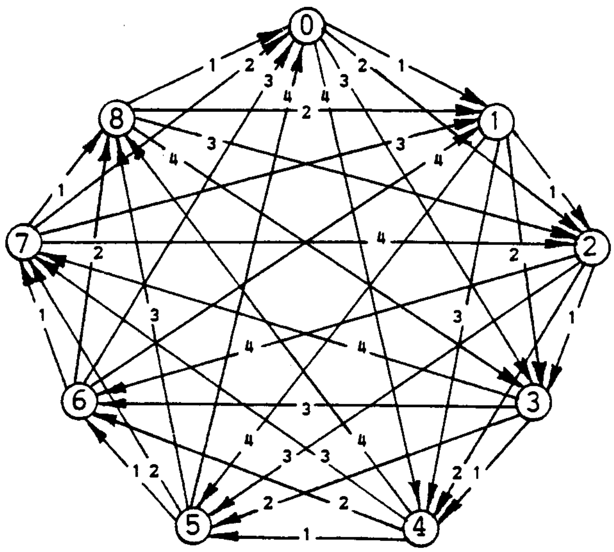
HC-decomposition:
(0,1,2,8,3,7,4,6,5,0)
(0,6,7,5,8,4,1,3,2,0)
(0,3,4,2,5,1,6,8,7,0)
(0,8,1,7,2,6,3,5,4,0)

τ-vector:
(1,21,9,5)

TYPE:LT_9(+1,−1,+1,−1)

Isomorphic to:
LT_9(+,+,−,+),LT_9(−,+,+,−),
LT_9(+,−,+,+),LT_9(−,+,−,+),
LT_9(−,−,+,−),LT_9(+,−,−,+),
LT_9(−,+,−,−).

FIGURE 1.11



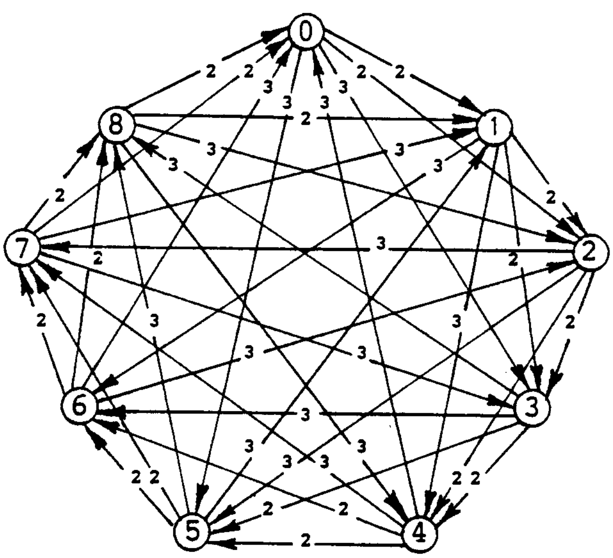
HC-decomposition:
(0,1,2,4,5,6,8,3,7,0)
(0,2,3,4,6,7,1,5,8,0)
(0,3,6,1,4,7,8,2,5,0)
(0,4,8,1,3,5,7,2,6,0)

τ-vector:
(9,9,9,9)

TYPE: $CT_9(1,2,3,4)$

Isomorphic to:
$CT_9(1,2,6,5)$, $CT_9(1,7,3,5)$,
$CT_9(8,7,6,5)$, $CT_9(8,7,3,4)$,
$CT_9(8,2,6,4)$.

FIGURE 1.12



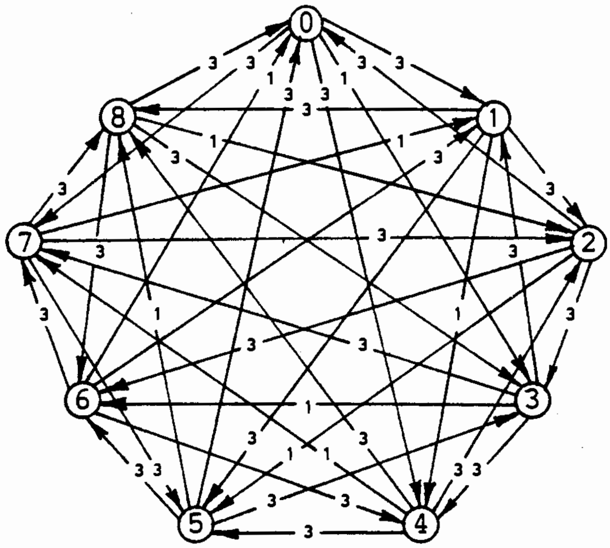
HC-decomposition:
(0,1,2,3,4,5,6,7,8,0)
(0,2,4,7,3,5,8,1,6,0)
(0,3,6,8,2,5,7,1,4,0)
(0,5,1,3,8,4,6,2,7,0)

τ-vector:
(0,18,18,0)

TYPE: $CT_9(1,2,3,5)$

Isomorphic to:
$CT_9(1,2,6,4)$, $CT_9(1,7,6,5)$,
$CT_9(8,7,6,4)$, $CT_9(8,7,3,5)$,
$CT_9(8,2,3,4)$.

FIGURE 1.13



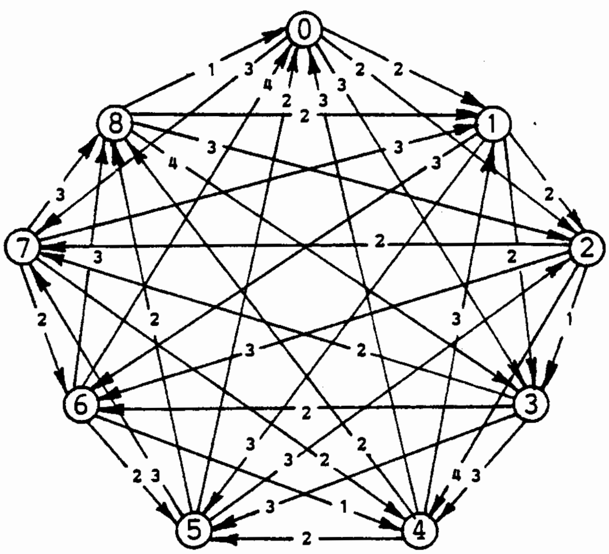
HC-decomposition:
(0,1,2,3,4,5,7,6,8,0)
(0,3,1,4,7,2,5,8,6,0)
(0,4,2,6,1,8,3,7,5,0)
(0,7,1,5,3,6,4,8,2,0)

τ-vector:
(9,0,27,0)

TYPE: $CT_9(1,7,3,4)$

Isomorphic to:
$CT_9(1,7,6,4)$, $CT_9(8,2,6,5)$,
$CT_9(8,2,3,5)$.

FIGURE 1.14

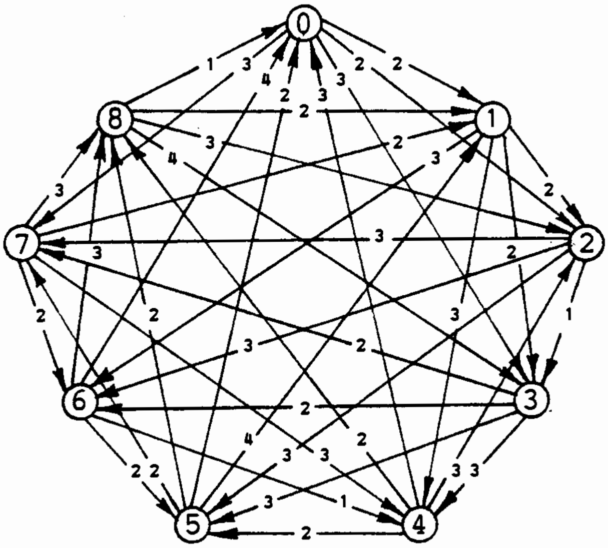


HC-decomposition:
(0,1,2,3,4,5,7,6,8,0)
(0,2,6,4,8,3,7,1,5,0)
(0,3,5,8,2,7,4,1,6,0)
(0,7,8,1,3,6,5,2,4,0)

τ-vector:
(3,15,15,3)

TYPE: UNKNOWN.

FIGURE 1.15



HC-decomposition:

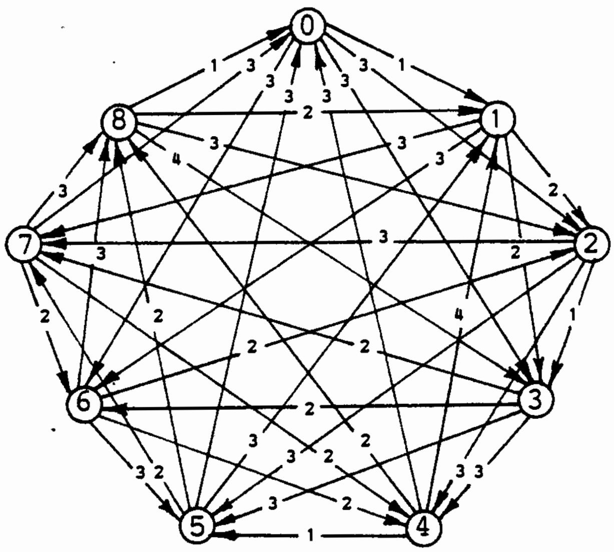
(0,1,2,3,4,5,7,6,8,0)
(0,2,7,1,6,4,8,3,5,0)
(0,3,7,8,2,6,5,1,4,0)
(0,7,4,2,5,8,1,3,6,0)

τ-vector:

(3,15,15,3)

TYPE: UNKNOWN.

FIGURE 1.16



HC-decomposition:

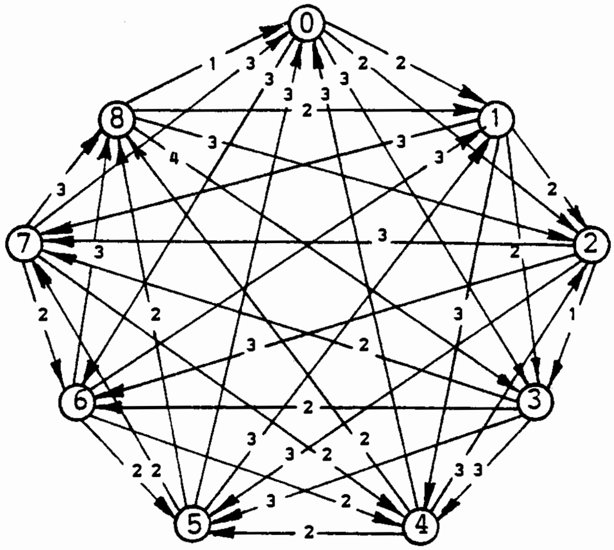
(0,1,2,3,4,5,7,6,8,0)
(0,2,4,1,7,8,3,6,5,0)
(0,3,5,1,6,4,8,2,7,0)
(0,6,2,5,8,1,3,7,4,0)

τ-vector:

(4,12,18,2)

TYPE: UNKNOWN.

FIGURE 1.17

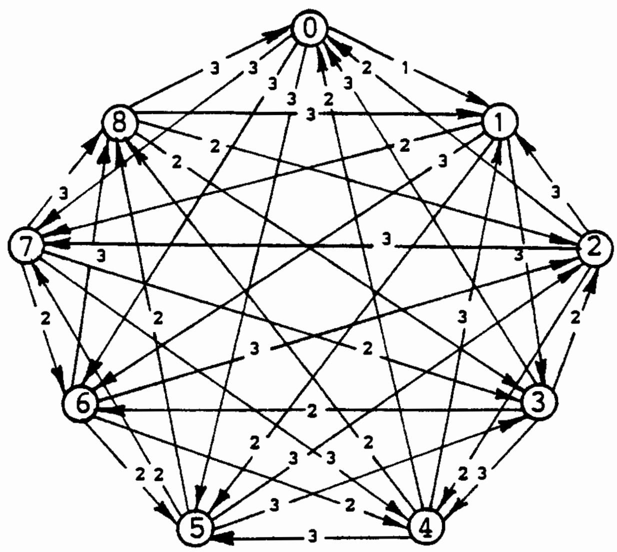


HC-decomposition:
(0,1,2,3,4,5,7,6,8,0)
(0,2,6,5,8,1,3,7,4,0)
(0,3,6,4,8,2,5,1,7,0)
(0,6,1,4,2,7,8,3,5,0)

τ-vector:
(2,15,18,1)

TYPE: UNKNOWN.

FIGURE 1.18

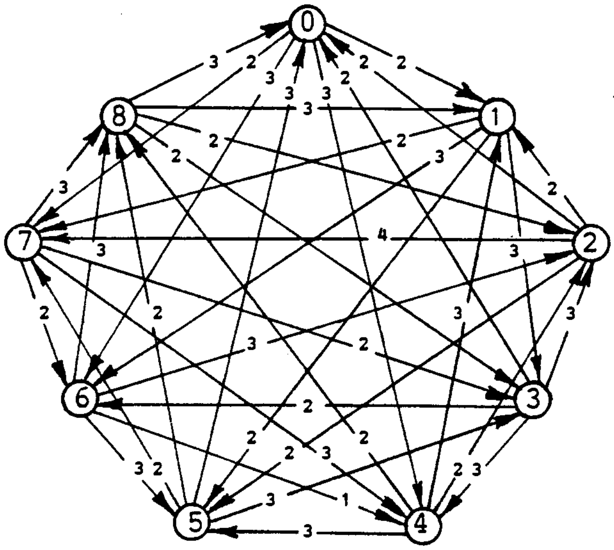


HC-decomposition:
(0,1,3,2,4,5,7,6,8,0)
(0,5,2,1,7,8,3,6,4,0)
(0,6,2,7,4,8,1,5,3,0)
(0,7,3,4,1,6,5,8,2,0)

τ-vector:
(1,16,19,0)

TYPE: UNKNOWN.

FIGURE 1.19



HC-decomposition:

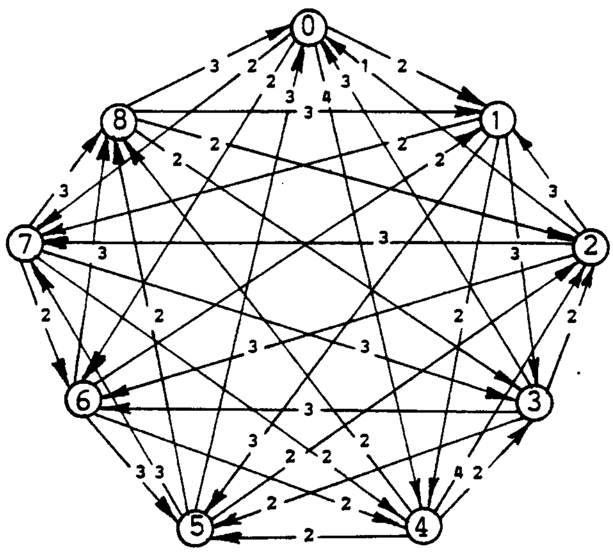
(0,1,3,2,5,7,6,4,8,0)
(0,4,2,1,7,8,3,6,5,0)
(0,6,8,2,7,4,1,5,3,0)
(0,7,3,4,5,8,1,6,2,0)

τ-vector:

(1,17,17,1)

TYPE: UNKNOWN.

FIGURE 1.20



HC-decomposition:

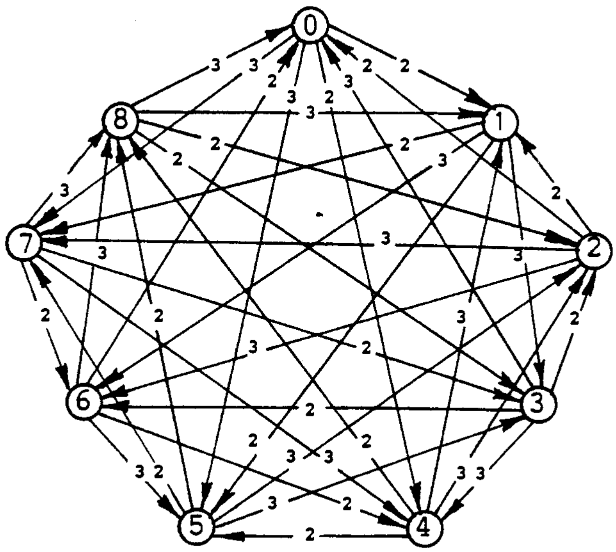
(0,1,3,2,6,4,5,7,8,0)
(0,4,2,1,7,6,5,8,3,0)
(0,6,1,4,8,2,7,3,5,0)
(0,7,4,3,6,8,1,5,2,0)

τ-vector:

(1,18,15,2)

TYPE: UNKNOWN.

FIGURE 1.21



HC-decomposition:

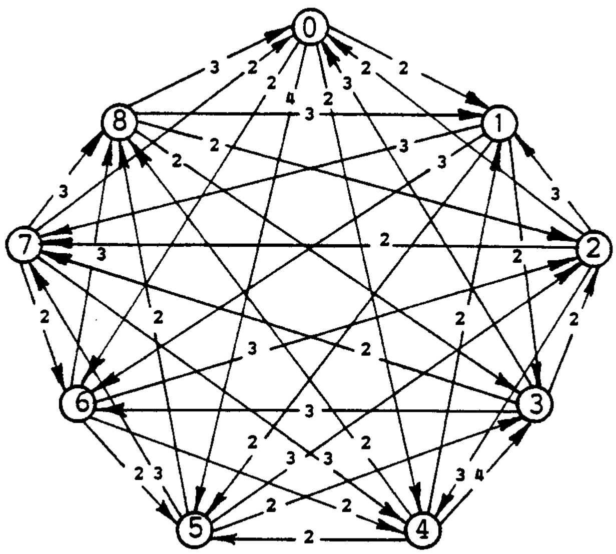
(0,1,3,2,6,4,5,7,8,0)
(0,4,1,6,5,8,2,7,3,0)
(0,5,2,1,7,4,8,3,6,0)
(0,7,6,8,1,5,3,4,2,0)

τ-vector:

(0,18,18,0)

TYPE: UNKNOWN.

FIGURE 1.22



HC-decomposition:

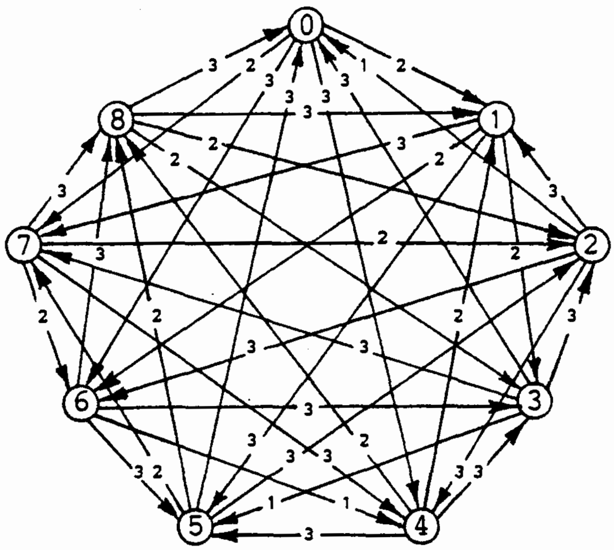
(0,1,3,2,4,5,7,6,8,0)
(0,4,1,6,5,2,7,8,3,0)
(0,5,8,1,7,4,3,6,2,0)
(0,6,4,8,2,1,5,3,7,0)

τ-vector:

(0,20,14,2)

TYPE: UNKNOWN.

FIGURE 1.23



HC-decomposition:
(0,1,3,2,4,5,7,6,8,0)
(0,4,1,7,2,6,5,8,3,0)
(0,6,3,7,4,8,1,5,2,0)
(0,7,8,2,1,6,4,3,5,0)

τ-vector:
(3,12,21,0)

TYPE: UNKNOWN.

FIGURE 1.24

This theorem implies that a regular tournament is doubly-regular if and only if there is exactly one non-zero entry in its τ-vector. For example, $CT_3(1)$ in Figure 1.5 and $CT_7(1,2,4)$ in Figure 1.8 are such tournaments.

Section III : Results on Path Numbers of Tournaments

Recall that the path number of a digraph G, pn(G), is the minimum number of arc-disjoint paths into which it can be decomposed. We will investigate upper and lower bounds for pn(G) (in particular, for tournaments) and compute the path numbers for some special tournaments. First we need to define the following.

DEFINITION 1.3.1 : The *excess* $x(v)$ of a vertex v in a digraph G is defined as $\max\{0, od(v)-id(v)\}$.

DEFINITION 1.3.2 : The *excess* $X(G)$ of a digraph G is defined as $\sum_{v \in G} x(v)$.

One observation here is that $X(RT_n)$ is 0 for any regular tournament RT_n and $X(NT_n)$ is $n/2$ for any near-regular tournament NT_n. Next, we define a special class of digraphs called consistent digraphs.

DEFINITION 1.3.3 : A digraph is *consistent* if $pn(G)=X(G)$.

Using this definition, we can rewrite Theorem 1.1.6 as

THEOREM 1.1.6 : Kelly's Conjecture holds if and only if NT_n is consistent for all even n.

Another observation is that if G' is the digraph obtained by reversing every arc in a digraph G, then we have the following lemma.

LEMMA 1.3.1 : For any digraph G, $pn(G)=pn(G')$ and $X(G)=X(G')$.

PROOF : Let P' be a minimum path partition of G'. Then each path $p' \epsilon P'$ gives rise to a unique path in G, simply by reversing the orientation of all arcs on the path. Let P be the collection of these paths in G. Since every arc in G is the reverse of some arc in G', and every arc in G' is in exactly one path of P', every arc in G must be in exactly one path in P. Thus P forms a path partition of G. Therefore, $pn(G) \leq |P| = |P'| = pn(G')$. Conversely, we can reverse the orientation of every path in any partition P of G to get a path partition P' of G'. This gives us $pn(G') \leq pn(G)$ and hence $pn(G)=pn(G')$. On the other hand, notice that

$$\sum_{v \epsilon G} od_G(v) = \sum_{v \epsilon G} id_G(v).$$

Let V^+ be the set of vertices v in G with $x_G(v)>0$ and V^- be the set of vertices v in G with $x_G(v)=0$. Then we can rewrite the above equality as

$$\sum_{v \epsilon V^+} od_G(v)-id_G(v) = \sum_{v \epsilon V^-} id_G(v)-od_G(v).$$

We know that $\sum_{v \epsilon V^+} od_G(v)-id_G(v) = X(G)$. Also for every $v \epsilon V^+$, $x_{G'}(v)=0$ and for every $v \epsilon V^-$, $x_{G'}(v) \geq 0$. Hence,

$$X(G') = \sum_{v \epsilon V^-} x_{G'}(v)$$

$$= \sum_{v \epsilon V^-} od_{G'}(v)-id_{G'}(v).$$

$$= \sum_{v \in V^-} id_G(v) - od_G(v)$$

$$= \sum_{v \in V^+} od_G(v) - id_G(v)$$

$$= X(G).$$

■

This lemma implies that G is consistent if and only if G' is consistent. For tournaments, we have a special name for G'.

DEFINITION 1.3.4 : The *complement* of a tournament T_n , denoted by $\bar{T}_n$, is obtained by reversing the orientation of every arc in T_n.

An example is given in Figure 1.25. A tournament T_n that is isomorphic to its complement $\bar{T}_n$ is said to be *self-complementary*. The example given in Figure 1.25 is self-complementary whereas the example given in Figure 1.26 is not.

The first result on the bounds for path numbers of digraphs is a theorem due to Alspach and Pullman [2].

THEOREM 1.3.2 : For any digraph G, $pn(G) \geq X(G)$.

PROOF : Every vertex $v \in G$ with $x(v)>0$ must be the initial vertex of at least $x(v)$ paths in any path partition of

(a) T_4 (b) $\bar{T}_4$

FIGURE 1.25

(a) T_4 (b) $\bar{T}_4$

FIGURE 1.26

G. This is because every path using v as an intermediate vertex uses up exactly one in-coming arc and one out-going arc. Also every path that originates at v uses up one out-going arc whereas every path that terminates at v uses up one in-coming arc. Thus there must be exactly od(v)-id(v) more paths starting from v than terminating at v. This implies that there are at least x(v) paths beginning at v. Since every path in a path partition begins at a unique vertex, we must have at least $\sum_{v \in G} x(v) = X(G)$

paths in any path partition of G. Therefore, $pn(G) \geq X(G)$.

∎

This theorem gives us an immediate lower bound for $pn(G)$. Notice that this allows us to quickly identify the path numbers for some digraphs. For example, the T_4 in Figure 1.26(a) can be decomposed into three paths as shown in Figure 1.27. Since $X(T_4)$ is also three, we must have $pn(T_4) = X(T_4) = 3$. Unfortunately, equality does not always hold. For instance, the regular tournament of order 3, RT_3 (see Figure 1.5), has $pn(RT_3)=2$ but $X(RT_3)=0$. In fact, $pn(RT_n)>X(RT_n)$ for all odd n, since $X(RT_n)=0$ and a path decomposition of size zero is impossible for any non-empty graph. Regarding a sufficient condition for equality to hold, Alspach and Pullman [2] gave the following theorem.

THEOREM 1.3.3 : If G is an acircuitous digraph, then $pn(G)=X(G)$.

PROOF : Let G be any acircuitous digraph and $P = \{p_1, p_2, \ldots, p_r\}$ be a minimum path partition of G with r equal to $pn(G)$. Suppose $pn(G)>X(G)$. Then there must exist a vertex $v \in G$ such that the number of paths in P starting at v is greater than $x(v)$ as otherwise we would have the total number of paths in P equal to $X(G)$. Now consider vertex v. Every path that begins at v uses up exactly one out-going arc and every path that

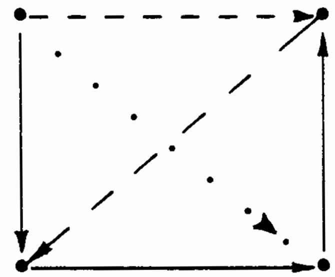


FIGURE 1.27

terminates at v uses up one in-coming arc. Since we use up more than $x(v)=od(v)-id(v)$ out-going arcs as the initial arcs of some paths, we must use at least one in-coming arc as the terminal arc of some path. Suppose p_i is one of the paths that begin at v and p_j is one of the paths that terminate at v. Then (p_i,p_j) forms a path because G is acircuitous. Hence $P\backslash\{p_i,p_j\}\cup(p_i,p_j)$ forms a path partition of G with size one less than that of P. This gives us a contradiction and thus we must have $pn(G)=X(G)$ for any acircuitous digraph G.

∎

This is, however, not a necessary condition as we have already seen in the previous example where $pn(T_4)=X(T_4)$ but T_4 has circuits. In fact, this theorem has very little use in determining the path number of tournaments because for each n there is only one tournament of order n that is acircuitous. This is a consequence of the following theorem given in Moon's book [15].

DEFINITION 1.3.5 : A tournament is *transitive* if, whenever u dominates v and v dominates w, then u dominates w. It is denoted by TT_n.

THEOREM 1.3.4 : The following statements are equivalent:

i) T_n is transitive

ii) Vertices of T_n can be labeled $\{v_1, v_2, \ldots, v_n\}$ such that vertex v_i dominates vertex v_j if and only if $i<j$

iii) T_n has score vector $(0,1, \ldots, n-1)$

iv) The score vector of T_n satisfies the equation

$$\sum_{i=1}^{n} s_i^2 = n \cdot (n-1) \cdot (n-2)/6$$

v) T_n contains no circuits

vi) T_n contains exactly $\binom{n}{k+1}$ paths of length k if $1 \le k \le n-1$

vii) T_n contains exactly $\binom{n}{k}$ transitive subtournaments T_k, if $1 \le k \le n$

viii) Each principal submatrix of the dominance matrix (that is tournament matrix) M contains a row and column of zeros

Nevertheless, Theorem 1.3.3 does give us an easy way of calculating the path number for acircuitous digraphs, and hence the path number of transitive tournaments TT_n which we will evaluate later in this section.

Next, we will look at upper bounds for path numbers of digraphs. Recall that our digraphs have no multiple arcs nor self-adjacent vertices, so the maximum number of arcs in any digraph G is $n\cdot(n-1)$, assuming that G has order n, and this gives us an obvious upper bound on path numbers. A better bound can be obtained by considering a digraph G as the union of two asymmetric digraphs of the same order since every pair of vertices has at most two arcs connecting them. This is because for asymmetric digraphs we have the following theorems, proved by Alspach and Pullman [2].

THEOREM 1.3.5 : If v is any vertex of an arbitrary digraph G then $pn(G) \le pn(G\backslash v)+\mu(v)$.

PROOF : Let t be the number of digons (a *digon* is a pair of arcs (u,w) and (w,u)) incident with v.

(i) If t=1 and $od(v)=id(v)=1$, then $\mu(v)=1$. Let (v,w) and (w,v) be the arcs incident with v. Now suppose $\mathcal{P}$ is an minimum path decomposition of $G\backslash v$ and p is a path in $\mathcal{P}$ that begins at, terminates at or passes through w. If p begins at w, then we can partition the arcs in G into $\mathcal{P}\backslash p \cup \{((v,w),p),(w,v)\}$ giving us a path partition of size $|P|+1$ or $pn(G\backslash v)+\mu(v)$. If p terminates at w, then we can partition the arcs in G into $P\backslash p \cup \{(p,(w,v)),(v,w)\}$. This is again of size $pn(G\backslash v)+\mu(v)$. Finally, if p uses w as an intermediate vertex, then w divides p into two parts p_1, p_2 where p_1 is the part from the beginning of p to w, and p_2 is the rest of p. Then $P\backslash p \cup \{(p_1,(w,v)),((v,w),p_2)\}$ forms a path partition of G with size $pn(G\backslash v)+\mu(v)$.

(ii) Otherwise, we can partition all the arcs in $G\backslash v$ (using P) into $pn(G\backslash v)$ paths and partition the arcs incident with v into $\min\{od(v),id(v)\}$ paths of length two and

$$\max\{od(v),id(v)\} - \min\{od(v),id(v)\}$$

paths of length 1. This gives us a path partition of G with

$$pn(G\backslash v) + \min\{od(v),id(v)\} + \max\{od(v),id(v)\} - \min\{od(v),id(v)\}$$

paths. This implies $pn(G)$ is again at most $pn(G\backslash v)+\mu(v)$.

Therefore, for any digraph G we have $pn(G) \leq pn(G\backslash v)+\mu(v)$.

∎

Using this theorem and Lemma 1.3.1 they calculated the following bound.

THEOREM 1.3.6 : For any asymmetric digraph G, $pn(G) \le \lfloor n^2/4 \rfloor$.

PROOF : First notice that

$$\lfloor (n-1)^2/4 \rfloor = \begin{cases} \lfloor n/2 \rfloor^2 & \text{if } n \text{ is odd,} \\ \lfloor n/2 \rfloor^2 - \lfloor n/2 \rfloor & \text{if } n \text{ is even,} \end{cases}$$

and

$$\lfloor n^2/4 \rfloor = \begin{cases} \lfloor n/2 \rfloor^2 + \lfloor n/2 \rfloor & \text{if } n \text{ is odd,} \\ \lfloor n/2 \rfloor^2 & \text{if } n \text{ is even.} \end{cases}$$

Therefore, $\lfloor n^2/4 \rfloor = \lfloor (n-1)^2/4 \rfloor + \lfloor n/2 \rfloor$.

Now we shall prove this theorem by induction on n. It is easy to check that for $n \le 3$, $pn(G) \le \lfloor n^2/4 \rfloor$. Suppose it is also true for any asymmetric digraph G of order up to n-1. Now consider an asymmetric digraph G of order n. By the induction hypothesis $pn(G\backslash v) \le \lfloor (n-1)^2/4 \rfloor$ for any $v \in G$, and by Theorem 1.3.5 we have

$$\begin{aligned} pn(G) &\le pn(G\backslash v) + \mu(v) \\ &\le \lfloor (n-1)^2/4 \rfloor + \mu(v). \end{aligned}$$

If $\mu(v) \le \lfloor n/2 \rfloor$ for some v in G, then we would have

$$\begin{aligned} pn(G) &\le \lfloor (n-1)^2/4 \rfloor + \lfloor n/2 \rfloor \\ &= \lfloor n^2/4 \rfloor \end{aligned}$$

and we are finished. Assume $\mu(v) > \lfloor n/2 \rfloor$ for all v in G. This implies that $|od(v)-id(v)| \geq 2$ for all v. In other words, for every v we either have $x_G(v)>0$ or $x_{G'}(v)>0$ (see page 33 for the definition of G'). Using this fact, we can assume that there are at least $\lceil n/2 \rceil$ vertices v with $x(v)>0$. Since otherwise, we can apply Lemma 1.3.1 and consider G' instead. Let u be one such vertex. Then we must have $od(u) > \lfloor n/2 \rfloor$. Since there are at least $\lceil n/2 \rceil$ vertices v with $x(v)>0$, there are at most $\lfloor n/2 \rfloor$ vertices v with $x(v)=0$. Thus if we let $W = \{ w \mid u \text{ dominates } w \text{ and } x(w)>0\}$, then $|W| \geq od(u) - \lfloor n/2 \rfloor$. Now let us construct a path decomposition of G as follows. Let P be a minimum path partition of $G\backslash u$. For each $w \in W$, we remove a path p_w from P, where p_w is a path with w as its initial vertex, and form the path $((u,w),p_w)$. This path must exist because $x(w) \geq 2$ and so $x_{G\backslash u}(w) \geq 2$ (see proof of Theorem 1.3.2). Let P' be the collection of these $|W|$ paths and P'' be the set of p_w that are removed from P. Now we have at most $\lfloor n/2 \rfloor$ out-going arcs and $\lfloor n/2 \rfloor$ in-coming arcs incident to u that we have to take care of. These arcs can be partitioned into $\min\{od(u)-|W|, id(u)\}$ paths of length 2 and $\max\{od(u)-|W|, id(u)\} - \min\{od(u)-|W|, id(u)\}$ paths of length 1. Let P^0 be this set of paths, clearly $|P^0| \leq \lfloor n/2 \rfloor$. Therefore $(P\backslash P'') \cup P' \cup P^0$ forms a path partition of G with cardinality

$$pn(G\backslash u) - |W| + |W| + \max\{od(u)-|W|, id(u)\}$$

which is less than or equal to $pn(G\backslash u) + \lfloor n/2 \rfloor$. Hence

$$\begin{aligned} pn(G) &\le pn(G\backslash u) + \lfloor n/2 \rfloor, \\ &\le \lfloor (n-1)^2/4 \rfloor + \lfloor n/2 \rfloor, \\ &= \lfloor n^2/4 \rfloor. \end{aligned}$$

Therefore, by induction, $pn(G) \le \lfloor n^2/4 \rfloor$ for any asymmetric digraph G.

■

This together with Theorem 1.3.2 implies that for any asymmetric digraph G, $X(G) \le pn(G) \le \lfloor n^2/4 \rfloor$ and consequently for any digraph G we have $X(G) \le pn(G) \le 2 \cdot \lfloor n^2/4 \rfloor$. In [2], Alspach and Pullman conjectured that we can in fact do better than this. They conjectured that the same upper bound for asymmetric digraphs will work for any digraph. This was later verified by O'Brien [16]. As for tournaments, we can improve our lower bound slightly to $\max\{\lfloor (n+1)/2 \rfloor, X(T_n)\}$, as shown by Alspach, Pullman and Mason [3].

THEOREM 1.3.7 : For any tournament T_n, $pn(T_n) \ge \lfloor (n+1)/2 \rfloor$.

PROOF : The total number of arcs in any T_n is $n \cdot (n-1)/2$ and the maximum number in each path in any path partition is $(n-1)$. Thus the minimum number of paths needed to cover every arc in T_n is $n/2$. Since $pn(T_n)$ is an integer, we must have $pn(T_n) \ge \lfloor (n+1)/2 \rfloor$.

■

This combines with Theorem 1.3.2 and Theorem 1.3.6 to give us the result.

COROLLARY 1.3.8 : For any tournament T_n,

$$\max\{\lfloor (n+1)/2 \rfloor, X(T_n)\} \leq pn(T_n) \leq \lfloor n^2/4 \rfloor.$$

They also examined which integers are possible path numbers for tournaments and came up with the following results [3].

THEOREM 1.3.9 : For any positive integer n,

i) if n is even, then there exists a tournament T_n with $pn(T_n)=k$ for every $k \epsilon [n/2, n^2/4]$, and

ii) if n is odd, then there exists a tournament T_n with $pn(T_n)=k$ for

a) every $k \epsilon [(n+1)/2, n-2]$, and

b) every even $k \epsilon [n-1, n^2/4]$.

The only case not covered in Theorem 1.3.9 is when both n and k are odd with $k \epsilon [n-1, n^2/4]$ and they conjectured the following.

CONJECTURE 2 :

There is no odd order tournament T_n with $pn(T_n) \epsilon [n-1, n^2/4]$ and $pn(T_n)$ odd.

Their results show that $\lfloor (n+1)/2 \rfloor$ is the best possible lower bound and $\lfloor n^2/4 \rfloor$ is the best possible upper bound for path numbers of tournaments. At the end of this section we shall look at some of the tournaments that give us these bounds. Now we turn our attention to the construction of some consistent tournaments. Since we can get one tournament from any other tournament of the same order by reversing the orientation of some arcs, all we need to know is what happens to the path number of a digraph when an arc is reversed. From this we can determine the path number of an arbitrary tournament by successively reversing the arcs of some consistent tournament. This sounds like a good idea but it is not always easy to implement. The following result is due to Alspach, Pullman and Mason [3].

THEOREM 1.3.10 : Suppose G is a consistent digraph and (v,w) is an arc of G with $od(v)-id(v) \leq 0$ and $od(w)-id(w) \geq 0$. If H is the digraph obtained from G by reversing (v,w), then H is consistent and $pn(H)=pn(G)+2$.

PROOF : First notice that $od_G(v)-id_G(v) \leq 0$ implies $x_G(v)=0$ and $od_G(w)-id_G(w) \geq 0$ implies $x_G(w)=od_G(w)-id_G(w)$. Since H is obtained from G by reversing (v,w), the degrees of any vertices other than v and w remain unchanged. Therefore, we have

$$X(H) = \sum_{u \in H} x_H(u)$$

$$= x_H(v) + x_H(w) + \sum_{u \in H\setminus\{v,w\}} x_H(u)$$

$$= x_H(v) + x_H(w) + \sum_{u \in G\setminus\{v,w\}} x_G(u)$$

Furthermore, by reversing (v,w), the out-degree of v and in-degree of w decreased by 1 and the in-degree of v and out-degree of w increased by 1, that is

$$\begin{aligned} x_H(v) &= \max\{0, od_H(v) - id_H(v)\} \\ &= \max\{0, (od_G(v)-1) - (id_G(v)+1)\} \\ &= \max\{0, od_G(v) - id_G(v) - 2\} \\ &= 0 \\ x_H(w) &= \max\{0, od_H(w) - id_H(w)\} \\ &= \max\{0, (od_G(w)+1) - (id_G(w)-1)\} \\ &= \max\{0, od_G(w) - id_G(w) + 2\} \\ &= x_G(w) + 2. \end{aligned}$$

Hence,

$$\begin{aligned} X(H) &= x_G(v) + x_G(w) + 2 + \sum_{u \in G\setminus\{v,w\}} x_G(u) \\ &= X(G) + 2. \end{aligned}$$

To construct a path partition of H, take any minimum path partition P of G. Let $p \in P$ be the path that covers the arc (v,w), then the removal of (v,w) splits p into two parts p_1, p_2 where p_1 is the path from the start of p to v and p_2 is the path from w to the end of p. Then $P' = (P\setminus p) \cup \{p_1, (w,v), p_2\}$ forms a path partition of H with $|P|+2$ paths. Thus

$$pn(H) \le pn(G)+2 \quad \text{because } P \text{ is minimum}$$

$$\begin{aligned} \text{but} \quad pn(G)+2 &= X(G)+2 \qquad \text{because } G \text{ is consistent} \\ &= X(H)-2+2 \\ &= X(H). \end{aligned}$$

This implies H is consistent; that is, $pn(H) = X(H) = X(G)+2 = pn(G)+2$.

■

What still remains to be understood is the case where $od(v)-id(v)>0$ or $od(w)-id(w)<0$, which I believe would be a relatively harder problem, since the solution to this problem gives us an indication of what would happen to path numbers when excess decreases. As we all know, regular and near-regular tournaments have the smallest excess among all tournaments. Therefore any knowledge about how decreases in excess affect path numbers would be a big step towards solving the path number problem for regular tournaments and hence Kelly's Conjecture. This concludes our discussion of arc-reversal.

Another possible direction to proceed is circuit-reversal, in particular, 3-circuits. This is because of a consequence of Ryser's result on 0-1 matrices (see Reid and Beineke [18,p.197]) implies that any tournament can be transformed into any other tournament with the same score vector by successively reversing 3-circuits. But so far, little progress has been made in this direction.

Next, we will look at yet another way of approaching the path number problem, namely, to build up consistent tournaments recursively. The following theorem suggests a way to construct consistent tournaments from smaller consistent tournaments. This theorem is derived from Theorem 1.3.5 which stated that $pn(G) \le pn(G \backslash v) + \mu(v)$, and so if there exists a $v \in G$ such that $X(G) = pn(G \backslash v) + \mu(v)$, then we must have $pn(G) = X(G)$.

NOTATION 1.3.1 : For any tournament T_n, denote by $\{V^+(T_n), V^-(T_n)\}$ a bipartition of the vertex set $V(T_n)$ of T_n, such that every $v \in V^+(T_n)$ has $x(v) \ge 0$ and every $v \in V^-(T_n)$ has $x(v) = 0$.

THEOREM 1.3.11 : Let T_n be a tournament of order n. If there exists a vertex $v \in V(T_n)$ such that

i) $T_n \backslash v$ is consistent and

ii) for some bipartition $\{V^+(T_n \backslash v), V^-(T_n \backslash v)\}$ of $T_n \backslash v$ we have v dominates every $u \in V^-(T_n \backslash v)$ and either $S = \phi$ or $|V^-(T_n \backslash v)| \ge |R|$,

where $R = \{u \mid u \in V^+(T_n \backslash v),$ u dominates $v\}$

and $S = \{u \mid u \in V^+(T_n \backslash v),$ v dominates $u\}$,

then T_n is consistent.

PROOF : Let us write V^+ and V^- instead of $V^+(T_n \backslash v)$ and $V^-(T_n \backslash v)$. We have two cases to consider.

(a) If $S = \phi$, then we have the situation shown in Figure 1.28. The excess of T_n is

$$X(T_n) = X(T_n \backslash v) + |V^+| + \max\{0, |V^-| - |V^+|\}$$

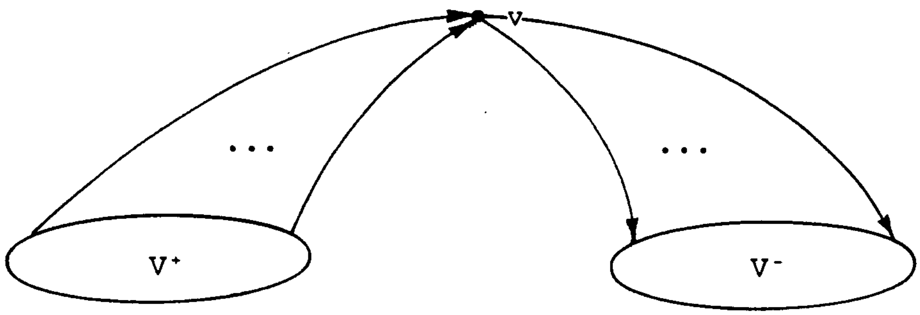


FIGURE 1.28

$$= X(T_n\backslash v) + \max\{|V^+|,|V^-|\}.$$

But since

$$\mu(v) = \max\ \{od(v),id(v)\} = \max\ \{|V^+|,|V^-|\},$$

by Theorem 1.3.5,

$$\begin{aligned} pn(T_n) &\leq pn(T_n\backslash v) + \mu(v) \\ &= X(T_n\backslash v) + \max\{|V^+|,|V^-|\} \\ &= X(T_n). \end{aligned}$$

Thus T_n is consistent.

(b) If $S\neq\phi$, then we must have $|V^-|\geq|R|$ by assumption and we have the situation shown in Figure 1.29. We can assume that all $u\epsilon S$ have $x(u)>0$, since otherwise we could consider the partition $\{V^+\backslash u, V^-\cup u\}$. Now since $T_n\backslash v$ is consistent, every $u\epsilon S$ must be the initial vertex of some paths in the minimum path partition of $T_n\backslash v$. We can then concatenate every arc (v,u), where $u\epsilon S$,

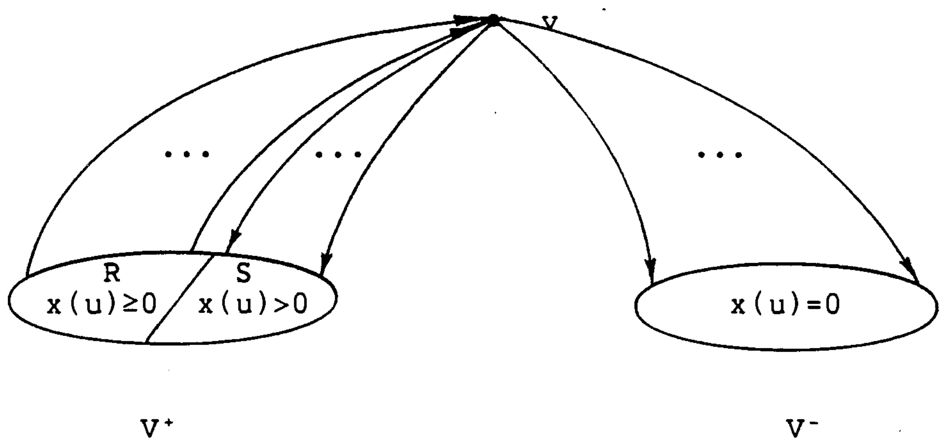


FIGURE 1.29

to one of these paths originating from u. The rest of the arcs incident with v can then be partitioned into $\max\{|R|,|V^-|\}=|V^-|$ paths of length at most two. This forms a path partition of T_n and we have

$$\begin{aligned} pn(T_n) &\le pn(T_n\backslash v) + |V^-| \\ &= X(T_n\backslash v) + |V^-| \\ &= X(T_n\backslash v)+|R|-|S|+(|V^-|+|S|-|R|) \\ &= X(T_n). \end{aligned}$$

Hence T_n is consistent.

■

By Lemma 1.3.1, we obtain the dual to this theorem.

THEOREM 1.3.11': Let T_n be a tournament of order n. If there exists a vertex $v \epsilon V(T_n)$ such that

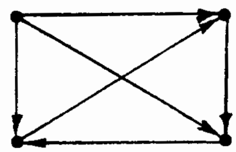
i) $T_n \backslash v$ is consistent and

ii) for some bipartition $\{V^+(T_n \backslash v), V^-(T_n \backslash v)\}$ of $T_n \backslash v$ we have v dominated by every $u \epsilon V^+(T_n \backslash v)$ and either $S=\phi$ or $|V^+(T_n \backslash v)| \geq |R|$, where $R=\{u | u \epsilon V^-(T_n \backslash v)$, u dominated by v$\}$ and $S=\{u | u \epsilon V^-(T_n \backslash v)$, v dominated by u$\}$,

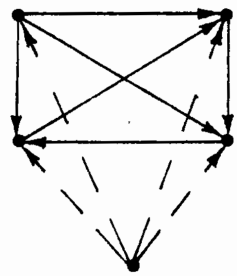
then T_n is consistent.

Using these two theorems, we can construct consistent tournaments by adding a *receiver* (a vertex with out-degree zero), a *transmitter* (a vertex with in-degree zero) or a vertex as described in (ii) of Theorems 1.3.11 and 1.3.11'. For example, given the tournament T_4 in Figure 1.30(a) (which is consistent), we can construct two non-isomorphic consistent tournaments T_5 and T_5' of order 5 as shown in Figures 1.30(b) and (c).

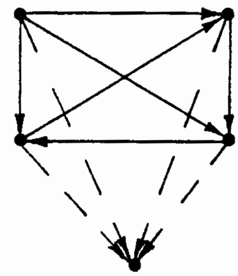
Finally, we will look at the path numbers for some special tournaments —— transitive tournaments, Walecki tournaments and circulant tournaments. We start with the transitive tournament since it is the easiest one to determine. We need only Theorems 1.3.3 and 1.3.4. We have the following result due to Alspach and Pullman [2].



(a) T_4



(b) T_5



(c) T_5'

FIGURE 1.30

THEOREM 1.3.12 : The path number of a transitive tournament TT_n of order n is $\lfloor n^2/4 \rfloor$.

PROOF : From Theorem 1.3.4, we know that TT_n is acircuitous and by Theorem 1.3.3, any acircuitous digraph G has $pn(G)=X(G)$ and thus $pn(TT_n)=X(TT_n)$. To find the excess of TT_n, we look at Theorem 1.3.4 again. It tells us that the score vector of TT_n is $(0,1,2, \ldots ,n-1)$. Therefore, for $i=0,1, \ldots ,\lfloor n/2 \rfloor-1$, we have

$$od(i) = s_i \leq \lfloor n/2 \rfloor -1$$

and so
$$\begin{aligned} id(i) &= n-1-od(i) \\ &\geq n - \lfloor n/2 \rfloor \\ &= \lceil n/2 \rceil . \end{aligned}$$

Hence $od(i)-id(i) < 0$

and $x(i) = 0$.

On the other hand, for $i=\lfloor n/2 \rfloor, \ldots ,n-1$

$$od(i) = s_i \geq \lfloor n/2 \rfloor$$

and $\quad id(i) = n-1-od(i)$

$$\le n - \lfloor n/2 \rfloor - 1$$

$$= \lceil n/2 \rceil - 1$$

so $\quad od(i)-id(i) \ge 0.$

Hence $\quad x(i) = od(i)-id(i)$

$$= i-(n-1-i)$$

$$= 2i-n+1.$$

Now we can compute $X(TT_n)$ obtaining

$$X(TT_n) = \sum_{i=0}^{n-1} x(i)$$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor - 1} x(i) + \sum_{i=\lfloor n/2 \rfloor}^{n-1} x(i)$$

$$= 0 + \sum_{i=\lfloor n/2 \rfloor}^{n-1} (2i-n+1)$$

$$= 2 \cdot \sum_{i=\lfloor n/2 \rfloor}^{n-1} i - \sum_{i=\lfloor n/2 \rfloor}^{n-1} (n-1)$$

$$= 2 \cdot \{n \cdot (n-1)/2 - \lfloor n/2 \rfloor \cdot (\lfloor n/2 \rfloor - 1)/2\} - \lceil n/2 \rceil \cdot (n-1)$$

$$= n \cdot (n-1) - \lfloor n/2 \rfloor^2 + \lfloor n/2 \rfloor - \lceil n/2 \rceil \cdot (n-1)$$

$$= (n-1) \cdot (n - \lceil n/2 \rceil) - \lfloor n/2 \rfloor^2 + \lfloor n/2 \rfloor$$

$$= (n-1) \cdot \lfloor n/2 \rfloor - \lfloor n/2 \rfloor^2 + \lfloor n/2 \rfloor$$

$$= n \cdot \lfloor n/2 \rfloor - \lfloor n/2 \rfloor^2$$

$$= \lfloor n/2 \rfloor \cdot (n - \lfloor n/2 \rfloor)$$

$$= \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$$

$$= \lfloor n^2/4 \rfloor.$$

Therefore,

$$pn(TT_n) = X(TT_n) = \lfloor n^2/4 \rfloor .$$

■

In fact, it is not hard to construct a path decomposition of the arc-set of TT_n of size $\lfloor n^2/4 \rfloor$. First we label the vertices $\{0,1,2, \ldots ,n-1\}$ as in Theorem 1.3.4(ii). Let P_i be the set of all arcs in TT_n with length i. Clearly, the P_i's partition the arc-set of TT_n into n-1 disjoint sets. Now let p_{ij} be the subset of P_i such that $(u,v) \epsilon p_{ij}$ if $u \equiv j \pmod i$, where $i=1, \ldots ,n-1$ and $j=0, \ldots ,i-1$. Then the set of all p_{ij}'s partition the arc-set of TT_n. Our claim is that each non-empty p_{ij} is a path in TT_n and there are altogether $\lfloor n^2/4 \rfloor$ of them, thus giving a path decomposition of TT_n with $\lfloor n^2/4 \rfloor$ paths.

To see that each non-empty p_{ij} forms a path, first notice that p_{ij} is empty if and only if $j+i>n-1$. Now for each non-empty p_{ij}, suppose $j<j+i< \ldots <j+ki \leq n-1$ are all the numbers in $\{0,1, \ldots ,n-1\}$ congruent to j modulo i. Then $p_{ij} = \{(j,j+i),(j+i,(j+i)+i), \ldots ,(j+(k-1)i,j+ki)\}$. Note that although j+ki is congruent to j modulo i, $(j+ki,(j+ki)+i)$ is not in p_{ij}. Therefore, p_{ij} is the path $(j,j+i, \ldots ,j+ki)$. Next, we will count the number of non-empty p_{ij}'s. Let n_i denote the number of non-empty p_{ij}'s in P_i, where $j=0, \ldots ,i-1$ and recall that p_{ij} is empty if and only if $i+j>n-1$ or $j>n-1-i$, so we must have

$$n_i = \min\{i, n-1-i+1\}$$
$$= \min\{i, n-i\}.$$

In other words,

$$n_i = \begin{cases} i & \text{when } 1 \le i \le \lfloor n/2 \rfloor \\ n-i & \text{when } \lfloor n/2 \rfloor + 1 \le i \le n-1. \end{cases}$$

Thus the total number of non-empty p_{ij}'s

$$= \sum_{i=1}^{n-1} n_i$$

$$= \sum_{i=1}^{\lfloor n/2 \rfloor} i + \sum_{i=\lfloor n/2 \rfloor+1}^{n-1} (n-i)$$

$$= \sum_{i=1}^{\lfloor n/2 \rfloor} i + \sum_{i=1}^{\lceil n/2 \rceil - 1} i$$

$$= \lfloor n/2 \rfloor \cdot (\lfloor n/2 \rfloor + 1)/2 + \lceil n/2 \rceil \cdot (\lceil n/2 \rceil - 1)/2$$

$$= \{\lfloor n/2 \rfloor \cdot (\lfloor n/2 \rfloor + 1) + \lceil n/2 \rceil \cdot (\lceil n/2 \rceil - 1)\}/2$$

$$= \{\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil + \lceil n/2 \rceil \cdot \lfloor n/2 \rfloor\}/2$$

$$= \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$$

$$= \lfloor n^2/4 \rfloor.$$

This shows us that TT_n can be decomposed into $\lfloor n^2/4 \rfloor$ arc-disjoint paths.

Notice that $pn(TT_n)$ coincides with the upper bound of path numbers of tournaments. We will now look at tournaments that give us the lower bound. Walecki tournaments are examples.

LEMMA 1.3.13 : Every Walecki tournament of even order n has path number $n/2$.

Lemma 1.3.13 follows by construction whereas for odd order Walecki tournaments it is considerably more complicated. Recall that every Walecki tournament $LT_n(s_0,s_1, \ldots ,s_{m-1})$ of order $n=2m+1$ is an arc-disjoint union of Hamilton circuits $\{C_0,C_1, \ldots ,C_{m-1}\}$. Now let $P=(x_0,x_1, \ldots ,x_m)$ be the directed walk constructed by taking an arc $(x_i,x_{i+1}) \epsilon C_i$. So if x_0 is chosen, then the rest of the x_i's are uniquely determined. Our goal is to select x_0 in such a way that all x_i's in P are distinct hence giving us a path which, together with the m Hamilton paths, $C_i \backslash (x_i,x_{i+1})$, form a path decomposition of $LT_n(s_0,s_1, \ldots ,s_{m-1})$. This decomposition has size $m+1$ or $(n+1)/2$ and by Theorem 1.3.7 it is minimum. To achieve this we need a few lemmas. The first lemma is the result of rotating $LT_n(s_0,s_1, \ldots ,s_{m-1})$. If we rotate $LT_n(s_0, s_1, \ldots ,s_{m-1})$ counter-clockwise, we will get $LT_n(s_1, s_2, \ldots , s_{m-1}, -s_0)$ and if we rotate it clockwise we will get $LT_n(-s_{m-1}, s_0, s_1, \ldots , s_{m-2})$. The second lemma is a special property relating arcs from consecutive Walecki circuits. For the next two lemmas and the following theorem, let us denote the number of negative s_j's in the set $\{s_0, s_1, \ldots ,s_{i-1}\}$, $i=1, \ldots ,m$, by k_i and the total number of negative s_j's in $\{s_0, \ldots ,s_{m-1}\}$ by k, clearly $k=k_m$.

LEMMA 1.3.14 : For every Walecki tournament,

$$LT_n(s_0,s_1, \ldots ,s_{m-1}),$$

where $n=2m+1$, we have

$$LT_n(s_0,s_1, \ldots ,s_{m-1}) \simeq LT_n(s_1,s_2, \ldots ,s_{m-1},-s_0)$$
$$\simeq LT_n(-s_{m-1},s_0, \ldots ,s_{m-2}).$$

PROOF : Define two permutations on the set of vertices $\{0,1, \ldots ,n-1\}$ of $LT_n(s_0,s_1, \ldots ,s_{m-1})$ as follows,

$$\sigma^+=(0)(1\ 2\ 3\ \ldots\ n-1)$$

and $$\sigma^-=(0)(n-1\ n-2\ \ldots\ 1).$$

We shall show that σ^+ is a domination preserving map mapping $LT_n(s_0,s_1, \ldots ,s_{m-1})$ to $LT_n(-s_{m-1},s_0, \ldots ,s_{m-2})$, and σ^- is a domination preserving map mapping $LT_n(s_0,s_1, \ldots ,s_{m-1})$ to $LT_n(s_1,s_2, \ldots ,s_{m-1},-s_0)$.

For each Walecki tournament $LT_n(t_0,t_1, \ldots , t_{m-1})$, we denote the j^{th} Walecki cycle (undirected) by C_{j-1} and the j^{th} Walecki circuit (oriented according to t_{j-1}) by $t_{j-1}C_{j-1}$. Then for $i=0, \ldots ,m-2$,

$$\begin{aligned}\sigma^+(s_iC_i)&=s_i\sigma^+(C_i)\\ &=s_i\sigma^+(<0,1+i,2+i,n-1+i, \ldots ,m+2+i,m+1+i,0>)\\ &=s_i<0,2+i,3+i,1+i, \ldots ,m+3+i,m+2+i,0>\\ &=s_iC_{i+1}\end{aligned}$$

which is exactly the $(i+2)^{nd}$ Walecki circuit of $LT_n(-s_{m-1},s_0, \ldots ,s_{m-2})$, that is, σ^+ preserves the orientation of each arc in s_iC_i. As for $i=m-1$, we have

$$\begin{aligned}\sigma^+(s_{m-1}C_{m-1})&=s_{m-1}\sigma^+(C_{m-1})\\ &=s_{m-1}\sigma^+(<0,m,m+1,m-1, \ldots ,n-2,1,n-1,0>)\end{aligned}$$

$=s_{m-1}<0,m+1,m+2,m, \ldots ,n-1,2,1,0>$

$=-s_{m-1}<0,1,2,n-1, \ldots ,m,m+2,m+1,0>$

$=-s_{m-1}C_0$

which is exactly the first Walecki circuit of $LT_n(-s_{m-1},s_0, \ldots ,s_{m-2})$. Again, σ^+ preserves the orientation of each arc in $s_{m-1}C_{m-1}$. Thus $LT_n(s_0,s_1, \ldots ,s_{m-1})$ is isomorphic to $LT_n(-s_{m-1},s_0, \ldots ,s_{m-2})$.

Similarly, we can use σ^- to show that $LT_n(s_0,s_1, \ldots ,s_{m-1})$ is isomorphic to $LT_n(s_1,s_2, \ldots ,s_{m-1},-s_0)$ by showing

$\sigma^-(s_iC_i)=s_iC_{i-1}$ for $i=1, \ldots ,m-1$,

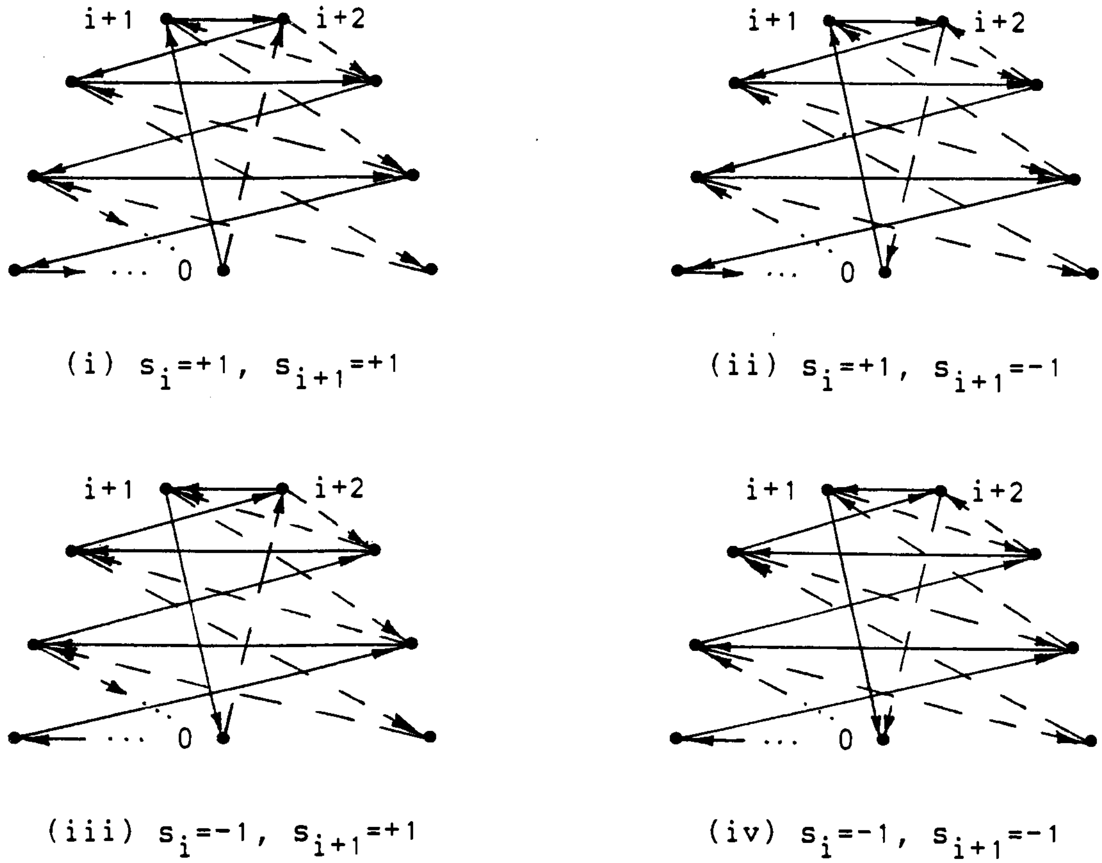
and $\sigma^-(s_0C_0)=-s_0C_{m-1}$. Thus σ^- is also a domination preserving map of the vertices. Hence $LT_n(s_0,s_1, \ldots ,s_{m-1})$ is isomorphic to $LT_n(s_1,s_2, \ldots ,s_{m-1},-s_0)$.

■

With this lemma, we can now assume that $s_0=s_{m-1}=+1$, since otherwise we can just rotate the sequence $(s_0,s_1, \ldots ,s_{m-1})$ as in Lemma 1.3.14 until we get "+1" for s_0 and s_{m-1}. Thus $k_1=0$ and $k_m=k_{m-1}=k$. Recall that $l(x,y)$, the length of the arc (x,y), is defined as y-x modulo n, but for Walecki tournaments we will use modulo n-1 instead.

LEMMA 1.3.15 : For $n=2m+1$, if $(x,y)\epsilon C_i \subset LT_n(s_0,s_1, \ldots ,s_{m-1})$ and $3\le l(x,y)\le 2m-1$, then $(y,z)\epsilon C_{i+1}$, where

$$z = x + 2 + (-1)^{l(x,y)}\cdot|1-k_{i+2}+k_i|.$$



(i) $s_i=+1$, $s_{i+1}=+1$ (ii) $s_i=+1$, $s_{i+1}=-1$

(iii) $s_i=-1$, $s_{i+1}=+1$ (iv) $s_i=-1$, $s_{i+1}=-1$

FIGURE 1.31

PROOF : Let us look at two consecutive Walecki circuits in an $LT_n(s_0,s_1, \ldots ,s_{m-1})$. We have the four cases as shown in Figure 1.31. From these four cases, we derived the following table which gives us the desired results:

	s_i	s_{i+1}	$l(x,y)$ even	$l(x,y)$ odd	$\|1-(k_{i+2}-k_i)\|$
(i)	+1	+1	z=x+3	z=x+1	1
(ii)	+1	-1	z=x+2	z=x+2	0
(iii)	-1	+1	z=x+2	z=x+2	0
(iv)	-1	-1	z=x+3	z=x+1	1

Since these are all the cases, we have the desired result that

$$z=x+2+(-1)^{l(x,y)}|1-k_{i+2}+k_i|.$$

∎

Using these lemmas, we can now show that odd order Walecki tournaments, $LT_n(s_0,s_1, \ldots ,s_{m-1})$, where $n=2m+1$ have path number $(n+1)/2$ or $m+1$ as mentioned earlier.

THEOREM 1.3.16 : For any Walecki tournament of order $n=2m+1$, $LT_n(s_0,s_1, \ldots ,s_{m-1})$, we can remove an arc (x_i,x_{i+1}) from each Walecki circuit C_i such that $P=(x_0,x_1, \ldots ,x_m)$ forms a path.

PROOF : Let $a = \lceil (m-1)/2 \rceil - k$ where k and k_i's are defined in the discussion immediately preceding Lemma 1.3.14.

Let

$$x_{2j} = x_0 + 2j + (-1)^{ma+1} \cdot (j-k_{2j}),$$
$$\text{for } j=0, \ldots ,\lceil (m-1)/2 \rceil,$$

and

$$x_{2j+1} = x_0 + (2j+1) + 2\cdot\lceil(m-1)/2\rceil + (-1)^{ma+1}\cdot(\lceil(m-1)/2\rceil - j - k + k_{2j+1}) + \rho(m,a), \quad \text{for } j=0, \ldots, \lfloor(m-1)/2\rfloor,$$

where $\rho(m,a) = \begin{cases} 1 & \text{if } m \text{ is even and } a \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$

The claim is that with an appropriate choice of x_0 such that $(x_0,x_1)\epsilon C_0$, then $(x_i,x_{i+1})\epsilon C_i$ for $i=1, \ldots, m-1$ and $P=(x_0,x_1, \ldots ,x_m)$ forms a path. To prove this claim, we need to prove the following:

i) There exists an x_0 such that $(x_0,x_1)\epsilon C_0$

ii) $x_{i+2} = x_i + 2 + (-1)^{\lambda(i)}\cdot|1-k_{i+2}+k_i|$
where $\lambda(i)=l(x_i,x_{i+1})$,

iii) $3 \le l(x_i,x_{i+1}) \le 2m-1$, $i=0, \ldots ,m-1$, and $m \ge 5$

iv) All x_i's are distinct

By lemma 1.3.15, (i), (ii) and (iii) imply that $(x_i,x_{i+1})\epsilon C_i$ for $i=0, \ldots ,m-1$ and (iv) implies P is indeed a path.

<u>Proof of (i) :</u> Finding x_0 amounts to finding an arc in C_0 with length $l(x_0,x_1)$. Since C_0 contains exactly one arc of each length from 1 to n-2, and (iii) $3 \le l(x_0,x_1) \le 2m-1$, then there must exist an arc $(u,v)\epsilon C_0$ such that $l(u,v)=l(x_0,x_1)$. Thus we can set $x_0=u$.

Proof of (ii) : We know that

$$x_{2j+2}-x_{2j} = 2 + (-1)^{ma+1}\cdot(1-k_{2j+2}+k_{2j}) \quad \text{for } j=0, \ldots, \lceil (m-1)/2\rceil -1,$$

and

$$x_{2j+3}-x_{2j+1} = 2 + (-1)^{ma+1}\cdot(-1+k_{2j+3}-k_{2j+1}) \quad \text{for } j=0, \ldots, \lfloor (m-1)/2\rfloor -1.$$

This implies

$$x_{i+2} = x_i + 2 + (-1)^{ma+1+i}\cdot(1-k_{i+2}+k_i) \quad \text{for } i=0, \ldots, m-2.$$

To prove (ii) now, it suffices to show that

$$\lambda(i) \equiv \begin{cases} ma+1+i \pmod 2 & \text{if } 1-k_{i+2}+k_i>0 \\ ma+i \pmod 2 & \text{if } 1-k_{i+2}+k_i<0. \end{cases}$$

We do not need to check the case where $1-k_{i+2}+k_i=0$, since $x_{i+2}=x_i+2$ when $1-k_{i+2}+k_i=0$ regardless of the values of m, a and i.

Now,

$$\begin{aligned}\lambda(2j) &= 2\cdot\lceil (m-1)/2\rceil + 1 \\ &\quad + (-1)^{ma+1}\cdot(\lceil (m-1)/2\rceil -2j-k+k_{2j+1}+k_{2j}) + \rho(m,a) \\ &= 2\cdot\lceil (m-1)/2\rceil + 1 \\ &\quad + (-1)^{ma+1}\cdot(a-2j+k_{2j}+k_{2j+1}) + \rho(m,a)\end{aligned}$$

whereas

$$\begin{aligned}\lambda(2j+1) &= 1 - 2\lceil (m-1)/2\rceil \\ &\quad + (-1)^{ma+1}\cdot(2j+1+k-k_{2j+1}-k_{2j+2}-\lceil (m-1)/2\rceil) - \rho(m,a) \\ &= 1 - 2\cdot\lceil (m-1)/2\rceil \\ &\quad + (-1)^{ma+1}\cdot(2j+1-k_{2j+1}-k_{2j+2}-a) - \rho(m,a).\end{aligned}$$

Recall that k_i denotes the number of negative 1's in the sequence $\{s_0, s_1, \ldots, s_{i-1}\}$ so that $0 \le (k_{i+2}-k_i) \le 2$. Also $\lambda(i)$ denotes $l(x_i, x_{i+1})$. Consider the following two cases:

(a) If $1-k_{i+2}+k_i>0$, then $k_{i+2}-k_i=0$ so we must have $k_i=k_{i+1}=k_{i+2}$. Hence,

$$\begin{aligned}
\lambda(2j) &\equiv 1+a+\rho(m,a) \pmod 2 \\
&\equiv 1+ma \pmod 2 \\
&\equiv ma+1+2j \pmod 2, \quad \text{and} \\
\lambda(2j+1) &\equiv 1+(1-a)+\rho(m,a) \pmod 2 \\
&\equiv a+\rho(m,a) \pmod 2 \\
&\equiv ma \pmod 2 \\
&\equiv ma+1+(2j+1) \pmod 2.
\end{aligned}$$

Therefore, $\lambda(i) \equiv ma+1+i \pmod 2$.

(b) If $1-k_{i+2}+k_i<0$, then $k_{i+2}-k_i=2$ and we must have $k_i+2=k_{i+1}+1=k_{i+2}$. Hence,

$$\begin{aligned}
\lambda(2j) &\equiv 1+(1+a)+\rho(m,a) \pmod 2 \\
&\equiv a+\rho(m,a) \pmod 2 \\
&\equiv ma \pmod 2 \\
&\equiv ma+2j \pmod 2, \quad \text{and} \\
\lambda(2j+1) &\equiv 1+a+\rho(m,a) \pmod 2 \\
&\equiv ma+(2j+1) \pmod 2.
\end{aligned}$$

Therefore, $\lambda(i) \equiv ma+i \pmod 2$.

Now (a) and (b) imply that

$$(-1)^{ma+1+i}\cdot(1-k_{i+2}+k_i) = (-1)^{\lambda(i)}\cdot|1-k_{i+2}+k_i|$$

and thus $x_{i+2} = x_i + 2 + (-1)^{\lambda(i)}\cdot|1-k_{i+2}+k_i|$.

Proof of (iii) : Let $k_0=0$ and

$$\delta_f=f+k-k_f-k_{f+1}, \qquad f=0, \ldots ,m-1.$$

Since $f,k \geq k_{f+1},k_f$, we have $\delta_f\geq 0$. To show that δ_f is bounded above by m-1, suppose $k\leq m-f-1$. Then

$$\delta_f = f+k-k_f-k_{f+1} \leq f+m-f-1-k_f-k_{f+1} \leq m-1.$$

Otherwise, if $k=m-f-1+c$ where $c>0$, then

$$\begin{aligned}\delta_f &= f+k-k_f-k_{f+1} \\ &\leq f+(m-f-1+c)-c-(c+1)\end{aligned}$$

because c is the least number of negative one's that must appear in the first f s_i's and thus $k_f\geq c$ and $k_{f+1}\geq c+1$, so that $\delta_f \leq m-c-2 \leq m-1$.

Therefore,

$$\lceil (m-1)/2\rceil-m+1 \leq \lceil (m-1)/2\rceil-\delta_f \leq \lceil (m-1)/2\rceil$$

$$-\lfloor (m-1)/2\rfloor \leq \lceil (m-1)/2\rceil-\delta_f \leq \lceil (m-1)/2\rceil.$$

Now,

$$\begin{aligned}l(x_{2j},x_{2j+1}) &= 2\cdot\lceil (m-1)/2\rceil + 1 \\ &\quad + (-1)^{ma+1}\cdot(\lceil (m-1)/2\rceil-2j-k+k_{2j+1}+k_{2j})+\rho(m,a) \\ &= 2\cdot\lceil (m-1)/2\rceil + 1 \\ &\quad + (-1)^{ma+1}\cdot(\lceil (m-1)/2\rceil-\delta_{2j}) + \rho(m,a)\end{aligned}$$

and

$$\begin{aligned}l(x_{2j+1},x_{2j+2}) &= 1 - 2\cdot\lceil (m-1)/2\rceil - \rho(m,a) \\ &\quad + (-1)^{ma+1}\cdot(2j+1+k-k_{2j+2}-k_{2j+1}-\lceil (m-1)/2\rceil) \\ &= 1 - 2\cdot\lceil (m-1)/2\rceil \\ &\quad + (-1)^{ma+1}\cdot(\delta_{2j+1}-\lceil (m-1)/2\rceil) - \rho(m,a).\end{aligned}$$

We have

$$2\cdot\lceil (m-1)/2\rceil + 1 - \lceil (m-1)/2\rceil + \rho(m,a)$$

$$\leq l(x_{2j},x_{2j+1})$$

$$\leq 2\cdot\lceil(m-1)/2\rceil + 1 + \lceil(m-1)/2\rceil + \rho(m,a)$$

or

$$\lceil(m-1)/2\rceil + 1 + \rho(m,a)$$

$$\leq l(x_{2j},x_{2j+1})$$

$$\leq 3\cdot\lceil(m-1)/2\rceil + 1 + \rho(m,a)$$

which implies $3 \leq l(x_{2j},x_{2j+1}) \leq 2m-1$ when $m\geq 5$.

As for $l(x_{2j+1},x_{2j+2})$, we have

$$1 - 2\cdot\lceil(m-1)/2\rceil - \rho(m,a) - \lceil(m-1)/2\rceil$$

$$\leq l(x_{2j+1},x_{2j+2})$$

$$\leq 1 - 2\cdot\lceil(m-1)/2\rceil - \rho(m,a) + \lceil(m-1)/2\rceil$$

or

$$1 - 3\cdot\lceil(m-1)/2\rceil - \rho(m,a)$$

$$\leq l(x_{2j+1},x_{2j+2})$$

$$\leq 1 - \lceil(m-1)/2\rceil - \rho(m,a).$$

In other words,

$$2m + 1 - 3\cdot\lceil(m-1)/2\rceil - \rho(m,a)$$

$$\leq l(x_{2j+1},x_{2j+2})$$

$$\leq 2m + 1 - \lceil(m-1)/2\rceil - \rho(m,a)$$

which in turn implies $3 \leq l(x_{2j+1},x_{2j+2}) \leq 2m-1$ when $m\geq 5$. Thus we have $3 \leq l(x_i,x_{i+1}) \leq 2m-1$ for $m\geq 5$.

<u>Proof of (iv) :</u> To show that all x_i's are distinct, it suffices to show that $x_0,x_2, \ldots ,x_{2\lceil m-1/2\rceil}, x_1,x_3, \ldots ,x_{2\lfloor m-1/2\rfloor+1}$ forms an increasing sequence and the difference between $x_{2\lfloor m-1/2\rfloor+1}$ and x_0 is less than or

equal to 2m-1. From the proof of (ii), we have

$$x_{i+2}-x_i = 2 + (-1)^{ma+1+i}\cdot(1-k_{i+2}+k_i).$$

Since $0 \le k_{i+2}-k_i \le 2$, we have $|1-k_{i+2}+k_i| \le 1$ and $1 \le x_{i+2}-x_i \le 3$. Furthermore,

$$x_1 = x_0 + 2\cdot\lceil(m-1)/2\rceil + 1 + (-1)^{ma+1}\cdot(\lceil(m-1)/2\rceil-k) + \rho(m,a),$$

and

$$\begin{aligned} x_{2\lceil m-1/2\rceil} &= x_0 + 2\cdot\lceil(m-1)/2\rceil + (-1)^{ma+1}\cdot(\lceil(m-1)/2\rceil-k_{2\lceil m-1/2\rceil}) \\ &= x_0 + 2\cdot\lceil(m-1)/2\rceil + (-1)^{ma+1}\cdot(\lceil(m-1)/2\rceil-k). \end{aligned}$$

Therefore, $x_1-x_{2\lceil m-1/2\rceil} = 1 + \rho(m,a)$,

that is, $1 \le x_1-x_{2\lceil m-1/2\rceil} \le 2$. So we have

$$x_0,\ x_2,\ \ldots\ ,x_{2\lceil m-1/2\rceil},\ x_1,\ x_3,\ \ldots\ ,x_{2\lfloor m-1/2\rfloor+1}$$

an increasing sequence. Finally, we will look at $x_{2\lfloor m-1/2\rfloor+1}$ which satisfies

$$\begin{aligned} x_{2\lfloor m-1/2\rfloor+1} &= x_0 + 2\cdot\lceil(m-1)/2\rceil + 2\cdot\lfloor(m-1)/2\rfloor + 1 \\ &\quad + (-1)^{ma+1}\cdot(\lceil(m-1)/2\rceil-\lfloor(m-1)/2\rfloor-k+k) + \rho(m,a) \\ &= x_0 + 2\cdot(m-1) + 1 \\ &\quad + (-1)^{ma+1}\cdot(\lceil(m-1)/2\rceil-\lfloor(m-1)/2\rfloor) + \rho(m,a) \\ &= \begin{cases} x_0+2\cdot(m-1)+1 & \text{if } m \text{ is odd} \\ x_0+2\cdot(m-1)+1-1+\rho(m,a) & \text{if } m \text{ is even} \end{cases} \\ &\le x_0+2m-1, \end{aligned}$$

that is , $x_{2\lfloor m-1/2\rfloor+1} - x_0 \le 2m-1$.

Thus $x_0,x_2,\ \ldots\ ,x_{2\lceil m-1/2\rceil},x_1,x_3,\ \ldots\ ,x_{2\lfloor m-1/2\rfloor+1}$ are all distinct. This finishes the proof of (iv).

As mentioned before, (i), (ii) and (iii) imply that $(x_i, x_{i+1}) \epsilon C_i$ for $i=0, \ldots, m-1$ and (iv) implies that $P=(x_0, x_1, \ldots, x_m)$ is a path, for any $m \geq 5$. For $m<5$, we have $n=3$, 5, 7 and 9 for which we have already seen in Section II that we can in fact remove a single arc from each Hamilton circuit to form a path. This completes our proof of Theorem 1.3.16.

■

For example, consider $LT_{21}(+1,+1,-1,-1,-1,-1,+1,-1,-1,+1)$. We have $n=21$, $m=10$, $\lfloor (m-1)/2 \rfloor = 4$, $\lceil (m-1)/2 \rceil = 5$ and $k=6$. To find P, we have to determine the value of x_0. As described in the proof of (i), finding x_0 is equivalent to finding an arc $(u,v) \epsilon C_0$ which has length $l(x_0, x_1)$. From the proof of (ii) we get

$$l(x_0, x_1) = 2 \cdot \lceil (m-1)/2 \rceil + 1 + (-1)^{ma+1} \cdot (\lceil (m-1)/2 \rceil - k) + \rho(m,a)$$

so for $LT_{21}(+1,+1,-1,-1,-1,-1,+1,-1,-1,+1)$ we must have

$$\begin{aligned} l(x_0, x_1) &= 10 + 1 - (5 - 6) + 1 \\ &= 13. \end{aligned}$$

The only arc in C_0 with length 13 is (15,8), so $x_0=15$, $x_1=8$. Hence we get $x_2=16$, $x_4=19$, $x_6=2$, $x_8=4$, $x_{10}=6$, $x_3=10$, $x_5=11$, $x_7=13$ and $x_9=14$. This path is illustrated in Figure 1.3.2.

Now we have shown that we can remove an arc from every Walecki circuit to form a path and obtain a path decomposition

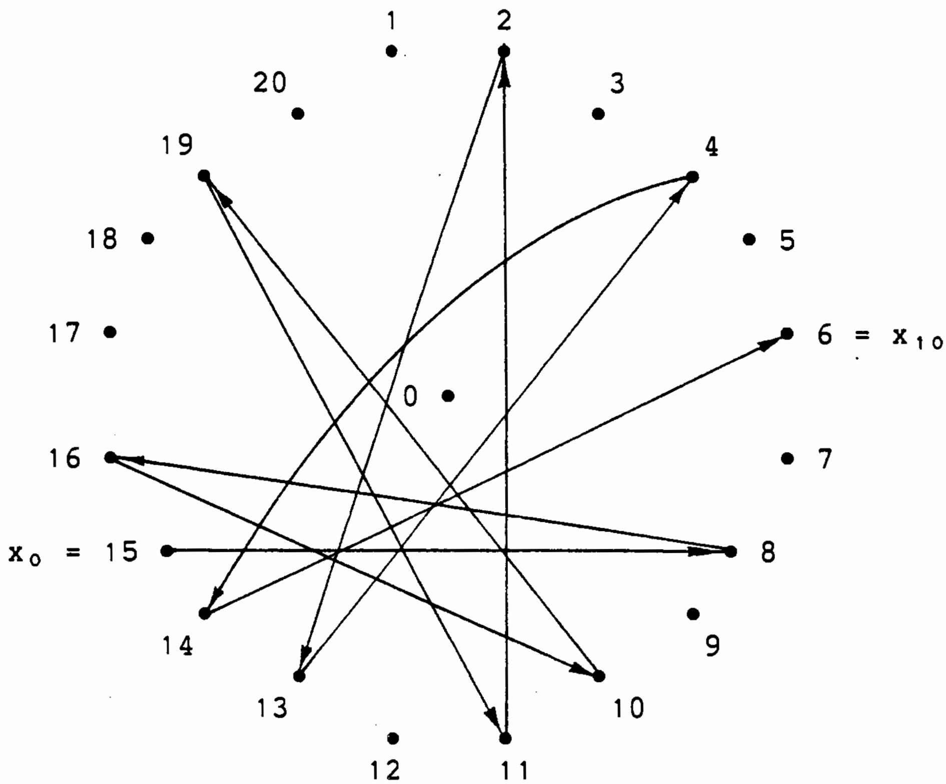


FIGURE 1.32

of size $(n+1)/2$ for $LT_n(s_0, s_1, \ldots, s_{m-1})$ where $n=2m+1$. Since Theorem 1.3.7 shows that this is indeed the lower bound for the path number of any tournament, we must have

$$pn(LT_n(s_0, s_1, \ldots, s_{m-1})) = (n+1)/2.$$

We shall state this as a corollary.

COROLLARY 1.3.17 : For every Walecki tournament $LT_n(s_0, s_1, \ldots, s_{m-1})$ with $n=2m+1$,

$$pn(LT_n(s_0, s_1, \ldots, s_{m-1})) = (n+1)/2.$$

Having seen the above result, a natural question to ask is the following: Is it true that if Kelly's Conjecture holds, then we can just remove one arc from each Hamilton circuit in an HC-decomposition and form a path? (This question was raised by B. Alspach.) If this is true, then we would have Kelly's Conjecture implying that every regular tournament of odd order has path number $(n+1)/2$. The answer to this question is no, because of the following example. Consider $CT_{11}(1,3,4,5,9)$. It has an HC-decomposition:

$$
\begin{array}{c}
(0,1,2,3,4,5,6,7,8,9,10,0)\\
(0,3,6,9,1,4,7,10,2,5,8,0)\\
(0,4,8,1,5,9,2,6,10,3,7,0)\\
(0,5,10,4,9,3,8,2,7,1,6,0)\\
(0,9,7,5,3,1,10,8,6,4,2,0)
\end{array}
$$

as described in Theorem 1.1.2. Each circuit is composed of arcs of a fixed length, so no matter which arc we choose from a circuit, we must have the total length of the path equal to the sum of the elements in the symbol set. The tournament $CT_{11}(1,3,4,5,9)$ has symbol set $\{1,3,4,5,9\}$ and its sum is 22 which is divisible by 11. Thus the arcs removed from the circuits cannot possibly form a path. However, there does exist another HC-decomposition of $CT_{11}(1,3,4,5,9)$ such that we can remove one arc from each circuit to get a single path. The following is one such decomposition:

$$
\begin{array}{c}
(0,\underline{1,2},3,4,5,6,7,8,9,10,0)\\
(0,3,1,4,\underline{2,5},9,7,10,8,6,0)\\
(0,4,9,1,\underline{5,8},2,6,10,3,7,0)\\
(0,5,3,6,9,2,7,1,10,4,\underline{8,0})\\
(\underline{0,9},3,8,1,6,4,7,5,10,2,0)
\end{array}
$$

with (1,2,5,8,0,9) being the path formed.

The last type of tournament that we are going to discuss here is circulant tournaments of prime order $p \geq 3$ with symbol set $\{1,2, \ldots ,p-1/2\}$. We have already seen in Theorem 1.1.2 that such tournaments are HC-decomposable. Here we will show that the above procedure of removing one arc from each circuit to form a path also applies to some HC-decomposition of circulant tournaments of prime order.

THEOREM 1.3.18 : Every circulant tournament of prime order $p \geq 3$ with symbol set $\{1,2, \ldots ,(p-1)/2\}$ has path number $(p+1)/2$.

PROOF : From Theorem 1.1.2 we know that CT_p can be partitioned into arc-disjoint circuits $C_0, C_1, \ldots, C_{(p-3)/2}$ where C_i contains all arcs with length $i+1$.

Let $(v_i, v_{i+1}) \epsilon C_i$. Then the claim is that $(v_0, v_1, \ldots, v_{(p-1)/2})$ forms a directed path. To show this, it is sufficient to show that $\sum_{k=i}^{i+j} k$ is not congruent to 0 modulo p. First, let us look at this sum:

$$\begin{aligned}\sum_{k=i}^{i+j} k &= \sum_{k=0}^{j} k + (j+1)\cdot i \\ &= j\cdot(j+1)/2 + (j+1)\cdot i \\ &= (j+1)\cdot(j/2 + i).\end{aligned}$$

We know that $i,j \leq (p-1)/2$, so that $1 \leq (j+1),(j/2+i) < p$ which implies $(j+1)$ and $(j/2+i)$ are both relatively prime to p. Thus p does not divide

$$\sum_{k=i}^{i+j} k = (j+i)\cdot(j/2+i)$$

and so all v_i's must be distinct and $(v_0, v_1, \ldots, v_{(p-1)/2})$ is a path. Together with $C_0\backslash(v_0,v_1), \ldots, C_{(p-3)/2}\backslash(v_{(p-3)/2},v_{(p-1)/2})$ we have a path partition of $CT_p(1,2, \ldots ,(p-1)/2)$ of size $(p+1)/2$ and by Theorem 1.3.7, $pn(CT_p) = (p+1)/2$.

■

Having seen Theorems 1.3.16 and 1.3.17, one would start to wonder if all regular tournaments of odd order n have path number $(n+1)/2$. This was conjectured by Alspach, Pullman and Mason [3].

CONJECTURE 3 :

Every regular tournament RT_n of odd order n has $pn(RT_n) = (n+1)/2$.

On the other hand, for even order near-regular tournaments we have already seen that every Walecki tournament of even order has path number $n/2$ and from Theorem 1.1.2 that every circulant tournament of prime order satisfies Kelly's Conjecture. Thus, by removing one vertex and all its incident arcs, as described in Lemma 1.1.5, we obtain an even order near-regular tournament NT_{n-1} which is an arc-disjoint union of $(n-1)/2$ Hamilton paths and therefore has path number $(n-1)/2$ (again by Theorem 1.3.7). This leads us to believe that every even order near-regular tournament NT_n has path number $n/2$. By combining the odd and even cases, we get the following conjecture.

<u>CONJECTURE 4 :</u>

Every regular or near-regular tournament of order n has path number $\lfloor (n+1)/2 \rfloor$.

Another interesting question concerning all even order tournaments was raised by O'Brien [17].

<u>CONJECTURE 5 :</u>

Every even order tournament is consistent.

The last two conjectures actually imply Kelly's Conjecture, so they are believed to be very hard problems, whereas Conjecture 3 should be slightly easier (relatively speaking), but apart from the results discussed here little is known.

CHAPTER 2 : PATH DECOMPOSITIONS OF COMPLETE UNDIRECTED GRAPHS

As mentioned in the introduction, the solution to the path number problem for complete graphs is a well known result (see Stanton, Cowan and James [20]). In this chapter, we will simply state this result and move on to a slightly different decomposition problem. First, let us look at the undirected analogue of Theorem 1.3.7.

LEMMA 2.1.1 : For any complete graph K_n of order n,

$$pn(K_n) \geq \lfloor (n+1)/2 \rfloor .$$

PROOF : There are $n \cdot (n-1)/2$ edges in K_n, and the longest path in any path decomposition is of length less than or equal to $n-1$. Thus the least number of paths needed to cover all edges of K_n is $n/2$. But $pn(K_n)$ must be an integer. Therefore, we must have $pn(K_n) \geq \lceil n/2 \rceil$ or $pn(K_n) \geq \lfloor (n+1)/2 \rfloor$.

■

From Lemma 1.3.13 and Corollary 1.3.17, we know that every Walecki tournament of order n has path number $\lfloor (n+1)/2 \rfloor$. So, by removing the orientation on every arc in any minimum path decomposition of LT_n, we get a path decomposition for K_n. This implies that there exists a path decomposition of size

$\lfloor (n+1)/2 \rfloor$ of K_n for all n. Hence by Lemma 2.1.1, for any complete graph K_n of order n we have $pn(K_n) = \lfloor (n+1)/2 \rfloor$ for all n. We shall state this result as a theorem.

THEOREM 2.1.2 : For every complete graph K_n, $pn(K_n)=\lfloor (n+1)/2 \rfloor$.

Since the path number problem is completely solved, we turn our attention to a slightly different version of a path decomposition problem. In this new problem, we are more concerned with the existence of a certain path partition than the size of the partition.

<u>CONJECTURE 6 :</u>

> The complete graph K_n can be decomposed into paths of lengths $m_1, m_2, \ldots, m_r$ if and only if $1 \le m_i \le n-1$ for $i=1, \ldots, r$ and $\sum_{i=1}^{r} m_i = \binom{n}{2}$.

This is what we referred to as "the path arboreal problem" in the introduction. The term "path arboreal" is due to Slater [19]. It is not hard to see that $1 \le m_i \le n-1$ for $i=1, \ldots, r$ and $\sum_{i=1}^{r} m_i = \binom{n}{2}$ are necessary conditions, however the sufficiency part is not as obvious. This problem is also stated by Tarsi in [21]. He formulated the general problem as follows.

CONJECTURE 7 :

Let $M=\{m_1,m_2, \ldots ,m_r\}$ be a sequence of natural numbers which satisfies $m_i \le n-1$ for $1 \le i \le r$ and $\sum_{i=1}^{r} m_i = \lambda \cdot \binom{n}{2}$. Then there exists a sequence of paths $P_1,P_2, \ldots ,P_r$ of lengths $m_1,m_2, \ldots ,m_r$ such that every edge of K_n belongs to exactly λ of them. Such a sequence of paths is called a $P_M[\lambda,n]$.

He also proved the following results based on techniques using Walecki's construction (see Lemma 1.1.1).

THEOREM 2.1.3 : Let n be odd or λ even, and $M=\{m_1,m_2, \ldots ,m_r\}$ a sequence of natural numbers with

$$1 \le m_i \le n-3 \quad \text{and} \quad \sum_{i=1}^{r} m_i = \lambda \cdot \binom{n}{2}.$$

Then there exists a $P_M[\lambda,n]$.

Now we return to the path arboreal problem. As we can see, it is just a special case of Conjecture 7 with $\lambda=1$ and as Theorem 2.1.3 suggests, if n is odd and all the m_i's are less than or equal to $n-3$, then we are done. What remains to be shown are the cases when (i) some $m_i \ge n-2$ for odd n, and (ii) n is even. For case (i), we can improve Theorem 2.1.3 to $m_i \le n-2$ as shown in Lemma 2.1.4, while Lemmas 2.1.5, 2.1.6 and 2.1.7 show us some partial results when we actually have some $m_i > n-2$, that is, $m_i=n-1$. As for case (ii), little is known. We begin by improving the result stated in Theorem 2.1.3.

LEMMA 2.1.4 : Given $n=2m+1$, and natural numbers $m_1, m_2, \ldots, m_r$ such that $m_i \leq n-2$ for $i=1, \ldots, r$ and $\sum_{i=1}^{r} m_i = \binom{n}{2}$, then K_n can be decomposed into paths $P_1, P_2, \ldots, P_r$ of lengths $m_1, m_2, \ldots, m_r$, respectively.

PROOF : Recall from Lemma 1.1.1 that for $n=2m+1$, K_n can be partitioned into m Hamilton cycles $\{C_0, C_1, \ldots, C_{m-1}\}$ such that

$$C_i = \langle 0, i+1, i+2, i+n-1, i+3, \ldots, i+m, i+m+2, i+m+1, 0 \rangle$$

Then $E=\langle C_0, C_1, \ldots, C_{m-1} \rangle$ forms an eulerian tour of K_n. Now suppose d is the length of the shortest cycle on this tour. If the given m_i's are all less than d, then we can just remove the first m_1 edges in E to form P_1 and the next m_2 edges in $E \backslash P_1$ to form P_2 and so on. Since $d>m_i$, for $i=1, \ldots, r$, $P_1, P_2, \ldots, P_r$ are all paths. Furthermore, $\sum_{i=1}^{r} m_i = \binom{n}{2}$ which is the total number of edges in E, therefore we have decomposed the edge-set of K_n into $P_1, P_2, \ldots, P_r$ such that each P_i is a path. To find the value of d, consider two consecutive C_i's (as depicted in Figure 2.1). We can see that every walk that starts at vertex $i+m+2, i+m+3, \ldots, i+n-1, i+1$ (the top row in

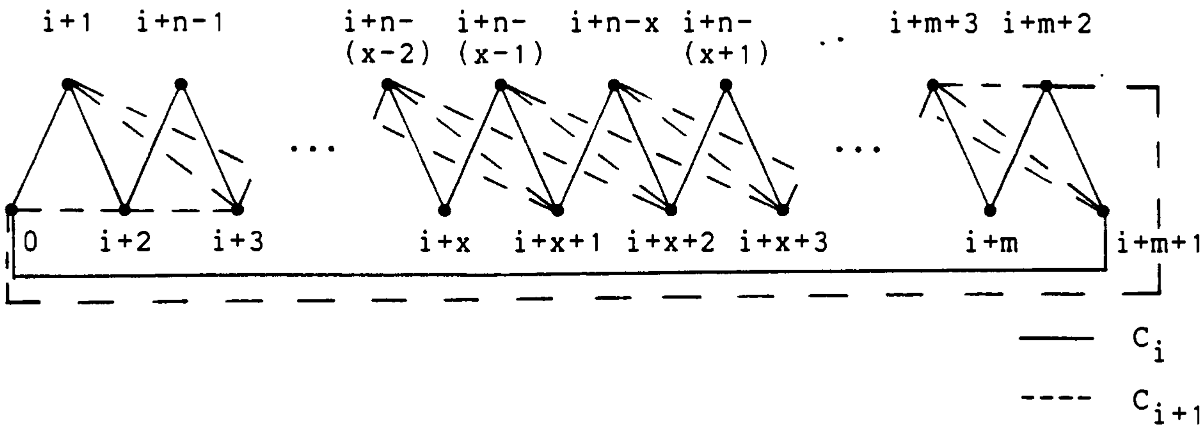


FIGURE 2.1

Figure 2.1) in C_i covers n-1 (in the case of i+1, it is n) distinct vertices, namely i+n-x, ... ,i+m+2 and i+x+2, ... ,i+m+1,0 by edges of C_i and i+1, i+n-1, ... ,i+n-(x-2) and 0, i+2, ... ,i+x+1 by edges of C_{i+1}, before encountering the first repeated vertex, that is, i+x+2. This is illustrated in Figure 2.2 (note that, modulo n-1 arithmetic is used with n-1 replacing 0). Thus every walk of length at most n-2 that starts at vertex i+j, $m+2 \le j \le n$, in C_i is a path. On the other hand, every walk that starts at vertex 0,i+2,i+3, ... ,i+m+1 (the bottom row) in C_i covers n-2 (in the case of 0 and i+2, it is n and n-1, respectively) distinct vertices as shown in Figure 2.3. Thus every walk of length at most n-3 that starts at vertex i+j, $2 \le j \le m+1$, in C_i is a path. From the above observations, we can see that d=n-2. For those m_i=n-2=d, we have a way to avoid choosing m_i edges which are cyclic. The first

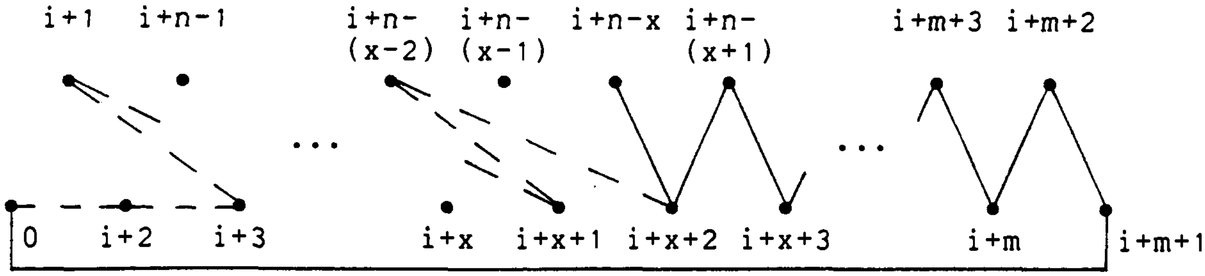


FIGURE 2.2

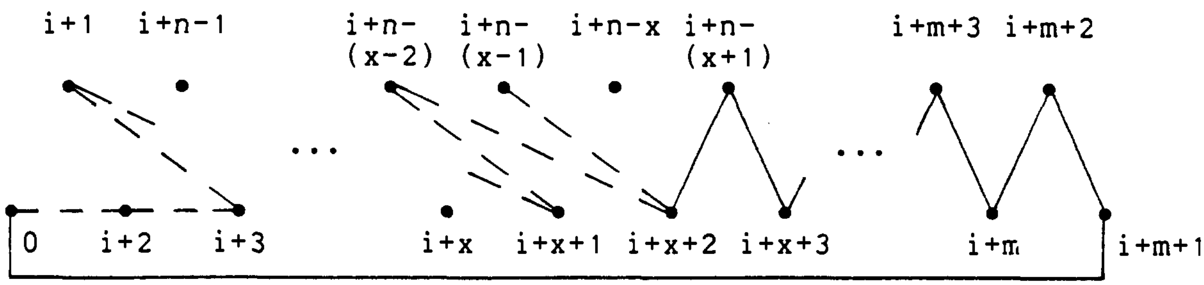


FIGURE 2.3

observation guarantees that if we begin choosing edges for P_1 at vertex $i+j$, $m+2\le j\le n$, along C_i, then the next $m_1=n-2$ edges will form a path. Hence, given $m_i\le n-2$, for $i=1, \ldots, r$, we rearrange the m_i's so that $m_1=m_2= \ldots =m_k=n-2$ and $m_i<n-2$ for $i=k+1, \ldots, r$. We can then remove the first m_1 edges on the tour for P_1, the next m_2 edges for P_2 and so on, since $\sum_{i=1}^{r} m_i = \binom{n}{2}$, all edges on the eulerian tour will belong to some P_i. Each P_i is a path because $P_{k+1}, \ldots, P_r$ are all of size less than n-2 which equals d. Also, $P_1, \ldots, P_k$ are all of size n-2 and except for P_1 which starts at

0 they all start at vertex $m+2(i-1) = (i-2)+(m+i)$ of C_{i-2} for $i=2, \ldots, k$. So by the above observation they are simple paths. Therefore, $P_1, P_2, \ldots, P_r$ are the required paths. This finishes the proof of the theorem. One last note. There are at most $m+1$ m_i's with value $n-2$, thus that last $(n-2)$-path P_k will start at vertex $(k-2)+(m+k)$ of C_{k-2} where $0 \le k-2 \le m-1$ and $2 \le k \le m+1$ so $m+2 \le m+k \le 2m+1$.

∎

Now suppose we have some $m_i=n-1$, we can relabel $\{m_1, m_2, \ldots, m_r\}$ such that $m_1 \ge m_2 \ge \cdots \ge m_r$ with $m_1=m_2= \ldots =m_k=n-1$. Then by pulling out one edge from each of the first k Walecki cycles and attaching it to C_{m-1}, we get the following lemma.

LEMMA 2.1.5 : Given $n=2m+1$, and $\{m_1, m_2, \ldots, m_r\}$ as described in Lemma 2.1.4. If $m_1=m_2= \ldots =m_k=n-1$ and $m_i < n-2k+1$ for $i=k+1, \ldots, r$, then K_n can be decomposed into edge disjoint paths $P_1, P_2, \ldots, P_r$ of lengths $m_1, m_2, \ldots, m_r$ respectively.

PROOF : First we construct the eulerian tour E as in the proof of Lemma 2.1.4. Then we remove the first k Walecki cycles to get $E'=E\backslash\{C_0, \ldots, C_{k-1}\}$. Now we can delete an edge from each of the k Hamilton cycles removed. In particular, $\langle m+1+(k-1),0\rangle$ from C_{k-1}, and $\langle m+2+i,m+1+i\rangle$ from C_i for $i=0, \ldots, k-2$. This gives us k Hamilton paths which we call P_1, P_2, ... ,P_k. Then we attach the deleted edges, which form the path $P=\langle 0,m+k,m+k-1, \ldots ,m+1\rangle$, to the end of E' to form $E''=\langle E',P\rangle$. E" is a trail that uses up the remaining edges. We have already seen in the proof of Lemma 2.1.4 that we can now remove edges from E" to form paths of lengths m_{k+1}, m_{k+2}, ... ,m_r as long as each m_i is less than or equal to n-2. However, this procedure fails when one of the P_i's, say P_j, requires edges from E' and P. This is because the shortest cycle in $\langle C_{m-1},P\rangle$ is of length $2\cdot(m-k)+2$ (see Figure 2.4) or simply n-2k+1. So in order to avoid this cycle, we need to have $m_i \leq n-2k$ for $i=k+1, \ldots ,r$. Then we can just remove the first m_{k+1} edges of E" for P_{k+1}, the next m_{k+2} edges for P_{k+2} and so on. Since $\sum_{i=1}^{r} m_i = \binom{n}{2}$, every edge in E is in exactly one P_i, and P_1, P_2, ... ,P_r are the required paths.

∎

In fact, we can state this result in a slightly stronger manner. Observe that the longest path that we can fit into the end of E" (that is, $\langle C_{m-1},P\rangle$) and uses up all of P is of length $2\cdot(m-k)+k$ or n-k-1 because every walk that begins at vertex i, $i\leq m-k$ or $i\geq m+k+1$, in C_{m-1} is a path (see Figure 2.4). The shortest such path is of length k because P has k edges in it. Therefore, if we can find m_i or a sum of m_i's with its value between k and

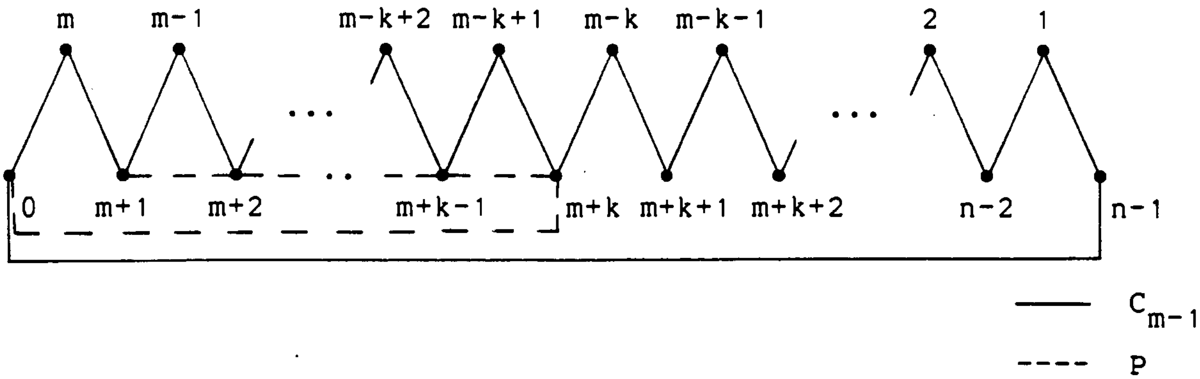


FIGURE 2.4

n-k-1, then we can rearrange the m_j's so that the above mentioned m_i or m_i's appears at the end of the sequence. Because this m_i or m_i's is the last and it must cover edges in C_{m-1} and all of P, by the above observation this P_i or P_i's must form a path or paths. This guarantees that all P_j's are paths. Lemma 2.1.5 can then be improved as follows.

LEMMA 2.1.6 : Given $n=2m+1$, if $m_1=m_2= \dots =m_k=n-1$, $m_i \le n-2$ for $i=k+1, \dots ,r$ and there exists $I \subseteq \{k+1, \dots ,r\}$ such that $\sum_{i\in I} m_i$ satisfies $k \le \sum_{i\in I} m_i \le n-k-1$, then K_n can be decomposed into edge-disjoint paths P_1, $P_2, \dots ,P_r$ of lengths $m_1,m_2, \dots ,m_r$ respectively.

Another way to improve the above result is to use a different choice of edges from the first k Walecki cycles. This is shown in the next lemma. Recall from the proof of Lemma 2.1.5 that if

$k=m$ then the remaining edges, one from each of the m Walecki cycles, form the path P. So we only need to consider the case where $k<m$ in the following lemma.

LEMMA 2.1.7 : Let $n=2m+1$ and $m > k \geq m/2$. If $m_1=m_2= \dots =m_k =n-1$, $m_i \leq n-2$ for $i=k+1, \dots ,r$ and there exists $I \subseteq \{k+1, \dots ,r\}$ such that $k \leq \sum_{i\in I} m_i \leq m+k$, then K_n can be decomposed into edge-disjoint paths $P_1, P_2, \dots ,P_r$ of lengths $m_1, m_2, \dots ,m_r$, respectively.

PROOF : As in the proof of Lemma 2.1.5, we construct the eulerian tour E of K_n and then remove the first k Walecki cycles to form the Hamilton paths P_1, P_2, ... ,P_k. But this time we will choose a different set of edges from these cycles. Let $s= \sum_{i\in I} m_i$, then our goal is to find a path of length s which will use up the k edges deleted from the k Walecki cycles and possibly some edges from the end of C_{m-1}. By doing so, the remaining portion of the tour can be used to form the rest fo the P_i's, since the corresponding m_i's are less than or equal to n-2, and the trail starts at the beginning of C_k.

Let $t=s-k$. Since $k \leq s \leq m+k$, we have $0 \leq t \leq m$. In fact, we can assume $t>0$, since otherwise, we can just follow the proof of Lemma 2.1.5 to get P, and P would be the required path. We can also assume that $m \geq 3$, for $m=1$ or 2 the answer is obvious. We have three cases:

a) If t is even, then $2 \le t \le m$ which implies $0 \le t/2-1 \le m/2-1 \le k-1$. We first remove $<i+1,i+2>$ from C_i for $i=t/2-1, \ldots, k-1$. Then if $t/2-1 \ge 1$, we remove $<(t/2-2)+m+1,0>$ from $C_{t/2-2}$ and furthermore if $t/2-1>1$, then we remove $<i+m+2,i+m+1>$ from C_i for $i=0, \ldots, t/2-3$. These k edges together with the last t edges of C_{m-1} form a path (as depicted in Figure 2.5(a)). One point that needs to be verified in this case is $m+t/2-1<2m+1-t/2=n-t/2$, but this follows from the fact that $t \le m < m+2$.

b) If t is odd and $m-(t+1)/2>k-1$, then we begin by removing $<i+m+2,i+m+1>$ from C_i for $i=0, \ldots, k-2$ and then $<(k-1)+m+1,0>$ from C_{k-1}. Again these k edges together with the last t edges of C_{m-1} form a path (see Figure 2.5(b)). Here we need to verify that $m+k<2m+1-(t+1)/2=n-(t+1)/2$. But this follows from the assumption that $m-(t+1)/2>k-1$.

c) Finally, if t is odd and $m-(t+1)/2 \le k-1$, then we choose $<i+m+2,i+m+1>$ from C_i for $i=0, \ldots, m-(t+1)/2-1$ and also $<0,1+(k-1)>$ from C_{k-1}.

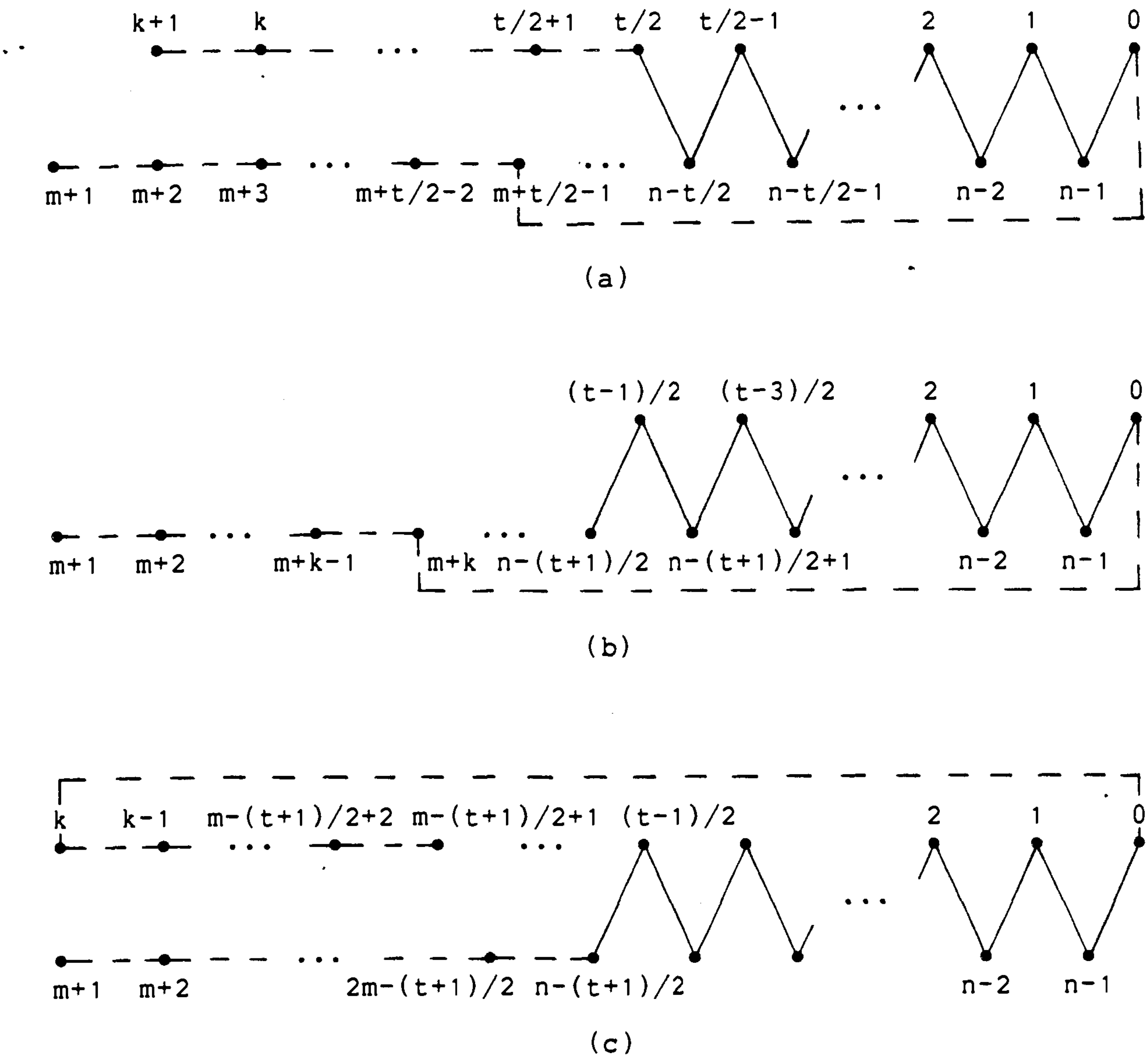


FIGURE 2.5

Now if $m-(t+1)/2<k-1$, then we delete $\langle i+1,i+2\rangle$ from C_i for $i=m-(t+1)/2, \ldots ,k-2$. These k edges will form a path with the last t edges from C_{m-1}. To show that, we need to verify that $m+1-(t+1)/2>(t-1)/2$, this follows from $t \le m < m+1$.(see Figure 2.5(c)).

∎

These three lemmas together tell us that for any k, $1 \le k \le m$, such that $m_1 = m_2 = \dots = m_k = n-1$, and $m_i \le n-2$ for $i=k+1, \dots, r$, if there exists $I \subseteq \{k+1, \dots, r\}$ satisfying $k \le \sum_{i \in I} m_i \le \max\{m+k, n-k-1\}$, then we can partition K_n into paths of lengths $m_1, m_2, \dots, m_r$, respectively. Unfortunately, there are examples that violate these conditions. For example, n=21, k=4, r=11 and $\{m_1, m_2, \dots, m_r\}$ = {20, 20, 20, 20, 19, 19, 19, 19, 18, 18, 18} where $20 \ge m_i \ge 21-4-1=16$ for all i and $\sum_{i=1}^{11} m_i = \binom{21}{2}$.

As mentioned earlier, not much is known about the even order case. The only result was stated by Tarsi [21].

LEMMA 2.1.8 : For even n, if $m_1 = m_2 = \dots = m_{r-1}$ adn $m_r \le m_{r-1} \le n-1$ with $\sum_{i=1}^{r} m_i = \binom{n}{2}$, then we can partition K_n into paths of lengths $m_1, m_2, \dots, m_r$ respectively.

The path arboreal problem on the whole is still very much open. Little has been done other than the few results stated here.

APPENDIX A : GLOSSARY OF SYMBOLS

$\cup$	union	
$\in$	an element of	
$\subset$	proper subset	
$\subseteq$	subset	
$\setminus$	set-theoretic difference	
$\lceil x \rceil$	least integer $\geq x$	
$\lfloor x \rfloor$	largest integer $\leq x$	
$\lvert S \rvert$	cardinality of set S	
$\equiv$	congruent	
$id(v)$	in-degree	2
$od(v)$	out-degree	2
$l(u,v)$	length of the arc (u,v)	3
$l\langle u,v\rangle$	length of the edge $\langle u,v\rangle$	3
$deg(v)$	degree of v	3
$\mu(v)$	$\max\{id(v),od(v)\}$	3
$pn(G)$	path number of G	4
$x(v)$	excess of v	33
$X(G)$	excess of G	33
M	tournament matrix	20
C_i	Walecki cycle/circuit	9
T_n	tournament of order n	3
$\bar{T}_n$	complement of T_n	35
RT_n	regular tournament of order n	3
NT_n	near-regular tournament of order n	3
LT_n	Walecki tournament of order n	11

APPENDIX B : LIST OF CONJECTURES AND THEOREMS

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